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Kenza ZEGHARI

Quantum mechanical aspects of Homogeneity and Isotropy: Applications to solid Inflation

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Liste de publications et participation aux conférences

Liste des publications et/ou brevets réalisées dans le cadre du projet de thèse:

1. A. Nicolis, F. Piazza and K. Zeghari, **"Rotating cosmologies: classical and quantum"**, JCAP 10 (2022) 059, arXiv:2204.04110 [hep-th].

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1. **"Rotating cosmology"** (poster), *Cosmo Rio 2022*, Rio de Janeiro, 22 – 26 Aug 2022.

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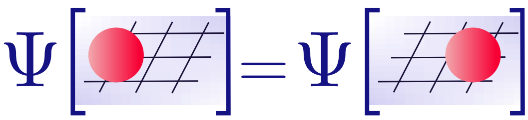
3. **"Cosmic rotation in solid inflation"** (poster + proceeding), *Rencontres de Moriond*, La Thuile, Italie, 23 - 30 Jan 2022.

4. **"Rotating inflation"** (séminaire), *Colloque nationale de Dark Energy, 5ème édition*, Institut Henri Poincaré, Paris, France, 13 - 14 Oct 2021.

Résumé

L'un des grands objectifs de cette thèse est de comprendre les symétries de l'espace-temps (*e.g.* l'homogénéité et l'isotropie) au niveau quantique, *i.e.* au niveau de la fonction d'onde de l'univers. Une première approche serait d'essayer de quantifier des métriques qui sont déjà symétriques. Ceci conduit aux modèles dits de *mini-superspace.* Cependant, nous savons que les fonctions d'onde qui sont homogènes et isotropes dans les théories des champs standards, (typiquement, celles de l'état fondamental), ont un support non nul sur *toutes* les configurations de champ, même celles qui ne sont pas homogènes et isotropes. La fonction d'onde est *e.g.* homogène si elle associe une même amplitude de probabilité pour toutes les versions translatées d'une configuration donnée. Schématiquement,



Au niveau de la gravité quantique, cependant, il n'est pas évident de définir les translations et les rotations en l'absence d'une métrique d'arrière-plan. En raison de leur complexité potentielle, nous avons abordé ces problèmes dans des modèles qui sont homogènes dès le niveau classique : les "not-so"-mini-superspaces, correspondants à la quantification des *métriques de Bianchi.* Dans ces configurations simplifiées, nous avons étudié comment imposer au moins l'*isotropie* au niveau quantique.

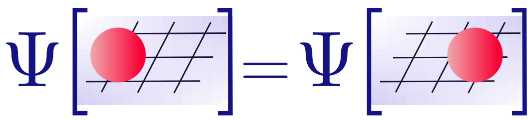
Durant cette quête, mes collaborateurs et moi-même avons découvert d'intéressantes solutions classiques d'*univers en rotation.* Nous montrons que cela n'est possible qu'en présence de matière générant des contraintes anisotropes. En particulier, nous avions considéré un "solide" comme source de matière [1]. La théorie effective de champ d'un solide implique un nombre (égal à la dimension spatiale) de champs scalaires qui "étiquettent" les éléments du volume infinitésimal. Nous passerons brièvement en revue les modèles d'*inflation solide* où le solide est utilisé pour générer l'expansion accélérée de l'univers. Ces modèles sont connus pour ne pas être très efficaces dans la dilution de l'anisotropie, en comparaison avec l'inflation standard. Tout en confirmant ce fait, l'étude au coeur de cette thèse trouve une autre caractéristique potentielle de l'inflation solide, à savoir une "rotation" des axes principaux de l'expansion. Une telle rotation n'est pas seulement un artefact de jauge comme dans le cas des modèles de Bianchi seuls ou couplés à des champs scalaires homogènes. En raison de la contrainte anisotrope générée par le solide, la rotation devient une véritable réelle quantité dynamique.

Le traitement quantique de ce modèle révèle des ambiguïtés intéressantes lorsqu'on opère une réduction des degrés de liberté type mini-superspace. La structure de l'opérateur Laplacien appliqué à la fonctionnelle d'onde semble dépendre intrinsèquement du nombre de champs présents dans la théorie, ou des symétries imposées classiquement avant quantification. Il en résulte une potentielle incohérence au niveau quantique : en effet, implémenter l'isotropie au niveau classique (cosmologies de Friedmann-Lemaître-Robertson-Walker) ne conduit pas nécessairement à la limite isotropique des modèles de Bianchi quantifiés.

Mots-clés : gravité quantique, cosmologie, équation de Wheeler-DeWitt, symmétries de l'espace-temps, modèles de Bianchi, inflation solide

Abstract

One broad aim of this thesis is to understand spacetime symmetries (*e.g.* homogeneity and isotropy) at the quantum level *i.e.* at the level of the wavefunction of the universe. One obvious possibility would be to try to quantize metrics which are already symmetric. This leads to the so-called *mini-superspace models*. However, we know that the wavefunctions that are homogeneous and isotropic in standard field theories (typically, those of the ground state), have support on *all* field configurations, even those which are *not* homogeneous and isotropic. The wavefunction is *e.g.* homogeneous if it associates a probability amplitude that is the same for all the translated versions of a given configuration. Schematically,



At the full quantum gravitational level, however, it is not clear how to define translations and rotations in the absence of a background metric. Because of their potential complexity, we have tackled these problems in models which are already homogeneous from the onset (*i.e.* the classical configurations are already homogeneous). These “not-so”-mini-superspace models correspond to quantization of *Bianchi metrics*. In these simplified setups we studied how to impose at least *isotropy* at the quantum level.

As a classical by-product of our quest, my collaborators and I have discovered interesting classical solutions of *rotating universes*. We show that this is possible only in the presence of matter with anisotropic stresses. In particular, we consider a “solid” as the matter source of a the cosmology we have studied [1]. The effective field theory of a solid involves a number (equal to the spatial dimension) of scalar fields that label the infinitesimal volume elements. We will briefly review models of *solid inflation* where the solid is used to ignite the accelerating expansion of the universe. These models are known for not being very efficient in diluting away anisotropy, as compared to standard inflation. While confirming this fact, the study at the core of this thesis finds another potential feature of solid inflation, namely a “rotation” of the principal axes of the expansion. Such a rotation is not just a gauge artifact as in the case of Bianchi models alone or coupled to homogeneous scalar fields. Due to the anisotropic stress generated by the solid, rotation becomes a real dynamical quantity.

The quantum counterpart of this model reveals interesting ambiguities in operating a mini-superspace-like truncations of the degrees of freedom. The structure of the Laplacian operator applied to the wavefunctional seems to intrinsically depend on the number of fields involved or the symmetries one imposes classically before quantization. This results in potential inconsistency at the quantum level: indeed, implementing isotropy at the classical level (Friedmann-Lemaître-Robertson-Walker cosmologies) does not necessarily lead to the isotropic limit of quantized Bianchi models.

Keywords: quantum gravity, cosmology, Wheeler-DeWitt equation, spacetime symmetries, Bianchi models, solid inflation

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Résumé long

A ce jour, deux théories fondamentales, la mécanique quantique et la relativité générale (RG), décrivent notre monde avec une grande précision.

La mécanique quantique est considérée comme étant le cadre standard pour décrire les forces fondamentales de notre univers. Elle est à la base de la *théorie quantique des champs* ou *quantum field theory* (QFT), qui a permis de décrire avec succès la physique des interactions électromagnétiques, fortes et faibles grâce au *modèle standard de la physique des particules* [2, 3].

Notre compréhension moderne de la gravité repose sur la théorie générale de la relativité d'Albert Einstein, qui modélise la gravité comme la dynamique de l'espace-temps et son interaction avec la matière régies par les *équations du champ d'Einstein* ou *Einstein field equations* [4–6].

Les deux théories sont couronnées de succès dans leurs prédictions, avec des confirmations expérimentales de leurs hypothèses. D'une part, la découverte du boson de Higgs en 2012 [7, 8] qui est une particule fondamentale du modèle standard et qui a été prédite dans les années 60 [9–14]. D'autre part, la détection d'ondes gravitationnelles en 2015 [15, 16], prédite un siècle plus tôt [17], s'ajoute aux nombreux tests que la RG a passés avec succès.

L'une des principales préoccupations des physiciens.ennes d'aujourd'hui est de formuler une théorie de la gravité quantique *i.e.* une théorie qui décrit le comportement quantique du champ gravitationnel.

Malgré de nombreuses tentatives dans cette quête, aucune théorie complète de la gravité quantique et aucune prédiction réussie n'ont encore été réalisées à ce jour.

Ce problème est cependant strictement lié au comportement de la théorie à haute énergie. Une façon simple de le voir est d'estimer l'amplitude de probabilité gravitationnelle $\mathcal{A}$ d'un processus $2 \to 2$ à l'énergie au centre de masse E. Sur le plan dimensionnel, en unités $\hbar = c = 1$, nous avons

$$\mathcal{A} \sim GE^2\,, \tag{0.1}$$

où G est la constante gravitationnelle. Il est clair que cette formule "arborescente" ou *tree-level* que l'on trouve dans la gravité d'Einstein n'a pas de sens à des énergies proches de l'échelle de Planck $M_P = G^{-1/2}$, puisqu'elle conduit à des valeurs potentiellement $\mathcal{A} > 1$. C'est là que les boucles de *gravitons* deviennent importantes et qu'une nouvelle théorie au-delà de la RG est nécessaire. Mais si nous restons à bonne distance de ces énergies, il est possible d'avoir un aperçu des effets quantiques de la gravité (d'Einstein) en inspectant les domaines physiques où ces échelles sont approchées - par exemple, près des singularités des trous noirs et pendant les toutes premières étapes de l'évolution de l'univers. Comme il semble peu probable que nous puissions un jour construire une expérience générant des énergies de l'ordre de $10^{-19} GeV$, nous pourrions envisager de tester la gravité quantique indirectement, en étudiant la physique de l'univers primordial.

En cosmologie, il est courant d'effectuer des calculs perturbatifs de gravité quantique à l'échelle de l'inflation primordiale, la courbure pouvant atteindre $\sim 10^{-5} M_P$. Dans ces calculs, le champ gravitationnel se mélange aux degrés de liberté de la matière (*inflaton*). Des corrélateurs à N points peuvent être calculés et le résultat est censé pouvoir être testé par observation en examinant les fluctuations du *fond diffus cosmique* ou *cosmic microwave background* (CMB) et les *structures à grande échelle* (LSS ou *large scale structures*) que nous observons dans l'univers aujourd'hui. Selon la théorie standard, ces inhomogénéités ont été ensemencées par les fluctuations primordiales générées pendant l'inflation. Par conséquent, toutes les fonctions de corrélation ultérieures qui décrivent la distribution des galaxies ou les fluctuations de température dans le CMB peuvent déterminer les fonctions de corrélation des champs quantiques calculées à la fin de l'inflation. L'étude de ces fonctions aiderait non seulement à comprendre la structure mathématique de l'inflation, mais nous permettrait également d'appréhender la physique des hautes énergies loin de notre portée, et peut-être même d'obtenir des indices sur une théorie de la gravité quantique.

L'étude des observables mécaniques quantiques en cosmologie est plus vivante que jamais avec le récent programme "*cosmological boostrap*" [18–20] qui vise à appliquer les règles générales de la *matrice S* analytique à la fonction d'onde cosmologique.

De manière encore plus audacieuse, plusieurs approches non perturbatives ont été appliquées à l'univers primordial. Dans ce cas, l'ensemble du champ métrique (et pas seulement ses perturbations) est décrit au niveau d'une *fonction d'onde* qui nous indique avec quelle probabilité une certaine configuration métrique peut être observée. Cette démarche a été initiée il y a plusieurs années, dans le contexte de la quantification canonique de la gravité, et a donné naissance à un tout nouveau domaine de recherche, la "cosmologie quantique".

Cosmologie quantique

La cosmologie quantique consiste à étudier les aspects quantiques de l'univers. L'un des ingrédients clés est l'équation de *Wheeler-DeWitt* (WdW) [21, 22], qui est l'équivalent de l'équation de Schrödinger en mécanique quantique standard, dérivée de la quantification canonique de la relativité générale et qui gouverne ce que l'on appelle la *fonction d'onde de l'univers*, Ψ, voir la Partie II Sec. 3. L'espace de Hilbert correspondant est un espace fonctionnel infini des métriques riemanniennes *spatiales* h_{ij} et tout champ de matière ϕ présent dans la théorie classique, appelé *superspace*.

L'équation de WdW est une équation fonctionnelle hyperbolique du second ordre qui génère l'invariance de la fonction d'onde Ψ par rapport à la paramétrisation temporelle. Elle souffre de nombreux problèmes, dont l'un provient du fait qu'il n'est pas évident de définir les conditions aux limites de Ψ.

Dans les années 80, beaucoup se sont attaqués à ce problème spécifique et deux propositions majeures ont été très populaires à l'époque (et sont encore étudiées aujourd'hui) : la *no-boundary proposal* de Hartle et Hawking [23, 24], et la *tunneling proposal* de Vilenkin [25, 26].

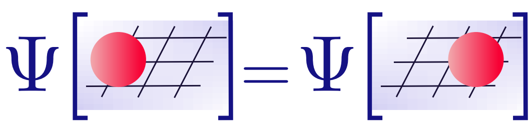
Symétries de l'espace-temps dans la gravité quantique

Étant donné le rôle important joué par l'homogénéité et l'isotropie en cosmologie, et du fait qu'il ne semble pas y avoir de problème à traiter le champ métrique en mécanique quantique, si nous restons à une énergie suffisamment basse, une question naturelle se pose : comment pouvons-nous définir l'homogénéité et l'isotropie au niveau quantique de façon totalement non perturbative ? Une métrique classique donnée peut avoir un certain nombre d'isométries, mais comment caractériser les symétries spatio-temporelles d'une fonction d'onde de métriques $\Psi[h_{ij}]$? Cette question a été l'inspiration principale de cette thèse. Bien qu'elle n'ait pas encore été complètement traitée, plusieurs résultats classiques et quantiques intéressants ont émergé dans la quête d'y répondre. Nous les résumerons bientôt, mais posons d'abord la question plus en détails.

En mécanique quantique, si un état dispose d'une certaine symétrie, il s'agit d'une propriété de la fonction d'onde. Ceci représente un véritable contraste avec la physique classique. Considérons l'exemple simple d'un champ scalaire ϕ dans l'espace de Minkowski. L'état d'un tel champ est *homogène* si la fonction d'onde Ψ satisfait

$$\Psi[\phi(\vec{x}\,)] \;=\; \Psi[\phi(\vec{x}-\vec{a}\,)]\,, \tag{0.2}$$

for every vector $\vec{a}$. Schématiqueùment,



Tout ceci est bien compris dans le cas de champs quantiques définis sur un espace-temps classique donné. Imposer une invariance de Ψ par translation implique que la *N-point correlation function* $\langle\phi(\vec{x}_1)\phi(\vec{x}_2)\ldots\phi(\vec{x}_n)\rangle$ ne dépend que des distances mutuelles, autrement dit la *geométrie de la N-point function*, et non de leur position globale, comme l'illustre la figure ci-dessous.

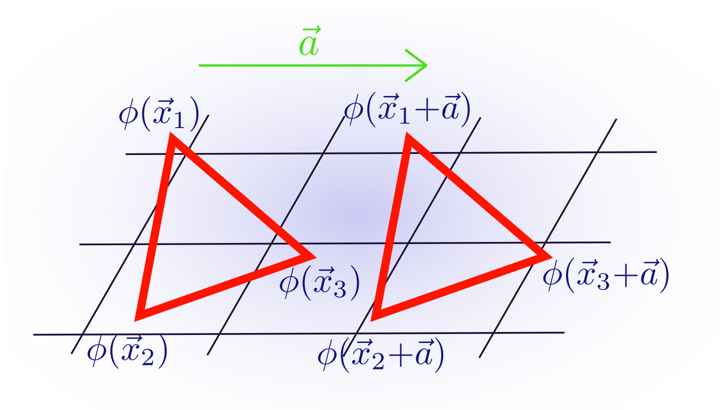


Figure 1.: Space-translation invariance of the spatial correlators.

Remarquez que l'homogénéité classique (*i.e.*, simplement, $\phi(\vec{x}\,) = \text{const.}$) a peu de liens avec les énoncés ci-dessus. Dans la théorie classique des champs, la restriction aux configurations présentant certaines symétries spatio-temporelles implique une troncature massive de l'espace des phases. Une telle troncature n'est pas impliquée par (0.2).

Pour autant que l'on sache, les configurations classiquement homogènes du type $\phi(\vec{x}) = \text{const.}$ pourraient même être absentes de l'ensemble quantique (0.9). En d'autres termes, Ψ pourrait s'annuler sur toute configuration homogène de la métrique, tout en restant elle-même homogène !

Cette troncature est précisément ce qui est habituellement fait en cosmologie quantique, où l'on quantifie une solution classique qui est *déjà* homogène et isotrope : les fameux *mini-superspaces*, Sec. 3.5. Cependant, nous savons que le véritable état quantique contient toutes les configurations possibles, qu'elles soient symétriques ou non. Par conséquent, cette approche correspond à une troncature drastique des degrés de liberté.

Mais quel est alors l'analogue de (0.9) pour la gravité ? Nous devrions pouvoir poser cette question - sans nécessairement y répondre - directement au niveau de la théorie effective à basse énergie. En gravité canonique, un état est une fonctionnelle de la métrique tridimensionnelle $h_{ij}(\vec{x})$ et des champs de matière. La propriété exposée dans l'équation (0.9) n'a manifestement pas de sens lorsqu'elle est appliquée au champ métrique lui-même, car il n'existe pas de notion de "métrique translatée $h_{ij}(\vec{x} - \vec{a})$" en l'absence d'un espace-temps classique de fond : les coordonnées $\vec{x}$ sont arbitraires et la *contrainte de quantité de mouvement* ou *momentum constraint* garantit que la fonction d'onde ne dépend que de quantités invariantes. Par conséquent, un état quantique n'est pas localisé "quelque part" dans l'espace. Alors, comment caractériser les translations pour une métrique générique (sans isométries) de manière indépendante des coordonnées ?

Il est clair que nous avons besoin d'un type de champs de matière qui puisse servir de "cadre de référence spatiale". Les champs ne seraient alors pas localisés sur un fond fixe, mais localisés les uns par rapport aux autres. Un champ de matière particulier se distingue à cet égard : *solides*. Les solides ont été utilisés comme champs de matière conduisant une phase d'inflation primordiale, dans un modèle cosmologique qui porte le nom de *inflation solide*, [27]. La théorie des champs effectifs de l'inflation solide diffère radicalement du scénario inflationniste standard Sec. 5, dans la mesure où les valeurs de l'espérance du vide des champs scalaires impliqués ne dépendent pas du temps (comme dans les modèles inflationnistes habituels), mais de *space*, Sec. 6. Cela signifie que les difféormorphismes spatiaux sont spontanément brisés. C'est cette particularité que nous pouvons utiliser pour définir des axes spatiaux par rapport auxquels nous "déplaçons" le champ gravitationnel.

En raison de la complexité de l'équation de Wheeler-DeWitt, la tâche initiale consistant à conserver tous les degrés de liberté infinis s'est avérée difficile. C'est pourquoi nous avons abordé ces problèmes dans des modèles qui sont déjà homogènes dès le départ (c'est-à-dire que les configurations classiques sont déjà homogènes) : les modèles de Bianchi. Leur quantification donne lieu à ce que nous avons appelé les "not-so"-mini-superspace, Sec. 3.6. Dans ces configurations simplifiées, nous avons étudié comment imposer au moins l'isotropie au niveau quantique.

Mais avant d'aborder le niveau quantique, examinons quelques solutions classiques intéressantes et inattendues que mes collaborateurs et moi-même avons découvertes dans le cadre de notre recherche. [1].

Cosmologies tournantes

Pour tenter de répondre aux questions que nous avons abordées ci-dessus, le principal modèle que nous avons étudié tout au long de la thèse est le modèle *Bianchi de type I* spatialement plat, couplé à un champ de matière solide. Nous trouvons que les solides brisent le groupe de symétrie $SL(d,\mathbb{R})$ de l'action gravitationnelle en $SO(d)$. Ceci a des conséquences importantes sur les solutions classiques. En particulier, le modèle présente des solutions classiques dont les axes d'expansion sont "en rotation".

Etant donné une métrique de Bianchi générale et spatialement plate,

$$ds^2 = -dt^2 + h_{ij}(t)dx^i dx^j \,, \tag{0.3}$$

il est habituel de choisir des coordonnées telles que h_{ij} soit diagonal. La forme diagonale est particulièrement utile lorsque l'on souhaite étudier les anisotropies. Celles-ci sont géométriquement caractérisées par des taux d'expansion dépendant de la direction. La forme diagonale permet donc d'extraire facilement les taux d'expansion des directions à partir des valeurs propres de la métrique.

Cependant, on peut se demander si h_{ij} doit rester diagonal à tout moment. En l'absence de *stress anisotropique* dans le secteur de la matière, nous pouvons montrer que c'est le cas, en utilisant les équations dynamiques (voir Sec. 3.6). Cependant, dans le cas où la source génère un stress anisotropique, la métrique spatiale ne reste pas diagonale à tout instant t, ce qui correspond à une "rotation" non triviale. Rotation de quoi par rapport à quoi ? Il existe de nombreuses façons de décrire le mouvement. L'une d'entre elles consiste à décomposer la dérivée de la métrique en ses vecteurs propres,

$$\dot{h}_{ij}(t) = \sum_{n=1}^{d} H_n(t)\ \hat{u}_i^{(n)}(t)\ \hat{u}_j^{(n)}(t)\,. \tag{0.4}$$

Les valeurs propres H_n sont des "taux de Hubble" dépendant de la direction, tandis que les vecteurs propres instantanés peuvent être considérés comme les principaux axes d'expansion, ou comme les axes principaux de la *courbure extrinsèque* $K_{ij} \propto \dot{h}_{ij}$ des hypersurfaces de constante t (voir Sec. 2.1 pour une revue de la décomposition $d+1$). Comme $\dot{h}_{ij}$ est symétrique, $\hat{u}_i^{(n)}(t)$ est, à tout moment, une base orthonormée par rapport au produit scalaire de Kronecker standard,

$$\hat{u}_i^{(n)}(t)\,\hat{u}_j^{(m)}(t)\,\delta_{ij} = \delta_{mn}\,. \tag{0.5}$$

Ainsi, la "rotation" signifie la dépendance temporelle de cet ensemble orthogonal d'axes principaux des cosmologies *Friedmann-Lemaître-Robertson-Walker* (FLRW) par rapport au système d'observateurs mobiles $x^i =$ const. qui sont en mouvement géodésique. Si l'on tente de définir de nouvelles coordonnées spatiales x'^i afin de "suivre la rotation", ces coordonnées n'étiquettent plus les observateurs géodésiques.

Dans les Secs. 8 et 9, nous revisitons l'approche habituelle du mini-superspace en mettant l'accent sur les symétries et les lois de conservation associées.

Ces considérations nous amènent à une conclusion logique. Les modèles d'inflation solide contiennent potentiellement un ingrédient qui n'a jamais été considéré auparavant. La rotation ! En effet, les auteurs de [28] avaient découvert que l'inflation solide est beaucoup moins efficace que l'inflation standard pour se débarrasser de l'anisotropie spatiale. Alors que dans l'inflation standard (en 2+1 dimensions), la densité d'énergie associée ρ_{aniso} s'échelonne comme A^{-4} (A étant le facteur d'échelle), dans l'inflation solide, l'échelonnement est supprimé par ce que l'on appelle *paramètre de slow-roll* ϵ, $A^{-2\epsilon}$, et $A^{-4\epsilon}$ selon le régime des anisotropies ($\xi \gg 1$, $\xi \ll 1$). De manière surprenante, la densité d'énergie de rotation s'échelonne comme l'anisotropie dans l'inflation standard *i.e.*A^{-4}. Par conséquent, la contribution de la rotation décroit beaucoup plus rapidement que celle provenant de l'anisotropie.

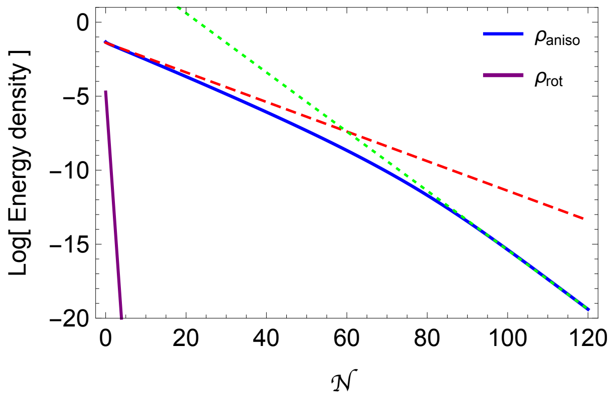


Figure 2.: The energy densities of the anisotropy ρ_{aniso} and the rotation ρ_{rot}, for $\epsilon = 0.05$. The red and green dashed lines correspond resp. to the $e^{-2\epsilon\mathcal{N}}$ and $e^{-4\epsilon\mathcal{N}}$ modes.

Traitement quantique

Dans la Sec. 11, nous procédons au traitement quantique du modèle, qui a été construit à l'origine dans le but d'étudier l'isotropie en mécanique quantique.

Nous trouvons des ambiguïtés intéressantes dans l'exploitation d'une troncature des degrés de liberté semblable à celle d'un mini-superspace. La structure de l'opérateur laplacien appliqué à la fonction d'onde semble dépendre intrinsèquement du nombre de champs impliqués ou des symétries imposées classiquement avant la quantification. En effet, pour un ordre spécifique des champs quantiques, l'opérateur de Laplace correspondant à l'action de Bianchi $(2+1)$ couplée à un champ scalaire ϕ est

$$\nabla_s^2 = \frac{1}{2A^2}\left[-\frac{A^2}{4}\partial_A^2 - \frac{3}{4}A\,\partial_A + \partial_\xi^2 + \frac{1}{\tanh\xi}\partial_\xi + \frac{1}{\sinh^2\xi}\partial_\theta^2 + \partial_\phi^2\right]. \tag{0.6}$$

où A est le facteur d'échelle, ξ et θ les paramètres décrivant les anisotropies et la rotation respectivement. Cependant, si nous considérons un espace FLRW isotrope au lieu de notre modèle de type Bianchi, nous obtenons

$$\nabla_{s,\text{F}LRW}^2 = \frac{1}{2A^2}\left[-\frac{A^2}{4}\partial_A^2 - \frac{1}{4}A\,\partial_A + \partial_\phi^2\right] \qquad (gravity + scalar, FLRW). \tag{0.7}$$

Les solutions classiques de FLRW étant des cas particuliers de modèles de Bianchi, on aurait pu espérer retrouver (0.7) en tant que limite isotrope de (0.6), par exemple lorsque l'état quantique Ψ ne dépend pas de ξ ou de θ. Cependant, nous voyons clairement que les deux opérateurs impliquent une dépendance différente de la fonction d'onde par rapport à A. Nous montrons que pour des paramétrisations plus générales de la métrique, la partie purement gravitationnelle du Laplacien est sensible au nombre total de champs impliqués, ou même simplement au nombre de symétries. L'exemple simple présenté ci-dessus montre que le mini-espace FLRW n'est même pas une troncature cohérente d'un espace de Bianchi !

Structure de la thèse

La thèse est organisée comme suit.

La première partie est consacrée à la dérivation du formalisme hamiltonien classique de la relativité générale et aux contraintes, les contraintes dites *Hamiltoniennes* et de *quantité de mouvement* ou *momentum*.

La partie II traite de la théorie quantique. Nous dérivons l'équation de Wheeler-DeWitt de la quantification canonique de la gravité, et discutons de certaines de ses caractéristiques très intrigantes. Nous introduisons également les modèles de mini-superspace, en particulier ceux dérivés de la quantification des modèles de Bianchi, les "not-so-mini-superspace". Nous détaillons également certaines de leurs caractéristiques géométriques classiques.

La partie III introduit l'inflation, où nous donnons un aperçu des modèles à déroulement lent et de l'inflation solide. Une brève comparaison entre les deux est donnée à la fin.

Enfin, la partie IV est consacrée au principal modèle étudié au cours de la thèse : les univers de Bianchi de type I couplés à des champs de matière solide. Nous décrivons en détail comment un tel modèle génère une rotation des axes principaux d'expansion. Nous fournissons également le traitement quantique du modèle, qui révèle des ambiguïtés intéressantes dans l'exploitation d'une troncature des degrés de liberté à la manière d'un mini-superspace.

Table of contents

Introduction

Contents

So far, two fundamental theories, *quantum mechanics* and *general relativity* (GR) describe our world with a high level of accuracy.

Quantum mechanics is believed to be the standard framework to describe the fundamental forces of our universe. It is the foundation for *quantum field theory* (QFT), which has provided a successful description of the physics of electromagnetic, strong and weak interactions through the *standard model of particle physics* [2, 3].

Our modern understanding of gravity relies on Albert Einstein's general theory of relativity, which models gravity as the dynamics of spacetime and its interaction with matter as governed by the *Einstein's field equations* [4–6].

Both theories are successful in their predictions, with experimental confirmations of their hypotheses. On the one hand, the discovery of the Higgs boson in 2012 [7, 8] , which is a fundamental particle in the Standard Model and which was predicted in the 60's [9–14]. On the other hand, the detection of gravitational waves in 2015 [15, 16], predicted one century earlier [17], adds up to the numerous tests GR has successfully passed.

One of the main concerns of physicists today is to formulate a quantum gravity theory *i.e.* a theory that describes the quantum behavior of the gravitational field. Despite many attempts for this quest, no complete quantum gravity theory and no successful prediction have ever been achieved yet.

This problem is however strictly related with the high energy behavior of the theory. One simple way to see this is to estimate the gravitational probability amplitude $\mathcal{A}$ of a $2 \to 2$ process at energy in the center of mass frame E. On dimensional grounds, in units $\hbar = c = 1$ we have

$$\mathcal{A} \sim GE^2 \,, \tag{0.8}$$

where G is the gravitational constant. Clearly this tree-level formula that one finds in Einstein gravity does not make sense at energies that approach the Planck scale $M_P = G^{-1/2}$, since it leads to potentially $\mathcal{A} > 1$. This is where loops of *gravitons* become important and a new theory beyond GR is needed. But if we keep at a safe distance from those energies it is possible to get a taste of the quantum effects of (Einstein) gravity by inspecting those physical arenas where these scales are approached - for example, near black holes singularities and during the very early stages of evolution of the universe. Since it seems unlikely that we will ever be able to build an experiment generating energies of the order $10^{-19}GeV$, we could instead consider testing quantum gravity indirectly, by studying the physics of the primordial universe.

It is common practice in cosmology to make perturbative quantum gravity calculations at the scales of primordial *inflation*, with curvature reaching as high as $\sim 10^{-5}M_P$. In those calculations the gravitational field mixes with the matter (*inflaton*) degrees of freedom. N-points correlators can be computed and the result is believed to be observationally testable by looking at the *cosmic microwave background* (CMB) fluctuations and the *large scale structures* that we see in the universe today. According to the standard lore, these inhomogeneities have been seeded by the primordial fluctuations generated during inflation. Hence, all later times correlations functions that describe the galaxies distributions or the temperature fluctuations in the CMB, can determine the correlation functions of quantum fields computed at the end of inflation. Studying these would not only help understanding the mathematical structure of inflation, but would also allow us to grasp physics at high energies far from our reach and perhaps hints on a quantum gravity theory.

The study of quantum mechanical observables in cosmology is as lively as ever with the recent "*cosmological boostrap* program" [18–20] that aims at applying the general rules of the analytic S-matrix to the cosmological wavefunction.

Even more venturously, several non-perturbative approaches have been applied to the primordial universe. In this case the entire metric field (and not just its perturbations) is described at the level of a *wavefunctional* that tells us with what probability a certain metric configuration can be observed. This actually was initiated several years ago, in the context of the canonical quantification of gravity, giving rise to a whole new field of research, *quantum cosmology*.

0.1. Quantum cosmology

Quantum cosmology is about studying quantum aspects of the universe. One key ingredient is the *Wheeler-DeWitt equation* (WdW) [21, 22], which is the "Schrödinger-equivalent" equation derived from the canonical quantization of general relativity and governs the so-called *wavefunction of the universe*, Ψ, see Part II Sec. 3. The corresponding Hilbert space is an infinite-dimensional functional space of the Riemannian *spatial metrics* h_{ij} and any matter fields ϕ present in the classical theory, called the *superspace*.

The WdW equation is a second-order hyperbolic functional equation generating time-reparametrisation invariance of the wavefunctional Ψ. It suffers from many issues, one of which stems from the fact that it is not clear how to set the boundary conditions for Ψ.

In the 80s, many tackled this specific issue and two major proposals were very popular at the time (and are still investigated now): the *no-boundary proposal* of Hartle and Hawking [23, 24], and the *tunneling proposal* of Vilenkin [25, 26].

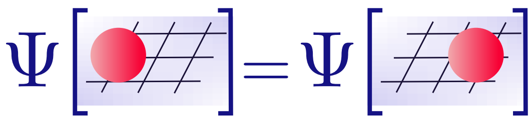
0.2. Spacetime symmetries in quantum gravity

Given the important role played by homogeneity and isotropy in cosmology, and given that there seems to be nothing wrong treating the metric field quantum mechanically, if we stay at sufficiently low energy, one natural question is: how can we define homogeneity and isotropy at the full quantum non-perturbative level? A given classical metric can have a number of isometries, but how do we characterize the spacetime symmetries of a wavefunctional of metrics $\Psi[h_{ij}]$? This question has been the driving inspiration for this thesis. Although it has not been fully addressed yet, while trying to do so, several interesting classical and quantum results have emerged. We will summarize them soon, but first let us pose this question in more details.

In quantum mechanics, if a state enjoys some symmetry, this is a property of the wavefunction. This represents an interesting twist on classical physics. Let us consider the simple example of a scalar field ϕ in Minkowski space. The state of such a field is *homogeneous* if the wavefunction Ψ satisfies

$$\Psi[\phi(\vec{x}\,)] \;=\; \Psi[\phi(\vec{x}-\vec{a}\,)]\,, \tag{0.9}$$

for every vector $\vec{a}$. Schematically,



This is all well understood in the case of quantum fields defined on a given classical spacetime. Imposing an invariance of Ψ under translation implies that the *N-point correlation function* $\langle\phi(\vec{x}_1)\phi(\vec{x}_2)\ldots\phi(\vec{x}_n)\rangle$ depends only on mutual distances, or let's say the *geometry of the N-point function* and not on their overall position, as shown in the figure below.

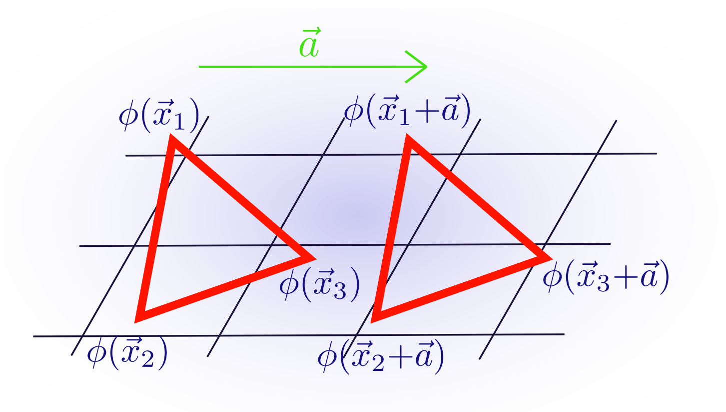


Figure 3.: Space-translation invariance of the spatial correlators.

Notice that classical homogeneity (*i.e.*, simply, $\phi(\vec{x}\,) = const.$) has little to do with the above statements. In classical field theory restricting to configurations with certain spacetime symmetries implies a massive truncation of the phase space. No such a truncation is implied by (0.9).

As far as we know, classically homogeneous configurations of the type $\phi(\vec{x}) = \text{const.}$ could even be absent from the quantum ensemble (0.9). In other words, Ψ could vanish on each and every homogeneous configuration, and still be homogeneous!

This truncation is precisely what is usually done in quantum cosmology where one quantizes a classical solution that is *already* homogeneous and isotropic: the so-called *mini-superspaces*, Sec. 3.5. However, we know that the true quantum state contains all possible configurations, whether they are symmetric or not. Therefore, this approach corresponds to a dramatic truncation of degrees of freedom.

But what is the analogue of (0.9) for gravity? We should be able to pose this question—albeit not necessarily to answer it—directly at the level of the low-energy effective theory. In canonical gravity a state is a functional of the three-dimensional metric $h_{ij}(\vec{x}\,)$ and of the matter fields. The property displayed in eq. (0.9) is clearly meaningless when applied to the metric field itself, because there is no such notion of "translated metric $h_{ij}(\vec{x}-\vec{a})$" in the absence of a background classical spacetime: the $\vec{x}$ coordinates are arbitrary, and the momentum constraint ensures that the wavefunction only depends on invariant quantities. Hence, a quantum state is not localized "somewhere" in space. So, how do we characterize translations for a generic metric (with no isometries) in a coordinate independent fashion?

Clearly, we need a type of matter fields that can serve as a "spatial reference frame". The fields then would be not localized on a fixed background, but localized with respect to one another. A particular matter field stands out for this purpose: *solids*. Solids have been used as matter fields driving a primordial inflationary phase, in a cosmological model that goes under the name of *solid inflation*, [27]. The effective field theory of solid inflation differs drastically from the standard inflationary scenario Sec. 5, in the fact that vacuum expectation values of the scalar fields involved do not depend on time (as in usual inflationary models), but on *space*, Sec. 6. This means that spatial diffeormorphisms are spontaneously broken. This is the particular feature that we can use to kind of define spatial axes with respect to which we "move" the gravitational field.

Because of the complexity of the Wheeler-DeWitt equation, the original task of keeping all the infinite degrees of freedom has proven hard. Therefore, we have tackled these problems in models which are already homogeneous from the onset (*i.e.* the classical configurations are already homogeneous): the *Bianchi models*. Their quantization yields to what we called the "not-so"-mini-superspaces, Sec. 3.6. In these simplified setups we studied how to impose at least isotropy at the quantum level.

But before going into the quantum, let us look at some interesting classical solutions that, as a by-product of our quest, my collaborators and I have discovered. [1].

0.3. Rotating cosmologies

In trying to answer the questions we addressed above, the main model we studied along the thesis is the spatially flat *Bianchi type I* model coupled to a solid matter field. We find that solids break the $SL(d,\mathbb{R})$ symmetry group of the gravitational action down to $SO(d)$. This has far consequences on the classical solutions. In particular, the model presents classical solutions whose axes of expansion are "rotating".

Given a general spatially flat Bianchi metric,

$$ds^2 = -dt^2 + h_{ij}(t)dx^i dx^j\,, \tag{0.10}$$

it is customary to choose coordinates such that h_{ij} is diagonal. The diagonal form is particularly useful when one wants to study anisotropies. These are geometrically characterized by direction-dependent expansion rates. Hence, the diagonal form allows to easily extract the direction expansion rates from the eigenvalues of the metric.

However, one may wonder, does h_{ij} need to stay diagonal at all times? In the absence of *anisotropic stress* in the matter sector, we can show that it does, using the dynamical equations (see Sec. 3.6). However, in the case where the source generates anisotropic stress, the spatial metric does not remain diagonal at all times, and this corresponds to a non trivial "rotation". Rotation of what with respect to what? There are many ways to describe the motion. One way to do so is decompose the derivative of the metric into its eigenvectors,

$$\dot{h}_{ij}(t) = \sum_{n=1}^{d} H_n(t)\ \hat{u}_i^{(n)}(t)\ \hat{u}_j^{(n)}(t)\,. \tag{0.11}$$

The eigenvalues H_n are direction-dependent "Hubble rates", while the instantaneous eigenvectors can be seen as the principal expansion axes, or as the principal axes of the *extrinsic curvature* $K_{ij} \propto \dot{h}_{ij}$ of the hypersurfaces of constant t (see Sec. 2.1 for a review of the $d+1$-decomposition). Because $\dot{h}_{ij}$ is symmetric, $\hat{u}_i^{(n)}(t)$ is, at any time, an orthonormal basis with respect to the standard Kronecker scalar product,

$$\hat{u}_i^{(n)}(t)\, \hat{u}_j^{(m)}(t)\, \delta_{ij} = \delta_{mn}\,. \tag{0.12}$$

So, "rotation" means time-dependence of this orthogonal set of principal axes of *Friedmann-Lemaître-Robertson-Walker* (FLRW) cosmologies with respect to the system of comoving observers $x^i =$ const. who are in geodesic motion. If one tried to define new spatial coordinates x'^i in order to "follow the rotation", such coordinates would not label geodesic observers any longer.

In Secs. 8 and 9, we revisit the usual mini-superspace approach by giving particular emphasis to symmetries and the associated conservation laws.

These considerations bring us to a logic conclusion. The models of solid inflation potentially contain an ingredient that has never been considered before. Rotation! Indeed, authors of [28] had discovered that solid inflation is much less efficient than standard inflation in getting rid of spatial anisotropy. While in standard inflation (in 2+1 dimensions), the associated energy density ρ_{aniso} scales as A^{-4} (A being the scale factor), in solid inflation, the scaling is suppressed by the so-called *slow-roll parameter* ϵ, $A^{-2\epsilon}$ and $A^{-4\epsilon}$ depending on the regime for anisotropies ($\xi \gg 1$, $\xi \ll 1$). Surprisingly, the energy density of rotation scales as the anisotropy in standard inflation *i.e.* A^{-4}. Consequently, the contribution of rotation from rotation decays much faster than that the one coming from anisotropy.

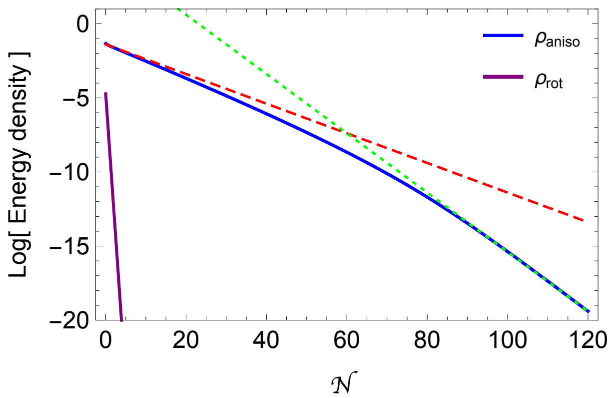


Figure 4.: The energy densities of the anisotropy ρ_{aniso} and the rotation ρ_{rot}, for $\epsilon = 0.05$. The red and green dashed lines correspond resp. to the $e^{-2\epsilon\mathcal{N}}$ and $e^{-4\epsilon\mathcal{N}}$ modes.

0.4. Quantum treatment

In Sec. 11, we proceed to the quantum treatment of the model, which was originally built in the purpose of studying isotropy quantum mechanically.

We find interesting ambiguities in operating a mini-superspace-like truncations of the degrees of freedom. The structure of the Laplacian operator applied to the wavefunctional seems to intrinsically depend on the number of fields involved or the symmetries one imposes classically before quantization. Indeed, for a specific ordering of the quantum fields, the corresponding Laplace operator of the Bianchi action $(2+1)$ coupled to a scalar field ϕ is

$$\nabla_s^2 = \frac{1}{2A^2}\left[-\frac{A^2}{4}\partial_A^2 - \frac{3}{4}A\ \partial_A + \partial_\xi^2 + \frac{1}{\tanh\xi}\partial_\xi + \frac{1}{\sinh^2\xi}\partial_\theta^2 + \partial_\phi^2\right]. \tag{0.13}$$

where A is the scale factor, ξ and θ the parameters describing the anisotropies and rotation resp. However, if we consider an isotropic FLRW space instead of our Bianchi-type model, we get

$$\nabla_{s,\text{F}LRW}^2 = \frac{1}{2A^2}\left[-\frac{A^2}{4}\partial_A^2 - \frac{1}{4}A\ \partial_A + \partial_\phi^2\right] \qquad (gravity + scalar, FLRW)\,. \tag{0.14}$$

Because classical FLRW solutions are special cases of Bianchi models, one might have hoped to recover (0.14) as an isotropic limit of (0.13), for instance when the quantum state Ψ does not depend on ξ or θ . However, we see clearly that both operators imply different dependence of the wavefunction on A . We show that for more general parametrisation of the metric, the pure gravitational part of the Laplacian is sensitive to the total number of fields involved, or even just on the number of symmetries. The simple example displayed above show that the FLRW mini-superspace is not even a consistent truncation of a Bianchi one!

Thesis outline

The thesis is organised as follow.

Part I is devoted to the derivation of the classical Hamiltonian formalism of general relativity and the constraints, the so-called *Hamiltonian* and *momentum* constraints.

Part II deals with the quantum theory. We derive the Wheeler-DeWitt equation from the canonical quantization of gravity, and discuss some of its puzzling features. We also introduce mini-superspace models, in particular those derived from the quantization of the Bianchi models, the not-so-minisuperspaces. We detail some of their classical geometrical features as well.

Part III is dedicated to inflation, where we provide an overview of slow-roll models and solid inflation. A brief comparison between the two is given at the end.

Finally, Part IV is devoted to the main model studied during the thesis: the Bianchi type I universes coupled to solid matter fields. We describe in detail how such a model generates a rotation of the main axes of expansion. We also provide the quantum treatment of the model, which reveals interesting ambiguities in operating a mini-superspace-like truncations of the degrees of freedom.

Part I.

Gravity: aspects of the classical theory

1. Lagrangian formulation

Contents

This section is a very quick reminder of some basics of the Lagrangian formulation. The purpose is to provide tools such as the variational computational techniques to derive the equations of motions, for those who might have forgotten the procedure or are new to the computations. For readers already familiar with these, this section can be skipped.

1.1. Newtonian mechanics

Let's commence with a very quick reminder of the Lagrangian formulation of Newtonian mechanics. The dynamics of a one-dimensional mechanical system is described with a *Lagrangian* $L(q, \dot{q})$ - a function of a generalized coordinate q and its velocity $\dot{q} \equiv \frac{dq}{dt}$ - and an *action functional* $S[q]$,

$$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}) \ , \tag{1.1}$$

that is the integration of the Lagrangian over a selected path $q(t)$. The equations of motion are derived from the extremization of the action, or equivalently, when $S[q]$ is *stationary*: under a variation of the path $\delta q(t)$, restricted by the boundary conditions:

$$\delta q(t_1) = \delta q(t_2) = 0 \ , \tag{1.2}$$

the action does not vary $\longrightarrow \ \delta S = 0$

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt\, \delta L \\ &= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \ \delta \dot{q} \right) \\ &= \left[\frac{\partial L}{\partial \dot{q}} \ \delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \end{aligned} \tag{1.3}$$

where, in the last step, we have integrated by parts. Because of the boundary conditions (1.2), and the variation $\delta q(t)$ being arbitrary in $t_1 < t < t_2$,

$$\delta S = 0 \quad \Longrightarrow \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \,. \tag{1.4}$$

This is the so-called *Euler-Lagrange equation* for a one-particle system and can easily be generalized to higher dimensions.

1.1.1. Field theory

Let's now formulate the field theory version of the Euler-Lagrange equation. For simplicity, we first consider the case of one single scalar-field $\phi(x^\alpha)$ on a curved manifold $\mathcal{M}$ bounded by a closed surface $\partial\mathcal{M}$. We define the *Lagrangian density* $L\left(\phi, \phi_{,\alpha}\right)$ [1], a scalar function of the field and its first derivatives. The action functional is then given by

$$S = \int_{\mathcal{M}} d^{d+1}x \,\sqrt{-g}\, L\left(\phi,\, \partial_\alpha \phi\right) \;, \tag{1.5}$$

where g is the determinant of the metric tensor, and $\sqrt{-g}\, d^{d+1}x$ [2] the $d+1$-volume element.

Following the same scheme we used for finite-dimensional systems, we introduce a variation $\delta\phi(x^\alpha)$ that is arbitrary within $\mathcal{M}$ but vanishes everywhere on $\partial\mathcal{M}$,

$$\delta\phi|_{\partial\mathcal{M}} = 0 \,. \tag{1.6}$$

Then we use the stationarity condition $\delta S = 0$ to derive the dynamical equations for ϕ:

$$\begin{aligned}
\delta S &= \int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\left[\frac{\partial L}{\partial \phi}\,\delta\phi + \frac{\partial L}{\partial \phi_{,\alpha}}\,\delta\phi_{,\alpha}\right] \\
&= \int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\left[\frac{\partial L}{\partial \phi}\,\delta\phi + \nabla_\alpha\left(\frac{\partial L}{\partial \phi_{,\alpha}}\,\delta\phi\right) - \nabla_\alpha\frac{\partial L}{\partial \phi_{,\alpha}}\,\delta\phi\right] \\
&= \int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\left[\frac{\partial L}{\partial \phi} - \nabla_\alpha\frac{\partial L}{\partial \phi_{,\alpha}}\right]\delta\phi + \oint_{\partial\mathcal{M}} d\Sigma_\alpha \frac{\partial L}{\partial \phi_{,\alpha}}\,\delta\phi \,,
\end{aligned} \tag{1.7}$$

where we have used the *Gauss-Stockes theorem* in the last step. The variation $\delta\phi$ being arbitrary within $\mathcal{M}$, we get:

$$\delta S = 0 \quad \Longrightarrow \quad \frac{\partial L}{\partial \phi} - \nabla_\alpha \frac{\partial L}{\partial \phi_{,\alpha}} = 0 \,. \tag{1.8}$$

This the Euler-Lagrange equation for a single scalar field ϕ. The same procedure is applied to vectors, tensors and spinors of any type. A well-known example worth mentioning is the (real) *Klein-Gordon* field ψ with Lagrangian density

$$L = -\frac{1}{2}\left(g^{\alpha\beta}\,\nabla_\alpha\phi\,\nabla_\beta\phi + m^2\phi^2\right) \;. \tag{1.9}$$

[1] Where $\phi_{,\alpha} = \frac{\partial\phi}{\partial x^\alpha}$.
[2] It is implied that we work on manifolds with Lorentzian signatures.

The corresponding Euler-Lagrange equation is given by

$$\left(\nabla^\alpha\nabla_\alpha - m^2\right)\phi = 0\,, \tag{1.10}$$

which turns out to be the curved-spacetime version of the well-known *Klein-Gordon equation*.

1.2. Action of general relativity

Now we finally introduce the action functional of general relativity. It contains a pure gravitational part $S_{\text{grav}}[g]$ depending on the metric field $g_{\alpha\beta}$, and a contribution from any matter fields ϕ, $S_{\text{matter}}[g,\phi]$ [3].

Let's first detail the pure gravitational part. We consider a $(d+1)$-dimensional[4] spacetime $\mathcal{M}$, whose boundary is denoted $\partial\mathcal{M}$.

The $(d+1)$-dimensional gravitational action itself contains the *Einstein-Hilbert action* S_{EH}, a boundary term S_{bound} and a *holographic renormalisation term* S_0 that is non-dynamical, in the sense that it does not affect the equations of motions but only changes the numerical value of the action.

$$S_{\text{grav}}[g] = \frac{1}{16\pi G}\left(S_{\text{EH}}[g] + S_{\text{bound}}[g] + S_0\right) \tag{1.11}$$

$$= \frac{1}{16\pi G}\left(\int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\,R + 2\epsilon\int_{\partial\mathcal{M}} d^d y\,\sqrt{\gamma}\,K + 2\epsilon\int_{\partial\mathcal{M}} d^d y\,\sqrt{\gamma}\,K_0\right) \tag{1.12}$$

where G is the gravitational constant, R the *Ricci scalar* which is the measure of the intrinsic curvature of $\mathcal{M}$. K is the trace of the *extrinsic curvature* of $\partial\mathcal{M}$, which measures the curvature of $\partial\mathcal{M}$ as perceived by an observer in $\mathcal{M}$ [5]. $\epsilon = \boldsymbol{n}\,.\,\boldsymbol{n}$, with $\boldsymbol{n}$ the normal vector of $\partial\mathcal{M}$. It is $+1$ where $\boldsymbol{n}$ is space-like and -1 where $\boldsymbol{n}$ is time-like [6]. γ is the determinant of the induced metric on $\partial\mathcal{M}$. Following the same conventions as Poisson in [30], the coordinates x^α are used for $\mathcal{M}$, and y^α for $\partial\mathcal{M}$.

The *Gibbons–Hawking–York* boundary term S_{bound} needs to be added to the Einstein–Hilbert action, when the underlying spacetime has a boundary. It is the counterterm that makes the variational problem well-defined, and allows to generate the right *Einstein's fields equations* [30], but only when the boundary is space-like or time-like. In the null-like case, see the boundary integral of [29].

The renormalization term S_0 allows to eliminate divergences coming from S_{EH} when the underlying spacetime is asymptotically-flat (see [30]).

For the rest of the thesis, we do not consider S_{bound} and S_0, and simply set theses to 0. The gravitational action is reduced to the Einstein-Hilbert action S_{EH}.

[3] For those willing a more detailed description of the theory, see [4, 6].

[4] The notation "$d+1$" will make more sense when we present the Hamiltonian formulation. d refers to the dimension of the spatial part, while the extra dimension is for time.

[5] We will detail all the hypersurfaces related objects as the extrinsic curvature in sect. 2.2.

[6] We do not consider the case where the boundary is null-like. Actually, the right boundary term in that case has been found recently in [29].

Stress-energy tensor

In GR, the symmetric stress-energy tensor $T_{\mu\nu}$ is a symmetric tensor that describes the density and flux of energy and momentum in spacetime, acting as the source of spacetime curvature. It is derived from the matter sector $S_{\rm matter}$,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}} \ . \tag{1.13}$$

In the case of a minimally-coupled scalar-field ϕ ,

$$\begin{aligned} S_{\rm matter} &= \int_{\mathcal{M}} dt \ L_{\rm matter} && (1.14) \\ &= \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi)\right) \ , && (1.15) \end{aligned}$$

where $V(\phi)$ is the potential. The corresponding stress-energy tensor is

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2}\partial^\alpha \phi \, \partial_\alpha \phi + V(\phi)\right) \ . \tag{1.16}$$

In the case of a perfect fluid in thermal equilibrium,

$$T_{\mu\nu} \ = (\rho + p) \, u_\mu \, u_\nu + p \, g_{\mu\nu} \ , \tag{1.17}$$

with ρ the mass-energy density, p the hydrostatic pressure and u^α the fluid's velocity satisfying $u^\alpha \, u_\alpha = -1$.

Einstein's field equations

The Einstein's field equations are derived from the variation of the action $S_{\rm grav} + S_{\rm matter}$ with respect to all the fields involved in the theory ($\delta g_{\mu\nu}$, $\delta\phi$, ...),

$$\begin{aligned} G_{\mu\nu} &= 8\pi G \, T_{\mu\nu} \ , && (1.18) \\ R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} &= 8\pi G \, T_{\mu\nu} \ , && (1.19) \end{aligned}$$

where $R_{\mu\nu}$ is the *Ricci tensor*, and R the Ricci scalar. These are contractions of the overall *Riemann curvature tensor*, $R^\alpha{}_{\mu\beta\nu}$.

2. Hamiltonian formulation of gravity

Contents

The Lagrangian and Hamiltonian formulations are two equivalent formalisms describing the dynamics of a given system. Each has its own advantages and relevance depending on one's purposes. For instance, the Lagrangian form makes it very simple to satisfy the conditions of special relativity by assuming that the action, or the time integral of the Lagrangian, is Lorentz invariant. There is no straightforward method for "relativizing" the Hamiltonian form.

On the other hand, the Hamiltonian formulation is particularly useful when one wants to extract the symmetries (gauge and physical) of a theory. These are encoded in functions called "constraints". These have been classified and well defined by Dirac in his constraints classification theory [31]. They allow to count the degrees of freedoms of the system. Also, it turns out that it is a suitable framework to derive the quantum theory, by elevating these constraints into operators acting on the *wavefunction* [21].

For now, we review basic features of Hamiltonian mechanics for a simple finite-dimensional system, in order to identify the key elements for a Hamiltonian formulation of general relativity. The quantum treatment will be dealt later in the quantum part.

2.1. Hamiltonian mechanics

The Hamiltonian formulation of any theory starts with the definition of the canonical momenta [1] . As we previously did in 1.1, we start with a simple one-particle system, bearing in mind that the generalisation to a N-dimensional is straight away. The canonical momentum p is defined by

$$p = \frac{\partial L}{\partial \dot{q}} \ . \tag{2.1}$$

When this relation is invertible[2] and give $\dot{q}$ as a function of p and q. The Hamiltonian is then given by the Legendre transformation

$$H(p,q) = p\,\dot{q} - L \ , \tag{2.2}$$

which can be used to recast the action

$$S = \int_{t_2}^{t^1} (p\dot{q} - H)\,dt \ . \tag{2.3}$$

As for the Lagrangian, we can use a variational principle to derive the equations of motions. We ask for the action to be stationary, as we vary q and p *independently*, with the usual boundary conditions $\delta q(t_1) = \delta q(t_2) = 0$. Hence,

$$\begin{aligned} \delta S &= \int_{t_2}^{t_1} \delta\,(p\,\dot{q} - H)\,dt & (2.4)\\ &= \int_{t_2}^{t_1} \left(p\,\delta\dot{q} + \delta p\,\dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p \right) dt & (2.5)\\ &= \Big[p\,\delta q \Big]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[-\left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q + \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p \right] dt. & (2.6) \end{aligned}$$

The variation between t_1 and t_2 being arbitrary, and the boundary conditions being fixed, we obtain the famous *Hamilton's equations*:

$$\delta S = 0 \quad \Longrightarrow \quad \dot{q} = \frac{\partial H}{\partial p} \ , \quad \dot{p} = -\frac{\partial H}{\partial q} \ , \tag{2.7}$$

which are easily generalised to a finite N-dimensional system. Notice that we can relate partial derivatives of the Hamiltonian and the Lagrangian through:

$$\dot{p} = \frac{dp}{dt} = \frac{d}{dt}\frac{\partial L}{\partial q} = \frac{\partial L}{\partial q} \quad \Longrightarrow \quad \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q} \tag{2.8}$$

[1] Actually, the Hamiltonian theory describes the whole dynamics by itself, there is no need for a Lagrangian on the onset. As we already stated, relativistic theories are better formulated in the Lagrangian formulation. However, their Hamiltonian counterpart enables to prescind particular features of the theory, such as the (gauge and physical) symmetries, and count the number of d.o.f.

[2] More details on this statement will be given in the section dedicated to the constrained Hamiltonian systems. General relativity is a constrained theory, and some of the conjugate momenta are not invertible!

We used the action-variational-principle approach to derive the equations of motion. Actually, these can also be obtained using the symplectic structure of the phase space, and the Poisson algebra. We provide an overview of the symplectic geometry in Appendix **??**.

The generalisation to fields is very subtle, and requires more mathematical material, as it involves *functionals* $H[p, q]$ of the field configurations q and the canonical momentum p. That is the purpose of next section, where we provide the needed geometrical tools to tackle the field formulation, in particular for general relativity.

2.2. The $d+1$ decomposition

The Hamiltonian formulation is particularly suitable when one wants to study the dynamics of a system and describe the time-evolution of any "initial conditions", equivalently known as the *initial-value problem*. For field theories, and especially for covariant field theories such as general relativity, the derivation of the Hamiltonian is not straightforward. We need to properly define what would be the "time" with respect to which we study the dynamics and determine the dynamical fields that depend on this time.

However, this means that we would have to break the manifest covariance and choose a preferred set of time in the overall manifold. This is possible by slicing the $(d+1)$-dimensional manifold into a family of space-like d-dimensional hypersurfaces Σ_t, one for each instant of "time".

Arnowitt, Deser and Misner used the geometrical decomposition of spacetime to construct the canonical formulation of gravity, in 1962 [32]. This decomposition is known as the *ADM formalism*, named after the authors.

Beyond its use in studying the dynamics, the slicing is particularly relevant in the formulation of the action principle, as it allows to (more or less) properly define the boundary conditions. Indeed, as stated in *gravitation* [4] Sec. 21.4, the boundary conditions are two faces of a sandwich of spacetime with given d-dimensional geometries, and the dynamics is derived by picking the $(d+1)$-dimensional geometry in between that extremizes the action.

Actually, there is another - and probably more accurate - way to study the dynamics of covariant theories, known as the *the covariant phase space method*. It was developed by [33–37] based on earlier work of [38]. The formalism provides an elegant way to construct the Hamiltonian formulation of Lagrangian field theories without breaking covariance [3] .

Let's proceed to the slicing of the $d+1$ spacetime into constant time t hypersurfaces Σ_t, and describe its dynamics through the time foliation. The first step is to re-write the Lagrangian as a function of appropriate variables as the metric, any spatial derivatives and first time derivatives.

Throughout the manuscript, the greek letters α, β... refer to the overall manifold, while the latin indices i, j, k... are for the hypersurfaces.

[3] The breakthrough that made the covariant phase space formalism popular was the 1987 article of Crnkovic and Witten [33], which gave explicit covariant constructions for the symplectic forms of scalar fields, Yang-Mills theory and General Relativity. See also [39–43] where the authors contributed to the developmnent of the formalism. See as well Halliwell's work regarding the boundary terms and total derivatives[44], where he also provided a nice historical overview of the formalism's evolution from the old Lagrange, Hamilton and Jacobi formalisms, to the construction of the non-covariant phase space formalism.

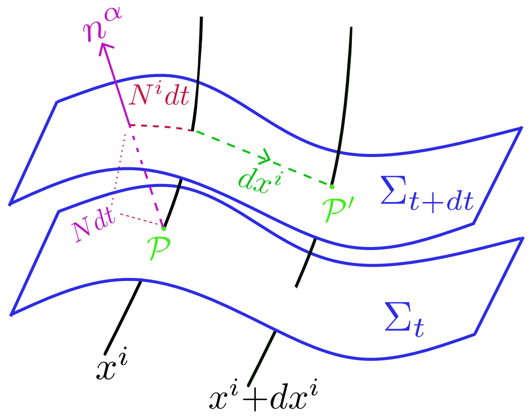


Figure 2.1.: An infinitesiaml thin volume of spacetime delimited by two constant time-slices Σ_t and Σ_{t+dt}. n^α is the unit normal vector to the hypersurfaces, and the curves x^i and $x^i + dx^i$ are constant space coordinates separated by a displacement dx^i. We consider a particle moving from point $\mathcal{P}$ at time t to $\mathcal{P}'$ a bit later at time $t + dt$

As showed in figure 2.1, consider an infinitesimally thin volume of spacetime, delimited by two hypersurfaces Σ_t and Σ_{t+dt}, resp. defined at a constant time t and a bit later time $t + dt$. We also draw two constant spatial trajectories x^i and $x^i + dx^i$, and the unit normal vector $\boldsymbol{n}$ to Σ_t. As we will see in more details later, the constant spatial trajectories are not necessarily *tangent* to the unit normal vector (when it is the case, the coordinates are said to be co-moving), there is a *shift* N^i that deviates the trajectory in the tangent direction, as illustrated in figure 2.1.

Now we follow a particle that moves from the position $\mathcal{P}$ on Σ_t to $\mathcal{P}'$ on Σ_{t+dt}. The proper interval ds between the two points can be computed by decomposing the path along the dashed trajectory in figure 2.1 then use the *generalised Pythagorean theorem*. The first piece is given by the proper time $d\tau = Ndt$ with N being the *lapse* which measures the difference between the proper time and the coordinate t. The spatial distance dl is given by the base-geometry of the hypersurfaces.

We compute ds using the Pythagorean theorem in the Lorentzian signature:

$$ds^2 = -d\tau^2 + dl^2 \tag{2.9}$$

which explicitly is

$$\begin{aligned} ds^2 &= -N^2 dt^2 + g_{ij}\left(N^i dt + dx^i\right)\left(N^j dt + dx^j\right) &(2.10)\\ &= \left(-N^2 + N^i N_i\right) dt^2 + 2N_i\, dt\, dx^i + g_{ij} dx^i dx^j \;. &(2.11)\end{aligned}$$

N and N^i are key objects in the decomposition. They define how the hypersurfaces are *stitched* to each other.

From (2.9), we introduce the following parametrisation for the metric field:

$$g_{\mu\nu} = \begin{pmatrix} sN^2 + N^k N_k & N_i \\ N_i & h_{ij} \end{pmatrix} \tag{2.12}$$

with N, N_i being resp. the lapse and shift; $h_{ij} = g_{ij}$ the metric tensor on the d-dimensional hypersurfaces Σ_t, also called the *first fundamental form*. These are also called the *ADM variables* [32]. s is either equal to $+1$ or -1 depending on the space-time's geometrical signature.

Let's give a detailed geometrical description of each of these elements. We define a set of coordinate basis vectors, $\boldsymbol{e}_t$ and $\{\boldsymbol{e}_i\}_{i=1}^{D}$; with $\boldsymbol{e}_i = \partial/\partial x^i$ tangent to Σ_t, and $\boldsymbol{e}_t = \partial/\partial t$ not tangent to Σ_t.

Any vector $\boldsymbol{v}$ lying in Σ_t is expressed as :

$$\boldsymbol{v} = v^i \boldsymbol{e_i} \ . \tag{2.13}$$

We also define the coordinate 1-forms basis $\{\boldsymbol{d}x^i\}_{i=1}^{d}$ dual to the tangent vector basis. A displacement in Σ_t

2.2.1. The first fundamental form h_{ij}

The spatial metric h_{ij} is a symmetric, linear operation at each point $p \in \Sigma_t$, $\quad h$: $T_p\Sigma_t \times T_p\Sigma_t \to \mathbb{R}$.

It allows to raise and lower the indices of vectors, tensors defined on the surface Σ_t, as long as Σ_t is a not null so that h_{ij} is non-singular. Its inverse is denoted h^{ij}, and is not automatically equal to g^{ij}.

h_{ij} specifies the geometry of each hypersurface Σ_t. The rest of the elements of the overall spacetime metric tensor $g_{tt} = -N^2 + N^k N_k$, $g_{ti} = N_i$ describe how these hypersurfaces are stitched to each other.

2.2.2. The shift N_i

The shift N_i, defined as

$$N_i = \boldsymbol{e}_t \,.\, \boldsymbol{e}_i \ , \tag{2.14}$$

measures how much a particle at constant spatial trajectory x^i deviates from a curve tangent to the normal vector of Σ_t. In particular, the spatial coordinates are comoving when $N_i = 0$.

It is a 1-form - linear mapping : $T_p\Sigma_t \to \mathbb{R}$ - on the space of vectors tangent to Σ_t, $N : T_p\Sigma_t \to \mathbb{R}$. Its contravariant form (vector tangent to Σ_t) is given by:

$$N^i = h^{ij} N_j \ . \tag{2.15}$$

2.2.3. The lapse N

First let's introduce the "future-directed" time-like unit vector normal to the hypersurfaces Σ_t, $\boldsymbol{n}$. Its squared norm is

$$s = \boldsymbol{n} \,.\, \boldsymbol{n} \tag{2.16}$$

where $s = +1$ if the hypersurfaces were defined in the Euclidean-signature, else $s = -1$ for Minkowski-signature. $\boldsymbol{n}$ being normal to these hypersurfaces, for every tangent vector $\boldsymbol{v} \in T_p\Sigma_t$, $\boldsymbol{n} \,.\, \boldsymbol{v} = 0$. Since the tangent vectors $\boldsymbol{v}$ are characterised by $v^t = 0$, therefore $n_i = 0$ and the only non-null component of the unit vector is n_t. This introduces the lapse N, related to the unit vector through

$$n_t = s\,N \;, \qquad N = -\boldsymbol{e}_t \,.\, \boldsymbol{n} \;. \tag{2.17}$$

Let's clarify the geometrical role of N. First notice that

$$sN^{-2} = N^{-2}\boldsymbol{n} \,.\, \boldsymbol{n} = \boldsymbol{d}t \,.\, \boldsymbol{d}t = g^{tt} \;. \tag{2.18}$$

which allows to find the relation between the lapse and the contravariant vector component n^t, by raising the index on $\boldsymbol{n}$.

$$n^t = N^{-1} \;. \tag{2.19}$$

This tells us that an observer moving orthogonal to Σ_t sees a difference between the proper time and the coordinate time t. This difference is measured by the lapse function N, through the relation $d\tau = Ndt$, $d\tau/dt = N$. In co-moving coordinates, $N = 1$.

La section sur les exemples de vecteurs, ajouter plutôt aux sections précédentes.

2.3. The inverse metric

From the previous results, we can derive another key element of the $d + 1$ decomposition, the inverse metric,

$$g^{\mu\nu} = \begin{pmatrix} \dfrac{s}{N^2} & -\dfrac{s\,N^i}{N^2} \\ -\dfrac{sN^i}{N^2} & h^{ij} + \dfrac{sN^iN^j}{N^2} \end{pmatrix} \tag{2.20}$$

For the rest of the chapter, we consider only the Lorentzian signature, $s = -1$.

2.4. The extrinsic curvature

Definition

So far, we have been able to construct the set of connectors, $\boldsymbol{n}$, N and N_i to glue these hypersurfaces together and thus construct the whole spacetime. Moreover, h_{ij} specifies their intrinsic geometry. There remains only one fundamental notion to define in order to finish completely the $d + 1$ decomposition, the *extrinsic curvature*. It is the geometrical object that measures how these hypersurfaces are *immersed* in the whole space-time, and how these are "perceived" (*i.e.* flat? curved?) by a particle living in the whole manifold.

The extrinsic curvature of a d-dimensional $\Sigma_t \subset \mathcal{M}$ at a point $\mathcal{P} \in \Sigma_t$ is the mapping $\mathbf{K} : T_p\Sigma_t \longrightarrow T_p\mathcal{M}$,

$$\mathbf{K}(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}}\boldsymbol{n}\text{[4]} . \tag{2.21}$$

$\nabla_{\boldsymbol{v}}$ is the $(d+1)$-dimensional covariant derivative. Since $\boldsymbol{v} \in T_p\Sigma_t$, the definition only takes into account the values of $\boldsymbol{n}$ on Σ_t. As a result, $\mathbf{K}$ depends only on Σ_t, and not on the lapse and the shift. Notice that

$$\boldsymbol{n} \,.\, \mathbf{K}(\boldsymbol{v}) = -\boldsymbol{n} \,.\, \nabla_{\boldsymbol{v}}\boldsymbol{n} = -\frac{1}{2}\nabla_{\boldsymbol{v}}\left(\boldsymbol{n} \,.\, \boldsymbol{n}\right) = 0 \,. \tag{2.22}$$

Thus $\mathbf{K}(\boldsymbol{v})$ is tangent to Σ_t, and is actually a mapping from $T_p\Sigma_t$ to itself. This immediately tells us that $\mathbf{K}(\boldsymbol{v})$ is a $d \times d$ matrix, with components K^i_j. Moreover, it is symmetric, as we can see in this simple check: consider two vectors $\boldsymbol{u}$ and $\boldsymbol{v} \in T_p\Sigma_t$, we have

$$\boldsymbol{u} \cdot \mathbf{K}(\boldsymbol{v}) - \boldsymbol{v} \cdot \mathbf{K}(\boldsymbol{u}) = -u^\alpha v^\beta \nabla_\beta n_\alpha + v^\alpha u^\beta \nabla_\beta n_\alpha \tag{2.23}$$
$$= 2v^\alpha u^\beta n_{[\alpha;\beta]} \tag{2.24}$$
$$= 2v^\alpha u^\beta n_{[\alpha,\beta]} \tag{2.25}$$
$$= 0 \,, \tag{2.26}$$

where at the end we used that $v^t = u^t = 0$ and $n_i = 0$. This tells us that $\boldsymbol{u} \,.\, \mathbf{K}(\boldsymbol{v}) = u^i h_{il} K^l_j v^j = u^i K_{ij} v^j$ is symmetric in $\boldsymbol{u}$ and $\boldsymbol{v}$, or equivalently K_{ij} is symmetric.

Notice that the first fundamental form h_{ij} is a pure characteristic of Σ_t and does not depend on $\mathcal{M}$, unlike the second fundamental form which essentially depends on $\mathcal{M}$, and on the way these hypersurfaces are immersed in.

To have a nice visual on the role of $\mathbf{K}$ and its definition (2.21), see Fig. 2.2.

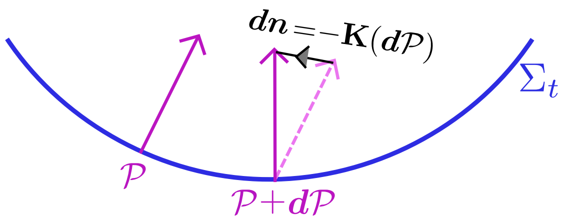


Figure 2.2.: The extrinsic curvature measures the rate of contraction of observers moving normal to the surface. Each unit normal vector $\boldsymbol{n}$ carries a unit interval of proper time normal to the hypersurface. The dashed arrow represents $\boldsymbol{n}$ at the fiducial point $\mathcal{P}$ after parallel transport to the nearby point $\mathcal{P} + \delta\mathcal{P}$

Consider $\mathcal{P} + \boldsymbol{d}\mathcal{P}$. As $\boldsymbol{n}$ does not change in length, the infinitesimal displacement $\boldsymbol{d}\mathcal{P}$ is

[4]The sign is chosen so that so that the trace K is positive when the normal vector is forward-directed.

a vector-valued 1-form lying on Σ_t. The apparent velocity of the normal vector observer at $\mathcal{P}$ relative to that at $\boldsymbol{d}\mathcal{P}$ is $\mathbf{K}(\boldsymbol{d}\mathcal{P})$. Therefore, the extrinsic curvature measures the rate of contraction or expansion of observers moving normal to the surface. In the case of a $d = 3$ dimensional *Friedman-Lemaître-Robertson-Walker* (FLRW) universe, this velocity is $H\boldsymbol{d}P$ and the extrinsic curvature is $K_{ij} = -Hh_{ij}$, with H being the *Hubble-Lemaître constant*. The anisotropic part measures the anisotropic expansion and the trace $K = -3H$ gives the *Hubble rate* [4].

2.4.1. Relation to the induced metric

Thanks to the definition of the extrinsic curvature, we approached a notion of “velocity” for the hypersurfaces Σ_t. Now let's build explicitly the relationship with the time rate of change of the metric $\dfrac{\partial h_{ij}}{\partial t} = \dot{h}_{ij}$ which, as we will see later, is one of the canonical variables and key elements in Einstein's equations of motion.

$$K_{ij} = -\nabla_j n_i = -n_{i,j} + \Gamma^{\alpha}_{ij} n_\alpha \,, \tag{2.27}$$

where the Γ's refer to the $(d+1)$-dimensional Christoffel symbols.

Since $n_i = 0$ and $n_t = -N$, we find

$$\begin{aligned} K_{ij} &= -N\,\Gamma^t_{ij} \\ &= -\frac{1}{2}Ng^{t\alpha}\left(g_{\alpha i,j} + g_{\alpha j,i} - g_{ij,\alpha}\right) \\ &= -\frac{1}{2}N\left[\left(-\frac{1}{N^2}\right)\left(N_{i,j} + N_{j,i} - \dot{h}_{ij}\right) + \frac{N^k}{N^2}\left(h_{ik,j} + h_{jk,i} - h_{ij,k}\right)\right]. \end{aligned} \tag{2.28}$$

Hence

$$\dot{h}_{ij} = -2NK_{ij} + N_{i,j} + N_{j,i} + N^k\left(h_{ik,j} + h_{jk,i} - h_{ij,k}\right) \tag{2.29}$$

With the change $N^k = h^{kl}N_l$, we can rewrite everything as covariant derivatives on Σ_t, using the spatial metric h_{ij} and the d-dimensional Christoffel symbols ${}^{(d)}\Gamma^l_{ij}$,

$$\dot{h}_{ij} = -2NK_{ij} + N_{i|j} + N_{j|i} \,. \tag{2.30}$$

where $|$ denote covariant derivatives on Σ_t.

2.5. Re-expressing the Lagrangian

Now that we have all the geometrical key elements for the $d+1$ decomposition, we can recast the Lagrangian of gravity as a function of these new variables.

We recall the pure gravity part of the action,

$$S_{\text{grav}} \equiv \frac{1}{16\pi G}\int_{\mathcal{M}} dt\; L_{\text{grav}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\;{}^{(d+1)}R \tag{2.31}$$

We re-express (2.31) with the ADM variables,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^k N_k & N_i \\ N_i & h_{ij} \end{pmatrix} . \tag{2.32}$$

First we deal with the volume element $\sqrt{-g}$. We have that

$$g = \det \begin{pmatrix} -N^2 + N^k N_k & N_i \\ N_i & h_{ij} \end{pmatrix} \tag{2.33}$$

$$= -N^2\, h + \det \begin{pmatrix} N^k N_k & N^k h_{ik} \\ N_i & h_{ij} \end{pmatrix} \tag{2.34}$$

$$= -N^2 h \; , \tag{2.35}$$

with $h = \det h_{ij}$. The second determinant in the second line vanishes, as the first row of the matrix is a linear combination of all the others (N^1 times the second row plus N^2 the third plus ... N^d times the last row). Hence:

$$\sqrt{-g} = N\sqrt{h} \tag{2.36}$$

Now we tackle the Ricci scalar ${}^{(d+1)}R$, which is much less trivial. We want to find a relation between ${}^{(d+1)}R$ and ${}^{(d)}R$ evaluated on the hypersurfaces Σ_t. It is provided by the *Gauss-Codazzi relations.*

2.5.1. Gauss-Codazzi relations

A nice trick [4, 45] is to use the *Gaussian normal coordinates i.e.* a coordinate system where $N = 1$ and $N_i = 0$ [5], as it simplifies a lot the Christoffel symbols. Then we derive expressions that are coordinate-independent, hence that apply to any coordinate system. Geometrically, choosing normal coordinate corresponds to consider geodesics orthogonal to Σ_t at spatial positions x^i, and taking the proper time as the coordinate for time.

First, we will use that in Gaussian normal coordinates, we have that

$$\Gamma^t_{ij} = -K_{ij} \tag{2.37}$$

$$\Gamma^i_{jk} = {}^{(d)}\Gamma^i_{jk} \tag{2.38}$$

$$\Gamma^i_{jt} = \frac{1}{2} h^{ik} \dot{h}_{kj} = -h^{ik} K_{kj} = K^i{}_j \tag{2.39}$$

where ${}^{(d)}\Gamma^i_{jk}$ is the d-dimensional Christoffel symbol, defined on Σ_t. The first relation comes from (2.28), and the last ones from the vanishing of all the crossed time-spatial component g^{0i}, g_{0i} .

[5] In differential geometry, normal coordinates at a point p on a manifold $\mathcal{M}$ are a local coordinate system in a neighborhood $\mathcal{U}$ of p on $\mathcal{M}$, defined by the exponential map $\exp_p : V \to U$, where V is a neighborhood of p on the tangent space $T_p\mathcal{M}$. It is a particular nice coordinate system as the Christoffel symbols vanish at the point p. Moreover, one can manage to write the metric as the Kronecker delta in the neighborhood of p, and the first partial derivatives of the metric at p vanish.

The components of the Riemann tensor are generally given by

$$R^{\rho}{}_{\sigma\mu\nu} = \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \ . \tag{2.40}$$

As we want to derive ${}^{(d)}R$, we consider only the components associated with the spatial part, *i.e.* i, j, k....

$$\begin{aligned} R^{i}{}_{jkl} &= {}^{(d)}\Gamma^{i}_{lj,k} - {}^{(d)}\Gamma^{i}_{kj,l} + {}^{(d)}\Gamma^{i}_{km}{}^{(d)}\Gamma^{m}_{lj} + K^{i}{}_{k}K_{lj} - {}^{(d)}\Gamma^{i}_{lm}{}^{(d)}\Gamma^{m}_{jk} - K^{i}{}_{l}K_{jk} \\ &= {}^{(d)}R^{i}_{jkl} + \left(K^{i}{}_{k}K_{lj} - K^{i}{}_{l}K_{jk}\right) \ . \end{aligned} \tag{2.41}$$

We lower the index using h_{ij}, we obtain the relation

$$R_{ijkl} = {}^{(d)}R_{ijkl} + \left(K_{ik}K_{jl} - K_{il}K_{jk}\right) \ . \tag{2.42}$$

By definition, $R_{ijkl} = R\left(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k, \boldsymbol{e}_l\right)$ and so does not depend on the choice of $\boldsymbol{e}_0$ *i.e.* on the lapse and shift, which means the relation is valid for any coordinate system. (2.42) is called the *first Gauss-Codazzi relation*.

There exists a second notable relation we can derive by taking the time t component of (2.41) in Gaussian normal coordinates

$$R^{t}{}_{ijk} = -K_{ki,j} + K_{ji,k} - K_{jm}{}^{(d)}\Gamma^{m}_{ki} + K^{(d)}_{km}\Gamma^{m}_{ji} \tag{2.43}$$

$$= K_{ji|k} - K_{ki|j} \ , \tag{2.44}$$

and rewrite the left hand side as

$$n_{\alpha}R^{\alpha}{}_{ijk} = K_{ji|k} - K_{ki|j} \ , \tag{2.45}$$

where $n_{\alpha} = (-1, \mathbf{0})$ in Gaussian normal coordinates. Therefore, (2.45) is coordinate-free and is called the *second Gauss-Codazzi relation*.

2.5.2. Final Lagrangian

$${}^{(d+1)}R = {}^{(d)}R + K^2 - K^{ij}K_{ij} + \text{total derivatives} \ . \tag{2.46}$$

$$K_{ij} \equiv -n_{i;j} = \frac{1}{2N}\left(-\dot{h}_{ij} + N_{i|j} + N_{j|i}{}^{6}\right) \tag{2.47}$$

$$K = h^{ij}K_{ij} \ . \tag{2.48}$$

The action becomes

$$S_{\text{grav}} = \frac{1}{16\pi}\int_{\mathcal{M}} dt\, d^d x\, N\sqrt{h}\left({}^{(d)}R + K^{ij}K_{ij} - K^2\right) \tag{2.49}$$

where we have integrated out the total derivative terms in (2.46).

Notice that the intrinsic curvature ${}^{(d)}R$ contains the spatial derivative of h and the extrinsic curvature K carries the time derivative $\dot{h}$. Hence ${}^{(d)}R$ plays the role of the potential while K provides the kinetic terms.

2.6. The gravitational Hamiltonian

Now we finally have all the tools to formulate the gravitational Hamiltonian.

As we saw in section 2.1, the very first step for this task is to identify the canonical variables - here h_{ij}, N, N_i - and define their conjugate momenta.

2.6.1. Primary constraints

We have,

$$\Pi^{ij} \equiv \frac{\delta L_{\text{grav}}}{\delta \dot{h}_{ij}} = -\frac{1}{16\pi}\left(K^{ij} - h^{ij}K\right)\sqrt{h}\;, \tag{2.50}$$

$$\Pi^0 \equiv \frac{\delta L_{\text{grav}}}{\delta \dot{N}} = 0\;, \tag{2.51}$$

$$\Pi^i \equiv \frac{\delta L_{\text{grav}}}{\delta \dot{N}_i} = 0\;. \tag{2.52}$$

As expected, the conjugate momentum of the spatial metric depends on the extrinsic curvature K_{ij} and its trace K, as the latter contain the time derivatives $\dot{h}_{ij}$. The Π^{ij} is a $d \times d$ *pseudo*-symmetric tensor, pseudo as it is not exactly a tensor because of the presence of the volume element $\sqrt{h}$ that does not allow the object to transform exactly as a tensor [7].

Since the Lagrangian does not depend on time derivatives of the lapse $\dot{N}$ and shift $\dot{N}_i$, the conjugate momenta associated to the shift N and the lapse N_i vanish. This is an important characteristic of GR, and is often encountered in the so-called *constrained Hamiltonian systems*. These were studied by Dirac in his constraints classification theory. General relativity is a constrained theory, and the vanishing conjugate momenta are called the *primary constraints* in Dirac's terminology. They restrict the solutions of the e.o.m. to subspaces of the overall phase space. In the $d = 4$ case, out of the 20 components of the metric and the conjugate momenta, only 4 are real degrees of freedom. We will discuss this in more details in the next section 2.8.

Now that we defined the conjugate momenta, we have to inverse the relations and express the "$\dot{q}$" variables with respect to their conjugate momenta "p". Hence we should re-express K and K^{ij} as functions of Π^{ij} and its trace $\Pi = \Pi^{ij}\,h_{ij}$. Notice that the primary constraints (2.51) and 2.52 are not invertible. This is one of the features of constrained Hamiltonian systems we mentioned earlier.

$$K = \frac{16\pi G}{(d-1)\sqrt{h}}\,\Pi\;, \tag{2.53}$$

which yields to

$$K^{ij} = \frac{16\pi G}{\sqrt{h}}\left(\frac{1}{d-1}\,\Pi\,h^{ij} - \Pi^{ij}\right)\;. \tag{2.54}$$

[7] See Weinberg [5]

The next step for building the Hamiltonian is to use the standard Legendre transform method. For the pure gravitational part,

$$H_{\text{grav}} = \int d^d x \left(\Pi^0 \dot{N} + \Pi^i \dot{N}_i + \Pi^{ij} \dot{h}_{ij}\right) - L_{\text{grav}} \,. \tag{2.55}$$

As $\Pi^0 = 0$, $\Pi^i = 0$ and

$$\dot{h}_{ij} = -2NK_{ij} + N_{i|j} + N_{j|i} \,, \tag{2.56}$$

we replace in the Hamiltonian

$$H_{\text{grav}} = \int d^d x \left(\Pi^0 \dot{N} + \Pi^i \dot{N}_i + \Pi^{ij} \dot{h}_{ij}\right) - L_{\text{grav}} \,. \tag{2.57}$$

$$= -\frac{1}{16\pi G} \int d^d x \sqrt{h} \left(K^{ij} - h^{ij} K\right) \left(-2N\, K_{ij} + N_{i|j} + N_{j|i}\right) - L_{\text{grav}} \,. \tag{2.58}$$

We are left with L_{grav}. From the reformulated Einstein-Hilbert action (2.49),

$$L_{\text{grav}} = \frac{1}{16\pi G} \int d^d x\, N \sqrt{h} \left({}^{(d)}R + K^{ij} K_{ij} - K^2\right) \,. \tag{2.59}$$

Using (2.59), the Hamiltonian simplifies into

$$H_{\text{grav}} = \frac{1}{16\pi G} \int d^d x \sqrt{h} \left[N \left(K^{ij} K_{ij} - K^2 - {}^{(d)}R\right) + 2N_{i|j} K^{ij} - 2N^i{}_{|i} K\right] \,. \tag{2.60}$$

From the expressions of K^{ij} (2.54) and K (2.53), we find

$$K^{ij} K_{ij} - K^2 = \frac{256\pi^2 G^2}{h} \left(\frac{1}{d-1}\Pi^2 - \Pi^{ij}\Pi_{ij}\right) \tag{2.61}$$

and replace everything in the Hamiltonian, we obtain

$$H_{\text{grav}} = 16\pi G \int d^d x \frac{N}{\sqrt{h}} \left(\frac{1}{d-1}\Pi^2 - \Pi^{ij}\Pi_{ij}\right) - \frac{1}{16\pi G} \int d^d x \sqrt{h}\, N\, {}^{(d)}R + 2 \int d^d x\, N_{i|j} \Pi^{ij} \,. \tag{2.62}$$

We re-express the last term using an integration by parts, for a purpose that will be explained a bit later.

$$\int d^d x\, N_{i|j} \Pi^{ij} = \int d^d x \sqrt{h}\, N_{i|j} \left(\Pi^{ij} \frac{1}{\sqrt{h}}\right) \tag{2.63}$$

$$= -\int d^d x \sqrt{h}\, N_i \left(\Pi^{ij} \frac{1}{\sqrt{h}}\right)_{|j} \tag{2.64}$$ [8]

which finally leads us to the gravitational Hamiltonian:

$$H_{\text{grav}} = 16\pi G \int d^d x \, \frac{N}{\sqrt{h}} \left(\frac{1}{d-1} \Pi^2 - \Pi^{ij}\Pi_{ij} \right) - \frac{1}{16\pi G} \int d^d x \, \sqrt{h} \, N \, {}^{(d)}R \\ - 2 \int d^d x \, \sqrt{h} \, N_i \left(\Pi^{ij} \frac{1}{\sqrt{h}} \right)_{|j} . \tag{2.65}$$

Of course, this is the pure gravitational part. If we had considered matter fields in the Lagrangian, there would have been an additional term in the Hamiltonian associated with them.

$$H = H_{\text{grav}} + H_{\text{matter}} . \tag{2.66}$$

2.6.2. Scalar field matter Lagrangian

Consider a very simple matter Lagrangian:

$$L_{\text{matter}} = \int_{\mathcal{M}} d^d x \, \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right) \tag{2.67}$$

$$= \int_{\mathcal{M}} d^d x \, \sqrt{h} \, N \left(\frac{1}{2} \frac{\dot{\phi}^2}{N^2} - \frac{N^i}{N^2} \dot{\phi} \, \phi_{,i} - \frac{1}{2} h^{ij} \phi_{,i} \phi_{,j} + \frac{1}{2} \frac{N^i N^j}{N^2} \phi_{,i} \phi_{,j} - V(\phi) \right) , \tag{2.68}$$

with ϕ a scalar field and $V(\phi)$ its potential.

The corresponding momentum conjugate is given by:

$$\Pi_\phi \equiv \frac{\delta L_{\text{matter}}}{\delta \dot{\phi}} = \frac{\sqrt{h}}{N} \left(\dot{\phi} - N^i \phi_{,i} \right) , \tag{2.69}$$

which leads to the matter Hamiltonian:

$$H_{\text{matter}} = \int d^d x \, \Pi_\phi \, \dot{\phi} - L_{\text{matter}} \tag{2.70}$$

$$= \int_{\mathcal{M}} d^d x \left[\sqrt{h} \, N \left(\frac{\Pi_\phi^2}{2h} + \frac{1}{2} h^{ij} \phi_{,i} \phi_{,j} + V(\phi) \right) + N^i \, \Pi_\phi \, \phi_{,i} \right] . \tag{2.71}$$

2.7. Hamiltonian and momentum constraints

So far, we were able to construct the Hamiltonian of gravity with a set of coordinates and their conjugate momenta $\{N, N_i, h_{ij}, \Pi^0, \Pi^i, \Pi^{ij}\}$. From the definition of the momenta conjugate, we derived equations called the primary constraints (2.51), (2.52) which reduce the set of the allowed coordinates in phase space. It turns out that there exist another set

[8] We introduced the volume element $\sqrt{h}$ in order to operate the covariant derivative on the surface Σ. We neglect the boundary term.

of equations that emerge when requiring that the primary constraints remain over time:

$$\dot{\Pi}^0 = \{H, \Pi^0\} = -\frac{\delta H}{\delta N} = 0 \,, \qquad \dot{\Pi}^i = \{H, \Pi^i\} = -\frac{\delta H}{\delta N_i} = 0 \,. \tag{2.72}$$

where $H = H_{\rm GR} + H_{\rm matter}$. These are called *secondary constraints*, according to Dirac's constraints terminology.

Using the gravitational Hamiltonian (2.65), we get:

$$\dot{\Pi}^0 = \frac{16\pi G}{\sqrt{h}}\left(\frac{1}{d-1}\Pi^2 - \Pi^{ij}\Pi_{ij}\right) - \frac{\sqrt{h}}{16\pi G}\,{}^{(d)}R + \frac{\delta H_{\rm matter}}{\delta N} = 0 \,, \tag{2.73}$$

$$\dot{\Pi}^i = -2\sqrt{h}\left(\Pi^{ij}\frac{1}{\sqrt{h}}\right)_{|j} + \frac{\delta H_{\rm matter}}{\delta N_i} = 0 \,. \tag{2.74}$$

[9]

In the case of a scalar field as in (2.70), the matter components of the secondary constraints would be

$$\frac{\delta H_{\rm matter}}{\delta N} = \frac{\sqrt{h}}{2}\left(\frac{\Pi_\phi^2}{h} + h^{ij}\phi_{,i}\phi_{,j} + 2V(\phi)\right) \,, \tag{2.75}$$

$$\frac{\delta H_{\rm matter}}{\delta N_i} = \Pi_\phi\, h^{ij}\phi_{,j} \,. \tag{2.76}$$

(2.73), (2.74) are constraints on the legal h_{ij} and Π^{ij}, or equivalently the spatial geometries and extrinsic curvatures of the spatial hypersurfaces Σ, as well as the matter fields and their conjugate momenta.

Every problem in mechanics starts with the question of the initial conditions. How many parameters do we need to set in order to fully determine the dynamics? In other words, how many degrees of freedom does the system have? In a standard one-dimensional mechanical system, one needs to fix p and q at a starting time t, then integrate two-first order equations forward in time in order to find all their values in the future times. The same happens in gravitation, one needs to fix the initial conditions among the set $\{N, N_i, h_{ij}, \Pi^0, \Pi^i, \Pi^{ij}, \phi, \Pi_\phi\}$. Only in this case, these should satisfy the constraints equations (2.51), (2.52), (2.73), (2.74). Hence, the constraints equations are also said to be constraints on the initial conditions.

In appendix **??**, we show how the secondary constraints are related to Einstein's equations of motion.

Back to the general Hamiltonian (2.66). As it is linear in N and N_i, we may recast it as

$$H = \int_\Sigma d^d x \left(N\mathcal{H} + N_i\mathcal{H}^i\right) \,, \tag{2.77}$$

with $\mathcal{H}$ and $\mathcal{H}^i$ the secondary constraints we derived earlier from $\dfrac{\delta H}{\delta N}$ and $\dfrac{\delta H}{\delta N_i}$

$$\mathcal{H} = \frac{16\pi G}{\sqrt{h}}\left(\frac{1}{d-1}\Pi^2 - \Pi^{ij}\Pi_{ij}\right) - \frac{1}{16\pi G}\sqrt{h}\,{}^{(d)}R + \frac{\sqrt{h}}{2}\left(\frac{\Pi_\phi^2}{h} + h^{ij}\phi_{,i}\phi_{,j} + 2V(\phi)\right) , \tag{2.78}$$

$$\mathcal{H}^i = -2\sqrt{h}\left(\Pi^{ij}\frac{1}{\sqrt{h}}\right)_{|j} + \Pi_\phi\, h^{ij}\phi_{,j} . \tag{2.79}$$

These are also known as the *Hamiltonian constraint* and the *momentum constraints* respectively. And we have that,

$$\mathcal{H} = 0 , \tag{2.80}$$
$$\mathcal{H}^i = 0 . \tag{2.81}$$

We may further rewrite the gravitational action in terms of the Hamiltonian,

$$S_{\text{grav}} = \int dt \int_\Sigma d^dx \left(\dot{N}\,\Pi^0 + \dot{N}_i\,\Pi^i - N\mathcal{H} - N_i\mathcal{H}^i\right) \tag{2.82}$$
$$= -\int dt \int_\Sigma d^dx \left(N\mathcal{H} + N_i\mathcal{H}^i\right) . \tag{2.83}$$

where Π^0 and Π^i vanish, according to the primary constraints (2.51), (2.52).

It is clear that the lapse and shift functions now act as *Lagrange multipliers*. The variation of (2.83) with respect to N and N_i, yields the Hamiltonian and momentum constraints respectively.

In section Sec. 2.8, we show how $\mathcal{H}$ and $\mathcal{H}^i$ are the *canonical generators* of the *gauge symmetries* (time and space diffeomorphisms) of the theory.

Now back to (2.77). Notice that if the constraints are satisfied, the gravitational Hamiltonian vanishes. Any allowed configuration of spatial geometries, extrinsic curvatures or matter fields, lead to an apparently non-dynamical Universe!

Actually, the vanishing of the gravitational Hamiltonian reflects that the evolution parameter given by the coordinate time t is not a physical quantity, but only a *free parameter*. As general relativity treats time and space on equal footing, the dynamics cannot be simply encoded in a singled out variable t. As we will discuss further in Sec. 3.4, we have to interpret the dynamics as a *relational* process between the fields involved in the theory.

2.8. More on the constraints of GR

In order to study GR constraints and show how these are actually the canonical generators of the gauge symmetries, we introduce their *smeared* versions

$$H(N) := \int_\Sigma d^d x\, N\, \mathcal{H}\ , \tag{2.84}$$

$$H(\vec{N}) := \int_\Sigma d^d x\, N^i\, \mathcal{H}_i\ , \tag{2.85}$$

where $\vec{N}$ is the usual shift vector we often refer to with its components N^i.

Smearing is particularly useful to compute integrals per part and to remove the Kronecker's delta functions "δ" in the Poisson brackets ([46]), and N, N^i in (2.84) and (2.85) are actually just labels for the smearing functions. According to Dirac's constrained systems theory Appendix. **??**, for every 1st-class constraint, the corresponding Lagrange multiplier is a *free* parameter *i.e.* it can be freely chosen. However, it is natural to think of the lapse and the shift as the gauge parameters of the diffeomorphism constraints [10].

As a first application of the operation above, we rewrite the smeared momentum constraint as

$$\begin{aligned} H(\vec{N}) &= \int d^d x\, N_i\, \mathcal{H}^i \\ &= -2 \int d^d x\, \sqrt{h}\ N_i\, \nabla_j \left(\frac{\Pi^{ij}}{\sqrt{h}} \right) \\ &= \text{b.t.} + 2 \int d^d x\, \nabla_j\, N_i\, \Pi^{ij}\ . \end{aligned} \tag{2.86}$$

In order to see how it acts on phase space, let's recall first some of the Poisson brackets rules. We formally define the Poisson brackets of two functions f, g on phase space with

$$\{f, g\} = \int_\Sigma d^d x\, \sqrt{h} \left(\frac{\delta f}{\delta h_{kl}(\boldsymbol{x})} \frac{\delta g}{\delta \Pi^{kl}(\boldsymbol{x})} - \frac{\delta g}{\delta \Pi^{kl}(\boldsymbol{x})} \frac{\delta f}{\delta h_{kl}(\boldsymbol{x})} \right)\ . \tag{2.87}$$

Hence the following Poisson bracket rules:

$$\{h_{ij}(\boldsymbol{x}), \Pi^{kl}(\boldsymbol{x}')\} = \frac{\delta_i^k \delta_j^l + \delta_i^l \delta_j^k}{2}\, \delta^{(d)}(\boldsymbol{x}, \boldsymbol{x}') = \delta_{(ij)}^{kl}\, \delta^{(d)}(\boldsymbol{x}, \boldsymbol{x}') \tag{2.88}$$

$$\{h_{ij}(\boldsymbol{x}), h_{kl}(\boldsymbol{x}')\} = 0 \tag{2.89}$$

$$\{\Pi^{ij}(\boldsymbol{x}), \Pi^{kl}(\boldsymbol{x}')\} = 0 \tag{2.90}$$

For a matter of clarity and simplification, we do not write the space components "$\boldsymbol{x}$" anymore, but they are all implicit in the integrals involved in the Poisson brackets with the condition $\delta^{(d)}(\boldsymbol{x}, \boldsymbol{x}')$. We therefore compute

$$\{H(\vec{N}), h_{ij}\} = -\nabla_i N_j - \nabla_j N_i\ . \tag{2.91}$$

We can show that the right side of (2.91) is actually the *Lie derivative* of the spatial

[10] Notice that summing (2.84) and (2.85) gives the total Hamiltonian.

metric h_{ij} along the shift vector $\vec{N}$, $\mathcal{L}_{\vec{N}}h_{ij}$. Hence,

$$\{H(\vec{N}), h_{ij}\} = -\mathcal{L}_{\vec{N}}h_{ij} \ , \tag{2.92}$$

and the constraint $H(\vec{N})$ acts as a generator of spatial diffeomorphisms.

We have to check if it is the same for the momentum conjugate. The Poisson bracket with Π^{ij} is similar to (2.91), but a bit more tricky to compute. Recall that Π^{ij} is a pseudo-tensor because of the factor $\sqrt{h}$ (see (2.50)), and hence does not transform exactly as a tensor. First we re-express it as a product of the volume element $\sqrt{h}$ and an exact tensor $\hat{\Pi}^{ij}$

$$\Pi^{ij} = \sqrt{h}\,\hat{\Pi}^{ij} \ . \tag{2.93}$$

We can then compute the Lie derivative

$$\left(\mathcal{L}_{\vec{N}}\sqrt{h}\,\hat{\Pi}^{ij}\right) = \left(\mathcal{L}_{\vec{N}}\sqrt{h}\right)\,\hat{\Pi}^{ij} + \sqrt{h}\left(\mathcal{L}_{N}\hat{\Pi}^{ij}\right) \ . \tag{2.94}$$

And since

$$\begin{aligned}\mathcal{L}_{\vec{N}}\sqrt{h} &= \frac{1}{2}\sqrt{h}\,h^{ij}\mathcal{L}_{\vec{N}}h_{ij} \\ &= \sqrt{h}\,\nabla^i N_i \ ,\end{aligned} \tag{2.95}$$

then

$$\left(\mathcal{L}_{\vec{N}}\sqrt{h}\,\hat{\Pi}^{ij}\right) = \sqrt{h}\left(\mathcal{L}_{\vec{N}} + \nabla^i N_i\right)\,\hat{\Pi}^{ij} \ . \tag{2.96}$$

As $\hat{\Pi}^{ij}$ is a tensor, we have that

$$\mathcal{L}_{\vec{N}}\hat{\Pi}^{ij} = N^k\nabla_k\hat{\Pi}^{ij} - \Pi^{ki}\nabla_k N^j - \Pi^{kj}\nabla_k N^i \ . \tag{2.97}$$

Using (2.97) in (2.94) and contracting with h_{ij}, we obtain

$$h_{ij}\left(\mathcal{L}_{\vec{N}}\sqrt{h}\,\hat{\Pi}^{ij}\right) = \nabla^k\left(N_k\,\Pi\right) - 2\Pi^{ij}\nabla_i N_j \ . \tag{2.98}$$

We then re-express the constraint (2.86)

$$\begin{aligned}H(\vec{N}) &= \text{b.t.} + 2\int_\Sigma d^dx\,\nabla_j\,N_i\,\Pi^{ij} \\ &= \text{b.t.} - \int_\Sigma d^dx\,h_{ij}\,\mathcal{L}_{\vec{N}}\Pi^{ij} \ .\end{aligned} \tag{2.99}$$

As $\mathcal{L}_{\vec{N}}\,\Pi^{ij}$ does not depend on h_{ij}, we finally get

$$\{H(\vec{N}), \Pi^{ij}\} = -\mathcal{L}_{\vec{N}}\Pi^{ij} \ , \tag{2.100}$$

and find a similar result for $H(\vec{N})$ when acting on Π^{ij}. Thus $H(\vec{N})$ is the generator of spatial diffeomorphisms on the phase space $\mathcal{P}$.

Now regarding the Hamiltonian constraint and its role as a generator of time-diffeomorphism, the computations are much more complicated[11]. As they are heavy, we only provide the global result and a very brief insight on the Lie algebra behind.

We find that

$$\mathcal{L}_{\vec{N}} h = \{h, H(\vec{N})\} \ , \tag{2.101}$$
$$\mathcal{L}_N h = \{h, H(N)\} + f(h) \ . \tag{2.102}$$

Unlike the spatial d-diffeomorphisms , time-diffeomorphism is actually quite complicated and not explicit in the constraints. As shown above, there is an additional term $f(h)$ that is a function of the metric. This feature actually comes from the fact that the constraints of GR satisfy a complicated algebra known as *Dirac's hypersurface deformation algebra* [47, 48].

$$\{H(\vec{N}), H(\vec{M})\} = -H(\mathcal{L}_{\vec{M}}\vec{N}) \ , \tag{2.103}$$
$$\{H(N), H(\vec{M})\} = -H(\mathcal{L}_{\vec{M}}N) \ , \tag{2.104}$$
$$\{H(N), H(M)\} = +H(\vec{V}) \quad \text{with} \quad V^i = h^{ij}\left(M\partial_j N - N\partial_j M\right) \ . \tag{2.105}$$

It is actually a *Lie algebroïd*, where there is a structure function - h^{ij} in (2.105) - instead of a usual structure constant in standard Lie algebras. $f(h)$ in (2.102) vanishes *on-shell i.e.* for configurations satisfying the equations of motion.

The above Poisson algebra describes *normal* and *tangential* deformations of a hypersurface. Time-diffeomorphisms generate displacements from one hypersurface Σ_1 to another hypersurface Σ_2 with a length proportional to N, while spatial-diffeomorphisms yield movements on one hypersurface itself along $\vec{N}$.

[11] Just by the looks, it is quite different from the momentum constraints. This is because general covariance was mathematically broken in the $d+1$-decomposition, but the physics is completely recovered.

Part II.

Quantum gravity and quantum cosmology

3. Quantization of gravity

Contents

3.1. Preliminaries on the quantum theory

Classical mechanics and quantum mechanics have very different mathematical frameworks. In the Hamiltonian formalism, we saw that classical systems with n degrees of freedom are points on a $2n$-dimensional manifold, the phase space $\mathcal{P}$. The fundamental structure on $\mathcal{P}$ is a symplectic form Ω_{ab}, which is a non-degenerate, closed 2-form on $\mathcal{P}$. Observables are real-valued functions on $\mathcal{P}$, and the dynamical evolution is given by a one-parameter family of canonical transformations generated by a Hamiltonian vector $h^i = \Omega^{ij}\nabla_j H$.

The dynamics is given by the Poisson bracket

$$\{f, g\} = \Omega^{ab}\nabla_a f \nabla_b g \; . \tag{3.1}$$

In quantum theory, a state is represented by a vector Ψ in an infinite dimensional Hilbert space H. An observable is represented as a self-adjoint operator $\hat{\mathcal{O}}$ acting on H. The dynamical evolution is given by a one-parameter family of unitary transformations on H generated by a Hamiltonian operator.

As stated by Wald in [49] chapter 2, the key issue in constructing a quantum theory that corresponds to a classical system is how to choose the Hilbert space and the self-adjoint operators corresponding to the classical observables "$\mathcal{O}$" of interest. In an attempt to preserve the formal structure of the classical structure, such as the *symmetries* of the classical theory, Dirac introduced his well-known *Poisson-bracket-commutator* relationship

$$\{A, B\} \longrightarrow \frac{1}{i\hbar}[\hat{A}, \hat{B}] \; , \tag{3.2}$$

which is a map upgrading classical observables A, B to quantum observables $\hat{A}, \hat{B}$. The Poisson bracket gives an algebraic structure on the space of the classical observables, whereas commutators provide a similar algebraic structure on quantum observables.

However, there is still a great issue regarding the choice of the representation of the quantum observables. Let H be a Hilbert space and V_α a collection of operators. (H, V_α) and (H', V'_α) are said to be *unitarily equivalent* if there exists a unitary map $U : \mathrm{H} \to \mathrm{H}'$ such that $U^{-1} V'_\alpha U = V_\alpha$ for all α. Hence, according to the *Stone-von Neumann theorem* [1], the two quantum theories are physically equivalent in the sense that given any $\Psi \in \mathrm{H}$, the state $U\Psi$ in the quantum theory (H', V'_α) has exactly the same physical properties as Ψ in the quantum theory (H, V_α).

Unfortunately, the Stone-von Neumann theorem does not provide a natural prescription on how to *represent* observables, except for position and momentum through the canonical commutation relation [2], and classical observables which are at most linear in the momentum (see [49]). For more general observables, *factor-ordering* ambiguities arise and it becomes difficult to find the right set satisfying (3.2). We particularly encounter this issue in defining the *Wheeler-DeWitt* equation, which is the quantum equation for gravity derived from its canonical quantization (see Sec. 3.3). Since gravity is an infinite dimensional system and involves fields, infinitely many unitarily inequivalent irreducible representations of the Weyl relations may exist. Actually, even in the *(not-so-)mini-superspace* example considered in Sec. 3.5 we show that these ambiguities appear already in finite dimensional systems.

3.2. Superspace

So the first step for constructing a quantum theory for gravity is to define the right Hilbert space with the right observables.

In the canonical quantization of gravity, we have to consider the space of all Riemannian d-metrics and matter configurations defined on the spatial hypersurfaces Σ

$$\mathrm{Riem}(\Sigma) := \{h_{ij}(x), \phi(x) \mid x \in \Sigma\} \, . \tag{3.5}$$

It is an infinite-dimensional space, but there are a finite number of degrees of freedom at each point $x \in \Sigma$. Moreover, two configurations that can be related to each other by a *diffeomorphism*, *i.e.* a coordinate transformation, are said to be equivalent since their intrinsic geometry is the same (this is a statement we will justify in next section). Therefore, by factoring out the diffeomorphisms on the spatial hypersurfaces, the relevant

[1] *Stone-von Neumann theorem*: Let $(\mathrm{H}, \hat{W}(y))$ and $(\mathrm{H}', \hat{W}'(y))$ be strongly continuous, irreducible, unitary representations of the Weyl relations

$$\hat{W}(y_1)\hat{W}(y_2) = e^{i\,\Omega(y_1,y_2)/2}\, \hat{W}(y_1 + y_2) \, , \tag{3.3}$$

$$\hat{W}^\dagger(y) = \hat{W}(-y) \, . \tag{3.4}$$

Then $(\mathrm{H}, \hat{W}(y))$ and $(\mathrm{H}', \hat{W}'(y))$ are unitarily equivalent. See chapter 2 of [49] for more details, and implications to equivalent unitarily representations.

[2] Consider a standard finite quantum mechanical system, $\mathrm{H} = L^2(\mathbb{R}^3)$. In the position representation, the position operator $\hat{X}^i$ is represented by a multiplication by x^i, and the momentum operator $\hat{P}_j$ by $-i\frac{\partial}{\partial x^j}$.

configuration space on which the quantum dynamics occurs is

$$\frac{\mathrm{Riem}(\Sigma)}{\mathrm{Diff}_0(\Sigma)}\ , \tag{3.6}$$

where the index 0 refers to the diffeomorphisms connected to the identity, and is called the *superspace* [21].

The state vector $\Psi[h_{ij},\phi]$ on the superspace is a functional of the spatial metrics h_{ij} and matter fields ϕ introduced in the classical theory, and is called the *wavefunction of the universe*. Unlike the case in standard quantum mechanics, Ψ does not explicitly depend on time. Time is measured *relationally*, in the sense that it is implicitly encoded by the mutual relations between the dynamical variables h_{ij}, ϕ. One typical quantity that one might want to calculate is, for example, the probability (amplitude) that the metric field be in some configuration *given that* the matter field attains some value. Hence the matter field plays the role of a *measuring apparatus*, as we would use a *clock* to measure time. These features will become clearer as we progress in the canonical quantization of gravity and derive the so-called *Wheeler-DeWitt equation* governing the wavefunction Sec. 3.3. In Sec. 3.5, we present one very well-known quantum cosmological model, the *mini-superspace*, that we use to highlight the relational property of the fields.

Note: we sometimes refer to the wavefunction of the universe as the "wavefunctional". The former denomination is the usual one we find in the litterature. In our work, we shorten it to wavefunctional

Note: we sometimes refer to the wavefunction of the universe as the "wavefunctional". The former denomination is the usual one we find in the litterature. In our work, we shorten it to wavefunctional

We can define a metric on this superspace, using the Hamiltonian of GR (2.65) or the recasted action (2.83):

$$\begin{aligned} H_{GR} = 16\pi G \int d^dx\, \frac{N}{\sqrt{h}} \left(\frac{1}{d-1}\Pi^2 - \Pi^{ij}\Pi_{ij} \right) - \frac{1}{16\pi G} \int d^dx\, \sqrt{h}\, N\, R^{(d)} \\ - 2 \int d^dx\, \sqrt{h}\, N_i \left(\Pi^{ij} \frac{1}{\sqrt{h}} \right)_{|j} \ . \end{aligned} \tag{3.7}$$

We can re-arrange the above as

$$\begin{aligned} H_{GR} = -16\pi G \int d^dx\, N\, G_{(ij)(kl)} \Pi^{ij}\Pi^{kl} - \frac{1}{16\pi G} \int d^dx\, \sqrt{h}\, N\, R^{(d)} \\ - 2 \int d^dx\, \sqrt{h}\, N_i \left(\Pi^{ij} \frac{1}{\sqrt{h}} \right)_{|j} \ , \end{aligned} \tag{3.8}$$

where

$$G_{(ij)(kl)} = \frac{1}{2h^{1/2}} \left(h_{ik}h_{jl} + h_{il}h_{jk} - \frac{2}{d-1} h_{ij}h_{kl} \right) \tag{3.9}$$

and is called the *DeWitt metric*. We recast it as

$$G^{AB}(x) := G_{(ij)(kl)}(x)\ , \tag{3.10}$$

where the indices A, B stand for pairs of symmetric indices (ij) and (kl) respectively, and are the independent components of the spatial metric h_{ij},

$$A, B \in \{11, 12, \dots 1d, 21, 22 \dots 2d, \dots dd\} \ . \tag{3.11}$$

In the case of $d = 3$ (which corresponds to a 4-dimensional overall spacetime $\mathcal{M}$) , the signature of the DeWitt metric is $(-, +, +, +, +, +)$. We can construct the inverse metric as well, using the following rule

$$G_{AB} \, G^{BC} = \delta_A^C \ , \tag{3.12}$$

$$G^{(ij)(kl)} \, G_{(kl)(mn)} = \frac{1}{2} \left(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m \right) \quad , \tag{3.13}$$

where δ_A^C is the identity matrix which, expressed in spatial indices $i, j, \dots$, is the symmetrized product of Kronecker delta's.

We find then

$$G_{AB} = G^{(ij)(kl)} = h^{1/2} \left(h^{ik} h^{jl} - h^{ij} h^{kl} \right) \ . \tag{3.14}$$

One last manipulation, we rewrite the total Hamiltonian and the action in (2.49) as

$$H = \int \frac{d^d x \, N}{2} \, G^{AB} \, \Pi_A \Pi_B + \dots \ , \tag{3.15}$$

$$S = \int \frac{dt \, d^d x}{2N} \, G_{AB} \, \dot{X}^A \dot{X}^B + \dots \ , \tag{3.16}$$

where we have set $8\pi G = 1$ to avoid confusion between the DeWitt metric and the gravitational constant. The left pieces are the non-dynamical terms: spatial curvature, fields potential etc.

The placement of "AB" on the upper side on (3.10) or equivalently on the bottom side in (3.14), is based on the fact that we consider the (covariant) spatial metric as a generalized coordinate in field space, $h_{ij} \sim X^A$.

If we add a matter field ϕ as in (2.68), the factors multiplying the dynamical pieces "$\dot{\phi}$" in the Lagrangian or "Π_ϕ" in the Hamiltonian, are contained in the DeWitt metric the same way we did for the gravitatonal part. The coordinates of the superspace X^A are then extended to matter fields $h_{ij}, \, \phi \sim X^A$.

3.3. The Wheeler-DeWitt equation

Now we introduce the fundamental quantum gravitational equations governing the quantum state Ψ. We perform the canonical quantization using the Hamiltonian formulation of GR (2.65). To keep it simple, we do not consider matter fields for now.

We recall the set of canonical coordinates and their conjugate momenta we used so far $\{N, N_i, h_{ij}, \Pi^0, \Pi^i, \Pi^{ij}\}$. According to the standard Dirac's quantization procedure [48, 50],

$$\Pi^0 \to -i \frac{\delta}{\delta N} \ , \quad \Pi^i \to -i \frac{\delta}{\delta N_i} \ , \quad \Pi^{ij} \to -i \frac{\delta}{\delta h_{ij}} \ . \tag{3.17}$$

The primary classical constraints (2.51), (2.52) become constraints on the wavefunction

$$\hat{\Pi}^0\Psi = 0 \quad \Rightarrow -i\frac{\delta\Psi}{\delta N} = 0 \,, \tag{3.18}$$

$$\hat{\Pi}^i\Psi = 0 \quad \Rightarrow -i\frac{\delta\Psi}{\delta N_i} = 0 \,, \tag{3.19}$$

which implies that Ψ does not depend on N and N_i.

Similarly, the momentum constraints (2.74) yield

$$\hat{\mathcal{H}}^i\Psi = 0 \quad \Rightarrow i\hbar\left[\frac{\delta\Psi}{\delta h_{ij}}\right]_{|j} = 0 \,. \tag{3.20}$$

The equation above reflects that ψ is invariant under spatial diffeomorphisms. It may not be straightforward to see it, for this reason we use the smeared constraint $H(\vec{N})$ we introduced in (2.86),

$$\hat{H}(\vec{N}) = \text{b.t.} - 2\int_\Sigma d^dx\, \nabla_j N_i\, \Pi^{ij} \;; \tag{3.21}$$

$$\hat{H}(\vec{N})\Psi[h_{ij}] = 0 \quad \Rightarrow i\hbar\int_\Sigma d^dx\, \nabla_j N_i \frac{\delta\Psi[h_{ij}]}{\delta h_{ij}} = 0 \,. \tag{3.22}$$

The variation of the functional Ψ along "the direction of $\nabla_j N_i$" in the superspace of the d-dimensional metrics has to be 0. Consequently, Ψ should be unchanged when the field configuration h_{ij} is transformed under the gauge transformation generated by $\hat{H}(\vec{N})$. As in the classical theory, we had that

$$\{H(\vec{N}), h_{ij}\} = -\mathcal{L}_{\vec{N}}\, h_{ij} \;, \tag{3.23}$$

therefore, the quantum version reads

$$\hat{H}(\vec{N})\,\Psi[h_{ij}] = 0 \implies \Psi[h_{ij} + \mathcal{L}_{\vec{N}}\, h_{ij}] = \Psi[h_{ij}] \,. \tag{3.24}$$ [3]

The physical wavefunctions are those invariant under the d-diffeomorphisms generated by the constraints. This is the equivalent of reducing the phase space in the classical theory, to the physical degrees of freedom by projecting on the constraint surface and then removing the gauge orbits.

Hence, quantizing the constraints yields a reduction of the superspace of all metrics to those up to diffeomorphisms, resulting in the (quotient) superspace $\frac{\text{Riem}(\Sigma)}{\text{Diff}_0(\Sigma)}$[4]. Therefore,

[3]We can do the analogy with a very basic finite particle system, where we impose that a given function f of two variables x, y to be invariant under y

$$\frac{\partial f(x,y)}{\partial y} = 0 \longrightarrow f(x,y) = f(x) \,. \tag{3.25}$$

Another way to rephrase it is to ask f to be invariant along the y direction, which translates as

$$f(x,y) = f(x, y+c) \,. \tag{3.26}$$

A more elaborate analogy with Maxwell's field theory is done in appendix **??**

[4]This is called a *strong imposition* of the constraints and concerns only 1rst-class constraints. The

$\Psi[h_{ij}]$ depends only on the *geometry* of the hypersurfaces Σ. As pointed by DeWitt in [21], in a finite universe, ψ would depend only on the *geometric invariants* of the $d-$dimensional space and so be expressed as a function of the independent invariants such as $\int_\Sigma d^dx\,\sqrt{h}$, $\int_\Sigma d^dx\,R\,\sqrt{h}$, $\int_\Sigma d^dx\,R^2\,\sqrt{h}$ and so on. These can be constructed out of products of the d-dimensional Riemann tensor and its covariant derivatives, with the topology of the d-space itself being separately specified. In an infinite universe, assymptotic coordinates should be taken into account.

Actually, for (3.24) to be a rigorous statement, we need to properly define a measure on the superspace. Unfortunately, to date, no measure of the type "$\mathcal{D}h(x)$" has been correctly constructed yet. First because it is a functional measure, and these are usually not clearly defined mathematically. But the difficulty lies essentially in the fact that the field itself is a metric field, and hence defines a dynamical background. Nevertheless, for some quantum cosmological models such as mini-superspace (Sec. 3.5), the issue is removed as we do not deal with fields anymore but simple variables instead. In this case, the system is reduced to quantum mechanical problem, hence the measure is not functional and is properly defined.

In ordinary quantum field theories defined on a fixed background, the ambiguity is usually solved. In Maxwell's field theory, we can construct a measure "$\mathcal{D}A_i(x)$" on its corresponding Hilbert space, with $A_i(x)$ being the spatial components of the potential vector $A_\mu(x)$. We provide an overview on its canonical analysis (classical and quantum) in Appendix **??**.

Loop quantum gravity [52] is a quantum gravity theory that based its formalism on *tetrad variables* [53]. These allow to bypass the problem of non-existence of a mathematically precise functional measure and hence constructing a well-defined Hilbert space.

Finally, the Hamiltonian constraint (3.15) yields

$$\hat{\mathcal{H}}\Psi = \left[-16\pi\, G_{(ij)(kl)}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - \frac{1}{16\pi}\sqrt{h}\,R^{(d)}\right]\Psi = 0\ . \tag{3.27}$$

The equation (3.27) is known as the *Wheeler-DeWitt* (WdW) *equation*. It is a second-order hyperbolic functional equation on each point $x \in \Sigma$ on superspace. It reflects time-reparametrisation invariance of Ψ.

In fact, "$G_{(ij)(kl)}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}$" is just a symbolic writing of the quantization of "$G_{(ij)(kl)}\Pi^{ij}\Pi^{kl}$", as we do not have a mathematical prescription on how to order the second-order derivatives. This is known as the *factor-ordering problem* [5], and is closely related to the fact that since we do not have a precisely defined measure, we do not know how to rigorously construct Hermitian operators. In Sec. 3.3.1, we elaborate a little on these ambiguities and provide a simple example to illustrate the issue.

If we consider a given matter field ϕ weakly coupled to gravity as in Sec. 2.6.2, the procedure is the same with this time additional terms in the Hamiltonian and momentum

2nd-class constraints must be solved at the classical level before quantization, unless one uses the Dirac bracket to get rid of the non-physical degrees of freedom [51].

[5] We can draw a parallel with what we said in the preliminaries section: the fact that in gravity we deal with an infinite dimensional system involving fields, infinitely many unitarily inequivalent irreducible representations of the Weyl relations exist, and this may be related to the factor-ordering ambiguities we encounter [49].

constraints.

$$\Pi_\phi \to -i\frac{\delta}{\delta\phi} \, , \tag{3.28}$$

Remark: all the statements above on the invariance of Ψ under time and spatial diffeomorphism of h_{ij} hold when adding matter fields ϕ.

3.3.1. On the hermiticity of operators

In a finite particle system, the ordering is determined by the *Hermiticity* of the quantum operators. Recall that an operator $\hat{\mathcal{O}}$ on a given finite-dimensional Hilbert space $\mathrm{H} = L^2(\mathbb{R}^d)$ is said to be Hermitian if $\hat{\mathcal{O}}$ it is equal to its self-adjoint $\hat{\mathcal{O}}^\dagger$, or equivalently

$$\langle \Psi, \hat{\mathcal{O}}\,\Phi\rangle = \langle \hat{\mathcal{O}}\,\Psi, \Phi\rangle \ \ , \tag{3.29}$$

for any two elements Ψ, Φ of the Hilbert space, and $\langle \cdot, \cdot\rangle$ the inner product in H. Written in terms of integrals of wavefunctions

$$\int d^dx\, \Psi^\star(\boldsymbol{x},t)\,\hat{\mathcal{O}}\,\Phi(\boldsymbol{x},t) = \int d^dx\,\left(\hat{\mathcal{O}}\,\Psi(\boldsymbol{x},t)\right)^\star \Phi(\boldsymbol{x},t) \, , \tag{3.30}$$

where "$\star$" denotes the complex conjugate.

Take for instance a $d=2$-dimensional free particle case, whose Hamiltonian is given by

$$H = \frac{p^2}{2m} \, , \tag{3.31}$$

m being the mass. The momentum p^i is quantized in the position representation as $\hat{p}^i = -i\hbar\frac{\partial}{\partial x_i}$, which corresponds to an Hermitian operator with respect to the integral defined in (3.34).

What about $\hat{p}^2$? Imposing $\hat{H}$ to be Hermitian implies a unique ordering of $\hat{p}^2$ depending on the chosen set of coordinates. Indeed, in Cartesian coordinates, the Hermitian Laplacian operator is simply

$$\hat{p}^2 = -\hat{\nabla}^2 = -\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \, , \tag{3.32}$$

with respect to the measure integral $dx\,dy$. While in spherical coordinates,

$$\hat{p}^2 = -\hat{\nabla}^2 = -\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] \, , \tag{3.33}$$

is the Hermitian Laplacian operator with respect to the measure $r\,dr\,d\theta$. Let's show explicitly for the"r" non-trivial piece of (3.33),

$$\int r\,dr\,d\theta\left(\Psi^\star\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\right)\Phi=\int dr\,d\theta\left(\Psi^\star\partial_r\Phi+r\,\Psi^\star\partial_r^2\Phi\right) \tag{3.34}$$

$$=\int dr\,d\theta\left(-\partial_r\Psi^\star\Phi+r\,\partial_r^2\Psi^\star\,\Phi+2\,\partial_r\Psi^\star\Phi\right) \tag{3.35}$$

$$=\int r\,dr\,d\theta\left(\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\Psi\right)^\star\Phi\,, \tag{3.36}$$

where we applied an integration per part in the second line, and used that Ψ and Φ cancel at the infinite boundaries.

This simple example teaches us that the form of the differential operator depends on the metric if we want such an operator to be Hermitian.

In the canonical quantization of gravity, one of the reasons it is difficult to construct such operators is that we do not have a well-defined measure in the functional version of (3.34). Moreover, it is not clear how to set the boundaries of the integral (3.34). In a finite particle system, we usually fix the wavefunctions to be 0 in the boundaries and hence ensure the convergence of any integral of the type (3.34). In particular, the inner product $\langle\Psi,\Psi\rangle=\int d^dx\ \Psi^\star(\boldsymbol{x},t)\,\Psi(\boldsymbol{x},t)$ is well-defined and $\mid\Psi(\boldsymbol{x},t)\mid^2$ evaluates a *probability density* in the system's configuration space.

The measure issue can be solved for a superspace reduced to the configurations which are homogeneous: the *mini-superspace* (see Sec. 3.5). In this case, the measure does not depend on $x\in\Sigma$, and hence is mathematically well-defined as we deal with "points" instead of "fields" on the reduced superspace. It also allows to define positive-definite probabilities and hence construct quantum observables.

3.4. Interpretation of the Wheeler-DeWitt equation

A bit of history. The WdW equation was first derived by DeWitt [21] starting from the Hamilton-Jacobi equation, which is an elegant alternative of Hamiltonian and Lagrangian mechanics, with the particularity that it presents the motion of a particle as a wave. The idea was to derive a quantum gravity theory the same way Schrödinger did with his famous equation, starting from the Hamilton-Jacobi equation of a particle in a central potential. Wheeler realized the importance of the wavefunction and the equation that governs it [22].

3.4.1. Probability measure

The WdW equation encodes time in a subtle way. For any solution of this equation, time can be extracted in a relational way, *e.g.*, from the relations among the fields in the theory. Already at the classical level, the structure of GR is relational. Indeed, the original Einstein-Hilbert action is time-reparametrisation invariant. General relativity treats space and time on an equal footing. There is no preferred observable for time, and time evolution is always measured with respect to some arbitrary variables. As we quote from [52], "General relativity describes the world in terms of relative evolution of partial observables[6], rather than in terms of evolution of degrees of freedom in time".

This translates in the quantum theory as a lack of a natural probabilistic interpretation of the wavefunctional. Indeed, there is no obvious equivalent of a conserved probability distribution as there is no external time parameter.

Recall that in a standard quantum mechanical system, the probability of finding a particle in a volume element d^dx of a Euclidean space V at a time t is given by

$$dP(\boldsymbol{x},t) = |\langle \boldsymbol{x}, \Psi\rangle|^2\, d^dx = |\Psi(\boldsymbol{x},t)|^2 d^dx \ , \tag{3.37}$$

according to the *Born rule*. The normalization of Ψ guarantees to find the particle somewhere in V,

$$\int_V d^dx\, P(\boldsymbol{x},t) = \int_V d^dx\, |\Psi(\boldsymbol{x},t)|^2 = \langle \Psi, \Psi\rangle = 1 \ . \tag{3.38}$$

Furthermore, the probability is *conserved* when it satisfies the "quantum continuity equation",

$$\frac{\partial}{\partial t}\int_V d^dx\, |\Psi(\boldsymbol{x},t)|^2 + \oint_{S=\partial V} d\boldsymbol{S}\cdot \boldsymbol{j}(\boldsymbol{x},t) = 0 \ , \tag{3.39}$$

where $\boldsymbol{J}$ is the probability density current given by

$$\boldsymbol{J} = -i\frac{\hbar}{2m}\left(\Psi^\star \boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^\star\right) \ . \tag{3.40}$$

How these find their analogues in quantum gravity? We could be tempted to use $|\Psi|^2$ directly as a probability measure as in (3.37), and define the probability of the universe to be in a configuration $\{h,\phi\}$ in the superspace $\mathcal{S}$ as

$$P(h,\phi) = \int_{\mathcal{S}} \mathcal{D}h\,\mathcal{D}\phi\,\sqrt{-G}\ |\Psi[h,\phi]|^2 \ , \tag{3.41}$$

where $\mathcal{D}h\,\mathcal{D}\phi\,\sqrt{-G}$ is the volume element in $\mathcal{S}$. This definition works well for a superspace reduced to the configurations which are homogeneous: the mini-superspaces. In these models, the fields - and hence the measure - do not depend on $x \in \Sigma$, which means we deal with a standard quantum mechanical system as above. Unfortunately, many issues arise. One important one is that in order to normalize Ψ, $\langle\Psi,\Psi\rangle = 1$, unphysical boundary conditions seem to be needed. For example, in the case of a minisuperspace model (see later for details) one would need to have Ψ to go rapidly to zero for large values of the scale factor. This seems in contradiction with the large universe that we observe today. Furthermore, while $|\Psi(\boldsymbol{x},t)|^2$ in (3.37) describes a *probability density* in a space of particle positions, in quantum cosmology the configurations space is a configuration of *fields*, and time is encoded in their mutual relations. Therefore, the fields cannot be considered as the analogues of particle positions and the probability conservation law (3.39) is not clearly recovered.

[6] By *partial observables* is meant observables which include time itself. It is used in [52] to differentiate two types of observables: those which can be predicted from a knowledge of an initial state (for instance the position of a particle at some time t, $q(t)$), and those which can be measured with a *measuring apparatus* as the position AND the time variable t, (q,t).

Some works [21, 54, 55] used the similarity of the WdW equation to the Klein-Gordon (KG) equation[7] (1.10) to define a conserved probability density current as in (3.40), conserved in the sense that $\nabla \boldsymbol{J} = 0$. However, since the (mini-)superspace has an indefinite metric signature, negative probabilities could rise. For those who considered the semi-classical limit[8] *i.e.* expansion of the wavefunction as a sum over the saddle points Ψ_n of its *path integral* version (see Sec.**??**), each component Ψ_n has a conserved KG-type current which flows very nearly along the direction of the classical trajectories in superspace [54–56]. This approach is interesting as it defines a kind of *time flow* in mini-superspace, and the resulting Ψ_n modes are very similar to the wavefunctions for coherent states in ordinary quantum mechanics.

Many other approaches were explored to find an appropriate measure[9]. All present pros and cons, and this reflects that we may need a better understanding of some concepts that have long been accepted. For instance, in the standard Copenhagen vision, measurements or observations are done by an apparatus which is external to the system. In quantum cosmology, the measurement apparatus is actually defined by the fields themselves. We may for instance construct probability amplitudes that the metric field be in some configuration *given that* the matter field attains some value. Thus we would be considering *conditional probabilities* rather than *absolute probabilities*[10].

3.5. Mini-superspace

One of the reasons the WdW equation we derived in (3.27) is hard to tackle, is that we deal with an infinite dimensional superspace. One can seek then for a truncation of the infinite degrees of freedom to a finite number, and hence work instead with a *mini-superspace* model, such as homogeneous geometries. The latter have been studied extensively and used as toy models for quantum cosmology, as they are easy to handle and at the same time seem to possess some predictive power [11] . However, the truncation seems to be drastic [12] and we know that at the quantum level, all the configurations for the field must be considered, even those which are not homogeneous. Nevertheless, the approximation can be useful to study quantum aspects of gravity, and more specifically - and that is at the core of the thesis - be a nice framework to characterise spacetime symmetries at the quantum level [13].

We first present one very well-known class of mini-superspace models: the quantized standard *Friedmann–Lemaître–Robertson–Walker* (FRLW) models, characterised by their spatial homogeneity and isotropy.

[7] In ordinary quantum field theory, fields obey the Klein-Gordon equation, but Ψ obeys the Schrödinger equation.

[8] Also known as the *WKB approximation*

[9] See the very nice review [56] and many references therein.

[10] See the precursor work of Page[57] on the conditional probabilities in quantum cosmology.

[11] See [58–61]

[12] As stated by Halliwell in [62] simultaneously setting configurations fields and their conjugate momenta to 0 violates the uncertainty principle.

[13] For a mathematical description of classical homogeneous and isotropic spacetimes, see appendix **??**.

The $(d+1)$-dimensional FLRW spatial metric is usually parameterized as

$$ds^2 = -dt^2 + A^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2\right] , \tag{3.42}$$

where $A(t)$ is the cosmic scale factor, $k = -1, 0, 1$ a curvature parameter whose values correspond resp. to the *open*, *flat* and *closed* case, t the *cosmic time* (that we define in next section), and $d\Omega$ the line element of the sphere $S^{(d-1)}$.

If coupled to a matter field ϕ, the model has only two d.o.f., the scale factor and the matter field. Therefore, in the quantized theory, the *mini-superspace coordinates* X^M are simply $\{A, \phi\}$. The usual infinite dimensional superspace is now reduced to a finite two-dimensional system. Instead of having a separate Wheeler-DeWitt equation for each point $x \in \Sigma$, we have a single equation for all of Σ.

Let's consider a general $(d+1)$-dimensional FLRW universe, coupled to a standard matter field ϕ. Using the parametrisation in (3.42), the action is given by

$$S = \int dt \left[-\frac{\dot{A}^2\, A^{d-2}\, d\,(d-1)}{2\,N} + \frac{A^d\, \dot{\phi}^2}{2N} + N\, A^d \left(\frac{d\,(d-1)\; k}{2A^2} - V(\phi)\right)\right] , \tag{3.43}$$

where $8\pi G = 1$. As the fields do not depend on their spatial positions, we integrated out the d-volume element. The canonical momenta and the Hamiltonian are then

$$\Pi_A = \frac{\partial L}{\partial \dot{A}} = -\frac{\dot{A}\, A^{d-2}\, d\,(d-1)}{N} \; ; \qquad \Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{A^2 \dot{\phi}}{N} \; ; \tag{3.44}$$

$$\begin{aligned} H &= \dot{A}\, \Pi_A + \dot{\phi}\, \Pi_\phi - L \\ &= N\left[-\frac{\Pi_A^2\, A^{2-d}}{2\, d\,(d-1)} + \frac{\Pi_\phi^2}{2A^d} - A^d\left(\frac{d\,(d-1)\; k}{2A^2} - V(\phi)\right)\right] \\ &= N\left[\frac{1}{2} G^{MN}\, \Pi_M\, \Pi_N - A^d\left(\frac{d\,(d-1)\; k}{2A^2} - V(\phi)\right)\right] , \end{aligned} \tag{3.45}$$

where in the last line we used the DeWitt metric as in (3.15), and Π_M, Π_N refer to the conjugate momenta of A and $\phi \longrightarrow \Pi_A$, Π_ϕ. The (inverse) DeWitt metric is now finite-dimensional and is given by $G^{MN} = \mathrm{diag}\big(-\frac{A^{2-d}}{d\,(d-1)}, \frac{1}{A^d}\big)$.

Under canonical quantization, (3.45) yields the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = \left[\frac{1}{2}\nabla^2 - A^d\left(\frac{d\,(d-1)\; k}{2A^2} - V(\phi)\right)\right]\Psi = 0 \, , \tag{3.46}$$

where the Laplacian operator ∇^2 is still to be defined, because of the factor-ordering ambiguity we mentioned in (3.27). Since the Hamiltonian constraint (3.46) is no longer a functional, as it involves simple variables A, ϕ, we could apply a "natural" choice of ordering and use the *Laplace-Beltrami operator* [21, 63] (which is the covariant generalization of the Laplacian for Riemannian geometries) of the corresponding mini-superspace metric G_{AB},

$$\nabla^2 = \frac{1}{\sqrt{-G}} \frac{\partial}{\partial X^A} \sqrt{-G}\, G^{AB} \frac{\partial}{\partial X^B} \,, \tag{3.47}$$

where $G = \det\{G_{AB}\} = d\,(d-1)\; A^{2(d-1)}$. We discuss this choice of ordering more in details at the end of the section.

Hence the operator is given by

$$\nabla^2 = \left[-\frac{1}{d\,(d-1)} \frac{1}{A^{d-1}} \frac{\partial}{\partial A} A \frac{\partial}{\partial A} + \frac{1}{A^d} \frac{\partial^2}{\partial \phi^2} \right] \,. \tag{3.48}$$

For instance, let's consider the $d = 2$ flat ($k = 0$) FLRW universe. The action is given by

$$S = \int dt \left[-\frac{\dot{A}^2}{N} + \frac{A^2\,\dot{\phi}^2}{2N} + N\,A^2\,V(\phi) \right] \,, \tag{3.49}$$

and the Hamiltonian reads

$$H = N \left[-\frac{\Pi_A^2}{4} + \frac{\Pi_\phi^2}{2A^2} - A^2 \left(\frac{k}{A^2} - V(\phi) \right) \right] \,. \tag{3.50}$$

The resulting WdW equation is

$$\hat{\mathcal{H}}\Psi = \left[\frac{1}{2}\nabla^2 - A^2\,V(\phi) \right] \Psi = 0 \,, \tag{3.51}$$

and the corresponding Laplace-Beltrami operator is

$$\nabla^2_{s,\text{FLRW}} = \left[-\frac{1}{2A} \frac{\partial}{\partial A} A \frac{\partial}{\partial A} + \frac{1}{A^2} \partial^2_\phi \right] \,, \tag{3.52}$$

Notice that it looks exactly like the spherical Laplacian (3.33) up to a factor $\frac{1}{2}$ we can remove by an appropriate rescaling of ϕ .

As the operator is particularly simple, we solve the WdW equation (3.46) for $V(\phi) = 0$.

$$\left[-\frac{\partial^2}{\partial\,(\ln A)^2} + \frac{\partial^2}{\partial \phi^2} \right] \Psi = 0 \,. \tag{3.53}$$

This corresponds to a one-dimensional wave equation whose solutions are given by

$$\Psi(A, \phi) = \Psi_+(\ln A - \phi) + \Psi_-(\ln A + \phi)^{14} \,. \tag{3.54}$$

It is interesting to compare the above wavefunctions with the classical results. By varying (3.49) (with $V(\phi)$= 0) with respect to N we obtain the Friedmann equation

$$H^2 \;\equiv\; \frac{\dot{A}^2}{A^2} \;\propto \dot{\phi}^2 \,, \tag{3.55}$$

[13]Notice that we used parenthesis instead of squared brakets for Ψ, as it is now a simple function of variables and not a functional.

(H being the so-called *Hubble constant*) which is easily integrated to give

$$\phi \;=\; \pm \ln A \;+\; \mathrm{cst}\;. \tag{3.56}$$

Naïvely, we recover the classical solutions if one of the branches of Ψ is very peaked around its argument and the other vanishes. Take, for instance, $\Psi_+(x) \sim \delta(x), \Psi_- = 0$. In this case we have

$$\Psi \sim \delta(\ln A - \phi + \mathrm{cst})\;. \tag{3.57}$$

The above is a wavefunction extremely peaked around some given classical spacetime. Clearly, we still do not have time but we can use the two variables A and ϕ relationally. In other words, we can measure ϕ with A and viceversa.

3.5.1. Measure, ordering

Regarding the inner product issues we discussed in Sec 3.4, in the mini-superspaces framework, we can build well-defined measures associated to the DeWitt metric, the latter being constructed with the coordinates of the field space. Moreover, these models provide a natural setup for the ordering of the Laplace operator and hence allow to define Hermitian operators with respect to the "standard" scalar product

$$\langle \Psi, \Phi \rangle = \int dA\, d\phi\, \sqrt{-G}\;\, \Psi(A,\phi)^\star \Phi(A,\phi)\;. \tag{3.58}$$

However, not all the issues are solved, such as the question of the boundary conditions. In order to normalize Ψ, $\langle \Psi, \Psi \rangle = 1$, we would impose Ψ to go rapidly to 0 for large values of A, $A \to \infty$ *i.e.* for an expanding universe, which is in contradiction with the observations. Moreover, the Laplace-Beltrami operator stays a choice of ordering among others, there is no prescription on how we should operate. Still, (3.47) presents nice features, as the fact that it is invariant under *field redefinition*. We discuss this point in more details in later sections. Other arguments supporting this choice of ordering are presented in [56], among which, the fact that is particularly convenient in the semi-classical expansion of the Wheeler-DeWitt equation (see [64] and references therein).

In next section, we introduce more general cosmological models which are homogeneous but not necessarilly isotropic, the *Bianchi models*. We describe some of their classical geometrical features, then we derive their quantum counterpart which provides a larger set of configurations fields.

3.6. Not-so-mini-superspace

Anisotropic cosmological models

A pictorial view of spatial homogeneity is that observers living on a same hypersurface see the same surroundings. Mathematically, this means that the hypersurfaces admit isometry groups. In particular, Bianchi studied the 3-dimensional case and classified the corresponding 3-dimensional Lie algebras into nine distinct ones according to their structure constants, resulting in nine types of *Bianchi cosmology*[15].

Precedent works [66, 67] and other approaches in [65] have shown that spatial homogeneity implies that spacetime consists of a family of space-like hypersurfaces. These hypersurfaces are geodesically parallel and define a *cosmic time* through a hypersurface-orthogonal timelike congruence of geodesics. Hence the metric can be put into the form

$$ds^2 = -dt^2 + h_{ij}(x^\mu)\, dx^i dx^j \ , \tag{3.59}$$

where t is the cosmic time. We can always rewrite the spatial metric in (3.59) as

$$h_{ij}(x^\mu)\ dx^i dx^j = h_{ij}(t)\ \boldsymbol{e}^i \otimes \boldsymbol{e}^j \ , \tag{3.60}$$

with $\boldsymbol{e}^i$ are the 1-forms associated with the isometry group. The simplest case is Bianchi type I, whose isometry group is the group of translations $\mathbb{R}^3$. It is an Abelian group, and hence, all of its structure constants are zero. Bianchi showed that all the spatial hypersurfaces are *flat* and we can choose $e^i = dx^i$, hence

$$ds^2 = -dt^2 + h_{ij}(t)\, dx^i dx^j \ . \tag{3.61}$$

The coordinates of the spatial part are for now arbitrary. But we can show that the Bianchi type I metric can always be put into the diagonal form if it is not coupled to a matter field generating *anisotropic stress*. See Sec. 3.6 for more details.

In the $d = 3$ case, we can write

$$ds^2 = -dt^2 + A^2(t)\, dx^2 + B^2(t)\, dy^2 + C^2(t)\, dz^2 \ . \tag{3.62}$$

If we couple the above with a standard scalar matter field ϕ, the mini-superspace coordinates would be $X^M \sim \{A, B, C, \phi\}$.

In the special case where $A(t) = B(t) = C(t)$, we recover the flat FLRW metric. The mini-superspace is then reduced to the d.o.f. $\{A, \phi\}$ we encountered in the previous section.

A very interesting (and more complicated) Bianchi model is Bianchi type IX, whose isometry group is $SO(3, \mathbb{R})$. In this case, the invariant 1-forms can be written as

$$\boldsymbol{e}^1 = -\sin\psi\ d\theta + \sin\theta\ \cos\psi\ d\varphi \ , \tag{3.63}$$

$$\boldsymbol{e}^2 = \cos\psi\ d\theta + \sin\theta\ \sin\psi\ d\varphi \ , \tag{3.64}$$

$$\boldsymbol{e}^3 = \cos\psi\ d\varphi + d\psi \ . \tag{3.65}$$

with (ψ, θ, φ) the Euler angles on the 3-sphere S^3 . The spatial hypersurfaces have the topology of S^3, but are not necessarily spherically symmetric. They can be squashed and twisted. In the special case where they are spherically symmetric, all the components of h_{ij} are equal and the Bianchi model corresponds to the *closed* FLRW universe. Bianchi IX played an important role in cosmology, particularly in the study of the initial singularity, and served as a foundation for the so-called *Mixmaster universe* [68–72]. The

[15] Actually different past works using different techniques lead to the same classification. See [65] for details and many references therein.

corresponding mini-superspace model has six independent coordinates, in addition to the matter fields coordinates.

The open FLRW universe can be recovered as well from Bianchi V, whose metric can be parametrised as (see [73])

$$ds^2 = -dt^2 + A^2(t)\, dx^2 + e^{2x}\left(B^2(t)\, dy^2 + C^2(t)\, dz^2\right) \,, \tag{3.66}$$

in the isotropic limit $A = B = C$. Because they are a generalisation of the three FLRW universes, Bianchi I, V and IX are considered as the most valid candidates among the other Bianchi models, according to current observations [65, 71, 73]. We shall restrict our interest to Bianchi type I, as it is the simplest one to use as a toy model for studying spacetime symmetries at the quantum level. It is the main gravitational universe we study along the thesis, and we provide a deeper analysis of its geometrical and algebraic features in later sections.

Quantum treatment

Before specializing in any cosmology, we may construct a general (not-so-)mini-superspace corresponding to an arbitrary homogeneous cosmology coupled to a standard scalar field as in (2.67). Let n be its dimension and its general coordinates denoted by $\left\{X^M\right\}$, $M = 1, 2, ..n$. We set $N_i = 0$ following (3.60). These types of models have an action of the following form

$$S = \int dt\, d^dx \left[\frac{1}{2N} G_{AB}(X)\dot{X}^A\dot{X}^B + N\sqrt{h}\left(\frac{{}^{(d)}R}{2} - V(\phi)\right)\right], \tag{3.67}$$

where G_{AB} is the “mini-supermetric” and is n-dimensional. The action (3.67) is equivalent to a simple point particle moving in a potential given by the second term in (3.67). We can even construct Christoffel symbols from the minisupermetric and hence geodesic equations, as well as an equivalent of a curvature tensor of the field configurations space[16] (see [21, 56]) .

The canonical momenta and the Hamiltonian are simply given by

$$\Pi_A = \frac{\partial L}{\partial \dot{X}^A} = \frac{G_{AB}\,\dot{X}^B}{N} \,, \tag{3.68}$$

$$H = \int d^dx\, N\left[\frac{1}{2}G^{AB}\,\Pi_A\Pi_B + \sqrt{h}\left(\frac{{}^{(d)}R}{2} - V(\phi)\right)\right] \tag{3.69}$$

$$= \int d^dx\, N\,\mathcal{H} \,, \tag{3.70}$$

with $\mathcal{H}$ the Hamiltonian constraint.

As the configuration space is finite-dimensional, the quantum theory is greatly simplified. Under canonical quantization, (3.69) yields the Wheeler-DeWitt equation

[16] We provide a deeper analysis about this *meta-universe* as we called it in Sec. 9.2 in the context of anisotropic cosmologies, which were the main geometrical models we used to study spacetime symmetries at the quantum level.

$$\hat{\mathcal{H}}\Psi = \left[-\frac{1}{2}\nabla^2 + \sqrt{h}\left(\frac{^{(d)}R}{2} - V(\phi)\right)\right]\Psi = 0\,, \tag{3.71}$$

where we used the Laplacian-Baltrami operator ∇^2 we introduced in (3.47).

Now let's apply the same construction for the $d = 2$ Bianchi type I cosmology. We introduce a new parametrisation for the metric,

$$ds^2 = -dt^2 + A^2(t)\, e^{\xi}\, dx^2 + A^2(t)\, e^{-\xi}\, dy^2\,, \tag{3.72}$$

where A is the usual scale factor, and ξ the shear. The isotropic limit is recovered when $\xi = 0$.

The action is given by

$$S = \int dt\left(-\frac{\dot{A}^2}{N} + \frac{A^2\,\dot{\xi}^2}{4N} + \frac{A^2\,\dot{\phi}^2}{2N} + N\,A^2\,V(\phi)\right)\,. \tag{3.73}$$

Following the same steps as above, the Hamiltonian reads

$$H = N\left[\frac{1}{2}G^{AB}\,\Pi_A\Pi_B - A^2\,V(\phi)\right]^{17} \tag{3.74}$$

$$= N\left[-\frac{\Pi_A^2}{4} + \frac{\Pi_\xi^2}{A^2} + \frac{\Pi_\phi^2}{2A^2} + N\,A^2\,V(\phi)\right]\,. \tag{3.75}$$

Hence,

$$\nabla^2_{\text{s, Bianchi}} = \frac{1}{A^2}\left[-\frac{A^2}{2}\partial_A^2 - A\,\partial_A + 2\,\partial_\xi^2 + \partial_\phi^2\right]\,. \tag{3.76}$$

The diagonalisability of the spatially flat homogeneous metric

We show how a homogeneous metric can be put in a diagonal form for any time t, in absence of matter source generating anisotropic stress.

Consider a general spatially flat homogeneous metric which, according to what we said in the previous section, can always be written as

$$ds^2 = -dt^2 + h_{ij}(t)\, dx^i dx^j\,. \tag{3.77}$$

In the literature, it is common to choose coordinates such that h_{ij} is diagonal. The diagonal form is particularly useful when one wants to study anisotropies. These are geometrically characterized by direction-dependent expansion rates. Hence, the diagonal form allows to easily extract them from the eigenvalues of the metric. However, one may wonder if $h_{ij}(t)$ needs to stay diagonal at all times? For that, we look at the dynamics. The spatial components of the Einstein equations for a metric in the form (3.77), in $d+1$

[17] Notice that in Bianchi type I, the d-volume element is completely integrated out as the fields do not depend on their spatial position in the hypersurfaces Σ.

dimensions are

$$\ddot{h}_{ij} + \frac{1}{2}\dot{h}_{ij}\, h^{kl}\, \dot{h}_{kl} - \dot{h}_{ik}\, h^{kl}\, \dot{h}_{lj} = 16\pi G \left(T_{ij} - \frac{1}{d-1} h_{ij}\, T^{\mu}{}_{\mu} \right) , \tag{3.78}$$

where $T_{\mu\nu}$ is the stress-energy tensor of an arbitrary matter source.

The spatial piece of (3.77), as well as the dynamical equation (3.78) are invariant under the general linear symmetry group $GL(d, \mathbb{R})$ [18]. Let's define initial conditions on $h_{ij}(t)$ and $\dot{h}_{ij}(t)$ at some given time t_0. We can choose Cartesian coordinates such that the metric is equal to identity δ_{ij} . As the rotation group $O(d, \mathbb{R})$ is the subgroup of $GL(d, \mathbb{R})$ under which the identity is invariant, $O^T\, \delta_{ij}\, O = \delta_{ij}$, we can bring $\dot{h}_{ij}(t_0)$ into the diagonal form by simply rotating the space axis, without affecting $h_{ij}(t_0)$. To sum up,

$$h_{ij}(t_0) = \delta_{ij} \; , \qquad \dot{h}_{ij}(t_0) = \mathrm{diag}\left(A^2(t_0), B^2(t_0), ... X^2(t_0) \right) . \tag{3.79}$$

Now, can this form be preserved? From the equation of motion (3.78), it is clear that the metric will stay diagonal at all times t, provided T_{ij} does not contain off-diagonal entries *i.e.* the matter source does not generate anisotropic stress.

3.7. Mini conclusion

One of the fundamental properties of quantum gravity we learned so far is the invariance of $\Psi[h_{ij}]$ under time and space reparametrisation of h_{ij}. Because of the absence of an external time parameter, the theory lacks of a natural probabilistic interpretation. Time is actually encoded in the mutual relations between the fields of the classical theory. For instance, in the mini-superspace framework Sec. 3.5, we saw that we could use the scale factor A and the scalar field ϕ relationally. One can be set to be/as the "clock" to parametrise the "time" evolution of the other.

The same analysis can be done for space. The momentum constraints ensure that the wavefunction only depends on geometrical invariants of the spatial hypersurfaces Σ. Hence, a quantum state is not localized "somewhere" in space. There is no background to define for instance translation or rotation of the spatial metric "$h_{ij}(\vec{x} - \vec{a})$" , " $h_{ij}(O.\vec{x})$", where the $\vec{x}$ coordinates are arbitrary. So how can we characterize these symmetries for a generic metric (with no isometries) in a coordinate-independent fashion?

Clearly, we need a type of matter fields that can serve as a "spatial reference frame", the same we did for time. The fields then would be not localized on a fixed background, but localized with respect to one another.

A particular matter field stands out for this purpose: *solids*. Solids have been used as matter fields driving a primordial inflationary phase, in a cosmological model that goes under the name of *solid inflation*, [27], see Sec. 6. The effective field theory of solid inflation differs drastically from the standard inflationary scenario, in the fact that vacuum expectation values of the scalar fields involved do not depend on time, but on *space*. This means that spatial diffeormorphisms are spontaneously broken. This is the particular feature we can use to kind of define spatial axes with respect to which we "move" the gravitational field.

[18] We discuss the symmetry features in more details in Sec. 9

Part III.

Inflation

4. Observational motivations

Contents

The Standard Model of Cosmology has successfully predicted many current observations of our universe. The *nucleosynthesis* of the light elements, the temperature and blackbody spectrum of the *cosmic microwave background* (CMB), and the redshift of light from galaxies revealing an expanding universe are examples of this successful history of predictions [74], [75]. However, there are still remaining unsolved mysteries. Some of them can be seen as initial value type of problems, such as the so-called *horizon* and *flatness problems*: how comes that the universe is so homogeneous and isotropic? How is it possible that the spatial curvature of the universe is so small? These are clearly problems that can be fixed by extremely fine tuned initial conditions at the Big Bang, which however look unnatural and unmotivated.

4.1. The Horizon problem

One fundamental principle in cosmology and modern astronomy is that the universe is homogeneous and isotropic, as we believe that there is no preferred location in space and no preferred direction to look in. A good observation in accordance to this statement is the *Cosmic Microwave Background* (CMB), which is a remnant from the very first moments of the universe.

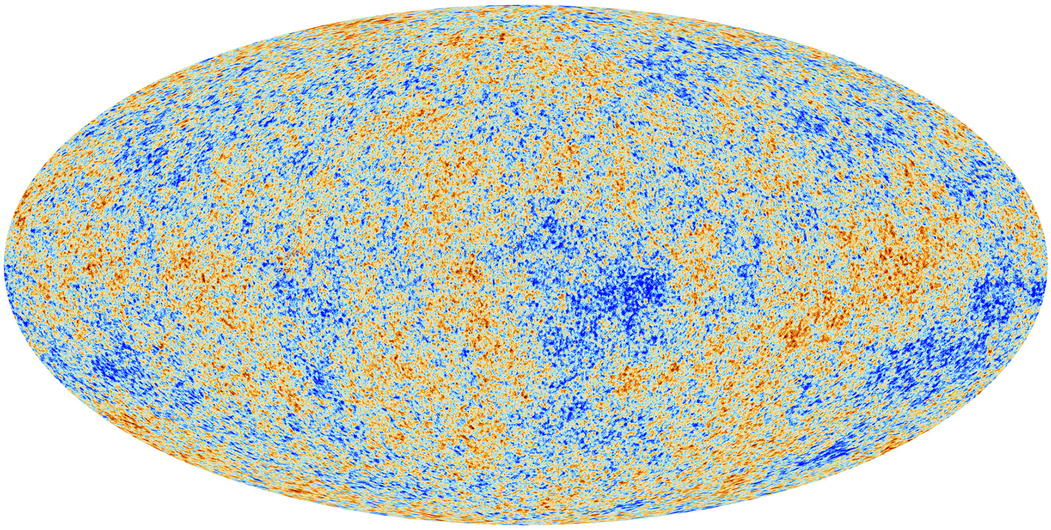


Figure 4.1.: The inhomogeneities of the Cosmic Microwave Background (CMB) as observed by Planck. Credits: *ESA and the Planck Collaboration*

This radiation has an approximately constant temperature at $2.73K$ with very small anisotropies [76]. This indicates that several regions of the universe must have been in thermal contact at some point in the past. However, if we consider the standard cosmological solutions (FRLW, Bianchi etc.), these regions cannot have been in causal (and hence thermal) contact [77], [78].

A *comoving particle horizon* is the causal horizon or the maximum distance a light ray can travel between time 0 and t

$$\tau \equiv \int_0^t \frac{\mathrm{d}t'}{A\,(t')} = \int_0^A \frac{\mathrm{d}A}{HA^2} = \int_0^A \,\mathrm{d}\ln A \left(\frac{1}{AH}\right) \,. \tag{4.1}$$

Here, we have expressed the comoving horizon as an integral of the *comoving Hubble radius*, $(AH)^{-1}$, which plays a crucial role in inflation.

We recall the energy-momentum tensor of a perfect fluid in thermal equilibrium

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha\, u_\beta + p\, g_{\alpha\beta} \;, \tag{4.2}$$

with ρ the mass-energy density, p the hydrostatic pressure and u^α the fluid's velocity satisfying $u^\alpha\, u_\alpha = -1$. Essentially all the perfect fluids relevant to cosmology obey the simple equation of state $p = w\,\rho$, with w a constant independent of time.

For a universe dominated by a fluid with equation of state w, we have

$$(AH)^{-1} = H_0^{-1} A^{\frac{1}{2}(1+3w)} , \tag{4.3}$$

where H_0 is the *Hubble-Lemaître parameter* measured at some given time. Notice the dependence of the exponent on the combination $(1+3w)$. Whether $(1+3w)$ is positive or negative, the Hubble radius is growing or shrinking. During the conventional Big Bang expansion ($w \gtrsim 0$), $(AH)^{-1}$ grows monotonically and the comoving horizon τ or the fraction of the universe in causal contact increases with time

$$\tau \propto A^{\frac{1}{2}(1+3w)}\text{[1]}. \tag{4.4}$$

This reveals that the comoving horizon grows monotonically with time, which implies that comoving scales entering the horizon today have been far outside the horizon at CMB decoupling. But the high homogeneity of the CMB tells us that the universe was extremely homogeneous at the time of last-scattering, on many regions that *a priori* are causally independent.

4.2. The flatness problem

The *flatness problem* refers to a cosmological *fine-tuning* problem. Such problems are typical of systems which are very sensitive to initial conditions. Small deviations from these values result in a significant change in future times/late-times state of the system. In cosmology, many parameters seem to be fine-tuned to very specific values to explain current observations. The flatness problem relates to the density of matter and energy in the universe. More precisely, the ratio of the density to the critical density at the present epoch, Ω_0, which constrains the curvature of the universe.

We define the curvature parameter as

$$\Omega_k \equiv \Omega - 1 = \frac{\rho - \rho_{\text{crit}}}{\rho_{\text{crit}}} , \quad \text{where} \quad \rho_{\text{crit}} \equiv 3H^2 . \tag{4.6}$$

We know from the equations of GR that spacetime is dynamical, curving in response to matter in the universe. However, observations show that the universe is very close to the flat configuration [79], [77]. How can we explain this?

Let's consider a standard 4-dimensional FLRW cosmological solution coupled to perfect fluids. One of the so-called *Friedmann Equation* is given by

$$H^2 = \frac{1}{3}\rho(A) - \frac{k}{A^2} , \tag{4.7}$$

derived from the 00th component of the Einstein field tensor.

[1] For radiation-dominated and matter-dominated universes, we find

$$\tau \propto \begin{cases} A & \text{RD} \\ A^{1/2} & \text{MD} \end{cases} \tag{4.5}$$

We define the *density parameter* and the *critical density*

$$\Omega(A) \equiv \frac{\rho(A)}{\rho_{\text{crit}}(A)} , \quad \rho_{\text{crit}}(A) \equiv 3H(A)^2 , \tag{4.8}$$

which are very often used cosmology. Using the above, we rewrite (4.7) as

$$1 - \Omega(A) = \frac{-k}{(AH)^2} . \tag{4.9}$$

In standard cosmology the comoving Hubble radius, $(\mathrm{AH})^{-1}$, grows with time and from (4.9), the quantity $|\Omega - 1|$ must thus diverge with time. Observations today show that $\Omega(A_0) \sim 1$ [77, 79]. The critical value $\Omega = 1$ is an unstable fixed point. This means that in standard Big Bang cosmology (without inflation) and using the FLRW models, the density parameter $\Omega(A)$ was very close to the spatially flat case $\Omega(A) = 1$ at early times. More specifically, the deviation from flatness at the Planck scale would be (see [77, 79–81])

$$|\Omega(A_{\text{pl}}) - 1| \leq \mathcal{O}\left(10^{-62}\right) [2] . \tag{4.10}$$

Why is $\Omega(A_0)$ so close to 1, and not much larger or smaller?

4.3. Inflation: Big Bang puzzles solved

The inflationary universe scenario, *i.e.* an early phase of exponential expansion of the universe solves these problems by drastically changing the past light cone in a such way that it removes the horizon problem, while also driving Ω close to unity and hence preserving the successes of the Big-Bang model.

Inflation was initially introduced to solve these particular initial conditions puzzles [82–84]. It was soon discovered that inflation could also provide a mechanism to seed the initial fluctuations that would eventually give rise to the CMB and the observed large-scale structures in the universe.

In describing the flatness and horizon problems, we used a lot the comoving Hubble radius $(AH)^{-1}$, which plays a fundamental role. Actually, both issues arise from the fact that in the conventional Big Bang cosmology, the comoving Hubble radius is *strictly increasing*. What if the opposite happened at early times? A decreasing Hubble radius would solve these puzzles at once, and this is the key (and elegant) idea of inflation.

Recall the comoving horizon we defined earlier

$$\tau = \int_0^A \mathrm{d}\ln A \left(\frac{1}{AH}\right) . \tag{4.13}$$

[2] Other scales can be computed, such as those of the *Grand Unification Theory* era, or the *Big Bang Nucléosynthesis*. We find

$$|\Omega(A_{\text{BBN}}) - 1| \leq \mathcal{O}\left(10^{-16}\right) , \tag{4.11}$$
$$|\Omega(A_{\text{GUT}}) - 1| \leq \mathcal{O}\left(10^{-55}\right) . \tag{4.12}$$

If particles are separated by distances greater than τ, they never could have communicated one with another. If they are separated by distances greater than $(AH)^{-1}$, they cannot communicate with each other *now*. Particles that are not in causal contact now ($\tau > (AH)^{-1}$), could have been in causal contact before. From (4.13), this could happen if the comoving Hubble radius in the early universe was much larger than it is now, which demands a decreasing phase of the Hubble radius. For an H that is approximately constant and A growing exponentially, the trick is done.

For a schematic picture, see the well-known Fig. 7 of [85].

Regarding the flatness and horizon problems, everything becomes trivial. First, according to

$$1 - \Omega(A) = \frac{-k}{(AH)^2} \,, \tag{4.14}$$

if the comoving Hubble radius decreases, then the universe tends towards flatness.

Second, a decreasing horizon means that large scales entering the present universe were *inside* the horizon before inflation. The needed causal structure at early times is recovered, and spatial homogeneity observed in the CMB is explained.

More than solving the Big Bang puzzles, the decreasing comoving horizon during inflation is the key feature required for the quantum generation of *cosmological perturbations*, that we do not discuss in the thesis but many references can be found in the literature.

More details and conditions for inflation are provided in the following section.

5. Slow-roll inflation

Contents

5.1. Scalar field dynamics

Inflation is a very intriguing phenomenon. Within a fraction of a second, the universe has expanded exponentially at an accelerating rate. In Einstein gravity, this requires a *negative* pressure source.

The simplest models of inflation involve a single scalar field ϕ, the *inflaton*. Without specifying its physical nature, we use ϕ as a clock to parametrize the time-evolution of the inflationary energy density. Let's consider the 4-dimensional case. The dynamics of a scalar field minimally coupled to gravity is determined by the action

$$S = S_{\text{grav}} + S_{\text{scalar}} \ , \tag{5.1}$$

$$= \frac{1}{2}\int d^4x\,\sqrt{-g}\;R - \int d^4x\,\sqrt{-g}\;\frac{1}{2}\left(g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + V(\phi)\right) \ , \tag{5.2}$$

The energy-momentum tensor for the scalar field is

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{scalar}}}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi + V(\phi)\right) \ . \tag{5.3}$$

Assuming the FLRW metric (3.42) for $g_{\mu\nu}$ and restricting to the case of a homogeneous field $\phi(t,\boldsymbol{x}) = \phi(t)$[1], the scalar energy-momentum tensor takes the form of a *perfect fluid*. We recall the energy-momentum tensor of a perfect fluid in thermal equilibrium

$$T_{\alpha\beta} = (\rho + p)\,u_\alpha\,u_\beta + p\,g_{\alpha\beta} \ , \tag{5.4}$$

with ρ the mass-energy density, p the hydrostatic pressure and u^α the fluid's velocity satisfying $u^\alpha\,u_\alpha = -1$. By matching (5.3) and (5.4), we find

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \ , \tag{5.5}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \ . \tag{5.6}$$

[1]Actually, the scalar field ϕ can take any form. The reason for this choice is that our universe is homogeneous and isotropic at large scales. Therefore, except from small fluctuations, we may assume that ϕ is homogeneous and isotropic.

The resulting equation of state is[2]

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \ . \tag{5.8}$$

Eq. (5.8) shows that a scalar field can lead to negative pressure, $w_\phi < 0$, and an accelerated expansion for $w_\phi < -\frac{1}{3}$, according to the equation

$$\frac{\ddot{A}}{A} = -\frac{4\pi G}{3}(\rho + 3p) \ . \tag{5.9}$$

The dynamics of the scalar field and the FLRW geometry is governed by

$$\ddot{\phi} + 3H\,\dot{\phi} + V_{,\phi} = 0 \ , \tag{5.10}$$

$$\frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = H^2 \ , \tag{5.11}$$

where $H = \frac{\dot{A}}{A}$ is the Hubble constant. For large values of the potential, the field undergoes a significant Hubble friction from the term $H\,\dot{\phi}$.

5.2. Slow-roll parameters

According to (5.16), the universe undergoes an accelerated expansion if $1 + 3w < 0$. In particular, if we want the universe to be close to the *de Sitter limit*[3] , $w = -1$, then the potential energy should dominate over the kinetic energy *i.e.* the field is in a "slow-roll" motion,

$$\frac{\dot{\phi}^2}{2} \ll V(\phi). \tag{5.12}$$

[2]The perfect fluids relevant to cosmology obey the equation of state $p = w\,\rho$, with w a constant independent of time. The conservation of energy yields

$$\rho \propto A^{-3(1+w)} \ , \tag{5.7}$$

with $A(t)$ the usual cosmic scale factor. The most familiar cosmological fluids are *dust*, whose equation of state is $w = 0$, and *radiation*, $w = \frac{1}{3}$ [74].

[3]Reminder, the *de Sitter* spacetime is the maximally symmetric vacuum solution of Einstein's field equations with a positive cosmological constant, corresponding to a positive vacuum energy density and negative pressure. It is one of the simplest models consistent with the observed accelerated expansion.

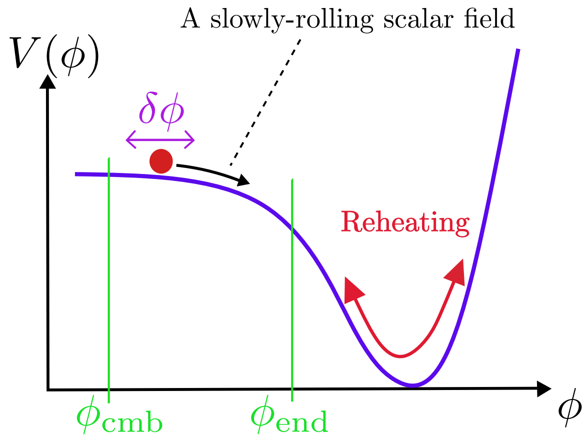


Figure 5.1.: The shape of a typical slow-roll potential $V(\phi)$. The accelerated expansion happens when $V(\phi)$ dominates over the kinetic energy $\frac{1}{2}\dot{\phi}^2$. Inflation ends at $\phi_{\rm end}$, when $\frac{1}{2}\dot{\phi}^2 \approx V(\phi)$. Quantum fluctuations of the field $\delta\phi$ create CMB fluctuations about 60 e-folds before the end of inflation. At reheating, the field oscillates around the minimum and its energy is converted into radiation.

Another condition is imposed to ensure that the accelerated expansion lasts for a sufficiently long period of time, namely that the second derivative of ϕ is small enough,

$$\left|\frac{\ddot{\phi}}{3H\dot{\phi}}\right| \ll 1 \,. \tag{5.13}$$

The two above conditions for slow-roll and long lasting inflation can also be expressed as conditions on the shape of the inflationary potential

$$\epsilon_{\rm v}(\phi) \equiv \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1 \,, \tag{5.14}$$

$$|\eta_{\rm v}(\phi)| \equiv \left|\frac{V_{,\phi\phi}}{V}\right| \ll 1 \,. \tag{5.15}$$

The parameters $\epsilon_{\rm v}$, $\eta_{\rm v}$ are called the *potential slow-roll parameters*. Other slow-roll parameters can be defined. We may recast (5.16) applied to ϕ as

$$\frac{\ddot{A}}{A} = -\frac{1}{6}\left(\rho_\phi + 3p_\phi\right) = H^2\left(1-\epsilon\right) \,, \tag{5.16}$$

and where we have set $8\pi G = 1$. Hence,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \,. \tag{5.17}$$

Acceleration occurs at $\epsilon < 1$, and the de Sitter limit is reached when $\epsilon \to 0$.

The equations of motions (5.10), (5.11) then become

$$H^2 \approx \frac{1}{3}V(\phi) \approx \text{ const. } , \tag{5.18}$$

$$\dot{\phi} \approx -\frac{V_{,\phi}}{3H} . \tag{5.19}$$

From (5.18), the spacetime is approximately de Sitter

$$A(t) \sim e^{Ht}. \tag{5.20}$$

Inflation ends when at least one of the two slow-roll conditions is violated

$$\eta_{\rm v}\left(\phi_{\rm end}\right) \approx 1 , \quad \text{or} \quad \epsilon_{\rm v}\left(\phi_{\rm end}\right) \approx 1 . \tag{5.21}$$

The number of e-folds before inflation ends is

$$\begin{aligned} N(\phi) &\equiv \ln \frac{a_{\rm end}}{a} \\ &= \int_t^{t_{\rm end}} H \text{ d}t \approx \int_{\phi_{\rm end}}^{\phi} \frac{V}{V_{,\phi}} \text{d}\phi, \end{aligned} \tag{5.22}$$

where we have used the slow-roll equations of motion (5.18) and (5.19).

In order to be solved, the horizon and flatness problems require that the total number $N_{\rm tot}$ of inflationary e-folds exceeds about 60 [4]

$$N_{\rm tot} \equiv \frac{a_{\rm end}}{a_{\rm start}} \gtrsim 60 . \tag{5.23}$$

The exact value depends on many parameters, including the energy scale of inflation and the *reheating*, which corresponds to a post-inflationary phase where the field begins to oscillate around the minimum of the potential and its energy is converted into radiations, ([86–88]). The fluctuations on the CMB occurred in $N_{\rm cmb} \approx 40 - 60$ before the end of inflation.

Inflation and observations

So far, we have presented all the setup that is used to solve the three puzzles of standard Big Bang theory we mentioned in the previous section. In fact, inflation goes beyond solving the mysteries of the primordial universe, and can be used to explain the large-scale structures.

It is important to stress that the flatness and horizon problems are not strict inconsistencies *per se*, but rather a questioning of the very constraining and improbable aspect of the initial data that would explain the observations. It is perfectly fine to assume that Ω was close to unity and that the universe started homogeneously over superhorizon distances, but with the right level of inhomogeneities to explain structure formation. But we would like for these features to be predicted by the model, rather than simply assumed in initial conditions.

[4] See [85, 86] and many references therein.

It turns out that inflation can solve the fine-tuning problem in an elegant way, while predicting the large-scale structures, through a quantum mechanical principle: *quantum fluctuations*. Around a "background" value, a scalar field ϕ can undergo quantum fluctuations $\delta\phi$. Due to these fluctuations, not all patches in the universe are inflated by the same "amount". During inflation, these inhomogeneities are stretched across the horizon scale and thus smoothed out. These become the seeds of the initial small inhomogeneities, and continue to grow after the end of inflation because of gravitational instability.

Although inflation is a very appealing theory in terms of the answers it provides, it has one main drawback: there are (too) many different models, depending on the potential $V(\phi)$ we consider. We could even introduce more than one scalar field, as long as the two slow-roll conditions are satisfied[5]. There are physical constraints and consistency relations that must be satisfied by the inflationary model, but still, the parameter space of the consistent models is still extensive, and it is hard to narrow down to a single satisfying inflationary model [79].

One could criticize this aspect, and say that we simply moved the fine-tuning problem elsewhere. However, inflation brings answers to many observational mysteries at once, and it may be worth to pursue in this direction. For instance, we could explore models which are drastically different than a standard scalar field inflation, such as *solid inflation*, where the fields involved in the model do not depend on time, but space. This yields whole different features in terms of the effective field theory. We provide an overview of the model in the following section.

[5]We cannot account for the many works done in the topic, but see [89–98] and references therein.

6. Solid inflation

Contents

We present one cosmological model that stands out from the standard inflationary scenario, *solid inflation*, where primordial inflation is driven by a "solid", [27]. The authors studied scalar fields whose dynamics and symmetry features are equivalent to those of a *cosmological solid*, and constructed the model relying on the framework of effective field theory. While slow roll inflation has been generalized in several ways (*e.g.* by changing the dynamics of the scalar of by adding more fields) without modifying the basic symmetry breaking pattern. In this sense, solid inflation belongs to a different universality class altogether. The different universality classes have been discussed in [99].

The effective field theory of solid inflation differs drastically from the standard inflationary scenario in the underlying symmetries and the symmetry breaking patterns. In solid inflation, the vacuum expectation values of the scalar fields involved do not depend on time (as in usual inflationary models) but on *space*. This means that spatial diffeormorphisms are spontaneously broken.

This seems to contradict two main important features we usually like to implement in an inflationary scenario.

- First, that the background is homogeneous and isotropic.
- Second, in an expanding universe physical quantities depend on time, among which one is defined as a physical "clock" that determines when inflation ends.

In fact, x-dependent matter fields can be compatible with homogeneous and isotropic cosmological solutions, as long as extra symmetries are imposed on these fields. For instance, FLRW cosmological solutions for the gravitational field are derived from an homogeneous and isotropic background stress-energy tensor. However, these can also be recovered from matter fields which are not homogeneous nor isotropic, if there are *internal symmetries* acting on the fields that can reabsorb the variations obtained after a translation or rotation has been applied. One simple example is that of a scalar field with a vacuum expectation value

$$\langle\phi\rangle = \alpha\, x\ . \tag{6.1}$$

The above configuration seems to break translation along x. However, if we impose an internal *shift symmetry*

$$\phi \longrightarrow \phi + a\ , \qquad a = \text{const}\ , \tag{6.2}$$

then (6.1) is invariant under the combined spatial translation and internal shift symmetry. Same goes for isotropy, but for that we need more fields than the simple scalar field configuration (6.1), and hence more symmetries. Cosmological solids are made up with these properties that we define in the following section.

6.1. Effective field theory for solids and fluids

We describe the mechanical degrees of freedom of a solid from an effective field theory standpoint. Consider a medium filling space. We neglect potential gravitational effects and consider a flat metric. Each volume element of the medium can be attached to a comoving label ϕ^I, with $I = 1, 2...d$, d corresponding to the spatial dimension. Fig. 6.1 illustrates how the fields "label" and follow a solid volume element.

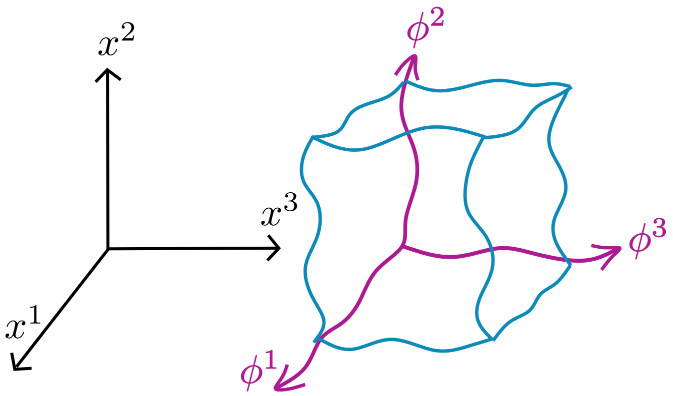


Figure 6.1.: A 3-dimensional solid volume element (in blue) labelled by the scalar fields ϕ^I, $I = 1, 2, 3$. (Credits: I thank Alberto Nicolis for letting me use his sketch of the solid volume element.)

Hence, the system is parametrized via $\phi^I(\boldsymbol{x}, t)$, whose expectation values are given by

$$\langle \phi^I \rangle = \alpha \, x^I \ . \tag{6.3}$$

The configuration above corresponds to a medium that is at rest, in equilibrium, at given external pressure. The parameter α measures the compression level.

As we would like the medium to be homogeneous and isotropic, but (6.3) breaks spatial translations and rotations, we impose two internal symmetries

$$\text{(internal translations = shifts)} \qquad \phi^I \longrightarrow \phi^I + a^I, \qquad a^I = \text{const} \ , \tag{6.4}$$

$$\text{(internal rotations)} \qquad \phi^I \longrightarrow O^I{}_J \phi^J, \qquad O^I{}_J \in SO(d) \ , \tag{6.5}$$

so that the background configurations (6.3) are invariant under combined spatial translation/internal shift symmetries, and combined spatial rotations/internal rotations.

Imposing a complete $SO(d)$ invariance on the solid is equivalent to considering a solid with no preferred axes, a "jelly". While some solids might present preferred axis that break $SO(d)$ (*e.g.* crystals), in most cases an effective $SO(d)$ invariance is restored at sufficiently large scales.

The only task left now is to construct a Lagrangian for the solid with the above properties. The shift forces each field to appear with at least one derivative. As the theory is defined on a gravitational background, we should recover diffeomorphism invariance. Thus, the derivatives should be contracted among themselves with the metric field. At lowest order in the derivative expansion, the object satisfying all these properties is

$$B^{IJ} \equiv g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J \;, \tag{6.6}$$

which is a spacetime-scalar, shift-invariant symmetric matrix with internal indices.

Then, for a generic solid in d spatial dimensions, the effective Lagrangian must be a function of the d independent $SO(d)$ invariants that one can build out of B . For instance,

$$X_1 = [B] \;, \qquad X_2 = [B^2] \qquad \ldots \qquad X_d = [B^d] \;, \tag{6.7}$$

where $[\cdots]$ denotes the trace of the matrix within. Or, following [27], one can take the trace of B and normalized versions of the traces of B^n:

$$X = [B] \;, \qquad Y_2 = \frac{[B^2]}{[B]^2} \qquad \ldots \qquad Y_d = \frac{[B^d]}{[B]^d} \;. \tag{6.8}$$

X enables to keep track of the "size" of the matrix B, while the other invariants $Y_2, ...Y_d$ are insensitive to an overall rescaling of B.

Therefore, to lowest order in derivatives, the action of a solid in $d+1$ dimensions then is

$$\begin{aligned} S_{\text{solid}} &= -\int dt\; L_{\text{solid}} \\ &= -\int dt\, d^dx\; h^{1/2}N\; F(X, Y_2, \ldots, Y_d) \;, \end{aligned} \tag{6.9}$$

where F is a generic function, related to the equation of state of the solid.

Aside: Perfect fluids

The general description above also applies to perfect fluids, but with infinitely many more internal symmetries. A fluid is characterized by the fact that we can move its volume element around in an *adiabatic* manner, without spending any energy. By contrast, if we want to move a solid's volume element, we would encounter stresses that push the volume element back to its rest position. Mathematically, this property corresponds to an invariance of the fluid's dynamics under internal volume-preserving diffeomorphisms

$$\phi^I \longrightarrow \xi^I(\phi) \;, \qquad \det\frac{\partial\xi^I}{\partial\phi^J} = 1 \;, \tag{6.10}$$

where $\dfrac{\partial\xi^I}{\partial\phi^J}$ is the associated Jacobian.

For instance, in the $d=3$ dimensional case, of the $SO(3)$ invariants (6.7), one particular combination survives: the determinant of B

$$\det B = \frac{1}{6}\left([B]^3 - [B]\,[B^2] + 2[B^3]\right) \,. \tag{6.11}$$

The general action for the fluid at lowest order in the derivative expansion, is given by

$$S_{\text{fluid}} = -\int dt\, d^d x\ h^{1/2} N \quad F(\det B)\,. \tag{6.12}$$

Therefore, the fluid is a solid, with (infinitely) many more symmetries.

6.2. Solid inflation and predictions

As mentioned before, in solid inflation time-translations are not broken. There are physical “clocks”, but played by the metric not the matter fields, as in usual inflationary models. More precisely, it is played by the gauge invariant observables made up of the scalars and of the metric, like the energy density or the pressure. This results in a symmetry breaking pattern and thus a different universality class than standard slow-roll inflation [99]. The different observational predictions are summarized in the figure below,

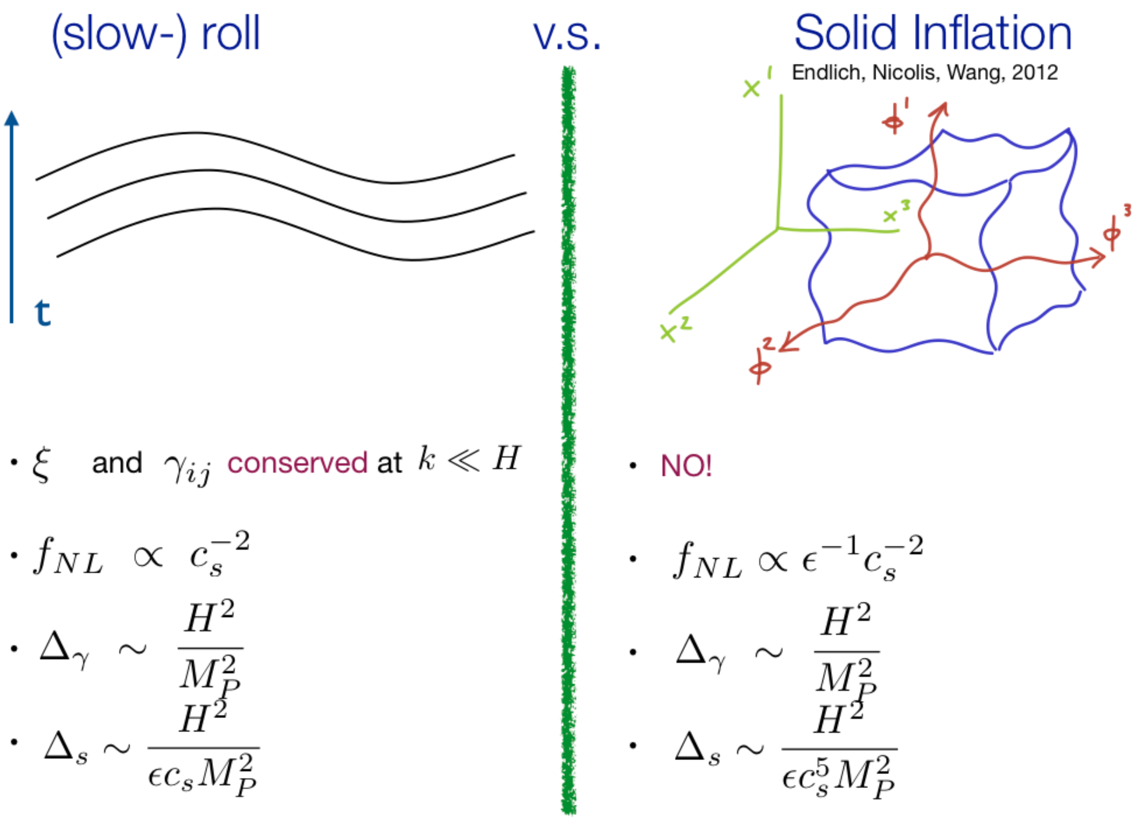


Figure 6.2.: Summary of the observational results. Credits: *Federico Piazza, talk at Atelier TUG Montpellier Oct 4-6 2022.*

where

- ξ is the adiabatic curvature perturbations,
- γ_{ij} are the gravitational waves,
- f_{NL} is a measure of non-Gaussianity, and c_s the speed of sound,
- ∇_γ power spectrum of the gravitational waves,
- and finally∇_s power spectrum of scalar perturbations.

All these quantities are evaluated when the modes exit the horizon. We do not explain in details their significance, but invite the interested readers to see [27, 99].

In the thesis, we have studied the Bianchi type I model coupled to a solid matter field, which we present in details in Part IV. In this framework, we also define a slow-roll parameter $\epsilon \equiv -\frac{\dot{H}}{H^2}$ which, under some conditions, parametrizes an inflationary expansion. Models of solid inflation are known for not being efficient in diluting away anisotropy [28]. While confirming this fact, we find another potential feature of solid inflation, namely a *rotation* of the principal axes of the expansion.

Part IV.

Homogeneity and Isotropy at the level of the wavefunctional

7. Introducing the model

Primordial inflation is extremely efficient at turning generic initial conditions into a highly homogeneous and isotropic spacetime [100]. Inhomogeneities are stretched out to unobservably large scales. Random initial anisotropies are also rapidly diluted away by the quasi-exponential expansion. In this thesis we point to another potential feature—"*rotation*"—that inflation can remove and that, to our knowledge, has never been previously considered.

In the homogeneous limit, anisotropies can be understood geometrically as a direction-dependent expansion rate. Given a general spatially flat homogeneous metric

$$ds^2 = -dt^2 + h_{ij}(t)dx^i dx^j \,, \tag{7.1}$$

it is customary to choose coordinates such that h_{ij} is diagonal, so that the different expansion rates can be easily read off. However, one may wonder, does h_{ij} need to stay diagonal at all times? In the absence of anisotropic stress in the matter sector, it is easy to see that it does.

To this end, consider the spatial components of the Einstein equations for a metric in the form (7.1) in $d+1$ dimensions,

$$\ddot{h}_{ij} + \frac{1}{2}\dot{h}_{kl}h^{kl}\dot{h}_{ij} - \dot{h}_{ik}h^{kl}\dot{h}_{lj} \;=\; 16\pi G\left(T_{ij} - \frac{1}{d-1}\,h_{ij}T^{\mu}{}_{\mu}\right) . \tag{7.2}$$

The metric maintains the form (7.1) under any general coordinate-independent linear transformation. The invariance of (7.2) under $GL(d,\mathbb{R})$ is clearly a remnant of the original diffeomorphism invariance. By using an element of such a group, at some given instant $t = t_0$ the metric can be made proportional to the identity. At the same time, a suitable rotation can diagonalize $\dot{h}_{ij}(t_0)$ without affecting $h_{ij}(t_0)$. From (7.2) it is then clear that the metric will stay diagonal at all times, provided T_{ij} does not contain off-diagonal entries.

In the presence of anisotropic stress in the matter fields, however, the spatial metric does not remain diagonal at all times, and this corresponds to a non trivial "rotation". Rotation of what with respect to what? There is more than one way to answer this question but the most physical approach is probably to focus on the *derivative*[1] of the metric, $\dot{h}_{ij}$, which can be decomposed as

$$\dot{h}_{ij}(t) = \sum_{n=1}^{d} H_n(t)\; \hat{u}_i^{(n)}(t)\; \hat{u}_j^{(n)}(t)\,. \tag{7.3}$$

The eigenvalues H_n are direction-dependent "Hubble rates", while the instantaneous eigenvectors can be seen as the principal expansion axes, or as the principal axes of the

[1] A mathematically more convenient choice is to decompose the metric itself, as we do in the rest of the thesis.

extrinsic curvature $K_{ij} \propto \dot{h}_{ij}$ of the hypersurfaces of constant t. Because $\dot{h}_{ij}$ is symmetric, $\hat{u}_i^{(n)}(t)$ is, at any time, an orthonormal basis with respect to the standard Kronecker scalar product,

$$\hat{u}_i^{(n)}(t)\, \hat{u}_j^{(m)}(t)\, \delta_{ij} = \delta_{mn}\,. \tag{7.4}$$

So, "rotation" means time-dependence of this orthogonal set of principal axes FLRW with respect to the system of comoving observers $x^i = \text{const.}$ who are in geodesic motion. If one tried to define new spatial coordinates x'^i in order to "follow the rotation", such coordinates would not label geodesic observers any longer.

In the rest of the thesis (Secs. 8 and 9), we revisit the usual mini-superspace approach sketched above by giving particular emphasis to symmetries and the associated conservation laws. Such a viewpoint is useful because it allows us to recast the above analysis in terms of which charges can be set to zero by a symmetry transformation. *If* the matter sector breaks certain symmetries of the gravitational action—as will be the case for a solid-driven cosmology—the angular momentum charges cannot be transformed to zero by a symmetry transformation. This, we take as the definition of having a "rotating cosmology". In Sec. 10 we try to understand in physical terms the associated rotation, by specializing to $2+1$ dimensions, and we display some explicit numerical solutions. We consider the quantum theory in Sec. 11.

8. Classical minisuperspace actions: the rules of the game

To study Friedmann Lemaître Robertson Walker (FLRW) cosmologies, one usually specializes the Einstein equations to the FLRW metric and to time-dependent matter fields, say in comoving coordinates and cosmic time, and tries to solve them. An alternative approach, is to write the action in "minisuperspace": that is, one substitutes the FLRW ansatz directly into the action, and interprets the resulting action as a functional of $a(t)$, of time-dependent matter fields, and of—famously—the lapse function $N(t)$. Indeed, retaining $N(t)$ as a degree of freedom is crucial to retain the Hamiltonian constraint as an equation of motion, which is nothing but the first Friedmann equation.

We would like to do the same for spatially flat, homogeneous, but anisotropic universes. Before attempting to do so, we must understand what the rules of the game are, in terms of how general an ansatz we should use, and in particular of how specializing to a given ansatz interferes with gauge invariance: why do we have to keep $N(t)$? Should we also keep the shifts $N^i(t)$ in general?

Neglecting for a moment the subtleties associated with gauge invariance and gauge fixing, we recall a general argument of Coleman's for classical field theories: if an action is invariant under a symmetry group G, and one is looking for solutions to the field equations that preserve a subgroup $H \subseteq G$ of such symmetries, one can plug into the action the most general ansatz that is invariant under H, and vary the action within that subspace of field configurations. The two variational problems (first vary and then impose the symmetries of the solution, first impose the symmetries of the solution and then vary) yield the same set of solutions [101].

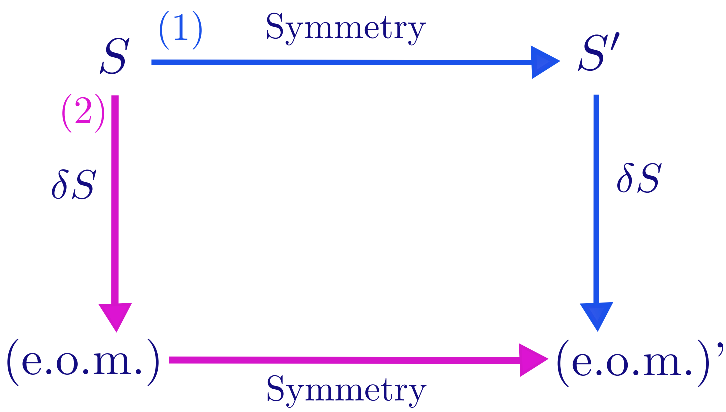


Figure 8.1.: Coleman's argument.

How does one adapt the above argument to gauge theories, and in particular to our case?

The answer is simple. For definiteness, let's consider directly the case of cosmological solutions in GR, which is what we are interested in. The symmetries of the action, G, are all the diffeomorphisms. The symmetries we would like our solution to preserve, $H \subset G$, are certain isometries, generated by some Killing vectors $\xi^\mu_H(x)$. However, in different gauges the Killing vectors take different forms. From the functional analysis viewpoint, which is at the basis of Coleman's argument, different-looking Killing vectors generate different symmetries. So, if we ask what is the most general field configuration that preserves a certain isometry $\xi^\mu_H(x)$, we must ask this with enough gauge-fixing to completely specify the functional form of $\xi^\mu_H(x)$.

For example, an FLRW spacetime has translational and rotational isometries. These take a particularly simple form if we decide to use the standard comoving coordinates:

$$\vec{x} \to \vec{x} + \vec{\epsilon}\,, \qquad \epsilon = \text{const} \tag{8.1}$$

$$\vec{x} \to O \cdot \vec{x}\,, \qquad O = \text{const}, \quad O^T \cdot O = \mathbb{1}\,. \tag{8.2}$$

So, applying Coleman's argument, we can plug into the action the most general metric preserving the above symmetries: translational invariance in the form above means that nothing can depend on $\vec{x}$. Rotational invariance in the form above means that spatial indices can only come from δ_{ij} or x^i. So, the most general metric with these properties is

$$g_{00} = g_{00}(t) \equiv -N^2(t)\,, \qquad g_{0i} = 0\,, \qquad g_{ij} = a^2(t)\delta_{ij}\,, \tag{8.3}$$

which is the standard minisuperspace ansatz. Coleman's argument guarantees that proceeding in this way is equivalent to looking for FLRW solutions at the level of the full Einstein equations.

The moral of this example is that, in order to apply Coleman's argument to a gauge theory, one should use an ansatz that is gauge-fixed enough so that all the symmetries one wants to preserve have a completely specified functional form, but not more. One should then use in the action the most general ansatz compatible with such symmetries.

9. Anisotropic cosmologies

Contents

Consider then the action for gravity and matter in $d+1$ dimensions,

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\;R \;+\; S_{\rm matter}\,. \tag{9.1}$$

We are interested in studying spatially flat, homogenous, but not necessarily isotropic solutions. “Spatially flat, homogeneous” means that there are d spacelike Killing vectors $\xi^\mu_{(1)}$, ..., $\xi^\mu_{(d)}$ with the same algebra as d-dimensional Euclidean translations—that is, they commute.

Following the logic of the last section, we must fix the gauge enough so that the functional form of these Killing vectors is completely specified. Similarly to the FLRW case, one can show that one can always choose x^1, ..., x^d coordinates such that

$$\xi_{(i)} = \partial_i\,, \qquad i = 1, \ldots, d\,, \tag{9.2}$$

that is, such that $\xi^\mu_{(i)}$ generates rigid translations along x^i.

By requiring that our ansatz be invariant under these Killing vectors, we end up with a metric that does not depend on $\vec{x} = (x^1, \ldots, x^d)$. In ADM variables,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^k N_k & N_j \\ N_i & h_{ij} \end{pmatrix}\,, \qquad g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{N^j}{N^2} \\ \dfrac{N^i}{N^2} & h^{ij} - \dfrac{N^i N^j}{N^2} \end{pmatrix} \tag{9.3}$$

where N, N^i and h_{ij} are only functions of time.

Notice that under a diffeomorphism of the form $x^i \to x^i + v^i(t)$, the shift transforms as $N^i \to N^i + \dot{v}^i$. So it is possible to get rid of $N^i(t)$ with a coordinate transformation. However, at face value this is incompatible with Coleman's argument: from the viewpoint of the variational problem, the only allowed truncations of configuration space are those that correspond to a symmetry requirement—that the ansatz preserve one or more of the symmetries of the action. Setting N^i to zero is not a statement of symmetry, unless we were trying to impose isotropy as well, which we are not. We should thus keep N^i around in the action, vary w.r.t it to get the associated equation of motion (the momentum constraint), and only then set it to zero. However, as we show in Appendix E, for the matter systems that we will consider, the momentum constraint itself guarantees that it is consistent to set N^i to zero directly at the level of the action and not worry about the momentum constraint any further. We will then do so, and postpone justifying in detail this course of action until the Appendix.

Notice that, after we set N^i to zero by a change of coordinates, the new coordinates are still Killing coordinates, in the sense that the Killing vector fields are still of the form ∂_i. The line element reduces to

$$ds^2 = -N^2(t)dt^2 + h_{ij}(t)dx^i dx^j \, . \tag{9.4}$$

Contrary to the momentum constraint, the Hamiltonian constraint is nontrivial, and so we have to keep the lapse N as a degree of freedom in the action. However, we will systematically set $N = 1$ in the equations of motion.

9.1. Symmetries of the gravitational action

For the ansatz (9.4), the action becomes

$$S = \frac{1}{16\pi G} \int \frac{dt}{4N} \sqrt{h} \left(h^{il} h^{jm} - h^{ij} h^{lm} \right) \dot{h}_{ij} \dot{h}_{lm} + \; S_{\text{matter}} \, . \tag{9.5}$$

Notice that eq. (9.4) does not correspond to a complete gauge-fixing: if we multiply $\vec{x}$ by a constant (in $\vec{x}$ and t) invertible matrix L,

$$x^i \to \tilde{x}^i = L^i{}_j x^j \; , \tag{9.6}$$

the metric still takes the form (9.4), with the same $N(t)$, but with a new $h_{ij}(t)$:

$$h_{ij} \to \tilde{h}_{ij} = (L^{-1})^k{}_i (L^{-1})^l{}_j \; h_{kl} \, . \tag{9.7}$$

General (invertible) linear transformations in d dimensions form the d^2-dimensional group $GL(d, \mathbb{R})$. In terms of its fundamental representation, it is useful to classify its generators ℓ_n in the following way:

1. *Dilations*: the associated generator is the identity matrix,

   $$(\ell^D)^{ij} = \delta^{ij} \; , \tag{9.8}$$

   which rescales h_{ij} by an overall constant.

2. *Rotations*: the associated generators are antisymmetric matrices, for instance of the form

   $$(\ell^R_{ab})^{ij} = \delta^i_a \delta^j_b - \delta^j_a \delta^i_b \; , \tag{9.9}$$

   which generate rotations in the (x^a, x^b) plane. We have $\frac{d(d-1)}{2}$ of them. The associated group, $SO(d)$, is the maximal subgroup of $SL(d, \mathbb{R})$.

3. *Diagonal shears*: the associated generators are diagonal, traceless matrices, for instance of the form

   $$\ell^{\text{diag}}_a = \text{diag}(0, \ldots, 0, +1, -1, 0, \ldots, 0) \; , \tag{9.10}$$

where the $+1$ is at the a-th position along the diagonal, so that ℓ_a^{diag} generates shears along the (x^a, x^{a+1}) directions. We have $d-1$ of them.

4. *Off-diagonal shears*: the associated generators are fully off-diagonal, symmetric matrices, for instance of the form

$$(\ell_{ab}^{\text{off}})^{ij} = \delta_a^i \delta_b^j + \delta_a^j \delta_b^i \,, \qquad a \neq b \,, \tag{9.11}$$

which generate shears along the $(x^a + x^b, x^a - x^b)$ directions. We have $\frac{d(d-1)}{2}$ of them.

Consider for now only the gravitational (i.e., Einstein-Hilbert) part of (9.5), and let's investigate how these generators act. Dilations rescale the spatial metric by a constant factor. Despite their being a symmetry of the equations of motion, they are *not* a symmetry of the action (9.5), and so there is no conserved charge associated with them. The reason is that in going from (9.1) to the reduced action (9.5), we have integrated over the spatial volume. In other words, of the invariant combination $d^dx\sqrt{h}$, we are left only with the volume element $\sqrt{h}$. As a result, the action is only invariant under special linear transformations $SL(d,\mathbb{R})$, that is, the subgroup of $GL(d,\mathbb{R})$ with unit determinant. All the other generators are symmetries of the Einstein-Hilbert part of the action (9.5), and so, neglecting matter for now, they correspond to conserved charges.

However, it turns out that for any given initial conditions, we can set to zero some of the charges by a suitable choice of coordinates, that is, by acting with the symmetry group itself. This is the the same logic that allows one to reduce a point-particle mechanics problem with a central potential in three spatial dimensions to a two-dimensional one: since the angular momentum vector $\vec{J}$ is conserved, motion happens on a fixed plane; aligning the x, y coordinates with such a plane corresponds to setting J_x and J_y to zero and keeping only a nonzero J_z.

For a generic symmetry group, there are arguments that indicate that one can always apply symmetry transformations so that the only charges that are turned on are those that make up the Cartan subalgebra, that is, the maximal abelian subgroup [102, 103]. We can confirm this general expectation in our specific case. We do so in Appendix C.5. For $SL(d,\mathbb{R})$, the Cartan subalgebra is spanned by the diagonal shears above. *If* the matter action is also invariant under $SL(d,\mathbb{R})$—a question that we will address soon—classical solutions will carry these $d-1$ conserved charges. If on the other hand the matter action is, for instance, only invariant under rotations, then the classical solutions will carry angular momentum charges, as many as the rank of $SO(d)$, which is the integer part of $d/2$.

Below we will study in some detail the $d=2$ case, so let's see explicitly how things work out there. Using the same numbering as in the general classification above, a convenient basis for the generators $GL(2,\mathbb{R})$ is

$$\ell_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \ell_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \ell_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \ell_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (d=2) \,. \tag{9.12}$$

In agreement with our discussion above, upon exponentiation, ℓ_1 generates a scale transformation, ℓ_2 a rotation, and ℓ_3 and ℓ_4 shears in the (x^1, x^2) plane, respectively along the x^1, x^2 and $x^1 \pm x^2$ directions. It can also be useful to characterize the generators in terms of (real) irreducible representations of the $SO(2) \subset GL(2,\mathbb{R})$ generated by ℓ_2: ℓ_1 is pure trace (helicity zero), ℓ_2 is antisymmetric (helicity zero), ℓ_3 and ℓ_4 are symmetric and traceless (helicity ± 2). In fact, under a 45° rotation, ℓ_3 and ℓ_4 transform into each other.

Consider now the most general time-dependent 2D spatial metric, $h_{ij}(t)$. It has three independent components, which we can parametrize conveniently as

$$h_{ij} = A^2 \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{\xi} & 0 \\ 0 & e^{-\xi} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} , \tag{9.13}$$

where the common scale factor A, the shear ξ, and the rotation angle θ are all functions of time. The factor of 2 in the definition of θ is conventional, and is useful for what follows. For the moment, suffice it to say that declaring θ to be an angular variable of period 2π is consistent with the fact that a rotation of $\frac{\theta}{2} = \pi$ is enough to map h_{ij} into itself (h_{ij} only has components of helicity zero and ± 2).

The gravitational action in these new variables becomes

$$S_{\rm grav} = \frac{1}{8\pi G_N} \int \frac{dt}{N} \left(-\dot{A}^2 + \frac{1}{4} A^2 (\dot{\xi}^2 + \sinh^2\xi\, \dot{\theta}^2) \right) . \tag{9.14}$$

In App. A we give the explicit transformations of the fields, whose infinitesimal versions read,

$$\ell_2 : \qquad \delta\theta = 1, \qquad \delta\xi = 0, \tag{9.15}$$
$$\ell_3 : \qquad \delta\theta = -\sin\theta \coth\xi, \qquad \delta\xi = \cos\theta, \tag{9.16}$$
$$\ell_4 : \qquad \delta\theta = +\cos\theta \coth\xi, \qquad \delta\xi = \sin\theta\, , \tag{9.17}$$

where we are suppressing the associated infinitesimal transformation parameters. One can check that the action (9.14) is invariant under these transformations. On the other hand, dilations simply rescale A and, as we have already emphasized, they are *not* a symmetry of (9.14).

9.2. Anisotropic cosmologies as trajectories in a meta-universe

It is illuminating to think of our variables A, ξ, θ as coordinates X^M in a $(2+1)$-dimensional meta-universe. Then, the gravitational action (9.14) is simply that of a free massless particle with trajectory $X^M(t)$ in so-called parametrized (or Polyakov) form,

$$S_{\rm grav} \propto \frac{1}{2} \int \frac{dt}{N} G_{MN}(X) \dot{X}^M \dot{X}^N , \tag{9.18}$$

where G_{MN} is the metric in this meta-universe:

$$ds^2_{\rm meta} = G_{MN}(X) dX^M dX^N = -dA^2 + \frac{A^2}{4} (d\xi^2 + \sinh^2\xi\, d\theta^2) . \tag{9.19}$$

Were it not for the relative factor of 4, this would be the metric of flat space in Milne coordinates. However, with the relative factor of 4, this is a more general (less symmetric) open FLRW cosmology. In fact, at fixed time-variable in the meta-universe, that is, at fixed A, this metric describes a two-dimensional hyperboloid, with ξ and θ playing the roles of the standard radial and angular coordinates (this is another reason why the factor of 2 in (9.13) is a convenient choice.) Related to this, notice that our symmetry group $SL(2,\mathbb{R}) \simeq SO(2,1)$ [1] spanned by the three generators l_2, l_3, and l_4 is nothing but the isometry group of the hyperboloid, with l_2 being the rotation and l_3 and l_4 the generalized translations.

Similar considerations apply in higher dimensions, albeit with less symmetric spatial sections in the meta-cosmology. To see this, let's start again from (9.5) and let's decompose h_{ij} into a common scale factor $A(t)$ and a unit-determinant matrix,

$$h_{ij}(t) = A^2(t)\, u_{ij}(t) \; , \qquad \det u = 1 \; . \tag{9.20}$$

Taking into account the unit-determinant condition on u, the gravitational part of the action becomes

$$S_{\text{grav}} = \frac{1}{16\pi G} \int \frac{dt}{N} \Big(- d(d-1) A^{d-2} \dot{A}^2 + A^d \, u^{il} u^{jm} \, \dot{u}_{ij} \dot{u}_{lm} \Big) \, . \tag{9.21}$$

Now, u is a symmetric matrix with unit determinant, and so it has $\frac{d(d+1)}{2} - 1$ independent components, exactly as many as the dimensions of the $SL(d,\mathbb{R})/SO(d)$ coset space. This is no accident: using the standard framework of nonlinear realizations [104, 105], it is useful to think of the most general $u_{ij}(t)$ as being the result of applying a suitable $SL(d,\mathbb{R})$ transformation to the identity matrix. However, in the language of spontaneous symmetry breaking, the identity matrix breaks $SL(d,\mathbb{R})$ down to its $SO(d)$ subgroup, and so we can obtain a generic $u_{ij}(t)$ by acting on the identity with the *broken* generators only (type 3 and 4 in the classification above), for instance exponentiated as

$$u_{ij}(t) = \big(e^{X^I(t)\, T_I}\big)^k{}_i \big(e^{X^I(t)\, T_I}\big)^l{}_j \, \delta_{kl} = \big(e^{2X^I(t)\, T_I}\big)_{ij} \; , \qquad I = 1, \dots, \tfrac{d(d+1)}{2} - 1 \; , \tag{9.22}$$

where the $X^I(t)$ are a set of "Goldstone fields", and we used the fact that the broken T_I generators in the fundamental representation correspond to symmetric matrices. The Goldstone fields can be thought of as coordinates on the coset $SL(d,\mathbb{R})/SO(d)$, and provide a parametrization of the most general positive definite, symmetric, unit-determinant u_{ij}. One can of course decide to put different coordinates on the coset space, and so from now on we can consider the X^I's to be a generic set of coset coordinates. They are the higher dimensional analogs of our ξ and θ variables above [2].

Plugging the parametrization of u_{ij} in terms of Goldstones into the gravitational action and defining the meta-cosmic time

$$X^0 \equiv A^{\frac{d}{2}} \; , \tag{9.23}$$

[1] Actually, it would have been wiser to express the correspondence (and the symmetries) in terms of Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1)$, to avoid any global topological ambiguities, but we preferred working with the Lie groups for convenient reasons.

[2] In fact, our parametrization in (9.13) is not the same as (9.22) specialized to $d = 2$, but is related to it by a nonlinear $(\xi, \theta) \leftrightarrow (X^1, X^2)$ change of variables.

we get

$$S_{\text{grav}} = \frac{1}{16\pi G}\int \frac{dt}{N}\Big(-4\tfrac{(d-1)}{d}(\dot{X}^0)^2 + \tfrac{1}{2}(X^0)^2\, g_{IJ}(\vec{X})\dot{X}^I\dot{X}^J\Big) \tag{9.24}$$

where g_{IJ} is the metric of the $SL(d,\mathbb{R})/SO(d)$ coset manifold in the X^I coordinate system, which is invariant under generic $SL(d,\mathbb{R})$ transformations [3].

Up to an overall constant, we thus have once again the action of a free massless particle, eq. (9.18), living in a meta-cosmology with metric

$$ds^2_{\text{meta}} = -(dX^0)^2 + (X^0)^2\tfrac{d}{8(d-1)}\, g_{IJ}(\vec{X})dX^I dX^J\ . \tag{9.25}$$

The scale factor of this meta-cosmology is linear in the cosmic time X^0, and the spatial sections are invariant under $SL(d,\mathbb{R})$. However, since these sections are $(\frac{d(d+1)}{2}-1)$-dimensional, and $SL(d,\mathbb{R})$ only has d^2-1 generators, for $d=3$ and above we don't have enough isometries to make the spatial sections maximally symmetric. Their geometry, for generic d, is nicely reviewed in [106].

Adding matter to (9.21) will correspond to adding more fields and potentially breaking some of the symmetries that we have discussed. In the meta-universe picture, this will amount to adding dimensions to the meta-universe, and to breaking some of the isometries by adding potentials for some of our point-particle's coordinates—in which case our particle will not follow geodesics anymore. Let's look at a few examples.

9.3. Time-dependent scalar matter

The simplest possibility is to consider a time-dependent scalar field $\phi(t)$ as matter, which is consistent with the isometry that we are trying to impose (spatial homogeneity). In fact, it is invariant under the full $SL(d,\mathbb{R})$ symmetry group we have been discussing at length. Considering for simplicity a free massless scalar in $d=2$, the action becomes

$$\begin{aligned} S &= \int\frac{dt}{N}\left[\frac{1}{8\pi G}\left(-\dot{A}^2 + \frac{1}{4}A^2(\dot{\xi}^2 + \sinh^2\!\xi\,\dot{\theta}^2)\right) + \frac{1}{2}A^2\dot{\phi}^2\right] \qquad (d=2) \\ &= \frac{1}{8\pi G}\int\frac{dt}{N}\left[-\dot{A}^2 + \frac{1}{4}A^2(\dot{\xi}^2 + \sinh^2\!\xi\,\dot{\theta}^2 + 2\dot{\phi}^2_{\text{Pl}})\right], \end{aligned} \tag{9.26}$$

where ϕ_{Pl} is simply ϕ measured in Planck units. As anticipated, we have effectively added one dimension to the meta-universe. Our fictitious particle still follows geodesics. Moreover, $SL(2,\mathbb{R})$ still corresponds to isometries of this enlarged meta-universe. In fact, we have a shift symmetry on ϕ_{Pl} as well, and so we have one more isometry. As a consequence, any solution of the equations of motion will conserve all $SL(2,\mathbb{R})$ charges as well as the shift-symmetry charge.

Following the discussion of the preceding subsection, for any given initial conditions we can perform a symmetry transformation and set to zero all the charges that are not in the Cartan subalgebra. For $SL(2,\mathbb{R})$, this means that, up to coordinate transformation, only the charge associated with ℓ_3 survives. Calling Q the charge associated with ϕ_{Pl}-shifts

[3] The $SL(d,\mathbb{R})$ isometries of the coset manifold do not fix the overall normalization of g_{IJ}. The normalization chosen in (9.24) matches what we had in $d=2$ if we declare "the" metric of the hyperbolic plane to be $d\xi^2 + \sinh^2\xi\, d\theta^2$.

and Q_a the charges associated with the ℓ_a's of $SL(2,\mathbb{R})$, we have that, with suitable normalizations (and with $N = 1$),

$$Q = A^2 \dot{\phi}_{\rm Pl} \tag{9.27}$$
$$Q_2 = A^2 \sinh^2\!\xi\, \dot{\theta} \tag{9.28}$$
$$Q_3 = A^2 (\dot{\xi} \cos\theta - \dot{\theta} \sin\theta \sinh\xi \cosh\xi) \tag{9.29}$$
$$Q_4 = A^2 (\dot{\xi} \sin\theta + \dot{\theta} \cos\theta \sinh\xi \cosh\xi) \tag{9.30}$$

Setting Q_2 and Q_4 to zero while keeping Q_3 nonzero corresponds to setting

$$\theta = 0 \; , \qquad \dot{\theta} = 0 \; . \tag{9.31}$$

This can be thought of as an initial condition, but, by conservation of Q_2 and Q_4, it will be true at all times. Also, we emphasize again that *all* initial conditions can be put in this form by a suitable coordinate transformation.

We can thus just forget about θ altogether, and focus on the $A, \xi, \phi_{\rm Pl}$ degrees of freedom. Their dynamics are completely determined by conservation laws: the Hamiltonian constraint and the conservation of Q and Q_3 read

$$H^2(t) = \frac{1}{4} \frac{Q_3^2 + 2Q^2}{A^4(t)} \; , \qquad H(t) \equiv \frac{\dot{A}}{A} \tag{9.32}$$
$$\dot{\xi}(t) = \frac{Q_3}{A^2(t)} \tag{9.33}$$
$$\dot{\phi}_{\rm Pl}(t) = \frac{Q}{A^2(t)} \; , \tag{9.34}$$

with constant Q and Q_3.

And so, in particular, the contribution of anisotropy (Q_3) to the Friedmann equation scales like A^{-4}, and so does the energy density of ϕ. The anisotropy itself, measured as the difference in the Hubble rate along the two principal axes,

$$\Delta H \equiv \frac{\frac{d}{dt}(A e^{\xi/2})}{A e^{\xi/2}} - \frac{\frac{d}{dt}(A e^{-\xi/2})}{A e^{-\xi/2}} = \dot{\xi} \; , \tag{9.35}$$

scales as A^{-2}.

We can generalize this analysis to $d > 2$ and to a scalar with a nonzero potential $V(\phi)$. We will have full $SL(d,\mathbb{R})$ symmetry, but no shift symmetry. By a judicious choice of coordinates, we can set to zero all the $SL(d,\mathbb{R})$ charges corresponding to rotations and to off-diagonal shears. We will thus be left with the $d-1$ charges associated with the diagonal shears. This means that, in this coordinates, the metric itself will be A^2 times a diagonal shear of the identity matrix, parametrized by $d-1$ "anisotropy Goldstones" $\xi^a(t)$, for instance as

$$h_{ij} = A^2 \, {\rm diag}\big(e^{\xi_1}, \, e^{\xi_2 - \xi_1}, \dots, \, e^{\xi_{d-1} - \xi_{d-2}}, \, e^{-\xi_{d-1}}\big) \; . \tag{9.36}$$

Since the generators in question commute, the coset submanifold spanned by these Goldstones is flat. So, for this reduced set of variables, up to factors the action must take the form (see eq. (9.24))

$$S \sim \int \frac{dt}{N}\big(- A^{d-2}\dot{A}^2 + A^d(\dot{\xi}^{a\,2} + \dot{\phi}_{\rm Pl}^2) - N^2 A^d\, V_{\rm Pl}(\phi_{\rm Pl})\big) \,, \tag{9.37}$$

where $V_{\rm Pl}$ is the potential for ϕ in Planck units. The story then is very similar to the $d = 2$ case, apart from the fact that now there is no shift-symmetry for ϕ and so both the dynamics of ϕ and those of the scale factor will be more complicated. Nonetheless, the dynamics of anisotropies is still completely determined by the conservation laws for the nonzero charges Q_a,

$$\dot{\xi}^a(t) = \frac{Q_a}{A^d(t)} \,, \qquad Q_a = \text{const} \,, \tag{9.38}$$

and so is their contributions to the Friedmann equation,

$$H^2 \supset \frac{1}{A^{2d}(t)} \cdot \sum_a Q_a^2 \,, \qquad Q_a = \text{const} \,. \tag{9.39}$$

All this is in agreement with standard results, but here we derived it using only symmetries and conservation laws.

9.4. Solid matter

We now consider what kind of matter can violate some of the $SL(d,\mathbb{R})$ symmetries of the gravitational action. We are particularly interested in the case in which the residual symmetries make up the $SO(d)$ rotation subgroup, so that solutions can be characterized in terms of conserved angular momentum charges—that is, they can be thought of as *rotating* solutions.

Ideally, we would like the matter action to be a function of h_{ij} that is only invariant under $SO(d)$, for instance because it is a function of the trace of h, the trace of h^2, etc. However, precisely because such $SO(d)$ invariants are not invariant under the full $SL(d,\mathbb{R})$, they cannot arise directly as the mini-superspace limit of diff-invariant quantities—$SL(d,\mathbb{R})$ is what is left of diff-invariance in mini-superspace.

Following [107], the solution is clear: introduce scalar fields playing the role of Stückelberg fields, with internal symmetries that have exactly the same algebra as the symmetries one wants to preserve (the d-dimensional Euclidean group, in our case), and then work in unitary gauge where these scalars are aligned with the coordinates. We now describe why, in our case, this is the same as considering a cosmological solid.

The low energy effective description of a solid or a fluid involves as many scalar fields ϕ^I as there are spatial dimensions (d), and a Lagrangian equipped with certain internal symmetries [108, 109]

$$\text{(internal translations = shifts)} \qquad \phi^I \longrightarrow \phi + a^I, \qquad a^I = \text{const} \tag{9.40}$$

$$\text{(internal rotations)} \qquad \phi^I \longrightarrow O^I{}_J\phi^J, \qquad O^I{}_J \in SO(d). \tag{9.41}$$

(We are for simplicity considering only solids that, at least at long distances, have isotropic ground states.) On top of those, the fluid Lagrangian is also invariant under internal volume-preserving diffeomorphisms. A straightforward way to implement these symmetries to lowest order in derivatives is through certain invariants built out of

$$B^{IJ} \equiv g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J \ , \tag{9.42}$$

which is a spacetime-scalar, shift-invariant symmetric matrix with internal indices.

Then, for a generic solid in d spatial dimensions, the effective Lagrangian must be a function of the d independent $SO(d)$ invariants that one can build out of B. For instance, following [27], one can take the trace of B and normalized versions of the traces of B^n:

$$X = [B] \ , \qquad Y_2 = \frac{[B^2]}{[B]^2} \qquad \dots \qquad Y_d = \frac{[B^d]}{[B]^d} \ , \tag{9.43}$$

where $[\cdots]$ denotes the trace of the matrix within. To lowest order in derivatives, the action of a solid in $d+1$ dimensions then is

$$S_{\text{solid}} = -\int dt \ h^{1/2} N F(X, Y_2, \dots, Y_d) \ , \tag{9.44}$$

where F is a generic function, related to the equation of state of the solid.

For our minisuperspace application, it is particularly convenient to work in so-called unitary gauge: recall that we are restricting to ansatze that are invariant under constant shifts of the spatial coordinates x^i. If we consider the field configuration

$$\phi^I(\vec{x}, t) = x^I \ , \tag{9.45}$$

we are formally breaking the spatial translations, but, given the shift symmetries (9.40), we are in fact preserving a diagonal combination of internal shifts and spatial translations. This makes this ϕ^I configuration compatible with the mini-superspace framework. Similarly, although spatial rotations are formally broken by the ϕ^I configuration above, thanks to the internal rotational symmetry (9.41), there is an unbroken diagonal combination of internal rotations and spatial ones. On the other hand, unless we demand that the solid action only be a function of $\det B$—in which case we would be describing a perfect *fluid* [108, 109]—the rest of $SL(d, \mathbb{R})$ is broken by (9.45). Notice that in unitary gauge, the matrix B^{IJ} defined in (9.42) reduces simply to

$$B^{IJ} = h^{IJ} \ . \tag{9.46}$$

The conclusion is that if we consider a generic solid in unitary gauge, we are effectively adding to the mini-superspace gravitational action a function of $h_{ij}(t)$ that is only invariant under the $SO(d)$ subgroup of $SL(d, \mathbb{R})$—precisely what we were looking for.

To make the discussion as concrete and as simple as possible, we now specialize to $d = 2$. The $d = 3$ case is analyzed in Appendix D. Using the same parametrization of h_{ij} as before, eq. (9.13), we get that our invariants in unitary gauge read

$$X = 2A^{-2}\cosh\xi \ , \qquad Y \equiv Y_2 = 1 - \frac{1}{2(\cosh\xi)^2} \ , \tag{9.47}$$

which, consistently with the residual rotational invariance, depend on A and ξ but not on θ. Since F in (9.44) is a generic function of X and Y, we can consider it directly as a generic function of A and ξ, $F = F(A, \xi)$. Notice that in the fluid case the matter action would only depend on $\det B^{IJ} = A^{-4}$, *i.e.*, $\partial_\xi F = 0$. In this limit the system would still be invariant under the full $SL(2, \mathbb{R})$ group.

In summary, including the gravitational part (9.14) as well, we consider the action

$$S = \int dt\, A^2 \Big[\frac{1}{N} \Big(-\frac{\dot{A}^2}{A^2} + \frac{\dot{\xi}^2}{4} + \frac{\dot{\theta}^2 \sinh^2 \xi}{4} \Big) - \, N F(A, \xi) \Big] \,, \tag{9.48}$$

where for simplicity we have chosen $8\pi G = 1$ units.

Now the only symmetry of this action is $SO(2)$, acting as a constant shift of θ. The associated charge is the angular momentum $J = \dot{\theta}\, A^2 \sinh^2 \xi$ (setting $N = 1$), and the dynamics of θ are completely determined by its conservation:

$$\dot{\theta} = \frac{J}{A^2 \sinh^2 \xi} \,, \qquad J = \text{const.} \tag{9.49}$$

Since we only have one symmetry generator—that associated with angular momentum—there is no way now to use the symmetries of the action to set the angular momentum to zero. This angular momentum is thus as physical as it can be, and a nonzero value for it can be taken to mean, by definition, that the universe is *rotating*.

Denoting from now on the derivatives of F with respect to A or ξ by subscripts, the other independent equations are the Hamiltonian constraint and the ξ equation of motion:

$$4H^2 = \dot{\xi}^2 + \frac{J^2}{A^4 \sinh^2 \xi} + 4F \tag{9.50}$$

$$\ddot{\xi} + 2H\dot{\xi} - \frac{J^2 \cosh \xi}{A^4 \sinh^3 \xi} + 2F_\xi = 0 \tag{9.51}$$

where $H \equiv \dot{A}/A$.

For both numerical and qualitative analyses, it proves useful to condense the above equations into a single second order one for $\xi(\mathcal{N})$, where $\mathcal{N} = \ln A$ is the number of e-folds and a prime denotes differentiation with respect to it:

$$\xi'' + 2\xi' - \frac{\xi'^3}{2} - \frac{\xi'}{H^2} \left(\frac{J^2}{2A^4 \sinh^2 \xi} - \frac{A F_A}{2} \right) - \frac{J^2 \cosh \xi}{H^2 A^4 \sinh^3 \xi} + \frac{2F_\xi}{H^2} = 0 \,. \tag{9.52}$$

With this notation, H^2 stands for

$$H^2 = \left(4 - \xi'^2\right)^{-1} \left(\frac{J^2}{A^4 \sinh^2 \xi} + 4F \right) . \tag{9.53}$$

10. Classical solutions: rotating universes

Contents

Solutions with $J \neq 0$, or, equivalently, $\dot{\theta} \neq 0$, are characterized by some "rotation". But rotation of what with respect to what? We should note that the solid's volume elements, which evolve in time following trajectories with $\phi^I = \text{const}$, in fact follow geodesics. Indeed, $x^i = \text{const}$ is always a geodesic for a metric of the form (9.4). Had we left the shifts N_i (the g_{0i} components of the metric) undetermined, the momentum constraint in the presence of a solid would enforce $N_i = 0$ in unitary gauge (see Appendix E). In other words, homogeneity of the cosmological solution seems to require that the solid's volume elements follow geodesics, as just stated. So there is no obvious sense in which the solid is rotating. One can also calculate the invariant vorticity associated with the solid's velocity field and easily check that it vanishes. In Appendix C, we compute the *peculiar velocities* in the Bianchi type I universe, and show how we can extract the rotating motion from them.

By looking at (9.13), it is rather clear that what is rotating here is the principal axes of expansion, with respect to the comoving frame $x^i = \text{const}$. This is the case, at least, when $\dot{\xi} \neq 0$. However, our system can also support rotating solutions with $\dot{\xi} = 0$.

In more standard cases, a constant ξ solution simply corresponds to an FLRW universe "in the wrong coordinates". That is, a constant ξ can be set to zero by a coordinate transformation, as a consequence of the $SL(2,\mathbb{R})$ symmetries. We can see this explicitly in our formalism as well, if we specialize (9.52) to non-rotating ($J = 0$), constant ξ ($\xi' = 0$) configurations. We obtain the condition

$$F_\xi(\xi, A) = 0\,. \tag{10.1}$$

Recall that now A is effectively playing the role of time. So, the above condition must be obeyed at all A's. *If* it is also obeyed for all values of ξ, then we are in the fluid limit, where the matter action only depends on the determinant of B^{IJ}, which in unitary gauge is A^{-4}. So, in this case we have an enhanced symmetry ($SL(2,\mathbb{R})$ vs. $SO(2)$) that allows us to rescale ξ to zero. This the standard case, that of a fluid-driven cosmology.[1]

In the presence of rotation, $J \neq 0$, it is still possible to fine-tune the action so that there are solutions with constant ξ. This time, by looking at (9.52), we see that we need

[1]There is also the possibility that the condition (10.1) is obeyed only at a specific $\xi = \xi_0$. In this case, the geometry is that of an FLRW cosmology, but the action for cosmological perturbations will likely be anisotropic.

to impose the condition

$$\frac{J^2 \cosh\xi}{A^4 \sinh^3\xi} = F_\xi \,. \tag{10.2}$$

By choosing $F = A^{-4} f(\xi)$, the scale factor A drops out of (10.2) and the equation becomes a condition for ξ only. The metric of the corresponding rotating solution is not that of an FLRW space, as one can check by explicitly calculating the spatial components of the Ricci tensor R_{ij} and verifying that they are *not* proportional to the metric h_{ij}. We are in the presence of a genuinely rotating, anisotropic solution. The scale factor behaves like that of a kinetically dominated universe ($p/\rho = 1$). We are away from the fluid limit and with just the $SO(2)$ symmetry of the solid we cannot set the constant ξ to zero. Equivalently, the solid is in an anisotropic state $Y \neq 1/2$. Loosely speaking, in this case the presence of angular momentum creates a centrifugal force that stabilizes the anisotropy at a certain constant value $\xi = \xi_0$.

10.1. A simple model for a solid

As an explicit example to use for a more in-depth study, we now specialize to the following solid Lagrangian,

$$F(X, Y) = X^\epsilon \,. \tag{10.3}$$

This simple model generates, in the limit of zero shear and rotation ($\xi = \theta = 0$), inflation with constant slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$ [27]. Equivalently, on an FLRW background, (10.3) enforces the constant equation of state $p/\rho = \epsilon - 1$. Using (9.47), eq. (9.52) reduces to

$$\begin{aligned} \xi'' - \xi'(\epsilon - 2) + \frac{\xi'^3(\epsilon-2)}{4} + \frac{\epsilon(4 - \xi'^2)\tanh\xi}{2} \\ + \frac{J^2 \, e^{-4\mathcal{N}}}{4H^2 \sinh^2\xi} \left[(\epsilon - 2)\xi' - \frac{4}{\tanh\xi} \right] = 0 \,. \end{aligned} \tag{10.4}$$

This equation can easily be solved numerically by substituting (9.53) in the second line. The qualitative behavior of solutions can also be captured in different limits, as we now show.

First, one should notice that the last term proportional to J^2 becomes very rapidly subdominant with respect to the other terms. So, it can be ignored to a very good approximation, and one can concentrate on just the first line of (10.4). We will do so from now on.

Second, after trial and error one discovers that the equation is more easily studied in the variable

$$y(\mathcal{N}) \equiv \log(\sinh\xi(\mathcal{N})) \,, \tag{10.5}$$

in terms of which the ξ eom reduces to

$$e^{2y}\Big[y'' - \tfrac{(2-\epsilon)}{4} y'^{\,3} - \tfrac{\epsilon}{2} y'^{\,2} + (2-\epsilon) y' + 2\epsilon \Big] + y'' + y'^{\,2} + (2-\epsilon) y' + 2\epsilon = 0 \,. \tag{10.6}$$

Despite this equation's looking complicated, it is quite easy to solve when y is large in absolute value, thanks to the exponential factor in front of the bracket: for large positive

y, the exponential is very large and the equation essentially reduces to the vanishing of the combination inside the brackets. For large negative y, the exponential is very small and the combination inside the bracket can be ignored. In both cases, the equation reduces to a *first*-order ODE for y'—y itself disappears from the equation.

It so happens that in both limits the least decaying attractor solutions as $\mathcal{N}$ increases evolve *linearly* in $\mathcal{N}$, with only slightly different rates,[2]

$$y'(\mathcal{N}) \simeq -\frac{\epsilon}{1-\epsilon/2} = -\epsilon + \mathcal{O}(\epsilon^2)\,, \qquad y \to +\infty \quad (\xi \gg 1) \tag{10.7}$$

$$y'(\mathcal{N}) \simeq \epsilon/2 - (1 - \sqrt{1 - 3\epsilon + \epsilon^2/4}) = -\epsilon + \mathcal{O}(\epsilon^2)\,, \qquad y \to -\infty \quad (\xi \ll 1)\,. \tag{10.8}$$

So, keeping only the leading order in ϵ, the least decaying solution for $\xi(\mathcal{N})$ has

$$\sinh \xi \sim e^{-\epsilon\mathcal{N}} \qquad \text{for } \xi \gg 1 \text{ or } \xi \ll 1\,, \tag{10.9}$$

with a transient behavior around $\xi \sim 1$ ($|y| \lesssim 1$), which our method is not able to resolve. Fig. 10.1 shows the validity of our approximation for different values of ϵ.

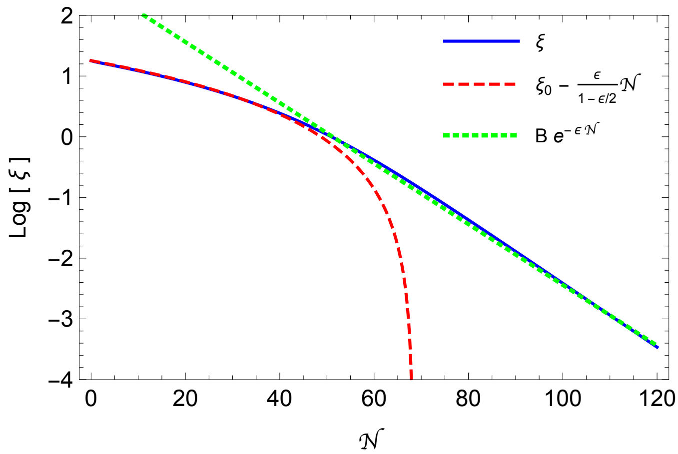


Figure 10.1.: We numerically integrate the system for a sufficient number of e-folds $\mathcal{N}$ so that ξ evolves from $\xi \gg 1$ to $\xi \ll 1$. As expected, the large-anisotropy regime is well approximated by (10.7), while the low-anisotropy regime by the dominant mode in (10.8), which we have approximated at small ϵ with $\sim e^{-\epsilon\mathcal{N}}$. Here $\epsilon = 0.05$.

Once the approximated solutions for ξ are found, we can deduce θ by integrating equation (9.49). By inspection, when the anisotropy has become small, $\xi \ll 1$, we obtain the asymptotic behavior

$$\theta' \sim e^{-2(1+\epsilon)\mathcal{N}}, \tag{10.10}$$

with θ approaching its asymptotic value exponentially fast.

[2]At $\xi \gg 1$ we find also the two behaviors $y'(\mathcal{N}) = \pm 2$. The growing mode corresponds to the vanishing of one of the two principal expansion rates. This is a repeller from the point of view of dynamical analysis. All configurations with $y'(\mathcal{N}) > 2$ are unstable and have expansion rates of opposite signs.

10.2. Energy densities

It is useful to interpret the terms in the Friedmann equation (9.50) as the energy densities associated with an isotropic background ($\xi = \dot{\theta} = 0$), with anisotropy, and with rotation,

$$\rho_{\rm bkg} = F|_{\xi=0} \tag{10.11}$$

$$\rho_{\rm aniso} = \rho_{\xi,\rm kin} + \rho_{\xi,\rm pot} \equiv \frac{\dot{\xi}^2}{4} + (F - F|_{\xi=0}) \tag{10.12}$$

$$\rho_{\rm rot} = \frac{J^2}{4A^4 \sinh^2\xi} \, , \tag{10.13}$$

hopefully with obvious notation. If we consider again our simple model (10.3) and work for simplicity to lowest-order in ϵ we can deduce the behaviors of these energy densities by looking at our explicit solutions.

Let's start with the isotropic background. It has

$$\rho_{\rm bkg} \sim e^{-2\epsilon\mathcal{N}} \, . \tag{10.14}$$

As a consistency check, notice that if this were the only contribution to the Friedmann equation, we would have $H \sim e^{-\epsilon\mathcal{N}}$, which corresponds to exactly constant $-\dot{H}/H^2 = \epsilon$, as advertised above for the model (10.3).

We can now assess the importance of anisotropy and rotation by comparing the behaviors of their energy densities to that of the background. For anisotropy, notice that ξ'^2 is well approximated by a constant at $\xi \gg 1$ and goes as $e^{-2\epsilon\mathcal{N}}$ when $\xi \ll 1$ (eqs. 10.7 and 10.8). As a consequence, the kinetic part of $\rho_{\rm aniso}$ goes through two different regimes (Fig. 10.2, left panel),

$$\rho_{\xi,\rm kin} \sim e^{-2\epsilon\mathcal{N}} \quad (\xi \gg 1) \, , \qquad \rho_{\xi,\rm kin} \sim e^{-4\epsilon\mathcal{N}} \quad (\xi \ll 1) \, . \tag{10.15}$$

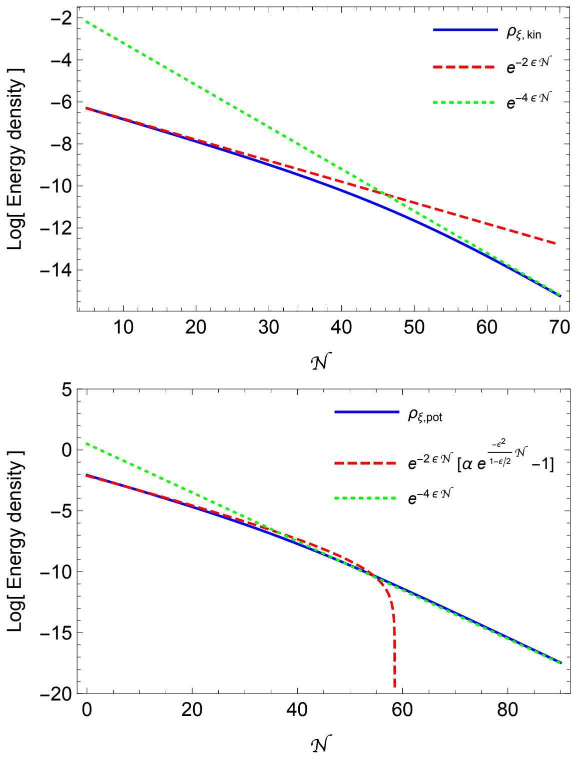


Figure 10.2.: On the left, the kinetic energy density $\rho_{\xi,\mathrm{kin}}$ as a function of the number of e-folds $\mathcal{N} = \ln A$, and on the right, the potential energy density $\rho_{\xi,\mathrm{pot}}$, for $\epsilon = 0.05$. As expected from the approximate solutions (10.7) and (10.8), there are two distinct modes corresponding to high and small values of ξ.

The potential part of ρ_{aniso} also has two distinct modes depending on the anisotropy regime. For small $\epsilon \ll 1$,

$$\rho_{\xi,\mathrm{pot}} \sim e^{-2\epsilon\mathcal{N}} \left(\alpha e^{-\frac{\epsilon^2}{1-\epsilon/2}\mathcal{N}} - 1 \right) \quad (\xi \gg 1)\,, \qquad \rho_{\xi,\mathrm{pot}} \sim e^{-4\epsilon\mathcal{N}} \quad (\xi \ll 1)\,, \tag{10.16}$$

with α some constant (see Fig. 10.2, right panel). So, as far anisotropies go, overall we have

$$\rho_{\xi,\mathrm{aniso}} \sim e^{-2\epsilon\mathcal{N}} \quad (\xi \gg 1)\,, \qquad \rho_{\xi,\mathrm{aniso}} \sim e^{-4\epsilon\mathcal{N}} \quad (\xi \ll 1)\,. \tag{10.17}$$

Then, the energy density associated with rotations behaves as

$$\rho_{\mathrm{rot}} \sim e^{-(4-2\epsilon)\mathcal{N}}\,. \tag{10.18}$$

We see that, in terms of energy densities, the contribution coming from rotation decays much faster than that coming from anisotropy (Fig. 10.3); however, the latter scales precisely as the isotropic background's energy density in the high-anisotropy regime, eq. (10.14), and decreases slightly faster in the low-anisotropy regime. In this simple model, and in stark contrast with more standard cosmological models with vanishing anisotropic stresses, the importance of anisotropies gets diluted very slowly by the Hubble expansion. This is in qualitative agreement with the conclusion of [28, 110], with a small quantitative discrepancy likely due to our working in $2+1$ dimensions.

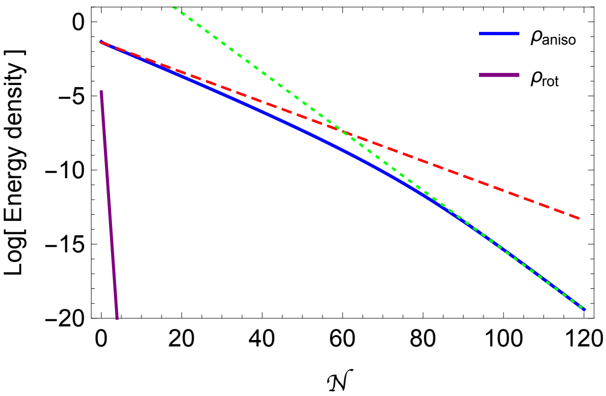


Figure 10.3.: The energy densities of the anisotropy $\rho_{\rm aniso}$ and the rotation $\rho_{\rm rot}$, for $\epsilon = 0.05$. The red and green dashed lines correspond resp. to the $e^{-2\epsilon\mathcal{N}}$ and $e^{-4\epsilon\mathcal{N}}$ modes.

11. Attempts at a quantum theory

Contents

Among the original motivations of this work was a better understanding of the two main pillars of cosmology—homogeneity and isotropy—in a fully quantum mechanical sense. We report some of our (persisting) confusion on the subject in the following subsection. Then we concentrate on the much more modest task of quantizing the mini superspace model that we are dealing with classically. This is a relatively simple quantum system, whose "diagonal version" (*i.e.* without rotation) has been throughly studied in the literature (see e.g. [111] and references therein). It seems to us that, even at this mini-superspace level, the rules of the game are not completely clear. The domain of validity of this approximation is not obvious, and neither is whether the corresponding truncation of degrees of freedom makes sense quantum mechanically.

11.1. Spacetime symmetries in quantum gravity

It is hard to overestimate the role of spacetime symmetries. In particle physics Poincaré invariance is nothing less than foundational. In cosmology an equally important role is played by homogeneity and isotropy. However, when dynamical gravity if fully taken into account, we do not seem to have a grasp of what spacetime symmetries even mean: a given classical metric can have a number of isometries, but how do we characterize the spacetime symmetries of a wavefunctional of metrics? One could argue that when (quantum) gravity is dynamical, spacetime symmetries are simply lost. But this conclusion is probably too rash. Presumably, there *is* a state of quantum gravity that corresponds to empty Minkowski spacetime and enjoys the symmetries of the Poincaré group. Also, perturbative calculations indicate that the quantum state of our universe exiting primordial inflation is homogeneous and isotropic to an extremely high degree. Is there any non-perturbative (*i.e.*, beyond cosmological perturbation theory) characterization of this?

Part of the confusion stems from the fact that in quantum mechanics symmetry is a property of the wavefunction. This represents an interesting twist on classical physics. Let us consider the simple example of a scalar field ϕ in Minkowski space. The state of such a field is *homogeneous* if the wavefunction Ψ satisfies[1]

$$\Psi[\phi(\vec{x}\,)] \;=\; \Psi[\phi(\vec{x}-\vec{a}\,)]\,, \tag{11.1}$$

for every vector $\vec{a}$. Notice that classical homogeneity (*i.e.*, simply, $\phi(\vec{x}\,) = \text{const.}$) has little to do with the above statement. In classical field theory restricting to configurations with certain spacetime symmetries implies a massive truncation of the phase space. No such a truncation is implied by (11.1). As far as we know, classically homogeneous configurations of the type $\phi(\vec{x}) = \text{const.}$ could even be absent from the quantum ensemble (11.1). In other words, Ψ could vanish on each and every homogeneous configuration, and still be homogeneous!

This is all well understood in the case of quantum fields on a given classical spacetime. But what is the analogue of (11.1) for gravity? We should be able to pose this question—albeit not necessarily to answer it—directly at the level of the low-energy effective theory. In canonical gravity a state is a functional of the three-dimensional metric $h_{ij}(\vec{x}\,)$ and of the matter fields. The property displayed in eq. (11.1) is clearly meaningless when applied to the metric field itself, because there is no such notion of "translated metric $h_{ij}(\vec{x}-\vec{a})$" in the absence of a background classical spacetime: the $\vec{x}$ coordinates are arbitrary, and the momentum constraint ensures that the wavefunction only depends on invariant quantities. So, how do we characterize translations for a generic metric (with no isometries) in a coordinate independent fashion?

While for translations we have nothing to say at the moment, for rotations we can use the mini-superspace approach that we have been using so far in the thesis, and try to make sense of it quantum-mechanically. As we have seen, mini-superspace corresponds to a truncation of degrees of freedom already in the classical configuration space. As we emphasized in Sec. 8, at the level of the classical variational problem this is a consistent truncation. At the quantum level, however, the question is more subtle: the degrees of freedom that we don't keep still have their quantum "life", which manifests itself in nontrivial correlation functions/variances, even if we assume that those degrees of freedom are in their vacuum. From the modern viewpoint, the sensible question seems to be whether the mini-superspace action (perhaps with renormalized couplings) corresponds to a consistent low-energy theory where all non-homogeneous degrees of freedom have been integrated out.

The answer appears to be 'no'. For a spatially infinite universe, the modes that are not constant in $\vec{x}$ form a gapless continuum, and there is no hope of ending up with a local derivative expansion upon integrating them out. For a spatially finite universe, say compactified on a torus, the Kaluza-Klein (KK) modes are gapped, and in principle one can integrate them out. However, one would like the physical size of the universe to be larger than the Hubble radius, in which case the KK are lighter than the Hubble scale, and cannot be ignored during the evolution of the universe—in particular, one can have cosmological KK particle production.

In conclusion, as far as quantum cosmology goes, the mini-superspace approach should not be thought of as a consistent truncation of the full theory in the same sense as low-energy effective field theory is. Rather, it should be thought of as a toy model that

[1] Equivalently, and perhaps more famously, any $n-$point correlator of the field $\langle\phi(\vec{x}_1)\phi(\vec{x}_2)\ldots\phi(\vec{x}_n)\rangle$ should only depend on the mutual distances among the points $\vec{x}_1,\ldots,\vec{x}_n$, *i.e.* on their intrinsic geometry, and not on their overall position.

drastically simplifies the theory while retaining some of the puzzling difficulties associated with gauge invariance.

With these qualifications in mind, we now proceed to consider the quantum mechanics of our models (and more general ones) in mini-superspace.

11.1.1. Mini-superspace approach

The gravitational part of the Hamiltonian in $d+1$ dimensions reads

$$H_{\rm grav} = N\ h^{-1/2}\left(\Pi^{ij}\Pi_{ij} - \frac{\Pi^2}{d-1}\right)\,, \tag{11.2}$$

where Π^{ij} is the momentum conjugate to h_{ij}, and indices are lowered by using h_{ij} itself. To the Hamiltonian above, we should add the matter part, which we will discuss below.

As is well known, gravity is a constrained system where the only equations governing the wave function are constraints. Variation with respect to N gives the Hamiltonian constraint, or Wheeler DeWitt (WdW) equation. When applied to the wavefuntion,

$$H\Psi = 0\,, \tag{11.3}$$

it enforces invariance under time reparameterizations.

One may wonder what happened to the momentum constraints, which are normally obtained by varying with respect to the shifts N_i. In the classical theory, when one loses some constraints because of a gauge choice implemented directly in the action, one should make sure to impose the corresponding equations independently. Momentum constraints represent the invariance of the wave function under spatial diffeomorphisms. As previously mentioned, the remnant of this GR symmetry here is the (finite dimensional) special linear group. In discussing classical solutions for cosmologies driven by a time-dependent scalar field, so that there is no anisotropic stress, we have used this group to set the angular momentum to zero. Can we thus interpret $SL(d,\mathbb{R})$ as a gauge symmetry? Should we impose "by hand" that $\Psi(h_{ij})$ be invariant under $SL(d,\mathbb{R})$? This sounds like the correct mini-superspace analog of the momentum constraint.

However, it is not difficult to see that this would kill the theory altogether. For example, in $2+1$ dimensions, the configuration space is the two dimensional hyperboloid and $SL(2,\mathbb{R}) \simeq SO(2,1)$ can take a point on the configuration space into any other point. Imposing the invariance of Ψ under $SL(2,\mathbb{R})$ would mean to set it to a constant on the entire hyperboloid. Apart from obvious normalization problems, this would mean dealing with a Hilbert space of just one state. One might be temped to impose that Ψ be invariant only under a compact subgroup of $SL(2,\mathbb{R})$—the rotations on the hyperboloid around the origin. This would solve the normalization problem and would leave some non-trivial dynamics to the system. But why should we restrict to this subgroup? We conclude that, although $SL(d,\mathbb{R})$ is morally the remnant of the full spatial gauge invariance of GR, we cannot interpret it as a gauge symmetry at the mini-superspace level.

In the case of a solid-driven cosmology, things appear to be better defined. working in unitary gauge we only have a residual $SO(d)$ invariance. Moreover, as far as spatial diffeomorphisms go, unitary gauge is a complete gauge-fixing. So, unless we decide to interpret the internal rotational symmetry (9.41) also as a gauge symmetry, we can now use the residual $SO(d)$ as a physical symmetry, and classify the physical states in terms of the associated angular momentum, without restricting only to states that are annihilated by it. It is then tempting to declare that the states that are *isotropic* in the quantum mechanical sense are those with zero angular momentum, since they are invariant under $SO(d)$. However, among these, there are those whose wave-function is peaked around isotropic cosmologies ($\xi = 0$, in the $d = 2$ case), and those whose wavefunction is perhaps peaked around some highly anisotropic cosmology, but for which the anisotropy direction has been averaged over. These are two very different cases, and only the former seems to be the quantum analog of an isotropic cosmology. Clearly, at the moment we only have a partial understanding of the quantum theory.

11.2. Ordering etc.

The WdW operator is the quantum version of the gravitational Hamiltonian (11.2), plus the matter part. Since h_{ij} and Π^{ij}, as quantum operators, don't commute, the expression (11.2) is ambiguous at the quantum level.

Using the wave-function representation of the quantum state, $\Psi = \Psi(h_{ij})$, the momentum conjugate to h_{ij} acts as a derivative:

$$\Pi^{ij} \longrightarrow -i\frac{\partial}{\partial h_{ij}} \,. \tag{11.4}$$

So the question is: how are we to order the h fields and the derivatives with respect to them when (11.2) acts on the wave function?

The most reasonable option seems to be that of building the *Laplace-Beltrami* operator of the corresponding superspace metric [21, 63]. In other words, whenever we are in the presence of a system (e.g. gravity + matter) whose Lagrangian/Hamiltonian can be cast in the form (see eq. (9.18))

$$L = \frac{1}{2N}\, G_{MN}(X)\dot{X}^M\dot{X}^N - NV(X) \quad\Longrightarrow\quad H = N\left(\frac{1}{2}\, G^{MN}\Pi_M\Pi_N + V\right) , \tag{11.5}$$

it is natural to identify the kinetic part of the Hamiltonian with the Laplacian associated with the metric G_{MN},

$$\nabla^2 = \frac{1}{\sqrt{-G}}\, \frac{\partial}{\partial X^M}\, \sqrt{-G}\; G^{MN}\frac{\partial}{\partial X^M} \,. \tag{11.6}$$

In the above expression the determinant of G_{MN} has been assumed to be negative because the conformal mode of the metric is a "time-like" direction in superspace (see e.g. eq. (9.37)).

This choice of ordering has important advantages. It gives an operator that is Hermitian with respect to the "standard" scalar product

$$\langle\psi_1, \psi_2\rangle = \int d^nX\sqrt{-G}\; \psi_1^*(X)\psi_2(X) \,, \tag{11.7}$$

once appropriate boundary conditions are chosen.[2] At the same time, and more impor-

[2] In order to interpret the WdW equation as a "time-evolution", one should use one of the fields, say, X^0,

tantly, the field-space Laplacian (11.6) is a well-defined prescription, invariant under field redefinitions. Other options (e.g. "push all derivatives to the right" $\sim G^{MN}\partial_M\partial_N$) are clearly dependent on a specific choice of coordinates in field space and therefore cannot be considered as valid alternatives.

11.3. (In-)consistent truncations

One puzzling, potentially interesting, property of the choice (11.6) is that it depends on the dimensions of the meta-universe, in the same way as the radial part of the standard Laplacian is different if we work, say, in two rather than three dimensions. Take, for example, the Hamiltonian (11.2) in 2+1 spacetime dimensions. The corresponding Laplacian operator reads (see App. F for details)

$$\nabla^2 = \frac{1}{\sqrt{h}}\left[(h_{ik}h_{jl} - h_{ij}h_{kl})\frac{\partial}{\partial h_{ij}}\frac{\partial}{\partial h_{kl}} + \frac{1}{4}h_{ij}\frac{\partial}{\partial h_{ij}}\right] \qquad \text{(pure gravity)}\,. \tag{11.8}$$

On the other hand, if we add a standard scalar field as in (9.26) we get

$$\nabla_s^2 = \frac{1}{\sqrt{h}}\left[(h_{ik}h_{jl} - h_{ij}h_{kl})\frac{\partial}{\partial h_{ij}}\frac{\partial}{\partial h_{kl}}\right] + \frac{2}{\sqrt{h}}\frac{\partial^2}{\partial\phi^2} \qquad \text{(gravity + scalar)}. \tag{11.9}$$

The point here is that adding a scalar field also modifies the piece of the Laplacian containing derivatives with respect to the metric.

Something similar happens if we change the number of degrees of freedom that we decide to keep in the metric itself. For instance, we can write the operator (11.9) more explicitly in the A, ξ and θ variables introduced in our general parametrization of a $d = 2$ spatial metric, eq. (9.13):

$$\nabla_s^2 = \frac{1}{2A^2}\left[-\frac{A^2}{4}\partial_A^2 - \frac{3}{4}A\,\partial_A + \partial_\xi^2 + \frac{1}{\tanh\xi}\partial_\xi + \frac{1}{\sinh^2\xi}\partial_\theta^2 + \partial_\phi^2\right]. \tag{11.10}$$

However, if we consider an isotropic FLRW space instead of our Bianchi-type model, then from (11.9), we get

$$\nabla_{s,\mathrm{FLRW}}^2 = \frac{1}{2A^2}\left[-\frac{A^2}{4}\partial_A^2 - \frac{1}{4}A\,\partial_A + \partial_\phi^2\right] \qquad (\text{gravity} + \text{scalar}, \text{FLRW})\,. \tag{11.11}$$

Because classical FLRW solutions are special cases of Bianchi models, one might have hoped to recover (11.11) in some isotropic limit of (11.10)—for instance when the state Ψ does not depend on ξ or θ. But this is clearly not the case, because (11.11) and (11.10) imply a different dependence of the wavefunction on A. In the structure of the Laplacian, first derivatives with respect to A are sensitive to the total number of fields involved, or even just on the number of symmetries that one wants to impose on the

as a time variable and think of $\psi(X^0, X^1, \dots)$ as a probability amplitude for the remaining variables, "evolving" in the time X^0. This approach would suggest a different scalar product than (11.7), the so-called "Klein-Gordon" scalar product [21, 112]. In this case the probability for the remaining fields $X^1, X^2, \dots$ is guaranteed to be conserved in "time" for all solutions of the WdW equation.

configuration space before quantization. In the beginning of this section we discussed how the mini-superspace approach does not seem a consistent truncation of the full theory. As an extreme example of this, we have just shown that the FLRW mini-superspace is not even a consistent truncation of a Bianchi one!

Conclusion and final remarks

The broad motivation of this thesis was (and is) to see how we can characterize the two great symmetries of cosmology - homogeneity and isotropy - at the quantum level. More specifically, at the level of the wavefunction of the universe.

The task turned out to be a huge undertaking, due to the complexity of the quantum theory when all the degrees of freedom of the classical theory are maintained. Therefore, we worked in a more modest framework. We considered metrics that are already homogeneous from the onset, *i.e.* Bianchi models, and study how to implement isotropy quantum mechanically.

The classical treatment of the model highlighted a new aspect of homogeneous cosmology, *rotation*, which is associated with anisotropy and with the way its main axes evolve in time. The effect is physical only in the presence of anisotropic stress in the energy momentum tensor. We have studied in some detail the case of solid matter, but it would be interesting to extend the analysis to other media. In particular, we have considered homogeneous and isotropic solids, characterized by a group of symmetry made up of internal translations and rotations [eqs. (9.40) and (9.41)]. As discussed in Sec. 9.4, this corresponds to breaking the $SL(d,\mathbb{R})$ symmetry of the gravitational action down to $SO(d)$. However, it would be interesting to consider also less symmetric solids that are not fully invariant under internal rotations (e.g. some "crystal" or "quasi-crystal"), for instance along the lines of [113]. In 2+1 dimensions, this would correspond to a matter Lagrangian that, in unitary gauge, explicitly depends on θ and not only on A and ξ (these parameters are defined in Sec. 9.1).

Rotation could be a feature of some relevance in other inflationary models beside solid inflation, such as chromo-natural [114] and gaugid [115] inflation. More generally, there could be traces of rotation even in more recent cosmic epochs, whenever the matter content of the universe features anisotropic stress. Although rotation is generally expected to rapidly dilute away like in the solid model considered in this thesis, this does not need to be always the case.

Finally, we have come back to the quantum mechanical description of these metrics and pointed to some questions/puzzles. Among them, the structure of the Laplacian in field space depends on the total number of fields involved. This directly affects the dependence of the wave function on the scale factor. One might object that we are not forced to use the Laplace-Beltrami operator as an ordering prescription for the Laplacian. However, it is hard to conceive other equally sensible choices than the one that is invariant under field redefinitions. A significant consequence of this feature is that we do not recover the Wheeler-DeWitt equation of the quantized FLRW models from the isotropic limit of the quantized Bianchi models. This reveals a potential inconsistency at the quantum level, the FLRW mini-superspace is not even a consistent truncation of a Bianchi one!

It is tempting to wonder whether, in a more realistic scenario, also long-wavelength, super-Hubble modes should be included in the degrees of freedom that affect the behavior of the scale factor. More generally, we would like to better understand how to operate a consistent truncation of degrees of freedom in quantum cosmology, in the same way we do in standard effective field theory: can we upgrade the mini-superspace approach to a systematic effective theory of quantum cosmology?

ANNEXES

A. Symmetries of the 2+1 action

The gravitational action (9.14) is invariant under $SL(2,\mathbb{R})$. By exponentiating the three generators ℓ_2, ℓ_3, and ℓ_4, we find that $\exp(\lambda\ell_2)$, $\exp(\epsilon\ell_3)$ and $\exp(\eta\ell_4)$ produce the coordinate transformations $\xi \to \xi'$, $\theta \to \theta'$, where, respectively,

$$\ell_2 : \qquad \xi' = \xi \tag{A.1}$$
$$\theta' = \theta + \lambda \tag{A.2}$$
$$\ell_3 : \qquad \cosh\xi' = \cosh\xi\cosh 2\epsilon + \sinh\xi\sinh 2\epsilon\cos\theta \tag{A.3}$$
$$\sin\theta' = \sin\theta\frac{\sinh\xi}{\sinh\xi'} \tag{A.4}$$
$$\ell_4 : \qquad \cosh\xi' = \cosh\xi\cosh 2\eta + \sinh\xi\sinh 2\eta\sin\theta \tag{A.5}$$
$$\cos\theta' = \cos\theta\frac{\sinh\xi}{\sinh\xi'}\,. \tag{A.6}$$

Now we want to show that, if the matter action does not depend on ξ and θ and thus respects all the $SL(2,\mathbb{R})$ symmetries, all rotating solutions—i.e., those with a nonzero $\dot\theta$—are equivalent to a non-rotating solution. Let us consider a non-rotating ("nr") solution first, with $\theta = 0$. By varying (9.14) we get that the eom for ξ simplifies to $(A^2\dot\xi)^\cdot = 0$. So

$$\theta_{\rm nr} = 0, \qquad \dot\xi_{\rm nr} = \frac{b}{A^2}, \tag{A.7}$$

is a solution of the system, for some constant b. The above non-rotating solution expresses the well-known result that, for a perfect fluid, the energy density associated with anisotropies $\sim \dot\xi^2$ scales as A^{-2d} in $d+1$ dimensions. We want to show that all other solutions with $\theta(t) \neq 0$ can be obtained by transforming the above. More generally, the eom for θ gives

$$\dot\theta = \frac{J}{A^2(\sinh\xi)^2}, \tag{A.8}$$

for some constant J, which represents the conserved charge associated with rotations, that is, the angular momentum.

Now, let us apply the transformation l_4 to the solution (A.7) with some parameter η. By using (A.5) and (A.6) we obtain a new solution $\xi(t)$, $\theta(t)$ with

$$\cosh\xi(t) = \cosh\xi_{\rm nr}(t)\cosh 2\eta \tag{A.9}$$
$$\theta(t) = \arccos\left(\frac{\sinh\xi_{\rm nr}(t)}{\sinh\xi(t)}\right) \tag{A.10}$$

Differentiating the second expression with respect to time and using (A.9) we find

$$\dot\theta = \frac{\dot\xi_{\rm nr}\sinh 2\eta}{1-(\cosh\xi_{\rm nr})^2(\cosh 2\eta)^2} = \frac{b\sinh 2\eta}{A^2(\sinh\xi)^2}\,, \tag{A.11}$$

where (A.7) and (A.9) have been used in the last step. It is clear that, by choosing η such that $b\sinh 2\eta = J$, any solution of the type (A.8) can be reproduced. One can then combine this with a constant rotation to fix the correct initial condition for $\theta(t)$.

B. Setting charges to zero by symmetry transformations

As we mentioned in the main text, there are general arguments that indicate that, given a continuous symmetry group G and its associated conservation laws, for generic initial conditions one can perform symmetry transformations that set to zero all charges apart from those in the Cartan subalgebra—the subalgebra that generates the maximal abelian subgroup of G [3].

In our specific cases, $G = SL(d, \mathbb{R})$ or $SO(d)$ acting on $h_{ij}(t)$ as

$$h \to g^T \cdot h \cdot g \ , \tag{B.1}$$

we can verify this claim explicitly. Let us rewrite the above transformation law at the infinitesimal level using $g = \exp(\alpha^a T_a) \simeq 1 + \alpha^a T_a$, where the T's are the generators of G and the α's are transformation parameters. We get

$$\delta h = \alpha^a (h \cdot T_a + T_a^T \cdot h) \ . \tag{B.2}$$

It is particularly convenient to work in the Hamiltonian formalism, where all symmetries $g \in G$ correspond to canonical transformations under which the Hamiltonian is invariant. In particular, for g to be a canonical transformation to begin with, it must act on the conjugate momenta p^{ij} as

$$p \to g^{-1} \cdot p \cdot (g^T)^{-1} \ , \tag{B.3}$$

so that

$$\delta p = -\alpha^a (p \cdot T_a^T + T_a \cdot p) \ . \tag{B.4}$$

Now, in phase-space the above infinitesimal canonical transformation must be generated by a phase-space function $Q = Q(h, p) = \alpha^a \tau_a(h, p)$ as

$$\delta h = \alpha^a \{h, \tau_a(h, p)\} \ , \qquad \delta p = \alpha^a \{p, \tau_a(h, p)\} \ , \tag{B.5}$$

where $\{\cdot, \cdot\}$ are the Poisson brackets. The τ_a's are the conserved charges associated with the generators T_a, expressed as functions of h_{ij} and p^{ij}. After trial and error, it is easy to convince oneself that they must take the form

$$\tau_a(h, p) = 2 \, \mathrm{tr}(h \cdot T_a \cdot p) \ . \tag{B.6}$$

Now the question is: assuming we start with generic initial conditions $h_{ij}(0)$, $p^{ij}(0)$, how many $\tau_a(h, p)$'s can we set to zero using only symmetry transformations?

There are two distinct cases:

- $G = SL(d, \mathbb{R})$: In this case we can perform a transformation that sets $h_{ij}(0)$ proportional to the identity matrix. After we do so, we have

$$\tau_a(h, p) \propto \mathrm{tr}(T_a \cdot p(0)) \ , \tag{B.7}$$

[3] This is, in fact, an upper bound on the number of nonzero charges that can survive. If the system's degrees of freedom make up a particularly small representation of G, one might be able to set even more charges to zero. For example, for a single particle in a central potential in d dimensions, the phase space is parametrized just by two d-vectors $\vec{q}$ and $\vec{p}$, and, up to rotations, for generic initial conditions there is only *one* nonzero angular momentum—that which acts in the $\vec{q}$-$\vec{p}$ plane.

and we are still allowed to perform $SO(d)$ rotations, since those are—by definition—the transformations in (B.1) that don't change the identity matrix. With these, we can make $p^{ij}(0)$ diagonal,

$$p(0) = \mathrm{d}iag(P^1, \ldots, P^d) \tag{B.8}$$

We thus get

$$\tau_a(h,p) \propto \sum_i (T_a)^i{}_i P^i \ , \tag{B.9}$$

and so, according to the classification of Sec. 9.1, we have that the only nonzero charges are those associated with the diagonal shears, which, indeed, make up the Cartan subalgebra of $SL(d,\mathbb{R})$.

- $G = SO(d)$: In this case the T_a generators are anti-symmetric and, using cyclicity of the trace and the fact that h and p are symmetric, we get

$$\tau_a(h,p) = \mathrm{tr}\big(T_a \cdot [\,p, h\,]\big) \ . \tag{B.10}$$

Now, the $[\,p, h\,]$ commutator is an anti-symmetric matrix, and can thus be block-diagonalized (at $t = 0$) by a suitable rotation in $SO(d)$. The resulting block-diagonal (anti-symmetric) matrix has $\lfloor d/2 \rfloor$ antisymmetric 2×2 blocks along the diagonal, and zeroes everywhere else. Plugging this structure into (B.10) and parametrizing the rotation generators as in (9.9), we see that the only nonzero charges that survive are those associated with the rotations that act within each of the aforementioned 2×2 blocks. These are precisely the generators of the Cartan subalgebra of $SO(d)$.

C. Peculiar velocities in general Bianchi type I

We compute the *peculiar velocities* in the three dimensional Bianchi type I spacetime, where the general metric in the co-moving coordinates is

$$ds^2 = -dt^2 + h_{ij}(t)dx^i dx^j \; . \tag{C.1}$$

The relation between the proper time τ and the co-moving coordinates is given by:

$$d\tau^2 = dt^2 - h_{ij}(t)\; dx^i dx^j \tag{C.2}$$

$$\Longrightarrow \; t'^2 = 1 + h_{ij}(t)\; x'^i x'^j \tag{C.3}$$

with " $'$ " denoting the derivative with respect to the proper time "$\frac{d}{d\tau}$".

The Killing vectors of Bianchi in the above gauge (C.1) are

$$\xi^j_{(i)} = (\partial_i)^j \; , \tag{C.4}$$

generating translations along x^i. The associated conserved charges along the directions $x^1 = x$, $x^2 = y$ are given by [4]:

$$P_{(i)} = x'^j \; (\xi_{(i)})_j \tag{C.5}$$

$$P_{(i)} = x'^j \; h_{jk} \; (\xi_{(i)})^k \; . \tag{C.6}$$

Hence,

$$P_x = x' \; h_{11} + y' \; h_{21} \; , \tag{C.7}$$

$$P_y = y' \; h_{22} + x' \; h_{12} \; . \tag{C.8}$$

From (C.5) we find,

$$x' = \frac{P_y \; h_{12} - P_x \; h_{22}}{{h_{12}}^2 - h_{11}h_{22}} \; , \tag{C.9}$$

$$y' = \frac{P_x \; h_{12} - P_y \; h_{11}}{{h_{12}}^2 - h_{11}h_{22}} \; . \tag{C.10}$$

Now we can derive the relation between the peculiar velocities $\dfrac{dx^i}{dt} = \dot{x}^i$ and the conserved

[4] Reminder: $\frac{d}{d\tau}\left(x'^j \; \xi_j\right) = x'^l \nabla_l \left(x'^j \; \xi_j\right) = x'^l \nabla_l x'^j . \; \xi_j + x'^l x'^j . \; \nabla_l \xi_j = 0$. The first term vanishes because of the geodesic equation $x'^l \nabla_l x'^j = 0$, and the second one cancels since it is the trace of the product of symmetric and anti-symmetric terms.

charges as follows:

$$\dot{x} = \frac{x'}{t'} = \frac{P_y\ h_{12} - P_x\ h_{22}}{(h_{12}\ ^2 - h_{11}h_{22})\sqrt{1 + h_{11}\ x'^2 + h_{22}\ y'^2 + 2h_{12}\ x'y'}}$$
$$= \frac{P_y\ h_{12} - P_x\ h_{22}}{(h_{12}\ ^2 - h_{11}h_{22})\left[1 + h_{11}\ \left(\frac{P_y\ h_{12} - P_x\ h_{22}}{h_{12}\ ^2 - h_{11}h_{22}}\right)^2 + h_{22}\ \left(\frac{P_x\ h_{12} - P_y\ h_{11}}{h_{12}\ ^2 - h_{11}h_{22}}\right)^2 + 2h_{12}\ \left(\frac{P_y\ h_{12} - P_x\ h_{22}}{h_{12}\ ^2 - h_{11}h_{22}}\right)\left(\frac{P_x\ h_{12} - P_y\ h_{11}}{h_{12}\ ^2 - h_{11}h_{22}}\right)\right]^{1/2}} \tag{C.11}$$

Conclusion: by contrast with the FLRW universe, we notice a rotation in the peculiar velocities. Indeed, let's set initial conditions $\dot{x}(t_0) = V_{0x}$, $\dot{y}(t_0) = V_{0y}$, and $h_{ij}(t_0) = \delta_{ij}$ [5]. Which yields the relations,

$$P_x = \frac{V_{0x}}{\sqrt{1 - V_{0x}^2 - V_{0y}^2}}\ , \tag{C.12}$$

$$P_y = \frac{V_{0y}}{\sqrt{1 - V_{0x}^2 - V_{0y}^2}}\ . \tag{C.13}$$

If $V_{0x} = 0$, then

$$\dot{x}(t) = \frac{V_{0y}\ h_{12}}{(h_{12}\ ^2 - h_{11}h_{22})\sqrt{1 - V_{0y}^2 - \frac{h_{11}h_{12}^2V_{0y}^2}{(h_{12}\ ^2 - h_{11}h_{22})^2} + \frac{h_{22}h_{11}^2V_{0y}^2}{(h_{12}\ ^2 - h_{11}h_{22})^2}}}$$
$$= \frac{V_{0y}\ h_{12}}{(h_{12}\ ^2 - h_{11}h_{22})\sqrt{1 - V_{0y}^2 - \frac{h_{11}V_{0y}^2}{h_{12}\ ^2 - h_{11}h_{22}}}}\ , \tag{C.14}$$

which means there is motion in the x direction even if it starts out with a velocity $V_{0x} = 0$, while in the FLRW case, $\dot{x}(t)$ remains null at all times t.

<u>In FLRW:</u>

$$\dot{x}(t) = \frac{P_x}{A^2(t)\sqrt{1 + P_x^2\ A^{-2}(t) + P_y^2\ A^{-2}(t)}}\ , \tag{C.15}$$

$$= \frac{V_{0x}}{A^2(t)\sqrt{1 + \frac{V_{0x}^2}{A^2(t)}(1 - A^2(t)) + \frac{V_{0y}^2}{A^2(t)}(1 - A^2(t))}}\ . \tag{C.16}$$

For an initial condition $V_{0x} = 0$, $\dot{x}(t)$ remains null at all times t.

[5] It is equivalent to setting $A(t_0) = 1$, $\xi(t_0) = 0$, $\theta(t_0) = 0$.

D. Rotating cosmologies in 3+1 dimensions

D.1. Parameterization of the metric

We still write the spatial metric, in matrix form, as

$$h(t) = R^T(t) \cdot D(t) \cdot R(t)\,, \tag{D.1}$$

where R is a rotation and D a diagonal matrix. We choose to write the rotation as

$$R(t) = e^{i\,\vec{\theta}(t)\cdot\vec{J}}\,, \tag{D.2}$$

where $(J_i)_{jk} = i\epsilon_{ijk}$ and $[J_i, J_j] = i\epsilon_{ijk}J_k$. We find that the time derivative of R can be written as

$$\dot{R} = iRP_+ = iP_-R\,, \tag{D.3}$$

where

$$P_\pm = \vec{E}_\pm \cdot \vec{J} \tag{D.4}$$

$$\equiv \dot{\vec{\theta}} \cdot \vec{J} + \frac{\sin|\vec{\theta}| - |\vec{\theta}|}{|\vec{\theta}|}\left[\dot{\vec{\theta}} \cdot \vec{J} - \frac{(\vec{\theta}\cdot\dot{\vec{\theta}})(\vec{\theta}\cdot\vec{J})}{|\vec{\theta}|^2}\right] \pm \left[\frac{1-\cos|\vec{\theta}|}{|\vec{\theta}|^2}\right]\left(\vec{\theta}\wedge\dot{\vec{\theta}}\right)\cdot\vec{J} \tag{D.5}$$

In the gravitational action

$$S_{grav} = \frac{1}{16\pi G}\int \frac{dt}{4N}\sqrt{h}\left(h^{il}h^{jm} - h^{ij}h^{lm}\right)\dot{h}_{ij}\dot{h}_{lm}\,, \tag{D.6}$$

there are two different structure. For that involving $h^{ij}\dot{h}_{ij}$, only the diagonal part survives,

$$h^{ij}\dot{h}_{ij} = \mathrm{T}r\left(D^{-1}\dot{D}\right)\,. \tag{D.7}$$

The other term is more complicated. We find

$$h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li} = -2\mathrm{T}r\left(P_-^2\right) + 2\mathrm{T}r\left(D^{-1}P_-DP_-\right) + \mathrm{T}r\left(D^{-1}\dot{D}D^{-1}\dot{D}\right)\,. \tag{D.8}$$

The first term represents the kinetic terms and the self interactions of the θ-fields. Because $\mathrm{T}r(J_lJ_m) = 2\delta_{lm}$, we get

$$\mathrm{T}r\left(P_-^2\right) = 2\vec{E}_-\cdot\vec{E}_- \tag{D.9}$$

The second term represents the coupling between the shear terms on the diagonal matrix and the angles. In order to calculate this we introduce the matrices

$$M_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\,,\quad M_2 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\,,\quad M_3 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\,. \tag{D.10}$$

So that we can write the diagonal piece as

$$D(t) = \alpha_i(t)M_i\,. \tag{D.11}$$

The coupling term above contains M and J matrices alternated inside the trace. We find

$$\mathrm{Tr}\,(J_i M_j J_k M_l) = \delta_{ik}|\epsilon_{ijl}| \tag{D.12}$$

where no summation over i is intended. We get

$$\mathrm{Tr}\left(D^{-1}P_- D P_-\right) = E_1^2\left(\alpha_2\alpha_3^{-1} + \alpha_3\alpha_2^{-1}\right) + E_2^2\left(\alpha_3\alpha_1^{-1} + \alpha_1\alpha_3^{-1}\right) + E_3^2\left(\alpha_1\alpha_2^{-1} + \alpha_2\alpha_1^{-1}\right) . \tag{D.13}$$

Finally the gravitational action reads

$$S = \frac{1}{16\pi G}\int \frac{dt}{2N}\big[- \dot\alpha_1\dot\alpha_2\alpha_3 - \dot\alpha_1\alpha_2\dot\alpha_3 - \alpha_1\dot\alpha_2\dot\alpha_3 - 2\alpha_1\alpha_2\alpha_3 E_-^2 \tag{D.14}$$

$$+ E_1^2\alpha_1(\alpha_2^2 + \alpha_3^2) + E_2^2\alpha_2(\alpha_1^2 + \alpha_3^2) + E_3^2\alpha_3(\alpha_1^2 + \alpha_2^2)\big] . \tag{D.15}$$

E. More on the solid's Lagrangian and Hamiltonian

As we discussed, in order to build the Lagrangian of a solid it is useful to introduce the Lorentz-scalar matrix $B^{IJ} \equiv \partial_\mu \phi^I \partial^\mu \phi^J$. In d spatial dimensions $I, J...$ take values between 1 and d and the Lagrangian is a general function of the invariants built with B^{IJ},

$$S_{\text{solid}} = -\int d^d x \, dt \, N\sqrt{h} \; F\left([B], [B^2], \ldots, [B^d]\right), \tag{E.1}$$

where $[\cdots]$ is shorthand for the trace.

By using the ADM form for the metric (9.3), we can perfor a useful $d+1$ decomposition of such a quantity:

$$B^{IJ} = -\frac{1}{N^2}\left(\dot{\phi}^I - N^i\partial_i\phi^I\right)\left(\dot{\phi}^J - N^j\partial_j\phi^J\right) + h^{ij}\partial_i\phi^I\partial_j\phi^J \tag{E.2}$$

$$\equiv -V^I V^J + b^{IJ}. \tag{E.3}$$

Note that on the second line we have implicitly defined $V^I = N^{-1}(\dot{\phi}^I - N^i\partial_i\phi^I)$ and $b^{IJ} = h^{ij}\partial_i\phi^I\partial_j\phi^J$.

The perfect fluid limit of (E.1) corresponds to F depending only on the determinant of B^{IJ},

$$S_{\text{fluid}} = -\int d^d x \, dt \, N\sqrt{h} \; F\left(\det B^{IJ}\right). \tag{E.4}$$

Such a determinant can be expressed as

$$\det B^{IJ} = \frac{1}{d!}\,\epsilon_{I_1 I_2 \ldots I_d}\,\epsilon_{J_i J_2 \ldots J_d}\left(-V^{I_1}V^{J_1} + b^{I_1 J_1}\right)\cdots\left(-V^{I_d}V^{J_d} + b^{I_d J_d}\right) \tag{E.5}$$

$$= \frac{1}{d!}\,\epsilon_{I_1 I_2 \ldots I_d}\,\epsilon_{J_i J_2 \ldots J_d}\left(-d\,V^{I_1}V^{J_1}b^{I_2 J_2}\cdots b^{I_d J_d} + b^{I_1 J_1}\cdots b^{I_d J_d}\right) \tag{E.6}$$

$$= (\det b^{IJ})\left[1 - (b^{-1})_{IJ}V^I V^J\right]. \tag{E.7}$$

In order to write the Hamiltonian of the solid one should calculate the conjugate momentum $\Pi_I = \partial\mathcal{L}/\partial\dot{\phi}_I$ and invert this relation in favor of $\dot{\phi}_I$. Such explicit calculation does not seem to be possible except in sporadic cases. One notable case is that of *non-relativistic fluid*, i.e., when $F = (\det B^{IJ})^{1/2}$. Then we get

$$\Pi_I = 2\sqrt{h}\,\det b\, F'(\det B^{IJ})(b^{-1})_{IJ}V^J \tag{E.8}$$

By using the ansatz $V^I = A\, b^{IJ}\Pi_J$ with A to be determined, the above relation can be inverted to give

$$V^I = \frac{b^{IJ}\Pi_J}{\sqrt{h\det b + b^{KL}\Pi_K\Pi_L}}, \tag{E.9}$$

i.e.

$$\dot{\phi}^I = N \frac{b^{IJ}\Pi_J}{\sqrt{h \det b + b^{KL}\Pi_K\Pi_L}} + N^i \partial_i \phi^I \,. \tag{E.10}$$

Now we can express the Hamiltonian of the non-relativistic fluid,

$$H_{\rm fluid} = \int d^d x \left(N \mathcal{H}^{\rm fluid} + N^i \mathcal{H}_i^{\rm fluid} \right) , \tag{E.11}$$

with

$$\mathcal{H}^{\rm fluid} = \sqrt{h \det b + b^{KL}\Pi_K \Pi_L} \,, \tag{E.12}$$

$$\mathcal{H}_i^{\rm fluid} = \Pi_I \partial_i \phi^I \,. \tag{E.13}$$

E.1. Momentum constraint

It is useful to look at the momentum constrain for the solid. In order to obtain its explicit form in the Hamiltonian formalism we should be able to invert for $\dot{\phi}_I$ as in (E.10), which does not seem to be doable in general. We can however calculate the momentum constraint in the Lagrangian formalism, by simply varying the action (E.1) with respect to N^i.

First we note that

$$\frac{\partial \left[B^n\right]}{\partial N^i} = \frac{n}{N} \left(B^{n-1} \right)^{IJ} \left(\partial_i \phi^I V^J + \partial_i \phi^J V^I \right) , \tag{E.14}$$

where $\left(B^{n-1}\right)^{IJ}$ is the $(n-1)th$ power of the matrix B^{IJ} (without tracing). This gives

$$\frac{\partial \mathcal{L}}{\partial N^i} = \sum_{n=1}^{d} n \sqrt{h}\, F_{[B^n]} \left[B^{n-1} \right]^{IJ} \left(\partial_i \phi^I V^J + \partial_i \phi^J V^I \right) = 0 \,. \tag{E.15}$$

A simple and general solution of the above equation is $V^I = 0$. That is,

$$\dot{\phi}^I - N^i \partial_i \phi^I = 0 \,. \tag{E.16}$$

In unitary gauge, $x^i = \phi^I$, this simply reduces to the condition $N^i = 0$.

We thus see that choosing the unitary gauge always implies $N^i = 0$ as a consequence of the momentum constraint, as anticipated in the main text.

F. Laplace-Beltrami operator in mini-superspace

We consider a general action of the type

$$S \propto \frac{1}{2}\int \frac{dt}{N}\, G_{AB}(X)\dot{X}^A\dot{X}^B\,, \tag{F.1}$$

and aim to calculate the Laplace-Beltrami operator associated with the metric G_{AB},

$$\nabla^2 = \frac{1}{\sqrt{-G}}\partial_A\,\sqrt{-G}\;G^{AB}\partial_B\,, \tag{F.2}$$

where $G = \det G_{AB}$. In the absence of matter fields and up to an irrelevant normalization factor, the metric G_{AB} can be read off eq. (9.5),

$$G_{AB} = h^{1/2}\left(h^{ik}h^{jl} - h^{ij}h^{kl}\right), \tag{F.3}$$

where $h = \det h_{ij}$ and (lower) capital latin letters A, B stand for pairs of (upper) symmetric indices, (ij) and (kl) respectively. The switch between lower and upper indexes is due to the fact that we are considering the (covariant) spatial metric as a generalized coordinate in field space, $h_{ij} \sim X^A$.

A simple scaling argument shows that

$$-G = h^{-3/2}\,, \tag{F.4}$$

up to a positive constant factor, which cancels in (F.2). The inverse of G_{AB}, on the other hand, is given in $d+1$ spacetime dimensions by

$$G^{BC} = \frac{1}{2\,h^{1/2}}\left(h_{km}h_{ln} + h_{kn}h_{ml} - \frac{2}{d-1}\,h_{kl}h_{mn}\right), \tag{F.5}$$

where, $B = (kl)$ and $C = (mn)$. One can check that by multiplying the RHS of (F.3) by that of (F.5) one obtains the identity matrix δ_A^C, which, expressed in spatial $i, j, \ldots$ indices, is just the symmetrized product of Kronecker delta's,

$$\frac{1}{2}\left(h^{ik}h^{jl} - h^{ij}h^{kl}\right)\,\left(h_{km}h_{ln} + h_{kn}h_{ml} - \frac{2}{d-1}\,h_{kl}h_{mn}\right) \;=\; \frac{1}{2}\left(\delta^i_m\delta^j_n + \delta^i_n\delta^j_m\right)\,. \tag{F.6}$$

There are now all the ingredients to calculate (F.2),

$$\nabla^2 = h^{3/4}\frac{\partial}{\partial h_{ij}}\;h^{-5/4}\left(h_{ik}h_{jl} - \frac{1}{d-1}\,h_{ij}h_{kl}\right) \tag{F.7}$$

$$= h^{-1/2}\left[\left(h_{ik}h_{jl} - \frac{1}{d-1}\,h_{ij}h_{kl}\right)\frac{\partial}{\partial h_{ij}}\frac{\partial}{\partial h_{kl}} + \frac{2d^2-2d-3}{4(d-1)}\,h_{ij}\frac{\partial}{\partial h_{ij}}\right] \tag{F.8}$$

For $d = 2$ this expression gives the pure-gravity Laplacian displayed in eq. (11.8). Alternatively, still in $d = 2$, we can parametrize h_{ij} in terms of the A, ξ, and θ variables and obtain

$$\nabla^2 = \frac{1}{2A^2}\left[-\frac{A^2}{4}\partial_A^2 - \frac{A}{2}\partial_A + \frac{1}{\tanh\xi}\partial_\xi + \partial_\xi^2 + \frac{1}{\sinh^2\xi}\partial_\theta^2\right]. \tag{F.9}$$

As commented in Sec. 11, the structure of the above operators changes when including additional matter degrees of freedom in the theory. This is why the pure-gravity Laplacian (F.9) is different than that in the presence of a scalar field (11.10).

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