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Cognition, Langage, Interaction

HOW INFORMAL KNOWLEDGE INFORMS US ABOUT THE TEACHING AND LEARNING OF MATHEMATICS

THÈSE

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COMMENT LES CONNAISSANCES INFORMELLES NOUS INFORMENT SUR L'ENSEIGNEMENT ET L'APPRENTISSAGE DES MATHEMATIQUES

ABSTRACT

This thesis raises questions on how knowledge constructed through previous experience influences the learning and teaching of mathematics. In our literature review, we first tackle the subject of conceptual development with a specific focus on intuitive conceptions, and we look at the intuitive conceptions that make it possible to grasp arithmetic concepts taught in elementary school. We then examine the characteristics of mathematical knowledge being acquired in school, as well as the factors influencing its acquisition. Afterward, we explore how informal arithmetic abilities serve as a basis for the development of principle-based arithmetic proficiency. Notably, we explore how the semantic context influences the representation of arithmetic problems and the strategies used to solve them. The last part presents an overview of different frameworks used to approach teachers' pedagogical competence.

The empirical research conducted in this thesis explores how informal knowledge influences students' solving processes on arithmetic problems and teachers' judgments about students' strategies. We conducted six experiments consisting of collective classroom studies, verbal reports, and retrospective think-aloud questionnaires with a total of 673 elementary school students, 36 teachers, and 36 lay adults. Our findings reveal that students have higher performance on problems that are easy to solve through a mental simulation of the encoded representation than on problems for which this mental simulation is challenging. The informal solving strategies that they use to find the solution do not require the use of arithmetic knowledge. Nevertheless, we demonstrate that students who have participated in an arithmetic intervention program working on the encoding of a problem's representation and its recoding when this initial representation leads to costly solving strategies increased the use of solving strategies that reflect the use of arithmetic principles. Yet, when we questioned teachers about the strategies students use to solve such problems, our findings revealed that they overlooked the difficulties that intuition-consistent problems can pose for students.

The role that intuitive conceptions play in the students' representational processes is then discussed, and a processing model about students' arithmetic problem solving strategies is proposed. The notion of 'intuitive blind spot' in teachers' diagnostic judgments is introduced as contrasting to the 'expert blind spot.' We conclude on some educational entailments based on our findings.

Keywords: informal knowledge; intuitive conceptions; analogical encoding; mathematical cognition; pedagogical content knowledge; teachers' cognition; arithmetic problem solving

RESUME

Cette thèse porte sur la manière dont les connaissances acquises grâce à l'expérience antérieure influencent l'apprentissage et l'enseignement des mathématiques. La revue de la littérature aborde dans un premier temps la question du développement conceptuel en mettant l'accent sur les conceptions intuitives sous-tendant les concepts arithmétiques enseignés à l'école. Nous examinons ensuite les facteurs qui influencent l'acquisition des notions mathématiques enseignées à l'école, puis la manière dont les compétences arithmétiques informelles peuvent venir en appui du développement des notions scolaires. Nous explorons notamment comment le contexte sémantique influence la construction de la représentation des problèmes arithmétiques et les stratégies mise en œuvre pour les résoudre. Un dernier chapitre présente une variété d'approches de la compétence pédagogique des enseignants.

La partie empirique de la thèse porte sur la façon dont les connaissances informelles influencent les processus de résolution des problèmes arithmétiques des élèves, ainsi que sur l'évaluation par les enseignants des stratégies mises en place par les élèves. Six expériences sont présentées, comprenant des études collectives en classe, le recueil de protocoles verbaux ainsi que de questionnaires auprès de 673 élèves du primaire, 36 enseignants et 36 adultes tout venant. Nos résultats montrent que les élèves ont une meilleure performance sur des problèmes dont la résolution est facilitée par une simulation mentale de la représentation encodée, par rapport aux problèmes pour lesquels cette simulation mentale est trop coûteuse pour être mise en œuvre. Ces stratégies informelles n'exigent pas l'utilisation de connaissances arithmétiques. Pourtant, nous montrons que les élèves qui ont participé à une intervention visant à travailler l'encodage de la représentation d'un énoncé et son recodage lorsque cette représentation initiale conduit à des stratégies de résolution coûteuses, utilisent plus fréquemment des stratégies reflétant l'utilisation de principes arithmétiques. En revanche, lorsque les enseignants sont interrogés sur les stratégies utilisées par les élèves pour résoudre ces problèmes, leurs réponses indiquent un manque de prise en compte des difficultés que des problèmes conformes à l'intuition peuvent poser aux élèves.

Nous discutons du rôle que jouent les conceptions intuitives dans les processus représentationnels des élèves et proposons un modèle des processus de résolution par les élèves. Nous introduisons également la notion d'angle mort de l'intuition dans les jugements des enseignants et concluons sur les implications pédagogiques de nos travaux.

Mots-clés : connaissances informelles ; conceptions intuitives ; l'encodage analogique ; cognition mathématique ; connaissances pédagogiques du contenu ; cognition des enseignants ; résolution de problèmes arithmétiques

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Related scientific output

Chapter 5

Experiments 1, 2 and 4

Gvozdic, K., & Sander, E. (2017). Solving additive word problems: Intuitive strategies make the difference. In B. C. Love, K. McRae, & V. M. Sloutsky (Eds.), Proceedings of the 39th Annual Conference of the Cognitive Science Society. London, UK: Cognitive Science Society.

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Chapter 6

Gvozdic, K., & Sander, E. (accepted). Learning to be an opportunistic word problem solver: Going beyond informal solving strategies. *ZDM Mathematics Education*.

Chapter 7

Gvozdic, K., & Sander, E. (2018). When intuitive conceptions overshadow pedagogical content knowledge: Teachers' conceptions of students' arithmetic word problem solving strategies. *Educational Studies in Mathematics*. *98*(2), 157-175.

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INTRODUCTION AND OVERVIEW

Introduction

Customarily the fields of psychology and mathematics education had divergent aims (De Corte, Greer, & Verschaffel, 1996). In psychology, the study of mathematical cognition would not address issues regarding educational applications, whereas mathematics educators would strive to find ways to change the educational practice. Yet, there is growing attention paid to developing specific lines of research on the cognitive and neurological processes involved in learning and teaching in school settings which would have translational aims and an evidence-based attitude (Davidesco & Milne, 2019; Higgins et al., 2019; Pasquinelli, Zalla, Gvozdic, Potier-Watkins, & Piazza, 2015). Indeed, numerous research programs have emerged sharing built upon a better understanding of the psychological mechanisms involved in the mathematics learning in order to change educational practices (e.g., Carpenter, Fennema, & Franke, 1996; Van den Heuvel-Panhuizen & Drijvers, 2014). There is a diversity of theoretical approaches shaping the intention with which such research is conceptualized (Lerman, 2006). This leads to sometimes prioritizing improved understanding, focusing on the processes of individual knowledge construction, and at other times prioritizing improved practice, focusing on teachers' mediation of the development of knowledge in students. Reframing classroom issues into a research question, as well as developing the adequate methodological design to tackle the issue, remain challenging endeavors. We believe that two conceptual shifts significantly contribute to establishing a dialogue and setting joint aims. One is the increasing importance given to situated approaches in psychology, and the other is revisiting the role of formalisms in education.

In the field of psychology, the nature of knowing and learning has varied historically, depending on different approaches. Greeno, Collins, and Resnick (1996) identified three general perspectives in the literature. The first is the behaviorist/empiricist perspective, where knowledge is "an organized accumulation of associations and components of skills" (p. 16). The second is the cognitivist/rationalist perspective, where knowing is emphasized as "the organization of information in cognitive structures and procedures" (p.16). The third is the situative/pragmatist-sociohistoric perspective, which sees knowledge as "being distributed among people and their environment, including objects, artifacts, tools, books, and the communities of which they are a part" (p.17). Nevertheless, in recent decades cognitivist and situative perspectives have grown closer together. This reconciliation can be seen in approaches that see cognition as embodied (Lakoff & Johnson, 1999). The relevance of this coupling can also be seen in the influence that context plays in making meaning of a situation (Barsalou, 1982). Bridging the gap is seen as a way to take into consideration the mental structures that are constructed, while also accounting for the flexibility, malleability and distributed nature of concepts (Vosniadou, 2007). In our view, adopting a situated cognitive perspective brings the field of psychology closer to constructing shared aims with educational research by acknowledging that cognitive processes are largely influenced by the context in which they develop and are modulated by the content that they operate on.

Nathan (2012) proposed an elaborate critique of a widely held belief in education and society, which takes the form of a *formalism first* approach to learning. The belief on which the formalisms first view is based on is that knowledge about the formalisms of a domain is a prerequisite to apply the knowledge. Formalisms are regarded in both their narrow sense, referring to specialized representational forms that are conventionally used in a field such as symbolic equations, and in their broad view, where they refer to scientific theories and formal principles. The prevalence of this view can be observed in teachers' predictions about student performance on arithmetic and algebra problems. Teachers consistently ranked problems presented in their formal form – symbolic equations – as easier for students than problems that were further away from their formal form – verbal story and word problems (Nathan, Koedinger, & Alibali, 2001; Nathan & Petrosino, 2003). This, however, is not always true when we look at student performance. Formalisms undoubtedly have a critical role in education, yet, adopting a formalism first view considers that "conceptual development proceeds from the formal to the applied" (Nathan, 2012, p. 128). This does not provide an adequate view of conceptual development, which is one of the main reasons why the formalism first approach is inappropriate. We, therefore, think that any attempt to conduct joint research in the field of psychology and educational sciences needs to have at least a moderate view of the role of formalisms in formal education.

We believe that studying teaching and learning needs to take into consideration both the development of the conceptual understanding of new content and how it relates to previously held knowledge, as well as the implicit beliefs held by educators and how they shape their practice. In order to tackle these very broad questions, we enter the field of analogical reasoning, which in our view provides a unifying theory for the informal, formal, and situated dimensions of school learning. In contrast to historical views of analogies as proportional relations, contemporary approaches see analogy making as a means to understand one thing by referring to something else (Holyoak & Thagard, 1995). Analogies make it possible to go beyond a singular experience of a situation by creating mental categories that can guide the interpretation of new situations (Hofstadter & Sander, 2013). For example, if one adopts a traditional Socratic point of view and states 'a teacher is like a midwife[1]', they are making an analogy. Providing such an analogy makes it possible to view teaching by referring to a different profession, that might be more familiar, and that lends its properties to 'a teacher'. This statement entails that the person doing the teaching only brings out the knowledge that is already implicit in students. Furthermore, to make such an analogy means that a person is not referring to a specific teacher, but a general category that instantiates the teaching profession (Glucksberg & Keysar, 1990).

THESIS OUTLINE

The first four chapters of this thesis present the theoretical background of research related to mathematics learning and teaching in the field of cognitive and educational psychology. In the first chapter, we tackle the subject of conceptual development with a specific focus on intuitive conceptions. The perspective that we adopt is that analogical reasoning is a central cognitive mechanism that organizes the conceptual system of learners and teachers by creating mental categories. Following this approach, we explain how intuitive conceptions can be regarded as analogical sources. We then describe the mechanisms that lead to inferences about unknown concepts. With this foundation, the second chapter focuses on domain-specific theories of intuitive conceptions. We look at the different intuitive conceptions that make it possible to grasp arithmetic concepts taught in elementary school. The third chapter of this thesis examines the characteristics of mathematical knowledge being acquired in school, as well as the different factors influencing its acquisition. We look at the main types of knowledge determining one's arithmetic competence – the knowledge of procedures and underlying principles. Afterward, we explore how informal arithmetic abilities serve as a basis for the development of principle-based arithmetic proficiency. The second part of this chapter focuses on the mediating role of situational knowledge for the use of different solving strategies and arithmetic principles. Notably, we explore how the semantic context can influence the representation of arithmetic problems and the strategies used to solve them. The last part of the third chapter explores some activities that can help solvers develop the capacity to use the most appropriate strategy on a given problem. The fourth chapter represents an overview of different approaches used to understand teachers' pedagogical competence. We

[1]A reminder that 'midwife' is a term describing a health professional, regardless of their gender.

present the different types of knowledge that have been identified to play an important role in teachers' judgments about student performance and address the question of what causes their judgment to be misaligned with students' actual performance. This extensive literature review raises questions about how knowledge constructed through previous experience influences the learning and teaching mathematics.

The empirical research conducted in this thesis explores how informal knowledge influences students' solving processes on arithmetic problems and teachers' judgments about students' strategies. In Chapters 5 to 7, we present six experiments that we have conducted with a total of 673 first and second-grade students in classroom contexts, with 36 elementary school teachers and 36 adults.

In Chapter 5, we studied the processes involved in solving arithmetic problems. We proposed that, based on the semantic relations described in the problems, different conceptions would guide the construction of a representation that solvers use to find the solution to a problem. Based on this representation a mental simulation would lead solvers to use informal strategies. When the mental simulation would bear low cost, then the informal strategy would easily provide the solvers with a numerical solution. However, when the mental simulation bears high cost, then solvers would need to use formal solving strategies. We expected to find higher performance rates on problems that are easy to simulate mentally than problems whose mental simulation bears high cost. Furthermore, we expected to observe more informal strategies on low cost mental simulation problems than on high cost mental simulation problems and more formal strategies on high cost mental simulation problems than on low cost mental simulation problems.

In Chapter 6, we present a research-based arithmetic intervention program that substituted the regular arithmetic curriculum in first-grade classes. The problem solving syllabus aimed to promote the use of the most adequate strategy for finding the solution on different problems, and namely to promote the use of formal strategies when they are the most appropriate choice. In our empirical assessment, we compared the performance of students who participated in the intervention to students from regular classes. We expected that students who were part of the intervention would have higher performance and use more formal solving strategies.

In Chapter 7 of this thesis, we investigated how teachers' prior knowledge influences their diagnostic judgments of student performance. We presented participants with arithmetic word problems for which there is strong empirical evidence regarding the strategies students use to solve them and their performance rates on such problems. The participants were asked to evaluate the relative difficulty of the problems that were presented and explain what makes certain problems more difficult than others. We expected that when a problem relates to prior-knowledge that is intuition-consistent, then teachers would be less successful in determining the difficulties it poses for students than on problems that are intuition-inconsistent.

LITERATURE REVIEW

CHAPTER 1 – DEVELOPMENT OF INTUITIVE CONCEPTIONS

MAIN APPROACHES TO INTUITIVE CONCEPTIONS

Widespread domains of intuitive conceptions

Through daily life experience, children come to develop intuitive understandings about the functioning of various phenomena and they come to acquire vast knowledge about concepts based on the interactions with their surroundings. For example, in the domain of biology children explain the behavior of animate and inanimate objects even though they have not encountered them before, by conceiving them as living and conscious entities like humans. These kinds of animist theories that children develop are characterized by their anthropocentric thinking (Piaget, 1960). Many terms have been used to describe the intuitive knowledge that is acquired without formal teaching about the topics that children come to intuitively understand: *intuitive* theories (Carey, 1985; McCloskey & Kargon, 1988), *naive analogies* (Hofstadter & Sander, 2013), *framework theories* (Vosniadou, 2012; Vosniadou, Vamvakoussi, & Skopeliti, 2008), *misconceptions* (Smith, diSessa, & Roschelle, 1993), *naive* or *"old" ideas* (diSessa, 1993), *lay cognition* (Shtulman, 2015), *folk theories* (Kempton, 1987), *theory theory* (Gopnik & Wellman, 1994), *intuitive rules* (Tirosh & Stavy, 1999), *mental models* (Vosniadou & Brewer, 1992), *cognitive construals* (Coley & Tanner, 2012). Whenever we talk about intuitive conceptions, we refer to knowledge that is acquired with little or no effort, no teaching and which is not necessarily aligned with the culturally and scientifically accepted notions. The different descriptions of intuitive conceptions often have the term "theory" attached to them, because, just like scientific theories, they lead to inferences which make it possible to provide explanations for the observed phenomena and make predictions about future outcomes (Gopnik & Wellman, 1994; Vosniadou, 2017). Yet, these initial understandings lack the explanatory power and internal consistencies of scientific theories and are not systematically challenged for falsification (Vosniadou, 2014). Intuitive conceptions are widespread across various domains, including physics, biology (Shtulman, 2006), mathematics and even psychology and pedagogy (Fischbein, 1987; Olson & Bruner, 1996; Shtulman, 2017). Studying them is mainly domain specific in regard to the ontological stances of the topic but addressing their characteristics and mechanisms of actions from a psychological point of view is common across all domains. This notably makes it possible for scholars to study the ways in which intuitive conceptions can influence the acquisition of new knowledge, since it is important to know when they facilitate the acquisition of a new concept, and when they are rather harmful. Several researchers have attempted to identify the mechanisms and structures underlying intuitive conceptions.

Intuitive theories

Carey's (1985) work initially investigated if children actually hold certain naïve theories about biology and physiological functioning. She initially focused on explaining if Piaget's animistic interpretation truly relies on a form of naïve biology, or if they rather rely on their naïve psychology to explain biological processes. Yet, whether we consider children's naïve biology or naïve psychology as a frame of reference, this does not undermine the fact that children hold intuitive conceptions that influence their way of thinking and the way in which they acquire new knowledge. In Susan Carey's (2009) view, intuitive theories are conceptual structures in which conceptual representations – symbolic content that is not a perceptual or sensory representation – are embedded. In her view, the acquisition of intuitive theories is not dependent on any previously held form of knowledge (including domain-specific core knowledge). She refrains from calling such representations *knowledge* since they are not required to be aligned with truthful facts. Secondly, a fundamental aspect of intuitive theories is their inferential role through which these initial representations are extended. This inferential capacity of the constructed conceptual representation can lead to the construction of new concepts that transcend the initial representations. In Carey's view, this is a process of conceptual change within which an individual concept is changed by transforming and restructuring the initial representation. In this way, two previously distinct representations can also be integrated. This conceptual change occurs when the initial representations can only provide partial interpretations for newly encountered concepts and require new representational machinery to provide stable predictions. For instance, young children's intuitive theories about biological functioning before 4 years of age mainly rely on their knowledge about humans. They do not make predictions about animal behavior based on similarities with other animals, but only based on human characteristics. However, between the ages of 4 and 10, children start making predictions about animal behaviors based on their similarities to other animals (Carey, 1985). In Carey's view, this is due to a fundamental restructuration in children's representation of biological functioning.

Knowledge in pieces

Another characterization of intuitive conceptions that is quite different from that of Susan Carey's has been advanced by Andrea diSessa (1993, 2017). In his view, in order to fully grasp what characterizes intuitive conceptions, it remains problematic to consider them as coherent, theory-like structures and at the same time to consider that they are fragmented (diSessa, 2009). He proposes a different set of mechanisms involved in the construction and overcoming of intuitive ideas which he terms *knowledge in pieces*. Through his work about intuitive ideas in physics, he explains that the knowledge that students initially expressed in the domain can be isolated into separate elements (diSessa, 1993). These elements are knowledge structures called *phenomenological primitives* (p-prims), which are relatively small, usually self-explanatory and mainly emerge from superficial interpretations of encountered phenomena. They constitute the essential elements with a minimal degree of abstraction, upon which a set of simple cognitive mechanisms act. On their own they are unstructured, meaning that there are no elements that have a more important role than other elements. Specific cues can lead to their recognition which triggers certain activations. The successive activation of distinct elements creates certain cueing priorities and influences future activations. Their appropriate activation is what ultimately leads to learning. When the p-prims can no longer satisfy their self-explanatory character, their function changes and they are integrated, encoded into more complex knowledge structures in which different kinds of relations can be established among them depending on the context, while at the same time continuing to exist as distinct intuitive elements at the p-prime level. In his example of learning mechanical physics, diSessa (diSessa, 1993) has listed more than 10 elements which represent p-prims that can interact through more than a dozen of heuristic principles which shift the function that these elements initially have in their naïve state.

Even though there are certain differences, diSessa's approach has commonalities with other widely spread frameworks of intuitive conceptions. Other authors have interpreted diSessa's proposals concerning p-prims as corresponding to the multiple sensory percepts that make up our experiential knowledge (Vosniadou, 2002; Vosniadou & Ioannides, 2002). The knowledge in pieces approach does admit the existence of conceptual networks where connections between p-prims are made following different experiences and interactions with the environment. The core distinction in these approaches seems to actually raise the question of the "grain size" of mental entities that are tracked (diSessa, 2009), which leads to different speculations about the coherence in one's intuitive conceptions.

INTUITIVE CONCEPTIONS AND ANALOGICAL REASONING

Useful and misleading aspects of intuitive conceptions

Whichever framework is adopted for understanding the emergence or the overcoming of intuitive conceptions, most authors agree that they constitute dynamic conceptual systems that have an influence on newly encountered knowledge. Some pioneering studies about the naïve theories in biology illustrate how children's intuitive conceptions are transposed when they need to make predictions about unknown entities. Following Piaget's (1960) claims that children have a strong tendency toward animistic and personifying tendencies, Inagaki and Hatano (1987) have explored how children actually use their knowledge about human beings to make educated guesses about animate objects (grasshoppers and plants) which might be less familiar to them. They illustrated in which ways using their intuitive conceptions about humans for predicting and explaining animal behavior can at times lead to reasonable predictions, while at other times the conclusions drawn from this animistic theory are misleading (Inagaki & Hatano, 1991). In their study, children were presented with situations that varied in their degree of similarity to animate objects' reactions and humans' behavior in specifically designed situations. In the first set of situations, humans and animate objects would react in the same way, while in the other two sets of situations they would have dissimilar reactions. If the children's predictions were the same as the ones made by at least a quarter of the adult population, they were deemed as reasonable. Indeed, when six-year-old children relied on the person analogy in situations where humans and animate objects would react similarly, they produced reasonable responses about their behavior. However, when they relied on the person analogy in dissimilar cases concerning the mental life of animate objects, they would produce unreasonable predictions. This goes to show that relying on intuitive conceptions has a certain domain of validity: using it to make predictions can at times provide answers that are compatible with the target domain, however it also has certain constraints since using it will, at other times, produce inferences not valid in the target domain. Furthermore, Inagaki and Hatano found that children did not rely on the personifying analogy in situations where it would lead children to make predictions about the reactions of animate objects contradictory to their existing knowledge about them. According to other studies conducted by these authors, one reason why the personifying analogy was not observed was that they used a different kind of analogy which was not envisioned in this study (Inagaki & Hatano, 1987). Nevertheless, their findings indicate that the constraints imposed by intuitive conceptions mainly arise when subjects lack factual knowledge about the target domain.

Intuitive conceptions as analogical sources

One explanation of why children rely on intuitive conceptions when they encounter new situations can be found in the processes underlying analogical reasoning. Scholars in this field put great focus on the influence that existing knowledge has on cognitive processing. In broad terms, current approaches to analogical reasoning in psychology consider it as a set of processes that make it possible to understand one thing by referring to something else (Holyoak & Thagard, 1995). These processes are considered to rely on mappings between an unknown or less familiar entity – a target – and a more familiar entity – a source (Gentner, 1989; Holyoak & Thagard, 1995). Even though in the field of analogical reasoning less research has addressed it, a first step in establishing this relationship relies on encoding the representation of the target (Chalmers, French, & Hofstadter, 1992; Sander, 2000). With it occurs the retrieval of the analog source. There exist different accounts for the similarities that lead to establishing the relations between the two (Holyoak & Morrison, 2005). Once the relations between the two are established, they enable one to draw inferences about the target entity. These inferences undergo an evaluation process, and when the inferences are compatible with other knowledge held about the target domain it can lead to learning by enhancing the target concept, whereas they can be disregarded when they are not. Following this evaluation, certain re-representational processes may occur through which not only is the representation of the target enhanced, but also the representation of the source.

However, one fact might be surprising when considering that analogical reasoning provides a relevant framework for understanding the influence that intuitive conceptions have on making inferences about newly encountered situations. When studying analogical reasoning, most studies are based on experimental paradigms involving an explicit comparison between the analog source and the target (Sander, 2000). Typically, when considering this preponderant experimental paradigm, the alignment of different concepts is put at the core through the solicited comparisons. This experimentally induced comparison process relies on the controlled sources of analogies and leaves almost no space for understanding the spontaneous evocation of different sources such as the influence that intuitive conceptions have even when they have not been explicitly solicited. Hofstadter and Sander (2013) have advocated that this evocation process of making analogical associations with sources that come to mind is a crucial aspect of analogical thinking where all previously acquired words, phrases and more broadly concepts can be mobilized for understanding a new situation. They, therefore, consider that, even though there is no deliberate mapping between two situations, relying on intuitive conceptions as sources in order to interpret an unfamiliar concept and make sense of a new situation falls directly within the process of analogy-making. Indeed, these *naïve analogies* which are used automatically and effortlessly provide a good foundation for understanding the influence of intuitive conceptions. For instance, in the aforementioned study about children's intuitive biology (Inagaki & Hatano, 1991), in order to explain the target behavior of animate objects children relied on their knowledge about human behavior as the source of analogy. Through this analogy, when they lacked factual knowledge, they made inferences about the target. Even though the children in the study were not asked to compare the animate objects to humans, their responses clearly indicated that they made predictions based on their knowledge about humans. Sometimes these inferences provided reasonable predictions and the analogy, therefore, has a domain of validity of the target entities, whereas at other times the inferences provided unreasonable predictions for which the analogy was not valid in the target domain.

Analogical encoding

Once intuitive conceptions are considered as sources of analogies, it makes it possible to address their influence on knowledge acquisition through insights of how prior knowledge influences the understanding of inherently incomplete input encountered in daily life. Ross and Bradshaw (1994) have conducted a set of studies investigating at which moment prior knowledge intervenes when understanding new situations. First, they presented adult participants with three target stories, each was ambiguous and had two possible interpretations. For example, they presented a story describing a protagonist's routines, the actions sequences and his relations to them, but did not give an explicit context in which the protagonist was found. The actions described were compatible with a jailbreak scenario as well as with a wrestling match. The targets were preceded by source stories. For every target story, there were two possible source stories, each of which could lead to one of the two possible interpretations of the target story. For the previous target example, one source story would describe a retirement party for a reporter who covered a jailbreak story, while another source story would describe a protagonist who's passionate about wrestling matches and Shakespeare. One version of each source story was presented to the participants, along with filler stories. After the target task, the participants were asked to describe what they thought the story was about. What they found was that there were significantly more participants who gave interpretations of the target stories consistent with the presented source stories than those who had inconsistent interpretations. The presentation of the source stories affected the interpretation of the target stories. The prior knowledge sharing superficial similarities affected the interpretation of target stories. Secondly, they looked directly at how participants processed the information of the target stories through their reading times. They constructed critical sentences, half of which were consistent with the first interpretation of the target story (and inconsistent with the second) and half of which were consistent with the second interpretation (and inconsistent with the first). The results revealed that the sentences consistent with the source story that was read by the participants had significantly shorter reading times. According to the previous literature, this was interpreted as a sign that these sentences were consistent with the participants' current understanding of the situation. These findings therefore reveal not only that previous knowledge affects how a target situation is understood, but that prior knowledge influences the actual processing of the target. Their findings endorse the view that specific instances of prior knowledge enrich the representation of a situation directly during the initial *encoding*. Intuitive conceptions, as cases of analogical sources, can therefore be considered to intervene during the initial encoding.

Indeed, the overextension of intuitive conceptions can be explained by the process of *analogical encoding* (Gentner, Loewenstein, & Thompson, 2003). Just like any form of analogical reasoning, analogical encoding considers that observing commonalities between different entities is essential for learning. In order to investigate this process, in one study participants were presented with a source example illustrating one explicitly stated principle about negotiation techniques followed by a target example illustrating the same principle in a different domain, but without any explicit explanations. They were encouraged to compare the two examples and express in detail the underlying commonalities between the compared materials. When the participants had to provide a solution to a test example that could be solved by three different negotiation principles, they indeed solved the example following the principle they had previously acquired in the comparison activity. The authors further demonstrated that the abstraction of the schema common to the two examples had transfer benefits to the test notably when there was a comparison training, and most importantly one aiming at analogical encoding, rather than studying the examples separately. The first specificity of analogical encoding demonstrated by these results is that the source does not have to be a rich set of knowledge or fully acquired in order to be transferred. It can indeed even be faulty, and it is through the comparison with the target that the source can be apprehended to a greater extent as well. This is the case of intuitive conceptions, which are usually faulty since they are not aligned with culturally accepted knowledge about specialized notions. The authors stress the importance that learning abstract schemas and transferring strategies to new cases depend on analogy making processes. Yet another important aspect of analogical encoding stresses that the initial encoding of a situation has an important influence when encountering new situations that it can relate to. Indeed, following the current approaches to analogical reasoning there is a growing number of studies that support the enrichment of not only the target but also the source analog.

Not only does the encoding of the target influence the abstraction of commonalities between different examples, but this encoding of the target's representation is that studies have paid much less attention to (Sander, 2000). Hofstadter and Sander (2013) have stressed the importance that the encoded characteristics of the situation have for evoking the situation in the future. This encoding includes both salient details specific to the situation, and also some salient abstract characteristics. Just like there can be multiple details specific to the situation that is encoded, there is more than one abstract structure that can be stored in memory during the encoding process. Each situation can have multiple abstract structures that are contained in it. In the previous study the test target situation ultimately had a wide variety of potential encodings, and it was the previously acquired structure that had the greatest influence on the encoding made by the participants. This encoding process highlights the importance that previously held knowledge has on how the representation of the target is constructed. Indeed, as an important process in analogical reasoning, the initial encoding of a representation is not only dependent on preexisting conceptual structures, but the encoded characteristics also influence the source that will be retrieved. Undeniably the different encodings of one representation can each lead to different inferences.

THE INFERENCES STEMMING FROM INTUITIVE CONCEPTIONS

Conceptual metaphors as a form of categorization

Metaphors have been seen as cases of analogies that provide a way of establishing correspondence between concepts that are usually seen as belonging to semantically different domains (Bowdle & Gentner, 2005). Taking a situated perspective on how people come about to establish and grasp abstract concepts, Lakoff and Johnson (1999) argued that conceptual metaphors play a fundamental role. They later put emphasis on the importance that familiar, embodied, human activities have in constructing them. Lakoff and Johnson's approach considers that the metaphors conveyed in language are not just figures of speech that have rhetoric qualities, but actually reflect the conceptual system of the speaker. In their view, conceptual metaphors are central cognitive mechanisms for structuring conceptions. They are defined as grounded, inference-preserving mappings across different domains. They reflect the physical, spatial system in which they are represented such as *knowing is seeing* which is later used in sayings such as "see what I mean" as a way of conveying the metaphorical content – act of knowing – rather than the literal expression – the act of seeing. Even though significant debate exists about the embodiment of metaphors, it should be acknowledged that conceptual metaphors could emerge from different levels of experience and are still useful for grasping abstract concepts (Gibbs, 2009). Hundreds of conceptual metaphors have been studied in detail by looking at the kinds of attributes that are used in order to conceptualize abstract terms, like physical closeness used to conceptualize similarity through expressions like "This is a big issue" or "It's a small matter, we can ignore it". Conceptual metaphors can therefore also provide a method for studying the underlying conceptions held by people.

Glucksberg and Keysar (1990) have proposed that categorization is the mechanism that accounts for how metaphorical content is understood in a communicative context. They consider that in order to understand a metaphor such as *My job is a jail*, one doesn't just access the numerous mental categories in which the source of the metaphorical analogy "jail" can be classified. They propose that one actually derives an abstract ad-hoc metaphoric category. This metaphoric category does not have a conventional name but is based on a set of properties that are exemplified by the source and can be attributed to the target, for example, as 'a set of unpleasant situations that are difficult to get out of'. It has the same structure and function as any ordinary mental category. Therefore, the target concept, in this example "my job", once assigned to this metaphoric category can be understood since, as a subordinate concept, it inherits the properties of the category.

Categorization makes it possible to make inferences

Categorization is indeed considered to be a key process in conceptual development. Eleanor Rosch (1978) qualified categorization as an automatic process through which perceived information is structured into a system creating a mental category that provides a maximum amount of information about the categorized entity with the least cognitive effort. The possibility to easily make inferences and predictions about a newly encountered entity simply by considering it as belonging to a certain category is indeed the most important quality of categorization (Murphy & Ross, 1994). Rosch (1978) described mental categories as having a gradual structure, meaning that certain entities can be considered more easily as belonging to a certain category than others. She describes the vertical dimension of a category that determines its level of inclusiveness: the more abstract the level of categorization is, the more inclusive it is; however, all levels of abstraction are useful. This dimension makes it possible for conceptual categories to have hierarchies. On the horizontal dimension, a category contains at its center a “prototype”, and the further we move away from it the less representative the entity is of the category. The prototype is not necessarily a specific instance of the category, but rather represents an image of what is considered the most prototypical item of the category. The prototype imposes constrains when evaluating other objects: an item that is inconsistent with the prototypical attributes of the category probably will not be recognized as a member of the same category. Yet, how we categorize an entity largely depends on the context in which categorization takes place (Barsalou, 1982), as well as the level of proficiency one has with the concepts at play (Chi, Feltovich, & Glaser, 1981).

Eleanor Rosch’s initial work mainly focused on the category structures regarding concrete objects or events, considering that a category is generally designated by a name. However, as we have seen, a mental category can also be defined by the set of properties it represents (Glucksberg & Keysar, 1990). Hofstadter and Sander (2013) have considered that categories outnumber words, since they are most often difficult to lexicalize and are sometimes represented in the form of proverbs or idiomatic expressions. Mental categories can even be constituted to represent experienced situations, grammatical patterns, and many different forms of abstract entities. In fact, they consider categorization to play a key role in the conceptual structuring of human thought which is carried out by analogy making (Hofstadter & Sander, 2013). They propose that the degree of abstractness of the source category is the crucial element that will make it possible to understand the target, which does not necessarily require the creation of a new metaphoric category. They consider that most concrete categories also exist in an abstract category form, which does not contain all the properties of the concrete category. In their view, new ad-hoc categories are only created when the concrete concept does not already exist in the form of an abstract category. This goes well with Glucksberg and Keysar’s descriptions of metaphoric categories which usually have an abstract character and provide an account as to how Lakoff and Johnson's conceptual metaphors can create categories for abstract concepts. The inferences that are drawn from a category are not just formal logical deductions, but more broadly they can simply be seen as a certain aspect of the activated concept that is brought to one's attention (Hofstadter & Sander, 2013).

Different conceptions lead to different inferences

Regarding intuitive conceptions, numerous studies have demonstrated that different conceptions entail different inferences. For instance Vosniadou and Brewer (1992) studied the conceptions that elementary school students have about the Earth's shape. They conducted individual interviews with first, third and fifth grade students in which they asked them questions about the Earth's shape and elicited drawings that would illustrate the shape of the Earth. Besides asking factual questions about the Earth's shape, which could simply have been facts that students retained without any theoretical implications, they also asked generative questions in which students had to explain different phenomena that would reflect their understanding about the Earth's surface. For instance, even if a student would say that the earth is a sphere, they would ask them what would happen if they walked in a straight line for many days or used different transportation modes. These questions would allow the researchers to detect inconsistencies about alternative notions that they associate with the concept of the earth's surface. Vosniadou and Brewer identified five alternative mental models that students have about the shape of the earth which did not correspond to the scientifically, culturally accepted sphere model. Each one of these models entailed different misconceptions concerning the inferences that can be generated from such models. Furthermore, students answered questions with generative inferences that were consistent with their mental model of the Earth's surface. For instance, students who believed that the earth was a flattened sphere considered that you can walk around the world and end up in the same spot, but explained the inconsistency about seeing a flat surface by describing it as a thick pancake with a flat surface and rounded ends.

In order to understand the prerequisites for grasping the model of Earth having a spherical shape (Vosniadou & Skopeliti, 2005) looked at how students spontaneously categorize Earth. Frank Keil (1989) has previously proposed that children classify knowledge according to different ontological categories and when these categories are theory-driven lead to inductive generalizations. Considering that at a certain stage children's conceptual knowledge is theory-based, Vosniadou and Skopeliti (2005) considered that categorizing the Earth as a physical object will lead to different inferences than if the Earth was categorized as a solar object. They presented first and fifth grade students with a categorization task in which they were asked to categorize the Earth, astronomical and physical objects as they considered it adequate. Their findings revealed that, when asked to group 'things that go with the Earth and things that do not', children categorized the Earth with other objects based on their theory of the Earth being either a solar object or a physical object. They were then asked the same set of questions as in Vosniadou and Brewer's (1992) study for which students would respond based on the inferences drawn from how they categorized the Earth. The results revealed a significant correlation between the students' categorization of the Earth and their responses concerning the shape of the Earth. When Earth is categorized as a physical object, the inferences regarding its solidity and lack of self-initiated movement apply, whereas when it is re-categorized physical-astronomical object it can be considered to obey the same inferences as the ones we make about other planets.

Based on these and similar findings, the framework theory (Vosniadou, 2014; Vosniadou et al., 2008) considers intuitive conceptions in physics and mathematics are structured into a coherent system and that when the principles stemming from intuitive conceptions are violated this can lead to conceptual change. Conceptual change occurs gradually, creating in the process fragmented or synthetic conceptions either by distorting scientific information or assimilating it to fit with the intuitive conceptions. This approach differentiates misconceptions, which emerge when previously held conceptions violate culturally accepted knowledge, from preconceptions, that were initially constructed through everyday experience. Notably, the framework theory considers that in order to achieve conceptual change and fully understand a scientifically accepted concept, such as the one of Earth being a sphere, an ontological category shift, such as re-categorizing the Earth from a physical to a solar object, has to take place first.

Persistence of intuitive conceptions

Vosniadou and Brewer (1992) suggest that conceptual change implies that intuitive knowledge is organized into theories and that these theories can change. However, the mere exposure to the correct knowledge is not sufficient in order for the intuitive theory to change. In fact, numerous researches have shown that intuitive conceptions are not easy to un-learn and that intuitive conceptions are robust and resistant to instruction. For example, before acquiring knowledge about biology, young children falsely consider inanimate objects that possess autonomous motion as living things. In school, extensive knowledge is acquired about metabolic processes and it could be assumed that distinction between living and non-living things would no longer depend on their mobility. However, it seems that this intuitive conception still continues to operate even when formal knowledge has been acquired. Babai, Sekal and Stavy (2010) presented 10^{th} grade students with a task where they had to classify items as living or non-living objects. They made a cross selection of items that were either living or non-living entities, moving or non-moving. The great majority of the students did manage to classify the objects correctly as living or non-living. However, the reaction times revealed that they took longer to classify plants, which are not mobile objects, as living than they did animals. It also took them longer to categorize as non-living mobile inanimate objects, such as celestial bodies, than it did for static ones, such as tools. A similar method revealed that this intuitive conception has also affected undergraduate students' judgments about living and non-living objects, and even biology professors' judgments, although to a significantly smaller extent (Goldberg & Thompson-Schill, 2009).

This persistence of intuitive conceptions has been demonstrated in a wide range of domains. Shtulman and Harrington (2016) have put to the test intuitive conceptions that contradict scientific knowledge in 10 different domains, ranging from fractions to astronomy. They wanted to see if the influence of intuitive conceptions contradictory to scientific knowledge diminishes with age, so they recruited two groups of participants, young adults around the age of 20, and adults about 45 years older than them. They expected that since older adults acquired knowledge about the relevant scientific theories earlier in life and had more opportunities to use them in their daily life, the intuitive conceptions would have faded in strength and relevance. They measured the reaction times on statements that were intuition-consistent, and those that were intuition-inconsistent. What they expected to observe was smaller lag between the reaction times on intuition consistent and inconsistent statements among older participants. However, it is not only that the lag in response times was still present, it even increased. Longer reaction times on intuition-inconsistent than consistent statements among professional scientists. There is no actual consensus on why intuitive conceptions are so resilient. There is even neural evidence that although participants do not exhibit behavioral proof of intuitive conceptions in physics as being influential, brain imaging reveals activations in areas associated with the detection of inconsistent information (Dunbar, Fugelsang, & Stein, 2007).

Chapter 2 – Intuitive comprehension of mathematics

Theories of intuitive conceptions in mathematics

Taking intuitive conceptions into account in school learning

Since before entering school students already have a rich source of analogies that they can use to understand the scientific concepts they are being taught, it is useful to understand how these sources might influence student learning. With this ambition, Tirosh and Stavy (Stavy & Tirosh, 2000; Tirosh & Stavy, 1999) have elaborated the intuitive rules theory in an attempt to explain the errors that occur on a variety of tasks in science and mathematics. Intuitive rules are considered to be self-evident, and more broadly, to have predictive and explanatory power since the intuitive rules can account for the different alternative conceptions observed in mathematics. They are supposed to be applicable in any context where the rule matches the situation. Tirosh and Stavy explain that these rules are used beyond their range of applicability. One such intuitive rule is 'Same A–same B' which leads to the reasoning that if two objects contain the same quantity of A, which is salient in a given task, then they will contain the same quantity of B, even if A and B are completely unrelated. This rule is typically observed in Piagetian conservations tasks where children are asked to compare two lengths that begin and finish in parallel points. In the case where both lines are straight it is indeed helpful in finding the answer. However, when children are comparing a straight and a wavy line, the misuse of this intuitive rule will lead to errors. Tirosh and Stavy noted that this intuitive rule is also observed and can predict student performance in different mathematics tasks regarding proportions, areas and perimeters of geometrical shapes. Tirosh and Stavy consider that identifying the intuitive rules which implicitly guide the students "enables researchers, teachers, and curriculum planners to foresee students' inappropriate reactions to specific situations, and this can help them plan appropriate sequences of instruction" (Tirosh & Stavy, 1999, p. 64).

Tacit models

As we have seen, children rely on intuitive conceptions in order to interpret unfamiliar concepts. When they start school, they are faced with scientific concepts for the first time in a formal setting and there exist different theories that propose a description of how these intuitive and scientific concepts interact. As we have previously seen, Tirosh and Stavy's (1999) intuitive rules are a task oriented theory that attempts to explain how intuitive rules systematically influence performance depending on the different features of the given tasks. Yet, before their proposal, Fischbein (1987) elaborated a theory that addresses the role of intuition in mathematics learning oriented towards the conceptualization of mathematical content.

Fischbein acknowledges that mathematics constitutes a coherent body of scientific knowledge, but the main consideration in his theory is that mathematics is an essentially human activity. He therefore distinguishes three components of mathematical knowledge: formal, algorithmic and intuitive (Fischbein, 1993). The formal component represents the axioms, definitions, theorems, and proofs that constitute the formal body of mathematical knowledge as a science, while the algorithmic component represents the solving procedures required to solve mathematical problems. The third intuitive component finds its roots in and is shaped by experience. This generates a system of automatized reactions and beliefs that Fischbein refers to in terms of intuitive cognition. Intuitions are characterized by their self-evident nature: their content is accepted without the need for any further justification. Most of the time they are automatically created and used tacitly.

Besides self-evidence, Fischbein (1987) described intuitive cognitions as having intrinsic certainty, perseverance, coerciveness, theory status, extrapolativeness, globality and implicitness. With all these characteristics, intuitive knowledge exceeds the mathematical facts encountered in a given situation and makes it possible to extrapolate information about mathematical notions that are not directly accessible, and moreover believe in its absoluteness. For instance, after noting that the opposite angles of two intersecting lines are equal, the universality of this property is accepted intuitively. In Fischbein’s view this is supported by considering that mental behavior, including mathematical activities, is based on the belief in the absoluteness of an empirical external reality that leads us to organize concepts following an internal consistency. This gives place to a large body of intuitions created by generalizing certain properties of certain elements to a whole category. Depending on their origin, they can be classified into primary and secondary intuitions, both being learned cognitive capacities. Primary intuitions are those that develop from personal experience independently of any systematic instruction, while secondary intuitions are those where an interpretation from a learned conception has been transformed into a belief. An example Fischbein gives for secondary intuitions is the equivalence of an infinite set and one of its sub-sets which becomes self-explanatory among mathematicians. Neither of the two types of intuitions are considered innate nor *a priori* by Fischbein; he rather proposes a continuum ranging from naturally acquired concepts to complex, counter-intuitive ones. The secondary intuitions can also vary in their degree of abstractness, and once such a mathematical, mainly counter-intuitive concept is no longer a mere formal acquisition, it can be considered a secondary intuition. As opposed to some other approaches to how intuitive conceptions are structured such as diSessa's approach, Fischbein considers that it is possible to develop *new* cognitive beliefs that are an outcome of systematic scientific training.

Fischbein (1987) proposed that different tacit, intuitive models are created in order to substitute notions that are intuitively unacceptable. They make it possible to encode in one's own terms data regarding the mathematical notion. One of the examples, which we will take a closer look later on, is seeing subtraction as the action of taking away. These models shape the intuitively acceptable cognitions and can be distinguished along different dimensions. One of these dimensions provides a classification of intuitive models regarding the types of similarities observed between the intuitive model and mathematical concept. This leads to the distinction between analogical and paradigmatic models. In both cases there are systematic similarities observed between the intuitive model and the original notion it substitutes. In the case of analogical models, Fischbein considers that the intuitive model and the original notion belong to different conceptual systems and the similarities are observed at the structural level, making it possible to make inferences. The analogies can be both intramathematical, numerical-algebraic symbols or geometrical representation, and extramathematical, mainly material representation of mathematical concepts. As for paradigmatic models, Fischbein considers that the intuitive model is an exemplar or sub-class of the original notion, defined by its function rather than its intrinsic attributes. The importance is given to the exemplars that are attached to a mathematical concept. These exemplars shape the meaning of the concept and those that are the most familiar for a person become the tacit model itself, influencing interpretations and solutions. By considering that there is a tendency to see a whole category of a concept through the particular example which becomes the paradigmatic intuitive model, Fischbein joins Rosch's prototype take on concepts.

These tacit models in mathematics act as mental models substituting for a complex, abstract notion, imposing their properties and constraints (Fischbein, 1989). By assigning properties of the intuitive models to mathematical entities, Fischbein considers that these analogies can become sources of misconceptions – for instance if we use graphic representations, we might intuitively tend to derive that each graphical function is intuitive. The intuitive models can at times be in accordance with formal justifications accepted by the scientific community, but at other times, incompatibilities between the two can arise. When there is a conflict between intuitive and formal components, an epistemological obstacle emerges leading to misconceptions and systematic mistakes. These misconceptions can arise in all aspects of mathematics, starting from the execution of algorithms in arithmetic calculation to interpreting arithmetic problems. The fact that there are conceptually based systematic mistakes makes it possible to conduct analyses through which the categories of mistakes are identified and classify the types of errors in order to avoid the misleading effects of seemingly evident statements.

Embodied mathematics

In opposition to abstract models such as formulas and functions, Fischbein considers that intuitive models can be perceived, represented or manipulated like concrete objects (Fischbein, 1987). In his view the sensory nature of intuitive conceptions comes from everyday experience. Lakoff and Núñez (2000) share the view that abstract properties of arithmetic are not directly accessible, yet they take a slightly different angle for explaining the origin of mathematical intuitive conceptions. Following Lakoff and Johnson (1999), Lakoff and Núñez (2000) take an embodied perspective for explaining how conceptual metaphors come to yield arithmetic abstract properties. They admit that there are certain simple cognitive capacities such as subitizing, approximation and perception of simple arithmetic relationships, that are essential for mathematical cognition, yet they view conceptual metaphors as the central cognitive mechanism through which basic arithmetic is extended to sophisticated mathematics. In their view mathematical cognition is grounded through language and action, which makes it possible to establish a conceptual mapping with a situated model that is used for guiding the acquisition of new arithmetic concepts.

They distinguish two types of metaphors: *grounding* metaphors, which yield basic, directly grounded ideas by projecting from everyday experience onto abstract notions (for instance piling objects and addition). One of the main cognitive gains that conceptual metaphors provide is that they preserve inference structures. In arithmetic grounding metaphors the metaphorizing capacity leads to extend arithmetic beyond subitizing and simple additions and subtractions made on this basis. All grounding metaphors extend arithmetic capacities beyond small amounts, and entail inferences not only about numbers, but about many different aspects of arithmetic as well. The first grounding metaphor that (Lakoff & Núñez, 2000) describe is that of *arithmetic as object collection*. In this grounding metaphor there is a precise mapping from the physical objects to the number domain. Collecting objects through piling them together and taking them apart acts as a source and arithmetic capacities based on subitizing are the target domain between which a mapping occurs. Correlating with these kinds of everyday activities makes it possible to conceptualize numbers as collections and make inferences about them based on everyday knowledge about object collections. For instance, this leads to inferences where the number size is like the size of a collection and therefore greater numbers are like bigger collections. The second grounding metaphor that they propose is *arithmetic as object construction* in which the source domain is not just a collection of objects, but object construction. Since constructing objects requires collecting parts together, this metaphor preserves the same inferences from the first grounding metaphor. However, it makes it possible to conceptualize numbers as wholes that are made up of parts and therefore this metaphor makes it possible to make inferences about fractions as well. The third grounding metaphor is the *measuring stick* metaphor in which the source domain is simply the use of a measuring stick in daily life activities. This source which represents a physical segment of a line in space has the characteristic of being unidimensional and continuous, therefore it entails different inferences from the previous two. Through this metaphor, numbers are conceived as physical segments, and when a unit length is fixed, then numbers can be associated with the segments that sequentially follow. Lastly, the fourth grounding metaphor is that of *arithmetic as motion along a path* with the source being moving straightforwardly from one point to another. One important inference that is drawn from this metaphor and different from the others is that zero is conceived as a point-location since the origin of a movement is in fact a point-location. It also provides an extension to negative numbers since it is possible to imagine that along the line a person is moving, it extends in the opposite direction as well, on which there are also different points. Lakoff and Johnson note that this metaphor is often observed in language when people ask for instance if two numbers are close or say that a result is around a number.

In this thesis manuscript, we will bear interest on mathematical concepts that are constructed through these grounding metaphors. However, it should be noted that Lakoff & Núñez also describe *linking* metaphors, which yield more sophisticated ideas linking arithmetic to other mathematical branches. These metaphors extend aspects of mathematics that have a direct grounding to branches that can only be indirectly grounded, such as the concept of limit.

Intuitive conceptions of arithmetic operations

Even tough Fischbein's and Lakoff and Núñez's theories cover a wide area of mathematics, from algebra to geometry, in the present chapter we will focus on different intuitive conceptions in arithmetic. With Fischbein's theory of tacit models and Lakoff and Núñez's conceptual metaphors we have seen some main theoretical contributions about intuitive conceptions in mathematics, but the different conceptions that we will present will not be limited to their proposals. Indeed, different lines of research in psychology (Sophian, 2008) as well as in didactics of mathematics (Blum, Artigue, Alessandra, Rudolf, & Van den Heuvel-panhuizen, 2019; Selter, Prediger, Nührenbörger, & Hußmann, 2012) converge to provide a better understanding of the primary forms that mathematical concepts take in the mind of learners and the gaps that exist between different conceptions and the mathematical object.

Subtraction and addition

Both Fischbein (1989) in his theory of tacit models and Lakoff and Núñez (2000) in their conceptual metaphor approach identify the most prominent intuitive conception of subtraction as *taking away*. In Fischbein's view where the intuitive model substitutes for the arithmetic operation, when subtraction is intuitively performed, we are calculating how much is left after taking away a subset from a larger set. This kind of action is also described in the first three grounding metaphors of arithmetic, where it's either a smaller collection that is taken away from a larger one, forming another object, either a shorter segment taken away from a larger segment. Accordingly, the minus sign in a formal subtraction would intuitively prompt this conception of taking away (van den Heuvel-Panhuizen & Treffers, 2009). The solution to a subtraction stemming from this model of subtraction would be referred to as the *remainder* (Usiskin, 2008). Fischbein (1993) proposed different inferences that are entailed from this conception. If one is guided by the primitive model of subtraction as taking-away, if we need to take away a quantity B from a quantity A (A – B), this can only be done if B < A. He proposes that if the opposite is true (B > A), then if a student sticks to this model of subtraction then they might proceed in several ways. Either they will take-away as much as possible, either they will reverse the subtraction into B – A. Indeed *bugs* reflecting systematic procedural mistakes that children make when they solve multidigit column subtractions have been documented (Brown & VanLehn, 1980; Resnick, 1983). Fischbein analyzed these mistakes on conceptual level and indeed found that they can be explained by the conception of subtraction as taking-away. When students haven't mastered the principle of borrowing, then this conception indeed explains the bug such as $\frac{\begin{array}{r}326\\-117\end{array}}{211}$ where the smaller is taken away from the larger and $\frac{\begin{array}{r}542\\-389\end{array}}{200}$ where students take out as much as possible. It also provides semantic explanations for bugs after students have learned the borrowing principle.

The intuitive conception of subtraction as taking away is the most widespread one since it is most often encountered in daily-life experiences (Fischbein, 1987; Selter et al., 2012). Yet it is not the only conception people can have of subtraction and sticking only to this conception has been described as too one-sided (van den Heuvel-Panhuizen & Treffers, 2009). Alternatively, subtraction has also been conceived as *determining the difference* (Selter et al., 2012). The fourth grounding metaphor of arithmetic as motion along a path explains this conception by referring to the source that the distance in moving from point A to B is the same as the distance when moving from B to A. When a subtraction is performed following this conception a person is calculating how much is needed to reach from one quantity to another. This conception has been also described in the comparison model of subtraction, where the answer to the subtraction is referred to as the *difference* (Usiskin, 2008). Indeed, when a subtraction corresponding to the determining the difference conception is performed, a distance is bridged from a smaller quantity to a larger quantity by adding on (van den Heuvel-Panhuizen & Treffers, 2009). This conception of subtraction can indeed provide additional explanations for the errors observed on subtraction problems. Indeed, Sander (2001) has analyzed the different bugs made by second and third grade students by taking into account both conceptions of subtraction for the semantic explanation of students' errors. He ran a simulation that would consider the procedural interpretation of the errors students made and later the semantic errors students made. The findings revealed that taking into account the different conceptions of subtraction could provide more accurate predictions of the errors students made. The verbal reports provided by students supported these findings and corresponded to both the take-away conception and that of determining the difference. This goes to show that taking into account students' intuitive conceptions can indeed lead to more accurate explanations for their mistakes.

Addition is probably the least empirically studied arithmetic operation regarding the intuitive conceptions that are involved in its conceptualization and the entailments that it leads to, which does not make this operation any less prone to them. The first three grounding metaphors of arithmetic entail a conception of addition as *putting together* since either collections, either physical segments can be put together in order to form larger ones. Yet, this conception of addition does not lead to appropriate inferences in all additive situations. Usiskin (2008) gives an example where the temperature was -4°C and increases by 15°. If the conception of addition as putting-together was to guide a solver in finding how much the temperature increased, then it would not provide the correct answer since putting together -4 and 15 would provide the answer 11, instead of the increase in 19°. In such cases, an alternative conception of addition would be more appropriate, the conception of a *shift* or *slide*. Lakoff and Núñez's (2000) description of the fourth grounding metaphor indeed entails such an alternative conception of addition, where we base addition on the source of the distance reached from point A to point B.

Multiplication and division

Fischbein (1987) described the intuitive conception of multiplication to be *repeated addition*. This entails that intuitively multiplication is considered to ‘make bigger’. Yet, when we multiply with a decimal number smaller than one, then multiplication in fact makes the initial quantity smaller. If one therefore sticks to the intuitive conception, multiplication involving a decimal smaller than one or a negative number has no intuitive meaning. It also does not view multiplication as commutative (Fischbein, Deri, Nello, & Marino, 1985). As for division, Fischbein et al. (1985) identified *sharing* as an intuitive conception, which considers that division cuts an object into equal parts. This partitive view of division sees division as ‘making smaller’ since each fragment would be smaller than the initial quantity. If dividing is assimilated with partitioning, it entails that the divisor must be smaller than the dividend and must be a whole number, and that the result must be smaller than the dividend. However, the latter is not the case if the divisor is a positive number smaller than 1. Alternatively, division can be conceived as a *measurement* where the search is for how many times one quantity is contained in a larger quantity. This quotative conceptions of division has less constraints, only that the dividend should be larger than the divisor. When the quotient (the result of the division) is a whole number, then this kind of division can be conceived as repeated subtraction. Lakoff and Núñez (2000) analysis also concur with these intuitive conceptions. They described that there are two ways of extending the metaphors which ground addition and subtraction to multiplication and division. The first extension is through an iterative process, entailing repeated addition as the conception on which multiplication is based and repeated subtraction as the conception on which division is based. The second extension is made through a pooling process in which collections or segments are fitted together or split apart, entailing a conception of multiplication that produces a pooled collection and division as splitting a collection through the sharing process. The inferences that can be drawn from each of these conceptions have indeed been shown to impact performance on multiplication and division problems.

A large study was conducted by Fischbein et al. (1985) with the aim to test the influence that the different inferences drawn from these conceptions. The study included 5th, 7th, and 9th grade students. They were presented with multiplication and division word problems whose numerical values were either compatible or incompatible with different inferences drawn from the different conceptions of the arithmetic operation. The students' task was to write down the operation used to solve the problem. The authors reported that students made fewer errors on problems where the inferences about the operands were compatible multiplication as making bigger (i.e. when it was a whole number), than when it was not compatible with their intuitive inference about the operand (i.e. when it was a decimal number smaller than one). For example, students made fewer errors on the problem "From 1 quintal of wheat you get 0.75 quintal of flour. How much flour do you get from 15 quintals of wheat?" (on average 16.67% of errors in writing down the operation) than on the problem "1 kilo of a detergent is used in making 15 kilos of soap. How much soap can be made from 0.75 kilo of detergent?" (on average 50.67% errors). As for partitive division, students made more errors when the problem violated the implicit rule that the dividend must be larger than the divisor such as "15 friends together bought 5 kg of cookies. How much did each one get?" (an average of 69.67% errors), where they would mostly invert the division that needs to be done (15/5 instead of 5/15) compared to problems where this violation was not made (on average only 6.3% of errors on the problem "With 75 roses you can make 5 equal bouquets. How many roses will be in each bouquet?". They also made more errors (on average 40.67%) when the operator was a decimal number (decimal divisor) such as "I spent 900 lire for 0.75 hg of cocoa. What is the price of 1 hg?". However, when the operator was a decimal number on quotative division problems, where this does not violate the constraints that the quotative conception entails, for example on the problem such as "The walls of a bathroom are 3 m high. How many rows of tile are needed to cover the walls if the width of each row is 0.15 m?", were on average easier for students (on average 53.44% correct operations written down) than on the case of partitive division (29% correct responses on average). The authors also noted that when no violations occurred, quotative division problems were harder than partitive division problems, yet experimental design did not allow to conclude further on these points.

Another study looked at how children solve multiplication problems considering the numbers and structure involved (De Corte, Verschaffel, & Van Collie, 1988). Taking into account the entailments of the intuitive conception of multiplication, the numerical values of the multiplier were either integers (compatible with the constraint of the intuitive conception), either decimal numbers smaller than one (incompatible with the constraint of the intuitive conception) or larger than one (compatible with the constraint of the intuitive conception, but harder for students to solve). The wording of the problems could describe an asymmetric case as in the problem "One pencil costs 12 Bfr. Ann buys 4 pencils. How much does she have to pay?" where 12 would be the multiplier since the asymmetrical relations would lead us to consider 4 times 12 Bfrs. The problem could also describe a symmetric case, where the roles played by the different quantities that needed to be multiplied could be interchangeable as in the problem "A hen-house has a length of 9 metres and a breadth of 4 metres. What is the area of that hen-house?". Concerning the type of number involved in the problem, the analyses conducted on the choice of appropriate strategy and free response indeed showed that problems where the multiplier was a decimal smaller than 1 were overall the hardest. This finding is indeed consistent with (Fischbein et al., 1985)'s findings advocating for the influence of the intuitive conception of multiplication as repeated addition and the constraints it imposes on the product of the multiplication – that it should make bigger, and on the nature of the multiplier – that it should be an integer. However, when the researchers looked at the interaction between the mentioned variables more precisely this effect was present in asymmetric problems but not on symmetric problems. Putting aside the possibility that the students mindlessly applied multiplication to the situations, the authors proposed that this could mean that when a problem does not require the attribution of the role of multiplier and multiplicand to the values, such as it is the case of symmetrical problems, maybe the intuitive conception does not impose its constraints. Alternatively, students' performance might have been influenced by another primitive model of multiplication, which is the *rectangular pattern*, which does not entail the same constraints as the repeated addition model. Indeed, conceptualizing the product of multiplication in a rectangular form has been proposed in the educational literature as a mean for separating multiplication from addition and is considered as a way of making the multiplication of fractions more intuitive (Freudenthal, 1973). The pooling extension towards multiplication of the grounding metaphors is indeed completely compatible with this view, except that Lakoff & Núñez didn't envision any specific rectangular form in their proposed extension.

Educational perspective regarding arithmetic intuitive conceptions

Many researchers concur that the introduction of mathematical concepts in initial instruction most often uses the intuitive conceptions of the school notions being taught (De Corte & Verschaffel, 1996; Fischbein, 1987; Freudenthal, 1973). According to Fischbein and collaborators (1985) "each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model. Identification of the operation needed to solve a problem […] takes place not directly but as mediated by the model" (p. 4). This implies that these intuitive conceptions are robust and will persist after instruction if the necessary measures aren't taken. Even though the influence of intuitive conceptions becomes modulated, their persistence is indeed supported by a variety of empirical data. Indeed, even when adults holding at least a secondary education degree correctly answer a mathematics task, the inferences drawn from the intuitive conceptions impact their performance (Vamvakoussi, Van Dooren, & Verschaffel, 2013). Namely, adults were presented with mathematical statements for the four arithmetic operations, each containing an unknown value. This was followed by a statement about the possible outcome of each operation which could either be mathematically true, for example "$5+2x$ can be greater than 5", or false "$1+10t$ is always greater than 1". What was manipulated experimentally was the congruency of the response with the assumed intuition. The statement "$5+2x$ can be greater than 5" was considered congruent since the response based on the intuition that addition makes bigger would lead to a correct judgment of the statement being true, while the statement "$1+10t$ is always greater than 1" was considered incongruent since the same intuition would lead to an incorrect judgment of the statement being true. The results showed that not only did adults make there fewer correct judgments on incongruent statements, but among these correct responses, participants took longer to make this correct judgment.

What may be even more surprising is that not only adults are influenced by the intuitive conceptions, but also populations of pre-service and in-service teachers. For example, one study presented pre-service teachers with quotative and partitive division word problems (Tirosh & Graeber, 1991). Half of the problems contained numerical values that respected the constraints imposed by the intuitive conception (for example "Five bottles contain 6.25 liters of rootbeer. How many liters of rootbeer are in each bottle?"). The other half of the problems violated the constraint, common to both conceptions of division, that the dividend needs to be greater than the divisor (for example “Peanuts are shipped in 5 pounds boxes. How much of a box is filled with .65 pounds of peanuts?”). Additionally, participants were presented with four numerical division statements (e.g. 6 ÷ 3), half of which violated the constraints of the intuitive conception (e.g. 2 ÷ 6), and were asked to write down word problems in order to determine the pre-service teachers’ preference and access to the different types of division problems. On average, pre-service teachers had significantly higher performance when solving partitive division problems (80.62%) than quotative division problems (56.5%), but overall they also had significantly more success on problems whose numerical values did not challenge the primitive models (85.87%) than on problems that violated the constraint (51%). Even though there was no interaction between these two variables, on partitive division problems 94% of the pre-service teachers successfully solved problems consistent with the inferences made from the intuitive conception, compared to 67% when this was not the case, and on quotitive division problems this was respectively observed 78% and 35% of the time. Among the correctly invented word problems, most teachers invented problems in line with the partitive conception of division, while problems in line with the measurement conception were mostly observed when the division in the given operation was a decimal (4 ÷ .5). The interviews conducted with a part of the population aiming to shed light on the difficulties and errors made by the participants are reported by the authors as support for the strong influence of the intuitive conception being division as sharing, even on measurement problems.

Whatever the theoretical approach we take, the fact that these intuitive conceptions persist in adulthood, even among future teachers, can be explained by their strong rooting in everyday life experience (Fischbein et al., 1985; Graeber, Tirosh, & Glover, 1989; Lakoff & Núñez, 2000). One basic factor why intuitive conceptions persevere is the same as their origin: experience. The previously mentioned authors would agree that most of the time in daily life when we are subtracting we are searching for the remainder, when we are adding we are putting different elements together, when we are multiplying we are making bigger and when we are dividing we are partitioning. It is of great educational importance for students to learn to detach the mathematical operation from their intuitive models in order to grasp these operations in their formal context (Fischbein, 1987). Fischbein describes that through teaching, students are first presented with mathematical concepts closely related to the intuitive models, which seems necessary in order to grasp the concept, and only later with abstract meaning. He therefore recommends that in order to prepare students to understand the formal meaning of the concept being taught, a first step should be to reveal relationships between closely related concepts and explicitly present their common underlying structures. For example, Fischbein (1987) stressed that "addition and subtraction are intuitively based on opposite practical operations" (p. 209) but are still deeply related. Sometimes a problem including addition is solved by subtraction and vice versa. Indeed, the conception of addition as putting-together and that of subtraction as taking-away could also be addressed through a single conception of a *part-whole* model (Sophian, 2008; Usiskin, 2008). And furthermore the slide conception of addition and difference conception of subtraction can also be joined in a single conception of *start-shift-finish* (Usiskin, 2008). In a similar manner, Fischbein (1987) proposes that multiplication and division, as inverse aspects of the same underlying structure, should be treated together and in relation to proportional reasoning.

Even though the primary intuitive conceptions are shown to persist, Fischbein (1987) considers that formal knowledge can also become intuitive, creating secondary intuitions. Both primary and secondary intuitions are indeed learned, however these secondary intuitions do not have "natural roots" as Fischbein describes it, in the sense that they will not develop through everyday experiences independently of any instruction. A secondary intuition is constructed once certain non-intuitive mathematical knowledge becomes self-explanatory, like for instance when expert mathematicians come to integrate the belief of the equivalence between infinite sets and one of their sub-sets.

CHAPTER 3 – CONCEPTUAL, PROCEDURAL, AND SITUATIONAL ASPECTS OF ARITHMETIC LEARNING

GOING FROM INFORMAL TO FORMAL ARITHMETIC

Distinguishing and measuring conceptual and procedural knowledge

Any mathematics educator must have encountered situations where they had the impression that students can do a certain task, but do not understand what they are doing, and at other times they might have had the impression that their students have the necessary knowledge to solve a task but yet do not know how to put their knowledge into practice or commit certain errors when they manage to do so. Influenced by these considerations, mathematics knowledge has been, in the broadest terms, distinguished into two categories: conceptual knowledge and procedural knowledge (Hiebert, 1986). These two types of knowledge are not always easy to separate, and can be considered as two ends of a continuum (Rittle-Johnson, Siegler, & Alibali, 2001).

A comprehensive review conducted by Crooks and Alibali (2014) identified 6 main characteristics crucial for defining conceptual knowledge in the domain of mathematics: connecting knowledge within the domain, knowing general principles including rules and definitions, knowing principles underlying procedures, knowing the categories around which information is organized, knowing the meanings of symbols, and knowing the structure of mathematics. This covers a very wide spectrum of knowledge that can be considered conceptual in nature, and different definitions cover at least one of these aspects, if not several. The first three characterizations of conceptual knowledge are the most commonly present in studies that address it and there is a general agreement that it can be either implicit or explicit (and as such verbalizable). Crooks and Alibali (2014) proposed to distinguish conceptual knowledge into two type, one general, not specific to any particular math problem, and one procedure-specific. The first category is *general principle knowledge* which is defined as “fundamental laws or regularities that apply within a problem domain” (Prather & Alibali, 2009, p. 222). This covers all the aforementioned main characteristics except for what would be the second type, *knowledge of principles underlying procedures*. This second type of conceptual knowledge captures knowing why a procedure is applied to certain problems and why each step of the procedure is relevant. Knowledge of principles underlying procedures would also mobilize knowledge about connections, since it addresses the conceptual underpinnings for why the different steps in the procedure are taken, and knowledge about categories, since the correct categorization of a problem has implications for selecting the solving procedure.

Procedural knowledge on the other hand is currently consensually described as "the ability to execute action sequences (i.e., procedures) to solve problems" (Rittle-Johnson, 2019, p. 126). In the past it has sometimes been oversimplified and loosely equated to knowledge memorized by rote (Baroody, Feil, & Johnson, 2007). Even though procedures are developed through practice, which might be why it was considered a product of rote memory, they cover a wide range of competency. Procedures, as the steps necessary for achieving a goal, can be "algorithms – a predetermined sequence of actions that will lead to the correct answer when executed correctly or (2) possible actions that must be sequenced appropriately to solve a given problem (e.g., equation-solving steps)" (Rittle-Johnson, 2019, p. 126). Even though procedural knowledge is tied to specific types of problems, contrary to conceptual knowledge, and as such less generalizable, it is a valuable skill in itself (Baroody et al., 2007).

A question that is most often raised bears on the relations between these two knowledge categories, in terms of their causal relations and developmental precedence. Previously there have been views which considered one to precede the other, or even that they develop independently from one another. A current dominant take is the *iterative* view which considers that the relations between conceptual and procedural knowledge are bidirectional (Rittle-Johnson, 2019; Rittle-Johnson et al., 2001). In this view, conceptual and procedural knowledge are intricately and dynamically related, and acquiring proficiency in either one can improve the other. Indeed, numerous empirical findings provide evidence that gains in one type of knowledge predict the gains in the other type (Baroody & Ginsburg, 1986; Rittle-Johnson et al., 2001; Schneider, Rittle-Johnson, & Star, 2011; for an overview see Rittle-Johnson, 2019). This bidirectional relationship between conceptual and procedural knowledge persists over time, yet it is not always symmetrical. Even though some studies have found that there was no difference in strength of one knowledge type predicting the other (Schneider et al., 2011), other studies focusing on instructional interventions found that conceptual instruction lead to greater gains in procedural knowledge than the other way around (Rittle-Johnson, Fyfe, & Loehr, 2016). Nevertheless, when procedural instruction is carefully structured it can have a significant impact on students' conceptual gains, therefore supporting the iterative account (Canobi, 2009).

With this kind of iterative relationship and with theoretical distinctions between conceptual from procedural knowledge varying among researchers, it is indeed challenging to make clear-cut distinctions when measuring these two categories of knowledge. Procedural knowledge is mostly measured through accuracy and procedure use on familiar tasks. On the other hand, there is a very large variability of tasks used to measure conceptual knowledge. Among the numerous tasks which exist in the literature, (Crooks & Alibali, 2014) identified *explanation of concepts* and *evaluation of examples* tasks to be well suited for measuring general principle knowledge. The first one involves asking participants, older children or adults, to provide verbal explicit definitions, not only for symbols, but also provide explanations for rules and elements of domain structures, like what it means for two operations to be inverse. Yet it should be noted that since conceptual knowledge can be both explicit and implicit, some researchers find tasks that ask participants to provide explicit verbalizations of a concept might underestimate the actual conceptual knowledge they hold (Greeno, 1993). Evaluating examples is on the other hand a more implicit measure in which performance demands are low. This kind of task can involve recognizing examples, definitions or statements of principles. As for measuring conceptual knowledge about the principles underlying procedures, Crooks and Alibali (2014) pointed out the task of *applying and justifying procedures,* as well as the *evaluation of procedures*. They suggest that the first one should not only require participants to solve a problem in one way, but also involve a non-performance-based task in order to have a comprehensive assessment. As for the evaluation of procedures, they suggest asking participants to consider and justify procedures that they did not generate themselves. Yet, as we can see from the description of these tasks aiming to separately measure conceptual and procedural knowledge, we have to acknowledge that it is difficult to evaluate one by excluding the other. Engaging with procedures to solve problems often requires conceptual knowledge to interpret the problem, and on the other hand making evaluations of the underlying conceptions requires understanding the steps involved in the solving process. Thus, as Rittle-Johnson (2019) summarizes, items don’t usually tap into a single category of knowledge, but are predominant measures of one type of knowledge over the other.

Informal arithmetic solving strategies

Before starting to acquire formal conceptual and procedural knowledge in schools, children already develop certain forms of early arithmetic. One informal ability that leads to early arithmetic is a child’s ability to find the cardinal value of a set by counting the elements that compose it

(Baroody & Ginsburg, 1986). Five to six-year-old children usually engage in these counting strategies without following a specific order in which the items are counted, however this is not a manifestation of them understanding the order-irrelevance principal of cardinality, but simply a form of a tagging rule which respects order-indifference. Indeed, children do not realize that if a different counting order is made, the cardinal designation of the set will remain the same (Baroody, 1984). Children initially use these counting procedures to find answers to arithmetic problems (Resnick, 1989). In fact, young children can use these counting procedures to find answers to arithmetic word problems before receiving any formal mathematics education (Carpenter & Moser, 1982). Before instruction on arithmetic operations, when solving addition problems first grade students indeed predominantly count all of the elements of the two sets in order to find the answer (Carpenter, Hiebert, & Moser, 1981). This is called the *counting all* strategy. On subtraction problems that describe a situation where one quantity is taken away from another, students mainly separate the set that needs to be taken away and then count the elements of the remaining set and this is called the *separating* strategy. Another strategy is the *matching* strategy in which students match the elements of a set one-to-one and count the unmatched elements (Carpenter et al., 1981). These strategies constitute children's initial informal solving procedures and when counting one by one is extended to adding or taking-away one by one, Baroody and Ginsburg (1986) consider this to represent children's early informal arithmetic.

This informal arithmetic is first manifest when using objects or even fingers, and these concrete computing procedures also take a mental form leading to the development of informal mental calculations (Baroody & Ginsburg, 1986; Carpenter et al., 1981; Resnick, 1989). One of the most basic forms of mental arithmetic procedures is the *counting on* strategy. In the first variation students will count on *from the first addend* and will keep track of how much they need to count on. For example, when they are calculating 2 + 4 with this strategy students will start from 2 and perform: 3(+1), 4(2), 5(+3), 6(+4). The cost of keeping track is reduced when students adopt a strategy of counting on *from the larger addend*. For the previous example this means that students will start from 4 and perform: 5(+1), 6(+2), finding the answer 6 more easily. This is also termed as the *min* strategy in the literature (Shrager & Siegler, 1998). In the case of subtraction, students develop *separating to* strategies in which they no longer count the remainder after taking out a sub-set but count the sub-set out by counting backwards from the largest quantity while keeping track. For example, if students needed to solve 6 – 4 they would start from 6 and count: 5(-1), 4(-

2), 3(-3), 2(-4) and find the answer 2. Another strategy is the *adding on* strategy where the child engages in a forward counting sequence starting from the smaller and ending with the larger number, while keeping track of the counting steps for giving the answer. In the previous case this would mean starting from 4 and counting up: 5(+1), 6(+2).

Some authors have suggested that the different mental arithmetic strategies mainly develop through pattern recognition where children realize that counting on from one of the addends is redundant with its cardinal designation (Baroody & Ginsburg, 1986). In this view, students adopt new strategies in order to reduce the work-load and gain in cognitive economy. Other research has attempted to describe the development of children's arithmetic abilities by proposing computational models for strategy choices that students make when solving problems (Shrager & Siegler, 1998; Siegler & Araya, 2005). These models focus on the strategies children use and their reinforcement, all the while attempting to outline basic cognitive mechanisms that are involved in choosing among alternative strategies. These models are therefore in line with theories about associative-learning and information-processing (Baroody, 2003). This body of research provides a description of the developmental trajectory of how children move away from counting strategies to retrieval strategies that resemble more closely adult-like performance.

The Strategy Choice and Discovery Simulation model (SCADS) (Shrager & Siegler, 1998) describes that initially the counting all, or as the authors call it *sum* strategy is used on initial trials, mobilizing a lot of attentional resources. When enough attentional resources are freed up, then new strategies can be discovered and if they meet the requirements of including both addends and providing the result which corresponds to their addition, then the new strategy is added to the repertoire. Later, strategies in which redundant processing is perceived are progressively eliminated from the strategy repertoire, such as the counting all strategy. This approach therefore puts a crucial value on receiving feedback about the use of a strategy in order to generalize its application and it assumes practically no conceptual input other than the requirements tested by the metacognitive system if we were to regard them as principles. The SCADS* model (Siegler & Araya, 2005) extends the previous models of strategy choice and discovery. On simple additions it also considers a *decomposition* strategy in which the problem 3 + 5 can be solved as 4 + 4. It also extends the SCADS model to include strategies used on inversion tasks such as 18 + 16 – 16 which require a solver to realize that there is no need to engage in the computation of + 16 – 16. Even though the model included additional basic cognitive mechanisms such as priming in the simulation, it still does not stipulate a role for conceptual understating. Indeed, both models predict that the combination of associative and metacognitive processes would be sufficient to generate adaptive choices in strategy use when they are available.

Yet, a different take considers the development of counting based strategies to be linked to the development of implicit conceptual knowledge about number principles (Resnick, 1989). Resnick (1992) considers that children's thinking about numbers is progressively done in four different contexts. They start in the object context where objects represent proto-quantities, then the second stage is a verbal context, the third a symbolic context where children think with numbers, and last an abstract context in which they grasp operators. She considers that children display knowledge of principles in each context before understanding it in the next one, and that the basis of this comes from the fact that each number is a composition of other numbers. This compositional character of numbers is already present in the object context and provides a basis for understating arithmetic principles. For example, she considers that when kindergarten children perform addition by verbally counting on instead of counting all, they are already manifesting an implicit appreciation of arithmetic principles. In this view informal arithmetic knowledge is transformed into formal mathematics by thinking about numbers in different contexts and the conceptual component of understanding arithmetic principles is the motor of change.

Formal arithmetic solving strategies

Knowing arithmetic principles

One thing that can be regarded as common to these different descriptions of the development of informal arithmetic strategies is the detection of regularities, which is exactly what defines *principles* (Prather & Alibali, 2009). The principles inherent to the domain of mathematics therefore take part in the construction of a learner's conceptualization of the domain. They may be explicitly taught or simply inferred from experience, which does not change the fact that learners can rely on these regularities to solve problems. In order to assess the relationship between the knowledge of these principles and the strategies students use, it is important to create opportunities for solvers to display behavior that would be characteristic of such principled knowledge. Truly grasping the conceptual knowledge that learners hold about arithmetic principles would mean testing them not only in symbolic contexts but also in object and verbal contexts, and comparing them in order to understand how the knowledge about arithmetic principles develops. We will now focus on two arithmetic principles extensively studied in the literature: commutativity and the complement principle between addition and subtraction.

Findings concur that applying mathematical principles can facilitate computation by reducing computational effort and increasing solution accuracy and speed when solving arithmetic problems (Baroody, Ginsburg, & Waxman, 1983; Baroody, Torbeyns, & Verschaffel, 2009). For example, the *commutativity principle* states that the order of the operands is irrelevant for operations that are commutative. This is the case of addition and multiplication and it allows numbers to be combined in any order to find the result. Most studies interested in understanding the use and development of the commutativity principle take the use of procedures that imply knowledge of it. A great proportion of studies examining commutativity would present participants with problems they need to solve and would examine if the used procedures are consistent with the commutative principle. For example, following such a design, when a child solves both 3 + 5 and 5 + 3 by counting-up from the larger addend, or even when they use the count all procedure but first count up to five, it is considered to imply knowledge of commutativity because the order of the addends is regarded as irrelevant (Prather & Alibali, 2009). Alternatively, other task designs might include the *looking back* paradigm. In this design, when a participant is solving a problem, they can also see the previous problem they solved, which could be consistent with an arithmetic principle applicable for solving the current problem (for example, being able to see $a + b$ while solving $b + a$). When participants would demonstrate a benefit in this case and not when the previous problem is not helpful, then this would be taken as a sign that children have knowledge of the arithmetic principle.

Another principle that is widely studied is the *complement principle* which describes the inverse relation between addition and subtraction. This refers to the consideration that if a + b = c, then c − b = a or c − a = b or that the difference c − b = ? can be efficiently determined by considering what can be added to b to make c (Baroody et al., 1983). Some scholars use the broader term of inversion to describe the subtraction as addition relation. For example, the problem ‘7 – 3 = ?’ can be solved by a *direct subtraction* strategy, where subtraction is straightforwardly used to solve the problem – the subtrahend is directly subtracted from the minuend). Alternatively, the problem can be solved as ‘3 + ? = 7’, *direct addition* strategy, where it is determined how much needs to be added to the minuend to reach the subtrahend. In both cases, the formal arithmetic *operation* that underlies the solution process is the same (Campbell, 2008). Switching between direct subtraction and indirect addition is made possible through the complement principle (Baroody et al., 2009).

Using arithmetic principles

Baroody et al. (1983) conducted a study in order to investigate if first, second and third graders have knowledge about arithmetic principles by evaluating if they are able to recognize opportunities where it is beneficial to use them. They presented children with games in which they were asked to find the answers to addition and subtraction problems. One of the sessions tested the commutativity principle. The students were first presented with a card containing an item in which the first addend was the larger addend, such as 13 + 6. The students were instructed to respond quickly since the experimenter timed their responses and were asked to describe their solution strategies to the experimenter. The answers were noted on the card and it was placed face up so that the student could see it when the next problem was presented. The following card was either a target item or a test item. On the target items, the calculation would be a commutative in regard to the previous item, in this case 6 + 13. On test items, the student could not rely on the commutativity principle to find the answer from the previous calculation they did. They found that most children from all three grade levels relied on the commutativity principle to find the answer to target items at least on three out of the four target items. This was concluded either from the student’s description of the strategy which corresponded to the commutative principle, either if the student looked at the previous card and responded within 3 seconds. It is interesting to note that students relied on the commutativity principle even when students previously relied on counting for finding the solution to the non-target items.

Another session in the same study, following the same design, tested the complement principle. In this case the first card would present a problem such as 9 + 9 while the target card would present a problem for which applying the complement principle would allow the student to avoid computing the solution, such as 18 – 9. The findings were less consistent among the three grade levels in this session. It was only in third grade that students recognized the utility of the complement principle in the majority of trials and found the solution to the target problems based on it. Almost all of them used the principle consistently after the first use. It was only about half as many first and second graders that used the principle on most of the trials. As opposed to what was found on the commutative principle, the majority of students who relied on the complement principle in the target problems did so after having used strategies on the previous item that are more developmentally advanced strategies compared to the counting strategy (such as computing with no evident counting or solutions provided from memory). It should be noted that all the problems involved doubles in this series, therefore it is not as surprising that students initially used the memory strategy. However, when students did use counting on from the initial item, it was only on a minority of cases that students would rely on the complement principle.

Many studies since have been interested in understanding how these principles develop. Following different accounts that children have access to arithmetic principles at different moments during development and that their access to the principle is influenced by the context in which these principles are assessed (Canobi, 2005, 2009; Resnick, 1992). One study aimed at understanding the patterns of individual differences in the development of children's additive concepts, namely the commutative and complement principles, by studying them in different contexts (Ching & Nunes, 2017). First grade children were presented with tasks to evaluate their additive reasoning and calculation abilities. The first one, the conceptual judgment task, presented students with a story problem such as "Mary has 3 fish and her mother gave her 5 more. How many fish does Mary have now?". A puppet would solve the base problem by counting very quickly to find the response. A target problem was then presented, and the experimenter would ask the students if the puppet would need to count again to solve the problem or could look back to the base problem to find the answer. The target problems included 12 test items, which were related to the previous base problem by the commutativity principle (for example "Mary has 5 fish and her mother gave her 3 more. How many fish does Mary have now?") or by the complement principle (for example "Mary has 8 fish and her mother took away 5 from her. How many fish does Mary have now?"). Among the target problems there were also an equal amount of control items that served as a correction for possible biases that might have occurred on the test items (for example if the participant responded that the test item could be solved by looking back simply because it contained the same numbers). Points were only awarded if the student responded correctly to both the test and control item. The problems were presented in a concrete context, where bricks represented the addends, and an abstract context, without bricks. In the second task students had to solve calculations and word problems. This was also repeated approximately 10 months later, in second grade. Control variables including demographic characteristics, working memory, procedural counting and general intelligence measures were also included at the first time of testing.

The authors conducted latent profile analyses in order to uncover groups of children with similar performance on the conceptual reasoning tasks (both concrete and abstract) and identified a four-class model as the best fit for their findings. The first group of students had best performance on problems involving the commutativity principle in the concrete context, the second group had high performance on the commutativity principle in both concrete and abstract tasks but not on the complement principle, the third group had low performance only on the complement principle when tested in the abstract context, and the last group had high performance on all four evaluations. All four categories also revealed significantly different mean scores on the calculation tasks in first grade. Additionally, the identified profiles in first grade predicted performance on calculation tasks in second grade beyond the effects of the control variables. The authors therefore interpret these findings as revealing a developmental trend where knowledge of commutativity and complement principles develops first within the concrete context. They also propose that the commutativity principle is understood before the complement principle, and that children acquire knowledge of the commutativity principle in an abstract context before they start to acquire knowledge of the complement principle. This indeed goes well with other studies suggest arithmetic principles are first acquired in concrete contexts before abstract ones (Resnick, 1992). They also support findings which indicate that not all arithmetic principles are acquired at the same time (Brissiaud & Sander, 2010; Robinson, Dubé, & Beatch, 2017). But it raises the question of what leads a solver to consider the application of an arithmetic principle as relevant for solving a problem.

Usefulness of applying arithmetic principles

Previous research provided evidence that the complement principle is involved in switching between indirect addition and direct subtraction strategies. Just as it is a general consideration that the application of arithmetic principles facilitates the computation required when solving problems, a switch between direct subtraction and indirect addition is considered to provide computational advantage. Namely it has been suggested that this computational advantage is gained on subtraction problems that have operands with a small difference between the subtrahend and the minuend such as ‘81 – 79 = ?’ (Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009). Even though students’ verbal reports of strategy use did not reveal a high use of indirect addition strategies for solving subtraction problem, their response times and accuracies seem to indicate that they might simply not report using different strategies. Peters, De Smedt, Torbeyns, Ghesquière and Verschaffel (2013) therefore established an experimental design in which the response times of the participants could reveal the use of different strategies on subtraction problems by creating linear regression models to assess the data. They presented fourth to sixth grade students with subtraction problems for which they varied the numerical values of the operands and the presentation format. The problems were either presented in their standard subtraction format (83 – 4 = ?) and the 'unusual' indirect addition format (4 + ? = 83). As for the numerical values they manipulated them on two dimensions: the size of the distance between the subtrahend (S) and the difference (D) (small or large), and the relative magnitude of the subtrahend compared to the difference (*S* < *D* or *S* > *D*). This yielded four problem types: (1) large distance S < D (34 – 8 = *26* or 8 + ? = 34), (2) large distance S > D (31 – 28 = *3* or 28 + ? = 31), (3) small distance with S < D (31 – 15 = *16* or 16 + ? = 31), and (4) small distance S > D (32 – 17 = *15* or 17 + ? = 32).

If students systematically used the direct subtraction strategy, then they should have longer response times when the subtrahend was large, and the problems was presented in the direct subtraction format. In opposition to that, if they mainly used the indirect addition strategy, then they should have longer reaction times when the difference was large. Alternatively, if the participants switch between the two strategies, then they should have the longest reaction times when the subtrahend has the value close to the minuend. And indeed, the model that best predicted the reaction times of the participants was the *switch* model. This model supposed that participants would switch between direct subtraction and indirect addition based on the relative size of the subtrahend and therefore predicted that problems with relatively small and those with relatively large subtrahends (problems for which the distance between *S* and *D* was large) would be solved faster than problems with intermediate size subtrahend (those for which the distance between the *S* and D was small). Secondly, the researchers compared students' reaction times on the two presentation formats in order to gather additional evidence for the changes in students' strategies. The results revealed a three-way interaction between the presentation format, the magnitude of S, and the numerical distance between S and D. For large distance problems, in the case of *S* > *D* (e.g. 83 – 79 = ?) problems were solved faster when presented in the indirect addition format than subtraction format, while in the case of S < D (e.g. 84 – 4 = ?) the problems were solved faster in the subtraction format than in the indirect addition format. The authors proposed that this was due to the mental *re*-arrangement of the problem into the opposite format. However, when the distance between *S* and *D* was small, the presentation format had no effect on the reaction times, since neither of the two strategies yield a clear computational advantage. The authors suggest that students rely on fast and quasi-automatic estimation processes in evaluating the relative size of the subtrahend and make a choice based on their knowledge of a strategy's efficiency on a specific problem. It should however be noted, that only the participants who were able to solve problems in their indirect addition format at a pre-test were included in this study, therefore this does not provide information about the development of strategy use, but it does provide strong evidence that when these strategies are available to students, they mobilize them in solving arithmetic problems. These findings confirmed that switching between direct subtraction and indirect addition solving strategies is largely determined by the magnitude of the subtrahend. Based on these findings it is also possible to have delimitation of the numerical factors which influence students' strategy preferences on subtraction problems. Furthermore, recent findings using non-verbal measures of reaction times as well as verbal reports on the looking back task suggest that third and fourth grade students do use their knowledge of the complement principle when changing between the indirect addition and direct subtraction strategies (Torbeyns, Peters, De Smedt, Ghesquière, & Verschaffel, 2016). Since applying several solving strategies can lead to more optimal solving strategies, gaining proficiency in the complement principle has important educational stakes.

Mediating role of situated knowledge

Influence of semantics on solving strategies

Along with conceptual and procedural knowledge, de Jong and Ferguson-Hessler (1996) identify *situational knowledge* or knowledge about situations as they appear. Such knowledge serves to create a representation of a problem from which conceptual and procedural knowledge can be invoked. Other authors agree that such situational knowledge should be considered in order to grasp the complex relations between procedural and conceptual knowledge (Baroody et al., 2007). Situational knowledge is often observed in the way math content is presented to participants. As we have seen, the presentation format of an arithmetic problem has an impact on the strategies used by students to solve arithmetic two-digit subtraction problems (Peters et al., 2013). The way of presenting a problem strongly impacts how it is encoded (Mcneil & Alibali, 2004).

The most robust effects of content on solving strategies have been observed in the semantic structure of word problems. Numerous studies have documented that semantic knowledge illustrated through the problem's statement can guide the choice of procedures put in place to solve a problem. Indeed, Carpenter et al. (1981) took into account the solving strategies that children use before instruction based on counting models and investigated if the use of these strategies differed based on the semantic characteristics of word problems. They identified two dimensions along which the word problems differed semantically: (1) description of action or static relationships, and (2) set inclusion relationship. The first dimension was based on the description of action (e.g. "Leroy had A pieces of candy. He gave B pieces to Jenny. How many pieces of candy did he have left?"), or static relationships between the quantities in the problem (e.g. "Some children were ice-skating. There was an A number of girls and a B number of boys. How many children were skating altogether?"). The second dimension was based on the set inclusion relationship, where either two quantities make up a third quantity, one of them being unknown (just as it is the case in the last example), or the sets described in the problem are completely separate (e.g. "Ralph has A pieces of gum. Jeff has B more pieces than Ralph. How many pieces of gum does Jeff have?"). According to the study, the most commonly used solving strategy modelled the relationship or action described in the problem. In the first example students mainly used separating strategy (direct subtraction), while in the second example, students mainly used the count-all strategy. Thus, the semantic structure of the problem was the main determinant of the solution strategy.

Riley, Greeno and Heller (1983) were also interested in the complexities of children's conceptual and procedural knowledge, namely how it is attributed if children have certain procedural or conceptual knowledge based on their problem-solving performance. They highlight previous findings that even when it appears as though children lack the understanding of a concept on one task, their performance on another task can reveal that they actually do have knowledge about the concept which was considered to lack. Riley et al. argue that children's skills in solving arithmetic word problems improve with their understanding of the semantic relationships, since problems with the same arithmetic structures but different wordings can lead to substantial differences in difficulties for children. Indeed, arithmetic word problems identify quantities and describe relationships among them. For problems involving one-step additions or subtractions,

Table 1: Categories of word problems taken from Riley, Greeno and Heller (1983)

Action	Static
CHANGE	COMBINE
Result unknown	Combine value unknown

1. Joe had 3 marbles.
   Then Tom gave him 5 more marbles.
   How many marbles does Joe have now?
2. Joe had 8 marbles.
   Then he gave 5 marbles to Tom.
   How many marbles does Joe have now?

Change unknown

3. Joe had 3 marbles.
   Then Tom gave him some more marbles.
   Now Joe has 8 marbles.
   How many marbles did Tom give him?

4. Joe had 8 marbles.
   Then he gave some marbles to Tom.
   Now Joe has 3 marbles.
   How many marbles did he give to Tom?

Start unknown

5. Joe had some marbles.
   Then Tom gave him 5 more marbles.
   Now Joe has 8 marbles.
   How many marbles did Joe have in the beginning?

6. Joe had some marbles.
   Then he gave 5 marbles to Tom.
   Now Joe has 3 marbles.
   How many marbles did Joe have in the beginning?

EQUALIZING

1. Joe has 3 marbles.
   Tom has 8 marbles.
   What could Joe do to have as many marbles as Tom?
   (How many marbles does Joe need to have as many as Tom?)
2. Joe has 8 marbles.
   Tom has 3 marbles.
   What could Joe do to have as many marbles as Tom?

1. Joe has 3 marbles.
   Tom has 5 marbles.
   How many marbles do they have altogether

Subset unknown

2. Joe and Tom have 8 marbles altogether.
   Joe has 3 marbles.
   How many marbles does Tome have?

COMPARE

Difference unknown

1. Joe has 8 marbles.
   Tom has 5 marbles.
   How many marbles does Joe have more than Tom?
2. Joe has 8 marbles.
   Tom has 5 marbles.
   How many marbles does Tom have less than Joe?

Compared quality unknown

3. Joe has 3 marbles.
   Tom has 5 more marbles than Joe.
   How many marbles does Tom have?
4. Joe has 8 marbles.
   Tom has 5 marbles less than Joe.
   How many marbles does Tom have?

Referent unknown

5. Joe has 8 marbles.
   He has 5 more marbles than Tom.
   How many marbles does Tom have?

6. Joe has 3 marbles.
   He has 5 marbles less than Tom.
   How many marbles does Tom have?

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different semantic relations involving increases, decreases, combinations or comparisons of different quantities have been widely used in the literature, and summarized by Riley, Greeno and Heller (1983) into one of the most well-known typologies of word problems (cf. Table 1).

Change problems describe sequential actions that cause an increase or decrease of one quantity over time, and the question can bear on either the initial quantity, the change, or the modified quantity (e.g. “Joe had 3 marbles. Then Tom gave him 5 more marbles. How many marbles does Joe have now?”). Combine problems contain either two distinct quantities that are combined into a whole, either the total quantity is presented along with one of the quantities that is combined, and the solver is asked to identify the second quantity (e.g., “Joe and Tom have 8 marbles altogether. Joe has 3 marbles. How many marbles does Tom have?”). Compare problems contain a relational statement and describe a comparison between two separate quantities, and the question applies either to the difference between the quantities, or to one of the compared quantities (e.g., “Joe has 3 marbles. Tom has 5 more marbles than Joe. How many marbles does Tom have?”). In equalizing problems there are two separate quantities, just as in compare problems, except that the question bears on how can one set be modified in order to be equal to the other (e.g., “Joe has 3 marbles. Tom has 8 marbles. How many marbles does Joe need to have as many as Tom?”). Numerous studies have provided evidence for the psychological validity of this classification scheme (Fuson, 1992; Verschaffel, Greer, & De Corte, 2007). These different categories are also widely recognized in the educational literature and can be found in what Vergnaud and Durand (1976) described as additive word problem categories in which two measures combine to make a third measure or transformations that operate on a quantity providing a new measure, and a similar classification, bearing different names, has been proposed by (Vergnaud, 1982).

A variety of studies have provided evidence that the semantic structures described in these problem categories have an influence on student performance. Although the underlying structure of certain problems is the same, they use different solving strategies. De Corte and Verschaffel (1987a) found that over the first school year, the semantic structure of subtraction as well as addition problems determined the material, verbal and mental solving strategies that students used to find the answers. A progressive level of internalization (going from material to mental strategies) was observed during the year, but the influence of the semantic structure continued to influence students’ strategies. An examination of the different material strategies used by students when solving addition problems revealed that students mostly used count-all strategies, and that the semantic structure lead students to use the counting-all strategy differently. Students would add blocks to the first set on the dynamic Change 1 problem but join two sets or not move them at all on the static Combine 1 problem. As for the verbal and material solution strategies they found that more students solve the addition problem by starting with the larger addend (compared to starting with the addend represented first in the problem) on Combine 1 problems than on Change 1 problems. Along with other findings (De Corte & Verschaffel, 1987b) the authors proposed that the semantics of the problem also influence the ease with which students interchange the two given quantities in the problem, since in the Change 1 problem the two quantities have different functions (the start set and change set), whereas in the Combine 1 problems both quantities are sub-sets. As for subtraction problems, the Change 2 problem was mainly solved through separating from or counting down material and verbal strategies, whereas when mental strategies were used, they were mostly direct subtraction strategies. On the Change 3 and Combine 2 problem the main material strategies were adding on or counting up strategies, while the matching strategy was the dominant one on the Compare 1 problem. As for mental solving strategies, students predominantly used indirect addition when solving these problems.

Indeed, the dominant solving strategies observed on each problem directly modeled the semantic structure of the problem. As for the difficulty of each problem category, Riley, Greeno, and Heller (1983) documented that most Change problems are easier and that students succeed on them earlier than on the vast majority of other problem categories. And even when the solution to a problem involves a simple addition, Compare problems 3 and 6 are more difficult for children than either Combine 1 problems or Change 1 problems. Indeed, compare problems are considered as more difficult and cognitively demanding than change problems (Verschaffel, De Corte, & Pauwels, 1992), while Combine problems have been less studied, probably because combine superset problems have a high success rate as of early age (Riley et al., 1983). Most of these problems can even be solved before students have acquired any formal instruction about the arithmetic operations (Carpenter et al., 1981). However, even though the difficulties are systematically observed, identifying which problems are more difficult than others does not provide information as to why a problem is difficult. Different theoretical and empirical attempts have followed in order to explain student performance on word problems.

Arithmetic word problem solving processes

Schema theory

Various descriptions of the problem solving processes children engage in have been put forward. Kintsch and Greeno (1985) followed the general principles of a discourse processing theory (van Dijk & Kintsch, 1983) in order to attempt to understand which processes take place when a word problem statement is read or heard. The main task in solving word problems according to Kintsch and Greeno's (1985) approach is comprehending the text of the problem which interacts with the selection of strategies. They focus on how solvers construct a mental representation with the necessary information from which the solving process can operate. This representation coordinates among two structures: a *text base*, where the linguistic cues in the text, namely the propositions describing the relations among the elements, are captured and form a propositional representation, and on the other hand a *problem model* in which the solver selects and infers the information, based on the schemata triggered by the text base propositions, that is needed for solving the problem. Based on the propositions contained or derived in this representation, a schema representing properties and relations in a general form can be activated. Each schema has a procedure associated with it, which are then put in place and operate on the numerical values in order to find the solution to the problem. For example, the propositions 'have more than' and 'have less than' in Compare problems activate a more-than or less-than schemata which contain a small set, large set and a difference set to which the numerical values presented in the problem are attributed. The strategy that is then triggered is the difference schema strategy though which the two sets are compared by the match-separate procedure. This procedure matches the objects in the small to the large set and counts the objects in the larger set that are not matched with the smaller set. Although it provides a solution to the Compare 1 & 2 problems, Kintsch and Greeno propose that a conversion into another schemata for the rest of the Compare problems is needed. This kind of problem schemata have been described to contain situational, conceptual and procedural knowledge (de Jong & Ferguson-Hessler, 1996).

However, subsequent studies revealed important content effects that influence children's solution processes and performance on word problems. For example, in one study the wording of Combine 2, Compare 3 and Change 5 problems was reformulated in such a way that made the semantic relations more explicit, without changing the underlying semantic or mathematical structure (De Corte, Verschaffel, & De Win, 1985). Even though the applicable schema was not modified through such a rewording, the difficulty and errors made by students were. In both first and second grade students succeeded better on the reworded problems than on their classical formulation. This goes to show that the semantic structure of word problems does more than activate a schema through which the problem is solved. In fact, Bassok, Chase and Martin (1998) have shown that knowledge about semantic relations inferred from real-world objects influences college students' reasoning about arithmetic even when they are not solving, but inventing problems. The participants were asked to create simple addition or division word problems that contain different object sets: either sets that have symmetrical relations, such as tulips and daffodils, either asymmetric sets, such as tulips and vases. The semantic (a)symmetry of the sets indeed influenced the mathematical structure of the problems that were created: when the sets had symmetric relations, the participants created direct addition problems more frequently than division problems, whereas in the asymmetric case participants created more division problems than addition problems. This kind of semantic alignment between the semantic relations of the entities in word problems and the arithmetic operation was also observed in textbooks for grades 1 to 8. Even more, presenting such categorically related or unrelated word-pairs as primes before addition problems influences the activation of addition facts (Bassok, Pedigo, & Oskarsson, 2008). Indeed, when presenting categorically related word-pairs, which are semantically aligned with addition, there was a priming effect not observed when they were misaligned. All these findings suggest that the semantic influence observed on word problem solving is related to more complex representational processes that schema activations. Thus, the schema theory does not account for all the representational processes that seem to be at play and are not based on text-based schema activations.

Mental models – the situation model

Alternatively, the model developed by Reusser (1985, 1990a) aimed to shift focus from the linguistic and situation comprehension cues a priori linked to solving strategies to the interaction between world/content knowledge and mathematical knowledge in the construction of the representation. The model called the *Situation-Problem-Solver* builds on Johnson-Laird's (1983) mental model theory, which considers that solvers construct a representation analogous to the described situation, not dependent on pre-developed schemas. This analogous representation of the real world preserves the structural relations between the perceived objects. Reusser (1985, 1990) proposed that solvers construct a non-mathematical representation based on their comprehension of the situation, through which a person can access the mathematical problem model and calculation strategies. This way, a textually based problem is transformed into an equation via the situation model. The difference with Kintsch and Greeno's (1985) model is that the activated schema is already a mathematical structure, whereas in Reusser's model the mathematization and calculation are separate part of the solving process. In Reusser's (1990) view the representation of the situation is a bridging element between linguistic input and mathematical output, where concrete actions become abstract actions through access to the mathematical model of the problem which itself is connected to a solution equation.

Thevenot (2010) had tested the relevance of the construction of a mental model of the situation as it is described by Johnson-Laird for arithmetic word problem solving. University students were presented with Compare 1 and Compare 2 problems, with the question of the problem preceding the rest of the text so that they would memorize the protagonists:

- “How many marbles does Louis have more than Jean? Louis has 33 marbles. John has 17 marbles” (Compare 1)
- “How many marbles does Marc have less than Paul? Marc has 14 marbles. Paul has 31 marbles” (Compare 2).

The participants were asked to solve the problems and then they were presented with an unexpected task asking them to recognize if the questions were presented ‘word for word’ in the problems that they had to solve. The questions were either: paraphrased while still describing the same relations (for example “How many marbles does Jean have less than Louis?” for the Compare 1 problem); inconsistent with the initial mental model but with the same linguistic expression (“How many marbles does Jean have more than Louis?”); inconsistent with both the initial mental model and linguistic expression (“How many marbles does Louis have less than Jean”); or the same as the original questions. The original questions were recognized the most often, but the paraphrased questions consistent with the initial mental model were more often recognized than any of the inconsistent questions. These findings illustrate that it is the representation of the situation that solvers encode when solving problems, and not necessarily the linguistic cues themselves. However, constructing a representation of the situation does not seem to be sufficient for finding the answer. For example, Schliemann, Araujo, Cassundé, Macedo and Nicéas (1998) found that teenage street-vendors who attended school irregularly were successful in solving certain word problems such as “A boy wants to buy chocolates. Each chocolate costs 50 cruzeiros. He wants to buy 3 chocolates. How much money does he need?” but not problems that elicit the same mental representation but involve different numerical values, such as “A boy wants to buy chocolates. Each chocolate costs 3 cruzeiros. He wants to buy 50 chocolates. How much money does he need?”. Among the younger street vendors, multiplication was almost never used to solve the first problem. The findings indicate that they actually used repeated addition, and therefore were not able to find the answer to the second problem by applying commutativity to the initially represented situation. Though it is potentially surprising, that solvers correctly represented the problems in the first case, but did not find the solution in the second, this is in line with numerous studies demonstrating that the strategies children use to solve arithmetic word problems when they have not acquired knowledge of the arithmetic operation are informal and situation-based (Brissiaud & Sander, 2010; Verschaffel, Greer, & De Corte, 2000).

Situation Strategy First framework

Brissiaud and Sander (2010) proposed the *Situation Strategy First* (SSF) framework which explains that solvers will initially construct a non-mathematical representation of the situation which leads to a situation-based solving strategy (either double-counting strategies, derived or known number facts, or simply trial-and-error). If this strategy is efficient, the solver will then provide a numerical answer. For example, when solving the Change 2 problems such as "Luc is playing with his 22 marbles at recess. During the recess, he loses 4 marbles. How many marbles does Luc have now?" solvers would use the situation-based strategy that mentally simulates taking away 4 from 22 (21 (1), 20 (2), 19 (3), 18 (4)). This is indeed the direct subtraction strategy, which in this case is efficient for finding the solution to the problem. On Change 3 problems, such as “Mary has 18 euros in her moneybox. For her birthday, she receives more euros and puts them in her moneybox. Now, she has 22 euros in her moneybox. How many euros did Mary get for her birthday?”, the situation model leads to the strategy of adding on and the use of indirect addition as an informal situation-based solving strategy through which the answer is easily provided. Since the situation-based strategy is efficient, these problems are called *Situation* (*Si) problems*. All problems sharing the same wording will initially lead to the same situation model and situation-based solving strategy, yet this strategy may not always lead efficiently to the solution. What the Situation Strategy First framework proposes is that when the situation-based strategy is not efficient, it is necessary to modify the initial representation in order to find the solution, either by using arithmetic principles, or by explicitly evoking the formal arithmetic operation (cf. Figure1).

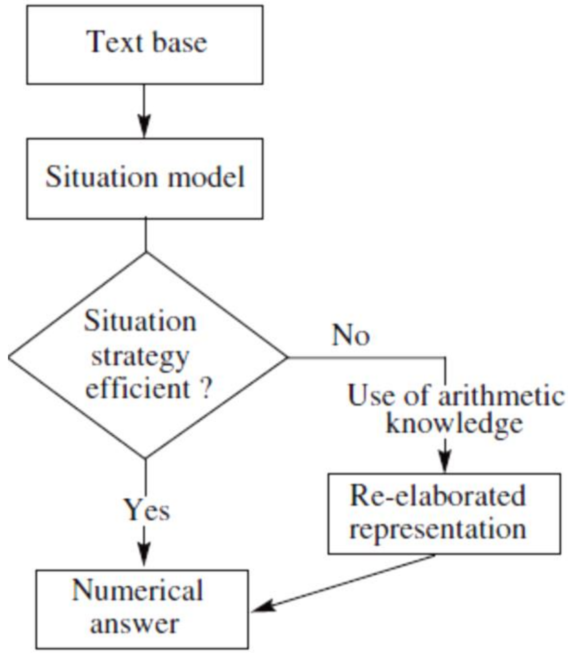


Figure 1: Architecture of the Situation Strategy First framework taken from Brissiaud and Sander (2010)

This is what would happen when the numerical values of the problem are changed and make the mental calculation modelling the described situation difficult to execute. Consider the previous examples, the Change 2 problem in which “Luc is playing with his 22 marbles at recess. During the recess, he loses 18 marbles. How many marbles does Luc have now?” finding the answer through the informal situation-based strategy would not be efficient, as it would require taking away 18 from 22. Yet, by applying relevant arithmetic knowledge this problem becomes easy to solve, which is why such problems are referred to as *Mental Arithmetic (MA) problems*. In this case the answer can be easily found if the complement principle is applied and the solution is found by performing indirect addition ‘18 + ? = 22’. This logic applies to the Change 3 problem “Mary has 3 euros in her moneybox. For her birthday, she receives more euros and puts them in her moneybox. Now, she has 22 euros in her moneybox. How many euros did Mary get for her birthday?”. The situation-based strategy of adding on is not efficient, but when the complement principle is applied the solver can find the answer by simply performing a direct subtraction of 22 – 3 = ?. The distinction between Si-problems and MA-problems therefore distinguishes the efficiency of different solving strategies.

The empirical assessment of the Situation Strategy First framework presented second grade students with Change 2, 3 and 5, as well as multiplication problems, and quotative and partitive division problems, at the beginning and the end of the school year (Brissiaud & Sander, 2010). The findings first revealed that at the beginning of the school year students performed 2.32 times better on additive Si-problems than on their MA counterparts, and 3.46 times better on the multiplicative problems. At the end of the school year, after having received more instruction on subtraction and having learned formal multiplication for the first time, this hierarchy of difficulty distinguishing Si- and MA problems remained prevalent. Additive Si-problems were on average 1.73 times easier than the corresponding MA-problems, and multiplicative problems were on average 2.15 times easier. A third experiment was conducted with third grade students at the beginning of the school year in order gain better insight into the hypothesized strategies following the Situation Strategy First framework. The participants were presented with the same problems and were asked to solve them and then write down the number sentence that corresponds to how they found the solution. On Si-problems, students predominantly wrote down number sentences which directly model the situation described in the text, while overall on MA-problems, number sentences which reflect the use of arithmetic principles were predominant. These findings confirm that the first step students engage in when solving arithmetic word problems is an attempt to simulate the situation described in the text. When this simulation is not possible, then they resort to using relevant arithmetic knowledge. Brissiaud and Sander (2010) further argued that contrasting Si- and MA-problems makes it possible to specify how using procedural knowledge depends on conceptual knowledge. The Situation Strategy First framework indeed considers that when students fail on MA-problems, it is not necessarily that they lack knowledge about the relevant procedures for solving the problem, but rather the conceptual arithmetic knowledge, i.e. the ability to apply arithmetic principles. So how do students come to succeed on problems where the informal solving strategy is costly? How do they succeed in applying the strategy that is most suitable?

Solving arithmetic problems adaptively

Adaptive strategy use

In order to acquire formal mathematics knowledge and attain proficiency, it is important to link new mathematical knowledge to preexisting informal knowledge (Baroody & Wilkins, 1999). Students' informal strategies are considered the basis for developing formal mathematics knowledge (Van den Heuvel-Panhuizen & Drijvers, 2014). Putting aside informal, situation-based strategies when they are inefficient and using a different solving strategy represents an important objective for mathematics education that is not easily achieved. This kind of process regarding strategy selection has received much attention in the many considerations about conceptual and procedural knowledge. In his reconsideration of procedural knowledge, Star (2005) draws attention to an important aspect worth considering in mathematics education – the notion of flexibility. He defines *deep procedural knowledge* as “knowledge of procedures that is associated with comprehension, flexibility, and critical judgment and that is distinct from (but possibly related to) knowledge of concepts.” This association of knowing multiple procedures as well as choosing the most appropriate one given a problem’s features has also been termed with *procedural flexibility* (Kilpatrick, Swafford, & Findell, 2001). However, the term ‘flexibility’ is sometimes used to simply refer to a smooth switch between different strategies. When it comes to selecting the most *appropriate* strategy, and not merely using multiple strategies, the term ‘adaptivity’ is more often emphasized (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009).

Flexibility in the use of different strategies is most valued in mathematics education when it is done in regards to the task characteristics and utilizes an understanding of numbers and arithmetic operations (Threlfall, 2009). On the other hand, Verschaffel et al. (2009) have proposed that task characteristics alone do not grasp the complexity involved in solving problems adaptively and identify three main factors that play an important role when operationalizing adaptive expertise: the task, the subject and the context. When operationalizing the adaptive use of strategies regarding task characteristics, the different strategies that can be used to solve additions and subtractions are distinguished. Their advantages and disadvantages are analyzed, and then define for which problem types each strategy provides the greatest gain, therefore considering it adaptive on that task (Blote, Van der Burg, & Klein, 2001). Taking into account the characteristics of the individual who is solving the task often requires observing participants’ performance when they can freely use the strategy they want and when they must use a particular strategy. Comparing participants' performance in terms of accuracy and speed on these two conditions provides insight into how well they master the different strategies (Torbeyns, Verschaffel, & Ghesquiere, 2005). Thirdly, the context in which problems are solved can also modulate the adaptive use of strategies. Indeed, solving problems within classroom settings doesn’t just emphasize the task goals, but also the social goals through the *didactical contract* (Brousseau, 1997). With these considerations, Verschaffel and collaborators (2009, p. 343) proposed that to be adaptive in one’s strategy choice means to consciously or unconsciously “select and use the most appropriate solution strategy on a given mathematical item or problem, for a given individual, in a given socio-cultural context." Even though the authors emphasize that 'the most appropriate strategy' does not simply refer to the strategy that will provide the quickest correct answer, most empirical research focuses on the efficiency and ease of execution as a determinant of appropriateness (Selter, 2009).

The construct of adaptive expertise has been considered to integrate both conceptual and procedural knowledge (Baroody, 2003). This integration had already been theoretisized outside of mathematics by Hatano, who considered that "flexibility and adaptability seem to be possible only when there is some corresponding conceptual knowledge to give meaning to each step of the skill and provide criteria for selection among alternative possibilities for each step within the procedures." (Hatano, 1982, p. 16). Indeed, learning to take into account the properties of a given situation in order to find the solution to a problem requires a conceptual change (Clément, 2009). More recent empirical findings are also favorable that achieving adaptivity by inventing new strategies or choosing between existing ones inextricably taps into conceptual knowledge (Blote et al., 2001; Rittle-Johnson et al., 2001; Verschaffel et al., 2007).

Comparison activities as a way of developing adaptive expertise

A promising path that could foster adaptive expertise in mathematics is working with different types of comparisons tasks (Rittle-Johnson & Star, 2011; Rittle-Johnson, Star, & Durkin, 2017). Comparison tasks favor analogical encoding which is not achieved when examples are separately studied (Gentner et al., 2003). There is a wide range of comparison activities that promote different aspects of mathematical proficiency. The comparison of problem categories with focus on how these problems differ can help students distinguish between problem categories that are easily confused (Vander-Stoep & Seifert, 1993). Comparison of incorrect methods with correct ones focusing on why one method works and the other doesn't has been shown to lead to less errors based on misconceptions (Durkin, 2009). While studying different examples that share the same underlying concept with focus on what is shared among them can lead to the acquisition of this concept (Hattikudur & Alibali, 2010). Achieving procedural flexibility in mathematics problem solving can also be achieved through comparison activities in which multiple procedures for solving the same problem, with focus on their efficiency, are compared (Rittle-Johnson & Star, 2007, 2009). In this kind of activity, students are presented with problems that have two worked out solving strategies. One of the worked-out examples illustrates a conventional solving strategy and the other one a shortcut method. By using specifically designed intervention materials the comparison task focuses students on understanding when and why is one method easier or more efficient than another. Engaging students in this task has been shown to benefit not only procedural knowledge and flexibility but also conceptual knowledge. This, however, is not the only form of comparison activities that can lead to greater mathematical proficiency. There have also been demonstrated benefits from comparing informal self-invented and formal procedures on problem solving that led to greater conceptual understanding (Hattikudur, Sidney, & Alibali, 2016).

Alibali, Phillips, and Fischer (2009) were interested in understanding what helps children shift from inefficient strategies for solving problems to more efficient ones. They considered that one main reason for failure is that they fail to accurately represent key features (Mcneil & Alibali, 2004), but also, that just as conceptual and procedural knowledge have a bidirectional relationship, they take that the problem representation and strategy use also affect each other in both directions. In their study they focused on how fourth-grade students solving of equation problems develops after participation in different types of lessons focused on the use of strategies that are new for students. Two lessons types focused on different common correct strategies through which the answer can be efficiently provided, one lesson focused on both of these strategies, and the last group did not receive any instruction. Prior to the lessons, students' problem solving and problem representation capacities were assessed. During the lessons, the students did indeed adopt the taught strategies, which was reflected in their better performance on problems after the lesson and on the post-test (with no significant difference between these two time-points), so it was possible to asses if this led to improvement of problem representations. Their findings revealed that indeed students had improved on their problem encodings, compared to their initial representations on the pre-test, after having learned the new equalizing strategy. However, this was not the case with the other, add-subtract strategy. According to the authors, it appears that the equalize strategy lead students to focus on elements of the problem (the 'sides' of the equation) that they usually fail to understand, whereas the add-subtract strategy encouraged students to simply use a different strategy on features that they already encoded well before this specific instruction. It seems that by learning new, more efficient strategies students can learn also to encode different features of the problem, leading to different representations. This favors the importance of re-representational processes in attaining adaptive expertise.

However, switching between two strategies in order to gain computational advantage involves a cognitive cost (Luwel, Schillemans, Onghena, & Verschaffel, 2009). The reason for this cost might lie in the re-representational processes that different lines of research have considered to support such a switch. Thevenot and Oakhill (2005) have demonstrated that when participants are engaged in a cognitively more demanding task, they construct an alternative mental representation of the situation. They presented participants with word problems that they had to solve such as "How many more marbles do John and Tom have altogether than Paul? John has 44 marbles. Tom has 24 marbles. Paul has 41 marbles.". The participants used an algorithmic strategy that would correspond to the initial description of the situation and would perform the calculation '(44 + 24) – 41'. Clearly, a less cognitively demanding strategy in regard to the mental computation would have been consider that Paul has 3 marbles less than John and calculate first '44 – 41' and then add Tom's 24 marbles. Yet, participants were more likely to construct this alternative representation and use this more efficient strategy only when the numbers were much higher ("John has 749 marbles. Tom has 323 marbles. Paul has 746 marbles"). Furthermore, empirical research has demonstrated that conceptual rewording of word problems, which aimed to highlight the underlying mathematical relations, improved children's performance, while this was not achieved with mere situational rewording (Vicente, Orrantia, & Verschaffel, 2007). This stresses the importance of *re*-representational processes that rely on conceptual understanding in order to overcome the tendency to apply informal strategies.

Semantic recoding

Achieving adaptive expertise via *re*-representation process that lead to a conceptual change in fact describes the process of *semantic recoding* (Gamo, Sander, & Richard, 2010): accessing a representation different from the one initially evoked by the situation. Gamo, Sander and Richard (2010) conducted a study with 4^{th} and 5^{th} graders on specific problems whose initial encoding usually leads to the use of costly strategies and wanted to see how the use of more efficient strategies can be achieved. In the first experiment they first gathered measures about students' strategy use on two categories of problems sharing the same formal structure (problems that can be solved by performing a one-step subtraction), but whose wording is known to influence two possible encodings leading to either a one-step strategy either a three-step strategy. For example, when a problem describes the relations between *numbers of elements* (e.g. "In the Richard family, there are 5 persons. When the Richards go on vacation with the Roberts, there are 9 persons at the hotel. In the Dumas family, there are 3 fewer persons than in the Richard family. The Roberts go on vacation with the Dumas. How many will they be at the hotel?"), then the privilege encoding of subtraction is *complementation* which leads solvers to calculate the size of all the different sets (e.g. 9 – 5 = 4; 5 – 3 = 2; 4 + 2 = 6), instead of using a more direct matching strategy (9 – 3 = 6). However, when a problem describes relations between different *ages* (e.g. "'Antoine took painting courses at the art school for 8 years and stopped when he was 17 years old. Jean began at the same age as Antoine and took the course for two years less. At what age did Jean stop?"), solvers privilege encoding situation as a comparison of different ages which leads solvers to only look for the difference between the quantities relative for finding the solution (e.g. 17 – 2 = 15). One group of the participants then took part in two training sessions. In the first session the number of elements problem was studied by comparing the two possible strategies. The students were asked to explain why the result was the same. They then had to solve an isomorphic problem with both strategies and the teacher led them to understand the notion of the *common part* and how it links to the matching strategy. The next session followed a similar design but started with problems describing heights and graphical representations highlighting the common part was introduced. One of the following isomorphic problems studied was a height problem and the second a number of elements problem. The results revealed that this led to the increase in use of the matching strategy which was more efficient since it only requires a single calculation. Yet, a question that remained unanswered was if this increase is actually due to a semantic recoding of the problem initially represented as a complementation into a comparison, making it possible to use the matching strategy. Alternatively, students could have only learned to use the one-step strategy and learned how to apply this new strategy. Therefore, in the second experiment, children were simply instructed to solve the different problems by using a one-step strategy. This however did not help them discover the matching strategy.

The authors conclude that the recoding the initial representation of a problem into an alternative one is important for producing the transfer of strategies. Semantic recoding could thus be of great importance in developing an adaptive expertise in elementary mathematics education because accessing a different representation and using arithmetic knowledge opens the path to the use of an efficient solving strategy on problems which cannot be easily solved through informal situation-based strategies. Therefore, this is a promising direction for designing pedagogical interventions that will foster students' *re*-representational processes. This kind of learning would also lead students to consider the abstract relations between the presented elements.

CHAPTER 4 – TEACHERS' CONCEPTIONS OF STUDENT LEARNING

CONCEPTUALIZING TEACHERS' KNOWLEDGE

Pedagogical content knowledge

For a long time, research on teaching bore on the content being taught and as such, was not really the subject of research in psychology. A shift was made when teaching effectiveness started to become the subject of research and policy making, and what was observed was how different teaching behaviors result in student achievement. Yet this did not give much importance to the subject matter and still did not bear on teachers' thought processes, but rather on their practice. Consequently, the question of how teachers' knowledge about the subject matter is transformed into the content of instruction did not receive great attention. Nor were the specific ways of translating this knowledge related to how students come understand or misconstruct the content. Shulman (1986) found that the lack of focus on this topic is a missing paradigm in research on teachers. He criticized the often-made sharp distinction between pedagogy and content and considered it essential to understand how teachers transform their understanding of knowledge that needs to be taught according to the curricular, into something that students can comprehend. He proposed that content knowledge is one category of knowledge that teachers hold, and it refers to how knowledge is organized in the mind of the teacher. Not merely the facts and concepts, but also the different underlying structures of the subject matter which focus on why it is worth knowing it and how it relates to other content. On the other hand, Shulman introduced a notion that has since become a very influential concept – *pedagogical content knowledge* (PCK).

Pedagogical content knowledge is subject matter knowledge for teaching. It still relies on content knowledge, but it is content knowledge that is most relevant for its teachability. This category of knowledge includes "the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations" (Shulman, 1986, p. 9) which all contribute to making the subject matter comprehensible to others. Furthermore, this also requires the teacher to understand what makes learning a specific topic difficult or easy. Pedagogical content knowledge, therefore, reduces the foremost importance given to mere content knowledge and draws the attention to knowledge that teachers have which differs from that of content specialists. Shulman proposes that this category of knowledge specific to teachers also requires them to know what are the preconceptions that students have about the subject matter at different stages in their development, especially since these preconceptions can often be misconceptions and require the teacher to apply strategies which will help students reorganize their knowledge. In fact, given the abundant number of studies focusing on exactly these aspects of learning, Shulman considers that this should be at the very heart of teachers' pedagogical understanding of the subject matter. Besides these two main categories Shulman (1987) also considered other categories to constitute teachers' knowledge: curriculum knowledge, knowledge of educational contexts, knowledge of learners and their characteristics, and knowledge of educational purposes and values.

Since then, many researchers working on teacher education bore focus on understanding PCK, proposing what distinguishes other categories of teacher knowledge from it, and how they relate. The different factors that influence teachers' classroom interactions are indeed contextual factors, teacher-centered considerations such as their general and subject-specific pedagogical beliefs, including pedagogical content knowledge, and lastly their judgments about students, namely their performance and comprehension (Wanlin & Crahay, 2012). Magnusson, Krajcik and Borko (1999) followed the work of Grossman (1990) and proposed that PCK is, in fact, the transformation of other knowledge categories – knowledge about the subject matter, pedagogy and context – into knowledge specific of the teaching profession which helps teachers understand how to help students learn the subject matter (Figure 2). This goes well with studies that have highlighted the involvement of reconstructive and interpretative processes in teachers' understanding of students' conceptions (Prediger, 2010). For this purpose, teachers need to make sense of "the analogies, metaphors, images, and logical constructs" that influence the acquisition of the notion being taught (Davis & Simmt, 2006, p. 300). Even though there is no clear consensus on the categories of knowledge that are necessary for teachers to faithfully address students' thinking processes, PCK is still one of the most prominent overarching models of teachers' competence.

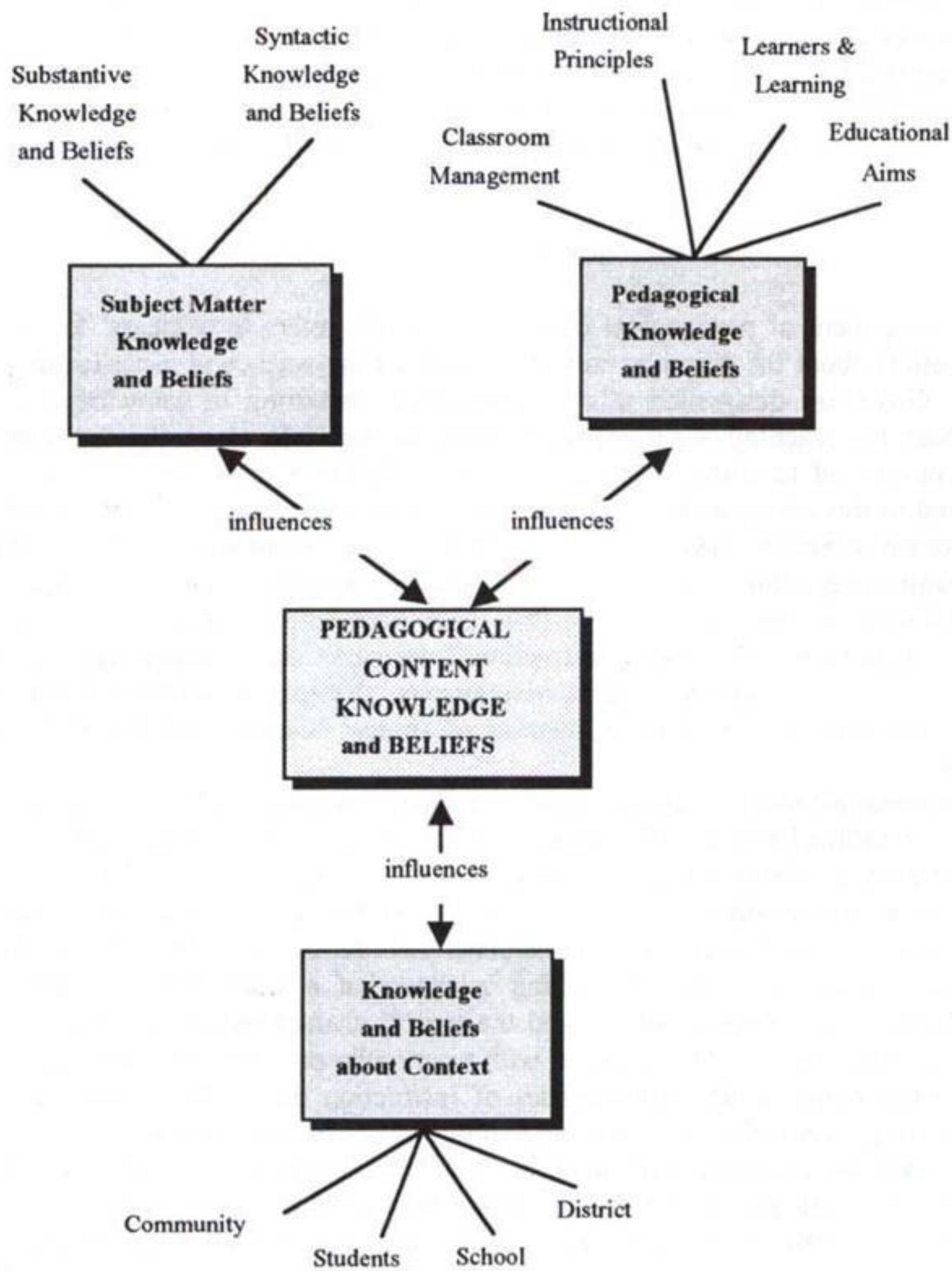


Figure 2: Magnusson, Krajcik and Borko's (1999) adaptation of Groosman's (1990) modeling of the relationship between PCK and other domains of teacher knowledge

Mathematical Knowledge for Teaching

The most prominent reconceptualization of PCK in the domain of mathematics education was proposed by researchers at Michigan University and is called *mathematical knowledge for teaching* (MKT) (Ball, Phelps, & Thames, 2008; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004). This taxonomy is an ad-hoc model shaped by their previous studies and their effort to conceptualize one of the primary elements in Shulman's PCK – knowledge of how students think about the content being taught. The MKT taxonomy is notably useful in order to gain insight into the multidimensionality of teachers' knowledge and the different contributions of content knowledge and PCK to teachers' knowledge. MKT makes a major distinction between *subject matter knowledge* and PCK (Figure 3). The latter does not contain any knowledge forms that are specific to students or teaching itself. It covers different aspects of mathematical knowledge varying in their degree of expertise. Within it we find *common content knowledge* which is used by adults and teachers in the same way and *specialized content knowledge* which is specific mathematical knowledge for teaching; however, it is related only to the underlying mathematical concepts and not pedagogical practice. On the other hand, their description of PCK refers to knowledge that teachers should hold in order to be good teachers in mathematics, and it is not something that all adults would intuitively know and contains several categories. They subdivide it into three categories *knowledge of content and teaching*, *knowledge of curriculum* and that which has received the most attention - knowledge of content and students (KCS). KCS focuses on teachers' understanding of how a student learns certain content and is defined as "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (Hill et al., 2008, p. 375). One main merit of MKT is that it stems from empirical research on the knowledge teachers need in order to teach mathematics. There is even an MKT task measuring teachers' mathematical knowledge for teaching that has been validated. It typically assesses all aspects of MKT apart from knowledge of content and curriculum and knowledge at the mathematical horizon. Items mainly relate to content regarding knowledge about number concepts, operations, and algebra and patterns (Hill et al., 2004, 2005).

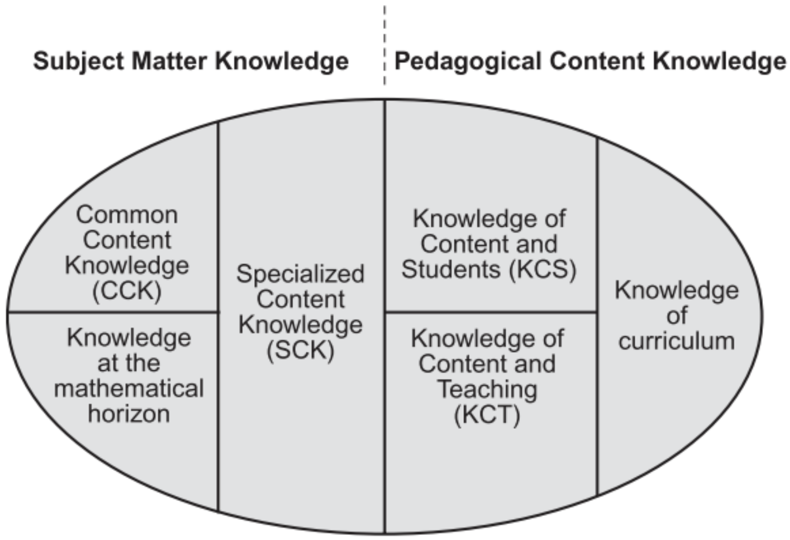


Figure 3: Mathematical knowledge for teaching taken from Hill et al. (2008)

Teachers' diagnostic competence

While MKT is a useful framework for looking into the construct of teacher knowledge, not all research in mathematics education adopts this approach. Nevertheless, when conceptualizing teachers' PCK in mathematics education, there some common characteristics have been identified, irrelevant of the adopted framework. A review by Depaepe, Verschaffel, and Kelchtermans (2013) investigated how PCK is pervaded in mathematics education research. They found that there are four common characteristics of how this kind of knowledge is conceptualized. A first commonality on which scholars agree is that PCK connects content knowledge to pedagogical knowledge. Second, it deals with the knowledge necessary to achieve the teaching objective. Third, it is specific to the content being taught and represents the teachers' pedagogical translation of it. And fourth, content knowledge is a prerequisite for teachers' PCK. Notably the kind of content knowledge considered important is that which allows teachers to be able to make connections among different mathematical ideas and to flexibly think in multiple ways about the specific concepts being taught. KCS involves teachers focusing on both the specific content being taught and learners' particularities. What Shulman described as central to PCK indeed finds its equivalence in this knowledge category. Hill and collaborators (2008) describe KCS as encompassing knowing how students typically learn the topic of study, what are the common mistakes they make and what misconceptions often arise. Investigating KCS requires tasks through which participants would need to rely on their knowledge of student thinking about certain mathematical notions and not their own knowledge about the subject itself. Following this definition, another crucial aspect of evaluating teachers' KCS is relying on empirical evidence about students' performance and how they learn.

Measuring and quantifying the conformity of teachers' judgments to objective data about student performance is also widely studied in research on teachers' *diagnostic competence* (Leuders, Philipp, & Leuders, 2018). Teachers' diagnostic competence are mobilized in *diagnostic activities* which "comprise the gathering and interpretation of information on the learning conditions, the learning process or the learning outcome, either by formal testing, by observation, by evaluating students' writings or by conducting interviews with students" (Leuders, Dörfler, Leuders, & Philipp, 2018, p. 4). Their aim is to assure teachers have valid knowledge about individual students and groups of students so that they can provide students with the necessary tools for learning. When given tasks of judging non-specific student groups, as opposed to judging the performance of familiar students, the focus bears on the requirements of the task. Diagnostic competences mobilize various categories of knowledge, including general and topic-specific categories for analyzing student thinking processes through relevant theoretical background (Prediger & Zindel, 2017). They are used in assessments on formal tests, but they are also commonly used in any classroom practice in order to assess students' learning processes. They are considered to be an important prerequisite for constructing and selecting activities that will correspond to students' abilities and help them overcome potential difficulties (Südkamp, Kaiser, & Möller, 2012). This important activity that teachers engage corresponds to what Shulman described as 'knowledge of students' (mis)conceptions' and 'knowledge of learners and their characteristics', which was further differentiated as KCS in Mathematical Knowledge for Teaching. Some studies have shown that after extensively working with teachers on understanding students' development in the subject matter, their judgments about the students' thought processes improved, and in their classroom practice also evolved in order to favor student development (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Yet, this correlation has not always been observed in later studies (Gabriele, Joram, & Park, 2016). Teachers' diagnostic judgments are nevertheless considered essential to student-centered teaching approaches (Davis & Simmt, 2006), the mechanisms of how the diagnostic competence impact students learning need more investigation.

Taking on a student's point of view

Studying Knowledge of Content and Students

One of the first studies that actually investigated teachers' views regarding their students' conceptions and learning processes in mathematics was conducted by Carpenter, Fennema, Peterson, and Carey (1988). They questioned first-grade teachers about the difficulty of addition and subtraction word problems, and about the strategies children use when solving them in a series of tasks. The tasks mainly relied on different semantic categories of word problems, for which there is a well-established hierarchy of difficulty and knowledge about the strategies students use. First, they asked teachers to invent word problems for given number sentences. Second, they presented pairs of problems and asked teachers to judge which one is more difficult for students. Third they were shown videos of students' strategies while solving problems and had to propose a description of how the students would solve related problems. The findings revealed that the teachers were able to propose word problems for which the provided number sentences directly models the solution, therefore indicating they are aware of the wide diversity of word problem categories. They were also able to identify the kind of strategies used by students when viewing videos of them solving problems – for instance viewing a student solve a problem through a counting strategy and identify it as such. However, after comparing two problems in order to assess their difficulty in the second task, distinctions between problem types were not the dominant explanatory factors when it came to explaining why one problem is more difficult than another. They often focused on keywords rather than on children's representations of the problem and therefore had difficulties in articulating why some problems are more difficult. The authors explain that the teachers did not categorize problems according to the strategies children use.

The influence of teachers' own knowledge

Along with understanding how teachers evaluate students' reasoning about the content, researchers have looked into how teachers' own knowledge about the task impacts their diagnostic competencies of student performance. Some scholars have proposed that having highly proficient content knowledge will impact teachers PCK (Nathan et al., 2001). This proposal has been defined in the *expert blind spot* hypothesis which states that "educators with advanced subject-matter knowledge of a scholarly discipline tend to use the powerful organizing principles, formalisms, and methods of analysis that serve as the foundation of that discipline as guiding principles for their students’ conceptual development and instruction, rather than being guided by knowledge of the learning needs and developmental profiles of novices” (Nathan & Petrosino, 2003, p. 906). As it has been proposed in Chapter 3, students arithmetic reasoning is considered to first develop in verbal contexts developing in a symbolic and abstract context, and the semantic context also influences the conceptual development in arithmetic. Yet, several studies have shown that pre-service teachers with an advanced subject matter knowledge (Nathan & Petrosino, 2003) as well as high-school teachers (Nathan & Koedinger, 2000a, 2000b) defend pedagogical approaches which focus on symbolic reasoning since they view this as ‘pure mathematics’, whereas when they do rely on a verbal context they considered it as ‘mathematical applications’. One study put this gap to the test in a study by asking pre-service teachers with advanced mathematics knowledge to rank the difficulty of different algebra and arithmetic problems (Nathan & Petrosino, 2003). The problems they used were either presented in their symbolic form, either in a verbal form (a story problem or word equation). The problems that they had to rank according to their expectations of ease or difficulty for students all shared the same mathematical relations. What they found was that pre-service teachers with advanced mathematics education deemed that students would be best at solving equations and worst at story problems. This contradicts the repeatedly observed findings that there are contexts in which students typically perform better on arithmetic word problems than on symbolic algebra problems. The authors interpret these findings indeed reflect the teachers’ ‘expert blind spot’ and propose teachers made judgments about student performance through a domain-centric lens.

Indeed, gaps between prospective teachers’ PCK and content knowledge are not rare (Depaepe et al., 2015). An overestimation of students' solution rates, where teachers consider certain problems to be easier than they actually are for students, than has been observed in other domains of mathematics for example studies bearing on graphs and function (Leinhardt, Zaslavsky, & Stein, 1990). Estimating the difficulty of a task is also studied in light of processes involved in social judgments by considering that the diagnostic competences of teachers are an ability to take on a students’ perspective. Making judgments about other people’s knowledge is not exclusive to teachers in educational settings and there already exist conceptual models of how people make judgments about the knowledge of others. For example, in Nickerson's (1999) model, one’s own knowledge plays the role of a default model and integrates what the person knows differentiates their own knowledge from that of others. It is only later on that the judgment is adjusted by one's knowledge about specific others in order to create a working model of others' knowledge on which one can rely when making judgments. This process of anchoring and adjustment is supported by the *availability heuristic* (Tversky & Kahneman, 1974) through which it could be considered that for experts, their extensive knowledge of the domain is so easily accessible that it overrules another one's perspective. In Nickerson's model this would mean that people tend to appraise their own knowledge as more common than it actually is and could thus explain the expert blind spot. For this reason, Ostermann and collaborators (2017) proposed a translation of this model for understanding mathematics teachers' judgments.

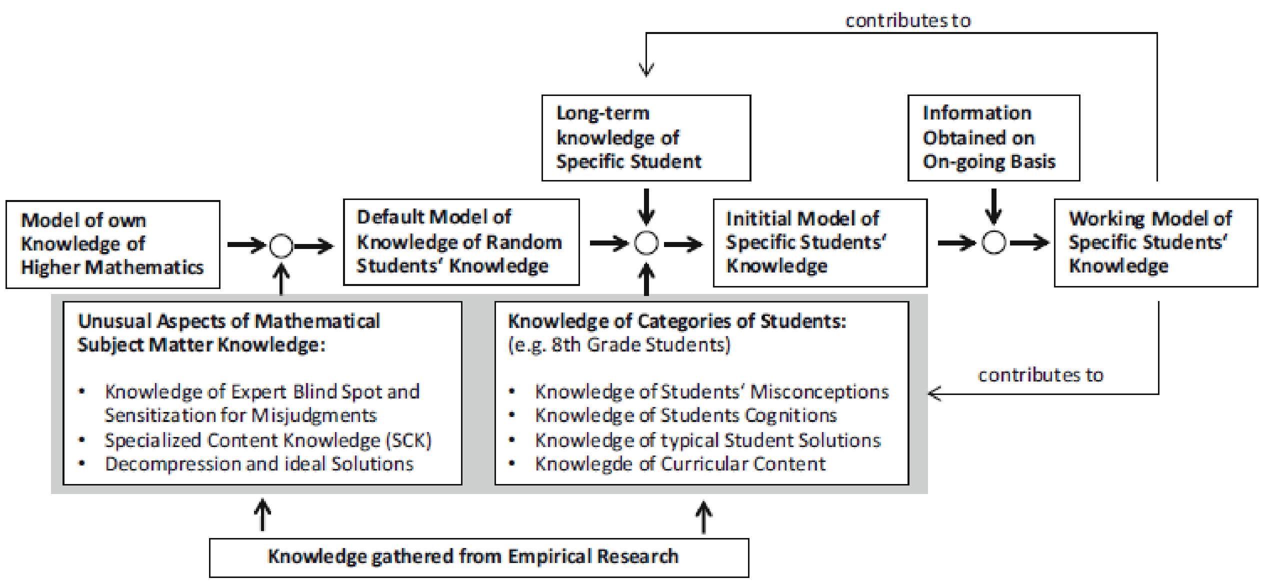


Figure 4: Ostermann, Leuders and Nuckles' (2017) model of teachers' judgments about what students know

Ostermann and collaborators consider that in making diagnostic judgments, teachers mobilize two main categories of MKT: specialized content knowledge and knowledge of content and students. The follow Nickerson's model and consider teachers' comprehensive expertise as the initial input (Figure 4). This initial input is then modulated by what is considered the *unusual* aspects of their knowledge – being aware of what content knowledge they as a teacher poses but not students. This is where their specialized content knowledge comes into play. Further on, knowledge of content and students influences the teachers' initial representation of specific groups of students. All of these elements contribute to constructing long-term knowledge about specific others. In any given situation, these elements constitute the foundation of the working model with which teachers make their diagnostic judgments. In the cases when teachers' judgments are overestimations of student performance, it is their extensive knowledge in the subject matter that anchors the diagnostic process and creates the illusion of simplicity of the task for students. Osterman et al. (2017) propose that being familiar with this tendency to overestimate students' performance may help teachers to select a more adequate task when teaching and evaluating.

Even though these lines of research are gaining more attention and stress the importance of understanding teachers' knowledge so that they can better address students' misconceptions the fact that these intuitive conceptions about the subject matter persist in the adult population has not yet gained interest. One of the few studies to take this into account proposed to combine Shulman's framework of PCK and Fischbein's theory of tacit models (Tsamir & Tirosh, 2008). They highlight that since students' intuitive, formal and algorithmic knowledge are often inconsistent and that this should be implemented and used to refine the tools for evaluating teachers' PCK and subject matter knowledge. Prospective and in-service teachers were questioned about some common mistakes that students make when multiplying and dividing fractions. They were presented with different calculations and were asked to first list the common mistakes students make and then to describe the possible sources for each of the mistakes. The analysis bore on problems for which algorithmic, formal and intuitively based mistakes have been identified in the literature. What the first observed was that the majority of the teachers stated that they didn't know what the sources of the mistakes are. Interestingly, when they did mention the mistake sources, the majority of the teachers mentioned algorithmically-based mistakes, which describe various computing 'bugs' and rarely intuitively based mistakes. It is true that the lack of reference to intuitively based mistakes could have been due to the algorithmic nature of the presented expressions. Additionally, in a study of pre-service teachers' knowledge of students' misconceptions regarding the division of fractions, teachers did not show awareness of the conceptual errors students made and did not identify the intuitive knowledge children hold (Tirosh, 2000). This contrasts with the finding that students make conceptually based errors when evaluating the numerical value of fractions (Stafylidou & Vosniadou, 2004). We believe that it is important to better understand how the widespread intuitive conceptions impact teachers' judgments of student strategies and more broadly their PCK.

AIMS AND OBJECTIVES

Aims and objectives

The empirical research conducted in this thesis explores how different facets of analogical reasoning are mobilized in arithmetic word problem solving and teaching. Intuitive conceptions constitute mental categories constructed on the basis of previous experience and these categories can be used to apprehend new concepts and situations (cf. Chapter 1). Yet, besides intuitive conceptions, there are other informal processes that influence school performance. Even before instruction young children solve arithmetic word problems with small numerical values based on their daily-life experience and by using a variety of informal strategies. Behind these informal strategies lay complex representational processes. Understanding these representational processes is part of a crucial category of teachers' knowledge and their diagnostic competence which is considered to greatly influence their practice (cf. Chapter 3).

When solving arithmetic word problems, the construction of a situation model is based on solvers' real-world knowledge and represents a step that enables solvers to construct a mathematical model and eventually note the corresponding number sentence or operation, making it possible to find the solution (Reusser, 1990b; Verschaffel et al., 2000). If we take the example of Change 2 and 3 problems from Riley et al.'s (1983) classification, the construction of the mathematical operation for solving the problem evolves in parallel with the described situation (Reusser 1985, 1990). For the Change 2 problem "Luc is playing with his 22 marbles at recess. During the recess, he loses 4 marbles. How many marbles does Luc have now?" solvers almost exclusively find the solution to the problem by taking away 4 from 22 (Brissiaud & Sander, 2010; De Corte & Verschaffel, 1987a), which corresponds to the mathematical operation of direct subtraction '22 – 4 = ?'. On the other hand, on Change 3 problem such as "Mary has 18 euros in her moneybox. For her birthday, she receives more euros and puts them in her moneybox. Now, she has 22 euros in her moneybox. How many euros did Mary get for her birthday?", most students find the solution by adding up from 18 to 22 (Brissiaud & Sander, 2010; De Corte & Verschaffel, 1987a), which corresponds to the mathematical operation of indirect addition. These informal situation-based strategies that straightforwardly model the situation are also related to different conceptions of subtraction. Namely, the Change 2 problems is consistent with the intuitive conception of subtraction as *taking away*, while the Change 3 problem is inconsistent with the intuitive conception of subtraction. The Change 3 problem is however within the conception of subtraction as *determining the difference*.

Furthermore, the numerical values embedded in the problem statements interfere with the difficulty of the problem, not only due to their size but due to their effect on the problem's representation. Specific problems can be efficiently solved with the strategy provided by the situation model, while at other times the informal solving strategy is inefficient. When the informal solving strategy is inefficient, solvers who succeed in finding the answer use formal strategies that require the use of conceptual arithmetic knowledge. This raises two questions. Firstly, what does it take to succeed in using a formal solving strategy when the initial representation of the problem leads to an inefficient informal solving strategy? Secondly, how is the situation model constructed when there are no sequences of actions described in the problem? If the construction of a situation model only occurs in parallel with a description of changes to a quantity, does one solve static problems by directly accessing the mathematical solution, as it would be suggested by Reusser's model? And if not, what provides the solver with an informal solving strategy on static problems? Within the educational context, another question that emerges is how do teachers interpret student performance on such tasks?

We propose that it is the process of mental simulation that leads to the use of an informal solving strategy. Namely, we propose that the mental simulation isn't simply situation-based, but that it operates on the encoded representation of the problem. This makes it possible for the mental simulation to provide solvers with an informal solving strategy even on problems which depict a static situation. Furthermore, we propose that encoding a problem's representation depends on both the format it is presented in and the arithmetic conception associated with the problem. When the problem is intuition-consistent, this conception will be used as a first resort for the encoding. However, when a problem is intuition-inconsistent, other available conceptions will be used in the encoding. The encoded representation provides a base on which the mental simulation can operate and delivers an informal strategy through which the solver can attempt to find the solution. When the mental simulation of the initial encoding is costly, in order to find the answer in a more efficient manner, the solver will need to resort to formal solving strategies. The solver will either engage with different arithmetic strategies, such as decomposition in order to facilitate the calculation, either in a recoding of the initial representation which can diminish the computational difficulty of the problem. We propose that in order to achieve this re-representation, the conception of the arithmetic operation used in the initial encoding will need to be put aside and accessing a different conception of the arithmetic operation will allow the solver to achieve a semantic recoding of the problem. The semantic recoding captures the importance of detaching from the problem's initial semantics and the arithmetic conception associated with it, in order to access a different encoding (cf. Chapter 3). This recoded representation will then make it possible for the solver to use a different, formal, solving strategy. Following this proposal, even when solvers have gained sufficient mathematical proficiency and can easily access a formal solving strategy that efficiently provides them with the numerical solution, they would have had to rely on an arithmetic conception that is not connected with the one exemplified by the problem. In teachers' judgments of students' solving strategies, it is their own process of encoding the problems' representations that will be used as a default model in order to reason about the strategies students use.

The empirical research conducted in this thesis explores how informal knowledge influences students' solving processes on arithmetic problems and teachers' judgments about students' strategies. We have conducted a series of studies among students and teachers aiming to investigate how different arithmetic conceptions influence the encoding of arithmetic word problems, both for the purpose of solving them, as well as for evaluating their difficulty for students. In Chapters 5 to 7, we present six experiments that we have conducted with a total of 673 first and second-grade students in classroom contexts, with 36 elementary school teachers and 36 adults.

In Chapter 5, we studied the processes involved in solving arithmetic problems. We proposed that, based on the semantic relations described in the problems, different conceptions would guide the construction of a representation that solvers use to find the solution to a problem. We proposed that students attempt to solve problems by performing a mental simulation upon the encoded representation. We measured the performance of second-grade students on different problem types in collective classroom studies. We expected to observe greater success on problems whose initial encoding could be easy to simulate mentally than on problems whose initial encoding would lead to a costly mental simulation. By using verbal reports, we then studied and analyzed the solving strategies students put in place. We expected to find more informal solving strategies on problems that are easy to simulate mentally and formal solving strategies when they succeed on high cost mental simulation problems.

In Chapter 6, we present a research-based arithmetic intervention program that substituted the regular arithmetic curriculum in first-grade classes. The problem solving syllabus aimed to promote the use of the most appropriate strategy for finding the solution to different problems. Through the use of arithmetic word problems, students worked on the encoding of a problem's representation and its recoding when this initial representation leads to costly solving strategies. We conducted a collective classroom experiment with students from the intervention group and students who followed the regular curriculum. We looked at both their performance and the strategies they wrote down. We expected that students from the intervention group would have higher performance and use formal strategies more frequently.

In Chapter 7, we investigated if the arithmetic conceptions that guide the encoding of arithmetic word problems in students will also impact teachers' diagnostic judgments. Based on the previous empirical evidence of students' solving strategies we selected arithmetic word problems that have the potential to mobilize both intuitive conceptions and informal solving strategies. The participants were asked to evaluate the relative difficulty of the problems that were presented and explain what makes certain problems more difficult than others. Since intuitive conceptions persist among the adult population, but it is only teachers who have specialized knowledge regarding pedagogical issues of understanding students' strategies, we compared the performance of elementary-school teachers to lay adults. We expected that when the content of the problem is intuition-consistent, then teachers will have performance like lay adults, since they will have a harder time explaining what difficulties the problem poses for students.

EMPIRICAL CONTRIBUTIONS

CHAPTER 5 – MENTAL SIMULATION IN THE DRIVING SEAT OF ARITHMETIC PROBLEM SOLVING [2]

INTRODUCTION

The current study investigated the processes behind students solving strategies on arithmetic problems. We propose that the informal strategies on arithmetic problems reflect the mental simulation of a specific problem encoding. The process of mental simulation is not unfamiliar in the cognitive science literature. According to Barsalou (1999, p. 586), mental simulation consists of the construction of "specific images of entities and events that go beyond particular entities and events experienced in the past". In this view, encountering a situation activates a simulation of actions and perceptions associated with it, which are not necessarily present in the perceptual surrounding. Landriscina (2013) pointed out that mental models can make information available to other cognitive subsystems by way of mental simulation. The involvement of dynamic and perceptual simulations in text comprehension and the processing of abstract concepts has great support (Barsalou & Wiemer-Hastings, 2005; Zwaan, Madden, Yaxley, & Aveyard, 2004). The theory of gestures as simulated action (Hostetter & Alibali, 2008) considers this close link between action and perception, and advocates even that the meaning of a sentence is simulated in physical terms through gestures, even when there is no physical movement described in the sentence. This simulation is characterized by the activation of motor and perceptual systems in the absence of external input (Hostetter & Alibali, 2018). Together these findings support the proposal that a mental simulation does not have to occur only in the presence of action sequences and provides a foundation for considering that on arithmetic word problem solving a mental simulation occurs on the encoded representation.

Mental simulation has actually been previously associated with arithmetic word problem solving (Brissiaud & Sander, 2010; Orrantia & Múñez, 2013). Orrantia and Múñez (2013) propose that during word problem solving, a simulation, which they refer to synonimously as an analog representation, is involved in the processing of magnitudes of the quantities contained in a problem. Magnitude processing takes a central role in this mental simulation given the importance

[2] Results from experiments 1, 2 and 4 have been published in: Gvozdic, K., & Sander, E. (2017). Solving additive word problems: Intuitive strategies make the difference. In B. C. Love, K. McRae, & V. M. Sloutsky (Eds.), Proceedings of the 39th Annual Conference of the Cognitive Science Society. London, UK: Cognitive Science Society.

of numerical magnitudes in the development of numerical capacities (Ansari, 2008) and the strong influence a reader's experience of the world has on their language comprehension (Zwaan et al., 2004). Their study does provide evidence for the construction of a mental representation of a described situation and illustrates its importance in preserving the relationship between the described elements. Yet, the mental simulation of the encoded magnitudes cannot not explain the different performance rates and strategies observed when semantically different problems contain the same numerical magnitudes. The Situation Strategy First framework on the other hand proposed a different characterization of the mental simulation taking place in arithmetic word problem solving. It proposed that solvers engage in a mental simulation of the depicted situation and provides the solver with situation-based solving strategies. The numerical magnitudes within the problem can influence the difficulty of the mental simulation, making the situation-based solving strategy computationally inefficient. The different situation-based and arithmetic strategies observed in a study testing the Situation Strategy First framework are in line with what would have been predicted by the Switch model (Peters et al., 2013) about performance on such calculations. The switch model describes which combination of the magnitude of the subtrahend, the numerical distance between the subtrahend and the minuend, and the presentation format of the arithmetic operation yield the switch from one strategy, direct subtraction or indirect addition, to the other, the computationally advantageous one. For example, the switch from direct subtraction used on the Si-problem to indirect addition on the MA-counterpart would have been predicted by the switch model based on the numerical characteristics. Together these findings point to the importance of how the problem is encoded as the determinant of the costliness posed for the mental simulation.

When students solve Si-problems the situation model already provides them with the optimal format for solving it, be it direct subtraction or indirect addition. Solvers are only in need of a correct encoding on which the mental simulation can operate. These problems mainly evaluate the execution of different solving strategies. On the other hand, finding the solution to MA-problems is not solely based on executing a solving strategy. In order to solve these problems in the most efficient manner, solvers need to resort to a different solving strategy than the informal one initially provided by the situation model.

The encoding and simulation of a representation

We propose that behind solvers' informal strategies on arithmetic problems lies the mental simulation of a specific problem encoding. This representation of the problem is encoded depending on the format of the presentation and the arithmetic conception embedded in the problem. As such, the encoded representation then initiates a mental simulation through which the solver attempts to find a solution. When the mental simulation bears low cost for the solver, they find the solution to the problem based on the initial encoding. This constitutes an informal solving strategy. On the other hand, when it is not possible for the solver to easily find an answer through the informal strategy, an arithmetic strategy needs to be used. When the encoded representation bears a high cost for the mental simulation, the solvers either rely on the initial encoding but apply computational strategies to render the calculation possible, either they re-represent the problem. In order to re-represent the problem, the solver would need to recode the initial representation, mobilizing conceptual arithmetic knowledge, leading to a recoded representation, which the solver can again attempt to mentally simulate. Recoding the representation of the problem is more favorable when it leads to a computational advantage and makes the problem easy to simulate mentally in this recoded representation. Previous findings on arithmetic word and non-word problems indeed fit well with this process. If we take the example of Change 2 problems on both low and high cost mental simulation problems, the solver would encode the representation with the help of the taking away conception of subtraction and engage in a mental simulation of this encoding. In the case of the low cost mental simulation problem such as “Luc is playing with his 22 marbles at recess. During the recess, he loses 4 marbles. How many marbles does Luc have now?”, the solver would easily find the numerical solution to the problem by the informal strategy of taking away 4 from 22. This informal strategy is made possible by mentally simulating the encoded representation. On high cost mental simulation problems such as “Luc is playing with his 22 marbles at recess. During the recess, he loses 18 marbles. How many marbles does Luc have now?”, the same encoded representation leads to mentally simulate taking away a large quantity, taking away 18 from 22. In this case, the mental simulation of the initial encoding is not computationally efficient for finding the answer. In order to access a more efficient solving strategy, a different conception of arithmetic needs to take part in the recoding – the conception of subtraction as searching for the difference. Therefore, what makes high cost mental simulation problems bear high cost is that their encoding would be influenced by a conception that does not allow access to the optimal solving strategy – i.e., the strategy that the initial encoding leads to cannot be easily performed in the same format as it is encoded in.

The same process would occur on non-word problems, for instance calculating '42 – 39'. The initial encoding of the representation would be in line with the taking away conception of subtraction, and the solver would engage its mental simulation. In this case, when the solver attempts to mentally simulate taking away 39 from 42, it would be rather costly to provide a numerical answer. If the solver sticks to this encoding, they would need to resort to different calculation strategies and decompose the subtraction into sub-steps in order to provide an answer. On the other hand, the solver could recode the problem relying on an alternative conception – that of searching for the difference – allowing them to mentally simulate a computationally favorable solving strategy. This proposal provides an account for the re-representational processes brought forward in the literature on arithmetic problem solving, such as the Situations Strategy First framework (Brissiaud & Sander, 2010) or the Switch model (Peters et al., 2013). It also gives an explanation for the processes involved in solving problems that do not describe action sequences.

One example of static problem would be the Compare 2 problem (cf. Table 2). In this case the encoding of the problem is influenced by the implicit selection of the determining the difference conception, which gives place to the mental simulation of the distance between the largest quantity (31) and the second operand: in the low cost mental simulation problem this mental simulation would be efficient. The students would describe their solving process as starting from 31 and counting down 30(**1**), 29(**2**), 28(**3**), 27(**4**), or counting up 28(**1**), 29(**2**), 30(**3**), 31(**4**), bearing the answer 4, and would be represented as '31 – ? = 27'. Yet in the high cost mental simulation problem, using the same informal strategy and mentally simulating this initial encoding through is a computationally costly strategy. When students would use this informal strategy, they would describe the same solving process: starting at 31 and counting down 30 (**1**), 29(**2**), 28(**3**), ... 5(**26**), 4(**27**), finding the answer 27. This would correspond to the *indirect subtraction strategy* '31 – ? = 4'. Nevertheless, by selecting the conception of subtraction as taking away, the solver will be able to take away 4 from 31 and provide the correct numerical answer to this problem. The result would no longer be the number of times a solver counted how much there is between the two quantities, but how much is left after taking away. We thus consider that the description corresponding to a number sentence that differs from the one that directly models the initial encoding indicates the use of an arithmetic strategy, which would be noted as '31 – 4 = ?' in this case.

PRESENT STUDIES

The studies presented in this chapter were designed to test the role of mental simulation following the problem's encoded representation, stipulating a strong influence of arithmetic conceptions solicited by the presented problem. Reusser (1990) described that the actions described in the problem lead to accessing the mathematical model of the problem. Therefore, in order to test if it is the influence of the arithmetic conception that influences the mental simulation of the encoded representation and not just the described situation, we used problems that do not describe actions. In a set of four experiments, we used word problems, whose wording does not favor a mental simulation of the described actions, and we used non-word problems, which do not contain contextual elements favorable for a mental simulation. Depending on different arithmetic conceptions hypothesized to be involved in the encodings, we built high and low cost mental simulation versions of each problem: a version that could be easily solved through the mental simulation of the initially encoded representation and a version for which a recoded representation would lead to the most efficient solving strategy.

In this chapter, we report the collective classroom studies conducted in order to measure the performance on different problems, as well as verbal protocols in order to get insight into the encoded representations and solving strategies. Our rationale was that problems encoded following arithmetic conceptions related to part-whole relations (namely subtraction as taking away and addition as combining quantities into a whole) would, through the mental simulation, be described as direct arithmetic operations. Whereas the mental simulation of problems encoded following the determining the difference conception would give place to indirect arithmetic operations as informal solving strategies. We predict different performance rates and strategies depending on the cost of the mental simulation, supporting the importance of the encoding which goes beyond propositional statements and numerical processing. On problems where the mental simulation of the initial encoding is inefficient and bears a high cost, we expect to find lower performance rates and formal solving strategies – those that do not reflect the initial encoding. We also expect to observe informal solving strategies that reflect the mental simulation of the initial encoding of the problem on low cost mental simulation problems, and formal strategies reflecting the mental simulation of a recoded representation mainly on high cost mental simulation problems.

Experiment 1

In the first experiment, we evaluated second-grade students' performance on arithmetic word problems that do not depict a change in the quantity appearing over time. We postulated which conception of arithmetic would be mobilized by each problem category based on its wording, proposing that arithmetic conceptions of part-whole relations would guide the encoding of Combine 1, Compare 4 and Compare 3 problems, whereas conceptions referring to differences would guide the encoding of Combine 2, Compare 1 and 2, Equalize 1 and 2. We went on to create two versions of each problem, one that would bear a low mental simulation cost, and one that would bear a high mental simulation cost. We predicted that students would succeed better on low cost than on high cost mental simulation problems because they can be easily solved through the informal strategy of mentally simulating the encoded representation. High cost mental simulation problems would be harder to solve because they would require the solvers to recode the initial representation of the problem by selecting a different conception of arithmetic and solve it by using formal strategies. Therefore, this re-representation process would cause these problems to be more difficult for students. Indeed, if this performance gap between low and high cost mental simulation problems would be observed on static problem categories it would testify to the importance of the initial encoding, as guided by the different arithmetic conceptions, which can be mentally simulated. It would also highlight the need for re-representational processes in order to use formal strategies when this initial encoding is not efficient.

Method

Participants. 341 second-grade students from 16 classes coming from 11 elementary schools from working-class neighborhoods in France participated in the study. The average age of the children in January was 7.60 years (*SD* = 0.33, 177 girls).

Material. There were 8 addition and subtraction problem types belonging to 3 major categories corresponding to Compare problems 1, 2, 3, 4, Combine problems 1, 2, and Equalizing problems 1 and 2 from Riley et al.'s (1983) classification (cf. Table 2). High and low cost mental simulation versions of each problem category were created. The number triplets involved in the data and the solution are (31, 27, 4), (33, 29, 4), (41,38, 3), and (42, 39, 3). The subtraction problems involved two numbers, a and b ($a > b$). The numerical values for a were either 42, 41, 33 or 31, while in order to differentiate between low and high cost mental simulation problems the values for b were either kept small (3 or 4) or were close to a (39, 38, 29 or 27). To create low cost mental simulation problems the small value of *b* was used for the Compare 4, while the *b* value close to *a* was used for Compare 1, 2, Equalizing 1, 2, and Combine 2 problems. To create high cost mental simulation problems, the alternative *b* value was respectively used for each problem, since it would make the mental simulation computationally costly. The addition problems Compare 3 and Combine 1 involved two numbers, *b* and *b′* ($b > b'$). Both *b* and *b′* had the same characteristics as *b* for subtraction problems, while the unknown value was equivalent to *a*. To create low cost mental simulation problems the value close to *a* (*b*) was presented first, while the small value (*b′*) was presented second. To create high cost mental simulation problems, they were presented in the opposite order, making the mental simulation costly.

The number size was not the determining factor for distinguishing the cost of the mental simulation. In the case of the low cost mental simulation Compare 4 problem there was a small *b* value, contrary to the rest of the subtraction problems where *b* was presented in the close to *a* value. Therefore if the determining factor of difficulty had the *b* value close to *a*, like in the majority of the problems, we should observe an inverse trend on the Compare 4 problem: we would observe higher success rates on the high cost mental simulation Compare 4 problem than on its low cost counterpart. Furthermore, in a previous study using word problems that could be contrasted by their ease of mental simulation (Brissiaud & Sander, 2010), the small *b* values, and the ones close to *a* were equally present in the low and high-cost mental simulation problems. Four contexts were used for the wording of each problem: marbles, euros, flowers, and fruits. Each problem was only presented in one context.

Design. There was a total of 16 problems: 8 problem categories in two different variants, one low and one high cost mental simulation. Children solved a total of 8 problems created by combining the 8 problem categories in either one of these versions. Each student, therefore, solved 4 low cost mental simulation problems and 4 high cost mental simulation problems. To control for the impact of position, numerical sets and context, 8 different problem sets were created. Another 8 problem sets were 'mirror' sets in which the low cost version of one problem would be presented in its high cost counterpart, while the high cost problem would be presented in its low cost counterpart. Thus, 16 groups of problem sets were created all together and counterbalanced across classrooms. Due to a technical error, the Compare 2 was not passed in two classes, since the Compare 1 problem was printed out instead of it. The responses to these instances were disregarded from the analyses.

Table 2: Example of the arithmetic word-problems for the number set (31, 27, 4)

Problem categories		Low cost mental simulation problems	High cost mental simulation problems
Comparison problems	Compare 1	There are 27 roses and 31 daisies in the bouquet. How many daisies are there more than roses in the bouquet?	There are 4 roses and 31 daisies in the bouquet. How many daisies are there more than roses in the bouquet?
	Compare 2	There are 31 oranges and 27 pears in the basket. How many pears are there less than oranges in the basket?	There are 31 oranges and 4 pears in the basket. How many pears are there less than oranges in the basket?
	Compare 3	James has 27 marbles. Steve has 4 marbles more than James. How many marbles does Steve have?	James has 4 marbles. Steve has 27 marbles more than James. How many marbles does Steve have?
	Compare 4	Anna has 31 euros. Susan has 4 euros less than Anna. How many euros does Susan have?	Anna has 31 euros. Susan has 27 euros less than Anna. How many euros does Susan have?
Equalizing problems	Equalizing 1	There are 27 oranges and 31 pears in the basket. How many oranges should we add to have as many oranges as we do pears?	There are 4 oranges and 31 pears in the basket. How many oranges should we add to have as many oranges as we do pears?
	Equalizing 2	There are 31 roses and 27 daisies in the bouquet. How many roses should we take away in order to have as many roses as we do daisies?	There are 31 roses and 4 daisies in the bouquet. How many roses should we take away in order to have as many roses as we do daisies?
Combine problems	Combine 2	Mary has 27 euros in her piggybank and she has euros in her pocket. In total, Mary has 31 euros. How many euros does Mary have in her pocket?	Mary has 4 euros in her piggybank and she has euros in her pocket. In total, Mary has 31 euros. How many euros does Mary have in her pocket?
	Combine 1	There are 27 blue marbles and 4 red marbles in Marc's bag. How many marbles are there in Marc's bag?	There are 4 blue marbles and 27 red marbles in Marc's bag. How many marbles are there in Marc's bag?

Design. There was a total of 16 problems: 8 problem categories in two different variants, one low and one high cost mental simulation. Children solved a total of 8 problems created by combining the 8 problem categories in either one of these versions. Each student, therefore, solved 4 low cost mental simulation problems and 4 high cost mental simulation problems. To control for the impact of position, numerical sets and context, 8 different problem sets were created. Another 8 problem sets were 'mirror' sets in which the low cost version of one problem would be presented in its high cost counterpart, while the high cost problem would be presented in its low cost counterpart. Thus, 16 groups of problem sets were created altogether and counterbalanced across classrooms. Due to a technical error, the Compare 2 was not passed in two classes, since the Compare 1 problem was printed out instead of it. The responses on these instances were disregarded from the analyses.

Procedure. The experiment was conducted in January and was administered in the students' classrooms. Each child received an 8 page booklet. There was a square in the middle of each page in which they wrote their answer. Each problem was read aloud twice to the whole classroom, reducing the demand on students' reading skills, and children then had one minute to write down the number that was the solution; the next problem was then read aloud.

Scoring. The solutions provided by the children were scored with 1 point when the numerical answer was exact or, in order to allow for mistakes in counting procedures, within the range of plus or minus one of the exact value. Admitting values of +/-1 of the exact answer has been used in previous studies (Brissiaud & Sander, 2010; Rittle-Johnson et al., 2016). Any other answers received 0 points.

Results

A first analysis was conducted in order to compare students' success rates on low and high cost mental simulation problems. The analyses were conducted using R software. Since the data points for the responses were binary and recorded in a repeated design (with low and high cost mental simulation problems), we conducted random-effects logistic regressions. We constructed a generalized linear mixed model (GLMM) with a binary distribution with the cost of mental simulation (low vs. high) as the fixed factors, while participants and problem categories were included as the random effects. The analyses showed a highly significant main effect of the cost of mental simulation on performance ($\beta = 1.05$, $z = 11.12$, $p < .001$). Students succeeded on average on 45.58% of the low cost mental simulation problems and on 27.10% of the high cost mental simulation problems. The low cost mental simulation problems had a 1.69 times higher success rate than high cost mental simulation problems.

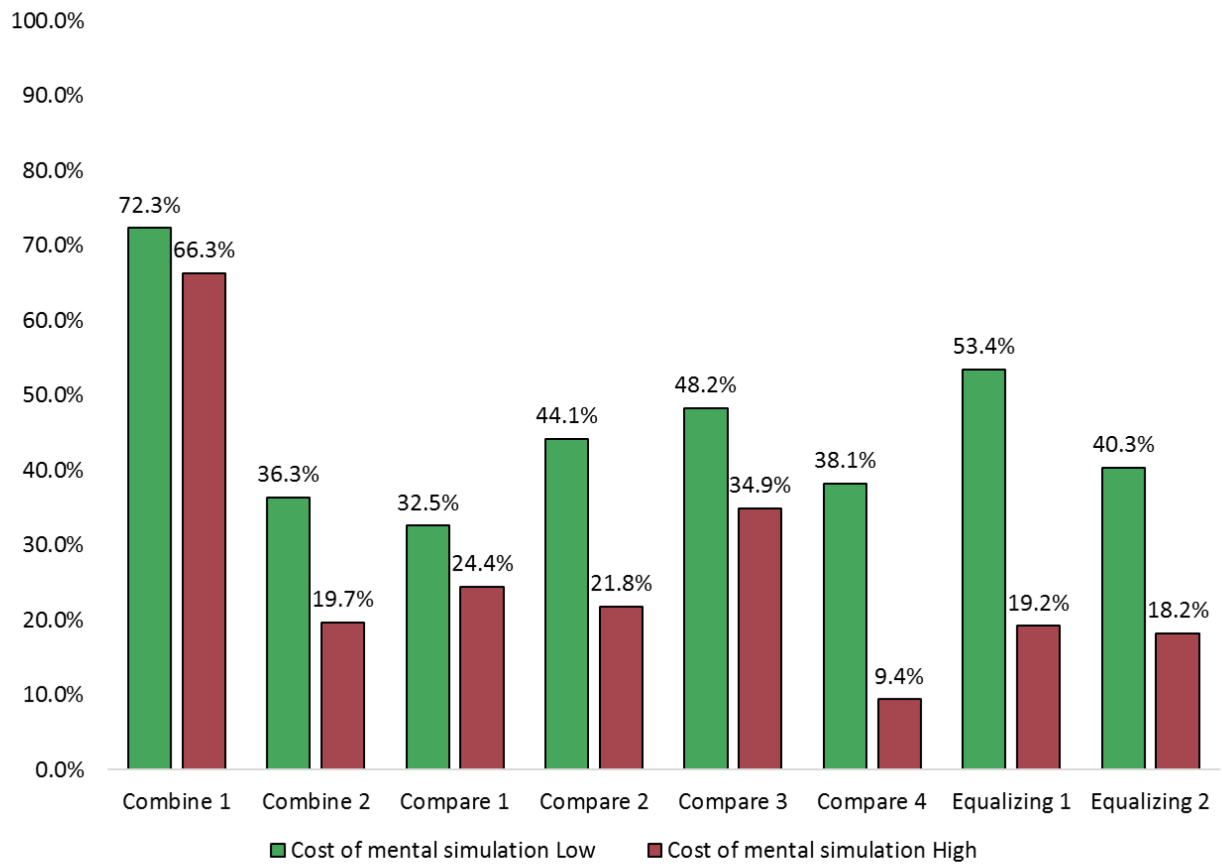


Figure 5: Performance per problem category

Furthermore, we tested our hypothesis on each problem category by conducting GLMMs with a binary distribution and the cost of the mental simulation as the fixed factor, and subjects as the random effect. The results revealed that the low cost mental simulation problems (Compare 2, 3, 4, Equalizing 1, 2, Combine 2) were significantly easier than the corresponding high cost mental simulation problems ($0.55 < \beta < 1.78$, $2.49 < z < 6.35$, $p < .01$). The performance rates are presented in Figure 5. On problem categories for which there was a significant difference, the low cost mental simulation problems were 1.38 to 4.07 times easier than their high cost counterparts. The Compare 1 problem seemed to be harder than others at this time of testing when only a tendency towards a difference was observed ($\beta = .4$, $z = 1.69$, $p = .09$): 32.5% success rate on low cost mental simulation problem and 24.4% on high cost mental simulation problem. While the Combine 1 problem seemed to be particularly easy, given that there was no effect of Problem type ($\beta = .28$, $z = 1.2$, $p > .1$), and the success rates on both problem types were highest (72.3% on the low cost mental simulation problem and 66.3% on the high cost mental simulation problem).

Discussion

The results revealed that the hypothesized efficiency of the mental simulation following the problems' encodings led to different performance rates among second-grade students, distinguishing high and low cost mental simulation problems for most problem categories. This was observed on almost all of the tested problems that do not contain scenarios with dynamic sequences that could favor the mental simulation and would not be possible if the encoded representation did not create a dynamic mental model that could be simulated. The two problem categories for which the distinction was not observed seemed to be particularly hard or particularly easy for students at this time of testing. The difficulty could stem from different reasons, certain of them being linguistic in nature. If this is the case, then we expect the efficiency of the mental simulation to be a relevant factor on Compare 1 problems later in the school year, when we expect students to have significantly higher performance on low cost mental simulation problems than on high cost ones.

Combine 1 problems, on the other hand, had a relatively high success rate. We proposed that they would be encoded with the help of the part-whole arithmetic conception, which is important for understanding commutativity (Resnick, 1989). The wording of the problem is also in coherence with this metaphor, since the problem describes two parts that are being combined into a whole. It, therefore, appears that these problems are easier for solvers to encode in a flexible way which makes these properties salient, and can then easily mentally simulate taking into account commutativity. These types of strategies, counting on from the larger quantity, are considered to imply knowledge of commutativity (Prather & Alibali, 2009).

Experiment 2

Findings from the previous experiment provide evidence for our proposal about the nature of the informal strategies and re-representational processes needed in order to access arithmetic strategies for facilitating the resolution of problems where the initial solving strategy is costly. In the following experiment, we replicated the same study design at the end of the school year, 6 months later, in order to gather confirmatory evidence about the persistence of problems' initial encodings even after students have received subsequent instruction in math classes. We expect that, even at the end of the school year, students will have better performance on the low cost mental simulation problems than on high cost ones since they will continue to encode the problems in the same way they initially did, leading to higher performance rates when the informal solving strategy is efficient. We expect that this difference in performance will also be present on Compare 1 problems.

Method

A total of 269 second-grade students from 13 classes coming from 7 elementary schools from working-class neighborhoods in France passed the second test, all of whom had passed the first experiment. The average age of these students in June was 8.02 years ($SD = 0.33$, 138 girls). The materials, design, and procedure were strictly identical to the first experiment.

Results

A first analysis was conducted in order to compare children's success rates on low and high cost mental simulation problems at the end of the school year, followed by analyses regarding the progression since the first experiment. We constructed a generalized linear mixed model (GLMM) with a binary distribution with the cost of mental simulation (low vs. high) as the fixed factors, while participants and problem categories were included as the random effects. In accordance with our hypotheses, the analyses revealed a highly significant main effect of the cost of mental simulation on performance ($\beta = 1.33$, $z = 12.22$, $p < .001$). Students succeeded on average on 59.8% of the low cost mental simulation problems and on 37.6% of the high cost mental simulation problems. The low cost mental simulation problems had a 1.59 times higher success rate than high cost mental simulation problems.

Furthermore, we tested our hypothesis on each problem category by conducting GLMMs with a binary distribution and the cost of the mental simulation as the fixed factor, and subjects as the random effect. for each of the eight problem categories. The results revealed that all of the low cost mental simulation problems were significantly easier than the corresponding high cost mental simulation problems ($0.51 < \beta < 1.8$, $2.05 < z < 6.82$, $p < .01$), except for the Combine 1 problem. This included the Compare 1 problem (44.7% success rate on the low cost mental simulation problem and 23.1% on the high cost mental simulation version) ($\beta = 0.99$, $z = 3.74$, $p < .001$). Indeed, on problem categories for which there was a significant difference, the low cost mental simulation problems were 1.36 to 2.62 times easier than their high cost counterparts at the end of the school year (cf Figure 6). The single exception remained the Combine 1 problem for which no significant difference was observed ($\beta = 0.24$, $z = 0.89$, $p = .35$), (77.4% success rate on low cost mental simulation and 72.8% on the high cost mental simulation problem).

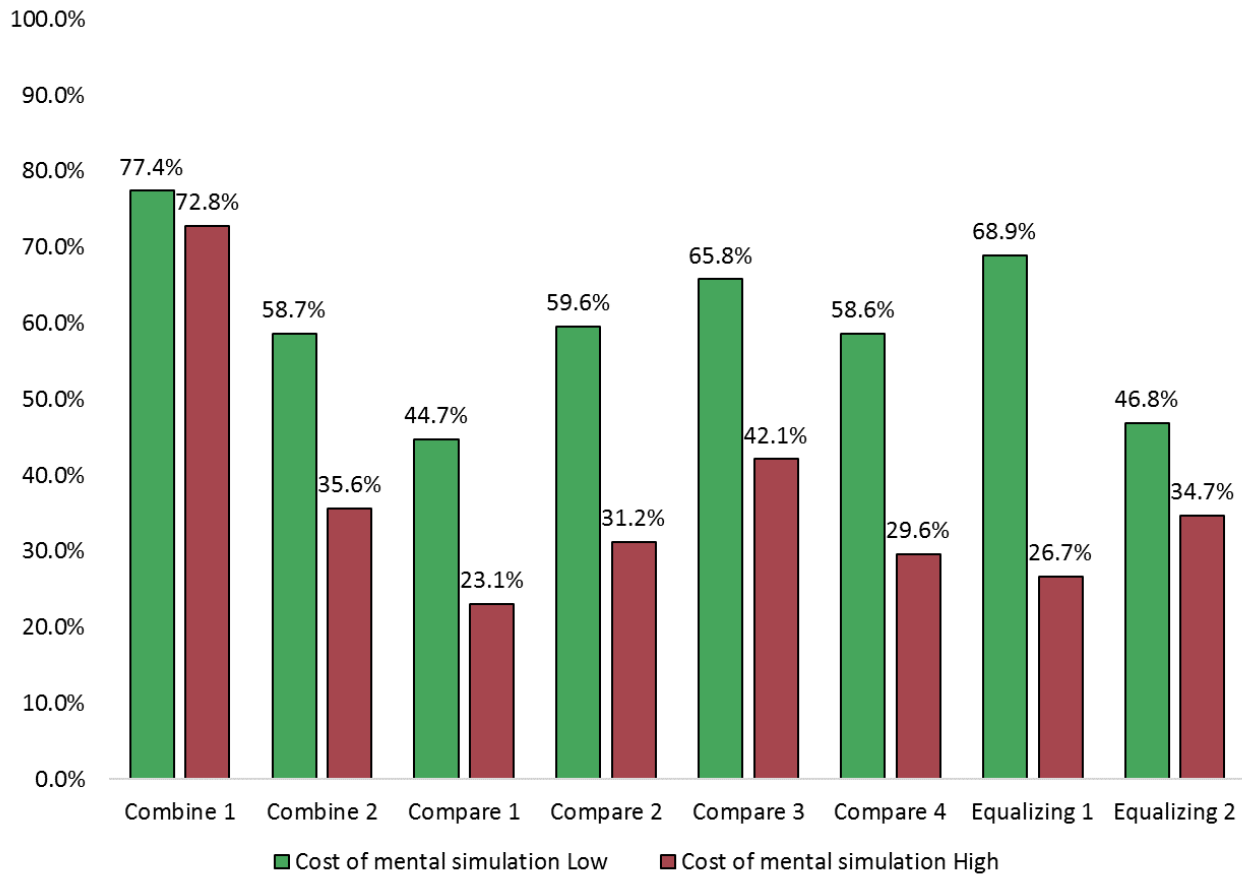


Figure 6: Performance per problem category in June

Discussion

The findings from this experiment testify to the role of mental simulation in solving arithmetic word problems influenced by the problem's encodings. Even though the wording of the problems did not describe action sequences, the presented problems differed in the efficiency of the mental simulation. Indeed, more school experience in mathematics was not sufficient to change how the presented problems are encoded as it was hypothesized according to different arithmetic conceptions. At the end of the school year, it was even on Compare 1 problems that students would solve the low cost mental simulation version more easily than the high cost one, which was not the case at the beginning of the year. The only case where there was no difference in the efficiency of the mental simulation following the encoded representation was on Combine 1 problems. These problems already had a markedly high success rate in the first experiment, as well as for the second one. Furthermore, as argued in the previous experiment, the wording of the problem is also more favorable for encoding a representation compatible with a part-whole conception of arithmetic, through which the commutativity principle can be easily applied. Overall these findings agree well with our proposal about the encoding and mental simulation processes on arithmetic problem solving, since it gives a central place to the problem's representation as a determinant for the efficiency of the mental simulation, instead of schema activations and computational strategies.

Experiment 3

In order to test if the encoded representations in our first two studies correspond indeed to those that were hypothesized, we went on to see if the gap between low and high cost mental simulation problems would prevail on students' performance on non-word problems. We expected that the non-word problems would also be represented as influenced by a specific conception and mentally simulated in the same way as word problems. We predict that the part-whole conception would be activated in the encoding of direct addition and direct subtraction problems, while the searching for the difference conception would influence the encoding of indirect addition and indirect subtraction problems. We then created versions of each problem for which the mental simulation would either be of low or high cost following these representations. If the participants successfully solved low cost mental simulation problems for direct arithmetic operations, then this indicates that they have the necessary strategies for solving high cost mental simulation problems of indirect arithmetic operations available to them, and vice versa for low-cost mental simulation problem of indirect operation. If having access to a solving strategy would have been the main factor that influences a problem's difficulty on high cost mental simulation problems, then we should observe no difference in performance rates between low and high cost mental simulation problems. However, since we expect that it is not the strategies that solvers lack, but rather that they attempt to mentally simulate the encoded representation and do not recode the problem, we expect higher performance on low cost mental simulation problems than on high cost mental simulation problems.

Method

Participants. 75 Grade 2 students from 4 classes coming from 2 different schools from working-class neighborhoods in France participated in the study. The test occurred in June and the average age of the children at the moment they passed the test was 8.01 years (*SD* = 0.34, 32 girls).

Material. There were 8 addition and subtraction arithmetic problems used in the experiment. They corresponded to all the computational possibilities in the first two experiments. Thus, number triplets involved in the data and the solution are (31, 27, 4), (33, 29, 4), (41,38, 3), and (42, 39, 3). Table 3 represents an overview of the used items.

Table 3: Example of the problems read out-loud for the number set (31, 27, 4)

	Problem type	
Mental calculation strategy	Low cost mental simulation	High cost mental simuation
Direct addition	27 + 4 =	4 + 27 =
Direct subtraction	31 - 4 =	31 - 27 =
Indirect addition	27 + *how much* = 31	4 + *how much* = 31
Indirect subtraction	31 - *how much* = 27	31 - *how much* = 4

Design. Children solved all 8 problems. To control for the impact of position, numerical sets and context, 4 different problem sets were created.

Procedure. The experiment was administered in the students' classrooms. Each child received an 8 page booklet. There was a square in the middle of each page in which they wrote their answer. Each arithmetic calculation was read aloud twice to the whole classroom and children had to write down the number that was the solution, corresponding to the same administration format as the arithmetic word problems in the first two experiments.

Scoring. The solutions provided by the children were scored with 1 point when the numerical answer was exact, or within the range of plus or minus one of the exact value, in order to take into account mistakes in counting. Any other answer received 0 points.

Results

We constructed a generalized linear mixed model (GLMM) with a binary distribution with the cost of mental simulation (low vs. high) as the fixed factors, while participants and calculation strategy presented were included as the random effects. In accordance with our hypotheses, the analyses revealed a highly significant main effect of the cost of mental simulation on performance ($\beta = 1.61$, $z = 7.25$, $p < .001$). Students succeeded on average on 67% of the low cost mental simulation problems and on 40.3% of the high cost mental simulation problems. The low cost mental simulation problems had a 1.66 times higher success rate than high cost mental simulation problems.

Furthermore, we tested our hypothesis on each calculation type by conducting GLMMs with a binary distribution and the cost of the mental simulation as the fixed factor, and subjects as the random effect. for each of the four calculation types. The results revealed that all of the low cost mental simulation problems were significantly easier than the corresponding high cost mental simulation problems ($1.29 < \beta < 2.87$, $2.72 < z < 4.08$, $p < .01$), except for direct addition. Low cost mental simulation problems were easier than their high cost counterparts 2.1 times on indirect addition problems, 2.16 times on indirect subtraction problems and 3.05 times easier for direct subtraction problems. Just as it was the case for the Combine 1 word problem in the first two experiments, direct addition problem was an exception and no difference was observed among the two problem types ($\beta = 0.59$, $z = 1.17$, $p > .01$), for which there seemed to be a ceiling effect (82.67% success rate on low cost mental simulation problems and 76% on high cost mental simulation problems).

Discussion

The findings support our proposal regarding the encodings of arithmetic problems and the mental simulations that they lead to. This third experiment converged with the evidence that even non-word arithmetic problems undergo a mental simulation of the encoded representation, since the low cost mental simulation problems were indeed much easier than the high cost ones. It has previously been demonstrated that switching flexibly between direct subtraction and indirect addition is not easy for students (Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2012). These findings do however point to the fact that it is not a lack of strategies that cause students to have significantly lower performance rates on high cost mental simulation problems, but rather the lack of conceptual knowledge that would make it possible for solvers to use a different solving strategy.

Our current proposal can provide an explanation for the kind of processes that need to be put in place in order to succeed in flexibly switching between different solving strategies. It is interesting to observe that the highest gap between performance rates on low and high cost mental simulation problems was on the direct subtraction problem. Indeed the minus sign of the direct subtraction is closely related to a taking away conception of subtraction (van den Heuvel-Panhuizen & Treffers, 2009), and might make it the most difficult for solvers to attempt to detach from the initially encoded representation and recode the problem in line with a different conception into a new representation that they would mentally simulate to determine the difference between the two quantities. On the other hand, the smallest, and non-significant, gap in performance on the low and high cost mental simulation problems was on direct addition problems. It might also be the case that this problem format is the most compatible for understanding part-whole relations, and is the most in coherence with the part-whole arithmetic conception, favoring a flexible encoding that the solver can mentally simulate in an efficient way. Yet more information about how solvers proceed with their resolution is needed in order to strengthen our proposal.

Experiment 4

A fourth experiment was conducted in order to provide evidence for the informal and formal solving strategies, based on the encodings of the problems' representations that are mentally simulated. For this purpose, we conducted verbal protocols concerning the strategies students actually use when solving arithmetic word problems which would reflect the way they encode a problem. The verbal protocols can reveal if low cost mental simulation problems are easier because they are actually solved through informal strategies, where the mental simulation operates on the initial encoding. It is also a good way to reveal if high cost mental simulation problems are indeed harder because students re-represent the problem in order to use formal strategies to solve it. Among the formal strategies, we can find (1) re-representation strategies, solving strategies which require the application of either the complement principle on subtraction problems, either the commutativity principle on addition problems; and 2) other types of strategies that differ from the mental simulation of the initial encoding. If informal and arithmetic strategies were indeed used to a different extent, this would provide confirmatory evidence that the difference in difficulty between low and high cost mental simulation problems actually results from an initial, non-flexible encoding of the problem.

We predicted that the informal strategies which directly simulate the initial encoding of the problem would be predominant for low cost mental simulation problems, but that the use of formal strategies would reflect a recoded representation and would be observed mainly on high cost mental simulation problems.

Method

Participants. 42 Grade 2 students from 4 classes coming from 2 different schools from working-class neighborhoods in France participated in the study. The test occurred in June and the average age of the children at the moment they passed the test was 7.93 years (*SD* = 0.26, 23 girls). None of the participants participated in the previous experiments.

Material & Design. The same material and design were used as in the first experiment.

Procedure. The procedure was identical to the first experiment, except that the test was conducted individually with each student in a separate classroom, and after writing down the numerical answer, each student was asked to explain aloud how he or she found the solution and their oral responses were recorded.

Scoring. The solutions provided by the children were scored with 1 point when the numerical answer was exact or, in order to allow for mistakes in counting procedures, within the range of plus or minus one of the exact value. The informal strategies corresponded to the mental simulation of the initial encoding, and the formal strategies corresponded to strategies that did not correspond to the initial encoding. Table 4 provides an overview of the informal solving strategies that directly correspond to the encoded representation and other number sentences that students used. Two separate coding were done, one for the informal strategy and one for the formal strategy. When the student provided an informal strategy, this was scored as 1 point for the informal strategy. When a student described a formal solving strategy, this was scored as 1 for the formal strategy. . No points were attributed if a student did not provide a correct response and/or did not describe any strategy after providing the correct answer (only 7.5% of the correct responses were not accompanied by a strategy description).

Two coders evaluated the solving strategies of 10 students by writing down the number sentence they considered corresponds to the descriptions children gave. The initially obtained inter-rater reliability was 98.75% with the Cohen's kappa score of 0.982, providing an almost perfect level of agreement. The only divergent case was when the student provided an accurate answer but reported number that was not stated in the problem (reported taking away 8 instead of 38). The coders attributed this to an error in number restitution (given that the student provided the correct answer and could have not done that had he used the reported value). This kind of explanation provided by students occurred twice in the overall population and the strategy was coded as if there was no error in the restitution of the numbers provided in the problem.

Table 4: Classification of strategies for each problem type

Problem category	Cost of mental simulation	Informal strategies	Formal strategies

Combine 1 [b + b'= .]	Low	27 + 4 = □		4 + 27 = □	
	High	4 + 27 = □		27 + 4 = □	
Combine 2 [b + . =a]	Low	27 + □ = 31	31 – □ = 27	31 – 27 = □	□ + 27 = 31
	High	4 + □ = 31	31 – □ = 4	31 – 4 = □	□ + 4 = 31
Compare 1 [b + □ =a]	Low	27 + □ = 31	31 – □ = 27	31 – 27 = □	□ + 27 = 31
	High	4 + □ = 31	31 – □ = 4	31 – 4 = □	□ + 4 = 31
Compare 2 [a - . =b]	Low	31 – □ = 27	27 + □ = 31	31 – 27 = □	□ + 27 = 31
	High	31 – □ = 4	4 + □ = 31	31 – 4 = □	□ + 4 = 31
Compare 3 [b + b'= .]	Low	27 + 4 = □		4 + 27 = □	
	High	4 + 27 = □		27 + 4 = □	
Compare 4 [a - b= .]	Low	31 – 4 = □		4 + □ = 31	31 – □ = 4
	High	31 – 27 = □		27 + □ = 31	31 – □ = 27
Equalizing 1 [b + . =a]	Low	27 + □ = 31	31 – □ = 27	31 – 27 = □	□ + 27 = 31
	High	4 + □ = 31		31 – 4 = □	□ + 4 = 31
Equalizing 2 [a - . =b]	Low	31 – □ = 27	27 + □ = 31	31 – 27 = □	□ + 27 = 31
	High	31 – □ = 4	4 + □ = 31	31 – 4 = □	□ + 4 = 31

Results

First, the experiment replicated the previous findings, confirming that low cost mental simulation problems were easier for children than high cost mental simulation problems. Respectively the overall success rates were 63.69% and 39.28%. A GLMM with a binary distribution, with the cost of the mental simulation as the fixed factor and participants and the problem categories as the random effects, confirmed this difference was significant ($\beta = 1.4$, z = 789.6, $p < .001$) The performance for each problem type is available in Figure 7

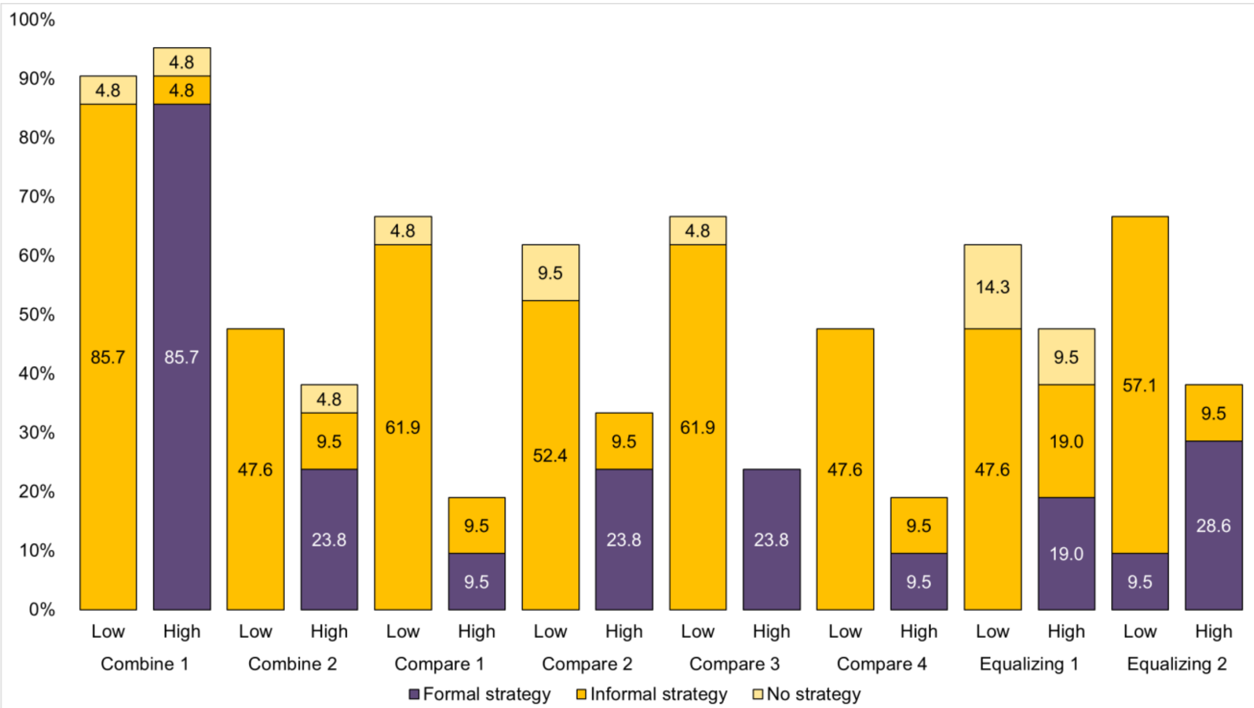


Figure 7: Rate of use of informal and formal strategies

We further conducted two GLMMs, one with the informal strategies and one with the formal strategies. Both were GLMMs with a binary distribution and the cost of mental simulation as the fixed factor and participants and the problem categories as the random effects. The average scores for each strategy type are presented in Figure 7. As predicted, both differences were significant. Informal strategies were used significantly more on low cost mental simulation problems than on high cost mental simulation problems ($\beta = 3.12$, z = 8.04, $p < .001$). Formal strategies were used significantly more on high cost mental simulation problems than on low cost ones ($\beta = -3.95$, z = -4.99, $p < .001$). The rate for both strategies obtained on each problem type among students that gave a correct numerical response are presented in Figure 8.

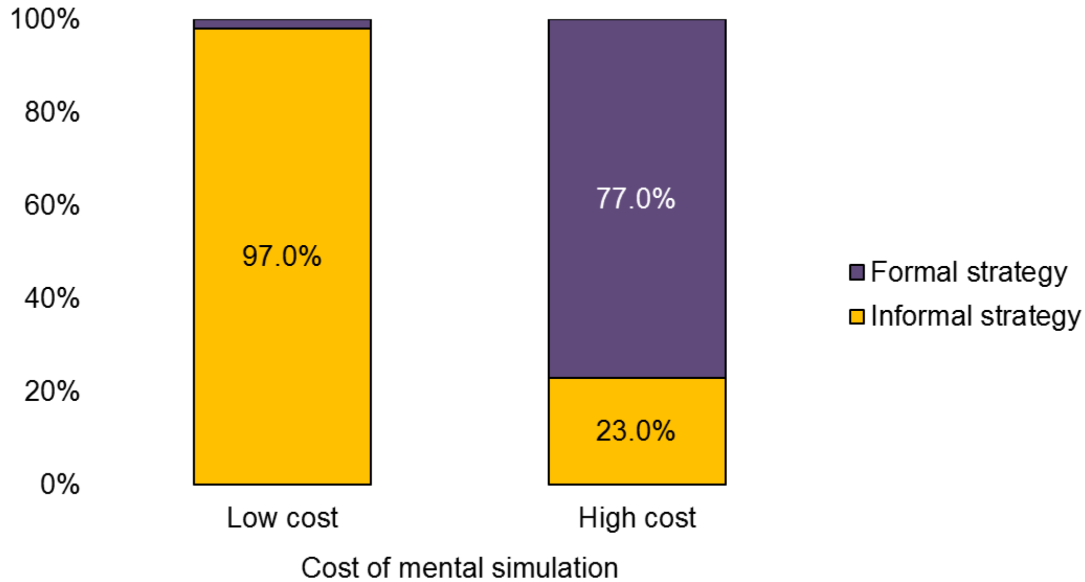


Figure 8: Rate of different strategy use on correctly solved problems with strategies

Discussion

The findings from the verbal protocols provided evidence for the hypothesized encoding of the presented word problems. Firstly, the analyses of the reported student strategies confirmed that informal solving strategies that directly simulate the encoded representation were predominant for low cost mental simulation problems, while arithmetic solving strategies revealing a recoded representation were predominant for high cost mental simulation problems. Secondly, when taking a closer look at the solving strategies used on each problem category, we can see through the frequency of informal solving strategies on low cost mental simulation problems, that the encoded representations did give place to solving strategies that correspond to the proposed arithmetic conceptions through which they are encoded, and almost never to the re-represented format. On the other hand, we can see through the frequency observed of arithmetic strategies that their use mainly relied on a re-representation of the initial encoding. It is not problematic that some students still stick with the initial encoding on high cost mental simulation problems and use informal strategies since their overall success rate was much lower than the one on low cost mental simulation problems. The informal strategy consisting in the mental simulation of the initial encoding remains possible, however, bears a high cost, and the initial encoding could have also given place to different calculation strategies following the same format. Together, findings from this experiment testify that the difference in difficulty between low and high cost mental simulation problems stems from the mental simulation of the encoded representation.

Discussion of chapter 5

Various accounts exist for the central role of a solver's problem representation for finding the solution to an arithmetic word problem (see Thevenot & Barrouillet, 2015). Notably it is proposed that the solving process unfolds in parallel with the described actions (Reusser, 1990b). Nevertheless, it remains unclear how this representation is encoded when the description of a problem does not involve a dynamic sequence of actions. The literature is also convergent about the existence of re-representational processes when the initial solving strategies are not efficient (Brissiaud & Sander, 2010; Peters et al., 2013). The present study investigated how the initial representation of a problem influences the efficiency of the mental simulation used to solve it, accounting for the informal strategies used for solving arithmetic problems, and provides evidence for the re-representational processes that take place when this mental simulation is costly. We proposed that the problem's presented format triggers different conceptions of arithmetic that determine the encoding of the arithmetic word and non-word problems. In order to test these proposals, we conducted classroom studies in four experiments with second-grade students. First, we demonstrated that the efficiency of the mental simulation does depend on the problem's encoded representation, suggesting that the low performance rates on problems that cannot be easily simulated have a high cost because they require the solver to re-represent the problem in order to attempt an easier mental simulation. Second, we provided evidence for the persistence of the aforementioned phenomenon: even after subsequent instruction during the school year, students still better solved problems that have low mental simulation cost than they did high cost ones. Third, we showed that the superiority induced by the efficiency of the mental simulation occurs even on non-word problems: the format of the problem triggers a specific encoding that appears to be the basis for a mental simulation, the efficiency of which determines the easiness of resolution. Fourth, we provided evidence for the initial encoded representations and re-representations that are mentally simulated by addressing students' verbal reports of the strategies they used. Overall, the experiments conducted in the present indicate that the process behind student's use of informal solving strategies is a non-mathematical mental simulation of the encoded representation, while the use of arithmetic strategies is dependent on the recoding of the initial representation done through the use of a different arithmetic conception.

The significant difference that was observed between low and high cost mental simulation problems fits with the previous findings on Change problems (Brissiaud & Sander, 2010), and confirms that the mental simulation does not only occur when the problem illustrates a dynamic sequence of events. Contrary to Kintsch & Greeno's (1985) schema theory of problem solving which considers that the "abstract-problem representation, the problem model, which contains the problem-relevant information from the text base in a form suitable for calculation strategies that yield the problem solution" (p. 111), our findings provide evidence that the abstract representation constructed based on a problem's textual input is not always suitable for finding the solution. Our studies illustrated that, even though the construction of magnitude-based mental representations provides an explanation for how the relationships between the described elements are preserved (Orrantia & Múñez, 2013), it cannot explain why problems with the same numeric values have such a difference in performance rates. And even though the Switch model (Peters et al., 2013) could have predicted that the high cost mental simulation problems are more efficiently solved if the solver uses a different strategy then the format they are encoded in, the processes behind this strategy switch have not been demonstrated. The systematic gap between success rates among low and high cost mental simulation problems could not be explained accumulating experience in problem solving, such as could be expected in certain strategy selection models in mental calculation, for instance the Strategy Choice and Discovery Simulation model (SCADS) (Shrager & Siegler, 1998) or SCADS* (Siegler & Araya, 2005). These models would predict that the combination of associative and metacognitive processes would be sufficient to generate adaptive choices in strategy use when they are available, and our results indicated that it is not the strategies that students lack, but rather their inability to select them with a given problem. Instead, our findings indicate that there are more complex processes involved in the encoding of an arithmetic problem and its mental simulation.

To a certain extent, our findings do challenge the traditional classification of arithmetic word problems according to which a problem's difficulty depends mostly on the problem category determined by their semantics (Riley et al., 1983). Yet, in all our experiments on word problems, we found that there was one problem category that was particularly easy for students, the Combine 1 problem, indicating that there are still important semantic factors at play. For this problem category, the success rates of the low cost mental simulation version were the highest in each experiment, and more importantly, the ratio between the success rates on low and high cost mental simulation problems was the lowest. We argued that the wording of this problem has semantic characteristics that are the closest to the part-whole conception of arithmetic, since it describes two parts joined together to form a larger quantity. The consistency between the semantic nature of the problem and the arithmetic conception may indeed allow the solver to encode a more flexible representation of the problem, such as the ones that usually undergo a recoding, allowing the solver to easily access different mental simulations of the problem. This was even observed on non-word problems involving direct addition.

A conceptual understanding necessary for achieving mathematical proficiency involves the comprehension of concepts, operations, and relations (Baroody et al., 2009) and it is crucial for flexible behavior in mathematics (Baroody & Dowker, 2003). As for the relation of conceptual knowledge to procedural knowledge, be it the concept-first view, inactivation view or iterative view, conceptual knowledge has a great importance in contributing to the increase of procedural knowledge (Rittle-Johnson & Schneider, 2014; Rittle-Johnson et al., 2001). Distinguishing the processes behind informal and arithmetic solving strategies through the efficiency of the mental simulation stresses that low cost mental simulation problems do not mobilize a rich conceptual knowledge of mathematics. Yet the procedures used on solving these problems can be useful for enriching the conceptual knowledge students have, which would in turn lead to an amelioration of their arithmetic knowledge. For example, working on students' informal strategies on low cost mental simulation problems can be a good tool for developing the students' repertoire of solving strategies, whereas working on high cost mental simulation problems would be a more appropriate tool to work on students' conceptual understanding of arithmetic. This should help students remediate to what was identified as a restricted conceptual representation of addition and subtraction (Stern, 1993). It is also a good way to not simply rely on the distinction between abstract and concrete representations, but actually connect representation to previous types of understanding (Lampinen & McClelland, 2018).

Several studies have documented the beneficial effect of learning by comparing strategies (Gentner et al., 2003; Rittle-Johnson & Star, 2007). As stated in the introduction, for the different problems that have been studied, there is a common arithmetic structure while it is only the encoded format stemming from the semantic characteristics and arithmetic conceptions that is different. There is evidence that studying problems sharing the same formal structure by comparing the solving procedures used to solve them can lead to the transfer of strategy use (Gamo et al., 2010). Also, comparing informal and formal procedures on problem solving in mathematics can lead to greater gains in conceptual knowledge among learners who do not like mathematics (Hattikudur et al., 2016). In this view, our findings suggest that there is a great interest in studying Combine problems in order to attain proficiency in different strategies, but also to work on the conceptual aspects behind the recoding into a different representation that comes easier to students on these than on other problems. This can further help students as a support for them to solve problems for which there is a greater gap between informal and arithmetic strategies. Additionally, since the wording of Change problems has a semantic structure that is the closest to the intuitive conception of taking away, we believe there would be more benefit of studying these problem categories after students have already gained certain procedural and conceptual efficiency which should help them more easily recode the initial encoding that was strongly influenced by the intuitive conception of subtraction.

CHAPTER 6 – OVERCOMING INFORMAL SOLVING STRATEGIES [3]

ARITHMETIC COMPREHENSION IN ELEMENTARY SCHOOL (ACE) INTERVENTION PROGRAM

In this chapter we will look at how working on the encoding of a problem's representation and its recoding has been put in place in an intervention program called 'Arithmetic Comprehension in Elementary school' (ACE). ACE is a math research-based teaching program developed in France with the objective to promote arithmetic teaching in first grade math classes (Fischer, Sander, Sensevy, Vilette, & Richard, 2018; Joffredo-Le Brun, Morellato, Sensevy, & Quilio, 2018; Vilette et al., 2017). It was designed to replace the regular arithmetic curriculum in math classes. Its elaboration followed the requirements of the French National Education's official program for teaching math. It encompassed 150 hours of arithmetic teaching, just like in regular mathematics classes. ACE was composed of four major teaching domains – number acquisition, word problem solving, estimation and mental calculation. Teachers of the participating classes took part in a professional development program. This training took place during two days at the beginning of the school year when they were also given the needed materials for implementing the program in their classrooms. They later attended 5 half-day sessions throughout the school year during which they met with the researchers leading the project and discussed their practice.

Previous studies have already supported the general effectiveness of the ACE program, testifying to long-term (Vilette et al., 2017) and cumulative benefits among students who participated (Fischer et al., 2018). All domains were treated in classes throughout the school year, however, the present study focuses on the twofold objectives specific to the word problem solving domain: introducing students to the analysis of the semantic content in arithmetic problem solving tasks, and working on semantic recoding in order to favor the re-representation of the situation. Overall, the problem solving domain represented about a quarter of the arithmetic curriculum. Although other subdomains of the program may have influenced student overall performance on arithmetic word problem solving, it is the problem solving domain that was specifically designed to develop students' adaptive expertise by using the most efficient strategies regardless of the semantic influence induced by the problem statement.

[3] The results presented in this chapter have been included in: Gvozdic, K., & Sander, E. (accepted). Learning to be an opportunistic word problem solver: Going beyond informal solving strategies. *ZDM Mathematics Education*.

Among the different approaches to mathematics instruction, the problem-solving approach considers mathematics to be a way of thinking and a search for patterns in order to find solutions to problems and focuses on reasoning and problem solving as a form of incidental learning (Baroody, 2003). It is different from approaches that focus mostly on skill learning where instruction is usually done without context. The problem solving curriculum within the ACE intervention was mainly inspired by findings from the Situation Strategy First framework (Brissiaud & Sander, 2010) and work on semantic recoding (Gamo et al., 2010). It started by engaging students in the analysis of the relations described in word problems, firstly with Combine problems – where two elements are combined to form a whole, and the question bears either on one of the parts or on the whole. Part-whole relations are essential for understanding commutativity, associativity and the complementary relation between addition and subtraction on problem solving (Resnick, 1989). Therefore, these problems, due to their static nature, were considered to optimally encourage the mental representation of part-whole relations, favorable for later generalizations and applications of arithmetic operations in different situations (Sophian, 2008). Secondly, compare problems, where two quantities are compared and the question bears on their difference, were introduced. Lastly, Change problems, depicting an action sequence leading to the change of one quantity over time, were studied. These problems were considered to most strongly solicit a mental simulation of the described situation since their semantic description already contains action verbs which describe a dynamic change. The semantic analyses of the problems discussed in class lead students to consider the abstract relations between the elements presented in problem solving tasks and work on the semantic encoding of the situation: after reading the problems the teachers would systematically discuss with the students what were the known and unknown quantities described in the problem, what were their relations and what needs to be done in order to find the unknown quantity. The number sentence which models the analyzed situation was then written down.

Two tools were introduced when working on the different problem categories in order to favor a conceptual generalization of the features encoded on the different problems: the number line and the number box (Figure 9). The number line was used to schematize part-whole relations (Verschaffel et al., 2007; Wolters, 1983). The semantic relations described in the problems were represented with arches on the number line: the largest arch would represent the largest quantity and two smaller arches would represent the two quantities that make up the largest one (Figure 9).

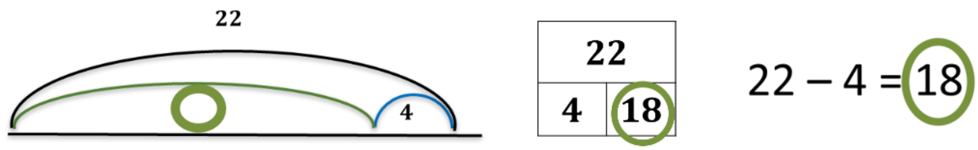
The number box designed for the ACE program was a rectangular representation divided into three parts – the bottom half divided into two parts and the upper half. The upper half would contain the largest quantity presented in the problem, while the lower half would contain the two quantities, that, when added together, would be equal to the largest value (Figure 9). This kind of representation was introduced in order to provide an abstract representation of the part-whole relations, which can be found in all one-step additive word problems (Riley & Greeno, 1988). When using both representational tools, the students would draw a blank circle in the position of the unknown quantity.

A key curricular phase was the semantic recoding phase. Teachers presented various high cost mental simulation problems. The relations described in the problem were analyzed, as it had been done in class until then, and the number sentence modeling the problem was written down along with the solution. The students were then urged to search if there is an easier way to find the solution. Rather than explicitly focusing on the equivalence of the arithmetic operations of direct subtraction and indirect addition for the various possible procedures, the informal and arithmetic strategy were then discussed with a focus on how they relate to the problem's semantics and solving strategies (see bottom part of Figure 9). Students were asked to choose which solving strategy was the easiest to perform in order to provide a correct response. Previous interventions have shown that creating and discussing student-invented solution procedures is beneficial for acquiring multiple solving strategies (Blote et al., 2001) and that the instruction on multiple procedures rather than single-procedure instruction favors conceptual understanding (Alibali & Rittle-Johnson, 1999). This specific kind of activity through which the different student-proposed strategies were discussed would allow students to reconsider their initial representation of the situation and informal strategies and to *re*-represent the situation in order to perform a more favorable solving procedure (Gamo, et al., 2010). For instance, a problem such as the one in which Luc loses 18 marbles from his 22 marbles would be recoded in such a way that the student would look for the number that should be added to 18 when trying to reach 22, instead of looking for the result after taking away 18 from 22 (Brissiaud & Sander, 2010; Peters et al., 2013). The semantic recoding activity was a key component of the problem solving domain in ACE since they were worked on in 30% of the lessons within the domain.

A. Low cost mental simulation problem

Luc is playing with his 22 marbles at recess. During the recess, he loses 4 marbles. How many marbles does Luc have now?

Informal situation-based solving strategy

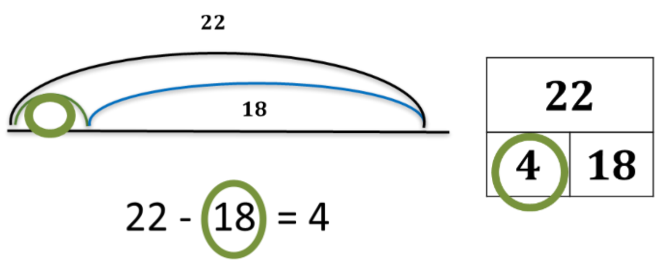


The informal situation-based strategy is to simulate the action described in the text: losing 4 marbles. It, therefore, consists in mentally counting down 4 from 22. This is then noted as the subtraction 22 – 4. In the case of this low cost mental simulation problem, this mental simulation is easy to perform.

B. High cost mental simulation problem

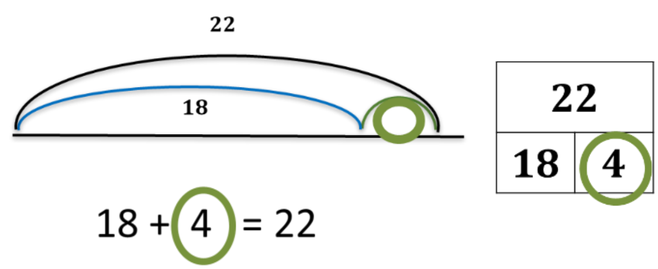
Luc is playing with his 22 marbles at recess. During the recess, he loses 18 marbles. How many marbles does Luc have now?

Informal situation-based solving strategy



The informal situation-based strategy is to simulate the action described in the problem: losing 18 marbles. It therefore consists in mentally counting down 18 from 22. This is noted as direct subtraction 22 – 18. In the case of the presented problem this is too costly to perform by mental simulation, so the students were asked if there is a different way in which they could find the solution.

Formal solving strategy



The formal arithmetic strategy relies on disengaging from the semantic context and changing how the situation is addressed, leading students to solve the problem by searching for the distance between 18 and 22. This entails recoding a direct subtraction situation into an indirect addition which is then noted as 18 + 4 = 22. Students are then asked which strategy they prefer to use on this problem.

Figure 9: Descriptions of the solving strategies studied in class

A. Working with the representational tools in the ACE problem solving domain.

B. Outcome of semantic recoding on a high cost mental simulation problem by comparing the informal and formal solving strategies.

AIM OF THE STUDY

The current study investigated the influence of the encoded representation on students' arithmetic word problem solving strategies and more specifically the ability to recode the initial representation when it is opportunistic. Since the solving procedures behind informal strategies can sometimes be costly, it is important to know how to use a different solving strategy when this is the case. Semantic analysis and recoding activities were considered to enhance the availability of formal strategies, and therefore higher performance as a reflection of the greater use of formal strategies. We first predicted that students would have better performance on problems for which the use of informal strategies easily provides a solution than on problems for which the informal strategy is costly. This first prediction was also suitable for a replication of the empirical findings from Chapter 5 and from Brissiaud and Sander's (2010) study in the business as usual (BAU) classes. Secondly, we predicted that students who participated in the ACE program would have better overall performance than the BAU group, since they had extensive practice in analyzing the abstract relationships between the elements presented in the problem and in practicing semantic encoding, allowing them to mentally simulate each problem more easily. Yet, when it comes to high cost mental simulation problems, having facilitated access to the encoded representation is not enough to provide an answer since solving the problem with an informal strategy is costly. On these problems, using formal strategies is efficient. Our third prediction was therefore that formal strategies would be used to a greater extent on high cost mental simulation problems than on low cost mental simulation problems in both groups, since it is on these problems that they provide a considerable gain in strategy efficiency. And fourth, we expected the ACE group to use formal strategies more frequently than the BAU group, since the extensive training in semantic recoding would have developed their ability to use the most appropriate solving strategy. We also expected that in the ACE group this greater use of formal strategies would be observed notably on the high cost mental simulation problems. In addition to their performance on formal word problems, ACE students were expected to have gained proficiency in part-whole reasoning more generally and were therefore expected to perform better on a transfer task regarding it.

METHOD

Participants

The current study was conducted in 10 first-grade classes with a total of 215 students. Seven students were eliminated from the analysis on the criteria that they did not successfully complete the current school year and were to be held back in the same grade the following school year. In total the analyses were conducted on 208 participants (5 ACE classes with 103 students, mean age = 7.05, SD = 0.28, girls = 60; 5 BAU classes with 105 students, mean age = 7.03, SD = 0.31, girls = 57). The elementary schools were from working-class neighborhoods in France. The experimental and control classes were paired according to the socio-economic status of the population and academic performance of the schools by the regional school inspector from a large sample of participating classes. The teachers who participated in the intervention program, as well as teachers from BAU classes who participated in the current study, did so on a voluntary basis.

Material

Arithmetic word problems

Six addition and subtraction problem categories were tested. According to Riley et al.'s (1983) classification, these problems were: Combine 1, Combine 2, Compare 1, Compare 2, Change 2, Change 3.

The subtraction problems involved two numbers, *a* and *b* (*a* > *b*). The numerical values for *a* were either 11, 12, 13, 21 or 22, while in order to differentiate between low and high cost mental simulation problems the values for *b* were either kept small (2, 3 or 4) or were close to *a* (8, 9, 18 or 19). To create low cost mental simulation problems, the small value of *b* was used for the Change 2 problem, while the *b* value close to *a* was used for Compare 2, Compare 3, Change 3 and Combine 2 problems (cf. Table 5). To create high cost mental simulation problems, the opposite *b* value was respectively used for each problem, since it would make the informal situation-based strategy costly to execute. The addition Combine 1 problem involved two numbers, *b* and *b'*. Both numbers had the same characteristics as *b* for subtraction problems, while the unknown value was equivalent to *a*. To create low cost mental simulation problems, the *b* value close to *a* was presented first, while the small *b* value (*b'*) was presented second. To create high cost mental simulation problem, they were presented in the opposite order. The high cost mental simulation problems created in this way required the application of the commutativity principle if they were to be solved in the most efficient manner. This material has been used and validated in previous research which has robustly demonstrated that the number size alone was not the determining factor for the difficulty of the mental simulation (Brissiaud & Sander, 2010; Chapter 5). Three semantic contexts were used for the wording of the problems: marbles, euros, and flowers.

Control tasks

Previous studies containing over 100 classes evaluating the global and specific gains of the ACE intervention showed no advantage of experimental classes as compared to BAU during pre-tests (Fischer et al., 2018; Vilette et al., 2017). In order to limit the researchers' intervention in the schools and not take time out of the teachers' curriculum, there was no pre-test in the present study. The current sample of ACE and BAU classes was chosen by the regional inspector from the same schools evaluated in the previous studies (Fischer et al., 2018; Vilette et al., 2017). She took into consideration the socio-economic status of the population as well the academic performance of the schools, with the aim of selecting classes that were the most comparable. We also deemed necessary to introduce certain control measures in the current study and used four exercises regarding aspects of mathematics not worked on in the ACE program. The tasks were taken from standardized tests available from the French Ministry of Education. In the first task students had to copy a figure following gridlines. In the second task they had to identify different geometric forms in a complex shape. In the third exercise students needed to identify the appropriate card following a set of rules. In the fourth exercise, students had four "identify the intruder" tasks among 5 different elements. Since the ACE program covered most of the mathematics curriculum, these tasks were inevitability quite marginal in terms of the evaluated content. We expected that the two groups would not differ in their performance.

Table 5: Problems used in the study and the corresponding informal and formal strategies.

Problem category	Cost of mental simulation	Problem statement	Informal strategies	Formal strategies
Combine 1	Low	There are 7 blue marbles and 4 red marbles in Marc's bag. How many marbles are there in Marc's bag?	7 + 4 = □	4 + 7 = □
	High	There are 4 blue marbles and 7 red marbles in Marc's bag. How many marbles are there in Marc's bag?	4 + 7 = □	7 + 4 = □
Combine 2	Low	Mary has 7 euros in her piggybank and she has euros in her pocket. In total, Mary has 11 euros. How many euros does Mary have in her pocket?	7 + □ = 11 11 – □ = 7	11 – 7 = □ □ + 7 = 11
	High	Mary has 4 euros in her piggybank and she has euros in her pocket. In total, Mary has 11 euros. How many euros does Mary have in her pocket?	4 + □ = 11 11 – □ = 4	11 – 4 = □ □ + 4 = 11
Compare 1	Low	There are 7 roses and 11 daisies in the bouquet. How many daisies are there more than roses in the bouquet?	7 + □ = 11 11 – □ = 7	11 – 7 = □ □ + 7 = 11
	High	There are 4 roses and 11 daisies in the bouquet. How many daisies are there more than roses in the bouquet?	4 + □ = 11 11 – □ = 4	11 – 4 = □ □ + 4 = 11
Compare 2	Low	Pierre has 11 marbles and Jack has 7 marbles. How many marbles does Jack have less than Pierre?	11 – □ = 7 7 + □ = 11	11 – 7 = □ □ + 7 = 11
	High	Pierre has 11 marbles and Jack has 4 marbles. How many marbles does Jack have less than Pierre?	11 – □ = 4 4 + □ = 11	11 – 4 = □ □ + 4 = 11
Change 2	Low	There are 11 flowers in the bouquet. Sophie takes out 4 flowers from the bouquet. How many flowers are in the bouquet now?	11 – 4 = □	4 + □ = 11 11 – □ = 4
	High	There are 11 flowers in the bouquet. Sophie takes out 7 flowers from the bouquet. How many flowers are in the bouquet now?	11 – 7 = □	7 + □ = 11 11 – □ = 7
Change 3	Low	Mary has 7 euros in her moneybox. For her birthday, she receives more euros and puts them in her moneybox. Now, she has 11 euros in her moneybox. How many euros did Mary get for her birthday?	7 + □ = 11	11 – 7 = □ □ + 7 = 11
	High	Mary has 4 euros in her moneybox. For her birthday, she receives more euros and puts them in her moneybox. Now, she has 11 euros in her moneybox. How many euros did Mary get for her birthday?	4 + □ = 11	11 – 4 = □ □ + 4 = 11

Transfer task

In order to measure students' reasoning about part-whole relations, a transfer task was created by adapting a classical Piagetian class inclusion task to a paper and pencil format. Students were presented with two exercises containing different groups of items:

1. 5 cats and 4 dogs;
2. 9 circles and 12 triangles.

The task was to count and write how many items were in each group. Then they were asked a class inclusion question for each exercise: 1. "Are there more cats or more animals?"; 2. "Are there more triangles or more shapes?". Students provided the answer by circling their choice in the provided booklet. Since the ACE word problem solving program had a strong focus on part-whole relations, we expected that it would also have an impact on students' conceptual understanding of part-whole relations outside the mathematics domain. The ACE group was therefore expected to have better performance than the BAU group.

Procedure

Students solved a total of 12 arithmetic word problems (one problem per category in the low cost mental simulation version and one problem per category in the high cost mental simulation version). To control for the impact of position, numerical sets and context, 5 different problem sets were created. The students performed all of the control tasks and the transfer task.

The experiment was administered in the students' classroom three to four weeks before the end of the school year. Each student received a 12 page booklet for solving the arithmetic word problems. There was a square in the middle of each page in which they wrote their answer and a rectangle for them to write their solving strategy. Each problem was read aloud twice to the whole classroom and children had 30 seconds to write down the number that was the solution and what procedure they undertook in order to find the answer. The arithmetic word problems were divided into two sets of 6 problems, between which the participants solved the first three control tasks. The fourth control task and the transfer task were administered after the second arithmetic word problem series.

Scoring

Arithmetic word problems

The solutions provided by the children were scored with 1 point when the numerical answer was exact or, in order to allow for mistakes in counting procedures, within the range of plus or minus one of the exact value. Admitting values of +/-1 of the exact answer has been used in previous studies (Brissiaud & Sander, 2010; Rittle-Johnson et al., 2016). Any other answers received 0 points. Depending on the category of the problem, the strategies that the students noted were classified according to Table 5. Two coders, blind to the experimental groups, categorized 10% of the strategies randomly sampled from the cases where the participants wrote down a solving strategy. The inter-rater reliability was 100%. When a student noted an formal strategy on a given problem, it was scored with 1 point, and 0 points for the informal strategy or otherwise (meaning that no points were attributed when the participant did not provide a strategy, or when he or she simply noted the numbers from the problem statement into mathematically incorrect equations).[4]

Control tasks

Each exercise was broken down into a set of steps or subtasks, which when scored according to well-defined scales could obtain one point per exercise. For instance, on the identify the intruder exercise containing four subtasks, each received 0.25 points for the correctly identified intruder. There was no ambiguity in the coding of the correct responses. A global score was computed by adding the different sub-scores and used in the analysis. The control task score could therefore range from 0 to 4.

Transfer task

In each exercise one point was attributed if the student responded by choosing the superordinate category as the response to the categorization question. Since there were two exercises in the transfer task, the score could thus range from 0 to 2.

[4] In order to display the distribution of the strategies among the correct answers in Figure 10, we counted the number of occurrences of informal strategies and the number of occurrences where no strategy was written down.

RESULTS

Analysis of control tasks

As expected, the t-test with the control task score as the dependent variable revealed no significant difference between the two groups ($t(205.84) = 1.88$, $p > .05$, $\eta^2 = .01$). Indeed, the average score of the ACE group was 2.76 (SD = 0.63) and of the BAU group was 2.59 (SD = 0.66).

Analysis of responses and strategies on arithmetic word problems

Since the data points for responses and strategies were binary and recorded in a repeated design (with low and high cost mental simulation problems), we conducted random effects logistic regressions. Unless mentioned otherwise, participants and problem categories were included as the random effects in the models. Firstly, we focused on the participants' responses. We investigated the performance of students from the BAU group by conducting a generalized linear mixed model (GLMM) with a binary distribution with the cost of mental simulation (low vs. high) as the fixed factors. The results replicated previous research about the costliness of the mental simulation influencing the difficulty of the problems, since high cost mental simulation problems were 1.42 times harder than low cost mental simulation ones ($\beta = 0.77$, $z = 5.44$, $p < .001$). We then analyzed the response performance of both groups, by conducting a GLMM with a binary distribution with cost of mental simulation and group (ACE vs. BAU) as the fixed factors. As predicted in the first hypothesis, low cost mental simulation problems were generally easier than high cost mental simulation problems ($\beta = 0.76$, $z = 5.45$, $p < .001$). Overall, performance on low cost mental simulation problems was 52.72%, while on high cost mental simulation problems it was 40.06%. Furthermore, as predicted in the second hypothesis, the ACE group performed better overall than the BAU group ($\beta = 1.22$, $z = 5.41$, $p < .001$). The interaction between the problem type and group was not significant ($\beta = -0.00$, $z = -0.02$, $p = .98$). Students from the ACE group succeeded on 63.43% of the low cost mental simulation problems and 50.48% of the high cost ones, while the students from the BAU group had an average success rate of 42.22% on low cost mental simulation problems and 29.84% on high cost mental simulation problems. The success rates for each problem category are displayed in Figure 10.

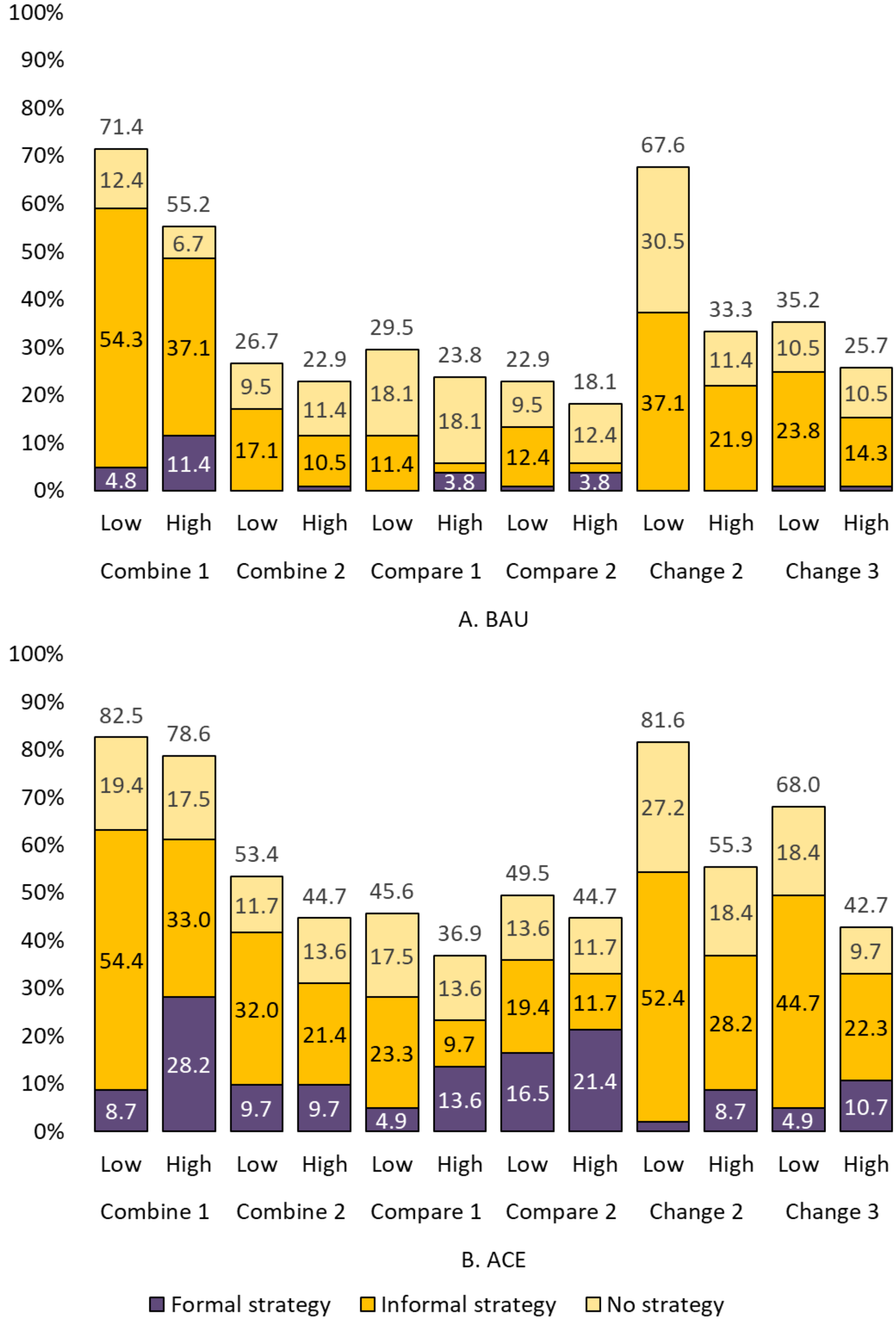


Figure 10: Success rates and distribution of strategy use per problem category and cost of mental simulation A. BAU group, B. ACE group

Further analyses were conducted on the strategies used by the students. The main analyses looked at the strategies used by students when they responded correctly to a problem. In such cases they wrote down a strategy 71.3% of the time in the ACE group and 61.67% of the time in the BAU group. With this data we conducted a GLMM with a binary distribution with the cost of the mental simulation and group as the fixed factors. Among these correct responses, there was an overall effect of the cost of mental significant ($\beta = -1.64$, $z = -7.24$, $p < .001$), confirming our third hypothesis that formal strategies are used significantly more often on high cost than on low cost mental simulation problems (cf. Figure 11). There was also an overall effect of group ($\beta = 1.44$, $z = 4.49$, $p < .001$), confirming our fourth hypothesis that the ACE group uses formal strategies significantly more than the BAU group (cf. Figure 11). Since the high cost mental simulation problems were those that would benefit the most from the use of formal strategies, in line with our fourth hypothesis, we further analyzed the strategies used on correctly solved high cost mental simulation problems. We conducted a GLMM with a binary distribution with the group as the fixed factor, and participants and the problem category as the random effects. As expected, the ACE group used formal strategies on high cost mental simulation problems significantly more than the BAU ($\beta = 1.32$, $z = 3.52$, $p < .001$). The rate of informal and formal strategies for each problem category can be found in Figure 10. Since the ACE group used more formal strategies than the BAU group, we were interested in understanding the way in which these strategies are used. For each problem category we conducted GLMMs with a binary distribution and the cost of the mental simulation as the fixed factor, and subjects as the random effect. On 4 out of 6 problems students used more formal strategies on high cost mental simulation problems than on low cost mental simulation problems ($2.4 < z < 3.8$, $p < .05$). Among the two remaining ones, there was a tendency to use more formal strategies on high cost mental simulation problems than on low cost ones for the Compare 2 problem ($\beta = -1.13$, $z = -1.74$, $p < .1$,), whereas there was no difference in the rate of use of formal strategies among the two cases for the Combine 2 problem.

Analysis of the far transfer tasks

We performed a variance analysis with the transfer score as the dependent variable. As expected, there was a significant difference between the two groups ($t(205.93) = 3.86$, $p < .001$), with the ACE group having an average score of 1.26 (SD = .97) and the BAU group 0.74 (SD = .97). This suggests that students in the ACE group developed a better understanding of part-whole relationships than the BAU group even outside the arithmetic domain.

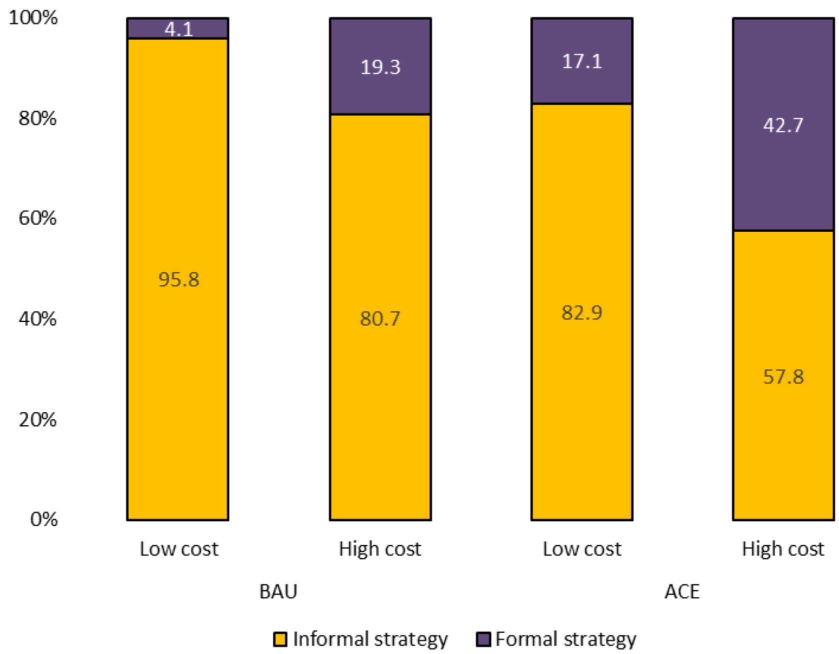


Figure 11: Rate of different strategy use on correctly solved problems

Discussion

The current study was designed to evaluate how lessons aiming for the development of adaptive expertise can promote the success on arithmetic word problems when the mental simulation does not provide an adequate strategy. Performance and strategies were compared among BAU first grade classes and those who participated in an arithmetic intervention program (ACE). The intervention was designed to teach students to be less dependent on informal solving strategies which directly model the described situation, and instead to favor formal strategies when they would be more efficient for providing the solution to the problem. This was done by engaging students in the semantic analysis of arithmetic word problems and studying why different strategies can be used to solve the problems through semantic recoding activities. Our findings first provided confirmatory evidence for our first hypothesis that problems which can easily be solved through informal situation-based strategies are easier than problems that bear a high cognitive cost if the same strategy is used. This replicated previous findings which highlight the difference between low and high cost mental simulation problems. In line with our second hypothesis, we also found that there was an overall higher performance in the ACE group. When looking at the strategies students used, in line with our third hypothesis we found that there were more formal strategies on high cost problems than on low cost problems. Furthermore, as predicted in our fourth hypothesis, students from the ACE group used more formal strategies than students from the BAU group. Lastly, our results indicated that students from the ACE group performed better when reasoning about part-whole relations.

Extending the research questions raised in the study, it would have been interesting to explore the strategies that students use when they fail to find the solution. Understanding if they fail to solve high cost mental simulation problems because they continue to use informal strategies would reflect that they do not manage to re-represent the problem. However, it should be noted that the analysis of the strategies students used on correctly solved problems was only available for about one third of all items, whereas this percentage was extremely low on problems that were not correctly solved. This does calls for caution when generalizing the interpretations of the findings, but it also leads us to consider alternative ways of collecting data on student strategies, such as verbal protocols, in order to dispose of higher rates of strategy use. It should also be noted that for most of the problems in the current study, indirect addition was the informal strategy, and therefore subtraction was the formal strategy. It is therefore possible that these confounding factors influenced student performance. Given that students had lower performance on high cost mental simulation problems, which mainly solicit the use of formal strategies, it is thus conceivable that students had lower performance on high cost mental simulation problems because they have more difficulties when using subtractions than indirect addition. Some previous research has demonstrated that when adults solve subtractions by means of indirect addition, they perform significantly better, even when it is not clear if using indirect addition provides computational gains (Torbeyns, Ghesquière, & Verschaffel, 2009). On the other hand, elementary school students hardly apply the indirect addition strategy for finding the solution to subtractions (Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009). Future research investigating semantic recoding should use experimental protocols that provide equal opportunities to use subtraction and indirect addition on low cost mental simulation problems, and symmetrically on high cost ones. This would make it possible to straightforwardly assess the importance of re-representational processes in developing adaptive expertise.

A noteworthy shortcoming of the current study is linked to the context in which it was conducted. The ACE word problem solving intervention was part of a larger research project that had high practical relevance. It was administered in the classes by teachers during regular school hours and was aligned with the learning objectives prescribed by the French National Education's official program for elementary school teaching. As for the current study, we compare the performance of classes that participated in the intervention program to classes that followed the regular curricular. We took precautions, first by pairing the experimental and control classes in the same way as it had previously been done and revealed no advantage of ACE classes (Fischer et al., 2018; Vilette et al., 2017), and second by using control tasks. Even though the difference between the two groups on the control tasks was non-significant, additional evidence would have been a better guarantee that the participating classes had comparable mathematics performance at the beginning of the school year. However, given the complex circumstances of implementing the intervention, there were time constraints on the researchers' intervention in the classrooms that precluded conducting a pre-test. This, therefore, gives the study high ecological validity but necessitated to make compromises in terms of its internal validity.

The present study provides insight into the solving processes on arithmetic word problems. As expected, students from the BAU classes very rarely used formal strategies on low cost mental simulation problems, contrary to the ACE group. However, informal solving strategies were observed on high cost mental simulation problems in both groups, but the ACE group used 2.2 times more formal strategies than the BAU group. It *is* possible to find the solution to high cost mental simulation problems through the mental simulation of the situation, yet the informal solving strategies are not the most efficient in this case, which might also have contributed to the lower performance rates in the BAU group on these problems. This stresses the importance of finding adequate means for guiding students to efficiently find the solution. The ACE intervention had the specificity of challenging students to set aside their informal strategies on arithmetic word problems: it provided students with the opportunity to use procedures in various contexts and not in isolation. Low cost mental simulation problems made it possible for students to work on their existing procedural knowledge mobilized in their informal solving strategies, while high cost mental simulation problems lead them to invent and search for alternative strategies. This might have favored the development of arithmetic conceptual knowledge while practicing known procedures, which contributes to knowledge retention (Baroody et al., 2007).

The findings from the study orientate towards a closer look at the semantic recoding approach to conceptual change in order to enhance students' use of formal strategies when solving word problems. Semantic recoding considers a change of representation to be the main key for going beyond informal situation-based strategies in favor of formal strategies (Brissiaud & Sander, 2010; Gamo et al., 2010). In the present study this approach seems to have encouraged the flexible shifting among different representations and favored adaptive expertise, since students from the ACE classes used formal strategies more than twice as much as the BAU classes, and more importantly the formal strategies in the intervention classes were more frequently used on most high cost problems, which were supposed to benefit from a semantic recoding.

This study provides indications on how using research findings involving an empirical and theoretical background about the cognitive processes involved can be of benefit to educational settings and provide fruitful gains to students solving strategies and learning outcomes. The main theoretical issue was to propose a way for students to develop an opportunistic take on solving strategies within the context of one-step additive and subtractive arithmetic word problems. Besides the educational material that was provided in the intervention program, another crucial aspect of the program's success was having teachers' who were adequately trained to foster students' mathematical development. It would, therefore, be promising to study the development of students' and teachers' conceptions side by side.

Chapter 7 – The intuitive blind spot in teachers' judgments [5]

Despite abundant studies in mathematics education conducted with students (e.g., Lerman, 2014; Thevenot & Barrouillet, 2015), teachers' views regarding their students' conceptions and learning processes have been much less studied. All the while, the importance of teachers' diagnostic judgments is considered essential to student-centered teaching approaches (Davis & Simmt, 2006; Ostermann, Leuders, & Nuckles, 2017; Prediger & Zindel, 2017). Intuitive conceptions in mathematics, such as tacit models (Fischbein, 1987, 1993) or intuitive rules (Tirosh & Stavy, 1999) are considered to have both an explanatory and predictive power on student performance, in the sense that their identification provides a prediction and an explanation for the difficulty of a task – easy if the application of intutive knowledge leads to a correct result and difficult if not. As such, Tsamir and Tirosh (2008) considered tacit models to provide an appropriate framework for studying mathematical thinking processes and identifying the sources of some common errors students make.

Furthermore, the strategies children use and the errors they make when solving word problems have been repeatedly investigated, providing an empirical basis to assess teachers' PCK by comparing teachers' predictions with empirical findings. Arithmetic word problems provide an appropriate methodological tool for studying the influence of intuitive conceptions. They mobilize both students' informal solving strategies and formal ones. Arithmetic word problems can also be consistent with intuitive conceptions, for example that of subtraction as taking away, and some were not. Thus, arithmetic word problems can be characterized by their adequacy with the intuitive conception, but also by their ease of mental simulation.

According to the intuitive rules theory, we would expect that intuition-consistent problems would be easy for students, while intuition-inconsistent problems would be more difficult. Yet, as empirical findings have shown (Chapters 5 & 6; Brissiaud & Sander, 2010) problems consistent with the intuitive conception are not always easier than the ones inconsistent with the intuitive conception. Any high cost mental simulation problem is harder for students to solve than any low cost mental simulation problem (Brissiaud & Sander, 2010). This holds true for high cost mental

[5] The results from this study have been published in: Gvozdic, K., & Sander, E. (2018). When intuitive conceptions overshadow pedagogical content knowledge: Teachers' conceptions of students' arithmetic word problem solving strategies. *Educational Studies in Mathematics. 98*(2), 157-175.

simulation problems consistent with the intuitive conception, which are harder than low cost mental simulation problems inconsistent with the intuitive conception.

AIM OF THE STUDY

In the present study, we investigated teachers' PCK on arithmetic word problems and how intuitive conceptions influence their assessment of students' solving strategies. We recruited teachers and non-teachers, in order to determine if teachers, guided by their *knowledge of content and students*, judge students' solving strategies differently than non-teachers; or reversely, to determine if in some contexts intuitive conceptions can overshadow teachers' PCK and lead them to judge students' performance like non-teachers do, making predictions based on the believed explanatory power of intuitive conceptions. The participants were presented with different arithmetic word problems and had to choose which ones they thought would have a higher success rate among second-grade students. Later, the participants were asked to explain their choices by describing what makes a given problem easier than another, which opened the route for explaining their view regarding children's strategies. As a control task, the participants also had to solve arithmetic word problems, in order to assess the use of informal and formal solving strategies amongst both populations.

Since the influence of intuitive conceptions in mathematics is robust and persists in adulthood, we expect that both teachers and non-teachers will be influenced by intuition-consistent content when judging students' performance and when assessing the strategies students undertake to solve problems. We expect that when judging a problem's difficulty for students, the widespread belief that problems consistent with the intuitive conception are easier than problems inconsistent with this conception should constrain both teachers' and non-teachers' judgments. Thus, we first hypothesize that, in order to judge a problem's difficulty for students, both populations will base their judgment on the consistency of the problem with the intuitive conception.

Secondly, in addition to intuitive conceptions being present in both populations, teachers possess KCS as part of their PCK, they should understand how students solve problems (Shulman, 1986), which leads us to believe that there will be a difference between the two populations in their understanding of the strategies students use. We expect that both populations will be inclined to judge intuition-consistent problems as "easy" and will provide explanations compatible with the intuitive conception while ignoring the influence of the efficiency of mental simulation (left arrows of Figure 12.a. and Figure 12.b.). In this case, teachers' PCK will be overshadowed by their believed explanatory power of intuitive conceptions. On the contrary, when a problem is intuition-inconsistent, we expect that, thanks to their PCK, teachers will be more apt than non-teachers in identifying the difficulty of the mental simulation of the problem and describing students' informal solving strategies (right arrows of Figure 12.a. and Figure 12.b.).

Thirdly, a control task was introduced in order to preclude the possibility that differences between teachers' and non-teachers' conceptions of children's arithmetic word problem solving strategies could be attributable to their own arithmetic word problem solving strategies. The overall privileged status given to informal strategies does not entail any assumptions regarding the specificity of strategy use for teachers compared to non-teachers. Thus, no differences were expected amongst the two populations in the rate of informal strategies on low cost mental simulation problems, nor formal strategies on high cost ones.

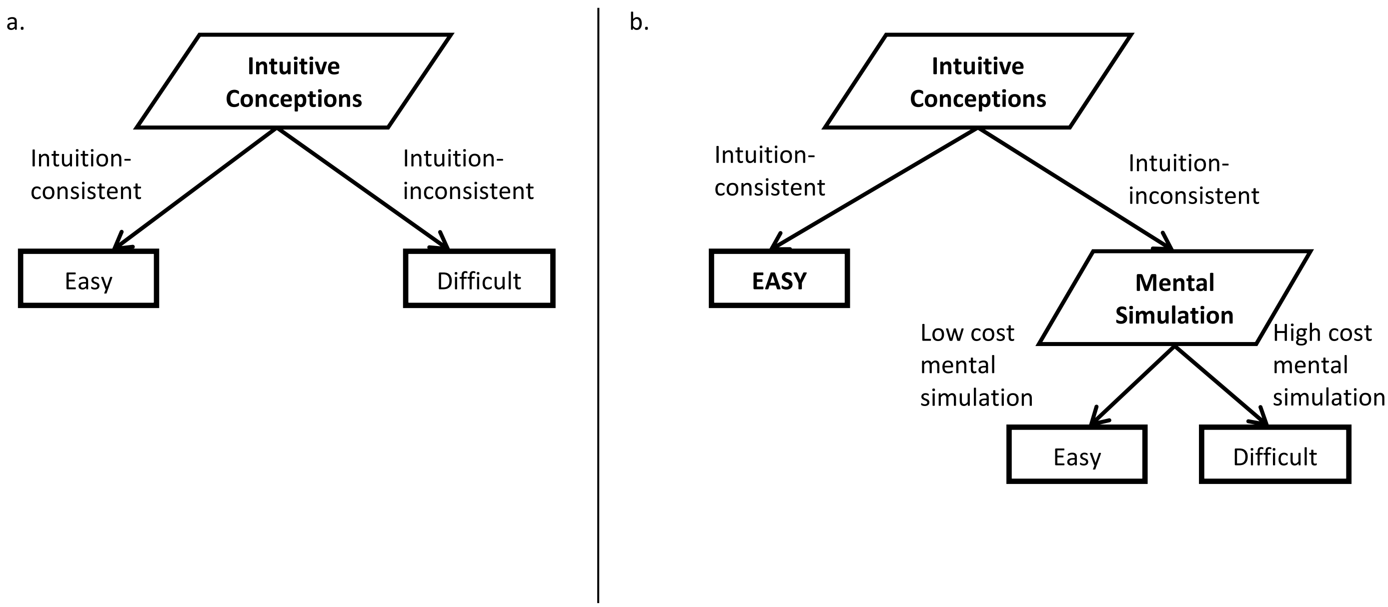


Figure 12: Non-teaching adults (a.) and teachers (b.) predictions about the determinants of problem difficulty and strategy use among students.

METHOD

Participants

In total, there were 72 participants in the study: 36 teachers (26 women, mean age = 41.75, *SD* = 9.23) and 36 non-teachers (26 women, mean age = 23.72, *SD* = 3.76). The teachers recruited for this study participated on a voluntary basis. All teachers were currently teaching in French elementary schools in working-class neighborhoods, with student populations from a variety of socioeconomic backgrounds. Teachers with a wide span of teaching experience were included in the study ranging from 1 to 40 years of experience (mean age of teaching experience = 9.23 years, *SD* = 13.25). The non-teachers were recruited in libraries, and all of them were enrolled or have already obtained a university degree. None of the participants – neither the teachers nor the non-teachers - majored in mathematics at university.

Since there is a greater percentage of woman than men in the French National Education (67.3% of overall teachers in the public sector are female, 83.4% female teachers in elementary school as documented for the school year 2016-2017 (Direction de l'évaluation, de la prospective et de la performance, 2017), our study included more woman (72.22%) than men in both populations.

Material

Three tasks involving subtraction word problems were presented. In the first task, the participants had to judge the difficulty of arithmetic word problems for students; in the second task, they justified their judgments and in the previous task, they solved arithmetic word problems and explained their own solving strategies.

The same items were used in the first two tasks. Each item contained two arithmetic word problems, that could each belong to one of the two following categories: the first category involved a semantic context consistent with the common intuitive conception of subtraction (taking away) and the second category involved a semantic context inconsistent with the intuitive conception (determining the distance)(Table 6). In each item, one of the paired problems was a low cost mental simulation problem and the other a high cost one. This pairing gave place to 4 items: two contained problems from different categories and two contained problems from the same category (Table 7). Based on previous empirical evidence for these problem categories (Brissiaud & Sander, 2010; Chapter 6), the low cost mental simulation problem was by far the easier problem in every item.

Two sets of number values on items were counterbalanced within and between subjects: (31, 27, 4) and (42, 39, 3). Each time, two of the values were provided within the text of the problem and the third one was the solution. The higher value was always in the text and the second given value determined the costliness of the mental simulation of the problem. Within each item, the set of numerical values was the same. Two contexts were used (marbles or euros) and held constant within subjects. The problems were presented in a random order, alternatively with problems from a different study.

Table 6: Examples of arithmetic word problems used in the experiment, sorted along the easiness of the mental simulation and the status regarding the consistency with the intuitive conception

Problem type	Problem category	Example of problem
Low cost mental simulation problem	Intuition-consistent	"Luc is playing with his 42 marbles at recess. During the recess, he loses 3 marbles. How many marbles does Luc have now?"
	Intuition-inconsistent	"Mary has 39 euros in her moneybox. For her birthday, she receives more euros and puts them in her moneybox. Now, she has 42 euros in her moneybox. How many euros did Mary get for her birthday?"
High cost mental simulation problem	Intuition-consistent	"Luc is playing with his 42 marbles at recess. During the recess, he loses 39 marbles. How many marbles does Luc have now?"
	Intuition-inconsistent	"Mary has 3 euros in her moneybox. For her birthday, she receives more euros and puts them in her moneybox. Now, she has 42 euros in her moneybox. How many euros did Mary get for her birthday?"

Table 7: Items constructed by pairing different types of word problems

Item	1	2	3	4
Low cost mental simulation problem	Intuition-consistent	Intuition-inconsistent	Intuition-consistent	Intuition-inconsistent
High cost mental simulation problem	Intuition-consistent	Intuition-consistent	Intuition-inconsistent	Intuition-inconsistent

In the third task, two problems – one intuition-consistent and one intuition-inconsistent – in either the low cost and high cost version, were presented, in a random order, in the different context than the one previously used. The participants also responded to a questionnaire concerning their age, gender and education.

Procedure

The participants passed each task individually with the experimenter. The material was presented to the participants in the form of a booklet with one item per page, and the experimenter interviewed them orally. Audio responses were recorded and later transcribed.

Task 1. The first task was constructed to assess if intuitive conceptions influence the participants' judgment of a problem's difficulty. Participants were asked to compare the paired problems in each item according to their difficulty for second-grade students. The experimenter asked them to judge which problem would have a higher success rate if presented to second-grade students. They responded by choosing between one of the two problems or by choosing the option that they have an equivalent success rate.

We did not proceed directly with a forced choice between the problems as not to influence the participants' decision processes.

Task 2. The second task was constructed in order to assess if there is a difference between the two populations in identifying the strategies students use to solve problems. A retrospective think-aloud technique (Sudman, Bradburn, & Schwarz, 1996) was used. The participants were first asked to describe the thought processes they underwent when making their judgment for all items in the first task, and afterwards to readdress the items for which they chose equal success rates. In the latter case, they were informed that one problem actually had a higher success rate and were asked to choose which one they thought it was and why.

Task 3. The third task was designed in order to document the strategies participants use when solving arithmetic word problems and determine if there is a difference between the two populations in their strategy use. All participants were asked to solve each problem and then describe orally the strategy they used to find the solution.

Scoring

In the first task, the *response on items* score was scored with 1 point when the participant chose the low cost mental simulation problem as having a higher success rate than the high cost one, and no points were given when the participant chose the high cost mental simulation problem. In cases where participants readdressed problems on which they initially judged the success rate as equal, the updated responses were rated using the same scoring.

On the second task, a congruency measure related the participants' justifications to the strategies students use when solving problems, similarly to Hill, Dean and Goffney (2007). An answer was considered as congruent when an individual chose the low cost mental simulation problem (with the empirically higher success rate among students) and gave a justification that revealed comprehension of students' modeling strategies, yielding 1 point to the *justification score*. An answer was considered incongruent when a participant chose the correct answer (low cost mental simulation problem) but did not provide a correct description of the informal strategies students put into place, meaning they did not demonstrate a comprehension of the difficulties formal strategies pose for students (see Table 8 for examples of congruent and incongruent justifications). In the cases where the participant chose the high cost mental simulation problem as having a higher success rate, it was considered incongruent because this judgment was in contradiction with empirical evidence about students' performance, and the justifications could therefore not be congruent with students' intuitive modeling strategies. When participants had to readdress items to which they responded by choosing the equivalent success rate, the justifications were updated and scored in the same way.

Regarding the third task, the participants received 1 point when they reported an informal strategy on a low cost mental simulation problem, while they received no points if they reported a formal strategy on a low cost mental simulation problem, and vice-versa for a high cost mental simulation problem. The points obtained on the low cost mental simulation problems were used to compute the informal strategy score, while the points obtained on the high cost mental simulation problems were used to compute the formal strategy score.

Table 8: Examples of participants' congruent and incongruent justifications

Item	Intuition-(in)consistency	Examples of congruent justifications	Examples of incongruent justifications
1	Both problems intuition-consistent	"We need to take away a small quantity from a large one, so I think that in the first case they will count with their fingers: 40, 39, 38. While for the second one, they will take away 39 from 42 and I think that's too much to count."	"The only difference is in one number, and that number is smaller in the first problem."
		"In both cases they will count downwards, therefore the first one will be easier. I don't think they will count from 39 to 42 in the second case."	"Subtracting a two-digit number is harder than subtracting a single digit number."

Item	Intuition-(in)consistency	Examples of congruent justifications	Examples of incongruent justifications
2	Intuition-consistent high cost mental simulation problem	"I think it is easier for children to count how much they will add to 39 to arrive to 42 than to take away 39 from 42. Indeed they could subtract in both cases, but this is easier."	"Many teachers teach subtraction by using indirect addition and that makes it easy."
		"The first problem has an additive structure, while in the second one we take away 39. The first problem is easier because we have to add a small amount, while the second one requires to imagine the problem differently in order for it to be easy."	"Counting forward is easier than subtracting. Adding something is more intuitive for children than losing."
3	Intuition-inconsistent high cost mental simulation problem	"In the first problem there are 42 marbles and we have to take away 3, so children will count backwards. While in the second one, the instructions are longer, and more importantly, children will start at 3 and will count up to 42, and it is complicated to count that much."	"In both cases the operation is 42-3, but I chose the first one as easier."
		"In the second problem children will count from 3 to 42 and it would be more complicated to do a mental subtraction that is easy to do in this particular case, while in the first one they will simply take away 3."	"I think doing a simple subtraction is easier than to have an unknown, to find the missing addendend." "It's easier to count-down a small amount" "They won't understand that they have to subtract."
4	Both problems intuition-inconsistent	"In one problem we start at 3 and go up to 42, while in the other one we only have to start at 39 and reaching 42 is easy."	"The numbers are smaller."
		"I don't think that children will think of subtracting in this problem, so I think that the first one is easier because there are fewer numbers to count."	"It is easier to do the operation with a number that has one digit than to do a subtraction with a two-digit number"

The participants' justifications on the second task and the strategies described by them in the third task were coded by two independent raters. The initially obtained inter-rater reliability for the justification scoring was 95.49% with the Cohen's kappa score of 0.91, providing an almost perfect level of agreement. The inter-rater reliability for the strategies used by the participants in the third task was 98.61% with Cohen's kappa score of 0.97. By discussion, the raters arrived to a mutual consent about coding the items on which they initially diverged.

Data analysis

The first set of analyses explored the responses on items in order to assess if intuitive conceptions influence the perceived difficulty of the problems. We compared judgments of both populations on the two problems where the high cost mental simulation problems was intuition-consistent (items where only the high cost mental simulation problem is intuition-consistent, and items where both problems are intuition-consistent) to problems where the high cost mental simulation problems was intuition-inconsistent (items where only the low cost mental simulation problem is intuition-consistent, and items where both problems are intuition-inconsistent). The average of such item groupings was computed. We predicted that participants would have more adequate judgments on the latter. We then tested if there were differences between the groups on each of the grouped items.

The second set of analyses explored the justifications of items in order to assess if intuitive conceptions overshadow teachers' PCK, biasing the teachers' capacities to identify the strategies children put into place. We measured if teachers' references to intuitive strategies in their justifications differed from those of non-teachers. We predicted that when the high cost mental simulation problem was intuition-consistent, the intuitive conception would overshadow teachers' PCK and they would not differ significantly from non-teachers. The average of the justification on items score was computed following this grouping, just as this was done for the responses on items. We tested for a group effect on items. Further on, we studied if there was an overall difference in the justifications of the groups on items where the high cost mental simulation problem was intuition-consistent and on items where the high cost mental simulation problem was intuition-inconsistent.

Finally, analyses were conducted in order to see if both populations resorted to intuitive modeling strategies when solving low cost mental simulation problems, and arithmetic strategies when solving high cost mental simulation problems. Non-teachers and teachers were not expected to differ in their strategy use.

RESULTS

Response on items analyses

The difference between the average of the response scores for both populations was in favor of problems where the high cost mental simulation problem was intuition-inconsistent (M = 0.89, SD = 0.23) compared to problems where the high cost mental simulation problem was intuition-consistent (M = 0.76, SD = 0.28) (Figure 13). A repeated measure GLM was conducted with the average scores of the responses as dependent variables and the group factor (teacher versus non-teacher) and counterbalanced factors as between-subject variables. As expected there was an overall significant effect of items ($F(1,56) = 15.446$, $p < .001$) and no main effect of group ($F(1,56) = 0.148$, $p = .702$) or any other counterbalanced factor, and no interaction between the items and the groups ($F(1,56) = .987$, $p = .325$). This lack of a significant difference between the two groups supports our hypothesis that the participants tend to consider intuition-consistent problems as easier for children.

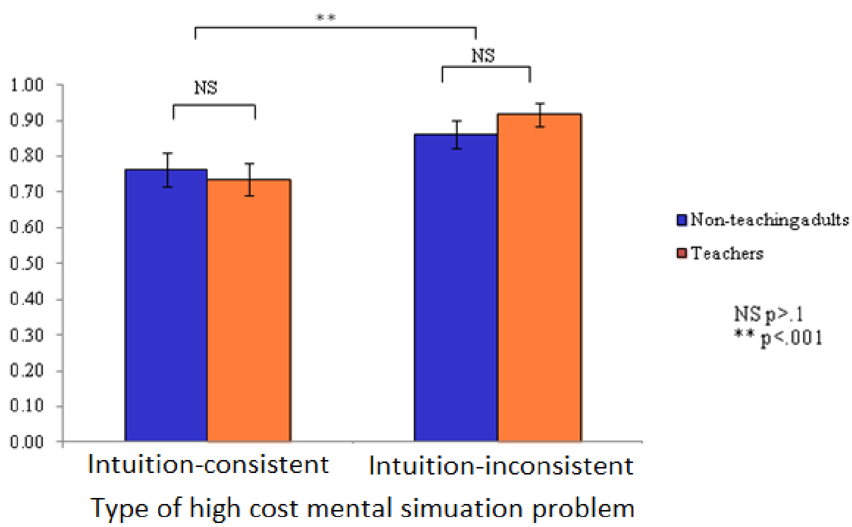


Figure 13: Average response on items

Univariate analyses of variance for the responses as the dependent variable and group and the counterbalanced factors as the between-subject variables were conducted in order to test for a difference between groups. The results revealed no significant effect of the group factor for items where the high cost mental simulation problem was intuition-consistent ($F(1,56) = .080$, $p = .779$) or where the high cost mental simulation problem intuition-inconsistent ($F(1.56) = 1.284$, p =

.262), confirming that there was no difference between the two populations when judging a problem's difficulty for students.

Justifications of items analyses

Testing for a group effect on items, we conducted a repeated measure GLM with justifications on items as repeated measures and the group and counterbalanced factors as between-subject variables. The results indicated a significant main effect of the group factor ($F(1, 56) = 6.959$, $p < .05$), a main effect of justifications of items ($F(3,56) = 11.179$, $p < .001$), and no significant interaction between justifications on items and the group factor ($F(3,56) = 1.549$, $p = .204$). As expected, this revealed that the justifications differed across the two populations and from one item to another.

Further on, studying the overall difference in the justifications of the groups, a repeated measure GLM was conducted with the average of the justification of items scores as dependent variables and the group and counterbalanced factors as between-subject variables. As expected, there was an overall significant effect of the items ($F(1,56) = 8.347$, $p = .004$) and a main effect of group ($F(1,56) = 6.959$, $p = .011$), and there was also a significant interaction between the items and the group ($F(1,56) = 3.907$, $p = .05$) (none of the counterbalanced factors had a significant effect) (Figure 14).

We further conducted univariate analyses of variance for the justification of items where the high cost mental simulation problem was intuition-consistent and items where the high cost mental simulation problem was intuition-inconsistent as the dependent variable, and group and the counterbalanced factors as the between-subject variables. There was a significant effect of group on items where the high cost mental simulation problem was intuition-inconsistent ($F(1,56) = 12.981$, $p < .001$), but no effect of group on the items where the high cost mental simulation problem was intuition-consistent ($F(1,56) = 0.638$, $p = .428$).

This thus reveals that teachers show a better understanding of students' problem-solving strategies than non-teachers only when the content they evaluate is intuition-inconsistent. This further supports our hypothesis that the consistency of a problem with the intuitive precludes teachers' use of their PCK.

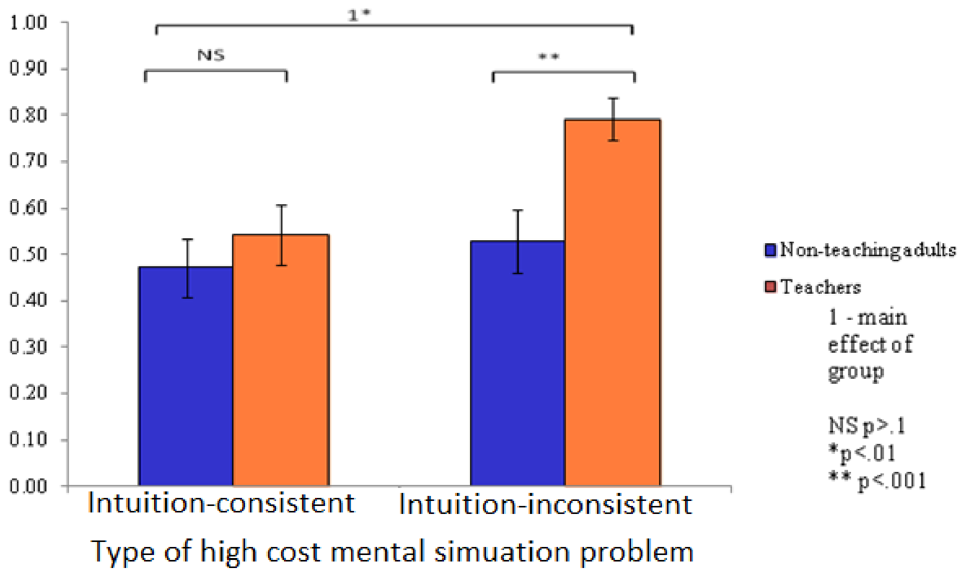


Figure 14: Average justification of items

Use of informal and formal solving strategies

A univariate analysis of variance was conducted for the informal and formal strategy score as the dependent variables, with group and the rest of the counterbalanced factors as between subject variables. As expected, no group effect ($F(1,56) = .344$, $p = .560$) was revealed for the informal strategy score nor formal one ($F(1,56) = 0.004$, $p = .949$) (see Figure 15).

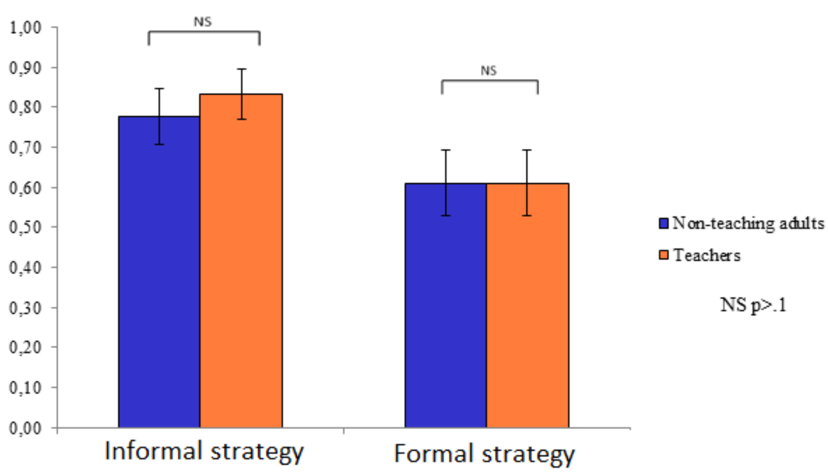


Figure 15: Informal and formal strategy scores

DISCUSSION

The study was conducted in order to investigate how mathematical intuitive conceptions influence teachers' understanding of students' processes. Despite the well documented literature on students' arithmetic word problem solving processes, research on the knowledge teachers hold about their students' solving strategies is scarce. In the present study, we hypothesized that teachers' diagnostic competences would be biased by intuitive conceptions. This would be manifest when the intuitive conception leads to contradictory predictions compared to predictions derived from assessing the efficiency of informal solving strategies. Despite the fact that only the latter would correctly predict the solver's behavior, we expected that intuition-consistent problems would overshadow teachers' PCK when they come into play. In order to test this hypothesis, teachers and non-teachers were asked to compare the difficulties of problems and describe the strategies they thought students use when solving these problems. The problems were either consistent or inconsistent with the intuitive conception of the arithmetic operation.

Problems were presented in pairs: a problem that is easy to solve by using informal modeling strategies (low cost mental simulation problems) and a problem that would be difficult to simulate through informal strategies, but would be easy to solve if arithmetic principles were applied (high cost mental simulation problems). As it has been empirically demonstrated, low cost mental simulation problems have higher success rates than high cost mental simulation problems (Brissiaud & Sander, 2010; Chapters 5 and 6). Problems were paired so that either both problems were intuition-consistent, only the low cost mental simulation problem was intuition-consistent, only the high cost mental simulation problem was intuition-consistent, or both problems were intuition-inconsistent.

In order to see if intuitive conceptions influence the judgment of a problem's difficulty, participants compared the difficulty of the paired problems for students and later provided a rationale for their choices, revealing if they understood the actual strategies students put in place. Finally, participants had to solve problems themselves, in order to verify that both populations use intuitive and arithmetic strategies to the same extent.

The analysis of the responses to the items revealed more accurate judgments on items where the problem that is the most difficult for children (the high cost mental simulation problem) was intuition-inconsistent. This means that the participants believed in the explanatory power of the intuitive conception when choosing which problem was easier, even when the intuition-consistent problem was actually more difficult for children. Because both groups shared the belief of the 'superiority' effect of the intuitive conception regarding a problem's difficulty, there was no demonstration of a benefit from PCK on this matter.

The study of justifications confirmed our predictions given that no difference was observed between the populations on items where the high cost mental simulation problem was intuition-consistent. However, teachers gave more congruent justifications than non-teachers about students' strategies on items where the high cost mental simulation problem was intuition-inconsistent. This means that teachers *were* able to accurately address the informal strategies primarily used by children, but *only* when the intuitive conception did not interfere with the task. Indeed, the findings suggest that teachers' PCK was overshadowed by mathematical intuitive conceptions. Additionally, there was no difference in the use of informal or formal strategies between the two populations when they solved problems, therefore the difference in performance could not have been influenced by preferential use of one strategy among one group.

The obstacles to understanding children's performance and strategies are not surprising since empirical research about arithmetic word problems often remains unrecognized in teacher education programs. Intuition-consistent problems for which the informal strategies are efficient favor students' successful resolutions. Yet, they do not challenge them to analyze the mathematical structure of the problem and apply relevant arithmetic knowledge. Thus their successful resolution is not as satisfactory as it might seem for assessing the mastery of the underlying mathematical notion. Some frameworks aiming to draw teachers' attention to specific aspects of student thinking that go beyond executing procedures can enhance teachers' abilities to make diagnostic judgments (e.g. Carpenter et al., 1988; Walkoe, 2014). Instructing teachers about the informal strategies induced by the problem's representation can therefore be a promising path in providing lessons that will both challenge students' intuitive conceptions and allow them to overcome their modeling strategies by applying arithmetic principles, since it can promote an adaptable behavior in strategy use for problem solving and favor conceptual understanding.

Another interesting observation is that non-teachers did demonstrate certain components of PCK in the response and justification tasks. If non-teachers did not possess any aspect of PCK, they might have simply made no predictions, nor provided explanations. Indeed, just as previous research outlined the importance of teachers' own knowledge in providing assumptions about what students know (Ostermann et al., 2017), it seems that this also enables non-teachers to make some diagnostic judgments. Furthermore, some theories have suggested that teaching is a natural cognitive ability (Strauss & Ziv, 2012), and even proposed that folk pedagogy stems from folk psychology (Olson & Bruner, 1996). Even though in Shulman’s conceptualization, PCK is a type of knowledge specific to teachers, this raises the question of what aspects of PCK actually rely on abilities that are shared among the general non-teaching populations.

GENERAL DISCUSSION AND CONCLUSION

Discussion

Synthesis of the main findings

The current thesis sought out to understand the role that different intuitive and informal knowledge play in mathematics teaching and learning. We investigated how different conceptual categories constructed through prior experience influence the representational processes in the course of arithmetic problem solving. We considered that the process of analogical encoding, in which features of a situation are put in correspondence to previously constructed mental categories, is a key process guiding the construction of the represented situation.

We conducted a series of empirical studies that were aimed at understanding how different conceptions mobilized in the encoding of arithmetic problems influence students' strategy use and teachers' judgments. In Chapter 5, we proposed that a crucial step in solving arithmetic word problems is the mental simulation of the encoded representation. We made predictions about the conceptions that will be mobilized in a problem's encoding and the efficiency of the mental simulation operating on this encoding. We then carried out a series of experiments that tested this effect on students' solving strategies. We found that when the mental simulation of the encoded representation had low cost, second graders almost exclusively used informal solving strategies and had higher performance. When the encoded representation led to a mental simulation that has high cost, students predominantly reported formal solving strategies that did not reflect the encoded representation.

In Chapter 6, we studied what it takes to succeed on high cost mental simulation problems. We proposed that succeeding on high cost mental simulation problems is facilitated when solvers rely on an arithmetic conception different than the one that guided the initial encoding. This was expected to lead to the semantic recoding of a problem's representation and make it possible to use a more efficient solving strategy than the one used based on the initial encoding. We compared the performance of first-grade students who participated in an arithmetic intervention program to students from business as usual classes. The intervention was focused on semantic encoding and recoding exercises by using arithmetic word problems. We found that students who had benefited from the intervention had higher performance on high cost mental simulation problems and used formal solving strategies more frequently.

In Chapter 7, we sought to explore if the conceptions that influence the encoding of arithmetic word problems also have an impact on teachers' judgments of student performance. Teachers' knowledge of content and students is considered to provide teachers with the necessary understanding of students' solving strategies. However, we observed that when the intuitive conception of subtraction is involved in the encoding of an arithmetic problem, teachers had a harder time understanding what makes it difficult for students to use a more efficient solving strategy.

THEORETICAL IMPLICATIONS

The role of intuitive conceptions in encoding

Following Fischbein's (1987) view, intuitive conceptions can take the form of an analogical or paradigmatic model, both characterized by the systematic similarities between the intuitive model and the notion it substitutes. In what he proposed, there are many similarities with our current take on analogical reasoning (Hofstadter & Sander, 2013). In Fischbein's analogical model, the original notion and the intuitive conception belong to different conceptual fields and are observed through similarities, while in his paradigmatic model, the exemplar becomes the prototype of the mathematical notion. In our view, they can both be seen as products of different categorization processes that support analogical reasoning. In Fischbein's analogical model, the intuitive model can be considered as a category to which the mathematical concept is assimilated, just as it is the case in paradigmatic models. Therefore, there would be no need for a differentiation between the analogical and paradigmatic models, since the primary determinant would be the degree of abstraction with which the categorization is done. Furthermore, Fischbein's description of formal and algorithmic knowledge stresses the importance of associating meanings and skills, which is also highlighted in the literature regarding conceptual and procedural knowledge in mathematics (Alibali & Rittle-Johnson, 1999; Rittle-Johnson et al., 2001). The intuitive models described by Fischbein are conceptual in nature but are rarely considered when we investigate how conceptual knowledge influences procedural knowledge. It would be interesting to explore how adding this component to the study of conceptual knowledge could inform both research and practice.

Going from action-based external modeling to reasoning based on part-whole relations is essential for understanding commutativity, associativity and complementary relations of addition and subtraction (Resnick, 1989). However, reasoning based on part-whole relations is certainly not an easy task, since students continue to perform poorly on problems that require the use of formal strategies reflecting the use of conceptual arithmetic knowledge. One obstacle in the learning and solving processes may be the intuitive conceptions of arithmetic, which operate in the background and conflict with the formal and the algorithmic components of mathematical activities (Fischbein, 1993). In the case of the intuitive conception of subtraction as *taking away*, this could mean that presenting a problem whose wording has a semantic structure that is the closest to this taking away conception, the problem should be more difficult for students to recode because it would be more challenging to change one's point of view when the content is intuition-consistent. The problem that would have the closest wording to this conception would be a Change 2 problem, such as those tested in Chapter 6 and by Brissiaud and Sander (2010). By looking at the data on Change problems, it is indeed the Change 2 – result unknown problem – that systematically had the highest performance gap on low and high cost mental simulation problems in Brissiaud and Sander's study, as well as among our BAU population. The observed performance in the ACE classes had the second-highest gap between low and high cost mental simulation problems. In the third experiment of Chapter 5, direct subtraction problems had the highest gap in performance rates, suggesting that the minus sign is closely related to the taking away conception of subtraction. In the case when the intuitive conception is mobilized in the encoding of a problem, the consistency between the semantic characteristics of the problem and the arithmetic conception may lead to an encoding that is more difficult for solvers to re-code. This provides an insight into the mediating effects that the problem semantics can have in the encoding of the problem, but it can also provide a way to look at how the conceptual intuitive knowledge might influence the persistence of certain strategies.

Towards a model of arithmetic problem solving

The current thesis proposed which conception would guide the arithmetic solving process by participating in the encoding. We observed that these different encodings had an impact on the difficulty of the problems since different encodings led to different strategies. These findings are in line with research proposing that the specific instances of prior knowledge enrich the representation of the situation as it is being processed (Gentner et al., 2003; Ross & Bradshaw, 1994). Our findings suggest that the abstraction processes which lead to the encoding of arithmetic problems, both word and non-word, involve the selection of an arithmetic conception. Furthermore, we proposed that it is this encoded representation that a solver will first attempt to mentally simulate in order to find a solution. When the mental simulation is efficient, this will lead the solvers to find a numerical solution. Finding the answer in this manner does not require any formal arithmetic knowledge, for this reason these are considered informal solving strategies. Yet, finding the solution to a problem by mentally simulating the encoded representation is not always simple. When the mental simulation is difficult to execute, the solvers will need to mobilize arithmetic knowledge. A modeling of this process is presented in Figure 16 and will be detailed in the following paragraphs.

In the present thesis, we have studied problems where a recoded representation would lead to a more efficient solving strategy. We proposed that when the mental simulation bears high cost, then in order to find a more optimal solving strategy, the solver would need to recode the initial representation of the problem. This means that the solver would have to rely on an arithmetic conception different than the one initially used and with its help, the solver would semantically recode the problem's representation. This recoded representation can undergo again the process of mental simulation, and in the case of the problems studied in the current thesis, this would efficiently lead to the numerical solution. Yet, it should be noted that sometimes, even when the informal strategy is inefficient, recoding the initial representation will not lead to an efficient solving strategy. In these cases, solvers would need to rely on formal solving strategies that do not require a recoding. For instance, if a problem contains numerical values which the Switch model described to have the longest reaction times (small distance with S < D (31 – 15 = *16* or 16 + ? = 31), and small distance S > D (32 – 17 = *15* or 17 + ? = 32)), then engaging in a recoding will not diminish the computational cost of the problem.

Nevertheless, there exist different accounts for the strategies students use which involve number manipulations that make finding the answer computationally easier to execute (e.g. Blöte, Klein, & Beishuizen, 2000). For example, solvers may use *decomposition* strategies in which they split both operands into tens and units and will operate on them separately, for example '45 -13 = ?' will be solved as '40 – 10 = 30, 5 – 3 = 2, 30 + 2 =32'. Solvers can also use *sequential* strategies, where they split one of the operands and sequentially solve the problem such as '45 – 10 = 35 – 3 = 32'. Alternatively, they can flexibly adapt the numbers based on their knowledge of the properties of arithmetic operations, for example, '45 – 9 = ?' will be solved as '45 – 10 = 35 + 1 = 36'. Such strategies can be undertaken instead of strategies requiring a conceptual recoding and can even be more beneficial in these cases. However, it still remains unknown what kind of an encoding determines if a solver will recode the initial representation or engage in number manipulation strategies.

It should, however, be noted that this proposal of arithmetic problem solving processes does not address certain aspects. Firstly, it does not address text comprehension processes but takes them as a pre-requisite for the solver to engage in the problem solving process. If the solver does not comprehend the vocabulary or mathematical symbols used in the problem, they cannot encode the problem. On the other hand, if the problem is comprehended, but there are no arithmetic conceptions at the solvers' disposal, the representation of the problem will not be able to evolve beyond the initial one, making only informal solving strategies available.

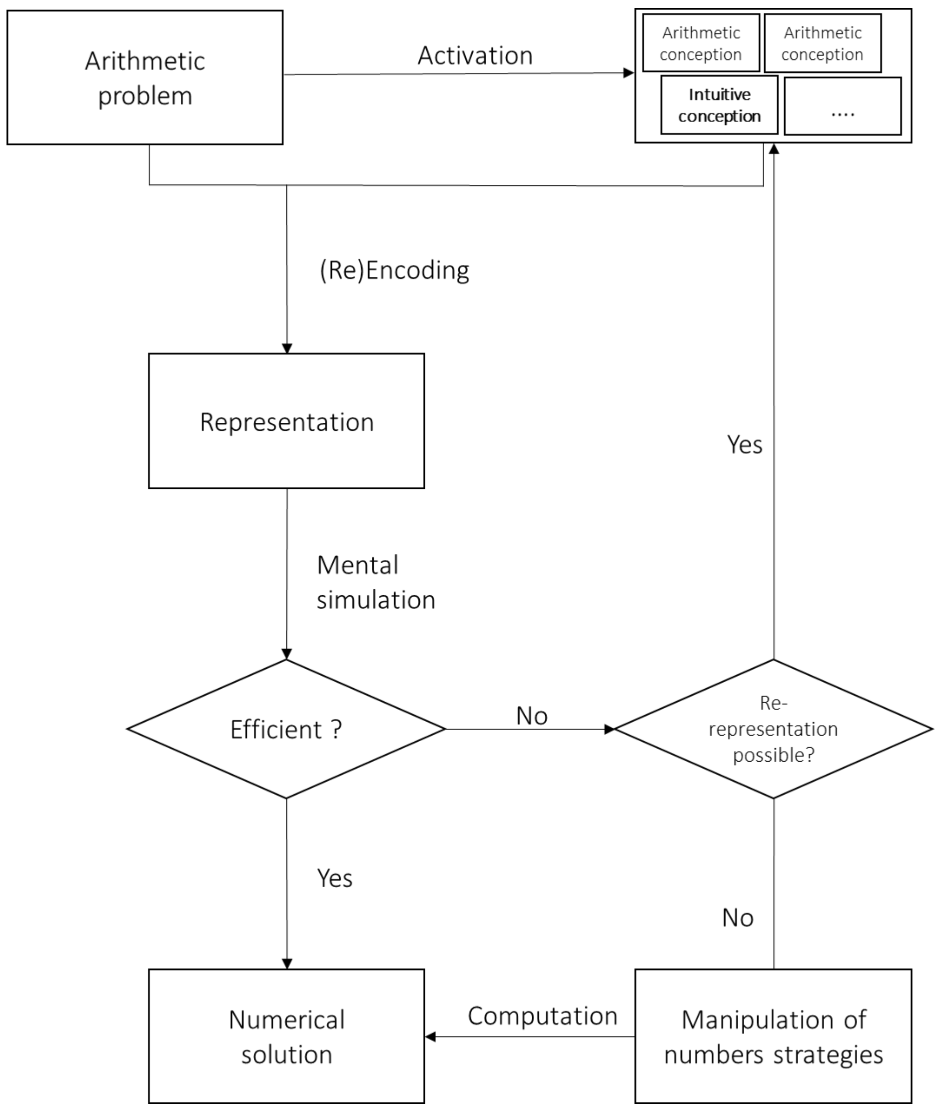


Figure 16: Modeling the encoding and recoding processes in arithmetic problem solving

Conceptualizing teachers' competences

The expert blind spot hypothesis advocates that educators with greater subject-matter knowledge seem to "view student development through a domain-centric lens" which causes them to inaccurately predict students' problem-solving behaviors (Nathan & Petrosino, 2003, p. 918). The authors proposed that teachers with high content knowledge do not draw on general principles of intellectual development when forming a model of student mathematical development, but draw on the ontological structure of the discipline, which overrides their underspecified PCK. However, elementary school teachers are rarely expert mathematicians (for instance, none of the teachers in Chapter 7 majored in mathematics prior to becoming teachers). In the present thesis, we showed that teachers' understanding of children's solving strategies on arithmetic word problems was overshadowed by the intuitive conception of the arithmetic operation. Teachers considered problems consistent with the intuitive conception to be easier for children, just like non-teachers, and did not have more success than non-teaching adults to identify the strategies children put into place when influenced by the intuitive conception.

Our findings highlighted that in a parallel way as described by the expert blind spot, teachers' PCK was overridden in some contexts when the intuitive conception was involved. We propose that teachers' PCK is subject to two effects of opposing origins but entailing similar consequences: expertise and intuitive conception. The first entails the expert blind spot phenomena, while the second entails an *intuitive blind spot*. Each one induces a non-flexible point of view concerning students' performance, operating as a guiding principle for assessing students' behavior and overshadowing teachers' PCK. Following the model of teachers' assumptions on what students know proposed by Ostermann and collaborators (2017) suggests that when the content is consistent with the intuitive conception, teachers simply regard this content as being easy and use the intuitive model as the working model for assessing the difficulty problems pose for students. Indeed, teachers who are subject to the intuitive blind spot do not necessarily lack subject matter knowledge or PCK. Their PCK may rather be underspecified, which would lead them to make predictions based on the perceived ease of content consistent with the intuitive conception. In this view, we believe that attention should also be drawn to simultaneously working on students' intuitive conceptions in classrooms but also drawing teachers' attention to the phenomena.

Open issues and educational entailments

About knowledge evaluation

While focusing on the processes involved in solving arithmetic tasks in this thesis, we have seen that students dispose of a wide variety of informal knowledge that they bring to class, present in both their conceptual and procedural learning. On the procedural side, students' informal arithmetic solving strategies are based on the early arithmetic abilities such as counting with objects but also mentally (Baroody & Ginsburg, 1986; Carpenter et al., 1981; Resnick, 1989). Likewise, the notions taught at school already find their roots and are shaped by previous experience, which has been conceptualized in the form of intuitive conceptions (Fischbein, 1987; Lakoff & Núñez, 2000). Our studies in Chapters 5 and 6 showed that the content of the problems determined what informal solving strategy would be used. Compare 1 problems would lead to the use of indirect addition or indirect subtraction as the informal solving strategy, whereas Compare 4 problems would lead to the use of direct subtraction as the informal solving strategy. Furthermore, we showed that there is a great gap in performance depending on whether the task requires the use of informal or formal arithmetic knowledge. When we would explain the stakes behind these different problem types to the teachers of the classes in which we conducted our experiments, they were often left asking if in their tests, they should mainly be using problems that require formal strategies.

The question that our teachers asked us, while it can be seen as an example of the formalisms first approach (Nathan, 2012) – considering it necessary to evaluate the use of formal principles but not that of informal strategies – is a valid concern. The answer to it requires consideration, since understanding what types of tasks are best suited for evaluating the learning objectives is an important and complex challenge, especially because the benefits that could be gained by working with students' informal strategies in the classroom should not be disregarded. In Chapter 6, we saw that students who participated in an arithmetic intervention during the school year used formal strategies to a greater extent than students who followed the regular first-grade arithmetic curriculum. Yet, the design of our study did not make it possible to determine to which degree progress on informal solving strategies contributed to the higher the use of formal strategies. A current view in cognitive psychology is that the relations between conceptual and procedural knowledge are bidirectional and that progressing on one also leads to progress on the other (Rittle-Johnson, 2019; Rittle-Johnson et al., 2001). Furthermore, connecting new mathematical knowledge to informal knowledge favors students' understanding (Baroody & Wilkins, 1999; Van den Heuvel-Panhuizen & Drijvers, 2014).

Nevertheless, an essential objective in mathematics education regards the selection of the most appropriate strategy for finding the solution to a problem (Threlfall, 2009; Verschaffel et al., 2009). If the aim is to evaluate the adaptive strategy use in mathematics, then introducing problems that would benefit the most from the use of formal strategies would be the optimal way to operationalize such an aim. Indeed, a correct answer to a problem that can easily be solved through the use of informal strategies is a poor cue for assessing the use of arithmetic principles, and this is even more the case when problems that are intuition-consistent, such as Change 2 problems, are used. For example, the problem "There are 21 flowers in the bouquet. Sophie takes out 3 flowers from the bouquet. How many flowers are in the bouquet now?" directly provides the encoding '21 – 3' and therefore does not challenge the solvers' conceptual understanding of the arithmetic notion. Yet, if 'Sophie takes out 19 flowerers' then it would be more beneficial to recode the problem by using arithmetic principles and solve it with the strategy '19 + ? = 21'. It is therefore important to take note that if students knowledge is assessed through problems that can be easily solved with informal solving strategies it is misleading to assume that a student is applying arithmetic knowledge, since the processes involved in finding the solution do not at all indicate that they are able to flexibly choose between different strategies. What does have higher pedagogical relevance, regarding adaptive expertise, is to use tasks on which different solving strategies would be revealing of the kinds of knowledge students use to solve the problems, such as the high cost mental simulation problems used in this thesis.

It should, however, be noted that this way of emphasizing adaptive expertise mainly concerns task characteristics (Threlfall, 2009). As we have seen in Chapter 3, along with the task characteristics, the subject and the context are also important in grasping the complexity of selecting the most appropriate solution (Verschaffel et al., 2009). A future perspective would be to also consider the specific classroom conditions and what kind of didactical contract is put in place between teachers and students. Indeed, an alternative interpretation to the use of informal strategies on problems that could benefit from a recoding, such as using indirect addition on the Change 2 problem instead of direct subtraction, may be that the teacher does not admit indirect addition as a correct formal strategy. Adding this information to the analyses would not compromise the robust effects concerning the gap between the use of informal and formal solving strategies, nor the relevance they have for evaluating students' knowledge. It would however provide more details about the factors influencing students' strategy choices.

About teachers' professional development

One thing that does remain unclear about the intuitive blind spot is whether it is caused by teachers' own intuitive conceptions about the concept, or it is triggered by the fact that the content itself is intuition-consistent. We have seen that intuitive conceptions do persist among the adult population (Tirosh & Graeber, 1991; Vamvakoussi et al., 2013). Intuition-consistent content impacts the accuracy and reaction times with which adults perform on various tasks (Dunbar et al., 2007; Goldberg & Thompson-Schill, 2009; Shtulman & Harrington, 2016). A study we are currently conducting also joins this line of research. In particular, we asked participants to create word problems that correspond to different subtractions such as '8 – 3 = ?'. They were instructed to think of two problems as different as possible. In the next step, we asked them if it is possible to imagine a subtraction problem whose wording describes a gain. This rule is inconsistent with the inferences that can be derived from the intuitive conception of subtraction as taking away (Fischbein, 1987). Change 5 problems from Riley, Greeno and Heller's (1983) classification with the start unknown typically correspond to this description (e.g., "Joe had some marbles. Then Tom gave him 5 more marbles. Now Joe has 8 marbles. How many marbles did Joe have in the beginning?"). Our preliminary findings reveal that 26 out of 87 teacher trainers, 35 out of 120 high-school teachers and 23 out of 56 pre-service teachers answered that it is not possible to propose such a problem. While the analysis regarding the categories of problems that the participants proposed is still ongoing, this testifies to the robustness that the intuitive conception has.

These studies focus on teachers' knowledge, but not their judgment of student performance and thinking processes, which is the main focus when assessing teachers' PCK. It is true that studies supporting the extent to which intuitive conceptions are mobilized in the adult and teacher population are not sufficient in order to solve the dilemma whether it is teachers' own intuitive conception of the intuition-consistent content that leads to the intuitive blind spot. Nevertheless, it does highlight that since intuitive conceptions persist, they continue to be a viable analogical source among teachers. As we have seen, it is during the processing of a situation itself that prior knowledge can influence how a situation is interpreted (Ross & Bradshaw, 1994). Furthermore, when the situation being processed is consistent with previously held knowledge, there is greater ease in processing the situation. This leads us to consider that, even though teachers are not solving a problem, they are evaluating a problem's difficulty for students, and this requires that a representation be encoded. When the content is intuition-consistent, then the intuitive conception guides the encoding of what is used as the initial model that guides teachers' assumptions of what students know as it has been described in Ostermann, Leuders and Nuckles's (2017) model. It is therefore at this initial stage that the intuitive blind spot casts a shadow on teachers' PCK and sensitizing teachers to its existence could be beneficial for strengthening teachers' diagnostic competence. This does not mean that we should aim to eliminate teachers' intuitive conceptions since as we have seen, they seem to be very resistant and persist until late in life (Shtulman & Valcarcel, 2012). The solution can rather be found in reinforcing knowledge about the intuitive blind spot and its mechanisms of action should be reinforced in teachers' PCK.

CONCLUSION

We want to conclude on some broader considerations regarding the educational practices that emerge from this thesis. First, a current topic in education revolves around differentiating one's teaching in order to take into account individual learning trajectories. A differentiated instruction focuses on student characteristics by recognizing the differences among students and responding proactively to them (Langa & Yost, 2007; Tomlinson, 2001). Considering individual differences in the classroom is not taken as an end in itself, but a means to: ensure that all students acquire the basic and necessary knowledge, fight against school failure, and help learners reach their potential (Forget, 2018). Therefore, to differentiate in one's pedagogical practice is not the same as to diversify since the teacher does not vary their instruction at random but strives to do so in response to the identified needs of the students. We believe that studying the psychological processes behind students' performance and behaviors in learning situations can provide a way of operationalizing such issues.

First, this thesis provided predictions about how students will apprehend the content being taught and what meaning it will have for them, which is very helpful for any approach that strives to address individual differences among students. Secondly, it makes it possible to determine the cognitive requirements of tasks and items used in instruction, as well as in the evaluation of student learning. For example, we relied on knowledge about the psychological mechanisms involved in the construction of a mental representation of a situation and the processes that act upon such representations. This led us to determine the costliness of different arithmetic problems. As a function of this costliness, we were able to determine which tasks reflect students' informal knowledge that they had even prior to instruction on arithmetic operations and which tasks reflect comprehension of the educational notions at stake. If in the latter case students persist in their use of informal strategies, it informs us they did not manage to go beyond the initial representation of the problem. Understanding the individual differences regarding the representations that students abstract from a situation could be useful in order to adapt the intervention that would follow as a way of helping students who face difficulties with the task. The framework that was proposed offers the possibility to assess if students always stick to the initial encoded representation on different problems. In this case it reveals that the intervention needs to focus more broadly on recoding activities through the comparison of informal to formal strategies. Yet, if a student succeeds in achieving such a recoding on one type of problem but not another which requires the same principle, then the teacher could rather focus on the comparison between the two different formal strategies in relation to the problem structure.

A second topic we want to address bears on the relation between findings related to the content itself and those regarding general processes for which the connections to the content being taught are sparse. The possibility of relying on general processes for promoting learning is debated. For instance, recent findings suggest that it is neither students' intrinsic nor extrinsic motivation that is correlated to academic achievement (Taylor et al., 2014). The motivational component that is most strongly negatively associated with school achievement over time is students' *amotivation*. Amotivation is not the opposite of intrinsic and extrinsic motivation; it is instead the lack of either kind of motivation: the lack of intention to act on a given task. Students who are amotivated tend to see their failures as a lack of self-competence (Leroy & Bressoux, 2016). A solution that seems to be a remedy to falling into the amotivational trap is encouraging students' active engagement with the tasks (Reeve, 2013). In such a view, it might be essential to consider the possibility of evaluating students who have particular difficulties with a subject matter mainly on tasks that mobilize informal knowledge, in order to favor their sense of competence. Nevertheless, the contribution that such amotivational antidotes bring to student achievement cannot substitute for other sources of progress that are more directly related to the content being taught: cross-disciplinary teaching practices such as those that promote active engagement and a focus on the content of the knowledge being acquired have every reason to be articulated with one another (Sander, Gros, Gvozdic, & Scheibling-Sève, 2018). For this purpose, adding a degree of refinement in the analysis of the content being taught and evaluated, as we aimed to in this thesis, is crucial for gaining insights into the representations that students generate about the content and finding ways to foster the development of initial representations, as well as their flexible adaptation to the demands of a task.

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RESUME EN FRANÇAIS

INTRODUCTION

Initialement, les domaines de la psychologie et de l'enseignement des mathématiques avaient des objectifs divergents (De Corte, Greer et Verschaffel, 1996). En psychologie, l'étude de la cognition mathématique n'abordait pas les questions relatives aux pratiques pédagogiques, tandis que les éducateurs en mathématiques visaient à trouver des moyens de modifier la pratique pédagogique. Les recherches portant sur les processus impliqués dans l'apprentissage, caractérisées par la poursuite d'un objectif translationnel tout en étant fondées sur des preuves, font néanmoins l'objet d'un intérêt croissant (Davidesco et Milne, 2019 ; Higgins et al, 2019 ; Pasquinelli, Zalla, Gvozdic, Potier-Watkins, et Piazza, 2015). En effet, de nombreux programmes de recherche ont émergé visant une meilleure compréhension des mécanismes psychologiques impliqués dans l'apprentissage des mathématiques afin de changer les pratiques éducatives (p. ex. Carpenter, Fennema, & Franke, 1996 ; Van den Heuvel-Panhuizen & Drijvers, 2014). Cette ligne de recherche a été empruntée par différentes approches théoriques, qui vont ainsi donner la priorité à différents aspects (Lerman, 2006). L'objectif est parfois avant tout de permettre une meilleure compréhension de l'apprentissage, en se focalisant sur les processus de construction des savoirs individuels, ou plutôt d'améliorer des pratiques, en se focalisant sur le rôle joué par les enseignants dans le développement des connaissances des élèves. Formuler une question de recherche à partir des enjeux de la classe et élaborer une méthodologie adéquate pour s'emparer de ce problème demeurent des défis importants. Nous pensons que deux changements conceptuels contribuent de manière significative à établir un dialogue et à poursuivre des objectifs communs. L'un concerne l'importance croissante accordée aux approches situées en psychologie, et l'autre a trait à l'évolution du rôle accordé aux formalisations dans l'éducation.

Dans le domaine de la psychologie, la conception du savoir et de l'apprentissage a évolué historiquement avec l'apparition de différentes approches. Greeno, Collins et Resnick (1996) ont ainsi dégagé trois perspectives générales dans la littérature. La première est la perspective behavioriste/empiriste, où le savoir est « une accumulation organisée d'associations et de composantes de compétences » (p. 16). La seconde est la perspective cognitiviste/rationaliste, où le savoir est perçu comme « l'organisation de l'information dans les structures et procédures cognitives » (p.16). La troisième est la perspective situationnelle/pragramtique-sociohistorique, qui voit le savoir comme étant « distribué entre les personnes et leur environnement, y compris les objets, les artefacts, les outils, les livres et les communautés dont ils font partie » (p.17). Néanmoins, au cours des dernières décennies, les perspectives cognitivistes et situationnelles se sont rapprochées. Cette réconciliation s'observe dans des approches qui considèrent la cognition comme incarnée (Lakoff & Johnson, 1999). La pertinence de ce couplage se manifeste aussi dans l'influence du contexte sur la signification d'une situation (Barsalou, 1982). Combler le fossé est considéré comme un moyen de prendre en considération les structures mentales qui sont construites, tout en tenant compte de la flexibilité, de la malléabilité et de la nature distribuée des concepts (Vosniadou, 2007). Selon nous, l'adoption d'une perspective cognitive située rapproche le champ de la psychologie de la construction d'objectifs communs avec les recherches en éducation, car elle reconnait que les processus cognitifs sont largement influencés par le contexte dans lequel ils se développent et sont modulés par le contenu sur lequel ils agissent.

Nathan (2012) a proposé une critique d'une croyance répandue dans l'éducation et la société, qui se manifeste à travers une approche formaliste de l'apprentissage. La croyance sur laquelle repose ce point de vue est que la connaissance des formalismes d'un domaine est une condition préalable à l'application de cette connaissance. Les formalismes sont considérés à la fois au sens étroit, se référant à des formes de représentations spécialisées qui sont conventionnellement utilisées dans un domaine tel que les équations symboliques, et à la fois au sens large, où ils se réfèrent à des théories scientifiques et des principes formels. La prévalence de ce point de vue peut être observée dans les prédictions des enseignants concernant les performances des élèves en arithmétique et en algèbre. Les enseignants classent systématiquement les problèmes présentés en fonction de leur structure formelle. En effet, ils considèrent les équations symboliques comme plus faciles pour les élèves que les problèmes plus éloignés de leur structure formelle, tels que les problèmes à énoncés verbaux et narratifs (Nathan, Koedinger, & Alibali, 2001 ; Nathan & Petrosino, 2003). Ce n'est toutefois pas toujours le cas lorsqu'on examine les performances des élèves. Les formalismes ont sans aucun doute un rôle critique dans l'éducation, pourtant, adopter une vision des *formalismes en premier abord* revient à considérer que « le développement conceptuel passe du formel à l'appliqué » (Nathan, 2012, p. 128). Cela ne donne pas une vision adéquate du développement conceptuel, ce qui est l'une des principales raisons pour lesquelles cette approche est inappropriée. Nous pensons donc que toute tentative de mener des recherches conjointes dans le domaine de la psychologie et des sciences de l'éducation doit avoir une vision au moins modérée du rôle des formalismes dans l'éducation scolaire.

RESUME DE L'ETAT DE L'ART (CHAPITRES 1 A 4)

Nous considérons premièrement que l'étude de l'enseignement et de l'apprentissage doit tenir compte à la fois du développement conceptuel et de la compréhension du nouveau contenu en lien avec les connaissances antérieures. Deuxièmement, nous considérons que cette étude doit tenir compte également des croyances implicites des enseignants et de la façon dont ils façonnent leur pratique. Afin d'aborder ces questions, nous nous appuyons sur le champ du raisonnement analogique qui, selon nous, fournit une théorie unificatrice des dimensions informelle, formelle et située de l'apprentissage scolaire. Contrairement à la vision historique des analogies en tant que relations proportionnelles, les approches contemporaines la considèrent comme un moyen de comprendre une chose en se référant à autre chose (Holyoak & Thagard, 1995). L'analogie est vu comme un processus par lequel on établit une correspondance entre une entité inconnue ou moins connue – une cible – et une entité plus connue – une source (Gentner, 1989 ; Holyoak & Thagard, 1995). Les analogies permettent de dépasser l'expérience singulière d'une nouvelle situation en se référant à des catégories mentales guidant son interprétation (Hofstadter & Sander, 2013). Par exemple, si l'on adopte un point de vue socratique traditionnel et que l'on déclare : « un·e enseignant·e est comme une sage-femme », on fait une analogie. Une telle analogie permet d'envisager l'enseignement en se référant à une profession différente, qui pourrait être familière, et qui confère ses propriétés à « l'enseignant ». Cette affirmation implique que la personne qui enseigne ne ferait ressortir que des connaissances qui sont déjà implicites chez les élèves. De plus, faire une telle analogie signifie qu'une personne ne fait pas référence à un enseignant en particulier, mais à une catégorie générale qui instancie la profession enseignante (Glucksberg et Keysar, 1990).

Grâce à cette capacité de faire des liens par analogie dans leur vie quotidienne, les enfants développent une compréhension intuitive du fonctionnement du monde qui les entourne (Carey, 2009; diSessa, 1993, 2017; Piaget, 1960; Shtulman, 2017). Ces *conceptions intuitives* sont considérées comme étant spécifiques à un domaine (ex : la biologie, les mathématiques, etc). Cependant, leurs caractéristiques et mécanismes d'action sur le plan psychologique sont communs aux différents domaines. Cela permet notamment aux chercheurs d'étudier comment les conceptions intuitives peuvent influencer l'acquisition de nouvelles connaissances, puisqu'il est important de savoir quand elles facilitent l'acquisition d'un nouveau concept, et quand elles lui sont plutôt néfastes. Le point de vue que nous adoptons est que les conceptions intuitives constituent des notions fortement familières qui sont utilisées pour appréhender de nouvelles connaissances, par analogie (Hofstadter & Sander, 2013). Lorsque nous considérons une connaissance intuitive comme une source d'analogie, cela rend possible de comprendre comment elles exercent une influence sur l'acquisition de connaissances ultérieures. Il est important de reconnaitre qu'une conception intuitive a un domaine de validité : l'utiliser pour faire des prédictions peut parfois fournir des réponses qui sont adaptées pour le domaine cible. Cette utilisation vient néanmoins avec certaines limites, car elle produira, à d'autres moments, des inférences non valides dans le domaine cible (Hatano & Inagaki, 1987). Les connaissances intuitives en tant que connaissances antérieures impactent la façon dont une situation cible est comprise, mais elles interviennent même directement dans le traitement de la cible en temps réel. En fait, les connaissances préalables enrichissent la représentation d'une situation dès *l'encodage* initial (Ross & Bradshaw, 1994).

Les conceptions intuitives ont été identifiées dans une diversité de domaines enseignés à l'école, y compris les mathématiques. Fischbein (1987), ainsi que Lakoff et Núñez (2000) ont par exemple identifié la conception *soustraire c'est enlever* comme étant la conception intuitive la plus répandue de la soustraction. Selon Fischbein, lorsque la soustraction est effectuée par le biais de l'utilisation de la conception intuitive, le calcul qui est effectué vise en fait à déterminer la quantité restante après avoir retiré un sous-ensemble d'un ensemble plus large. Ce type d'action est également décrit dans les métaphores conceptuelles décrivant l'arithmétique comme une collection d'objets, où la plus petite quantité est retirée d'une collection plus grande, formant un autre objet (Lakoff & Núñez, 2000). Par conséquent, le signe « moins » d'une soustraction formelle déclencherait intuitivement cette conception *soustraire c'est enlever* (van den Heuvel-Panhuizen & Treffers, 2009). Dans ces cas, c'est le modèle intuitif qui se substitue à l'opération arithmétique. La solution à une soustraction découlant de ce modèle de soustraction serait appelée *le reste* (Usiskin, 2008). Fischbein (1993) a proposé différentes inférences à partir de cette conception. Si l'on est guidé par ce modèle intuitif de soustraction, si l'on a besoin d'extraire une quantité B d'une quantité A (A – B), cela ne peut se faire que si B < A. Il propose que si cela n'est pas respecté (B > A), alors un élève qui tient à ce modèle de soustraction pourrait procéder de plusieurs manières. Soit, il enlèvera autant d'unités que possible, soit il inversera la soustraction en B - A. En outre, une conception alternative de la soustraction, moins répandue, existe, celle de l'*écart*. Ces deux conceptions peuvent conduire à des réponses correctes, mais peuvent aussi amener à commettre des erreurs (Sander, 2001).

Les notions enseignées à l'école trouvent donc déjà leurs racines et sont façonnées par les expériences antérieures, prenant la forme de conceptions intuitives (Fischbein, 1987 ; Lakoff & Núñez, 2000). De même, les élèves disposent d'une grande variété de connaissances informelles avec lesquelles ils arrivent en classe, qui se manifestent dans leurs apprentissages tant conceptuels que procéduraux. Quant aux connaissances procédurales, les stratégies informelles mises en place par les élèves sont fondées sur les habiletés arithmétiques précoces telles que compter en utilisant des objets ou compter mentalement (Baroody et Ginsburg, 1986 ; Carpenter et al., 1981 ; Resnick, 1989). Une habileté informelle qui mène à une forme précoce d'arithmétique réside dans la capacité de l'enfant à trouver la valeur cardinale d'un ensemble en comptant les éléments qui le composent (Baroody et Ginsburg, 1986). Les enfants de cinq à six ans s'engagent généralement dans des stratégies de comptage sans respecter un ordre précis dans lequel les objets sont comptés. Pourtant cela ne signifie pas nécessairement qu'ils comprennent le principe de la cardinalité, qui consiste à ne pas respecter l'ordre. Cette arithmétique informelle se manifeste d'abord par l'utilisation d'objets ou de doigts, et ces procédures concrètes mènent au développement du calcul mental informel (Baroody & Ginsburg, 1986 ; Carpenter et al, 1981 ; Resnick, 1989).

Dans l'apprentissage des mathématiques à l'école élémentaire, la résolution de problèmes arithmétiques à énoncés verbaux permet de donner du sens et d'appliquer en situation réelle des concepts et des opérations arithmétiques. De nombreuses recherches empiriques s'intéressent aux processus mobilisés lors de tâches de résolution de problèmes, nous renseignant sur les stratégies mises en place (Brissiaud & Sander, 2010 ; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993 ; Riley, Greeno, & Heller, 1983 ; Verschaffel & De Corte, 1997). La lecture d'un problème arithmétique à énoncé verbal donne lieu à l'élaboration d'une représentation mentale qui tient compte des relations entre les éléments décrits dans le problème (Reusser, 1990). Avant d'être scolarisés en primaire, les enfants réussissent à résoudre certains problèmes arithmétiques en ayant recours à des stratégies de résolution informelles (Verschaffel & De Corte, 1997). Ces stratégies informelles consistent en des procédures qui ne mobilisent pas les formalisations mathématiques mais reflètent les situations décrites dans l'énoncé, et excluent l'application de principes arithmétiques tels que la commutativité (Verschaffel & De Corte, 1997). On pourrait penser qu'une fois les compétences arithmétiques acquises durant la scolarisation, ces stratégies informelles font place à d'autres, plus formelles. Cependant, des travaux sur les stratégies utilisées par les élèves de CE1 et CE2 ont mis en évidence que les stratégies informelles persistent, y compris suite à l'enseignement en classe de stratégies plus formelles (Brissiaud & Sander, 2010).

Le cadre théorique de la Primauté des Stratégies de Situation[6] (Brissiaud & Sander, 2010) décrit les processus de résolution de problèmes arithmétiques et souligne l'importance de la représentation que se fait un individu tentant de résoudre un problème. Notamment, le cadre prédit que la résolution d'un problème fasse appel à des processus qui modélisent les actions décrites dans l'énoncé et représentent donc des stratégies informelles. Dans le cas des problèmes additifs, les individus vont essayer en premier lieu de faire un comptage soit en avançant, soit à rebours, selon ce qui est décrit dans le texte. Cependant, cette stratégie informelle n'est pas toujours optimale et peut engager des procédures de calcul couteuses. Dans ce cas résoudre le problème nécessite de recourir à l'arithmétique mentale, et donc de mobiliser des stratégies de résolutions formelles. Le cadre PSS a permis de mettre en avant que les stratégies informelles continuent à être utilisées en premier recours même si elles sont couteuses et même lorsque les connaissances arithmétiques qui auraient facilité leur résolution sont acquises en classe.

Malgré le grand nombre de recherches qui ont étudié les conceptions présentes chez les élèves et les processus par lesquels les élèves acquièrent des connaissances en mathématiques, les conceptions des enseignants à l'égard de leurs élèves ont été beaucoup moins étudiées. Les connaissances intuitives persistent malgré les apprentissages scolaires et continuent à se manifester dans des contextes spécifiques. Par exemple, les adultes prennent plus de temps pour évaluer la justesse d'opérations de soustraction et de division lorsque les problèmes présentés sont incompatibles avec les connaissances intuitives des opérations que l'inverse, suggérant un effet facilitateur (Vamvakoussi, Van Dooren, & Verschaffel, 2013). Par ailleurs, Tirosh & Graeber (1991) ont mis en évidence que les connaissances intuitives persistaient même chez des enseignants en formation : ceux-ci réussissent nettement mieux des problèmes compatibles avec la connaissance intuitive de la division (94% de réussite), par rapport aux problèmes non compatibles avec cette connaissance intuitive (67% de réussite). En outre, lorsque des enseignants en formation devaient décrire les sources des erreurs commises par les élèves sur des problèmes de division, les enseignants décrivaient majoritairement des sources algorithmiques et rarement des sources intuitives (Tsamir & Tirosh, 2008).

[6] Situation Strategy First en anglais

Il est possible que les conceptions intuitives en arithmétique aient un impact sur la Connaissance Pédagogique du Contenu[7] (CPC) des enseignants (Shulman, 1986). Cette CPC s'appuie sur la connaissance du contenu qui est enseigné, mais c'est la connaissance du contenu qui est la plus pertinente pour son enseignement. Par ailleurs, enrichir sa CPC implique aussi que l'enseignant comprenne ce qui rend l'apprentissage d'une personne difficile ou facile. La CPC, par conséquent, réduit l'importance accordée à la connaissance du contenu et attire l'attention sur les connaissances des enseignants, qui ne se confondent pas avec celles des spécialistes du domaine. Shulman propose que cette catégorie de connaissances propres aux enseignants exige également qu'ils sachent quelles sont les idées préconçues que les élèves ont sur le sujet à différentes étapes de leur développement. Et cela d'autant plus que ces connaissances antérieures peuvent être inappropriées et demandent des enseignants qu'ils mettent en œuvre des stratégies qui aideront les élèves à réorganiser leurs connaissances.

Resume du chapitre 5 – La simulation mental au volant de la resolution des problemes arithmetiques [8]

La présente étude porte sur les processus qui sous-tendent la résolution de problèmes arithmétiques. Nous proposons que les stratégies informelles utilisées pour résoudre les problèmes arithmétiques reflètent la simulation mentale de *l'encodage* d'un énoncé. Cet encodage est influencé par la forme de présentation du problème et de la conception arithmétique associée. Ainsi, la représentation initialement encodée va déclencher une simulation mentale par laquelle l'individu tente de trouver une solution. Lorsque la simulation mentale est peu coûteuse pour l'individu, il trouve la solution au problème sur la base de cet encodage initial. Il s'agit d'une stratégie de résolution informelle. Ces problèmes sont donc *concordants avec la simulation mentale*. D'autre part, lorsqu'il n'est pas possible pour l'individu de trouver facilement une réponse par le biais de la simulation mentale, une stratégie formelle doit être utilisée. Ces problèmes sont donc *discordants avec la simulation mentale*. Lorsque la représentation encodée porte un coût élevé pour la simulation mentale, les individus soit s'appuient sur le codage initial mais appliquent des stratégies de calcul pour rendre le calcul possible, soit ils recodent la représentation du

[7] Pedagogical Content Knowledge en anglais

[8] Les résultats des expériences 1, 2 et 4 ont été publiés dans : Gvozdic, K., & Sander, E. (2017). Solving additive word problems: Intuitive strategies make the difference. In B. C. Love, K. McRae, & V. M. Sloutsky (Eds.), Proceedings of the 39th Annual Conference of the Cognitive Science Society. London, UK: Cognitive Science Society.

problème. Pour créer une nouvelle représentation du problème, l'individu devrait recoder la représentation initiale, en mobilisant les connaissances arithmétiques conceptuelles, ce qui conduirait à une représentation recodée, que l'individu pourrait à nouveau essayer de simuler mentalement. Le recodage de la représentation du problème est plus favorable lorsqu'il conduit à un avantage computationnel et rend cette représentation recodée facile à simuler mentalement.

Si nous prenons l'exemple des problèmes de Transformation 2 de la typologie de Riley, Greeno, et Heller (1983) tel que « Luc joue avec ses 22 billes à la récréation. Pendant la récréation, il perd 4 billes. Combien de billes Luc a-t-il maintenant ? », l'individu encoderait la représentation à l'aide de la conception *soustraire c'est enlever*. On s'engagerait ensuite dans la simulation mentale de cet encodage, qui aurait un faible coût. Dans le cas de ce problème concordant avec la simulation mentale, l'individu trouverait facilement la réponse par la stratégie informelle qui consiste à enlever 4 billes à 22. Cependant, pour les problèmes tels que « Luc joue avec ses 22 billes à la récréation. Pendant la récréation, il perd 18 billes. Combien de billes Luc a-t-il maintenant ? », l'individu encoderait la représentation à l'aide de la même conception *soustraire c'est enlever*, mais sa simulation mentale aurait un coût élevé car elle conduit à enlever une grande quantité de billes, 18 de 22. Dans le cas de ce problème discordant avec la simulation mentale, l'encodage initial n'est pas efficace pour trouver la réponse. Pour accéder à une stratégie de résolution plus efficace, une conception différente de l'arithmétique doit participer au recodage - la conception de la soustraction comme *écart*. Par conséquent, ce qui fait qu'un problème soit discordant avec la simulation mentale, c'est que son encodage initial serait influencé par une conception qui ne permet pas l'accès à la stratégie de résolution optimale, c'est-à-dire que la stratégie à laquelle mène l'encodage initial ne peut pas être facilement réalisée dans le même format que celui dans lequel il était encodé.

Reusser (1990) a proposé que les actions décrites dans un problème conduisent au modèle mathématique du problème. Brissiaud et Sander (2010) ont émis l'hypothèse que c'est la simulation de la situation décrite dans le texte d'un problème qui mène une solution. Cependant, ce que nous proposons est que la simulation mentale se fait à partir de la représentation encodée et pas uniquement à partir de la situation décrite par l'énoncé d'un problème. Par conséquent, afin de tester si la simulation mentale opère bien sur la représentation encodée, telle qu'elle est influencée par la conception arithmétique mobilisée par le problème, nous avons utilisé des problèmes qui ne décrivent pas des séquences d'actions. En fonction des différentes conceptions arithmétiques supposées être impliquées dans les encodages, nous avons construit des problèmes concordants et discordants avec la simulation mentale.

Dans une première expérience, nous avons évalué les performances des élèves de CE1 sur les problèmes arithmétiques à énoncés verbaux qui ne décrivent pas un changement dans la quantité qui apparaît au fil du temps. Nous avons postulé la conception de l'arithmétique mobilisée pour chaque catégorie de problème en fonction de sa formulation. Nous avons ensuite créé deux versions de chaque problème, un concordant et l'autre discordant avec la simulation mentale. Nous avions prédit que les élèves réussiraient mieux les problèmes concordants avec la simulation mentale que les problèmes discordants. Les problèmes discordants avec la simulation mentale seraient plus difficiles à résoudre parce qu'ils exigeraient que les individus recodent la représentation initiale du problème en s'appuyant sur une conception différente. Par conséquent, ce processus de re-représentation rendrait ces problèmes plus difficiles. En effet, si cet écart de performance entre les problèmes concordant et discordants avec la simulation mentale était observé sur des catégories de problèmes statiques, il témoignerait de l'importance de l'encodage initial, guidé par les différentes conceptions arithmétiques, qui peuvent être simulées mentalement. Il soulignerait également la nécessité du processus de re-représentation afin d'utiliser des stratégies formelles lorsque cet encodage initial n'est pas efficace.

Méthode

Participants. Seize classes de CE1 ont participé à l'étude, comportant 314 élèves. L'âge moyen des enfants en janvier était de 7,60 ans (écart-type = 0,33, 177 filles).

Matériaux. Il y avait 8 types de problèmes additifs appartenant à 3 grandes catégories correspondant aux problèmes de Comparaison 1, 2, 3, 4, Combinaison 1, 2 et Égalisation 1 et 2 de la classification de Riley et al. (1983) (voir Tableau 2). Des versions concordantes et discordantes avec la simulation mentale ont été créées. Les triplets impliqués dans les données et la solution sont (31, 27, 4), (33, 29, 4), (41, 38, 3), et (42, 39, 3). La taille des nombres n'était pas le facteur décisif pour déterminer la concordance avec la simulation mentale. Quatre contextes ont été utilisés pour la formulation de chaque problème : billes, euros, fleurs et fruits. Chaque problème a été présenté dans un seul contexte.

Design. Il y avait un total de 16 problèmes : 8 catégories de problèmes dans deux variantes différentes – concordants ou discordants avec la simulation mentale. Les élèves ont résolu un total de 8 problèmes créés en combinant les 8 catégories de problèmes dans l'une ou l'autre de ces versions. Chaque élève a donc résolu 4 problèmes concordants avec la simulation mentale et 4 problèmes discordants avec la simulation mentale. Pour contrôler l'effet de l'ordre, les valeurs numériques et le contexte, 8 ensembles de problèmes différents ont été créés. Huit autres ensembles de problèmes étaient des ensembles 'miroirs' dans lesquels la concordant était présentée dans son équivalent discordants, tandis que le problème discordant était présenté dans son équivalent concordant. Ainsi, 16 groupes d'ensembles de problèmes ont été créés et contrebalancés parmi les différentes classes.

Procédure. L'expérience a été menée en janvier et a été administrée dans les salles de classe des élèves. Chaque enfant a reçu un livret de 8 pages. Il y avait un carré au milieu de chaque page dans lequel ils écrivaient leur réponse. Chaque problème a été lu deux fois à haute voix à toute la classe, ce qui a réduit la charge de travail des élèves en lecture, et les élèves ont ensuite eu une minute pour noter le chiffre qui était la solution ; le problème suivant a ensuite été lu à haute voix.

Notation. Lorsque résultat était exact ou à +/-1, 1 point était attribué à l'élève. Toute autre réponse recevait 0 point.

Résultats

Une première analyse a été effectuée afin de comparer les taux de réussite des élèves sur des problèmes concordants et discordants avec la simulation mentale. Nous avons effectué des régressions logistiques à effets aléatoires. Nous avons construit un modèle linéaire mixte généralisé (MLMG) avec une distribution binaire avec la concordance de la simulation mentale (concordant vs. discordant) comme facteurs fixes, tandis que les participants et les catégories de problèmes ont été inclus comme effets aléatoires. Les analyses ont montré un effet principal significatif de la concordance avec la simulation mentale sur la performance (β = 1,05, z = 11,12, p < 0,001). Les étudiants ont réussi en moyenne sur 45,58 % des problèmes concordants et 27,10 % des problèmes discordants avec la simulation mentale. Les problèmes concordants avec la simulation mentale avaient un taux de réussite 1,69 fois plus élevé que les problèmes discordants.

Tableau 2 en français : Exemples d'énoncés de problèmes utilisés dans l'étude

Catégorie de problème		Problèmes concordants avec la simulation mentale	Problèmes discordants avec la simulation mentale
Comparison	Comparaison 1	Il y a 27 roses et 31 marguerites dans le bouquet. Combien y a-t-il de marguerites de plus que de roses ?	Il y a 4 roses et 31 marguerites dans le bouquet. Combien y a-t-il de marguerites de plus que de roses ?
	Comparaison 2	Il y a 31 oranges et 27 poires dans le panier. Combien y a-t-il de poires de moins que d' oranges ?	Il y a 31 oranges et 4 poires dans le panier. Combien y a-t-il de poires de moins que d'oranges ?
	Comparaison 3	Pierre a 27 billes. Jacques a 4 billes de plus que Pierre. Combien de billes Jacques a-t-il ?	Pierre a 4 billes. Jacques a 27 billes de plus que Pierre. Combien de billes Jacques a-t-il ?
	Comparaison 4	Anne a 31 euros. Carole a 4 euros de moins que Anne. Combien d'euros Carole a-t-elle ?	Anne a 31 euros. Carole a 27 euros de moins que Anne. Combien d'euros Carole a-t-elle ?
Egalisation	Egalisation 1	Dans le panier il y a 27 oranges et il y a 31 poires. Combien d'oranges doit-on ajouter pour avoir autant d'oranges que de poires ?	Dans le panier il y a 4 oranges et il y a 31 poires. Combien d'oranges doit-on ajouter pour avoir autant d'oranges que de poires ?
	Egalisation 2	Dans le bouquet il y a 31 roses et il y a 27 marguerites. Combien de roses doit-on enlever pour avoir autant de roses que de marguerites ?	Dans le bouquet il y a 31 roses et il y a 4 marguerites. Combien de roses doit-on enlever pour avoir autant de roses que de marguerites ?
Combinaison	Combinaison 2	Dans sa tirelire Céline a 27 euros et elle a des euros dans sa poche. Au total, Céline a 31 euros. Combien d'euros Céline a-t-elle dans sa poche ?	Dans sa tirelire Céline a 4 euros et elle a des euros dans sa poche. Au total, Céline a 31 euros. Combien d'euros Céline a-t-elle dans sa poche ?
	Combinaison 1	Dans le sac de Léo il y a 27 billes rouges et 4 billes bleues. Combien de billes y a-t-il dans le sac de Léo ?	Dans le sac de Léo il y a 4 billes rouges et 27 billes bleues. Combien de billes y a-t-il dans le sac de Léo ?

Discussion

Dans cette première expérience, les résultats ont montré que l'efficacité de la simulation mentale issue de l'encodage proposé a conduit à des taux de réussite différents, ce qui a permis de distinguer les problèmes concordants des problèmes discordants avec la simulation mentale. Ceci a été observé sur presque tous les problèmes testés, qui ne contiennent pas de scénarios avec des séquences d’actions qui pourraient favoriser la simulation mentale de la situation. Cette simulation mentale ne serait possible que si la représentation encodée ne créait pas un modèle mental dynamique qui pourrait être simulé. Nous avons mené ensuite trois autres expériences en classes de CE1. Dans ces autres expériences, nous avons fourni des preuves de la persistance du phénomène de la simulation mentale de la représentation encodée : même après l'enseignement ultérieur durant l'année scolaire, les élèves résolvent encore mieux les problèmes concordants avec la simulation mentale que ceux qui sont discordants. Ensuite, nous avons montré dans une troisième expérience que la facilité induite par la concordance avec la simulation mentale s’observe aussi sur des problèmes arithmétiques non-verbaux : le format du problème déclenche un encodage spécifique qui semble être la base d'une simulation mentale, dont l'efficacité détermine la facilité de résolution. Dernièrement, nous avons étudié les stratégies réellement utilisées par les élèves en récoltant des protocoles verbaux dans une quatrième expérience. Les résultats reflétaient les stratégies correspondant à l’encodage initiale lorsque le problème était concordant avec la simulation mentale, et les stratégies formelles correspondant à une représentation recodée du problème. Dans l'ensemble, les expériences menées indiquent que le processus responsable des stratégies de résolution informelles est une simulation mentale non-mathématique de la représentation encodée, tandis que l'utilisation de stratégies formelles dépend du recodage de la représentation initiale, effectué par le biais d'une autre conception arithmétique. Ces résultats confirment que la simulation mentale ne se produit pas seulement lorsque le problème illustre une séquence dynamique d'événements.

RESUME DU CHAPITRE 6 – VERS LE DEPASSEMENT DES STRATEGIES INFORMELLES [9]

Les recherches ont mis en évidence que la comparaison de différentes procédures lors de la résolution de problèmes a des effets bénéfiques pour les apprentissages (Rittle-Johnson & Star, 2011). En outre, la comparaison entre procédures informelles et formelles s'est avérée favorable à une conceptualisation adéquate (Hattikudur, Sidney, & Alibali, 2016). L'accès à une représentation autre que celle initialement évoquée par la situation a été rendue possible par des activités de comparaison en classe (Gamo, Sander, & Richard, 2010). Ce type de re-représentation décrit le processus de *recodage sémantique*. Ce processus pourrait être d'une grande importance dans le développement d'une « l'expertise adaptative », souligné comme un des principaux défis dans l'enseignement des mathématiques élémentaires (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009, p.1). Ainsi le recodage sémantique peut être mobilisé pour aider les élèves à aller au-delà de l'utilisation des stratégies informelles et à faire appel à des connaissances arithmétiques, nécessaires pour résoudre les problèmes discordants avec la simulation mentale.

La recherche-action Arithmétique et Compréhension à l'École Élémentaire (ACE) a élaboré un programme d'enseignement ayant pour objectif de favoriser l'apprentissage arithmétique à l'école primaire. Le programme comportait quatre grands domaines d'enseignement : un travail autour de situations « fil rouge », l'estimation, le calcul mental et la résolution de problèmes. Le domaine de la résolution de problème avait deux objectifs principaux : initier les élèves à l'analyse sémantique dans les tâches de résolution de problèmes et travailler sur le recodage sémantique afin de développer des stratégies optimales de résolution de problèmes.

Le programme commençait en engageant les élèves dans l'analyse des relations décrites dans les énoncés, d'abord sur des problèmes de Combinaison – dans lesquels deux éléments sont regroupés pour former un tout et la question porte sur un des éléments ou sur le tout ; en poursuivant avec des problèmes de Comparaison, dans lesquelles deux quantités sont comparées et la question porte sur la différence entre les deux. Ces problèmes encouragent la construction d'une représentation mentale des relations partie-tout, qui est considérée comme favorable pour la généralisation ultérieure et l'application des opérations arithmétiques dans d'autres contextes (Sophian, 2007). Les problèmes de Transformation étaient étudiés ensuite – où une quantité évolue

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[9] Les résultats de cette étude ont été acceptés pour publication dans : Gvozdic, K., & Sander, E. (accepted). Learning to be an opportunistic word problem solver: Going beyond informal solving strategies. *ZDM Mathematics Education*.

au cours du temps. Dans la résolution de problèmes comme dans l'ensemble du programme ACE, les situations étudiées étaient modélisées à l'aide de deux outils afin de présenter schématiquement les relations entre les éléments du problème : un schéma-ligne et un schéma-boîte (cf. Figure 17).

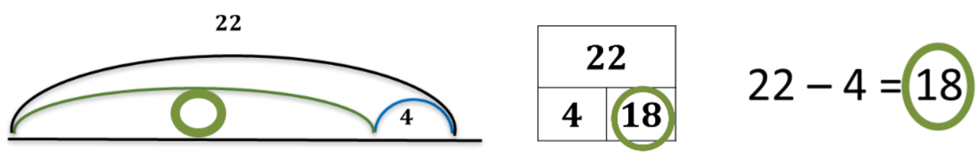
Le recodage sémantique constituait une étape clé. Les enseignants présentaient divers problèmes discordants avec la simulation mentale et incitaient leurs élèves à trouver la stratégie la plus courte pour les résoudre. Les stratégies étaient comparées et discutées en classe (cf. Figure 9 – problème discordant), et les élèves devaient choisir quelle procédure permettraient de trouver la réponse le plus facilement. Cette comparaison permettait aux élèves d'aller au-delà de leur encodage initial de la situation et de se re-représenter la situation afin de mettre en œuvre une procédure de résolution plus favorable (Gamo et al., 2010).

Dans cette étude nous avons cherché à évaluer l'efficacité d'activités de classe centrées sur le recodage sémantique. Les performances des élèves sur les problèmes concordants et discordants avec la simulation mentale étaient évalués, ainsi que leurs stratégies de résolution. La prédiction était que les élèves des classes ACE auraient des performances supérieures à celle des élèves des classes tout venants (TV), et notamment que les élèves ACE utiliseraient plus de stratégies formelles.

C. Problème concordant avec la simulation mentale

Nicolas va en récréation avec ses 22 billes. Pendant la récréation, il perd 4 billes. Combien Nicolas a-t-il de billes maintenant ?

Stratégie informelle

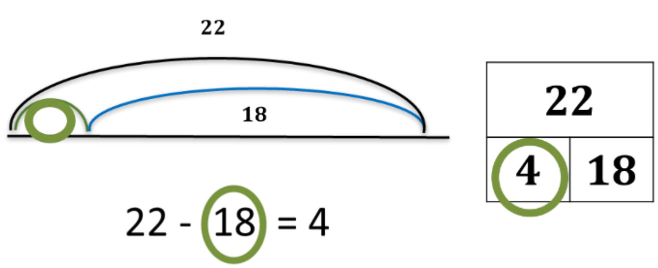


La stratégie informelle est de simuler mentallement la représentation encodé à partir de la situation décirt dans le problème – enlever 4 billes. Par conséquent, la stratégie informelle est de partiri de 22 et de compter à rebours de 4. Cela est noté comme une soustraction directe '22 – 4'. Dans le cas d'un problème concordant avec la simulation mentale, cette stratégie est facile à mettre en place.

A. Problème discordant avec la simulation mentale

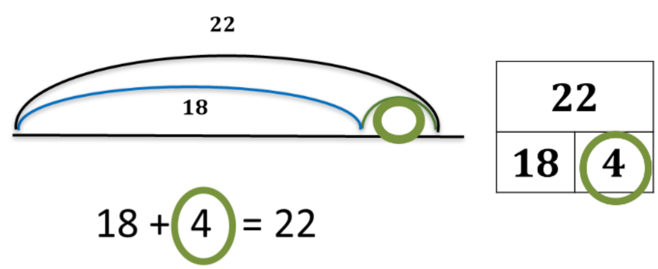
Luc is playing with his 22 marbles at recess. During the recess, he loses 18 marbles. How many marbles does Luc have now?

Stratégie infromelle



La stratégie informelle est de simuler mentallement la représentation encodé à partir de la situation décirt dans le problème – enlever 18 billes. Par conséquent, la stratégie informelle est de partiri de 22 et de compter à rebours de 18. Cela est noté comme une soustraction directe '22 – 18'. Dans le cas d'un problème discordant avec la simulation mentale, cette stratégie est difficile à mettre en place. On demandait aux élèves de chercher s'il y avait un autre moyen de trovuer la solution.

Stratégie formelle



La stratégie formelle consiste à se désengager du contexte sémantique et à changer la façon dont la situation est abordée, ce qui amène les élèves à résoudre le problème en recherchant l'écart entre 18 et 22. Ceci implique le recodage d'une situation de soustraction directe en une addition à trou qui est alors notée comme 18 + 4 = 22. On demande ensuite aux élèves quelle stratégie ils préfèrent utiliser pour résoudre ce problème.

Figure 9 en français : Descriptions des stratégies de résolution travaillées en classe

Méthode

Dix classes de CP ont participé à l’étude, comportant 208 élèves (5 classes ACE : 103 étudiants, âge moyen = 7.05, *ET* = 0.3, 60 filles ; 5 classes TV : 105 étudiants, âge moyen = 7.03, *ET* = 0.31, 57 filles).

Six catégories de problèmes à structure additive ont été présentées aux élèves. Chaque catégorie de problème était présentée dans sa version concordant et dans sa version discordant avec la simulation mentale. Les élèves résolvaient donc 12 problèmes arithmétiques, avec des contrebalancements de l’ordre de présentation, des valeurs numériques et de l’habillage des énoncés. Afin de s’assurer de la comparabilité des deux populations, les élèves devaient aussi résoudre des tâches contrôles, qui contenaient des exercices de géométrie, domaine non-travaillé dans le programme ACE.

Les passations expérimentales avaient lieu 3 à 4 semaines avant la fin de l’année scolaire. Chaque élève recevait un carnet pour écrire ses résultats et les opérations réalisées. Un autre carnet contenait les tâches de géométrie. Chaque problème était lu à voix haut à l’ensemble de la classe et répété deux fois.

Lorsque résultat était exact ou à +/-1, 1 point était attribué à l’élève. Selon la catégorie du problème, les stratégies que les élèves ont notées ont été classées selon le Tableau 2. Lorsqu'un élève notait une stratégie formelle sur un problème donné, il obtenait 1 point, et 0 point pour la stratégie informelle ou autre. L’ensemble des tâches contrôles étaient notés sur 4 points, conformément à une grille de codage.

Tableau 5 en français : Enoncés de problèmes avec les stratégies

Catégorie de problème	Concordance avec la simulation mentale	Enoncé de problème	Stratégie informelle	Stratégie formelle
Combinaison 1	Concordant	Dans le sac de Léo il y a 7 billes rouges et 4 billes bleues. Combien de billes y a-t-il dans le sac de Léo ?	7 + 4 = □	4 + 7 = □
	Discordant	Dans le sac de Léo il y a 4 billes rouges et 7 billes bleues. Combien de billes y a-t-il dans le sac de Léo ?	4 + 7 = □	7 + 4 = □
Combinaison 2	Concordant	Dans sa tirelire Céline a 7 euros et elle a des euros dans sa poche. Au total, Céline a 11 euros. Combien d'euros Céline a-t-elle dans sa poche ?	7 + □ = 11 11 – □ = 7	11 – 7 = □ □ + 7 = 11
	Discordant	Dans sa tirelire Céline a 4 euros et elle a des euros dans sa poche. Au total, Céline a 11 euros. Combien d'euros Céline a-t-elle dans sa poche ?	4 + □ = 11 11 – □ = 4	11 – 4 = □ □ + 4 = 11
Comparaison 1	Concordant	Il y a 7 roses et 11 marguerites dans le bouquet. Combien y a-t-il de marguerites de plus que de roses ?	7 + □ = 11 11 – □ = 7	11 – 7 = □ □ + 7 = 11
	Discordant	Il y a 4 roses et 11 marguerites dans le bouquet. Combien y a-t-il de marguerites de plus que de roses ?	4 + □ = 11 11 – □ = 4	11 – 4 = □ □ + 4 = 11
Comparaison 2	Concordant	Pierre a 11 billes. Jacques a 7 billes. Combien Jacques a-t-il de billes de moins que Pierre ?	11 – □ = 7 7 + □ = 11	11 – 7 = □ □ + 7 = 11
	Discordant	Pierre a 11 billes. Jacques a 4 billes. Combien Jacques a-t-il de billes de moins que Pierre ?	11 – □ = 4 4 + □ = 11	11 – 4 = □ □ + 4 = 11
Transformation 2	Concordant	Dans un bouquet, il y a 11 fleurs au total. Sophie enlève 4 fleurs du bouquet. Combien de fleurs y a-t-il dans le bouquet maintenant ?	11 – 4 = □	4 + □ = 11 11 – □ = 4
	Discordant	Dans un bouquet, il y a 11 fleurs au total. Sophie enlève 7 fleurs du bouquet. Combien de fleurs y a-t-il dans le bouquet maintenant ?	11 – 7 = □	7 + □ = 11 11 – □ = 7
Transformation 3	Concordant	Marie a 7 euros dans sa tirelire. Pour son anniversaire, elle reçoit d'autres euros et elle les met dans sa tirelire. Maintenant, elle a 11 euros. Combien Marie a-t-elle reçu d'euros pour son anniversaire ?	7 + □ = 11	11 – 7 = □ □ + 7 = 11
	Discordant	Marie a 4 euros dans sa tirelire. Pour son anniversaire, elle reçoit d'autres euros et elle les met dans sa tirelire. Maintenant, elle a 11 euros. Combien Marie a-t-elle reçu d'euros pour son anniversaire ?	4 + □ = 11	11 – 4 = □ □ + 4 = 11

Résultats

Nous avons analysé la performance des deux groupes, en exécutant un MLMG avec une distribution binaire avec concordance avec la simulation mentale et le groupe (ACE vs TV) comme facteurs fixes. Comme prévu, les problèmes concordants avec la simulation mentale étaient généralement plus faciles à résoudre que les problèmes discordants ($\beta = 0{,}76$, $z = 5{,}45$, $p < {,}001$). De plus, le groupe ACE a obtenu de meilleurs résultats globaux que le groupe TV ($\beta = 1{,}22$, $z = 5{,}41$, $p < {,}001$). Les étudiants du groupe ACE ont réussi 63,43 % des problèmes concordants avec la simulation mentale et 50,48 % des problèmes discordants, tandis que les étudiants du groupe TV ont obtenu un taux de réussite moyen de 42,22 % pour les problèmes concordants et 29,84 % pour les problèmes discordants.

Nous avons également effectué des analyses sur les stratégies utilisées par les élèves. Les principales analyses ont porté sur les stratégies utilisées par les élèves lorsqu'ils répondent correctement à un problème. Dans de tels cas, ils ont noté une stratégie 71,3 % du temps dans le groupe ACE et 61,67 % du temps dans le groupe TV. Avec ces données, nous avons effectué un MLMG avec une distribution binaire avec la concordance avec la simulation mentale et le groupe comme facteurs fixes. Parmi ces bonnes réponses, il y avait un effet global de la concordance avec signification mentale ($\beta = -1{,}64$, $z = -7{,}24$, $p < 0{,}001$), confirmant notre hypothèse que les stratégies formelles sont utilisées significativement plus souvent sur des problèmes discordants que sur des problèmes concordants. Il y a également eu un effet global du groupe ($\beta = 1{,}44$, $z = 4{,}49$, $p < 0{,}001$), confirmant notre hypothèse que le groupe ACE utilise des stratégies formelles significativement plus que le groupe TV. Puisque les problèmes discordants avec la simulation mentale étaient ceux qui bénéficieraient le plus de l'utilisation de stratégies formelles, nous avons analysé davantage les stratégies utilisées pour résoudre correctement les problèmes discordants avec la simulation mentale. Nous avons effectué un MLMG avec une distribution binaire avec le groupe comme facteur fixe et les participants et la catégorie de problèmes comme effets aléatoires. Comme prévu, le groupe ACE a utilisé des stratégies formelles sur les problèmes discordants beaucoup plus que le groupe TV ($\beta = 1{,}32$, $z = 3{,}52$, $p < 0{,}001$).

Discussion

Cette étude nous renseigne sur des processus de résolution des problèmes à énoncés verbaux. Comme prévu, les étudiants des classes TV ont très rarement utilisé des stratégies formelles sur des problèmes concordants avec la simulation mentale, contrairement au groupe

ACE. Cependant, des stratégies de résolution informelles ont été observées sur des problèmes discordants dans les deux groupes, mais le groupe ACE a utilisé 2,2 fois plus de stratégies formelles que le groupe BAU. Il est possible de trouver la solution à des problèmes discordants par la simulation mentale, mais les stratégies de résolution informelles ne sont pas les plus efficaces dans ce cas, ce qui contribuait à la baisse des taux de performance dans le groupe TV sur ces problèmes. Ceci souligne l'importance de trouver des moyens appropriés pour guider les élèves dans la recherche de la stratégie la plus efficace. L'intervention ACE avait la particularité de mettre les élèves au défi d'écarter l'utilisation des stratégies informelles en leur donnant l'occasion d'utiliser des procédures dans divers contextes. Les problèmes concordants ont permis aux élèves de travailler sur leurs connaissances procédurales, tandis que les problèmes discordants les ont amenés à inventer et à rechercher des stratégies alternatives. Cela aurait pu favoriser le développement de connaissances conceptuelles arithmétiques tout en pratiquant des procédures connues, ce qui contribue à la rétention des connaissances (Baroody et al., 2007).

Les résultats de l'étude s'orientent vers un examen plus approfondi du recodage sémantique qui vise un changement conceptuel afin d'améliorer l'utilisation des stratégies formelles par les élèves pour résoudre les problèmes à énoncés verbaux. Le recodage sémantique considère le changement de représentation comme la clé principale pour aller au-delà des stratégies informelles basées sur des situations et passer à des stratégies formelles (Brissiaud & Sander, 2010 ; Gamo et al., 2010). Dans cette étude, le recodage sémantique semble avoir favorisé le changement flexible entre les différentes représentations, puisque les étudiants des classes ACE utilisaient plus de deux fois plus de stratégies formelles que les classes BAU, et surtout, les stratégies formelles des classes d'intervention étaient plus fréquemment utilisées pour les problèmes les plus coûteux, qui devaient bénéficier d'un recodage sémantique.

RESUME DU CHAPITRE 7 – LES CONNAISSANCES PEDAGOGIQUES DES ENSEIGNANTS DANS L'ANGLE MORT DE L'INTUITION [10]

Cette recherche a évalué si un contenu compatible avec la connaissance intuitive influence les interprétations que font les enseignants des stratégies utilisées par les élèves pour résoudre les problèmes arithmétiques. Les connaissances intuitives sont présentes chez les adultes tout venants et chez les enseignants, cependant seuls les enseignants possèdent des CPC. Nous avons recruté les deux populations afin de déterminer si les enseignants, guidés par leurs CPC, évaluent différemment les stratégies de résolution mises en place par les élèves que des adultes tout venants, ou inversement pour déterminer si leurs connaissances intuitives masquent leur CPC. Nous avons présenté des paires de problèmes arithmétiques à énoncés verbaux. La tâche était de choisir lequel des deux problèmes était mieux réussi par les élèves et de justifier leur choix. Une hypothèse était que les enseignants identifieraient mieux que les adultes tout venants les stratégies mises en place par les élèves grâce à leur CPC. En revanche, nous avons également postulé que les connaissances intuitives biaiseraient leur compréhension de ces stratégies. Plus précisément, nous avons prédit que les CPC des enseignants ne pourraient pas s'exprimer sur les énoncés compatibles avec la connaissance intuitive et qu'ils auraient des difficultés à identifier correctement les stratégies mises en place par les élèves. Au contraire, nous avons fait l'hypothèses que lorsque le problème n'est pas compatible avec la connaissance intuitive les enseignants mobiliseraient leurs CPC et identifieraient mieux que les adultes tout venants les stratégies des élèves.

Méthode

L'étude a été réalisée auprès de 36 professeurs d'école élémentaire et de 36 adultes tout venants (âge moyen de 32.67 ans, 52 femmes, et les groupes étaient appariés par sexe).

Les items étaient construits par appariement de variantes d'un même énoncé, issues de l'étude de Brissiaud et Sander (2010), pour lesquelles les taux de réussite et stratégies déployées par les élèves sont connus : la variante concordante étant toujours mieux réussie que la variante discordante. Deux types de problèmes étaient utilisés, un énoncé dont le contexte était compatible avec la connaissance intuitive de la soustraction, et un deuxième dont le contexte sémantique n'était pas compatible avec la connaissance intuitive.

Cet appariement a conduit à la construction de 4 items :

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[10] Les résultats de cette étude ont été publiés dans : Gvozdic, K., & Sander, E. (2018). When intuitive conceptions overshadow pedagogical content knowledge: Teachers' conceptions of students' arithmetic word problem solving strategies. Educational Studies in Mathematics. 98(2), 157-175.

- Un item où les deux problèmes appariés étaient du même type, compatibles tous deux avec la connaissance intuitive,
- Un item où le problème concordant n'était pas compatible avec la connaissance intuitive alors que le problème discordant l'était,
- Un item où le problème concordant était compatible avec la connaissance intuitive et le problème discordant non compatible,
- Un item où les deux problèmes appariés n'étaient pas compatibles avec la connaissance intuitive.

Dans un premier temps les participants se prononçaient sur la difficulté relative des énoncés proposés dans chaque item. Ensuite les participants devaient justifier chaque choix. Nous avons analysé les justifications pour les items à propos desquels les participants ont choisi correctement que le problème concordant était mieux réussi. Lorsque les justifications étaient congruentes avec les stratégies mises en place par les élèves, on attribuait un score de 1, et de 0 lorsqu'elles étaient non-congruentes.

Conformément à nos hypothèses, nous avions prédit que pour les items dans lesquels le problème discordant était compatible avec la connaissance intuitive, il n'y aurait pas de différence entre les deux populations. Au contraire, lorsque le problème discordant n'était pas compatible avec la connaissance intuitive, notre prédiction était que les enseignants réussiraient mieux que les adultes tout venants à identifier les stratégies des élèves.

Résultats

Nous avons cherché à identifier un éventuel effet groupe sur les performances des participants, en effectuant un Modèle Linaire Généralisé (MLG) à mesure répétées sur le score des justifications, avec le facteur groupe et les facteurs contrebalancés comme variables inter-sujets. Les résultats ont indiqués un effet principal du facteur groupe ($F(1, 56) = 6.959$, $p <.05$), un effet principal des justifications ($F(3,56) = 11.179$, $p < .001$), et pas d'interaction entre les justifications et le facteur groupe ($F(3,56) = 1.549$, $p = .204$)**.** Ces résultats montrent que les justifications différaient d'une population à l'autre et d'un item à l'autre.

Nous avons aussi étudié s'il y avait une différence entre les deux populations sur la justification des items lorsque le problème discordant était compatible avec la connaissance intuitive et ceux pour lesquels le problème discordant n'était pas compatible avec la connaissance intuitive. Un MLG a été réalisé sur la moyenne des justifications aux items comme variable dépendante et la variable groupe et les variables contrebalancés comme variables inter-sujets. Comme prédit, un effet global significatif des items ($F(1,56) = 8.347$, $p < .01$) a été observé, ainsi qu'un effet principal de groupe ($F(1,56) = 6.959$, $p = .011$). Une interaction significative entre les items et les groupes $F(1,56) = 3.907$, $p = .05$) a également été notée.

Discussion

L'analyse des justifications des participants a montré qu'il n'y avait pas de différence dans les jugements des populations lorsque le problème discordant était compatible avec la connaissance intuitive de la soustraction. Les enseignants avaient toutefois de meilleures performances que les adultes tout venants sur les items pour lesquelles les problèmes discordants n'étaient pas compatibles avec les connaissances intuitives. Cela indique que les enseignants identifient plus correctement les stratégies des élèves pour résoudre les problèmes lorsque leurs connaissances intuitives n'entrent pas en compétition avec leur CPC.

Le fait que les enseignants aient des difficultés à accéder aux stratégies de résolution mises en place par les élèves n'est pas forcement surprenant compte tenu du fait que les connaissances issues des recherches sur la résolution des problèmes arithmétiques à énoncés verbaux ne sont toujours pas prises en compte dans les manuels scolaires ou dans les formations des enseignants, même si leurs implications sont conséquentes sur le plan éducatif. Cette étude montre l'importance d'intégrer l'enseignement sur les connaissances intuitives en mathématiques dans la formation des enseignants. Cela pourrait aider les enseignants à être moins sujets aux biais observés ici, et également contribuer à un travail pour faire évoluer les connaissances intuitives de leurs élèves.

RESUME DE LA DISCUSSION GENERALE

La présente thèse visait à comprendre le rôle que les connaissances informelles jouent dans l'enseignement et l'apprentissage des mathématiques. Nous avons examiné comment différentes catégories conceptuelles construites à partir des expériences antérieures influencent les processus représentationnels qui ont lieu au cours de la résolution de problèmes arithmétiques. Nous avons considéré que le processus d'encodage analogique, par lequel les caractéristiques d'une situation sont mises en correspondance avec des catégories mentales préalablement construites, est un processus clé qui guide la construction d'une représentation mentale de la situation.

Nous avons mené une série d'études visant à montrer comment les différentes conceptions mobilisées dans l'encodage des problèmes arithmétiques influencent les stratégies de résolution de problèmes par des élèves et les jugements des enseignants sur les stratégies utilisées par les élèves. Dans le Chapitre 5, nous avons proposé que la simulation mentale de la représentation encodée est une étape cruciale dans la résolution de problèmes arithmétiques à énoncés verbaux. Nous avons fait des prédictions concernant les conceptions qui seront mobilisées dans l'encodage d'un problème et concernant l'efficacité de la simulation mentale opérant sur cet encodage. Nous avons ensuite effectué une série d'expériences qui ont testé l'effet que la concordance ou discordance avec la simulation mentale a sur les stratégies de résolution des élèves. Nous avons constaté que les élèves de CE1 utilisaient quasi exclusivement des stratégies de résolution informelles et avaient des meilleures performances sur les problèmes concordants pour lesquels la simulation mentale de la représentation encodée était peu coûteuse. Lorsque la représentation encodée menait à une simulation mentale coûteuse, ce qui était le cas sur les problèmes discordants, les élèves ont surtout signalé des stratégies de résolution formelles qui ne correspondaient pas à la représentation initialement encodée, mais reflétaient une représentation recodée.

Dans le Chapitre 6, nous avons étudié comment les élèves parviennent à résoudre des problèmes dont la simulation mentale a un coût élevé. Nous avons proposé que la réussite à des problèmes discordants est facilitée lorsque les individus utilisent des stratégies qui s'appuient sur une conception arithmétique différente de celle qui était impliquée dans l'encodage initial. Le recodage sémantique de la représentation initiale du problème devait permettre l'utilisation d'une stratégie de résolution plus efficace. Nous avons comparé les performances et les stratégies des élèves de CP qui ont participé à une recherche-action en arithmétique à ceux des élèves des classes tout venants. L'enseignement dans l'intervention arithmétique s'est concentré sur des exercices d'encodage et de recodage sémantique en utilisant des problèmes arithmétiques à énoncés verbaux. Nous avons constaté que les élèves qui ont bénéficié de l'intervention avaient de meilleures performances sur les problèmes discordants, pour lesquels la simulation mentale était coûteuse, et ces élèves utilisaient plus fréquemment des stratégies de résolution formelle.

Dans le Chapitre 7, nous avons cherché à déterminer si les conceptions qui influencent l'encodage des problèmes arithmétiques influencent également le jugement des enseignants sur les performances des élèves et les stratégies qu'ils utilisent. Les Connaissances Pédagogiques du Contenu (CPC) des enseignants sont censées leur fournir des moyens pour comprendre les stratégies de résolution des problèmes des élèves. Cependant, nous avons observé que lorsque la conception intuitive de la soustraction est impliquée dans l'encodage d'un problème arithmétique, les enseignants ont plus de difficulté à comprendre ce qui rend le problème difficile pour les élèves, c'est-à-dire le coût que porte l'utilisation des stratégies formelles. Dans cette thèse nous avons identifié certaines conceptions devant guider la résolution de problèmes arithmétiques en participant à leur encodage. Nous avons observé que ces différents encodages avaient un impact sur la difficulté des problèmes puisque différents encodages conduisaient à des stratégies différentes. Ces résultats suggèrent que des connaissances préalables enrichissent la représentation mentale d'une situation telle qu'elle est perçue (Gentner et al., 2003 ; Ross & Bradshaw, 1994). Nos résultats suggèrent que les processus d'abstraction qui conduisent à l'encodage de problèmes arithmétiques, qu'ils soient ou non des problèmes verbaux, impliquent la sélection d'une conception arithmétique. De plus, nous avons proposé que c'est cette représentation encodée qu'un individu tentera de simuler mentalement afin de trouver une solution. Lorsque la simulation mentale est efficace, cela amène les individus à trouver une solution numérique. Trouver la réponse de cette manière ne nécessite pas de connaissances arithmétiques formelles. Pour cette raison, quand la stratégie de résolution reflète la simulation mentale, ces stratégies sont considérées comme des stratégies de résolution informelles. Pourtant, trouver la solution à un problème en simulant mentalement la représentation encodée n'est pas toujours simple. Lorsque la simulation mentale est difficile à exécuter, les individus devront mobiliser des connaissances arithmétiques. Ce processus de résolution est décrit dans la Figure 16.

Répercussions théoriques

Dans cette thèse, nous avons étudié des problèmes où une représentation recodée conduirait à une stratégie de résolution plus efficace. Nous avons proposé que lorsque la simulation mentale a un coût élevé, les individus doivent recoder leur représentation initiale du problème pour trouver une stratégie de résolution plus optimale. Cela signifie que les individus doivent s'appuyer sur une conception arithmétique différente de celle utilisée dans l'encodage initiale et qu'avec son aide, ils recodent sémantiquement la représentation du problème. Cette représentation recodée peut subir à nouveau le processus de simulation mentale et, dans le cas des problèmes étudiés dans cette thèse, cela conduit efficacement à la solution numérique. Cependant, il convient de noter que parfois, même lorsque la stratégie informelle est inefficace, le recodage de la représentation initiale ne conduit pas à une stratégie de résolution efficace. Dans ces cas, les individus devraient avoir recours à des stratégies de résolution formelles qui n'exigent pas un recodage.

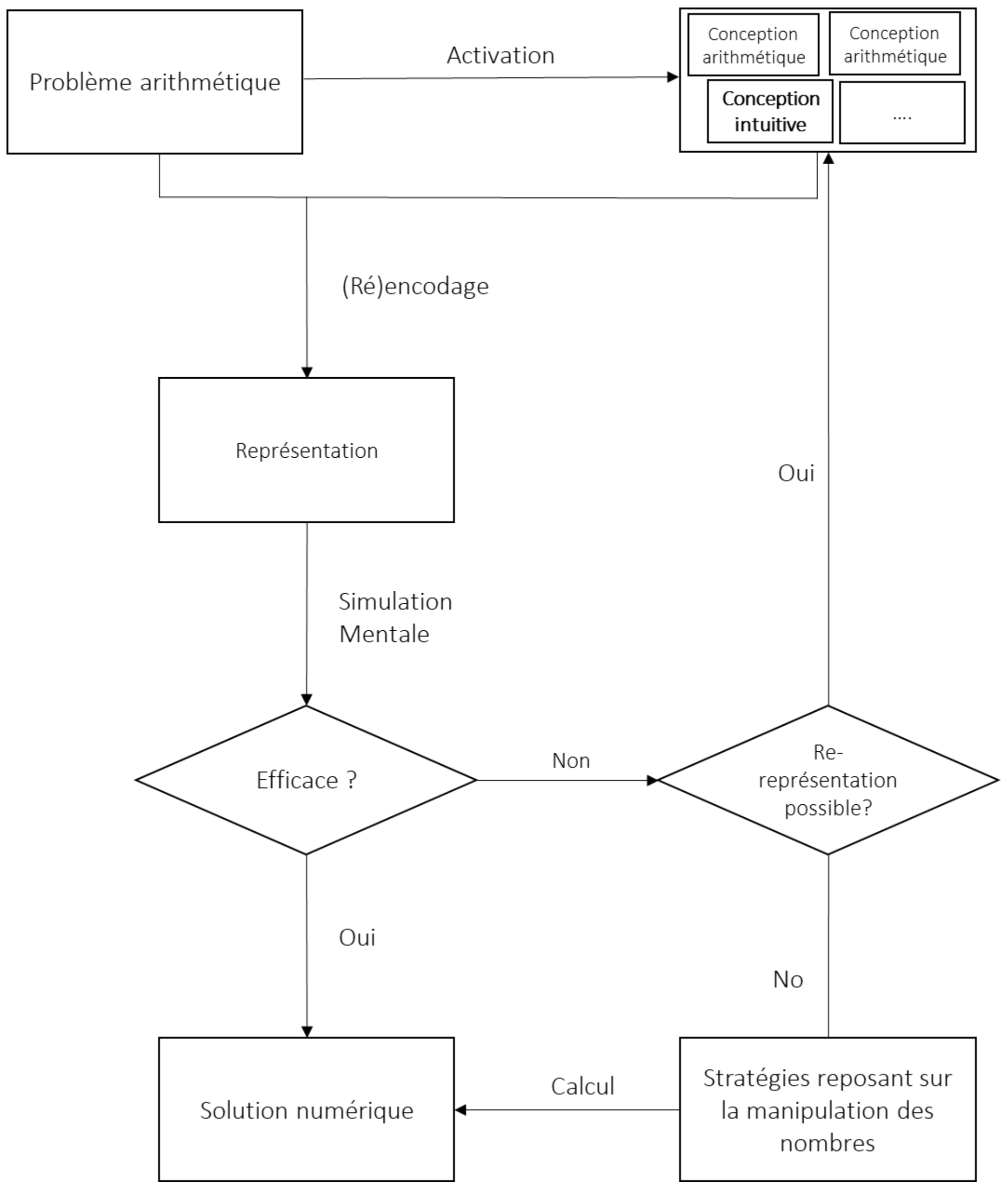

Figure 16 en français : Modélisation des processus d'encodage et de recodage dans la résolution de problèmes arithmétiques

Pour comprendre la commutativité, l'associativité et la complémentarité entre l'addition et la soustraction, il est essentiel d'adopter un raisonnement fondé sur des relations partie-tout (Resnick, 1989). Cependant, s'engager dans un raisonnement fondé sur des relations partie-tout n'est pas une tâche facile, puisque les élèves continuent à avoir de mauvaises performances sur les problèmes qui nécessitent l'utilisation de stratégies formelles reflétant l'utilisation de connaissances arithmétiques. Les conceptions intuitives de l'arithmétique, qui opèrent en arrièreplan et entrent en conflit avec les composantes formelle et algorithmique des activités mathématiques (Fischbein, 1993), peuvent constituer un obstacle dans l'apprentissage et la résolution. Dans le cas de la conception intuitive *soustraire c'est enlever*, cela pourrait signifier qu'en présentant un problème dont la formulation a une structure sémantique qui est la plus proche de cette conception, le problème devrait être plus difficile à recoder pour les élèves, car il serait plus difficile de changer son point de vue lorsque le contenu est conforme à leur intuition.

Les problèmes qui se rapprocheraient le plus de cette conception seraient les problèmes de type Transformation 2, comme ceux testés dans le Chapitre 6 et par Brissiaud et Sander (2010). En effet, la formulation dans ce type de problème porte sur la quantité qui reste après avoir ôté. En examinant les performances pour chaque problème, c'est en effet sur les problèmes Transformation 2 que l'on observait systématiquement l'écart de réussite entre les problèmes concordants et discordants avec la simulation mentale la plus élevée. C'était le cas dans l'étude de Brissiaud et Sander, ainsi que dans notre population d'élèves de classes tout-venant dans le Chapitre 6. Dans la troisième expérience du Chapitre 5, les problèmes de soustraction directe présentaient un écart de réussite sur les problèmes les plus élevés, ce qui suggère que le signe moins est étroitement lié à la conception de la soustraction. Dans le cas où la conception intuitive est mobilisée dans l'encodage d'un problème, la compatibilité entre les caractéristiques sémantiques du problème et la conception arithmétique peut conduire à un encodage qui sera plus difficile à recoder par les individus. Cela donne un aperçu des effets médiateurs que la sémantique du problème peut avoir dans l'encodage du problème, mais cela permet aussi d'illustrer comment la connaissance intuitive peut influencer la persistance de certaines stratégies.

Par ailleurs, nous avons observé dans cette thèse que les conceptions intuitives impactent non seulement les stratégies de résolution des élèves, mais également les jugements des enseignants. Les recherches antérieures ont mis en évidence que les enseignants ayant une expertise élevée du contenu ne s'appuient pas sur les principes du développement des élèves pour juger la difficulté des exercices mathématiques (Nathan et Petrosino, 2003, p. 918). À la place, leurs jugements reflétaient plutôt leur vision de la complexité ontologique de la discipline, ce qui les amenait à faire des prédictions inexactes sur la façon dont les élèves résolvent les problèmes. Ce phénomène était intitulé par les chercheurs *l'angle mort de l'expertise*. Cependant, quand il s'agit de l'école élémentaire, les enseignants sont rarement des experts mathématiciens (par exemple, aucun des enseignants qui a participé dans l'étude du Chapitre 7 ne s'est spécialisé en mathématiques avant de devenir enseignant). Dans la présente thèse, nous avons montré que la compréhension qu'ont les enseignants des stratégies des élèves en résolution de problèmes arithmétiques à énoncés verbaux était éclipsée par la conception intuitive de la soustraction. Les enseignants considéraient que les problèmes liés à la conception intuitive étaient plus faciles pour les élèves, tout comme le pensaient les non-enseignants, et ils n'avaient pas plus de succès que les adultes tout venants pour identifier les stratégies mises en place par les élèves dans ce contexte.

Nos résultats ont mis en lumière le fait que, parallèlement à ce qui est prédit par *l'angle mort de l'expertise*, les CPC des enseignants ont été occultées dans certains contextes lorsque la conception intuitive entrait en jeu. Nous proposons que les CPC des enseignants soient influencées par deux facteurs d'origines opposées mais aux conséquences similaires : l'expertise et la conception intuitive. Le premier implique le phénomène de l'angle mort de l'expertise, tandis que le second implique l'angle mort de l'intuition. Chacun induit un point de vue non flexible quant aux évaluations des processus des élèves, servant comme principe qui guide l'évaluation du comportement des élèves et éclipsant les CPC des enseignants. Si on suit le modèle de décision que Ostermann et ses collaborateurs (2017) ont proposé pour les processus par lesquels passe le jugement des enseignants, cela suggère que lorsqu'un contenu est cohérent avec la conception intuitive, les enseignants considèrent simplement ce contenu comme étant facile. Les enseignants utilisent donc le modèle intuitif pour évaluer les difficultés que le contenu pose aux élèves. Néanmoins, quand un contenu tombe dans l'angle mort de l'intuition, les enseignants ne manquent pas nécessairement de connaissance sur ce contenu ni de CPC. Leurs CPC peuvent être plutôt sous-spécifiés, ce qui les amèneraient à faire des prédictions basées sur la facilité perçue du contenu compatible avec la conception intuitive. Dans cette optique, nous pensons qu'il convient également d'attirer l'attention sur le fait qu'il faut simultanément travailler sur les conceptions intuitives des élèves dans les salles de classe tout en attirant l'attention des enseignants sur ce phénomène.

Répercussions éducatives

L'un des aspects qu'il reste à élucider au sujet de l'angle mort de l'intuition, concerne son origine. Provient-il de la conception intuitive propre à l'enseignant, ou du fait que le contenu lui-même soit conforme à cette conception intuitive ? Nous avons vu que les conceptions intuitives persistent dans la population adulte (Tirosh & Graeber, 1991 ; Vamvakoussi et al., 2013). Un contenu conforme à l'intuition a un impact sur la performance et le temps de réaction avec lequel les adultes effectuent diverses tâches (Dunbar et al., 2007 ; Goldberg & Thompson-Schill, 2009 ;

Shtulman & Harrington, 2016). Bien que la persistance des connaissances intuitives ne soit pas suffisante pour déterminer si l’angle mort de l’intuition provient de la conception intuitive des enseignants ou de la cohérence du contenu avec l'intuition, ces études soulignent que les conceptions intuitives continuent d'être une source d'analogie utilisable par les enseignants. Comme nous l'avons vu, c'est au cours du traitement d'une situation que les connaissances préalables peuvent influencer la façon dont une situation est interprétée (Ross & Bradshaw, 1994). De plus, lorsque la situation en cours de traitement est conforme aux connaissances antérieures, il est plus facile d'interpréter la situation. Cela nous conduit à considérer que, même si les enseignants ne résolvent pas un problème, ils évaluent sa difficulté pour les élèves, ce qui exige qu'une représentation soit encodée. Lorsque le contenu est conforme à une connaissance intuitive, celle-ci guide l'encodage qui lui-même influencera les hypothèses des enseignants sur ce que les élèves savent. C'est donc à ce stade initial que l'angle mort de l'intuition jette une ombre sur la CPC des enseignants. Sensibiliser les enseignants à son existence pourrait être bénéfique pour renforcer leurs compétences diagnostiques.

Quant aux élèves et à l’évaluation de leurs connaissances, bien que nous nous soyons centrés sur les processus impliqués dans la résolution de problèmes arithmétiques, nous avons vu que les élèves disposent d'une variété de connaissances informelles, à la fois dans leur apprentissages conceptuel et procédural. Nos études dans les Chapitres 5 et 6 ont montré que le contenu des problèmes détermine la stratégie informelle utilisée. Le problème Comparaison 1 conduirait à l'utilisation de l'addition à trou ou de la soustraction à trou comme stratégie de résolution informelle, tandis que le problème Comparaison 4 conduirait à l'utilisation de la soustraction directe comme stratégie de résolution informelle. De plus, nous avons montré qu'il y a une grande différence de performance selon que la tâche nécessite l'utilisation de connaissances arithmétiques formelles ou informelles.

Nos observations amènent à réfléchir au type de tâche qui convient le mieux pour évaluer les objectifs de l'apprentissage. Dans le chapitre 6, le paradigme utilisé ne nous permettait pas de nous prononcer sur la mesure dans laquelle le progrès sur les stratégies informelles contribuait à l'utilisation accrue des stratégies formelles. Cependant, il ne faut pas négliger les avantages qui peuvent résulter du travail en classe sur les stratégies informelles des élèves. Le point de vue dominant en psychologie cognitive est que les relations entre les connaissances conceptuelles et procédurales sont bidirectionnelles et que progresser sur l'une mène aussi au progrès sur l'autre

(Rittle-Johnson, 2019 ; Rittle-Johnson et al., 2001). De plus, lier les nouvelles connaissances mathématiques aux connaissances informelles favorise la compréhension des élèves (Baroody et Wilkins, 1999 ; Van den Heuvel-Panhuizen et Drijvers, 2014).

Cependant, un objectif primordial de l'enseignement des mathématiques est de choisir la stratégie la plus appropriée pour trouver la solution à un problème (Threlfall, 2009 ; Verschaffel et al., 2009). Si l'objectif est d'évaluer l'utilisation des stratégies adaptées aux problèmes, alors il est préférable d'utiliser des problèmes qui bénéficieraient le plus de l'utilisation de stratégies formelles. En effet, une réponse correcte à un problème qui peut facilement être résolu par l'utilisation de stratégies informelles ne constitue pas une bonne indication lorsqu'on cherche à évaluer l'utilisation des principes arithmétiques, et c'est d'autant plus le cas si ces problèmes sont compatibles avec des connaissances intuitives, comme les problèmes Transformation 2.

Par exemple, le problème « Il y a 21 fleurs dans le bouquet. Sophie retire 3 fleurs du bouquet. Combien y a-t-il de fleurs dans le bouquet maintenant ? » fournit directement l'encodage '21 - 3' et ne prouve donc pas la compréhension conceptuelle de la connaissance arithmétique formelle. Pourtant, si « Sophie enlève 19 fleurs », il serait plus avantageux de recoder le problème en utilisant des principes arithmétiques et de le résoudre avec la stratégie '19 + ? = 21'. Il est donc important de noter que si les connaissances des élèves sont évaluées par le biais de problèmes qui peuvent être facilement résolus par des stratégies de résolution informelles, il est trompeur de supposer que ces élèves ont appliqué des connaissances arithmétiques. Ce qui est plus pertinent d'un point de vue pédagogique, en ce qui concerne la sélection de la stratégie la plus appropriée pour résoudre un problème, c'est d'utiliser des tâches pour lesquelles différentes stratégies de résolution révéleraient la compréhension ou non des principes arithmétiques, comme les problèmes discordants avec la simulation mentale pour lesquels l'utilisation de la stratégie informelle n'est pas optimale.