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Stationary critical points for a phase field model in fracture mechanics

Points critiques stationnaires pour un modèle champ de phase en mécanique de la rupture

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THESE DE DOCTORAT

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Titre : Points critiques stationnaires pour un modèle champ de phase en mécanique de la rupture
Mots clés : Calcul des variations, équations aux dérivées partielles, fonctionnelle d'Ambrosio–Tortorelli, fonctionnelle de Mumford–Shah, variations internes.

Résumé : Cette thèse est consacrée à la fonctionnelle de Mumford-Shah, utilisée pour modéliser les fissures fragiles dans un matériau élastique, et plus particulièrement à son approximation par la fonctionnelle d'Ambrosio–Tortorelli. Nous nous intéressons notamment à l'analyse asymptotique des points critiques de la fonctionnelle d'Ambrosio–Tortorelli, afin d'étendre aux points critiques la convergence des minimiseurs globaux obtenue par Γ-convergence. Nous commençons par présenter les fonctionnelles de Mumford–Shah et d'Ambrosio–Tortorelli et le principe de l'approximation champ de phase. Ensuite, en dimension 1, nous établissons la convergence des points critiques dans le cas où l'on rajoute à la fonctionnelle d'Ambrosio–Tortorelli une condition d'obstacle. En dimension quelconque, la convergence des points critiques est connue sous l'hypothèse supplémentaire que l'énergie d'Ambrosio–Tortorelli converge. En dimension 2, nous montrons ce résultat en ne supposant qu'une partie de la convergence de l'énergie. Enfin, nous étudions le système parabolique associé aux équations vérifiées par les points critiques d'Ambrosio–Tortorelli. Nous établissons l'existence, les conditions d'unicité et la régularité des solutions de ce système, et obtenons en particulier leur régularité maximale en toute dimension.

Title : Stationary critical points for a phase field model in fracture mechanics
Keywords : Calculus of variations, partial differential equations, Ambrosio–Tortorelli functional, Mumford–Shah functional, inner variations.

Abstract : This thesis is devoted to the Mumford–Shah functional, used to modelize brittle fracture in an elastic material, and more specifically to its approximation by the Ambrosio–Tortorelli functional. We focus on the asymptotic analysis of the critical points of the Ambrosio–Tortorelli functional with the aim of extending to critical points the convergence result for global minimizers obtained through Γ-convergence. We start by presenting the Mumford–Shah and Ambrosio–Tortorelli functionals and the phase-field approximation method. Next, in dimension 1, we establish the convergence of critical points when an obstacle constraint is added to the Ambrosio–Tortorelli functional. In arbitrary dimension, the convergence of critical points is known under the additional assumption that the whole Ambrosio–Tortorelli energy converges. In dimension 2, we prove this result by assuming only that the phase field energy converges. Finally, we study the parabolic system associated to the equations satisfied by the critical points of Ambrosio–Tortorelli. We establish the existence, sufficient conditions for uniqueness and the regularity of solutions to this system and, in particular, prove maximal regularity of the solutions in arbitrary dimension.

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Si je devais résumer ma thèse (≪aujourd'hui avec vous≫), je dirais que c'est d'abord des rencontres. Au cours de ma thèse, et plus généralement durant mon parcours académique, des rencontres, il y en aura eu beaucoup. Je profite de ces remerciemments pour toutes les évoquer. Tâchons de n'en oublier aucune.

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Je me permets à présent de remonter dans le temps, et de revenir sur les rencontres antérieures à mes trois années de thèse. La plupart de ces rencontres se sont transformées en amitiés et ont, de fait, à un moment ou à un autre, entendu parler de ma thèse.

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La POFM c'est aussi et surtout les élèves. Des élèves brillants, doués, motivés, travailleurs et passionnés. Les élèves que tout enseignant rêve d'avoir. J'ai eu beaucoup de chance en étant encadrant de plusieurs équipes de France aux Olympiades Internationales, à savoir les équipes des années 2020, 2021, 2022 et 2024. Je garde précieusement tous les souvenirs construits pendant nos séjours à Oslo, Bucarest, Bath ou encore...Marne-la-Vallée.

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On arrive désormais aux années lycée, moment où je suis tombé amoureux des mathématiques. Là aussi, j'aurai fait une rencontre déterminante pour la suite de mon parcours académique, celle de ma professeure de mathématiques Mme Collette, qui m'aura donné le goût des mathématiques, poussé à toujours vouloir progresser, fait découvrir les olympiades et qui s'était donné pour mission que je puisse les explorer dans les meilleures conditions possibles. Bénéficier d'un tel soutien de son professeur est la meilleure chose que je puisse souhaiter à quiconque découvre les mathématiques.

C'est donc au lycée que j'ai découvert les Olympiades de mathématiques, ce qui a constitué une excellente porte d'entrée dans le monde des mathématiques, en révélant son caractère ludique mais exigeant, deux caractéristiques qui, pour quelqu'un comme moi, sont suffisantes pour provoquer une addiction. En arrivant à ma première Olympiade Internationale, à Hong Kong, je m'attendais à trouver une atmosphère de concurrence, de rivalité, où tous les participants des différents pays se

regardent du coin de l'oeil avec méfiance. Je n'aurais pas pu me tromper plus. Que cela soit aux Olympiades Internationales, aux stages de la préparation française ou au sein de l'équipe du Luxembourg, toutes les personnes que j'ai rencontrées étaient heureuses de pouvoir parler de leur passion avec leurs homologues, tandis que l'aspect «compétition » était relayé au second plan. Cette atmosphère de camaraderie, je l'ai retrouvée à chaque étape de mes études, malgré le caractère concurrentiel de ces étapes. Les personnes que j'ai rencontrées à ces compétitions sont restées mes amis jusqu'à aujourd'hui, parfois en étant des camarades d'études ou de thèse. C'est en m'entraînant à leur côté pour préparer ces compétitions que j'ai le plus progressé. Beaucoup de mes idées et de mes manières de voir les mathématiques, ce sont eux qui me les ont transmises. Je les remercie pour leurs vifs encouragements et soutien tout au long de mon parcours : Baptiste, Joachim, Olivier, Rémi et Yakob, rencontrés aux stages de la préparation française et Marie P., Rodrigue et Savinien de l'équipe nationale belge. Je suis flatté d'avoir gardé parmi mes amis des personnes de votre trempe.

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Chapter 1

Introduction

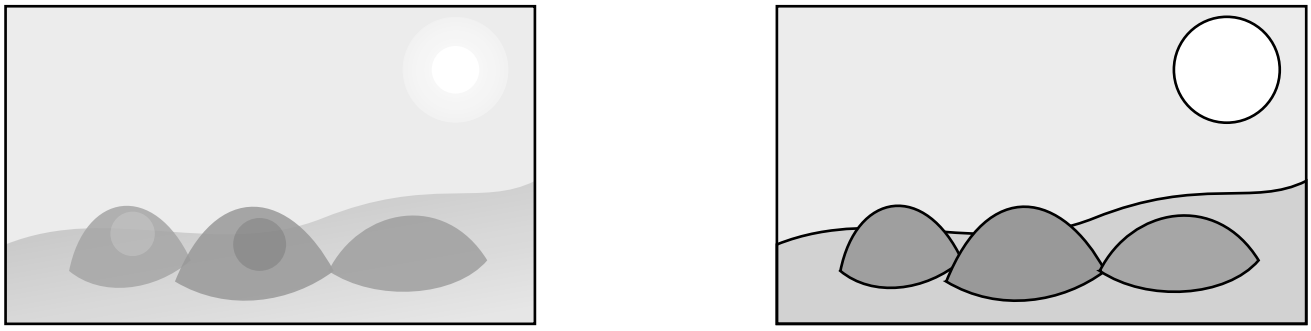
Cette thèse a pour but d'étudier la convergence des points critiques d'une approximation variationnelle d'un modèle de rupture fragile. Nous nous concentrons sur le modèle donné par la fonctionnelle de Mumford–Shah et son approximation champ de phase donnée par la fonctionnelle d'Ambrosio-Tortorelli. Si la convergence des minimiseurs approchés vers un minimiseur de Mumford–Shah est bien connue, elle ne constitue pas un résultat utilisable pour les implémentations numériques. Nos travaux ont pour objectif, d'une part de prolonger les différents résultats dans cette direction dans le cadre spécifique de la dimension 1 et de la dimension 2, et d'autre part d'étudier le système parabolique associé aux équations vérifiées par les points critiques d'Ambrosio–Tortorelli.

On commence par introduire la fonctionnelle de Mumford–Shah et son interprétation en mécanique. On poursuit avec une introduction à la fonctionnelle d'Ambrosio–Tortorelli. Cette approximation variationnelle s'inspire d'un modèle "champ de phase" d'Allen-Cahn, que l'on présente également. On définit ensuite la notion de point critique que nous utiliserons pour ces fonctionnelles, avant de présenter les résultats déjà établis dans la littérature. Enfin, on présentera succintement les résultats donnés dans les chapitres 2,3 et 4.

1.1 La fonctionnelle de Mumford–Shah

1.1.1 Un modèle pour la segmentation d'image

La fonctionnelle de Mumford–Shah a été introduite pour la première fois par D.Mumford et J.Shah en 1989 dans [MS89], dans le cadre du problème de segmentation d'image pour un ordinateur. Considérons une image correspondant à la projection d'une scène en trois dimensions sur une pellicule $\Omega \subset \mathbb{R}^2$ en deux dimensions. En chaque point x de la pellicule, on peut représenter par une fonction $f(x)$ l'intensité de la lumière capturée, qui donne à l'image des constrastes et différentes nuances de couleurs. Lorsque l'ordinateur numérise la pellicule, l'ensemble Ω est séparé en régions $R_1, \dots, R_k$, qui correspondent aux différents objets de la scène, et ces régions sont délimitées clairement par un ensemble Γ de dimension 1, correspondant aux contours nets des objets.



(a) L'image de la pellicule

(b) L'image numérisée

Figure 1.1: Un exemple de segmentation d'image

Reproduire numériquement les contrastes et les couleurs dans chaque région de l'image d'origine revient à déterminer une fonction u définie sur Ω qui approche la nuance de gris f. Comme on s'attend à ce que les variations de couleur soient faibles à l'intérieur de chaque région mais fortes à l'interface Γ entre chaque objet, on s'attend à ce que la fonction u varie très peu à l'intérieur de chaque régions et soit discontinue en chaque point de Γ.

Avec cette interprétaiton en tête, on peut définir la fonctionnelle de Mumford–Shah. Si Ω est un ouvert de $\mathbb{R}^d$ et $f \in L^\infty(\Omega)$, la fonctionnelle de Mumford–Shah s'écrit

$$\mathcal{MS}(u,\Gamma) = \int_{\Omega\setminus\Gamma} |\nabla u|^2 dx + \mathcal{H}^{d-1}(\Gamma) + \lambda \int_\Omega (u-f)^2 dx,$$

avec Γ un sous-ensemble fermé de Ω, la mesure $\mathcal{H}^{d-1}$ la mesure de Hausdorff de dimension $d-1$, le nombre λ un réel strictement positif et $u \in H^1(\Omega \setminus \Gamma)$. Déterminer l'intensité idéale u et la frontière numérique Γ revient donc à minimiser l'énergie $\mathcal{MS}(u,\Gamma)$.

Lorsqu'on regarde la compétition entre chacun des trois termes, on remarque qu'un éventuel minimiseur u doit être aussi proche que possible de f (le terme $\int_\Omega (u-f)^2$ est d'ailleurs parfois appelé *terme de fidélité*), mais doit très peu varier au sein de chaque région, tandis que l'interface Γ doit être aussi petite que possible. Pour cette raison, l'image numérisée aura donc des traits bien délimités mais moins de nuances et de différences de couleurs au sein de chaque région, ce qui lui donne en somme un effet cartoon.

1.1.2 La minimisation de la fonctionnelle $\mathcal{MS}$

Minimiser la fonctionnelle $\mathcal{MS}$ n'a rien d'évident, principalement parce que l'application $\Gamma \mapsto \mathcal{H}^{d-1}(\Gamma)$ n'est pas semi-continue inférieurement. De plus, l'ensemble des paires admissibles (u,Γ) ne possède pas de propriétés de compacité adéquates. La résolution de ce problème a donné lieu à plusieurs généralisations pour des classes variées de problèmes aux discontinuités libres, c'est-à-dire les problèmes pour lesquels l'ensemble des points de discontinuité fait partie des inconnues du problème (voir [Amb90]).

Le problème de minimisation a été résolu par [DGCL89a] en 1989. La première idée consiste à remplacer la paire (u,Γ) par une fonction u qui porte la discontinuité. Dans ce but, Ambrosio et De Giorgi ont introduit dans [DGA88] l'espace $SBV^2(\Omega)$ (voire Definition 1.5.4), un espace fonctionnel contenant des fonctions très peu régulières et qui possède de bonnes propriétés de compacité. Ces propriétés de compacité ont été étudiées par Ambrosio dans [Amb89], [Amb95] et sont résumées dans [AFP00, Section 4].

La fonctionnelle de Mumford–Shah sur l'espace $SBV^2(\Omega)$ s'écrit

$$\mathcal{MS}_2(u) = \int_{\Omega\setminus J_u} |\nabla u|^2 + \mathcal{H}^{d-1}(J_u) + \lambda \int_\Omega (u-f)^2. \tag{1.1.1}$$

La variable Γ est remplacée par l'ensemble J_u des points de discontinuité de la fonction u. En utilisant les propriétés de compacité de $SBV^2(\Omega)$, on montre l'existence d'un minimiseur u_m de $\mathcal{MS}_2$ dans $SBV^2(\Omega)$ pour lequel $\mathcal{H}^{d-1}(\overline{J_{u_m}}) = \mathcal{H}^{d-1}(J_{u_m})$. On prouve enfin (voir [DGCL89b, Lemme 5.1]) que la paire $(u_m, \overline{J_{u_m}})$ est un minimiseur de $\mathcal{MS}$.

L'étude des minimiseurs de $\mathcal{MS}$ est la source d'une abondante littérature (voir [AFP00], [Dav05], [Lem16] et [DLF25a]). On pourra mentionner la célèbre conjecture formulée par Mumford et Shah :

Conjecture : Soit (U,K) un minimiseur de $\mathcal{MS}$. Alors, à un ensemble $\mathcal{H}^{d-1}-$ négligeable près, K est localement l'union finie d'arcs de classe $C^{1,1}$. De plus, lorsque $d=2$, ces arcs ne peuvent se rejoindre qu'en leur extrémité, par groupe de trois, et en formant des angles de 120°.

1.1.3 Un modèle pour les fractures fragiles

La fonctionnelle de Mumford–Shah a été réintroduite par Francfort et Marigo [FM98] en mécanique de la rupture fragile, dans la suite des travaux fondateurs de Griffith [Gri21]. L'ouvert borné $\Omega \subset \mathbb{R}^d$ représente cette fois-ci un matériau élastique soumis à une contrainte à son bord $\partial\Omega$, contrainte que

l'on représente par une fonction $g \in L^\infty(\mathbb{R}^d)$. Le déplacement du matériau est représenté par une fonction u et la fissure par un sous-ensemble $J_u \subset \overline{\Omega}$. Dans le cas d'un mouvement quasi-statique et irréversible à l'état d'équilibre, l'énergie associée au système est la somme de l'énergie élastique du matériau en dehors de la fracture et de l'énergie de surface engendrée par la fissure.

En ce qui concerne l'énergie de surface, Francfort et Marigo ont proposé la forme suivante

$$E_s(\Gamma) = \int_\Gamma k(x) d\mathcal{H}^{d-1}_{|\Gamma},$$

où $k(x)$ est la ténacité du matériau et est proportionnelle à l'énergie nécessaire pour créer une fissure au point x.

L'énergie élastique est quant à elle modélisée par l'intégrale de type Dirichlet suivante :

$$E_e(u, J_u) = \int_{\Omega \setminus J_u} A(x)\nabla u \cdot \nabla u dx,$$

où $A(x)$ est une matrice symétrique représentant le module d'élasticité du materiel au point x.

Nous nous plaçons désormais dans le cas spécifique où l'élasticité et la résistance du matériau sont uniformes, de sorte que les fonctions A et k sont constantes sur Ω. Pour simplifier, on supposera $A(x) = I_d$ et $k(x) = 1$.

Dans ce cas, on retrouve bien la fonctionnelle de Mumford–Shah :

$$MS(u) = \int_{\Omega \setminus J_u} |\nabla u|^2 + \mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega \cap \{u \neq g\}), \tag{1.1.2}$$

On reconnaît le terme de Dirichlet, correspondant à l'énergie élastique, tandis que le second terme représente l'énergie de surface, qui prend en compte la possibilité pour le matériau de se fissurer sur le bord $\partial\Omega$.

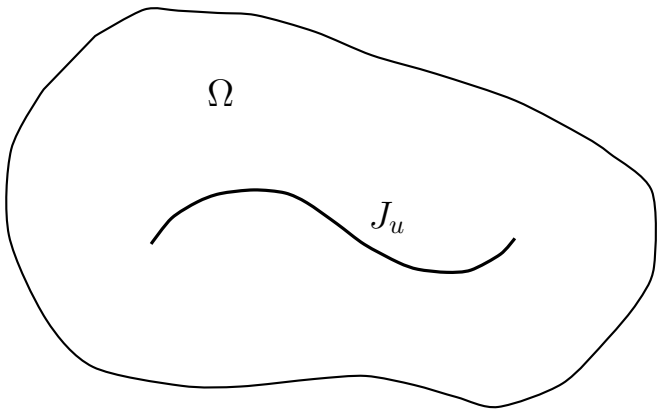


Figure 1.2: Fracture J_u

La question de l'existence de minimiseurs pour MS, qui a été discutée dans la sous-section précédente, appelle la question de leur implémentation à l'aide de méthodes numériques. Puisque l'ensemble de discontinuité de u est une inconnue du problème, les méthodes classiques de type éléments finis échouent à calculer correctement les minimiseurs. Pour surmonter cet obstacle, on adopte une approche variationnelle : on approche MS par une fonctionnelle plus régulière pour laquelle on calcule un minimiseur, et on montre que ce minimiseur est proche de la cible de départ. L'historique de cette approche est donné dans la section suivante.

1.2 Approximation champ de phase

1.2.1 Sur la Γ-convergence

L'approximation champ de phase repose fortement sur la $\Gamma-$convergence, dont la définition Definition 1.5.1 se trouve dans la section notations. Il s'agit d'un mode de convergence pour des fonction-

nelles, introduit par De Giorgi dans [DG75] et [DGF75], et dont l'objectif est de garantir la convergence des minimiseurs.

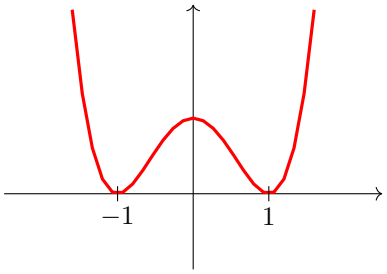
1.2.2 Le modèle d'Allen-Cahn

L'une des applications les plus célèbres de la méthode de la Γ-convergence est le problème d'approximation de la fonctionnelle de périmètre, résolu par Modica et Mortola dans [MM77]. L'objectif est de déterminer, parmi tous les ensembles d'aire m inclus dans Ω, celui dont le périmètre est minimal. Le périmètre de cette forme correspond à l'interface entre deux phases non miscibles A et B dans un domaine borné Ω à l'état d'équilibre.

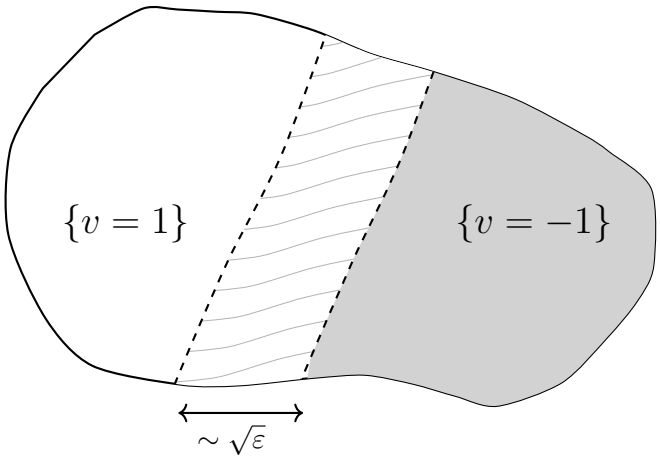
Pour décrire le système précédent, on introduit la fonction $v(x)$ représentant la proportion de chaque liquide dans le mélange en chaque point $x \in \Omega$ et prenant des valeurs entre -1 et 1, ces deux valeurs extrêmes correspondant à chacune des deux phases. La condition du volume fixe pour chacun des liquides se traduit par la contrainte $\int_\Omega v = c$. Dans le modèle introduit par Allen et Cahn dans [AC79], l'énergie associée au système contient deux termes. La densité du premier terme est le potentiel à deux puits $W(x) = (1-x^2)^2$, qui favorise la non-miscibilité des deux phases. Cette densité n'est cependant pas suffisante pour décrire à elle seule tout le système, dans la mesure où de nombreux équilibres instables minimisent l'énergie associée. Le second terme, appelée énergie de capillarité, pénalise les variations trop brusques entre les deux phases et correspond à une énergie de Dirichlet. L'énergie totale renormalisée associée au système est alors la suivante :

$$AC_\varepsilon(v) = \int_\Omega \left(\frac{1}{\varepsilon} W(v(x)) + \varepsilon |\nabla v|^2 \right) dx.$$

Le premier terme est petit lorsque $v(x) \in \{-1, 1\}$ tandis que le second terme est petit si les variations de v entre les deux états sont lentes. On s'attend donc à ce qu'un minimiseur de AC divise le domaine Ω en deux régions $\{v = -1\}$ et $\{v = 1\}$, séparées par un interface d'épaisseur $\sqrt{\varepsilon}$ dans lequel la fonction v opère la transition de phase.



(a) Le potentiel à deux puits W



(b) Transition entre deux phases

Figure 1.3: Modèle d'Allen-Cahn

Comme dit plus haut, la surface de la transition de phase doit être la plus petite possible afin de minimiser le terme de Dirichlet dans AC_ε, et son aire dépend d'un paramètre ε. Il est donc naturel de penser que, lorsque ε tend vers 0, le système limite obtenu est une division de Ω en deux régions avec un interface $\partial\{v = -1\}$ de périmètre minimal. Ce résultat a été prouvé dans [MM77] (voir aussi [Mod87], [Ste88]), sous la forme de la $\Gamma-$convergence de la fonctionnelle AC_ε vers la fonctionnelle de périmètre. On dit aussi que la fonctionnelle d'Allen-Cahn est une régularisation champ de phase de la fonctionnelle de périmètre.

1.2.3 La fonctionnelle d'Ambrosio–Tortorelli

Dans la même veine que le modèle d'Allen-Cahn pour la transition de phase, Ambrosio et Tortorelli ont proposé dans [AT92] la régularisation champ de phase suivante pour la fonctionnelle MS, définie pour $(u,v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ par

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\right) dx$$

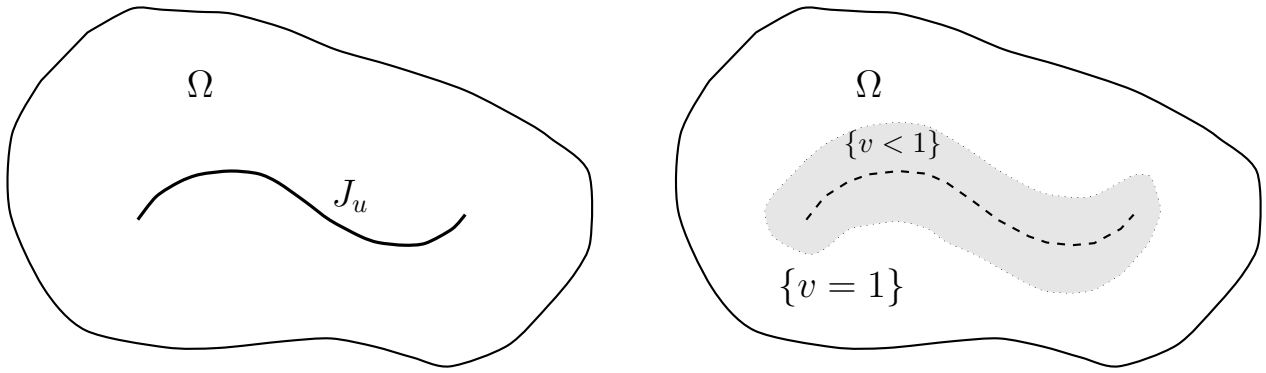
où $0 < \eta_\varepsilon \ll \varepsilon$ est un petit paramètre. Le premier terme de AT_ε est une approximation de l'énergie de Dirichlet de l'énergie de Mumford–Shah, tandis que le second terme, appelé terme champ de phase, qui ne dépend que de la variable v, est une régularisation de l'énergie de surface de MS.

Sous réserve de conditions suffisantes sur la contrainte au bord g et grâce au petit paramètre η_ε garantissant l'ellipticité du système, la méthode directe du calcul des variations permet d'établir l'existence d'un minimiseur de AT_ε à ε fixé. Autrement dit, le problème suivant

$$\mathrm{Min}\big\{ AT_\varepsilon(u,v), (u,v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)], (u_{|\partial\Omega}, v_{|\partial\Omega}) = (g,1) \text{ p.p} \big\}$$

admet au moins une solution. Un argument classique de troncature permet de prouver que les minimiseurs vérifient le principe du maximum, c'est-à-dire $0 \leqslant v \leqslant 1$ et $|u| \leqslant \|g\|_\infty$ sur Ω. On peut voir que les trois termes de AT_ε sont en compétition : le potentiel $\dfrac{(1-v)^2}{4\varepsilon}$ favorise les fonctions v proches de 1, tandis que le terme $v^2|\nabla u|^2$ favorise plutôt des fonctions v prenant des valeurs proches de 0. Enfin, le terme $\varepsilon|\nabla v|^2$ empêche les transitions trop brutales de l'état $\{v=0\}$ vers l'état $\{v=1\}$.

Cette approximation possède également une interprétation mécanique. Au lieu d'une fracture nette, la fonctionnelle AT_ε est l'énergie d'un matériau présentant un endommagement intermédiaire, représenté par la variable v prenant des valeurs dans l'intervale $[0,1]$. L'état $\{v=1\}$ correspond à un matériau sain tandis que l'état $\{v=0\}$ correspond à un état totalement endommagé. La fissure J_u est remplacée par une région $\{v \sim 0\}$ d'épaisseur $\sqrt{\varepsilon}$ dans laquelle le matériau est partiellement endommagé.



(a) Une fracture dans un matériau élastique

(b) Approximation champ de phase de la fracture

Dans [AT92], il est établi que la fonctionnelle d'Ambrosio–Tortorelli Γ-converge vers la fonctionnelle MS. Puisque AT_ε admet au moins un minimiseur $(u_\varepsilon, v_\varepsilon)$, le théorème fondamental de la $\Gamma-$convergence assure la convergence de $(u_\varepsilon, v_\varepsilon)$ vers un minimiseur $(u,1)$ de MS.

Dans l'expression de MS, le terme de Dirichlet ne mesure pas ce qu'il se passe sur la fracture J_u. Cela se retrouve dans l'énergie d'Ambrosio–Tortorelli, puisque dans la région $\{v \sim 0\}$, le terme $(\eta_\varepsilon + v^2)|\nabla u|^2$ est petit. Dans cette région, les deux termes $\varepsilon|\nabla v|^2$ et $\dfrac{(1-v)^2}{4\varepsilon}$ apportent la même contribution asymptotique lorsque ε tend vers 0. Ce principe est appelé *équipartition de l'énergie*.

Permettons-nous une brève digression sur ce principe d'équipartition de l'énergie. Pour qui connaît l'inégalité arithmético-géométrique $a^2 + b^2 \geqslant 2ab$, il est très tentant de l'appliquer à l'énergie champ de phase pour obtenir

$$\int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\right) \geqslant \int_\Omega |1-v||\nabla v|.$$

Cette manipulation est en fait la première étape de la preuve de la $\Gamma-$liminf de AT_ε vers MS. La propriété de la $\Gamma-$limsup assure que cette inégalité est optimale, ce qui justifie que les deux termes $\varepsilon|\nabla v|^2$ et $\dfrac{(1-v)^2}{4\varepsilon}$ sont en fait proches du cas d'égalité de l'inégalité $a^2+b^2 \geqslant 2ab$, qui correspond au cas où $a=b$. C'est pourquoi on peut s'attendre à ce que, asympotiquement, $\varepsilon|\nabla v|^2 \sim \dfrac{(1-v)^2}{4\varepsilon}$.

1.2.4 A propos de l'implémentation numérique

Comme mentionné précédemment, calculer les minimiseurs globaux de MS représente un véritable challenge à cause de la présence de l'ensemble de discontinuité. Les méthodes naïves comme la méthode de Galerkin sont alors inefficaces, à moins d'adapter le pas de discrétisation autour de la fracture, comme dans [Neg03].

L'approximation des minimiseurs de MS par des minimiseurs de AT_ε est donc aussi le coeur de plusieurs méthodes de simulation numérique (voir [BFM08, Section 8]). Le principal enjeu est de déterminer un minimiseur global de AT_ε. Comme celle-ci est strictement convexe et coercive par rapport à chaque variable u et v, les minimiseurs de $AT_\varepsilon(u,\cdot)$ et $AT_\varepsilon(\cdot,v)$ sont biens définis et peuvent être obtenus par une descente de gradient. Il est alors tentant d'implémenter une méthode de minimisation alternée sur AT_ε.

Toutefois, à cause du terme $v^2|\nabla u|^2$ dans l'expression de AT_ε, la fonctionnelle n'est pas convexe par rapport à la variable (u,v), donc le couple (u,v) obtenu avec l'algorithme décrit plus haut n'est pas nécessairement un minimiseur global, ni même un minimiseur local. D'après [Bou07, Theorem 2], le couple obtenu est seulement un point critique de AT_ε. Il est donc naturel d'étudier les limites possibles de la suite $(u_\varepsilon, v_\varepsilon)$ de points critiques et de déterminer si ces limites sont elles-mêmes des minimiseurs ou des points critiques pour MS, dans le but d'étendre le théorème de la $\Gamma-$convergence de [AT92] pour les points critiques.

1.3 Points critiques d'Ambrosio-Tortorelli

Dans cette section, on décrit la notion de point critique pour la fonctionnelle d'Ambrosio–Tortorelli et pour la fonctionnelle de Mumford–Shah et les résultats présents dans la littérature concernant leur convergence.

1.3.1 Points critiques d'Ambrosio–Tortorelli pour des conditions de Dirichlet

On pose

$$\mathcal{A}_g := \{(u,v) \in [H^1(\Omega)\times(H^1(\Omega)\cap L^\infty(\Omega))]^2 :\ u=g \text{ et } v=1 \text{ sur } \partial\Omega\} \tag{1.3.1}$$

et on considère la restriction de l'énergie d'Ambrosio–Tortorelli $AT_\varepsilon : \mathcal{A}_g \to \mathbb{R}$ à l'ensemble $\mathcal{A}_g$:

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\right) dx \quad \text{pour tout } (u,v)\in\mathcal{A}_g, \tag{1.3.2}$$

où $0<\eta_\varepsilon \ll \varepsilon$ est un petit paramètre.

Un point critique de la fonctionnelle AT_ε est un couple $(u_\varepsilon, v_\varepsilon)\in\mathcal{A}_g$ tel que

$$\frac{d}{dt}\Big|_{t=0} AT_\varepsilon(u_\varepsilon + t\psi, v_\varepsilon + t\varphi) = 0 \quad \text{pour tout } (\psi,\varphi)\in H^1_0(\Omega)\times(H^1_0(\Omega)\cap L^\infty(\Omega)). \tag{1.3.3}$$

Il s'agit des zéros de la différentielle de Gateaux de AT_ε, que l'on appellera aussi *variations externes*.

Un point critique $(u_\varepsilon, v_\varepsilon)$ de AT_ε est alors une solution faible du système elliptique suivant :

$$\begin{cases} -\operatorname{div}\left((\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon\right) = 0 & \text{sur } \Omega, \\ -\varepsilon\Delta v_\varepsilon + \dfrac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon|\nabla u_\varepsilon|^2 = 0 & \text{sur } \Omega, \\ u = g, \quad v = 1 & \text{sur } \partial\Omega. \end{cases} \tag{1.3.4}$$

Bien que dans ce manuscrit, on ne s'intéresse qu'à des conditions de Dirichlet au bord, d'autres conditions sur le bord peuvent être considérées, comme des conditions de Neumann $\partial_n v = 0$ sur le bord $\partial\Omega$.

1.3.2 Points critiques de Mumford–Shah

Alors que la notion de point critique pour la fonctionnelle AT est relativement standard, la notion de point critique pour l'énergie MS est plus délicate ([AFP00, Section 7.4]).

Par exemple on ne peut pas se contenter de considérer uniquement les variations externes de MS. Si φ est de classe $\mathcal{C}_c^\infty(\Omega)$, alors $J_{u+t\varphi} = J_u$. La dérivée au sens de Gateaux définie sur les fonctions régulières ne permet donc pas d'observer ce qu'il se passe près de la fracture J_u.

Pour observer ce qu'il se passe au bord de la fracture, on définit les variations externes sur l'espace des fonctions tests $SBV^2(\Omega)$:

Definition 1.3.1. *Soit $\varphi \in SBV^2(\Omega)$ tel que $J_\varphi \subset J_u$. La* première variation externe de MS *est définie par*

$$dMS(u)[\varphi] := \lim_{t\to 0} \frac{MS(u+t\varphi) - MS(u)}{t}. \tag{1.3.5}$$

Dans cette définition, les fonctions $\varphi \in SBV^2(\Omega)$ doivent vérifier la contrainte $J_\varphi \subset J_u$ afin que la limite du taux d'accroissement existe. Ce type de variations n'apporte donc toujours pas d'information sur la fracture elle-même.

On considère également les variations *internes* de MS. Contrairement aux variations externes, qui appliquent une déformation à la variable u, les variations internes appliquent une déformation à l'ensemble Ω considéré. Rappelons la définition du flot associé à un champ de vecteurs :

Definition 1.3.2. *Soit $X \in \mathcal{C}_c^1(\mathbb{R}^d;\mathbb{R}^d)$ un champ de vecteurs satisfaisant $X \cdot \nu = 0$ sur $\partial\Omega$. L'*application de flot $\Phi_t(x)$ *associée à X, définie pour tout x dans $\mathbb{R}^d$, est l'unique solution du problème de Cauchy*

$$\begin{cases} \dfrac{d}{dt}\Phi_t(x) = X(\Phi_t(x)) & \text{pour tout } t \in \mathbb{R}, \\ \Phi_0(x) = x. \end{cases}$$

Grâce au théorème de Cauchy–Lipschitz, on peut montrer que pour tout $x \in \mathbb{R}^d$, l'application $t \in \mathbb{R} \mapsto \Phi_t(x)$ est globalement bien définie, que $(t,x) \mapsto \Phi_t(x)$ appartient à $\mathcal{C}^1(\mathbb{R}\times\mathbb{R}^d;\mathbb{R}^d)$, et que $\{\Phi_t\}_{t\in\mathbb{R}}$ forme un groupe à un paramètre de $\mathcal{C}^1$-difféomorphismes de $\mathbb{R}^d$ avec $\Phi_0 = I_d$. De plus, pour tout $t > 0$, on vérifie que Φ_t est un $\mathcal{C}^1$-difféomorphisme de $\overline{\Omega}$ qui préserve à la fois Ω et $\partial\Omega$.

On peut alors définir les variations internes de MS de la façon suivante :

Definition 1.3.3. *Soit $X \in \mathcal{C}_c^1(\mathbb{R}^d;\mathbb{R}^d)$ tel que $X \cdot \nu = 0$ sur $\partial\Omega$, et soit $\{\Phi_t\}_{t\in\mathbb{R}}$ son flot associé. Pour tout $u \in SBV^2(\Omega)$, la* première variation interne de MS *en u dans la direction X est définie par*

$$\delta MS(u)[X] := \lim_{t\to 0} \frac{MS(u\circ\Phi_t^{-1}) - MS(u)}{t}.$$

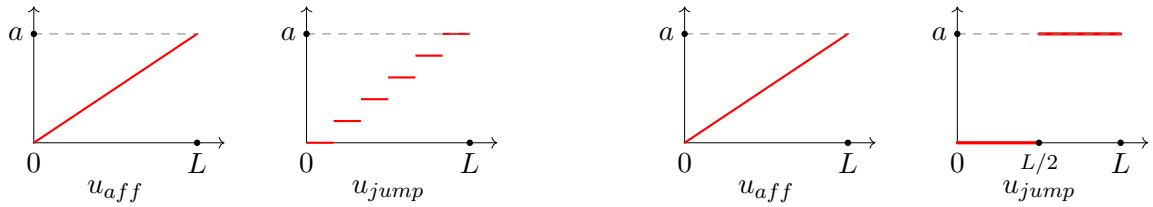
On peut désormais définir la notion de point critique pour la fonctionnelle de Mumford–Shah :

Definition 1.3.4. *Une fonction $u \in SBV^2(\Omega)$ est un* point critique *de MS si elle satisfait les deux conditions suivantes :*

$$\begin{cases} dMS(u)[\varphi] = 0 & \text{pour tout } \varphi \in SBV^2(\Omega) \text{ tel que } J_\varphi \subset J_u, \\ \delta MS(u)[X] = 0 & \text{pour tout } X \in \mathcal{C}_c^1(\mathbb{R}^d;\mathbb{R}^d) \text{ tel que } X\cdot\nu = 0 \text{ sur } \partial\Omega. \end{cases}$$

En calculant les variations internes et externes de MS, voir par exemple [BMR24, Appendix A], on obtient qu'un point critique de MS est une fonction $u \in SBV^2(\Omega)$ qui satisfait :

$$\begin{cases} -\operatorname{div}\nabla u = 0 \text{ dans } \mathcal{D}'(\Omega) \text{ ;} \\ \displaystyle\int_\Omega \nabla u\cdot\nabla\varphi = 0 \quad \text{pour tout } \varphi \in SBV^2(\Omega) \text{ tel que } J_\varphi \subset J_u \text{ ;} \\ \displaystyle\int_\Omega \left(|\nabla u|^2 I_d - 2\nabla u\otimes\nabla u\right) : DX + \int_{J_u} (I_d - \nu_{J_u}\otimes\nu_{J_u}) : DX \, d\mathcal{H}^{d-1}_{|J_u} \\ \qquad\qquad = -2\displaystyle\int_{\partial\Omega} (\nabla u\cdot\nu)(X\cdot\nabla g)\, d\mathcal{H}^{d-1} \text{ pour tout } X \in \mathcal{C}_c^1(\mathbb{R}^d;\mathbb{R}^d) \text{ avec } X\cdot\nu = 0 \text{ sur } \partial\Omega. \end{cases} \tag{1.3.6}$$



(a) Conditions de Neumann $v'(0) = v'(L) = 0$ (b) Conditions de Dirichlet $v(0) = v(L) = 1$

Figure 1.5: Points critiques limites pour MS en dimension un

On insiste sur le fait que, dans la première équation, il s'agit de la divergence du gradient approché ∇u, et non de la différentielle Du en entier. La deuxième équation peut-être interprétée comme une condition de Neumann homogène sur u au bord du saut J_u (voir [DLF25a, Proposition 2.5.2]). Dans la troisième condition, le second terme du membre de gauche correspond à l'opérateur de divergence tangentielle par rapport à J_u.

Remarquons qu'en dimension 1, les points critiques de Mumford–Shah sont les fonctions constantes par morceaux (avec un nombre arbitraire mais fini de morceaux) et la fonction affine vérifiant la condition de Dirichlet.

1.3.3 Résultats présents dans la littérature

Les points critiques pour une approximation variationnelle ont déjà été étudiés dans divers cadres. Concernant le modèle d'Allen-Cahn, on peut obtenir la convergence des points critiques sous l'hypothèse minimale

$$\limsup_{\varepsilon \to 0} AC_\varepsilon(u_\varepsilon) < +\infty$$

dans différents contextes (voir [HT00, Ton02, Ton05] pour le cas scalaire et [Bet25] pour le cas vectoriel). On trouve des résultats similaires portant sur la fonctionnelle de Ginzburg-Landau pour la supraconductivité, pour lesquels nous renvoyons à [BBH94, BBO01, LR01, SS07]. Une présentation générale des extensions de la Γ-convergence pour des fonctionnelles dépendant de petits paramètres peut être trouvée dans [Bra14].

Concernant la fonctionnelle d'Ambrosio–Tortorelli, la première extension du théorème de Γ-convergence a été réalisée par Francfort, Le et Serfaty dans [FLS09] dans le cas particulier de la dimension un. En dimension un, le matériau élastique peut être identifié à une barre de longueur L, fixée à une extrémité et tirée à l'autre. La fonction u représente alors la hauteur de la barre et doit satisfaire

$$u(0) = 0 \qquad \text{et} \qquad u(L) = a > 0.$$

Francfort, Le et Serfaty considèrent le cas des conditions au bord de Neumann

$$v'(0) = v'(L) = 0$$

pour la variable de champ de phase v. Ils établissent la convergence d'une solution $(u_\varepsilon, v_\varepsilon)$ de (3.1.6) sous la seule hypothèse

$$\limsup_{\varepsilon \to 0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty.$$

Le point limite $(u, 1)$ apparaît alors comme un point critique de la fonctionnelle unidimensionnelle MS, qui n'est pas nécessairement un minimiseur global. Cependant, ils mettent en évidence un phénomène de sélection concernant les points limites possibles $(u, 1)$. En effet, la fonction u doit posséder de fortes propriétés de symétrie et ne peut qu'être soit la fonction affine joignant $(0, 0)$ à (L, a), soit une fonction constante par morceaux croissante, dont chaque morceau est de même longueur. Une telle configuration est appelée *l'escalier parfait* (*the perfect staircase*) et n'a que très peu de restrictions sur le nombre de marches.

Une sélection similaire subsiste lorsque l'on considère des conditions de Dirichlet

$$v(0) = v(L) = 1,$$

comme cela a été démontré par Babadjian, Millot et Rodiac dans [BMR23]. Dans le cas des conditions de Dirichlet, le résultat est encore plus discriminant, puisque seuls deux points critiques possibles de la fonctionnelle MS peuvent être atteints via l'approximation de champ de phase AT_ε, à savoir la fonction affine et la fonction constante par morceaux croissante comportant deux morceaux de longueur $L/2$. Nous souhaitons également mentionner qu'une convergence similaire a été établie par Le dans [Le10] pour la fonctionnelle unidimensionnelle AT_ε munie d'un terme de fidélité.

Au-delà des critères de sélection, les travaux [FLS09] et [BMR23] démontrent tous les deux un principe d'équipartition de l'énergie,

$$\left| \varepsilon |v_\varepsilon'|^2 - \frac{(1-v_\varepsilon)^2}{4\varepsilon} \right| \to 0 \quad \text{dans } L^1(0,L),$$

qui constitue une propriété partagée par les minimiseurs globaux.

Tous ces résultats reposent sur l'introduction de la quantité appelée *discrépance* ou *désaccord* (*discrepancy* en anglais) :

$$d_\varepsilon = \frac{1}{2} \left\{ \frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon |\nabla v_\varepsilon|^2 - (\eta_\varepsilon + v_\varepsilon^2) |\nabla u_\varepsilon|^2 \right\}. \tag{1.3.7}$$

En dimension un, la fonction d_ε est constante. Cela implique directement une borne uniforme

$$|\nabla v_\varepsilon| \leqslant \frac{C}{\varepsilon},$$

qui est cruciale pour établir l'équipartition de l'énergie. La convergence de cette fonction vers 0 est également au cœur du travail de [HT00] pour le modèle d'Allen-Cahn. Cette quantité apparaît naturellement lorsque l'on dérive par rapport à r l'énergie AT_ε restreinte aux boules de rayon r.

En ce qui concerne la convergence de l'énergie complète $AT_\varepsilon(u_\varepsilon, v_\varepsilon)$ vers $MS(u)$, [BMR23] montre la convergence suivante

$$\left(\varepsilon |v_\varepsilon'|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} \right) dx \overset{\star}{\rightharpoonup} \alpha \delta_{L/2} \qquad \text{dans } \mathcal{M}([0,L]).$$

Des résultat analogues sont également montrés dans [FLS09] et [Le10] : l'énergie champ de phase à la limite se concentre en les potentiels points de fissure. De plus, dans cette convergence, le réel α ne peut prendre que les valeurs 0 et 1. Lorsque u est la solution constante par morceau, c'est-à-dire lorsque la barre se fissure, on a nécessairement $\alpha = 1$. On pourrait s'attendre à ce que la réciproque soit vraie, et que lorsque $\alpha = 1$, u soit nécessairement constante par morceau et ne puisse pas être la solution affine. Cette dernière implication reste cependant ouverte dans les travaux précédents.

En dimension $N \geqslant 2$, la convergence des points critiques a été établie dans [BMR24], sous l'hypothèse supplémentaire de convergence de l'énergie

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \to MS(u).$$

Il y est montré qu'une suite de points critiques $(u_\varepsilon, v_\varepsilon)$ converge vers un couple $(u, 1)$ avec $u \in SBV^2(\Omega)$ vérifiant les conditions 1 et 3. Ainsi la fonction limite u est presque un point critique.

L'hypothèse additionnelle de convergence de l'énergie ne présente malheureusement pas d'interprétation physique. Une hypothèse analogue de convergence de l'énergie a été utilisée dans [Le11, Le15, LS19] afin d'étudier la convergence des points critiques de la fonctionnelle d'Allen-Cahn vers des surfaces minimales, ainsi que la manière dont la stabilité se transmet à la limite. Cependant, cette convergence de l'énergie n'est pas toujours vérifiée dans le cas général. Par exemple, il est démontré dans [HT00, Section 6.3] que des interfaces avec multiplicités peuvent apparaître comme limites d'une suite de points critiques de la fonctionnelle d'Allen-Cahn. Par conséquent, ces points critiques ne satisfont pas l'hypothèse de convergence de l'énergie.

1.4 Présentation des resultats

1.4.1 Convergence des points critiques pour Ambrosio–Tortorelli avec obstacle

Le premier chapitre de cette thèse porte sur la convergence des points critiques de la fonctionnelle d'Ambrosio–Tortorelli avec un terme d'obstacle en dimension un. Ce chapitre à fait l'object d'un article à paraître dans la revue *Interfaces and free boundary*, voir [Rak25].

Dans [FL03], Francfort et Larsen proposent un modèle pour décrire l'évolution quasi-statique d'une fracture fragile utilisant la fonctionnelle de Mumford-Shah. Le modèle part d'une version discrétisée de l'évolution de la fissure, où le temps continu est remplacé par des instants discrets $t_0 < t_1 < \dots$. À chaque nouvel instant t_{i+1}, on minimise une nouvelle énergie de Mumford–Shah qui tient compte de la fracture obtenue à l'instant t_i.

L'irréversibilité au cours du temps se traduit par la condition selon laquelle la fissure $\Gamma(t_i)$ est incluse dans la fissure $\Gamma(t_{i+1})$. La fonctionnelle de Mumford-Shah est alors minimisée à l'instant t_{i+1} parmi toutes les fissures Γ contenant $\Gamma(t_i)$, avec une condition de Dirichlet variable g_{i+1} sur le bord $\partial\Omega$:

$$(u(t_{i+1}),\Gamma(t_{i+1})) = \underset{(u,\Gamma)}{\mathrm{Min}} \left\{ \int_{\Omega\setminus\Gamma} |\nabla u|^2 + \mathcal{H}^{d-1}(\Omega\setminus\Gamma) + \mathcal{H}^{d-1}(\partial\Omega \cap \{u \neq g_{i+1}\}),\ \Gamma(t_i)\subset\Gamma,\ u\in H^1(\Omega\setminus\Gamma) \right\}.$$

Dans [Gia05], Giacomini démontre qu'il est possible de construire une évolution quasi-statique des minimiseurs de AT_ε qui approche une évolution quasi-statique de MS. La condition d'irréversibilité est appliquée à la variable v et se traduit par la minimisation de AT_ε à l'instant t_{i+1} parmi tous les couples (u,v) tels que $v \leqslant v_{t_i,\varepsilon}$, où $(u_{t_i,\varepsilon}, v_{t_i,\varepsilon})$ désigne un minimum de AT_ε à l'instant t_i :

$$(u_{t_{i+1},\varepsilon}, v_{t_{i+1},\varepsilon}) = \mathrm{Argmin}\left\{AT_\varepsilon(u,v), (u,v)\in H^1(\Omega)\times[H^1(\Omega)\cap L^\infty(\Omega)], v\leqslant v_{t_i,\varepsilon}\right\}.$$

La question est alors de savoir si le résultat de [Gia05] peut être étendu aux points critiques. Puisqu'en grande dimension les résultats de convergence des points critiques de AT_ε nécessitent des hypothèses plus fortes, une telle extension ne peut pour l'instant être étudiée qu'en dimension un. L'article [Rak25] constitue une première étape dans cette direction : on y étudie la convergence des points critiques de la fonctionnelle AT_ε définie sur les couples (u,v) de $H^1([0,L])$ satisfaisant une condition d'obstacle $v\leqslant v_{0,\varepsilon}$, où $(u_{0,\varepsilon}, v_{0,\varepsilon})$ est un point critique de AT_ε. Nous utilisons le même cadre que dans [BMR23], à savoir des conditions de Dirichlet sur les variables (u,v).

Plus concrètement, on définit

$$\overline{AT_\varepsilon}(u,v) := \begin{cases} AT_\varepsilon(u,v) & \text{si } v\leqslant v_{0,\varepsilon}\ , \\ +\infty & \text{sinon,} \end{cases} \qquad \forall\,(u,v)\in H^1(0,L). \tag{1.4.1}$$

La condition d'obstacle $v\leqslant v_{0,\varepsilon}$ induit, dans les équations d'Euler–Lagrange de (u,v), un second membre sur l'équation en v. Un point critique $(u_\varepsilon, v_\varepsilon)$ de (2.2.5) est en effet une solution faible du système suivant

$$\begin{cases} -\varepsilon v_\varepsilon'' + \frac{v_\varepsilon - 1}{\varepsilon} + v_\varepsilon |u_\varepsilon'|^2 = \mu_\varepsilon \leqslant 0 & \text{sur } (0,L) \\ \left[u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)\right]' = 0 & \text{sur } (0,L) \\ v_\varepsilon \leqslant v_{0,\varepsilon} & \text{sur } (0,L)\ , \\ u_\varepsilon(0) = 0,\ u_\varepsilon(L) = a_1 < a_0\ , \\ v_\varepsilon(0) = v_\varepsilon(L) = 1. \end{cases} \tag{1.4.2}$$

où μ_ε est une mesure négative et a_1 est une condition de bord différente de la condition de bord a_0 utilisée pour construire $(u_{0,\varepsilon}, v_{0,\varepsilon})$.

On prouve deux résultats. Le premier résultat est un résultat de régularité sur la paire $(u_\varepsilon, v_\varepsilon)$. On prouve en effet que $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{1,1}(\Omega)$. Cette régularité est en général la régularité optimale pour un problème d'obstacle (voir [FRRO22, Section 5.3]). On prouve ensuite que la suite $(u_\varepsilon, v_\varepsilon)_{\{\varepsilon>0\}}$, sous l'hypothèse d'une énergie finie $\limsup_{\varepsilon\to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$, admet une valeur d'adhérence $(u, 1)$ dans $L^2(\Omega)^2$. On montre que la fonction u ne peut prendre que deux formes, à savoir une fonction affine u_{aff} et une solution croissante et constante par morceau u_{jump}, et avec deux morceaux.

En revanche, contrairement à ce qui est observé par [BMR23] pour les points critiques de AT_ε, on perd la symétrie de la solution, puisque le point de discontinuité x_{min} de la solution u_{jump} n'est pas nécessairement $L/2$. On peut toutefois montrer que $\frac{1}{4}L \leqslant x_{min} \leqslant \frac{3}{4}L$. La condition d'obstacle induit une certaine irréversibilité du système entre le temps 0 et le temps L, en revanche on ne sait pas écarter la situation où $u = u_{jump}$ au temps 0 et $u = u_{aff}$ au temps 1, notamment si $v_\varepsilon < v_{0,\varepsilon}$ pour tout ε. Enfin, on prouve un résultat de convergence de l'énergie et d'équipartition de l'énergie similaire à celui de [BMR23].

Notre étude s'appuie sur les mêmes outils que [BMR23] et [FLS09], à savoir les variations de la discrepancy et une estimation du type $|v'_\varepsilon| \leqslant C/\varepsilon$. Pour cela, on montre d'abord que, loin de la fracture la fonction v_ε vérifie $v_\varepsilon \geqslant 1 - b\varepsilon$, à la manière de [HT00]. Pour obtenir l'estimée sur v'_ε, on applique à la première équation de (2.2.7) une fonction test φ_ε dépendant de ε jouant le rôle d'approximation de l'unité.

Finalement, on montre que la fonction limite u est point critique d'une fonctionnelle de Mumford–Shah avec obstacle, dont on montre qu'elle est également limite de $\overline{AT}_\varepsilon$ au sens de la $\Gamma-$convergence.

1.4.2 Convergence des points critiques sous hypothèse de convergence du terme champ de phase en dimension 2

Le deuxième chapitre de cette thèse porte sur la convergence des points critiques d'Ambrosio–Tortorelli en dimension deux, et a fait l'objet d'une prépublication [BRR26].

Dans ce chapitre, on affaiblit l'hypothèse de convergence de l'énergie utilisée dans [BMR24]. Plus précisément, on établit la convergence des points critiques de AT_ε sous l'hypothèse que seul le terme de champ de phase converge, c'est-à-dire

$$\int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon} \right) dx \to \mathcal{H}^{d-1}(J_u).$$

Ce résultat est restreint à la dimension $d = 2$, ce qui correspond à la dimension utile pour les applications en segmentation d'image et pour le modèle de fracture fragile antiplanaire. On établit également que, lorsque J_u est C^1, dans le cas de la convergence complète de l'énergie de [BMR24] en dimension 2, la seconde condition de (1.3.6) est également vérifiée, signifiant que la fonction u limite est un point critique de Mumford–Shah.

Nous renvoyons à la section précédente pour la définition des points critiques d'Ambrosio–Tortorelli et des points critiques de Mumford–Shah.

Tout comme dans [BMR24], on prouve ce résultat de convergence en passant à la limite dans les variations internes d'Ambrosio–Tortorelli. Celles-ci sont définies de façon similaire à celles de la fonctionnelle de Mumford–Shah. D'après [BMR24, Theorem 3.1], si $(u_\varepsilon, v_\varepsilon)$ est un point critique de AT_ε au sens de (3.2.1), alors $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^\infty(\Omega)]^2$. Cette régularité s'étend jusqu'au bord si la condition de Dirichlet g est suffisamment régulière. Cela garantit qu'un point critique $(u_\varepsilon, v_\varepsilon)$ annule également

les variations internes, c'est-à-dire (voir [BMR24, Lemma A.3]) qu'il vérifie

$$\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)\left(2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \mathrm{Id}\right) : DX\, dx + \int_\Omega \left[2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathrm{Id}\right] : DX\, dx = 2(\eta_\varepsilon + 1)\int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X \cdot \nabla g)\, d\mathcal{H}^1 \quad (1.4.3)$$

pour tout $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ tel que $X \cdot \nu = 0$ sur $\partial\Omega$.

Il s'agit alors de passer à la limite dans les variations internes (3.2.4). Dans le cas de [BMR24] où l'on suppose la convergence de toute l'énergie, on trouve une convergence terme à terme vers les variations internes de Mumford–Shah. Dans le cas où l'on suppose seulement la convergence du terme champ de phase, cette convergence terme à terme n'a plus de raison d'être. Plus précisément, on trouve que

$$(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \overset{\star}{\rightharpoonup} \nabla u \otimes \nabla u \mathcal{L}^2 + \mu \qquad \text{dans } \mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

Notre premier résultat est de montrer que la mesure de défaut μ ci-dessus est en fait singulière par rapport à $\mathcal{L}^2$. Ce résultat se démontre à partir d'une méthode de "blow-up" sur l'équation en u_ε. On utilise notamment que la fonctionnelle $AT_\varepsilon(\cdot, v)$ est convexe et coercive, ce qui implique que si $(u_\varepsilon, v_\varepsilon)$ est un point critique de AT_ε, alors u_ε est un minimiseur de $AT_\varepsilon(\cdot, v_\varepsilon)$.

Dans un second temps, on étudie la convergence du terme champ de phase des variations internes. La convergence de l'énergie champ de phase implique l'équipartition de l'énergie $\varepsilon|\nabla v_\varepsilon|^2 \sim \dfrac{(1-v_\varepsilon)^2}{4\varepsilon}$. Cette étude est très proche de la méthode déjà employée par [BMR24] et passe notamment par l'étude du varifold approché V_ε (voir notations) associé à AT_ε. On prouve que $(\|V_\varepsilon\|)_{\{\varepsilon>0\}}$ se concentre sur l'ensemble J_u.

Enfin, lorsqu'on combine les résultats de convergence précédente, il faut encore identifier les compensations des différentes mesures de défaut. L'outil principal est un résultat de De Philippis et Rindler [DPR16, Corollary 1.13], permettant de justifier que la mesure singulière et la mesure absolument continue par rapport à $\mathcal{H}^1_{|J_u}$ sont toutes deux des matrices non inversibles. La dimension deux est alors nécessaire ici pour obtenir la forme des matrices à partir de leur trace et de leur déterminant.

1.4.3 Le système parabolique d'Ambrosio–Tortorelli

Le dernier chapitre de cette thèse se concentre sur les équations du flot-gradient associé à la fonctionnelle d'Ambrosio–Tortorelli. Il fait l'objet d'une pré-publication, voir [Rak26].

Une approche possible pour la minimisation (locale) de MS consiste à utiliser une méthode de descente de gradient, ce qui revient à déterminer le flot de gradient de MS. Bien que la définition d'une notion adaptée de gradient pour MS soit délicate, un flot de gradient généralisé peut être construit par discrétisation en temps au moyen d'un schéma d'Euler implicite. En faisant tendre le pas de temps vers zéro, toute courbe limite obtenue est appelée *mouvement minimisant* (*minimizing movement*) pour la fonctionnelle de Mumford–Shah (voir [DG93], [Amb95]).

De manière analogue à [Gob98], où un flot de gradient pour la fonctionnelle Mumford–Shah en dimension 1 a été obtenu au moyen d'une approximation non locale, l'approche proposée dans [FP04] consiste à dériver un flot de gradient pour MS comme limite, lorsque $\varepsilon \to 0$, des flots gradient de AT_ε. Le flot gradient de AT_ε est défini comme un couple (u, v) appartenant à $\left[L^\infty_{loc}([0,+\infty); L^2(\Omega)) \cap L^2_{loc}([0,+\infty); H^1(\Omega))\right]^2$ satisfaisant

$$\begin{cases} \partial_t u - \operatorname{div}\big((\eta_\varepsilon + v^2)\nabla u\big) = 0 & \text{dans } (0,+\infty)\times\Omega, \\ \partial_t v - \varepsilon\Delta v + \dfrac{v-1}{4\varepsilon} + v|\nabla u|^2 = 0 & \text{dans } (0,+\infty)\times\Omega, \\ (u,v) = (g,1) & \text{sur } (0,+\infty)\times\partial\Omega, \\ (u(0,\cdot), v(0,\cdot)) = (u_0, v_0) & \text{dans } \Omega. \end{cases} \quad (1.4.4)$$

La même approche a été utilisée dans [BM14] afin d'obtenir un flot gradient unilatéral pour MS, incorporant une condition d'irréversibilité sur la fissure, laquelle ne peut que croître au cours du temps. L'étude du flot de gradient de AT_ε devient un sujet central afin de mieux comprendre le flot gradient de MS.

Dans ce chapitre, on prouve l'existence d'une solution faible $(u_\varepsilon, v_\varepsilon)$ du système (4.1.2) en toute dimension. On donne également une classe d'unicité pour ce système. On établit ensuite que $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^\infty((0,+\infty)\times\Omega)^2$. Cette régularité est étendue jusqu'à $[0,\infty)\times\overline{\Omega}$ lorsque la condition initiale (u_0, v_0) et la condition au bord g satisfont des hypothèses adéquates de régularité et de compatibilité.

La régularité optimale du problème elliptique associé à (4.1.2) a été établie dans [BMR24], en s'appuyant sur la théorie des applications harmoniques et les espaces de Morrey–Campanato, méthode notamment utilisée dans [Riv93] pour montrer la régularité des applications harmoniques à valeurs dans un tore. Afin d'adapter cette approche au cadre parabolique, nous utilisons des arguments issus de [Lie96], complétés par la méthode de [Yin97]. L'argument principal consiste à démontrer la régularité Höldérienne du couple (u, v). Le théorème de De Giorgi–Nash–Moser fournit la régularité höldérienne de la fonction u. En utilisant la première équation de (4.1.2), nous en déduisons une estimation de type Campanato–Morrey sur ∇u. La seconde équation de (4.1.2) devient alors une équation de la chaleur avec un terme source appartenant à un espace de Morrey. Dans le cas elliptique, cette étape découle directement du théorème de De Giorgi–Nash–Moser. Pour montrer que v est Höldérienne, il suffit, d'après un lemme de Campanato parabolique, d'établir une estimation de décroissance de la norme L^2 de v moins sa valeur moyenne sur des cylindres paraboliques. Pour obtenir cette estimation, nous considérons w, une extension "de la chaleur" de v. Alors que w satisfait l'estimation souhaitée, la distance de l'erreur $v - w$ à sa valeur moyenne peut être contrôlée grâce à une inégalité de type Poincaré pour les solutions d'une équation de la chaleur avec un terme source dans un espace de Morrey.

En particulier, l'adaptation de la stratégie elliptique nécessite une inégalité de type Caccioppoli contrôlant la décroissance du gradient spatial ∇u, ainsi qu'une inégalité de type Poincaré contrôlant la distance de v à sa moyenne parabolique par la norme L^2 de son gradient, que nous prouvons également.

1.5 Quelques notations

On rappelle brièvement la définition de la $\Gamma-$convergence ainsi que celle de l'espace $SBV^2(\Omega)$.

La Γ-convergence. La Γ-convergence est un mode de convergence des fonctionnelles, introduit pour la première fois par De Giorgi dans [DG75] et [DGF75], qui assure la convergence des minimiseurs. Nous en rappelons la définition :

Definition 1.5.1. *Soient F et $(F_\varepsilon)_{\varepsilon>0}$ une famille de fonctionnelles définies sur l'espace métrique (E,d). On dit que F_ε Γ-converge vers F si les trois conditions suivantes sont satisfaites :*

(i) ***Γ-liminf :*** *Pour tout $x \in E$ et toute suite $(x_\varepsilon)_{\varepsilon>0}$ telle que $x_\varepsilon \longrightarrow x$, on a $\liminf\limits_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \geqslant F(x)$.*

(ii) ***Γ-limsup :*** *Pour tout $x \in E$, il existe une suite $(x_\varepsilon)_{\varepsilon>0}$ telle que $x_\varepsilon \longrightarrow x$ et $\lim\limits_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = F(x)$.*

Ces axiomes suffisent pour impliquer le théorème fondamental de la Γ-convergence :

Theorem 1.5.2. *Supposons que la suite $(F_\varepsilon)_{\varepsilon>0}$ Γ-converge vers F. Soit $(x_\varepsilon)_{\varepsilon>0}$ une suite d'éléments de E telle que, pour tout $\varepsilon > 0$, x_ε soit un minimiseur de F_ε. On suppose qu'il existe $x \in E$ et une sous-suite de (x_ε) qui converge vers x.*

Alors il existe une sous-suite (non réindexée) (x_ε) et un point $x \in E$ tels que $x_\varepsilon \to x$ et que x soit un minimiseur de F et que $F_\varepsilon(x_\varepsilon) \longrightarrow F(x)$.

La Γ-convergence s'applique à des espaces métriques ne possédant pas de bonnes propriétés de compacité, ce qui est notamment le cas des espaces fonctionnels. On pourra trouver une présentation plus détaillée des propriétés de la Γ-convergence dans [Bra02].

Espace des fonctions spéciales à variation bornée : L'espace $SBV^2(\Omega)$ de fonctions sur lequel est définie la fonctionnelle de Mumford–Shah avec laquelle nous allons travailler est un espace assez grand pour accueillir des fonctions très peu régulières et assez spécifique pour posséder des propriétés de compacités propices à l'analyse fonctionnelle.

Avant de le définir, on rappelle qu'on note $\mathcal{L}^d$ la mesure de Lebesgue dans $\mathbb{R}^d$, et $\mathcal{H}^k$ la mesure de Hausdorff de dimension k.

Un ensemble $K \subset \mathbb{R}^d$ qui est $\mathcal{H}^{d-1}-$mesurable de mesure finie est dit $d-1$-rectifiable s'il existe une famille $(f_i)_{i\in I}$ au plus dénombrable de fonctions lipschitz telles que $\mathcal{H}^{d-1}(E \setminus \cup_{i\in I} f(\mathbb{R}^{d-1})) = 0$.

On rappelle également la définition de l'espace $BV(\Omega)$ des fonctions à variations bornées.

Definition 1.5.3. *Une fonction u de $L^1(\Omega)$ est une **fonction à variation bornée** si sa dérivée au sens des distributions Du est une mesure de Radon finie sur Ω. Autrement dit, pour tout $\varphi \in \mathcal{C}_0^1(\Omega)$:*

$$\int_\Omega u\nabla\varphi = -\int_\Omega \varphi dDu$$

On note $BV(\Omega)$ l'espace des fonctions à variation bornée.

On s'intéressera notamment au sous-espace suivant des fonctions à variations bornées.

Definition 1.5.4. *On note*

$$SBV^2(\Omega) = \{u \in L^1(\Omega), Du = \nabla u\mathcal{L}^d + \nu_u(u^+ - u^-)\mathcal{H}^{d-1}_{|J_u}, \nabla u \in L^2(\Omega;\mathbb{R}^d), \mathcal{H}^{d-1}(J_u) < \infty\} \quad (1.5.1)$$

le sous-espace de $BV(\Omega)$ composée des fonctions dont la dérivée au sens des distributions Du ne comporte pas de partie Cantor. La densité ∇u est appelée gradient approché, l'ensemble J_u est un ensemble $d-1-$rectifiable appelé ensemble des sauts et ν_u est la normale approchée à J_u.

Chapter 2

Critical points of the one dimensional Ambrosio-Tortorelli functional with an obstacle condition

We consider a family of critical points of the Ambrosio-Tortorelli energy with an obstacle condition on the phase field variable. This problem can be interpreted as a time discretization of a quasistatic evolution problem where the obstacle at step n is defined as the solution at step $n-1$. The obstacle condition now reads as an irreversibility condition (the crack can only increase in time). The questions tackled here are the regularity of the critical points, the properties inherited from the obstacle sequence, the position of the limit points and the equipartition of the phase field energy. The limits of such critical points turn out to be critical points of the Mumford-Shah energy that inherit the possible discontinuities induced by the obstacle sequence.

2.1 Introduction

The Mumford-Shah functional, introduced by D.Mumford and J.Shah in 1989 in [MS89] in the context of image segmentation, is also used in fracture mechanics to describe brittle fractures [FM98]. If Ω is a bounded open subset of $\mathbb{R}^d$, it is defined, for any $u \in SBV^2(\Omega)$, as

$$MS(u) = \int_\Omega |\nabla u|^2 + \mathcal{H}^{d-1}(J_u).$$

In the spirit of the Allen-Cahn model for phase transition ([Mod87], [Ste88]), Ambrosio and Tortorelli proposed in [AT92] the following variational phase field regularization, defined for $(u,v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ by

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\right) dx$$

where $0 < \eta_\varepsilon \ll \varepsilon$ is a small parameter that ensures ellipticity. This regularization has also a mechanical interpretation. The phase field variable v can be seen as a damage variable taking values in the interval $[0,1]$, between the state $\{v=1\}$ where the material is sane and the state $\{v=0\}$ where the material is broken. This regularized formulation is also at the basis of numerical simulations (see [BFM08]).

In [AT92], a $\Gamma-$convergence result of AT_ε to MS is proved by suitably extending MS as a two variable functional :

$$MS(u,v) = \begin{cases} MS(u) & \text{if } v \equiv 1\ , \\ +\infty & \text{otherwise.} \end{cases}$$

Under suitable boundary conditions, the existence of minimizers for AT_ε for a fixed value of ε is obtained via the direct method of the Calculus of Variation. The Γ-convergence then ensures that a converging sequence $(u_\varepsilon, v_\varepsilon)$ of minimizers of AT_ε converges toward a minimizer of MS.

Concerning the numerical implementation, the term $v^2|\nabla u|^2$ implies a lack of convexity for the functional AT_ε. Consequently, numerical methods might fail to converge toward a minimizer of AT_ε. Using the fact that the functional remains separately strictly convex, [BFM08] proposes to perform an alternate minimization algorithm. It is proven in [Bou07, Theorem 1] that the sequence of iterates converge toward a critical point of AT_ε, but not always to a minimizer. This motivates to investigate the convergence of critical points of the Ambrosio-Tortorelli functional as $\varepsilon \to 0$ and compare this convergence with the original goal to minimize the Mumford-Shah functional.

A critical point for the Ambrosio-Tortorelli functional is a pair $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ such that

$$\frac{d}{dt}\,|_{t=0}\; AT_\varepsilon(u_\varepsilon + t\psi, v_\varepsilon + t\varphi) = 0 \quad \forall\, (\psi, \varphi) \in\; H_0^1(\Omega)^2.$$

with suitable conditions at the boundary $\partial\Omega$ (for example Dirichlet conditions, as in [BMR24]).

The Γ-convergence provides little help to study the convergence of critical points. The extension of the fundamental theorem of $\Gamma-$convergence to the convergence of critical points has been studied in various settings, see [HT00], [Ton02], [Ton05] for the Allen-Cahn functional and [BBH94], [BBO01], [LR01], [SS07] for the Ginzburg-Landau functional. For the Ambrosio-Tortorelli functional, the convergence has been established in all dimensions in [BMR24], with the additional hypothesis of the convergence of the energy $AT_\varepsilon(u_\varepsilon, v_\varepsilon)$ to $MS(u, 1)$. In dimension one, this hypothesis is no longer required, and it is proved in [FLS09] and [BMR23] that a sequence of critical points for the one-dimensional Ambrosio-Tortorelli converges to critical points of MS. A similar result is proved in [Le10] for the one-dimensional Mumford-Shah functional adapted to image segmentation. (See also [BI23] in cohesive fracture).

In [FL03], a model for the quasi-static evolution of a brittle fracture using the Mumford-Shah functional is proposed. The model starts with a discretized version of the evolution of the crack, where the irreversibility through time is translated into the condition that the crack $K(t_i)$ is included in the crack $K(t_{i+1})$, and the Mumford-Shah functional is then minimized at time t_{i+1} over all the cracks K containing $K(t_i)$, and with a variable Dirichlet condition g_i at the boundary $\partial\Omega$. In [Gia05], it is proved that a quasi-static evolution of AT_ε approximates a quasi-static evolution for MS using minimizers of AT_ε. The irreversibility condition translates into the minimization of AT_ε at time t_{i+1} among all couples (u, v) such that $v \leqslant v_{t_i,\varepsilon}$, where $(u_{t_i,\varepsilon}, v_{t_i,\varepsilon})$ is the minimum of AT_ε at time t_i.

A natural question is whether the result of [Gia05] can be extended to critical points. Since in high dimension, results on the convergence of critical points of AT_ε require stronger hypothesis, such an extension can be investigated only in dimension one for the moment. This paper is the first step in that direction, namely, we study the convergence of the critical points of the functional AT_ε defined on the pairs (u, v) in $H^1([0, L])$ satisfying $v \leqslant v_{0,\varepsilon}$, where $(u_{0,\varepsilon}, v_{0,\varepsilon})$ is a critical point of AT_ε. We use the same setting as [BMR23] (Dirichlet conditions on the variables (u, v)).

In Section 2, we define the one-dimensional functional $\overline{AT}_\varepsilon$ at study, which corresponds to the functional AT_ε restricted to the pairs (u, v) satisfying $v \leqslant v_{0,\varepsilon}$, we recall the results established by [BMR23] and state the main result in Theorem 2.2.7 : the critical points of $\overline{AT}_\varepsilon$ with the irreversibility condition $v \leqslant v_{0,\varepsilon}$ converge to critical point of MS. Theorem 2.2.8, proved in the appendix, deals with the $\Gamma-$convergence of the sequence $\overline{AT}_\varepsilon$. Theorem 2.2.7 can then be seen as an extension of Theorem 2.2.8 for critical points.

When we study the convergence of a functional with a phase field term, a natural question is the validity of an equipartition of the energy principle, i.e whether the terms of the phase field energy share the same limit. This principle, as well as the convergence of the Dirichlet approximation term is also stated in Theorem 2.2.7.

In Section 3, we establish the properties concerning the critical points of $\overline{AT}_\varepsilon$, namely the regularity and the variations of v_ε. In Section 4, we study the asymptotic behaviour of the sequence $(u_\varepsilon, v_\varepsilon)$ and prove Theorem 2.2.7.

Similarly to [FLS09], [BMR23] and [Le10], the key point to the main estimates is the study of the quantity called *discrepancy* (Equation 2.4.5) associated to $\overline{AT}_\varepsilon$ (which can be interpreted as the Hamiltonian of the functional). In [FLS09], [BMR23], this discrepancy is constant, which is no

longer the case in our situation. However, we are able to control the variations of the discrepancy in Lemma 2.4.7 to obtain analogous estimates and results as in [BMR23].

2.2 Notations and statement of the results

2.2.1 Notations and preliminary results

Throughout the paper, L is a positive number. Unless otherwise stated, letters like C, m and M will denote constants that do not depend on any other parameter and that may change from line to line. We also denote by $I_r(x)$ the interval $(x-r, x+r)$.

For any open subset $\Omega \subset (0, L)$, we denote by

$$||f||_{\infty,\Omega} := \sup_{x\in\Omega} |f(x)| \,, \quad [\,f\,]_{1/2,\Omega} := \sup_{x\neq y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^{1/2}} \,, \quad ||f||_{C^{0,1/2},\Omega} := ||f||_{\infty,\Omega} + [\,f\,]_{1/2,\Omega}.$$

We also denote by

$$\|f\|_1 = \int_0^L |f|.$$

The one-dimensional Ambrosio-Tortorelli functional, is defined as

$$AT_\varepsilon(u,v) := \int_0^L \left((\eta_\varepsilon + v^2)|u'|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon |v'|^2 \right) dx, \qquad \forall (u,v) \in H^1(0,L)^2,$$

where η_ε is a positive parameter such that $\eta_\varepsilon/\varepsilon \to 0$. Before defining the one-dimensional Mumford-Shah functional, we recall the definition of the space $SBV^2(0,L)$, which is the space of the functions $u \in L^1$ such that their distributional derivative is a Radon measure which writes as

$$Du = u' d\mathcal{L} + (u^+ - u^-)\nu_u \mathcal{H}^0_{|J_u} \text{ with } \int_0^L |u'|^2 < \infty,\ \mathcal{H}^0(J_u) < \infty.$$

The one-dimensional Mumford-Shah functional MS defined on $SBV^2(0,L) \times L^2(0,L)$ reads as

$$MS(u,v) := \begin{cases} \displaystyle\int_0^L |u'|^2 + \mathcal{H}^0(J_u) & \text{if } v \equiv 1 \,, \\ +\infty & \text{otherwise.} \end{cases} .$$

We define a critical point of AT_ε as follows

Definition 2.2.1. *Let $a_0 \in \mathbb{R}$. A critical point of AT_ε is a pair $(u,v) \in H^1(0,L)^2$ that satisfies for all $(\psi,\varphi) \in \mathcal{C}_c^\infty(0,L)$,*

$$\begin{cases} dAT_\varepsilon(u,v)[(\psi,\varphi)] := \lim_{t\to 0} \dfrac{AT_\varepsilon(u+t\psi, v+t\varphi) - AT_\varepsilon(u,v)}{t} = 0 \,, \\ u(0) = 0,\ u(L) = a_0 \,, \\ v(0) = v(L) = 1. \end{cases}$$

A critical point $(u_{0,\varepsilon}, v_{0,\varepsilon})$ of AT_ε satisfies the following equations on $(0,L)$

$$\begin{cases} -\varepsilon v''_{0,\varepsilon} + \frac{v_{0,\varepsilon}-1}{4\varepsilon} + v_{0,\varepsilon}|u'_{0,\varepsilon}|^2 = 0 & \text{in } (0,L) \,, \\ \left[u'_{0,\varepsilon}(\eta_\varepsilon + v^2_{0,\varepsilon})\right]' = 0 & \text{in } (0,L) \,, \\ u_{0,\varepsilon}(0) = 0,\ u_{0,\varepsilon}(L) = a_0 \,, \\ v_{0,\varepsilon}(0) = v_{0,\varepsilon}(L) = 1. \end{cases} \tag{2.2.1}$$

In the sequel, we consider a sequence $(u_{0,\varepsilon}, v_{0,\varepsilon})$ of critical points of AT_ε satisfying the following energy bound

$$\limsup_{\varepsilon>0} AT_\varepsilon(u_{0,\varepsilon}, v_{0,\varepsilon}) < C. \tag{2.2.2}$$

In [BMR23, Theorem 1.1], the following has been established :

Theorem 2.2.2. *Up to a subsequence, $(u_{0,\varepsilon}, v_{0,\varepsilon}) \to (u_0, 1)$ in $L^2(0,L)^2$, where either $u_0 = u_{jump} \equiv a_0 1_{[L/2,L]}$ or $u_0(x) = u_{aff}(x) = a_0 x/L$.*

When integrating the second equation of (2.2.1), we obtain a constant $c_{0,\varepsilon}$ such that

$$u'_{0,\varepsilon}(\eta_\varepsilon + v_{0,\varepsilon}^2) = c_{0,\varepsilon}.$$

[BMR23, Lemma 2.4] states that

Lemma 2.2.3. *Up to a subsequence, the sequence*

$$c_{0,\varepsilon} \to c_0; \tag{2.2.3}$$

with c_0 a real number satisfying $c_0 \in \{0, a_0/L\}$.

Theorem 2.2.2 enlightens a selection phenomenon : the accumulation points of the sequence $(u_{0,\varepsilon}, v_{0,\varepsilon})$ are specific critical points of MS. Indeed, critical points of MS are either the affine function u_{aff} or the piecewise constant functions. It is also proved in [BMR23, Theorem 1.2] that u_{jump} can be reached by a sequence of critical points of AT_ε. The crack of the material at time 0 then corresponds to J_{u_0}, which is, according to [BMR23, Theorem 1.1]

$$\Gamma_0 = J_{u_0} = \emptyset \qquad \text{or} \qquad \Gamma_0 = J_{u_0} = \left\{\frac{L}{2}\right\}. \tag{2.2.4}$$

The Ambrosio-Tortorelli functional with obstacle is defined as

$$\overline{AT_\varepsilon}(u,v) := \begin{cases} AT_\varepsilon(u,v) & \text{if } v \leqslant v_{0,\varepsilon}\ , \\ +\infty & \text{else,} \end{cases} \qquad \forall\, (u,v) \in H^1(0,L). \tag{2.2.5}$$

2.2.2 Euler-Lagrange equations

In order to account for the constraint on the phase field variable, we introduce the following definition for constrained critical points.

Definition 2.2.4. *A critical point of $\overline{AT}_\varepsilon$ is a pair $(u,v) \in H^1(0,L)^2$ that satisfies $v \leqslant v_{0,\varepsilon}$ and, for all $(\psi,\varphi) \in \mathcal{C}_c^\infty(0,L)$ with $\varphi \geqslant 0$,*

$$\begin{cases} d\overline{AT}_\varepsilon(u,v)[(\psi,\varphi)] := \lim\limits_{t\to 0^+} \dfrac{\overline{AT}_\varepsilon(u,v) - \overline{AT}_\varepsilon(u - t\psi, v - t\varphi)}{t} \leqslant 0\ , \\ \\ u(0) = 0,\ u(L) = a_1\ , \\ \\ v(0) = v(L) = 1. \end{cases}$$

Note that the Dirichlet condition at L is different and carries the evolution in time of the material. In the sequel, we denote

$$O_\varepsilon = \{v_\varepsilon < v_{0,\varepsilon}\} \quad \text{and} \quad F_\varepsilon = \{v_\varepsilon = v_{0,\varepsilon}\}. \tag{2.2.6}$$

Since $v_\varepsilon, v_{0,\varepsilon} \in H^1(0,L)$, their difference is continuous by Sobolev embedding so O_ε and F_ε are respectively open and closed in $[0,L]$.

The Euler-Lagrange equations of a critical point $(u_\varepsilon, v_\varepsilon)$ of $\overline{AT}_\varepsilon$ are given by the following lemma.

Lemma 2.2.5. *Let $(u_\varepsilon, v_\varepsilon)$ be a critical point of $\overline{AT}_\varepsilon$. There exists a non-positive Radon measure μ_ε supported on F_ε such that*

$$\begin{cases} -\varepsilon v_\varepsilon'' + \frac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon |u_\varepsilon'|^2 = \mu_\varepsilon \leqslant 0 & \text{in } (0,L) \\ \left[u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)\right]' = 0 & \text{in } (0,L) \\ v_\varepsilon \leqslant v_{0,\varepsilon} & \text{in } (0,L)\,, \\ u_\varepsilon(0) = 0,\ u_\varepsilon(L) = a_1\,, \\ v_\varepsilon(0) = v_\varepsilon(L) = 1. \end{cases} \tag{2.2.7}$$

Proof. Let $(u_\varepsilon, v_\varepsilon)$ be a critical point of $\overline{AT}_\varepsilon$. Take $\varphi = 0$ in Definition 2.2.4. Similarly to [FLS09, Introduction], the equation satisfied by u_ε reads as

$$\left[u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)\right]' = 0. \tag{2.2.8}$$

This means that there exists a constant c_ε such that

$$u_\varepsilon' = \frac{c_\varepsilon}{(\eta_\varepsilon + v_\varepsilon^2)} \quad \text{a.e in } (0,L). \tag{2.2.9}$$

As v_ε is continuous by Sobolev embedding, the left hand side is also continuous. Similarly to [FLS09], we obtain that u_ε' has a constant sign, corresponding to the sign of c_ε, meaning that u_ε is monotone on $[0,L]$.

We next move on to the equation for v_ε. Taking $\psi = 0$ in Definition 2.2.4, we obtain the following inequality, where v_ε'' denotes the distributional derivative of v_ε', which exists as $v_\varepsilon \in H^1(0,L)$,

$$\left\langle \varphi\,,\, v_\varepsilon |u_\varepsilon'|^2 - \varepsilon v_\varepsilon'' + \frac{v_\varepsilon - 1}{4\varepsilon} \right\rangle \leqslant 0 \quad \forall\, \varphi \in C^\infty(0,1), \varphi \geqslant 0 \tag{2.2.10}$$

which we rewrite in the following way : there exists a non positive Radon measure distribution μ_ε such that

$$-\varepsilon v_\varepsilon'' + \frac{v_\varepsilon - 1}{\varepsilon} + v_\varepsilon |u_\varepsilon'|^2 = \mu_\varepsilon \leqslant 0. \tag{2.2.11}$$

Taking φ compactly supported on $\{v_\varepsilon < v_{0,\varepsilon}\}$ gives the reversed inequality of (2.2.10). This means that the measure μ_ε is supported on the closed set F_ε. □

Using (2.2.11), we see that v_ε'' is also a Radon measure, so that $v_\varepsilon' \in BV(0,L)$, which is embedded in $L^\infty(0,L)$, and that $v_\varepsilon \in C^{0,1}(0,L)$. In section 3.1, we provide a better regularity result on v_ε. This allows us to establish an explicit formula for μ_ε at the end of section 3.1 (Corollary 2.3.3).

2.2.3 Statement of the theorems

We consider a sequence $(u_\varepsilon, v_\varepsilon)$ of critical points of $\overline{AT}_\varepsilon$. We suppose that the sequence $(u_\varepsilon, v_\varepsilon)$ satisfies the following energy bound

$$\limsup_{\varepsilon \to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \mathcal{C}. \tag{2.2.12}$$

Our first result concerns the regularity and is proved in section 3.

Theorem 2.2.6. *Let $(u_\varepsilon, v_\varepsilon)$ be a critical point of $\overline{AT}_\varepsilon$ in $H^1(0,L) \times H^1(0,L)$. Then $v_\varepsilon \in C^{1,1}([0,L])$.*

We now state the main result of this work concerning the limit points of $(u_\varepsilon, v_\varepsilon)$.

Theorem 2.2.7 (Convergence of the critical points)**.** *Let $(u_\varepsilon, v_\varepsilon)$ be a family of critical points of $\overline{AT}_\varepsilon$ satisfying* (2.2.12)*. Then there exists a real number $x_{min} \in \left[\frac{1}{4}L, \frac{3}{4}L, \right]$ and a subsequence such that*

(*i*) $(u_\varepsilon, v_\varepsilon) \to (u_1, 1)$ *in* $L^2(0,L)$, *with either* $u_1(x) = u_{jump}(x) = a_1 1_{[x_{min},L]}(x)$ *or* $u_1(x) = u_{aff}(x) = \dfrac{a_1 x}{L}$. *Moreover, if* $|F_\varepsilon| \neq 0$ *and* $|a_1| \geqslant a_0$, *then* $u_1 = u_{jump}$.

(*ii*) $\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon(v_\varepsilon')^2\right) dx \overset{\star}{\rightharpoonup} \alpha\delta_{x_{min}}$ *in* $\mathcal{M}([0,L])$ *with* $\alpha = 0$ *or* $\alpha = 1$. *Moreover, if* $u_1 = u_{jump}$ *then* $\alpha = 1$.

(*iii*) $(\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 dx \overset{\star}{\rightharpoonup} |u_1'|^2 dx$ *in* $\mathcal{M}([0,L])$.

(*iv*) *(Equipartition of the energy principle)* $\left|\dfrac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v_\varepsilon'|^2\right| \to 0$ *in* $L^1(0,L)$.

(*v*) $\varepsilon|v_\varepsilon'|^2 dx \overset{\star}{\rightharpoonup} \dfrac{\alpha}{2}\delta_{x_{min}}$, *and* $\dfrac{(1-v_\varepsilon)^2}{4\varepsilon} dx \overset{\star}{\rightharpoonup} \dfrac{\alpha}{2}\delta_{x_{min}}$ *in* $\mathcal{M}([0,L])$.

Remark 2.2.1. Theorem 2.2.7 extends the following $\Gamma-$ convergence result, which we prove in the appendix:

Theorem 2.2.8 (Γ convergence of $\overline{AT}_\varepsilon$)**.** *The functional* $\overline{AT}_\varepsilon$ Γ*-converges to the functional* $\overline{MS}$, *defined for every* $(u,v) \in L^2(0,L)^2$ *as*

$$\overline{MS}(u,v) := \begin{cases} \displaystyle\int_0^L |u'|^2 + \mathcal{H}^0(J_u \cup \Gamma_0) & \text{if } v = 1\,, \\ +\infty & \text{else}\,. \end{cases}$$

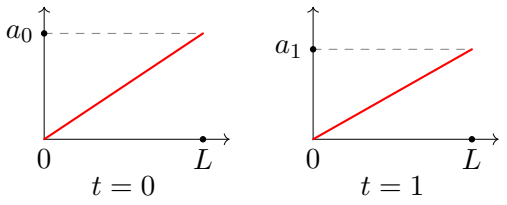
where Γ_0 *is defined by* (2.2.4).

While the Γ-convergence implies that a converging sequence of minimizers of $\overline{AT}_\varepsilon$ converges to a minimizer of $\overline{MS}(u)$, Theorem 2.2.7 acts as a selection criteria : the limit of critical points of $\overline{AT}_\varepsilon$ are critical points of $\overline{MS}$ that have at most one discontinuity point. Observe that we recover a similar selection phenomenon as in [BMR23, Theorem 1.2].

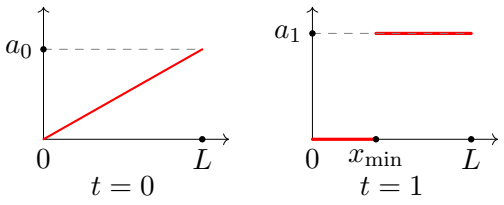
Considering (2.2.8), it is natural to define the crack at time 1 by

$$\Gamma_1 = \Gamma_0 \cup J_{u_1} \tag{2.2.13}$$

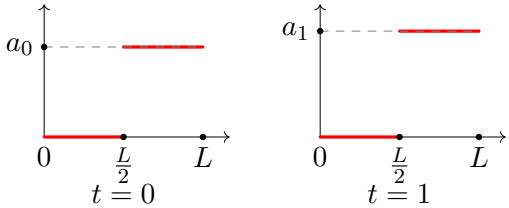
The different possible values for Γ_1 according to Theorem 2.2.7 are represented in Figure 2.1.



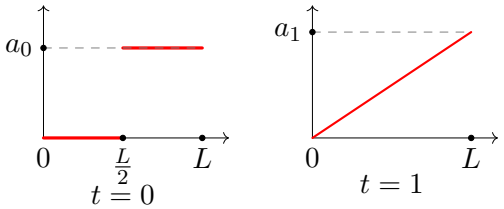
(a) $\Gamma_0 = \emptyset$, $J_{u_1} = \emptyset$ and $\Gamma_1 = \emptyset$



(b) $\Gamma_0 = \emptyset$, $J_{u_1} = \{x_{\min}\}$ and $\Gamma_1 = \{x_{\min}\}$



(c) $\Gamma_0 = \{L/2\}$, $J_{u_1} = \{L/2\}$ and $\Gamma_1 = \{L/2\}$



(d) $\Gamma_0 = \{L/2\}$, $J_{u_1} = \emptyset$ and $\Gamma_1 = \{L/2\}$

Figure 2.1: Possible values of Γ_0, J_{u_1} and Γ_1.

Remark 2.2.2. The item (iv) of Theorem 2.2.7 is called the *equipartition of energy principle* and is to be compared with the equality case of the arithmetico-geometric inequality

$$\frac{(1-v)^2}{4\varepsilon} + \varepsilon|v'|^2 \geqslant |1-v||v'|$$

used in the proof of Gamma-convergence of AT_ε to MS.

Remark 2.2.3. Let us stress that the implication in (ii) is not an equivalence, and that the converse is still an open question. However, the fact that $\alpha > 0$ is equivalent to the existence of a so-called $v-$jump for the sequence (v_ε) defined as follows :

Definition 2.2.9. *A point $x \in [0, L]$ is a $v-$jump if there exists a sequence (x_ε) such that $x_\varepsilon \to x$ and $v_\varepsilon(x_\varepsilon) \to 0$.*

It turns out that a $v-$jump does not necessarily lead to a discontinuity point in the limit. The variations and the symmetry of the function $v_{0,\varepsilon}$ around $L/2$ imply that the only possible $v-$jump for $(v_{0,\varepsilon})$ is $L/2$. In the case where $L/2$ is a $v-$jump for $(v_{0,\varepsilon})$, the obstacle condition implies that $L/2$ is also a $v-$jump for v_ε which means that $x_{min} = L/2$. We can then explicit the four possible outcomes for Γ_1, represented on Figure 2.1.

1. The case (a) on Figure 2.1 corresponds to the situation where $u_0(x) = \dfrac{a_0 x}{L}$ and $u_1(x) = u_{aff}$.

2. In the case (b), there are two possibilities : if $(v_{0,\varepsilon})$ does not have a $v-$jump, there is no restriction on the value of $x_{min} \in [L/4, 3L/4]$. However, if $(v_{0,\varepsilon})$ has a $v-$jump, which does not necessarily translates to $u_0 = u_{jump}$, then according to the above discussion, $x_{min} = L/2$.

3. In the case (c), $u_0 = u_{jump}$. Hence, $L/2$ is a $v-$jump for $(v_{0,\varepsilon})$. It implies again that $x_{min} = L/2$.

4. In the case (d), which might be the less intuitive, $u_0 = u_{jump}$ so similarly $L/2$ is a $v-$jump for (v_ε) and $\alpha > 0$. However, as it is still open whether this implies that $u_1 = u_{jump}$, the outcome where $u_1 = u_{aff}$ can not be excluded so far.

Remark 2.2.4. A similar result holds for a sequence $(u_\varepsilon, v_\varepsilon)$ of critical points of $\overline{AT}_\varepsilon$ with Neumann conditions on v_ε, which is the framework of [FLS09, Theorem 2.2]. More precisely, if Γ_0 and Γ_1 are defined as in (2.2.4) and (2.2.13), [FLS09, Theorem 2.2] states that $\Gamma_0 = \emptyset$ or Γ_0 is a regular subdivision of $(-L, L)$. It can be proved that

- in the former case, $\Gamma_1 = \{x_0, \dots, x_{n+1}\}$, where $x_0 = 0, x_{n+1} = L$ and $x_i \in \left[\dfrac{(2i-1)L}{2n}, \dfrac{(2i+1)L}{2n}\right]$ for $1 \leqslant i \leqslant n$ or $\{x_1, \dots, x_n\}$ where $x_i \in \left[\dfrac{(i-1)L}{n}, \dfrac{iL}{n}\right]$ for $1 \leqslant i \leqslant n$. Observe that, as for Theorem 2.2.7, the symmetry of the "perfect staircase" observed in [FLS09, Remark 2.3] is lost at time 1.

- in the latter case, $\Gamma_1 = \Gamma_0$.

We would like to close this section by explaining what happens if we intend to iterate Theorem 2.2.7 for $t = 2$. Denote by $(u_{1,\varepsilon}, v_{1,\varepsilon})$ a sequence of critical points at time 1, i.e satisfying $v_{1,\varepsilon} \leqslant v_{0,\varepsilon}$, $\mu_{1,\varepsilon}$ the right-hand side of the first equation of (2.2.7) and $(u_{2,\varepsilon}, v_{2,\varepsilon})$ a sequence of critical points at time 2, i.e satisfying $v_{2,\varepsilon} \leqslant v_{1,\varepsilon}$. The right hand-side in the first equation of (2.2.7) corresponding to $v_{2,\varepsilon}$ is a non positive measure $\mu_{2,\varepsilon}$. The expression of $\mu_{2,\varepsilon}$ now depends both on $\{v_{0,\varepsilon} = v_{1,\varepsilon}\}$ and $\{v_{1,\varepsilon} = v_{2,\varepsilon}\}$, so that we do not always have a similar result as Corollary 2.3.3 in the case $|\{v_{1,\varepsilon} = v_{2,\varepsilon}\}| \neq 0$. As a consequence, the behavior of $(u_{2,\varepsilon}, v_{2,\varepsilon})$ is less predictable, in the sense that we do not have an analogous of item (i) of Theorem 2.2.7. A possible reason to that phenomenon is that, as we consider critical points and not only minimizers, we do not have an energy balance at our disposal, which is crucial in the iteration argument proposed in [Gia05, Theorem 2.2].

2.3 Critical points of $\overline{AT}_\varepsilon$

2.3.1 $C^{1,1}$ regularity of the critical points

In this subsection, we investigate the regularity of v_ε. More precisely, we establish the following proposition :

Proposition 2.3.1. *Let $(u_\varepsilon, v_\varepsilon)$ be a critical point of $\overline{AT}_\varepsilon$ in $H^1(0,L)\times H^1(0,L)$. Then $v_\varepsilon \in C^{1,1}([0,L])$.*

The methods of the proof are similar to those of the classical obstacle problem, see e.g. [FRRO22, Theorem 5.5]. Set $w_\varepsilon = v_{0,\varepsilon} - v_\varepsilon$, which is a non-negative function on $[0,L]$. Note that by Sobolev embedding, $v_\varepsilon \in C^{0,1/2}([0,L])$, so that, using the equation (2.2.9), $u_\varepsilon \in C^{1,1/2}([0,L])$. In particular, u'_ε is bounded on $[0,L]$. On $(0,L)$, w_ε satisfies the following equation

$$\begin{aligned}
\varepsilon w''_\varepsilon &= \varepsilon(v''_{0,\varepsilon} - v''_\varepsilon) = \frac{v_{0,\varepsilon} - 1 - (v_\varepsilon - 1)}{4\varepsilon} + v_{0,\varepsilon}|u'_{0,\varepsilon}|^2 - v_\varepsilon|u'_\varepsilon|^2 + \mu_\varepsilon \\
&= \frac{w_\varepsilon}{4\varepsilon} + (v_{0,\varepsilon} - v_\varepsilon)|u'_{0,\varepsilon}|^2 + v_\varepsilon(|u'_{0,\varepsilon}|^2 - |u'_\varepsilon|^2) + \mu_\varepsilon \\
&= \frac{w_\varepsilon}{4\varepsilon} + w_\varepsilon|u'_{0,\varepsilon}|^2 + v_\varepsilon\left(\frac{c_{0,\varepsilon}^2}{(\eta_\varepsilon + v_{0,\varepsilon}^2)^2} - \frac{c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2}\right) + \mu_\varepsilon
\end{aligned}$$

Since $\mu_\varepsilon \leqslant 0$, we obtain the following expressions

$$\begin{cases}
\varepsilon w''_\varepsilon \leqslant w_\varepsilon\left(\dfrac{1}{4\varepsilon} + |u'_{0,\varepsilon}|^2\right) + v_\varepsilon\left(\dfrac{c_{0,\varepsilon}^2}{(\eta_\varepsilon + v_{0,\varepsilon}^2)^2} - \dfrac{c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2}\right) & \text{on } [0,L], \\
\varepsilon w''_\varepsilon = w_\varepsilon\left(\dfrac{1}{4\varepsilon} + |u'_{0,\varepsilon}|^2\right) + v_\varepsilon\left(\dfrac{c_{0,\varepsilon}^2}{(\eta_\varepsilon + v_{0,\varepsilon}^2)^2} - \dfrac{c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2}\right) & \text{on } O_\varepsilon.
\end{cases} \tag{2.3.1}$$

Since $v_{0,\varepsilon}$ is smooth, it is sufficient to prove that w_ε is in $C^{1,1}([0,L])$. We first need to prove that w_ε grows at most quadratically around ∂O_ε (Lemma 2.3.2). Then, we can use Schauder estimates on (2.3.1) to ensure that w'_ε is continuous at ∂O_ε.

Lemma 2.3.2. *There exists a constant $C_\varepsilon > 0$ only depending on $\|u'_{0,\varepsilon}\|_{\infty,[0,L]}, \|u'_\varepsilon\|_{\infty,[0,L]}$ and ε such that for any x such that $d(x,\partial O_\varepsilon) \leqslant \frac{1}{3}d(x,\{0,L\})$, one has*

$$0 \leqslant w_\varepsilon(x) \leqslant C_\varepsilon\, d(x,\partial O_\varepsilon)^2.$$

Proof. Since $0 \leqslant v_\varepsilon \leqslant v_{0,\varepsilon} \leqslant 1$, w_ε is bounded. As u'_ε and $u'_{0,\varepsilon}$ are also bounded on $[0,L]$, the first inequality of the system (2.3.1) gives

$$\varepsilon w''_\varepsilon \leqslant \frac{1}{4\varepsilon}w_\varepsilon + |u'_{0,\varepsilon}|^2 w_\varepsilon + v_\varepsilon(|u'_{0,\varepsilon}|^2 - |u'_\varepsilon|^2) \leqslant C_\varepsilon, \tag{2.3.2}$$

where this inequality is written in the distributional sense.

Fix any $z \in (0,L)$. The mean value formula applied to the superharmonic function $\varepsilon w_\varepsilon - C_\varepsilon\dfrac{|z-\cdot|^2}{2}$ yields, for any r such that $I_r(z) \subset [0,L]$,

$$\varepsilon w_\varepsilon(z) \geqslant \frac{1}{2r}\int_{I_r(z)}\left(\varepsilon w_\varepsilon(y) - C_\varepsilon\frac{|z-y|^2}{2}\right)dy \geqslant \frac{\varepsilon}{2r}\int_{I_r(z)} w_\varepsilon(y)dy - C_\varepsilon r^2 \tag{2.3.3}$$

Secondly, for any $y \in O_\varepsilon$, the second equation of (2.3.1) yields

$$\varepsilon w''_\varepsilon(y) \geqslant -v_\varepsilon(y)\frac{c_\varepsilon^2}{(\eta_\varepsilon + |v_\varepsilon|^2)^2} \geqslant -\frac{c_\varepsilon^2}{\eta_\varepsilon^2} = -C'_\varepsilon.$$

Fix then $z \in O_\varepsilon$. The mean value formula applied to the subharmonic function $\varepsilon w_\varepsilon + C'_\varepsilon \frac{|z-\cdot|^2}{2}$ yields for any r such $I_r(z) \subset O_\varepsilon$,

$$\varepsilon w_\varepsilon(z) \leqslant \frac{1}{2r}\int_{I_r(z)} \left(\varepsilon w_\varepsilon(y) + C'_\varepsilon \frac{|z-y|^2}{2}\right) dy \leqslant \frac{\varepsilon}{2r}\int_{I_r(z)} w_\varepsilon(y)dy + C'_\varepsilon r^2. \tag{2.3.4}$$

Now fix $x \in O_\varepsilon$ such that $r := d(x, \partial O_\varepsilon)$ satisfies $r \leqslant \frac{1}{3}d(x,\{0,L\})$ and take any $x' \in \overline{I_r(x)} \cap \partial O_\varepsilon$. Then $I_{2r}(x') \subset I_{3r}(x) \subset (0,L)$ and $x' \in F_\varepsilon$ so, using (2.3.3),

$$0 = \varepsilon w_\varepsilon(x') \geqslant \frac{1}{4r}\int_{I_{2r}(x')} \varepsilon w_\varepsilon(y)dy - C_\varepsilon r^2. \tag{2.3.5}$$

Then, since $w_\varepsilon \geqslant 0$ and $I_r(x) \subset I_{2r}(x')$, we get

$$\varepsilon w_\varepsilon(x) - C'_\varepsilon r^2 \overset{(2.3.4)}{\leqslant} \frac{\varepsilon}{2r}\int_{I_r(x)} w_\varepsilon(y)dy \leqslant \frac{\varepsilon}{2r}\int_{I_{2r}(x')} w_\varepsilon(y)dy \overset{(2.3.5)}{\leqslant} 2C_\varepsilon r^2.$$

which is the desired inequality. □

We are now ready to prove Proposition 2.3.1.

Proof of Proposition 2.3.1. First, we focus on the regularity on $(0,L)$. Let $x_1 \in O_\varepsilon$ and take $x_0 \in \partial O_\varepsilon$ such that $r := |x_0 - x_1| = d(x_1, \partial O_\varepsilon)$ and suppose that $r < \frac{1}{3}d(x_1,\{0,L\})$. Notice that the right-hand side of the second equation of (2.3.1) belongs to $C^{0,1/2}([0,L])$. Applying Schauder estimates ([GT01, Theorem 4.6]) on $I_r(x_1) \subset O_\varepsilon$, there exists a constant $C_\varepsilon > 0$ independent of x_1, r and w_ε such that

$$\begin{aligned} &r\|w'_\varepsilon\|_{\infty, I_{r/2}(x_1)} + r^2\|w''_\varepsilon\|_{\infty, I_{r/2}(x_1)} \\ &\quad\leqslant C_\varepsilon \left(\underbrace{\|w_\varepsilon\|_{\infty, I_r(x_1)}}_{:=I} + r^2 \underbrace{\left|\left| w_\varepsilon\left(\frac{1}{4\varepsilon} + |u'_{0,\varepsilon}|^2\right)\right|\right|_{C^{0,1/2}, I_r(x_1)}}_{:=II} + r^2 \underbrace{\|v_\varepsilon(|u'_{0,\varepsilon}|^2 - |u'_\varepsilon|^2)\|_{C^{0,1/2}, I_r(x_1)}}_{:=III} \right). \end{aligned} \tag{2.3.6}$$

We now estimate the terms in the right hand side of (2.3.6).

I : Using Lemma 2.3.2, one has $\|w_\varepsilon\|_{\infty, I_r(x_1)} \leqslant C_\varepsilon\, d(x_1, \partial O_\varepsilon)^2 = C_\varepsilon r^2$.

II : We split the norm $\|\cdot\|_{C^{0,1/2}}$ into the norm $\|\cdot\|_\infty$ and the semi-norm $[\,\cdot\,]_{1/2}$. Concerning the norm $\|\cdot\|_\infty$, one has

$$\left|\left| w_\varepsilon\left(\frac{1}{4\varepsilon} + |u'_\varepsilon|^2\right)\right|\right|_{\infty, I_r(x_1)} \leqslant 2\left(\frac{1}{4\varepsilon} + \|u'_\varepsilon\|^2_{\infty,[0,L]}\right) = C_\varepsilon,$$

where $C_\varepsilon > 0$ is a constant which depends on $\|u'_\varepsilon\|_{\infty,[0,L]}$. Concerning the semi-norm $[\,\cdot\,]$, we use the inequality $[|u'_\varepsilon|^2]_{1/2} \leqslant 2\|u'_\varepsilon\|_\infty [u'_\varepsilon]_{1/2}$. This yields

$$\begin{aligned} \left[w_\varepsilon\left(\frac{1}{4\varepsilon} + |u'_\varepsilon|^2\right)\right]_{1/2, I_r(x_1)} &\leqslant \|w_\varepsilon\|_{\infty, I_r(x_1)} \left[\frac{1}{4\varepsilon} + |u'_\varepsilon|^2\right]_{1/2, I_r(x_1)} + [w_\varepsilon]_{1/2, I_r(x_1)} \left|\left| \frac{1}{4\varepsilon} + |u'_\varepsilon|^2\right|\right|_{\infty, I_r(x_1)} \\ &\leqslant 2\|u'_\varepsilon\|_{\infty,[0,L]}[u'_\varepsilon]_{1/2,[0,L]} + [w_\varepsilon]_{1/2,[0,L]} \left|\left| \frac{1}{4\varepsilon} + |u'_\varepsilon|^2\right|\right|_{\infty,[0,L]} \\ &\leqslant C_\varepsilon \end{aligned}$$

where $C_\varepsilon > 0$ is a constant depending on $\|u'_\varepsilon\|_{C^{0,1/2},[0,L]}$ and $\|w_\varepsilon\|_{C^{0,1/2},[0,L]}$.

III : Similarly, one has

$$\|v_\varepsilon(|u'_{0,\varepsilon}|^2-|u'_\varepsilon|^2)\|_{\infty,I_r(x_1)}\leqslant\|u'_{0,\varepsilon}\|^2_{\infty,[0,L]}+\|u'_\varepsilon\|^2_{\infty,[0,L]},$$

and

$$\begin{aligned}
&[v_\varepsilon(|u'_{0,\varepsilon}|^2-|u'_\varepsilon|^2)]_{1/2,I_r(x_1)}\\
\leqslant\ &\|v_\varepsilon\|_{\infty,I_r(x_1)}([|u'_{0,\varepsilon}|^2]_{1/2,I_r(x_1)}+|u'_\varepsilon|^2_{1/2,I_r(x_1)}]+[v_\varepsilon]_{1/2,I_r(x_1)}(\|u'_{0,\varepsilon}\|^2_{\infty,I_r(x_1)}+\|u'_\varepsilon\|^2_{\infty,I_r(x_1)})\\
\leqslant\ &2\|u'_{0,\varepsilon}\|_{\infty,[0,L]}[u'_{0,\varepsilon}]_{1/2,[0,L]}+2\|u'_\varepsilon\|_{\infty,[0,L]}[u'_\varepsilon]_{1/2,[0,L]}+[v_\varepsilon]_{1/2,[0,L]}(\|u'_{0,\varepsilon}\|^2_{\infty,[0,L]}+\|u'_\varepsilon\|^2_{\infty,[0,L]})\\
\leqslant\ &C_\varepsilon
\end{aligned},$$

where $C_\varepsilon>0$ is a constant depending only on $\|u'_{0,\varepsilon}\|_{C^{0,1/2},[0,L]}$, $\|u'_\varepsilon\|_{C^{0,1/2},[0,L]}$ and $\|v'_\varepsilon\|_{C^{0,1/2},[,L]}$. It follows from (2.3.6) that

$$r\|w'_\varepsilon\|_{\infty,I_{r/2}(x_1)}+r^2\|w''_\varepsilon\|_{\infty,I_{r/2}(x_1)}\leqslant C_\varepsilon r^2.$$

In particular, $|w'_\varepsilon(x_1)|\leqslant\|w'_\varepsilon\|_{\infty,I_{r/2}(x_1)}\leqslant C_\varepsilon\, d(x_1,\partial O_\varepsilon)$. This means that $w'_\varepsilon(x_1)$ tends to 0 as $d(x_1,\partial O_\varepsilon)\to 0$. Moreover, as w_ε vanishes on F_ε, $w'_\varepsilon=0$ a.e on F_ε (see [GT01, Lemma 7.7]). So w'_ε admits a representative on $(0,L)$ that is zero on F_ε and is continuous on $(0,L)$, so that $w_\varepsilon\in C^1(0,L)$.

From (2.3.2), we also have

$$\|w''_\varepsilon\|_{\infty,(0,L)}\leqslant C_{2,\varepsilon},$$

where $C_{2,\varepsilon}>0$ is a constant depending on $\|u'_{0,\varepsilon}\|_{\infty,[0,L]}$, from which we deduce that w'_ε is Lipschitz on $(0,L)$.

To recover the regularity at the boundary points 0 and L, we see that w'_ε is uniformly continuous on $(0,L)$(because it is Lipschitz) so it admits a continuous extension on $[0,L]$. This implies that w_ε is differentiable at 0 and L with $w'_\varepsilon(0)=\lim\limits_{x\to0^+}w'_\varepsilon(x)$ and $w'_\varepsilon(L)=\lim\limits_{x\to L^-}w'_\varepsilon(x)$. We deduce that $w_\varepsilon\in C^1([0,L])$. Then, as w'_ε is Lipschitz on $(0,L)$, its extension to $[0,L]$ is Lipschitz as well so that $w_\varepsilon\in C^{1,1}([0,L])$. Hence $v_\varepsilon\in C^{1,1}([0,L])$. □

The following result gives an explicit pointwise formula for μ_ε.

Corollary 2.3.3. *Let μ_ε be the right-hand side in* (2.2.11)*. We have the following expression for μ_ε,*

$$\mu_\varepsilon=\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2}((c_\varepsilon)^2-(c_{0,\varepsilon})^2)1_{F_\varepsilon}=v_{0,\varepsilon}(|u'_\varepsilon|^2-|u'_{0,\varepsilon}|^2)1_{F_\varepsilon}\tag{2.3.7}$$

In particular, if $|F_\varepsilon|\neq0$, then $|c_\varepsilon|\leqslant c_{0,\varepsilon}$.

Proof. As $v_\varepsilon\in C^{1,1}([0,1])$, μ_ε is absolutely continuous with respect to the Lebesgue measure. Moreover, as $v_\varepsilon=v_{0,\varepsilon}$ on F_ε, we have $v'_\varepsilon=(v_{0,\varepsilon})'$ and $v''_\varepsilon=(v_{0,\varepsilon})''$ a.e on F_ε (see [GT01, Lemma 7.7]). Hence

$$\begin{aligned}
\mu_\varepsilon &= \left(-\varepsilon v''_\varepsilon+\frac{v_\varepsilon-1}{4\varepsilon}+v_\varepsilon|u'_\varepsilon|^2\right)1_{F_\varepsilon} &&= \left(-\varepsilon v''_{0,\varepsilon}+\frac{v_{0,\varepsilon}-1}{4\varepsilon}+v_{0,\varepsilon}|u'_\varepsilon|^2\right)1_{F_\varepsilon}\\
&= \left(-v_{0,\varepsilon}|u'_{0,\varepsilon}|^2+v_{0,\varepsilon}|u'_\varepsilon|^2\right)1_{F_\varepsilon} &&= v_{0,\varepsilon}(|u'_\varepsilon|^2-|u'_{0,\varepsilon}|^2)1_{F_\varepsilon}\\
&= v_{0,\varepsilon}\left(\frac{c_\varepsilon^2}{(v_\varepsilon^2+\eta_\varepsilon)^2}-\frac{c_{0,\varepsilon}^2}{(v_{0,\varepsilon}^2+\eta_\varepsilon)^2}\right)1_{F_\varepsilon} &&= \frac{v_{0,\varepsilon}}{(v_{0,\varepsilon}^2+\eta_\varepsilon)^2}(c_\varepsilon^2-c_{0,\varepsilon}^2)1_{F_\varepsilon}
\end{aligned}.$$

As $\mu_\varepsilon\leqslant0$, provided that $|F_\varepsilon|\neq0$, one has $c_\varepsilon^2\leqslant c_{0,\varepsilon}^2$, and $|c_\varepsilon|\leqslant c_{0,\varepsilon}$. □

2.3.2 Shape of v_ε

The aim of this section is to take advantage of the ODE (2.2.11) satisfied by v_ε to determine its variations. We prove that v_ε admits a unique global minimum, denoted by $x_\varepsilon \in [L/4, 3L/4]$ and that v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$. We start with the following lemma on the possible positions of the critical points :

Lemma 2.3.4. *Let x_ε be a critical point of v_ε. If $x_\varepsilon \in F_\varepsilon$, then $x_\varepsilon = L/2$.*

Proof. Note that statement is non empty, by Rolle's Theorem, as $v_\varepsilon(0) = v_\varepsilon(L)$. Suppose $x_\varepsilon \in F_\varepsilon \setminus \{0, L\}$. Then

$$0 = v_\varepsilon'(x_\varepsilon) = \lim_{h\to 0^+} \frac{v_\varepsilon(x_\varepsilon + h) - v_\varepsilon(x_\varepsilon)}{h} = \lim_{h\to 0^+} \frac{v_\varepsilon(x_\varepsilon + h) - v_{0,\varepsilon}(x_\varepsilon)}{h}$$
$$\leqslant \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(x_\varepsilon + h) - v_{0,\varepsilon}(x_\varepsilon)}{h} = v_{0,\varepsilon}'(x)$$

and

$$0 = v_\varepsilon'(x_\varepsilon) = \lim_{h\to 0^+} \frac{v_\varepsilon(x_\varepsilon) - v_\varepsilon(x_\varepsilon - h)}{h} = \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(x_\varepsilon) - v_\varepsilon(x_\varepsilon - h)}{h}$$
$$\geqslant \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(x_\varepsilon) - v_{0,\varepsilon}(x_\varepsilon - h)}{h} = v_{0,\varepsilon}'(x)$$

so that $v_{0,\varepsilon}'(x_\varepsilon) = 0$, which implies $x_\varepsilon = L/2$ by [BMR23, Proposition 2.1]. If $x_\varepsilon = 0$, then, using again [BMR23, Proposition 2.1],

$$0 > v_{0,\varepsilon}'(0) = \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(h) - v_{0,\varepsilon}(0)}{h} \geqslant \lim_{h\to 0^+} \frac{v_\varepsilon(h) - 1}{h} = v_\varepsilon'(0)$$

and if $x_\varepsilon = L$,

$$0 < v_{0,\varepsilon}'(L) = \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(L) - v_{0,\varepsilon}(L - h)}{h} \leqslant \lim_{h\to 0^+} \frac{1 - v_\varepsilon(L - h)}{h} = v_\varepsilon'(L)$$

so that 0 and L are not critical points of v_ε. □

The following proposition establishes the variations of v_ε on $[0, L]$.

Proposition 2.3.5. *The function v_ε admits a unique local minimum at x_ε on $[0, L]$. This local minimum is also a global minimum and is either located at $L/2$ or it belongs to the connected component of O_ε containing $L/2$.*

Observe that while $v_{0,\varepsilon}$ is symmetric with respect to $L/2$, the previous proposition states that it is not necessarily the case for v_ε.

Proof. According to Lemma 2.3.4, a local minimum of v_ε is either $L/2$ or in O_ε. The proof is divided in two steps. In the first step, we prove that any connected component of O_ε contains at most one critical point y. Using this first step, we prove in the second step that the only connected component of O_ε that contains a local minimum is (if it exists) the connected component containing $L/2$.

Step 1 : Any connected component of O_ε contains at most one critical point.

Take (β, γ) a non-empty connected component of O_ε and suppose that it contains a critical point of v_ε. Assume that $v_\varepsilon(\beta) \leqslant v_\varepsilon(\gamma)$ (the case $v_\varepsilon(\beta) \geqslant v_\varepsilon(\gamma)$ can be treated similarly). As $\beta, \gamma \in F_\varepsilon$, we also have $v_{0,\varepsilon}(\beta) \leqslant v_{0,\varepsilon}(\gamma)$. As $v_{0,\varepsilon}$ is decreasing on $[0, L/2]$ and increasing on $[L/2, L]$ ([BMR23, Proposition 2.1]), we have in particular that $\gamma > L/2$ and $v_{0,\varepsilon}'(\gamma) > 0$.

Assume that $\{v_\varepsilon' = 0\} \cap (\beta, \gamma)$ is non-empty. First, γ cannot be critical points of v_ε, as

$$v_\varepsilon'(\gamma) = \lim_{h\to 0} \frac{v_\varepsilon(\gamma) - v_\varepsilon(\gamma - h)}{h} \geqslant \lim_{h\to 0} \frac{v_{0,\varepsilon}(\gamma) - v_{0,\varepsilon}(\gamma - h)}{h} = v_{0,\varepsilon}'(\gamma) > 0.$$

Hence, the supremum y of $\{v_\varepsilon' = 0\} \cap (\beta, \gamma)$ belongs to (β, γ). First, suppose that $y > \frac{\beta+\gamma}{2}$. The Cauchy problem

$$\begin{cases} -\varepsilon v_\varepsilon'' + \dfrac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon |u_\varepsilon'|^2 &= 0 \qquad \text{on } (\beta, \gamma), \\ v_\varepsilon(y) &= v_\varepsilon(y), \\ v_\varepsilon'(y) &= 0, \end{cases}$$

admits a unique solution on the maximal interval (β, γ) (by Cauchy-Lipschitz Theorem). But the function $v_\varepsilon(2y-\cdot)$, defined on $(2y-\gamma, \gamma)$, is also a solution of the Cauchy problem on $(2y-\gamma, \gamma) \subset (\beta, \gamma)$, so that by uniqueness, v_ε is symmetric around y on $(2y-\gamma, \gamma)$ (see Figure 2.2a). By continuity, v_ε is symmetric around y on $[2y-\gamma, \gamma]$ and

$$v_{0,\varepsilon}(2y-\gamma) \geqslant v_\varepsilon(2y-\gamma) = v_\varepsilon(\gamma) = v_{0,\varepsilon}(\gamma).$$

Since $2y - \gamma \in (\beta, \gamma)$, it follows from the variations of $v_{0,\varepsilon}$ (see [BMR23, Proposition 2.1]) that $v_{0,\varepsilon}(2y-\gamma) < \max(v_{0,\varepsilon}(\beta), v_{0,\varepsilon}(\gamma)) = v_{0,\varepsilon}(\gamma)$. This contradiction shows that

$$y \leqslant \frac{\beta+\gamma}{2}. \tag{2.3.8}$$

Now, suppose that there exists another critical point z of v_ε on (β, γ). As y is supposed to be the largest critical point, $z < y$. Similarly to the previous argument of symmetry, $v_\varepsilon(2y - \cdot)$ is a solution of the Cauchy problem on $(\beta, 2y-\beta) \subset (\beta, \gamma)$ so v_ε is symmetric around y on $[\beta, 2y-\beta]$. It follows that $2y - z$ is also critical point of v_ε on (β, γ), which implies $2y - z = y$, hence $y = z$.

Finally, y is the unique critical point of v_ε on (β, γ).

Step 2 : (β, γ) contains a local minimum if and only if $L/2 \in (\beta, \gamma)$.

We distinguish two cases.

Case 1 : Suppose that $\dfrac{L}{2} \notin (\beta, \gamma)$ and that y is a local minimum of v_ε. As v_ε' does not vanish on $(\beta, \gamma) \setminus \{y\}$, v_ε is decreasing on (β, y) and increasing on (y, γ).

On one hand, the symmetry of v_ε around y yields :

$$v_\varepsilon'(\beta) = \lim_{x \to \beta^+} v_\varepsilon'(x) = - \lim_{x \to (2y-\beta)^-} v_\varepsilon'(x) = -v_\varepsilon'(2y-\beta) < 0.$$

But on the other hand :

$$v_\varepsilon'(\beta) = \lim_{h \to 0} \frac{v_\varepsilon(\beta) - v_\varepsilon(\beta-h)}{h} \geqslant \lim_{h \to 0} \frac{v_{0,\varepsilon}(\beta) - v_{0,\varepsilon}(\beta-h)}{h} = v_{0,\varepsilon}'(\beta) > 0.$$

Thus, it follows that (β, γ) does not contain any local minimum.

Case 2 : Suppose that $L/2 \in (\beta, \gamma)$. Since

$$v_\varepsilon(L/2) \leqslant v_{0,\varepsilon}(L/2) < \min(v_{0,\varepsilon}(\beta), v_{0,\varepsilon}(\gamma)) = \min(v_\varepsilon(\beta), v_\varepsilon(\gamma)),$$

v_ε admits at least one local minimum on (β, γ). By uniqueness, this local minimum is y.

This concludes the proof.

□

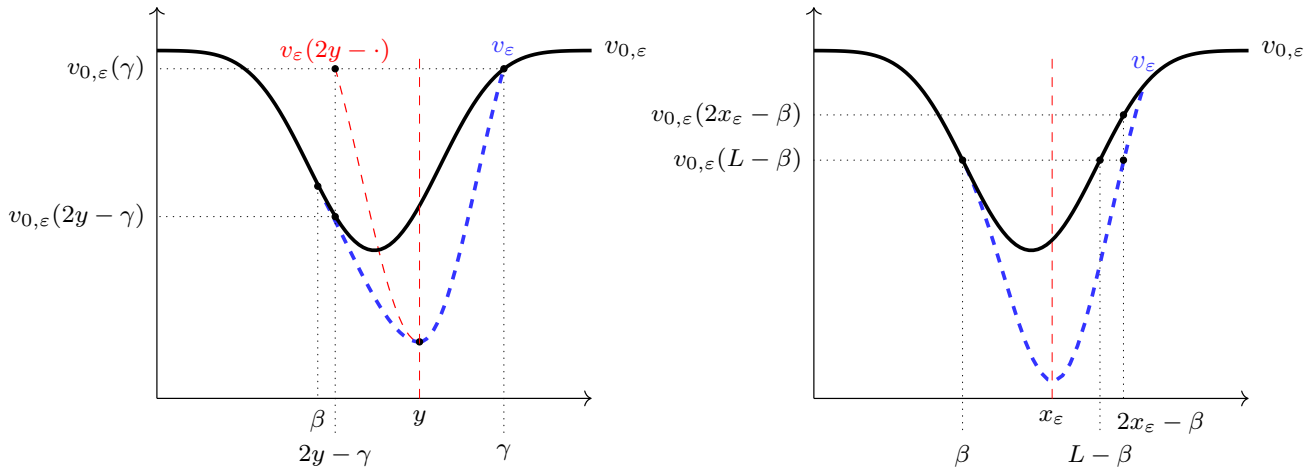
Corollary 2.3.6. *The function v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$.*

Proof. Let us prove that v_ε is non-increasing on $[0, x_\varepsilon]$. Let $a, b \in [0, x_\varepsilon]$ such that $a < b$. If $v_\varepsilon(a) < v_\varepsilon(b)$, then $v_\varepsilon(a) < \min(v_\varepsilon(0), v_\varepsilon(b))$, so v_ε admits a local minimum on $(0, b)$, which contradicts Proposition 2.3.5. So $v_\varepsilon(a) \geqslant v_\varepsilon(b)$ and v_ε is non increasing on $[0, x_\varepsilon]$. Similarly, v_ε is non decreasing on $[x_\varepsilon, L]$. □

Notice that the sequence (x_ε) of global minimum of v_ε is bounded so, up to an extraction, we can suppose that

$$x_\varepsilon \longrightarrow x_{min} \in [0, L]. \tag{2.3.9}$$

The previous proof implies the following on x_{min} :



(a) $y \leqslant \dfrac{\gamma+\beta}{2}$ (b) $x_\varepsilon \leqslant \dfrac{3}{4}L$

Figure 2.2: Critical points of v_ε

Corollary 2.3.7. *Let x_ε denote the global minimum of v_ε and let x_{min} be the limit of a converging subsequence of (x_ε). Then $x_{min} \in [L/4, 3L/4]$.*

Proof. Suppose that $x_\varepsilon \in (\beta, \gamma)$ the connected component of O_ε containing $L/2$. Suppose also that $v_\varepsilon(\beta) \leqslant v_\varepsilon(\gamma)$. According to (2.3.8), $x_\varepsilon \leqslant \dfrac{\gamma+\beta}{2} \leqslant \dfrac{3}{4}L < L$. On the other hand, the symmetries of $v_{0,\varepsilon}$ around $L/2$ imply

$$v_{0,\varepsilon}(2x_\varepsilon - \beta) \geqslant v_\varepsilon(2x_\varepsilon - \beta) = v_\varepsilon(\beta) = v_{0,\varepsilon}(\beta) = v_{0,\varepsilon}(L-\beta).$$

From the variations of $v_{0,\varepsilon}$ (see [BMR23, Proposition 2.1]) and the inequality $v_{0,\varepsilon}(2x_\varepsilon - \beta) \geqslant v_{0,\varepsilon}(\beta)$, we deduce that $2x_\varepsilon \geqslant L/2$. Then, from the the variations of $v_{0,\varepsilon}$ on $[L/2, L]$ and the inequality $v_{0,\varepsilon}(2x_\varepsilon - \beta) \geqslant v_{0,\varepsilon}(L-\beta)$, we deduce that $2x_\varepsilon - \beta \geqslant L - \beta$, so $x_\varepsilon \geqslant L/2$ (see Figure 2.2b). The case where $v_\varepsilon(\beta) \geqslant v_\varepsilon(\gamma)$ can be treated similarly. □

In particular, x_{min} is not localized at the boundaries of $[0, L]$.

2.4 Asymptotic analysis

In this section, we prove Theorem 2.2.7. The proofs of the several items of Theorem 2.2.7 are disseminated within the subsections : (i) is proved in subsection 4.3, (ii) in subsection 4.5 and subsection 4.7, (iii) in subsection 4.4 and (iv) and (v) in subsection 4.6.

2.4.1 Preliminary estimates

The following estimates are classical and can be found in [FLS09], [BMR23] and [Le10].

First, notice that the energy bound (2.2.12) provides the following inequality on c_ε :

$$\mathcal{C} \geqslant \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \geqslant \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 = \int_0^L |c_\varepsilon u_\varepsilon'| = |c_\varepsilon||u_\varepsilon(L) - u_\varepsilon(0)| = |c_\varepsilon a_1|.$$

where we used that u_ε' is of constant sign (see (2.2.9)). It follows that $|c_\varepsilon| \leqslant C/|a_1|$. The sequence (c_ε) is bounded, so, up to an extraction, we can suppose that it converges to a constant c_1. The following proposition establishes the connection between c_1 and the limit function of (u_ε).

Proposition 2.4.1. *One has $v_\varepsilon \to 1$ in $L^2(0, L)$ and, up to an extraction, there exists $u_1 \in BV(0, L)$ such that $u_\varepsilon \to u_1$ in $L^1(0, L)$. Moreover, $u_\varepsilon' \to c_1$ a.e in $[0, L]$ with $|Du_1|(0, L) \leqslant |a_1|$. We also have that $c_1 \in [0, a_1/L]$ if $a_1 \geqslant 0$ and $c_1 \in [a_1/L, 0]$ if $a_1 \leqslant 0$.*

Proof. The convergence of v_ε comes from the energy bound (2.2.12) :

$$\int_0^L (1-v_\varepsilon)^2 \leqslant 4\varepsilon \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \mathcal{C}\varepsilon \to 0.$$

As u_ε is monotone, $u_\varepsilon(0) = 0$ and $u_\varepsilon(L) = a_1$, it is of constant sign and $\int_0^L |u_\varepsilon| = |\int_0^L u_\varepsilon| \leqslant |a_1|L$. So (u_ε) is bounded in $L^1(0,L)$. Secondly, as u'_ε is of constant sign,

$$|Du_\varepsilon|(0,L) = \int_0^L |u'_\varepsilon| = |\int_0^L u'_\varepsilon| = |a_1|.$$

So the total variation of u_ε is also bounded. Then, the sequence (u_ε) is bounded in $BV(0,L)$ and we can extract a converging subsequence in $L^1(0,L)$ to a function $u_1 \in BV(0,L)$. The lower semi-continuity of the total variation implies

$$|Du_1|(0,L) \leqslant \liminf_{\varepsilon\to 0} |Du_\varepsilon|(0,L) \leqslant |a_1|.$$

Using equation (2.2.9), as one can extract a converging subsequence of (v_ε) that converges to 1 almost everywhere, one gets $u'_\varepsilon \to c_1$ almost everywhere. If $u_\varepsilon \geqslant 0$, then by Fatou's lemma :

$$c_1 L = \int_0^L \lim_{\varepsilon\to 0} u'_\varepsilon \leqslant \liminf_{\varepsilon\to 0} \int_0^L u'_\varepsilon = a_1.$$

If $u_\varepsilon \leqslant 0$, then similarly

$$-c_1 L = \int_0^L \lim_{\varepsilon\to 0} (-u'_\varepsilon) \leqslant \liminf_{\varepsilon\to 0} \int_0^L (-u'_\varepsilon) = -a_1.$$

We obtain that if $a_1 \geqslant 0$, then $c_1 \in [0, a_1/L]$ and if $a_1 \leqslant 0$, then $c_1 \in [a_1/L, 0]$. □

2.4.2 Estimates on v_ε

In this section we establish the two main estimates on v_ε, namely :

Proposition 2.4.2. *Let $x \in [0,L]$. Suppose that $v_\varepsilon \geqslant \alpha > 0$ over $I_r(x)$ for all ε small enough. Then there exists $\varepsilon_0 > 0$ and $b, M > 0$ depending on α such that on $I_{r/2}(x)$*

$$|\mu_\varepsilon| \leqslant M, \qquad |u'_\varepsilon| \leqslant M, \qquad v_\varepsilon \geqslant 1 - b\varepsilon, \quad \forall \varepsilon < \varepsilon_0. \tag{2.4.1}$$

Corollary 2.4.3. *For any compact set $K \subset [0,L]\backslash\{x_{min}\}$, there exists $\varepsilon_0 > 0$ and $b, M > 0$ depending on K such that* (2.4.1) *holds on K.*

Proposition 2.4.4. *If $c_0 > 0$, then,*

$$\|v'_\varepsilon\|_\infty \leqslant \frac{C}{\varepsilon}.$$

If $|c_1| > 0$, then,

$$\|u'_\varepsilon\|_\infty \leqslant \frac{C}{\varepsilon}, \qquad \inf_{(0,L)} v_\varepsilon \geqslant m\sqrt{\varepsilon}.$$

Note that the first estimate relies on a condition concerning c_0 and not c_1. The proof of Proposition 2.4.2 and its corollary is inspired by the proof of the Proposition 3.12 of [Le10] and the proof of Proposition 3.2 of [HT00].

Proof of Proposition 2.4.2. Suppose that the assertion is false for every b. We will reach a contradiction for b large enough. To this end, fix $b > 0$ and suppose on the contrary that, for infinitely many ε, $\inf_{I_{r/2}(x)} v_\varepsilon < 1 - b\varepsilon$.

First we can already bound u'_ε and μ_ε uniformly in ε on $I_r(x)$, namely :

$$|u'_\varepsilon| \leqslant \frac{|c_\varepsilon|}{\alpha^2+\eta_\varepsilon} \leqslant M \tag{2.4.2}$$

Concerning μ_ε, since $v_{0,\varepsilon} = v_\varepsilon$ on F_ε, one has by Corollary 2.3.3 on $I_r(x)$:

$$-\mu_\varepsilon = |\mu_\varepsilon| = \frac{v_{0,\varepsilon}}{(v_{0,\varepsilon}^2+\eta_\varepsilon)^2}|c_\varepsilon^2 - c_{0,\varepsilon}^2|1_{F_\varepsilon} \leqslant \frac{1}{(\alpha^2+\eta_\varepsilon)^2}M1_{F_\varepsilon} \leqslant M. \tag{2.4.3}$$

Let ζ_ε be a smooth function such that $\zeta_\varepsilon(x-r) = \zeta_\varepsilon(x+r) = \alpha$, $\zeta_\varepsilon = 1 - b/2\varepsilon$ on $I_{r/2}(x)$, $\zeta_\varepsilon \leqslant 1$ on $I_r(x)$ and $\|\zeta''_\varepsilon\| = O(1)$.

Set $g_\varepsilon = v_\varepsilon - \zeta_\varepsilon$. Then $g_\varepsilon(x+r), g_\varepsilon(x-r) \geqslant 0$, while $\inf\limits_{I_{r/2}(x)} g_\varepsilon \leqslant (1-b\varepsilon)-(1-(b\varepsilon)/2) = -(b\varepsilon)/2 < 0$ since $\inf\limits_{I_{r/2}(x)} v_\varepsilon \leqslant 1 - b\varepsilon$. In particular, g_ε reaches its minimum at a point x_1 of $(x-r, x+r)$ and this minimum satisfies $g_\varepsilon(x_1) \leqslant -(b\varepsilon)/2$. Then :

$$\begin{aligned} 0 \;&\leqslant\; \varepsilon g''_\varepsilon(x_1) \;=\; \varepsilon v''_\varepsilon(x_1) - \varepsilon\zeta''_\varepsilon(x_1) \;=\; \frac{v_\varepsilon(x_1)-1}{4\varepsilon} + v_\varepsilon(x_1)|u'_\varepsilon(x_1)|^2 - \mu_\varepsilon(x_1) - \varepsilon\zeta''_\varepsilon(x_1) \\ &=\; g_\varepsilon(x_1)\left(\frac{1}{4\varepsilon} + |u'_\varepsilon(x_1)|^2\right) + \zeta_\varepsilon(x_1)\left(\frac{1}{4\varepsilon} + |u'_\varepsilon(x_1)|^2\right) - \frac{1}{4\varepsilon} - \varepsilon\zeta''_\varepsilon(x_1) - \mu_\varepsilon(x_1) \end{aligned}$$

We deduce, using (2.4.2) and (2.4.3),

$$\begin{aligned} -b/8 \geqslant -\frac{b\varepsilon}{2}\left(\frac{1}{4\varepsilon} + |u'_\varepsilon(x_1)|^2\right) &\geqslant \frac{1-\zeta_\varepsilon(x_1)}{4\varepsilon} + \varepsilon\zeta''_\varepsilon(x_1) - \zeta_\varepsilon(x_1)|u'_\varepsilon(x_1)|^2 + \mu_\varepsilon(x_1) \\ &\geqslant -\varepsilon\|\zeta''_\varepsilon\|_\infty - M^2 - M \geqslant -2M^2 \end{aligned}$$

which is absurd for b large enough. □

Proof of Corollary 2.4.3. Let K be a compact set of $[0, L] \setminus \{x_{min}\}$ and take $\delta > 0$ such that $(x_{min} - \delta, x_{min} + \delta) \subset [0, L] \setminus K$. Then, for $\varepsilon > 0$ small enough, $x_\varepsilon \in (x_{min} - \delta/2, x_{min} + \delta/2)$ so using Corollary 2.3.6, v_ε is non-increasing on $(x_{min} - \delta, x_{min} - \delta/2)$. The L^2 convergence of v_ε then implies

$$\frac{\delta}{2}(1 - v_\varepsilon(x_{min} - \delta))^2 \leqslant \int_{x_{min}-\delta}^{x_{min}-\delta/2} (1 - v_\varepsilon)^2 \to 0. \tag{2.4.4}$$

It follows that for ε small enough, $v_\varepsilon(x_{min} - \delta) \geqslant 1/2$ and then $v_\varepsilon \geqslant 1/2$ on $[0, x_{min} - \delta]$. Similarly, $v_\varepsilon \geqslant 1/2$ on $[x_{min} + \delta, L]$. Hence $v_\varepsilon \geqslant 1/2$ on K.

Let $x \in K$ and r_x such that $I_{r_x}(x) \subset (0, x_{min} - \delta) \cup (x_{min} + \delta, L)$. Then $v_\varepsilon \geqslant 1/2$ on $I_{r_x}(x)$. According to Proposition 2.4.2, there exists $\varepsilon_x > 0$ and constants b_x, M_x such that

$$v_\varepsilon \geqslant 1 - b_x\varepsilon \quad , \quad |u'_\varepsilon| \leqslant M_x \quad , \quad |\mu_\varepsilon| \leqslant M_x \quad \forall \varepsilon < \varepsilon_x.$$

As K is compact, there exists a finite covering of K of the form $K \subset \bigcup_{i=1}^n I_{r_{x_i}/2}(x_i)$. Taking $\varepsilon_0 = \min\limits_{1\leqslant i\leqslant n} \varepsilon_{x_i}, b = \max\limits_{1\leqslant i\leqslant n} b_{x_i}$ and $M = \min\limits_{1\leqslant i\leqslant n} M_{x_i}$ yields Corollary 2.4.3. □

Before proving Proposition 2.4.4, we introduce the so-called discrepancy of $\overline{AT}_\varepsilon$.

Definition 2.4.5. *We denote by d_ε, also named as the discrepancy of $\overline{AT}_\varepsilon$ the following quantity :*

$$d_\varepsilon := \frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2. \tag{2.4.5}$$

Remark that

$$\int_0^L |d_\varepsilon| = \int_0^L \left|\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon(v'_\varepsilon)^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2\right| \leqslant AT_\varepsilon(v_\varepsilon, u_\varepsilon) < C.$$

As in [FLS09], [BMR23] and [Le10], we use the variations of the discrepancy to infer estimates on u_ε and v_ε. The following proposition gives the expression of the derivative of the discrepancy :

Proposition 2.4.6. *There holds*

$$d'_\varepsilon = 2v'_{0,\varepsilon}\mu_\varepsilon. \tag{2.4.6}$$

Consequently, d_ε is non-decreasing on $[0, L/2]$ and non-increasing on $[L/2, L]$.

Proof. Note that $d_\varepsilon \in C^{0,1}([0, L])$, so that we can define its derivative almost everywhere but here in the sense of $\mathcal{D}'$, which reads as :

$$\begin{aligned}
\left\{\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon(v'_\varepsilon)^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2\right\}' &= 2v'_\varepsilon\left[\frac{v_\varepsilon - 1}{4\varepsilon} - \varepsilon v''_\varepsilon\right] - 2(v_\varepsilon^2 + \eta_\varepsilon)u''_\varepsilon u'_\varepsilon - 2v_\varepsilon v'_\varepsilon|u'_\varepsilon|^2 \\
&= 2v'_\varepsilon[\mu_\varepsilon - v_\varepsilon|u'_\varepsilon|^2] - 2(v_\varepsilon^2 + \eta_\varepsilon)u''_\varepsilon u'_\varepsilon - 2v_\varepsilon v'_\varepsilon|u'_\varepsilon|^2 \\
&= 2v'_\varepsilon\mu_\varepsilon - 2u'_\varepsilon\underbrace{[(\eta_\varepsilon + v_\varepsilon^2)u'_\varepsilon]'}_{=0} = 2v'_\varepsilon\mu_\varepsilon = 2v'_{0,\varepsilon}\mu_\varepsilon
\end{aligned}$$

where the last equality uses the fact that $v_\varepsilon = v_{0,\varepsilon}$ on the support of μ_ε. Since $\mu_\varepsilon \leqslant 0$ and $v'_{0,\varepsilon} \leqslant 0$ on $[0, L/2]$ while $v'_{0,\varepsilon} \geqslant 0$ on $[L/2, L]$, we deduce that d_ε is non-decreasing on $[0, L/2]$ and non-increasing on $[L/2, L]$. □

We use this expression to prove the following estimate on the discrepancy.

Lemma 2.4.7. *Let d_ε be the discrepancy associated to $(u_\varepsilon, v_\varepsilon)$, defined by* (2.4.5)*. If $c_0 > 0$, then :*

$$\|d_\varepsilon\|_\infty \leqslant \frac{C}{\varepsilon}$$

The hypothesis on c_0 comes from the necessity, in the proof, to bound from below $v_{0,\varepsilon}(L/2)$. As stated in Lemma 2.2 of [BMR23], if $c_0 > 0$, then $v_{0,\varepsilon}(L/2) \geqslant C\sqrt{\varepsilon}$.

Proof. The proof of Lemma 2.4.7 consists in estimating d_ε at its local extremum $L/2$ and controlling the integral of the derivative d'_ε to prove that the difference $|d_\varepsilon(x) - d_\varepsilon(L/2)|$ is bounded by C/ε.

We start by estimating $d_\varepsilon(L/2)$. We distinguish two cases. Suppose first that $d_\varepsilon(L/2) \geqslant 0$. Then we have

$$0 \leqslant d_\varepsilon(L/2) \leqslant \frac{(1 - v_\varepsilon(L/2))^2}{4\varepsilon} \leqslant \frac{1}{4\varepsilon}.$$

Suppose now that $d_\varepsilon(L/2) \leqslant 0$. Recall that $\|d_\varepsilon\|_1$ is bounded by a constant C independent from ε. Then, because $L/2$ is a global maximum of d_ε, d_ε is non-positive and

$$C \geqslant \int_0^L |d_\varepsilon| = -\int_0^L d_\varepsilon \geqslant L(-d_\varepsilon(L/2))$$

so that $|d_\varepsilon(L/2)| \leqslant C$. Both cases imply that $|d_\varepsilon(L/2)| \leqslant C/\varepsilon$.

Now we prove that the total variation of d_ε is bounded by C/ε. By Corollary 2.3.3 :

$$\begin{aligned}
\int_0^L |d'_\varepsilon| &= 2\int_0^L \left|v'_{0,\varepsilon}\mu_\varepsilon\right| = 2\int_0^L \left|v'_{0,\varepsilon}\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2 + \eta_\varepsilon]^2}((c_\varepsilon)^2 - (c_{0,\varepsilon})^2)1_{F_\varepsilon}\right| \\
&= 2((c_{0,\varepsilon})^2 - (c_\varepsilon)^2)\int_0^L |v'_{0,\varepsilon}|\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2 + \eta_\varepsilon]^2} \leqslant 2C\int_0^L |v'_{0,\varepsilon}|\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2 + \eta_\varepsilon]^2}.
\end{aligned}$$

Using that $v'_{0,\varepsilon} \leqslant 0$ on $[0, L/2]$ and $v'_{0,\varepsilon} \geqslant 0$ on $[L/2, L]$, along with the estimate $v_{0,\varepsilon}(L/2) \geqslant C\sqrt{\varepsilon}$, we have :

$$\begin{aligned}
\int_0^{L/2} |v'_{0,\varepsilon}|\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2 + \eta_\varepsilon]^2} &= -\int_0^{L/2} v'_{0,\varepsilon}\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2 + \eta_\varepsilon]^2} \\
&= \frac{1}{2}\left[\frac{1}{(v_{0,\varepsilon})^2 + \eta_\varepsilon}\right]_0^{L/2} = \frac{1}{2(v_{0,\varepsilon}^2(L/2) + \eta_\varepsilon)} - \frac{1}{2(1+\eta_\varepsilon)} \leqslant \frac{C}{\varepsilon}.
\end{aligned}$$

Similarly, we obtain that $\int_{L/2}^{L} |v'_{0,\varepsilon}| \frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2} \leqslant \frac{C}{\varepsilon}$, so that $\int_0^L |d'_\varepsilon| \leqslant C/\varepsilon$.

We deduce that for a.e x in $[0,L]$, one has

$$|d_\varepsilon(x)| \leqslant |d_\varepsilon(L/2)| + \left|\int_{L/2}^{x} d'_\varepsilon\right| \leqslant |d_\varepsilon(L/2)| + \int_0^L |d'_\varepsilon| \leqslant C/\varepsilon,$$

which gives the desired result. □

We are now in position to complete the proof of Proposition 2.4.4 :

Proof of Proposition 2.4.4. First we focus on the estimate on v_ε. For a.e x in $[0,L]$, one has

$$0 \leqslant \varepsilon|v'_\varepsilon|^2 = \frac{(1-v_\varepsilon)^2}{4\varepsilon} - d_\varepsilon - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \leqslant \frac{1}{4\varepsilon} - d_\varepsilon \leqslant \frac{C}{\varepsilon},$$

hence the result. Now we focus on the estimate on u'_ε. One has

$$|c_\varepsilon u'_\varepsilon| = (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 = \frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 - d_\varepsilon \leqslant \frac{C}{\varepsilon}.$$

Dividing by $|c_\varepsilon|$, we obtain the desired result. Concerning v_ε, we have

$$\eta_\varepsilon + v_\varepsilon^2 = \frac{c_\varepsilon}{u'_\varepsilon} \geqslant C\varepsilon.$$

Since $\eta_\varepsilon \ll \varepsilon$, we conclude that $v_\varepsilon \geqslant m\sqrt{\varepsilon}$. □

In [FLS09] and [BMR23], the first estimate of Proposition 2.4.4 is true without any assumption on the slope c_0. This comes from the fact that the discrepancy d_ε is constant in their case, so an estimate as in Lemma 2.4.7 follows. However, the hypothesis $c_0 > 0$ is not that restrictive in our case, since $c_0 = 0$ implies that $c_1 = 0$ as well, as long as $|F_\varepsilon| \neq 0$ (see Corollary 2.4.9).

2.4.3 Possible values for the limit slope

The goal of this subsection is to establish the following property :

Proposition 2.4.8. $c_1 \in \{0, \frac{a_1}{L}\}$.

Corollary 2.4.9. *If $|a_1| > a_0$ and $|F_\varepsilon| \neq 0$ for infinitely many values of ε, then $c_1 = 0$.*

Proof of Corollary 2.4.9. If $|F_\varepsilon| \neq 0$, then by Corollary 2.3.3 $|c_\varepsilon| \leqslant |c_{0,\varepsilon}|$. Hence, by [BMR23, Lemma 2.1] $|c_1| \leqslant c_0 \leqslant a_0/L < |a_1|/L$, so Proposition 2.4.8 yields $c_1 = 0$. □

The proof of Proposition 2.4.8 is similar to that of [FLS09, Lemma 4.4].

Proof of Proposition 2.4.8. We already know by Proposition 2.4.1 that c_1 is within the interval $[0, a_1/L]$ (or $[a_1/L, 0]$).

Suppose now that $|c_1| > 0$. The proof is divided into two steps. In the first step, we integrate (2.2.11) over sets of the form

$$K_{\varepsilon,A} := \{v_\varepsilon \leqslant A\sqrt{\varepsilon}\}$$

to obtain an estimate on $|K_{\varepsilon,A}|$. In the second step, we split the integral $\int_0^L u'_\varepsilon$ into three terms which we study using the previous estimate and Proposition 2.4.4.

Step 1 : We estimate $|K_{\varepsilon,A}|$. Fix $A > 0$. We integrate the inequality

$$-\varepsilon v''_\varepsilon + \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{4\varepsilon} \leqslant 0$$

over the set $\{v_\varepsilon \leqslant A\sqrt{\varepsilon}\}$. We obtain

$$\int_{K_{\varepsilon,A}} \varepsilon v_\varepsilon'' \geqslant \int_{K_{\varepsilon,A}} \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \int_{K_{\varepsilon,A}} \frac{v_\varepsilon - 1}{4\varepsilon}. \tag{2.4.7}$$

According to Proposition 2.3.5, the set $K_{A,\varepsilon}$ is an interval that we denote by $[a_\varepsilon, b_\varepsilon]$. Then we have, using Proposition 2.4.4,

$$|\int_{a_\varepsilon}^{b_\varepsilon} \varepsilon v_\varepsilon''| = \varepsilon |v_\varepsilon'(b_\varepsilon) - v_\varepsilon'(a_\varepsilon)| \leqslant C.$$

Next, recall by Proposition 2.4.4 that $v_\varepsilon^2 \geqslant m\varepsilon \ll \eta_\varepsilon$. We can then bound from below the right-hand side of (2.4.7) by

$$\int_{K_{\varepsilon,A}} \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \int_{K_{\varepsilon,A}} \frac{v_\varepsilon - 1}{4\varepsilon} \geqslant \int_{K_{\varepsilon,A}} \frac{v_\varepsilon c_1^2/2}{2v_\varepsilon^4} - \frac{|K_{\varepsilon,A}|}{4\varepsilon} \geqslant C\frac{|K_{\varepsilon,A}|}{A^3\varepsilon^{3/2}} - \frac{|K_{\varepsilon,A}|}{4\varepsilon} \geqslant C\frac{|K_{\varepsilon,A}|}{A^3\varepsilon^{3/2}}.$$

Combining the two bounds, we obtain $|K_{\varepsilon,A}| \leqslant CA^3\varepsilon^{3/2}$. This ends Step 1.

Step 2 : In this step, we investigate the weight of u_ε' in each of the three regimes $\{v_\varepsilon \leqslant A\sqrt{\varepsilon}\}, \{A\sqrt{\varepsilon} \leqslant v_\varepsilon \leqslant 1/2\}$ and $\{v_\varepsilon \geqslant 1/2\}$. Indeed, one has :

$$a_1 = \int_0^L u_\varepsilon' = \underbrace{\int_{K_{\varepsilon,A}} u_\varepsilon'}_{:=I_1} + \underbrace{\int_{\{A\sqrt{\varepsilon}\leqslant v_\varepsilon \leqslant 1/2\}} u_\varepsilon'}_{:=I_2} + \underbrace{\int_{\{v_\varepsilon \geqslant 1/2\}} u_\varepsilon'}_{:=I_3}.$$

First, thanks to the previous estimate on $K_{\varepsilon,A}$ and Proposition 2.4.4,

$$I_1 \leqslant \int_{K_{\varepsilon,A}} |u_\varepsilon'| \leqslant \frac{C}{\varepsilon}|K_{\varepsilon,A}| \leqslant CA^3\varepsilon^{1/2}.$$

As $u_\varepsilon' \to c_1$ and $v_\varepsilon \to 1$ a.e, we have $u_\varepsilon' 1_{v_\varepsilon \geqslant 1/2} \to c_1$ a.e. As

$$\left|\frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} 1_{\{v_\varepsilon \geqslant 1/2\}}\right| \leqslant 2|c_1| \times 4,$$

we can apply the dominated convergence theorem to obtain

$$I_3 \to \int_0^L c_1 = c_1 L.$$

For I_2, the energy bound (2.2.2) gives :

$$C \geqslant \int_0^L \frac{(1-v_\varepsilon)^2}{4\varepsilon} \geqslant \int_{\{v_\varepsilon \leqslant 1/2\}} \frac{(1-v_\varepsilon)^2}{4\varepsilon} \geqslant \frac{1}{16\varepsilon}|\{v_\varepsilon \leqslant 1/2\}|$$

On $\{A\sqrt{\varepsilon} \leqslant v_\varepsilon \leqslant 1/2\}$, we recover

$$|u_\varepsilon'| = \frac{|c_\varepsilon|}{\eta_\varepsilon + v_\varepsilon^2} \leqslant \frac{|c_\varepsilon|}{v_\varepsilon^2} \leqslant \frac{2|c_1|}{A^2\varepsilon}.$$

Thus

$$|I_2| \leqslant |\{A\sqrt{\varepsilon} \leqslant v_\varepsilon \leqslant 1/2\}| \times \frac{2|c_1|}{A^2\varepsilon} \leqslant \frac{C}{A^2}$$

Combining the estimates on I_1, I_2 and I_3, we obtain

$$|a_1| \leqslant \frac{C}{A^2} + |c_1|L$$

By letting A tending to ∞, we obtain $|c_1|L \geqslant |a_1|$. Combining with $|c_1| \leqslant |a_1|/L$, we get the desired result. □

We can now conclude the proof of (i) in Theorem 2.2.7.

Proof of (i) in Theorem 2.2.7. Let $\delta > 0$. We have already seen with Corollary 2.4.3 that $v_\varepsilon \to 1$ uniformly on $[0, L] \setminus I_\delta(x_{min})$ for every $\delta > 0$. This combined with (2.2.9) implies that $u'_\varepsilon \to c_1$ uniformly on $[0, L] \setminus I_\delta(x_{min})$ for every $\delta > 0$. Then we get that :

$$u_\varepsilon(x) = \int_0^x u'_\varepsilon(t)dt \to c_1 x,$$

uniformly with respect to $x \in [0, x_{min} - \delta]$. Similarly, $u_\varepsilon(x) \to a_1 - c_1(L - x)$ uniformly with respect to $x \in [x_{min} + \delta, L]$. Using that $c_1 = 0$ or $c_1 = a_1/L$, we see that the limit function u_1 of (u_ε) (in the point-wise convergence sense) is either $u_1 = u_{aff}$ or $u_1 = u_{jump}$. Moreover, as u_ε is monotone and u_ε is uniformly bounded by a_1 on $[0, L]$, by dominated convergence, we get that $u_\varepsilon \to u_1$ in $L^2(0, L)$. The additionnal conclusion corresponds to Corollary 2.4.9. □

2.4.4 Convergence of the Dirichlet energy

In this subsection, we prove the following, which proves part (iii) of Theorem 2.2.7. The proof is similar to [FLS09, section 6] and [BMR23, section 3].

Lemma 2.4.10. $(\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 dx \overset{\star}{\rightharpoonup} |u'_1|^2 dx$ *weak⋆ in* $\mathcal{M}([0, L])$.

Proof. Let φ be a test-function in $\mathcal{C}^\infty([0, L])$. According to Proposition 2.4.1 :

$$\int_0^L (v_\varepsilon^2 + \eta_\varepsilon)|u'_\varepsilon|^2\varphi = c_\varepsilon \int_0^L u'_\varepsilon\varphi = -c_\varepsilon \int_0^L u_\varepsilon\varphi' + c_\varepsilon a_1\varphi(L) \to -c_1 \int_0^L u_1\varphi' + c_1 a_1\varphi(L).$$

If $c_1 = 0$, then $u_1 = u_{jump}$ and $u'_1 = 0$ so

$$-c_1 \int_0^L u_1\varphi' + c_1 a_1\varphi(L) = 0 = \int_0^L |u'_{jump}|^2\varphi.$$

If $c_1 = a_1/L$, then $u_1 = u_{aff}$ and $u'_1 = a_1/L$, so by integration by part,

$$-c_1 \int_0^L u_1\varphi' + c_1 a_1\varphi(L) = c_1 \int_0^L u'_{aff}\varphi = \int_0^L |u'_{aff}|^2\varphi.$$

□

2.4.5 Convergence of the phase-field term

In this subsection, we prove the first part of (ii) in Theorem 2.2.7, namely

Lemma 2.4.11. $\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|v'_\varepsilon|^2\right) dx \overset{\star}{\rightharpoonup} \alpha\delta_{x_{min}}$ *weak⋆ in* $\mathcal{M}([0, L])$ *for some* $\alpha \in \mathbb{R}$.

The computation of the weight α is made in section subsection 2.4.7.

We start by noticing that the energy bound (2.2.2) implies that the measure $\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|v'_\varepsilon|^2\right) dx$ is bounded, thus we can extract a converging subsequence such that

$$\left(\frac{(1 - v_\varepsilon)^2}{4\varepsilon} + \varepsilon|v'_\varepsilon|^2\right) dx \overset{\star}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}([0, L])$$

for some non negative $\nu \in \mathcal{M}([0, L])$. In [BMR23] and in [FLS09], the proof that ν is atomic relies on the following estimate (see [BMR23, Lemma 3.1] and [FLS09, Lemma 6.1]): if K is a compact subset of $[0, L] \setminus \{x_{min}\}$, then

$$\int_K \left(\frac{(1 - v_{0,\varepsilon})^2}{4\varepsilon} + \varepsilon|v'_{0,\varepsilon}|^2\right) dx \leqslant C_K \varepsilon^{1/4}.$$

The proof of this estimate relies on an integration by part of the PDE satisfied by $v_{0,\varepsilon}$ on K, and strongly uses the estimate $\|v'_{0,\varepsilon}\|_\infty \leqslant C/\varepsilon$. In our case, such an estimate $\|v'_\varepsilon\|_\infty \leqslant C/\varepsilon$ is proved only in the case where $c_0 > 0$.

We are going to show the following result, which provides a sharper estimate (an order 1 on ε instead of an order $1/4$) :

Proposition 2.4.12. *Let K be a compact subset of $[0, L] \setminus \{x_{min}\}$. There exists $\varepsilon_0 > 0$ and a constant C_K such that :*

$$\int_K \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) \leqslant C_K \varepsilon, \quad \forall \varepsilon < \varepsilon_0.$$

The proof is divided in three steps. In the first one, we prove the result for a compact set K of $(0, L) \setminus \{x_{min}\}$:

Lemma 2.4.13. *Let K be a compact subset of $(0, L) \setminus \{x_{min}\}$. There exists $\varepsilon_0 > 0$ with the following property : for any positive test-function φ supported in K, there exists a constant $C_{K,\varphi}$ such that :*

$$\int_K \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) \varphi \leqslant C_{K,\varphi} \varepsilon, \quad \forall \varepsilon < \varepsilon_0. \tag{2.4.8}$$

Moreover, for every compact subset K of $(0, L) \setminus \{x_{min}\}$, there exists $\varepsilon_0 > 0$ and a constant C_K such that

$$\int_K \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) \leqslant C_K \varepsilon, \quad \forall \varepsilon < \varepsilon_0. \tag{2.4.9}$$

In the second one, we use Lemma 2.4.13 to obtain a sharper estimate on $\|v'_\varepsilon\|_\infty$ on $(0, L) \setminus \{x_{min}\}$.

Lemma 2.4.14. *Let K be a connected compact subset of $(0, L) \setminus \{x_{min}\}$. Then there exists $\varepsilon_0 > 0$ and a constant C_K such that,*

$$\sup_K \varepsilon |v'_\varepsilon|^2 \leqslant C_K \varepsilon^{1/3}, \quad \forall \varepsilon < \varepsilon_0.$$

Finally, we use Lemma 2.4.14 to prove Proposition 2.4.12, in a similar way than [BMR23, Lemma 3.1] and [FLS09, Lemma 6.1].

Proof of Lemma 2.4.13. Let us first prove (2.4.9) from the first statement of Lemma 2.4.13. Take K a compact subset of $(0, L) \setminus \{x_{min}\}$. Consider a compact subset K_1 of $(0, L) \setminus \{x_{min}\}$ satisfying $K \subset \overset{\circ}{K_1}$ and take a continuous function φ supported on K_1, satisfying $0 \leqslant \varphi \leqslant 1$ and such that $\varphi = 1$ on K. Then there exists $C_{K_1,\varphi}$ and $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$,

$$\int_K \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) \leqslant \int_{K_1} \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) \varphi \leqslant C_{K_1,\varphi} \varepsilon,$$

which proves (2.4.9).

We now turn to the proof of (2.4.8). The argument is inspired from [HT00, Proposition 4.1] to prove a monotonicity formula for the Allen-Cahn functional.

Take φ a smooth and positive test function supported on K. First, using Corollary 2.4.3, there exists $\varepsilon_0 > 0$ and $b, M > 0$ such that

$$u'_\varepsilon \leqslant M, \quad |\mu_\varepsilon| \leqslant M, \quad v_\varepsilon \geqslant 1 - b\varepsilon \quad \text{on } K, \ \forall \varepsilon < \varepsilon_0.$$

Then

$$\int_K \frac{(1-v_\varepsilon)^2}{4\varepsilon} \varphi \leqslant \int_K \frac{b^2 \varepsilon^2}{4\varepsilon} \varphi \leqslant C_{K,\varphi} \varepsilon. \tag{2.4.10}$$

It remains to estimate $\int_K \varepsilon |v'_\varepsilon|^2 \varphi$. To do so, we compare $\varepsilon |v'_\varepsilon|^2$ with its second derivative (in the distributional sense). One has

$$[\varepsilon |v'_\varepsilon|^2]' = 2\varepsilon v''_\varepsilon v'_\varepsilon = 2v'_\varepsilon \left(\frac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon |u'_\varepsilon|^2 - \mu_\varepsilon \right).$$

Note that $v_\varepsilon'\mu_\varepsilon \in L^1(0,L)$, so it admits a derivative in the distributional sense and then

$$[\varepsilon|v_\varepsilon'|^2]'' = \underbrace{2v_\varepsilon''\left(\frac{v_\varepsilon-1}{4\varepsilon}+v_\varepsilon|u_\varepsilon'|^2\right)}_{=A_\varepsilon}\underbrace{-2(v_\varepsilon'\mu_\varepsilon)'}_{=B_\varepsilon}+2\frac{v_\varepsilon'^2}{\varepsilon}+\underbrace{2|v_\varepsilon'|^2|u_\varepsilon'|^2}_{=C_\varepsilon}+\underbrace{4v_\varepsilon'v_\varepsilon u_\varepsilon''u_\varepsilon'}_{=D_\varepsilon}.$$

It follows that

$$2\int_K \varepsilon|v_\varepsilon'|^2\varphi = \varepsilon^2\int_K(\varepsilon(v_\varepsilon')^2)''\varphi - \varepsilon^2\left(\int_K A_\varepsilon\varphi + \langle\, B_\varepsilon\,,\,\varphi\,\rangle + \int_K C_\varepsilon\varphi + \int_0^L D_\varepsilon\varphi\right). \tag{2.4.11}$$

We now study each terms of the right hand side.

$\underline{\int_0^L(\varepsilon(v_\varepsilon')^2)''\varphi}$: As $\varphi = 0$ outside of K, then φ and φ' vanish at 0 and L. Applying (2.4.10) to φ'' yields

$$\lim_{\varepsilon\to0}\int_0^L(\varepsilon|v_\varepsilon'|^2)''\varphi = \lim_{\varepsilon\to0}\int_0^L\varepsilon|v_\varepsilon'|^2\varphi'' = \lim_{\varepsilon\to0}\int_0^L\left(\varepsilon|v_\varepsilon'|^2+\frac{(1-v_\varepsilon)^2}{4\varepsilon}\right)\varphi'' \to \int_0^L\varphi''d\nu.$$

Hence,

$$\varepsilon^2\int_K(\varepsilon(v_\varepsilon')^2)''\varphi = \varepsilon^2\int_0^L(\varepsilon(v_\varepsilon')^2)''\varphi \leqslant C_\varphi\varepsilon^2. \tag{2.4.12}$$

$\underline{\int_K A_\varepsilon\varphi}$:

$$A_\varepsilon = \frac{1}{\varepsilon}\underbrace{\left(\frac{v_\varepsilon-1}{4\varepsilon}+v_\varepsilon|u_\varepsilon'|^2\right)^2}_{\geqslant0}+\frac{1}{\varepsilon}\underbrace{(-\mu_\varepsilon)}_{\leqslant M}\left(\underbrace{\frac{v_\varepsilon-1}{\varepsilon}}_{\geqslant -b}+\underbrace{v_\varepsilon|u_\varepsilon'|^2}_{\geqslant0}\right)\geqslant\frac{-bM}{\varepsilon}.$$

so that

$$-\varepsilon^2\int_K A_\varepsilon\varphi \leqslant C_{K,\varphi}\varepsilon. \tag{2.4.13}$$

$\underline{\langle\, B_\varepsilon\,,\,\varphi\,\rangle}$: We use Cauchy-Schwarz inequality and the fact that $|\mu_\varepsilon|\leqslant M$ to get that

$$\left|2\varepsilon^2\langle\,(v_\varepsilon'\mu_\varepsilon)'\,,\,\varphi\,\rangle\right| = 2\varepsilon^2\left|\int_0^L v_\varepsilon'\mu_\varepsilon\varphi'\right| \leqslant 2\varepsilon^{3/2}\underbrace{\|\mu_\varepsilon\|_{L^2(0,L)}}_{\leqslant C}\left|\left|\varepsilon^{1/2}|v_\varepsilon'|\varphi'\right|\right|_{L^2(0,L)}. \tag{2.4.14}$$

On the other hand,

$$\left|\left|\varepsilon^{1/2}|v_\varepsilon'|\varphi'\right|\right|^2_{L^2(0,L)} = \int_0^L\varepsilon|v_\varepsilon'|^2|\varphi'|^2 \leqslant \int_0^L\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon}+\varepsilon|v_\varepsilon'|^2\right)|\varphi'|^2 \to \int_0^L|\varphi'|^2d\nu,$$

from which we deduce that $\|\varepsilon^{1/2}(v_\varepsilon')\varphi'\|^2_{L^2(0,L)}\leqslant C_\varphi$. (2.4.14) then yields

$$\left|\varepsilon^2\,\langle\, B_\varepsilon\,,\,\varphi\,\rangle\right| \leqslant C_\varphi\varepsilon^{3/2}. \tag{2.4.15}$$

$\underline{\int_K C_\varepsilon\varphi}$: $C_\varepsilon\geqslant0$, so

$$-\varepsilon^2\int_K C_\varepsilon\varphi \leqslant 0. \tag{2.4.16}$$

$\underline{\int_K D_\varepsilon\varphi}$: We have $u_\varepsilon'' = \left(\frac{c_\varepsilon}{v_\varepsilon^2+\eta_\varepsilon}\right)' = -2c_\varepsilon\frac{v_\varepsilon'v_\varepsilon}{(v_\varepsilon^2+\eta_\varepsilon)^2}$, so that

$$D_\varepsilon = -8v_\varepsilon'v_\varepsilon c_\varepsilon\frac{v_\varepsilon'v_\varepsilon}{(v_\varepsilon^2+\eta_\varepsilon)^2}\cdot\frac{c_\varepsilon}{v_\varepsilon^2+\eta_\varepsilon} = -8c_\varepsilon^2|v_\varepsilon'|^2\underbrace{\frac{v_\varepsilon^2}{(v_\varepsilon^2+\eta_\varepsilon)^3}}_{\leqslant C}.$$

where we used that $1 - b\varepsilon \leqslant v_\varepsilon \leqslant 1$ on K. As (c_ε) is bounded, we can write :

$$|D_\varepsilon| \leqslant M|v_\varepsilon'|^2.$$

Thus

$$|\varepsilon^2 \int_K D_\varepsilon \varphi| \leqslant M\varepsilon \int_K \varepsilon |v_\varepsilon'|^2 \varphi. \tag{2.4.17}$$

Injecting (2.4.16) and (2.4.17) in (2.4.11) yields the following inequality, which is true for every ε small enough and every positive smooth test function φ :

$$(1 - M\varepsilon) \int_K \varepsilon |v_\varepsilon'|^2 \varphi \leqslant \varepsilon^2 \int_K (\varepsilon |v_\varepsilon'|^2)'' \varphi - \varepsilon^2 \left(\int_K A_\varepsilon \varphi + \langle B_\varepsilon, \varphi \rangle \right) \tag{2.4.18}$$

Now combining with (2.4.12),(2.4.13) and (2.4.15), we obtain

$$(1 - M\varepsilon) \int_K \varepsilon |v_\varepsilon'|^2 \varphi \leqslant C\varepsilon^2 + C\varepsilon + C\varepsilon^{3/2} \leqslant C_{K,\varphi}\varepsilon \tag{2.4.19}$$

Which concludes the proof. □

We now move on to the proof of Lemma 2.4.14.

Proof of Lemma 2.4.14. Let K be a connected compact subset of $(0, L) \setminus \{x_{min}\}$. Let K_1 be a connected compact subset of $(0, L) \setminus \{x_{min}\}$ such that $K \subset \overset{\circ}{K_1}$. According to Corollary 2.4.3, there exists $\varepsilon_1 > 0$ and $b_1, M_1 > 0$ such that on K_1,

$$|\mu_\varepsilon| \leqslant M_1, \quad |u_\varepsilon'| \leqslant M_1, \quad v_\varepsilon \geqslant 1 - b_1\varepsilon, \quad \forall\, \varepsilon < \varepsilon_1.$$

The strategy is the following : given and $y \in K$, we consider a mollifier ρ_ε (i.e ρ is positive smooth function, compactly supported on $[-1, 1]$ with integral 1 and $\rho_\varepsilon = \frac{1}{\varepsilon^\alpha}\rho\left(\frac{\cdot}{\varepsilon^\alpha}\right)$, whith $\alpha > 0$ to be determined). Then we take $\rho_\varepsilon(\cdot - y)$ as a test function in the inequality (2.4.18) and we prove the two estimates

$$\int_{K_1} \rho_\varepsilon(x - y)\varepsilon |v_\varepsilon'(x)|^2 dx \leqslant C_{K_1}\varepsilon^{1-\alpha} \tag{2.4.20}$$

and

$$\left| \int_{K_1} \rho_\varepsilon(x - y)\varepsilon |v_\varepsilon'(x)|^2 dx - \varepsilon |v_\varepsilon'(y)|^2 \right| \leqslant C_{K_1}\varepsilon^{\alpha/2}. \tag{2.4.21}$$

Combining (2.4.20) and (2.4.21), it follows that for $\alpha = 2/3$, one has

$$\varepsilon |v_\varepsilon'(y)|^2 \leqslant \int_{K_1} \rho_\varepsilon(x - y)\varepsilon |v_\varepsilon'(x)|^2 dx + \left| \int_{K_1} \rho_\varepsilon(x - y)\varepsilon |v_\varepsilon'(x)|^2 dx - \varepsilon |v_\varepsilon'(y)|^2 \right| \leqslant C_{K_1}\varepsilon^{1/3}.$$

Observe that the choice of K_1 implies the existence of a constant $\delta > 0$ such that $d(y, \partial K_1) > \delta$ for any $y \in K$. Hence, for every $\varepsilon > 0$ such that $\varepsilon^\alpha < \delta$, the function $\rho_\varepsilon(\cdot - y)$ is supported on K_1. By taking $\varepsilon_0 < \min(\delta^{1/\alpha}, \varepsilon_1)$, this ensures the existence of ε_0 such that the estimates in the sequel are valid for any $y \in K$ and any $\varepsilon < \varepsilon_0$.

We start by proving (2.4.20). One already has from inequalities (2.4.18), (2.4.13) and (2.4.14) applied to K_1 and ρ_ε that

$$(1 - M_1\varepsilon) \int_{K_1} \rho_\varepsilon(x - y)\varepsilon |v_\varepsilon'|^2(x) dx \leqslant \underbrace{\varepsilon^2 \int_{K_1} (\varepsilon |v_\varepsilon'|^2)'' \rho_\varepsilon}_{:= I_1} + \underbrace{\varepsilon \int_{K_1} b_1 M_1 \rho_\varepsilon}_{:= I_2} - \underbrace{\varepsilon^2 \int_{K_1} (v_\varepsilon' \mu_\varepsilon)' \rho_\varepsilon}_{:= I_3}.$$

$\underline{I_1}$: Using Lemma 2.4.13,

$$\left| \int_{K_1} \varepsilon |v_\varepsilon'|^2 \rho_\varepsilon'' \right| = \varepsilon^{-2\alpha} \left| \int_{K_1} \varepsilon |v_\varepsilon'|^2 \frac{1}{\varepsilon^\alpha} \rho'' \left(\frac{x}{\varepsilon^\alpha} \right) \right| \leqslant \varepsilon^{-3\alpha} \|\rho''\|_\infty \int_{K_1} \varepsilon |v_\varepsilon'|^2 \leqslant C_{K_1}\varepsilon^{1-3\alpha}. \tag{2.4.22}$$

Hence,

$$I_1 \leqslant C_{K_1}\varepsilon^{3-3\alpha} \tag{2.4.23}$$

$\underline{I_2:}$

$$|I_2| = \left|\varepsilon\, b_1\, M_1 \int_{K_1} \rho_\varepsilon\right| \leqslant b_1\, M_1\, |K_1|\, \varepsilon^{1-\alpha}\|\rho\|_\infty \leqslant C_{K_1}\varepsilon^{1-\alpha}. \tag{2.4.24}$$

$\underline{I_3:}$ Using the Cauchy-Schwarz inequality, (2.4.3) and Lemma 2.4.13,

$$|I_3| \leqslant \left|\varepsilon^2 \int_{K_1} v_\varepsilon' \mu_\varepsilon \rho_\varepsilon'\right| \leqslant \varepsilon^2 \underbrace{\left(\int_{K_1} \mu_\varepsilon^2\right)^{1/2}}_{\leqslant C_{K_1}} \varepsilon^{-1/2} \left(\int_{K_1} \varepsilon |v_\varepsilon'|^2 |\rho_\varepsilon'|^2\right)^{1/2}$$

$$\leqslant C_{K_1}\varepsilon^2\varepsilon^{-2\alpha-1/2}\|\rho'\|_\infty \underbrace{\left(\int_{K_1} \varepsilon|v_\varepsilon'|^2\right)^{1/2}}_{\leqslant C_{K_1}\varepsilon^{1/2}}. \tag{2.4.25}$$

And thus,

$$|I_3| \leqslant C_{K_1}\varepsilon^{2-2\alpha}. \tag{2.4.26}$$

Combining (2.4.23), (2.4.24) and (2.4.26) yields (2.4.20).

We now move on to the proof of (2.4.21). First, using that $\int_{K_1} \rho_\varepsilon(x-y) = 1$, we rewrite (2.4.21) as

$$\int_{K_1} \rho_\varepsilon(x-y)(\varepsilon|v_\varepsilon'|^2(x)dx - \varepsilon|v_\varepsilon'|^2(y))dx.$$

Then, for all x in K_1, considering the fact that $|v_\varepsilon'|^2$ is $C^{0,1}([0,L])$ and using the Cauchy-Schwarz inequality and Lemma 2.4.13,

$$|\varepsilon|v_\varepsilon'(x)|^2 - \varepsilon|v_\varepsilon'(y)|^2| = 2\varepsilon\left|\int_y^x v_\varepsilon''(t)v_\varepsilon'(t)dt\right| \leqslant 2\varepsilon\varepsilon^{-1}\left|\int_y^x \varepsilon^2|v_\varepsilon''(t)|^2dt\right|^{1/2} \varepsilon^{-1/2} \underbrace{\left|\int_y^x \varepsilon|v_\varepsilon'(t)|^2dt\right|^{1/2}}_{\leqslant C_{K_1}\varepsilon^{1/2}}.$$

By Corollary 2.4.3,

$$\|\varepsilon v_\varepsilon''\|_{L^\infty(K_1)} = \|\frac{v_\varepsilon - 1}{4\varepsilon} - \mu_\varepsilon + v_\varepsilon|u_\varepsilon'|^2\|_{L^\infty(K_1)} \leqslant b_1 + M_1 + M_1^2 \leqslant C$$

so that

$$\left|\int_y^x \varepsilon^2 v_\varepsilon''(t)^2\right|^{1/2} \leqslant C|x-y|^{1/2}.$$

We deduce

$$\int_{K_1} \rho_\varepsilon(x-y)\left|\varepsilon|v_\varepsilon'(x)|^2 - \varepsilon|v_\varepsilon'(y)|^2\right| dx \leqslant C_{K_1}\int_{K_1} |x-y|^{1/2}\frac{1}{\varepsilon^\alpha}\rho\left(\frac{x-y}{\varepsilon^\alpha}\right)dx.$$

Changing variable $s = \dfrac{x-y}{\varepsilon^\alpha}$, we rewrite the integral as

$$\int_{(K_1-y)/\varepsilon^\alpha} \varepsilon^{\alpha/2}|s|^{1/2}\rho(s)ds \leqslant C\varepsilon^{\alpha/2}.$$

Finally,

$$\int_{K_1} \rho_\varepsilon(x-y)\left|\varepsilon|v_\varepsilon'(x)|^2 - \varepsilon|v_\varepsilon'(y)|^2\right| dx \leqslant C_{K_1}\varepsilon^{\alpha/2}.$$

Combining (2.4.20) and (2.4.21) finishes the proof. □

We are now in position to prove Proposition 2.4.12.

Proof of Proposition 2.4.12. It is sufficient to prove the inequality for $K = [\,0, x_{min} - \delta] \cup [x_{min} + \delta, L]$ for any $\delta > 0$. In the sequel, we fix $\delta > 0$. We adapt the argument used by [BMR23, Lemma 3.1] and [FLS09, Lemma 6.3]. Multiplying both sides of the equation (2.2.11) by $v_\varepsilon - 1$ and integrating by parts over $[0, x_{min} - \delta]$ yields :

$$\int_0^{x_{min}-\delta} \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon |v_\varepsilon'|^2 \right) = \varepsilon v_\varepsilon'(x_{min}-\delta)(v_\varepsilon(x_{min}-\delta)-1) + \int_0^{x_{min}-\delta} \mu_\varepsilon(v_\varepsilon - 1) + \int_0^{x_{min}-\delta} c_\varepsilon^2 \frac{v_\varepsilon(1-v_\varepsilon)}{(\eta_\varepsilon + v_\varepsilon^2)^2}.$$

We now bound each term on the right-hand side. Let $K' = K \cap [L/8, 7L/8]$ which is a compact subset of $[0, L] \setminus \{x_{min}\}$. According to Lemma 2.4.14, $|v_\varepsilon'| \leqslant C_{K'}\varepsilon^{-1/3}$ on K'. Using Corollary 2.4.3, there exists a constant $b > 0$ such that $v_\varepsilon \geqslant 1 - b\varepsilon$ on K. This implies that μ_ε is uniformly bounded on K (see the inequality (2.4.3)). Consequently :

$$\varepsilon \underbrace{|v_\varepsilon'(x_{min}-\delta)|}_{\leqslant C_{K'}\varepsilon^{-1/3}} \underbrace{|v_\varepsilon(x_{min}-\delta)-1|}_{\leqslant b\varepsilon} \leqslant C_\delta \varepsilon^{5/3}.$$

We also have

$$\left| \int_0^{x_{min}-\delta} \underbrace{\mu_\varepsilon}_{\leqslant M} \underbrace{(v_\varepsilon - 1)}_{\geqslant -b\varepsilon} \right| \leqslant C_\delta \varepsilon,$$

as well as

$$\left| \int_0^{x_{min}-\delta} c_\varepsilon^2 (1-v_\varepsilon) \frac{v_\varepsilon}{(\eta_\varepsilon + v_\varepsilon^2)^2} \right| \leqslant c_\varepsilon^2 \int_0^{x_{min}-\delta} \underbrace{|1-v_\varepsilon|}_{\leqslant b\varepsilon} \underbrace{\frac{1}{v_\varepsilon^3}}_{\leqslant C} \leqslant C_\delta \varepsilon.$$

We deduce that

$$\int_0^{x_{min}-\delta} \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v_\varepsilon'|^2 \right) \leqslant C_\delta \varepsilon.$$

We proceed similarly on $[x_{min} + \delta, L]$ to conclude. □

Once we get Proposition 2.4.12, we can adapt the proof of Lemma 2.4.14 to obtain the following :

Corollary 2.4.15. *Let K be a connected compact set of $[\,0, L\,] \setminus \{x_{min}\}$. Then there exists a constant C_K such that for all $y \in K$:*

$$\sup_K \varepsilon |v_\varepsilon'|^2 \leqslant C_K \varepsilon^{1/3}.$$

Proof. Fix $y \in K$. Take ρ_ε a mollifier, with parameter ε^α as in the proof of Lemma 2.4.14 and $\alpha > 0$ to be determined. The same argument used to establish (2.4.21) yields

$$\left| \int_K \rho_\varepsilon(x-y)\varepsilon |v_\varepsilon'(x)|^2 dx - \varepsilon |v_\varepsilon'(y)|^2 \right| \leqslant C_K \varepsilon^{\alpha/2}.$$

On the other hand, using Proposition 2.4.12,

$$\begin{aligned} \int_K \rho_\varepsilon(x-y)\varepsilon |v_\varepsilon'(x)|^2 dx &\leqslant \int_K \rho_\varepsilon(x-y) \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v_\varepsilon'(x)|^2 \right) dx \\ &\leqslant \frac{\|\rho\|_\infty}{\varepsilon^\alpha} \int_K \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v_\varepsilon'(x)|^2 \right) dx \leqslant C_K \varepsilon^{1-\alpha}. \end{aligned}$$

Combining both estimates for $\alpha = 2/3$ yields the desired result. □

The Proposition 2.4.12 implies that ν is supported only on $\{x_{min}\}$ and thus $\nu = \alpha \delta_{x_{min}}$ for some α.

2.4.6 Equipartition of the energy

In this subsection, our goal is to prove the following result, which corresponds to the item (iv) of Theorem 2.2.7, and which implies (v).

Proposition 2.4.16. *We have*

$$\lim_{\varepsilon\to 0}\int_0^L \left|\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon(v_\varepsilon')^2\right| = 0. \tag{2.4.27}$$

We recover here a classical result in the case of the minimizers of AT_ε, as it corresponds to the equality case of the arithmetico-geometric inequality used in the proof of the Γ-convergence of AT_ε to MS. It is used in [FLS09] and [BMR23] to compute the value of the weight α associated to the limit Dirac-measure of the phase field term of AT_ε. An equipartition of the energy result also exists for the Ambrosio-Tortorelli functional with an additional fidelity term, see [Le10].

In [FLS09] and [Le10], the Neumann conditions on (2.2.11) imply that the difference $\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v_\varepsilon'|^2$ is positive. In [BMR23] and [FLS09], the proof of the equipartition of the energy hinges on the fact that d_ε is constant and tends to $-c_0^2$.

In our case, recalling the equality :

$$\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v_\varepsilon'|^2 = d_\varepsilon + c_\varepsilon u_\varepsilon',$$

the first ingredient is to prove that $d_\varepsilon + c_\varepsilon u_\varepsilon' \to 0$ at the boundary point 0 and L of $(0,L)$. It rests on Corollary 2.4.15. The second ingredient is to bound d_ε' to control the variation of v_ε on the full interval. To deal with the signs of each terms and get rid of the absolute values, we need to separate our integrals along $L/2$ and x_ε.

Proof of Proposition 2.4.16. Corollary 2.4.15 implies in particular that $\varepsilon|v_\varepsilon'|^2(0), \varepsilon|v_\varepsilon'|^2(L) \to 0$ as $\varepsilon \to 0$, by recalling that $x_{min} \in [L/4, 3L/4]$ (Corollary 2.3.7). Then we have

$$d_\varepsilon(0) + c_\varepsilon u_\varepsilon'(0) = -\varepsilon|v_\varepsilon'|^2(0) \to 0 \tag{2.4.28}$$

and similarly, $d_\varepsilon(L) + c_\varepsilon u_\varepsilon'(L) \to 0$.

By Proposition 2.4.6, $d_\varepsilon \in C^{0,1}([0,L])$, so we can write $d_\varepsilon(x) = d_\varepsilon(0) + \int_0^x d_\varepsilon'$ on $[0, L/2]$ and $d_\varepsilon(x) = d_\varepsilon(L) + \int_L^x d_\varepsilon'$ on $[L/2, L]$, with $d_\varepsilon' \geqslant 0$ on $[0, L/2]$ and $d_\varepsilon' \leqslant 0$ on $[L/2, L]$. However, the sign of $u_\varepsilon'(x) - u_\varepsilon'(0)$ depends on the position of x with respect to x_ε. So we split the integral in the left-hand side of (2.4.27) along x_ε and later along $L/2$.

$$\begin{aligned}
\int_0^L \left|\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v_\varepsilon'|^2\right| &= \int_0^{x_\varepsilon} \left|d_\varepsilon(0) + c_\varepsilon u_\varepsilon'(0) + \int_0^x d_\varepsilon'(t)dt + c_\varepsilon(u_\varepsilon'(x) - u_\varepsilon'(0))\right| dx \\
&+ \int_{x_\varepsilon}^L \left|d_\varepsilon(L) + c_\varepsilon u_\varepsilon'(L) + \int_L^x d_\varepsilon'(t)dt + c_\varepsilon(u_\varepsilon'(x) - u_\varepsilon'(L)\right| dx \\
&\leqslant \underbrace{x_\varepsilon|d_\varepsilon(0) + c_\varepsilon u_\varepsilon'(0)|}_{:=I} + \underbrace{\int_0^{x_\varepsilon}\left|\int_0^x d_\varepsilon'\right|}_{:=II} + \underbrace{\int_0^{x_\varepsilon}|c_\varepsilon(u_\varepsilon'(x) - u_\varepsilon'(0))|dx}_{:=III} \\
&+ \underbrace{(L-x_\varepsilon)|d_\varepsilon(L) + c_\varepsilon u_\varepsilon'(L)|}_{:=IV} + \underbrace{\int_{x_\varepsilon}^L\left|\int_x^L d_\varepsilon'\right|}_{:=V} + \underbrace{\int_{x_\varepsilon}^L|c_\varepsilon(u_\varepsilon'(x) - u_\varepsilon'(L))|dx}_{:=VI}
\end{aligned}$$

We deal separately with $I + IV$, $II + V$ and $III + VI$.

$\underline{I + IV}$: This tends to 0 according to (2.4.28).

$\underline{II+V:}$ We prove that

$$II+V \leqslant \int_0^{L/2}\left(\int_0^x d'_\varepsilon\right)dx - \int_{L/2}^{L}\left(\int_x^L d'_\varepsilon\right)dx + \left|\frac{L}{2}-x_\varepsilon\right|\left|\int_0^L d'_\varepsilon\right|$$

Suppose that $x_\varepsilon \geqslant L/2$ (the case $x_\varepsilon \leqslant L/2$ can be treated in a similar way). Recall that $d'_\varepsilon \geqslant 0$ on $[0, L/2]$ and $d'_\varepsilon \leqslant 0$ on $[L/2, L]$. Then $II+V$ can be rewritten as

$$\begin{aligned}
&\int_0^{x_\varepsilon}\left|\int_0^x d'_\varepsilon\right|dx + \int_{x_\varepsilon}^{L}\left|\int_x^L d'_\varepsilon\right|dx\\
&= \int_0^{L/2}\left|\int_0^x \underbrace{d'_\varepsilon}_{\geqslant 0}\right|dx + \int_{L/2}^{x_\varepsilon}\left|\int_0^L d'_\varepsilon - \int_x^L \underbrace{d'_\varepsilon}_{\leqslant 0}\right|dx + \int_{x_\varepsilon}^{L}\left|\int_x^L \underbrace{d'_\varepsilon}_{\leqslant 0}\right|dx\\
&\leqslant \int_0^{L/2}\left(\int_0^x d'_\varepsilon\right)dx - \int_{L/2}^{x_\varepsilon}\left(\int_x^L d'_\varepsilon\right)dx + \int_{L/2}^{x_\varepsilon}\left|\int_0^L d'_\varepsilon\right| - \int_{x_\varepsilon}^{L}\left(\int_x^L d'_\varepsilon\right)dx\\
&= \int_0^{L/2}\left(\int_0^x d'_\varepsilon\right)dx - \int_{L/2}^{L}\left(\int_x^L d'_\varepsilon\right)dx + \left|\frac{L}{2}-x_\varepsilon\right|\left|\int_0^L d'_\varepsilon\right|.
\end{aligned}$$

Now, as $u'_\varepsilon(0) = u'_\varepsilon(L) = \dfrac{c_\varepsilon}{\eta_\varepsilon+1}$, by (2.4.28) $d_\varepsilon(0)$ and $d_\varepsilon(L)$ both converge to $-c_1^2$. Then

$$\left|\frac{L}{2}-x_\varepsilon\right|\left|\int_0^L d'_\varepsilon\right| = \left|\frac{L}{2}-x_\varepsilon\right||d_\varepsilon(L)-d_\varepsilon(0)| \to 0.$$

We study the remaining terms separately, recalling the expressions (2.4.6) and (2.3.7).

$$\begin{aligned}
\int_0^{L/2}\int_0^x d'_\varepsilon dx &\leqslant 2(c_{0,\varepsilon}^2 - c_\varepsilon^2)\int_0^{L/2}\int_0^x -\frac{v'_{0,\varepsilon}v_{0,\varepsilon}}{(v_{0,\varepsilon}^2+\eta_\varepsilon)^2}dx\\
&= (c_{0,\varepsilon}^2 - c_\varepsilon^2)\int_0^{L/2}\left[\frac{1}{v_{0,\varepsilon}^2+\eta_\varepsilon}\right]_0^x dx\\
&\leqslant c_{0,\varepsilon}^2\int_0^{L/2}\left(\frac{1}{v_{0,\varepsilon}^2(x)+\eta_\varepsilon} - \frac{1}{v_{0,\varepsilon}^2(0)+\eta_\varepsilon}\right)dx\\
&= \int_0^{L/2}\frac{c_{0,\varepsilon}^2}{v_{0,\varepsilon}^2(x)+\eta_\varepsilon}dx - \frac{c_{0,\varepsilon}^2}{v_{0,\varepsilon}^2(0)+\eta_\varepsilon}\cdot\frac{L}{2}\\
&= \int_0^{L/2} c_{0,\varepsilon}u'_{0,\varepsilon}(x)dx - \frac{c_{0,\varepsilon}^2}{1+\eta_\varepsilon}\cdot\frac{L}{2}
\end{aligned}$$

Similarly, one has

$$-\int_{L/2}^{L}\left(\int_x^L d'_\varepsilon\right)dx \leqslant \int_{L/2}^{L} c_{0,\varepsilon}u'_{0,\varepsilon}(x)dx - c_{0,\varepsilon}^2\frac{1}{1+\eta_\varepsilon}\cdot\frac{L}{2}.$$

Summing both inequalities yields

$$\begin{aligned}
\int_0^{L/2}\left(\int_0^x d'_\varepsilon\right)dx - \int_{L/2}^{L}\left(\int_x^L d'_\varepsilon\right)dx &\leqslant c_{0,\varepsilon}\underbrace{\int_0^L u'_{0,\varepsilon}(x)dx}_{u_{0,\varepsilon}(L)-u_{0,\varepsilon}(0)} - Lc_{0,\varepsilon}^2\frac{1}{1+\eta_\varepsilon}\\
&= c_{0,\varepsilon}\left(a_0 - L\frac{c_{0,\varepsilon}}{1+\eta_\varepsilon}\right) \to c_0(a_0 - Lc_0) = 0,
\end{aligned}$$

where we used that $c_0 \in \{0, a_0/L\}$ (see [BMR23, Lemma 2.4]). We conclude that $II + V \to 0$.

$\underline{III + VI}$: Recall that v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$. Then, $c_\varepsilon u'_\varepsilon$ is itself non-decreasing on $[0, x_\varepsilon]$ and non-increasing on $[x_\varepsilon, L]$. This allows us to get rid of the absolute values in the expressions, so that $III + VI$ rewrites as :

$$\begin{aligned} III + VI &= \int_0^{x_\varepsilon} c_\varepsilon(u'_\varepsilon(x) - u'_\varepsilon(0))dx + \int_{x_\varepsilon}^{L} c_\varepsilon(u'_\varepsilon(x) - u'_\varepsilon(L))dx \\ &= \int_0^{L} c_\varepsilon u'_\varepsilon(x)dx - c_\varepsilon(x_\varepsilon u'_\varepsilon(0) + (L - x_\varepsilon)u'_\varepsilon(L)) \;=\; c_\varepsilon(u_\varepsilon(L) - u_\varepsilon(0)) - Lc_\varepsilon \frac{c_\varepsilon}{1+\eta_\varepsilon} \\ &\to \; c_1(a_1 - c_1 L) = 0 \end{aligned}$$

where we used Proposition 2.4.8. In conclusion, we get $III + VI \to 0$.

This completes the proof of the equipartition of the energy. □

2.4.7 Computation of the weight of the limit measure

In this section, we compute the weight α of the Dirac mass $\delta_{x_{min}}$, which completes the proof of the item (ii) in Theorem 2.2.7.

To do so, we use the equipartition principle, as it is done in [FLS09, section 6] and [BMR23, section 3].

Proof of (ii). The equipartition principle ensures that :

$$\left| \int_0^L \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) - |v'_\varepsilon|(1-v_\varepsilon) \right| = \int_0^L \left| \frac{1-v_\varepsilon}{2\sqrt{\varepsilon}} - \sqrt{\varepsilon} v'_\varepsilon \right|^2 \leqslant \int_0^L \left| \frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon |v'_\varepsilon|^2 \right| \to 0,$$

so that it is enough to evaluate $\displaystyle\int_0^L |v'_\varepsilon|(1-v_\varepsilon)$. Recall that v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$, where x_ε denotes the global minimum of v_ε. We have :

$$\int_0^L |v'_\varepsilon|(1-v_\varepsilon) = \int_0^{x_\varepsilon} -v'_\varepsilon(1-v_\varepsilon) + \int_{x_\varepsilon}^L v'_\varepsilon(1-v_\varepsilon) = \frac{1}{2}\left[(1-v_\varepsilon)^2\right]_0^{x_\varepsilon} + \frac{1}{2}\left[-(1-v_\varepsilon)^2\right]_{x_\varepsilon}^L = (1-v_\varepsilon(x_\varepsilon))^2.$$

So the value of α depends on the limit of $v_\varepsilon(x_\varepsilon)$.

First, note that $\alpha \geqslant 0$ as the measures $\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|v'_\varepsilon|^2\right)dx$ are non negative. Suppose that $\alpha > 0$. We prove that $v_\varepsilon(x_\varepsilon) \to 0$. Suppose that it is not the case and that there exists a subsequence such that $v_\varepsilon(x_\varepsilon) \geqslant \beta > 0$. Then $v_\varepsilon \geqslant \beta$ on $[0, L]$. According to Proposition 2.4.2, it implies that $v_\varepsilon(x_\varepsilon) \geqslant 1 - b\varepsilon$. But then

$$\left[(1-v_\varepsilon)^2\right]_0^{x_\varepsilon} + \left[-(1-v_\varepsilon)^2\right]_{x_\varepsilon}^L \to 0$$

which contradicts that $\alpha > 0$. Hence, $v_\varepsilon(x_\varepsilon) \to 0$ and

$$\left[(1-v_\varepsilon)^2\right]_0^{x_\varepsilon} + \left[-(1-v_\varepsilon)^2\right]_{x_\varepsilon}^L \to 2$$

We conclude that if $\alpha \neq 0$, then $\alpha = 1$. □

2.5 Appendix : Proof of Theorem 2.2.8

The proof of the $\Gamma-$convergence of $\overline{AT}_\varepsilon$ is an adaptation the proof of the classical Γ-convergence result of AT_ε to MS (see [AT92]). The main difficulty is to take into account the obstacle Γ_0 in the proof of the Γ-limsup. The argument is inspired by [Gia05, Lemma 6.2].

2.5.1 Proof of the Γ-liminf

Lemma 2.5.1. *Let $(u_\varepsilon, v_\varepsilon)$ be a sequence in $L^2(0,L)^2$ converging to (u,v) in $L^2(0,L)^2$. Then*

$$\overline{MS}(u,v) \leqslant \liminf_{\varepsilon\to+\infty} \overline{AT_\varepsilon}(u_\varepsilon, v_\varepsilon).$$

Proof. We suppose that $\liminf_{\varepsilon\to\infty} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$. We extract a subsequence for which the energy converges to the liminf. From now on, we keep the notations $(u_\varepsilon, v_\varepsilon)$ for simplicity but we only work with this subsequence.

Since $u_\varepsilon \in H^1(0,L) \subset L^\infty(0,L)$, we can write $\|u_\varepsilon\|_\infty \leqslant M$. We also have by a maximum principle (i.e replacing v_ε by a better competitor) that $0 \leqslant v_\varepsilon \leqslant 1$. Now, applying the arithmetico-geometric inequality to the second member of the energy yields

$$\mathcal{C} \geqslant \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \geqslant \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 + \int_0^L |v_\varepsilon'|(1-v_\varepsilon). \tag{2.5.1}$$

We first focus on the second term of the right-hand-side. Fix $0 < a \leqslant b < 1$. The coarea formula allows us to compute

$$\int_0^L |v_\varepsilon'|(1-v_\varepsilon) = \int_0^1 (1-t)\mathcal{H}^0(\partial^\star\{v_\varepsilon < t\})dt \geqslant \int_a^b (1-t)\mathcal{H}^0(\partial^\star\{v_\varepsilon < t\})dt. \tag{2.5.2}$$

Set $f(t) = t - t^2/2$. Then there exists $t_\varepsilon \in (a,b)$ such that

$$\int_a^b f'(t)\mathcal{H}^0(\partial^\star\{v_\varepsilon < t\})dt \geqslant (f(b)-f(a))\mathcal{H}^0(\partial^\star\{v_\varepsilon < t_\varepsilon\}). \tag{2.5.3}$$

Set $X_\varepsilon = \{v_\varepsilon < t_\varepsilon\}$ and $Y_\varepsilon = \{v_{0,\varepsilon} < t_\varepsilon\}$. Note that since $v_\varepsilon \leqslant v_{0,\varepsilon}$, one has $Y_\varepsilon \subset X_\varepsilon$. Combining (2.5.1), (2.5.2) and (2.5.3) yields

$$\mathcal{H}^0(\partial^\star X_\varepsilon) \leqslant C. \tag{2.5.4}$$

Using the variations of $v_{0,\varepsilon}$ (see [BMR23, Proposition 2.1], $\mathcal{H}^0(\partial^\star Y_\varepsilon) \leqslant 2$ and Y_ε is an open interval $(c^\varepsilon, d^\varepsilon)$. Moreover, since $t_\varepsilon < b$, one has

$$|Y_\varepsilon| \subset |X_\varepsilon| \leqslant |\{1-v_\varepsilon > 1-b\}| \leqslant \frac{1}{(1-b)^2}\int_0^L (1-v_\varepsilon)^2 \leqslant \frac{C}{(1-b)^2}\varepsilon \to 0. \tag{2.5.5}$$

Because $v_\varepsilon \in H^1(0,L)$, v_ε is continuous and X_ε is open set and can be written as the union of a countable number of open intervals and $\partial^\star X_\varepsilon = \partial X_\varepsilon$. By (2.5.4), up to an extraction, there exists an integer m such that for all ε,

$$X_\varepsilon = \bigcup_{i=1}^m (a_i^\varepsilon, b_i^\varepsilon).$$

There exists an index i_0 such that $(c^\varepsilon, d^\varepsilon) \subset (a_{i_0}^\varepsilon, b_{i_0}^\varepsilon)$. Since $|X_\varepsilon|, |Y_\varepsilon| \to 0$, one has $b_i^\varepsilon - a_i^\varepsilon \to 0$ and since $[0,L]$ is compact, up to another extraction, the sequences $(a_i^\varepsilon), (b_i^\varepsilon)$ converge to the same limit z_i. In particular, $c^\varepsilon, d^\varepsilon \to z_{i_0}$.

We will prove that $\Gamma_0 \cup J_u \subset \cup_i\{z_i\}$, with Γ_0 being empty or $\{L/2\}$. We first prove that $J_u \subset \cup_i\{z_i\}$, which is given if we prove that $u \in H^1(z_i, z_{i+1})$. Take an index $i \in \{1,\dots,m\}$ and take $z_i < \alpha < \beta < z_{i+1}$. Let us show that $u_\varepsilon \rightharpoonup u$ in $H^1(\alpha,\beta)$.

First, $\|u_\varepsilon\|_2$ is bounded. Second, thanks to the energy bound (2.2.2), and since for ε large enough, one has $[\alpha,\beta] \subset [0,L] \setminus X_\varepsilon \subset [0,L] \setminus \{v_\varepsilon < b\}$:

$$C \geqslant \int_\alpha^\beta (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \geqslant b\int_\alpha^\beta |u_\varepsilon'|^2 = b\|u_\varepsilon'\|_2^2.$$

So (u_ε) is bounded in $H^1(0,L)$. Thus, up to an extraction, $u_\varepsilon \rightharpoonup \hat{u}$ in $H^1(\alpha,\beta)$. But according to Rellich embedding, up to a subsequence, $u_\varepsilon \to \hat{u}$ in $L^2(\alpha,\beta)$. Thus $\hat{u} = u$ and

$$\int_\alpha^\beta |u'|^2 \leqslant \liminf_{\varepsilon\to 0}\int_\alpha^\beta |u_\varepsilon'|^2 \leqslant \liminf_{\varepsilon\to 0}\int_{b_i^\varepsilon}^{a_{i+1}^\varepsilon} |u_\varepsilon'|^2. \tag{2.5.6}$$

Letting α and β tend to z_i and z_{i+1} respectively shows that $u \in H^1(z_i, z_{i+1})$ with

$$\int_{z_i}^{z_{i+1}} |u'|^2 \leqslant \liminf_{\varepsilon\to 0} \int_{b_i^\varepsilon}^{a_{i+1}^\varepsilon} |u_\varepsilon'|^2. \tag{2.5.7}$$

Similarly, we obtain

$$\int_0^{z_1} |u'|^2 \leqslant \liminf_{\varepsilon\to 0} \int_0^{a_1^\varepsilon} |u_\varepsilon'|^2 \qquad \text{and} \qquad \int_{z_m}^{L} |u'|^2 \leqslant \liminf_{\varepsilon\to 0} \int_{b_m^\varepsilon}^{L} |u_\varepsilon'|^2. \tag{2.5.8}$$

Combining (2.5.8) with (2.5.7) for $i = 1, \dots, m$, we deduce that $u \in SBV^2(0, L)$ with $J_u \subset \cup_{i=1}^m \{z_i\}$ and with

$$\int_0^L |u'|^2 \leqslant \liminf_{\varepsilon\to 0} \int_{[0,L]\setminus X_\varepsilon} |u_\varepsilon'|^2. \tag{2.5.9}$$

The same argument applied to the sequences $(u_{0,\varepsilon}, v_{0,\varepsilon})$ proves that $J_{u_0} \subset \{z_{i_0}\}$. Recalling (2.2.4), this proves that $\Gamma_0 \subset \cup_i \{z_i\}$. We then have

$$\mathcal{H}^0(J_u \cup \Gamma_0) \leqslant m = \frac{1}{2}\mathcal{H}^0(\partial^\star\{v_\varepsilon < t_\varepsilon\})$$

Combining this inequality with (2.5.2) and (2.5.3) yields

$$\liminf_{\varepsilon\to 0} \int_0^L |v_\varepsilon|(1 - v_\varepsilon) \geqslant (f(b) - f(a)) \lim_{\varepsilon\to 0} \mathcal{H}^0(\partial^\star X_\varepsilon) = 2(f(b) - f(a))\mathcal{H}^0(J_u \cup \Gamma_0). \tag{2.5.10}$$

Combining (2.5.10) with (2.5.1) and letting a and b tending to 0 and 1 respectivelygives

$$\liminf_{\varepsilon\to 0} \int_0^L \left(\frac{(1 - v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v_\varepsilon'|^2 \right) \geqslant \mathcal{H}^0(J_u \cup \Gamma_0). \tag{2.5.11}$$

Now focus on the first term of $\overline{AT}_\varepsilon$. Indeed,

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \geqslant \int_{[0,L]\setminus X_\varepsilon} v_\varepsilon^2 |u_\varepsilon'|^2 \geqslant a^2 \int_{[0,L]\setminus X_\varepsilon} |u_\varepsilon'|^2.$$

and using (2.5.9),

$$\liminf_{\varepsilon\to 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \geqslant a^2 \liminf_{\varepsilon\to 0} \int_{[0,L]\setminus X_\varepsilon} |u_\varepsilon'|^2 \geqslant a^2 \int_0^L |u'|^2.$$

Since this inequality is true for every $a < 1$, letting $a \to 1$ leads to

$$\liminf_{\varepsilon\to 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \geqslant \int_0^L |u'|^2. \tag{2.5.12}$$

Summing (2.5.11) and (2.5.12), gives $\liminf_{\varepsilon\to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \geqslant \overline{MS}(u)$. □

2.5.2 Proof of the Γ-limsup

Lemma 2.5.2. *Let $u \in L^2((0, L))$. Then there exists a sequence $(u_\varepsilon, v_\varepsilon)$ converging in $L^2 \times L^2$ to $(u, 1)$ such that*

$$\limsup_{\varepsilon\to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \overline{MS}(u).$$

In the sequel, we denote by $MM_\varepsilon^V(v) := \displaystyle\int_V \frac{(1 - v)^2}{4\varepsilon} + \varepsilon|v'|^2$ and $MM_\varepsilon(v) := MM_\varepsilon^{[0,L]}(v)$.

We suppose that u is in $SBV^2(0, L)$, so that $\overline{MS}(u) < \infty$. The proof goes the following way : we approach $J_u \setminus \Gamma_0$ with the classical recovery sequence $(w_\varepsilon, z_\varepsilon)$ used the proof of the $\Gamma-$limsup of AT_ε to

MS (Proposition 2.5.3). This sequence might not satisfy the obstacle condition $w_\varepsilon \leqslant v_{0,\varepsilon}$ so we apply the modification $w_{1,\varepsilon} = \min(w_\varepsilon, v_{0,\varepsilon})$ (Lemma 2.5.4). In a neighbourhood W of Γ_0, we might not have $\limsup_{\varepsilon\to 0} MM_\varepsilon^V(w_{1,\varepsilon}) \leqslant \mathcal{H}^0(\Gamma_0)$. For this reason we construct a sequence $w_{2,\varepsilon}$ and a cutoff function φ_ε that both vanish around Γ_0(Lemma 2.5.5). Finally, setting $v_\varepsilon = \min(w_{1,\varepsilon}, w_{2,\varepsilon})$ and $u_\varepsilon = \varphi_\varepsilon z_\varepsilon$, we prove that $(u_\varepsilon, v_\varepsilon)$ satisfies the requirements of Lemma 2.5.2.

Proof. Let us start by recalling the construction of a recovery sequence $(w_\varepsilon, z_\varepsilon)$ for $J_u \setminus \Gamma_0$. This construction can be found in [AT92].

Proposition 2.5.3. *Let $d(x)$ denote the distance from x to the set $J_u \setminus \Gamma_0$ and $\alpha_\varepsilon = \sqrt{\eta_\varepsilon \varepsilon}$. We set*

$$h_\varepsilon(t) = 1 - e^{-\frac{t-\alpha_\varepsilon}{\varepsilon}},$$

Then the pair $(w_\varepsilon, z_\varepsilon)$ given by

$$w_\varepsilon = \begin{cases} h_\varepsilon(d(x)) & \text{if } d(x) > \alpha_\varepsilon \\ 0 & \text{if } 0 \leqslant d(x) \leqslant \alpha_\varepsilon \end{cases} \tag{2.5.13}$$

and

$$z_\varepsilon = \begin{cases} u(x) & \text{if } d(x) > \alpha_\varepsilon \\ \left(\frac{2d(x)}{\alpha_\varepsilon} - 1\right) u(x) & \text{if } \frac{\alpha_\varepsilon}{2} < d(x) \leqslant \alpha_\varepsilon \\ 0 & \text{if } d(x) \leqslant \frac{\alpha_\varepsilon}{2} \end{cases} \tag{2.5.14}$$

satisfies

$$(w_\varepsilon, z_\varepsilon) \overset{L^2(0,L)^2}{\longrightarrow} (1, u) \tag{2.5.15}$$

and, for all neighbourhood W of $J_u \setminus \Gamma_0$

$$\limsup_{\varepsilon\to 0} MM_\varepsilon^W(w_\varepsilon) \leqslant \mathcal{H}^0(J_u \setminus \Gamma_0) \tag{2.5.16}$$

and,

$$\limsup_{\varepsilon\to 0} \int_W (\eta_\varepsilon + w_\varepsilon^2)|z_\varepsilon'|^2 \leqslant \int_W |u'|^2. \tag{2.5.17}$$

Now we adapt this recovery sequence to satisfy the obstacle condition :

Lemma 2.5.4. *Let*

$$w_{1,\varepsilon} = \min(w_\varepsilon, v_{0,\varepsilon}) \tag{2.5.18}$$

Then $w_{1,\varepsilon} \leqslant v_{0,\varepsilon}$,

$$w_{1,\varepsilon} \overset{L^2(0,L)}{\longrightarrow} 1, \tag{2.5.19}$$

and for any neighbourhood W_1 of $J_u \setminus \Gamma_0$ not containing Γ_0,

$$\limsup_{\varepsilon\to 0} MM_\varepsilon^{W_1}(w_{1,\varepsilon}) \leqslant \mathcal{H}^0(J_u \setminus \Gamma_0) \tag{2.5.20}$$

and,

$$\limsup_{\varepsilon\to 0} \int_{W_1} (\eta_\varepsilon + w_{1,\varepsilon}^2)|z_\varepsilon'|^2 \leqslant \int_{W_1} |u'|^2. \tag{2.5.21}$$

Proof of Lemma 2.5.4. First, by definition of (2.5.18), $w_{1,\varepsilon} \leqslant v_{0,\varepsilon}$.

Second, according to (2.5.15), $w_\varepsilon \to 1$ in $L^2(0,L)$ and according to [BMR23, Theorem 1.1 (i)], $v_{0,\varepsilon} \to 1$ in $L^2(0,L)$, so that $w_{1,\varepsilon} \to 1$ in $L^2(0,L)$ as well.

Third, take W_1 a neighbourhood of $J_u \setminus \Gamma_0$ not containing Γ_0. Using [BMR23, Theorem 1.1 (iv)], we have that $\limsup_{\varepsilon\to 0} MM_\varepsilon^{W_1}(v_{0,\varepsilon}) \to 0$, so using (2.5.16) and (2.5.17) :

$$\limsup_{\varepsilon\to 0} MM_\varepsilon^{W_1}(w_{1,\varepsilon}) \leqslant \limsup_{\varepsilon\to 0} \left(\underbrace{MM_\varepsilon^{W_1}(v_{0,\varepsilon})}_{\to 0} + MM_\varepsilon^{W_1}(w_\varepsilon) \right) \leqslant \mathcal{H}^0(J_u \setminus \Gamma_0)$$

and

$$\limsup_{\varepsilon\to 0}\int_{W_1}(\eta_\varepsilon+w_{1,\varepsilon}^2)|z_\varepsilon'|^2\leqslant\limsup_{\varepsilon\to 0}\int_{W_1}(\eta_\varepsilon+w_\varepsilon^2)|z_\varepsilon'|^2\leqslant\int_{W_1}|u'|^2.$$

□

Now we deal with Γ_0. Let $\mathcal{C}>0$ be such that

$$\limsup MM_\varepsilon(v_{0,\varepsilon})\leqslant\mathcal{C}. \tag{2.5.22}$$

We construct a function $w_{2,\varepsilon}$ and a cut-off function φ_ε that vanish around Γ_0. The construction is given in the following lemma (which is the analogue of [Gia05, Lemma 6.2]) :

Lemma 2.5.5. *Set $r_\varepsilon=\sqrt[4]{\eta_\varepsilon/\varepsilon}, s_\varepsilon=2r_\varepsilon, t_\varepsilon=\sqrt{s_\varepsilon}$ and $k_\varepsilon=t_\varepsilon/s_\varepsilon$.*

There exists a cut-off function φ_ε such that

$$\varphi_\varepsilon=\begin{cases}1 & \text{if} \quad s_\varepsilon\leqslant v_{0,\varepsilon}\\ 0 & \text{if} \quad v_{0,\varepsilon}\leqslant r_\varepsilon\end{cases} \tag{2.5.23}$$

and

$$\int_{\{r_\varepsilon\leqslant v_{0,\varepsilon}\leqslant s_\varepsilon\}}\eta_\varepsilon|\varphi_\varepsilon'|^2\to 0 \tag{2.5.24}$$

Second, set

$$w_{2,\varepsilon}=\begin{cases}v_{0,\varepsilon} & \text{if} \quad t_\varepsilon\leqslant v_{0,\varepsilon}\\ \frac{k_\varepsilon}{k_\varepsilon-1}(v_{0,\varepsilon}-t_\varepsilon)+t_\varepsilon & \text{if} \quad s_\varepsilon\leqslant v_{0,\varepsilon}\leqslant t_\varepsilon\\ 0 & \text{if} \quad v_{0,\varepsilon}\leqslant s_\varepsilon\end{cases} \tag{2.5.25}$$

Then $w_{2,\varepsilon}\leqslant v_{0,\varepsilon}$

$$w_{2,\varepsilon}\overset{L^2(0,L)}{\to}1, \tag{2.5.26}$$

and, for some constant $\mathcal{C}>0$,

$$\limsup_{\varepsilon\to 0}MM_\varepsilon(w_{2,\varepsilon})-MM_\varepsilon(v_{0,\varepsilon})\leqslant\mathcal{C}\left(\frac{2k_\varepsilon}{(k_\varepsilon-1)^2}+\frac{t_\varepsilon}{2(k_\varepsilon-1)(1-t_\varepsilon)^2}+\frac{t_\varepsilon}{2(1-t_\varepsilon)^2}\right). \tag{2.5.27}$$

Remark 2.5.1. Observe that the right hand side of (2.5.27) tends to 0 with ε as $t_\varepsilon\to 0$ and $k_\varepsilon\to+\infty$.

Remark 2.5.2. The transition from the two states of φ_ε is done on a domain on which $w_{2,\varepsilon}$ is equal to zero.

Remark 2.5.3. The set $\{v_{0,\varepsilon}<t_\varepsilon\}$ might be empty. In this case, $w_{2,\varepsilon}=v_{0,\varepsilon}$ and $\varphi_\varepsilon=1$.

We postpone the proof of Lemma 2.5.5 to the end of the proof of Lemma 2.5.2 and now we construct the sequence $(u_\varepsilon,v_\varepsilon)$ satisfying the requirements of Lemma 2.5.2, using Lemma 2.5.4 and Lemma 2.5.5.

Set

$$v_\varepsilon=\min(w_{1,\varepsilon},w_{2,\varepsilon}), \tag{2.5.28}$$

and

$$u_\varepsilon=\varphi_\epsilon z_\varepsilon. \tag{2.5.29}$$

$\underline{v_\varepsilon\leqslant v_{0,\varepsilon}}$: This is a consequence of Lemma 2.5.4 and Lemma 2.5.5.

$\underline{(u_\varepsilon,v_\varepsilon)\to(u,1)\text{ in }L^2(0,L)^2}$: The fact that $v_\varepsilon\to 1$ in $L^2(0,L)$ is a consequence of (2.5.19) and (2.5.26). Concerning the convergence of u_ε, one has

$$\|u_\varepsilon-u\|_{L^2}\leqslant\|u_\varepsilon-z_\varepsilon\|_{L^2}+\|z_\varepsilon-u\|_{L^2}.$$

One already has from (2.5.15) that $\|z_\varepsilon-u\|_{L^2}\to 0$. On the other hand, as u belongs to $SBV^2((0,L))$, it is a bounded function and using the definition (2.5.14) of z_ε and the definition (2.5.23) of φ_ε,

$$\int_0^L |\varphi_\varepsilon z_\varepsilon - z_\varepsilon|^2 = \int_0^L |z_\varepsilon|^2|\varphi_\varepsilon - 1|^2 \leqslant \int_0^L |u|^2|\varphi_\varepsilon - 1|^2 \leqslant \int_{\{v_{0,\varepsilon}\leqslant s_\varepsilon\}} |u|^2 \leqslant \|u\|_\infty^2 |\{v_{0,\varepsilon} \leqslant s_\varepsilon\}|$$

But the energy bound (2.5.22) yields

$$\mathcal{C} \geqslant \int_{\{v_{0,\varepsilon}\leqslant s_\varepsilon\}} \frac{(1-v_{0,\varepsilon})^2}{4\varepsilon} \geqslant \frac{(1-t_\varepsilon)^2}{4\varepsilon} |\{v_{0,\varepsilon} \leqslant s_\varepsilon\}| \tag{2.5.30}$$

Thus $|\{v_{0,\varepsilon} \leqslant s_\varepsilon\}| \to 0$ and $\|u_\varepsilon - z_\varepsilon\|_{L^2} \to 0$.

$\limsup_{\varepsilon\to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \overline{MS}(u)$:

Fix W_1 a neighbourhood of $J_u \setminus \Gamma_0$ and W_2 a neighbourhood of Γ_0 so that $W_1 \cap W_2 = \emptyset$. Then one has

$$v_\varepsilon = w_{1,\varepsilon} \text{ on } W_1 \quad ; \quad u_\varepsilon = \begin{cases} z_\varepsilon & \text{on } W_1, \\ \varphi_\epsilon u & \text{on } W_2, \\ u & \text{outside of } W_1 \sqcup W_2. \end{cases} \tag{2.5.31}$$

Indeed, outside of W_2, $v_{0,\varepsilon} \geqslant 1 - M\varepsilon > t_\varepsilon$ (see Corollary 2.4.3 applied to $\mu_\varepsilon = 0$ and $x_{min} = L/2$) so, outside of W_2, $w_{2,\varepsilon} = v_{0,\varepsilon}$ and $v_\varepsilon = \min(w_{1,\varepsilon}, v_{0,\varepsilon}) = w_{1,\varepsilon}$ and φ_ε is equal to 1 outside of W_2. Similarly, for ε small enough, $\{d \leqslant \alpha_\varepsilon\} \subset W_1$, so $z_\varepsilon = u$ outside of W_1.

Concerning the convergence of $MM_\varepsilon(v_\varepsilon)$, by (2.5.20) and (2.5.27).

$$\begin{aligned} \limsup_{\varepsilon\to 0} MM_\varepsilon(v_\varepsilon) &\leqslant \limsup_{\varepsilon\to 0} MM_\varepsilon^{W_1}(w_{1,\varepsilon}) + \limsup_{\varepsilon\to 0} MM_\varepsilon(w_{2,\varepsilon}) \\ &\leqslant \mathcal{H}^0(J_u \setminus \Gamma_0) \\ &\quad + \limsup_{\varepsilon\to 0} \left(MM_\varepsilon(v_{0,\varepsilon}) + \mathcal{C}\left(\frac{2k_\varepsilon}{(k_\varepsilon-1)^2} + \frac{t_\varepsilon}{2(k_\varepsilon-1)(1-t_\varepsilon)^2} + \frac{t_\varepsilon}{2(1-t_\varepsilon)^2} \right) \right) \end{aligned}$$

Then, according to [BMR23, Theorem 1.1 (iii),(iv)], $MM_\varepsilon(v_{0,\varepsilon}) \to \mathcal{H}^0(\Gamma_0)$. Thus, using Remark 2.5.1,

$$\limsup_{\varepsilon\to 0} MM_\varepsilon(v_\varepsilon) \leqslant \mathcal{H}^0(J_u \setminus \Gamma_0) + \mathcal{H}^0(\Gamma_0) = \mathcal{H}^0(J_u \cup \Gamma_0).$$

Now we deal with the convergence of the elastic energy $\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2$. According to (2.5.21),

$$\limsup_{\varepsilon\to 0} \int_{W_1} (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \leqslant \limsup_{\varepsilon\to 0} \int_{W_1} (\eta_\varepsilon + w_{1,\varepsilon}^2)|z_\varepsilon'|^2 \leqslant \int_{W_1} |u'|^2,$$

and using the expression (2.5.31) of u_ε,

$$\limsup_{\varepsilon\to 0} \int_{[0,L]\setminus(W_1\sqcup W_2)} (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \leqslant \limsup_{\varepsilon\to 0} \int_{[0,L]\setminus(W_1\sqcup W_2)} (\eta_\varepsilon + 1)|u'|^2 = \int_{[0,L]\setminus(W_1\sqcup W_2)} |u'|^2.$$

It remains to deal with

$$\limsup_{\varepsilon\to 0} \int_{W_2} (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2.$$

To compute $\int_{W_2} (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2$, we recall the definition (2.5.23) of φ_ε and we apply the inequality $|p+q|^2 \leqslant 2(|p|^2 + |q|^2)$ to $p = \varphi_\varepsilon' z$ and $q = \varphi_\varepsilon z'$ on the set $\{r_\varepsilon \leqslant v_{0,\varepsilon} \leqslant s_\varepsilon\}$. Namely,

$$\begin{aligned} \int_{W_2} (\eta_\varepsilon + v_\varepsilon^2)|(\varphi_\varepsilon u)'|^2 &= \int_{\{s_\varepsilon\leqslant v_{0,\varepsilon}\}\cap W_2} (\eta_\varepsilon + v_\varepsilon)^2|\varphi_\varepsilon' u + \varphi_\varepsilon u'|^2 + \int_{\{r_\varepsilon\leqslant v_{0,\varepsilon}\leqslant s_\varepsilon\}} (\eta_\varepsilon + v_\varepsilon)^2|\varphi_\varepsilon' u + \varphi_\varepsilon u'|^2 \\ &\leqslant \int_{\{s_\varepsilon\leqslant v_\varepsilon\}\cap W_2} (\eta_\varepsilon + 1)|u'|^2 + 2\int_{\{r_\varepsilon\leqslant v_\varepsilon\leqslant s_\varepsilon\}} \eta_\varepsilon \underbrace{|\varphi_\varepsilon|^2}_{\leqslant 1} |u'|^2 + 2\int_{\{r_\varepsilon\leqslant v_\varepsilon\leqslant s_\varepsilon\}} \eta_\varepsilon |\varphi_\varepsilon'|^2 |u|^2 \\ &\leqslant \int_{\{s_\varepsilon\leqslant v_\varepsilon\}\cap W_2} (\eta_\varepsilon + 1)|u'|^2 + 2\eta_\varepsilon \|u'\|_{L^2} + 2\|u\|_\infty^2 \int_{\{r_\varepsilon\leqslant v_\varepsilon\leqslant s_\varepsilon\}} \eta_\varepsilon |\varphi_\varepsilon'|^2. \end{aligned}$$

The limsup of this last term is smaller than $\int_{W_2} |u'|^2$, the second term goes to 0 and the third term goes to 0 according (2.5.24), which concludes the proof of the $\Gamma-$limsup. □

We now turn to the proof of the lemma Lemma 2.5.5 :

Proof of Lemma 2.5.5. We start with the construction of a cut-off function φ_ε satisfying (2.5.23) and (2.5.24). Divide the interval $[r_\varepsilon, s_\varepsilon]$ into h_ε intervals $I_1, \dots, I_{h_\varepsilon}$ of the same size, such that $\frac{\eta_\varepsilon}{\varepsilon} \cdot \frac{h_\varepsilon}{(s_\varepsilon - r_\varepsilon)^2} \to 0$ (which is possible as $\frac{\eta_\varepsilon}{\varepsilon} \cdot \frac{1}{(s_\varepsilon - r_\varepsilon)^2} \to 0$, take for example $h_\varepsilon = \left\lfloor \left(\frac{\eta_\varepsilon}{\varepsilon} \cdot \frac{1}{(s_\varepsilon - r_\varepsilon)^2} \right)^{-1/2} \right\rfloor$). Note that by the energy bound (2.5.22) :

$$\mathcal{C} \geqslant \sum_{j=1}^{h_\varepsilon} \int_{I_j} \varepsilon |v'_{0,\varepsilon}|^2.$$

So there exists an interval $I_\varepsilon = [\alpha_\varepsilon, \beta_\varepsilon]$ of length $(s_\varepsilon - r_\varepsilon)/h_\varepsilon$ such that

$$\int_{I_\varepsilon} \varepsilon |v'_{0,\varepsilon}|^2 \leqslant \frac{\mathcal{C}}{h_\varepsilon}.$$

Define $\varphi_\varepsilon = \min\left(\frac{(v_{0,\varepsilon} - \alpha_\varepsilon)^+}{\beta_\varepsilon - \alpha_\varepsilon}, 1 \right)$. Then φ_ε takes the desired values on $\{v_{0,\varepsilon} \leqslant r_\varepsilon\}$ and $\{v_{0,\varepsilon} \geqslant s_\varepsilon\}$ and

$$\eta_\varepsilon \int_{\{r_\varepsilon \leqslant v_{0,\varepsilon} \leqslant s_\varepsilon\}} |\varphi'_\varepsilon|^2 = \eta_\varepsilon \int_{\{\alpha_\varepsilon \leqslant v_{0,\varepsilon} \leqslant \beta_\varepsilon\}} \frac{|v'_{0,\varepsilon}|^2}{(\beta_\varepsilon - \alpha_\varepsilon)^2} \leqslant \frac{\eta_\varepsilon}{\varepsilon} \frac{\mathcal{C}}{h_\varepsilon} \frac{h_\varepsilon^2}{(s_\varepsilon - r_\varepsilon)^2} \to 0$$

which proves (2.5.24).

Now we turn to the proof of (2.5.26) and (2.5.27). Observe that $w_{2,\varepsilon} \leqslant v_{0,\varepsilon}$ as $v_{0,\varepsilon}$ is positive and on $\{s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon\}$, according to (2.5.25) :

$$w_{2,\varepsilon} = v_{0,\varepsilon} + \frac{1}{k_\varepsilon - 1}(v_{0,\varepsilon} - t_\varepsilon) \leqslant v_{0,\varepsilon}.$$

To prove (2.5.26), observe that $w_{2,\varepsilon}$ and $v_{0,\varepsilon}$ are taking values in $[0,1]$, so that

$$\int_0^L |w_{2,\varepsilon} - v_{0,\varepsilon}|^2 = \int_{\{v_{0,\varepsilon} \leqslant t_\varepsilon\}} |w_{2,\varepsilon} - v_{0,\varepsilon}|^2 \leqslant 4 |\{v_{0,\varepsilon} \leqslant t_\varepsilon\}|.$$

Then, using the energy bound (2.5.22), we see that

$$\mathcal{C} \geqslant \int_{\{v_{0,\varepsilon} \leqslant t_\varepsilon\}} \frac{(1 - v_{0,\varepsilon})^2}{4\varepsilon} \geqslant \frac{(1 - t_\varepsilon)^2}{4\varepsilon} |\{v_{0,\varepsilon} \leqslant t_\varepsilon\}|. \tag{2.5.32}$$

So $|\{v_{0,\varepsilon} \leqslant t_\varepsilon\}| \to 0$ and $\|w_{2,\varepsilon} - v_{0,\varepsilon}\|_2 \to 0$, which proves (2.5.26).

We turn to the proof of (2.5.27). We compute the difference in energy, starting with the difference between the Dirichlet energies. The expression (2.5.25) of $w_{2,\varepsilon}$ and the energy bound (2.5.22) yields

$$\begin{aligned} \int_0^L \varepsilon |w'_{2,\varepsilon}|^2 - \varepsilon |v'_{0,\varepsilon}|^2 &= \int_{\{s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon\}} \varepsilon \left(\frac{k_\varepsilon^2}{(k_\varepsilon - 1)^2} - 1 \right) |v'_{0,\varepsilon}|^2 - \int_{\{v_{0,\varepsilon} \leqslant s_\varepsilon\}} \varepsilon |v'_{0,\varepsilon}|^2 \\ &\leqslant \left(\frac{k_\varepsilon^2}{(k_\varepsilon - 1)^2} - 1 \right) \int_{\{s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon\}} \varepsilon |v'_{0,\varepsilon}|^2 \leqslant \mathcal{C} \left(\frac{k_\varepsilon^2}{(k_\varepsilon - 1)^2} - 1 \right) \end{aligned}.$$

On the other hand

$$\begin{aligned}
&\int_0^L \frac{(1-w_{2,\varepsilon})^2}{4\varepsilon} - \frac{(1-v_{0,\varepsilon})^2}{4\varepsilon} \\
&= \frac{1}{4\varepsilon}\int_{\{v_{0,\varepsilon}\leqslant t_\varepsilon\}} (v_{0,\varepsilon}-w_{2,\varepsilon})(2-v_{0,\varepsilon}-w_{2,\varepsilon}) \\
&= \frac{1}{4\varepsilon}\int_{\{s_\varepsilon\leqslant v_{0,\varepsilon}\leqslant t_\varepsilon\}} \frac{1}{k_\varepsilon-1}\underbrace{(t_\varepsilon-v_{0,\varepsilon})}_{\leqslant t_\varepsilon}\underbrace{(2-v_{0,\varepsilon}-w_{2,\varepsilon})}_{\leqslant 2} + \frac{1}{4\varepsilon}\int_{\{v_{0,\varepsilon}\leqslant s_\varepsilon\}} \underbrace{v_{0,\varepsilon}}_{\leqslant s_\varepsilon}\underbrace{(2-v_{0,\varepsilon})}_{\leqslant 2} \\
&\leqslant \frac{t_\varepsilon}{k_\varepsilon-1}\frac{|\{v_{0,\varepsilon}\leqslant t_\varepsilon\}|}{2\varepsilon} + \frac{s_\varepsilon}{k_\varepsilon-1}\frac{|\{v_{0,\varepsilon}\leqslant s_\varepsilon\}|}{2\varepsilon}
\end{aligned}$$

But the energy bound (2.5.22) yields, in a similar fashion than (2.5.32), for any $\alpha > 0$,

$$\mathcal{C} \geqslant \int_{\{v_{0,\varepsilon}\leqslant\alpha\}} \frac{(1-v_{0,\varepsilon})^2}{4\varepsilon} \geqslant \frac{(1-\alpha)^2}{4\varepsilon}|\{v_{0,\varepsilon}\leqslant\alpha\}|.$$

Hence, the last term is smaller than for

$$\mathcal{C}\frac{t_\varepsilon}{2(k_\varepsilon-1)(1-t_\varepsilon)^2} + \mathcal{C}\frac{s_\varepsilon}{2(1-s_\varepsilon)^2} \leqslant \mathcal{C}\left(\frac{t_\varepsilon}{2(k_\varepsilon-1)(1-t_\varepsilon)^2} + \frac{t_\varepsilon}{2(1-t_\varepsilon)^2}\right).$$

Summing both estimates yields the desired result. This achieves the proof of Lemma 2.5.5. □

2.5.3 Critical Points of $\overline{MS}$

Theorem 2.2.8 ensures that the minimizers of $\overline{AT}_\varepsilon$ converge to a pair $(1,u)$ that minimizes $\overline{MS}$ if we have the additional assumption of an energy bound on (AT_ε).

A natural extension is to ask whether the critical points of $\overline{AT}_\varepsilon$ also converge to critical points of $\overline{MS}$. Here, we prove that the critical points of $\overline{MS}$ are either affine or piecewise constant.

Take u a minimizer of $\overline{MS}$. First, u has a finite number of discontinuity points, that we denote by $\{x_1,\dots,x_d\}$. Let φ in $SBV^2(0,L)$ with $J_\varphi \subset J_u$ and take $u+t\varphi$ as a competitor for t a real number. Then, $J_{u+t\varphi} = J_u$ so one has

$$\int_0^L |u'|^2 \leqslant \int_0^L |u'+t\varphi'|^2,$$

which rewrites, after dividing by t, letting t tend successively to 0^+ and 0^-, as :

$$\int_0^L u'\varphi' = 0.$$

So u is affine on each interval (x_i, x_{i+1}). In the case where u admits at least one discontinuity point, say x_i, and take φ equal to 0 outside (x_i,x_{i+1}) and equal to $\dfrac{x-x_{i+1}}{x_i-x_{i+1}}$ on (x_i,x_{i+1}) (where x_{i+1} can be replaced by L). Then $\varphi \in SBV^2(0,L)$ with $\varphi' = 1_{(x_i,x_{i+1})}$, $J_\varphi = \{x_i\}$ and

$$u(x_{i+1}) - u(x_i) = \int_{x_i}^{x_{i+1}} u' = \int_0^L u'\varphi' = 0,$$

so that $u(x_i) = u(x_{i+1})$, i.e u is piecewise constant.

Hence, critical points of $\overline{AT}_\varepsilon$ converge to critical points of $\overline{MS}$. However, there is a selection criterion as limit functions of critical points of AT_ε must be non decreasing and admit at most one discontinuity point.

Chapter 3

Critical points of the two-dimensional Ambrosio-Tortorelli functional with convergence of the phase-field energy

We consider a family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ of critical points of the Ambrosio-Tortorelli functional. Assuming a uniform energy bound, the sequence $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ converges in $L^2(\Omega)$ to a limit $(u, 1)$ as $\varepsilon \to 0$, where u is in $SBV^2(\Omega)$. It was previously shown that if the full Ambrosio-Tortorelli energy associated to $(u_\varepsilon, v_\varepsilon)$ converges to the Mumford-Shah energy of u, then the first inner variation converges as well. In particular, u is a critical point of the Mumford-Shah functional in the sense of inner variations. In this work, focusing on the two-dimensional setting, we extend this result under the sole convergence of the phase-field energy to the length energy term in the Mumford-Shah functional.

3.1 Introduction

The Mumford-Shah (MS) functional is a prototypical energy used in free-discontinuity problems where the competition between a volume and a surface energy leads to concentration on a co-dimension one discontinuity set. It has been historically introduced in [MS89] in the context of image segmentation, and has also been used in fracture mechanics in [FM98] to describe the propagation of brittle cracks.

In order to present this energy, we need to introduce some notation. Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^d$ with $d \geq 1$, and let $g \in H^{\frac{1}{2}}(\partial\Omega)$ be a Dirichlet boundary data. The energy space, denoted by $SBV^2(\Omega)$, is made of all functions $u \in BV(\Omega)$ with vanishing Cantor part such that the approximate gradient ∇u belongs to $L^2(\Omega; \mathbb{R}^d)$ and $\mathcal{H}^{d-1}(J_u) < \infty$, where $J_u \subset \Omega$ is the jump set of u and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. In order to account for possible boundary discontinuities, we introduce the extended jump set $\widehat{J_u}$ of u defined by

$$\widehat{J_u} := J_u \cup (\partial\Omega \cap \{u \neq g\}), \tag{3.1.1}$$

where, with a slight abuse of notation, we still denote by u the inner trace of u on $\partial\Omega$. We define the Mumford-Shah functional $MS : SBV^2(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} MS(u) &= \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega \cap \{u \neq g\}) \\ &= \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{d-1}(\widehat{J_u}) \quad \text{for all } u \in SBV^2(\Omega). \end{aligned} \tag{3.1.2}$$

Using the direct method in the calculus of variations, it can be proven that this functional admits minimizers as a consequence of Ambrosio's compactness and lower semicontinuity results, see [AFP00, DGCL89b, CL90]. The description of the jump set is of particular interest in connexion with the Mumford-Shah conjecture which stipulates that, in dimension $d = 2$, the jump set J_u is made of finitely many $\mathcal{C}^1$ curves intersecting at triple junction type points. In particular, the regularity (or lack of regularity) of the jump set represents a very interesting topic of research also from the application

point of view for the description of edges in image segmentation and cracks in mechanics. We refer to the monographs [AFP00, Dav05, DLF25b] for related results in that direction.

If one is interested into the numerical approximation of minimizers of the Mumford-Shah energy, we face a difficulty related to the fact that the discontinuity set is not a priori known. It would require a high mesh size precision in the spatial region where u is discontinuous which might lead to too costly numerical schemes. This is why it is convenient to approximate the MS energy by a more regular functional. In the spirit of the Allen-Cahn model for phase transitions, see e.g. [Mod87, Ste88], Ambrosio and Tortorelli proposed in [AT92] the following variational phase-field regularization of the MS energy. In our context, we set

$$\mathcal{A}_g := \{(u,v) \in [H^1(\Omega) \times (H^1(\Omega) \cap L^\infty(\Omega))] : \ u = g \text{ and } v = 1 \text{ on } \partial\Omega\} \tag{3.1.3}$$

and we define the Ambrosio-Tortorelli (AT) energy $AT_\varepsilon : \mathcal{A}_g \to \mathbb{R}$ by

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\right) dx \quad \text{for all } (u,v) \in \mathcal{A}_g, \tag{3.1.4}$$

where $0 < \eta_\varepsilon \ll \varepsilon$ is a small parameter that ensures ellipticity. This regularization has a mechanical interpretation. The phase-field variable v can be seen as a damage variable taking values in the interval $[0,1]$. The region $\{v = 1\}$ corresponds to completely sane material, whereas where $\{v = 0\}$ the material is totally damaged. In [AT92], in a slightly different context (i.e., in absence of Dirichlet boundary conditions), a Γ-convergence result of AT_ε to $\widetilde{MS}$ is proved by suitably extending MS as a two variables functional :

$$\widetilde{MS}(u,v) = \begin{cases} MS(u) & \text{if } u \in SBV^2(\Omega) \text{ and } v \equiv 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1.5}$$

The existence of minimizers in $\mathcal{A}_g$ for AT_ε for a fixed value of ε is obtained via the direct method in the calculus of variations. The fundamental theorem of Γ-convergence then ensures that a subsequence $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ of minimizers of AT_ε converges in $[L^2(\Omega)]^2$ to $(u,1)$ where u a minimizer of MS. This motivates the use of AT_ε, with $\varepsilon > 0$ small, to numerically approximate minimizers of the MS functional. This regularized formulation is indeed at the basis of numerical simulations, see e.g. [BFM08].

However, concerning the numerical implementation, we observe that the term $v^2|\nabla u|^2$ makes the functional AT_ε non-convex. Consequently, numerical methods might fail to converge to a minimizer of AT_ε. For example, using the fact that AT_ε remains separately strictly convex, it was proposed in [BFM08] to perform an alternate minimization algorithm. It is proven in [Bou07, Theorem 1] that the sequence of iterates converges to a critical point of AT_ε, but nothing guarantees this critical point to be a minimizer. It motivates the study of the convergence of critical points of the AT functional as $\varepsilon \to 0$, and it raises the question to know if their limits correspond to critical points of the MS functional in some sense.

We start by recalling that a critical point for the AT functional is a pair $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g$ such that

$$\frac{d}{dt}\Big|_{t=0} AT_\varepsilon(u_\varepsilon + t\psi, v_\varepsilon + t\varphi) = 0 \quad \text{for all } (\psi,\varphi) \in H_0^1(\Omega) \times (H_0^1(\Omega) \cap L^\infty(\Omega)). \tag{3.1.6}$$

Tools of Γ-convergence provide little help to study the convergence of critical points. The extension of the fundamental theorem of Γ-convergence to the convergence of critical points has been studied in various settings, see e.g. [HT00, Ton02, Ton05, Bet25] for the Allen-Cahn functional and [BBH94, BBO01, LR01, SS07] for the Ginzburg-Landau functional, see also [Bra14]. Concerning the AT functional, the general convergence of critical points to critical points of the MS energy is established in dimension $d = 1$ in [FLS09, Le10, BMR23]. We also refer to [BI23] for the convergence of critical points of a phase-field approximation of cohesive fracture energies in 1D. In dimension $d \geqslant 2$, the convergence of critical points has been established in [BMR24], with the additional assumption of convergence of the energy

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \to MS(u).$$

A similar assumption of convergence of energy was made in [Le11, Le15, LS19] to study the convergence of critical points of the Allen-Cahn functional to minimal surfaces, and to understand how stability passes to the limit. However it is not always true that convergence of the energy holds. For example, in [HT00, Section 6.3] it is proved that interfaces with multiplicities can be limits of a sequence of critical points of the Allen-Cahn functional. Consequently these critical points do not satisfy the energy convergence assumption.

In this article we weaken the assumption of convergence of energy made in [BMR24]. Our main result Theorem 2.2.7, establishes the convergence of critical points under the assumption that only the phase-field term converges, i.e.,

$$\int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\right) dx \to \mathcal{H}^{d-1}(\widehat{J_u}).$$

This result is restricted to dimension $d = 2$, which is meaningful from the point of view of applications either in image segmentation, or in the anti-plane brittle fracture model.

Let us close this introduction by some notations and definitions used throughout the paper.

Linear algebra

Let $\mathbb{M}^{d\times d}$ be the set of real $d \times d$ matrices, and $\mathbb{M}^{d\times d}_{\text{sym}}$ the set of real symmetric $d \times d$ matrices. Given two vectors a and b in $\mathbb{R}^d$, we denote by $a \cdot b \in \mathbb{R}$ their inner product and by $a \otimes b = a^T b \in \mathbb{M}^{d\times d}$ their tensor product. We will use the Frobenius inner product of matrices A and $B \in \mathbb{M}^{d\times d}$, defined by $A : B = \mathrm{Tr}(A^T B)$ and the associated norm $|A| = \mathrm{Tr}(A^T A)^{1/2}$.

Measures

The Lebesgue measure in $\mathbb{R}^d$ is denoted by $\mathcal{L}^d$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$.

If $X \subset \mathbb{R}^d$ is a locally compact set and Y an Euclidean space, we denote by $\mathcal{M}(X;Y)$ the space of Y-valued bounded Radon measures in X endowed with the norm $\|\mu\| = |\mu|(X)$, where $|\mu|$ is the variation of the measure μ. If $Y = \mathbb{R}$, we simply write $\mathcal{M}(X)$ instead of $\mathcal{M}(X;\mathbb{R})$. By Riesz representation theorem, $\mathcal{M}(X;Y)$ can be identified with the topological dual of $\mathcal{C}_0(X;Y)$, the space of continuous functions $f : X \to Y$ such that $\{|f| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$. The weak* topology of $\mathcal{M}(X;Y)$ is defined using this duality.

Functional spaces

We use standard notation for Lebesgue, Sobolev and Hölder spaces. Given a bounded open set $\Omega \subset \mathbb{R}^N$, the space of functions of bounded variation is defined by

$$BV(\Omega) = \{u \in L^1(\Omega) : \ Du \in \mathcal{M}(\Omega;\mathbb{R}^d)\}.$$

We shall also consider the subspace $SBV(\Omega)$ of special functions of bounded variation made of functions $u \in BV(\Omega)$ whose distributional derivative can be decomposed as

$$Du = \nabla u \mathcal{L}^d + (u^+ - u^-)\nu_u \mathcal{H}^{d-1} \llcorner J_u.$$

In the previous expression, ∇u is the Radon-Nikodým derivative of Du with respect to $\mathcal{L}^d$, and it is called the approximate gradient of u. The Borel set J_u is the (approximate) jump set of u. It is a countably $\mathcal{H}^{d-1}$-rectifiable subset of Ω oriented by the (approximate) normal direction of jump $\nu_u : J_u \to \mathbf{S}^{d-1}$, and $u^\pm$ are the one-sided approximate limits of u on J_u according to ν_u. Finally we define

$$SBV^2(\Omega) = \{u \in SBV(\Omega) : \ \nabla u \in L^2(\Omega;\mathbb{R}^d) \text{ and } \mathcal{H}^{d-1}(J_u) < \infty\}.$$

We say that a Lebesgue measurable set $E \subset \Omega$ has finite perimeter in Ω if its characteristic function $\mathbf{1}_E \in BV(\Omega)$. We denote by $\partial^\star E$ its reduced boundary. We refer to [AFP00] for a detailed description of the space BV.

Varifolds

Let us recall several basic ingredients of the theory of varifolds (see e.g. [Sim83]). We denote by $\mathbf{G}_{d-1}$ the Grassmannian manifold of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$. The set $\mathbf{G}_{d-1}$ is as usual identified with the set of all orthogonal projection matrices onto $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$, i.e., $d \times d$ symmetric matrices A such that $A^2 = A$ and $\mathrm{tr}(A) = d-1$, in other words, matrices of the form

$$A = \mathrm{Id} - e \otimes e$$

for some $e \in \mathbf{S}^{d-1}$.

A $(d-1)$-varifold in X (a locally compact subset of $\mathbb{R}^d$) is a bounded Radon measure on $X \times \mathbf{G}_{d-1}$. The class of $(d-1)$-varifold in X is denoted by $\mathbf{V}_{d-1}(X)$. The mass of $V \in \mathbf{V}_{d-1}(X)$ is simply the measure $\|V\| \in \mathbb{M}(X)$ defined by $\|V\|(B) = V(B \times \mathbf{G}_{d-1})$ for all Borel sets $B \subset X$. We define the first variation of an $(d-1)$-varifold in V in an open set $U \subset \mathbb{R}^d$ by

$$\delta V(\varphi) = \int_{U \times \mathbf{G}_{d-1}} D\varphi(x) : A dV(x, A) \quad \text{for all } \varphi \in \mathcal{C}^1_c(U; \mathbb{R}^d).$$

We say that an $(d-1)$-varifold is stationary in U if $\delta V(\varphi) = 0$ for all φ in $\mathcal{C}^1_c(U; \mathbb{R}^d)$. We recall that such a varifold satisfies the monotonicity formula

$$\frac{\|V\|(B_\varrho(x_0))}{\varrho^{d-1}} = \frac{\|V\|(B_r(x_0))}{r^{d-1}} + \int_{(B_\varrho(x_0) \setminus B_r(x_0)) \times \mathbf{G}_{d-1}} \frac{|P_{A^\perp}(x - x_0)|^2}{|x - x_0|^{d+1}} dV(x, A) \tag{3.1.7}$$

for all $x_0 \in U$ and $0 < r < \varrho$ with $B_\varrho(x_0) \subset U$, where $P_{A^\perp}$ is the orthogonal projection onto the one-dimensional space $A^\perp$ (see [Sim83, paragraph 40]).

3.2 Formulation of the problem and statements of the main result

Throughout the paper, Ω is a bounded open subdomain of $\mathbb{R}^2$ with $\mathcal{C}^{2,1}$ boundary. We denote by ν the outward unit normal field on $\partial\Omega$.

We recall that the Ambrosio-Tortorelli functional is defined for $(u, v) \in \mathcal{A}_g$ by (3.1.4), where $\mathcal{A}_g$ is given by (3.1.3) and $\eta_\varepsilon > 0$ is a positive small parameter such that $\eta_\varepsilon/\varepsilon \to 0$. We also recall that the Mumford-Shah functional is defined by (3.1.2). A critical point $(u_\varepsilon, v_\varepsilon)$ of AT_ε with a prescribed Dirichlet boundary condition is defined in (3.1.6). It corresponds to a zero of the outer variations of AT_ε. It is immediate to check that $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g$ is a weak solution of the following system of elliptic partial differential equations:

$$\begin{cases} -\varepsilon \Delta v_\varepsilon + \dfrac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon |\nabla u_\varepsilon|^2 = 0 & \text{in } \Omega, \\ \mathrm{div}\,((\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon) = 0 & \text{in } \Omega, \\ u = g, \quad v = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.2.1}$$

Whereas the notion of critical points for the AT functional is fairly standard, the notion of critical points for the MS energy is more involved, see [AFP00, Section 7.4]. We first observe that considering the outer variations of MS is not sufficient to obtain a proper notion of critical point for MS. Indeed, outer variations of u with respect to a smooth direction leaves the surface energy $\mathcal{H}^1(J_u)$ unchanged, so that such variations do not bring any information on the discontinuity set J_u. In order to complete the equations derived from the outer variations, we consider *inner variations* of the MS energy. Such variations correspond to deformations of the domain Ω. To define the inner variations of MS up to the boundary, in addition to requiring $\partial\Omega$ to be of class $\mathcal{C}^{2,1}$ we also ask the boundary data g to belong to the Hölder space $\mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$. These assumptions are needed to invoke the boundary regularity results of [BMR24, Theorem 3.2] which ensures that the weak solutions $(u_\varepsilon, v_\varepsilon)$ of (3.2.1) are actually classical solutions and belong to $[\mathcal{C}^{2,\alpha}(\overline{\Omega})]^2$. Note that, it is necessary, in this type of variational problems to have some regularity results at our disposal in order to relate inner and outer variations, cf. e.g. [LS19, Section 2].

Definition 3.2.1. *Let $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ be a vector field satisfying $X \cdot \nu = 0$ on $\partial\Omega$. The* flow map *$\Phi_t(x)$ of X, defined for every x in $\mathbb{R}^2$, is the unique solution of the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}\Phi_t(x) = X(\Phi_t(x)) & \text{for all } t \in \mathbb{R}, \\ \Phi_0(x) = x. \end{cases}$$

Thanks to the Cauchy-Lipschitz Theorem, it can be shown that for all $x \in \mathbb{R}^2$, the map $t \in \mathbb{R} \mapsto \Phi_t(x)$ is globally well-defined, that $(t,x) \mapsto \Phi_t(x)$ belongs to $\mathcal{C}^1(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^2)$ and that $\{\Phi_t\}_{t\in\mathbb{R}}$ is a one-parameter group of $\mathcal{C}^1$-diffeomorphisms of $\mathbb{R}^2$ with $\Phi_0 = \mathrm{Id}$. Furthermore, for each $t > 0$, we can check that Φ_t is a $\mathcal{C}^1$-diffeomorphism of $\overline{\Omega}$ which preserves both Ω and $\partial\Omega$.

In the sequel, we consider an arbitrary extension G of $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$, which we assume to belong to $\mathcal{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$. As already observed in [BMR24], the notion of inner variations introduced below a priori depends on the choice of the extension G. However, we do not explicitly highlight this dependence in order not to overburden notation.

Definition 3.2.2. *Let $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ be such that $X\cdot\nu = 0$ on $\partial\Omega$ and $\{\Phi_t\}_{t\in\mathbb{R}}$ be its flow map. For all $u \in SBV^2(\Omega)$, the* first inner variation of MS *at u in the direction X is defined by*

$$\delta MS(u)[X] := \lim_{t\to 0} \frac{MS(u\circ\Phi_t^{-1} - G\circ\Phi_t + G) - MS(u)}{t}.$$

In order to define outer variations, we introduce the following notation. Given $\varphi \in SBV^2(\Omega)$, we set $\widehat{J_\varphi} = J_\varphi \cup (\partial\Omega \cap \{\varphi \neq g\})$ where we still denote by φ the inner trace of φ on $\partial\Omega$. In other words, setting $\hat{\varphi} := \varphi \mathbf{1}_\Omega + G\mathbf{1}_{\mathbb{R}^2\setminus\Omega} \in SBV^2_{\mathrm{loc}}(\mathbb{R}^2)$, then $\widehat{J_\varphi} = J_{\hat{\varphi}}$.

Definition 3.2.3. *Let $\varphi \in SBV^2(\Omega)$ be such that $\widehat{J_\varphi} \subset \widehat{J_u}$. The* first outer variation of MS *is defined by*

$$dMS(u)[\varphi] := \lim_{t\to 0} \frac{MS(u+t\varphi) - MS(u)}{t}. \tag{3.2.2}$$

Observe that tests function are taken in $SBV^2(\Omega)$ and not only in $\mathcal{C}^\infty_c(\Omega)$. The assumption $\widehat{J_\varphi} \subset \widehat{J_u}$ ensures the existence of the limit in (3.2.2).

Definition 3.2.4. *A function $u \in SBV^2(\Omega)$ is a* critical point *of MS if it satisfies the following two conditions:*

$$\begin{cases} dMS(u)[\varphi] = 0 & \text{for all } \varphi \in SBV^2(\Omega) \text{ such that } \widehat{J_\varphi} \subset \widehat{J_u}, \\ \delta MS(u)[X] = 0 & \text{for all } X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2) \text{ such that } X\cdot\nu = 0 \text{ on } \partial\Omega. \end{cases}$$

Computing the inner and the outer variations of MS, cf. e.g. [BMR24, Appendix A], we find that a critical point of MS satisfies

$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx = 0 \quad \text{for all } \varphi \in SBV^2(\Omega) \text{ such that } \widehat{J_\varphi} \subset \widehat{J_u} \tag{3.2.3}$$

and

$$\int_\Omega (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : DX \, dx + \int_{\widehat{J_u}} (\mathrm{Id} - \nu_u \otimes \nu_u) : DX \, d\mathcal{H}^1$$
$$= -2\int_{\partial\Omega} (\nabla u \cdot \nu)(X\cdot\nabla g)\, d\mathcal{H}^1 \text{ for all } X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2) \text{ with } X\cdot\nu = 0 \text{ on } \partial\Omega.$$

Note that if $\widehat{J_u}$ is smooth enough, (3.2.3) translates into $\Delta u = 0$ in $\Omega \setminus J_u$ and $\partial_{\nu_u} u = 0$ on $\widehat{J_u}$ (see [AFP00, Section 7.4]).

One can also define the inner variations of AT_ε.

Definition 3.2.5. *Let $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ be such that $X \cdot \nu = 0$ on $\partial\Omega$ and let Φ_t be the flow map associated to X. The first inner variation of AT_ε at (u,v) is defined by*

$$\delta AT_\varepsilon(u,v)[X] = \lim_{t\to 0} \frac{AT_\varepsilon(u\circ\Phi_t^{-1} - G\circ\Phi_t^{-1} + G, v\circ\Phi_t^{-1}) - AT_\varepsilon(u,v)}{t}.$$

According to [BMR24, Theorem 3.1], if $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g$ is a critical point of AT_ε in the sense of (3.2.1), then $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^\infty(\Omega)]^2$. Moreover, since we assumed that the boundary data g belongs to $\mathcal{C}^{2,\alpha}(\partial\Omega)$ and $\partial\Omega$ to be of class $\mathcal{C}^{2,1}$, we deduce from [BMR24, Theorem 3.2] that $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^{2,\alpha}(\overline{\Omega})]^2$. This regularity property ensures that $(u_\varepsilon, v_\varepsilon)$ is also a zero of the inner variations, i.e., $\delta AT_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$, cf. e.g. [LS19, Corollary 2.3] or [BMR24, Proposition 4.2]. Computing explicitly the inner variations of AT_ε (see [BMR24, Lemma A.3]), we find that a critical point $(u_\varepsilon, v_\varepsilon)$ of AT_ε also satisfies

$$\begin{aligned}\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)\left(2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \mathrm{Id}\right) : DX\, dx \\ + \int_\Omega \left[2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathrm{Id}\right] : DX\, dx \\ = 2(\eta_\varepsilon + 1)\int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X\cdot\nabla g)\, d\mathcal{H}^1 \quad (3.2.4)\end{aligned}$$

for all $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ such that $X\cdot\nu = 0$ on $\partial\Omega$.

We are now in position to state the main result of this work.

Theorem 3.2.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary of class $\mathcal{C}^{2,1}$ and $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε in the sense of (3.2.1). Suppose that the sequence $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ satisfies the energy bound*

$$\sup_{\varepsilon>0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty. \quad (3.2.5)$$

Then, up to extraction, we have that

$$(u_\varepsilon, v_\varepsilon) \xrightarrow[\varepsilon\to 0]{[L^2(\Omega)]^2} (u, 1) \text{ with } u \in SBV^2(\Omega) \text{ and } \mathrm{div}(\nabla u) = 0 \text{ in } \mathcal{D}'(\Omega). \quad (3.2.6)$$

If we further assume the following phase-field energy convergence

$$\int_\Omega \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right) dx \to \mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega \cap \{u \neq g\}) = \mathcal{H}^1(\widehat{J_u}), \quad (3.2.7)$$

then

$$\begin{aligned}\int_\Omega (|\nabla u|^2\mathrm{Id} - 2\nabla u\otimes\nabla u) : DX\, dx + \int_{\widehat{J_u}} (\mathrm{Id} - \nu_u\otimes\nu_u) : DX\, d\mathcal{H}^1 \\ = -2\int_{\partial\Omega} (\nabla u\cdot\nu)(X\cdot\nabla g)\, d\mathcal{H}^1 \text{ for all } X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2) \text{ with } X\cdot\nu = 0 \text{ on } \partial\Omega. \quad (3.2.8)\end{aligned}$$

Note that the first point of (3.2.6) follows from the compactness part of the Γ-convergence theory for AT_ε, see [AT92, BMR24]. The second point of (3.2.6) is also a rather direct consequence of the Γ-convergence theory for AT_ε. Indeed from this theory we obtain the weak convergence $(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \rightharpoonup \nabla u$ in $L^2(\Omega;\mathbb{R}^2)$, and we can pass to the limit in the second equation of (3.2.1) to obtain $\mathrm{div}(\nabla u) = 0$ in $\mathcal{D}'(\Omega)$. Note that this condition differs from (3.2.3) because test functions are smooth, and not discontinuous across $\widehat{J_u}$. Hence, this condition does not suffice to claim that u is a critical point of MS for the outer variations because the Neumann condition on $\widehat{J_u}$, even in a weak sense, is not satisfied. We leave as an open problem to determine if (3.2.3) is satisfied by limits of critical points of the AT energy satisfying the assumptions of Theorem 2.2.7, see also Remark 3.4.2. Finally, the validity of condition (3.2.8) actually shows that u a critical point of MS for the inner variations.

We point out that, whereas the main assumption in [BMR24, Theorem 1.2] was

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \xrightarrow[\varepsilon\to 0]{} MS(u)$$

we are able to obtain the same conclusion with the weaker assumption (3.2.7). However our result is restricted to the two-dimensional case. We would like to stress that this result does not amount to proving the convergence of the whole energy from the only convergence of the phase-field variable. In fact, we will see in Proposition 3.4.1 that the elastic term $(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2$ of AT_ε does not necessarily converge to the elastic term $|\nabla u|^2$ of MS in the sense of measures, and that a singular defect measure might arise. In particular, the lack of strong $L^2(\Omega;\mathbb{R}^2)$-convergence of $\{(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon\}_{\varepsilon>0}$ to ∇u is an obstacle to pass to the limit in the second inner variation as in [BMR24, Theorem 1.3].

To complete this section, let us explain the strategy of proof of Theorem 3.2.6 and the organization of the paper. As expected, one attempts to pass to the limit in (3.2.4) as $\varepsilon \to 0$. To do so, we first use some consequences of the compactness in the Γ-convergence theory for AT_ε to MS, and of the convergence of the phase-field energy (3.2.7) in Section 3.3. They lead to the so-called equi-partition of the (phase-field) energy principle and the identification of the limiting measure of the phase-field density as the one-dimensional Hausdorff measure restricted to the jump set $\widehat{J_u}$. We also explain how to associate a natural varifold to the phase-field variable v_ε following the ideas of [Ton02]. In Section 3.4, we present one of our new ingredients which consists in a finer analysis of the sequence $\{(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \otimes \nabla u_\varepsilon\}_{\varepsilon>0}$. Assumption (3.2.5) implies that this sequence is bounded, so that, up to an extraction, there exists a matrix-valued measure μ such that

$$(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mu \quad \text{ in } \mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{3.2.9}$$

In Propositions 3.4.1 and 3.4.2, we perform a fine analysis of the defect measure μ by showing that its Lebesgue-absolutely continuous part is given by $\nabla u \otimes \nabla u \mathcal{L}^2 \llcorner \Omega$. This is achieved by means of a blow-up argument around convenient approximate differentiability points of u, which allows us to reduce to the case where the limit function u is actually an affine function. Then the convexity of AT_ε with respect to the phase-field variable enables one to use the equivalence between minimality and criticality with respect to v and then, improve in that specific case the weak L^2-convergence of the term $\{\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon\}_{\varepsilon>0}$ to ∇u into a strong L^2-convergence. We observe that the blow-up argument requires the L^2-approximate differentiability of BV functions which holds only in dimension 2. This is one of the reason why our proof is restricted to the dimension $N = 2$. Finally Section 3.5 completes the proof of the inner variations convergence thanks to a result of De Phillipis-Rindler [DPR16] on the singular part of measures satisfying a linear PDE. To this aim, we introduce the stress-energy tensor

$$T_\varepsilon := (\eta_\varepsilon + v_\varepsilon^2)(2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \mathrm{Id}) + 2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathrm{Id}.$$

According to (3.2.4), T_ε is a divergence measure matrix-valued function in Ω. The energy bound (3.2.5) ensures that $T_\varepsilon \overset{\star}{\rightharpoonup} T$ weakly* as measures in Ω, for some $T \in \mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ which remains divergence free. Applying the result of [DPR16] implies that the singular part of T has a polar taking values into the set of singular matrices. In particular, it ensures that the singular part of T with respect to the Lebesgue measure must be absolutely continuous with respect to $\mathcal{H}^1 \llcorner \widehat{J_u}$. The density of T with respect to $\mathcal{H}^1 \llcorner \widehat{J_u}$ is then characterized by means of a blow-up argument. We then employ the previous information on the absolutely continuous part of μ with respect to the Lebesgue measure to deduce that the limit map u in Theorem 3.2.6 does satisfy (3.2.8).

3.3 Preliminary results

We first show that the energy bound assumption (3.2.5) provides compactness properties leading to (3.2.6).

Lemma 3.3.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{A}_g$ be a sequence of critical points of AT_ε satisfying (3.2.5). Then, up to a subsequence (not relabeled), there exists $u \in SBV^2(\Omega)$ such that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $[L^2(\Omega)]^2$,*

$$\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2(\Omega;\mathbb{R}^2). \tag{3.3.1}$$

and

$$\mathrm{div}(\nabla u) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. First of all, the energy bound immediately implies that $v_\varepsilon \to 1$ in $L^2(\Omega)$.

Next according to the maximum principle (see [BMR24, Lemma 3.1]), we have $0 \le v_\varepsilon \le 1$ in Ω, and $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$. In particular, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega)$.

Let $0 < a < b < 1$, we use Young's inequality and the coarea formula to get that

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \ge \int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|^2 \, dx + \int_\Omega (1 - v_\varepsilon) |\nabla v_\varepsilon| \, dx \tag{3.3.2}$$

$$\ge \int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|^2 \, dx + \int_a^b (1-t) \mathcal{H}^1(\partial^\star \{v_\varepsilon > t\}) \, dt. \tag{3.3.3}$$

By the mean value Theorem, there exists $t_\varepsilon \in (a, b)$ such that the set $A_\varepsilon = \{v_\varepsilon > t_\varepsilon\}$ has finite perimeter in Ω and

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \ge t_\varepsilon^2 \int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \left[\frac{(1-a)^2}{2} - \frac{(1-b)^2}{2} \right] \mathcal{H}^1(\partial^\star A_\varepsilon).$$

Let $\hat{u}_\varepsilon := u_\varepsilon \mathbf{1}_{A_\varepsilon}$. According to [AFP00, Theorem 3.84], $\hat{u}_\varepsilon \in SBV^2(\Omega)$ with $\nabla \hat{u}_\varepsilon = \nabla u_\varepsilon \mathbf{1}_{A_\varepsilon}$ and $J_{\hat{u}_\varepsilon} \subset \partial^\star A_\varepsilon$ so that, using $t_\varepsilon > a$,

$$\int_\Omega |\nabla \hat{u}_\varepsilon|^2 \, dx + \mathcal{H}^1(J_{\hat{u}_\varepsilon}) \le C_{a,b},$$

for some constant $C_{a,b} > 0$. Moreover, recalling that $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$, it follows that the sequence $\{\hat{u}_\varepsilon\}_{\varepsilon>0}$ is also bounded in $L^\infty(\Omega)$. We are thus in position to apply Ambrosio's compactness Theorem (Theorem 4.8 in [AFP00]) which ensures, up to the extraction of a subsequence (not relabelled), the existence of $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ such that $\hat{u}_\varepsilon \to u$ in $L^2(\Omega)$, $\|u\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$ and $\nabla \hat{u}_\varepsilon \rightharpoonup \nabla u$ in $L^2(\Omega; \mathbb{R}^2)$.

According to the Chebychev inequality, we have

$$\mathcal{L}^2(\Omega \setminus A_\varepsilon) \le \frac{1}{(1-t_\varepsilon)^2} \int_\Omega (1 - v_\varepsilon)^2 \, dx \le \frac{1}{(1-a)^2} \int_\Omega (1 - v_\varepsilon)^2 \, dx \to 0,$$

where we used the energy bound (3.2.5). As a consequence,

$$\|u_\varepsilon - u\|_{L^2(\Omega)} \le \|u_\varepsilon - \hat{u}_\varepsilon\|_{L^2(\Omega)} + \|\hat{u}_\varepsilon - u\|_{L^2(\Omega)} \le 2\|g\|_{L^\infty(\partial\Omega)} \mathcal{L}^2(\Omega \setminus A_\varepsilon) + \|\hat{u}_\varepsilon - u\|_{L^2(\Omega)} \to 0,$$

hence $u_\varepsilon \to u$ in $L^2(\Omega)$. Similarly, for all $\varphi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^2)$, we have

$$\int_\Omega (\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon - \nabla u) \cdot \varphi \, dx = \int_\Omega (\nabla \hat{u}_\varepsilon - \nabla u) \cdot \varphi \, dx + \int_{\Omega \setminus A_\varepsilon} \sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \cdot \varphi \, dx + \int_{A_\varepsilon} (\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon - \nabla u_\varepsilon) \cdot \varphi \, dx. \tag{3.3.4}$$

On the one hand, we have

$$\int_\Omega (\nabla \hat{u}_\varepsilon - \nabla u) \cdot \varphi \, dx \to 0. \tag{3.3.5}$$

On the other hand, thanks to the Cauchy-Schwarz inequality,

$$\left| \int_{\Omega \setminus A_\varepsilon} \sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \cdot \varphi \, dx \right| \le \|\varphi\|_{L^\infty(\Omega)} \left\| \sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \right\|_{L^2(\Omega)} \mathcal{L}^2(\Omega \setminus A_\varepsilon)^{1/2} \to 0, \tag{3.3.6}$$

where we used once more the energy bound. Finally, since $v_\varepsilon > t_\varepsilon > a > 0$ in A_ε and $\{\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega; \mathbb{R}^2)$, we infer that $\{\|\nabla u_\varepsilon\|_{L^2(A_\varepsilon)}\}_{\varepsilon>0}$ is bounded. Thus, using again the Cauchy-Schwarz inequality and the energy bound, we get that

$$\left| \int_{A_\varepsilon} (\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon - \nabla u_\varepsilon) \cdot \varphi \, dx \right| \le \|\varphi\|_{L^\infty(\Omega)} \|\nabla u_\varepsilon\|_{L^2(A_\varepsilon)} \left\| \sqrt{\eta_\varepsilon + v_\varepsilon^2} - 1 \right\|_{L^2(\Omega)} \to 0. \tag{3.3.7}$$

Gathering (3.3.4)–(3.3.7) together with the boundedness of $\{\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega; \mathbb{R}^2)$, we conclude that $\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$.

Since $\sqrt{\eta_\varepsilon + v_\varepsilon^2} \to 1$ in $L^2(\Omega)$ and $\operatorname{div}((\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon) = 0$ in $\mathcal{D}'(\Omega)$, we deduce that $\operatorname{div}(\nabla u) = 0$ in $\mathcal{D}'(\Omega)$. □

The phase-field energy convergence assumption (3.2.7) has several consequences, the main one being the so called equi-partition of the energy (see [BMR24, Proposition 5.1]).

Proposition 3.3.2. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε satisfying the assumptions of Theorem 3.2.6. We define the discrepancy*

$$\xi_\varepsilon = \varepsilon|\nabla v_\varepsilon|^2 - \frac{(1-v_\varepsilon)^2}{4\varepsilon}.$$

Then $\xi_\varepsilon \to 0$ in $L^1(\Omega)$ and setting

$$w_\varepsilon = \Phi(v_\varepsilon) \quad \text{with } \Phi(t) = t - t^2/2, \tag{3.3.8}$$

then

$$|\nabla w_\varepsilon|\mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mathcal{H}^1 \llcorner \widehat{J_u} \quad \text{weakly}^\star \text{ in } \mathcal{M}(\overline{\Omega}).$$

The equi-partition of the energy is crucial in [BMR24] to pass to the limit in (3.2.4). Combining [AFP00, Proposition 1.80] and (3.2.7), we find the following convergence.

Proposition 3.3.3. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε satisfying the assumptions of Theorem 3.2.6. Then*

$$\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mathcal{H}^1 \llcorner \widehat{J_u} \quad \text{weakly}^\star \text{ in } \mathcal{M}(\overline{\Omega}).$$

The energy bound also implies a weak-convergence result for the boundary terms, which corresponds to [BMR24, Lemma 4.1].

Lemma 3.3.4. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε satisfying the assumptions of Theorem 3.2.6. Up to a further subsequence (not relabeled), $\partial_\nu u_\varepsilon \rightharpoonup \nabla u \cdot \nu$ in $L^2(\partial\Omega)$. Moreover, there exists a non-negative boundary Radon measure $m \in \mathcal{M}(\partial\Omega)$ such that*

$$\left[|\partial_\nu u_\varepsilon|^2 + \varepsilon|\partial_\nu v_\varepsilon|^2\right]\mathcal{H}^1 \llcorner \partial\Omega \overset{\star}{\rightharpoonup} m \quad \text{weakly}^\star \text{ in } \mathcal{M}(\partial\Omega).$$

The results concerning the measure μ and the passage to the limit in the inner variations of AT_ε find a better formulation in terms of varifolds. We now recall the expression of the varifold associated to the phase-field function v_ε, and some of the convergence results established in [BMR24].

Definition 3.3.5. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence satisfying $\sup_\varepsilon AT_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$ and let w_ε be defined by (3.3.8). We introduce the varifold $V_\varepsilon \in \mathcal{M}(\overline{\Omega} \times \mathbf{G}_1)$ by*

$$\langle V_\varepsilon, \varphi\rangle = \int_{\Omega\cap\{\nabla w_\varepsilon \neq 0\}} \varphi\left(x, \mathrm{Id} - \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\right)|\nabla w_\varepsilon|\, dx \quad \text{for all } \varphi \in \mathcal{C}(\overline{\Omega} \times \mathbf{G}_1). \tag{3.3.9}$$

Definition 3.3.9 implies that the weight measure of V_ε is given by $\|V_\varepsilon\| = |\nabla w_\varepsilon|\mathcal{L}^2 \llcorner \Omega$. In [BMR24, Section 5] the following convergence result of Proposition 3.3.6 is established. It is an immediate consequence of the energy bound (3.2.5), the weak* compactness of bounded sequences of Radon measures and the Disintegration Theorem (see e.g. [AFP00, Theorem 2.28])

Proposition 3.3.6. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence satisfying $\sup_\varepsilon AT_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$, and let V_ε be the varifold defined in (3.3.9). Up to a subsequence (not relabeled), there exists a varifold $V \in \mathcal{M}(\overline{\Omega} \times \mathbf{G}_1)$ such that*

$$V_\varepsilon \overset{\star}{\rightharpoonup} V \quad \text{weakly}^\star \text{ in } \mathcal{M}(\overline{\Omega} \times \mathbf{G}_1). \tag{3.3.10}$$

Moreover, there exists a weak* $(\mathcal{H}^1 \llcorner \widehat{J_u})$-measurable mapping $x \mapsto V_x \in \mathcal{M}(\mathbf{G}_1)$ of probability measures such that $V = (\mathcal{H}^1 \llcorner \widehat{J_u}) \otimes V_x$, i.e.

$$\int_{\overline{\Omega}\times\mathbf{G}_1} \varphi(x,A)\, dV(x,A) = \int_{\widehat{J_u}} \left(\int_{\mathbf{G}_1} \varphi(x,A)\, dV_x(A)\right) d\mathcal{H}^1(x) \quad \text{for all } \varphi \in \mathcal{C}(\overline{\Omega} \times \mathbf{G}_1).$$

Using the disintegrated structure of V, we have an alternative expression of the first variation of the varifold V.

Proposition 3.3.7. *Let $V \in \mathcal{M}(\overline{\Omega} \times G_1)$ be the varifold as in Proposition 3.3.6. For $\mathcal{H}^1$-a.e $x \in \widehat{J_u}$, we define the first moment of V_x as*

$$\overline{A}(x) := \int_{\mathbf{G}_1} A \, dV_x(A). \tag{3.3.11}$$

Then, the matrix $\overline{A}(x) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ is an orthogonal projector for $\mathcal{H}^1$-a.e x in $\widehat{J_u}$ and[1]

$$\delta V(X) = \int_{\widehat{J_u}} \overline{A}(x) : DX(x) \, d\mathcal{H}^1(x) \quad \text{for all } X \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^2). \tag{3.3.12}$$

Moreover, for $\mathcal{H}^1$-a.e $x \in \widehat{J_u}$, we have $V_x = \delta_{\overline{A}(x)}$.

Proof. According to [BMR24, Lemma 5.2], for $\mathcal{H}^1$-a.e $x \in \widehat{J_u}$, the matrix $\overline{A}(x) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ satisfies $\mathrm{Tr}(\overline{A}(x)) = 1$ and $\rho(\overline{A}(x)) = 1$, where ρ denotes the spectral radius. In dimension two, those conditions imply that the eigenvalues of $\overline{A}(x)$ are exactly 0 and 1, which ensures that $\overline{A}(x)$ is an orthogonal projector.

Concerning the representation of the first variation of V, we consider a test function $X \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^2)$. According to Proposition 3.3.6 with $\varphi(x, A) = DX(x) : A$ together with Fubini's Theorem, we infer that

$$\begin{aligned}\delta V(X) = \int_{\overline{\Omega}\times\mathbf{G}_1} DX(x) : A \, dV(x, A) &= \int_{\widehat{J_u}} \left(\int_{\mathbf{G}_1} DX(x) : A \, dV_x(A) \right) d\mathcal{H}^1(x) \\ &= \int_{\widehat{J_u}} \left(\int_{\mathbf{G}_1} A \, dV_x(A) \right) : DX(x) \, d\mathcal{H}^1(x),\end{aligned}$$

which corresponds to (3.3.12) by definition (3.3.11) of $\overline{A}$.

Since any element $A \in \mathbf{G}_1$ satisfies $|A| = 1$ and V_x is a probability measure over $\mathbf{G}_1$ for $\mathcal{H}^1$-a.e. $x \in \widehat{J_u}$, we infer that the variance of V_x is zero, i.e.

$$\int_{\mathbf{G}_1} |\overline{A}(x) - A|^2 \, dV_x(A) = 0,$$

and thus, $A = \overline{A}(x)$ for V_x-a.e $A \in \mathbf{G}_1$. As a consequence, V_x is a probability measure concentrated at $\{\overline{A}(x)\}$, hence the Dirac measure at $\overline{A}(x)$. □

In higher dimension, Proposition 3.3.7 remains true under the hypothesis (3.2.7). A direct consequence is that the probability measure V_x is actually a Dirac mass concentrated at $\overline{A}(x)$.

3.4 Analysis of the defect measure

The aim of this section is to prove the following result, which is independent of the phase-field energy convergence assumption.

Proposition 3.4.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of critical points of the Ambrosio-Tortorelli functional such that*

$$\sup_{\varepsilon>0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty.$$

Then, there exist a subsequence (not relabeled) and a non-negative measure $\sigma \in \mathcal{M}(\Omega)$ which is singular with respect to the Lebesgue measure, such that

$$(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega + \sigma \quad \text{weakly}^\star \text{ in } \mathcal{M}(\Omega). \tag{3.4.1}$$

[1] the first variation of a varifold is usually defined on an open set. We extend this definition to treat the boundary here.

Proof. Since the families $\{(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2\}_{\varepsilon>0}$ and $\{\varepsilon|\nabla v_\varepsilon|^2 + (1-v_\varepsilon)^2/4\varepsilon\}_{\varepsilon>0}$ are bounded in $L^1(\Omega)$ by the energy bound (3.2.5), up to a subsequence (not relabelled), there exist nonnegative measures λ_1 and $\lambda_2 \in \mathcal{M}(\Omega)$ such that

$$(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \lambda_1 \quad \text{weakly}^\star \text{ in } \mathcal{M}(\Omega)$$

and

$$\left(\varepsilon|\nabla v_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon}\right)\mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \lambda_2 \quad \text{weakly}^\star \text{ in } \mathcal{M}(\Omega).$$

On the other hand, since from Lemma 3.3.1 we know that $\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$, it results from the weak lower semi-continuity of the norm that $\lambda_1 \geqslant |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega$. Thus the measure $\sigma = \lambda_1 - |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega$ is nonnegative, and it remains to show that it is singular with respect to $\mathcal{L}^2$. To this aim, by Lebesgue's differentiation Theorem, it is enough to prove that

$$\frac{d\lambda_1}{d\mathcal{L}^2} = |\nabla u|^2 \quad \mathcal{L}^2\text{-a.e. in } \Omega. \tag{3.4.2}$$

We write $\lambda_1 = \frac{d\lambda_1}{d\mathcal{L}^2}\mathcal{L}^2 + \lambda_1^s$ with λ_1^s which is singular with respect to the Lebesgue measure. Let $x_0 \in \Omega$ be

(i) such that $\frac{d\lambda_1}{d\mathcal{L}^2}(x_0)$ and $\frac{d\lambda_2}{d\mathcal{L}^2}(x_0)$ exist and are finite;

(ii) a Lebesgue point of u, ∇u and $\frac{d\lambda_1}{d\mathcal{L}^2}$;

(iii) such that

$$\lim_{\rho\to 0} \frac{1}{\pi\rho^2} \int_{B_\rho(x_0)} \frac{|u(y) - u(x_0) - \nabla u(x_0)\cdot(y-x_0)|^2}{\rho^2}\, dy = 0; \tag{3.4.3}$$

(iv) such that $\frac{d\lambda_1^s}{d\mathcal{L}^2}(x_0) = 0$.

Note that $\mathcal{L}^2$ almost every points x_0 in Ω fullfil these properties by the Lebesgue differentiation Theorem, the L^2-differentiability a.e. of BV functions cf. [EG15, Theorem 1, Section 6.1] and the fact that λ_1^s is singular with respect to $\mathcal{L}^2$. We point out that the L^2-differentiability a.e. of BV functions holds only in dimension 2 which is one of the reasons of our dimensional restriction.

Let $R > 0$ be such that $B_R(x_0) \subset\subset \Omega$, and $\{\varrho_j\}_{j\in\mathbb{N}} \searrow 0^+$ be an infinitesimal sequence of radii such that $\rho_j < R$ for all $j \in \mathbb{N}$ and

$$\lambda_1(\partial B_{\rho_j}(x_0)) + \lambda_2(\partial B_{\rho_j}(x_0)) = 0. \tag{3.4.4}$$

For all y in B_1, we define the rescaled functions.

$$\hat{v}_{\varepsilon,j}(y) = v_\varepsilon(x_0 + \rho_j y), \quad \hat{u}_{\varepsilon,j}(y) = \frac{u_\varepsilon(x_0 + \rho_j y) - u(x_0)}{\rho_j}.$$

Then, since $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $[L^2(\Omega)]^2$,

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0} \int_{B_1} |\hat{v}_{\varepsilon,j} - 1|^2\, dy = \lim_{j\to\infty}\lim_{\varepsilon\to 0} \frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)} |v_\varepsilon - 1|^2\, dx = 0,$$

while

$$\begin{aligned}
&\lim_{j\to\infty}\lim_{\varepsilon\to 0} \int_{B_1} |\hat{u}_{\varepsilon,j}(y) - \nabla u(x_0)\cdot y|^2\, dy \\
&\qquad = \lim_{j\to\infty} \int_{B_1} \left|\frac{u(x_0+\rho_j y) - u(x_0)}{\rho_j} - \nabla u(x_0)\cdot y\right|^2 dy \\
&\qquad\qquad = \lim_{j\to\infty} \frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)} \frac{|u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)|^2}{\rho_j^2}\, dx = 0,
\end{aligned}$$

where we used (3.4.3). Moreover, for all $\zeta \in \mathcal{C}_c(B_1)$,

$$\begin{aligned}\lim_{j\to\infty}\lim_{\varepsilon\to 0}\int_{B_1}(\eta_\varepsilon+\hat v_{\varepsilon,j}^2)|\nabla \hat u_{\varepsilon,j}|^2\zeta\,dy&\\ &=\lim_{j\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}(\eta_\varepsilon+v_\varepsilon(x)^2)|\nabla u_\varepsilon(x)|^2\zeta\left(\frac{x-x_0}{\rho_j}\right)dx\\ &=\lim_{j\to\infty}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)d\lambda_1(x).\end{aligned}$$

Condition (iv) implies that

$$\frac{1}{\rho_j^2}\left|\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)d\lambda_1^s\right|\leqslant\|\zeta\|_{L^\infty(B_1)}\frac{\lambda_1^s(B_{\rho_j}(x_0))}{\rho_j^2}\underset{j\to+\infty}{\longrightarrow}0.$$

Hence, using item (ii),

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)d\lambda_1(x)&=\lim_{j\to\infty}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)\frac{d\lambda_1}{d\mathcal{L}^2}(x)\,dx\\ &=\frac{d\lambda_1}{d\mathcal{L}^2}(x_0)\int_{B_1}\zeta\,dy.\end{aligned}$$

We next observe that, thanks to (3.4.4), (i) and (iv),

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi\rho_j^2}\int_{B_{\rho_j}(x_0)}(\eta_\varepsilon+v_\varepsilon^2)|\nabla u_\varepsilon|^2\,dx=\lim_{j\to\infty}\frac{\lambda_1(B_{\rho_j}(x_0))}{\pi\rho_j^2}=\frac{d\lambda_1}{d\mathcal{L}^2}(x_0)$$

and

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi\rho_j^2}\int_{B_{\rho_j}(x_0)}\left(\varepsilon|\nabla v_\varepsilon|^2+\frac{(v_\varepsilon-1)^2}{4\varepsilon}\right)dx=\lim_{j\to\infty}\frac{\lambda_2(B_{\rho_j}(x_0))}{\pi\rho_j^2}=\frac{d\lambda_2}{d\mathcal{L}^2}(x_0),$$

while, changing variables

$$\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}(\eta_\varepsilon+v_\varepsilon^2)|\nabla u_\varepsilon|^2\,dx=\int_{B_1}(\eta_\varepsilon+\hat v_{\varepsilon,j}^2)|\nabla\hat u_{\varepsilon,j}|^2\,dy,$$

and

$$\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\left(\varepsilon|\nabla v_\varepsilon|^2+\frac{(v_\varepsilon-1)^2}{4\varepsilon}\right)dx=\frac{1}{\rho_j}\int_{B_1}\left((\varepsilon/\rho_j)|\nabla\hat v_{\varepsilon,j}|^2+\frac{(\hat v_{\varepsilon,j}-1)^2}{4(\varepsilon/\rho_j)}\right)dy.$$

Using a diagonal extraction argument together with the separability of $\mathcal{C}_c(B_1)$, we can find an infinitesimal sequence $\varepsilon_j\to 0$ such that, setting $\hat\varepsilon_j:=\varepsilon_j/\rho_j$, $\hat v_j:=\hat v_{\varepsilon_j,j}$, $\hat u_j:=\hat u_{\varepsilon_j,j}$ and $\hat u(y)=\nabla u(x_0)\cdot y$, then

$$\begin{cases}\hat\varepsilon_j\to 0,\\ (\hat u_j,\hat v_j)\to(\hat u,1)\text{ in }[L^2(B_1)]^2,\\ (\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\mathcal{L}^2\llcorner\Omega\rightharpoonup\frac{d\lambda_1}{d\mathcal{L}^2}(x_0)\mathcal{L}^2\llcorner\Omega\quad\text{weakly}^\star\text{ in }\mathcal{M}(B_1),\end{cases}\tag{3.4.5}$$

and

$$\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\,dy\to\pi\frac{d\lambda_1}{d\mathcal{L}^2}(x_0),\quad\frac{1}{\rho_j}\int_{B_1}\left(\hat\varepsilon_j|\nabla\hat v_j|^2+\frac{(\hat v_j-1)^2}{4\hat\varepsilon_j}\right)dy\to\pi\frac{d\lambda_2}{d\mathcal{L}^2}(x_0).$$

By the classical compactness argument of the Ambrosio-Tortorelli functional (see Lemma 3.3.1), the previous convergences imply that $(\eta_{\varepsilon_j}+\hat v_j^2)^{1/2}\nabla\hat u_j\rightharpoonup\nabla\hat u=\nabla u(x_0)$ weakly in $L^2(B_1;\mathbb{R}^2)$. Now using that $(u_\varepsilon,v_\varepsilon)$ is a critical point of the Ambrosio-Tortorelli functional, we infer that $\operatorname{div}((\eta_{\varepsilon_j}+\hat v_j^2)\nabla\hat u_j)=0$ in B_1. Note that this partial differential equation is equivalent to the minimality property

$$\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\,dy\leq\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla z|^2\,dy\quad\text{for all }z\in\hat u_j+H_0^1(B_1).$$

Let $\varphi \in \mathcal{C}_c^\infty(B_1)$ be a cut-off function with $0 \le \varphi \le 1$ in Ω. Since $\hat{u} \in H^1(B_1)$, we are allowed to take $z = \varphi\hat{u} + (1-\varphi)\hat{u}_j$ as competitor which leads to

$$\int_{B_1} (\eta_{\varepsilon_j} + \hat{v}_j^2)|\nabla \hat{u}_j|^2\, dy \le \int_{B_1} (\eta_{\varepsilon_j} + \hat{v}_j^2)|\varphi\nabla\hat{u} + (1-\varphi)\nabla\hat{u}_j + \nabla\varphi(\hat{u}_j - \hat{u})|^2\, dy.$$

Expanding the square and using that $0 \le \hat{v}_j \le 1$, and that $0 \le \varphi^2 \le \varphi$, we find that

$$\begin{aligned}\int_{B_1} (\eta_{\varepsilon_j} + \hat{v}_j^2)|\nabla \hat{u}_j|^2\, dy \;\le\; & \int_{B_1} \big(\varphi(\eta_{\varepsilon_j}+1)|\nabla\hat{u}|^2 + (1-\varphi)(\eta_{\varepsilon_j} + \hat{v}_j^2)|\nabla\hat{u}_j|^2\big)\, dy \\ & + \int_{B_1} |\nabla\varphi|^2(\hat{u}_j - \hat{u})^2\, dy \\ & + 2\int_{B_1} (\eta_{\varepsilon_j} + \hat{v}_j^2)(\hat{u}_j - \hat{u})(\varphi\nabla\hat{u} + (1-\varphi)\nabla\hat{u}_j)\cdot\nabla\varphi\, dy.\end{aligned}$$

Reorganizing the terms in the right-hand side and recalling that $\nabla\hat{u} = \nabla u(x_0)$, we obtain that

$$\begin{aligned}\int_{B_1} (\eta_{\varepsilon_j} + \hat{v}_j^2)|\nabla \hat{u}_j|^2\varphi\, dy \;\le\; & (\eta_{\varepsilon_j}+1)|\nabla u(x_0)|^2 \int_{B_1} \varphi\, dy + \int_{B_1} |\nabla\varphi|^2(\hat{u}_j - \hat{u})^2\, dy \\ & + 2\int_{B_1} (\eta_{\varepsilon_j} + \hat{v}_j^2)(\hat{u}_j - \hat{u})(\varphi\nabla\hat{u} + (1-\varphi)\nabla\hat{u}_j)\cdot\nabla\varphi\, dy.\end{aligned}$$

Since $\hat{v}_j \to 1$ in $L^2(B_1)$, $\hat{u}_j \to \hat{u}$ in $L^2(B_1)$ and $\{(\eta_{\varepsilon_j} + \hat{v}_j^2)\nabla\hat{u}_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(B_1;\mathbb{R}^2)$, we can pass to the limit and get that

$$\limsup_{j\to\infty} \int_{B_1} (\eta_{\varepsilon_j} + \hat{v}_j^2)|\nabla\hat{u}_j|^2\varphi\, dy \le |\nabla u(x_0)|^2 \int_{B_1} \varphi\, dy.$$

Combining this inequality with the weak $L^2(B_1;\mathbb{R}^2)$ convergence of $\{(\eta_{\varepsilon_j} + \hat{v}_j^2)^{1/2}\nabla\hat{u}_j\}_{j\in\mathbb{N}}$ to $\nabla\hat{u} = \nabla u(x_0)$, we infer that $(\eta_{\varepsilon_j} + \hat{v}_j^2)^{1/2}\nabla\hat{u}_j \to \nabla u(x_0)$ strongly in $L^2_{\rm loc}(B_1)$. Recalling the last convergence in (3.4.5), we deduce that $\frac{d\lambda_1}{d\mathcal{L}^2}(x_0) = |\nabla u(x_0)|^2$, hence we obtain (3.4.2). □

Remark 3.4.1. If $\nabla u \in L^\infty(\Omega;\mathbb{R}^2)$, we have an alternative proof of the previous result based on Anzellotti's duality pairing (see [Anz83]). Indeed, recalling that $\operatorname{div}(\nabla u) = 0$ in $\mathcal{D}'(\Omega)$, we can define as in [Anz83, Definition 1.4] the distribution $[\nabla u \cdot Du] \in \mathcal{D}'(\Omega)$ by

$$\langle [\nabla u \cdot Du], \varphi\rangle := -\int_\Omega u\nabla u\cdot\nabla\varphi\, dx \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\Omega). \tag{3.4.6}$$

By [Anz83, Theorem 1.5 and Theorem 2.4], $[\nabla u \cdot Du]$ extends to a bounded Radon measure in Ω whose absolutely continuous part is given by

$$[\nabla u \cdot Du]^a = |\nabla u|^2\mathcal{L}^2 \llcorner \Omega. \tag{3.4.7}$$

We now make the connexion between the measures λ_1 and $[\nabla u \cdot Du]$. Indeed, let $\varphi \in \mathcal{C}_c^\infty(\Omega)$, taking $u_\varepsilon\varphi$ as test function in the first equation of (3.2.1) yields

$$\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2\varphi\, dx = -\int_\Omega u_\varepsilon(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon\cdot\nabla\varphi\, dx.$$

Since $\{u_\varepsilon\}_{\varepsilon>0}$ and $\{v_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\infty(\Omega)$ and $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in strongly in $[L^2(\Omega)]^2$, we have $u_\varepsilon \to u$ and $(\eta_\varepsilon + v_\varepsilon^2)^{1/2} \to 1$ strongly in $L^4(\Omega)$. Recalling that $(\eta_\varepsilon + v_\varepsilon^2)^{1/2}\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$ and $(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2\mathcal{L}^2\llcorner\Omega \rightharpoonup \lambda_1$ weakly* in $\mathcal{M}(\Omega)$, we obtain

$$\int_\Omega \varphi\, d\lambda_1 = -\int_\Omega u\nabla u\cdot\nabla\varphi\, dx. \tag{3.4.8}$$

In orther words, $\lambda_1 = [\nabla u \cdot Du]$, and thus (3.4.7) implies that $\lambda_1 - |\nabla u|^2\mathcal{L}^2\llcorner\Omega$ is a singular measure with respect to $\mathcal{L}^2$.

Remark 3.4.2. Anzellotti's pairing (3.4.6) is still well-defined if we assume only $\nabla u \in L^2(\Omega;\mathbb{R}^2)$ and $\operatorname{div}(\nabla u) = 0$. In this remark we show that this pairing can be used to give a weak formulation to the criticality condition $\partial_{\nu_u} u = 0$ on J_u, valid for critical points of the MS energy when J_u is smooth enough. We recall that this criticality condition comes from (3.2.3). Now, formally we have that $[\nabla u \cdot Du] = \nabla u \cdot Du = |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega + (u^+ - u^-)\nabla u \cdot \nu_u \mathcal{H}^1 \llcorner J_u$. Hence a possible weak formulation of (3.2.3) could be

$$\begin{cases} \operatorname{div}(\nabla u) = 0 \text{ in } \mathcal{D}'(\Omega) \\ [\nabla u \cdot Du] = |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega. \end{cases}$$

Notice that, in light of (3.4.8), the second condition is fulfilled when $(\eta_\varepsilon + v_\varepsilon^2)^{1/2}\nabla u_\varepsilon \to \nabla u$ strongly in $L^2(\Omega)$ which is the case in the context of [BMR24].

Proposition 3.4.1 allows us to compute the density of μ, the weak* limit of $(\eta_\varepsilon+v_\varepsilon^2)\nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \llcorner \Omega$, with respect to $\mathcal{L}^2$.

Proposition 3.4.2. *There exists a measure $\mu^s \in \mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ which is singular with respect to the Lebesgue measure such that*

$$(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mu = \nabla u \otimes \nabla u \mathcal{L}^2 \llcorner \Omega + \mu^s \quad \textit{weakly}^\star \textit{ in } \mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{3.4.9}$$

Proof. Let σ be defined as in Proposition 3.4.1. Since the family $\{(v_\varepsilon^2 + \eta_\varepsilon)\nabla u_\varepsilon \otimes \nabla u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^1(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ thanks to the energy bound (3.2.5), we infer that, up to a subsequence, $(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mu$ weakly* in $\mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ for some $\mu \in \mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$. Writing the Lebesgue decomposition of μ as $\mu = \mu^a + \mu^s$, we claim that $\mu^a = \nabla u \otimes \nabla u \mathcal{L}^2 \llcorner \Omega$.

For $i = 1, 2$, we denote by $\sigma_i \in \mathcal{M}(\Omega)$ the nonnegative measure such that

$$(v_\varepsilon^2 + \eta_\varepsilon)|\partial_i u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} |\partial_i u|^2 \mathcal{L}^2 \llcorner \Omega + \sigma_i \text{ weakly}^\star \text{ in } \mathcal{M}(\Omega).$$

Let σ be the singular measure provided by Proposition 3.4.1. As $\sigma = \sigma_1 + \sigma_2$ with σ_1 and σ_2 non negative, we infer that σ_1 and σ_2 are absolutely continuous with respect to σ. Since σ is singular with respect to $\mathcal{L}^2$, we infer that both σ_1 and σ_2 are singular with respect to $\mathcal{L}^2$ as well. In addition, since $(\eta_\varepsilon + v_\varepsilon^2)|\partial_i u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mu_{ii}^a + \mu_{ii}^s$, we obtain, by uniqueness of the Lebesgue decomposition, that $\mu_{ii}^s = \sigma_i$ and $\mu_{ii}^a = |\partial_i u|^2 \mathcal{L}^2 \llcorner \Omega$.

Let us write $(v_\varepsilon^2+\eta_\varepsilon)\partial_1 u_\varepsilon\, \partial_2 u_\varepsilon \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \partial_1 u\, \partial_2 u \mathcal{L}^2 \llcorner \Omega + \sigma_{12}$ weakly* in $\mathcal{M}(\Omega)$ for some $\sigma_{12} \in \mathcal{M}(\Omega)$. For every test function $\varphi \in \mathcal{C}_c(\Omega)$, Proposition 3.4.1 and Lemma 3.3.1 yield

$$\begin{aligned} \int_\Omega \left|\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\nabla u_\varepsilon - \nabla u\right|^2 \varphi\, dx &= \int_\Omega (v_\varepsilon^2 + \eta_\varepsilon)|\nabla u_\varepsilon|^2 \varphi\, dx + \int_\Omega |\nabla u|^2 \varphi\, dx - 2\int_\Omega \sqrt{v_\varepsilon^2 + \eta_\varepsilon}\nabla u_\varepsilon \cdot \nabla u \varphi\, dx \\ &\underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega |\nabla u|^2 \varphi\, dx + \int_\Omega \varphi\, d\sigma + \int_\Omega |\nabla u|^2 \varphi\, dx - 2\int_\Omega |\nabla u|^2 \varphi\, dx \\ &= \int_\Omega \varphi\, d\sigma. \end{aligned}$$

Thus, we have $|\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\nabla u_\varepsilon - \nabla u|^2 \mathcal{L}^2 \llcorner \overset{\star}{\rightharpoonup} \sigma$ weakly* in $\mathcal{M}(\Omega)$. Hence

$$\begin{aligned} 2\left|(\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\partial_1 u_\varepsilon - \partial_1 u)(\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\partial_2 u_\varepsilon - \partial_2 u)\right| & \\ \leqslant \left(\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\partial_1 u_\varepsilon - \partial_1 u\right)^2 + \left(\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\partial_2 u_\varepsilon - \partial_2 u\right)^2 & \\ = \left|\sqrt{v_\varepsilon^2 + \eta_\varepsilon}\nabla u_\varepsilon - \nabla u\right|^2 \overset{\star}{\rightharpoonup} \sigma. & \end{aligned} \tag{3.4.10}$$

Next, expanding the left-hand side of (3.4.10) and using Lemma 3.3.1, we obtain for all $\varphi \in \mathcal{C}_c(\Omega)$,

$$\int_\Omega \left(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_1 u_\varepsilon - \partial_1 u\right)\left(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_2 u_\varepsilon - \partial_2 u\right)\,\varphi\,dx$$
$$= \int_\Omega (v_\varepsilon^2+\eta_\varepsilon)\partial_1 u_\varepsilon\,\partial_2 u_\varepsilon\,\varphi\,dx - \int_\Omega \sqrt{v_\varepsilon^2+\eta_\varepsilon}\,\partial_1 u_\varepsilon\,\partial_2 u\,\varphi\,dx - \int_\Omega \sqrt{v_\varepsilon^2+\eta_\varepsilon}\,\partial_2 u_\varepsilon\,\partial_1 u\varphi\,dx$$
$$+ \int_\Omega \partial_1 u\,\partial_2 u\varphi\,dx$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \partial_1 u\,\partial_2 u\,\varphi + \int_\Omega \varphi\,d\sigma_{12} - \int_\Omega \partial_1 u\,\partial_2 u\,\varphi\,dx = \int_\Omega \varphi\,d\sigma_{12}. \quad (3.4.11)$$

By using (3.4.10) and (3.4.11) along with a measure theoreric argument (see e.g. in [Sch95, Lemma 5.1]), we conclude that $|\sigma_{12}| \leqslant \frac{1}{2}\sigma$, from which we deduce that σ_{12} is absolutely continuous with respect to σ. Hence, σ_1, σ_2 and σ_{12} are singular with respect to $\mathcal{L}^2$ thanks to Proposition 3.4.1. We have thus established that

$$\mu = \mu^a + \mu^s = \nabla u \otimes \nabla u \mathcal{L}^2 \llcorner \Omega + \begin{pmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2 \end{pmatrix},$$

with σ_1, σ_2 and σ_{12} singular with respect to $\mathcal{L}^2$. By uniqueness of the Lebesgue decomposition, we infer that

$$\mu^a = \nabla u \otimes \nabla u \mathcal{L}^2 \llcorner \Omega, \quad \mu^s = \begin{pmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2 \end{pmatrix}$$

which completes the proof of the proposition. □

Remark 3.4.3. Propositions 3.4.1 and 3.4.2 show that one should not expect the strong L^2-convergence of $\{\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon\}_{\varepsilon>0}$ to ∇u. This is an obstacle to be able to pass to the limit in the second inner variation as done in [BMR24, Theorem 1.3].

3.5 Proof of Theorem 3.2.6

In this subsection, we prove Theorem 3.2.6 by passing to the limit in (3.2.4). Recall the definition of μ (3.4.9). Observe that $\mu_{12} = \mu_{21}$ and that, passing to the trace, $(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2\mathcal{L}^2\llcorner\Omega \rightharpoonup \mu_{11} + \mu_{22}$. As in [BMR24, Lemma 5.1], we have the following limit conservation law involving μ, the first moment $\overline{A}$ of V, and the boundary measure $m \in \mathcal{M}(\partial\Omega)$ defined in Lemma 3.3.4.

Lemma 3.5.1. *For all vector field $X \in \mathcal{C}_c^1(\mathbb{R}^2;\mathbb{R}^2)$, we have*

$$\langle(\mu_{11}+\mu_{22})\mathrm{Id} - 2\mu, DX\rangle + \int_{\widehat{J_u}} \overline{A} : DX\,d\mathcal{H}^1$$
$$= -\int_{\partial\Omega}(X\cdot\nu)\,dm + \int_{\partial\Omega}|\partial_\tau g|^2(X\cdot\nu)\,d\mathcal{H}^1 - 2\int_{\partial\Omega}(\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\,d\mathcal{H}^1, \quad (3.5.1)$$

where τ is a unit tangent vector to $\partial\Omega$ and $\partial_\tau g = \nabla g\cdot\tau$ is the tangential derivative of g on $\partial\Omega$.

Proof. Let $X \in \mathcal{C}_c^1(\mathbb{R}^2;\mathbb{R}^2)$ be an arbitrary vector field. By [BMR24, Proposition 4.2],

$$\int_\Omega (2\sqrt{\eta_\varepsilon+v_\varepsilon^2}\nabla u_\varepsilon \otimes \sqrt{\eta_\varepsilon+v_\varepsilon^2}\nabla u_\varepsilon - (\eta_\varepsilon+v_\varepsilon^2)|\nabla u_\varepsilon|^2\mathrm{Id}) : DX\,dx$$
$$+ \int_\Omega \left[2\varepsilon\nabla v_\varepsilon\otimes\nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathrm{Id}\right] : DX\,dx$$
$$= \int_{\partial\Omega}\left[(\eta_\varepsilon+1)|\partial_\nu u_\varepsilon|^2 + \varepsilon|\partial_\nu v_\varepsilon|^2 - (\eta_\varepsilon+1)|\partial_\tau g|^2\right](X\cdot\nu)d\mathcal{H}^1$$
$$+ 2(\eta_\varepsilon+1)\int_{\partial\Omega}\partial_\nu u_\varepsilon(X\cdot\tau)(\partial_\tau g)\,d\mathcal{H}^1, \quad (3.5.2)$$

We now study separately each term of this expression.

Using the definition (3.2.9) of the measure μ, we have

$$\int_\Omega (2\sqrt{\eta_\varepsilon+v_\varepsilon^2}\nabla u_\varepsilon \otimes \sqrt{\eta_\varepsilon+v_\varepsilon^2}\nabla u_\varepsilon - (\eta_\varepsilon+v_\varepsilon^2)|\nabla u_\varepsilon|^2\mathrm{Id}) : DX\, dx \xrightarrow[\varepsilon\to 0]{} \langle 2\mu - (\mu_{11}+\mu_{22})\mathrm{Id}, DX\rangle. \quad (3.5.3)$$

Next, thanks to the equi-partition of the energy (3.3.2) and the varifold convergence (3.3.10), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \left[2\varepsilon\nabla v_\varepsilon\otimes\nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon|\nabla v_\varepsilon|^2\right)\right] : DX\, dx\\
=\ &\lim_{\varepsilon\to 0}\int_{\{|\nabla v_\varepsilon|\neq 0\}\cap\Omega} 2\varepsilon|\nabla v_\varepsilon|^2\left(\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|}\otimes\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|}-\mathrm{Id}\right) : DX\, dx\\
=\ &\lim_{\varepsilon\to 0}\int_{\{|\nabla w_\varepsilon|\neq 0\}\cap\Omega} |\nabla w_\varepsilon|\left(\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\otimes\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}-\mathrm{Id}\right) : DX\, dx \qquad (3.5.4)\\
=\ &-\lim_{\varepsilon\to 0}\int_{\overline{\Omega}\times\mathbf{G}_1} A : DX(x)\, dV_\varepsilon(x,A)\\
=\ &-\int_{\overline{\Omega}\times\mathbf{G}_1} A : DX(x)\, dV(x,A) = -\int_{\widehat{J_u}} \overline{A} : DX d\mathcal{H}^1.
\end{aligned}$$

Finally, according to the regularity properties of $\partial\Omega$ and g, Lemma 3.3.4 yields

$$\int_{\partial\Omega}\left[(\eta_\varepsilon+1)|\partial_\nu u_\varepsilon|^2+\varepsilon|\partial_\nu v_\varepsilon|^2-(\eta_\varepsilon+1)|\partial_\tau g|^2\right](X\cdot\nu)\, d\mathcal{H}^1 \xrightarrow[\varepsilon\to 0]{} \int_{\partial\Omega}(X\cdot\nu)\, dm - \int_{\partial\Omega}(X\cdot\nu)|\partial_\nu g|^2\, d\mathcal{H}^1 \quad (3.5.5)$$

and

$$2(\eta_\varepsilon+1)\int_{\partial\Omega}(\partial_\nu u_\varepsilon)(X\cdot\tau)\partial_\tau g\, d\mathcal{H}^1 \xrightarrow[\varepsilon\to 0]{} 2\int_{\partial\Omega}(\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\, d\mathcal{H}^1. \quad (3.5.6)$$

Combining (3.5.3), (3.5.4),(3.5.5), (3.5.6) yields (3.5.1). □

Our strategy is then to analyse each term of the left-hand-side of (3.5.1) to recover (3.2.8). We now set

$$T := (\mu_{11}+\mu_{22})\,\mathrm{Id} - 2\mu + \overline{A}\mathcal{H}^1\llcorner\widehat{J_u} \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{2\times 2}). \quad (3.5.7)$$

We observe that (3.5.1) rewrites as

$$\langle T, DX\rangle = -\int_{\partial\Omega}(X\cdot\nu)\, dm + \int_{\partial\Omega}|\partial_\tau g|^2(X\cdot\nu)\, d\mathcal{H}^1 - 2\int_{\partial\Omega}(\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\, d\mathcal{H}^1 \quad \text{for all } X\in\mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2), \quad (3.5.8)$$

or still

$$-\mathrm{div}(T) = -\nu m\llcorner\partial\Omega + |\partial_\tau g|^2\nu\mathcal{H}^1\llcorner\partial\Omega - 2(\nabla u\cdot\nu)(\partial_\tau g)\tau\mathcal{H}^1\llcorner\partial\Omega \quad \text{in } \mathcal{D}'(\mathbb{R}^2;\mathbb{R}^2). \quad (3.5.9)$$

Let us write the Lebesgue-Besicovitch decomposition for μ and T,

$$\begin{cases}\mu = \mu^a+\mu^j+\mu^c,\\ T = T^a+T^j+T^c,\end{cases} \quad (3.5.10)$$

where μ^a and T^a are absolutely continuous with respect to $\mathcal{L}^2$, μ^j and T^j are absolutely continuous with respect to $\mathcal{H}^1 \llcorner \widehat{J_u}$, and μ^c and T^c are singular with respect to both $\mathcal{L}^2$ and $\mathcal{H}^1 \llcorner \widehat{J_u}$. We observe that, according to Proposition 3.4.2,

$$\begin{cases} T^a = (\mu^a_{11} + \mu^a_{22})\mathrm{Id} - 2\mu^a = (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u)\mathcal{L}^2, \\ T^j = (\mu^j_{11} + \mu^j_{22})\mathrm{Id} - 2\mu^j + \overline{A}\mathcal{H}^1 \llcorner \widehat{J_u}, \\ T^c = (\mu^c_{11} + \mu^c_{22})\mathrm{Id} - 2\mu^c. \end{cases} \tag{3.5.11}$$

Let Θ be the density of T^j with respect to $\mathcal{H}^1 \llcorner \widehat{J_u}$, so that

$$T^j = \Theta \mathcal{H}^1 \llcorner \widehat{J_u} = (\mu^j_{11} + \mu^j_{22})\mathrm{Id} - 2\mu^j + \overline{A}\mathcal{H}^1 \llcorner \widehat{J_u}. \tag{3.5.12}$$

We now establish algebraic properties of the measures T^j and T^c. The theory developed in [DPR16] aims precisely at describing singular measures satisfying a linear PDE. It states that the polar of the singular part of the measure belongs to the wave-cone of the linear operator.

Lemma 3.5.2. *For $\mathcal{H}^1$-a.e. $x \in \widehat{J_u}$, the matrix $\Theta(x) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ is an orthogonal projector. Moreover, $T^c = 0$ and there exists a scalar measure $\lambda^c \in \mathcal{M}(\overline{\Omega})$ such that $\mu^c = \lambda^c \mathrm{Id}$.*

Proof. According to (3.5.9), the limiting stress-energy tensor is a measure T (extended by zero in $\mathbb{R}^2 \setminus \overline{\Omega}$) which satisfies the linear PDE $\mathrm{div}(T) \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}^2)$. The wave cone of the divergence operator (acting on symmetric matrices) is defined by

$$\Lambda := \bigcup_{\xi \in \mathbb{R}^2,\, |\xi|=1} \{A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} :\ A\xi = 0\}$$

and it is immediate to check that it corresponds to the set of singular matrices. Hence according to [DPR16, Corollary 1.13] and using that T^j and T^c are singular to each other and both singular with respect to $\mathcal{L}^2$, one finds that

$$\mathrm{rank}\,(\Theta) \leq 1 \quad \mathcal{H}^1\text{-a.e. on } \widehat{J_u} \tag{3.5.13}$$

and

$$\mathrm{rank}\left(\frac{dT^c}{d|T^c|}\right) \leq 1 \quad |T^c|\text{-a.e. in } \overline{\Omega}. \tag{3.5.14}$$

Since the matrix-valued measure $(\mu_{11} + \mu_{22})\mathrm{Id} - 2\mu$ has zero trace, it follows from (3.5.11) together with Proposition 3.3.7 that

$$\mathrm{Tr}\,(\Theta) = \mathrm{Tr}(\overline{A}) = 1 \quad \mathcal{H}^1\text{-a.e. on } \widehat{J_u}.$$

Hence, recalling (3.5.13), for $\mathcal{H}^1$-a.e. $x \in \widehat{J_u}$, the eigenvalues of $\Theta(x) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ are exactly 0 and 1 so that $\Theta(x)$ is indeed an orthogonal projector.

Using (3.5.11) and (3.5.14), it follows that for $|T^c|$-a.e. $x \in \overline{\Omega}$, $\frac{dT^c}{d|T^c|}(x) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ is a trace free matrix with zero determinant, which implies that $\frac{dT^c}{d|T^c|}(x) = 0$. It thus follows that $T^c = 0$ and using again (3.5.11),

$$\mu^c = \frac{\mu^c_{11} + \mu^c_{22}}{2}\mathrm{Id},$$

which completes the proof of the result by setting $\lambda^c := \frac{\mu^c_{11} + \mu^c_{22}}{2}$. □

Note that, in the previous argument, we crucially used the fact that the dimension is set to be equal to two.

Reporting the information obtained in (3.5.7), (3.5.11) and Lemma 3.5.2 inside (3.5.8) yields for all $X \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^2)$,

$$\begin{aligned} \int_\Omega (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : DX\, dx + \int_{\widehat{J_u}} \Theta : DX\, d\mathcal{H}^1 \\ = -\int_{\partial\Omega} (X \cdot \nu)\, dm + \int_{\partial\Omega} |\partial_\tau g|^2 (X \cdot \nu)\, d\mathcal{H}^1 - 2\int_{\partial\Omega} (\nabla u \cdot \nu)(X \cdot \tau)\partial_\tau g\, d\mathcal{H}^1. \end{aligned} \tag{3.5.15}$$

We already know that for $\mathcal{H}^1$-a.e. $x \in \widehat{J_u}$, the matrix $\Theta(x)$ is an orthogonal projector. We now make this information more precise by showing that it actually consists in the orthogonal projection onto the (approximate) tangent space to the rectifiable set $\widehat{J_u}$ at x. We proceed in two steps, by distinguishing interior points in Ω to boundary points $\partial\Omega$.

We first consider interior points. The proof of the following result is similar to that of [BMR24, Lemma 5.3] with several simplications due to the fact that we are here working in dimension two.

Lemma 3.5.3. *For $\mathcal{H}^1$-a.e $x \in \widehat{J_u} \cap \Omega = J_u$, one has $\Theta(x) = \mathrm{Id} - \nu_u(x) \otimes \nu_u(x)$.*

Proof. We perform a blow-up argument on (3.5.15). Let $x_0 \in J_u = \widehat{J_u} \cap \Omega$ be such that

(i) J_u admits an approximate tangent space at x_0, which is given by $T_{x_0} J_u = \nu_u(x_0)^\perp$.

(ii) $\lim_{\rho \to 0} \frac{\mathcal{H}^1(J_u \cap B_\rho(x_0))}{2\pi\rho} = 1$;

(iii) $\lim_{\rho \to 0} \frac{1}{\rho} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx = 0$;

(iv) x_0 is a Lebesgue point of Θ with respect to $\mathcal{H}^1 \llcorner J_u$.

It turns out that $\mathcal{H}^1$-a.e points x_0 in $J_u \cap \Omega$ fulfill those four items as a consequence of the Besicovitch Differentiation Theorem (see [AFP00, Theorem 2.22]), the $\mathcal{H}^1$-rectifiability of J_u (see [AFP00, Theorem 2.63]), and the fact the measures $|\nabla u|^2 \mathcal{L}^2 \llcorner \Omega$ and $\mathcal{H}^1 \llcorner J_u$ are singular to each other.

Let $\zeta \in \mathcal{C}_c(\mathbb{R}^2; \mathbb{R}^2)$ be a test vector-field such that $\mathrm{Supp}(\zeta) \subset B_1$ and define $\varphi_\delta(x) = \zeta\left(\frac{x - x_0}{\delta}\right)$ for every $\delta > 0$ small enough such that $B_\delta(x_0) \subset\subset \Omega$. Note that $\mathrm{Supp}(\varphi_\delta) \subset \Omega$ so that φ_δ can be taken as a test function in (3.5.15). It leads to

$$\int_{J_u \cap B_\delta(x_0)} \Theta : D\varphi_\delta \, d\mathcal{H}^1 = - \int_{B_\delta(x_0)} (|\nabla u|^2 \mathrm{Id} - 2 \nabla u \otimes \nabla u) : D\varphi_\delta \, dx, \tag{3.5.16}$$

which rewrites as

$$\begin{aligned} \frac{1}{\delta} \int_{J_u \cap B_\delta(x_0)} \Theta(x) : D\zeta\left(\frac{x - x_0}{\delta}\right) d\mathcal{H}^1(x) \\ = -\frac{1}{\delta} \int_{B_\delta(x_0)} (|\nabla u|^2 \mathrm{Id} - 2 \nabla u \otimes \nabla u) : D\zeta\left(\frac{x - x_0}{\delta}\right) dx. \end{aligned} \tag{3.5.17}$$

Since x_0 satisfies item (iii), one has

$$\begin{aligned} \left| \frac{1}{\delta} \int_{B_\delta(x_0)} (|\nabla u|^2 \mathrm{Id} - 2 \nabla u \otimes \nabla u) : D\zeta\left(\frac{x - x_0}{\delta}\right) dx \right| \\ \leqslant C \|D\zeta\|_{L^\infty(B_1)} \frac{1}{\delta} \int_{B_\delta(x_0)} |\nabla u|^2 \, dx \longrightarrow 0. \end{aligned} \tag{3.5.18}$$

On the other hand, by item (iv) we have

$$\begin{aligned} \left| \frac{1}{\delta} \int_{J_u \cap B_\delta(x_0)} (\Theta(x) - \Theta(x_0)) : D\zeta\left(\frac{x - x_0}{\delta}\right) d\mathcal{H}^1(x) \right| \\ \leqslant C \|D\zeta\|_{L^\infty(B_1)} \frac{1}{\delta} \int_{J_u \cap B_\delta(x_0)} |\Theta(x) - \Theta(x_0)| \, d\mathcal{H}^1(x) \longrightarrow 0. \end{aligned} \tag{3.5.19}$$

Combining (3.5.17), (3.5.18), (3.5.19) and item (ii) yields

$$0 = \Theta(x_0) : \lim_{\delta \to 0} \frac{1}{\delta} \int_{J_u \cap B_\delta(x_0)} D\zeta\left(\frac{x - x_0}{\delta}\right) d\mathcal{H}^1(x) = \int_{T_{x_0} J_u \cap B_1} \Theta(x_0) : D\zeta(x) d\mathcal{H}^1(x).$$

Recalling that $\Theta(x_0)$ is an orthonormal projector, there exists a unit vector $e(x_0) \in \mathbb{R}^2$ such that $\Theta(x_0) = \mathrm{Id} - e(x_0) \otimes e(x_0)$. In particular, the measure $W = (\mathcal{H}^1 \llcorner T_{x_0} J_u) \otimes \delta_{\Theta(x_0)} \in \mathcal{M}(B_1 \times \mathbf{G}_1)$ is a 1-varifold and the previous computation shows that W is stationary in B_1, i.e., $\delta W = 0$ in B_1. Hence, according to the monotonicity formula (3.1.7), for all $0 < r < \rho < 1$,

$$\frac{\mathcal{H}^1(T_{x_0} J_u \cap B_\rho)}{\rho} - \frac{\mathcal{H}^1(T_{x_0} J_u \cap B_r)}{r} = \int_{T_{x_0} J_u \cap (B_\rho \setminus B_r)} \frac{|e(x_0) \cdot y|^2}{|y|^3} \, d\mathcal{H}^1(y).$$

Since the left-hand side of the previous equality vanishes, we obtain

$$\int_{T_{x_0} J_u \cap (B_\rho \setminus B_r)} \frac{|e(x_0) \cdot y|^2}{|y|^3} d\mathcal{H}^1(y) = 0.$$

We deduce that that $y \cdot e(x_0) = 0$ for $\mathcal{H}^1$-a.e. $y \in T_{x_0} J_u \cap B_1$, which implies that $T_{x_0} J_u = e(x_0)^\perp$ and $e(x_0) = \pm \nu_u(x_0)$. □

We next address the case of boundary points following the ideas of [BMR24, Lemma 5.4].

Lemma 3.5.4. *For $\mathcal{H}^1$-a.e x in $\widehat{J_u} \cap \partial\Omega$, we have $\Theta(x) = \mathrm{Id} - \nu(x) \otimes \nu(x) = \mathrm{Id} - \nu_u(x) \otimes \nu_u(x)$, where ν is the outward unit normal to $\partial\Omega$.*

Proof. Let $x_0 \in \widehat{J_u} \cap \partial\Omega$ be such that :

(i) x_0 is a Lebesgue point of Θ with respect to $\mathcal{H}^1 \llcorner \widehat{J_u} \cap \partial\Omega$;

(ii) $\widehat{J_u}$ admits an approximate tangent space at x_0 which is given by $T_{x_0} \widehat{J_u} = \nu_u(x_0)^\perp$;

(iii) $\nu_u(x_0) = \nu(x_0)$;

(iv) $\lim_{\rho \to 0} \frac{\mathcal{H}^1(\widehat{J_u} \cap B_\rho(x_0))}{2\pi\rho} = 1$;

(v) $\lim_{\rho \to 0} \frac{1}{\rho} \int_{B_\rho(x_0) \cap \Omega} |\nabla u|^2 \, dx = 0$;

(vi) $m(\{x_0\}) = 0$.

Let us first prove that $\mathcal{H}^1$-a.e points x in $\widehat{J_u} \cap \partial\Omega$ fullfils those conditions. The validity of item (i) follows from the Besicovitch Differentiation Theorem (see [AFP00, Theorem 2.22]). Then, the rectifiability of $\widehat{J_u}$, the locality of the approximate tangent space (see [AFP00, Theorem 2.85 and Remark 2.87]) and the Besicovitch-Mastrand-Mattila Theorem (see [AFP00, Theorem 2.63]) ensure that item (ii), item (iii) and item (iv) are also satisfied. The validity of item (v) is a direct consequence of the fact that the measures $|\nabla u|^2 \mathcal{L}^2 \llcorner \Omega$ and $\mathcal{H}^1 \llcorner \widehat{J_u}$ are singular to each other. Finally, since m is a finite measure, it follows that its set of atoms is at most countable, hence $\mathcal{H}^1$-negligible which ensure that item (vi) holds.

We now prove that $\nu(x_0)$ belongs to the kernel of $\Theta(x_0)$, which, combined with Lemma 3.5.2 together with item (iii), implies that $\Theta(x_0) = \mathrm{Id} - \nu_u(x_0) \otimes \nu_u(x_0) = \mathrm{Id} - \nu(x_0) \otimes \nu(x_0)$.

Let $\psi \in \mathcal{C}^1_c(B_1)$ be a scalar test function, and $\zeta \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^2)$ a test vector field. For $\delta > 0$, set

$$\varphi_\delta = \psi \left(\frac{\cdot - x_0}{\delta} \right) \zeta \in \mathcal{C}^1_c(B_\delta(x_0); \mathbb{R}^2),$$

and notice that

$$D\varphi_\delta = \psi \left(\frac{\cdot - x_0}{\delta} \right) D\zeta \ + \zeta \otimes \frac{1}{\delta} \nabla\psi \left(\frac{\cdot - x_0}{\delta} \right).$$

Using φ_δ as a test function in (3.5.15) yields

$$\begin{aligned}\int_{B_\delta(x_0)\cap\Omega}(|\nabla u|^2-2\nabla u\otimes\nabla u):D\varphi_\delta\,dx+\int_{\widehat{J_u}\cap B_\delta(x_0)}\Theta:D\varphi_\delta\,d\mathcal{H}^1\\ =-\int_{\partial\Omega}(\varphi_\delta\cdot\nu)\,dm+\int_{\partial\Omega}|\partial_\tau g|^2(\varphi_\delta\cdot\nu)\,d\mathcal{H}^1\\ -2\int_{\partial\Omega\cap B_\delta(x_0)}(\nabla u\cdot\nu)\partial_\tau g(\tau\cdot\varphi_\delta)\,d\mathcal{H}^1. \quad (3.5.20)\end{aligned}$$

Observe that

$$\begin{aligned}\left|\int_{B_\delta(x_0)\cap\Omega}(|\nabla u|^2-2\nabla u\otimes\nabla u):(D\zeta)\psi\left(\frac{\cdot-x_0}{\delta}\right)dx\right|\\ \leqslant C\|\psi\|_{L^\infty(B_1)}\|D\zeta\|_{L^\infty(\mathbb{R}^2)}\int_{B_\delta(x_0)\cap\Omega}|\nabla u|^2\,dx\xrightarrow[\delta\to0]{}0\end{aligned}$$

and, thanks to item (v)

$$\begin{aligned}\left|\int_{B_\delta(x_0)\cap\Omega}(|\nabla u|^2-2\nabla u\otimes\nabla u):\zeta\otimes\frac{1}{\delta}\nabla\psi\left(\frac{\cdot-x_0}{\delta}\right)dx\right|\\ \leqslant C\|\nabla\psi\|_{L^\infty(B_1)}\|D\zeta\|_{L^\infty(\mathbb{R}^2)}\frac{1}{\delta}\int_{B_\delta(x_0)\cap\Omega}|\nabla u|^2dx\xrightarrow[\delta\to0]{}0.\end{aligned}$$

We then deduce that

$$\int_{B_\delta(x_0)\cap\Omega}(|\nabla u|^2-2\nabla u\otimes\nabla u):D\varphi_\delta\,dx\xrightarrow[\delta\to0]{}0. \quad (3.5.21)$$

Next, we have

$$\left|\int_{\widehat{J_u}\cap B_\delta(x_0)}\Theta:(D\zeta)\psi\left(\frac{\cdot-x_0}{\delta}\right)d\mathcal{H}^1\right|\leqslant C\|\psi\|_{L^\infty(B_1)}\|D\zeta\|_{L^\infty(\mathbb{R}^2)}\mathcal{H}^1(\widehat{J_u}\cap B_\delta(x_0))\xrightarrow[\delta\to0]{}0. \quad (3.5.22)$$

and according to item (i)

$$\begin{aligned}\left|\int_{\widehat{J_u}\cap B_\delta(x_0)}(\Theta(x)-\Theta(x_0)):\zeta(x)\otimes\frac{1}{\delta}\nabla\psi\left(\frac{x-x_0}{\delta}\right)d\mathcal{H}^1\right|\\ \leqslant C\|\nabla\psi\|_{L^\infty(B_1)}\|\zeta\|_{L^\infty(\mathbb{R}^2)}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)}|\Theta(x)-\Theta(x_0)|\,d\mathcal{H}^1(x)\xrightarrow[\delta\to0]{}0. \quad (3.5.23)\end{aligned}$$

Combining (3.5.22), (3.5.23), the continuity of ζ and item (ii) yield

$$\begin{aligned}\lim_{\delta\to0}\int_{\widehat{J_u}\cap B_\delta(x_0)}\Theta:D\varphi_\delta\,d\mathcal{H}^1&=\lim_{\delta\to0}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)}\Theta(x):\zeta(x)\otimes\nabla\psi\left(\frac{x-x_0}{\delta}\right)d\mathcal{H}^1(x)\\ &=\lim_{\delta\to0}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)}\Theta(x_0):\zeta(x)\otimes\nabla\psi\left(\frac{x-x_0}{\delta}\right)d\mathcal{H}^1(x)\\ &=\lim_{\delta\to0}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)}\Theta(x_0):\zeta(x_0)\otimes\nabla\psi\left(\frac{x-x_0}{\delta}\right)d\mathcal{H}^1(x)\\ &=\int_{T_{x_0}\widehat{J_u}\cap B_1}\Theta(x_0):\zeta(x_0)\otimes\nabla\psi(y)\,d\mathcal{H}^1(y). \quad (3.5.24)\end{aligned}$$

Finally item (vi) leads to

$$\begin{aligned}\left|-\int_{\partial\Omega}(\varphi_\delta\cdot\nu)\,dm+\int_{\partial\Omega}|\partial_\tau g|^2(\varphi_\delta\cdot\nu)\,d\mathcal{H}^1-2\int_{\partial\Omega\cap B_\delta(x_0)}(\nabla u\cdot\nu)\partial_\tau g(\tau\cdot\varphi_\delta)\,d\mathcal{H}^1\right|\\ \leqslant\|\zeta\|_{L^\infty(\mathbb{R}^2)}\|\psi\|_{L^\infty(B_1)}\left[m(B_\delta(x_0))+\int_{B_\delta(x_0)\cap\partial\Omega}(|\nabla u\cdot\nu|+|\nabla g|^2)\,d\mathcal{H}^1\right]\xrightarrow[\delta\to0]{}0. \quad (3.5.25)\end{aligned}$$

Gathering (3.5.20), (3.5.21), (3.5.24), (3.5.25) leads to

$$\int_{T_{x_0}\widehat{J_u}\cap B_1} \Theta(x_0) : \zeta(x_0)\otimes\nabla\psi\, d\mathcal{H}^1 = 0. \tag{3.5.26}$$

We now specify (3.5.26) for ζ such that $\zeta(x_0)=\nu(x_0)$. Denoting by $\tau(x_0)$ an orthonormal vector to $\nu(x_0)$ we decompose $\nabla\psi$ as

$$\nabla\psi = (\partial_{\tau(x_0)}\psi)\tau(x_0) + (\partial_{\nu(x_0)}\psi)\nu(x_0).$$

Since $\mathrm{Supp}(\psi)\subset B_1$, then $\int_{T_{x_0}\widehat{J_u}\cap B_1}\partial_{\tau(x_0)}\psi\, d\mathcal{H}^1=0$ and (3.5.26) becomes

$$\int_{T_{x_0}\widehat{J_u}\cap B_1} \Theta(x_0) : (\nu(x_0)\otimes\nu(x_0))\partial_{\nu(x_0)}\psi\, d\mathcal{H}^1 = 0 \quad \text{for all } \psi\in\mathcal{C}^1_c(B_1).$$

We conclude that

$$\nu(x_0)\cdot(\Theta(x_0)\nu(x_0)) = \Theta(x_0):(\nu(x_0)\otimes\nu(x_0)) = 0$$

and thus, since $\Theta(x_0)$ is a projector, we obtain $\Theta(x_0)\nu(x_0)=0$, as claimed. □

We are now in position to complete the proof of our main result.

Proof of Theorem 3.2.6. According to (3.5.15), Lemma 3.5.3 and Lemma 3.5.4, we have that for all $X\in\mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$,

$$\begin{aligned}\int_\Omega (|\nabla u|^2 - 2\nabla u\otimes\nabla u) : DX\, dx + \int_{\widehat{J_u}} (\mathrm{Id}-\nu_u\otimes\nu_u) : DX\, d\mathcal{H}^1 \\ = -\int_{\partial\Omega}(X\cdot\nu)\, dm + \int_{\partial\Omega}|\partial_\tau g|^2(X\cdot\nu)\, d\mathcal{H}^1 - 2\int_{\partial\Omega}(\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\, d\mathcal{H}^1.\end{aligned}$$

Specifying to vector fields $X\in\mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ such that $X\cdot\nu=0$ on $\partial\Omega$ leads to

$$\int_\Omega (|\nabla u|^2 - 2\nabla u\otimes\nabla u) \,:\, DX\, dx + \int_{\widehat{J_u}} (\mathrm{Id}-\nu_u\otimes\nu_u) \,:\, DX\, d\mathcal{H}^1 \,=\, -2\int_{\partial\Omega}(\nabla u\cdot\nu)X\cdot\nabla g\, d\mathcal{H}^1,$$

which completes the proof of Theorem 3.2.6. □

Remark 3.5.1. The assumption of convergence of the energy can also be used to pass to the limit in the second inner variation in general, cf. [Le11, Le15, LS19, BMR24]. Here the sole convergence of the phase field energy is not sufficient a priori to pass to the limit in the second inner variation of the AT energy.

Thanks to the varifold convergence and the equi-partition of energy, we first observe as in [BMR24, Corollary 5.1] that if $\{(u_\varepsilon,v_\varepsilon)\}_{\varepsilon>0}$ is a family of critical points of AT_ε satisfying the assumptions of Theorem 3.2.6 then, up to a subsequence

$$\nabla w_\varepsilon\otimes\nabla w_\varepsilon\mathcal{L}^2\llcorner\Omega \rightharpoonup (\mathrm{Id}-\overline{A})\mathcal{H}^1\llcorner J_u \quad \text{weakly* in } \mathcal{M}(\Omega;\mathbb{M}^{2\times2}_{\mathrm{sym}})$$

and

$$\varepsilon\nabla v_\varepsilon\otimes\nabla v_\varepsilon\mathcal{L}^2\llcorner\Omega \rightharpoonup \frac{1}{2}(\mathrm{Id}-\overline{A})\mathcal{H}^1\llcorner J_u \quad \text{weakly* in } \mathcal{M}(\Omega;\mathbb{M}^{2\times2}_{\mathrm{sym}}).$$

Recalling that $(\mathrm{Id}-\overline{A})\mathcal{H}^1\llcorner J_u = \nu_u\otimes\nu_u\mathcal{H}^1\llcorner J_u - ((\mathrm{Tr}(\mu^j))\mathrm{Id}-2\mu^j)$

Using Propositions 3.4.1 and 3.4.2, the expression of the second inner variation of the AT energy computed in [BMR24, Lemma A.3], we find that if $X \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^2)$,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X] \;=\; & \int_\Omega |\nabla u|^2((\mathrm{div}X)^2 - \mathrm{Tr}((DX)^2))\,dx + \langle \mathrm{Tr}(\mu^s), (\mathrm{div}X)^2 - \mathrm{Tr}((DX)^2)\rangle \\
& -4\int_\Omega ((\nabla u \otimes \nabla u) : DX)\mathrm{div}X\,dx - 4\langle \mu^s, (\mathrm{div}X)DX\rangle \\
& +4\int_\Omega (\nabla u \otimes \nabla u) : (DX)^2\,dx + 4\langle \mu^s, (DX)^2\rangle \\
& +2\int_\Omega |DX^T\nabla u|^2\,dx + 2\langle \mu^s, (DX)(DX)^T\rangle \\
& +\int_{J_u} ((\mathrm{div}X)^2 - \mathrm{Tr}((DX)^2))\,d\mathcal{H}^1 \\
& -2\int_{J_u} (\nu_u \otimes \nu_u) : (DX)\mathrm{div}X\,d\mathcal{H}^1 + 2\langle (\mathrm{Tr}(\mu^j))\mathrm{Id} - 2\mu^j, (DX)\mathrm{div}X\rangle \\
& +2\int_{J_u} (\nu_u \otimes \nu_u) : (DX)^2\,d\mathcal{H}^1 - 2\langle (\mathrm{Tr}(\mu^j))\mathrm{Id} - 2\mu^j, (DX)^2\rangle \\
& +\int_{J_u} |DX^T\nu_u|^2\,d\mathcal{H}^1 - \langle (\mathrm{Tr}(\mu^j))\mathrm{Id} - 2\mu^j, (DX)(DX)^T\rangle.
\end{aligned}$$

Thus it does not seem to have any compensation phenomenon to get rid-off the terms involving the singular measure μ, and stability of the the second inner variation might not be satisfied.

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Chapter 4

Existence, uniqueness and regularity of solutions to the parabolic Ambrosio-Tortorelli system

We investigate the existence, uniqueness, and regularity of the gradient flow of the Ambrosio–Tortorelli functional, viewed as an initial-boundary value problem, in arbitrary dimension. Using a time-discrete Euler scheme,for any initial data, we establish the existence of a weak gradient flow satisfying a maximum principle. We also identify a functional space in which uniqueness holds. We further show that such gradient flow is smooth in the interior of the space-time domain. Under additional assumptions on the initial data, the boundary datum and the space boundary, we prove optimal regularity for the solution, up to the space-time boundary.

4.1 Introduction

The Mumford–Shah functional was introduced by D. Mumford and J. Shah in 1989 in their celebrated paper [MS89], in the context of image segmentation. It was later given a meaningful interpretation in fracture mechanics to describe brittle fractures in elastic materials (see [FM98]). If Ω is a bounded open subset of $\mathbb{R}^d$, it can be defined for functions in $SBV^2(\Omega)$, the space of Special Bounded Variation functions, defined by

$$SBV^2(\Omega) = \{u \in L^1(\Omega), Du = \nabla u \mathcal{L}^d + (u^+ - u^-)\nu_u \mathcal{H}^{d-1}_{|J_u}, \nabla u \in L^2(\Omega;\mathbb{R}^d), \mathcal{H}^{d-1}(J_u) < +\infty\},$$

where $\mathcal{L}^d$ is the Lebesgue measure, $\mathcal{H}^{d-1}$ the $d-1$-Hausdorff measure, ∇u is the approximate gradient of u, the set J_u is the jump set of u, the vector ν_u is the approximate normal to J_u and u^+ and u^- are the approximate one-sided limits across J_u. For any $u \in SBV^2(\Omega)$, the Mumford Shah functional is defined by

$$MS(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{d-1}(J_u).$$

For an elastic material, the function u represents the displacement and J_u denotes a part of the crack set. Computing local minimizers of the Mumford–Shah functional numerically is challenging, as the geometric properties of the discontinuity set J_u are part of the unknown.

A possible approach to the (local) minimization of MS is to use a steepest descent method, which amounts to determining the gradient flow of MS. While defining a suitable notion of gradient for MS is delicate, a generalized gradient flow can be constructed via time discretization using an implicit Euler scheme. Letting the time step tend to zero, any limit curve is called a minimizing movement for the Mumford–Shah functional (see [DG93], [Amb95], [CD97]).

The Mumford-Shah functional admits various variational regularization in $L^2(\Omega)$ using De Giorgi's Γ-convergence (see [AT90], [AT92], [Gob98]). The most common regularization of the Mumford-Shah functional is probably the Ambrosio and Tortorelli functional, proposed in [AT92] and defined for

$(u,v) \in (H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)])^2$ by

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{\varepsilon}\right) dx \tag{4.1.1}$$

where $0 < \eta_\varepsilon \ll \varepsilon$ is a small parameter ensuring ellipticity. This approximation follows the spirit of the Allen–Cahn model for phase transitions ([Mod87], [Ste88]) and forms the basis for many numerical simulations (see [BFM08]). The phase-field variable v can be interpreted mechanically as a damage variable taking values in $[0,1]$, where $\{v = 1\}$ corresponds to a sane material and $\{v = 0\}$ to a fully damaged state. The existence of minimizers for AT_ε follows from the direct method in the calculus of variations. Using the Γ-convergence result of [AT92], one can approximate a global minimizer of MS by considering a convergent sequence $(u_\varepsilon, v_\varepsilon)$ of minimizers of AT_ε. However, this approach provides no information about local minimizers. Extensions of the Gamma-convergence result to general critical points of AT_ε can be found in [FLS09], [BMR24], [BRR26].

Similarly to [Gob98], where a gradient flow for the one-dimensional Mumford–Shah functional was obtained via a non local approximation, the approach proposed in [FP04] consists in deriving a gradient flow for MS as the limit, as $\varepsilon \to 0$, of the gradient flows of AT_ε. The gradient flow of AT_ε with respect to L^2 is defined as a pair (u,v) in $\left[L^\infty((0,+\infty); L^2(\Omega)) \cap L^2((0,+\infty); H^1(\Omega))\right]^2$ satisfying

$$\begin{cases} \partial_t u - \operatorname{div}\big((\eta_\varepsilon + v^2)\nabla u\big) = 0 & \text{in } (0,+\infty)\times\Omega, \\ \partial_t v - \varepsilon\Delta v + \dfrac{v-1}{4\varepsilon} + v|\nabla u|^2 = 0 & \text{in } (0,+\infty)\times\Omega, \\ (u,v) = (g,1) & \text{on } (0,+\infty)\times\partial\Omega, \\ (u(0,\cdot), v(0,\cdot)) = (u_0, v_0) & \text{in } \Omega. \end{cases} \tag{4.1.2}$$

The same approach was used in [BM14] to obtain a unilateral gradient flow for MS, incorporating an irreversibility condition on the crack, which can only grow in time. Consequently, studying the L^2 gradient flow of AT_ε becomes a center of interest in order to understand the gradient flow of MS.

The existence of weak solutions to the gradient flow system (4.1.2) was investigated in [FP04], where a uniqueness class for strong solutions was also identified, in dimensions 2 and 3. The existence of a strong gradient flow in dimension 2 was studied in [Cor24] using a Galerkin spatial discretization. The main difficulty in proving the existence of weak solutions for the gradient flow of AT_ε lies in the presence of a quadratic gradient term in the nonlinear part, which makes standard compactness arguments inefficient (see [Eva90]). In the Galerkin method, the space of test functions allowed in the Galerkin method is dense in $D(\Omega)$ only for the H^1 topology, which is not enough to pass to the limit the term $v|\nabla u|^2$ without stronger estimates on $|\nabla u|^2$. The issue raised by the presence of a quadratic gradient is also discussed in [Xu94] in the context of flows of incompressible fluids with temperature-dependent viscosity, and is overcome by using a time-discretization approach. We adopt the same approach in this work.

The time-discretization method was also used in [BM14] to construct a gradient flow for the variable u, coupled with an obstacle condition on the damage variable v that enforces irreversibility of the evolution (see also [Gia05]).

In this paper, we prove the existence of a weak gradient flow for AT_ε with initial–boundary conditions in any dimension. The existence is established in section 4.3. In view of applications to the study of a gradient flow for the Mumford–Shah functional, we also derive a uniform energy bound and a maximum principle satisfied by the constructed weak solution (u,v). We further provide, in section 4.4, a uniqueness condition for a "strong " gradient flow, which depends on the dimension. Under suitable boundary conditions, the solutions of the weak gradient flow appear to be regular up to the boundary, as shown in section 4.5.

The optimal regularity of the elliptic counterpart of (4.1.2) was established in [BMR24], relying on fine elliptic estimates involving Morrey–Campanato spaces, with the same method as [Riv93] for harmonic maps with value in a revolution torus. To adapt this approach to the parabolic setting, we use arguments from [Lie96], completed by the method of [Yin97]. The main argument is to prove Hölder regularity for the pair (u,v). De Giorgi-Nash-Moser theorem provides the Hölder regularity for the function u. Using the first equation of (4.1.2), we deduce a Campanato–Morrey estimate on ∇u

(see Lemma 4.5.2). The second equation of (4.1.2) becomes a heat equation with a source term in a Morrey space. In the elliptic case, this part is a direct consequence of De Giorgi-Nash Moser theorem. To prove that v is Hölder, according to a parabolic Campanato lemma (see [Lie96, Lemma 4.3]), it is sufficient to prove a decay estimate on the L^2 norm of v minus its average on parabolic cylinders. To obtain this estimate, we consider w a "heat "-extension of v. While w satisfies the desired estimate, the distance of the error $v - w$ to its mean value can be controled using a Poincare-type inequality for solutions of a heat equation with a source term in a Morrey space (see Lemma 4.5.3). The latter does not hold in general for functions in $L^2((0,+\infty);H^1(\Omega))$ and relies on the shape of the equation satisfied by the pair (u,v).

While the method of [Xu94] relies strongly on the specific structure of the system considered, the method presented in section 4.5 can be adapted to other parabolic system with a source term in L^1. In particular, a similar regularity result can be expected for the gradient flow of the Ambrosio-Tortorelli functional with a fidelity term, which is the functional studied in [Cor24] in the special case of a time-independent fidelity term. Another type of system in which this approach could provide regularity is the time evolution for a harmonic map flow with value in a revolution torus, for which the regularity in the elliptic case is proved in [Riv93].

We close this introduction with some notations and definitions used in the rest of the paper.

Standard notations. In the rest of the article, $d \geqslant 1$ is a positive integer. Let $B_R(x_0)$ be the ball of radius R centered at x. We denote by ω_d the Lebesgue measure of $B_1(0)$. For any open subset $\Omega \in \mathbb{R}^d$ and $p \geqslant 1$, for a measurable function $f : \Omega \to \mathbb{R}$, we denote by

$$\|f\|_{\infty,\Omega} = \sup_{x\in\Omega} |f(x)|, \quad \|f\|_p = \left(\int_\Omega |f|^p\right)^{1/p}, \quad \|f\|_{H^1} = \left(\int_\Omega |f(x)|^2 + \int_\Omega |\nabla f|^2\right)^{1/2}.$$

We also use the notation $\lfloor x \rfloor$ for the largest integer smaller than x and $\binom{n}{a,b,c} = \dfrac{n!}{a!b!c!}$ for the multinomial coefficient.

Parabolic framework. We introduce some tools from the theory of parabolic equations. To take into account the specificities of the parabolic framework, we introduce parabolic cylinders, which are analogous to open balls in the elliptic framework: if $X_0 = (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ and $R > 0$, the parabolic cylinder $Q_R(X_0)$ or $Q_R(t_0, x_0)$ is defined by

$$Q_R(X_0) = (t_0 - R^2, t_0] \times B_R(x_0) \subset \mathbb{R} \times \mathbb{R}^d.$$

While $Q_R(X_0)$ is not an open set, it is the convenient set that fits the translation invariance properties of the parabolic equations. For any interval $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$, we call the parabolic boundary the set

$$\partial_\sqcup(I \times \Omega) := (I \times \partial\Omega) \cup (\inf I \times \Omega).$$

In particular, $\partial_\sqcup Q_R(t_0, x_0) = ((t_0 - R^2, t_0) \times \partial B_r(x_0)) \cap (\{t_0 - R^2\} \times B_R(x_0))$.

The parabolic distance d_p on $\mathbb{R} \times \mathbb{R}^d$ is defined by

$$d_p((t,x);(s,y)) = |t-s|^{1/2} + |x-y| \quad \text{for all } (t,x),(s,y) \in \mathbb{R} \times \mathbb{R}^d.$$

The regularity of solutions of parabolic equations is studied in terms of the oscillation of u in parabolic cylinders and is naturally described in terms of parabolic Hölder spaces. For any real number $\alpha \in (0,1)$, the Hölder space $\mathcal{C}^{\alpha/2,\alpha}_{par}(I \times \Omega)$ is defined as

$$\mathcal{C}^{\alpha/2,\alpha}_{par}(I \times \Omega) = \left\{ f \in \mathcal{C}^0(I \times \Omega), \sup_{X \neq Y \in I\times\Omega} \frac{|f(X) - f(Y)|}{d_p(X,Y)^\alpha} < \infty \right\}.$$

The above quantity defines a semi-norm for f, also denoted by

$$[f]_{\alpha, I\times\Omega} = \sup_{X \neq Y \in I\times\Omega} \frac{|f(X) - f(Y)|}{d_p(X,Y)^\alpha},$$

so that the natural norm associated to $C^{\alpha/2,\alpha}_{par}(I\times\Omega)$ is denoted by

$$\|.\|_\alpha = \|.\|_{\infty,I\times\Omega} + [.]_{\alpha,I\times\Omega}.$$

The notion of parabolic space can be extended to parameters larger than 1. For any nonintegral positive number ℓ (not necessarily an integer), we define the space $\mathcal{C}^{\ell/2,\ell}_{par}(I\times\Omega)$ as the space of continuous functions which admit derivatives $\partial_t^s\partial_x^r$ for any integers s and r such that $2r+s<\ell$. More precisely,

$$\mathcal{C}^{\ell/2,\ell}_{par}(I\times\Omega) = \left\{ f\in\mathcal{C}(I\times\Omega),\ \sum_{2r+s\leqslant\lfloor\ell\rfloor}\|\partial_t^r\partial_x^s f\|_{\infty,I\times\Omega} + \sum_{2r+s=\lfloor\ell\rfloor}[\partial_t^r\partial_x^s f]_{\ell-\lfloor\ell\rfloor,I\times\Omega}<+\infty \right\}. \quad (4.1.3)$$

Observe that a function $f\in\mathcal{C}^{\alpha/2,\alpha}_{par}((0,T)\times\Omega)$ is $\alpha/2-$Hölder with respect to the time variable and $\alpha-$Hölder with respect to the space variable, following the idea that a solution to a parabolic solution is twice as regular in the space variable than in the time variable.

Finally, for a parabolic cylinder $Q_R(t_0,x_0)$ and a function $f\in L^1(Q_R(t_0,x_0))$, the average of f on $Q_R(t_0,x_0)$ is denoted by

$$\{f\}_{R,t_0,x_0} := \frac{1}{|Q_R(t_0,x_0)|}\iint_{Q_R(t_0,x_0)} f = \frac{1}{\omega_d r^{d+2}}\iint_{Q_R(t_0,x_0)} f.$$

Parabolic Morrey spaces. Given $I\subset\mathbb{R}$ an interval and $\Omega\subset\mathbb{R}^d$, $p\geqslant 1$ and $\alpha\in(0,1)$, we define the space of Morrey functions with parameters p,α by

$$\mathcal{L}^{p,d+\alpha}(I\times\Omega) = \left\{ f\in L^p(I;L^p(\Omega));\ \sup_{Q_\rho(t_0,x_0)\subset I\times\Omega}\frac{1}{\rho^{d+\alpha}}\iint_{Q_\rho(t_0,x_0)}|f(t,x)|^p dx\,dt<+\infty \right\}.$$

The space $\mathcal{L}^{p,d+\alpha}(I\times\Omega)$ can be endowed with the norm

$$\|f\|_{\mathcal{L}^{p,d+\alpha}} = \sup_{Q_\rho(t_0,x_0)\subset I\times\Omega}\left(\frac{1}{\rho^{d+\alpha}}\iint_{Q_\rho(t_0,x_0)}|f(t,x)|^p dx\,dt\right)^{1/p}.$$

4.2 Statement of the results

In the sequel, we consider an integer $d\geqslant 1$ and $\Omega\subset\mathbb{R}^d$ a bounded open domain with a Lipschitz boundary $\partial\Omega$. The sequence $(\eta_\varepsilon)_{\varepsilon>0}$ is a sequence of positive numbers such that $\eta_\varepsilon\ll\varepsilon$. The Ambrosio-Tortorrelli functional is defined as (4.1.1). Let u_0,v_0,g be three functions satisfying the following conditions:

(H.1) $(u_0,v_0)\in[H^1(\Omega)\cap L^\infty(\Omega)]^2$ and $g\in H^1_{loc}([0,\infty);H^1(\Omega))\cap L^\infty([0,\infty)\times\Omega)$,

(H.2) $g(0,\cdot)=u_0$ and $0\leqslant v_0\leqslant 1$ almost everywhere on Ω,

(H.3) $(u_0-g(0,\cdot),v_0-1)\in H^1_0(\Omega)$,

(H.4) $g(t)\in L^\infty(\partial\Omega)\cap H^{1/2}(\partial\Omega)$ for almost every $t\in(0,+\infty)$.

The system of equations corresponding to the gradient flow of the functional AT_ε writes as :

$$\begin{cases} \partial_t u - \operatorname{div}((\eta_\varepsilon+v^2)\nabla u) = 0 & \text{on } (0,+\infty)\times\Omega \\ \partial_t v - \varepsilon\Delta v + \frac{v-1}{4\varepsilon} + v|\nabla u|^2 = 0 & \text{on } (0,+\infty)\times\Omega \\ (u,v)=(g,1) & \text{on } (0,+\infty)\times\partial\Omega \\ (u(0,\cdot),v(0,\cdot))=(v_0,u_0) & \text{on } \Omega \end{cases} \quad (4.2.1)$$

We define the following notion of weak solution for the system (4.2.1) :

Definition 4.2.1. *Let u_0, v_0, g be functions satisfying conditions $(H.1), (H.2), (H.3)$ and $(H.4)$. Denote by*

$$\mathcal{F} = [L^2_{loc}([0,+\infty); H^1(\Omega)) \cap H^1_{loc}([0,+\infty); L^2(\Omega)) \cap L^\infty([0,+\infty)\times\Omega)]^2 \tag{4.2.2}$$

A pair $(u,v) \in \mathcal{F}$ is a weak solution of (4.2.1) if for every every $\varphi \in \mathcal{C}^\infty_c((0,+\infty)\times\Omega)$,

$$\int_0^{+\infty}\int_\Omega \left[\partial_t u(t,x)\varphi(t,x) + (\eta_\varepsilon + v^2)\nabla u(t,x)\nabla\varphi(t,x)\right] dxdt = 0, \tag{4.2.3}$$

and

$$\int_0^{+\infty}\int_\Omega \Big[\partial_t v(t,x)\varphi(t,x) + \varepsilon\nabla v(t,x)\cdot\nabla\varphi(t,x) + \frac{v(t,x)-1}{4\varepsilon}\varphi(t,x) + v(t,x)|\nabla u(t,x)|^2\varphi(t,x)\Big] dxdt = 0. \tag{4.2.4}$$

Our first main statement, which is proved in section 4.3, is the following :

Theorem 4.2.2. *Let u_0, v_0, g be functions satisfying conditions $(H.1), (H.2), (H.3)$ and $(H.4)$.*

The system (4.2.1) admits a weak solution (u,v) in the sense of Definition 4.2.1. Moreover, the pair (u,v) satisfies the following uniform energy bound for every $t > 0$,

$$\begin{aligned} AT_\varepsilon(u(t),v(t)) + \int_0^t \left(\|\partial_t u(s)\|_2^2 + \|\partial_t v(s)\|_2^2\right) ds \\ \leqslant AT_\varepsilon(u_0,v_0) + \int_0^t \|\partial_t g(s)\|_2^2 ds + \int_0^t\int_\Omega (\eta_\varepsilon + v(s)^2)|\partial_t\nabla g(s)|^2 dxds \\ + 2\int_0^t\int_\Omega (\eta_\varepsilon + v(s)^2)\nabla u(s)\cdot\nabla(\partial_t g(s))dxds, \end{aligned} \tag{4.2.5}$$

and it satisfies the maximum principle

$$0 \leqslant v \leqslant 1, \quad |u| \leqslant \max(\|g\|_{\infty,[0,+\infty)\times\mathbb{R}^d}, \|u_0\|_{\infty,\Omega}) \text{ a.e on } (0,+\infty)\times\Omega. \tag{4.2.6}$$

To our knowledge, this notion of weak solution is not sufficient to grant the uniqueness of the solution of (4.2.1) (see [FP04, Remark 2.3]). However, inspired from the proof of [FP04, Theorem 2.3, Step 3], we obtain the following notion of uniqueness for the system (4.2.1):

Theorem 4.2.3. *Let u_0, v_0, g be functions satisfying conditions $(H.1), (H.2), (H.3)$ and $(H.4)$. Suppose that the boundary $\partial\Omega$ is of class $\mathcal{C}^2$.*

Let (u_1,v_1) and (u_2,v_2) be weak solutions of (4.2.1) in the sense of Definition 4.2.1. Suppose that $\nabla u_2, \nabla u_1 \in L^{d+2}((0,T); L^{d+2}(\Omega))$ for every $T > 0$. Then $(u_1,v_1) = (u_2,v_2)$.

Our final statement tackles the regularity of the weak solution (u,v). We prove that the pair is smooth in $(0,+\infty)\times\Omega$. The regularity up to the boundary requires additional properties on $\partial\Omega$ and on the initial and boundary conditions (u_0,v_0) and g, especially compatibility conditions on the derivatives of g and (u_0,v_0) at $\{0\}\times\partial\Omega$. To understand the necessary conditions, we can look briefly at the expected compatibility satisfied by a pair (u,v) in $\mathcal{C}^{1,2}_{par}([0,+\infty)\times\overline{\Omega})$, focusing on the equation in u. If $(u,v) \in \mathcal{C}^{1,2}_{par}([0,+\infty)\times\overline{\Omega})$, then we must have

$$\partial_t u(0,x) = \lim_{t\to} \operatorname{div}((\eta_\varepsilon + v^2(t,x))\nabla u(t,x)) = \operatorname{div}((\eta_\varepsilon + v_0(x)^2)\nabla u_0(x)) \quad \text{for any } x \in \Omega,$$

and this equality must be satisfied for any $x \in \partial\Omega$, yielding the condition

$$\partial_t g(0,x) = \operatorname{div}((\eta_\varepsilon + v_0(x)^2)\nabla u_0(x)) \quad \text{for any } x \in \partial\Omega.$$

To obtain the compatibility conditions of superior order, we differentiate with respect to t in the equations of (4.1.2) and use again that $\partial_t^k u(0,x) = \lim_{t\to 0}\partial_t^k u(t,x)$, and a similar reasoning for the function v. The following definition describes the quantity that is expected to correspond to $(\partial_t^k u(0,\cdot), \partial_t^k v(0,\cdot))$ in the case where $(u,v) \in \mathcal{C}^{k,2k}([0,+\infty)\times\overline{\Omega})$.

Definition 4.2.4. *Let $k \geqslant 0$ be an integer and let $(u_0, v_0) \in [\mathcal{C}^{k+1,1}(\overline{\Omega})]^2$. We define recursively the sequence $(\partial_t^{(\ell)} u_0, \partial_t^{(\ell)} v_0)_{\ell \leqslant k} : \Omega \to \mathbb{R}$ by*

$$(\partial_t^{(0)} u_0, \partial_t^{(0)} v_0) = (u_0, v_0)$$

$$(\partial_t^{(1)} u_0, \partial_t^{(1)} v_0) = \left(div((\eta_\varepsilon + v_0^2)\nabla u_0), \varepsilon \Delta v_0 + \frac{1 - v_0}{4\varepsilon} - v_0 |\nabla u_0|^2 \right)$$

and for every $1 \leqslant \ell \leqslant k-1$,

$$\partial_t^{(\ell+1)} u_0 \;=\; \sum_{a+b+c=\ell} \binom{k}{a,b,c} \partial_t^{(a)} v_0 \partial_t^{(b)} v_0 \Delta \partial_t^{(c)} u_0 \;+\; 2 \sum_{a+b+c=\ell} \binom{k}{a,b,c} \partial_t^{(a)} v_0 \nabla \partial_t^{(b)} v_0 \;\cdot\; \nabla \partial_t^{(c)} u_0,$$

$$\partial_t^{(\ell+1)} v_0 = \varepsilon \Delta \partial_t^{(\ell)} v_0 - \frac{1}{4\varepsilon} \partial_t^{(\ell)} v_0 - \sum_{a+b+c=\ell} \binom{k}{a,b,c} \partial_t^{(a)} v_0 \nabla \partial_t^{(b)} u_0 \cdot \nabla \partial_t^{(c)} u_0.$$

Notice that if $(u, v) \in \mathcal{C}^{k,2k}([0, +\infty) \times \overline{\Omega})$ is a solution of (4.2.1), then one has the equality $(\partial_t^{(\ell)} u_0, \partial_t^{(\ell)} v_0) = (\partial_t^{(\ell)} u(0, \cdot), \partial_t^{(\ell)} v(0, \cdot))$ on $\overline{\Omega}$.

Theorem 4.2.5. *Suppose that Ω has a Lipschitz boundary $\partial\Omega$ and let u_0, v_0, g be functions satisfying conditions $(H.1), (H.2), (H.3)$ and $(H.4)$. Let (u, v) be a weak solution of (4.2.1) and suppose that it also satisfies (4.2.5) and (4.2.6). Then $(u, v) \in [\mathcal{C}^\infty((0, \infty) \times \Omega)]^2$.*

Moreover, suppose that there exists an integer $k \geqslant 1$ and a real number $\beta \in (0, 1)$ satisfying the following conditions:

(i) $(u_0, v_0) \in \mathcal{C}^{k+1,1}(\Omega)$,

(ii) $g \in \mathcal{C}_{par}^{\lfloor k/2 \rfloor + \beta/2, 2\lfloor k/2 \rfloor + \beta}([0, +\infty) \times \Omega)$,

(iii) $\partial\Omega \in \mathcal{C}^{k,\beta}$,

(iv) for any integer $\ell \leqslant \lfloor k/2 \rfloor + 1$,

$$(\partial_t^{(\ell)} u_0, \partial_t^{(\ell)} v_0) = (\partial_t^{(\ell)} g(0, x), 0).$$

Then $(u, v) \in \mathcal{C}_{par}^{\lfloor k/2 \rfloor + \beta/2, 2\lfloor k/2 \rfloor + \beta}([0, +\infty) \times \overline{\Omega})$.

Notice that the regularity stated in Theorem 4.2.5 is non-trivial. Indeed, the second equation of (4.2.1) is of the form

$$\partial_t v - \varepsilon \Delta v = f \in L^\infty((0, +\infty) \times L^1(\Omega)),$$

for which the standard theory does not apply. To prove the regularity for the solution of (4.2.1), we use arguments from the theory of parabolic differential equations (see [Lie96]). The key point to start a bootstrap argument is to obtain parabolic Hölder regularity for the pair (u, v). This is achieved using a parabolic version of Caccioppoli and Poincaré's inequalities to prove that ∇u belongs to a suitable Morrey space.

In the sequel, we adopt the following notation given a function $g \in L^\infty(\partial\Omega) \cap H^{1/2}(\partial\Omega)$:

$$\mathcal{A}_g = \{(u, v) \in [H^1(\Omega) \cap L^\infty(\Omega)]^2, u_{|\partial\Omega} = g, v_{|\partial\Omega} = 1\}.$$

A consequence of the direct method in calculus of variation is the following result:

Proposition 4.2.6. *Let $\delta > 0$. Let $(\tilde{u}, \tilde{v}) \in [H^1(\Omega) \cap L^\infty(\Omega)]^2$ with $0 \leqslant \tilde{v} \leqslant 1$ a.e in Ω and $g \in L^\infty(\partial\Omega) \cap H^{1/2}(\partial\Omega)$. The problem*

$$\min \left\{ AT_\varepsilon(u,v) + \frac{1}{\delta} \int_\Omega \left(|u-\tilde{u}|^2 + |v-\tilde{v}|^2 \right), (u,v) \in \mathcal{A}_g \right\}$$

admits a solution (u,v) which satisfies the following maximum principle :

$$|u| \leqslant \max(\|g\|_{\infty,\partial\Omega}, \|\tilde{u}\|_{\infty,\Omega}), \qquad 0 \leqslant v \leqslant 1 \quad \text{a.e in } \Omega.$$

The proof of the maximum principle follows from a classical truncation argument.

4.3 Existence of solutions

The object of section 4.3 is to prove Theorem 4.2.2, namely the existence of a weak solution to (4.2.1) by using the finite difference approximation in time, in a similar fashion as in [Xu94] and [BM14]. The Euler scheme is defined as followed :

Definition 4.3.1. *Let $\delta > 0$ and $(u_0, v_0) \in (H^1(\Omega) \cap L^\infty(\Omega))^2$ be a pair such that $(u_0, v_0) = (g(0,\cdot), 1)$ on $\partial\Omega$ and such that $0 \leqslant v_0 \leqslant 1$. The sequence $((u_i^\delta, v_i^\delta))_{i \geqslant 0}$ is defined recursively by $(u_0^\delta, v_0^\delta) = (u_0, v_0)$ and for $i \geqslant 1$,*

$$(u_{i+1}^\delta, v_{i+1}^\delta) \in Argmin \left\{ AT_\varepsilon(u,v) + \frac{1}{\delta} \int_\Omega \left(|u-u_i^\delta|^2 + |v-v_i^\delta|^2 \right), (u,v) \in \mathcal{A}_{g((i+1)\delta)} \right\}.$$

The pair $(\overline{u}^\delta(t), \overline{v}^\delta(t)) = \sum_{i=0}^\infty (u_i^\delta(t), v_i^\delta(t)) 1_{t \in [i\delta,(i+1)\delta)}(t)$ aims to be a time approximation of a weak solution to (4.2.1).

Let us highlight that, as the functional is not strictly convex, the minimizing pair $(u_{i+1}^\delta, v_{i+1}^\delta)$ might not be unique, which explains why the functional space $\mathcal{F}$ is not a uniqueness class for Definition 4.2.1. The lack of uniqueness is also an issue for the numerical approximation, as the classic gradient scheme might not converge to the minimizer (u_i^δ, v_i^δ). See [BM14] for an alternative algorithm used for numerical experiments.

Observe that this definition implies that the pair (u_i^δ, v_i^δ) is a weak distributional solution of the following Euler-Lagrange equation for any index $i \geqslant 1$:

$$\begin{cases} \dfrac{u_i^\delta - u_{i-1}^\delta}{\delta} - \operatorname{div}((\eta_\varepsilon + |v_i^\delta|^2)\nabla u_i^\delta) = 0 & \text{in } \mathcal{D}'(\Omega), \\ \dfrac{v_i^\delta - v_{i-1}^\delta}{\delta} - \varepsilon\Delta v_i^\delta + \dfrac{v_i^\delta - 1}{4\varepsilon} + v_i^\delta |\nabla u_i^\delta|^2 = 0 & \text{in } \mathcal{D}'(\Omega), \\ (u_i^\delta - g(i\delta,\cdot), v_i^\delta - 1) \in [H_0^1(\Omega)]^2 \end{cases} \tag{4.3.1}$$

4.3.1 Study of the sequence (u_i, v_i)

In this subsection, we fix a positive number $\delta > 0$ and consider the sequence $((u_i^\delta, v_i^\delta))_{i \geqslant 0}$ defined in Definition 4.3.1. In order to pass to the limit in the sequence $((u_i^\delta, v_i^\delta))_{i \geqslant 0}$ as $\delta \to 0$, we need compactness properties, which are the issue of this subsection. More precisely, we provide a uniform energy bound on $(AT_\varepsilon(u_i^\delta, v_i^\delta))_{i \geqslant 0}$. We drop the subscript δ and write simply the sequence as (u_i, v_i).

We also define

$$g_i(x) = g(x, i\delta) \quad \forall x \in \mathbb{R}^d. \tag{4.3.2}$$

A consequence of Proposition 4.2.6 is the discrete maximum principle :

Proposition 4.3.2. *For any index $i \geqslant 1$, the pair (u_i, v_i) satisfies*

$$0 \leqslant v_i \leqslant 1, \quad |u_i| \leqslant \max(\|g\|_{\infty,(0,\infty)\times\Omega}, \|u_0\|_{\infty,\Omega}). \tag{4.3.3}$$

We now establish the discrete energy bound on $(AT_\varepsilon(u_i, v_i))_{i\geqslant 0}$.

Proposition 4.3.3. *For any $N \geqslant 1$,*

$$AT_\varepsilon(u_N, v_N) + \sum_{i=1}^{N} \frac{1}{\delta}\int_\Omega \left(|u_i - u_{i-1}|^2 + |v_i - v_{i-1}|^2\right) \leqslant AT_\varepsilon(u_0, v_0) + \sum_{i=1}^{N} \frac{1}{\delta}\int_\Omega |g_i - g_{i-1}|^2 + \sum_{i=1}^{N}\left[\int_\Omega (\eta_\varepsilon + v_{i-1}^2)(|\nabla(g_i - g_{i-1})|^2 + 2\nabla u_{i-1}\cdot\nabla(g_i - g_{i-1}))\right]. \quad (4.3.4)$$

Proof. Take the pair $(u, v) = (u_{i-1} + g_i - g_{i-1}, v_{i-1})$, which is an element of $\mathcal{A}_{g_i}$, as a competitor. The construction of the pair (u_i, v_i) in Definition 4.3.1 provides

$$\begin{aligned} AT_\varepsilon(u_i, v_i) &+ \frac{1}{\delta}\int_\Omega \left(|u_i - u_{i-1}|^2 + |v_i - v_{i-1}|^2\right) \\ &\leqslant AT_\varepsilon(u, v) + \frac{1}{\delta}\int_\Omega \left(|u - u_{i-1}|^2 + |v - v_{i-1}|^2\right) = AT_\varepsilon(u_{i-1}, v_{i-1}) + \frac{1}{\delta}\int_\Omega |g_i - g_{i-1}|^2 \\ &\quad + \int_\Omega (\eta_\varepsilon + v_{i-1}^2)\left(|\nabla(g_i - g_{i-1})|^2 + 2\nabla u_{i-1}\cdot\nabla(g_i - g_{i-1})\right). \end{aligned} \quad (4.3.5)$$

Summing (4.3.5) for $i = 1, \dots, N$ and simplifying yields the desired inequality. □

4.3.2 Convergence with delta

In the subsection, we let $\delta \to 0$ to obtain a solution of (4.2.1). We define the respectively piecewise constant and affine functions on $\mathbb{R}_+ \times \Omega$:

$$(\overline{u}_\delta(t), \overline{v}_\delta(t), \overline{g}_\delta(t)) = (u_i, v_i, g_i) \qquad \text{for } t \in [i\delta, (i+1)\delta),$$

$$(\hat{u}_\delta(t), \hat{v}_\delta(t), \hat{g}_\delta(t)) = \left((u_i, v_i, g_i) + \frac{t - i\delta}{\delta}(u_{i+1} - u_i, v_{i+1} - v_i, g_{i+1} - g_i)\right) \quad \text{for } t \in [i\delta, (i+1)\delta).$$

Note that $(\overline{u}_\delta, \overline{v}_\delta, \overline{g}_\delta) \in [L^\infty_{loc}([0,\infty); H^1(\Omega)]^3$ and $(\hat{u}_\delta, \hat{v}_\delta, \hat{g}_\delta) \in [H^1_{loc}([0,\infty); H^1(\Omega))]^3$. For any time $t > 0$ let $t_\delta = \delta\left\lfloor \frac{t}{\delta}\right\rfloor$. The maximum principle Proposition 4.3.2 rewrites as

$$0 \leqslant \overline{v}_\delta, \hat{v}_\delta \leqslant 1, \quad |\overline{u}_\delta|, |\hat{u}_\delta| \leqslant \max(\|g\|_{\infty,(0,+\infty)\times\Omega}, \|u_0\|_{\infty,\Omega}), \quad (4.3.6)$$

while Proposition 4.3.3 rewrites as

$$\begin{aligned} AT_\varepsilon(\overline{u}_\delta(t), \overline{v}_\delta(t)) &+ \int_0^{t_\delta} \left(\|\partial_t\hat{u}_\delta(s)\|_2^2 + \|\partial_t\hat{v}_\delta(s)\|_2^2\right) ds \leqslant AT_\varepsilon(u_0, v_0) + \int_0^{t_\delta} \|\partial_t\hat{g}_\delta(s)\|_2^2 ds \\ &\quad + \int_0^{t_\delta}\int_\Omega (\eta_\varepsilon + \overline{v}_\delta(s)^2)\left(|\partial_t\nabla\hat{g}_\delta(s)|^2 + 2\nabla\overline{u}_\delta(s)\cdot\nabla(\partial_t\hat{g}_\delta(s)\right) dxds. \end{aligned} \quad (4.3.7)$$

Considering the Euler–Lagrange equations (4.3.1), the pair (u_i, v_i) satisfies for a.e $t > 0$

$$\begin{cases} \partial_t\hat{u}_\delta(t) - \operatorname{div}((\eta_\varepsilon + \overline{v}_\delta^2(t))\nabla\overline{u}_\delta(t)) = 0 & \text{in } \mathcal{D}'(\Omega), \\ \partial_t\hat{v}_\delta(t) - \varepsilon\Delta\overline{v}_\delta(t) + \frac{\overline{v}_\delta(t)-1}{4\varepsilon} + \overline{v}_\delta(t)|\nabla\overline{u}_\delta(t)|^2 = 0 & \text{in } \mathcal{D}'(\Omega), \\ (\overline{u}_\delta(t) - g_i(t), v_i^\delta(t) - 1) \in H_0^1(\Omega). \end{cases} \quad (4.3.8)$$

Let $\varphi \in \mathcal{D}((0,\infty)\times\Omega)$. For any $t > 0$, the function $\varphi(t,\cdot)$ is in $\mathcal{D}(\Omega)$, so

$$\int_\Omega \partial_t\hat{u}_\delta(t,x)\varphi(t,x)dx + \int_\Omega (\eta_\varepsilon + \overline{v}_\delta^2(t,x))\nabla\overline{u}_\delta(t,x)\cdot\nabla\varphi(t,x)dx = 0, \quad (4.3.9)$$

$$\int_\Omega \partial_t \hat{v}_\delta(t,x)\varphi(t,x)dx + \int_\Omega \Big[\varepsilon \nabla \overline{v}_\delta(t,x)\cdot\nabla\varphi(t,x) + \frac{\overline{v}_\delta(t,x)-1}{4\varepsilon}\varphi(t,x) + \overline{v}_\delta(t,x)|\nabla\overline{u}_\delta(t,x)|^2\varphi(t,x)\Big]dx = 0 \quad (4.3.10)$$

Integrating (4.3.9) and (4.3.10) over $(0,\infty)$, we deduce that

$$\int_0^\infty\int_\Omega \partial_t\hat{u}_\delta(t,x)\varphi(t,x)dxdt + \int_0^\infty\int_\Omega (\eta_\varepsilon + \overline{v}_\delta^2(t,x))\nabla\overline{u}_\delta(t,x)\cdot\nabla\varphi(t,x)dxdt = 0, \quad (4.3.11)$$

$$\int_0^\infty\int_\Omega \partial_t\hat{v}_\delta(t,x)\varphi(t,x)dxdt + \int_0^\infty\int_\Omega \Big[\varepsilon\nabla\overline{v}_\delta(t,x)\cdot\nabla\varphi(t,x) + \frac{\overline{v}_\delta(t,x)-1}{4\varepsilon}\varphi(t,x) + \overline{v}_\delta(t,x)|\nabla\overline{u}_\delta(t,x)|^2\varphi(t,x)\Big]dxdt = 0. \quad (4.3.12)$$

As (4.3.11) and (4.3.12) are true for any $\varphi \in \mathcal{D}((0,\infty)\times\Omega)$, the space of admissible test functions $\mathcal{C}_c^\infty((0,+\infty)\times\Omega)$ can be extended by density to $L^\infty_{loc}((0,+\infty)\times\Omega)\cap H^1_{loc}((0,+\infty);L^2(\Omega))\cap L^2((0,+\infty);H^1_0(\Omega))$.

Let us first recall the following standard interpolation result for the sequence $(\overline{g}_\delta,\hat{g}_\delta)$.

Proposition 4.3.4. *Suppose that $g \in H^2_{loc}((0,+\infty);H^1(\Omega))$. Then*

1. *$\overline{g}_\delta(t) \longrightarrow g(t)$ strongly in $H^1(\Omega)$ for almost every $t>0$,*
2. *$\overline{g}_\delta \longrightarrow g$ strongly in $L^2_{loc}((0,+\infty);H^1(\Omega))$*
3. *$\partial_t\hat{g}_\delta(t) \longrightarrow \partial_t g(t)$ strongly in $H^1(\Omega)$ for almost every $t>0$.*
4. *$\partial_t\hat{g}_\delta \longrightarrow \partial_t g$ strongly in $L^2_{loc}((0,+\infty);H^1(\Omega))$,*

We now prove uniform estimates for $(\overline{u}_\delta,\overline{v}_\delta)$ and $(\hat{u}_\delta,\hat{v}_\delta)$ derived from (4.3.7).

Corollary 4.3.5. *For any $T>0$, there exists a constant $C_{g,T}>0$, independent of δ, such that*

$$\sup_{t\in[0,T]}\|\overline{u}_\delta(t)\|_{H^1(\Omega)} + \sup_{t\in[0,T]}\|\overline{v}_\delta(t)\|_{H^1(\Omega)} + \int_0^T\int_\Omega \left(\|\partial_t\hat{u}_\delta(s)\|_2^2 + \|\partial_t\hat{v}_\delta(s)\|_2^2\right)ds < C_{g,T}. \quad (4.3.13)$$

Proof. Fix $t>0$ and set $t_\delta = \left\lceil \frac{t}{\delta}\right\rceil$. Using that $2ab \leqslant a^2+b^2$ in the right-hand side of (4.3.7),

$$\begin{aligned}
&AT_\varepsilon(\overline{u}_\delta(t),\overline{v}_\delta(t)) + \int_0^{t_\delta}\left(\|\partial_t\hat{u}_\delta(s)\|_2^2 + \|\partial_t\hat{v}_\delta(s)\|_2^2\right)dxds\\
&\leqslant AT_\varepsilon(u_0,v_0) + \int_0^{t_\delta}\|\partial_t\hat{g}_\delta(s)\|_2^2 ds + \int_0^{t_\delta}\int_\Omega(\eta_\varepsilon+\overline{v}_\delta(s)^2)\left(|\partial_t\nabla\hat{g}_\delta(s)|^2 + 2\nabla\overline{u}_\delta(s)\cdot\nabla\partial_t\hat{g}_\delta(s)\right)dxds\\
&\leqslant AT_\varepsilon(u_0,v_0) + \int_0^{t_\delta}\|\partial_t\hat{g}_\delta(s)\|_2^2 ds + 2(1+\eta_\varepsilon)\int_0^{t_\delta}\int_\Omega|\partial_t\nabla\hat{g}_\delta(s)|^2dxds + \int_0^{t_\delta}\int_\Omega(\eta_\varepsilon+\overline{v}_\delta^2(s))|\nabla\overline{u}_\delta(s)|^2dxds.
\end{aligned}$$

From Proposition 4.3.4, the sequence $(\|\partial_t\hat{g}_\delta\|_{L^2((0,T);H^1(\Omega))})_\delta$ is uniformly bounded, so there exists a constant $C_{g,T}$ such that

$$AT_\varepsilon(\overline{u}_\delta(t),\overline{v}_\delta(t)) + \int_0^{t_\delta}\left(\|\partial_t\hat{u}_\delta(s)\|_2^2 + \|\partial_t\hat{v}_\delta(s)\|_2^2\right)ds \leqslant AT_\varepsilon(u_0,v_0) + C_{g,T} + \int_0^{t_\delta} AT_\varepsilon(\overline{u}_\delta(s),\overline{v}_\delta(s))ds. \quad (4.3.14)$$

Applying Gronwall's lemma in (4.3.14) yields for all $t \in [0,T]$

$$AT_\varepsilon(\overline{u}_\delta(t),\overline{v}_\delta(t)) \leqslant (AT_\varepsilon(u_0,v_0)+C_{g,T})e^T, \tag{4.3.15}$$

and injecting it in (4.3.14) implies in turn for all $t \in [0,T]$

$$\int_0^{t_\delta} \left(\|\partial_t\hat{u}_\delta(s)\|_2^2+\|\partial_t\hat{v}_\delta(s)\|_2^2\right) ds \leqslant AT_\varepsilon(u_0,v_0)+C_{g,T}+\int_0^t AT_\varepsilon(\overline{u}_\delta(s),\overline{v}_\delta(s))ds \leqslant C_{g,T}, \tag{4.3.16}$$

which implies (4.3.13). □

We are now ready to pass to the limit in (4.3.11), (4.3.12) and (4.3.7).

Proposition 4.3.6. *There exists a pair* $(u,v) \in \left(L^\infty_{loc}(\mathbb{R}_+;H^1(\Omega)) \cap H^1_{loc}(\mathbb{R}_+;L^2(\Omega))\right)^2$ *and a (not relabeled) subsequence* $(\overline{u}_\delta,\overline{v}_\delta,\hat{u}_\delta,\hat{v}_\delta)$ *satisfying :*

1. $(\hat{u}_\delta,\hat{v}_\delta) \to (u,v)$ *strongly in* $(C^0([0,T];L^2(\Omega))^2$ *for every* $T>0$,
2. $(\partial_t\hat{u}_\delta,\partial_t\hat{v}_\delta) \rightharpoonup (\partial_t u,\partial_t v)$ *weakly in* $(L^2([0,T];L^2(\Omega)))^2$,
3. $(\overline{u}_\delta,\overline{v}_\delta) \longrightarrow (u,v)$ *strongly in* $[L^2_{loc}((0,+\infty);L^2(\Omega))]^2$ *and almost everywhere on* $(0,+\infty)\times\Omega$,
4. $(\overline{u}_\delta(t),\overline{v}_\delta(t)) \rightharpoonup (u(t),v(t))$ *weakly in* $[H^1(\Omega)]^2$ *for almost every* $t>0$,
5. $(\overline{u}_\delta,\overline{v}_\delta) \rightharpoonup (u,v))$ *weakly in* $[L^2_{loc}((0,+\infty);H^1(\Omega))]^2$,
6. $(\overline{u}_\delta,\overline{v}_\delta) \overset{\star}{\rightharpoonup} (u,v)$ *weakly-*$\star$ *in* $[L^\infty_{loc}((0,+\infty)\times\Omega)]^2$,
7. $(u(t)-g(t),v(t)-1) \in H^1_0(\Omega)$ *for almost every* $t>0$,

Proof. **Step 1 :** There exists a pair $(u,v) \in C^0([0,+\infty);L^2(\Omega))$ and a converging subsequence $(\hat{u}_\delta,\hat{v}_\delta)$ such that $(\hat{u}_\delta,\hat{v}_\delta) \to (u(t),v(t))$ strongly in $(C^0([0,T];L^2(\Omega))^2$ for every $T>0$.

Let us show that, for any $T>0$, the sequence $(\hat{u}_\delta,\hat{v}_\delta)_\delta$, satisfies the hypothesis of the Ascoli theorem.

- For any fixed time t, the sets $\{\|\hat{u}_\delta(t)\|_2^2,\delta>0\}$ and $\{\|\hat{v}_\delta(t)\|_2^2,\delta>0\}$ are relatively compact in $L^2(\Omega)$ for the strong topology. Indeed, according to Proposition 4.3.2,

$$\sup_{[0,T]}\|\hat{u}_\delta(t)\|_{\infty,\Omega} \leqslant \max(\|u_0\|_{\infty,\Omega},\|g\|_{\infty,(0,+\infty)\times\Omega}), \quad \sup_{[0,T]}\|\hat{v}_\delta(t)\|_{\infty,\Omega} \leqslant 1,$$

  and according to Corollary 4.3.5,

$$\sup_{[0,T]}\|\nabla\hat{u}_\delta(t)\|_{L^2(\Omega)} \leqslant \sup_{[0,T]}\|\nabla\overline{u}_\delta(t)\|_{L^2(\Omega)} \leqslant C_{T,g}, \tag{4.3.17}$$

$$\sup_{[0,T]}\|\nabla\hat{v}_\delta(t)\|_{L^2(\Omega)} \leqslant \sup_{[0,T]}\|\nabla\overline{v}_\delta(t)\|_{L^2(\Omega)} \leqslant C_{T,g}.$$

  Hence, the sets $\{\|\hat{u}_\delta(t)\|_2^2,\delta>0\}$ and $\{\|\hat{v}_\delta(t)\|_2^2,\delta>0\}$ are bounded in $[H^1(\Omega)]^2$, so they are compact in $[L^2(\Omega)]^2$ according to the Rellich-Theorem.

- The sequence $(\hat{u}_\delta,\hat{v}_\delta)_\delta$ is equi-continuous for the $L^2(\Omega)$ norm. This is a consequence of (4.3.13) and Jensen inequality : for any $t<s$,

$$\|\hat{u}_\delta(t)-\hat{u}_\delta(s)\|_2^2 = \left\|\int_t^s \partial_t\hat{u}_\delta\right\|_2^2 \leqslant (s-t)\int_t^s\|\partial_t\hat{u}_\delta\|_2^2 \leqslant (s-t)C_{g,T}. \tag{4.3.18}$$

  A similar inequality occurs for v.

Hence, according to Ascoli's theorem, the sets $\{(\hat{u}_\delta, \hat{v}_\delta), \delta > 0\}$ is a compact subset of $C^0([0,T]; L^2(\Omega))$ for any $T > 0$. Using a diagonal extraction argument, we deduce the existence of a pair $(u, v) \in [C^0([0,+\infty); L^2(\Omega))]^2$ and a (not relabelled) converging subsequence $(\hat{u}_\delta, \hat{v}_\delta)$ to (u, v) strongly in the space $[C^0([0,+\infty); L^2(\Omega))]^2$. This proves item (1).

Step 2 : $(\partial_t \hat{u}, \partial_t \hat{v}) \rightharpoonup (\partial_t u, \partial_t v)$ weakly in $[L^2_{loc}((0,\infty); L^2(\Omega))]^2$.

We prove that $\partial_t \hat{u}_\delta \rightharpoonup \partial_t u$ weakly in $[L^2([0,T]; L^2(\Omega)]^2$. Indeed, we first have $(\partial_t \hat{u}_\delta, \partial_t \hat{v}_\delta) \rightharpoonup (\partial_t u, \partial_t v)$ in $[\mathcal{D}'((0,T)\times\Omega)]^2$. Using Corollary 4.3.5, the sequence $(\partial_t \hat{u}_\delta, \partial_t \hat{v}_\delta)$ is bounded in $[L^2_{loc}((0,+\infty); L^2(\Omega))]^2$, so it admits a converging subsequence to an element in $[L^2_{loc}((0,+\infty); L^2(\Omega))]^2$. This element has to agree with $(\partial_t u, \partial_t v)$. This proves item (2). In particular, $(u, v) \in [H^1((0,T); L^2(\Omega))]^2$.

Step 3 : We now prove item (3).

Fix a time $t > 0$. We prove that $\overline{u}_\delta(t) \to u(t)$ strongly in $L^2(\Omega)$. Indeed, let $i = \lfloor t/\delta \rfloor$. Observe that with Proposition 4.3.2 and the strong convergence $\hat{u}_\delta(t) \to u(t)$ in $L^2(\Omega)$,

$$\begin{aligned}\|\overline{u}_\delta(t) - u(t)\|_2 = \|\hat{u}_\delta(i\delta) - u(t)\|_2 &\leqslant \|\hat{u}_\delta(i\delta) - \hat{u}_\delta(t)\|_2 + \|\hat{u}_\delta(t) - u(t)\|_2 \\ &\leqslant C_{g,T}\sqrt{\delta} + \|\hat{u}_\delta(t) - u(t)\|_2 \to 0.\end{aligned}$$

Using Proposition 4.3.2 and item (1), we can apply Lebesgue convergence theorem to conclude that $\overline{u}_\delta \to u$ strongly in $L^2_{loc}((0,+\infty)\times\Omega)$. We also deduce that there exists a (not relabelled) subsequence $\overline{u}_\delta$ that converges to u almost everywhere on $(0,+\infty)\times\Omega$. A similar argument holds for $\overline{v}_\delta$, which concludes (3).

Step 4 : We prove items (4) and (5).

Recall from Proposition 4.3.2 and (4.3.17) that the sequence $(\hat{u}_\delta(t))$ is bounded in $H^1(\Omega)$. Hence, it admits a weakly-converging subsequence in $H^1(\Omega)$ and then, again due to Rellich theorem, a strong converging subsequence in $L^2(\Omega)$. Thanks to item (1), we deduce that $\hat{u}_\delta(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega)$. We deduce that $t \mapsto u(t) \in H^1(\Omega)$ is weakly measurable as the limit of weak measurable functions. Since $H^1(\Omega)$ is separable, according to Pettis theorem (see [Sho97, Theorem III.1.1]), this map is in fact strongly measurable. We can integrate this convergence in time using Lebesgue dominated convergence theorem and (4.3.17) to conclude item (5). A similar argument applies to $\overline{v}_\delta$.

Step 5 : The pair (u, v) belongs to $[L^\infty_{loc}((0,+\infty); H^1(\Omega)) \cap H^1_{loc}((0,+\infty); L^2(\Omega))]^2$.

Using again Proposition 4.3.2 and by the weak lower semi-continuity of $\|.\|_{H^1(\Omega)}$, we deduce that for all $t \in [0,T]$,

$$\|u(t)\|_{H^1(\Omega)} \leqslant \liminf_{\delta\to 0} \|\overline{u}_\delta(t)\|_{H^1(\Omega)} \leqslant C_{g,T},$$

which proves that $u \in L^\infty_{loc}((0,+\infty); H^1(\Omega))$. Morevoer, since $\partial_t u \in L^2_{loc}([0,\infty); L^2(\Omega))$, we deduce that $u \in H^1_{loc}((0,\infty); L^2(\Omega))$. The similar reasoning applies to $\overline{v}_\delta$.

Step 6 : Item (6) is a consequence of Proposition 4.3.2 and (4.3.6).

Step 7 : We prove that $(u(t) - g(t), v(t) - 1) \in H^1_0(\Omega)$ for almost every $t > 0$.

Indeed, according to Step 4 and Proposition 4.3.4, for almost every $t > 0$, $\overline{u}_\delta(t) - \overline{g}_\delta(t) \rightharpoonup u(t) - g(t)$ weakly in $H^1(\Omega)$. Since $\overline{u}_\delta(t) - \overline{g}_\delta(t) \in H^1_0(\Omega)$, we deduce that $u(t) - g(t) \in H^1_0(\Omega)$ as well. The similar reasoning allows to conclude that $v(t) - 1 \in H^1_0(\Omega)$ for almost every $t > 0$ as well. This proves item (7).

Finally, as $(\overline{u}_\delta, \overline{v}_\delta \to (u, v)$ strongly in $L^2([0,T]; L^2(\Omega))$, we can extract a subsequence that converges a.e on $[0,T]\times\Omega$, which proves (5). □

We can easily deduce the maximum principle for the pair (u, v), which we state in the following proposition.

Proposition 4.3.7. *Recall that $0 \leqslant v_0(x) \leqslant 1$ for all x. Then $0 \leqslant v(t,x) \leqslant 1$. The pair (u, v) satisfies*

$$0 \leqslant v \leqslant 1, \quad \|u\|_\infty \leqslant \max(\|g\|_\infty, \|u_0\|_{\infty,\Omega}).$$

Proof. This is a consequence of (4.3.6) and item (3) of Proposition 4.3.6. □

We now prove that the pair (u, v) is a solution of (4.2.1) in the sense of Definition 4.2.1.

Proposition 4.3.8. *The pair (u, v) is a solution of (4.2.1).*

Proof. In order to pass to the limit in (4.3.11) and especially in the second equation (via weak-strong convergence) (4.3.12), we must prove the strong convergence of the term $\nabla\overline{u}_\delta$ in $L^2(\Omega)$ in the equation (4.2.1). Our proof is inspired from the method of [Xu94]. Namely we multiply the second equation of (4.3.11) by $\overline{u}_\delta - u$ and obtain the strong convergence of $\nabla(\overline{u}_\delta - u)$ in $L^2_{\rm loc}((0,+\infty)\times\Omega)$. Here are the details. Let $\varphi \in \mathcal{C}_c^\infty((0,+\infty)\times\Omega)$ with $\varphi \geqslant 0$. According to Proposition 4.3.7 and Corollary 4.3.5, the function $\varphi(\overline{u}_\delta - u)$ then belongs to $L^\infty_{loc}((0,+\infty)\times\Omega) \cap H^1_{loc}((0,+\infty); L^2(\Omega)) \cap L^2((0,+\infty); H^1_0(\Omega))$. Applying it as a test function in (4.3.11) yields

$$\int_0^{+\infty}\int_\Omega \partial_t\hat{u}_\delta(\overline{u}_\delta - u)\varphi + \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)\varphi\nabla\overline{u}_\delta\cdot\nabla(\overline{u}_\delta - u) \\ + \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)(\overline{u}_\delta - u)\nabla\overline{u}_\delta\cdot\nabla\varphi = 0.$$

By weak-strong convergence, the first and third terms satisfy

$$\int_0^{+\infty}\int_\Omega \partial_t\hat{u}_\delta(\overline{u}_\delta - u)\varphi + \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)(\overline{u}_\delta - u)\nabla\overline{u}_\delta\cdot\nabla\varphi \to 0,$$

hence

$$\int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)\varphi\nabla\overline{u}_\delta\cdot\nabla(\overline{u}_\delta - u) \to 0. \tag{4.3.19}$$

According to Proposition 4.3.6 (item (3) and (5)), $\eta_\varepsilon + \overline{v}_\delta^2$ converges strongly to $\eta_\varepsilon + v^2$ in $L^2_{loc}((0,+\infty); L^2(\Omega))$ and $\nabla\overline{u}_\delta \rightharpoonup \nabla u$ weakly in $L^2_{loc}((0,+\infty); L^2(\Omega))$, which implies

$$\int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)\nabla\overline{u}_\delta\cdot\nabla\varphi \longrightarrow \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + v^2)\nabla u\nabla\varphi. \tag{4.3.20}$$

Combining (4.3.19) and (4.3.20), we deduce

$$\lim_{\delta\to 0}\int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)\varphi|\nabla\overline{u}_\delta|^2 = \lim_{\delta\to 0}\int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)\varphi\nabla\overline{u}_\delta\cdot\nabla u = \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + v^2)\varphi|\nabla u|^2,$$

This implies that

$$\begin{aligned}\eta_\varepsilon\int_0^{+\infty}\int_\Omega|\nabla(\overline{u}_\delta - u)|^2\varphi &\leqslant \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)|\nabla(\overline{u}_\delta - u)|^2\varphi \\ &= \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)|\nabla\overline{u}_\delta|^2\varphi - 2\int_0^{+\infty}\int_\Omega(\eta + \overline{v}_\delta^2)\varphi\nabla u\cdot\nabla\overline{u}_\delta \\ &\quad + \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)|\nabla u|^2\varphi \longrightarrow 0,\end{aligned}$$

hence

$$\lim_{\delta\to 0}\int_0^{+\infty}\int_\Omega|\nabla(\overline{u}_\delta - u)|^2\varphi\,dt\,dx = 0. \tag{4.3.21}$$

We deduce that $\nabla\overline{u}_\delta \to \nabla u$ strongly in $L^2_{\rm loc}((0,+\infty)\times\Omega)$.

We are now ready to pass to the limit in each equation. Let $\varphi \in \mathcal{C}_c^\infty((0,+\infty)\times\Omega)$ be a test function. For any δ,

$$0 = \int_0^{+\infty}\int_\Omega\partial_t\hat{u}_\delta\varphi + \int_0^{+\infty}\int_\Omega(\eta_\varepsilon + \overline{v}_\delta^2)\nabla\overline{u}_\delta\nabla\varphi.$$

Using Proposition 4.3.6 item (2), $\partial_t\hat{u}_\delta \rightharpoonup \partial_t u$ in $L^2_{loc}((0,+\infty); L^2(\Omega))$, hence,

$$\int_0^{+\infty}\int_\Omega \partial_t \hat{u}_\delta \varphi \to \int_0^{+\infty}\int_\Omega \partial_t u \varphi \tag{4.3.22}$$

Combining (4.3.22) and (4.3.20), we deduce that u satisfies

$$0 = \int_0^{+\infty}\int_\Omega \partial_t u\varphi + \int_0^{+\infty}\int_\Omega (\eta_\varepsilon + v^2)\nabla u \nabla \varphi. \tag{4.3.23}$$

Now consider (4.3.12).

$$0 = \int_0^{+\infty}\int_\Omega \partial_t \hat{v}_\delta \varphi + \int_0^{+\infty}\int_\Omega \varepsilon \nabla \overline{v}_\delta \cdot \nabla \varphi + \frac{\overline{v}_\delta - 1}{4\varepsilon}\varphi + \overline{v}_\delta |\nabla \overline{u}_\delta|^2 \varphi.$$

Using that $\partial_t \hat{v}_\delta \rightharpoonup \partial_t v$ in $L^2_{loc}((0,+\infty); L^2(\Omega))$, we have

$$\int_0^{+\infty}\int_\Omega \partial_t \hat{v}_\delta \varphi \to \int_0^{+\infty}\int_\Omega \partial_t v \varphi \tag{4.3.24}$$

Using that $\overline{v}_\delta \rightharpoonup v$ in $L^2_{loc}((0,+\infty); H^1(\Omega))$ (Proposition 4.3.6 item (5)),

$$\int_0^{+\infty}\int_\Omega \varepsilon \nabla \overline{v}_\delta \cdot \nabla \varphi + \frac{\overline{v}_\delta - 1}{4\varepsilon}\varphi \to \int_0^{+\infty}\int_\Omega \varepsilon \nabla v \cdot \nabla \varphi + \frac{v-1}{4\varepsilon}\varphi. \tag{4.3.25}$$

Finally, since $\overline{v}_\delta \overset{\star}{\rightharpoonup} v$ in $L^\infty((0,+\infty); L^\infty(\Omega))$ (Proposition 4.3.6 item (6)) and $|\nabla \overline{u}_\delta|^2$ converges to $|\nabla u|^2$ in $L^1_{loc}((0,+\infty)\times\Omega)$ (see (4.3.21)), we deduce by Lebesgue dominated convergence that

$$\int_0^{+\infty}\int_\Omega \overline{v}_\delta |\nabla \overline{u}_\delta|^2 \varphi \to \int_0^{+\infty}\int_\Omega v|\nabla u|^2 \varphi. \tag{4.3.26}$$

Combining (4.3.24), (4.3.25) and (4.3.26) yields (4.2.4). □

To conclude the proof of Theorem 4.2.2, we now prove the energy bound (4.2.5).

Proposition 4.3.9. *For almost every $t > 0$, one has*

$$\begin{aligned} AT_\varepsilon(u(t), v(t)) + \int_0^t \left(\|\partial_t u(s)\|_2^2 + \|\partial_t v(s)\|_2^2\right) ds \leqslant AT(u_0, v_0) \\ + \int_0^t \|\partial_t g(s)\|_2^2 ds + \int_0^t \int_\Omega (\eta_\varepsilon + v^2(s)) \left(|\partial_t \nabla g(s)|^2 + 2\nabla u(s)\cdot \nabla \partial_t g(s)\right) dxds. \end{aligned}$$

Proof. Recall that $(\overline{u}_\delta, \overline{v}_\delta, \overline{g}_\delta)$ satisfy (4.3.7)

$$\begin{aligned} AT_\varepsilon(\overline{u}_\delta(t), \overline{v}_\delta(t)) + \int_0^{t_\delta} \left(\|\partial_t \hat{u}_\delta(s)\|_2^2 + \|\partial_t \hat{v}_\delta(s)\|_2^2\right) ds \leqslant AT_\varepsilon(u_0, v_0) + \int_0^{t_\delta} \|\partial_t \hat{g}_\delta(s)\|_2^2 ds \\ + \int_0^{t_\delta}\int_\Omega (\eta_\varepsilon + \overline{v}_\delta(s)^2) \left(|\partial_t \nabla \hat{g}_\delta(s)|^2 ds + 2\nabla \overline{u}_\delta(s)\cdot \nabla(\partial_t \hat{g}_\delta(s)\right) dxds. \end{aligned}$$

We pass to the limit in each term.

First, since for almost every $t > 0$, $\overline{v}_\delta(t) \rightharpoonup v(t)$ weakly in $H^1(\Omega)$ and $\overline{v}_\delta(t) \to v(t)$ strongly in $L^2(\Omega)$ (according to Proposition 4.3.6 items (3) and (4)), we obtain that

$$\int_\Omega \varepsilon |\nabla v(t)|^2 + \frac{(1-v(t))^2}{4\varepsilon} \leqslant \liminf_{\delta\to 0} \int_\Omega \varepsilon |\nabla \overline{v}_\delta(t)|^2 + \frac{(1-\overline{v}_\delta)^2}{4\varepsilon}. \tag{4.3.27}$$

Since, for almost every $t > 0$, $\overline{u}_\delta(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega)$ and $\overline{v}(t) \to v(t)$ strongly in $L^2(\Omega)$ (according again to Proposition 4.3.6 items (3) and (4)),

$$\int_\Omega (\eta_\varepsilon + v(t)^2)|\nabla u(t)|^2 \leqslant \liminf_{\delta\to 0} \int_\Omega (\eta_\varepsilon + \overline{v}_\delta(t)^2)|\nabla \overline{u}_\delta(t)|^2. \tag{4.3.28}$$

Secondly, using that $(\partial_t \hat{u}_\delta, \partial_t \hat{v}_\delta) \rightharpoonup (\partial_t v, \partial_t u)$ weakly in $[L^2((0,t)\times\Omega)]^2$ (according to Proposition 4.3.6 item (2)) and that $1_{(0,t_\delta)} \to 1_{(0,t)}$ strongly in $L^2((0,T); L^2(\Omega))$,

$$\int_0^t \left(\|\partial_t u(s)\|_2^2 + \|\partial_t v(s)\|_2^2\right) ds \leqslant \liminf_{\delta\to 0} \int_0^t \left(\|\partial_t \hat{u}_\delta(s)\|_2^2 + \|\partial_t \hat{v}_\delta(s)\|_2^2\right) 1_{(0,t_\delta)}(s)\, ds. \tag{4.3.29}$$

Combining (4.3.27), (4.3.28) and (4.3.29), we deduce that for almost every $t > 0$,

$$AT_\varepsilon(u(t), v(t)) + \int_0^t \left(\|\partial_t u(s)\|_2^2 + \|\partial_t v(s)\|_2^2\right) ds \leqslant \liminf_{\delta\to 0} AT_\varepsilon(\overline{u}_\delta(t), \overline{v}_\delta(t)) + \int_0^{t_\delta} \left(\|\partial_t \hat{u}_\delta(s)\|_2^2 + \|\partial_t \hat{v}_\delta(s)\|_2^2\right) ds. \tag{4.3.30}$$

On the other hand, using that $\partial_t \hat{g}_\delta \to \partial_t g$ strongly in $L^2((0,t)\times\Omega)$ (according to Proposition 4.3.4 item (3)) and that $1_{(0,t_\delta)} \to 1_{(0,t)}$ strongly in $L^2((0,t); L^2(\Omega))$, we deduce that

$$\lim_{\delta\to 0} \int_0^{t_\delta} \|\partial_t \hat{g}_\delta(s)\|_2^2 ds = \int_0^t \|\partial_t g(s)\|_2^2 ds. \tag{4.3.31}$$

Using that $\partial_t \nabla \hat{g} \to \partial_t \nabla g$ strongly in $L^2((0,t); L^2(\Omega))$, that $\overline{v}_\delta \to v$ strongly in $L^2((0,t)\times\Omega)$ and that $\nabla \overline{u} \rightharpoonup \nabla u$ weakly in $L^2((0,t)\times\Omega)$,

$$\lim_{\delta\to 0} \int_0^{t_\delta} \int_\Omega (\eta_\varepsilon + \overline{v}_\delta^2(s)) |\partial_t \nabla \hat{g}_\delta(s)|^2 \, dxds = \int_0^{t_\delta} \int_\Omega (\eta_\varepsilon + v^2(s)) |\partial_t \nabla g(s)|^2 dxds, \tag{4.3.32}$$

$$\lim_{\delta\to 0} \int_0^{t_\delta} \int_\Omega 2(\eta_\varepsilon + \overline{v}_\delta^2(s)) \nabla \overline{u}_\delta(s) \cdot \nabla \partial_t \hat{g}_\delta(s)\, dxds = \int_0^{t_\delta} \int_\omega 2(\eta_\varepsilon + v^2(s)) \nabla u(s) \cdot \nabla \partial_t g(s)\, dxds. \tag{4.3.33}$$

Combining (4.3.30), (4.3.31), (4.3.32), (4.3.33), we deduce the desired property. □

Remark : In the case where $(u, v) \in \mathcal{C}^{1,2}_{par}([0,\infty)\times\overline{\Omega})$, the pair $(\partial_t u - \partial_t g, \partial_t v)$ is in $\mathcal{F}$. Using $\partial_t u - \partial_t g$ as a test function in (4.2.3) and $\partial_t v$ in (4.2.4), we can obtain an equality in (4.2.5).

4.4 Uniqueness

In this section, we prove **??**. This is merely an adaptation of the proof of [FP04, Theorem 2.3, Step 3].

Recall the Gagliardo-Nirenberg inequality (see [Nir59]) :

Lemma 4.4.1. *Suppose that the boundary $\partial\Omega$ is $\mathcal{C}^2$. Let p, r, s and $\alpha \in (0,1)$ be positive numbers satisfying the relation*

$$\frac{1}{p} = \alpha\left(\frac{1}{r} - \frac{1}{d}\right) + \frac{1-\alpha}{s}.$$

There exists a constant C_G (depending on p, r, s and α) such that for any $f \in L^s(\Omega)$ such that $\nabla f \in L^r(\Omega)$,

$$\|f\|_p \leqslant C_G \left(\|\nabla f\|_r^\alpha \|f\|_s^{1-\alpha} + \|f\|_s\right).$$

Proof of **??**. Let (u_1, v_1) and (u_2, v_2) be two solutions of (4.2.1) in the sense of Definition 4.2.1 and set $u = u_1 - u_2$ and $v = v_1 - v_2$. Substracting the equations satisfied by u_1 and u_2 and taking $u1_[0,t]$ as a test function leads to

$$\int_0^t \int_\Omega \partial_t u u + \int_0^t \int_\Omega \nabla u \cdot [(\eta_\varepsilon + v_1^2)\nabla u_1 - (\eta_\varepsilon + v_2^2)\nabla u_2) = 0 \quad \text{for all t ¿0.} \tag{4.4.1}$$

Differentiating (4.4.1) with respect to t leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\eta_\varepsilon\|\nabla u(t)\|_2^2&=\int_\Omega (v_2^2(t)\nabla u_2(t)-v_1^2(t)\nabla u_1(t))\cdot\nabla u(t)\\&=\int_\Omega (v_2(t)^2\nabla u_2(t)-v_1(t)^2(\nabla u(t)+\nabla u_2(t)))\cdot\nabla u(t)\\&=-\int_\Omega v_1(t)^2|\nabla u(t)|^2-\int_\Omega v(t)(v_1(t)+v_2(t))\nabla u_2(t)\cdot\nabla u(t)\\&\leqslant-\int_\Omega v(t)(v_1(t)+v_2(t))\nabla u_2(t)\cdot\nabla u(t).\end{aligned}\tag{4.4.2}$$

We proceed similarly for v : substracting the equations satisfied by v_1 and v_2 and taking $v1_[0,t]$ as a test function leads to, after differentiating with respect to t,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2+\varepsilon\|\nabla v(t)\|_2^2+\frac{1}{4\varepsilon}\|v(t)\|_2^2&=\int_\Omega (v_2(t)|\nabla u_2(t)|^2-v_1(t)|\nabla u_1(t)|^2)v(t)\\&=\int_\Omega (v_2(t)|\nabla u_2(t)|^2-(v(t)+v_2(t))|\nabla u_1(t)|^2)v(t)\\&=-\int_\Omega v(t)^2|\nabla u_1(t)|^2-\int_\Omega v_2(t)v(t)\nabla u(t)\cdot(\nabla u_2(t)+\nabla u_1(t))\\&\leqslant-\int_\Omega v_2(t)v(t)\nabla u(t)\cdot(\nabla u_2(t)+\nabla u_1(t)).\end{aligned}\tag{4.4.3}$$

Summing (4.4.2) and (4.4.3), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_2^2+\|v(t)\|_2^2\right)+\varepsilon\|\nabla v(t)\|_2^2+\eta_\varepsilon\|\nabla u(t)\|^2\\&\qquad\leqslant-\int_\Omega v(t)(v_1(t)+2v_2(t))\nabla u_2(t)\nabla u(t)-\int_\Omega v(t)v_2(t)\nabla u(t)\cdot\nabla u_1(t).\end{aligned}\tag{4.4.4}$$

When we apply Lemma 4.4.1 to $v(t)$, with $p=2\dfrac{d+2}{d}$, $r=s=2$ and $\alpha=\dfrac{d}{d+2}$, we find

$$\|v(t)\|_{2\frac{d+2}{d}}\leqslant C_G\|\nabla v(t)\|_2^{\frac{d}{d+2}}\|v(t)\|_2^{\frac{2}{d+2}}+C_G\|v(t)\|_2.$$

Since $0\leqslant v_1,v_2\leqslant 1$, using twice Young's inequality for a given $\delta>0$ to be chosen later and Hölder inequality for the pair $(1+d/2,1+2/d)$, there exist $C_\delta,C'_\delta>0$ such that

$$\begin{aligned}&\left|\int_\Omega (v_1(t)+2v_2(t))v(t)\nabla u(t)\cdot\nabla u_2(t)\right|\\&\leqslant 3\left(C_\delta\int_\Omega v(t)^2|\nabla u_2(t)|^2+\delta\int_\Omega|\nabla u(t)|^2\right)\\&\leqslant 3C_\delta\|\nabla u_2(t)\|_{d+2}^2\|v(t)\|_{\frac{2(d+2)}{d}}^2+3\delta\int_\Omega|\nabla u(t)|^2\\&\leqslant 3C_\delta\|\nabla u_2(t)\|_{d+2}^2C_G^2\left(\|\nabla v(t)\|_2^{\frac{d}{d+2}}\|v(t)\|_2^{\frac{2}{d+2}}+\|v(t)\|_2\right)^2+3\delta\int_\Omega|\nabla u(t)|^2\\&\leqslant 6C_\delta C_G^2\left(\|\nabla u_2(t)\|_{d+2}^2\|\nabla v(t)\|_2^{\frac{2d}{d+2}}\|v(t)\|_2^{\frac{4}{d+2}}+\|\nabla u_2(t)\|_{d+2}^2\|v(t)\|_2^2\right)+3\delta\int_\Omega|\nabla u(t)|^2\\&\leqslant 6C_\delta C_G^2\left(C'_\delta\|\nabla u_2(t)\|_{d+2}^{2\frac{d+2}{2}}\|v(t)\|_2^2+\delta\|\nabla v(t)\|_2^2\right)\\&\quad+6C_\delta C_G^2\|\nabla u_2(t)\|_{d+2}^2\|v(t)\|_2^2+3\delta\int_\Omega|\nabla u(t)|^2.\end{aligned}$$

Similarly, we get that

$$\left|\int_\Omega v(t)v_2(t)\nabla u(t)\cdot\nabla u_1(t)\right| \leqslant C_{G,\delta}\left(\|\nabla u_1(t)\|_{d+2}^{2\frac{d+2}{2}}\|v(t)\|_2^2+\|\nabla u_1(t)\|_{d+2}^2\|v(t)\|_2^2+\delta\|\nabla v(t)\|_2^2\right)+\delta\int_\Omega|\nabla u(t)|^2.$$

For δ small enough, we can simplify (4.4.4) into

$$\frac{d}{dt}\left(\|u(t)\|_2^2+\|v(t)\|_2^2\right)\leqslant C\left(\|\nabla u_2(t)\|_{d+2}^{d+2}+\|\nabla u_1(t)\|_{d+2}^{d+2}+\|\nabla u_2(t)\|_{d+2}^2+\|\nabla u_1(t)\|_{d+2}^2\right)\|v(t)\|_2^2. \tag{4.4.5}$$

Since $\nabla u_2,\nabla u_1\in L^{d+2}((0,T);L^{d+2}(\Omega))$ for any $T>0$, and since $x^2\leqslant 1+x^{d+2}$ for any positive real number x, we have that for any $T>0$,

$$\int_0^T\left(\|\nabla u_2(t)\|_{d+2}^{d+2}+\|\nabla u_2(t)\|_{d+2}^2\right)dt\leqslant\int_0^T\|\nabla u_2(t)\|_{d+2}^{d+2}dt+\int_0^T\left(1+\|\nabla u_2(t)\|_{d+2}^{d+2}\right)dt<+\infty.$$

$$\int_0^T\left(\|\nabla u_1(t)\|_{d+2}^{d+2}+\|\nabla u_1(t)\|_{d+2}^2\right)dt\leqslant\int_0^T\|\nabla u_1(t)\|_{d+2}^{d+2}dt+\int_0^T\left(1+\|\nabla u_1(t)\|_{d+2}^{d+2}\right)dt<+\infty.$$

We are then in position to apply Gronwall's lemma to (4.4.5) to conclude that $\|u(t)\|_2^2+\|v(t)\|_2^2=0$ for any t, leading to $u=v=0$. This concludes the proof. □

4.5 Regularity of solutions to the Ambrosio-Tortorelli equations

We prove Theorem 4.2.5, i.e., that the weak solutions constructed in section 4.3 are in fact smooth.

We recall the outline of the proof of the inner regularity. The main step is to prove that $(u,v)\in\mathcal{C}^{\alpha/2,\alpha}(Q_R(t_0,x_0))$ for a real number α and a parabolic cylinder $Q_R(t_0,x_0)\subset(0,+\infty)\times\Omega$. For the function u, this is a consequence of DeGiorgi–Nash–Moser theorem.

For the function v, as in the elliptic framework, the parabolic Hölder spaces can be linked with the Morrey-Campanato spaces (see [GM12, Section 5.1], and the Hölder condition can be rewritten as a control of the distance of the function to its average, via the following lemma, which is stated in [Lie96, Lemma 4.3]

Lemma 4.5.1. *Let $f\in L^1(Q_{2R}(t_0,x_0))$. Suppose that there exist $\alpha\in(0,1)$ and a constant $H>0$ such that, for any $Y\in Q_R(t_0,x_0)$ and any $r\in(0,R)$,*

$$\iint_{Q_r(Y)}|f(X)-\{f\}_{t,Y}|dX\leqslant Hr^{d+2+\alpha}.$$

Then there is a constant $C(d,\alpha)$ such that

$$\sup_{Q_R(t_0,x_0)}\frac{|f(X)-f(X')|}{d_p(X,X')^\alpha}\leqslant C(d,\alpha)H.$$

The proof of this estimate of (4.5.1) for the function v follows the following steps :

- **Step 1 :** We prove that $\nabla u\in\mathcal{L}^{2,d+2\alpha}(Q_R(t_0,x_0))$. In the elliptic case, this is a consequence of DeGiorgi–Nash–Moser theorem. In the parabolic case, this is the object of Lemma 4.5.2.
- **Step 2 :** We introduce the "heat"-extension w of v on a smaller cylinder $Q_r(t_0,x_0)$, which corresponds to the solution of the heat equation $\partial_t w-\varepsilon\Delta w=0$ on $Q_r(t_0,x_0)$ with $w=v$ on $\partial_\sqcup Q_r(t_0,x_0)$. The function w satisfies

$$\iint_{Q_\rho(t_0,x_0)} |w - \{w\}_{\rho,t_0,x_0}|^2 \leqslant C\left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_0,x_0)} |w - \{w\}_{r,t_1,x_1}|^2$$

for any $\rho \leqslant r$.

- **Step 3 :** The error function $v_2 = v - w$ is the solution of a heat equation with a source term in a Morrey space. As $v_2 \in L^2((0,+\infty); H_0^1(\Omega))$ we can use it as a test function to estimate $\iint_{Q_r(t_0,x_0)} |\nabla v_2|^2$. Finally we can control $\|v_2 - \{v_2\}_{\rho,t_0,x_0}\|_{2,Q_\rho(t_0,x_0)}$ by $\|\nabla v_2\|_{2,Q_\rho(t_0,x_0)}$ using a Poincare-type inequality. The proof of the Poincare inequality for solutions of heat equation with a source term in a Morrey space is the object of Lemma 4.5.3.

To address the regularity at the boundary $\partial_\sqcup[(0,+\infty)\times\Omega]$, we need to consider the bottom boundary $\{0\}\times\Omega$ and the lateral boundary $(0,+\infty)\times\partial\Omega$. In each case, the strategy is to extend the pair (u,v) beyond the boundary and then prove the inner regularity on the extended domain. The resulting equations for the extended pairs contain non constant coefficients and additional source terms. The strategy of the proof remains the same as the strategy below for the inner regularity, although the equations involve more terms (see (4.5.36), (4.5.41) and (4.5.42)). The lemmas Lemma 4.5.2 and Lemma 4.5.3 are stated in a general way to encapsulate all the equations.

The section is divided into three parts. In the first subsection, we establish preliminary results concerning parabolic equations with a source term that belongs to a Morrey space, namely a Campanato-Morrey and a Poincaré inequality. The second section is dedicated to the proof of the inner Hölder continuity of the pair (u,v). The third section is dedicated to the proof of the Hölder continuity of the pair (u,v) at the boundary $((0,+\infty)\times\partial\Omega)\cup(\{0\}\times\Omega)$.

4.5.1 Preliminary results - Parabolic equations and Morrey space

In this subsection, we give several results concerning estimates in a the Morrey space of solutions of the heat equation.

The Morrey-Campanato inequality

The following lemma connects the parabolic Hölder continuity with a Morrey-Campanato. In other word, for a function f satisfying a PDE with a source term in a Morrey space and which is $\alpha-$Hölder, we expect the gradient ∇f to satisfy a Morrey-Campanato estimate. In the elliptic case, this estimate is a consequence of the elliptic form of De-Giorgi-Nash-Moser theorem (see [BMR24, Theorem 3.1]). In our case, the estimates follow from a Caccioppoli-type estimate. This lemma will be applied to the function u to prove that $|\nabla u| \in \mathcal{L}^{2,d+2\alpha}(Q_R)$.

Lemma 4.5.2. *(Morrey-Campanato inequality) Let $(t_0,x_0) \in (0,+\infty)\times\Omega$ and $R>0$ be such that $Q_R = Q_R(t_0,x_0) \subset\subset (0,T)\times\Omega$. Let $\alpha\in(0,1)$ and let $A:\mathbb{R}^d\to\mathbb{M}_d(\mathbb{R})$, $B:\mathbb{R}^d\to\mathbb{R}^d$ be measurable functions and $h\in L^\infty(Q_R), h_1\in\mathcal{C}^1(Q_R)$ and $h_2\in L^\infty(Q_R;\mathbb{R}^d)$ such that*

$$0 < \lambda I_d \leqslant A(x) \leqslant \Lambda I_d, \quad |B(x)| \leqslant \beta$$

for some positive constants λ,Λ,β.

Let $f\in L^2((0,T);H^1(\Omega))\cap H^1((0,T);L^2(\Omega))\cap\mathcal{C}^{\alpha/2,\alpha}_{par}((0,T)\times\Omega)$ be a weak solution of

$$\partial_t f - div(A\nabla f) + B\cdot\nabla f = h + h_1 div(h_2) \quad \text{in } Q_R(t_0,x_0). \tag{4.5.1}$$

meaning that for any test function $\varphi\in H_0^1(Q_1\cap L^\infty(Q_1)$,

$$\begin{aligned}\iint_{Q_R}\partial_t f(t,x)\varphi(t,x)dxdt + \iint_{Q_R} A(x)\nabla f(t,x)\cdot\nabla\varphi(t,x)dxdt \\ + \iint_{Q_R} B(x)\cdot\nabla f(t,x)\varphi(t,x)dxdt = \iint_{Q_R} h(t,x)\varphi(t,x)dxdt \\ - \iint_{Q_R} h_1(t,x)h_2(t,x)\cdot\nabla\varphi(t,x)dxdt - \iint_{Q_R} h_2(t,x)\cdot\nabla h_1(t,x)\varphi(t,x)dxdt.\end{aligned} \tag{4.5.2}$$

There exists $C_{CM} > 0$ such that for any $(t,x) \in (0,+\infty)\times\Omega$ and any $0<\rho<R$ such that $Q_\rho(t,x) \subset Q_R(t_0,x_0) \subset\subset (0,+\infty)\times\Omega$ we have

$$\iint_{Q_\rho(t_0,x_0)} |\nabla f(t,x)|^2 dxdt \leqslant C_{CM}\left(\frac{\rho}{R}\right)^{d+2\alpha} R^{d+2}. \tag{4.5.3}$$

where C_{CM} depends on $\alpha, d, R, \lambda, \Lambda, \beta, \|h_1\|_{C^1}, \|h_2\|_\infty, [f]_\alpha, \|\mu\|_{\mathcal{L}^{1,d+2\alpha}}$.

This means that ∇f is locally in a parabolic Morrey space $\mathcal{L}^{2,d+2\alpha}$.

Proof. By translation and dilation, we can assume that $(t_0,x_0)=(0,0)$ and $R=1$. We set $Q_1 = Q_1(0,0)$. We can also assume that $\rho \in (0,\frac{1}{4}]$ since if $1/4<\rho<1$ the relation(4.5.3) holds with $C = 4^{-(d+2\alpha)}\iint_{Q_1}|\nabla f|^2$. We further suppose that $\iint_{Q_1} f = 0$ since $f - \frac{1}{|Q_1|}\iint_{Q_1} f$ is a solution to the same equation. Since $f \in \mathcal{C}^{\alpha/2,\alpha}_{par}((0,T)\times\Omega)$, for any $(t,x)\in Q_1(0,0)$,

$$|f(t,x)-f(0,0)|^2 \leqslant [f]_\alpha(|t|^{\alpha/2}+|x|^\alpha).$$

For $0<\rho\leqslant 1/4$, we choose a cut-off function $\zeta \in \mathcal{C}^\infty_c((-1,1)\times B_1(0))$ such that $\zeta\equiv 1$ in Q_ρ and

$$\mathrm{Supp}(\zeta) \subset (-4\rho^2,4\rho^2)\times B_{2\rho}(0), \quad 0\leqslant\zeta\leqslant 1, \quad |\partial_t\zeta| \leqslant \frac{C}{\rho^2}, \quad |\nabla\zeta| \leqslant \frac{2}{\rho}. \tag{4.5.4}$$

Let $f_0 = f(0,0)$. Since f is bounded on Q_1, the function $\varphi = \zeta^2(f-f_0)$ is in $H^1_0(Q_1)\cap L^\infty(Q_1)$. We test (4.5.2) with $\varphi=\zeta^2(f-f_0)$ and we obtain

$$\begin{aligned}
&\iint_{Q_1} f\partial_t[\zeta^2(f-f_0)] + \iint_{Q_1} A\nabla f\cdot\nabla(\zeta^2(f-f_0)) + \iint_{Q_1}(B\cdot\nabla f)\zeta^2(f-f_0)\\
&= \iint_{Q_1} h\zeta^2(f-f_0) - \iint_{Q_1}(h_2\cdot\nabla f)\zeta^2 h_1 - 2\iint_{Q_1}(h_2\cdot\nabla\zeta)(f-f_0)\zeta h_1 - \iint_{Q_1}(h_2\cdot\nabla h_1)\zeta^2(f-f_0).
\end{aligned} \tag{4.5.5}$$

Let us denote by

$$I = \iint_{Q_1} f\partial_t[\zeta^2(f-f_0)], \quad II = \iint_{Q_1} A\nabla f\cdot\nabla(\zeta^2(f-f_0)), \quad III = \iint_{Q_1}(B\cdot\nabla f)\zeta^2(f-f_0),$$

$$IV = \iint_{Q_1} h\zeta^2(f-f_0), \quad V = \iint_{Q_1}(h_2\cdot\nabla f)\zeta^2 h_1,$$

$$VI = 2\iint_{Q_1}(h_2\cdot\nabla\zeta)(f-f_0)\zeta h_1, \quad VII = \iint_{Q_1}(h_2\cdot\nabla h_1)\zeta^2(f-f_0).$$

We treat each term separately.

$$\begin{aligned}
I = \iint_{Q_1}\partial_t f\zeta^2(f-f_0) &= \iint_{Q_1}\frac{1}{2}\zeta^2\partial_t|f-f_0|^2 = -\iint_{Q_1}|f-f_0|^2\zeta\partial_t\zeta\\
&\leqslant \sup_{Q_{2\rho}}|f-f_0|^2\iint_{Q_{2\rho}}|\partial_t\zeta| \leqslant C[f]^2_\alpha\rho^{2\alpha}\frac{1}{\rho^2}|Q_{2\rho}| \leqslant C[f]^2_\alpha\rho^{d+2\alpha}.
\end{aligned} \tag{4.5.6}$$

For the second term, we use Young's inequality.

$$\begin{aligned}
II = \iint_{Q_1} A\nabla f\cdot\nabla(\zeta^2(f-f_0)) &= \iint_{Q_R} A\nabla f\cdot\nabla f\zeta^2 + 2\iint_{Q_1}(A\nabla f\cdot\nabla\zeta)\zeta(f-f_0)\\
&\geqslant \iint_{Q_{2\rho}}\lambda|\nabla f|^2 - \delta\iint_{Q_{2\rho}}\Lambda^2|\nabla f|^2 - C_\delta\sup_{Q_{2\rho}}|f-f_0|^2\iint_{Q_{2\rho}}|\nabla\zeta|^2\\
&\geqslant \iint_{Q_{2\rho}}\lambda|\nabla f|^2 - \delta\iint_{Q_{2\rho}}\Lambda^2|\nabla f|^2 - C_\delta[f]^2_\alpha\rho^{d+2\alpha}.
\end{aligned} \tag{4.5.7}$$

III and V can be dealt with using the same Young's inequality,

$$III = \left| \iint_{Q_1} (B \cdot \nabla f)\zeta^2 (f - f_0) \right| \leqslant \delta \iint_{Q_{2\rho}} |\nabla f|^2 \zeta^2 + C_\delta \beta^2 \sup_{Q_{2\rho}} |f - f_0|^2 \iint_{Q_{2\rho}} 1$$
$$\leqslant \delta \iint_{Q_{2\rho}} |\nabla f|^2 \zeta^2 + C_\delta \beta^2 [f]_\alpha^2 \rho^{d+2+2\alpha}. \quad (4.5.8)$$

$$V = \left| \iint_{Q_1} (h_2 \cdot \nabla f)\zeta^2 h_1 \right| \leqslant \delta \iint_{Q_{2\rho}} |\nabla f|^2 \zeta^2 + C_\delta \|h_1\|_\infty^2 \|h_2\|_\infty^2 \iint_{Q_{2\rho}} 1$$
$$\leqslant \delta \iint_{Q_{2\rho}} |\nabla f|^2 \zeta^2 + C_\delta \|h_1\|_\infty^2 \|h_2\|_\infty^2 \rho^{d+2}, \quad (4.5.9)$$

Moreover,

$$IV = \iint_{Q_1} h\zeta^2 (f - f_0) \leqslant \sup_{Q_{2\rho}} |f - f_0| \iint_{Q_{2\rho}} |h| \leqslant [f]_\alpha \|h\|_\infty \rho^{d+2+\alpha}, \quad (4.5.10)$$

$$VI = 2 \iint_{Q_1} h_2 \cdot \nabla\zeta (f - f_0)\zeta h_1 \leqslant \|h_1\|_\infty \|h_2\|_\infty \sup_{Q_{2\rho}} |f - f_0| \iint_{Q_{2\rho}} |\nabla\zeta|$$
$$\leqslant \|h_1\|_\infty \|h_2\|_\infty [f]_\alpha \rho^{d+1+\alpha}, \quad (4.5.11)$$

$$VII = \iint_{Q_1} h_2 \cdot \nabla h_1 \zeta^2 (f - f_0) \leqslant \|h_2\|_\infty \|\nabla h_1\|_\infty \sup_{Q_{2\rho}} |f - f_0| \iint_{Q_{2\rho}} 1$$
$$\leqslant \|h_2\|_\infty \|\nabla h_1\|_\infty [f]_\alpha \rho^{d+2+\alpha}. \quad (4.5.12)$$

Gathering all the estimates (4.5.6), (4.5.7), (4.5.8), (4.5.9), (4.5.10),(4.5.11), (4.5.12) into (4.5.5) yields, for $\delta > 0$ small enough (depending only on λ, Λ),

$$\begin{aligned}
\iint_{Q_\rho} |\nabla f|^2 &\leqslant C_{\lambda,\delta} \iint_{Q_{2\rho}} (\lambda - \delta(\Lambda^2 + 2)) |\nabla f|^2 \zeta^2 \\
&\leqslant C[f]_\alpha^2 \rho^{d+2\alpha} + C_\delta [f]_\alpha^2 \rho^{d+2\alpha} + C_\delta \beta^2 [f]_\alpha^2 \rho^{d+2+2\alpha} + C_\delta \|h_1\|_\infty^2 \|h_2\|_\infty^2 \rho^{d+2} \\
&\quad + [f]_\alpha \|h\|_\infty \rho^{d+2+\alpha} + \|h_1\|_\infty \|h_2\|_\infty [f]_\alpha \rho^{d+1+\alpha} + \|h_2\|_\infty \|\nabla h_1\|_\infty [f]_\alpha \rho^{d+2+\alpha} \\
&\leqslant C_{CM} \rho^{d+2\alpha}.
\end{aligned}$$

This proves the proposition. □

Remark : Although it is not useful to our present paper, Lemma 4.5.2 remains true if we replace the function $h \in L^\infty(Q_R)$ by a function $\mu \in \mathcal{L}^{1,d+\beta}$ for $\beta \in (0,1)$.

The Poincaré inequality

In this subsection, we prove a Poincaré inequality for the spatial gradient. The proof is similar to the proof of [Yin97, Lemma 2.6] and [Str81, Lemma 4]. In our case, this lemma will be applied to the function $v_2 = v - w$, where w is the "heat"-extension of v.

Lemma 4.5.3. *Let $(t_0, x_0) \in (0, +\infty) \times \Omega$ and $R > 0$ be such that $Q_R = Q_R(t_0, x_0) \subset\subset (0, T) \times \Omega$. Let $A : \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ and $B : \mathbb{R}^d \to \mathbb{R}^d$ be measurable functions satisfying,*

$$0 < \lambda I_d \leqslant A(x) \leqslant \Lambda I_d, \quad |B(x)| \leqslant \beta$$

for some positive constants λ, Λ, β. Let $\alpha \in (0,1)$ and $\mu \in \mathcal{L}^{1,d+2\alpha}(Q_R(t_0,x_0))$, $g,h \in \mathcal{C}^0(\overline{B_R(x_0)}) \cap \mathcal{C}^1(B_R(x_0))$ and $\mu_1, \mu_2 \in L^2(Q_R(t_0,x_0);\mathbb{R}^d)$ be such that

$$\sup_{Q_r(t_1,x_1)\subset Q_R(t_0,x_0)} \frac{1}{r^d} \iint_{Q_r(t_1,x_1)} |\mu_i|^2 < C_i \tag{4.5.13}$$

for positive constants C_1 and C_2.

Let $(t_1,x_1) \in Q_{R/2}(t_0,x_0)$. Let $f \in H^1((0,T)\times\Omega)$ be a weak solution of the equation

$$\partial_t f - div(A\nabla f) + B\cdot\nabla f = -div((g-g(x_1))\mu_1) + h\cdot\mu_2 + \mu \quad in\ Q_R(t_0,x_0),$$

meaning that for any test function $\varphi \in L^2((t_0-R^2,t_0);H_0^1(B_R(x_0)) \cap L^\infty(Q_R)$,

$$\begin{aligned}\iint_{Q_R} \partial_t f(t,x)\varphi(t,x)dxdt + \iint_{Q_R} A(x)\nabla f(t,x)\cdot\nabla\varphi(t,x)dxdt \\ + \iint_{Q_R} B(x)\cdot\nabla f(t,x)\varphi(t,x)dxdt = \iint_{Q_R}(g(t,x)-g(x_1))\mu_1(t,x)\cdot\nabla\varphi(t,x)dxdt \\ + \iint_{Q_R} h(t,x)\cdot\mu_2(t,x)\varphi(t,x)dxdt + \iint_{Q_R}\mu(t,x\varphi(t,x)dxdt. \end{aligned}\tag{4.5.14}$$

There exists a constant $C_P>0$ such that for any $Q_r(t_1,x_1)\subset Q_{R/2}(t_0,x_0)$,

$$\begin{aligned}\iint_{Q_r(t_1,x_1)} |f-\{f\}_{r,t_1,x_1}|^2 \\ \leqslant C_P r^2\left(\iint_{Q_{2r}(t_1,x_1)} |\nabla f|^2 + C_1[g]_1^2 r^{d+2} + C_2\|h\|_\infty r^{d+2} + \|\mu\|^2_{\mathcal{L}^{1,d+2\alpha}} r^{d+2\alpha}\right).\end{aligned}$$

Proof. Let $\overline{Q_r(t_1,x_1)} \subset Q_R(t_0,x_0)$ with $2r<R$. We will write $Q_r = Q_r(t_1,x_1), Q_{2r} = Q_{2r}(t_1,x_1), B_r = B_r(x_1)$ and $B_{2r} = B_{2r}(x_1)$.

Let $\zeta : \mathbb{R}^d \to \mathbb{R}$ be a smooth cut-off function such that $0\leqslant\zeta\leqslant 1$ with $\zeta = 1$ on B_r and $\zeta = 0$ outside B_{2r}, and $|\nabla\zeta| \leqslant 2/r$. We insist that ζ depends only on the spatial variable. Denote by

$$f_r^\zeta = \frac{\iint_{Q_{2r}} f(t,x)\zeta(x)dxdt}{\iint_{Q_{2r}}\zeta(x)dxdt} = \frac{\iint_{Q_{2r}} f(t,x)\zeta(x)dxdt}{4r^2\int_{B_{2r}}\zeta(x)dx},$$

$$f_{r,t}^\zeta = \frac{\int_{B_{2r}} f(t,x)\zeta(x)dx}{\int_{B_{2r}}\zeta(x)dx} \text{ for } t\in(t_1-4r^2,t_1).$$

From [FKS82, Theorem 1.5], we deduce the following Poincaré-Wirtinger inequality : there exists a constant $C_\star$ such that for any $t\in(t_1-r^2,t_1)$,

$$\begin{aligned}\int_{B_r(x_0)} \left|f(t,x)-f_{r,t}^\zeta\right|^2 dx \leqslant \int_{B_{2r}(x_0)} \left|f(t,x)-f_{r,t}^\zeta\right|^2\zeta\,dx \\ \leqslant C_\star r^2\int_{B_{2r}(x_0)}|\nabla f(t,x)|^2\zeta\,dx \leqslant C_\star r^2\int_{B_{2r}(x_0)}|\nabla f(t,x)|^2\,dx.\end{aligned}\tag{4.5.15}$$

Let $s,t\in(t_1-4r^2,t_1)$. Denote by $\varphi(\tau,x) = \zeta(x)1_{(s,t)}(\tau)(f_{r,t}^\zeta - f_{r,s}^\zeta)$. The function φ belongs to $L^2((t_1-R^2,t_1);H_0^1(B_R(x_1)))\cap L^\infty(Q_R)$. Using φ as a test function in (4.5.14), we obtain the following equation :

$$\begin{aligned}\iint_{Q_{2r}}\partial_t f\varphi + \iint_{Q_{2r}} A\nabla f\cdot\nabla\varphi + \iint_{Q_{2r}} B\cdot\nabla f\varphi \\ = \iint_{Q_{2r}}(g-g(x_1))\mu_1\cdot\nabla\varphi + \iint_{Q_{2r}} h\mu_2\varphi + \iint_{Q_{2r}}\mu\,\varphi.\end{aligned}\tag{4.5.16}$$

Let us denote

$$I = \iint_{Q_{2r}} \partial_t f\varphi, \quad II = \iint_{Q_{2r}} A\nabla f \cdot \nabla\varphi, \quad III = \iint_{Q_{2r}} B \cdot \nabla f\varphi,$$
$$IV = \iint_{Q_{2r}} (g - g(x_1))\mu_1 \cdot \nabla\varphi, V = \iint_{Q_{2r}} h\mu_2\varphi, \quad VI = \iint_{Q_{2r}} \mu\,\varphi.$$

We treat each term separately.

$$\begin{aligned} I = \iint_{Q_{2r}} \partial_t f\varphi &= \int_{B_{2r}} \int_s^t \partial_t f(\tau, x)\zeta(x)(f^\zeta_{r,t} - f^\zeta_{r,s})d\tau dx \\ &= \int_{B_{2r}} (f^\zeta_{r,t} - f^\zeta_{r,s})(f(t,x)\zeta(x) - f(s,x)\zeta(x))dx \\ &= (f^\zeta_{r,t} - f^\zeta_{r,s})\left(f^\zeta_{r,t}\int_{B_{2r}} \zeta - f^\zeta_{r,s}\int_{B_{2r}} \zeta\right) \\ &\geqslant (f^\zeta_{r,t} - f^\zeta_{r,s})^2 r^d \omega_d. \end{aligned} \tag{4.5.17}$$

Let $\delta > 0$ be small enough. Using the condition $A \leqslant \Lambda I_d$ and Young's inequality,

$$\begin{aligned} II = \iint_{Q_{2r}} A(x)\nabla f(\tau, x) \cdot \nabla\varphi(\tau, x)dxd\tau &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 \iint_{Q_{2r}} |\nabla\zeta|^2 + C_\delta \iint_{Q_{2r}} \Lambda^2|\nabla f|^2 \\ &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 \left(\frac{2}{r}\right)^2 (2r)^{d+2} + C_\delta \iint_{Q_{2r}} |\nabla f|^2. \end{aligned} \tag{4.5.18}$$

$$\begin{aligned} III = \iint_{Q_{2r}} B\nabla f \cdot \varphi &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 \iint_{Q_{2r}} |\zeta|^2 + C_\delta\beta^2 \iint_{Q_{2r}} |\nabla f|^2 \\ &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 r^{d+2} + C_\delta\beta^2 \iint_{Q_{2r}} |\nabla f|^2 \end{aligned} \tag{4.5.19}$$

By (4.5.13),

$$\begin{aligned} V = \iint_{Q_{2r}} h \cdot \mu_2\varphi &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 \frac{1}{r^2} \iint_{Q_{2r}} |\zeta|^2 + C_\delta r^2\|h\|^2_\infty \iint_{Q_{2r}} |\mu_2|^2 \\ &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 r^d + C_\delta C_2\|h\|^2_\infty 4r^{d+2} \end{aligned} \tag{4.5.20}$$

From the mean value theorem and Young's inequality and (4.5.13),

$$\begin{aligned} IV = \iint_{Q_{2r}} (g - g(x_1))\mu_1 \cdot \nabla\varphi &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 \iint_{Q_{2r}} |\nabla\zeta|^2 + C_\delta[g]^2_1 4r^2 \iint_{Q_{2r}} |\mu_1|^2 \\ &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 r^d + C_\delta C_1[g]^2_1 4r^{d+2}. \end{aligned} \tag{4.5.21}$$

Finally,

$$\begin{aligned} VI = \iint_{Q_{2r}} \mu\,\varphi \leqslant |f^\zeta_{r,t} - f^\zeta_{r,s}| \iint_{Q_{2r}} |\mu| &\leqslant |f^\zeta_{r,t} - f^\zeta_{r,s}|\|\mu\|_{\mathcal{L}^{1,d+2\alpha}} r^{d+2\alpha} \\ &\leqslant \delta(f^\zeta_{r,t} - f^\zeta_{r,s})^2 r^d + C_\delta\|\mu\|^2_{\mathcal{L}^{1,d+2\alpha}} r^{d+2+2\alpha}. \end{aligned} \tag{4.5.22}$$

Combining (4.5.17), (4.5.18), (4.5.19) and (4.5.21), we find that there exists a constant $C > 0$ depending on λ, β, Λ such that

$$(1-5\delta)(f^\zeta_{r,t} - f^\zeta_{r,s})^2 r^d \leqslant C\left[\iint_{Q_{2r}} |\nabla f|^2 + \|\mu\|^2_{\mathcal{L}^{1,d+2\alpha}} r^{d+2+2\alpha} + C_1[g]_1^2 r^{d+2} + C_2\|h\|_\infty r^{d+2}\right]. \quad (4.5.23)$$

Recall that, for any $c \in \mathbb{R}$, one has

$$\frac{1}{|Q_r|}\iint_{Q_r} |f - \{f\}_{r,t_1,x_1}|^2 = \frac{1}{|Q_r|}\iint_{Q_r} |f-c|^2 + |\{f\}_{r,t_1,x_1} - c|^2 \leqslant \frac{1}{|Q_r|}\iint_{Q_r} |f-c|^2. \quad (4.5.24)$$

Applying (4.5.24) to $c = f^\zeta_r$ and recalling (4.5.15), one has

$$\begin{aligned}
\iint_{Q_r} |f - \{f\}_{r,t_1,x_1}|^2 &\leqslant \iint_{Q_r} |f(x,t) - f^\zeta_r|^2 dx\, dt \\
&\leqslant 2\iint_{Q_r} |f(x,t) - f^\zeta_{r,t}|^2 dx\, dt + 2\iint_{Q_r} |f^\zeta_{r,t} - f^\zeta_r|^2 dx\, dt \\
&\leqslant 2C_\star r^2 \int_{t_1-r^2}^{t_1}\int_{B_{2r}(x_1)} |\nabla f(x,t)|^2 dxdt + 2\iint_{Q_{2r}} |f^\zeta_{r,t} - f^\zeta_r|^2 dx\, dt \\
&\leqslant 2Cr^2 \iint_{Q_{2r}} |\nabla f(t,x)|^2 dx\, dt + 2|B_{2r}(x_1)| \int_{t_1-4r^2}^{t_1} |f^\zeta_{r,t} - f^\zeta_r|^2 dt. \quad (4.5.25)
\end{aligned}$$

Considering that

$$\frac{1}{4r^2}\int_{t_1-4r^2}^{t_1} f^\zeta_{r,s} ds = \frac{\int_{t_1-4r^2}^{t_1}\int_{B_{2r}(x_1)} f(s,x)\zeta(x) dx ds}{\int_{t_1-4r^2}^{t_1}\int_{B_{2r}(x_1)} \zeta(x) dx ds} = f^\zeta_r,$$

we deduce from Jensen's inequality and (4.5.23) that

$$\begin{aligned}
|B_{2r}(x_1)| \int_{t_1-4r^2}^{t_1} |f^\zeta_{r,t} - f^\zeta_r|^2 dt &= |B_{2r}(x_1)| \int_{t_1-4r^2}^{t_1} \left|\frac{1}{4r^2}\int_{t_1-4r^2}^{t_1} (f^\zeta_{r,s} - f^\zeta_{r,t}) ds\right|^2 dt \\
&\leqslant |B_{2r}(x_1)| \int_{t_1-4r^2}^{t_1} \frac{1}{4r^2}\int_{t_1-4r^2}^{t_1} |f^\zeta_{r,t} - f^\zeta_{r,s}|^2 ds dt \\
&\leqslant C_P r^2 \left[\iint_{Q_{2r}} |\nabla f|^2 + \|\mu\|^2_{\mathcal{L}^{1,d+2\alpha}} r^{d+4\alpha}\right. \\
&\qquad \left. + C_1[g]_1^2 r^{d+2} + C_2\|h\|_\infty r^{d+2}\right]. \quad (4.5.26)
\end{aligned}$$

Combining (4.5.25) and (4.5.26) concludes. □

4.5.2 Proof of inner regularity

We prove the following :

Proposition 4.5.4. *For all $T > 0$, there exists a positive number $\alpha \in (0,1)$ depending only on T such that, for any $Q_R(t_0,x_0) \subset (0,T)\times\Omega$, the pair (u,v) belongs to $C^{\alpha/2,\alpha}_{par}(Q_R(t_0,x_0))$.*

Corollary 4.5.5. *The pair (u,v) belongs to $\mathcal{C}^\infty((0,+\infty)\times\Omega)$.*

Proof of Corollary 4.5.5. Let $T > 0$. According to Proposition 4.5.4, there exists a real number α such that $(\overline{u},\overline{v}) \in \mathcal{C}^{\alpha/2,\alpha}_{par,loc}((0,T)\times\Omega)$, so $(u,v) \in \mathcal{C}^{\alpha/2,\alpha}_{par,loc}((0,T)\times\Omega)$.

Since u is a weak solution of

$$\partial_t u - \operatorname{div}((\eta_\varepsilon + v^2)\nabla u) = 0,$$

according to parabolic Schauder Theorem (see [Kry96, Theorem 8.11]), the function u is in $\mathcal{C}^{1+\alpha/2,2+\alpha}_{par,loc}((0,T)\times\Omega)$ (see (4.1.3)), which means that $\nabla\overline{u}\in\mathcal{C}^{1+\alpha/2,1+\alpha/2}_{par,loc}((0,T)\times\Omega)$. Since v is a weak solution of

$$\partial_t v-\varepsilon\Delta v=\frac{v-1}{4\varepsilon}-v|\nabla u|^2,$$

with a source term in $\mathcal{C}^{1+\alpha/2,1+\alpha/2}_{par,loc}((0,T)\times\Omega)$, according to Schauder Theorem again, the function v is in $\mathcal{C}^{1+\alpha/2,2+\alpha}_{par,loc}((0,T)\times\Omega)$. We can conclude from a classical bootstrap argument that the pair (u,v) belongs to $\mathcal{C}^\infty((0,T)\times\Omega)$. □

Proof of Proposition 4.5.4. Let $2R>0$ and $Q_{2R}(t_0,x_0)\subset\subset(0,T)\times\Omega$ be a parabolic cylinder inside Ω.

According to De Giorgi-Nash-Moser Theorem (see [Imb26, Theorem 3.2.3], [DiB93, Section IV, Theorem 1.2]) there exists a positive real number $\alpha\in(0,1)$ such that $u\in C^{\alpha/2,\alpha}_{par,loc}((0,T)\times\Omega)$. We now prove that $v\in C^{\alpha/2,\alpha}_{par,loc}((0,T)\times\Omega)$. Let $(t_0,x_0)\in(0,T)\times\Omega$ and $R>0$ such that $Q_{2R}(t_0,x_0)\subset(0,T)\times\Omega$. According to Lemma 4.5.2, the function $|\nabla u|^2$ belongs to $\mathcal{L}^{1,d+2\alpha}(Q_{2R}(t_0,x_0))$. By setting

$$\mu=\frac{1-v}{4\varepsilon}+v|\nabla u|^2,$$

the equation of v in (4.2.4) rewrites as

$$\partial_t v-\varepsilon\Delta v=\mu\qquad\text{in } Q_{2R}(t_0,x_0).$$

We claim that there exists a constant C_1 such that for any parabolic cylinder $Q_r(t_1,x_1)\subset Q_R(t_0,x_0)$,

$$\iint_{Q_r(t_1,x_1)}|v-\{v\}_{r,t_1,x_1}|^2\leqslant C_1 r^{d+2+2\alpha}. \tag{4.5.27}$$

Suppose that (4.5.27) is true. We deduce that for any $Q_r(t_1,x_1)\subset Q_{R/2}(t_0,x_0)$, according to Cauchy-Schwarz inequality,

$$\begin{aligned}\iint_{Q_r(t_1,x_1)}|v-\{v\}_{r,t_1,x_1}|&\leqslant|Q_r|^{1/2}\left(\iint_{Q_r(t_1,x_1)}|v-\{v\}_{r,t_1,x_1}|^2\right)^{1/2}\\&\leqslant\omega_d^{1/2}r^{(d+2)/2}\sqrt{C_1}r^{(d+2+2\alpha)/2}\leqslant C(\alpha,d)r^{d+2+\alpha}.\end{aligned}$$

According to [Lie96, Lemma 4.3], we conclude that $[v]_{\alpha,Q_R(t_0,x_0)}\leqslant C(\alpha,d)$. Combining with the fact that $v\in L^\infty(Q_R(t_0,x_0))$, we deduce that $v\in C^{\alpha/2,\alpha}_{par}(Q_R(t_0,x_0))$, as desired.

We now turn to the proof of (4.5.27). Fix $t_1,x_1\in Q_R(t_0,x_0)$, so that $Q_R(t_1,x_1)\subset Q_{2R}(t_0,x_0)$. In order to prove (4.5.27), we prove the following intermediate inequality : there exists a constant C_2 such that for any $\rho<r\leqslant R$, one has

$$\iint_{Q_\rho(t_1,x_1)}|v-\{v\}_{\rho,t_1,x_1}|^2\leqslant C_2\left[\left(\frac{\rho}{r}\right)^{d+4}\iint_{Q_r(t_1,x_1)}|v-\{v\}_{r,t_1,x_1}|^2+r^{d+2+2\alpha}\right]. \tag{4.5.28}$$

Using the iteration lemma [Lie96, Lemma 4.3] and the fact that $0\leqslant v\leqslant 1$, we deduce that

$$\begin{aligned}\iint_{Q_r(t_1,x_1)}|v-\{v\}_{r,t_1,x_1}|^2&\leqslant C_2 r^{d+2+2\alpha}\left(\frac{1}{R^{d+2+2\alpha}}\iint_{Q_R(t_1,x_1)}|v-\{v\}_{2r,t_1,x_1}|^2+1\right)\\&\leqslant r^{d+2+2\alpha}\left(\frac{1}{R^{d+2+2\alpha}}4R^{d+2}+C\right),\end{aligned}$$

which corresponds to (4.5.27).

Fix $0 < \rho < r \leqslant R$. Let w be the unique weak solution in $H^1(Q_{2r}(t_1, x_1))$ of the equation

$$\begin{cases} \partial_t w - \varepsilon \Delta w &= 0, \quad \text{on } Q_{2r}(t_1, x_1), \\ w = v & \quad \text{on } \partial_{\sqcup} Q_{2r}(t_1, x_1), \end{cases}$$

meaning that for any test function $\varphi \in H_0^1(Q_{2r}(t_1, x_1))$,

$$\iint_{Q_{2r}(t_1,x_1)} \partial_t w \varphi + \varepsilon \iint_{Q_{2r}(t_1,x_1)} \nabla w \cdot \nabla \varphi = 0.$$

According to the maximum principle (see [Lie96, Lemma 3.15]), $|w| \leqslant v \leqslant 1$. Moreover, w satisfies the following Campanato estimate (see [Lie96, Lemma 4.5]) : there exists $C > 0$ such that for any $\rho < r$,

$$\iint_{Q_\rho(t_1,x_1)} |w - \{w\}_{\rho,t_1,x_1}|^2 \leqslant C \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_1,x_1)} |w - \{w\}_{r,t_1,x_1}|^2. \tag{4.5.29}$$

Let $v_2 = v - w$. Notice that for a.e $t \in (t_1 - (2r)^2, t_1)$, the function $v_2(t, \cdot)$ is zero at $\partial B_{2r}(x_1)$. The function $v_2 \in L^\infty(Q_{2r}(t_1, x_1)) \cap L^2((t_1 - 4r^2, t_1); H_0^1(B_{2r}(x_0))) \cap H^1((t_1 - 4r^2, t_1); L^2(B_{2r}(x_0)))$ satisfies for any test function $\varphi \in L^2((t_1 - 4r^2, t_1); H_0^1(B_{2r}(x_0))) \cap H^1((t_1 - 4r^2, t_1); L^2(B_{2r}(x_0)))$,

$$\iint_{Q_{2r}(t_1,x_1)} \partial_t v_2 \varphi + \varepsilon \iint_{Q_{2r}(t_1,x_1)} \nabla v_2 \cdot \nabla \varphi = \iint_{Q_{2r}(t_1,x_1)} \mu \varphi \tag{4.5.30}$$

Taking $v_2(t, \cdot)$ as a test function in (4.5.30), we obtain

$$\iint_{Q_{2r}(t_1,x_1)} \partial_t v_2 v_2 + \iint_{Q_{2r}(t_1,x_1)} \varepsilon |\nabla v_2|^2 = \iint_{Q_{2r}(t_1,x_1)} \mu v_2. \tag{4.5.31}$$

Observe that $v_2(t_1 - 4r^2, \cdot) = 0$ on $B_{2r}(x_1)$, hence

$$\begin{aligned} \iint_{Q_{2r}(t_1,x_1)} \partial_t v_2 v_2 &= \int_{B_{2r}(x_1)} \int_{t_1-4r^2}^{t_1} \partial_t v_2 v_2 \\ &= \int_{B_{2r}(x_1)} \left[\frac{1}{2}|v_2(t_1, x)|^2 - \frac{1}{2}|v_2(t_1 - 4r^2, x)|^2\right] dx = \int_{B_{2r}(x_1)} \frac{1}{2}|v_2(t_1, x)|^2 \geqslant 0. \end{aligned} \tag{4.5.32}$$

On the other hand, using that $|v_2| = |v - w| \leqslant 2$,

$$\iint_{Q_{2r}(t_1,x_1)} \mu v_2 \leqslant 2 \iint_{Q_{2r}(t_1,x_1)} |\mu| \leqslant C \|\mu\|_{\mathcal{L}^{1,d+2\alpha}} r^{d+2\alpha}. \tag{4.5.33}$$

Injecting (4.5.32) and (4.5.33) into (4.5.31) yields

$$\iint_{Q_{2r}(t_1,x_1)} |\nabla v_2|^2 \leqslant \frac{1}{\varepsilon} C \|\mu\|_{\mathcal{L}^{1,d+2\alpha}} r^{d+2\alpha}. \tag{4.5.34}$$

Now for any $\rho < r$, using twice that $(a+b)^2 \leqslant 2(a^2+b^2)$, twice Lemma 4.5.3, (4.5.29) and (4.5.34), there exist constants $C_P, C > 0$ such that

$$
\begin{aligned}
&\iint_{Q_\rho(t_1,x_1)} |v - \{v\}_{\rho,t_1,x_1}|^2 \\
&\leqslant 2 \iint_{Q_\rho(t_1,x_1)} |w - \{w\}_{\rho,t_1,x_1}|^2 + 2 \iint_{Q_\rho(t_1,x_1)} |v_2 - \{v_2\}_{\rho,t_1,x_1}|^2 \\
&\leqslant 2C \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_1,x_1)} |w - \{w\}_{r,t_1,x_1}|^2 + 2C_P\rho^2 \iint_{Q_{2\rho}(t_1,x_1)} |\nabla v_2|^2 + 2C_P\rho^{d+2\alpha} \\
&\leqslant 4C \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_1,x_1)} |v - \{v\}_{r,t_1,x_1}|^2 + 4C \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_1,x_1)} |v_2 - \{v_2\}_{r,t_1,x_1}|^2 \\
&\quad + 2C_P r^2 \iint_{Q_{2r}(t_1,x_1)} |\nabla v_2|^2 + 2C_P r^{d+2+2\alpha} \\
&\leqslant 4C \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_1,x_1)} |v - \{v\}_{r,t_1,x_1}|^2 + 4C_P r^2 \iint_{Q_{2r}(t_1,x_1)} |\nabla v_2|^2 \\
&\quad + 2C_P r^2 \iint_{Q_{2r}(t_1,x_1)} |\nabla v_2|^2 + 2C_P r^{d+2+2\alpha} \\
&\leqslant 4C \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_1,x_1)} |v - \{v\}_{r,t_1,x_1}|^2 + Cr^{d+2+2\alpha}.
\end{aligned}
$$

which proves (4.5.28) and concludes the proof of Proposition 4.5.4. □

4.5.3 Regularity at the boundary

In this section, we prove the global regularity property, which corresponds to the second part of Theorem 4.2.5.

Description of the extension in time

We start by extending the pair (u, v) on $(-1, +\infty) \times \Omega$ by the time-constant pair (u_0, v_0) on $(-1, 0]$. We consider $(\overline{u}, \overline{v})$ the extension of the pair (u, v) on the negative time, defined by

$$
(\overline{u}(t,x), \overline{v}(t,x)) = \begin{cases} (u(t-1,x), v(t-1,x)) & \text{if } t > 1 \\ (u_0(x), v_0(x)) & \text{if } t \leqslant 1. \end{cases} \tag{4.5.35}
$$

The pair $(\overline{u}, \overline{v}) \in \mathcal{F}$ satisfies the following system :

$$
\begin{cases}
\partial_t \overline{u} - \operatorname{div}((\eta_\varepsilon + \overline{v}^2)\nabla \overline{u}) = -\operatorname{div}((\eta_\varepsilon + v_0^2)\nabla u_0) 1_{\{t \leqslant 1\}}, & \text{in } \mathcal{D}'((0,\infty)\times\Omega) \\
\partial_t \overline{v} - \varepsilon\Delta \overline{v} + \frac{\overline{v}-1}{4\varepsilon} + \overline{v}|\nabla \overline{u}|^2 = (-\varepsilon\Delta v_0 + \frac{v_0-1}{4\varepsilon} + v_0|\nabla u_0|^2) 1_{\{t\leqslant 1\}}, & \text{in } \mathcal{D}'((0,\infty)\times\Omega) \\
\overline{v}(t,x) = 1 & \text{on } (0,\infty)\times\partial\Omega \\
\overline{u}(t,x) = g(t-1,x) 1_{\{t>1\}} + g(0,x) 1_{\{t\leqslant 1\}} & \text{on } (0,\infty)\times\partial\Omega
\end{cases} \tag{4.5.36}
$$

We will adopt the following notations :

$$
U_0 = -\operatorname{div}((\eta_\varepsilon + v_0^2)\nabla u_0) 1_{\{t\leqslant 1\}}, \quad V_0 = \left(-\varepsilon\Delta v_0 + \frac{v_0 - 1}{4\varepsilon} + v_0|\nabla u_0|^2\right) 1_{\{t\leqslant 1\}}. \tag{4.5.37}
$$

Notice that if we suppose that $(u_0, v_0) \in \mathcal{C}^{1,1}(\Omega)$, then $U_0, V_0 \in L^\infty((0,+\infty)\times\Omega)$.

Description of the reflexion method

We extend the pair $(\overline{u}, \overline{v})$ on a neighbourhood of Ω. Studying the regularity of $(\overline{u}, \overline{v})$ at the boundary $\partial\Omega$ then amounts to study the inner regularity of the extended pair. The construction of this extension relies on a reflection method across $\partial\Omega$ presented in [Sch06]. This is the same method as in the proof of [BMR24, Theorem 3.2], to show the regularity up to the boundary in the elliptic case. In the next

paragraph, we describe the method and define the variables $(\hat{u}, \hat{v})$ which corresponds to the extension of $(\overline{u}, \overline{v})$.

There exists a $\delta > 0$ such that the projection π_Ω on $\partial\Omega$ is well defined and smooth (of class $C^{1,1}$) on the tubular neighbourhood $\{d(x, \partial\Omega) < \delta\}$. We define

$$\begin{cases} U := \{x \in \mathbb{R}^d, d(x, \partial\Omega) < \delta\}, \\ U^{in} := \Omega \cap U, \\ U^{ex} := U \setminus \overline{\Omega}. \end{cases}$$

We define σ the geosdesic reflection across $\partial\Omega \cap U$, which satisfies $\sigma_\Omega(x) := 2\pi_\Omega(x) - x$ for all $x \in U_\delta$. The function σ is an involutive $C^{1,1}$ diffeomorphism onto its image and satisfies $\sigma(x) = x$ for $x \in \partial\Omega$.

Now consider $\widetilde{\Omega} = U \cup \sigma(U)$, so that σ satisfies $\sigma(\widetilde{\Omega}) = \widetilde{\Omega}$ with $\sigma(\widetilde{\Omega} \cap \Omega) = \widetilde{\Omega} \setminus \overline{\Omega}$ and $\sigma(\widetilde{\Omega} \setminus \overline{\Omega}) = \widetilde{\Omega} \cap \Omega$. The equality $\sigma \circ \sigma(x) = x$ yields $D\sigma(x) D\sigma(\sigma(x)) = I_d$, so $[D\sigma(x)]^{-1} = D\sigma(\sigma(x))$ for all $x \in \widetilde{\Omega}$. On the other hand, for $x \in \partial\Omega$, the matrix $D\sigma(x)$ is the reflection across the hyperplane $T_x(\partial\Omega)$, so $[D\sigma(x)]^T = D\sigma(x) = D\sigma(x)^{-1}$ for $x \in \partial\Omega \cap \widetilde{\Omega}$.

For all $x \in \overline{\Omega}$, let

$$j(x) := \begin{cases} 1 & \text{if } x \in \widetilde{\Omega} \cap \Omega \\ |\det D\sigma(x)| & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases}$$

and

$$M(x) = \begin{cases} 1 & \text{if } x \in \widetilde{\Omega} \cap \Omega \\ j(x)[D\sigma(\sigma(x))]^T D\sigma(\sigma(x)) & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases}$$

Note that j and M are functions of class $\mathcal{C}^{1,1}$ on $\widetilde{\Omega}$. Moreover, provided $\delta > 0$ is small enough, there exists $\lambda > 0$ such that

$$M(x) \geqslant \lambda I_d \qquad \text{for all } x \in \widetilde{\Omega}. \tag{4.5.38}$$

This extension in space being done, we define the extended pairs $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ and $(\widetilde{u}_\varepsilon, \widetilde{v}_\varepsilon)$ on $(0, \infty) \times \widetilde{\Omega}$:

$$(\hat{u}(t,x), \hat{v}(t,x)) := \begin{cases} (\overline{u}(t,x), \overline{v}(t,x)) & \text{if } x \in \widetilde{\Omega} \cap \Omega \\ (\overline{u}(t, \sigma(x)), \overline{v}(t, \sigma(x)) & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases} \tag{4.5.39}$$

and

$$(\widetilde{u}(t,x), \widetilde{v}(t,x)) := \begin{cases} (\overline{u}(t,x), \overline{v}(t,x)) & \text{if } x \in \widetilde{\Omega} \cap \Omega \\ (2\overline{g}(t, \pi(x)) - \overline{u}(t, \sigma(x)), 2 - \overline{v}(t, \sigma(x))) & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases} \tag{4.5.40}$$

The pairs $(\hat{u}, \hat{v})$ and $(\widetilde{u}, \widetilde{v})$ belong to $\mathcal{F}$. Finally, we set $\widetilde{g}(t,x) := \overline{g}(t, \pi(x)) \in \mathcal{C}^{1,1}((0,T) \times \widetilde{\Omega})$.

The following lemma establishes the equations satisfied by the pairs $(\hat{u}, \hat{v})$ and $(\widetilde{u}, \widetilde{v})$.

Lemma 4.5.6. *Recall the definition* (4.5.37) *of* (U_0, V_0) *The functions* $\widetilde{u}, \widetilde{v}, \hat{u}, \hat{v}$ *as defined in* (4.5.39) *and* (4.5.40) *satisfy*

$$\begin{aligned} & j\,\partial_t \widetilde{u} - div((\eta_\varepsilon + \hat{v}_\varepsilon^2) M \nabla \widetilde{u}) \\ & = 2j\,\partial_t \widetilde{g} 1_{\widetilde{\Omega} \setminus \overline{\Omega}} - 2div(1_{\widetilde{\Omega} \setminus \overline{\Omega}} (\eta_\varepsilon + \hat{v}^2) M \nabla \widetilde{g}) + \frac{j}{2}(U_0 1_{\widetilde{\Omega} \cap \Omega} - U_0 \circ \sigma 1_{\widetilde{\Omega} \setminus \Omega}) \quad \text{in } \mathcal{D}'((0,\infty) \times \widetilde{\Omega}), \end{aligned} \tag{4.5.41}$$

$$\begin{aligned} & j\,\partial_t \widetilde{v} - div(M \nabla \widetilde{v}) \\ & = (1_{\widetilde{\Omega} \cap \Omega} - 1_{\widetilde{\Omega} \setminus \overline{\Omega}}) \left(\frac{j}{4\varepsilon}(1 - \hat{v}) - (M \widetilde{v} \nabla \widetilde{u} \cdot \nabla \widetilde{u}) \right) + \frac{j}{2}(V_0 1_{\widetilde{\Omega} \cap \Omega} - V_0 \circ \sigma 1_{\widetilde{\Omega} \setminus \Omega}) \quad \text{in } \mathcal{D}'((0,\infty) \times \widetilde{\Omega}). \end{aligned} \tag{4.5.42}$$

Proof. Let $\varphi \in \mathcal{C}_c^\infty((0,\infty)\times\widetilde{\Omega})$. We decompose $\varphi = \varphi^s + \varphi^a$ as the sum of its symmetric and anti-symmetric part, with

$$\varphi^s(x) = \frac{1}{2}(\varphi(x)+\varphi(\sigma(x))), \quad \text{and} \quad \varphi^a(x) = \frac{1}{2}(\varphi(x)-\varphi(\sigma(x))).$$

The functions φ^s and φ^a belong to $\mathcal{C}^{1,1}((0,\infty)\times\Omega))$ and satisfy $\varphi^s(t,\sigma(x)) = \varphi^s(t,x)$ and $\varphi^a(t,\sigma(x)) = -\varphi^a(t,x)$ for all $(t,x)\in(0,\infty)\times\widetilde{\Omega}$.

Proof of (4.5.41) : In the expression

$$\int_0^\infty \int_{\widetilde{\Omega}} j\partial_t\widetilde{u}\varphi + \int_{\widetilde{\Omega}} (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{u}\cdot\nabla\varphi,$$

we consider separately the terms with φ^s and the terms with φ^a :

$$\begin{aligned}\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} j\partial_t\widetilde{u}\,\varphi^s + \int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{u}\cdot\nabla\varphi^s \\ = 2\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\partial_t\widetilde{g}\varphi^s + (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{g}\cdot\nabla\varphi^s\right] \\ - \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\partial_t(\overline{u}\circ\sigma)\varphi^s + (\eta_\varepsilon+\hat{v}^2)M\nabla(\overline{u}\circ\sigma)\cdot\nabla\varphi^s\right]. \quad (4.5.43)\end{aligned}$$

Using $\varphi^s\circ\sigma = \varphi^s$ and changing the variable :

$$\begin{aligned}\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j(\partial_t\overline{u}\circ\sigma)(\varphi^s\circ\sigma) + (\eta_\varepsilon+\hat{v}^2)M\nabla(\overline{u}\circ\sigma)\cdot\nabla(\varphi^s\circ\sigma)\right] \\ = \int_{\widetilde{\Omega}\cap\Omega} \left[\partial_t\overline{u}\varphi^s + (\eta_\varepsilon+\overline{v}^2)\nabla\overline{u}\cdot\nabla\varphi^s\right]. \quad (4.5.44)\end{aligned}$$

From (4.5.43) and (4.5.44), we deduce

$$\int_0^\infty \int_{\widetilde{\Omega}} \left[j\partial_t\widetilde{u}\,\varphi^s + (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{u}\cdot\nabla\varphi^s\right] = 2\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\partial_t\widetilde{g}\,\varphi^s + (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{g}\cdot\nabla\varphi^s\right]. \quad (4.5.45)$$

We now focus on the part with φ^a. Notice that $\varphi^a = 0$ on $(0,\infty)\times\partial\Omega$, so that

$$\begin{aligned}\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} j\partial_t\widetilde{u}\,\varphi^a + \int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{u}\cdot\nabla\varphi^a \\ = 2\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\partial_t\widetilde{g}\varphi^a + (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{g}\cdot\nabla\varphi^a\right] \\ - \int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\partial_t(u\circ\sigma)\varphi^a + (\eta_\varepsilon+\hat{v}^2)M\nabla(u\circ\sigma)\cdot\nabla\varphi^a\right]. \quad (4.5.46)\end{aligned}$$

Using that $\varphi^a\circ\sigma = -\varphi^a$ and changing the variable :

$$\begin{aligned}&\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\partial_t(u\circ\sigma)\varphi^a + (\eta_\varepsilon+\hat{v}^2)M\nabla(u\circ\sigma)\cdot\nabla\varphi^a\right] \\ &= -\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\partial_t(u\circ\sigma)(\varphi^a\circ\sigma) + (\eta_\varepsilon+\hat{v}^2)M\nabla(u\circ\sigma)\cdot\nabla(\varphi^a\circ\sigma)\right] \\ &= -\int_0^\infty \int_{\widetilde{\Omega}\cap\Omega} \left[\partial_t\overline{u}\varphi^a + (\eta_\varepsilon+\overline{v}^2)\nabla\overline{u}\cdot\nabla\varphi^a\right] \\ &= -\int_0^\infty \int_{\widetilde{\Omega}\cap\Omega} U_0\varphi^a,\end{aligned}$$

where we used (4.5.36) in the last equality. We deduce from (4.5.46) that

$$\begin{aligned}
&\int_0^\infty \int_{\widetilde{\Omega}} j\,\partial_t\widetilde{u}\,\varphi^a + (\eta_\varepsilon+\hat{v}^2)M\nabla\tilde{u}\cdot\nabla\varphi^a \\
&= \int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} j\,\partial_t\widetilde{u}\varphi^a + \int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon+\hat{v}_\varepsilon^2)M\nabla\widetilde{u}\cdot\nabla\varphi^a \\
&\quad + \int_0^\infty \int_{\widetilde{\Omega}\cap\Omega} \left[\partial_t\overline{u}\varphi^a + (\eta_\varepsilon+\overline{v}^2)\nabla u\cdot\nabla\varphi^a\right] \\
&= 2\int_0^\infty \int_{\widetilde{\Omega}\setminus\overline{\Omega}} \left[j\,\partial_t\widetilde{g}\,\varphi^a + (\eta_\varepsilon+\hat{v}^2)M\nabla\widetilde{g}\cdot\nabla\varphi^a\right] + \int_0^\infty \int_{\widetilde{\Omega}\cap\Omega} U_0\varphi^a + 0.
\end{aligned} \tag{4.5.47}$$

We observe that

$$\begin{aligned}
\int_0^\infty \int_{\widetilde{\Omega}\cap\Omega} U_0\varphi^a &= \frac{1}{2}\int_0^\infty \int_{\widetilde{\Omega}\cap\Omega} U_0(\varphi-\varphi\circ\sigma) \\
&= \frac{1}{2}\int_0^\infty \int_{\widetilde{\Omega}\cap\Omega} U_0\varphi - \frac{1}{2}\int_0^\infty \int_{\widetilde{\Omega}\setminus\Omega} U_0\circ\sigma\varphi j = \int_0^\infty \int_{\widetilde{\Omega}} \frac{j}{2}(U_0 1_{\widetilde{\Omega}\cap\Omega} - U_0\circ\sigma 1_{\widetilde{\Omega}\setminus\Omega})\varphi.
\end{aligned} \tag{4.5.48}$$

Combining (4.5.45), (4.5.47) and (4.5.48) , we obtain (4.5.41). The same computation holds for (4.5.42). □

In the sequel, we rewrite (4.5.41) as

$$\begin{aligned}
\partial_t\widetilde{u} - \operatorname{div}\left(\frac{(\eta_\varepsilon+\widetilde{v}^2)}{j}M\nabla\widetilde{u}\right) - (\eta_\varepsilon+\widetilde{v}^2)M\frac{\nabla j}{j^2}\cdot\nabla\widetilde{u} \\
= 2\partial_t\widetilde{g}\,1_{\widetilde{\Omega}\setminus\overline{\Omega}} - \frac{2}{j}\operatorname{div}(1_{\widetilde{\Omega}\setminus\overline{\Omega}}(\eta_\varepsilon+\widetilde{v}^2)M\nabla\widetilde{g}) + \widetilde{U}_0
\end{aligned} \tag{4.5.49}$$

and (4.5.42) as

$$\partial_t\widetilde{v} - \operatorname{div}(H\nabla\widetilde{v}) - J\cdot\nabla\widetilde{v} = \mu \tag{4.5.50}$$

where, keeping in mind the definition (4.5.37) of (U_0, V_0), we set

$$\widetilde{U}_0 = \frac{1}{2}U_0 1_{\widetilde{\Omega}\cap\Omega} - \frac{1}{2}U_0\circ\sigma 1_{\widetilde{\Omega}\setminus\overline{\Omega}}, \quad \widetilde{V}_0 = \frac{1}{2}V_0 1_{\widetilde{\Omega}\cap\Omega} - \frac{1}{2}V_0\circ\sigma 1_{\widetilde{\Omega}\setminus\overline{\Omega}} \tag{4.5.51}$$

$$H = \frac{M}{j}, \quad J = M\frac{\nabla j}{j^2}, \quad \mu = (1_{\widetilde{\Omega}\cap\Omega} - 1_{\widetilde{\Omega}\setminus\overline{\Omega}})\left(\frac{1-\widetilde{v}}{4\varepsilon} - \frac{1}{j}(M\widetilde{v}\nabla\widetilde{u}\cdot\widetilde{u})\right) + \widetilde{V}_0. \tag{4.5.52}$$

As $\partial\Omega$ is of class $\mathcal{C}^{2,1}$, for any $x_0 \in \widetilde{\Omega}$ and $R > 0$ such that $B_{2R}(x_0) \subset \widetilde{\Omega}$, the functions H and J are in $\mathcal{C}^0(B_{2R}(x_0))$. In particular, we can introduce positive constants c_j, C_j and λ so that $C_j > j(x) \geqslant c_j > 0$ and $H(x) \geqslant \lambda I_d > 0$ for all $x \in B_{2R}(x_0)$.

Proof of the regularity

The aim of this subsection is to prove the following proposition.

Proposition 4.5.7. *Let $(t_0, x_0) \in (0,+\infty)\times\widetilde{\Omega}$ and $Q_{2R}(t_0,x_0) \subset\subset (0,+\infty)\times\widetilde{\Omega}$. There exists $\alpha\in(0,1)$ and a constant C depending only on $d(x_0,\partial\widetilde{\Omega}), \eta_\varepsilon, \alpha$ such that $(\widetilde{u},\widetilde{v}) \in \mathcal{C}_{par}^{\alpha/2,\alpha}(Q_R(t_0,x_0))$.*

From this result and a bootstrap argument, we can prove Theorem 4.2.5, as follows :

Proof of Theorem 4.2.5. Fix any $T > 0$. Let us start with the case $k = 1$. Gathering Proposition 4.5.4 and Proposition 4.5.7, we deduce that there exists a real number $\alpha \in (0,1)$ such that $u, v \in \mathcal{C}^{\alpha/2,\alpha}_{par}([0,T] \times \overline{\Omega})$.

Considering the equation for u (4.2.3), as the conditions (i), (ii) and (iii) for $k = 1$ of Theorem 4.2.5 and the compatibility condition (iv) for $\ell = 1$ are fulfilled, we deduce from the fact that $v \in \mathcal{C}^{\alpha/2,\alpha}_{loc}([0,T] \times \overline{\Omega})$ and from [LSU68, Section IV, Theorem 5.2] that $u \in \mathcal{C}^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$. It implies that $\dfrac{1-v}{4\varepsilon} + v|\nabla u|^2 \in \mathcal{C}^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$. Considering the equation for v (4.2.4) and again as the conditions (i), (ii), (iii) and (iv) of Theorem 4.2.5 are satisfied, we can apply [LSU68, Section IV, Theorem 5.2] and deduce that $v \in \mathcal{C}^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$.

In particular, $v \in \mathcal{C}^{\beta/2,\beta}([0,T] \times \overline{\Omega})$. Back to the equation (4.2.3) in u, we deduce the same way as above that $u \in \mathcal{C}^{1+\beta/2,2+\beta}([0,T] \times \overline{\Omega})$. Back to the equation (4.2.4) in v, we deduce similarly as above that $v \in \mathcal{C}^{1+\beta/2,2+\beta}([0,T] \times \overline{\Omega})$.

For $k \geqslant 2$, one can iterate the above argument : let $m = \lfloor k/2 \rfloor$ and suppose that we have $(u,v) \in \mathcal{C}^{m-1+\beta/2,2m-2+\beta}([0,T] \times \overline{\Omega})$. Considering the equation for u (4.2.3), as the conditions (i), (ii) and (iii) for $k = m$ of Theorem 4.2.5 and the compatibility condition (iv) for $\ell \leqslant m$ are fulfilled, we deduce from the fact that $v \in \mathcal{C}^{m-1+\beta/2,2m-2+\beta}_{loc}([0,T] \times \overline{\Omega})$ and from [LSU68, Section IV, Theorem 5.2] that $u \in \mathcal{C}^{m+\beta/2,2m+\beta}([0,T] \times \overline{\Omega})$. It implies that $\dfrac{1-v}{4\varepsilon} + v|\nabla u|^2 \in \mathcal{C}^{m-1+\beta/2,2m-2+\beta}([0,T] \times \overline{\Omega})$. Considering the equation for v (4.2.4) and again as the conditions (i), (ii) and (iii) of Theorem 4.2.5 are satisfied, we can apply [LSU68, Section IV, Theorem 5.2] and deduce that $v \in \mathcal{C}^{m+\beta/2,2m+\beta}([0,T] \times \overline{\Omega})$. □

As for the inner regularity, the starting point is to be able to establish the existence of α such that $|\nabla \widetilde{u}|^2 \in \mathcal{L}^{1,d+2\alpha}(Q_R(t_0,x_0))$, which is performed in the following lemma.

Lemma 4.5.8. *There exists a real number $\alpha \in (0,1)$ such that $\widetilde{u} \in \mathcal{C}^{\alpha/2,\alpha}_{par}(Q_R(t_0,x_0))$. Moreover, one has $|\nabla \widetilde{u}|^2 \in \mathcal{L}^{1,d+2\alpha}(Q_R(t_0,x_0))$ and $\mu \in \mathcal{L}^{1,d+2\alpha}(Q_R(t_0,x_0))$.*

Proof. According to (4.5.38), $\exists \lambda, C > 0$ such that

$$\frac{(\eta_\varepsilon + \widetilde{v}^2)}{j} M \geqslant \frac{\eta_\varepsilon}{C} \lambda I_d > 0, \quad \left| (\eta_\varepsilon + \widetilde{v}^2) M \frac{\nabla j}{j^2} \right| \leqslant \Lambda,$$

$$\left| 2\partial_t \widetilde{g} 1_{\widetilde{\Omega} \setminus \overline{\Omega}} \right|, |\widetilde{U}_0| \leqslant C, \quad \frac{2}{j} \in \mathcal{C}^{1,1}(\widetilde{\Omega}), \quad \left| 1_{\widetilde{\Omega} \setminus \overline{\Omega}} (\eta_\varepsilon + \widetilde{v}^2) M \nabla \widetilde{g} \right| \leqslant C. \tag{4.5.53}$$

Let $T > 0$. Considering the equation (4.2.3) satisfied by u , according to [DiB93, Theorem IV.1.2], there exists a real number $\alpha \in (0,1)$ such that $\overline{u} \in \mathcal{C}^{\alpha/2,\alpha}_{par}([0,T] \times \overline{\Omega})$. Using that $\sigma \in \mathcal{C}^{1,1}(\widetilde{\Omega})$ and the expression of $\widetilde{u}$, we deduce that $\widetilde{u} \in \mathcal{C}^{\alpha/2,\alpha}_{par}(Q_R(t_0,x_0))$.

Considering the condition (4.5.53), the right hand-side of (4.5.49) satisfies the hypothesis of Lemma 4.5.2 with

$$A = \frac{\eta_\varepsilon + \widetilde{v}^2}{j} M, \quad B = -(\eta_\varepsilon + \widetilde{v}^2) A \frac{\nabla j}{j^2}, \quad h = 2\partial_t \widetilde{g} 1_{\widetilde{\Omega} \setminus \overline{\Omega}} + \widetilde{U}_0,$$

$$h_1 = \frac{2}{j}, \quad h_2 = 1_{\widetilde{\Omega} \setminus \overline{\Omega}} (\eta_\varepsilon + \hat{v}^2) M \nabla \widetilde{g}.$$

We deduce that $\nabla \widetilde{u} \in \mathcal{L}^{2,d+2\alpha}(Q_R(t_0,x_0))$.

We conclude that $\mu \in \mathcal{L}^{1,d+2\alpha}(Q_R(t_0,x_0))$. Observe that $\frac{1-\widetilde{v}}{4\varepsilon}$ is bounded thanks to Proposition 4.3.7, that $\widetilde{V}_0$ is bounded because $j \geqslant c_j$ and that $(A\hat{v}\nabla \hat{u} \cdot \hat{u})/j \in \mathcal{L}^{1,d+2\alpha}(Q_R(t_0,x_0))$. We conclude that $\mu \in \mathcal{L}^{1,d+2\alpha}(Q_R(t_0,x_0))$. □

The main difficulty of Proposition 4.5.7 is to prove that $\widetilde{v} \in \mathcal{C}^{\alpha/2,\alpha}_{par}(Q_R(t_0,x_0))$. The proof of Proposition 4.5.7 follows the same method as the proof of Proposition 4.5.4, although it is slightly more technical due to the presence of non constant terms in (4.5.50). As in the proof of the inner regularity, it is sufficient to prove a uniform decay estimate of the Campanato norm, namely :

Lemma 4.5.9. *Let $\alpha \in (0,1)$ be given by Lemma 4.5.8. There exists a constant $C_1 > 0$ such that for any $(t_1, x_1) \in Q_R(t_0, x_0)$ and any $\rho < r \leqslant R$ with $Q_r(t_1, x_1) \subset Q_R(t_0, x_0)$,*

$$\iint_{Q_\rho(t_1,x_1)} |\widetilde{v} - \{\widetilde{v}\}_{\rho,t_1,x_1}|^2 \leqslant C_1 \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r(t_1,x_1)} |\widetilde{v} - \{\widetilde{v}\}_{r,t_1,x_1}|^2 + C_1 r^{d+2+2\alpha}.$$

We briefly recall how this estimate leads to the proof of Proposition 4.5.7.

Proof of Proposition 4.5.7. Suppose that Lemma 4.5.9 is satisfied. Fix $(t_1, x_1) \in Q_R(t_0, x_0)$, so that $Q_R(t_1, x_1) \subset Q_{2R}(t_0, x_0)$.

Using the iteration lemma [Lie96, Lemma 4.6] and the fact that $0 \leqslant \widetilde{v} \leqslant 1$ allows to conclude that for any $Q_r(t_1, x_1) \subset Q_R(t_1, x_1)$,

$$\begin{aligned}\iint_{Q_r(t_1,x_1)} |\widetilde{v} - \{\widetilde{v}\}_{r,t_1,x_1}|^2 &\leqslant C_1 r^{d+2+2\alpha} \left(\frac{1}{R^{d+2+2\alpha}} \iint_{Q_R(t_1,x_1)} |\widetilde{v} - \{\widetilde{v}\}_{2r,t_1,x_1}|^2 + 1 \right) \\ &\leqslant C_1 r^{d+2+2\alpha} \left(\frac{1}{R^{d+2+2\alpha}} 4R^{d+2} + 1 \right).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we deduce that

$$\iint_{Q_r(t_1,x_1)} |\widetilde{v} - \{v\}_{r,t_1,x_1}| \leqslant |Q_r|^{1/2} \left(\iint_{Q_r(t_1,x_1)} |\widetilde{v} - \{\widetilde{v}\}_{r,t_1,x_1}|^2 \right)^{1/2} = C(\alpha, d) r^{d+2+\alpha}.$$

Hence, according to Lemma 4.5.1, we deduce that $[\widetilde{v}]_{\alpha, Q_R(t_0,x_0)} \leqslant C(\alpha, d)$. Recalling that $\widetilde{v}$ is bounded, we deduce that $\widetilde{v} \in C^{\alpha/2,\alpha}_{par}(Q_R(t_0, x_0))$. □

The rest of the subsection is devoted to the proof of Lemma 4.5.9. Let us briefly explain the outline of the proof. We fix a $(t_1, x_1) \subset Q_R(t_0, x_0)$, so that $Q_R(t_1, x_1) \subset Q_{2R}(t_0, x_0))$ and we fix $r < R$. We consider a "heat"-extension w of $\widetilde{v}$ on $Q_r(t_1, x_1)$. The difference $v_2 = \widetilde{v} - w$ satisfies a parabolic PDE on $Q_r(t_1, x_1)$ with homogeneous initial and boundary Dirichlet conditions, so that we can apply v_2 as a test function to it and estimate $\iint_{Q_r(t_1,x_1)} |\nabla v_2|^2$.

The source term of the equation in v_2 depends on the function $\nabla\widetilde{v}$, so that we need a preliminary estimate of the energy $\iint_{Q_r(t_1,x_1)} |\nabla\widetilde{v}|^2$, which we state before getting to the proof of Lemma 4.5.9.

Lemma 4.5.10. *There exists a constant $C_\star$ depending only on R such that for any $Q_r(t_1, x_1) \subset Q_R(t_0, x_0)$, one has*

$$\iint_{Q_r(t_1,x_1)} |\nabla\widetilde{v}|^2 \leqslant C_\star r^d.$$

Proof. Let $\zeta \in \mathcal{C}^\infty_c(\mathbb{R} \times \Omega)$ be a test function such that $0 \leqslant \zeta \leqslant 1$, $\mathrm{Supp}(\zeta) \subset (t_1 - (2r)^2, t_1 + (2r)^2) \times B_{2r}(x_1)$ and $\zeta = 1$ on $Q_r(t_1, x_1)$, with

$$|\nabla\zeta| \leqslant \frac{2}{r}, \quad |\partial_t\zeta| \leqslant \frac{C}{r^2}.$$

Applying $\widetilde{v}\zeta^2 \in L^\infty_{loc}((0,+\infty) \times \widetilde{\Omega}) \cap H^1((0,+\infty); L^2(\widetilde{\Omega})) \cap L^2((0,+\infty); H^1_{loc}(\widetilde{\Omega}))$ as a test function to (4.5.50) yields

$$\iint_{Q_{2r}} \zeta^2 \partial_t\widetilde{v}\widetilde{v} + \iint_{Q_{2r}} H\nabla\widetilde{v} \cdot \nabla\widetilde{v}\zeta^2 + 2\iint_{Q_{2r}} H\nabla\widetilde{v} \cdot \nabla\zeta\, \widetilde{v}\zeta = \iint_{Q_{2r}} \mu\widetilde{v}\zeta^2 - \iint_{Q_{2r}} J \cdot \nabla\widetilde{v}\widetilde{v}\zeta^2. \quad (4.5.54)$$

Set

$$I = \iint_{Q_{2r}} \zeta^2 \partial_t\widetilde{v}\widetilde{v}, \quad II = 2\iint_{Q_{2r}} H\nabla\widetilde{v} \cdot \nabla\zeta\widetilde{v}\zeta,$$

$$III = \iint_{Q_{2r}} \mu\widetilde{v}\zeta^2, \quad IV = \iint_{Q_{2r}} J\cdot\nabla\widetilde{v}\widetilde{v}\zeta^2.$$

We estimate each term separately. Using that $0 \leqslant \widetilde{v} \leqslant 1$,

$$I = \iint_{Q_{2r}} \zeta^2\partial_t\widetilde{v}\widetilde{v} = \iint_{Q_{2r}} \zeta^2\partial_t\frac{1}{2}|\widetilde{v}|^2 = \int_{Q_{2r}} |\widetilde{v}|^2\zeta\partial_t\zeta \leqslant \int_{Q_{2r}} |\partial_t\zeta| \leqslant Cr^d. \tag{4.5.55}$$

Using Young's inequality,

$$\begin{aligned} II = 2\iint_{Q_{2r}} H\nabla\widetilde{v}\cdot\nabla\zeta\widetilde{v}\zeta \leqslant \delta\iint_{Q_{2r}} |\nabla\widetilde{v}|^2\zeta^2 + C_\delta\|H\|^2_{\infty,\widetilde{\Omega}}\iint_{Q_r}|\widetilde{v}|^2|\nabla\zeta|^2 \\ \leqslant \delta\iint_{Q_{2r}} |\nabla\widetilde{v}|^2\zeta^2 + C_\delta\|H\|^2_{\infty,\widetilde{\Omega}}r^d. \end{aligned} \tag{4.5.56}$$

and

$$\iint_{Q_{2r}} J\cdot\nabla\widetilde{v}\widetilde{v}\zeta^2 \leqslant \delta\iint_{Q_{2r}} |\nabla\widetilde{v}|^2\zeta^2 + C_\delta\|J\|^2_{\infty,\widetilde{\Omega}}\iint_{Q_{2r}}|\widetilde{v}|^2 \leqslant \delta\iint_{Q_{2r}} |\nabla\widetilde{v}|^2\zeta^2 + C_\delta\|J\|^2_{\infty,\widetilde{\Omega}}r^{d+2}. \tag{4.5.57}$$

Finally, recalling $\mu \in \mathcal{L}^{1,d+2\alpha}$,

$$IV = \iint_{Q_{2r}} \mu\widetilde{v}\zeta^2 \leqslant \iint_{Q_{2r}} |\mu| \leqslant \|\mu\|_{\mathcal{L}^{1,d+2\alpha}}r^{d+2\alpha}. \tag{4.5.58}$$

Recalling that $H(x) \geqslant \lambda I_d$ for all $x \in \widetilde{\Omega}$, with $\lambda > 0$, we can choose δ small enough such that, combining (4.5.55), (4.5.56), (4.5.58) and (4.5.57) into (4.5.54),

$$\iint_{Q_r(t_1,x_1)} |\nabla\widetilde{v}|^2 \leqslant \iint_{Q_{2r}} |\nabla\widetilde{v}|^2\zeta^2 \leqslant Cr^d + Cr^{d+2} + Cr^{d+2\alpha} \leqslant C_\star r^d.$$

This concludes the lemma. □

We now turn to the proof of Lemma 4.5.9.

Proof of Lemma 4.5.9. Fix $r < R$ such that $Q_r(t_1,x_1) \subset Q_R(t_0,x_0)$. Denote by $H_1 = H(x_1)$ and $J_1 = J(x_1)$. Let w be the (unique weak) solution in $H^1((t_1-(2r)^2)\times B_{2r}(x_1))$ of the following equation

$$\begin{cases} \partial_t w - \operatorname{div}(H_1\nabla w) + J_1\nabla w &= 0, \quad \text{on } Q_{2r}(t_1,x_1) \\ w &= \widetilde{v} \quad \text{on } \partial_\sqcup Q_{2r}(t_1,x_1) \end{cases}$$

Notice that $\hat{w} : (t,x) \mapsto w(t, x+J_1t)$ is the solution of $\partial_t\hat{w} - \operatorname{div}(H_1\nabla\hat{w}) = 0$, so w will have the same properties as the classic heat equation (maximum principle and Campanato estimates). In particular, $|w| \leqslant \widetilde{v} \leqslant 1$.

Let $v_2 = \widetilde{v} - w$. The function v_2 is a weak solution of the following equation

$$\begin{cases} \partial_t v_2 - \operatorname{div}(H_1\nabla v_2) - J_1\nabla v_2 &= \mu + \operatorname{div}((H-H_1)\nabla\widetilde{v}) + (J-J_1)\cdot\nabla\widetilde{v}, \quad \text{on } Q_{2r}(t_1,x_1) \\ v_2 &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial_\sqcup Q_{2r}(t_1,x_1). \end{cases} \tag{4.5.59}$$

The rest of the proof is divided into two steps :

- **Step 1 :** We prove that

$$\iint_{Q_{2r}(t_1,x_1)} |\nabla v_2|^2 \leqslant C_2\cdot 2r^{d+2+2\alpha}.$$

This is done by multiplying (4.5.59) by v_2 which vanishes at the boundary.

- **Step 2 :** We conclude using Poincare's inequality

$$\iint_{Q_r(t_1,x_1)} |v_2 - \{v_2\}_{r,t_1,x_1}|^2 \leqslant Cr^2 \iint_{Q_{2r}} |\nabla v_2|^2.$$

Step 1 : Taking v_2 as a test function in (4.5.59). We obtain that

$$\iint_{Q_{2r}} \partial_t v_2 v_2 + \iint_{Q_{2r}} H_1 \nabla v_2 \cdot \nabla v_2 + \iint_{Q_{2r}} J_1 \cdot \nabla v_2 v_2 = \iint_{Q_{2r}} \mu v_2 - \iint_{Q_{2r}} (H - H_1)\nabla \widetilde{v} \cdot \nabla v_2 - \iint_{Q_{2r}} (J - J_1) \cdot \nabla \widetilde{v} v_2. \quad (4.5.60)$$

Let us denote this equation by $I + II + III = IV + V + VI$. We estimate each term separately.

$$I = \iint_{Q_{2r}} \partial_t v_2 v_2 = \int_{B_{2r}} \frac{1}{2} |v_2(t_1, x)|^2 \geqslant 0. \quad (4.5.61)$$

Recall that $H \geqslant c > 0$ on $\widetilde{\Omega}$. In particular,

$$II = \iint_{Q_{2r}} H_1 |\nabla v_2|^2 \geqslant c \iint_{Q_{2r}} |\nabla v_2|^2. \quad (4.5.62)$$

Using Young's inequality and the fact that $|v_2| = |\widetilde{v} - w| \leqslant 2$,

$$III = \iint_{Q_{2r}} J_1 \nabla v_2 v_2 \leqslant \delta \iint_{Q_{2r}} |\nabla v_2|^2 + C_\delta \|J\|_{\infty,\widetilde{\Omega}} \iint_{Q_{2r}} |v_2|^2 \leqslant \delta \iint_{Q_{2r}} |\nabla v_2|^2 + C_\delta \|J\|_{\infty,\widetilde{\Omega}} (2r)^{d+2}. \quad (4.5.63)$$

$$IV = \iint_{Q_{2r}} \mu v_2 \leqslant 2 \iint_{Q_{2r}} |\mu| \leqslant 2\|\mu\|_{\mathcal{L}^{1,d+2\alpha}} (2r)^{d+2\alpha}. \quad (4.5.64)$$

Using Young's inequality, Lemma 4.5.10 and the fact that $H \in \mathcal{C}^{1,1}(\widetilde{\Omega})$,

$$V = \iint_{Q_{2r}} (H - H_1)\nabla \widetilde{v} \cdot \nabla v_2 \leqslant \delta \iint_{Q_{2r}} |\nabla v_2|^2 + C_\delta \iint_{Q_{2r}} |H - H_1|^2 |\nabla \widetilde{v}|^2 \leqslant \delta \iint_{Q_{2r}} |\nabla v_2|^2 + C_\delta \|\nabla H\|^2_{\infty,\widetilde{\Omega}} r^{d+2}. \quad (4.5.65)$$

Using that $J \in \mathcal{C}^{1,1}(\widetilde{\Omega})$,

$$VI = \iint_{Q_{2r}} (J - J_1) \cdot \nabla \widetilde{v} v_2 \leqslant \iint_{Q_{2r}} |v_2|^2 + \iint_{Q_{2r}} |J - J_1|^2 |\nabla \widetilde{v}|^2 \leqslant 4(2r)^{d+2} + \|\nabla J\|^2_{\infty,\widetilde{\Omega}} (2r)^2 \iint_{Q_{2r}} |\nabla \widetilde{v}|^2 \leqslant Cr^{d+2}. \quad (4.5.66)$$

Combining (4.5.61), (4.5.62), (4.5.63), (4.5.64), (4.5.65) and (4.5.66) with $\delta > 0$ small enough yields Step 1.

Step 2 : We now conclude. Let $\rho < r$. Observe that equation (4.5.59) satisfies the requirements of Lemma 4.5.3. On the other hand, the function w satisfies the following Campanato estimate : there exists a constant C_{cam} such that

$$\iint_{Q_\rho} |w - \{w\}_\rho|^2 \leqslant C_{cam} \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r} |w - \{w\}_r|^2.$$

Using twice that $|a+b|^2 \leqslant 2a^2+2b^2$, twice Poincare's inequality and Step 1, there exists a constant $C_P > 0$ such that

$$\begin{aligned}
\iint_{Q_\rho} |\widetilde{v} - \{\widetilde{v}\}_{\rho,t_1,x_1}|^2 &\leqslant 2 \iint_{Q_\rho} |w - \{w\}_{\rho,t_1,x_1}|^2 + 2 \iint_{Q_\rho} |v_2 - \{v_2\}_{\rho,t_1,x_1}|^2 \\
&\leqslant 2C_{Cam} \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r} |w - \{w\}_{r,t_1,x_1}|^2 + C_P \rho^2 \iint_{Q_{2\rho}} |\nabla v_2|^2 \\
&\leqslant 4C_{Cam} \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r} |\widetilde{v} - \{\widetilde{v}\}_{r,t_1,x_1}|^2 \\
&\qquad + 4C_{Cam} \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r} |v_2 - \{v_2\}_r|^2 + C_P r^2 \iint_{Q_{2r}} |\nabla v_2|^2 \\
&\leqslant 4C_{Cam} \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r} |\widetilde{v} - \{\widetilde{v}\}_{r,t_1,x_1}|^2 \\
&\qquad + 4C_{Cam} \cdot 1 \cdot C_P r^2 \iint_{Q_{2r}} |\nabla v_2|^2 + C_P r^2 \iint_{Q_{2r}} |\nabla v_2|^2 \\
&\leqslant C_1 \left(\frac{\rho}{r}\right)^{d+4} \iint_{Q_r} |\widetilde{v} - \{\widetilde{v}\}_{r,t_1,x_1}|^2 + C_2 r^{d+2+2\alpha}.
\end{aligned}$$

This concludes the proof of Lemma 4.5.9 and the proof of Proposition 4.5.7. □

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