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Espaces de fonctions inhomogènes ou anisotropes: modélisation et estimation de spectres multifractals

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Résumé

Cette thèse se compose de trois parties. La première porte sur certains espaces de Besov inhomogènes. La deuxième étudie les propriétés de régularité des fonctions dans des espaces anisotropes de type Besov. La troisième adopte une approche numérique et se concentre sur les propriétés des séries « non-différentiables » de Riemann.

La première partie porte sur la régularité des traces de fonctions dans des espaces fonctionnels inhomogènes récemment introduits par Barral-Seuret. Ces espaces généralisent les espaces de Besov classiques et dépendent d'un « environnement » ξ (que l'on peut voir comme une mesure de probabilité) et de réels positifs p et q. Cet environnement ξ introduit une contrainte dans la définition, permettant aux fonctions d'avoir des comportements qui diffèrent selon la région de $\mathbb{R}^D$. Le premier résultat établit une équivalence entre ces espaces et des espaces definis via une formule de type Sobolev–Slobodeckij dépendant de ξ, les confirmant comme une généralisation des espaces de Besov classiques. Sous des hypothèses naturelles sur ξ, nous mettons ensuite en évidence le comportement multifractal prévalent des fonctions dans ces espaces. L'analyse multifractale consiste à déterminer l'exposant de Hölder ponctuel d'une fonction en chaque point, et à estimer la dimension de Hausdorff des ensembles de points de même exposant. Introduite notamment par Frisch et Parisi dans leurs travaux de 1984 sur la turbulence, la multifractalité est à présent vue comme une propriété importante en analyse, en probabilités et en systèmes dynamiques, entre autres. Dans ce contexte, les propriétés sont étudiées sur des ensembles dits prévalents, une notion d'analyse fonctionnelle qui formalise l'idée de « presque toutes fonctions » dans des espaces de dimension infinie. Si le comportement multifractal prévalent était connu pour des espaces de Besov classiques, son étude est plus délicate dans le cadre inhomogène. Cette thèse étudie aussi les traces de fonctions dans ces espaces inhomogènes, à savoir la restriction d'une fonction à des sous-espaces de dimension inférieure. C'est une question naturelle en physique, où les mesures expérimentales ne sont que des traces unidimensionnelles de phénomènes tridimensionnels tels que les écoulements turbulents. Contrairement aux espaces de Besov classiques, la régularité des traces dans ces espaces inhomogènes peut dépendre fortement de la région de la restriction. Les techniques combinent théorie de la mesure, analyse fonctionnelle, outils d'ondelettes et éléments de systèmes dynamiques.

La deuxième partie étudie les propriétés de régularité des fonctions dans des contextes anisotropes. L'anisotropie, qui apparaît lorsque les propriétés physiques multi-dimensionnelles varient selon la direction, apparaît dans de nombreuses applications comme l'hydrologie et l'imagerie médicale. Nous montrons que les fonctions dans des espaces anisotropes de type Besov présentent une régularité maximale pour une structure d'anisotropie spécifique, ce qui

fournit donc un cadre optimal pour leur analyse.

La troisième partie adopte une approche numérique décrite comme mathématiques expérimentales. Elle vise à approximer le spectre de Legendre de certaines fonctions afin d'émettre des hypothèses sur leur spectre de singularités. Le spectre de Legendre fournit une borne supérieure de ce dernier et coïncide pour de larges classes de fonctions et processus stochastiques. Cette méthode est appliquée à des variantes de la fonction non-différentiable de Riemann, qui ont récemment suscité un regain d'intérêt par leur apparition en modélisation de la turbulence dans des travaux récents de Banica-Vega. Des résultats récents ne fournissent que des bornes pour le spectre de singularités et des preuves numériques visent à éclairer le comportement réel entre celles-ci.

Inhomogeneous or anisotropic function spaces: modelling and estimation of multifractal spectra

Abstract

This thesis consists of three parts. The first investigates certain inhomogeneous Besov spaces. The second studies regularity properties of functions in anisotropic Besov-type spaces. The third adopts a numerical perspective and focuses on the properties of Riemann's "non-differentiable" series.

The first part focuses on the regularity of traces of functions belonging to inhomogeneous function spaces recently introduced by Barral-Seuret. These spaces generalise classical Besov spaces and depend on an "environment" ξ (which should be thought of as a probability measure) and two positive real parameters p and q. The environment ξ introduces a constraint into the definition, allowing functions to display different behaviours in different regions of $\mathbb{R}^D$. A first result establishes an equivalence between these spaces and spaces defined by a Sobolev-Slobodeckij-type formula depending on ξ, reinforcing that these new spaces are genuine analogues of classical Besov spaces. Under natural hypotheses on ξ, we then identified the prevalent multifractal behaviour of functions in these spaces. Multifractal analysis consists of determining the pointwise Hölder exponent of a function around each point, and estimating the Hausdorff dimension of the sets of points having the same exponent. Introduced essentially by Frisch and Parisi in a seminal 1984 paper on turbulence, multifractality is now recognised as an important property in analysis, probability theory and dynamical systems, among others. It is customary to prove properties on prevalent sets of functions, a notion from functional analysis formalising the idea of "almost every function" in infinite-dimensional spaces. While prevalent multifractal behaviour was already known for classical Besov spaces, the corresponding Hausdorff dimensions are significantly easier to compute in that setting. This thesis also studies traces of functions in inhomogeneous Besov spaces, meaning the restriction of a function defined on $\mathbb{R}^D$ to d-dimensional subspaces. This question naturally arises in physics, where experimental measurements are only one-dimensional traces of phenomena such as the three-dimensional turbulent flows. In contrast to standard Besov spaces, the regularity of the traces in inhomogeneous Besov spaces may strongly depend on the region where the restriction is taken. The techniques rely on measure theory, functional analysis, wavelet methods, and some elements of dynamical systems.

The second part studies the regularity properties of functions in anisotropic settings. Anisotropy, which arises when multidimensional physical properties vary with direction, appears in many applications such as hydrology and medical imaging. We show that functions in anisotropic Besov-type spaces exhibit maximal regularity with respect to a specific anisotropy structure, which therefore provides an optimal framework for their analysis.

The third part presents numerical results described as "experimental mathematics".

The goal is to approximate the Legendre spectrum of certain functions to conjecture their singularity spectrum. The Legendre spectrum is known to provide an upper bound for the singularity spectrum, and for large classes of functions and stochastic processes, coincides with it. We applied this technique to several variants and extensions of Riemann's non-differentiable function, which recently attracted renewed attention due to their unexpected appearance in turbulence modelling in recent work of Banica-Vega. Recent results provide upper and lower bounds for the singularity spectrum that do not coincide, and numerical evidence may help determine what to expect between these values.

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Chapter 1

Introduction en français

Dans cette thèse, nous nous intéressons principalement à l'analyse multifractale des fonctions. L'analyse multifractale décrit et étudie les propriétés locales des ensembles, des mesures ou des fonctions. L'étude des propriétés multifractales des fonctions, qui sera le sujet principal de cette thèse, est un domaine récent des mathématiques, dont l'importance est apparue dès les années 1980, lorsque les physiciens ont cherché à quantifier les variations locales de la vitesse d'un fluide dans le contexte d'une turbulence pleinement développée. Des expériences ont montré que la vitesse mesurée d'un écoulement turbulent présente de fortes irrégularités qui dépendent du temps [53, 68], ce qui signifie que les exposants ponctuels décrivant le signal correspondant varient dans le temps. Les écoulements turbulents ne sont pas non plus spatialement homogènes, et la régularité ponctuelle de la vitesse semble varier considérablement d'un point à l'autre.

La notion de multifractalité a été introduite par Mandelbrot, même s'il ne l'appelait pas ainsi. Dans les années 1970 [130, 131, 132], Mandelbrot a introduit les cascades multiplicatives, qui sont des mesures aléatoires pouvant être utilisées comme modèle pour la dissipation d'énergie [133]. Kahane et Peyriere [105] ont étudié leur convergence et prouvé que ces cascades sont des mesures multifractales, ce qui signifie qu'une réalisation de la mesure peut avoir des dimensions locales qui dépendent de l'endroit autour duquel on se concentre.

La relation entre la multifractalité et les propriétés d'échelle globales de la vitesse a été découverte pour la première fois par U. Frisch et G. Parisi [66] (voir également [1, 8] pour les travaux reliant la multifractalité et la turbulence). Depuis lors, les notions multifractales se sont révélées utiles et pertinentes dans divers domaines mathématiques : théorie géométrique de la mesure, analyse réelle [42, 86, 160], les systèmes dynamiques et la théorie ergodique, les probabilités, l'approximation diophantique [59, 161, 162, 163], pour n'en citer que quelques-uns.

La prise en compte des propriétés multifractales présente également un intérêt en pratique. L'existence d'algorithmes numériques stables permettant d'estimer des quantités multifractales a permis de nombreuses avancées dans le domaine du traitement du signal et de l'image [175, 176] et d'autres domaines scientifiques [2, 99, 177].

Nous étudierons principalement la caractérisation du comportement local d'une fonction autour d'un point, ce qui est nécessaire dans diverses applications telles que la mécanique des fluides et le traitement d'images. Nous fournirons également des informations complémentaires issues de la régularité globale de cette fonction.

Nous rappellerons la définition et les propriétés de l'exposant de régularité ponctuelle et du spectre de singularités pour les fonctions et les fonctions d'ensembles dans la section 1.1, du formalisme en ondelettes, qui sera utilisé dans la thèse, dans la section 1.2 et d'ensembles prévalent et des catégorie de Baire pour les ensembles dans la section 1.3. La section 1.4 rappelle d'abord les définitions des espaces de Besov classiques et les résultats sur les propriétés multifractales des fonctions dans ces espaces, puis donne quelques extensions de ces espaces et en particulier les espaces de Besov inhomogènes qui constituent le sujet principal de la première partie de cette thèse, ainsi que des résultats génériques au sens de Baire dans ces espaces. Elle présente également une extension des espaces de Sobolev–Slobodeckij dans un environnement inhomogène et donne des résultats du chapitre 4 reliant ces espaces de Sobolev inhomogènes aux espaces de Besov inhomogènes, prolongeant ainsi les résultats des espaces de Besov et de Sobolev classiques. La section 1.5 présente les propriétés multifractales prévalentes des fonctions dans les espaces de Besov inhomogènes du chapitre 4, étendant le résultat générique au sens de Baire. La section 1.6 rappelle d'abord les résultats connus sur les traces des fonctions dans les espaces de Besov et en donne leur extension aux traces des fonctions dans les espaces de Besov inhomogènes, qui sont prouvées au chapitre 5. La section 1.7 introduit les espaces de type Besov dotés de structures anisotropes et présente des résultats sur la régularité des fonctions en fonction de l'anisotropie, exposés au chapitre 6. La section 1.8 introduit les considérations du chapitre 7 relevant du domaine des « mathématiques expérimentales » et l'objectif de l'estimation numérique des propriétés multifractales de la fonction non différentiable de Riemann, avec des résultats théoriques et des conjectures.

1.1 Régularité ponctuelle

Nous nous concentrons sur l'analyse multifractale des fonctions, qui consiste à décrire les comportements ponctuels des fonctions $f : \mathbb{R}^D \to \mathbb{R}$, afin de quantifier la « rugosité » de f (ou de son graphique) près d'un point x_0.

Definition 1.1. *Étant donné $x_0 \in \mathbb{R}^D$, $H \in \mathbb{R}_+$, une fonction $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^D)$ appartient à $\mathcal{C}^H(x_0)$ s'il existe un polynôme P_{x_0} de degré inférieur à $\lfloor H \rfloor$, une constante $C > 0$ et un voisinage V de x_0 tels que*

$$\forall x \in V, \quad |f(x) - P_{x_0}(x - x_0)| \leq C \ |x - x_0|^H . \tag{1.1}$$

L'exposant de Hölder ponctuel de f en x_0 est alors

$$h_f(x_0) = \sup\{H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0)\}. \tag{1.2}$$

L'exposant de Hölder ponctuel minimal de f est $h_f^{\min} = \sup\{H \in \mathbb{R} : f \in \mathcal{C}^H(\mathbb{R})\}$.

Certaines fonctions ou certains processus aléatoires ont un unique exposant de Hölder constant, l'un des plus notables étant la fonction de Weierstrass

$$W_{a,b}(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$$

avec $0 < a < 1$ et $b > 1$ tel que $ab > 1$. Une contrepartie stochastique à ce phénomène est apparue avec le mouvement brownien $(B_t)_{t\geq 0}$, modélisant le mouvement des particules observé par Brown en 1827. Il est connu que la fonction de Weierstrass a un exposant de Hölder constant $h_{W_{a,b}}(x) = \frac{-\log(a)}{\log(b)}$, tandis que le mouvement brownien satisfait presque sûrement $h_B(x) = 1/2$.

Bien que l'exposant de Hölder ponctuel décrive le comportement local, il présente certaines lacunes. En effet, il ne décrit pas le comportement oscillatoire au voisinage du point étudié. Considérons d'abord la fonction « cusp », définie par $x \mapsto |x|^\gamma$, où γ est un réel positif. Cette fonction a un exposant de Hölder ponctuel en 0 égal à γ, mais ne présente aucun phénomène d'oscillation autour de 0. De même, la fonction « chirp », définie par $x \mapsto |x|^\gamma \sin(1/|x|^\beta)$, où γ et β sont deux réels positifs, a également un exposant de Hölder égal à γ en 0, mais présente des oscillations rapides autour de 0. Cela montre que l'exposant de Hölder ponctuel ne permet pas de distinguer ces deux comportements de nature très différente. En outre, il n'est pas stable sous l'action d'opérateurs pseudo-différentiels d'ordre 0, et ces considérations seront abordées dans le dernier chapitre de cette thèse.

Cependant, comme pour la vitesse ci-dessus, les fonctions peuvent avoir un exposant de Hölder qui varie considérablement d'un point à l'autre. Il est donc particulièrement important de comprendre la distribution géométrique de ces singularités. Nous nous concentrerons donc sur la dimension de Hausdorff des ensembles de niveau de points avec un exposant de Hölder donné et le spectre de singularités σ_f d'une fonction f.

Definition 1.2. *Les ensembles de niveau de $x \mapsto h_f(x)$ sont $E_f(h) := \{x \in \mathbb{R}^D : h_f(x) = h\}$, et le spectre de singularités (ou multifractal) associé σ_f de f est l'application*

$$\sigma_f : h \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H (E_f(h)) \in [0, D] \cup \{-\infty\}. \tag{1.3}$$

Ici, $\dim_H$ représente la dimension de Hausdorff avec $\dim_H (\emptyset) = -\infty$ (notez que $E_f(h) = \emptyset$ pour $h < 0$).

Ces quatités peuvent être étendue à des fonctions telles que des mesures de la manière suivante. L'ensemble $\mathcal{B}(\mathbb{R}^D)$ représente les ensembles boréliens de $\mathbb{R}^D$. Le diamètre d'un ensemble E est noté $|E|$ et sa mesure de Lebesgue D-dimensionnel est $\mathcal{L}^D(E)$.

Definition 1.3. *L'ensemble des fonctions d'ensembles de Hölder est*

$$\mathcal{H}(\mathbb{R}^D) = \{\xi : \mathcal{B}(\mathbb{R}^D) \to \mathbb{R}_+ \cup \{+\infty\} : \exists C, s > 0, \forall E \in \mathcal{B}(\mathbb{R}^D), \xi(E) \leq C|E|^s\}.$$

Lorsque $\xi(E) \leq C|E|^s$ pour tout E avec $0 \leq |E| \leq 1$, ξ est dit s-Hölder. Le support topologique de $\xi \in \mathcal{H}(\mathbb{R}^D)$ est $\text{supp}(\xi) = \{x \in \mathbb{R}^D : \xi(B(x, r)) > 0, \forall\, r > 0\}$. Le support d'une fonction d'ensembles ξ est un ensemble de Borel A lorsque $\text{supp}(\xi) = A$. Par abus de

notation, nous noterons

$$\mathcal{H}([0,1]^D) = \{\xi \in \mathcal{H}(\mathbb{R}^D) : \mathrm{supp}(\xi) = [0,1]^D\}. \tag{1.4}$$

Pour $\xi \in \mathcal{H}(\mathbb{R}^D)$, $s > 0$ et $E \subset \mathbb{R}^D$, définissons

$$\xi^s(E) = \xi(E)^s \quad \text{et} \quad \xi^{(+s)}(E) = \xi(E)|E|^s,$$

et si ξ est s_0-Hölder, définissons pour tout $s \in (0, s_0)$,

$$\xi^{(-s)}(E) = \begin{cases} 0 & \text{si } |E| = 0, \\ \xi(E)|E|^{-s} & \text{si } 0 < |E| < \infty. \end{cases}$$

Remarquons que ξ^s, $\xi^{(+s)}$ et $\xi^{(-s)}$, définis comme ci-dessus, appartiennent toujours à $\mathcal{H}(\mathbb{R}^D)$.

Nous aurons besoin de l'hypothèse supplémentaire selon lesquelles les fonctions d'ensemble $\xi \in \mathcal{H}(\mathbb{R}^D)$ sont décroissantes par rapport à l'inclusion, c'est-à-dire qu'elles sont des *capacités*. C'est naturel puisque, dans de nombreuses situations, les fonctions d'ensembles considérées sont des mesures ou des puissances de mesures. Nous définissons donc les ensembles de capacités

$$\mathcal{C}(\mathbb{R}^D) = \{\xi \in \mathcal{H}(\mathbb{R}^D) : \forall E, F \in \mathcal{B}(\mathbb{R}^D), E \subset F \Rightarrow \xi(E) \leq \xi(F)\}$$
$$\mathcal{C}([0,1]^D) = \mathcal{H}([0,1]^D) \cap \mathcal{C}(\mathbb{R}^D).$$

Les fonctions d'ensembles considérées dans cette thèse vérifient la propriété (P) qui suit.

Definition 1.4. *Une fonction d'ensembles $\xi \in \mathcal{H}(\mathbb{R}^D)$ satisfait la propriété (P) s'il existe $C, s_1, s_2 > 0$ tels que*

$$\text{pour tout } r > 0, \text{ pour tout } x \in \mathrm{supp}(\xi), \quad C^{-1} r^{s_2} \leq \xi(B(x,r)) \leq C r^{s_1}. \tag{1.5}$$

En particulier, un tel ξ est s_1-Hölder. Nous ne considérerons que les fonctions d'ensembles satisfaisant la propriété (P) sur $\mathbb{R}^D$ (ou $[0,1]^D$), ce qui implique que ces fonctions auront pour support $\mathbb{R}^D$ (ou $[0,1]^D$). Rappelons qu'une mesure borélienne finie ξ est dite doublante si, pour une certaine constante $C_\xi \geq 1$, pour tout $x \in \mathbb{R}^D$ et $r > 0$,

$$\xi(B(x,r)) \leq \xi(B(x,2r)) \leq C_\xi\, \xi(B(x,r)).$$

Comme nous travaillons avec des fonctions d'ensembles (pour lesquelles on n'a pas en général que $\xi(B(x,r)) \leq \xi(B(x,2r))$), et pas nécessairement avec des mesures, nous devons adapter cette notion à notre situation.

Definition 1.5. *Soit Φ l'ensemble des fonctions décroissantes $\phi : (0,1] \to \mathbb{R}_+$ telles que $\lim_{r \to 0^+} \frac{\phi(r)}{\log(r)} = 0$.*
Une fonction d'ensembles $\xi \in \mathcal{H}(\mathbb{R}^D)$ est presque doublante *sur un ensemble A lorsqu'il existe $\phi \in \Phi$ tel que pour tout $x \in A$, $r \in (0,1]$, $R \in [r, 2r]$ et $x' \in B(x,r)$,*

$$e^{-\phi(r)} \xi(B(x',r)) \leq \xi(B(x,R)) \leq e^{\phi(r)} \xi(B(x',r)). \tag{1.6}$$

La fonction d'ensembles ξ est doublante lorsque (1.6) est vérifiée avec une fonction constante ϕ.

Intuitivement, (1.6) implique que le fait de déplacer légèrement le centre et le rayon d'une boule garde sa masse ξ contrôlée. Une mesure doublante satisfait (1.6) avec une fonction constante ϕ. Les exemples de fonctions d'ensembles doublantes incluent les mesures de Gibbs (1-périodiques) ξ, voir la Définition 1.25 et [30, 47, 147], et leurs puissances $\xi^s(E) := \xi(E)^s$ pour $s > 0$.

Pour tout entier $j \geq 0$, soit Λ_j^D l'ensemble des cubes dyadiques de taille 2^{-j} dans $[0,1]^D$ et $\Lambda^D = \bigcup_{j\in\mathbb{N}} \Lambda_j^D$.

Definition 1.6. *Une fonction d'ensembles $\xi \in \mathcal{H}(\mathbb{R}^D)$ satisfait la propriété (P_2) s'il existe $C, s_2 > 0$ et $\phi \in \Phi$ (voir Définition 1.5) tels que pour tous $j, j' \in \mathbb{N}$ avec $j \leq j'$, pour tous les $\lambda, \widetilde{\lambda} \in \Lambda_j^D$ tels que $\partial\lambda \cap \partial\widetilde{\lambda} \neq \emptyset$, et $\lambda' \in \Lambda_{j'}^D$ tels que $\lambda' \subset \lambda$:*

$$\xi(\widetilde{\lambda}) \leq C 2^{\phi(2^{-j})} 2^{(j'-j)s_2} \xi(\lambda').$$

En particulier, ξ est presque doublante dans ce cas [19]. Alors que la masse ξ de chaque cube dyadique est contrôlée par s_1 et s_2 par (P), la propriété (P_2) peut être considérée comme une sorte de propriété quasi-Bernoulli faible pour ξ.

Nous désignerons souvent $\xi \in \mathcal{H}(\mathbb{R}^D)$ par un *environnement* dans la suite pour des raisons que nous expliquerons après. Les propriétés multifractales d'une fonction d'ensembles peuvent être considéré de manière similaire aux fonctions étudiées au début de la section. Pour étudier sa régularité ponctuelle, nous utilisons les dimensions locales et le spectre de singularités associé, définies comme suit.

Definition 1.7. *Soit $\xi \in \mathcal{H}(\mathbb{R}^D)$. Pour $x \in \operatorname{supp}(\xi)$, la dimension locale inférieure de ξ en x est*

$$\underline{h}_\xi(x) = \liminf_{r\to 0^+} \frac{\log \xi(B(x,r))}{\log(r)}. \tag{1.7}$$

Le spectre de singularités de ξ est la dimension de Hausdorff des ensembles de niveau de la dimension locale

$$\sigma_\xi \ : \ h \in \mathbb{R} \mapsto \dim_H \left(\underline{E}_\xi(h)\right) \quad \text{où} \quad \underline{E}_\xi(h) = \{x \in \operatorname{supp}(\xi) \ : \ \underline{h}_\xi(x) = h\}.$$

L'ensemble $E_\xi(h)$ est l'ensemble des points x où la dimension locale (1.7) est une limite qui est égale à h.

Definition 1.8. *La fonction d'échelle de $\xi \in \mathcal{H}([0,1]^D)$ avec $\operatorname{supp}(\xi) \cap [0,1]^D \neq \emptyset$ est*

$$\tau_\xi(q) = \liminf_{j\to+\infty} \frac{1}{-j} \log_2 \sum_{\lambda\in\Lambda_j^D} \xi(\lambda)^q,$$

où $0^q = 0$ par convention.

Les dimensions locales minimale et maximale ainsi que la dimension de ξ sont définies respectivement par

$$h_\xi^{\min} = \min\{\underline{h}_\xi(x) : x \in \operatorname{supp}(\xi)\}, \quad h_\xi^{\max} = \max\{\underline{h}_\xi(x) : x \in \operatorname{supp}(\xi)\},$$
$$\dim(\xi) = \inf\{\dim_H(E) : \xi(E) = 1\}.$$

1.2 Formalisme en ondelettes

Tout au long de cette thèse, nous décomposons des fonctions sur des bases d'ondelettes. En effet, les coefficients d'ondelettes permettent de caractériser la régularité de Hölder ou des espaces classiques tels que les espaces de Besov. Au début des années 1900, Haar fut le premier à construire une base d'ondelettes orthonormée sans toutefois l'appeler « ondelette ». La première base d'ondelettes de régularité arbitrairement grande a été construite par Strömberg [164] en 1983. Lemarié et Meyer, en 1986 [120], ont généralisé la construction de bases d'ondelettes orthonormées de classe C^∞ en dimension supérieure. Mallat [128] a introduit, en 1989, une méthode pour construire des bases d'ondelettes à partir de l'analyse multirésolution en une dimension, et Meyer l'a généralisée à des dimensions supérieures en 1990 [136] et a introduit une méthode par tensorisation. En 1992, Daubechies a construit des ondelettes à support compact [48].

Rappelons la décomposition sur une base d'ondelettes. Soit $D \in \mathbb{N}^*$ et $1 \leq p, q \leq +\infty$. Une fonction $\psi : \mathbb{R}^D \to \mathbb{R}$ a $n+1$ moments nuls lorsque, pour tout multi-indice $\alpha \in \mathbb{N}^D$ tel que $\sum_{i=1}^D \alpha_i \leq n$, $\int_{\mathbb{R}^D} x_1^{\alpha_1} \dots x_D^{\alpha_D} \psi(x) dx = 0$.

Soit $L^D = \{0,1\}^D \setminus \{0^D\}$, et considérons $\{\varphi, \psi_\lambda := \psi^l(2^j \cdot - k)\}_{\lambda=(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^D\times L^D}$ une base d'ondelettes de $L^2(\mathbb{R}^D)$ générée par une famille d'ondelettes $(\varphi, \psi^l)_{l\in L^D}$, telle que chaque $f \in L^2(\mathbb{R}^D)$ puisse s'écrire [128, 136]

$$f \overset{L^2}{=} \sum_{\lambda\in\mathbb{Z}\times\mathbb{Z}^D\times L^D} c_\lambda \psi_\lambda \quad \text{avec} \quad c_\lambda = 2^{Dj} \int_{\mathbb{R}^D} f(x)\psi_\lambda(x)dx.$$

Pour $k = (k_1, \dots, k_D) \in \mathbb{Z}^D$, $j \in \mathbb{Z}$, nous identifions souvent $\lambda = (j,k,l)$ au cube dyadique $\prod_{i=1}^D [k_i 2^{-j}, (k_i+1)2^{-j})$, et en particulier nous écrivons $\xi(\lambda)$ pour $\xi(\prod_{i=1}^D [k_i 2^{-j}, (k_i+1)2^{-j}))$.

Pour tout $f \in L^p(\mathbb{R}^D)$, $1 \leq p \leq \infty$, f peut être écrit [136]

$$f = \sum_{k\in\mathbb{Z}^D} \beta(k)\varphi(\cdot - k) + \sum_{j\geq 0} \sum_{\lambda\in\{j\}\times\mathbb{Z}^D\times L^D} c_\lambda \psi_\lambda, \text{ avec } \beta(k) = \int_{\mathbb{R}^D} f(x)\varphi(x-k)dx. \tag{1.8}$$

Sans perte de généralité pour notre problème, nous travaillerons principalement avec des fonctions à support dans $[0,1]^D$ et nous écrirons

$$f = \beta(0)\varphi + \sum_{j\geq 0} \sum_{\lambda\in\Lambda_j^D\times L^D} c_\lambda \psi_\lambda. \tag{1.9}$$

En effet, toute fonction $f \in L^p(\mathbb{R}^D)$ peut être écrite sous la forme d'une somme $f = g + \sum_{\ell\in\mathbb{Z}^D} f_\ell(\cdot + \ell)$, où g est une fonction aussi régulière que les ondelettes et f_ℓ peut être écrite sous la forme (1.9). Comme nous ne nous intéressons qu'aux propriétés locales et multifractales, et en utilisant le fait que la dimension de Hausdorff d'une union dénombrable est le supremum de la dimension de tous les ensembles, le spectre de singularités d'une fonction localement bornée $f \in L^p(\mathbb{R}^D)$ est obtenu comme le supremum de ceux des f_ℓ.

Dans cette thèse, nous utiliserons une famille d'ondelettes $(\varphi, \psi^l)_{l\in L^D}$ construite à partir d'un processus de tensorisation tel que décrit dans [136, Chapitres 2 et 3]. Pour tout entier

arbitraire $N \geq 1$, il existe des fonctions à support compact $\psi^0 \in C^N(\mathbb{R})$ (appelées fonctions d'échelle) et $\psi^1 \in C^N(\mathbb{R})$ (appelées ondelettes mères), avec ψ^1 ayant au moins $N+1$ moments nuls et telles que $(\psi^0, \psi^1_{j,k} := \psi^1(2^j \cdot -k))_{j,k\in\mathbb{Z}}$ forment une base orthogonale de $L^2(\mathbb{R})$. Dans ce cas, l'ondelette est dite N-régulière. Introduisons les notations

$$0^D := (0, \dots, 0), \quad 1^D := (1, \dots, 1), \quad L^D = \{0,1\}^D \backslash 0^D.$$

Une base orthogonale de $L^2(\mathbb{R}^D)$ est obtenue par tensorisation comme suit : pour $\lambda = (j,k,l) \in \mathbb{Z} \times \mathbb{Z}^D \times L^D$, définissons l'ondelette tensorisée

$$\psi_\lambda = \prod_{i=1}^{d} \psi^{l_i}_{j,k_i}$$

avec $k = (k_1, \dots, k_D)$ et $l = (l_1, \dots, l_D) \in L^D$, et en posant $\varphi = \psi^{0^D}$.

Ainsi, pour chaque $f \in L^p(\mathbb{R}^D)$, on a la décomposition (1.8) (notons que nous utilisons une normalisation L^∞ pour le coefficient d'ondelette $(c_\lambda)_{\lambda\in\Lambda}$).

Proposition 1.9. [83, 92] *Soient $\gamma > 0$ et $N = \lfloor \gamma + 1 \rfloor$, et $\varphi, \psi \in C^N(\mathbb{R})$ des fonctions au moins $N+1$-régulier. Soit $f : [0,1]^D \to \mathbb{R}$ une fonction localement bornée avec des coefficients d'ondelettes $\{c_\lambda\}_{\lambda\in\Lambda^D\times L^D}$, et soit $x \in [0,1]^D$.*

Si $f \in C^\gamma(x)$, alors il existe une constante $M < \infty$ telle que pour tout $\lambda = (j,k,l) \in \Lambda^D \times L^D$,

$$|c_\lambda| \leq M\ 2^{-j\gamma} \left(1 + |2^j x - k|\right)^\gamma . \tag{1.10}$$

Inversement, si (1.10) est vérifié pour $f \in \bigcup_{\varepsilon>0} C^\varepsilon([0,1]^D)$, alors $f \in \bigcap_{\eta>0} C^{\gamma-\eta}(x)$.

1.3 Prévalence et généricité de Baire

Definition 1.10. *Soit X un espace métrique complet, et $A \subset X$. Un ensemble A est de première catégorie lorsqu'il peut s'écrire $A = \bigcup_{n\in\mathbb{N}} A_n$, où pour tout $n \in \mathbb{N}$, A_n est inclus dans un ensemble fermé dont l'intérieur est vide dans X.*

Un ensemble est de deuxième catégorie, ou résiduel, si son complément est de première catégorie, c'est-à-dire un ensemble qui contient une intersection dénombrable d'ensembles ouverts denses.

Une propriété est typique dans l'ensemble X si elle est vérifiée pour tous les éléments de l'ensemble de deuxième catégorie $A \subset X$.

L'étude des propriétés des fonctions typiques remonte au début des années 1930 [13, 102, 135], avec l'exemple classique qu'une fonction typique dans $C([0,1])$ est nulle part dérivable. Jaffard s'est intéressé au spectre de singularités des fonctions typiques dans un espace de Besov [90]. Barral-Seuret ont prouvé dans [19] que les fonctions typiques dans un espace de Besov inhomogène ont un spectre de singularités spécifique.

Cependant, les notions d'être de première catégorie et de mesure de Lebesgue nulle se recoupent mais ne correspondent pas. L'ensemble des nombres rationnels $\mathbb{Q}$ est de première

catégorie et a une mesure de Lebesgue nulle, mais l'ensemble des nombres de Liouville est de seconde catégorie et a une mesure de Lebesgue nulle, tandis que l'ensemble des nombres diophantiens a une mesure de Lebesgue pleine et est de première catégorie [81].

Cette différence entre catégorie de Baire et mesure de Lebesgue justifie qu'on s'intéresse à la notion de « mesure de Lebesgue pleine » pour les espaces de fonctions, qui correspond à la notion d'ensembles prévalent. Christensen [44] et Hunt, Sauer et York [81] ont proposé indépendamment le concept d'ensembles timides et prévalents pour remplacer la notion de mesure de Lebesgue pour les ensembles universellement mesurables dans tout espace vectoriel topologique réel ou complexe E doté de son σ-algèbre de Borel $\mathcal{B}(E)$. Dans cette thèse, toutes les mesures de Borel ϑ sur $(E, \mathcal{B}(E))$ seront automatiquement complétées, c'est-à-dire que pour tout ensemble $A \subset E$ et $B \in \mathcal{B}(E)$ tels que la différence symétrique $A \Delta B$ est incluse dans un certain $D \in \mathcal{B}(E)$ avec $\vartheta(D) = 0$, on pose $\vartheta(A) := \vartheta(B)$.

Definition 1.11. *Un ensemble $A \subset E$ est universellement mesurable s'il est mesurable pour toute mesure de Borel (complétée) sur E. Un ensemble universellement mesurable $A \subset E$ est dit Haar-nul, ou timide, s'il existe une mesure complétée finie ϑ qui est positive sur un sous-ensemble compact K de E et telle que*

$$\text{pour tout } x \in E, \quad \vartheta(A + x) = 0. \tag{1.11}$$

La mesure ϑ utilisée pour montrer qu'un sous-ensemble est Haar-nul est appelée transversale. Plus généralement, un ensemble qui est inclus dans un ensemble universellement mesurable Haar-nul est également appelé Haar-nul. Enfin, le complément dans E d'un sous-ensemble Haar-null S, noté $S^{\complement}$, est appelé prévalent.

Hunt a été le premier à utiliser la prévalence dans les espaces fonctionnels [80]. Fraser-Hyde [61], Bayart-Heurteaux [21] et Balka-Darji-Elekes [12] ont étudié la dimension des images des fonctions continues.

Theorem 1.12 (Fraser-Hyde [61])**.** *Soit $d \in \mathbb{N}$. L'ensemble*

$$\{f \in C([0,1]^d) : \dim_H(\{(x, f(x)) : x \in [0,1]^d\}) = d + 1\}$$

est un sous-ensemble prévalent de $C([0,1]^d)$.

Fraysse-Jaffard-Kahane [63, 64] et Aubry-Maman-Seuret [10] ont étudié, respectivement, la régularité prévalente des fonctions et les traces des fonctions dans les espaces de Besov, voir les Théorèmes 1.14 et 1.15. Olsen a étudié la dimension multifractale et la dimension L^q des mesures prévalentes sur des ensembles compacts de $\mathbb{R}^D$ [142, 143].

1.4 Des espaces homogènes aux espaces inhomogènes

Dans les années 1980, Frisch et Parisi ont été les premiers à identifier un lien entre la multifractalité et les propriétés multi-échelles de la vitesse [66]. En se basant sur le travail

de Kolmogorov en 1941, ils ont proposé de considérer la fonction d'échelle ζ heuristiquement définie par la relation

$$\forall p > 0, \quad S_p(h) := \int |v(x+h) - v(x)|^p dx \underset{h\to 0}{\sim} |h|^{\zeta(p)}.$$

Jaffard [90] a mis en évidence un lien entre ζ et les normes de type Besov, montrant que $\zeta(p) = \sup\{\tau : v \in B^{\tau/p}_{p,p}\}$ et que le champ de vitesse peut être analysé à travers la régularité des fonctions dans $\bigcap_{\varepsilon>0,p>0} B^{(\zeta(p)-\varepsilon)/p}_{p,p}$.

Rappelons la définition des espaces de Besov [136].

Definition 1.13. *Soient $p, q \in (0, +\infty]$, $s > D/p$. Considérons un entier $n > s$ et des ondelettes $(\varphi, \psi^l)_{l\in L^D}$ à support compact, qui appartiennent à $C^n(\mathbb{R}^D)$ et ont au moins n moments nuls.*

Une fonction f, écrite comme dans (1.8), appartient à $B^s_{p,q}(\mathbb{R}^D)$ si et seulement si

$$\|\beta\|_{\ell^p(Z^D)} + \left\|(\varepsilon_j^{s,p})_{j\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})} < \infty, \quad \text{où} \quad \varepsilon_j^{s,p} = \left\|\left(2^{j(s-d/p)} c_\lambda\right)_{\lambda\in\Lambda_j^D\times L^D}\right\|_{\ell^p}. \tag{1.12}$$

Des résultats sur la régularité ponctuelle des fonctions dans des espaces classiques tels que les espaces de potentiel de Bessel $H^{s,p}(\mathbb{R}^D)$, de Sobolev–Slobodeckij $W^{s,p}(\mathbb{R}^D)$ ou de Besov $B^s_{p,q}(\mathbb{R}^D)$ ont été obtenus dans [9, 63, 87, 90, 98]. Il en existe deux types :

- l'identification d'une borne supérieure universelle pour le spectre de singularités de toutes les fonctions dans un espace donné, comme le résultat prouvé par Jaffard [87]

  Theorem 1.14. *Soient $p, q \in (0, \infty]$, $s > D/p$. Pour chaque $f \in B^s_{p,q}(\mathbb{R}^D)$, on a*

  $$\sigma_f(h) \le \begin{cases} \min\{p(h - (s - D/p)), D\} & \text{si } h \ge s - D/p, \\ -\infty & \text{si } h < s - D. \end{cases}$$

- le calcul de la valeur explicite du spectre d'un ensemble générique de fonctions au sens de la prévalence [63] ou des catégories de Baire [90]

  Theorem 1.15. *Soient $p, q \in (0, \infty]$, $s > D/p$. Il existe un ensemble de deuxième catégorie de Baire et un ensemble prévalent de fonctions $f \in B^s p, q(\mathbb{R}^D)$ satisfaisant*

  $$\sigma_f(h) = \begin{cases} p(h - (s - D/p)) & \text{si } h \in [s - D/p, s], \\ -\infty & \text{sinon.} \end{cases}$$

Une caractéristique clé des fonctions typiques dans les espaces classiques $W^{s,p}(\mathbb{R}^D)$ et $B^s_{p,q}(\mathbb{R}^D)$ est que leur spectre de singularités est toujours affine croissant. C'est une contradiction apparente avec le fait que les spectres estimés sur des données réelles à l'aide du formalisme multifractal ont toujours une forme strictement concave, avec une partie croissante *et* une partie décroissante.

Jaffard [90] a montré que, étant donné une fonction concave *admissible* $\tau : \mathbb{R}^+ \to \mathbb{R}$, les fonctions typiques au sens de Baire dans l'intersection $\bigcap_{\varepsilon>0,p>0} B^{(\tau(p)-\varepsilon)/p}_{p,q,loc}(\mathbb{R}^D)$ ont des spectres concaves croissants. Fraysse [62] a obtenu que les fonctions prévalentes dans de

tels espaces possèdent le même spectre. Ces résultats ont réfuté une idée communément admise dans la communauté des physiciens, à savoir qu'une mesure présentant un spectre de singularités non concave résulte nécessairement d'un processus en cascade multiplicatif sous-jacent.

Afin de prendre en compte les spectres de singularités non affines, on peut se référer aux généralisations naturelles existantes des espaces de Sobolev et de Besov classiques, les espaces de Sobolev et de Besov à poids, qui peuvent être définis de différentes manières.

Pour $h \in \mathbb{R}^D$ et $f : \mathbb{R}^D \to \mathbb{R}$, considérons l'opérateur de différence finie $\Delta_h f : x \in \mathbb{R}^D \mapsto f(x+h) - f(x)$. Puis, pour $n \geq 2$, définissons $\Delta_h^n f = \Delta_h(\Delta_h^{n-1} f)$.

Soit $\rho : \mathbb{R} \to \mathbb{R}$ une fonction mesurable, positive presque partout, appelée poids. Sous certaines hypothèses sur le comportement de ρ données dans [7], les espaces de Besov à poids $B^\rho_{p,q}(\mathbb{R}^D)$ sont constitués de fonctions $f : \mathbb{R}^D \to \mathbb{C}$ telles que

$$\int_{|h|\leq 1} \rho(|h|)^{-q} \, \|\Delta_h^n f\|^q_{L^p(\mathbb{R}^D)} \, \frac{dh}{|h|^D} < \infty.$$

Une fonction f appartenant à ces espaces présente un comportement ponctuel homogène, car le poids dans l'intégrale ne dépend que de $|h|$. Ces espaces ont une version généralisée appelée *espace de Besov de régularité généralisée* $B^{\gamma,N}_{p,q}$, avec γ, N certaines séquences, étudiée par Farkas-Leopold [58]. Moura [138] et Loosveldt-Nicolay [124] ont donné une définition alternative. Kreit-Nicolay les a reliés aux espaces généralisés de Hölder-Zygmund $\Lambda^\gamma_{\sigma,N}$ [114, 115]. Ces espaces de Besov de régularité généralisée ont été mis en relation avec des traces de fonction dans des espaces de Besov sur des ensembles fractals par Caetano-Haroske [34] et étudiés par Kleyntssens-Nicolay [111].

Une deuxième version des espaces de Besov à poids existe dans laquelle le poids ne détermine pas le comportement en échelle des oscillations de la fonction f, mais le comportement local de f elle-même et peut être relié aux espaces de Besov standard, voir les resultats de Edmunds, Haroske et Triebel [54, 76, 167]. Elle est définie comme suit : pour un poids $\rho \in L^1_{\text{loc}}(\mathbb{R})$, une fonction $f \in L^1_{\text{loc}}(\mathbb{R}^D)$ appartient à $B^s_{p,q}(\mathbb{R}^D, \rho)$ si

$$\int_{|h|\leq 1} |h|^{-sq} \, \|\rho \, \Delta_h^n f\|^q_{L^p(\mathbb{R}^D)} \, \frac{dh}{|h|^D} < \infty.$$

Même si le comportement multifractal des fonctions dans de tels espaces n'a pas encore été étudié, elles ont une caractérisation explicite par ondelettes donnée par Triebel [171, Chapitre 6] qui peut être liée aux espaces introduits ci-après.

Une autre direction est celle des espaces de Lebesgue et Sobolev à exposant variable, qui sont définis à travers les espaces d'Orlicz à poids [52, 67] comme suit. Soit $p : \mathbb{R}^D \to [1, \infty)$ une fonction mesurable bornée, appelée un exposant variable. L'*espace de Lebesgue à exposant variable* $L^{p(\cdot)}(\mathbb{R}^D)$ est constitué de toutes les fonctions mesurables $f : \mathbb{R}^D \to \mathbb{R}$ telles que

$$\int_{\mathbb{R}^D} |f(x)|^{p(x)} \, dx < +\infty.$$

L'*espace de Sobolev à exposant variable* $W^{k,p(\cdot)}(\mathbb{R}^D)$, $k \in \mathbb{N}$, est constitué de toutes les fonc-

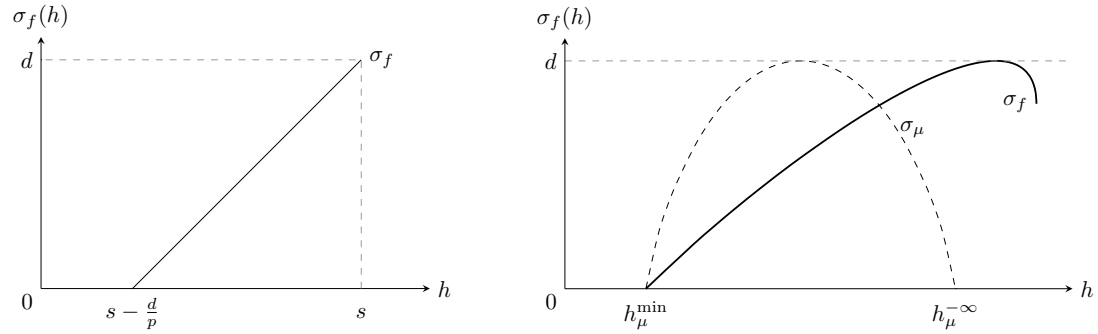


Figure 1.1: Cette figure illustre le spectre de singularités typique des espaces de Besov standard, par opposition à celui des espaces de Besov inhomogènes, afin de motiver leur étude. **Gauche:** Spectre de singularités d'une fonction typique $f \in B^s_{p,q}(\mathbb{R}^D)$. **Droite:** Spectre de singularités d'une fonction typique $f \in B^\xi_{1,q}(\mathbb{R}^D)$. Le graphique en pointillés représente le spectre de singularités (initial) de ξ. Lorsque $p = \infty$ et que f est typique dans $B^\xi_{\infty,q}(\mathbb{R}^D)$, $\sigma_f = \sigma_\xi$.

tions de $L^{p(\cdot)}(\mathbb{R}^D)$ satisfaisant $f^{(k)} \in L^{p(\cdot)}(\mathbb{R}^D)$. L'exposant variable p est en général au moins continu, l'hypothèse habituelle étant la continuité log-Hölder. Alors que les fonctions typiques dans $W^{k,p}(\mathbb{R}^D)$ présentent des spectres affines sans parties décroissantes [90], les espaces $W^{k,p(\cdot)}(\mathbb{R}^D)$ n'ont pas encore été étudiés.

Barral et Seuret ont introduit les espaces de Besov inhomogènes suivants.

Definition 1.16. *Soit $\xi \in \mathcal{H}([0,1]^D)$ satisfaisant (P) de la Définition 1.4 et (P_2) de la Définition 1.6 pour certains $C > 0$ et $0 < s_1 < s_2$. Soit $p, q \in [1, +\infty]$, considérons un entier $n \geq \left\lfloor s_2 + \frac{D}{p} \right\rfloor + 1$ et des ondelettes $(\varphi, \psi^l)_{l \in L^D}$ à support compact, qui appartiennent à $C^n(\mathbb{R}^D)$ et ont au moins n moments nuls.*

Une fonction f, écrite comme dans (1.9), appartient à $B^\xi_{p,q}([0,1]^D)$ si et seulement si $f \in L^p([0,1]^D)$ et $|f|_{\xi,p,q} < +\infty$, où

$$|f|_{\xi,p,q} = \left\| (\varepsilon_j^{\xi,p})_{j \in \mathbb{N}} \right\|_{\ell^q(\mathbb{N})}, \quad \text{où} \quad \varepsilon_j^{\xi,p} = \left\| \left(\frac{c_\lambda}{\xi(\lambda)} \right)_{\lambda \in \Lambda_j^D \times L^D} \right\|_{\ell^p(\Lambda_j^D \times L^D)}. \tag{1.13}$$

Puis

$$\widetilde{B}^\xi_{p,q}([0,1]^D) = \bigcap_{0 < \varepsilon < \min(1, s_1)} B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D). \tag{1.14}$$

Nous considérerons pour cette introduction que $\xi \in \mathcal{H}([0,1]^D)$ satisfait (P) et (P_2), bien que les résultats présentés au chapitre 4 puissent être considérés dans un cadre plus général.

Ces espaces ne sont pas spatialement homogènes dans le sens où, dans l'intégrale ci-dessus, les oscillations $\Delta_h^n f(x)$ de f sont comparées localement à la masse ξ d'une petite boule de rayon h proche de x, et cette masse peut fortement dépendre de x. Ils fournissent un cadre naturel dans lequel les fonctions typiques ont des comportements multifractals similaires à ceux observés dans les données réelles [19].

En prenant $\xi = (\mathcal{L}^D)^{\frac{s}{D}-\frac{1}{p}}$ avec $s > D/p$, on voit d'après la définition que $B^{\xi}_{p,q}([0,1]^D) = B^s_{p,q}([0,1]^D)$. De plus, lorsque $p = q = \infty$, $B^{\xi,\psi}_{\infty,\infty}([0,1]^D)$ est l'espace des fonctions f pour lesquelles

$$C = \sup_{\lambda \in \mathbb{Z}\times\mathbb{Z}^D\times L^D} \frac{|c_\lambda|}{\xi(\lambda)} < +\infty.$$

Cela peut être réécrit sous la forme $|c_\lambda| \leq C\,\xi(\lambda)$. Rappelons que les espaces de Hölder $C^\alpha([0,1]^D)$ sont caractérisés par le fait que $|c_\lambda| \leq C\,|\lambda|^\alpha$ pour tout λ, les espaces $B^{\xi}_{p,q}([0,1]^D)$ peuvent être compris comme une généralisation inhomogène de $C^\alpha([0,1]^D)$, puisque la valeur de $\xi(\lambda)$ varie beaucoup en fonction de λ. Cette interprétation explique pourquoi ξ est appelé un environnement.

Il est prouvé dans [19] que la définition de $B^{\xi}_{p,q}([0,1]^D)$ et $\widetilde{B}^{\xi}_{p,q}([0,1]^D)$ est indépendante du choix de ψ dès lors que ξ est doublante et les normes $\|\cdot\|_{L^p} + |\cdot|_{\xi,p,q}$ sont équivalentes pour tout choix de ψ.

Remark 1.17. Remarquons que, par construction, la famille $\left\{B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D)\right\}_{0<\varepsilon<\min(1,s_1)}$ satisfait, pour $0 < \varepsilon < \varepsilon' < \min(1, s_1)$, $B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D) \hookrightarrow B^{\xi^{(-\varepsilon')}}_{p,q}([0,1]^D)$. De plus, $\widetilde{B}^{\xi}_{p,q}([0,1]^D)$ est un espace de Fréchet [19].

Barral-Seuret ont prouvé dans [19] que les Théorèmes 1.14 et 1.15 dans le cadre des catégories de Baire pouvaient être étendus aux espaces de Besov inhomogènes, donnant :

Theorem 1.18. *Soit ξ une puissance de mesure telle que définie dans [19], soit $p, q \in [1, +\infty]$. Considérons les applications*

$$\tau_{\xi,p}(t) = \begin{cases} \frac{p-t}{p}\ \tau_\xi\left(\frac{p}{p-t}\ t\right) & \text{si } t \in (-\infty, p) \\ \tau'_\xi(+\infty)t & \text{si } t \in [p, +\infty). \end{cases}$$

Pour tout $f \in \widetilde{B}^{\mu}_{p,q}([0,1]^d)$,

$$\sigma_f(h) \leq \begin{cases} \tau^*_{\xi,p}(h) & \text{si } h \leq \tau'_{\xi,p}(0) \\ d & \text{si } h > \tau'_{\xi,p}(0). \end{cases}$$

De plus, les fonctions typiques $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^d)$ satisfont $\sigma_f = \tau^*_{\xi,p}$.

Rappelons que pour un entier k et $p \in [1, +\infty]$, l'espace de Sobolev classique $W^{k,p}(\mathbb{R}^D)$ est défini comme suit :

$$W^{k,p}(\mathbb{R}^D) = \left\{f \in L^p(\mathbb{R}^D) : \forall |\alpha| \leqslant k,\ D^\alpha f \in L^p(\mathbb{R}^D)\right\}.$$

C'est un espace de Banach lorsqu'il est doté de la norme $\|f\|_{W^{k,p}(\mathbb{R}^D)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\mathbb{R}^D)}$. Lorsque $s \in \mathbb{R}_+\backslash\mathbb{N}$ et $p \in [1, +\infty]$, l'espace de Sobolev–Slobodeckij $W^{s,p}(\mathbb{R}^D)$ est l'ensemble des fonctions $f : \mathbb{R}^D \to \mathbb{R}$ telles que

$$\|f\|_{W^{s,p}(\mathbb{R}^D)} = \|f\|_{W^{\lfloor s\rfloor,p}(\mathbb{R}^D)} + \sup_{|\alpha|=\lfloor s\rfloor} |D^\alpha f|_{W^{s-\lfloor s\rfloor,p}(\mathbb{R}^D)} < +\infty$$

où, lorsque $\theta \in (0,1)$,

$$|f|_{W^{\theta,p}(\mathbb{R}^D)} = \left(\iint_{\mathbb{R}^D \times \mathbb{R}^D} \frac{|f(x+h) - f(x)|^p}{|h|^{\theta p + D}} dx dh \right)^{1/p}. \tag{1.15}$$

Nous introduisons maintenant de nouveaux espaces de Sobolev inhomogènes dans un environnement ξ en modifiant la formule ci-dessus afin d'obtenir une classe de fonctions dont le spectre de singularités n'est pas nécessairement affine. Pour ce faire, nous interprétons dans (1.15) le terme $|h|^{\theta}$ au dénominateur comme, à une constante près, la puissance de la mesure de Lebesgue à D dimensions, notée $\mathcal{L}^D$, de la boule $B(x,h)$, et nous le remplaçons par la masse ξ de cette boule. Cela introduit une inhomogénéité spatiale dans (1.15), incluant les espaces de Sobolev comme cas particulier.

Soit $B[x,y]$ la boule euclidienne de diamètre le segment $[x,y]$.

Definition 1.19. *Soit $\xi \in \mathcal{H}(\mathbb{R}^D)$ satisfaisant (P) et (P_2) avec des exposants $0 < s_1 < s_2$, et considérons $p \geq 1$ ainsi qu'un entier $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$.*

Une fonction f appartient à $W^{\xi,p}(\mathbb{R}^D)$ si et seulement si $\|f\|_{W^{\xi,p}(\mathbb{R}^D)} < +\infty$, où $\|f\|_{W^{\xi,p}(\mathbb{R}^D)} = \|f\|_{L^p(\mathbb{R}^D)} + |f|_{W^{\xi,p}(\mathbb{R}^D)}$ et

$$|f|_{W^{\xi,p}(\mathbb{R}^D)} := \left(\iint_{\mathbb{R}^{2d}} \frac{|\Delta_h^n f(x)|^p}{\xi(B[x, x+nh])^p |h|^{2d}} dx dh \right)^{1/p}.$$

Rappelons que, pour tout $p \in [1, +\infty]$ et $s \in \mathbb{R}_+ \backslash \mathbb{N}$ avec $s > D/p$, on a $W^{s,p} = B^s_{p,p}$, voir [167]. Dans le chapitre 4, nous montrons que cela peut être étendu aux espaces de Sobolev et de Besov inhomogènes de la manière suivante :

Theorem 1.20. *Soit $p \geq 1$. Si $\xi \in \mathcal{H}(\mathbb{R}^D)$ avec $\operatorname{supp}(\xi) = \mathbb{R}^D$ satisfait (P) et (P_2) avec des exposants $0 < s_1 < s_2$, alors*

$$B^{\xi}_{p,p}(\mathbb{R}^D) \subset W^{\xi,p}(\mathbb{R}^D) \subset \widetilde{B}^{\xi}_{p,p}(\mathbb{R}^D).$$

1.5 Propriétés multifractales prévalentes dans $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$

Tout d'abord, rappelons la définition suivante.

Definition 1.21. *Une fonction d'ensembles $\xi \in \mathcal{H}([0,1]^D)$ satisfait le formalisme multifractal fort (SMF) si, pour tout $h \geq 0$, $\sigma_{\xi}(h) = \tau^*_{\xi}(h) = \dim_H(E_{\xi}(h))$.*

Dans le chapitre 4, nous étendons le Théorème 1.15 pour la prévalence aux espaces de Besov inhomogènes et donc le Théorème 1.18 au cas de la prévalence lorsque $p = \infty$. Nous montrons les propriétés multifractales prévalentes qui s'appliquent pour des environnements qui sont des capacités.

Theorem 1.22. *Soit $\xi \in \mathcal{C}([0,1]^D)$ une capacité presque doublante satisfaisant (P), (P_2) et vérifiant le SMF. Pour tout $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$,*

$$\sigma_f(h) \leq \begin{cases} \sigma_{\xi}(h) & \text{si } h \leq \tau'_{\xi}(0) \\ d & \text{si } h > \tau'_{\xi}(0). \end{cases} \tag{1.16}$$

De plus, il existe un ensemble prévalent de fonctions $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ satisfaisant $\sigma_f = \sigma_\xi$.

Remarquons que ceci reste vrai pour $f \in B^{\xi}_{\infty,q}([0,1]^D) \subset \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. En d'autres termes, le spectre de singularités prévalent dans $\widetilde{B}^{\xi}_{\infty,q}$ ou $B^{\xi}_{\infty,q}$ coïncide avec celui de ξ.

Ces résultats du chapitre 4 ont été soumis et sont disponibles sur Hal et arXiv [152].

1.6 Résultats prévalents des traces

Le chapitre 5 est consacré à l'analyse multifractale des traces de fonctions dans des espaces de Besov inhomogènes $B^{\xi}_{p,q}(\mathbb{R}^D)$ sur des sous-espaces affines de dimension d de $\mathbb{R}^D$, pour ξ une mesure produit de mesures de Gibbs sur $\mathbb{R}^D$, voir la Définition 1.25. Cet intérêt pour les traces provient de problèmes pratiques. Les physiciens ont pu mesurer la vitesse d'un fluide turbulent dans une direction, ce qui correspond à la trace unidimensionnelle de la vitesse du fluide, un phénomène à quatre dimensions (un phénomène dans $\mathbb{R}^3$ dépendant également du temps).

Soient $0 < d < D$ deux entiers fixes et $d' := D - d$. Pour $a \in \mathbb{R}^{d'}$, nous notons $\mathcal{C}_a := \{(x,a) \ : \ x \in \mathbb{R}^d\}$ le sous-espace affine de dimension d de $\mathbb{R}^D$ passant par $(0,a)$. La trace d'une fonction f sur $\mathcal{C}_a$ est définie par

$$f_a := f_{|\mathcal{C}_a} : \begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x,a). \end{array} \tag{1.17}$$

Cette notion a été considéré pour des fonctions dans des espaces de Besov standard par Aubry-Maman-Seuret et Jaffard [10, 34, 85, 129] et des espaces de Besov à poids par Besoy-Haroske-Triebel [26], par exemple. Des résultats universels et génériques au sens de la prévalence et des catégories de Baire sur la régularité ponctuelle des traces de fonctions appartenant à des espaces classiques ont été obtenus par Aubry-Maman-Seuret [10, 129], donnant les deux résultats suivants :

Theorem 1.23. *Soit $D \in \mathbb{N}^*$, $0 < p, s < \infty$, avec $s > D/p$, et soit $q \in (0,\infty]$. Si $f \in B^s_{p,\infty}(\mathbb{R}^D)$, alors pour Lebesgue presque tous $a \in \mathbb{R}^{d'}$, $f_a \in \bigcap_{s'<s} B^{s'}_{p,\infty}(\mathbb{R}^d)$.*

De plus, si $f \in B^s_{p,q}(\mathbb{R}^D)$ et $q < p$ (resp. $q = p$), alors pour Lebesque presque tous $a \in \mathbb{R}^{d'}$, $f_a \in B^s_{p,qp/(p-q)}(\mathbb{R}^d)$ (resp. $f_a \in B^s_{p,\infty}(\mathbb{R}^d)$).

Theorem 1.24. *Soit $0 < p < \infty$, $0 < q \leq \infty$ et $d/p < s$. Il existe un ensemble de deuxième catégorie et un ensemble prévalent de fonctions $f \in B^s_{p,q}(\mathbb{R}^D)$ tels que pour Lebesgue presque tous $a \in \mathbb{R}^{d'}$,*

$$\sigma_{f_a}(h) = \begin{cases} p(h - (s - d/p)) & \text{si } h \in [s - d/p, s], \\ -\infty & \text{sinon.} \end{cases}$$

Dans le chapitre 5, nous considérerons les capacités μ et ν définies comme des puissances des mesures de Gibbs. Soit $\mathcal{L}^D$ la mesure de Lebesgue D-dimensionnel.

Definition 1.25. *Soit $\varphi : \mathbb{R}^D \to \mathbb{R}$ une fonction de Hölder continue $\mathbb{Z}^D$-périodique (appelée potentiel). Alors, la suite de mesures définies sur $\mathbb{R}^D$ dont les densités sont données par*

$$\nu_n(dx) = \frac{\exp(S_n\varphi(x))}{\int_{[0,1]^D}\exp(S_n\varphi(t))\mathcal{L}^D(dt)}\mathcal{L}^D(dx), \quad \text{où } S_n\varphi(x) = \sum_{k=0}^{n-1}\varphi(2^k x),$$

converge faiblement vers une mesure $\mathbb{Z}^D$-invariant ν_φ, appelée mesure de Gibbs associée à φ.

Ces mesures ont fait l'objet d'études approfondies [82, 144, 145, 146, 147, 159]. Pour l'instant, nous donnons quelques propriétés de ces mesures : Soit un potentiel fixe $\varphi : \mathbb{R}^D \to \mathbb{R}$,

1. La mesure de Gibbs ν_φ est doublante, satisfait la SMF et $\mathrm{supp}(\nu_\varphi) = \mathbb{R}^D$.
2. Pour tout $r \in \mathbb{R}$, en posant
$$h^r_{\nu_\varphi} = \tau'_{\nu_\varphi}(r), \tag{1.18}$$
on a $\sigma_{\nu_\varphi}(h^r_{\nu_\varphi}) = rh^r_{\nu_\varphi} - \tau_{\nu_\varphi}(r)$. De plus, $h^{\min}_{\nu_\varphi} = h^{+\infty}_{\nu_\varphi}$ et $h^{\max}_{\nu_\varphi} = h^{-\infty}_{\nu_\varphi}$.
3. Pour tout $r \in \mathbb{R}$, la mesure de Gibbs $\nu_{r\varphi}$ est appelée mesure auxiliaire associée à φ et r (la mesure auxiliaire est associée au potentiel $r\varphi$) et possède des propriétés multifractales similaires à celles de ν_φ. Alors $\nu_{r\varphi}(\underline{E}_{\nu_\varphi}(h^r_{\nu_\varphi})) = 1$, c'est-à-dire que pour x presque partout dans $\nu_{r\varphi}$, $\underline{h}_{\nu_\varphi}(x) = h^r_{\nu_\varphi}$.

Le deuxième élément découle de (1.18) et de considérations simples sur la transformée de Legendre. Puisque ν_φ est $\mathbb{Z}^D$-périodique, nous identifierons ν_φ avec sa restriction à $[0,1]^D$. Le point 3 indique que la mesure auxiliaire $\nu_{r\varphi}$ est concentrée sur les points x qui ont une dimension locale pour ν_φ égale à $h^r_{\nu_\varphi}$. De plus amples détails sur les mesures auxiliaires sont rappelés au chapitre 5.

Definition 1.26. *L'ensemble des capacités de Gibbs $\mathscr{E}_D \subset \mathcal{C}(\mathbb{R}^D)$ est l'ensemble des puissances positives des mesures de Gibbs ν, c'est-à-dire*

$$\mathscr{E}_D = \{\nu^s \ : \ \nu \text{ une mesure de Gibbs }, s > 0\}. \tag{1.19}$$

Notre objectif est d'étudier les propriétés multifractales des traces de $f \in \widetilde{B}^\xi_{p,q}(\mathbb{R}^D)$ lorsque ξ est un produit de capacités de Gibbs. Plus précisément, si $D = d + d'$, pour $\mu_\varphi \in \mathscr{E}_d$, associé à un potentiel $\varphi : R^d \to \mathbb{R}$, et $\nu_{\widetilde{\varphi}} \in \mathscr{E}_{d'}$, associé à un potentiel $\widetilde{\varphi} : \mathbb{R}^{d'} \to \mathbb{R}$, nous considérerons la capacité $\xi = \mu_\varphi \times \nu_{\widetilde{\varphi}}$ sur $[0,1]^D$, c'est-à-dire pour $A \in \mathcal{B}([0,1]^d)$, $B \in \mathcal{B}([0,1]^{d'})$, $\xi(A \times B) := \mu_\varphi(A) \cdot \nu_{\widetilde{\varphi}}(B)$. Dans la suite, nous omettrons de mentionner φ et $\widetilde{\varphi}$, et nous écrirons simplement μ, ν, ainsi que ν_r pour les mesures auxiliaires associées à ν.

Le Théorème 1.22 s'applique à $\xi = \mu \times \nu$, et on constate qu'il existe un ensemble prévalent de fonctions $f \in \widetilde{B}^\xi_{p,q}([0,1]^D)$ tel que $\sigma_f = \tau^*_\xi$. Le spectre de ξ est déterminé en fonction de ceux de μ et ν.

Proposition 1.27. *Soit $\mu \in \mathscr{E}_d$ et $\nu \in \mathscr{E}_{d'}$ avec $D = d + d'$. Soit ξ la capacité produit sur $[0,1]^D$ définie par $\xi = \mu \times \nu$. La capacité produit ξ satisfait la SMF et vérifie $\tau_\xi = \tau_\mu + \tau_\nu$ et*

$$\text{pour tout } r \in \mathbb{R}, \quad \sigma_\xi\left(h^r_\mu + h^r_\nu\right) = \sigma_\mu\left(h^r_\mu\right) + \sigma_\nu\left(h^r_\nu\right).$$

Pour les espaces de Besov standard, l'opérateur de trace $f \mapsto f_a$ est une application de $B^s_{p,q}(\mathbb{R}^D)$ sur $B^{s-d'/p}_{p,q}(\mathbb{R}^d)$, ce qui reflète une perte de régularité lors de la prise de traces.

Cependant, presque-toutes traces f_a appartiennent à $B^s_{p,q}(\mathbb{R}^d)$, voir 1.23. Notre premier résultat montre que le même phénomène se produit pour les espaces de Besov inhomogènes $\widetilde{B}^{\xi}_{p,q}([0,1]^D)$.

Theorem 1.28. *Soient $p,q \in [1,+\infty]$. Soient $\mu \in \mathscr{E}_d$ et $\nu \in \mathscr{E}_{d'}$ avec $D = d + d'$. Soit ξ la capacité produit sur $[0,1]^D$ définie par $\xi = \mu \times \nu$.*

Pour toute $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$, pour tout $r \in \mathbb{R}$ et ν_r-presque tout $a \in [0,1]^{d'}$,

$$f_a \in \widetilde{B}^{\mu^{(+\Gamma_{\nu,r})}}_{p,1}([0,1]^d) \quad \text{avec } \Gamma^p_{\nu,r} = h^r_\nu + \frac{\dim(\nu_r)}{p}.$$

Par convention, $\frac{\dim(\nu_r)}{\infty} = 0$. Remarquons que la trace f_a appartient à un espace dont l'indice de régularité dépend de la régularité locale de ν en a.

À partir de maintenant, **nous ne considérons que le cas $p = \infty$.** Nous trouvons une borne supérieure pour le spectre des traces de toutes les fonctions $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$.

Theorem 1.29. *Pour toute $f \in \widetilde{B}^{\xi}_{\infty,q}$ et $a \in [0,1]^{d'}$, pour tout $h \in \mathbb{R}_+$, on a*

$$\sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h^{\min}_\nu) & \text{si } h \leq h^0_\mu + h^{\min}_\nu, \\ d & \text{si } h > h^0_\mu + h^{\min}_\nu. \end{cases} \tag{1.20}$$

Pour toute $f \in \widetilde{B}^{\xi}_{\infty,q}$, pour tout $r \in \mathbb{R}$ et ν_r-presque tout $a \in [0,1]^{d'}$, pour tout $h \in \mathbb{R}_+$, on a

$$\sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h^r_\nu) & \text{si } h \leq h^0_\mu + h^r_\nu, \\ d & \text{si } h > h^0_\mu + h^r_\nu. \end{cases} \tag{1.21}$$

Le théorème concernant le comportement multifractal prévalent dans $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ est le suivant. Il dépend de l'existence d'une ondelette satisfaisant (R) détaillée au chapitre 3.

Theorem 1.30. *Soient $\mu \in \mathscr{E}_d$, $\nu \in \mathscr{E}_{d'}$, et considérons la capacité produit $\xi = \mu \times \nu \in \mathcal{C}([0,1]^D)$, avec $D = d + d'$. Appelons C, s_1, s_2 les nombres réels tels que (1.5) soit satisfait.*

Soit $n \geq \left\lfloor s_2 + \frac{D}{p} \right\rfloor + 1$. Supposons qu'il existe une ondelette à support compact $\psi \in C^n(\mathbb{R})$ ayant (au moins) n moments nuls et satisfaisant la propriété (R).

Soit $(r_n)_{n \geq \mathbb{N}}$ une suite dénombrable de nombres réels. Il existe un ensemble prévalent de fonctions $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ tel que pour tout $n \in \mathbb{N}$, pour ν_{r_n}-presque tout $a \in (0,1]^{d'}$,

$$\sigma_{f_a}(h) = \begin{cases} \sigma_\mu(h - h^{r_n}_\nu) & \text{si } h \in \operatorname{supp}(\sigma_\mu) + h^{r_n}_\nu \\ -\infty & \text{sinon.} \end{cases}$$

Le spectre de singularités σ_{f_a} de la trace f_a coïncide avec celui de μ translaté par h^r_ν, qui dépend de la régularité de ν en a.

Les résultats du chapitre 5 correspondent à un article soumis, disponible sur HAL et arXiv [154], et utilisent les propriétés sur les ondelettes présentées dans le chapitre 3, également soumis et disponible [153].

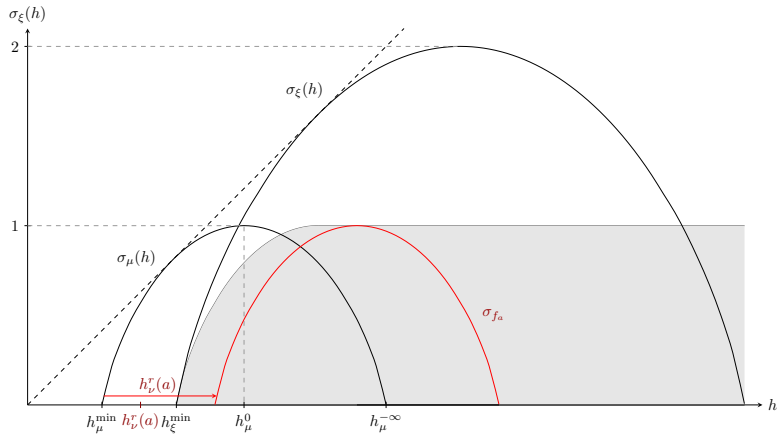


Figure 1.2: Exemple des bornes complètes pour $\sigma_{f_a}(h)$ pour presque toute $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ et ν_r-presque tout $a \in [0,1]^{d'}$, avec $\xi = \mu \times \nu$ où μ et ν ont un même spectre de singularité.

1.7 Anisotropie dans des espaces de type Besov

Nous considerons maintenant les propriétés de régularité des fonctions dans des contextes anisotropes. L'anisotropie, qui se produit lorsque les lois de la physique ne s'appliquent pas de la même manière selon les directions, apparaît dans un large éventail d'applications telles que la géophysique, l'hydrologie [148], l'imagerie médicale (ostéoporose, tissus musculaires, mammographies,...) [27, 29]. Une première question réside dans l'introduction des notions globales et locales de régularité anisotrope, qui englobent et étendent les espaces de régularité (isotropes), tels que les espaces de Sobolev ou de Besov, et la notion classique de régularité ponctuelle de Hölder. Pour modéliser l'anisotropie, des champs aléatoires anisotropes auto-similaires ont été introduits et étudiés[22, 27, 45, 55], et certaines des propriétés caractérisant l'anisotropie peuvent être analysées à l'aide d'espaces fonctionnels bien adaptés à l'anisotropie, tels que les espaces de Besov anisotropes ou les espaces de Besov à régularité dominante mixte. Ces espaces possèdent une littérature fournie, voir par exemple [25, 31, 56, 158, 165, 168, 169].

L'analyse de la régularité des signaux et des fonctions peut être effectuée dans un cadre isotrope à l'aide de l'analyse par ondelettes, comme présenté dans la section 1.2. Cependant, les ondelettes classiques ne sont pas bien adaptées à l'analyse des signaux anisotropes en raison de leur principe de construction basé sur des dilatations isotropes. Un exemple simple illustrant ce problème est un signal plutôt lisse dans une direction mais rugueux dans la direction orthogonale, comme des couches ou des rayures dans une texture, que les méthodes d'ondelettes isotropes ne permettent pas d'analyser correctement.

De nombreux auteurs ont abordé ce problème en proposant différentes solutions selon la définition de l'espace anisotrope et la base anisotrope utilisée, et les ondelettes ont été adaptées de multiples façons pour « détecter » et résoudre l'anisotropie, comme par exemple les ridgelets et les curvelets de Candes, Donoho et Demanet [36, 37, 38, 39, 40], les ban-

delets [119, 127], les aniset [94], les shearlets développés dans [71, 117, 118], et les ondelettes anisotropes abordées par de nombreux auteurs, tel que Ben Slimane, Ben Braiek, Bownik, Farkas, Garrigos, Hochmuth, Haroske, Liu, Tabacco, Tamási, Triebel and Yang [24, 32, 57, 69, 75, 79, 123, 139, 170, 171]. Cependant, ces ondelettes nécessitent de connaître l'anisotropie à l'avance et, si les paramètres d'anisotropie ne sont pas fournis par le contexte du signal considéré, il est impossible de le caractériser sans « essayer » plusieurs bases anisotropes, ce qui est difficilement réalisable en pratique.

Dans [3], il a été démontré que tout espace de Besov anisotrope sur $\mathbb{R}^2$ peut être « presque » caractérisé par une transformation en ondelettes dite hyperbolique. Cela permet de détecter et de caractériser l'anisotropie d'un signal par un poids spécifique dans les coefficients d'ondelettes. Cela a conduit à un algorithme efficace pour la classification d'images et la détection d'anisotropie appliqué à des textures synthétiques et réelles (voir [6, 155]). Schäfer-Ullrich-Vedel a ensuite généralisé cette caractérisation à des dimensions supérieures et aux espaces de Triebel-Lizorkin et Sobolev [157]. Vybiral [172] a montré que les espaces de Besov à régularité dominante mixte peuvent également être caractérisé par une transformée en ondelettes hyperbolique, et Esser-Loosveldt-Vedel [55] ont utilisé cette base d'ondelettes hyperboliques pour introduire les espaces de Besov à poids tensorisés qui, à l'aide d'un paramètre r décroissant de 1 à 0, correspondent à des espaces de Besov à régularité dominante mixte lorsque $r = 0$ et à des espaces de Besov hyperboliques isotropes lorsque $r = 1$.

Cette caractérisation à travers une base d'ondelettes hyperboliques est liée à des approximation non linéaire de fonctions en EDP [73, 103] et à des applications pratiques telles que l'analyse des textures anisotropes [155]. L'approximation non linéaire d'une fonction dans certains espaces de potentiel de Bessel peut être effectuée à l'aide d'une fonction dans des espaces de Besov à poids tensorisés.

Notre objectif, dans le chapitre 6, est d'introduire une version anisotrope des espaces de Besov à poids tensorisés, ainsi que d'expliquer comment ils peuvent être considérés du point de vue des espaces de Besov hyperboliques anisotropes et des espaces de Besov à régularité dominante mixte. Ces espaces possèdent une structure anisotrope similaire à celle des espaces de Besov anisotropes et des espaces de Besov à régularité dominante mixte, avec un paramètre qui contrôle l'effet réticulé présent dans les espaces de Besov à régularité dominante mixte, mais absent des espaces de Besov anisotropes. De plus, ils généralisent les espaces de Besov à poids tensorisés isotropes d'Esser-Loosveldt-Vedel. Nous montrons également comment la régularité d'une fonction dans un espace de Besov anisotrope à poids tensorisé change en tenant compte de différentes directions d'anisotropie. Ce chapitre constitue un point de départ visant à fournir un fondement théorique à la méthode d'estimation des paramètres d'anisotropie des textures telle qu'elle est présentée dans [155].

Nous considerons une base d'ondelettes hyperboliques comme le système $\{\psi_{\overline{j},\overline{k}} : \overline{j} \in \mathbb{N}^2_{-1}, \overline{k} \in \mathbb{Z}^2\}$ de produits tensoriels de deux bases d'ondelettes unidimensionnelles, voir le chapitre 6 et [51] pour plus de détails. Cette base d'ondelettes permet d'étudier la régularité et les oscillations d'une fonction présentant des comportements différents dans les deux directions

orthogonales, ce qui n'est pas possible avec les méthodes d'ondelettes classiques à dilatation isotrope.

De manière similaire aux espaces de Besov classiques, les espaces de Besov hyperboliques anisotropes $\widetilde{B}^{s,\overline{\alpha}}_{p,q}(\mathbb{R}^2)$ sont des espaces de fonctions dont les oscillations à chaque échelle sont contrôlées dont la régularité dans une direction est contrôlée par s/α_1 et une régularité dans la direction orthogonale est contrôlée par s/α_2.

D'un autre coté, les espaces de Besov à régularité dominante mixte $S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2)$ sont des ensemble de fonctions dont la régularité également contrôlée par s/α_1 et s/α_2 dans chaque direction tout en présentent une structure similaire à celle des textures réticulées telles que les textiles.

Afin de relier ces deux espaces entre eux et de décrire une famille de processus gaussiens anisotropes, Esser-Loosveldt-Vedel [55] a introduit les espaces de Besov à poids tensorisés. Ces espaces ont une structure anisotrope similaire à celle des espaces de Besov hyperboliques anisotropes et des espaces de Besov à régularité dominante mixte, avec un paramètre qui contrôle l'effet de réticule. Donnons maintenant une version anisotrope de ces derniers espaces.

Nous utiliserons les notations suivantes. Posons $\mathbb{N}_{-1} := \mathbb{N}\cup\{-1\}$. Les indices multivariés sont composés avec une barre, comme $\overline{k}, \overline{j}$ ou $\overline{\alpha}$. Nous posons également $x_+ := \max\{x, 0\}$ pour $x \in \mathbb{R}$ et $\overline{x}_+ := ((x_1)_+, (x_2)_+)$ dans le cas multidimensionnel. Dans la suite, le multi-indice $\overline{\alpha} = (\alpha_1, \alpha_2) > 0$ représente un indice d'anisotropie et est tel que $\alpha_1 + \alpha_2 = 2$. La notation $\overline{j}/\overline{\alpha}$ désigne toujours $(j_1/\alpha_1, j_2/\alpha_2)$ et donc $\overline{j}_+/\overline{\alpha}$ désigne toujours $((j_1)_+/\alpha_1, (j_2)_+/\alpha_2)$. Pour $r \in [0, 1]$, considérons la fonction

$$\theta_r(\overline{j}_+/\overline{\alpha}) := (1 + r)\max(\overline{j}_+/\overline{\alpha}) + (1 - r)\min(\overline{j}_+/\overline{\alpha}).$$

Definition 1.31. *Soient $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $0 \le r \le 1$ et un indice d'anisotropie $\overline{\alpha} \in \mathbb{R}^2$. Considérons la base d'ondelettes hyperboliques.*

Une fonction $f \in \mathcal{S}'(\mathbb{R}^2)$ appartient à l'espace de Besov anisotropes à poids tensorisés $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ si $\|f\|_{TB^{s,\overline{\alpha},r}_{p,q}} < +\infty$, où

$$\|f\|_{TB^{s,\overline{\alpha},r}_{p,q}} := \left\|(\varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}})_{\overline{j}\in\mathbb{N}^2_{-1}}\right\|_{\ell^q}, \quad \text{avec} \quad \varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}} = \left\|\left(2^{\theta_r(\overline{j}_+/\overline{\alpha})s}\, 2^{-\|\overline{j}_+\|_1/p}\, c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2}\right\|_{\ell^p} \tag{1.22}$$

avec la modification habituelle pour $q = +\infty$, $p = +\infty$.

Ces espaces peuvent être reliés aux espaces de Besov hyperboliques anisotropes et aux espaces de Besov à régularité dominante mixte comme suit.

Remark 1.32. L'espace de Besov anisotropes à poids tensorisés correspond aux espaces de Besov à régularité dominante mixte lorsque $r = 0$

$$S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2) = \left\{f \in \mathcal{S}'(\mathbb{R}^2) : \left\|\left(\left\|\left(2^{\|\overline{j}_+/\overline{\alpha}\|_1 s}\, 2^{-\|\overline{j}_+\|_1/p}\, c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2}\right\|_{\ell^p}\right)_{\overline{j}\in\mathbb{N}^2_{-1}}\right\|_{\ell^q} < \infty\right\},$$

et aux espaces de Besov hyperboliques anisotropes (avec régularité 2s) lorsque $r = 1$

$$B^{2s,\overline{\alpha}}_{p,q}(\mathbb{R}^2) = \left\{f \in \mathcal{S}'(\mathbb{R}^2) : \left\|\left(\left\|\left(2^{\|\overline{j}_+/\overline{\alpha}\|_\infty s}\, 2^{-\|\overline{j}_+\|_1/p}\, c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2}\right\|_{\ell^p}\right)_{\overline{j}\in\mathbb{N}^2_{-1}}\right\|_{\ell^q} < \infty\right\}.$$

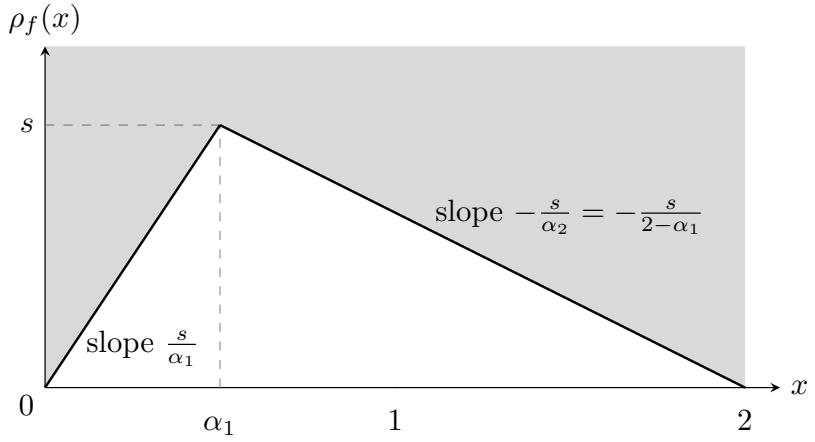


Figure 1.3: La fonction de régularité $\rho_{f,r}$ en fonction de l'anisotropie $(x, 2-x)$. Le graphe de la fonction $\rho_{f,r}$ est inclus dans la partie grise pour toutes les fonctions f et, pour les fonctions typiques dans $TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$, $\rho_{f,r}$ est représenté en noir.

Une propriété importante de $TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$ est que ces espaces sont des espaces de Banach comme montré dans le Chapitre 6. Nous étudions comment la régularité d'une fonction dans un espace de Besov anisotropes à poids tensorisés se comporte par rapport à d'autres directions d'anisotropie.

Definition 1.33. *Soient* $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, *et un indice d'anisotropie* $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. *Pour* $f \in TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$, *définissons la fonction*

$$\forall x \in (0,2), \quad \rho_{f,r}(x) := \sup\{s' \in \mathbb{R} : f \in TB_{p,q}^{s',(x,2-x),r}(\mathbb{R}^2)\}.$$

En considérant une fonction $f \in TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$, cette fonction présente généralement sa régularité maximale lorsqu'elle est étudiée en tenant compte d'un indice d'anisotropie $\overline{\alpha}$, et sa régularité ne fait que diminuer si l'on considère d'autres indices d'anisotropie.

Theorem 1.34. *Soient* $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, *et un indice d'anisotropie* $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$.

Pour tout $f \in TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$, *on a*

$$\rho_{f,r}(x) \geq \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{1.23}$$

Pour tout f *dans un ensemble de deuxième catégorie de Baire dans* $TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$, *on a*

$$\rho_{f,r}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{1.24}$$

Cela peut être interprété comme suit : si l'on calcule la régularité d'une fonction pour différents indices d'anisotropie, on peut en déduire son paramètre d'anisotropie comme étant celui pour lequel la fonction présente la plus grande régularité. Cette méthode a été utilisée dans [155] pour évaluer le paramètre d'anisotropie d'une texture simulée à partir de

processus gaussiens anisotropes et la régularité correspondante. En effet, pour des textures anisotropes simulées à partir de processus gaussiens anisotropes avec une régularité et un indice d'anisotropie particuliers, ils ont estimé la régularité $s(\overline{\alpha})$ pour plusieurs indices d'anisotropie $\overline{\alpha}$ et ont récupéré l'anisotropie et la régularité utilisées pour la simulation. La régularité estimée pour plusieurs indices d'anisotropie forme une fonction triangulaire similaire à la figure 1.3.

En particulier, dans le cas des espaces de Besov hyperboliques anisotropes, si pour une fonction f, le maximum de $\rho_{f,1}$ est connu, alors on peut déterminer complètement $\rho_{f,1}$ comme suit.

Theorem 1.35. *Soit* $r = 1$ *et* $\rho_{f,1}(x) = \sup\{s' \in \mathbb{R} : f \in B_{p,q}^{2s',(x,2-x)}(\mathbb{R}^2)\}$. *Considérons une fonction* f *telle qu'il existe un indice d'anisotropie* $\overline{\alpha}$ *pour lequel* $\rho_{f,1}(\alpha_1) = s = \max_{y\in(0,2)} \rho_{f,1}(y)$ *pour* $s \geq 0$. *Alors*

$$\rho_{f,1}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{1.25}$$

Les théorèmes 1.34 et 1.35 impliquent tous deux que si une fonction f satisfait $\rho_{f,1}(\alpha_1) = s = \max_{y\in(0,2)} \rho_{f,1}(y)$ pour un indice d'anisotropie $\overline{\alpha}$ et un $s \geq 0$, alors $\rho_{f,1}(x) = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$. Ainsi, si l'on considère le cadre des espaces de Besov hyperboliques anisotropes, il suffit d'obtenir (numériquement) deux points qui ne sont alignés ni avec $(0,0)$ ni avec $(2,0)$ dans le plan $(x, \rho_{f,1}(x))$ comme dans la figure 1.3. Cela donnera la régularité maximale par régression linéaire, ce qui signifie qu'il n'est pas nécessaire de tester toutes les anisotropies, ce qui simplifie la méthode utilisée dans [155].

Ce chapitre présente une étude préliminaire qui nécessite des approfondissements présentés au chapitre 6.

1.8 Série de Riemann

La détermination de la régularité des séries trigonométriques a été un sujet d'importance majeure au cours des deux derniers siècles. Ces séries ont fourni des exemples et des contre-exemples de fonctions ayant des propriétés spécifiques et les avancées théoriques réalisées dans ce domaine ont eu des implications majeures en analyse mathématique, voir par exemple Jaming-Kellay-Saba-Wang, Kahane-Lemarie, Korner and Zygmund [100, 101, 104, 113, 179]. Un exemple typique est fourni par les *fonctions de Weierstrass*

$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(\pi b^n x), \quad \text{où} \quad 0 < a < 1 \text{ et } b > 1$$

qui ont permis de déterminer s'il existe des fonctions continues nulle part dérivables. Weierstrass a prouvé que c'est le cas si b est un entier et $ab > 1 + 3\pi/2$ [173], condition qui a ensuite été affinée par Hardy [74] pour devenir $b > 1$ et $ab > 1$. Un autre exemple clé a été proposé

par Riemann : il a émis l'hypothèse que la fonction

$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2} \tag{1.26}$$

est continue et nulle part dérivable. Cette fonction s'est avérée beaucoup plus difficile à traiter que les fonctions de Weierstrass : Dans [74], Hardy a prouvé qu'elle n'était dérivable nulle part, sauf peut-être aux rationnels de la forme $\frac{2p+1}{2q+1}$, et ce n'est qu'en 1969 que Gerver [70] a montré que la fonction de Riemann était dérivable à ces rationnels. Ses propriétés de régularité complètes n'ont été entièrement dérivées que dans les années 1990, d'abord avec la détermination dans [86] de sa régularité ponctuelle exacte en chaque point, puis par un développement asymptotique aux points de dérivabilité, voir [97], qui a mis en évidence son *comportement oscillatoire* en ces points. Un corollaire important est que la fonction de Riemann est un exemple de *fonction multifractale*. Ces questions de régularité ne sont pas complètement résolues pour plusieurs variantes et généralisations de la fonction de Riemann (1.26). Considérons la famille

$$\mathcal{S}_{b,s}(x,t) = \sum_{n=1}^{\infty} \frac{e^{i\pi(n^b x + nt)}}{n^s}, \quad \text{and} \quad \mathcal{Q}_{b,s}(x) = Im(\mathcal{S}_{b,s}(x,0)) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^b x)}{n^s} \tag{1.27}$$

où $b \geq 2$ est un entier et $s \in \mathbb{R}$ (la fonction de Riemann est $\mathcal{Q}_{2,2}$). Divers cas de ces fonctions ont été étudiés : le cas $b = 2$ se rencontre dans les solutions explicites de plusieurs EDP linéaires, telles que l'équation de Schrödinger périodique unidimensionnelle ou l'équation de la plaque supportée bidimensionnelle, de sorte que leur analyse donne un aperçu des propriétés de régularité de ces EDP ; en outre, elles jouent un rôle important en mécanique des fluides en raison de leur présence dans la dynamique de l'écoulement binormal : Dans [14], il est démontré qu'elles décrivent la trajectoire temporelle d'un filament de vortex.

Dans [150], Hervé Queffelec a étudié les propriétés de régularité des fonctions

$$\sum \frac{\sin(\pi P(n)x)}{P(n)}$$

où P est une fonction polynomiale à coefficients entiers. Dans le cas particulier de $P(n) = n^s$ correspondant à $\mathcal{Q}_{s,s}$ et plus généralement $\mathcal{S}_{b,s}(\cdot, 0)$, ces résultats ont été complétés par Boichu dans [28]. Il a montré que

$$h^{\min}_{\mathcal{Q}_{b,s}} = \frac{s-1}{b},$$

Si $s < 1$, alors les fonctions $\mathcal{Q}_{b,s}$ ne sont plus bornées, de sorte que la notion habituelle de régularité ponctuelle fournie par l'exposant de Hölder ne peut plus être utilisée. Calderón et Zygmund ont introduit dans [35] une notion de régularité ponctuelle adaptée aux fonctions dans $L^p_{\text{loc}}(\mathbb{R})$.

Definition 1.36. *Soit $f \in L^p_{loc}(\mathbb{R})$ avec $p \geq 1$, $x_0 \in \mathbb{R}$ et $\alpha \geq -1/p$. La fonction f appartient à $T^p_\alpha(x_0)$ lorsqu'il existe un polynôme P_{x_0} de degré inférieur à α et $C, R > 0$ tels que*

$$\forall r \in (0,R), \quad \left(\frac{1}{r}\int_{x_0-r}^{x_0+r} |f(x) - P_{x_0}(x-x_0)|^p dx\right)^{\frac{1}{p}} \leq Cr^\alpha. \tag{1.28}$$

Le p-exposant de f en x_0 est $h_f^p(x_0) = \sup\{\alpha : f \in T_\alpha^p(x_0)\}$.

Des résultats utilisant cette notion plus faible de régularité ponctuelle ont été obtenus par Seuret-Ubis [160]. Néanmoins, notre compréhension des propriétés de régularité de cette famille de fonctions reste partielle: leur spectre de singularités n'est que partiellement connu et, dans de nombreux cas, seules des bornes supérieures et inférieures approximatives sont connues [42]. Une façon de faire des estimations précises sur leurs valeurs exactes consiste à appliquer un *formalisme multifractal*, voir la section 7.2, qui établit une relation entre le spectre de singularités et des quantités calculables numériquement. Jusqu'à présent, ces techniques ont principalement été utilisées pour la classification et la sélection de modèles. Bien que la portée exacte de cette formule ne soit pas connue, elle s'est avérée valable pour un grand nombre de modèles mathématiques [19, 23, 90], de sorte que l'élaboration du formalisme multifractal d'une manière numériquement précise fournirait des « hypothèses » des valeurs correctes de leur spectre de singularités. Il convient de noter que jusqu'à présent, la détermination numérique des spectres de Legendre des fonctions mathématiques n'a été effectuée que dans les cas où leur spectre de singularités était théoriquement connu, voir par exemple [5], dans le but de tester sa précision numérique, ce qui a souvent conduit à des améliorations ou à des variantes de la méthode ; son objectif se situe donc à la croisée des mathématiques appliquées et du traitement du signal. Notre objectif ici est différent : il s'agit de fournir aux mathématiciens des conjectures fiables concernant leur spectre de singularités, ce qui relève donc du domaine des *mathématiques expérimentales*.

Les fonctions dont la régularité ponctuelle varie de manière erratique d'un point à l'autre apparaissent naturellement dans de nombreux domaines des mathématiques. La série de Riemann $\mathcal{R}$ a été le premier exemple déterministe dont on a démontré qu'il appartenait à ce type ; plus récemment, il a été démontré que la fonction de Brjuno [96] partageait les propriétés de régularité de la fonction de Riemann : dans les deux cas, leur exposant de régularité ponctuelle est directement lié aux exposants d'irrationalité diophantiens. Des exemples stochastiques sont fournis par les modèles en cascade, voir [17] et les références qui y sont citées, ou les processus de Lévy, voir [88, 149]. Ces modèles, dont les propriétés théoriques sont parfaitement comprises, fournissent des études de cas importantes sur lesquelles de nouveaux algorithmes numériques peuvent être testés et dont la précision peut être remise en question. Ils sont utilisés dans plusieurs domaines d'application tels que la turbulence [41, 65] ou la finance [11].

De plus, les fonctions de Riemann présentent un intérêt particulier en raison de la présence de singularités oscillantes dans leurs graphiques, similaires à la fonction « chirp » de la section 1.1. De tels comportements se retrouvent dans divers phénomènes physiques, voir [60]. Pour les signaux ou les fonctions sans telles singularités, les exposants de Hölder ou p-exposants fournissent toutes les informations sur la régularité de la fonction ainsi que sur la régularité de ses intégrales fractionnaires. Cependant, en présence de telles singularités, l'étude des p-exposants d'une fonction et de sa primitive fournit des informations sur le type

de comportement oscillatoire qu'elle présente.

Dans le chapitre 7, après avoir rappelé certaines notions et propriétés de la régularité ponctuelle des séries de Fourier et de leurs dérivées fractionnaires et primitives, nous proposons une comparaison des résultats connus et des conjectures avec l'estimation numérique des quantités de régularité de $\mathcal{Q}_{b,s}$. Nous fournissons le résultat connu sur la régularité de $\mathcal{Q}_{2,s}$ pour $s > 1/2$ ainsi que certaines conjectures formulées précédemment et les comparons à l'estimation numérique pour $s = 2$ et pour $1/2 < s < 1$ (nous choisirons $s = 0,7$). Nous confrontons également le résultat théorique pour $\mathcal{Q}_{b,s}$, avec $b = 3$ et $b = 4$, aux résultats estimés numériquement. Enfin, nous présentons l'étude numérique de $\mathcal{S}_{2,2}(x,t)$ afin de conjecturer son comportement de régularité.

Dans ce chapitre, nous présentons plusieurs résultats partiels concernant la détermination numérique du spectre multifractal de généralisations de la fonction de Riemann ; toutefois, cette étude préliminaire doit être corroborée par des recherches numériques supplémentaires avant qu'un article ne soit finalisé.

Chapter 2

Introduction

In this thesis, we are primarily interested in the multifractal analysis of functions. Multifractal analysis describes and studies the local properties of sets, measures or functions. The study of multifractal properties of functions, which will be the main topic of this thesis, is a recent field of mathematics, whose importance became apparent back in the 1980s, when physicists sought to quantify the local variations of the velocity of a fluid in the context of fully developed turbulence. Experiments have shown that the measured velocity of a turbulent flow exhibits strong irregularities that depend on time [53, 68], meaning that the pointwise exponents describing the corresponding signal vary in time. Turbulent flows are also not spatially homogeneous, and the pointwise regularity of the velocity seems to differ widely from point to point.

The notion of multifractal was brought by Mandelbrot even though he did not call it that way. In the 1970s [130, 131, 132], Mandelbrot introduced the multiplicative cascades, which are random measures that can be used as a model for energy dissipation [133]. Kahane and Peyriere [105] studied their convergence and proved that these cascades are multifractal measures, meaning that a realisation of the measure can have local dimensions that depend on the location around which one focuses.

The relationship between multifractality and the global scaling properties of the velocity was first discovered by U. Frisch and G. Parisi [66] (see also [1, 8] for work linking multifractality and turbulence). Since then, multifractal notions have proved to be useful and relevant in various mathematical domains: geometric measure theory, real analysis [42, 86, 160], dynamical systems and ergodic theory, probability, diophantine approximation [59, 161, 162, 163], to name a few.

The consideration of multifractal properties is also of interest in application. The existence of stable numerical algorithms to estimate multifractal quantities allowed many breakthrough methods in signal and image processing [175, 176] and other fields [2, 99, 177].

We will mainly study the characterisation of the local behaviour of a function around a point, which is necessary in the various applications such as in fluid mechanic and image processing. We also give complementary information provided by global regularity of such

function.

We will recall the definition and properties of pointwise regularity exponent and singularity spectrum for functions and set functions in Section 2.1, of the wavelet formalism that will be used in the thesis in Section 2.2 and of prevalent set and Baire category for set in Section 2.3. Section 2.4 recalls first the definitions of classical Besov spaces and result on multifractal properties of function in these spaces, then gives some extensions of this spaces and in particular the inhomogeneous Besov spaces that are the main topic of the first part of this thesis along with Baire generic results in these spaces. It also introduces a extension of Sobolev–Slobodeckij spaces in an inhomogeneous environment and gives results from Chapter 4 linking this inhomogeneous Sobolev spaces to inhomogeneous Besov spaces extending the results of classical Besov and Sobolev spaces. Section 2.5 present the prevalent multifractal properties of function in inhomogeneous Besov spaces from Chapter 4, extending the Baire generic result. Section 2.6 first recalls known result on traces of function in Besov spaces and gives their extension to the traces of function in inhomogeneous Besov spaces, which are proved in Chapter 5. Section 2.7 introduces Besov-type spaces with anisotropic structures and presents results on regularity of function depending on the anisotropy showed in Chapter 6. Section 2.8 introduce the consideration of Chapter 7 falling in the scope of "experimental mathematics" and the goal of numerical estimation of multifractal properties of Riemann's non-differentiable function with theoretical results and conjecture.

2.1 Pointwise regularity

We focus on the multifractal analysis of functions, which consists in describing the pointwise behaviour of functions $f : \mathbb{R}^D \to \mathbb{R}$, in order to quantify the "roughness" of f (or of its graph) near a point x_0.

Definition 2.1. *Given $x_0 \in \mathbb{R}^D$, $H \in \mathbb{R}_+$, a function $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^D)$ belongs to $\mathcal{C}^H(x_0)$ if there exist a polynomial P_{x_0} of degree less than $\lfloor H \rfloor$, a constant $C > 0$, and a neighbourhood V of x_0 such that*

$$\forall x \in V, \quad |f(x) - P_{x_0}(x - x_0)| \leq C \ |x - x_0|^H . \tag{2.1}$$

The pointwise Hölder exponent of f at x_0 is then

$$h_f(x_0) = \sup\{H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0)\}. \tag{2.2}$$

The minimal pointwise Hölder exponent of f is $h_f^{\min} = \sup\{H \in \mathbb{R} : f \in \mathcal{C}^H(\mathbb{R})\}$.

Some functions or random processes have a single constant Hölder exponent, one of the most notable is the Weierstrass function

$$W_{a,b}(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$$

with $0 < a < 1$ and $b > 1$ such that $ab > 1$. A stochastic counterpart to this phenomenon emerged with the Brownian motion $(B_t)_{t\geq 0}$, modelling the movement of particles observed

by Brown in 1827. It is known that the Weierstrass function has a constant Hölder exponent $h_{W_{a,b}}(x) = \frac{-\log(a)}{\log(b)}$, while Brownian motion satisfies almost surely $h_B(x) = 1/2$.

Although the pointwise Hölder exponent describes local behaviour, it has certain shortcomings. Indeed, the oscillating behaviour in the neighbourhood of the point under study is not described. First, consider the "cusp" function, defined by $x \mapsto |x|^\gamma$, where γ is a positive real number. This function has a pointwise Hölder exponent at 0 equal to γ, but does not exhibit any oscillation phenomenon at 0. At the same time, the "chirp" function, defined by $x \mapsto |x|^\gamma \sin(1/|x|^\beta)$, where γ and β are two positive real numbers, also has a Hölder exponent equal to γ at 0 but exhibits rapid oscillations around 0. This shows that the pointwise Hölder exponent does not distinguish between these two behaviours that are very different in nature. Additionally, it is not stable under the action of pseudo-differential operators of order 0 and such considerations will take place in the last chapter of this thesis.

However, similarly to the velocity above, functions may have a Hölder exponent that vary greatly from one point to another. Therefore, seeking to understand the geometric distribution of these singularities is of particular importance. We will therefore focus on the Hausdorff dimension of level sets of points with a given Hölder exponent and the singularity spectrum σ_f of a function f.

Definition 2.2. *The level sets of $x \mapsto h_f(x)$ are $E_f(h) := \{x \in \mathbb{R}^D : h_f(x) = h\}$, and the associated singularity (or multifractal) spectrum σ_f of f is the mapping*

$$\sigma_f : h \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H (E_f(h)) \in [0, D] \cup \{-\infty\}. \tag{2.3}$$

Here, $\dim_H$ stands for the Hausdorff dimension with $\dim_H (\emptyset) = -\infty$ (note that $E_f(h) = \emptyset$ for $h < 0$).

Such quatities can be extended to set functions such as measures in the following way. The set $\mathcal{B}(\mathbb{R}^D)$ stands for the Borel sets in $\mathbb{R}^D$. The diameter of a set E is written $|E|$ and its D-dimensional Lebesgue measure is $\mathcal{L}^D(E)$.

Definition 2.3. *The set of Hölder set functions is*

$$\mathcal{H}(\mathbb{R}^D) = \{\xi : \mathcal{B}(\mathbb{R}^D) \to \mathbb{R}_+ \cup \{+\infty\} : \exists C, s > 0, \forall E \in \mathcal{B}(\mathbb{R}^D), \xi(E) \leq C|E|^s\}.$$

When $\xi(E) \leq C|E|^s$ for all E with $0 \leq |E| \leq 1$, ξ is said to be s-Hölder. The topological support of $\xi \in \mathcal{H}(\mathbb{R}^D)$ is $\mathrm{supp}(\xi) = \{x \in \mathbb{R}^D : \xi(B(x, r)) > 0, \forall\, r > 0\}$. A set function ξ is fully supported on a Borel set A when $\mathrm{supp}(\xi) = A$. By a small abuse of notations we will denote

$$\mathcal{H}([0, 1]^D) = \{\xi \in \mathcal{H}(\mathbb{R}^D) : \mathrm{supp}(\xi) = [0, 1]^D\}. \tag{2.4}$$

For $\xi \in \mathcal{H}(\mathbb{R}^D)$, $s > 0$, and $E \subset \mathbb{R}^D$, define

$$\xi^s(E) = \xi(E)^s \quad \text{and} \quad \xi^{(+s)}(E) = \xi(E)|E|^s,$$

and if ξ is s_0-Hölder, define for all $s \in (0, s_0)$,

$$\xi^{(-s)}(E) = \begin{cases} 0 & \text{if } |E| = 0, \\ \xi(E)|E|^{-s} & \text{if } 0 < |E| < \infty. \end{cases}$$

Observe that ξ^s, $\xi^{(+s)}$ and $\xi^{(-s)}$, as defined above, still belongs to $\mathcal{H}(\mathbb{R}^D)$.

We will need the additional assumptions that the set functions $\xi \in \mathcal{H}(\mathbb{R}^D)$ are decreasing with respect to the embedding relationship, i.e. they are *capacities*. This is natural since, in many situations, the set functions that are considered are measures or powers of measures. Hence, we define the sets of capacities

$$\mathcal{C}(\mathbb{R}^D) = \{\xi \in \mathcal{H}(\mathbb{R}^D) : \forall E, F \in \mathcal{B}(\mathbb{R}^D), E \subset F \Rightarrow \xi(E) \leq \xi(F)\}$$
$$\mathcal{C}([0,1]^D) = \mathcal{H}([0,1]^D) \cap \mathcal{C}(\mathbb{R}^D).$$

The set functions considered in this thesis will verify the following property (P).

Definition 2.4. *A set function $\xi \in \mathcal{H}(\mathbb{R}^D)$ satisfies property (P) if there exist $C, s_1, s_2 > 0$ such that*

$$\text{for all } r > 0, \text{ for all } x \in \operatorname{supp}(\xi), \quad C^{-1} r^{s_2} \leq \xi(B(x,r)) \leq C r^{s_1}. \tag{2.5}$$

In particular, such a ξ is s_1-Hölder. We will only consider set functions satisfying property (P) on $\mathbb{R}^D$ (or $[0,1]^D$), implying that these functions will be fully supported on $\mathbb{R}^D$ (or $[0,1]^D$). Recall that a finite Borel measure ξ is said to be doubling if for some constant $C_\xi \geq 1$, for every $x \in \mathbb{R}^D$ and $r > 0$,

$$\xi(B(x,r)) \leq \xi(B(x,2r)) \leq C_\xi\, \xi(B(x,r)).$$

Since we work with set functions (for which one does not have in full generality $\xi(B(x,r)) \leq \xi(B(x,2r))$), and not necessarily with measures, we need to adapt this notion to our situation.

Definition 2.5. *Let Φ be the set of non increasing functions $\phi : (0,1] \to \mathbb{R}_+$ such that $\lim_{r \to 0^+} \frac{\phi(r)}{\log(r)} = 0$.*
A set function $\xi \in \mathcal{H}(\mathbb{R}^D)$ is almost doubling *on a set A when there exists $\phi \in \Phi$ such that for every $x \in A$, $r \in (0,1]$, $R \in [r, 2r]$ and $x' \in B(x,r)$,*

$$e^{-\phi(r)} \xi(B(x',r)) \leq \xi(B(x,R)) \leq e^{\phi(r)} \xi(B(x',r)). \tag{2.6}$$

The set function ξ is doubling when (2.6) holds with a constant function ϕ.

Intuitively, (2.6) implies that slightly moving the centre and the radius of a ball keep its ξ-mass controlled. A doubling measure satisfies (2.6) with a constant function ϕ. Examples of doubling set functions include (1-periodic) Gibbs measures ξ, see Definition 2.25 and [30, 47, 147], and their powers $\xi^s(E) := \xi(E)^s$ for $s > 0$.

For every integer $j \geq 0$, let Λ_j^D be the set of dyadic cubes of side length 2^{-j} in $[0,1]^D$ and $\Lambda^D = \bigcup_{j \in \mathbb{N}} \Lambda_j^D$.

Definition 2.6. *A set function $\xi \in \mathcal{H}(\mathbb{R}^D)$ satisfies property (P_2) if there exist $C, s_2 > 0$ and $\phi \in \Phi$ (see Definition 2.5) such that for all $j, j' \in \mathbb{N}$ with $j \leq j'$, for all $\lambda, \widetilde{\lambda} \in \Lambda_j^D$ such that $\partial\lambda \cap \partial\widetilde{\lambda} \neq \emptyset$, and $\lambda' \in \Lambda_{j'}^D$ such that $\lambda' \subset \lambda$:*

$$\xi(\widetilde{\lambda}) \leq C 2^{\phi(2^{-j})} 2^{(j'-j)s_2} \xi(\lambda').$$

In particular, ξ is almost doubling in this case [19]. While the ξ-mass of each dyadic cube is controlled by s_1 and s_2 by (P), Property (P_2) can be seen as a kind of weak quasi-Bernoulli property of ξ.

We will often refer to $\xi \in \mathcal{H}(\mathbb{R}^D)$ as an *environment* here after for reasons we will later explain. The multifractal properties of a set function can be considered similarly to functions considered at the begining of the section. To study its pointwise regularity, we use local dimensions and the associated singularity spectrum, defined as follows.

Definition 2.7. *Let $\xi \in \mathcal{H}(\mathbb{R}^D)$. For $x \in \operatorname{supp}(\xi)$, the lower local dimension of ξ at x is*

$$\underline{h}_\xi(x) = \liminf_{r \to 0^+} \frac{\log \xi(B(x,r))}{\log(r)}. \tag{2.7}$$

The singularity spectrum of ξ is the Hausdorff dimension of the level sets of the local dimension

$$\sigma_\xi \ : \ h \in \mathbb{R} \mapsto \dim_H \left(\underline{E}_\xi(h)\right) \quad \text{where} \quad \underline{E}_\xi(h) = \{x \in \operatorname{supp}(\xi) \ : \ \underline{h}_\xi(x) = h\}.$$

The set $E_\xi(h)$ consists of the points x where the local dimension (2.7) is a limit and equals h.

Definition 2.8. *The scaling function of $\xi \in \mathcal{H}([0,1]^D)$ with $\operatorname{supp}(\xi) \cap [0,1]^D \neq \emptyset$ is*

$$\tau_\xi(q) = \liminf_{j \to +\infty} \frac{1}{-j} \log_2 \sum_{\lambda \in \Lambda_j^D} \xi(\lambda)^q,$$

where $0^q = 0$ as convention.

The minimal and maximal local dimension and the dimension of ξ are defined respectively by

$$h_\xi^{\min} = \min\{\underline{h}_\xi(x) : x \in \operatorname{supp}(\xi)\}, \quad h_\xi^{\max} = \max\{\underline{h}_\xi(x) : x \in \operatorname{supp}(\xi)\},$$
$$\dim(\xi) = \inf\{\dim_H(E) : \xi(E) = 1\}.$$

2.2 Wavelet formalism

Throughout this thesis, we decompose functions on wavelet bases. Indeed, wavelet coefficients allow the characterisation of Hölder regularity or classical spaces such as Besov spaces. In the early 1900's, Haar was the first to construct an orthonormal wavelet basis without calling it wavelet. The first wavelet basis of arbritrary high regularity was contructed by Strömberg [164] in 1983. Lemarié and Meyer in 1986 [120] generalised the construction of C^∞ orthonormal wavelet bases of and higher dimension. Mallat [128] introduced, in 1989, a method to construct wavelet basis from multiresolution analysis in one dimension and Meyer

generalised it to higher dimension in 1990 [136] and introduced a tensorisation method. In 1992, Daubechies constructed compactly supported wavelets [48].

Let us recall the wavelet basis decomposition. Let $D \in \mathbb{N}^*$ and $1 \leq p, q \leq +\infty$. A function $\psi : \mathbb{R}^D \to \mathbb{R}$ has $n+1$ vanishing moments when for every multi-index $\alpha \in \mathbb{N}^D$ such that $\sum_{i=1}^D \alpha_i \leq n$, $\int_{\mathbb{R}^D} x_1^{\alpha_1} \dots x_D^{\alpha_D} \psi(x) dx = 0$.

Let $L^D = \{0,1\}^D \backslash \{0^D\}$, and consider $\{\varphi, \psi_\lambda := \psi^l(2^j \cdot - k)\}_{\lambda=(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^D\times L^D}$ a wavelet basis of $L^2(\mathbb{R}^D)$ generated by a family of wavelets $(\varphi, \psi^l)_{l\in L^D}$, such that every $f \in L^2(\mathbb{R}^D)$ can be written [128, 136]

$$f \overset{L^2}{=} \sum_{\lambda\in\mathbb{Z}\times\mathbb{Z}^D\times L^D} c_\lambda \psi_\lambda \quad \text{with} \quad c_\lambda = 2^{Dj} \int_{\mathbb{R}^D} f(x)\psi_\lambda(x)dx.$$

For $k = (k_1, \dots, k_D) \in \mathbb{Z}^D$, $j \in \mathbb{Z}$, we often identify $\lambda = (j,k,l)$ with the dyadic cube $\prod_{i=1}^D [k_i 2^{-j}, (k_i+1)2^{-j})$, and in particular we write $\xi(\lambda)$ for $\xi(\prod_{i=1}^D [k_i 2^{-j}, (k_i+1)2^{-j}))$.

For any $f \in L^p(\mathbb{R}^D)$, $1 \leq p \leq \infty$, f can be written [136]

$$f = \sum_{k\in\mathbb{Z}^D} \beta(k)\varphi(\cdot - k) + \sum_{j\geq 0} \sum_{\lambda\in\{j\}\times\mathbb{Z}^D\times L^D} c_\lambda \psi_\lambda, \text{ with } \beta(k) = \int_{\mathbb{R}^D} f(x)\varphi(x-k)dx. \tag{2.8}$$

Without loss of generality for our problem, we will mostly work with functions supported in $[0,1]^D$ and will write

$$f = \beta(0)\varphi + \sum_{j\geq 0} \sum_{\lambda\in\Lambda_j^D\times L^D} c_\lambda \psi_\lambda. \tag{2.9}$$

Indeed, any function $f \in L^p(\mathbb{R}^D)$ can be written as a sum $f = g + \sum_{\ell\in\mathbb{Z}^D} f_\ell(\cdot + \ell)$, where g is a function as regular as the wavelets and f_ℓ can be written as (2.9). Since we are interested only in local and multifractal properties, and using that the Hausdorff dimension of a countable union is the supremum of the dimension of all sets, the singularity spectrum of a locally bounded function $f \in L^p(\mathbb{R}^D)$ is obtained as the supremum of those of the f_ℓ.

In this thesis, we will use a family of wavelets $(\varphi, \psi^l)_{l\in L^D}$ built through a tensorisation process as described in [136, Chapter 2 and 3]. For an arbitrary integer $N \geq 1$, there exist compactly supported functions $\psi^0 \in C^N(\mathbb{R})$ (called the scaling function) and $\psi^1 \in C^N(\mathbb{R})$ (called the mother wavelet), with ψ^1 having at least $N+1$ vanishing moments and such that $(\psi^0, \psi^1_{j,k} := \psi^1(2^j \cdot - k))_{j,k\in\mathbb{Z}}$ form an orthogonal basis of $L^2(\mathbb{R})$. In this case, the wavelet is said to be N-regular. Let us introduce the notations

$$0^D := (0, \dots, 0), \quad 1^D := (1, \dots, 1), \quad L^D = \{0,1\}^D \backslash 0^D.$$

An orthogonal basis of $L^2(\mathbb{R}^D)$ is obtained by tensorisation as follows: for $\lambda = (j,k,l) \in \mathbb{Z}\times\mathbb{Z}^D\times L^D$, let us define the tensorised wavelet

$$\psi_\lambda = \prod_{i=1}^d \psi^{l_i}_{j,k_i}$$

with $k = (k_1, \dots, k_D)$ and $l = (l_1, \dots, l_D) \in L^D$, and setting $\varphi = \psi^{0^D}$.

Thus for every $f \in L^p(\mathbb{R}^D)$, one has the decomposition (2.8) (observe that we use an L^∞ normalisation for the wavelet coefficient $(c_\lambda)_{\lambda\in\Lambda}$.)

Proposition 2.9. [83, 92] *Suppose that $\gamma > 0$ and $N = \lfloor \gamma + 1 \rfloor$. Suppose that $\varphi, \psi \in C^N(\mathbb{R})$ is at least $N+1$-regular. Let $f : [0,1]^D \to \mathbb{R}$ be a locally bounded function with wavelet coefficients $\{c_\lambda\}_{\lambda\in\Lambda^D\times L^D}$, and let $x \in [0,1]^D$.*

If $f \in C^\gamma(x)$, then there exists a constant $M < \infty$ such that for all $\lambda = (j,k,l) \in \Lambda^D \times L^D$,

$$|c_\lambda| \leq M\ 2^{-j\gamma}\left(1 + |2^j x - k|\right)^\gamma. \tag{2.10}$$

Conversely, if (2.10) holds for $f \in \bigcup_{\varepsilon>0} C^\varepsilon([0,1]^D)$, then $f \in \bigcap_{\eta>0} C^{\gamma-\eta}(x)$.

2.3 Prevalence and Baire genericity

Definition 2.10. *Let X be a complete metric space, and $A \subset X$. A set A is of the first category when it can be written as $A = \bigcup_{n\in\mathbb{N}} A_n$, where for all $n \in \mathbb{N}$, A_n is included in a closed set with empty interior in X.*

A set if of second category, or residual, if its complement is of first category, meaning a set that contains a countable intersection of dense open sets.

A property is typical in the set X if it holds for all elements of second category set $A \subset X$.

The study of the properties of typical functions dates back to the early 1930s [13, 102, 135], with the classical example that a typical function in $C([0,1])$ is nowhere differentiable. Jaffard looked at the singularity spectrum of typical functions in a Besov space [90]. Barral-Seuret proved in [19] that typical function in an inhomogeneous Besov space have a specific singularity spectrum.

However, the notions of being of first category or Legesgue measure zero intersect but does not correspond. The set of rationals $\mathbb{Q}$ is of first category and Lebesgue measure zero but the set of Liouville numbers is of second category and Lebesgue measure zero while the set of diophantine numbers is of full Lebesgue measure and of first category [81].

This difference between Baire category and Lebesgue measure justifies the interest in looking at a notion of "full Lebesgue measure" for function spaces, which is the notion of prevalent set. Christensen [44] and Hunt, Sauer and York [81] proposed indepentently the concept of shy and prevalent sets to supersede the notion of Lebesgue measure for universally measurable sets in any real or complex topological vector space E endowed with its Borel σ-algebra $\mathcal{B}(E)$. In this thesis, all Borel measures ϑ on $(E, \mathcal{B}(E))$ will be automatically completed, i.e. for every set $A \subset E$ and $B \in \mathcal{B}(E)$ such that the symmetric difference $A\Delta B$ is included in some $D \in \mathcal{B}(E)$ with $\vartheta(D) = 0$, one sets $\vartheta(A) := \vartheta(B)$.

Definition 2.11. *A set $A \subset E$ is universally measurable if it is measurable for any (completed) Borel measure on E. A universally measurable set $A \subset E$ is called Haar-null, or shy,*

if there exists a finite completed measure ϑ that is positive on some compact subset K of E and such that

$$\text{for every } x \in E, \quad \vartheta(A+x) = 0. \tag{2.11}$$

The measure ϑ used to show that some subset is Haar-null is called transverse. More generally, a set that is included in a Haar-null universally measurable set is also called Haar-null. Finally, the complement in E of a Haar-null subset S, denoted $S^{\complement}$, is called prevalent.

Hunt pioneered the use of prevalence in function spaces [80]. Fraser-Hyde [61], Bayart-Heurteaux [21] and Balka-Darji-Elekes [12] studied the dimension of images of continuous functions.

Theorem 2.12 (Fraser-Hyde [61])**.** *Let $d \in \mathbb{N}$. The set*

$$\{f \in C([0,1]^d) : \dim_H\big(\{(x, f(x)) : x \in [0,1]^d\}\big) = d+1\}$$

is a prevalent subset of $C([0,1]^d)$.

Fraysse-Jaffard-Kahane [63, 64] and Aubry-Maman-Seuret [10] studied, respectively, the prevalent regularity of functions and traces of functions in Besov spaces, see Theorems 2.15 and 2.24 below. Olsen studied the multifractal and L^q-dimension of prevalent measures on compact sets of $\mathbb{R}^D$ [142, 143].

2.4 From homogeneous to inhomogeneous spaces

In the 1980s, Frisch and Parisi first identified a connection between multifractality and multi-scale properties of the velocity [66]. Based on works by Kolmogorov in 1941, they proposed to consider the scaling function ζ heuristically defined via the relation

$$\forall p > 0, \quad S_p(h) := \int |v(x+h) - v(x)|^p dx \underset{h\to 0}{\sim} |h|^{\zeta(p)}.$$

Jaffard [90] highlighted a connection between ζ and Besov-type norms, showing that $\zeta(p) = \sup\{\tau : v \in B^{\tau/p}_{p,p}\}$ and that the velocity field can be analysed through the regularity of functions in $\bigcap_{\varepsilon>0, p>0} B^{(\zeta(p)-\varepsilon)/p}_{p,p}$.

Let us recall the definition of Besov spaces [136].

Definition 2.13. *Let $p, q \in (0, +\infty]$, $s > D/p$. Consider an integer $n > s$ and wavelets $(\varphi, \psi^l)_{l\in L^D}$ compactly supported, that belong to $C^n(\mathbb{R}^D)$ and have at least n vanishing moments.*

A function f, written as (2.8), belongs to $B^s_{p,q}(\mathbb{R}^D)$ if and only if

$$\|\beta\|_{\ell^p(Z^D)} + \big\|(\varepsilon_j^{s,p})_{j\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})} < \infty, \quad \text{where} \quad \varepsilon_j^{s,p} = \big\|\big(2^{j(s-d/p)} c_\lambda\big)_{\lambda\in\Lambda_j^D\times L^D}\big\|_{\ell^p}. \tag{2.12}$$

Results on the pointwise regularity of functions in classical spaces such as Bessel potential spaces $H^{s,p}(\mathbb{R}^D)$, Sobolev-Slobodeckij $W^{s,p}(\mathbb{R}^D)$ or Besov $B^s_{p,q}(\mathbb{R}^D)$ spaces have been obtained in [9, 63, 87, 90, 98]. They are of two types:

- the identification of a universal upper bound for the singularity spectrum of all function in a given space, such as the result proved by Jaffard [87]

  Theorem 2.14. *Let $p, q \in (0, \infty]$, $s > D/p$. For every $f \in B^s_{p,q}(\mathbb{R}^D)$, one has*

  $$\sigma_f(h) \leq \begin{cases} \min\{p(h - (s - D/p)), D\} & \text{if } h \geq s - D/p, \\ -\infty & \text{if } h < s - D. \end{cases}$$

- the computation of the explicit value for the spectrum of a generic set of functions in the sense of prevalence [63] or Baire categories [90]

  Theorem 2.15. *Let $p, q \in (0, \infty]$, $s > D/p$. There is a Baire second category set and a prevalent set of functions $f \in B^s p, q(\mathbb{R}^D)$ satisfying*

  $$\sigma_f(h) = \begin{cases} p(h - (s - D/p)) & \text{if } h \in [s - D/p, s], \\ -\infty & \text{else.} \end{cases}$$

A key feature of typical functions in the classical spaces $W^{s,p}(\mathbb{R}^D)$ and $B^s_{p,q}(\mathbb{R}^D)$ is that their singularity spectrum is always affine increasing. This is an apparent contradistinction with the fact that the spectra estimated on real data through the multifractal formalism always have a strictly concave shape, with an increasing *and* a decreasing part.

Jaffard [90] showed that given an *admissible* concave function $\tau : \mathbb{R}^+ \to \mathbb{R}$, Baire typical functions in the intersection of Besov spaces $\bigcap_{\varepsilon>0, p>0} B^{(\tau(p)-\varepsilon)/p}_{p,q,loc}(\mathbb{R}^D)$ have concave increasing spectra. Fraysse [62] obtained that prevalent functions in such spaces possess the same spectrum. These results disproved a commonly acknowledged idea in the physics community, namely that a measurement with a nonconcave singularity spectrum necessarily is the consequence of an underlying multiplicative cascade process.

In order to consider non-affine singularity spectra, one can look at the existing natural generalizations of the classical Sobolev and Besov spaces, the weighted Sobolev and Besov spaces, which have various possible definitions.

For $h \in \mathbb{R}^D$ and $f : \mathbb{R}^D \to \mathbb{R}$, consider the finite difference operator $\Delta_h f : x \in \mathbb{R}^D \mapsto f(x + h) - f(x)$. Then, for $n \geq 2$, set $\Delta^n_h f = \Delta_h(\Delta^{n-1}_h f)$.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a measurable function, positive almost everywhere, called a weight. Under some hypotheses on the behaviour of ρ given in [7], weighted Besov spaces $B^\rho_{p,q}(\mathbb{R}^D)$ consist of functions $f : \mathbb{R}^D \to \mathbb{C}$ such that

$$\int_{|h|\leq 1} \rho(|h|)^{-q} \left\|\Delta^n_h f\right\|^q_{L^p(\mathbb{R}^D)} \frac{dh}{|h|^D} < \infty.$$

A function f belonging to such spaces presents a homogeneous pointwise behaviour as the weight in the integral depends only $|h|$. These spaces have a generalised version called *Besov space of generalised smoothness* $B^{\gamma,N}_{p,q}$, with γ, N some sequences, studied by Farkas-Leopold [58]. Moura [138] and Loosveldt-Nicolay [124] gave alternative definition. Kreit-Nicolay linked them to generalized Hölder-Zygmund $\Lambda^\gamma_{\sigma,N}$ spaces [114, 115]. Such Besov spaces of generalised

smoothness have been linked to traces of functions in Besov spaces on fractal sets by Caetano-Haroske [34] and studied by Kleyntssens-Nicolay [111].

A second version of weighted Besov spaces exists where the weight does not dictate the scale behaviour of the oscillations of the function f but the local behaviour of f itself and can be linked to standard Besov spaces, see results from Edmunds, Haroske and Triebel [54, 76, 167]. It is defined as follows: for a weight $\rho \in L^1_{\mathrm{loc}}(\mathbb{R})$, a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^D)$ belongs to $B^s_{p,q}(\mathbb{R}^D, \rho)$ if

$$\int_{|h|\leq 1} |h|^{-sq} \, \|\rho \, \Delta_h^n f\|^q_{L^p(\mathbb{R}^D)} \, \frac{dh}{|h|^D} < \infty.$$

Even if the multifractal behaviour of functions in such spaces has not been studied yet, they have an explicit characterisation through wavelets given by Triebel [171, Chapter 6] that can be linked to the spaces introduced hereafter.

Another direction is the variable exponent Lebesgue and Sobolev spaces, which are defined through the weighted Orlicz spaces [52, 67] as follows. Let $p : \mathbb{R}^D \to [1, \infty)$ be a measurable bounded function, called a variable exponent. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^D)$ consist of all measurable functions $f : \mathbb{R}^D \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^D} |f(x)|^{p(x)} \, dx < +\infty.$$

The *variable exponent Sobolev space* $W^{k,p(\cdot)}(\mathbb{R}^D)$, $k \in \mathbb{N}$, consist of all functions of $L^{p(\cdot)}(\mathbb{R}^D)$ satisfying that $f^{(k)} \in L^{p(\cdot)}(\mathbb{R}^D)$. The variable exponent p is usually at least continuous, the usual hypothesis being log-Hölder continuous. While typical functions in $W^{k,p}(\mathbb{R}^D)$ exhibit affine spectra with no decreasing parts [90], the spaces $W^{k,p(\cdot)}(\mathbb{R}^D)$ have not been studied yet.

Barral and Seuret introduced the following inhomogeneous Besov spaces.

Definition 2.16. *Let $\xi \in \mathcal{H}([0,1]^D)$ satisfying (P) of Definition 2.4 and (P_2) of Definition 2.6 for some $C > 0$ and $0 < s_1 < s_2$. Let $p, q \in [1, +\infty]$, consider an integer $n \geq \left\lfloor s_2 + \frac{D}{p} \right\rfloor + 1$ and wavelets $(\varphi, \psi^l)_{l \in L^D}$ compactly supported, that belong to $C^n(\mathbb{R}^D)$ and have at least n vanishing moments.*

A function f, written as (2.9), belongs to $B^\xi_{p,q}([0,1]^D)$ if and only if $f \in L^p([0,1]^D)$ and $|f|_{\xi,p,q} < +\infty$, where

$$|f|_{\xi,p,q} = \left\| (\varepsilon_j^{\xi,p})_{j \in \mathbb{N}} \right\|_{\ell^q(\mathbb{N})}, \quad \text{where} \quad \varepsilon_j^{\xi,p} = \left\| \left(\frac{c_\lambda}{\xi(\lambda)} \right)_{\lambda \in \Lambda_j^D \times L^D} \right\|_{\ell^p(\Lambda_j^D \times L^D)}. \tag{2.13}$$

Then

$$\widetilde{B}^\xi_{p,q}([0,1]^D) = \bigcap_{0<\varepsilon<\min(1,s_1)} B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D). \tag{2.14}$$

We will consider for this introduction that $\xi \in \mathcal{H}([0,1]^D)$ satisfies (P) and (P_2), although results presented can be seen in a more general setting in Chapter 4.

These spaces are not spatially homogeneous in the sense that, in the above integral, the oscillations $\Delta_h^n f(x)$ of f are locally compared to the ξ-mass of a small ball of radius h close

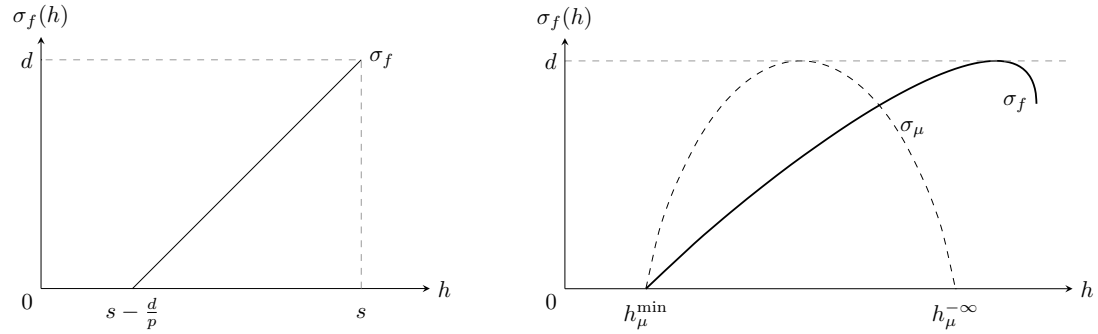


Figure 2.1: This figure illustrates the typical singularity spectrum in standard Besov spaces opposed to those in inhomogeneous Besov spaces, to motivates there consideration. **Left:** Singularity spectrum of a typical function $f \in B^s_{p,q}(\mathbb{R}^D)$. **Right:** Singularity spectrum of a typical function $f \in B^\xi_{1,q}(\mathbb{R}^D)$. The dashed graph represents the (initial) singularity spectrum of ξ. When $p = \infty$ and f is typical in $B^\xi_{\infty,q}(\mathbb{R}^D)$, $\sigma_f = \sigma_\xi$.

to x, and this mass may strongly depend on x. They provide a natural framework in which typical functions have multifractal behaviours similar to those observed in real data [19].

Taking $\xi = (\mathcal{L}^D)^{\frac{s}{D}-\frac{1}{p}}$ with $s > D/p$, one sees from the definition that $B^\xi_{p,q}([0,1]^D) = B^s_{p,q}([0,1]^D)$. Also, when $p = q = \infty$, $B^{\xi,\psi}_{\infty,\infty}([0,1]^D)$ is the space of those functions f for which

$$C = \sup_{\lambda \in \mathbb{Z}\times\mathbb{Z}^D\times L^D} \frac{|c_\lambda|}{\xi(\lambda)} < +\infty.$$

This can be rewritten as $|c_\lambda| \leq C\,\xi(\lambda)$. Recalling that the Hölder spaces $C^\alpha([0,1]^D)$ are characterised by the fact that $|c_\lambda| \leq C\,|\lambda|^\alpha$ for every λ, the spaces $B^\xi_{p,q}([0,1]^D)$ can be understood as an inhomogeneous generalisation of $C^\alpha([0,1]^D)$, since the value of $\xi(\lambda)$ varies a lot depending on λ. This interpretation explains that ξ is called an environment.

It is proved in [19] that the definition of $B^\xi_{p,q}([0,1]^D)$ and $\widetilde{B}^\xi_{p,q}([0,1]^D)$ is independent of the choice of ψ as soon as ξ is doubling and the norms $\|\cdot\|_{L^p} + |\cdot|_{\xi,p,q}$ are equivalent for any choice of ψ.

Remark 2.17. Observe that, by construction, the family $\left\{B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D)\right\}_{0<\varepsilon<\min(1,s_1)}$ satisfies that, for $0 < \varepsilon < \varepsilon' < \min(1, s_1)$, $B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D) \hookrightarrow B^{\xi^{(-\varepsilon')}}_{p,q}([0,1]^D)$. Moreover, $\widetilde{B}^\xi_{p,q}([0,1]^D)$ is a Fréchet space [19].

Barral-Seuret proved in [19] that Theorems 2.14 and 2.15 in the Baire category setting could be extended to inhomogeneous Besov spaces, giving:

Theorem 2.18. *Let ξ be a power of measure as defined in [19], let $p, q \in [1, +\infty]$. Consider the mappings*

$$\tau_{\xi,p}(t) = \begin{cases} \frac{p-t}{p}\,\tau_\xi\left(\frac{p}{p-t}\,t\right) & \text{if } t \in (-\infty, p) \\ \tau'_\xi(+\infty)t & \text{if } t \in [p, +\infty). \end{cases}$$

For all $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^d)$,

$$\sigma_f(h) \leq \begin{cases} \tau^*_{\xi,p}(h) & \text{if } h \leq \tau'_{\xi,p}(0) \\ d & \text{if } h > \tau'_{\xi,p}(0). \end{cases}$$

Furthermore, typical functions $f \in \widetilde{B}^{\xi,}_{p,q}([0,1]^d)$ *satisfy* $\sigma_f = \tau^*_{\xi,p}$.

Recall that for an integer k and $p \in [1,+\infty]$, the classical Sobolev space $W^{k,p}(\mathbb{R}^D)$ is defined as:

$$W^{k,p}(\mathbb{R}^D) = \left\{ f \in L^p(\mathbb{R}^D) : \forall |\alpha| \leqslant k,\ D^\alpha f \in L^p(\mathbb{R}^D) \right\}.$$

It is a Banach space when endowed with the norm $\|f\|_{W^{k,p}(\mathbb{R}^D)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\mathbb{R}^D)}$. When $s \in \mathbb{R}_+\backslash\mathbb{N}$, and $p \in [1,+\infty]$, the Sobolev-Slobodeckij space $W^{s,p}(\mathbb{R}^D)$ is the set of functions $f : \mathbb{R}^D \to \mathbb{R}$ such that

$$\|f\|_{W^{s,p}(\mathbb{R}^D)} = \|f\|_{W^{\lfloor s \rfloor,p}(\mathbb{R}^D)} + \sup_{|\alpha|=\lfloor s \rfloor} |D^\alpha f|_{W^{s-\lfloor s \rfloor,p}(\mathbb{R}^D)} < +\infty$$

where, when $\theta \in (0,1)$,

$$|f|_{W^{\theta,p}(\mathbb{R}^D)} = \left(\iint_{\mathbb{R}^D \times \mathbb{R}^D} \frac{|f(x+h) - f(x)|^p}{|h|^{\theta p + D}} dx dh \right)^{1/p}. \tag{2.15}$$

We now introduce new inhomogeneous Sobolev spaces in a ξ-environment by modifying the above formula in order to obtain a class of functions with a singularity spectrum not necessarily affine. To do so, we interpret in (2.15) the term $|h|^\theta$ in the denominator as, up to a constant, the power of the D-dimensional Lebesgue measure, denoted by $\mathcal{L}^D$, of the ball $B(x,h)$, and we replace it by the ξ-mass of this ball. This introduces spatial inhomogeneity in (2.15), including Sobolev spaces as special case.

Let $B[x,y]$ denote the Euclidean ball of diameter $[x,y]$.

Definition 2.19. *Let* $\xi \in \mathcal{H}(\mathbb{R}^D)$ *satisfy* (P) *and* (P_2) *with exponents* $0 < s_1 < s_2$, *and consider* $p \geq 1$ *as well as an integer* $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$.

A function f *belongs to* $W^{\xi,p}(\mathbb{R}^D)$ *if and only if* $\|f\|_{W^{\xi,p}(\mathbb{R}^D)} < +\infty$, *where* $\|f\|_{W^{\xi,p}(\mathbb{R}^D)} = \|f\|_{L^p(\mathbb{R}^D)} + |f|_{W^{\xi,p}(\mathbb{R}^D)}$ *and*

$$|f|_{W^{\xi,p}(\mathbb{R}^D)} := \left(\iint_{\mathbb{R}^{2d}} \frac{|\Delta^n_h f(x)|^p}{\xi(B[x,x+nh])^p |h|^{2d}} dx dh \right)^{1/p}.$$

Recall that, for any $p \in [1,+\infty]$ and $s \in \mathbb{R}_+\backslash\mathbb{N}$ with $s > D/p$, one has $W^{s,p} = B^s_{p,p}$, see [167]. In Chapter 4, we show that this can be extended to inhomogeneous Sobolev and Besov spaces in the following way:

Theorem 2.20. *Let* $p \geq 1$. *If* $\xi \in \mathcal{H}(\mathbb{R}^D)$ *with* $\mathrm{supp}(\xi) = \mathbb{R}^D$ *satisfies* (P) *and* (P_2) *with exponents* $0 < s_1 < s_2$, *then*

$$B^{\xi}_{p,p}(\mathbb{R}^D) \subset W^{\xi,p}(\mathbb{R}^D) \subset \widetilde{B}^{\xi}_{p,p}(\mathbb{R}^D).$$

2.5 Prevalent multifractal properties in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$

First, let us recall the following definition.

Definition 2.21. *A set function $\xi \in \mathcal{H}([0,1]^D)$ satisfies the strong multifractal formalism (SMF) if for every $h \geq 0$, $\sigma_\xi(h) = \tau^*_\xi(h) = \dim_H(E_\xi(h))$.*

In Chapter 4, we extend Theorem 2.15 for the prevalence to inhomogeneous Besov spaces and thus Theorem 2.18 to the prevalence case when $p = \infty$. We show prevalent multifractal properties that hold for environments that are capacities.

Theorem 2.22. *Let $\xi \in \mathcal{C}([0,1]^D)$ be an almost doubling capacity satisfying (P), (P_2) and verifying the SMF. For all $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$,*

$$\sigma_f(h) \leq \begin{cases} \sigma_\xi(h) & \text{if } h \leq \tau'_\xi(0) \\ d & \text{if } h > \tau'_\xi(0). \end{cases} \tag{2.16}$$

Also, there is a prevalent set of functions $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ satisfying $\sigma_f = \sigma_\xi$.

Remark that this result stays true for $f \in B^{\xi}_{\infty,q}([0,1]^D) \subset \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. In other words, the prevalent singularity spectrum in $\widetilde{B}^{\xi}_{\infty,q}$ or $B^{\xi}_{\infty,q}$ coincides with that of ξ.

These result of Chapter 4 are submitted and available on Hal and arXiv [152].

2.6 Prevalent results of traces

Chapter 5 is devoted to the multifractal analysis of traces of functions in inhomogeneous Besov spaces $B^{\xi}_{p,q}(\mathbb{R}^D)$ on d-dimensional affine subspaces of $\mathbb{R}^D$, for ξ a product measure of Gibbs measures on $\mathbb{R}^D$, see Definition 2.25. Such an interest on traces comes from practical problems. Physicists have been able to measure the velocity of a turbulent fluid along one direction, which correspond to the one-dimensional trace of the fluid's velocity, a four-dimensional phenomenon (a phenomenon in $\mathbb{R}^3$ also depending on time).

Let $0 < d < D$ be two fixed integers and $d' := D - d$. For $a \in \mathbb{R}^{d'}$, we denote $\mathcal{C}_a := \{(x,a) \ : \ x \in \mathbb{R}^d\}$ the d-dimensional affine subspace of $\mathbb{R}^D$ passing by $(0,a)$. A trace of a function f on $\mathcal{C}_a$ is defined by

$$f_a := f_{|\mathcal{C}_a} : \begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x,a). \end{array} \tag{2.17}$$

This notion has been considered for functions in standard Besov spaces by Aubry-Maman-Seuret and Jaffard [10, 34, 85, 129] and weighted Besov spaces by Besoy-Haroske-Triebel [26] for example. Universal and generic results in the sense of prevalence and Baire's categories on the pointwise regularity of traces of functions belonging to classical spaces have been obtained by Aubry, Maman and Seuret [10, 129] giving the two next results:

Theorem 2.23. *Let $D \in \mathbb{N}^*$, $0 < p, s < \infty$, with $s > D/p$, and let $q \in (0, \infty]$. If $f \in B^s_{p,\infty}(\mathbb{R}^D)$, then for Lebesgue-almost all $a \in \mathbb{R}^{d'}$, $f_a \in \bigcap_{s'<s} B^{s'}_{p,\infty}(\mathbb{R}^d)$.*

Moreover, if $f \in B^s_{p,q}(\mathbb{R}^D)$ and $q < p$ (resp. $q = p$), then for Lebesgue-almost all $a \in \mathbb{R}^{d'}$, $f_a \in B^s_{p,qp/(p-q)}(\mathbb{R}^d)$ (resp. $f_a \in B^s_{p,\infty}(\mathbb{R}^d)$).

Theorem 2.24. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $d/p < s$. There exist a Baire second category set and a prevalent set of functions $f \in B^s_{p,q}(\mathbb{R}^D)$ such that for Lebesgue-almost all $a \in \mathbb{R}^{d'}$,*

$$\sigma_{f_a}(h) = \begin{cases} p(h - (s - d/p)) & \text{if } h \in [s - d/p, s], \\ -\infty & \text{else.} \end{cases}$$

In Chapter 5, we will consider capacities μ and ν defined as powers of Gibbs measures. Let $\mathcal{L}^D$ stand for the D-dimensional Lebesgue measure.

Definition 2.25. *Let $\varphi : \mathbb{R}^D \to \mathbb{R}$ be a $\mathbb{Z}^D$-periodic Hölder continuous function (called a potential). Then the sequence of measures defined on $\mathbb{R}^D$ whose densities are given by*

$$\nu_n(dx) = \frac{\exp(S_n\varphi(x))}{\int_{[0,1]^D} \exp(S_n\varphi(t))\mathcal{L}^D(dt)}\mathcal{L}^D(dx), \quad \text{where } S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(2^k x),$$

converges weakly to a $\mathbb{Z}^D$-invariant measure ν_φ, called the Gibbs measure associated with φ.

These measures have been extensively studied [82, 144, 145, 146, 147, 159]. For the moment, we give few properties of these measures: Given a fixed potential $\varphi : \mathbb{R}^D \to \mathbb{R}$,

1. The Gibbs measure ν_φ is doubling, satisfies the SMF and $\mathrm{supp}(\nu_\varphi) = \mathbb{R}^D$.
2. For every $r \in \mathbb{R}$, setting
$$h^r_{\nu_\varphi} = \tau'_{\nu_\varphi}(r), \tag{2.18}$$
one has $\sigma_{\nu_\varphi}(h^r_{\nu_\varphi}) = rh^r_{\nu_\varphi} - \tau_{\nu_\varphi}(r)$. Also, $h^{\min}_{\nu_\varphi} = h^{+\infty}_{\nu_\varphi}$ and $h^{\max}_{\nu_\varphi} = h^{-\infty}_{\nu_\varphi}$.
3. For every $r \in \mathbb{R}$, the Gibbs measure $\nu_{r\varphi}$ is called the auxiliary measure associated with φ and r (the auxiliary measure is associated with the potential $r\varphi$), and enjoys multifractal properties similar to those of ν_φ. Then $\nu_{r\varphi}(\underline{E}_{\nu_\varphi}(h^r_{\nu_\varphi})) = 1$, i.e. for $\nu_{r\varphi}$-a.e. x, $\underline{h}_{\nu_\varphi}(x) = h^r_{\nu_\varphi}$.

The second item follows from (2.18) and simple considerations on the Legendre transform. Since ν_φ is $\mathbb{Z}^D$-periodic, we will identify ν_φ with its restriction to $[0,1]^D$. Item 3 states that the auxiliary measure $\nu_{r\varphi}$ is concentrated on those points x which have a local dimension for ν_φ equal to $h^r_{\nu_\varphi}$. More details on auxiliary measures are recalled in Chapter 5.

Definition 2.26. *The set of Gibbs capacity $\mathscr{E}_D \subset \mathcal{C}(\mathbb{R}^D)$ is the set of positive powers of Gibbs measures ν, i.e.*

$$\mathscr{E}_D = \{\nu^s \ : \ \nu \text{ a Gibbs measure }, s > 0\}. \tag{2.19}$$

Our goal is to study the multifractal properties of the traces of $f \in \widetilde{B}^{\xi}_{p,q}(\mathbb{R}^D)$ when ξ is a product of Gibbs capacities. More precisely, if $D = d + d'$, for $\mu_\varphi \in \mathscr{E}_d$, associated to a potential $\varphi : \mathbb{R}^d \to \mathbb{R}$, and $\nu_{\widetilde{\varphi}} \in \mathscr{E}_{d'}$, associated to a potential $\widetilde{\varphi} : \mathbb{R}^{d'} \to \mathbb{R}$, we will consider the capacity

$\xi = \mu_\varphi \times \nu_{\widetilde{\varphi}}$ on $[0,1]^D$, i.e. for $A \in \mathcal{B}([0,1]^d)$, $B \in \mathcal{B}([0,1]^{d'})$, $\xi(A \times B) := \mu_\varphi(A) \cdot \nu_{\widetilde{\varphi}}(B)$. In the following, we will omit the mention to φ and $\widetilde{\varphi}$, and write simply μ, ν, and also ν_r for the auxiliary measures associated with ν.

Theorem 2.22 applies to $\xi = \mu \times \nu$, and one finds that there exists a prevalent set of functions $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$ such that $\sigma_f = \tau^*_\xi$. The spectrum of ξ is determined in terms of those of μ and ν.

Proposition 2.27. *Let $\mu \in \mathscr{E}_d$ and $\nu \in \mathscr{E}_{d'}$ with $D = d + d'$. Let ξ be the product capacity on $[0,1]^D$ defined by $\xi = \mu \times \nu$. The product capacity ξ satisfies the SMF and verifies $\tau_\xi = \tau_\mu + \tau_\nu$ and*

$$\text{for all } r \in \mathbb{R}, \quad \sigma_\xi\left(h^r_\mu + h^r_\nu\right) = \sigma_\mu\left(h^r_\mu\right) + \sigma_\nu\left(h^r_\nu\right).$$

For standard Besov spaces, the trace operator $f \mapsto f_a$ maps $B^s_{p,q}(\mathbb{R}^D)$ to $B^{s-d'/p}_{p,q}(\mathbb{R}^d)$, which reflects a loss of regularity when taking traces. However, almost all traces f_a belongs to $B^s_{p,q}(\mathbb{R}^d)$, see Theorem 2.23. Our first result shows that the same phenomenon occurs for inhomogeneous Besov spaces $\widetilde{B}^{\xi}_{p,q}([0,1]^D)$.

Theorem 2.28. *Let $p, q \in [1, +\infty]$. Let $\mu \in \mathscr{E}_d$ and $\nu \in \mathscr{E}_{d'}$ with $D = d + d'$. Let ξ be the product capacity on $[0,1]^D$ defined by $\xi = \mu \times \nu$.*

For every $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$, for all $r \in \mathbb{R}$ and ν_r-almost all $a \in [0,1]^{d'}$,

$$f_a \in \widetilde{B}^{\mu(+\Gamma_{\nu,r})}_{p,1}([0,1]^d) \quad \text{with } \Gamma^p_{\nu,r} = h^r_\nu + \frac{\dim(\nu_r)}{p}.$$

By convention $\frac{\dim(\nu_r)}{\infty} = 0$. Observe that the trace f_a belongs to a space whose regularity index depends on the local regularity of ν at a.

From now on, **we only consider the case $p = \infty$.** We find an upper-bound for the spectrum of the traces of all functions $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$.

Theorem 2.29. *For all $f \in \widetilde{B}^{\xi}_{\infty,q}$ and $a \in [0,1]^{d'}$, for every $h \in \mathbb{R}_+$, one has*

$$\sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h^{\min}_\nu) & \text{if } h \leq h^0_\mu + h^{\min}_\nu, \\ d & \text{if } h > h^0_\mu + h^{\min}_\nu. \end{cases} \tag{2.20}$$

For all $f \in \widetilde{B}^{\xi}_{\infty,q}$, for every $r \in \mathbb{R}$ and ν_r-almost all $a \in [0,1]^{d'}$, for every $h \in \mathbb{R}_+$, one has

$$\sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h^r_\nu) & \text{if } h \leq h^0_\mu + h^r_\nu, \\ d & \text{if } h > h^0_\mu + h^r_\nu. \end{cases} \tag{2.21}$$

The theorem regarding the prevalent multifractal behaviour in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ is the following. It depends on the existence of a wavelet satisfying (R) detailed in Chapter 3.

Theorem 2.30. *Let $\mu \in \mathscr{E}_d$, $\nu \in \mathscr{E}_{d'}$, and consider the product capacity $\xi = \mu \times \nu \in \mathcal{C}([0,1]^D)$, with $D = d + d'$. Call C, s_1, s_2 the real numbers such that (2.5) is satisfied.*

Let $n \geq \left\lfloor s_2 + \frac{D}{p} \right\rfloor + 1$. Assume that there exists a compactly supported wavelet $\psi \in C^n(\mathbb{R})$ having (at least) n vanishing moments and satisfying the property (R) of Chapter 3.

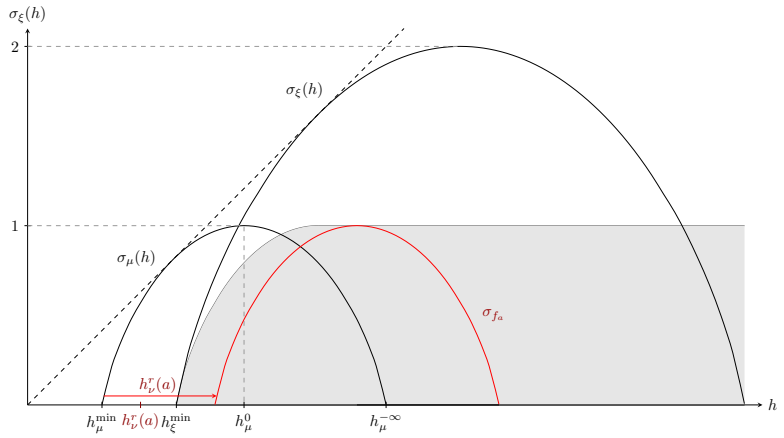


Figure 2.2: Example of complete bounds for $\sigma_{f_a}(h)$ for almost all $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ and ν_r-almost all $a \in [0,1]^{d'}$ for $\xi = \mu \times \nu$ with μ and ν have the same known singularity spectrum.

Let $(r_n)_{n\geq\mathbb{N}}$ be a countable sequence of real numbers. There exists a prevalent set of function $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ such that for every $n \in \mathbb{N}$, for ν_{r_n}-almost all $a \in (0,1]^{d'}$,

$$\sigma_{f_a}(h) = \begin{cases} \sigma_\mu(h - h_\nu^{r_n}) & \text{if } h \in \operatorname{supp}(\sigma_\mu) + h_\nu^{r_n} \\ -\infty & \text{else.} \end{cases}$$

The singularity spectrum σ_{f_a} of the trace f_a coincides with that of μ translated by h_ν^r, which depends on the regularity of ν in a.

Result of Chapter 5 correspond to a submitted article available on HAL and arXiv [154] and use properties of wavelet presented in Chapter 3 also submitted and available [153].

2.7 Anisotropy in Besov-type Spaces

We now consider regularity properties of function in anisotropic settings. Anisotropy, which happens when physics does not act the same in different directions, arises in a wide range of applications such as geophysics, hydrology [148], medical imaging (osteoporosis, muscular tissues, mammographies,...) [27, 29]. A first issue lies in introducing global and local notions of anisotropic regularity, which encompass and extend (isotropic) regularity spaces, such as Sobolev or Besov spaces, and the classical notion of pointwise Hölder regularity. To model anisotropy, anisotropic self-similar random fields have been introduced and studied [22, 27, 45, 55], and some of the properties characterising anisotropy can be analysed with functional spaces well-adapted to anisotropy, such as anisotropic Besov spaces or Besov with dominating mixed smoothness. These spaces have an extensive literature, see for example [25, 31, 56, 158, 165, 168, 169].

Signal and function regularity analysis can be done in an isotropic setting with wavelet analysis as introduced in Section 2.2. However, classical wavelets are not well-suited for the analysis of anisotropic signals due to their construction principle based on isotropic dilation.

A simple example that illustrates this problem is a signal which is rather smooth in one direction but rough in the orthogonal direction, such as layers or stripes in a texture for which the isotropic wavelet methods does not properly analysed it.

Many authors addressed this problem with different solutions depending on the definition of anisotropic space and the anisotropic basis used, and wavelets have been adapted in many different ways to "detect" and resolve anisotropy, such as, for instance, ridgelets and curvelets studied by Candes, Donoho and Demanet [36, 37, 38, 39, 40], bandelets [119, 127], aniset [94], shearlets developped in [71, 117, 118], and anisotropic wavelets adressed by many authors, such as Ben Slimane, Ben Braiek, Bownik, Farkas, Garrigos, Hochmuth, Haroske, Liu, Tabacco, Tamási, Triebel and Yang [24, 32, 57, 69, 75, 79, 123, 139, 170, 171]. However, these wavelets require to know the anisotropy in advance and if anisotropy parameters are not provided by the context of the signal considered, characterising it cannot be done without "trying out" several anisotropic bases, which is hardly implementable in practice.

In [3], it has been shown that any anisotropic Besov space on $\mathbb{R}^2$ can "almost" be characterised with a so-called hyperbolic wavelet transform. This allows the detection and characterisation of the anisotropy of a signal by a specific weight in the wavelet coefficients. This has led to an efficient algorithm for image classification and anisotropy detection applied to both synthetic and real textures (see [6, 155]). Schäfer-Ullrich-Vedel later generalised such characterisation to higher dimension and for Triebel-Lizorkin and Sobolev spaces [157]. Vybiral [172] showed that Besov spaces with dominating mixed smoothness can also be characterised with an hyperbolic wavelet transform, and Esser-Loosveldt-Vedel [55] used this hyperbolic wavelet basis to introduce the weighted tensorized Besov spaces, which, with the help of a parameter r decreasing from 1 to 0, correspond to mixed dominating smoothness Besov spaces when $r = 0$ and isotropic hyperbolic Besov spaces when $r = 1$.

This characterisation through an hyperbolic wavelet basis is tied to consideration in nonlinear approximation of functions in PDE [73, 103] and practical applications such as analysis of anisotropic textures [155]. The nonlinear approximation of function in some Bessel potential spaces can be done with function in weighted tensorized Besov spaces.

Our aim, in Chapter 6, is to introduce an anisotropic version of the weighted tensorised Besov spaces, as well as to explain how they can be considered from the viewpoint of anisotropic hyperbolic Besov spaces and mixed dominating smoothness Besov spaces. These spaces have an anisotropic structure similar to anisotropic hyperbolic Besov and Besov with dominating mixed smoothness, with a parameter that controls the reticulated effect present in mixed dominating smoothness Besov spaces and not in anisotropic Besov spaces. Moreover, they generalise the isotropic weighted tensorised Besov spaces of Esser-Loosveldt-Vedel. We also present how the regularity of a function in an anisotropic weighted tensorised Besov spaces changes with the consideration of different anisotropy directions. This chapter is a starting point that aims at providing theoretical background to the method for estimating anisotropic parameters of textures as done in [155].

Consider a hyperbolic wavelet basis as the system $\{\psi_{\overline{j},\overline{k}} : \overline{j} \in \mathbb{N}^2_{-1}, \overline{k} \in \mathbb{Z}^2\}$ of tensor

products of two unidimensional wavelet bases, see Chapter 6 and [51] for details. This wavelet basis allows the study of the regularity and the oscillations of a function with different behaviours across the two orthogonal directions, which cannot be done with classical wavelet methods having isotropic dilation.

Similarly to classical Besov spaces, the anisotropic hyperbolic Besov spaces $\widetilde{B}^{s,\overline{\alpha}}_{p,q}(\mathbb{R}^2)$ are spaces of function with controlled oscillations over scales with regularity in one direction controlled by s/α_1 and regularity in the orthogonal direction controlled by s/α_2.

On the other hand, the Besov space with dominating mixed smoothness $S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2)$ is a set of functions with regularity also controlled by s/α_1 and s/α_2 in each direction while presenting a structure similar to reticulated textures such as textiles.

To link these two spaces to each other and describe a family of anisotropic Gaussian processes, Esser-Loosveldt-Vedel [55] introduced the so-called weighted tensorized Besov spaces. Such spaces have an anisotropic structure similar to anisotropic hyperbolic Besov spaces and Besov spaces with dominating mixed smoothness with a parameter which controls the reticulated effect. Let us now give an anisotropic version of these last spaces.

We will use the following notation. Put $\mathbb{N}_{-1} := \mathbb{N} \cup \{-1\}$. Multivariate indices are typeset with a bar, such as $\overline{k}, \overline{j}$ or $\overline{\alpha}$. We also put $x_+ := \max\{x, 0\}$ for $x \in \mathbb{R}$ and $\overline{x}_+ := ((x_1)_+, (x_2)_+)$ in the multidimensional case. In the following, the multi-index $\overline{\alpha} = (\alpha_1, \alpha_2) > 0$ stands for an anisotropy index and is such that $\alpha_1 + \alpha_2 = 2$. The notation $\overline{j}/\overline{\alpha}$ shall always stand for $(j_1/\alpha_1, j_2/\alpha_2)$ and thus $\overline{j}_+/\overline{\alpha}$ shall always stand for $((j_1)_+/\alpha_1, (j_2)_+/\alpha_2)$. For $r \in [0, 1]$, consider the function

$$\theta_r(\overline{j}_+/\overline{\alpha}) := (1 + r)\max(\overline{j}_+/\overline{\alpha}) + (1 - r)\min(\overline{j}_+/\overline{\alpha}).$$

Definition 2.31. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$ and an anisotropy index $\overline{\alpha} \in \mathbb{R}^2$. Consider the hyperbolic wavelet basis.*

A function $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to the anisotropic weighted tensorised Besov space $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ if $\|f\|_{TB^{s,\overline{\alpha},r}_{p,q}} < +\infty$, where

$$\|f\|_{TB^{s,\overline{\alpha},r}_{p,q}} := \left\|(\varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}})_{\overline{j}\in\mathbb{N}^2_{-1}}\right\|_{\ell^q}, \quad \text{with} \quad \varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}} = \left\|\left(2^{\theta_r(\overline{j}_+/\overline{\alpha})s}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2}\right\|_{\ell^p} \tag{2.22}$$

with the usual modification for $q = +\infty$, $p = +\infty$.

Such spaces can be linked back to anisotropic hyperbolic Besov spaces and Besov spaces with dominating mixed smoothness as follows.

Remark 2.32. The anisotropic weighted tensorised Besov space corresponds to the Besov spaces with dominating mixed smoothness when $r = 0$

$$S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2) = \left\{f \in \mathcal{S}'(\mathbb{R}^2) : \left\|\left(\left\|\left(2^{\|\overline{j}_+/\overline{\alpha}\|_1 s}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2}\right\|_{\ell^p}\right)_{\overline{j}\in\mathbb{N}^2_{-1}}\right\|_{\ell^q} < \infty\right\},$$

and to anisotropic hyperbolic Besov spaces (with regularity 2s) when $r = 1$

$$B^{2s,\overline{\alpha}}_{p,q}(\mathbb{R}^2) = \left\{f \in \mathcal{S}'(\mathbb{R}^2) : \left\|\left(\left\|\left(2^{\|\overline{j}_+/\overline{\alpha}\|_\infty s}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2}\right\|_{\ell^p}\right)_{\overline{j}\in\mathbb{N}^2_{-1}}\right\|_{\ell^q} < \infty\right\}.$$

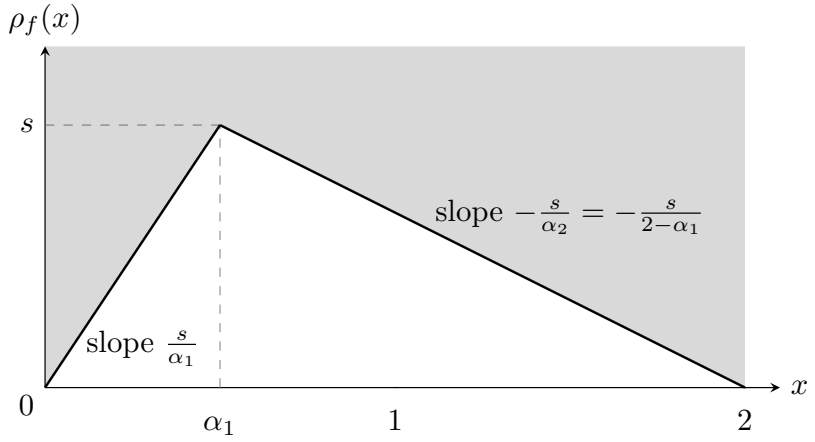


Figure 2.3: The regularity function $\rho_{f,r}$ depending on the anisotropy $(x, 2-x)$. The graph of the function $\rho_{f,r}$ is included in the grey part for all functions f and, for typical functions in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, $\rho_{f,r}$ is represented in black.

An important property of $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ is that these spaces are Banach spaces as showed in Chapter 6. We investigate how the regularity of a function in an anisotropic weighted tensorised Besov space behaves in comparison to other anisotropy directions.

Definition 2.33. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, and an anisotropy index $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. For $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, define the function*

$$\forall x \in (0,2), \quad \rho_{f,r}(x) := \sup\{s' \in \mathbb{R} : f \in TB^{s',(x,2-x),r}_{p,q}(\mathbb{R}^2)\}.$$

Considering a function $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, typically, this function shows its largest regularity when studied considering an anisotropy index $\overline{\alpha}$ and its regularity only decreases if we consider other anisotropy indices.

Theorem 2.34. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, and an anisotropy index $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$.*

For any $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, we have

$$\rho_{f,r}(x) \geq \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{2.23}$$

For any f in a Baire second category set in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, we have

$$\rho_{f,r}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{2.24}$$

This can be interpreted that if one computes the regularity of a function for different anisotropy indices, one can deduce its anisotropy parameter as the one where the function has the largest regularity. This method was used in [155] to evaluate the anisotropy parameter of texture simulated from anisotropic Gaussian processes and the corresponding regularity. Indeed, for anisotropic textures simulated from anisotropic Gaussian processes with particular regularity and anisotropy index, they estimated the regularity $s(\overline{\alpha})$ for multiple anisotropy

indices $\overline{\alpha}$ and recovered the anisotropy and regularity used for the simulation. The estimated regularity for multiple anisotropy index forms a triangular-shaped function similar to Figure 6.1.

In particular, in the case of anisotropic hyperbolic Besov spaces, if for a function f, the maximum of $\rho_{f,1}$ is known, then one can completely determine $\rho_{f,1}$ as follows.

Theorem 2.35. *Let $r = 1$ and $\rho_{f,1}(x) = \sup\{s' \in \mathbb{R} : f \in B_{p,q}^{2s',(x,2-x)}(\mathbb{R}^2)\}$. Consider a function f such that there exists an anisotropy index $\overline{\alpha}$ for which $\rho_{f,1}(\alpha_1) = s = \max_{y\in(0,2)} \rho_{f,1}(y)$ for $s \geq 0$. Then*

$$\rho_{f,1}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{2.25}$$

Both Theorems 2.34 and 2.35 imply that if a function f satisfies $\rho_{f,1}(\alpha_1) = s = \max_{y\in(0,2)} \rho_{f,1}(y)$ for an anisotropy index $\overline{\alpha}$ and a $s \geq 0$, then $\rho_{f,1}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}$. Thus, if one consider the setting of anisotropic hyperbolic Besov spaces, one simply has to obtain (numerically) two points that aren't aligned with either $(0,0)$ or $(2,0)$ in the plane $(x, \rho_{f,1}(x))$ as in Figure 2.3. This will give the maximum regularity by linear regression, which means that not all the anisotropies need to be tested, simplifying the method used in [155].

This chapter presents a preliminary study that need further extensions presented in Chapter 6.

2.8 Riemann Fourier Series

The determination of the regularity of trigonometric series has been a topic of major importance during the last two centuries. These series have supplied examples and counterexamples of functions having specific properties and the theoretical advances made on this subject have had major implications in mathematical analysis, see e.g. Jaming-Kellay-Saba-Wang, Kahane-Lemarie, Korner and Zygmund [100, 101, 104, 113, 179]. A typical example is supplied by *Weierstrass functions*

$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(\pi b^n x), \quad \text{where} \quad 0 < a < 1 \text{ and } b > 1$$

which settled the question of determining if there exist continuous nowhere differentiable functions. Weierstrass proved that it is the case if b is an integer and $ab > 1 + 3\pi/2$ [173], later sharpened to the condition $b > 1$ and $ab > 1$ by Hardy [74]. Another key example was proposed by Riemann: he conjectured that the function

$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2} \tag{2.26}$$

is continuous and nowhere differentiable. This function proved much harder to be dealt with than the Weierstrass functions: In [74], Hardy proved that it is nowhere differentiable except perhaps at the rationals of the form $\frac{2p+1}{2q+1}$, and it is only in 1969 that Gerver [70] who showed that Riemann's function is differentiable at those rationals. Its full regularity properties were

completely derived only in the 1990s, first with the determination in [86] of its exact pointwise regularity at every point, and then by an asymptotic expansion at differentiability points, see [97], which put in evidence its *oscillatory behavior* at these points. An important byproduct is that Riemann's function is an example of *multifractal function*. Such regularity issues are not completely settled for several variants and generalisations of the seminal Riemann's function (2.26). Let us consider the family

$$\mathcal{S}_{b,s}(x,t) = \sum_{n=1}^{\infty} \frac{e^{i\pi(n^b x + nt)}}{n^s}, \quad \text{and} \quad \mathcal{Q}_{b,s}(x) = Im(\mathcal{S}_{b,s}(x,0)) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^b x)}{n^s} \tag{2.27}$$

where $b \geq 2$ is an integer and $s \in \mathbb{R}$ (Riemann's function is $\mathcal{Q}_{2,2}$). Various instances of these functions have been studied: the case $b = 2$ is met in explicit solutions of several linear PDEs, such as one-dimensional periodic Schrödinger equation or two-dimensional supported plate equation, so that their analysis gives insight on the regularity properties of these PDEs; furthermore, they play an important role in fluid mechanics because of their occurrence in the dynamics of the binormal flow: In [14], it is shown that they describe the temporal trajectory of a vortex filament.

In [150], Hervé Queffelec investigated the regularity properties of the functions

$$\sum \frac{\sin(\pi P(n)x)}{P(n)}$$

where P is a polynomial function with integer coefficients. In the particular case of $P(n) = n^s$ corresponding to $\mathcal{Q}_{s,s}$ and more generally $\mathcal{S}_{b,s}(\cdot, 0)$, these results were complemented by Boichu in [28]. He showed that

$$h^{\min}_{\mathcal{Q}_{b,s}} = \frac{s-1}{b},$$

If $s < 1$, then the functions $\mathcal{Q}_{b,s}$ are no more bounded, so that the usual notion of pointwise regularity supplied by the Hölder exponent can no more be used. Calderón and Zygmund introduced in [35] a notion of pointwise regularity adapted to functions in $L^p_{\text{loc}}(\mathbb{R})$.

Definition 2.36. *Let $f \in L^p_{loc}(\mathbb{R})$ with $p \geq 1$, $x_0 \in \mathbb{R}$ and $\alpha \geq -1/p$. The function f belongs to $T^p_\alpha(x_0)$ when there exists a polynomial P_{x_0} of degree less than α and $C, R > 0$ such that*

$$\forall r \in (0,R), \quad \left(\frac{1}{r} \int_{x_0-r}^{x_0+r} |f(x) - P_{x_0}(x-x_0)|^p dx \right)^{\frac{1}{p}} \leq C r^\alpha. \tag{2.28}$$

The p-exponent of f at x_0 is $h^p_f(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}$.

Partial results using this weaker notion of pointwise regularity have been obtained by Seuret-Ubis [160]. Nonetheless, our understanding of the regularity properties of this family of functions is partial; their singularity spectrum are only partially known, and in many instances, only loose upper and lower bounds are known [42]. One way to make precise guesses on their exact values is through the application of a *multifractal formalism*, see Section 7.2, which yields a relationship between the singularity spectrum and numerically computable quantities. Up to now, these techniques have mainly been used for classification and model

selection. Though the precise range of validity of this formula in not known, it has been proved to hold for a large number of mathematical models [19, 23, 90], so that working out the multifractal formalism in a numerically precise way would supply "guesses" of the correct values of their singularity spectrum. Note that up to now, the numerical determination of Legendre spectra of mathematical functions was performed only in cases where their singularity spectrum were theoretically known, see e.g. [5], with the aim of testing its numerical accuracy, often leading to improvements or variants of the method; therefore its purpose lies at the cross-road of applied mathematics and signal processing. Our purpose here is different: it is to supply mathematicians with reliable conjectures concerning their singularity spectrum, and therefore falls in the field of *experimental mathematics*.

Functions, whose pointwise regularity varies erratically from one point to another, appear naturally in many areas of mathematics. The Riemann Fourier series $\mathcal{R}$ was the first deterministic example shown to be of this type; more recently, the Brjuno function [96], was shown to share regularity properties with the Riemann function: in both cases their pointwise regularity exponent is directly related with Diophantine irrationality exponents. Stochastic examples are supplied by cascade models, see [17] and references therein, or Lévy processes, see [88, 149]. Such models, whose theoretical properties are fully understood, yield important case studies on which new numerical algorithms can be tested, and their accuracy can be challenged. They are used in several fields of applications such as turbulence [41, 65] or finance [11].

Furthermore, Riemann's functions are of particular interest because of the presence of oscillating singularities in their graphs similar to the "chirp" function of Section 2.1. Such behaviors are met in various physical phenomena see [60]. For signals or function without such singularities, the Hölder or the p-exponents provide all the information of the regularity of the function as well as the regularity of its fractional integrals. However, in the presence of such singularities, studying the p-exponents of a function and its primitive yields informations on the type of oscillatory behaviour that it displays.

In Chapter 7, after recalling some notions and properties of pointwise regularity of Fourier series and their fractional derivatives and primitives, we propose a comparison of known results and conjectures to numerical estimation of the regularity quantities of $\mathcal{Q}_{b,s}$. We provide the known result on the regularity of $\mathcal{Q}_{2,s}$ for $s > 1/2$ as well as some conjectures previously formulated and compare those to numerical estimation for $s = 2$ and for $1/2 < s < 1$ (we will pick $s = 0.7$). We also confront theoretical result for $\mathcal{Q}_{b,s}$, with $b = 3$ and $b = 4$, to numerically estimated results. Finally, we give the numerical study of $\mathcal{S}_{2,2}(x, t)$ in order to conjecture its regularity behaviour.

In this chapter, we present several partial results concerning the numerical determination of the multifractal spectrum of generalizations of the Riemann function; however this preliminary study needs to be conforted by additional numerical invertigations before an article is finalized.

Chapter 3

A non-vanishing property for tensor products of wavelets

3.1 Introduction

In this chapter, we study certain cancellation properties of wavelets and of their periodized versions, with a particular focus on Daubechies wavelets.

A fundamental question in wavelet analysis is the localization of the zeros of a wavelet $\psi : \mathbb{R} \to \mathbb{R}$. Fix (ϕ, ψ) a couple of scaling function and its associated wavelet, so that the family of dilations and translations

$$\{\psi_{j,k}(\cdot) := 2^{j/2}\psi(2^j \cdot -k)\}_{j,k\in\mathbb{Z}}$$

of ψ forms an orthonormal basis of $L^2(\mathbb{R})$. For a function $f \in L^2(\mathbb{R})$, its wavelet coefficients with respect to this basis are denoted by

$$c_k = \int_{\mathbb{R}} f(x)\phi(x-k)dx \quad \text{and} \quad d_{j,k} = \int_{\mathbb{R}} f(x)\,\psi_{j,k}(x)\,dx.$$

For a sufficiently regular function $f : \mathbb{R} \to \mathbb{R}$, written as

$$f = \sum_{k\in\mathbb{Z}} c_k\phi(\cdot - k) + \sum_{j\geq 0}\sum_{j,k\in\mathbb{Z}} d_{j,k}\,\psi_{j,k}$$

where the equality holds pointwise, the fact that all $\psi_{j,k}(x) = 0$ for every $j \geq 0$ and $k \in \mathbb{Z}$ at some point $x \in \mathbb{R}$ would imply that none of the wavelet coefficients $d_{j,k}$ contribute to the value of $f(x)$, which is a degenerate situation.

This observation leads to the following natural question: if $\psi(x) = 0$, does there necessarily exist an integer $k \in \mathbb{Z}$ such that $\psi(x-k) \neq 0$? If this were not the case, as observed before, the point x would exhibit a remarkable degeneracy, since all coefficients $d_{0,k}$ would be irrelevant to the value of $f(x)$. Such a phenomenon would be unexpected, and we conjecture that it does not occur for general wavelets. We provide numerical evidence showing that this property fails for the first 45 Daubechies wavelets.

Obtaining a rigorous analytical proof of this observation remains an open problem, we

have not seen such a proof in the literature. Results closed in this question were obtained in [151], where the distribution of zeros of finite sums of wavelets is investigated, highlighting how their structure constrains where these zeros can occur and [108, 109, 110, 166], where the location of zeros of polynomials associated with Daubechies orthogonal and biorthogonal wavelets is analyzed, showing they are constrained (notably within the unit disk) to ensure perfect reconstruction.

One easily shows that under a very weak assumption (see below), for a fixed point $x \in \mathbb{R}$, it is impossible to have $\psi_{j,k}(x) = 0$ for all $(j,k) \in \mathbb{Z}^2$. Consequently, even in the extreme situation where $\psi(x-k) = 0$ for all $k \in \mathbb{Z}$, there must exist at least one pair (j_x, k_x) such that $\psi_{j_x,k_x}(x) \neq 0$. It is therefore natural to ask whether the scale parameter j_x can be chosen uniformly bounded with respect to $x \in \mathbb{R}$.

The main focus of this chapter is to investigate whether analogous phenomena arise in the setting of tensor-product wavelets with compact supports. We prove that, under a mild and natural assumption, for every point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, there exist infinitely many pairs $(j,k) \in \mathbb{N} \times \mathbb{Z}^d$ such that

$$\psi(2^j x_i - k_i) = 0 \quad \text{simultaneously for all } i = 1, \dots, d.$$

We make precise the notion of "infinitely many" in this context and show, moreover, that the translation parameters k_i can be chosen independently of x.

We introduce a property that, according to the previous considerations, is verified by many compactly supported wavelets.

Definition 3.1. *A real function $\psi : \mathbb{R} \to \mathbb{R}$ satisfies the property (R) if:*
(R_1) *$\psi \in \mathcal{C}^1(\mathbb{R})$, and there is an integer $K_\psi \in \mathbb{N}^*$ such that $\overline{\mathrm{supp}(\psi)} = [0, K_\psi]$.*
(R_2) *The cardinality of the set $\mathcal{Z}(0,0) := \psi^{-1}(\{0\}) \bigcap [0, K_\psi]$ of zeros of ψ is finite.*
(R_3) *for all $x \in [0,1]$, $S(x) := \sum_{k=0}^{K_\psi - 1} |\psi(x+k)| > 0$.*

This property is satisfied by at least the Daubechies' wavelet of order up to 45 [48, 49], see Section 3.3. It is an interesting question to prove (not only numerically) that large classes of wavelets, including Daubechies', satisfy (R), we will study it in an upcoming work. Property (R) implies the following fact, which is the main result of the chapter.

Theorem 3.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function satisfying property (R), and let*

$$G : x \in \mathbb{R} \mapsto \sum_{p \in \mathbb{Z}} \psi(x - pK_\psi). \tag{3.1}$$

be the periodized version of ψ.

There is $\alpha > 0$ with the following property: for every $d \in \mathbb{N}^*$, there exists an integer $N(d)$ and a sequence of multi-integer $(\overline{p_j} = (p_{j,1}, \dots, p_{j,d}))_{j \in \mathbb{N}}$ such that for every $x \in \mathbb{R}^d$ and

$J \in \mathbb{N}$. there is an integer $j \in \{J, J+1, ..., J+N(d)\}$ such that

$$\prod_{i=1}^{d} |G(2^j x_i - p_{j,i})| \geq \alpha^d. \tag{3.2}$$

In words, Theorem 3.2 asserts that, for any dimension d, there exists a sequence of multi-indices $(\overline{p_j})_{j\in\mathbb{N}}$, depending on the function ψ only, such that for every $x \in \mathbb{R}^d$, the translated and dilated versions of the periodized function

$$G^1_{j,p_{j,i}} := G(2^j \cdot - p_{j,i})$$

do not vanish simultaneously at the coordinates x_i for infinitely many values of j. More precisely, this non-vanishing property occurs at least once every $N(d)+1$ scales j, hence very regularly, and not only the translates do not vanish at x, but we deduce from the proof that they all are uniformly bounded by below. The independence with respect to x of the sequence $(\overline{p_j})_{j\in\mathbb{N}}$ is important and noticeable.

Beyond the intrinsic interest of the question itself, Theorem 3.2 plays a crucial role in the analysis of the regularity of traces of functions. More precisely, let $d, d' \in \mathbb{N}^*$ and $D = d + d'$. For every $a \in \mathbb{R}^{d'}$, call $H_a = \mathbb{R}^d \times \{a\}$ the horizontal affine subspace of $\mathbb{R}^D$. The trace of a continuous function $f : \mathbb{R}^D \to \mathbb{R}$ along H_a is then $f_a : y \in \mathbb{R}^d \mapsto f_a(y) := f(y, a)$, which is also continuous. It is proved in [10, 154] that in a homogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ or in an inhomogeneous Besov space $B^\xi_{p,q}(\mathbb{R}^d)$, there is a prevalent set of functions whose multifractal properties can be explicitly determined for most of the parameters $a \in \mathbb{R}^{d'}$. These results are based on the fact that given a specific function $f : \mathbb{R}^D \to \mathbb{R}$ with wavelet coefficients $(d_{j,\mathbf{k}})_{j\in\mathbb{N},\mathbf{k}\in\mathbb{Z}^D}$, the wavelet coefficients $(d_{j,\mathbf{k}'}(a))_{j\in\mathbb{Z},\mathbf{k}'\in\mathbb{Z}^d}$ of its trace f_a depend explicitly on G. Without entering into too many details, when for instance ψ is a compactly supported wavelet with $\overline{\mathrm{supp}(\psi)} = [0, N]$ and when

$$d_{j,k} = \begin{cases} 2^{-j\alpha} & \text{if } \mathbf{k} = (k_1..., k_D) \text{ is such that } \begin{cases} \forall i \in \{1, ..., d\}, k_i = p_{j,i} \mod N \\ \forall i \in \{d+1, ..., D\}, k_i = 0 \end{cases} \\ 0 & \text{otherwise,} \end{cases}$$

then one has

$$d_{j,k}(a) = 2^{-j\alpha} \prod_{i=1}^{d} |G(2^j a_i - p_{j,i})|.$$

Then, in order to estimate the regularity properties of f_a, it is mandatory to estimate $d_{j,k}(a)$, and the non-vanishing property (3.2) is key.

The rest of this note is devoted to the proof of Theorem 3.2 in Section 3.2, and to the numerical verification of the property (R) in Section 3.3 for Daubechies wavelets. Observe that Theorem 3.2 does not require ψ to be a wavelet, however for our application purposes it is needed to verify it for compactly supported wavelets.

3.2 Proof of Theorem 3.2

We assume that the function ψ satisfies Property (R) given in Definition 3.1.

Since ψ is compactly supported on $[0, K_\psi]$, the function G (recall (3.1)) is K_ψ-periodic, with $G(0) = G(K_\psi) = 0$ and $G \in \mathcal{C}^1(\mathbb{R})$.

Definition 3.3. *For every $\overline{p_j} = (p_{j,1}, \dots, p_{j,d}) \in \mathbb{Z}^d$, the function $G^d_{j,\overline{p_j}} : \mathbb{R}^d \to \mathbb{R}$ is defined by*

$$\text{for every } x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad G^d_{j,\overline{p_j}}(x) = \prod_{i=1}^{d} G(2^j x_i - p_{j,i}). \tag{3.3}$$

Note that with this notation, $G^d_{j,\overline{p_j}}$ can be rewritten as $\prod_{i=1}^d G^1_{j,p_{j,i}}$.

Our goal is to prove Theorem 3.2, which asserts that the integers $\overline{p_j}$ can be chosen such that, for every x, at least one of the functions $(G^d_{J+\ell,\overline{p_{J+\ell}}})_{\ell \in \{0,\dots,N(d)\}}$ is non-zero and bounded below (in absolute value) by a constant depending on α at x. Moreover, this occurs infinitely often.

We begin with some preliminary remarks and definitions.

Since $\psi \in \mathcal{C}^1(\mathbb{R})$, ψ' is continuous and bounded on $[0, K_\psi]$. Consequently, G' is bounded on $\mathbb{R}$ as a periodized version of ψ. We denote

$$M_G := \max_{x \in [0, K_\psi]} |G'(x)|.$$

For the remainder of this section, we fix $d \in \mathbb{N}^*$ and consider an increasing family of integers $N(1), \dots, N(d)$, to be specified later.

Definition 3.4. *For every integers j and p, let $\mathcal{Z}(j,p)$ be the set of zeros of $G(2^j x - p)$ on $[0, K_\psi]$. The ω-neighborhood of $\mathcal{Z}(j,p)$ is*

$$\mathcal{N}_\omega \mathcal{Z}(j,p) = \bigcup_{z \in \mathcal{Z}(j,p)} B(z, \omega), \quad \text{where } B(z,\omega) = (z - \omega, z + \omega).$$

We start with a basic lemma, which shows that for any $x \in [0, K_\psi]$, at each scale j, there exists an integer p such that $|G_{j,p}(x)|$ is uniformly bounded below.

Lemma 3.5. *There exists $\eta > 0$ such that for every $x \in \mathbb{R}$ and every j, there exists $p_{j,x} \in \{0, \dots, K_\psi - 1\}$ satisfying*

$$|G^1_{j,p_x}(x)| = |G(2^j x - p_x)| \geq \eta. \tag{3.4}$$

Proof. The function S, defined in (R_3), is continuous on $[0,1]$, hence bounded and attains its bounds. Let $\widetilde{\eta} = \min_{x \in [0,1]} S(x) > 0$.

By construction, for $x \in [0, K_\psi]$, we have $G(x) = \psi(x)$. Thus, for $x \in [0,1]$, $S(x) = \sum_{p=0}^{K_\psi - 1} |\psi(x+p)| = \sum_{p=0}^{K_\psi - 1} |G(x+p)| \geq \widetilde{\eta}$. Therefore, for all $x \in [0, K_\psi]$, there exists $p_x \in \{0, \dots, K_\psi - 1\}$ such that

$$|G(x - p_x)| \geq \frac{\widetilde{\eta}}{K_\psi} =: \eta. \tag{3.5}$$

The K_ψ-periodicity of G yields (3.4) for $j = 0$.

For $j \geq 1$, we apply the above result to $x' = 2^j x$ to obtain (3.4). □

We define the following sequence of integers: let $N = \#\mathcal{Z}(0,0)$ and set

$$N(1) = 2(N+1) \quad \text{and for } d' \geq 2, \quad N(d') = 2(N(d'-1)+1)^{d'} 2^{N(d'-1)}. \tag{3.6}$$

Recall the parameter η from Lemma 3.5, and define

$$\varepsilon_d := \frac{\eta}{2^{N(d)+1} M_G}, \quad \varepsilon'_d := \min_{x,y \in \mathcal{Z}(0,0)} \frac{|x-y|}{4 \times 2^{N(d)}}, \quad \text{and} \quad \omega := \min\{\varepsilon_d, \varepsilon'_d\}. \tag{3.7}$$

Since $\mathcal{Z}(0,0)$ is a finite set of cardinality N, we denote its elements by

$$z_1 = 0 < z_2 < \cdots < z_{N-1} < z_N = K_\psi.$$

We make the following observations:

- For every j, the elements of $\mathcal{Z}(j,p)$ are precisely those real numbers that can be written as $2^{-j}(kK_\psi + p + z_i)$, where $z_i \in \mathcal{Z}(0,0)$ and $k \in \mathbb{Z}$.
- For every $j \in \{0, \ldots, N(d)\}$, two elements of $\mathcal{Z}(j,p)$ are separated by at least $4\varepsilon'_d$.
- By our choice of ω, for every $j \in \{0, \ldots, N(d)\}$, each set $\mathcal{N}_\omega \mathcal{Z}(j,p)$ is a union of disjoint open intervals in $\mathbb{R}$, as any two elements of $\mathcal{Z}(j,p)$ are separated by at least 4ω.

We first consider the one-dimensional case.

Lemma 3.6. *There exists $\alpha > 0$ such that the following holds: for every integers $0 \leq J \leq N(d) - N(1)$ and $p_J \in \mathbb{Z}$, there exists a sequence $(p_j)_{j \geq J}$ such that for every $x \in \mathbb{R}$ and every $j \in \{J, \ldots, N(d) - N(1)\}$, there is an integer $\ell \in \{0, \ldots, N(1)\}$ with $|G^1_{j+\ell, p_{j+\ell}}(x)| \geq \alpha$.*

Proof. We work on the interval $I = [2^{-J} p_J, 2^{-J}(p_J + K_\psi)]$. Since all functions $(G^1_{j,p})_{j \geq J, p \in \mathbb{Z}}$ are $2^{-J}K_\psi$-periodic, the result extends directly to $[0,1]$.

By the above remark, $\mathcal{Z}(J, p_J) \cap I = \{2^{-J}(z_1 + p_J), \ldots, 2^{-J}(z_N + p_J)\}$ and this set has cardinality N. For simplicity, we introduce the notation:

$$\text{for } m \in \{1, ..., N\}, \quad S_m := B(2^{-J}(z_m + p_J), \omega) \tag{3.8}$$

$$\text{and} \quad S_0 := I \setminus \mathcal{N}_\omega \mathcal{Z}(J, p_J), \tag{3.9}$$

so that for every $x \in I$, there exists a unique index $m \in \{0, \ldots, N\}$ such that $x \in B(2^{-J}(z_m + p_J), \omega)$.

(i) If $x \in S_0$, then this set is a finite union of closed intervals on which $|G^1_{J,p_J}|$ is non-zero. Thus, $|G^1_{J,p_J}(x)| \geq \min\{|G^1_{J,p_J}(x)| : x \in B(2^{-J}(z_0 + p_J), \omega)\} =: \widetilde{\alpha} > 0$. Importantly, by auto-similarity and periodicity, $\widetilde{\alpha}$ does not depend on J nor on p_J.

(ii) If $x \in S_m$ for some $m \in \{1, \ldots, N\}$, then by Lemma 3.5, there exists p_{J+m} such that (3.4) holds for $G^1_{J+m, p_{J+m}}$ at the point $2^{-J}(z_m + p_J)$, i.e., $|G^1_{J+m, p_{J+m}}(2^{-J}(z_m + p_J))| \geq \eta$. By the mean value theorem,

$$|G^1_{J+m, p_{J+m}}(x)| \geq \eta - |x - 2^{-J}(z_m + p_J)| 2^{J+m} M_G \geq \frac{\eta}{2},$$

due to our choice of ω in (3.7) and the restriction $J + m \leq N(d)$.

This allows us to fix p_j for all $j \in \{J, \dots, J+N\}$, ensuring the conclusion of Lemma 3.6 with $\alpha = \min(\eta/2, \widetilde{\alpha})$ for these integers.

Assume now that p_j is constructed for all $j \in \{J, \dots, J+k(N+1)-1\}$ for some $k \geq 1$. We then set $p_{J+k(N+1)} = 0$ and apply the above argument to $G^1_{J+k(N+1),0}$ to obtain the values of p_j for $j \in \{J+k(N+1), \dots, J+(k+1)(N+1)-1\}$.

This process can be repeated until $J+(k+1)(N+1)-1 \leq N(d)$, providing the entire sequence $(p_j)_{J \leq j \leq N(d)}$. Finally, with $N(1) = 2(N+1)$ as in (3.6), for every $x \in [0, K_\psi]$ and every $J \leq j \leq N(d)-N(1)$, the sequence of integers $\{j, j+1, \dots, j+N(1)\}$ contains a subset of the form $j \in \{J+k(N+1), \dots, J+(k+1)(N+1)-1\}$ on which it is guaranteed that one of the j's satisfies $|G^1_{j,p_j}(x)| \geq \alpha$. □

We now outline the approach for the two-dimensional case, which serves as the base case for an induction on the dimension. The key idea is to ensure that one coordinate satisfies $|G^1_{j,p_1}(x_i)| > \alpha$ and to apply Lemma 3.6 to the other coordinate.

Lemma 3.7. *For the same α as in Lemma 3.6, the following property holds: for every integer $0 \leq J \leq N(d)-N(2)$ and $\overline{p_J} = (p_{J,1}, p_{J,2}) \in \mathbb{Z}^2$, there is a sequence $(\overline{p_j} = (p_{j,1}, p_{j,2}))_{j \geq J}$ such that for every $x \in \mathbb{R}^2$, for every $j \in \{J, ..., N(d)-N(2)\}$, there is an integer $\ell \in \{0, ..., N(2)\}$ with $|G^2_{j+\ell,\overline{p_{j+\ell}}}(x)| \geq \alpha^2$.*

Proof. We work on $I_2 = [2^{-J}p_{J,1}, 2^{-J}(p_{J,1}+K_\psi)] \times [2^{-J}p_{J,2}, 2^{-J}(p_{J,2}+K_\psi)]$ and the results are extended on $\mathbb{R}^2$ by periodicity. Fix $\overline{p_J} = (p_{J,1}, p_{J,2})$. We partition I_2 into the following $(N+1)^2$ disjoint subsets defined as follows: for $(m_1, m_2) \in \{0, 1, 2, ..., N\}^2$,

$$S_{m_1,m_2} = S_{m_1} \times S_{m_2},$$

where the notation (3.8) is used again.

• **Case $(m_1, m_2) = (0, 0)$:** we apply the argument (i) of the proof of Lemma 3.6 to each coordinate to get that if $x = (x_1, x_2) \in S_{0,0}$, $|G^2_{J,(p_{J,1},p_{J,2})}(x_1, x_2)| = |G_{J,p_{J,1}}(x_1) G_{J,p_{J,2}}(x_2)| \geq \alpha^2$.

• **Case $m_1 \neq 0$ and $m_2 \neq 0$:** we call $\ell = m_1 + (m_2-1)N$. We apply Lemma 3.6 to each coordinate $i \in \{1, 2\}$ with $G^1_{J,p_{J,i}}$ and $j = J+\ell$ to obtain two integers $p_{J+\ell,i}$, $i = 1, 2$, such that if $x = (x_1, x_2) \in S_{m_1,m_2}$, then $|G^2_{J+\ell,(p_{J+\ell,1},p_{J+\ell,2})}(x_1, x_2)| = |G^1_{J,p_{J+\ell,1}}(x_1) G^1_{J,p_{J+\ell,2}}(x_2)| \geq \alpha^2$.

This requires N^2 generations j to treat all situations in this case, and we set $\overline{p_{J+\ell}} = (p_{J+\ell,1}, p_{J+\ell,2})$ for $\ell \in \{1, ..., N^2\}$.

• **Case $m_1 \neq 0$ and $m_2 = 0$:** We start with $m_1 = 1$. Up to now, the $\overline{p_j}$ are found for $J \leq j \leq J+N^2$.

For $1 \leq \ell \leq N(1)$, choose $p_{J+N^2+\ell,1}$ such that (3.4) holds true for $j = J+N^2+\ell$ and z_{m_1}. The same argument as in item (ii) in the proof of Lemma 3.6 gives that for $x_1 \in B(2^{-J}(z_1+p_J), \omega)$, $|G^1_{j+N^2+\ell,p_{j+N^2+\ell,1}}(x_1)| \geq \eta/2$ for all $j \in \{j = J+N^2+1, j+N^2+N(1)\}$. This treats the first coordinate, i.e. we ensure that $G^1_{j+N^2+\ell,p_{j+N^2+\ell,1}}(x_1)$ is not zero and uniformly bounded by below.

To deal with the second coordinate, we apply Lemma 3.6 to $G^1_{J,p_{J,2}}$ with $j = J+N^2+1$: for every $x_2 \in [0, K_\psi]$, there exist integers $(p_{j+\ell,2})_{\ell\in\{0,\dots,N(1)\}}$ such that $|G^1_{j+\ell,p_{j+\ell,2}}(x_2)| \geq \alpha$ for some $\ell \in \{0, ..., N(1)\}$.

So, setting $\overline{p_j} = (p_{j,1}, p_{j,2})$ with the above values and $j \in \{J+N^2+1, ..., J+N^2+N(1)\}$, we ensured that for every $x \in S_{1,m_2}$, for at least one of the integers $j \in \{J + N^2 + 1, ..., J + N^2 + N(1)\}$ one has $|G^2_{j,\overline{p_j}}(x)| \geq \alpha^2$.

We apply the same method N times to treat all situations $S_{m_1,0}$, $m_1 \in \{1, ..., N\}$. This requires $N \cdot N(1)$ generations.

• **Case $m_1 = 0$ and $m_2 \neq 0$:** This situation is symmetric to the first one, and requires again to fix the values of $\overline{p_j}$ on $N \cdot N(1)$ generations.

Gathering the above, one needs the generations $J \leq j \leq J + N^2 + 2N \cdot N(1)$ to ensure that for every $x \in [0, K_\psi 2^{-J}]^2$, $|G^2_{j,\overline{p_j}}(x)| \geq \alpha^2$ for (at least) one of them.

The argument then runs similarly as in Lemma 3.6: assume that $\overline{p_j}$ is built for all $j \in \{J, ..., J + k(N^2 + 4N \cdot N(1)) - 1\}$, for some $k \geq 0$.

We then fix $\overline{p_{J+k(N^2+4N\cdot N(1))}} = (0, ..., 0)$, and apply to $G_{J+k(N^2+4N\cdot N(1)),(0,\dots,0)}$ the above argument to get the values of $\overline{p_j}$ for $j \in \{J + k(N^2 + 4N \cdot N(1)), ..., J + (k + 1)(N^2 + 4N \cdot N(1)) - 1\}$.

We complete the sequence $\overline{p}_j$ until j reaches $N(d)$. This ensures that $G^2_{j,\overline{p_j}}(x)$ is regularly non-zero (at most once every $2(2N^2 + 4N \cdot N(1))$ integers j).

Recalling that $N(2)$ was fixed in (3.6), observe that

$$N(2) = 2(N(1) + 1)^2 2^{N(1)} \geq 2(2N^2 + 4N \cdot N(1)),$$

Lemma 3.7 is proved. □

We are now ready for the full induction.

Lemma 3.8. *For the same α as in Lemma 3.6, the following property holds for every $d' \in \{1, 2, \dots, d\}$: for every integer $0 \leq J \leq N(d) - N(d')$ and $\overline{p_J} = (p_{J,1}, \dots, p_{J,d'}) \in \mathbb{Z}^{d'}$, there exists a sequence $(\overline{p_j} = (p_{j,1}, \dots, p_{j,d'}))_{j\geq J}$ such that for every $x \in \mathbb{R}^{d'}$ and every $j \in \{J, \dots, N(d) - N(d')\}$, there is an integer $\ell \in \{0, \dots, N(d')\}$ with $|G^{d'}_{j+\ell,\overline{p_{j+\ell}}}(x)| \geq \alpha^{d'}$.*

Observe in particular that for $d' = d$, Lemma 3.8 applies only to $J = j = 0$. The arguments are essentially those of Lemma 3.6, extended to higher dimensions.

Proof. Assume that the conclusion holds for all $1 \leq d'' < d'$.

We work on $I_{d'} = \prod_{i=1}^{d'} [2^{-J} p_{J,i}, 2^{-J}(p_{J,i} + K_\psi)]$, and the results are extended by periodicity. Fix $\overline{p_J} = (p_{J,1}, \dots, p_{J,d'})$. The set $I_{d'}$ is partitioned into the $(N + 1)^{d'}$ disjoint subsets

$$S_{m_1,\dots,m_{d'}} = S_{m_1} \times \cdots \times S_{m_{d'}},$$

where $(m_1, \dots, m_{d'}) \in \{0, 1, 2, \dots, N\}^{d'}$.

• **Case** $(m_1,\dots,m_{d'}) = (0,\dots,0)$**:** We apply item (i) of the proof of Lemma 3.6 to each coordinate. If $x = (x_1,\dots,x_{d'}) \in S_{0,\dots,0}$, then

$$|G^{d'}_{J,(p_{J,1},\dots,p_{J,d'})}(x_1,\dots,x_{d'})| = |G^1_{J,p_{J,1}}(x_1)\cdots G^1_{J,p_{J,d'}}(x_{d'})| \geq \alpha^{d'}.$$

• **Case** $m_i \neq 0$ **for all** $i \in \{1,\dots,d'\}$**:** As in Lemma 3.7, for each d'-uplet $(m_1,\dots,m_{d'})$, we apply Lemma 3.6 to each coordinate $i \in \{1,\dots,d'\}$ with $G^1_{J,p_{J,i}}$ and $j = J+\ell$, where $\ell = \sum_{i=1}^{d'}(m_i-1)N^{i-1} - 1$, to obtain d' integers $p_{J+\ell,i}$, $i = 1,\dots,d'$, such that if $x = (x_1,\dots,x_{d'}) \in S_{m_1,\dots,m_{d'}}$, then

$$|G^{d'}_{J+\ell,(p_{J,1},\dots,p_{J,d'})}(x_1,\dots,x_{d'})| = |G^1_{J,p_{J,1}}(x_1)\cdots G^1_{J,p_{J,d'}}(x_{d'})| \geq \alpha^{d'}.$$

This requires $N^{d'}$ generations j.

• **There exists at least one non-zero index** m_i **and** $(m_1,\dots,m_{d'}) \neq (0,\dots,0)$**:** There are $(N+1)^{d'} - N^{d'} - 1$ such cases. Assume that all values of $\overline{p_j}$ have been constructed up to generation $\widetilde{J}$. We explain how to handle one of these cases $(m_1,\dots,m_{d'})$.

The idea is first to fix the integers $p_{j,i}$ for those indices i such that $m_i \neq 0$ using Lemma 3.6, and then to treat the other indices via the induction hypothesis.

Let $i_1,\dots,i_k$ be the indices i for which $m_i \neq 0$. For $1 \leq \ell \leq 2N(d'-k)$ and all $n \in \{1,\dots,k\}$, choose $p_{\widetilde{J}+\ell,i_n}$ such that (3.4) holds for $j = \widetilde{J}+\ell$, $G^1_{\widetilde{J}+\ell,p_{\widetilde{J}+\ell,i_n}}$, and $z_{m_{i_n}}$. As in Lemma 3.7, for each $x_{i_n} \in B(2^{-J}(z_{i_n}+p_{J,i_n}),\omega)$, we have $|G^1_{\widetilde{J}+\ell,p_{\widetilde{J}+\ell,i_n}}(x_{i_n})| \geq \eta/2$. Thus, for all such integers $j \in \{\widetilde{J}+1,\dots,\widetilde{J}+2N(d'-k)\}$,

$$\prod_{i\in\{i_1,\dots,i_k\}} |G^1_{j,p_{j,i}}(x_i)| \geq \left(\frac{\eta}{2}\right)^k.$$

To handle the other coordinates $i'_1,\dots,i'_{d'-k}$, we apply the induction hypothesis to $\prod_{i\in\{i'_1,\dots,i'_{d'-k}\}} G^1_{\widetilde{J},p_{\widetilde{J},i}}$ with $j = \widetilde{J}+1$: there exists a finite sequence of multi-integers $(p_{\widetilde{J}+\ell,i'_1},\dots,p_{\widetilde{J}+\ell,i'_{d'-k}})_{\ell\in\{0,\dots,N(d'-k)\}}$ such that for every

$$x' = (x'_{i'_1},\dots,x'_{i'_{d'-k}}) \in \widetilde{I} := \prod_{i\in\{i'_1,\dots,i'_{d'-k}\}} [2^{-J}p_{J,i}, 2^{-J}(p_{J,i}+K_\psi)],$$

we have

$$\prod_{i\in\{i'_1,\dots,i'_{d'-k}\}} |G^1_{\widetilde{J}+\ell,p_{j,i}}(x'_i)| \geq \alpha^{d'-k},$$

for some $\ell \in \{0,\dots,N(d'-k)\}$.

Thus, setting $\overline{p_j} = (p_{j,1},\dots,p_{j,d'})$ with the above values and $j \in \{\widetilde{J}+1,\dots,\widetilde{J}+N(d'-k)\}$, we ensure that for every $x \in S_{m_1,\dots,m_{d'}}$, one of the functions satisfies $|G^{d'}_{j,\overline{p_j}}(x)| \geq \left(\frac{\eta}{2}\right)^k \alpha^{d'-k} \geq \alpha^{d'}$ for at least one $j \in \{\widetilde{J}+1,\dots,\widetilde{J}+N(d'-k)\}$.

We apply the same method $(N+1)^{d'} - N^{d'} - 1$ times to address all situations, each requiring at most $N(d'-1)$ values $\overline{p_j}$ to be fixed.

At this step, we have fixed a number, say $\widetilde{N}$, of values of $\overline{p_j}$. We note that (3.6) ensures

that

$$N(d') = 2N(d'-1)^{d'}2^{N(d'-1)}N^{d'} \geq 2(N^{d'} + N(d'-1)(N+1)^{d'}) \geq 2\widetilde{N}.$$

Finally, we complete the sequence $\overline{p}_j$ similarly until j reaches $N(d)$.

We complete the sequence $\overline{p_j}$ as follows: assume that for all $j \in \{J, \dots, J + kN(d')\}$ for some $k \geq 0$. The above construction provides a method to fix $\overline{p_j}$ over the next $N(d')$ integers, i.e., for $j \in \{J + kN(d') + 1, ..., J + (k+1)N(d')\}$.

The construction ensures that Lemma 3.8 holds for dimension d', completing the induction. □

We are now in a position to complete the proof of Theorem 3.2 and to construct the sequence $\overline{p_j}$.

• **The case $J = 0$:** This is precisely Lemma 3.8 in dimension d applied to $G^d_{(0,\dots,0)}$. This gives the values of the multi-integers $\overline{p_j}$ for $j = 1, \dots, N(d)$.

• **The case $J = kN(d)$ with $k \geq 1$:** For every $j \in \{J+1, \dots, J+N(d)\}$, set $\overline{p_j} = \overline{p_{j-J}}$.

Let $x = (x_1, \dots, x_d) \in [0, 2^{-J}K_\psi]^d$. We apply Lemma 3.8 to $G^d_{0,\overline{p_J}}$ at the point $x' = (2^J x_1, \dots, 2^J x_d)$. There exists $j_{x'} \in \{0, \dots, N(d)\}$ such that $|G^d_{j_{x'},\overline{p_{j_{x'}}}}(x')| \geq \alpha^d$. But $G^d_{j_{x'},\overline{p_{j_{x'}}}}(x') = G^d_{j+j_{x'},\overline{p_{j_{x'}}}}(x)$, hence the result for $x = (x_1, \dots, x_d) \in [0, 2^{-j}K_\psi]^d$.

The $2^{-J}K_\psi$-periodicity of the functions $G^d_{J+\ell}$ allows us to extend the results to all $x \in \mathbb{R}^d$.

This concludes the proof of Theorem 3.2.

3.3 Numerical proof of (R)

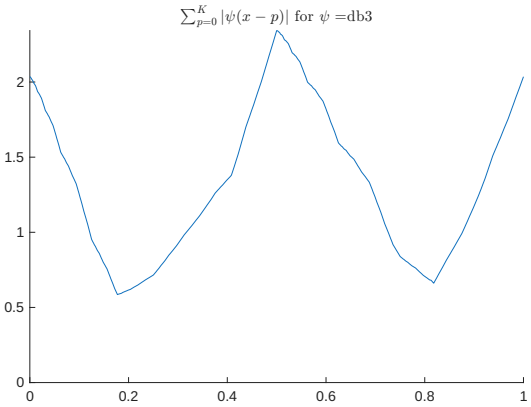
We plot the graph of the function S (see Definition 3.1) for several Daubechies wavelets, specifically, the graph of S associated with $\psi = \text{db3}$ and $\psi = \text{db7}$ are given in Figure 3.1, and that of $\psi = \text{db45}$ is shown in Figure 3.2. The plots were generated using the Wavelet Toolbox of MATLAB, and they clearly illustrate the non-vanishing property of S, provided that the errors in estimating ψ are controlled. Let us now discuss a few observations regarding these results.

Let ψ denote the Daubechies wavelet of order p, and φ its associated scaling function. The relation between ψ and φ is

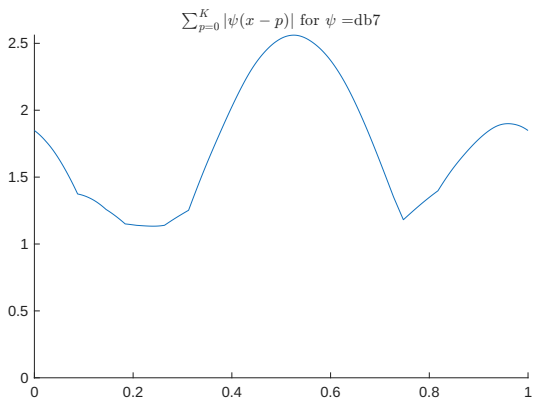
$$\begin{aligned} \varphi(x) &= \sqrt{2}\sum_{k\in\mathbb{Z}} h_k\, \varphi(2x-k) \\ \psi(x) &= \sqrt{2}\sum_{k\in\mathbb{Z}} g_k\, \varphi(2x-k), \end{aligned}$$

where $\{h_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are the low-pass and high-pass filters. The scaling function is computed via the cascade iteration

$$\varphi_{n+1} = T\varphi_n, \qquad (Tf)(x) = \sqrt{2}\sum_k h_k f(2x-k).$$



(a) 'db3' $\in C^1(\mathbb{R})$ with 3 vanishing moments



(b) 'db7' $\in C^2(\mathbb{R})$ with 7 vanishing moments

Figure 3.1: Daubechies wavelets 'db3' and 'db7' verify (R).

An approximation of the wavelet, used by the standard algorithms, is given by

$$\psi_n(x) = \sqrt{2} \sum_k g_k \, \varphi_n(2x - k).$$

The errors made when replacing ψ by ψ_n are of two types: the truncation errors and the roundoff errors.

Concerning the truncation error, it is known [46, 49] that if $\varphi \in C^s(\mathbb{R})$, then

$$\|\varphi - \varphi_n\|_{C^r} \leq C 2^{-ns}, \quad \text{and so} \quad \|\psi - \psi_n\|_{C^r} \leq C 2^{-ns}.$$

The constant C is determined by the initial regularity of φ_0 used in the cascade algorithm and remains bounded by a small, controlled value.

Furthermore, floating-point perturbations inherent in computational implementations, affecting both the estimation of filter coefficients and all subsequent calculations, are on the order of 10^{-16}. Even after 15 iterations, standard stability analysis of stationary subdivision schemes demonstrates that the cumulative error remains at most on the order of 10^{-14}.

Collectively, provided that the wavelet belongs to C^2, the error after $n = 15$ iterations of the cascade algorithm to estimate φ and ψ is less than 10^{-9}. This result ensures the confidence provided by Figures 3.1 and 3.2 regarding the non-vanishing property of S for Daubechies wavelets of reasonable order.

It is also worth noting that as the number p of vanishing moments of the Daubechies wavelet ψ increases, the trade-off between the global regularity of ψ and the computational cost due to the larger filter banks becomes more challenging. Consequently, increasing the number of vanishing moments degrades the conditioning and reduces the number of reliable digits. Nevertheless, even for relatively high orders, the cascade algorithm in double precision remains stable and yields accurate approximations suitable for standard numerical applications. As illustrated in Figure 3.2, the estimated function S (obtained after 12 iterations of the cascade algorithm) is sufficiently bounded away from zero, ensuring its strict positivity.

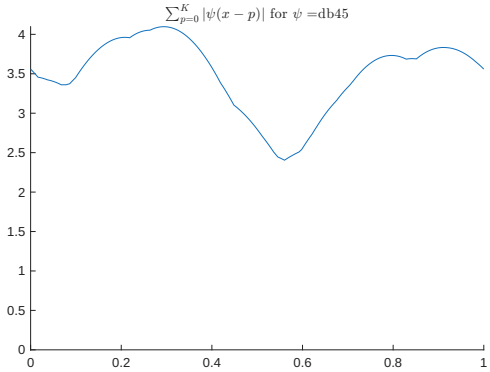


Figure 3.2: Function S associated with 'db45': it is strictly positive.

Chapter 4

Inhomogeneous Besov Spaces

4.1 Introduction

The interest in multifractal properties dates back to the 1980s, when physicists sought to quantify the local variations in the velocity of a turbulent fluid. Indeed, turbulent flows are not spatially homogeneous, and the pointwise regularity of the velocity seems to differ widely from point to point. The relationship between multifractality and the global scaling properties of the velocity was first discovered by U. Frisch and G. Parisi [66]. Since then, multifractal notions have proved to be useful and relevant in various mathematical domains: geometric measure theory, real analysis [42, 86, 160], dynamical systems and ergodic theory, diophantine approximation [59, 161, 162, 163], to name a few.

This chapter focuses on the multifractal analysis of functions, which consists in describing the pointwise behaviours of functions $f : \mathbb{R}^d \to \mathbb{R}$, computing their singularity spectrum σ_f, and, when possible, relating σ_f with the scaling function associated with f. Given $x_0 \in \mathbb{R}^d$, $H \in \mathbb{R}_+$, a function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $\mathcal{C}^H(x_0)$ if there exist a polynomial P_{x_0} of degree less than $\lfloor H \rfloor$, a constant $C > 0$, and a neighbourhood V of x_0 such that

$$\forall x \in V, \quad |f(x) - P_{x_0}(x - x_0)| \leq C|x - x_0|^H. \tag{4.1}$$

The *pointwise Hölder exponent* of $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ at x_0 is then

$$h_f(x_0) = \sup\{H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0)\}. \tag{4.2}$$

The level sets of $x \mapsto h_f(x)$ are $E_f(h) := \{x \in \mathbb{R}^d : h_f(x) = h\}$, and the associated *singularity* (or *multifractal*) *spectrum* σ_f of f is the mapping

$$\sigma_f : h \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H(E_f(h)) \in [0, d] \cup \{-\infty\}. \tag{4.3}$$

Here, $\dim_H$ stands for the Hausdorff dimension with $\dim_H (() \emptyset) = -\infty$ (note that $E_f(h) = \emptyset$ for $h < 0$). This singularity spectrum describes the size of the level sets $E_f(h)$ of the singularities of f in terms of Hausdorff dimension and provides a geometrical description of their distribution.

Results on the pointwise regularity of functions in classical spaces such as Bessel po-

tential spaces $H^{s,p}(\mathbb{R}^d)$, Sobolev-Slobodeckij $W^{s,p}(\mathbb{R}^d)$ or Besov $B^s_{p,q}(\mathbb{R}^d)$ spaces have been obtained in [9, 63, 87, 90, 98]. They are of two types: first, the identification of a universal upper bound for the singularity spectrum of all function in a given space, and second, when possible, the computation of the explicit value for the spectrum of a generic set of functions in the sense of prevalence or Baire categories.

Let us emphasize that for large classes of functions f (this holds similarly for Borel measures), the singularity spectrum of f also satisfies a so-called *multifractal formalism*, meaning that σ_f coincides with the Legendre transform τ_f^* of a scaling function $\tau_f : \mathbb{R} \to \mathbb{R}$ associated with f, i.e.

$$\sigma_f(h) = \tau_f^*(h) := \inf_{q\in\mathbb{R}} hq - \tau_f(q). \tag{4.4}$$

The scaling function τ_f, defined using wavelets (see definitions below and [92, 95]), describes the statistical scaling properties of f. This relationship is key, since when properly defined, τ_f is numerically estimable on real data, see [95]. Such estimation algorithms have generated breakthrough methods in signal and image processing [175, 176] and other scientific fields [99, 177]. When (4.4) holds, one says that the *multifractal formalism* holds for f.

A key feature of typical functions in the classical spaces $W^{s,p}(\mathbb{R}^d)$ and $B^s_{p,q}(\mathbb{R}^d)$ is that their singularity spectrum is always affine increasing. This is an apparent contradistinction with the fact that the spectra estimated on real data through the multifractal formalism always have a strictly concave shape, with an increasing *and* a decreasing part.

Jaffard [90] showed that given an *admissible* concave function $\tau : \mathbb{R}^+ \to \mathbb{R}$, Baire typical functions in the intersection of Besov spaces $\bigcap_{\varepsilon>0,p>0} B^{(\tau(p)-\varepsilon)/p}_{p,q,loc}(\mathbb{R}^d)$ have concave increasing spectra. Fraysse [62] obtained that prevalent functions in such spaces possess the same spectrum.

In order to consider non-affine singularity spectra, one can look at the existing natural generalizations of the classical Sobolev and Besov spaces, the weighted Sobolev and Besov spaces, which have various possible definitions.

For $h \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$, consider the finite difference operator $\Delta_h f : x \in \mathbb{R}^d \mapsto f(x+h) - f(x)$. Then, for $n \geq 2$, set $\Delta_h^n f = \Delta_h(\Delta_h^{n-1} f)$.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a measurable function, positive almost everywhere, called a weight. Under some hypotheses on the behaviour of ρ given in [7], weighted Besov spaces $B^\rho_{p,q}(\mathbb{R}^d)$ consist of functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\int_{|h|\leq 1} \rho(|h|)^{-q} \, \|\Delta_h^n f\|^q_{L^p(\mathbb{R}^d)} \, \frac{dh}{|h|^d} < \infty.$$

A function f belonging to such spaces presents a homogeneous pointwise behaviour as the weight in the integral depends only $|h|$. These spaces have a generalised version called *Besov space of generalised smoothness* $B^{\gamma,N}_{p,q}$, with γ, N some sequences, studied by Farkas-Leopold [58]. Moura [138] and Loosveldt-Nicolay [124] gave alternative definition. Kreit-Nicolay linked them to generalized Hölder-Zygmund $\Lambda^\gamma_{\sigma,N}$ spaces [114, 115]. Such Besov spaces of generalised smoothness have been linked to traces of functions in Besov spaces on fractal sets by Caetano-

Haroske [34] and studied by Kleyntssens-Nicolay [111].

A second version of weighted Besov spaces exists where the weight does not dictate the scale behaviour of the oscillations of the function f but the local behaviour of f itself and can be linked to standard Besov spaces, see results from Edmunds, Haroske and Triebel [54, 76, 167]. It is defined as follows: for a weight $\rho \in L^1_{\mathrm{loc}}(\mathbb{R})$, a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $B^s_{p,q}(\mathbb{R}^d, \rho)$ if

$$\int_{|h|\leq 1} |h|^{-sq} \, \|\rho \, \Delta_h^n f\|^q_{L^p(\mathbb{R}^d)} \, \frac{dh}{|h|^d} < \infty.$$

Even if the multifractal behaviour of functions in such spaces has not been studied yet, they have an explicit characterisation through wavelets given by Triebel [171, Chapter 6] that can be linked to the spaces introduced hereafter.

Another direction is the variable exponent Lebesgue and Sobolev spaces, which are defined through the weighted Orlicz spaces [52, 67] as follows. Let $p : \mathbb{R}^d \to [1, \infty)$ be a measurable bounded function, called a variable exponent. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^d)$ consist of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} |f(x)|^{p(x)} \, dx < +\infty.$$

The *variable exponent Sobolev space* $W^{k,p(\cdot)}(\mathbb{R}^d)$, $k \in \mathbb{N}$, consist of all functions of $L^{p(\cdot)}(\mathbb{R}^d)$ satisfying that $f^{(k)} \in L^{p(\cdot)}(\mathbb{R}^d)$. The variable exponent p is usually at least continuous, the usual hypothesis being log-Hölder continuous. While typical functions in $W^{k,p}(\mathbb{R}^d)$ exhibit affine spectra with no decreasing parts [90], the spaces $W^{k,p(\cdot)}(\mathbb{R}^d)$ have not been studied yet.

This motivated the introduction by Barral and Seuret [19] of *inhomogeneous* Besov spaces, in which Baire typical functions may have as singularity spectrum any admissible continuous concave curve, prescribed in advance. For a Hölder set function μ (which can be thought as a measure for the moment, see Definition 4.1 below), the *inhomogeneous* Besov space $B^\mu_{p,q}(\mathbb{R}^d)$ consists of all functions f in $L^p(\mathbb{R}^d)$ such that

$$\int_{|h|\leq 1} |h|^{-dq/p} \left\| \frac{\Delta_h^n f}{\mu\left(B(\cdot + nh/2, nh/2)\right)} \right\|^q_{L^p(\mathbb{R}^d)} \frac{dh}{|h|^d} < +\infty \tag{4.5}$$

These spaces are not spatially homogeneous in the sense that, in the above integral, the oscillations $\Delta_h^n f(x)$ of f are locally compared to the μ-mass of a small ball of radius h close to x, and this mass may strongly depend on x. They provide a natural framework in which typical functions have multifractal behaviours similar to those observed in real data [19].

The present chapter has two objectives. First, we propose a generalised version of Sobolev-Slobodeckij spaces in an inhomogeneous environment governed by almost doubling set functions, and relate them to the already defined standard and inhomogeneous function spaces. Second, we obtain prevalent generic results for the multifractal properties of functions in inhomogeneous Sobolev and Besov spaces.

We start by some definitions. The diameter of a set E is denoted by $|E|$ and its d-dimensional Lebesgue measure is $\mathcal{L}^d(E)$.

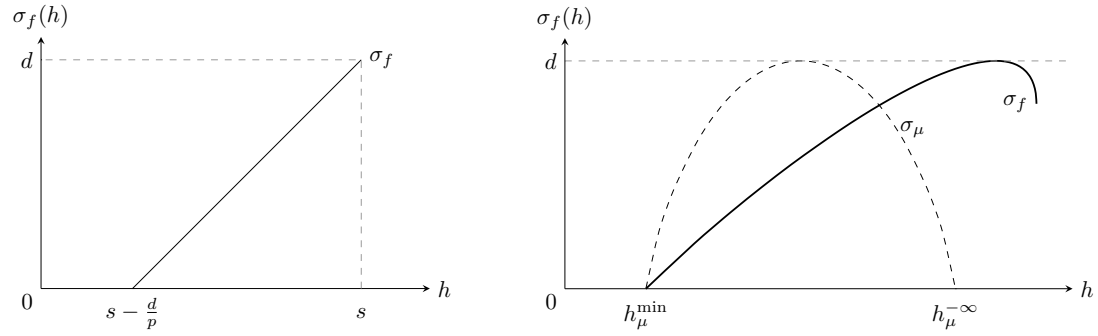


Figure 4.1: This figure illustrates the typical singularity spectrum in standard Besov spaces, as opposed to those in inhomogeneous Besov spaces, to motivate their consideration. **Left:** Singularity spectrum of a typical function $f \in B^s_{p,q}(\mathbb{R}^d)$. **Right:** Singularity spectrum of a typical function $f \in B^\mu_{1,q}(\mathbb{R}^d)$. The dashed graph represents the (initial) singularity spectrum of μ. When $p = \infty$ and f is typical in $B^\mu_{\infty,q}(\mathbb{R}^d)$, $\sigma_f = \sigma_\mu$.

Definition 4.1. *The set of Hölder set functions is*

$$\mathcal{H}(\mathbb{R}^d) = \{\mu : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{+\infty\} : \exists C, s > 0, \forall E \in \mathcal{B}(\mathbb{R}^d), \mu(E) \leq C|E|^s\}.$$

When $\mu(E) \leq C|E|^s$ for all E with $0 \leq |E| \leq 1$, μ is said to be s-Hölder.

The topological support of $\mu \in \mathcal{H}(\mathbb{R}^d)$ is $\mathrm{supp}(\mu) = \{x \in [0,1]^d : \mu(B(x,r)) > 0, \forall\, r > 0\}$. A set function μ is fully supported on a Borel set A when $\mathrm{supp}(\mu) = A$. By a small abuse of notations we will denote

$$\mathcal{H}([0,1]^d) = \{\mu \in \mathcal{H}(\mathbb{R}^d) : \mathrm{supp}(\mu) = [0,1]^d\}. \tag{4.6}$$

For $\mu \in \mathcal{H}(\mathbb{R}^d)$, $s > 0$, and $E \subset \mathbb{R}^d$, define

$$\mu^s(E) = \mu(E)^s \quad \text{and} \quad \mu^{(+s)}(E) = \mu(E)|E|^s,$$

and if μ is s_0-Hölder, define for all $s \in (0, s_0)$,

$$\mu^{(-s)}(E) = \begin{cases} 0 & \text{if } |E| = 0, \\ \mu(E)|E|^{-s} & \text{if } 0 < |E| < \infty. \end{cases}$$

Observe that μ^s, $\mu^{(+s)}$ and $\mu^{(-s)}$, as defined above, still belongs to $\mathcal{H}(\mathbb{R}^d)$.

In the second part of the chapter, we will need the additional assumptions that the set functions $\mu \in \mathcal{H}(\mathbb{R}^d)$ are decreasing with respect to the embedding relationship, i.e. they are *capacities*. This is natural since in many situations, the set functions, that are considered, are measures or powers of measures. Hence, we define the sets of capacities

$$\mathcal{C}(\mathbb{R}^d) = \{\mu \in \mathcal{H}(\mathbb{R}^d) : \forall E, F \in \mathcal{B}(\mathbb{R}^d), E \subset F \Rightarrow \mu(E) \leq \mu(F)\}$$
$$\mathcal{C}([0,1]^d) = \mathcal{H}([0,1]^d) \cap \mathcal{C}(\mathbb{R}^d).$$

Definition 4.2. *A set function $\mu \in \mathcal{H}(\mathbb{R}^d)$ satisfies property (P) if there exist $C, s_1, s_2 > 0$ such that*

$$\text{for all } r > 0, \text{ for all } x \in \mathrm{supp}(\mu), \quad C^{-1} r^{s_2} \leq \mu(B(x,r)) \leq C r^{s_1}. \tag{4.7}$$

In particular, such a μ is s_1-Hölder. We will only consider set functions satisfying property (P) on $\mathbb{R}^d$ (or $[0,1]^d$), implying that these functions will be fully supported on $\mathbb{R}^d$ (or $[0,1]^d$). Recall that a finite Borel measure μ is said to be doubling if for some constant $C_\mu \geq 1$, every $x \in \mathbb{R}^d$, $r > 0$,

$$\mu(B(x,r)) \leq \mu(B(x,2r)) \leq C_\mu\, \mu(B(x,r)).$$

Since we work with set functions (for which one does not have in full generality $\mu(B(x,r)) \leq \mu(B(x,2r))$), and not necessarily with measures, we need to adapt this notion to our situation.

Definition 4.3. *Let Φ be the set of non increasing functions $\phi : (0,1] \to \mathbb{R}_+$ such that $\lim_{r\to 0^+} \frac{\phi(r)}{\log(r)} = 0$.*
A set function $\mu \in \mathcal{H}(\mathbb{R}^d)$ is almost doubling *on a set A when there exists $\phi \in \Phi$ such that for every $x \in A$, $r \in (0,1]$, $\rho \in [r,2r]$ and $x' \in B(x,r)$,*

$$e^{-\phi(r)}\mu(B(x',r)) \leq \mu(B(x,\rho)) \leq e^{\phi(r)}\mu(B(x',r)). \tag{4.8}$$

The set function μ is doubling when (4.8) holds with a constant function ϕ.

Intuitively, (4.8) implies that slightly moving the centre and the radius of a ball keep its μ-mass controlled. A doubling measure satisfies (4.8) with a constant function ϕ. Examples of doubling set functions include (1-periodic) Gibbs measures μ [30, 47, 147] and their powers $\mu^s(E) := \mu(E)^s$ for $s > 0$.

We will refer to $\mu \in \mathcal{H}(\mathbb{R}^d)$ as an *environment*, and we now describe how environments are used to define inhomogeneous Sobolev and Besov spaces.

Recall that for an integer k and $p \in [1,+\infty]$, the classical Sobolev space $W^{k,p}(\mathbb{R}^d)$ is defined as:

$$W^{k,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : \forall |\alpha| \leq k,\ D^\alpha f \in L^p(\mathbb{R}^d)\}.$$

It is a Banach space with the norm $\|f\|_{W^{k,p}(\mathbb{R}^d)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\mathbb{R}^d)}$. When $s \in \mathbb{R}_+\backslash\mathbb{N}$, and $p \in [1,+\infty]$, the Sobolev-Slobodeckij space $W^{s,p}(\mathbb{R}^d)$ is the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} = \|f\|_{W^{\lfloor s\rfloor,p}(\mathbb{R}^d)} + \sup_{|\alpha|=\lfloor s\rfloor} |D^\alpha f|_{W^{s-\lfloor s\rfloor,p}(\mathbb{R}^d)} < +\infty$$

where, when $\theta \in (0,1)$,

$$|f|_{W^{\theta,p}(\mathbb{R}^d)} = \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x+h)-f(x)|^p}{|h|^{\theta p+d}} dxdh\right)^{1/p}. \tag{4.9}$$

We now introduce new inhomogeneous Sobolev spaces in a μ-environment by modifying the above formula in order to obtain a class of functions with a singularity spectrum not necessarily affine. To do so, we interpret in (4.9) the term $|h|^\theta$ in the denominator as, up to a constant, the power of the d-dimensional Lebesgue measure, denoted by $\mathcal{L}^d$, of the ball $B(x,h)$, and we replace it by the μ-mass of this ball. This introduces spatial inhomogeneity in (4.9), including Sobolev spaces as special case.

Let $B[x,y]$ denote the Euclidean ball of diameter $[x,y]$.

Definition 4.4. *Let $\mu \in \mathcal{H}(\mathbb{R}^d)$ satisfy property (P) with exponents $0 < s_1 < s_2$, and consider $p \geq 1$ as well as an integer $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$.*

A function f belongs to $W^{\mu,p}(\mathbb{R}^d)$ if and only if $\|f\|_{W^{\mu,p}(\mathbb{R}^d)} < +\infty$, where $\|f\|_{W^{\mu,p}(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu,p}(\mathbb{R}^d)}$ and

$$|f|_{W^{\mu,p}(\mathbb{R}^d)} := \left(\iint_{\mathbb{R}^{2d}} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x, x+nh])^p |h|^{2d}} dx dh \right)^{1/p}. \tag{4.10}$$

Consider $s, s_1, s_2 > 0$ such that $s + d/p \notin \mathbb{N}$ and similarly for s_1 and s_2. For $\mu = (\mathcal{L}^d)^{\frac{s}{d} - \frac{1}{p}}$, it follows from the definition that $W^{\mu,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$. When μ satisfies (P) with $0 < s_1 < s_2$, the following embeddings holds

$$W^{s_2 + \frac{d}{p}, p}(\mathbb{R}^d) \hookrightarrow W^{\mu,p}(\mathbb{R}^d) \hookrightarrow W^{s_1 + \frac{d}{p}, p}(\mathbb{R}^d), \tag{4.11}$$

see Remark 4.23. In particular, for an s-Hölder set function $\mu \in \mathcal{H}(\mathbb{R}^d)$, one has $W^{\mu,p}(\mathbb{R}^d) \hookrightarrow W^{s + \frac{d}{p}, p}(\mathbb{R}^d)$.

Our first purpose is to relate these new inhomogeneous Sobolev spaces to the inhomogeneous Besov spaces in a multifractal environment introduced by Barral and Seuret in [19]. To define these spaces, already mentioned in (4.5), they first extended the classical definition of L^p moduli of smoothness.

Definition 4.5. *For every $\mu \in \mathcal{H}(\mathbb{R}^d)$, for every $n \in \mathbb{N}^*$, $h \in \mathbb{R}^d \backslash \{0\}$ and $x \in \operatorname{supp}(\mu)$, set*

$$\Delta_h^{\mu,n} f(x) = \frac{\Delta_h^n f(x)}{\mu(B[x, x+nh])}.$$

For $p \in [1, +\infty]$, for any open set $\Omega \subset \mathbb{R}^d$, the μ-adapted n-th order L^p modulus of smoothness of f is defined for any $t > 0$ by

$$\omega_n^\mu(f, t, \Omega)_p = \sup_{t/2 \leq h \leq t} \left\| \Delta_h^{\mu,n} f \right\|_{L^p(\Omega_{h,n})},$$

where $\Omega_{h,n} = \{x \in \Omega : B[x, x+nh] \subset \Omega\}$.

Observe that it may happen, e.g. when μ has a bounded support, that $\mu(B[x, x+nh]) = 0$ even if $x \in \operatorname{supp}(\mu)$. In this case we write $\Delta_h^{\mu,n} f(x) = 0$. Inhomogeneous Besov spaces in μ-environment are defined in [19] as follows.

Definition 4.6. *Let $\mu \in \mathcal{H}(\mathbb{R}^d)$ with $\operatorname{supp}(\mu) = \mathbb{R}^d$ satisfying (P). Let $1 \leq p, q \leq +\infty$. Let n be an integer such that $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$.*

The Besov space in μ-environment $B^\mu_{p,q}(\mathbb{R}^d)$ is the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B^\mu_{p,q}(\mathbb{R}^d)} < +\infty$ with

$$|f|_{B^\mu_{p,q}(\mathbb{R}^d)} := \left\| (2^{jd/p} \omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p)_{j \in \mathbb{N}} \right\|_{l^q(\mathbb{N})} < +\infty. \tag{4.12}$$

We will also consider the spaces

$$\widetilde{B}^\mu_{p,q}(\mathbb{R}^d) := \bigcap_{0 < \varepsilon < \min(1, s_1)} B^{\mu^{(-\varepsilon)}}_{p,q}(\mathbb{R}^d). \tag{4.13}$$

If $\mathrm{supp}(\mu) = [0,1]^d$, *we use* $\omega_n^\mu(f, 2^{-j}, (0,1)^d)_p$ *in equation* (4.12) *to define the space* $B^\mu_{p,q}([0,1]^d)$ *and* $\widetilde{B}^\mu_{p,q}([0,1]^d)$ *similarly as in* (4.12) *and* (4.13).

Taking $\mu = (\mathcal{L}^d)^{\frac{s}{d}-\frac{1}{p}}$, one sees [19] that $B^\mu_{p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d)$ when $s > d/p$. Our first result concerns embeddings between inhomogeneous Sobolev and inhomogeneous Besov spaces.

Theorem 4.7. *Let* $p \geq 1$. *If* $\mu \in \mathcal{H}(\mathbb{R}^d)$ *with* $\mathrm{supp}(\mu) = \mathbb{R}^d$ *satisfies property* (P) *with exponents* $0 < s_1 < s_2$, *then*

$$B^\mu_{p,p}(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d). \tag{4.14}$$

If in addition μ *is almost doubling, then*

$$B^\mu_{p,p}(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d) \subset \widetilde{B}^\mu_{p,p}(\mathbb{R}^d). \tag{4.15}$$

If μ is doubling, then, for $p \geq 1$, $B^\mu_{p,p}(\mathbb{R}^d) = W^{\mu,p}(\mathbb{R}^d)$ (see Remark 4.27), which is the analog of the standard equality for usual Sobolev-Slobodeckij and Besov spaces. This property and the previous embeddings emphasise that formula (4.10) is indeed natural and generalises in an inhomogeneous way the standard function spaces.

Remark 4.8. One could get similar embeddings with $[0,1]^d$ in (4.15).

Let us now proceed with prevalent results. In [18], Barral and Seuret identified the typical multifractal behaviour, in the Baire categories sense, of functions in $B^\mu_{p,q}(\mathbb{R}^d)$, when μ belongs to a large class of set functions satisfying not only (P) but also an extra condition (P_2), see Definition 4.19. This typical behaviour is intimately related to that of μ, and the value of p. Barral and Seuret used this space to solve the so-called Frisch-Parisi conjecture, which consists in proving the existence of function spaces in which typical functions satisfy the multifractal formalism [18, 19, 87] for any possible admissible spectrum. To achieve this, they prove that the $B^\mu_{p,q}(\mathbb{R}^d)$ spaces admit a wavelet characterization. The second part of this chapter will also use such a characterization to obtain the prevalent multifractal properties in $B^\mu_{\infty,q}(\mathbb{R}^d)$.

Let $L^d = \{0,1\}^d \setminus \{0^d\}$, and consider $\{\Phi, \Psi_\lambda := \Psi^l(2^j \cdot - k)\}_{\lambda=(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^d\times L^d}$ generated by a family of wavelets $(\Phi, \Psi^l)_{l\in L^d}$ (see Section 4.2.2 for details), such that every $f \in L^2(\mathbb{R}^d)$ can be written

$$f \overset{L^2}{=} \sum_{\lambda\in\mathbb{Z}\times\mathbb{Z}^d\times L^d} c_\lambda \Psi_\lambda \quad \text{with} \quad c_\lambda = 2^{dj}\int_{\mathbb{R}^d} f(x)\Psi_\lambda(x)dx. \tag{4.16}$$

For $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, $j \in \mathbb{Z}$, we often identify the triplet $\lambda = (j,k,l)$ with the dyadic cube $\prod_{i=1}^d [k_i 2^{-j}, (k_i+1)2^{-j})$, and in particular we write $\mu(\lambda)$ for $\mu(\prod_{i=1}^d [k_i 2^{-j}, (k_i+1)2^{-j}))$.

Write $\Lambda_j = \{\lambda = (j,k,l) : k \in \mathbb{Z}^d, l \in L^d\}$ for the set of the dyadic cubes of generation j. If the function f has a support in $[0,1]^d$, we will write

$$f = \langle f, \Phi\rangle\ \Phi + \sum_{j\geq 0}\sum_{\lambda\in\Lambda_j} c_\lambda \Psi_\lambda. \tag{4.17}$$

Definition 4.9. *Let $\mu \in \mathcal{H}(\mathbb{R}^d)$ with $\mathrm{supp}(\mu) = \mathbb{R}^d$ satisfying (P) with $0 < s_1 < s_2$ and $p \in [1, +\infty]$. Consider an integer $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$. Consider wavelets $(\Phi, \Psi^l)_{l \in L^d}$ that belong to $C^n(\mathbb{R}^d)$ and have at least n vanishing moments (see Section 4.2.2 for details).*

A function f belongs to $b^{\mu,\Psi}_{p,q}(\mathbb{R}^d)$ if $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q,\Psi} < +\infty$, where

$$|f|_{\mu,p,q,\Psi} = \left\|(\varepsilon_j^{\mu,p})_{j\in\mathbb{N}}\right\|_{l^q(\mathbb{N})} \text{ with } \varepsilon_j^{\mu,p} = \left\| \left(\frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j \times L^d} \right\|_{l^p(\Lambda_j \times L^d)}. \tag{4.18}$$

Then define

$$\widetilde{b}^{\mu,\Psi}_{p,q}(\mathbb{R}^d) = \bigcap_{0<\varepsilon<\min(1,s_1)} b^{\mu^{(-\varepsilon)},\Psi}_{p,q}(\mathbb{R}^d). \tag{4.19}$$

Similarly, when $\mathrm{supp}(\mu) = [0,1]^d$, a function f belongs to $b^{\mu,\Psi}_{p,q}([0,1]^d)$ if $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q,\Psi} < +\infty$, with the convention that $\frac{c_\lambda}{\mu(\lambda)} = 0$ as soon as $\mu(\lambda) = 0$, and $\widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$ is defined by replacing $\mathbb{R}^d$ by $[0,1]^d$ in (4.19).

Remark that $b^{\mu,\Psi}_{p,q}(\mathbb{R}^d) \subset \widetilde{b}^{\mu,\Psi}_{p,q}(\mathbb{R}^d)$ and similarly on $[0,1]^d$. In particular, when $p = q = \infty$, $b^{\mu,\Psi}_{\infty,\infty}(\mathbb{R}^d)$ is the space of those functions f for which

$$C = \sup_{\lambda \in \mathbb{Z}\times\mathbb{Z}^d\times L^d} \frac{|c_\lambda|}{\mu(\lambda)} < +\infty.$$

This can be rewritten as $|c_\lambda| \leq C\,\mu(\lambda)$, so the space $b^{\mu,\Psi}_{\infty,\infty}(\mathbb{R}^d)$ can be understood as a inhomogeneous generalisation of Hölder $C^\alpha(\mathbb{R}^d)$ spaces, which are known to be characterised by the fact that $|c_\lambda| \leq C\,|\lambda|^\alpha$ for every λ. This justifies, once again, the name "environment" used for μ, since the set function may have various local behaviours and the constraint $|c_\lambda| \leq C\,\mu(\lambda)$ is highly dependent on μ and the location of the cube λ.

When $p < \infty$, the space $b^{\mu,\Psi}_{p,\infty}(\mathbb{R}^d)$ can be seen as a inhomogeneous generalisation of Caldéron and Zygmund $T^\alpha_p(\mathbb{R}^d)$ spaces. Let us also mention that the spaces $C^\alpha(\mathbb{R}^d)$ and $T^\alpha_p(\mathbb{R}^d)$ have been generalised by Kreit, Loosveldt and Nicolay [114, 115, 116, 125, 126] with the spaces $C^\gamma(\mathbb{R}^d)$ and $T^\gamma_{p,q}(\mathbb{R}^d)$ with γ a sequence. The space $C^\gamma(\mathbb{R}^d)$ is characterise by $|c_{\lambda_j}| \leq C\gamma_j$ for every $\lambda \in \Lambda_j$ (for a sequence $\gamma = (\gamma_j)$ verifying, for all j, $C^{-1}\gamma_j \leq \gamma_{j+1} \leq C\gamma_j$.)

Remark 4.10. The spaces $\widetilde{b}^{\xi}_{p,q}(\mathbb{R}^d)$ and $\widetilde{b}^{\xi}_{p,q}([0,1]^d)$ are Fréchet space [19]. A translation-invariant complete metric that induces the same topology as the family $\left\{ \|\cdot\|_{b^{\mu^{(-\varepsilon)},\Psi}_{p,q}} := \|\cdot\|_{L^p} + |\cdot|_{\xi^{(-\varepsilon)},p,q,\Psi} \right\}_{0<\varepsilon<\min(1,s_1)}$ can be defined by

$$\forall f, g \in \widetilde{b}^{\mu,\Psi}_{p,q}, \quad d(f,g) = \sum_{n>\max(1,s_1^{-1})} 2^{-n} \frac{\|f-g\|_{b^{\mu^{(-1/n)},\Psi}_{p,q}}}{1 + \|f-g\|_{b^{\mu^{(-1/n)},\Psi}_{p,q}}}. \tag{4.20}$$

The key property, relating the two parts of the present work, is the following: Assume that the set function $\mu \in \mathcal{H}(\mathbb{R}^d)$ is in addition a capacity, i.e. for every Borel sets $E, F \in \mathcal{B}(\mathbb{R}^d)$ such that $E \subset F$, $\mu(E) \leq \mu(F)$. Then, it is proved in [19] that, **provided an extra condition on the capacity μ (condition (P_2) that we give below), $\widetilde{b}^{\mu,\Psi}_{p,q}(\mathbb{R}^d)$ and $\widetilde{B}^{\mu}_{p,q}(\mathbb{R}^d)$ coincide (resp. $\widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$ and $\widetilde{B}^{\mu}_{p,q}([0,1]^d)$)**, and in particular do not depend on the choice of Ψ satisfying the condition of Definition 4.9, see Theorem 4.22 for the

precise statement. So, when working with suitable set functions μ, the prevalent multifractal behaviour in $B^{\mu}_{p,q}(\mathbb{R}^d)$ coincides with that in $b^{\mu}_{p,q}(\mathbb{R}^d)$.

Furthermore, since multifractal properties are local, we will focus on behaviours of functions on the cube $[0,1]^d$, and will consider functions written under the form (4.17). This is not a restriction, since any function f written as in (4.16) can be decomposed as a (countable) sum

$$f = \widetilde{f} + \sum_{\ell \in \mathbb{Z}^d} f_\ell(. + \ell),$$

where $\widetilde{f}$ is regular (corresponding to the low frequency part of f) and f_ℓ has a decomposition of the form (4.17). Then, each function $f_\ell(. + \ell)$ has an interesting behaviour only on the cube $\ell + [0,1]^d$, and the multifractal properties of f will be deduced from the juxtaposition of those of the f_ℓ.

From now on, we focus on set functions $\mu \in \mathcal{H}([0,1]^d)$ and on functions belonging to $b^{\mu,\Psi}_{p,q}([0,1]^d)$ and $\widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$ written as (4.17).

Our goal is to obtain the prevalent multifractal behaviour in $b^{\mu,\Psi}_{\infty,q}(\mathbb{R}^d)$ in larger classes of capacities than in [19]. Recall that the prevalence theory, proposed by Christensen [44] and Hunt, Sauer and York [81] independently, supersedes the notion of Lebesgue for universally measurable sets in any real or complex topological vector space E endowed with its Borel σ-algebra $\mathcal{B}(E)$. All Borel measures ϑ on $(E, \mathcal{B}(E))$ will be automatically completed, i.e. for every set $A \subset E$ and $B \in \mathcal{B}(E)$ such that the symmetric difference $A \Delta B$ is included in some $D \in \mathcal{B}(E)$ with $\vartheta(D) = 0$, one sets $\vartheta(A) := \vartheta(B)$.

Definition 4.11. *A set $A \subset E$ is universally measurable if it is measurable for any (completed) Borel measure on E. A universally measurable set $A \subset E$ is called Haar-null, or shy, if there exists a finite completed measure ϑ that is positive on some compact subset K of E and such that*

$$\text{for every } x \in E, \quad \vartheta(A + x) = 0. \tag{4.21}$$

The measure ϑ used to show that some subset is Haar-null is called transverse. More generally, a set that is included in a Haar-null universally measurable set is also called Haar-null. Finally, the complement in E of a Haar-null subset S, denoted $S^{\complement}$, is called prevalent.

Hunt pioneered the use of prevalence in function spaces [80]. Bayart-Heurteaux [21] and Balka-Darji-Elekes [12] studied the dimension of images of continuous functions. Fraysse-Jaffard-Kahane [63, 64] and Aubry-Maman-Seuret [10] studied, respectively, the prevalent regularity of functions and traces of functions in Besov spaces. Olsen studied the multifractal and L^q-dimension of prevalent measures on compact sets of $\mathbb{R}^d$ [142, 143].

Let us give the last definitions needed to state our second main theorem.

Definition 4.12. *Let $\mu \in \mathcal{H}(\mathbb{R}^d)$. For $x \in \operatorname{supp}(\mu)$, the lower local dimension of μ at x is defined by*

$$\underline{h}_\mu(x) = \liminf_{r \to 0^+} \frac{\log \mu(B(x,r))}{\log(r)}. \tag{4.22}$$

The singularity spectrum of μ is then the Hausdorff dimension of the level sets of the local dimension

$$\sigma_\mu \ : \ h \in \mathbb{R} \mapsto \dim_H(\underline{E}_\mu(h)) \text{ where } \underline{E}_\mu(h) = \{x \in \mathrm{supp}(\mu) \ : \ \underline{h}_\mu(x) = h\}.$$

We also defined $E_\mu(h)$ as the set of points x where the local dimension (4.22) is a limit and equals h.

Finally, the scaling function of $\mu \in \mathcal{H}([0,1]^d)$ is defined by

$$\tau_\mu(q) = \liminf_{j \to +\infty} -\frac{1}{j} \log_2 \sum_{\lambda \in \Lambda_j \cap [0,1]^d} \mu(\lambda)^q. \tag{4.23}$$

The convention $0^q = 0$ is adopted. When $\mathrm{supp}(\mu) = [0,1]^d$, the singularity spectrum of μ is bounded above by the Legendre transform τ_μ^* of τ_μ (recall formula (4.4)) : for every $h \geq 0$, $\sigma_\mu(h) \leq \tau_\mu^*(h)$, see [33, 90, 122].

Definition 4.13. *A set function $\mu \in \mathcal{H}([0,1]^d)$ satisfies the strong multifractal formalism (SMF) if for every $h \geq 0$, $\sigma_\mu(h) = \tau_\mu^*(h) = \dim_H(E_\mu(h))$.*

Our main theorem regarding prevalent multifractal properties holds for environments that are capacities. This additional (decreasing) property is in fact natural in the functional setting. Indeed, looking back to formula (4.5), it is intuitive that if the oscillation of f on an interval I is controlled by $\mu(I)$, then the oscillations on a subinterval $J \subset I$ shall be controlled by $\mu(J) \leq \mu(I)$, since the oscillation on J is necessarily smaller than that on I. **We consider the case** $p = \infty$, the case $p < \infty$ would require a finer study.

Theorem 4.14. *Let $\mu \in \mathcal{C}([0,1]^d)$ be an almost doubling capacity satisfying (P) and verifying the SMF. For all $f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$,*

$$\sigma_f(h) \leq \begin{cases} \sigma_\mu(h) & \text{if } h \leq \tau'_\mu(0) \\ d & \text{if } h > \tau'_\mu(0). \end{cases} \tag{4.24}$$

Also, there is a prevalent set of functions $f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ satisfying $\sigma_f = \sigma_\mu$.

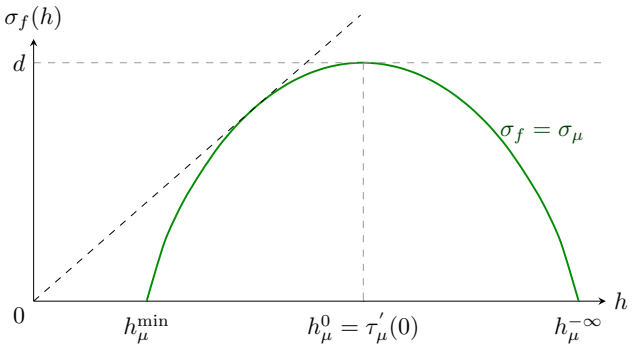


Figure 4.2: Example of singularity spectrum of a prevalent set of functions in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ for a measure μ with known singularity spectrum.

Remark that this result stays true for $f \in b^{\mu,\Psi}_{\infty,q}([0,1]^d) \subset \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$. In other words, the prevalent singularity spectrum in $\widetilde{b}^{\mu,\Psi}_{\infty,q}$ or $b^{\mu,\Psi}_{\infty,q}$ coincides with that of μ. One directly deduces a similar result for $\widetilde{B}^{\mu,\Psi}_{\infty,q}([0,1]^d)$.

Corollary 4.15. *If μ satisfies (P_2) given in Definition 4.19, then Theorem 4.14 holds when replacing $\widetilde{b}^{\mu}_{\infty,q}([0,1]^d)$ by $\widetilde{B}^{\mu}_{\infty,q}([0,1]^d)$ and $W^{\mu,\infty}([0,1]^d)$.*

This simply follows from the fact that, as said before, provided that (P_2) is satisfied, $\widetilde{b}^{\mu}_{\infty,q}([0,1]^d)$ and $\widetilde{B}^{\mu}_{\infty,q}([0,1]^d)$ coincide.

We shall prove Theorem 4.14 for $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$, the proof for $b^{\mu,\Psi}_{\infty,q}([0,1]^d)$ is simpler, we will explain where to adapt our proof to get the simpler case.

In [19], Barral and Seuret obtained the same singularity spectrum as in Theorem 4.14 for Baire typical functions, but only for two large classes of capacities (a class of measure with a multifractal spectrum prescribed in advance and Gibbs capacities defined as power of Gibbs measures). Here we treat a larger class of capacities, and complete this result by confirming that prevalent and Baire typical regularity coincide.

It is certainly interesting to extend Theorem 4.14 to the case $p < +\infty$.

Organisation of the chapter

Section 4.2 gives the necessary definitions and the properties of measures and capacities used to define a *multifractal environment*, then presents the wavelet multifractal formalism and the wavelet leaders formalism developed by Jaffard in [92, 95]. Section 4.3 contains the proof of our first main Theorem 4.7. It also proves that $B^{\mu}_{p,p}(\mathbb{R}^d) = W^{\mu,p}(\mathbb{R}^d)$ for μ doubling. In Section 4.4, the proof of Theorem 4.14 is given. For this, using a wavelet construction, we introduce a *saturating function* $\mathscr{G}_q$ of $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ having a prescribed multifractal behaviour coinciding with that of μ. This $\mathscr{G}_q$ is a key object we build to show prevalent results on the spectrum of functions of $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$.

4.2 Additional definitions and properties on capacities, multifractal formalism and wavelets

Let $\Lambda = \bigcup_{j\in\mathbb{Z}} \Lambda_j$ be the set of all dyadic cubes. For $x \in \mathbb{R}^d$, $\lambda_j(x)$ is the dyadic cubes λ of generation j such that $x \in \lambda$, with the abuse of notation identifying $\lambda = (j, k = (k_1, ..., k_d), l) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d$ with the corresponding dyadic cube $\prod_{i=1}^d [k_i 2^{-j}, (k_i+1)2^{-j})$.

For $j \in \mathbb{Z}$, $\lambda \in \Lambda_j$ and $N \in \mathbb{N}^*$, $N\lambda$ stands for the cube with same centre as λ and radius equal to $N2^{-j-1}$ in $(\mathbb{R}^d, \|\cdot\|_\infty)$. For instance, 3λ is the union of those $\lambda' \in \Lambda_j$ such that $\partial\lambda \cap \partial\lambda' \neq \emptyset$ where $\partial\lambda$ is the boundary of λ.

4.2.1 Capacity and formalism

For any set Borel $A \subset \mathbb{R}^d$ and capacity μ, we will denote $\mu|_A = \mu(.\cap A)$. To describe the local behaviour of $\mu \in \mathcal{H}([0,1]^d)$, one needs, in addition to the lower local dimension (4.22), another local dimension.

Definition 4.16. *Let $\mu \in \mathcal{H}([0,1]^d)$. For $x \in \mathrm{supp}(\mu)$, the upper local dimension of μ at x is*

$$\overline{h}_\mu(x) = \limsup_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log(r)}.$$

Whenever $\underline{h}_\mu(x) = \overline{h}_\mu(x)$, the common limit is denoted by $h_\mu(x)$. For $\alpha \in \mathbb{R}$,

$$\underline{E}_\mu(\alpha) = \{x \in \mathrm{supp}(\mu) : \underline{h}_\mu(x) = \alpha\},\ \overline{E}_\mu(\alpha) = \{x \in \mathrm{supp}(\mu) : \overline{h}_\mu(x) = \alpha\}$$
$$\text{and} \quad E_\mu(\alpha) = \underline{E}_\mu(\alpha) \cap \overline{E}_\mu(\alpha).$$

Remark 4.17. Note that the local dimensions are sometimes defined as

$$\underline{h}_\mu(x) = \liminf_{j\to+\infty} \frac{\log_2 \mu(\lambda_j(x))}{-j} \text{ and } \overline{h}_\mu(x) = \limsup_{j\to+\infty} \frac{\log_2 \mu(\lambda_j(x))}{-j}, \tag{4.25}$$
$$\text{or} \quad \underline{h}_\mu(x) = \liminf_{j\to+\infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j} \text{ and } \overline{h}_\mu(x) = \limsup_{j\to+\infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j}.$$

However, this chapter mainly considers doubling or almost doubling set functions for which all previous notions of local dimensions (4.22) and (4.16), level sets, singularity spectrum and scaling function do not depend on whether dyadic cubes or centred balls are considered.

Recall that the Legendre spectrum $h \mapsto \tau_\mu^*(h)$ is increasing on the interval $h \le \tau_\mu'(0^+)$ and is decreasing on $h \ge \tau_\mu'(0^-)$. The following Proposition 4.18 can be deduced from [15, 16, 19, 122, 141].

Proposition 4.18. *Let $\mu \in \mathcal{H}([0,1]^d)$ with $supp(\mu) = [0,1]^d$, such that μ obeys the SMF (Definition 4.12). Then for every $h \in \mathbb{R}^+$:*

(i) One has $\sigma_\mu(h) = \dim_H(\underline{E}_\mu(h)) = \dim_H(\overline{E}_\mu(h)) = \tau_\mu^*(h)$.
(ii) When $h \le \tau_\mu'(0^+)$, $\dim_H\big(\underline{E}_\mu^{\le}(h)\big) = \tau_\mu^*(h)$.
(iii) When $h \ge \tau_\mu'(0^-)$, $\dim_H\big(\overline{E}_\mu^{\ge}(h)\big) = \tau_\mu^*(h)$.
(iv) When $\tau_\mu'(0^+) \le h \le \tau_\mu'(0^-)$, $\sigma_\mu(h) = d$.

All the set functions we work with satisfy the property (P) given in Definition 4.2. Observe that (P) implies that for all $j \in \mathbb{N}$ and $\lambda \in \Lambda_j$,

$$C^{-1}2^{-js_2} \le \mu(\lambda) \le C2^{-js_1}. \tag{4.26}$$

Property (P) and (4.26) are not equivalent in general, but it is the case when μ is almost doubling (with slightly different constant C and exponents s_1 and s_2), which will always be the case for us.

Definition 4.19. *A set function $\mu \in \mathcal{H}(\mathbb{R}^d)$ satisfies property (P_2) if there exist $C, s_2 > 0$ and $\phi \in \Phi$ (see Definition 4.3) such that for all $j, j' \in \mathbb{N}$ with $j \le j'$, for all $\lambda, \widetilde{\lambda} \in \Lambda_j$ such*

that $\partial\lambda \cap \partial\widetilde{\lambda} \neq \emptyset$*, and* $\lambda' \in \Lambda_{j'}$ *such that* $\lambda' \subset \lambda$*:*

$$\mu(\widetilde{\lambda}) \leq C 2^{\phi(2^{-j})} 2^{(j'-j)s_2} \mu(\lambda').$$

In particular, μ is almost doubling in this case. While the μ-mass of each dyadic cube is controlled by s_1 and s_2 by (P), Property (P_2) can be seen as a kind of weak quasi-Bernoulli property of μ.

Lemma 4.20. *Any doubling set function* $\mu \in \mathcal{H}(\mathbb{R}^d)$ *satisfies Property* (P_2)*.*

Proof. Let $j, j' \in \mathbb{N}$ with $j \leq j'$, $\lambda, \widetilde{\lambda} \in \Lambda_j$ such that $\partial\lambda \cap \partial\widetilde{\lambda} \neq \emptyset$, and $\lambda' \in \Lambda_{j'}$ such that $\lambda' \subset \lambda$. Calling $y_{\lambda'}$ the centre of λ' and $y_{\widetilde{\lambda}}$ the one of $\widetilde{\lambda}$, one has $\widetilde{\lambda} \subset B(y_{\lambda'}, 2 \cdot 2^{-j})$.

Since μ is doubling, (4.8) holds for a constant function ϕ, so there exists a C_μ such that for $\rho \in [r, 2r]$ and $x' \in B(x, r)$,

$$C_\mu^{-1} \mu(B(x', r)) \leq \mu(B(x, \rho)) \leq C_\mu \, \mu(B(x', r)). \tag{4.27}$$

Remark that $|y_{\lambda'} - y_{\widetilde{\lambda}}| < 3 \times 2^{-(j+1)}$. So, applying inequality (4.27) first between $\widetilde{\lambda}$ and $B\big(\frac{y_{\lambda'} - y_{\widetilde{\lambda}}}{2}, 2^{-j}\big)$, and then from this last ball to $B(y_{\lambda'}, 2 \cdot 2^{-j})$, one gets $\mu(\widetilde{\lambda}) \leq C_\mu^2 \, \mu(B(y_{\lambda'}, 2 \cdot 2^{-j}))$. Then applying it $j' - j + 2$ times to pass from radius 2^{-j+1} to $2^{-j'-1}$ yields finally

$$\mu(\widetilde{\lambda}) \leq C_\mu^4 \, C_\mu^{j'-j} \mu(\lambda') = C_\mu^4 \, 2^{(j'-j)\log_2(C_\mu)} \mu(\lambda'). \tag{4.28}$$

One concludes that μ satisfies (P_2) with $C = C_\mu^4$ and $s_2 = \log_2(C_\mu)$. □

4.2.2 Multi-resolution wavelet analysis

Let $d \in \mathbb{N}^*$ and $1 \leq p, q \leq +\infty$. A mapping $\psi : \mathbb{R}^d \to \mathbb{R}$ has $n+1$ vanishing moments when for every multi-index $\alpha \in \mathbb{N}^d$ such that $\sum_{i=1}^d \alpha_i \leq n$, $\int_{\mathbb{R}^d} x_1^{\alpha_1} \dots x_d^{\alpha_d} \psi(x) dx = 0$.

For an arbitrary integer $N \geq 1$, there exist compactly supported functions $\Psi^0 \in C^N(\mathbb{R})$ (called the scaling function) and $\Psi^1 \in C^N(\mathbb{R})$ (called the mother wavelet), with Ψ^1 having at least $N+1$ vanishing moments and such that $(\Psi^1_{j,k} : x \mapsto \Psi^1(2^j x - k))_{j,k \in \mathbb{Z}}$ form an orthogonal basis of $L^2(\mathbb{R})$. In this case, the wavelet is said to be N-regular. Let us introduce the notations

$$0^d := (0, \dots, 0), \quad 1^d := (1, \dots, 1), \quad L^d = \{0,1\}^d \backslash 0^d.$$

An orthogonal basis of $L^2(\mathbb{R}^d)$ is obtained by tensorization as follows (see [136, Chapter 2 and 3] for a general construction): for $\lambda = (j, k, l) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d$, let us define the tensorised wavelet

$$\Psi_\lambda = \prod_{i=1}^d \Psi^{l_i}_{j,k_i}$$

with $k = (k_1, \dots, k_D)$ and $l = (l_1, \dots, l_D) \in L^d$, and setting $\Phi = \Psi^{0^d}$.

Thus for every $f \in L^2(\mathbb{R}^d)$, one has the decomposition (4.16) (observe that we use an L^∞ normalisation for the wavelet coefficient $(c_\lambda)_{\lambda \in \Lambda}$.)

Proposition 4.21. *[83, 92] Suppose that $\gamma > 0$ and $N = \lfloor \gamma + 1 \rfloor$. Suppose that $\Phi, \Psi \in C^N(\mathbb{R})$ is at least $N+1$-regular. Let $f : [0,1]^d \to \mathbb{R}$ be a locally bounded function with wavelet coefficients $\{c_\lambda\}_{\lambda \in \Lambda \times L^d}$, and let $x \in [0,1]^d$.*

If $f \in C^\gamma(x)$, then there exists a constant $M < \infty$ such that for all $\lambda = (j,k,l) \in \Lambda \times L^d$,

$$|c_\lambda| \leq M \; 2^{-j\gamma} \left(1 + |2^j x - k|\right)^\gamma . \tag{4.29}$$

Conversely, if (4.29) holds for $f \in \bigcup_{\varepsilon>0} C^\varepsilon([0,1]^d)$, then $f \in \bigcap_{\eta>0} C^{\gamma-\eta}(x)$.

The *inhomogeneous Besov* spaces introduced in Section 4.1 and those of Definition 4.9 are related thanks to the following Theorem proved in [19].

Theorem 4.22. *Let $\mu \in \mathcal{C}(\mathbb{R}^d)$, $0 < s_1 \leq s_2$ and $n \geq r \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$. Suppose that (P) and (P_2) holds for μ with the exponents (s_1, s_2) and that $B^\mu_{p,q}(\mathbb{R}^d)$ has been constructed using the μ-adapted n-th order L^p moduli of smoothness. Consider a n-regular wavelet Ψ.*

For every $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 1$ such that for all $f \in L^p(\mathbb{R}^d)$,

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q,\Psi} \leq C_\varepsilon(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B^{\mu^{(+\varepsilon)}}_{p,q}(\mathbb{R}^d)}),$$

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B^\mu_{p,q}(\mathbb{R}^d)} \leq C_\varepsilon(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu^{(+\varepsilon)},p,q,\Psi}).$$

Moreover, when μ is doubling (with Lemma 4.20), $B^\mu_{p,q}(\mathbb{R}^d) = B^{\mu,\Psi}_{p,q}(\mathbb{R}^d)$ and the norms $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q,\Psi}$ and $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B^\mu_{p,q}(\mathbb{R}^d)}$ are equivalent. Also, this equivalence is independent of the choice of $n \geq r$ and Ψ r-regular.

Remark 4.23. [19, Remark 2.17] When μ satisfies (P) with $0 < s_1 < s_2$, one has $(\mathcal{L}^d)^{\frac{s_2}{d}} \leq \mu \leq (\mathcal{L}^d)^{\frac{s_1}{d}}$ (in the sense that for every $B(x,r) \subset \mathbb{R}^d$, $(\mathcal{L}^d)^{\frac{s_2}{d}}(B(x,r)) \leq \mu(B(x,r)) \leq (\mathcal{L}^d)^{\frac{s_1}{d}}(B(x,r))$) inducing the embeddings

$$B^{s_2+\frac{d}{p}}_{p,q}(\mathbb{R}^d) \hookrightarrow B^\mu_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_1+\frac{d}{p}}_{p,q}(\mathbb{R}^d). \tag{4.30}$$

When $\alpha \geq \beta$, one has

$$B^{\mu^{(+\alpha)}}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{\mu^{(+\beta)}}_{p,q}(\mathbb{R}^d) \quad \text{and} \quad \widetilde{B}^{\mu^{(+\alpha)}}_{p,q}(\mathbb{R}^d) \hookrightarrow \widetilde{B}^{\mu^{(+\beta)}}_{p,q}(\mathbb{R}^d) \tag{4.31}$$

Similarly, we can replace $\mathbb{R}^d$ by $[0,1]^d$ in all the previous embeddings.

4.3 Embeddings between inhomogeneous Sobolev and Besov spaces.

4.3.1 Preliminary

We start by extending [19, lemma 2.3] for the μ-adapted L^p modulus of smoothness to obtain the following lemma.

Lemma 4.24. *Let $\varepsilon > 0$, $p \in [1, +\infty)$, $n \in \mathbb{N}^*$ and $\mu \in \mathcal{H}(\mathbb{R}^d)$ be an almost doubling set function. There exists a constant $C = C(\varepsilon, d, n, p)$ such that for all $f \in L^p(\mathbb{R}^d)$ and $0 < t \leq 1$,*

the following inequality holds:

$$\omega_n^{\mu}(f,t,\mathbb{R}^d)_p^p \leq C\ t^{-d} \int_{t\leq|h|\leq 4nt} \int_{x\in\mathbb{R}^d} \left|\Delta_h^{\mu^{(+\varepsilon)},n} f(x)\right|^p dxdh. \tag{4.32}$$

Proof. The approach follows the lines of the proof of [46, inequality (3.3.17)] and [19, Lemma 2.3]. For any $h, y \in \mathbb{R}^d$, recall the following equality (see (3.3.19) in [46]):

$$\Delta_h^n f(x) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} [\Delta_{ky}^n f(x+kh) - \Delta_{h+ky}^n f(x)]. \tag{4.33}$$

First, one has

$$\begin{aligned}\omega_n^{\mu}(f,t,\mathbb{R}^d)_p^p &= \sup_{t/2\leq|h|\leq t} \left\|\Delta_h^{\mu,n} f\right\|_{L^p(\mathbb{R}^d)}^p = \sup_{t/2\leq|h|\leq t} \int_{x\in\mathbb{R}^d} |\Delta_h^{\mu,n} f(x)|^p dx \\ &= \sup_{t/2\leq|h|\leq t} \int_{x\in\mathbb{R}^d} \left|\frac{\Delta_h^n f(x)}{\mu(B[x,x+nh])}\right|^p dx.\end{aligned}$$

By (4.33), for $t/2 \leq |h| \leq t$ and $y \in \mathbb{R}^d$, one has

$$\begin{aligned}&\int_{x\in\mathbb{R}^d} \left|\frac{\Delta_h^n f(x)}{\mu(B[x,x+nh])}\right|^p dx \\ &= \int_{x\in\mathbb{R}^d} \left|\frac{\sum_{k=1}^n (-1)^k \binom{n}{k} [\Delta_{ky}^n f(x+kh) - \Delta_{h+ky}^n f(x)]}{\mu(B[x,x+nh])}\right|^p dx \\ &\leq C_{n,p} \sum_{k=1}^{n} \int_{x\in\mathbb{R}^d} \left|\frac{\Delta_{ky}^n f(x+kh)}{\mu(B[x,x+nh])}\right|^p + \left|\frac{\Delta_{h+ky}^n f(x)}{\mu(B[x,x+nh])}\right|^p dx.\end{aligned} \tag{4.34}$$

In particular, the last inequality holds for $y \in B(0,3t)\backslash B(0,2t)$.
For a fixed $t/2 \leq |h| \leq t$ and $y \in B(0,3t)\backslash B(0,2t)$, for every $k \in \{1,\cdots,n\}$, $|h| \leq |n(h+ky)| \leq 8n^2|h|$ and $|h| \leq |k(h+ny)| \leq 8n^2|h|$. Hence,

$$\begin{aligned} B[x,x+n(h+ky)] &\subset B\left(x+nh/2, 8n^2|h|\right) \\ \text{and}\quad B[x+kh,x+k(h+ny)] &\subset B\left(x+nh/2, 8n^2|h|\right). \end{aligned}$$

Lemma 4.25. *There exists $C_{\varepsilon,n} > 0$ that depends only on n and ε such that*

$$\mu^{(+\varepsilon)}\left(B[x,x+n(h+ky)]\right) \leq C_{\varepsilon,n}\ \mu\left(B\left(x+nh/2, 8n^2|h|\right)\right), \tag{4.35}$$

$$\mu^{(+\varepsilon)}\left(B[x+kh,x+k(h+ny)]\right) \leq C_{\varepsilon,n}\ \mu\left(B\left(x+nh/2, 8n^2|h|\right)\right), \tag{4.36}$$

$$\mu^{(+\varepsilon)}\left(B\left(x+nh/2, 8n^2|h|\right)\right) \leq C_{\varepsilon,n}\ \mu\left(B[x,x+nh]\right). \tag{4.37}$$

Proof. The key property is that the above balls have comparable diameters (with a constant depending on n only), and the almost doubling property of μ ensures that their μ-mass is similar up to a small exponential error term.

Let us prove (4.37), and we will use the same argument to prove the rest. Remark that $B[x,x+nh] = B\left(x+nh/2, n|h|/2\right)$ and $B\left(x+nh/2, 8n^2|h|\right)$ have the same centre and $8n^2|h| \leq 2^K n|h|/2$ for $K = \lfloor \log_2(n) \rfloor + 5$. Using the almost doubling property (4.8) applied K times, one obtains

$$\mu(B(x+nh/2, 8n^2|h|)) \leq e^{\sum_{k=0}^{K-1} \phi(2^k n|h|/2)} \mu(B(x+nh/2, n|h|/2)). \tag{4.38}$$

Using that ϕ is non-increasing, we have

$$\mu(B(x+nh/2, 8n^2|h|)) \leq e^{K\phi(n|h|/2)}\mu(B[x, x+nh]).$$

Since $\lim_{r\to 0^+} \frac{\phi(r)}{\log(r)} = 0$, there exists an $0 < R_\varepsilon < 1$ such that for all $0 < r \leq R_\varepsilon$, $0 \leq \phi(r) \leq -\frac{\varepsilon}{K}\log(r)$. So depending on the value of $|h|$, two cases can be separated:

- If $0 < \frac{n|h|}{2} \leq R_\varepsilon$, then $e^{K\phi(n|h|/2)} \leq 2^\varepsilon\ (n|h|)^{-\varepsilon}$.
- If $R_\varepsilon < \frac{n|h|}{2}$, then, since ϕ is non-increasing, $\phi(n|h|/2) \leq \phi(R_\varepsilon)$. So $e^{K\phi(n|h|/2)} \leq e^{K\phi(R_\varepsilon)} \leq R_\varepsilon^{-\varepsilon} \leq \left(\frac{n}{R_\varepsilon}\right)^\varepsilon\ (n|h|)^{-\varepsilon}$.

Summarising the above, for some $C_{\varepsilon,n}$ large enough, one has

$$\mu(B(x+nh/2, 8n^2|h|)) \leq C_{\varepsilon,n}(16\,n^2|h|)^{-\varepsilon}\mu(B[x, x+nh]),$$

leading to (4.37).

The same argument gives (4.35) and (4.36), we give them quickly. Using the almost doubling property (4.8) at most K times and that ϕ is non-increasing gives inequalities similar to (4.38), which in turn yields that for $k \in \{1, \dots, n\}$,

$$\begin{aligned}\mu(B[x, x+n(h+ky)]) &\leq e^{K\phi(n|h+ky|/2)}\mu(B(x+nh/2, 8n^2|h|)),\\ \mu(B[x+kh, x+k(h+ny)]) &\leq e^{K\phi(nk|y|/2)}\mu(B(x+nh/2, 8n^2|h|))\end{aligned}$$

Then, the same discussion based on the value of $|h|$ gives

$$\begin{aligned}\mu^{(+\varepsilon)}(B[x, x+n(h+ky)]) &\leq C_{\varepsilon,n}\ \mu(B(x+nh/2, 8n^2|h|)),\\ \mu^{(+\varepsilon)}(B[x+kh, x+k(h+ny)]) &\leq C_{\varepsilon,n}\ \mu(B(x+nh/2, 8n^2|h|)).\end{aligned}$$

□

Proof of Lemma 4.24 (continued). Combining (4.35), (4.36) and (4.37) gives

$$\begin{aligned}\max\left(\mu^{(+2\varepsilon)}(B[x, x+n(h+ky)]), \mu^{(+2\varepsilon)}(B[x+kh, x+k(h+ny)])\right)&\\ \leq C_{\varepsilon,n}\ \mu(B(x+nh/2, 8n^2|h|))&\end{aligned}$$

for some uniform constant $C_{\varepsilon,n}$. Using this suitably in (4.34) leads to

$$\begin{aligned}\left\|\Delta_h^{\mu,n} f\right\|_{L^p}^p &\leq C_{\varepsilon,n,p}\sum_{k=1}^n \int_{x\in\mathbb{R}^d} \left|\frac{\Delta_{ky}^n f(x+kh)}{\mu^{(+2\varepsilon)}(B[x+kh, x+k(h+ny)])}\right|^p\\ &\quad + \left|\frac{\Delta_{h+ky}^n f(x)}{\mu^{(+2\varepsilon)}(B[x, x+n(h+ky)])}\right|^p dx\\ &\leq C_{\varepsilon,n,p}\sum_{k=1}^n \int_{x\in\mathbb{R}^d} \left|\Delta_{ky}^{\mu^{(+2\varepsilon)},n} f(x+kh)\right|^p + \left|\Delta_{h+ky}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p dx\\ &\leq C_{\varepsilon,n,p}\sum_{k=1}^n \int_{x\in\mathbb{R}^d} \left|\Delta_{ky}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p + \left|\Delta_{h+ky}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p dx.\end{aligned}$$

Defining $C_d = \mathcal{L}^d(B(0,3)\backslash B(0,2))$, an integration with respect to y over $B(0,3t)\backslash B(0,2t)$

yields

$$\begin{aligned} & C_d\ t^d\ \left\|\Delta_h^{\mu,n} f\right\|_{L^p}^p \\ & \leq C_{\varepsilon,n,p} \sum_{k=1}^{n} \int_{2t\leq|y|\leq 3t} \int_{x\in\mathbb{R}^d} \left|\Delta_{ky}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p + \left|\Delta_{h+ky}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p dxdy \\ & \leq C_{\varepsilon,n,p} \sum_{k=1}^{n} \int_{2kt\leq|y|\leq 3kt} \int_{x\in\mathbb{R}^d} \left|\Delta_{y}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p + \left|\Delta_{h+y}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p dxdy, \end{aligned}$$

where the change of variable $y' = ky$ was applied to each term. By observing that $t \leq |h+y| \leq 4nt$, it gives

$$C_d\ t^d\ \left\|\Delta_h^{\mu,n} f\right\|_{L^p}^p \leq 2n\ C_{\varepsilon,n,p} \int_{t\leq|y|\leq 4nt} \int_{x\in\mathbb{R}^d} \left|\Delta_{y}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p dxdy.$$

The previous upper bound is independent of h, so one concludes by

$$\omega_n^{\mu}(f,t,\mathbb{R}^d)_p^p \leq \frac{2n\ C_{\varepsilon,n,p}}{C_d\ t^d} \int_{t\leq|y|\leq 4nt} \int_{x\in\mathbb{R}^d} \left|\Delta_{y}^{\mu^{(+2\varepsilon)},n} f(x)\right|^p dxdy.$$

□

Remark 4.26. Let $p \in [1,+\infty)$, $n \in \mathbb{N}^*$. If $\mu \in \mathcal{H}(\mathbb{R}^d)$ is doubling, the ε can be dropped in (4.32) and we obtain that there exists a constant $C = C(d,n,p)$ such that for all $f \in L^p(\mathbb{R}^d)$ and $0 < t \leq 1$:

$$\omega_n^{\mu}(f,t,\mathbb{R}^d)_p^p \leq C\ t^{-d} \int_{t\leq|h|\leq 4nt} \int_{x\in\mathbb{R}^d} \left|\Delta_{h}^{\mu,n} f(x)\right|^p dxdh. \tag{4.39}$$

4.3.2 Embedding of inhomogeneous Besov spaces into inhomogeneous Sobolev spaces

Let us explain our approach to get the first embedding (4.14) of Theorem 4.7. Recall that $B_{p,p}^{\mu}(\mathbb{R}^d)$ and $W^{\mu,p}(\mathbb{R}^d)$ are defined using μ-adapted n-th order finite difference operator and moduli of smoothness with n an integer such that $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$

To prove (4.14), we first need to obtain that $\|\cdot\|_{L^p(\mathbb{R}^d)} + |\cdot|_{W^{\mu,p}(\mathbb{R}^d)} \leq C(\|\cdot\|_{L^p(\mathbb{R}^d)} + |\cdot|_{B_{p,p}^{\mu}(\mathbb{R}^d)})$ to get $B_{p,p}^{\mu}(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d)$.

Let $f \in B_{p,p}^{\mu}(\mathbb{R}^d)$. We begin by decomposing the semi-norm defined in (4.10) as follows

$$\begin{aligned} & \iint_{\mathbb{R}^{2d}} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x,x+nh])^p |h|^{2d}} dxdh \\ & \leq \sum_{j\in\mathbb{Z}} \int_{2^{-j}\leq|h|\leq 2^{-j+1}} \int_{x\in\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x,x+nh])^p |h|^{2d}} dxdh \\ & = I_1 + I_2 \end{aligned}$$

where

$$I_1 = \sum_{j\in\mathbb{N}} \int_{2^{-j}\leq|h|\leq 2^{-j+1}} \int_{x\in\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x,x+nh])^p |h|^{2d}} dxdh$$

$$I_2 = \int_{|h|\geq 2} \int_{x\in\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x,x+nh])^p |h|^{2d}} dxdh.$$

The term I_1 can be bounded above as follows

$$\begin{aligned}
I_1 &\leq \sum_{j\in\mathbb{N}} \int_{2^{-j}\leq|h|\leq 2^{-j+1}} \int_{x\in\mathbb{R}^d} 2^{2jd} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x,x+nh])^p} dxdh \\
&\leq \sum_{j\in\mathbb{N}} \int_{2^{-j}\leq|h|\leq 2^{-j+1}} \int_{x\in\mathbb{R}^d} 2^{2jd} |\Delta_h^{\mu,n} f(x)|^p dxdh \\
&\leq \sum_{j\in\mathbb{N}} \int_{2^{-j}\leq|h|\leq 2^{-j+1}} 2^{2jd} \left\|\Delta_h^{\mu,n} f\right\|_{L^p(\mathbb{R}^d)}^p dh \\
&\leq \sum_{j\in\mathbb{N}} \int_{2^{-j}\leq|h|\leq 2^{-j+1}} 2^{2jd}\ \omega_n^\mu(f, 2^{-j+1}, \mathbb{R}^d)_p^p dh \\
&\leq C_d \sum_{j\in\mathbb{N}} 2^{2jd}\ 2^{-jd} \omega_n^\mu(f, 2^{-j+1}, \mathbb{R}^d)_p^p = C_d \sum_{j\in\mathbb{N}} 2^{d(j-1)} \omega_n^\mu(f, 2^{-(j-1)}, \mathbb{R}^d)_p^p.
\end{aligned}$$

We next proceed with I_2. For every $n \in \mathbb{N}^*$ and $x, h \in \mathbb{R}^d$

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh).$$

So the convexity of $t \mapsto |t|^p$ yields

$$|\Delta_h^n f(x)|^p \leq C_{n,p}\ \sum_{k=0}^{n} \binom{n}{k} |f(x+kh)|^p.$$

Moreover, as μ satisfy property (P) from Definition 4.2 and $nh/2 > 1$, one has

$$C^{-1} \leq \mu(B[x,x+nh]).$$

Further,

$$\begin{aligned}
I_2 &\leq C_{n,p}\ \sum_{k=0}^{n} \binom{n}{k} \int_{2\leq|h|} \frac{1}{|h|^{2d}} \int_{x\in\mathbb{R}^d} |f(x+kh)|^p dxdh \\
&\leq C_{n,p}\ \sum_{k=0}^{n} \binom{n}{k} \int_{2\leq|h|} \frac{1}{|h|^{2d}} \|f\|_{L^p(\mathbb{R}^d)}^p dh \\
&\leq C_{n,p,d}\ \sum_{k=0}^{n} \binom{n}{k} \|f\|_{L^p(\mathbb{R}^d)}^p \leq C_{n,p,d}\ \|f\|_{L^p(\mathbb{R}^d)}^p .
\end{aligned}$$

Combining the previous inequalities on I_1 and I_2, with the subadditivity of $t \mapsto t^{1/p}$, one gets

$$\begin{aligned}
|f|_{W^{\mu,p}(\mathbb{R}^d)} &\leq C_{n,p,d}^{1/p} \|f\|_{L^p(\mathbb{R}^d)} + C_d^{1/p} \left(\sum_{j\in\mathbb{N}} 2^{d(j-1)} \omega_n^\mu(f, 2^{-(j-1)}, \mathbb{R}^d)_p^p \right)^{1/p} \\
&\leq C_{n,p,d}(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_{p,p}^\mu(\mathbb{R}^d)}).
\end{aligned}$$

We conclude that $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu,p}(\mathbb{R}^d)} \leq C(\|f\|_{L^p(\mathbb{R}^d)} + \|f\|_{B_{p,p}^\mu(\mathbb{R}^d)})$ i.e. $B_{p,p}^\mu(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d)$.

4.3.3 Embedding of inhomogeneous Sobolev spaces into inhomogeneous Besov spaces

We prove now the converse embedding in (4.15) with the additional assumption that $\mu \in \mathcal{H}(\mathbb{R}^d)$ is almost doubling. To prove this result, we will show the following property:

For $\varepsilon \in (0,1)$, there exist a constant $C_\varepsilon > 1$ such that for all $f \in L^p(\mathbb{R}^d)$,

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B^\mu_{p,p}(\mathbb{R}^d)} \leq C_\varepsilon(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu^{(+\varepsilon)},p}(\mathbb{R}^d)}). \tag{4.40}$$

This yields $W^{\mu,p}(\mathbb{R}^d) \subset \bigcap_{0<\varepsilon<\min(s_1,1)} B^{\mu^{(-\varepsilon)}}_{p,p}(\mathbb{R}^d)$.

Let us prove (4.40) for a fixed $\varepsilon \in (0,1)$. Let $f \in L^p(\mathbb{R}^d)$. By definition

$$\left\|(2^{jd/p}\omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p)_{j\in\mathbb{N}}\right\|^p_{l^p(\mathbb{N})} = \sum_{j\in\mathbb{N}} 2^{jd}\omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p^p.$$

Then, using Lemma 4.24, we obtain, with $K_n = \lfloor \log_2(4n) \rfloor = 2 + \lfloor \log_2(n) \rfloor$,

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} 2^{jd}\omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p^p \\
&\leq C_{d,n,p,\varepsilon} \sum_{j\in\mathbb{N}} 2^{2jd} \int_{2^{-j}\leq|h|\leq 4n2^{-j}} \int_{\mathbb{R}^d} \left|\Delta_h^{\mu^{(+\varepsilon)},n} f(x)\right|^p dxdh \\
&\leq C_{d,n,p,\varepsilon} \sum_{k=1}^{K_n} \sum_{j\in\mathbb{N}} 2^{2jd} \int_{2^{-j+k}\leq|h|\leq 2^{-j+k+1}} \int_{\mathbb{R}^d} \left|\Delta_h^{\mu^{(+\varepsilon)},n} f(x)\right|^p dxdh \\
&\leq C_{d,n,p,\varepsilon} \sum_{k=1}^{K_n} \sum_{j\in\mathbb{N}} 2^{2jd} \int_{2^{-j+k}\leq|h|\leq 2^{-j+k+1}} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p} dxdh.
\end{aligned}$$

For a fixed $j \in \mathbb{N}$, for all $k \in \{1, \ldots, K_n\}$, $2^{j-K_n-1} \leq 2^{j-k-1}$. Hence, for $|h| \leq 2^{-j+k+1}$, we have $2^j \leq 2^{K_n+1}\,|h|^{-1}$. It follows that

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} 2^{jd}\omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p^p \\
&\leq 2^{K_n+1}\, C_{d,n,p,\varepsilon} \sum_{k=1}^{K_n} \sum_{j\in\mathbb{N}} \int_{2^{-j+k}\leq|h|\leq 2^{-j+k+1}} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p} \frac{dxdh}{|h|^{2d}} \\
&\leq C_{d,n,p,\varepsilon} \int_{0\leq|h|\leq 1} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p |h|^{2d}} dxdh \\
&\leq C_{d,n,p,\varepsilon} \int_{|h|\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p |h|^{2d}} dxdh.
\end{aligned}$$

We conclude by $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B^\mu_{p,p}(\mathbb{R}^d)} \leq C_{d,n,p,\varepsilon}(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu^{(+\varepsilon)},p}(\mathbb{R}^d)})$.

Remark 4.27. Using Remark 4.26 instead of Lemma 4.24 above, one gets that $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B^\mu_{p,p}(\mathbb{R}^d)} \leq C_{d,n,p}(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu,p}(\mathbb{R}^d)})$ and the ε can be dropped in (4.40). So we obtain that for every $p \geq 1$, $B^\mu_{p,p}(\mathbb{R}^d) = W^{\mu,p}(\mathbb{R}^d))$, as claimed after Theorem 4.7.

4.4 Prevalent spectrum in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$

We fix from now on $q \in [1,+\infty]$ and an almost doubling capacity $\mu \in \mathcal{C}([0,1]^d)$ with $\mathrm{supp}(\mu) = [0,1]^d$ satisfying (P) and verifying the SMF (Definition 4.12). The local behaviour of elements of $b^{\mu,\Psi}_{p,q}([0,1]^d)$ and $\widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$ is described via their pointwise Hölder exponent, whose study is based on the characterisation developed by Jaffard in [95].

Consider an integer $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$ and wavelets $(\Phi, \{\Psi^l\}_{l \in L^d})$ that belong to $C^n(\mathbb{R}^d)$ and have at least n vanishing moments.

Definition 4.28. *Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for $p \in [1,+\infty]$. Let $(c_\lambda)_{\lambda \in \Lambda \times L^d}$ be the wavelet coefficients of f associated with Ψ. The wavelet leader of f associated with $\lambda \in \Lambda$ is defined as*

$$L^f_\lambda = \sup\{|c_{\lambda'}| : \lambda' = (j,k,l) \in \Lambda \times L^d, \lambda' \subset 3\lambda\}. \tag{4.41}$$

Notation. Consider a function $f \in \widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$. Following Definition 4.28, for $x \in [0,1]^d$, the wavelet leader of f at scale j at x is denoted by

$$L^f_j(x) := L^f_{\lambda^d_j(x)},$$

with $\lambda^d_j(x) \in \Lambda^d_j$ the unique dyadic cube of generation j containing x (recall the definitions of Section 4.2). The mention of f in the leaders might be omitted when the function is clear from the context.

The pointwise Hölder exponent can be expressed as follows [95, Prop. 4].

Proposition 4.29. *If $f \in \mathcal{C}^\varepsilon(R^d)$ for some $\varepsilon > 0$, then for every $x_0 \in \mathbb{R}^d$, $h_f(x_0) < r$ if and only if $\liminf_{j \to +\infty} \dfrac{\log L^f_j(x_0)}{\log(2^{-j})} < r$, and in this case*

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log L^f_j(x_0)}{\log(2^{-j})}.$$

4.4.1 A saturating function in $\widetilde{b}^{\mu,\Psi}_{\infty,q}$

To identify the prevalent properties of functions in $\widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$, the classical strategy is to perturb a countable family of prevalent sets with a *saturating function* whose multifractal structure is the one claimed to be generic. Let us consider the saturating function of [19, Section 6].

Definition 4.30. *Let us define, for every $\lambda = (j,k,l) \in \Lambda_j \times L^d$,*

$$g_\lambda = \frac{1}{j^{\frac{2}{q}}}\ \mu(\lambda^d) \tag{4.42}$$

with the convention $\frac{2}{\infty} = 0$. The saturating function is $\mathscr{G}_q := \sum_{\lambda \in \Lambda \times L^d} g_\lambda \Psi_\lambda$.

From [19, Lemma 6.4], recalling the scaling function τ_μ (4.23), one has

Proposition 4.31. *The function $\mathscr{G}_q$ belongs to $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ and*

$$\sigma_{\mathscr{G}_q} = \tau^*_{\mu,\infty} = \sigma_\mu.$$

The fact that $\mathscr{G}_q \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ is directly deduced from (4.18) which gives that for every $\varepsilon \in (0, \min(1,s_1))$, $|\mathscr{G}_q|_{\mu^{(-\varepsilon)},\infty,q,\Psi} = \sum_{j=1}^{\infty} 2^{-j\varepsilon+2\log_2(j)}$ that converges. The singularity spectrum is deduced through the key property that for every $x \in (0,1)^d$, $h_{\mathscr{G}_q}(x) = h_\mu(x)$.

4.4.2 Lower bound of the pointwise Hölder exponent in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$

Let us begin with the following result.

Lemma 4.32. *For all $f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$, for every $x \in [0,1]^d$, $h_f(x) \geq \underline{h}_\mu(x)$.*

Proof. Let us write the wavelet decomposition $f = c_{0,0^d,0^d}\Psi_{0,0^d,0^d} + \sum_{\lambda\in\Lambda\times L^d} c^f_\lambda \Psi_\lambda$. Let $\varepsilon \in (0,\min(1,s_1))$. Recall Definition 4.9. Since $f \in b^{\mu^{(-\varepsilon)},\Psi}_{\infty,q}([0,1]^d)$, for every $j\in\mathbb{N}$, $\varepsilon_j^{\mu^{(-\varepsilon)},\infty} < +\infty$ and one has

$$\sup_{j\in\mathbb{N}} \sup_{\lambda\in\Lambda_j\times L^d} \left|\frac{c^f_\lambda}{\mu(\lambda)\ 2^{j\varepsilon}}\right| \leq C.$$

For a fixed $x\in[0,1]^d$, one has

$$L^f_j(x) = \sup_{\lambda\subset 3\lambda^d_j(x)} |c^f_\lambda| \leq C \sup_{\lambda_{j'}\subset 3\lambda^d_j(x)} |\mu(\lambda_{j'})| 2^{j'\varepsilon}.$$

Recall that μ satisfy property (P) with $0<s_1<s_2$. Hence, setting $M = \left\lfloor \frac{s_2}{s_1}\right\rfloor + 1$, (4.26) gives that for every $j' > Mj$, if $\lambda_{j'} \subset 3\lambda^d_j(x)$

$$\mu(\lambda_{j'})\ 2^{j'\varepsilon} \leq \widetilde{C}\, 2^{-j'(s_1-\varepsilon)} \leq \widetilde{C}\, 2^{-j(s_2-\varepsilon)} \leq \widetilde{C}^2\, \mu(\lambda^d_j(x))\ 2^{j\varepsilon}.$$

For $j \leq j' \leq Mj$, using that μ is a capacity, one has

$$\mu(\lambda_{j'})\ 2^{j'\varepsilon} \leq \mu(\lambda_{j'})\ 2^{Mj\varepsilon} \leq \mu(3\lambda^d_j(x))\ 2^{Mj\varepsilon}.$$

The almost-doubling property of μ gives that there exist $\phi_\mu : [0,1) \to \mathbb{R}_+$, such that $\lim_{j\to+\infty} \frac{\phi_\mu(r)}{\log(r)} = 0$, for which (4.8) applies. Considering $r = 2^{-j}$, one has $\mu(3\lambda^d_j(x)) \leq e^{\phi_\mu(2^{-j})}\, \mu(\lambda^d_j(x))$. Combining this with the last two inequalities gives, for large js,

$$L^f_j(x) \leq C\ \max(\widetilde{C}^2\, 2^{j\varepsilon}, e^{\phi_\mu(2^{-j})}\, 2^{Mj\varepsilon})\ \mu(\lambda^d_j(x)) \leq C\, e^{\phi_\mu(2^{-j})}\, 2^{Mj\varepsilon}\, \mu(\lambda^d_j(x)).$$

Hence, for all $j\in\mathbb{N}$ large enough

$$\log L^f_j(x) \leq \log C + \phi_\mu(2^{-j}) + \log\mu(\lambda^d_j(x)) + \log\left(2^{Mj\varepsilon}\right)$$

$$\text{and}\quad \frac{\log L^f_j(x)}{\log(2^{-j})} \geq \frac{\log C}{\log(2^{-j})} + \frac{\phi_\mu(2^{-j})}{\log(2^{-j})} + \frac{\log\mu(\lambda^d_j(x))}{\log(2^{-j})} - M\varepsilon.$$

Taking the $\liminf_{j\in\mathbb{N}}$ on both sides gives $h_f(x) \geq \underline{h}_\mu(x) - M\varepsilon$. Since this holds for all $\varepsilon\in(0,\min(1,s_1))$, one concludes that $h_f(x) \geq \underline{h}_\mu(x)$. □

Remark 4.33. In Lemma 4.32, one can replace $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ by $b^{\mu,\Psi}_{\infty,q}([0,1]^d)$, by simply setting ε to zero in all the computations.

4.4.3 Proof of Theorem 4.14

The key result to obtain the prevalent value of the singularity spectrum of functions in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ in Theorem 4.14 is the following:

Proposition 4.34. *The set*

$$\mathcal{F}^\mu = \left\{ f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d) : \forall x \in [0,1]^d,\ h_f(x) \leq \overline{h}_\mu(x) \right\} \tag{4.43}$$

is prevalent in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$.

The proof of Proposition 4.34 is postponed to Subsection 4.4.3.1. For the moment, we admit it and explain how this allows us to deduce Theorem 4.14.

Proof of Theorem 4.14. Using Lemma 4.32 and Proposition 4.34, for all $f \in \mathcal{F}^\mu$, for every $x \in [0,1]^d$,

$$\underline{h}_\mu(x) \leq h_f(x) \leq \overline{h}_\mu(x). \tag{4.44}$$

Fix $h \in \mathbb{R}_+$.

Step 1: we start with *the upper bound* $\sigma_f \leq \sigma_\mu$. By (4.44), one gets

$$E_f(h) \subset \{x \in \mathbb{R}^d : h_f(x) \leq h\} \subset \{x \in \mathbb{R}^d : \underline{h}_\mu(x) \leq h\} = \underline{E}^{\leq}_\mu(h),$$
$$E_f(h) \subset \{x \in \mathbb{R}^d : h_f(x) \geq h\} \subset \{x \in \mathbb{R}^d : \overline{h}_\mu(x) \geq h\} = \overline{E}^{\geq}_\mu(h).$$

We now apply Proposition 4.18 to μ which satisfies the SMF, therefore

- if $h \leq \tau'_\mu(0^+)$, *(ii)* gives
$$\sigma_f(h) = \dim_H(E_f(h)) \leq \dim_H\big(\underline{E}^{\leq}_\mu(h)\big) = \tau^*_\mu(h) = \sigma_\mu(h),$$
- if $\tau'_\mu(0^+) \leq h \leq \tau'_\mu(0^-)$, with *(iv)*, one has $\sigma_f(h) \leq d = \sigma_\mu(h)$,
- if $h \geq \tau'_\mu(0^-)$, *(iii)* gives
$$\sigma_f(h) = \dim_H(E_f(h)) \leq \dim_H\big(\overline{E}^{\geq}_\mu(h)\big) = \tau^*_\mu(h) = \sigma_\mu(h).$$

Hence one gets the upper bound.

Step 2: let us move to *the lower bound* $\sigma_f \geq \sigma_\mu$. For all $x \in E_\mu(h)$, by Definition 4.16, one has $\underline{h}_\mu(x) = \overline{h}_\mu(x) = h_\mu(x)$. Thus, by (4.44), one deduces $h_\mu(x) = h_f(x)$ and $E_\mu(h) \subset E_f(h)$. Finally, Proposition 4.18 gives $\sigma_f(h) = \dim_H(E_f(h)) \geq \dim_H(E_\mu(h)) = \sigma_\mu(h)$. □

4.4.3.1 Prevalence property of the auxiliary set $\mathcal{F}^\mu$

We want to show that $(\mathcal{F}^\mu)^\complement = \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d) \backslash \mathcal{F}^\mu$ is shy. The set $(\mathcal{F}^\mu)^\complement$ can be written as

$$(\mathcal{F}^\mu)^\complement = \left\{ f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d) : \exists x \in [0,1]^d,\ h_f(x) > \overline{h}_\mu(x) \right\}.$$

We simplify the problem by including the complement of $\mathcal{F}^\mu$ in a countable union of

simpler ancillary sets. For $p \in \mathbb{N}^*$, $\gamma > 0$ and $M = (M_1, M_2) \in (\mathbb{N}^*)^2$, let

$$\mathcal{O}^{p,\gamma,M} = \left\{ f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d) : \begin{array}{c} \exists x \in [0,1]^d, \forall \lambda_j = (j,k,l) \in \Lambda \times L^d, \\ \left|c^f_{\lambda_j}\right| \leq M_1\ 2^{-j\gamma} \left(1 + |2^j x - k|\right)^\gamma \\ \forall j > M_2,\ \mu(\lambda_j(x)) \geq 2^{-j(\gamma - 1/p)} \end{array} \right\}.$$

Remembering Proposition 4.21, the sets $\mathcal{O}^{p,\gamma,M}$ shall be understood as the set of functions in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ such that for some $x \in [0,1]^d$, $f \in C^\gamma(x)$ and simultaneously the μ-mass of the dyadic cubes $\lambda_j(x)$ is bounded below. This two properties will allow us to control $\frac{c^f_{\lambda_j(x)}}{\mu(\lambda_j(x))}$.

Proposition 4.35. *One has* $(\mathcal{F}^\mu)^\complement \subset \bigcup_{p\in\mathbb{N}^*} \bigcup_{\gamma\in\mathbb{Q}_+} \bigcup_{M\in(\mathbb{N}^*)^2} \mathcal{O}^{p,\gamma,M}$.

Proof. We write $(\mathcal{F}^\mu)^\complement = \bigcup_{p\in\mathbb{N}^*} \mathcal{F}_p$, where

$$\mathcal{F}_p = \left\{ f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d) : \exists x \in [0,1]^d,\ h_f(x) - \overline{h}_\mu(x) > 2/p \right\}.$$

Supposing that $f \in \mathcal{F}^\mu_p$, there exists $\gamma \in \mathbb{Q}_+$ and $x \in [0,1]^d$ such that $h_f(x) > \gamma$ and $\overline{h}_\mu(x) \leq \gamma - \frac{2}{p}$. Using (4.29) of Proposition 4.21, one has

$$\begin{aligned} h_f(x) > \gamma &\Longrightarrow f \in C^\gamma(x) \\ &\Longrightarrow \exists M_1 \in \mathbb{N}, \forall \lambda_j = (j,k,l) \in \Lambda \times L^d, \left|c^f_{\lambda_j}\right| \leq M_1\ 2^{-j\gamma} \left(1 + |2^j x - k|\right)^\gamma. \end{aligned}$$

In addition, by (4.25), one has

$$\begin{aligned} \overline{h}_\mu(x) \leq \gamma - \frac{2}{p} &\Longrightarrow \exists M_2 \in \mathbb{N},\ \sup_{j>M_2} \frac{\log \mu(\lambda_j(x))}{\log 2^{-j}} \leq \gamma - \frac{1}{p} \\ &\Longrightarrow \exists M_2 \in \mathbb{N},\ \forall j > M_2,\ \mu(\lambda_j(x)) \geq 2^{-j(\gamma - 1/p)}. \end{aligned}$$

Hence, $f \in \mathcal{O}^{p,\gamma,M}$ with $M = (M_1, M_2)$. One gets that if $f \notin \mathcal{F}^\mu$, then there is $p \in \mathbb{N}^*$, $f \in \mathcal{F}^\mu_p$. Then, for $p \in \mathbb{N}^*$, there is $\gamma \in \mathbb{Q}_+$ and $M = (M_1, M_2) \in (\mathbb{N}^*)^2$ such that $f \in \mathcal{O}^{p,\gamma,M}$. □

To use the prevalence theory, it remains to prove the universal measurability of $\mathcal{O}^{p,\gamma,M}$. Analytic sets in Polish spaces are defined as continuous images of Borel sets. Unfortunately, as we work with $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ which is not separable, we cannot use the Polish spaces characterisation and have to consider a more general definition, from Choquet [43], valid for any Hausdorff topological space X endowed with a Borel σ-algebra $\mathcal{B}(X)$. Choquet's capacitability theorem [43, Theorem 30.1] with the following definition of analyticity [43, Theorem 5.1] give the right setting, as was observed in [10].

Definition 4.36. *For a compact topological space K, the collection of its closed subsets is written $\mathcal{K}$. Let $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ be the collection of countable intersections of countable unions of sets that are Cartesian products of a Borel set in X and a closed set in K.*

A set $A \subset X$ is said to be analytic if there exists a compact space K and $\mathcal{T} \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ such that

$$A = \pi(\mathcal{T}) \tag{4.45}$$

with $\pi : X \times K \to X$ the canonical projection map $\pi(x,y) = x$.

Then Choquet's capacitability theorem yields the following theorem.

Theorem 4.37. *Every analytic set of a Hausdorff topological space is universally measurable.*

Theroem 4.37 states the equivalence between the universal measurability of $\mathcal{O}^{p,\gamma,M}$ and its analyticity.

Proposition 4.38. *Let $p \in \mathbb{N}^*$, $M \in (\mathbb{N}^*)^2$ and $\gamma \in \mathbb{Q}_+$. The set $\mathcal{O}^{p,\gamma,M}$ is universally measurable in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$.*

Proof. We are going to prove the analyticity of $\mathcal{O}^{p,\gamma,M}$ by taking the right sets X, K and $\mathcal{T}$ in Definition 4.36, then writing it as $\pi(\mathcal{T})$. Denote $X := \left(\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d), d\right)$ with d defined in (4.20) and $K := ([0,1]^d, \|\cdot\|_\infty)$.

Step 1: For $\lambda_j = (j,k,l) \in \Lambda \times L^d$, let us introduce the maps $\phi_{\lambda_j}, \phi : X \times K \to \mathbb{R}$ defined as

$$\phi_{\lambda_j} : (f,x) \mapsto M_1\, 2^{-j\gamma} \left(1 + |2^j x - k|\right)^\gamma - \left|c^f_{\lambda_j}\right|$$
$$\phi : (f,x) \mapsto \inf_{\lambda \in \Lambda \times L^d} \phi_\lambda(f,x).$$

Lemma 4.39. *Each ϕ_{λ_j} is continuous on $X \times K$.*

Proof. Let $\varepsilon > 0$. Recalling (4.20), for any $f, g \in X$ such that $d(f,g) < \varepsilon$, one gets that, for any $n := \left\lfloor \max(1, s_1^{-1}) \right\rfloor + 1$,

$$2^{-n} \frac{\|f-g\|_{b^{\mu(-1/n),\Psi}_{\infty,q}}}{1 + \|f-g\|_{b^{\mu(-1/n),\Psi}_{\infty,q}}} < \varepsilon.$$

This induces that $\|f-g\|_{b^{\mu(-1/n),\Psi}_{\infty,q}} < 2^n\varepsilon$. By definition of $\|\cdot\|_{b^{\mu(-\varepsilon),\Psi}_{\infty,q}}$ in Remark 4.10, $\|f-g\|_\infty < 2^n\varepsilon$.

The application $f \in X \mapsto c^f_{\lambda_j} = \left\langle f, \Psi_{\lambda_j} \right\rangle \in \mathbb{R}$ is a scalar product between f and a fixed wavelet Ψ_{λ_j}. With the classical remark that $\left\langle f, \Psi_{\lambda_j} \right\rangle \le \|f\|_\infty \|\Psi_{\lambda_j}\|_1$, $f \mapsto c^f_{\lambda_j}$ is continuous. So the mapping $(f,x) \mapsto c^f_{\lambda_j}$ is continuous and so is ϕ_{λ_j} as a sum of continuous functions. □

Step 2: Denote

$$\mathcal{D}_\mu(j, \gamma - 1/p) := \left\{ \lambda_j \in \Lambda_j : \frac{\log \mu(\lambda_j)}{\log(2^{-j})} \le \gamma - \frac{1}{p} \right\}. \tag{4.46}$$

Let us introduce

$$\mathcal{T} := \phi^{-1}([0,+\infty)) \bigcap \left(X \times \bigcap_{j \ge M_2} \bigcup_{\lambda \in \mathcal{D}_\mu(j,\gamma-1/p)} \lambda \right).$$

Lemma 4.40. *The set $\mathcal{T}$ belongs to $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$.*

Proof. First, it is straightforward from the definition that, for all $\lambda \in \Lambda$, $X \times \lambda \in \mathcal{B}(X) \times \mathcal{K}$, so taking a countable intersection of compact sets,

$$X \times \bigcap_{j \ge M_2} \bigcup_{\lambda \in \mathcal{D}_\mu(j,\gamma-1/p)} \lambda \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}.$$

Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ and consider

$$F_{\lambda_j}(n,m) := \{f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d) : c^f_{\lambda_j} \in 2^{-n}[m,m+1]\}.$$

Since $2^{-n}[m,m+1]$ is a closed interval, $F_{\lambda_j}(n,m)$ is closed as the preimage of a closed set by the continuous function $f \mapsto c^f_{\lambda_j}$. Next, let us define

$$X_{\lambda_j}(n,m) := \left\{x \in [0,1]^d : \sup_{f \in F_{\lambda_j}(n,m)} \phi_{\lambda_j}(f,x) \geq 0\right\}.$$

Observe that when f ranges in $\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$, all possible values of $c^f_{\lambda_j} \in 2^{-n}[m,m+1]$ are reached. Hence, one can rewrite $X_{\lambda_j}(n,m)$ as

$$X_{\lambda_j}(n,m) = \left\{x \in [0,1]^d : \max_{c \in 2^{-n}[m,m+1]} M_1\, 2^{-j\gamma}\left(1+|2^j x-k|\right)^\gamma - |c| \geq 0\right\}.$$

The max can be in turn rewritten

$$X_{\lambda_j}(n,m) = \left\{x \in [0,1]^d : \begin{array}{c} \exists c \in 2^{-n}[m,m+1], \\ M_1 2^{-j\gamma}\left(1+|2^j x-k|\right)^\gamma - |c| \geq 0 \end{array}\right\}.$$

Denoting the mapping $v : (x,c) \in \mathbb{R}^d \times \mathbb{R} \mapsto M_1\, 2^{-j\gamma}\left(1+|2^j x-k|\right)^\gamma - |c| \in \mathbb{R}$, one has, for $\widetilde{\pi}(x,c) = x$ the projection on the first coordinate,

$$X_{\lambda_j}(n,m) = \widetilde{\pi}\big(v^{-1}(\mathbb{R}_+) \cap ([0,1]^d \times 2^{-n}[m,m+1])\big).$$

The mapping $v : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is continuous, so $v^{-1}(\mathbb{R}_+) \cap ([0,1]^d \times 2^{-n}[m,m+1])$ is compact. As the projection is continuous, $X_{\lambda_j}(n,m)$ is also compact.

Lemma 4.41. *One has*

$$\phi^{-1}_{\lambda_j}([0,+\infty)) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{Z}} F_{\lambda_j}(n,m) \times X_{\lambda_j}(n,m).$$

Proof. First, we prove that $\phi^{-1}_{\lambda_j}([0,+\infty)) \subset \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{Z}} F_{\lambda_j}(n,m) \times X_{\lambda_j}(n,m)$.

Let $(f,x) \in \phi^{-1}_{\lambda_j}([0,+\infty))$. Then for every $n \in \mathbb{N}$, there is an integer $m \in \mathbb{Z}$ such that $c^f_{\lambda_j} \in 2^{-n}[m,m+1]$ and $f \in F_{\lambda_j}(n,m)$. In addition, $\phi_{\lambda_j}(f,x) \geq 0$, so $\sup_{g \in F_{\lambda_j}(n,m)} \phi_{\lambda_j}(g,x) \geq 0$ and $x \in X_{\lambda_j}(n,m)$. Hence $(f,x) \in F_{\lambda_j}(n,m) \times X_{\lambda_j}(n,m)$.

Next, let us prove $\phi^{-1}_{\lambda_j}([0,+\infty)) \supset \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{Z}} F_{\lambda_j}(n,m) \times X_{\lambda_j}(n,m)$.

Let $(f,x) \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{Z}} F_{\lambda_j}(n,m) \times X_{\lambda_j}(n,m)$. Then for all $n \in \mathbb{N}$, there exist an $m \in \mathbb{Z}$ such that $f \in F_{\lambda_j}(n,m)$ and $x \in X_{\lambda_j}(n,m)$.

One has $c^f_{\lambda_j} \in 2^{-n}[m,m+1]$ and $\sup_{g \in F_{\lambda_j}(n,m)} \phi_{\lambda_j}(g,x) \geq 0$. Hence

$$\begin{aligned}\left|\sup_{g \in F_{\lambda_j}(n,m)} \phi_{\lambda_j}(g,x) - \phi_{\lambda_j}(f,x)\right| &\leq \sup_{g \in F_{\lambda_j}(n,m)} |\phi_{\lambda_j}(g,x) - \phi_{\lambda_j}(f,x)| \\ &\leq \sup_{g \in F_{\lambda_j}(n,m)} \left|c^g_{\lambda_j} - c^f_{\lambda_j}\right| \leq 2^{-n}.\end{aligned}$$

As, this result holds for all $n \in \mathbb{N}$, one deduces that $\phi_{\lambda_j}(f,x) \geq 0$ □

Hence, $\phi^{-1}_{\lambda_j}([0,+\infty)) \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ and $\phi^{-1}([0,+\infty)) = \bigcap_{\lambda \in \Lambda \times L^d} \phi^{-1}_\lambda([0,+\infty))$ also

belongs to $(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$. This proves that the set $\mathcal{T}$ indeed belongs to $(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$. □

Step 3: By Definition 4.36, the canonical projection map $\pi : (x,y)\in X\times K\to x\in X$ gives that

$$\pi(\mathcal{T}) := \left\{ f\in X : \begin{array}{c} \exists x\in K, (f,x)\in \phi^{-1}([0,+\infty)), \\ \forall j > M_2,\ \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)} \end{array} \right\}$$

is analytic in the space $(X,\mathcal{B}(X))$.

Step 4: To conclude, notice that one can rewrite $\mathcal{O}^{p,\gamma,M} = \pi(\mathcal{T})$. This yields that $\mathcal{O}^{p,\gamma,M}$ is analytic and thus universally measurable. □

As the sets $\mathcal{O}^{p,\gamma,M}$ are universally measurable, prevalence theory is now used to show the shyness of $(\mathcal{F}^{\mu})^{\complement}$ through the shyness of $\mathcal{O}^{p,\gamma,M}$. To do so, the classical method is to use a finite-dimensional subspace $\mathcal{P}$ and the Lebesgue measure restricted to a compact subset of $\mathcal{P}$ as transverse measure.

Let d_1 be an integer such that

$$d < d_1/2p. \tag{4.47}$$

Definition 4.42. *Consider the saturating function $\mathscr{G}_q$ whose wavelet coefficients are g_λ (see (4.42)). Let $\mathcal{P} = \mathrm{Vect}_{i=1,\dots,d_1}(\mathscr{G}_q^{(i)}) \subset \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$ where $\mathscr{G}_q^{(i)}$ is the wavelet series whose wavelet coefficients $g_\lambda^{(i)}$ are defined in the following way: for every $\lambda=(j,k,l)\in\Lambda\times\{0,1\}^d$,*

$$g_\lambda^{(i)} = \begin{cases} g_\lambda & \text{if } j \pmod{d_1} = i, \\ 0 & \text{else.} \end{cases} \tag{4.48}$$

Take an arbitrary $f\in\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$, define the function f^β by

$$\begin{array}{ccc} \mathbb{R}^{d_1} & \to & f+\mathcal{P} \\ \beta & \mapsto & f^\beta = f + \sum_{i=1}^{d_1}\beta_i\cdot\mathscr{G}_q^{(i)}. \end{array} \tag{4.49}$$

Denote c_λ^f (resp. c_λ^β) the wavelet coefficient of f (resp. f^β) and L_λ^f (resp. L_λ^β) the associated wavelet leader.

Proposition 4.43. *For any function $f\in\widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$, the set $\{\beta\in\mathbb{R}^{d_1} : f^\beta\in\mathcal{O}^{p,\gamma,M}\}$ has d_1-dimensional Lebesgue measure $\mathcal{L}_{d_1}$ zero.*

Proof. Let us rewrite $\left\{\beta\in\mathbb{R}^{d_1} : f^\beta\in\mathcal{O}^{p,\gamma,M}\right\}$ as

$$\mathcal{B}_f := \left\{ \beta\in\mathbb{R}^{d_1} : \begin{array}{c} \exists x\in[0,1]^d, \forall\lambda_j=(j,k,l)\in\Lambda\times L^d, \\ \left|c_{\lambda_j}^\beta\right| \leq M_1\ 2^{-j\gamma}\left(1+|2^jx-k|\right)^\gamma, \\ \forall j > M_2,\ \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)} \end{array} \right\}.$$

We start with a technical lemma.

Lemma 4.44. *The mapping* $\left\{ \begin{array}{ccc} \mathbb{R}^{d_1} & \longrightarrow & \mathbb{R} \\ \beta & \longmapsto & \mathbb{1}_{\mathcal{B}}(\beta) \end{array} \right.$ *is measurable.*

Proof. We will use the notation and function from the proof of Lemma 4.40.

Let $\Phi : (\beta, x) \mapsto \inf_{\lambda \in \Lambda \times L^d} \phi_\lambda(f^\beta, x)$. The map Φ is Borel on $\mathbb{R}^{d_1} \times [0,1]^d$ as an infimum of countably many continuous functions. One sees that

$$\mathcal{B}_f = \left\{ \beta \in \mathbb{R}^{d_1} : \begin{array}{c} \exists x \in [0,1]^d,\ \Phi(\beta, x) \geq 0, \\ \forall j > M_2,\ \mu(\lambda_j(x)) \geq 2^{-j(\gamma - 1/p)} \end{array} \right\}.$$

This set can be rewritten as

$$\mathcal{B}_f = \widetilde{\pi}\Big(\Phi^{-1}([0,+\infty)) \bigcap \Big(\mathbb{R}^{d_1} \times \bigcap_{j \geq M_2} \bigcup_{\lambda \in \mathcal{D}_\mu(j, \gamma - 1/p)} \lambda\Big)\Big), \tag{4.50}$$

where $\widetilde{\pi}(\beta, x) = \beta$ is the canonical projection on the first coordinate and $\mathcal{D}_\mu(j, \gamma - 1/p)$ is defined by (4.46). Since the set in brackets in (4.50) is a Borel set, $\mathcal{B}_f$ is analytic by Definition 4.36. By Theorem 4.37, it is also Lebesgue-measurable. □

We are ready to prove that the set $\mathcal{B}_f$ has d_1-dimensional Lebesgue measure $\mathcal{L}_{d_1}$ zero. For any $\lambda_0 := (j_0, k_0, l_0) \in \Lambda$, denote

$$\mathcal{B}_{f,\lambda_0} = \left\{ \beta \in \mathbb{R}^{d_1} : \begin{array}{c} \exists x \in \lambda_0, \forall \lambda_j = (j,k,l) \in \Lambda \times L^d, \\ \left|c^\beta_{\lambda_j}\right| \leq M_1\ 2^{-j\gamma} \left(1 + |2^j x - k|\right)^\gamma, \\ \forall j > M_2,\ \mu(\lambda_j(x)) \geq 2^{-j(\gamma - 1/p)} \end{array} \right\}.$$

so that

$$\mathcal{B}_f = \limsup_{j_0 \to \infty} \bigcup_{(j_0,k_0,l_0) \in \Lambda^d_{j_0}} \mathcal{B}_{f,\lambda_0} = \bigcap_{J \in \mathbb{N}^*} \bigcup_{j_0 \geq J} \bigcup_{(j_0,k_0,l_0) \in \Lambda^d_{j_0}} \mathcal{B}_{f,\lambda_0}. \tag{4.51}$$

Let us show that $\mathcal{L}_{d_1}(\mathcal{B}_f) = 0$ by establishing an upper bound for each $\mathcal{L}_{d_1}(\mathcal{B}_{f,\lambda_0})$ and using the Borel-Cantelli lemma. One can ensure that j_0 is large enough so that $j_0 \geq M_2 + d_1$. Suppose that β and $\widetilde{\beta}$ belong to a same $\mathcal{B}_{f,\lambda_0}$. There exist $x_\beta, x_{\widetilde{\beta}} \in \lambda_0 \subset \mathbb{R}^d$ such that for every $j \in \mathbb{N}^*$,

$$\left|c^\beta_{\lambda_j(x_\beta)}\right| \leq M_1\ 2^{-j\gamma} \left(1 + |2^j x_\beta - k_{j,\beta}|\right)^\gamma \tag{4.52}$$

and similarly for $c^{\widetilde{\beta}}_{\lambda_j(x_{\widetilde{\beta}})}$ with $k_{j,\beta}$ and $k_{j,\widetilde{\beta}}$ the multi-integers associated respectively to $\lambda_j(x_\beta)$ and $\lambda_j(x_{\widetilde{\beta}})$.

For every $j \in \{j_0 - d_1, \ldots, j_0 - 1\}$, one has $\lambda_j(x_\beta) = \lambda_j(x_{\widetilde{\beta}})$. Let us denote $\widetilde{\lambda}_j = \lambda_j(x_\beta)$.

Using $x_\beta, x_{\widetilde{\beta}} \in \lambda_0 \subset \widetilde{\lambda}_j$ and (4.52), one deduces that there is a constant $C > 0$ depending on M, j_0, d_1 and γ such that, for every $j \in \{j_0 - d_1, \ldots, j_0 - 1\}$, $\left|c^\beta_{\widetilde{\lambda}_j}\right| \leq C\ 2^{-j\gamma}$ and $\left|c^{\widetilde{\beta}}_{\widetilde{\lambda}_j}\right| \leq C\ 2^{-j\gamma}$.

From all this, we deduce that, for $j \in \{j_0 - d_1, \ldots, j_0 - 1\}$ and i such that $j \pmod{d_1} \equiv i$,

$$\begin{aligned} \left|(\beta_i - \widetilde{\beta}_i)\ g^{(i)}_{\widetilde{\lambda}_j}\right| &= \left|\left(c_{\widetilde{\lambda}_j} + \beta_i\ g^{(i)}_{\widetilde{\lambda}_j}\right) - \left(c_{\widetilde{\lambda}_j} + \widetilde{\beta}_i\ g^{(i)}_{\widetilde{\lambda}_j}\right)\right| \\ &\leq \left|c^\beta_{\widetilde{\lambda}_j}\right| + \left|c^{\widetilde{\beta}}_{\widetilde{\lambda}_j}\right| \leq 2C 2^{-(j_0 - d_1)\gamma}. \end{aligned}$$

So, by (4.42) and (4.48), $\left|\beta_i - \widetilde{\beta}_i\right| \leq 2C \frac{2^{-(j_0 - d_1)\gamma}}{j^{-2/q}\ \mu(\widetilde{\lambda}_j)}$. As for every $j \in \{j_0 - d_1, \ldots, j_0 - 1\}$,

one has $j > M_2$, then $\mu(\widetilde{\lambda}_j) \geq 2^{-j(\gamma-1/p)}$, and we deduce that $\mu(\widetilde{\lambda}_j) \geq 2^{-j_0(\gamma-1/p)}$. Going through all $j \in \{j_0 - d_1, \dots, j_0 - 1\}$ induce that for every $i \in \{1, \dots, d_1\}$, one has

$$\begin{aligned}\left|\beta_i - \widetilde{\beta}_i\right| &\leq 2C \; \frac{2^{-(j_0-d_1)\gamma}}{j^{-2/q}\; 2^{-j_0(\gamma-1/p)}} \leq 2C \; j_0^{2/q} \; \frac{2^{-(j_0-d_1)\gamma}}{2^{-j_0(\gamma-1/p)}} \\ &\leq 2C \; 2^{d_1\gamma} \; 2^{-j_0\left(\frac{1}{p}-\frac{2}{q}\frac{\log_2(j_0)}{j_0}\right)}.\end{aligned}$$

Considering j_0 large enough, for every $i \in \{1, \dots, d_1\}$, $\left|\beta_i - \widetilde{\beta}_i\right| \leq C2^{-j_0/2p}$. This means that the distance between two elements of $\mathcal{B}_{\lambda_0}$ is at most $2^{-j_0/2p}$ up to a universal constant. Hence, the volume of $\mathcal{B}_{f,\lambda_0}$ is given by

$$\mathcal{L}_{d_1}(\mathcal{B}_{\lambda_0}) \leq C \; 2^{-j_0 \cdot d_1/2p}.$$

Summing over all the 2^{dj_0} dyadic cubes for $k_0 \in \{0, \dots, 2^{j_0}\}^d$ at scale j_0 gives

$$\mathcal{L}_{d_1}\left(\bigcup_{k_0 \in \mathbb{Z}^d_{j_0}} \mathcal{B}_{\lambda_0}\right) \leq C2^{j_0(d-d_1/2p)}.$$

By our choice (4.47), $d < d_1/2p$ so the Borel-Cantelli lemma implies that

$$\mathcal{L}_{d_1}\left(\limsup_{j_0 \to \infty} \bigcup_{k_0 \in \mathbb{Z}^d_{j_0}} \mathcal{B}_{\lambda_0}\right) = 0.$$

Finally by (4.51), $\mathcal{L}_{d_1}(\mathcal{B}) = 0$, which is the claim of Proposition 4.43. □

We can now conclude regarding the shyness of the sets $\mathcal{O}^{p,\gamma,M}$.

Proposition 4.45. *For all $p \in \mathbb{N}^*$, $M \in (\mathbb{N}^*)^2$ and $\gamma \in \mathbb{Q}_+$, $\mathcal{O}^{p,\gamma,M}$ is shy.*

Proof. Let $\vartheta = G_{\#}\mathcal{L}_{d_1}$ be the push-forward measure of the d_1-dimensional Lebesgue measure by

$$G : \left\{ \begin{array}{ccc} [0,1]^{d_1} & \to & \mathcal{P} \\ \beta & \mapsto & \sum_{i=1}^{d_1} \beta_i \cdot \mathscr{G}_q^{(i)}. \end{array} \right. \tag{4.53}$$

This measure is supported by a compact set of $\mathcal{P} \subset \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$.

Fix any $f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$. Using Proposition 4.43, for ϑ-almost all $F \in \mathcal{P}$, $f + F \notin \mathcal{O}^{p,\gamma,M}$. Hence for any $f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$, the set $\{f + \mathcal{O}^{p,\gamma,M}\}$ has ϑ-measure equal to 0, i.e., $\vartheta(\{f + \mathcal{O}^{p,\gamma,M}\}) = 0$. Since it is true for any $f \in \widetilde{b}^{\mu,\Psi}_{\infty,q}([0,1]^d)$, the set $\mathcal{O}^{p,\gamma,M}$ is shy. □

By Proposition 4.35, $(\mathcal{F}^\mu)^\complement$ is shy as subset of countable unions and intersections of shy sets. This yields that $\mathcal{F}^\mu$ is prevalent, hence Proposition 4.34.

4.5 Extensions

We have identified the singularity spectrum for a prevalent set of functions in the case $p = \infty$. This leaves the finite case $p < \infty$ for which the conjectured result is the analog of the result by Barral-Seuret [19].

Conjecture 4.46. *Let μ be an almost doubling capacity on $[0,1]^d$ satisfying (P) and verifying the SMF, let $p,q \in [1,+\infty]$. Consider the mappings*

$$\tau_{\mu,p}(t) = \begin{cases} \frac{p-t}{p}\ \tau_\mu\left(\frac{p}{p-t}\ t\right) & \text{if } t \in (-\infty, p) \\ d & \text{if } t \in [p,+\infty) \end{cases}$$

and $\theta_p : \alpha \in [\tau'_\mu(+\infty), \tau'_\mu(-\infty)] \longmapsto \alpha + \frac{\tau^*_\mu(\alpha)}{p}$. For all $f \in \widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$,

$$\sigma_f(h) \leq \begin{cases} \sigma_\mu(\theta^{-1}(h)) = \tau^*_{\mu,p}(h) & \text{if } h \leq \tau'_{\mu,p}(0) \\ d & \text{if } h > \tau'_{\mu,p}(0). \end{cases}$$

Also, there is a prevalent set of $\widetilde{b}^{\mu,\Psi}_{p,q}([0,1]^d)$ satisfying $\sigma_f = \sigma_\mu(\theta^{-1}(\cdot)) = \tau^*_{\mu,p}$.

A second possible extension is to relax the hypothesis of Theorem 4.14 to include capacities that are not almost doubling, such as the Bernoulli multinomial measures and multiplicative cascade.

Chapter 5

Prevalence of traces in inhomogeneous Besov spaces

5.1 Introduction

In the context of turbulence, the study of local variations in the velocity of turbulent fluids led to the introduction of multifractal descriptions of signals and functions. Turbulent flows are indeed far from being spatially homogeneous, and the pointwise regularity of the velocity may vary significantly from one point to another; this is typical of multifractal functions and measures. A fundamental step was then made by Frisch and Parisi, who uncovered a striking connection between multifractality and the global scaling properties of the velocity [66]. This connection, now known as the *multifractal formalism*, provided a unifying framework that subsequently stimulated the development of multifractal analysis in a wide range of mathematical settings, including geometric measure theory, real analysis [42, 86, 160], dynamical systems and ergodic theory, and Diophantine approximation [59, 161, 162], among others. In parallel, the development of numerical methods for the estimation of multifractal characteristics led to significant advances in signal and image processing [175, 176], and in other scientific fields [2, 99].

It is important to emphasise that, in practice, measurements of turbulent velocity fields are typically performed along one-dimensional sets, rather than over full three-dimensional regions of $\mathbb{R}^3$. As a consequence, the multifractal properties of turbulent flows are often inferred from one-dimensional traces, which naturally raises the question of how accurately such traces reflect the multifractal structure of the underlying three-dimensional flow. This problem is, in fact, quite general and applies to all three-dimensional phenomena for which only partial measurements are available.

Our main objective is to rigorously examine the validity of this inference procedure of multifractality for functions in inhomogeneous Besov spaces $B^{\xi}_{p,q}$. We establish precise relationships between the regularity properties of functions in $B^{\xi}_{p,q}$ and those of their traces restricted to affine subspaces of dimension $d < D$, see (5.4) for the definition of traces.

Let us now give precise definitions. Multifractal analysis consists of describing the pointwise behaviour of functions $f : \mathbb{R}^D \to \mathbb{R}$, computing the singularity spectrum σ_f, and, when possible, relating σ_f to the scaling function associated with f. Given $x_0 \in \mathbb{R}^D$, $H \in \mathbb{R}_+$, a locally bounded function $f : \mathbb{R}^D \to \mathbb{R}$ belongs to $\mathcal{C}^H(x_0)$ if there exist a polynomial P_{x_0} of degree less than $\lfloor H \rfloor$, a constant $C > 0$, and a neighborhood V of x_0 such that

$$\forall x \in V, \quad |f(x) - P_{x_0}(x - x_0)| \leq C\,|x - x_0|^H.$$

The *pointwise Hölder exponent* of $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^D)$ at x_0 is then

$$h_f(x_0) = \sup\big\{H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0)\big\}. \tag{5.1}$$

The associated *singularity* (or *multifractal*) *spectrum* σ_f of f is the mapping

$$\sigma_f : h \mapsto \dim_H\big(E_f(h)\big), \quad \text{where } E_f(h) := \big\{x \in \mathbb{R}^D : h_f(x) = h\big\}. \tag{5.2}$$

Here, $\dim_H$ stands for the Hausdorff dimension with the convention $\dim_H(\emptyset) = -\infty$ (note that $E_f(h) = \emptyset$ for $h < 0$). The singularity spectrum provides a geometrical description of the distribution of the singularities of f.

Let us emphasise that for large classes of functions f (similarly, of Borel measures), the singularity spectrum of f satisfies a so-called *multifractal formalism*, meaning that σ_f coincides with the Legendre transform τ_f^* of a scaling function $\tau_f : \mathbb{R} \to \mathbb{R}$ associated with f, i.e.

$$\sigma_f = \tau_f^*(h) := \inf_{q \in \mathbb{R}} (hq - \tau_f(q)). \tag{5.3}$$

The scaling function τ_f, defined using wavelets (see below and [92, 95]), describes the statistical scaling properties of f. When (5.3) holds, one says that the *multifractal formalism* holds for f. This relationship is key, since when properly defined, τ_f has proved to be numerically estimable on real data, see for instance [95].

Results on the pointwise regularity of functions belonging to classical spaces such as Sobolev $W^{s,p}(\mathbb{R}^D)$ or Besov $B^s_{p,q}(\mathbb{R}^D)$ spaces have been obtained in [9, 63, 87, 90, 98]. They are of two types: first, identify a universal upper bound for the singularity spectrum of all functions in a given space, and second, when possible, compute the explicit value for the spectrum of a generic set of functions in the sense of prevalence or Baire categories.

A remarkable feature of generic functions in $W^{s,p}(\mathbb{R}^D)$ and $B^s_{p,q}(\mathbb{R}^D)$ is that their singularity spectrum is invariably affine increasing [62, 90]. This stands in stark contrast to the fact that spectra estimated from real-world data via the multifractal formalism consistently exhibit a strictly concave profile, featuring both an increasing and a decreasing part [2, 99, 175]. Numerous generalisations of these classical Sobolev and Besov spaces have been studied extensively [7, 52, 58, 62, 67, 90, 115, 124, 138, 167]; for further details, see also [152]. However, in none of these cases do generic functions exhibit a concave singularity spectrum with both increasing and decreasing components.

Motivated by these observations, Barral and Seuret [19] introduced inhomogeneous Besov spaces $B^\xi_{p,q}(\mathbb{R}^D)$, where Baire-typical functions may admit any prescribed admissible

continuous concave singularity spectrum. The precise definition of these spaces will be addressed later. The inhomogeneity in these spaces is governed by a Hölder capacity ξ, which can be interpreted as a measure with strongly location-dependent mass distribution. This framework naturally captures the multifractal behaviour observed in empirical data.

This chapter deals with the multifractal analysis of traces of functions in inhomogeneous Besov spaces $B^{\xi}_{p,q}(\mathbb{R}^D)$ on d-dimensional affine subspaces of $\mathbb{R}^D$, when ξ is the product of Gibbs measures on $\mathbb{R}^D$. Let $0 < d < D$ be two fixed integers and $d' := D - d$. For $a \in \mathbb{R}^{d'}$, we denote $\mathcal{C}_a := \{(x, a) : x \in \mathbb{R}^d\}$ the horizontal d-dimensional affine subspace of $\mathbb{R}^D$ passing by $(0^d, a)$. The trace of a function f on $\mathcal{C}_a$ is defined by

$$f_a := f|_{\mathcal{C}_a} : \begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x, a). \end{array} \tag{5.4}$$

This notion has been considered for functions in standard Besov spaces by Aubry-Maman-Seuret and Jaffard [10, 34, 85, 129] and weighted Besov spaces by Besoy-Haroske-Triebel [26], for example. Universal and generic results in the sense of prevalence and Baire categories on the pointwise regularity of traces of functions belonging to classical spaces have been obtained by Aubry, Maman and Seuret [10, 129]. This chatper presents two main results: first, for any function f in an inhomogeneous Besov space $B^{\xi}_{p,q}(\mathbb{R}^D)$, we establish that all its traces f_a belong to another specific inhomogeneous Besov space, which we characterise explicitly, while simultaneously demonstrating that most traces exhibit higher regularity than anticipated (Theorem 5.12). Second, we prove that for a prevalent set of functions in $B^{\xi}_{p,q}(\mathbb{R}^D)$, the multifractal properties of the traces f_a can be determined (Theorem 5.14). Notably, unlike in standard Besov spaces where a unique singularity spectrum is typically observed for most traces f_a, inhomogeneous spaces admit a richer structure: depending on the choice of a (within large parameter sets that we describe), the traces may display distinct multifractal behaviours.

The present chapter describes two kinds of results: first, for any function f in an inhomogeneous Besov space, we show that all its traces f_a belong to an inhomogeneous Besov space that we identify, and simultaneously that most traces are more regular than expected (Theorem 5.12). Second, we prove that for a prevalent set of functions, the multifractal properties of the traces can be explicitly determined (Theorem 5.14). In particular, contrary to standard Besov spaces where there is only one well-identified singularity spectrum for most of the traces f_a, in inhomogeneous spaces, depending on the choice of a (in large sets of parameters that we describe), the traces may have distinct multifractal behaviours.

Let us now give precise statements. Let $\mathcal{B}(\mathbb{R}^D)$ stand for the Borel sets in $\mathbb{R}^D$, the diameter of a set E is denoted by $|E|$ and its D-dimensional Lebesgue measure is $\mathcal{L}^D(E)$.

Definition 5.1. *Let $D \geq 1$. The set of Hölder set functions on $\mathbb{R}^D$ is*

$$\mathcal{H}(\mathbb{R}^D) = \big\{\xi : \mathcal{B}(\mathbb{R}^D) \to \ \mathbb{R}_+ \cup \{+\infty\} : \exists C, s > 0, \ \forall E, F \in \mathcal{B}(\mathbb{R}^D), \xi(E) \leq C|E|^s\big\}.$$

The set of Hölder capacities is then

$$\mathcal{C}(\mathbb{R}^D) = \big\{\xi : \mathcal{H}(\mathbb{R}^D) \to \ \mathbb{R}_+ \cup \{+\infty\} : \forall E, F \in \mathcal{B}(\mathbb{R}^D), E \subset F \Rightarrow \xi(E) \leq \xi(F)\big\}.$$

When $\xi(E) \le C|E|^s$ for all E, ξ is said to be s-Hölder. The topological support $\mathrm{supp}(\nu)$ of $\nu \in \mathcal{H}(\mathbb{R}^D)$ is the set of points $x \in \mathbb{R}^D$ for which $\nu(B(x,r)) > 0$ for every $r > 0$. A capacity ν is fully supported on A when $\mathrm{supp}(\nu) = A$. We mainly look at capacities supported on $[0,1]^D$, but our results can be extended to $\mathbb{R}^D$.

For $\xi \in \mathcal{B}(\mathbb{R}^D)$, $s > 0$, and $E \in \mathcal{B}(\mathbb{R}^D)$, define the set functions ξ^s, $\xi^{(+s)}$ and $\xi^{(-s)}$ by

$$\xi^s(E) = \xi(E)^s \quad \text{and} \quad \xi^{(+s)}(E) = \xi(E)|E|^s,$$

and when ξ is s_0-Hölder, set for all $s \in (0, s_0)$, $\xi^{(-s)}(E) = \begin{cases} 0 & \text{if } |E| = 0, \\ \xi(E)|E|^{-s} & \text{if } 0 < |E| < \infty. \end{cases}$

Observe that if $\xi \in \mathcal{C}(\mathbb{R}^D)$, then ξ^s and $\xi^{(+s)}$ are still Hölder capacities.

Recall that $\xi \in \mathcal{C}(\mathbb{R}^D)$ is doubling if for some $C_\xi \ge 1$, for every $x \in \mathbb{R}^D$ and $r > 0$,

$$\xi(B(x,2r)) \le C_\xi \, \xi(B(x,r)). \tag{5.5}$$

Remark 5.2. A doubling s-Hölder capacity $\xi \in \mathcal{C}([0,1]^d)$ satisfies, for $C > 0$, for all $r > 0$,

$$C^{-1} r^{\log_2(C_\xi)} \le \xi(B(x,r)) \le C r^s. \tag{5.6}$$

The multifractal properties of a set function are based on local dimensions. The convention $0^q = 0$ is adopted.

Definition 5.3. *Let $\xi \in \mathcal{H}(\mathbb{R}^D)$. The lower local dimension of ξ at $x \in \mathrm{supp}(\xi)$ is*

$$\underline{h}_\xi(x) = \liminf_{r \to 0^+} \frac{\log \xi(B(x,r))}{\log(r)}. \tag{5.7}$$

The singularity spectrum of ξ is given by

$$\sigma_\xi \ : \ h \in \mathbb{R} \mapsto \dim_H \underline{E}_\xi(h) \quad \textit{where} \quad \underline{E}_\xi(h) = \{x \in \mathrm{supp}(\xi) \ : \ \underline{h}_\xi(x) = h\}.$$

For every integer $j \ge 1$, let Λ_j^D be the set of dyadic cubes of side length 2^{-j} in $[0,1]^D$. The scaling function of $\xi \in \mathcal{H}(\mathbb{R}^D)$ with $\mathrm{supp}(\xi) \cap [0,1]^D \neq \emptyset$ is defined by

$$\tau_\xi(q) = \liminf_{j \to +\infty} \frac{1}{-j} \log_2 \sum_{\lambda \in \Lambda_j^D} .\xi(\lambda)^q,$$

The minimal and maximal local dimension and the dimension of ξ are defined by

$$h_\xi^{\min} = \min\{\underline{h}_\xi(x) : x \in \mathrm{supp}(\xi)\}, \quad h_\xi^{\max} = \max\{\underline{h}_\xi(x) : x \in \mathrm{supp}(\xi)\}, \tag{5.8}$$

$$\dim(\xi) = \inf\{\dim_H(E) : \xi(E) = 1\}. \tag{5.9}$$

This chapter considers capacities μ and ν defined as powers of Gibbs measures. Let us recall how they are defined, see [147].

Definition 5.4. *Let $\varphi : \mathbb{R}^D \to \mathbb{R}$ be a $\mathbb{Z}^D$-periodic Hölder continuous function (called a potential). The sequence of measures defined on $\mathbb{R}^D$ whose densities are given by*

$$\nu_n(dx) = \frac{\exp(S_n\varphi(x))}{\int_{[0,1]^D} \exp(S_n\varphi(t)) \mathcal{L}^D(dt)} \mathcal{L}^D(dx), \quad \textit{where } S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(2^k x),$$

converges weakly to the $\mathbb{Z}^D$-invariant Gibbs measure ν_φ associated with φ.

These measures have been extensively studied [82, 144, 145, 146, 147, 159]. We recall in Section 5.2 their main multifractal properties. For the moment, we need the following ones: Given a fixed potential $\varphi : \mathbb{R}^D \to \mathbb{R}$:

1. The Gibbs measure ν_φ is doubling, and $\mathrm{supp}(\nu_\varphi) = \mathbb{R}^D$.
2. For every $h \geq 0$, $\sigma_{\nu_\varphi}(h) = \tau^*_{\nu_\varphi}(h) := \inf_{r\in\mathbb{R}}(rh - \tau_{\nu_\varphi}(r))$.
3. For every $r \in \mathbb{R}$, setting
$$h^r_{\nu_\varphi} = \tau'_{\nu_\varphi}(r), \tag{5.10}$$
one has $\sigma_{\nu_\varphi}(h^r_{\nu_\varphi}) = rh^r_{\nu_\varphi} - \tau_{\nu_\varphi}(r)$. Also, $h^{\min}_{\nu_\varphi} = h^{+\infty}_{\nu_\varphi}$ and $h^{\max}_{\nu_\varphi} = h^{-\infty}_{\nu_\varphi}$.
4. For every $r \in \mathbb{R}$, the Gibbs measure $\nu_{r\varphi}$ is called the auxiliary measure associated with φ and r (the auxiliary measure is associated with the potential $r\varphi$), and enjoys multifractal properties similar to those of ν_φ. Then
$$\nu_{r\varphi}(\underline{E}_{\nu_\varphi}(h^r_{\nu_\varphi})) = 1, \quad \text{i.e. for } \nu_{r\varphi}\text{-a.e. } x, \quad \underline{h}_{\nu_\varphi}(x) = h^r_{\nu_\varphi}. \tag{5.11}$$

Item (2) is referred to as the multifractal formalism for measures, and is analogous to the formalism (5.3) for functions. Item (3) follows from (5.10) and simple considerations on the Legendre transform. Item (4) states that the auxiliary measure $\nu_{r\varphi}$ is concentrated on those points x whose local dimension for ν_φ equals $h^r_{\nu_\varphi}$, see Theorem 5.22 for details on auxiliary measures. Since ν_φ is $\mathbb{Z}^D$-periodic, we identify ν_φ with its restriction to $[0,1]^D$.

Definition 5.5. *The set of Gibbs capacities $\mathscr{E}_D \subset \mathcal{C}(\mathbb{R}^D)$ is*
$$\mathscr{E}_D = \{\nu^s : \nu \text{ a Gibbs measure}, s > 0\}.$$

To state our results, we recall the definition of inhomogeneous Besov spaces. Write $L^D = \{0,1\}^D \setminus \{0^D\}$, and consider the family $\{\phi, \psi_\lambda := \psi^l(2^j \cdot -k)\}_{\lambda=(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^D\times L^D}$ generated by a collection of wavelets $(\phi, \psi^l)_{l\in L^D}$ (see Section 5.2.4 for details), such that every $f \in L^2(\mathbb{R}^D)$ can be written
$$f \overset{L^2}{=} \sum_{\lambda\in\mathbb{Z}\times\mathbb{Z}^D\times L^D} c_\lambda\psi_\lambda \quad \text{with} \quad c_\lambda = 2^{Dj}\int_{\mathbb{R}^D} f(x)\psi_\lambda(x)dx.$$
For $k = (k_1, \dots, k_D) \in \mathbb{Z}^D$, $j \in \mathbb{Z}$, we often identify $\lambda = (j,k,l)$ with the dyadic cube $\prod_{i=1}^D [k_i 2^{-j}, (k_i+1)2^{-j})$, and in particular we write $\xi(\lambda)$ for $\xi(\prod_{i=1}^D [k_i 2^{-j}, (k_i+1)2^{-j}))$.

For any $f \in L^p(\mathbb{R}^D)$, $1 \leq p \leq \infty$, f can be written
$$f = \sum_{k\in\mathbb{Z}^D} \beta(k)\phi(\cdot - k) + \sum_{j\geq 0}\sum_{\lambda\in\{j\}\times\mathbb{Z}^D\times L^D} c_\lambda\psi_\lambda, \quad \text{with} \quad \beta(k) = \int_{\mathbb{R}^D} f(x)\phi(x-k)dx. \tag{5.12}$$
Without loss of generality, we consider functions supported in $[0,1]^D$ and write
$$f = \beta(0)\phi + \sum_{j\geq 0}\sum_{\lambda\in\Lambda_j^D\times L^D} c_\lambda\psi_\lambda. \tag{5.13}$$

Indeed, any function $f \in L^2(\mathbb{R}^D)$ can be written as a sum $f = g + \sum_{\ell\in\mathbb{Z}^D} f_\ell(\cdot + \ell)$, where g is a function as regular as the wavelets and f_ℓ can be written as (5.13). Since we are interested only in local and multifractal properties, and using the fact that the Hausdorff

dimension of a countable union is the supremum of the dimension of all sets, the singularity spectrum of a locally bounded function $f \in L^2(\mathbb{R}^D)$ is obtained as the supremum of those of the f_ℓ.

So, from now on, we will work with functions defined on $[0,1]^D$ written as (5.13) and with a set function ξ with $\operatorname{supp}(\xi) = [0,1]^D$. By a slight abuse of notation, we write $\mathcal{H}([0,1]^D) = \{\xi \in \mathcal{H}(\mathbb{R}^D) : \operatorname{supp}(\xi) = [0,1]^D\}$ and $\mathcal{C}([0,1]^D) = \mathcal{H}([0,1]^D) \cap \mathcal{C}(\mathbb{R}^D)$.

Barral and Seuret introduced the following inhomogeneous Besov spaces.

Definition 5.6. *Let $\xi \in \mathcal{H}([0,1]^D)$ satisfying that for some $C > 0$ and $0 < s_1 < s_2$,*

$$\text{for all } x \in [0,1]^D, \ \text{ for all } r > 0, \quad C^{-1} r^{s_2} \le \xi(B(x,r)) \le C r^{s_1}. \tag{5.14}$$

Let $p \in [1, +\infty]$, consider an integer $n \ge \lfloor s_2 + D/p + 1 \rfloor$ and wavelets $(\phi, \psi^l)_{l \in L^D} \in C^n(\mathbb{R}^d)$ compactly supported, having at least n vanishing moments (see Section 5.2.4). A function f written as (5.13) belongs to $B^\xi_{p,q}([0,1]^d)$ if and only if $f \in L^p([0,1]^d)$ and $|f|_{\xi,p,q} < +\infty$, where

$$|f|_{\xi,p,q} = \left\| (\varepsilon_j^{\xi,p})_{j \in \mathbb{N}} \right\|_{\ell^q(\mathbb{N})}, \quad \text{with} \quad \varepsilon_j^{\xi,p} = \left\| \left(\frac{c_\lambda}{\xi(\lambda)} \right)_{\lambda \in \Lambda_j^D \times L^D} \right\|_{\ell^p(\Lambda_j^D \times L^D)}. \tag{5.15}$$

Then define

$$\widetilde{B}^\xi_{p,q}([0,1]^D) = \bigcap_{0 < \varepsilon < \min(1,s_1)} B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D). \tag{5.16}$$

Formula (5.15) implies that $|c_\lambda| \le C\, \xi(\lambda)$, for every cube λ. Recalling that the Hölder spaces $C^\alpha([0,1]^D)$ are characterised by the fact that $|c_\lambda| \le C\, |\lambda|^\alpha$ for every λ, the spaces $B^\xi_{p,q}([0,1]^D)$ can be understood as an inhomogeneous generalisation of $C^\alpha([0,1]^D)$, since the value of $\xi(\lambda)$ varies a lot depending on λ. This interpretation justifies the term environment for ξ.

Taking $\xi = (\mathcal{L}^D)^{\frac{s}{D} - \frac{1}{p}}$ with $s > D/p$, it implies that $B^\xi_{p,q}([0,1]^D) = B^s_{p,q}([0,1]^D)$. Also, when $p = q = \infty$, $B^{\xi,\psi}_{\infty,\infty}([0,1]^D)$ is the space of those functions f for which

$$C = \sup_{\lambda \in \mathbb{Z} \times \mathbb{Z}^D \times L^D} \frac{|c_\lambda|}{\xi(\lambda)} < +\infty.$$

It is proved in [19] that the definition of $B^\xi_{p,q}([0,1]^D)$ and $\widetilde{B}^\xi_{p,q}([0,1]^D)$ is independent of the choice of ψ as soon as ξ is doubling, and that the norms $\|\cdot\|_{L^p} + |\cdot|_{\xi,p,q}$ are equivalent for any choice of ψ. Since this will be our situation, we omit the dependence on ψ in the notations.

Remark 5.7. By construction, the family of Banach spaces $\left\{B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D)\right\}_{0<\varepsilon<\min(1,s_1)}$ satisfies that, for $0 < \varepsilon < \varepsilon' < \min(1, s_1)$, $B^{\xi^{(-\varepsilon)}}_{p,q}([0,1]^D) \hookrightarrow B^{\xi^{(-\varepsilon')}}_{p,q}([0,1]^D)$. Hence, in (5.16), the intersection can be taken on any interval of the form $(0, \varepsilon)$ with $\varepsilon < \min(1, s_1)$.

Also, $\widetilde{B}^\xi_{p,q}([0,1]^d)$ is a Fréchet space [19], and a translation-invariant complete metric inducing the same topology as $\left\{\|\cdot\|_{B^{\xi^{(-\varepsilon)}}_{p,q}} := \|\cdot\|_{L^p} + |\cdot|_{\xi^{(-\varepsilon)},p,q}\right\}_{0<\varepsilon<\min(1,s_1)}$ is

$$\forall f, g \in \widetilde{B}^\xi_{p,q}([0,1]^d), \quad \mathscr{D}(f,g) = \sum_{n > \max(1, s_1^{-1})} 2^{-n} \frac{\|f - g\|_{B^{\xi^{(-1/n)}}_{p,q}}}{1 + \|f - g\|_{B^{\xi^{(-1/n)}}_{p,q}}}. \tag{5.17}$$

We recall the multifractal properties of prevalent functions in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ [152]. We will come back soon to the notion of prevalence; for the moment, the reader shall have in mind that this is a natural substitute for "almost every element" in infinite-dimensional spaces.

Definition 5.8. *The capacity $\xi \in \mathcal{C}([0,1]^D)$ is said to satisfy the strong multifractal formalism (SMF) if for every $h \geq 0$, $\sigma_\xi(h) = \tau^*_\xi(h) = \dim_H E_\xi(h)$, where $E_\xi(h)$ is the set of points x where the local dimension (5.7) is a limit and equals h.*

Theorem 5.9 ([152])**.** *If $\xi \in \mathcal{C}([0,1]^D)$ is doubling and satisfies the SMF, then there exists a prevalent set of functions $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ such that $\sigma_f = \tau^*_\xi = \sigma_\xi$.*

Our objective is to study the multifractal properties of the traces of $f \in \widetilde{B}^{\xi}_{p,q}(\mathbb{R}^D)$ when ξ is a product of Gibbs measures. More precisely, if $D = d + d'$, for $\mu_\varphi \in \mathscr{E}_d$ (resp. $\nu_{\widetilde{\varphi}} \in \mathscr{E}_{d'}$) associated to a potential $\varphi : \mathbb{R}^d \to \mathbb{R}$, (resp. $\widetilde{\varphi} : \mathbb{R}^{d'} \to \mathbb{R}$), we consider the capacity on $\mathbb{R}^D$

$$\xi = \mu_\varphi \times \nu_{\widetilde{\varphi}}$$

i.e. for $A \in \mathcal{B}([0,1]^d)$, $B \in \mathcal{B}([0,1]^{d'})$, $\xi(A \times B) := \mu_\varphi(A) \cdot \nu_{\widetilde{\varphi}}(B)$. We omit the mention to φ and $\widetilde{\varphi}$, and write simply μ, ν, and also ν_r for the auxiliary measures associated with ν.

Theorem 5.9 applies to $\xi = \mu \times \nu$: there is a prevalent set of functions $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$ for which $\sigma_f = \tau^*_\xi$. The spectrum of ξ is determined in terms of μ and ν, see Section 5.2.3.

Proposition 5.10. *Let $\mu \in \mathscr{E}_d$ and $\nu \in \mathscr{E}_{d'}$ with $D = d + d'$. Let $\xi = \mu \times \nu$ be the product capacity on $[0,1]^D$. The product capacity ξ satisfies the SMF and verifies $\tau_\xi = \tau_\mu + \tau_\nu$.*

For standard Besov spaces, the trace operator $f \mapsto f_a$ maps $B^s_{p,q}(\mathbb{R}^D)$ to $B^{s-d'/p}_{p,q}(\mathbb{R}^d)$, which reflects a loss of regularity when taking traces. However, for $\mathcal{L}^{d'}$-a.e. a, $f_a \in B^s_{p,q}(\mathbb{R}^d)$. Our first result shows that the same phenomenon occurs for inhomogeneous spaces.

Theorem 5.11. *Let $p, q \in [1, +\infty]$. Let $\mu \in \mathscr{E}_d$ and $\nu \in \mathscr{E}_{d'}$ with $D = d + d'$. Let ξ be the product capacity on $[0,1]^D$ defined by $\xi = \mu \times \nu$.*

For all $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$ and all $r \in \mathbb{R}$, for ν_r-almost all $a \in [0,1]^{d'}$, one has

$$f_a \in \widetilde{B}^{\mu^{(+\Gamma_{\nu,r})}}_{p,1}([0,1]^d) \quad \text{with } \Gamma^p_{\nu,r} = h^r_\nu + \dim(\nu_r)/p.$$

By convention $\alpha/\infty = 0$. Observe that f_a belongs to a space whose regularity depends on the local regularity of ν at a, since by (5.11), ν_r-a.e. points a are such that $h_\nu(a) = h^r_\nu$.

For the next results, **we focus on the case $p = \infty$.** First, an upper-bound for the singularity spectrum of traces of all functions $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ is obtained.

Theorem 5.12. *For all $f \in \widetilde{B}^{\xi}_{\infty,q}$ and $a \in [0,1]^{d'}$, one has for every $h \in \mathbb{R}_+$*

$$\sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h^{\min}_\nu) & \text{if } h \leq h^0_\mu + h^{\min}_\nu, \\ d & \text{if } h > h^0_\mu + h^{\min}_\nu. \end{cases} \tag{5.18}$$

For all $f \in \widetilde{B}^{\xi}_{\infty,q}$, for every $r \in \mathbb{R}$ and ν_r-almost all $a \in [0,1]^{d'}$, one has for every $h \in \mathbb{R}_+$

$$\sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h^r_\nu) & \text{if } h \leq h^0_\mu + h^r_\nu, \\ d & \text{if } h > h^0_\mu + h^r_\nu. \end{cases} \tag{5.19}$$

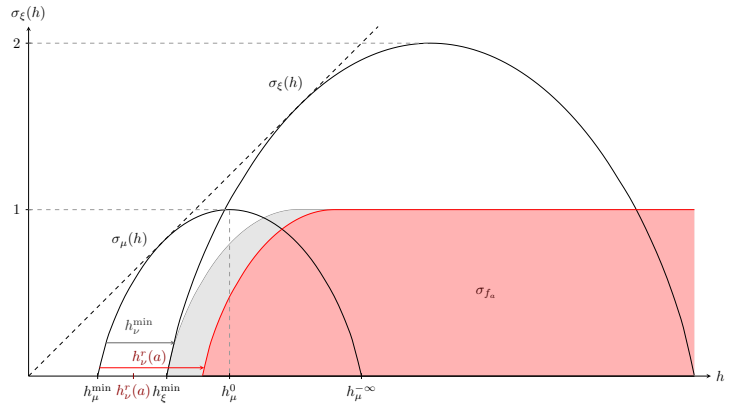


Figure 5.1: Example of upper-bound for $\sigma_{f_a}(h)$ for all $a \in [0,1]^{d'}$ in grey, and for ν_r-almost all $a \in [0,1]^{d'}$ in red for $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ where $\xi = \mu \times \nu$ with μ and ν have the same known singularity spectrum.

These upper bounds can be sharpened and turned into equalities for prevalent sets of functions in $\widetilde{B}^{\xi}_{\infty,q}$. As quickly mentioned above, the prevalence theory is used to supersede the notion of Lebesgue measure for Borel sets in any real or complex topological vector space E endowed with its Borel σ-algebra $\mathcal{B}(E)$. Christensen [44] and Hunt, Sauer and York [81] proposed this notion independently. In this chapter, all Borel measures ϑ on $(E, \mathcal{B}(E))$ will be automatically completed, i.e. for every set $A \subset E$, one sets $\vartheta(A) := \vartheta(B)$ if $B \in \mathcal{B}(E)$ and the symmetric difference $A \Delta B$ is included in some $D \in \mathcal{B}(E)$ with $\vartheta(D) = 0$.

Definition 5.13. *A set $A \subset E$ is universally measurable if it is measurable for any (completed) Borel measure on E. A universally measurable set $A \subset E$ is Haar-null if there exists a finite completed measure ϑ that is positive on some compact subset K of E and such that*

$$\text{for every } x \in E, \quad \vartheta(A + x) = 0. \tag{5.20}$$

The measure ϑ used to show that some subset is Haar-null is called transverse. More generally, a set that is included in a Haar-null universally measurable set is also called Haar-null.

Finally, the complement in E of a Haar-null subset S, denoted $S^{\complement}$, is called prevalent.

Hunt pioneered the use of prevalence in function spaces [80]. More recently, Bayart and Heurteaux [21] and Balka, Darji and Elekes [12] studied the dimension of images of continuous function, Fraysse, Jaffard and Kahane [63, 64] and then Aubry, Maman and Seuret [10] studied, respectively, the prevalent regularity of functions and the trace function f_a of function f in Besov spaces. Olsen studied the multifractal and L^q-dimension of prevalent measure on a compact of $\mathbb{R}^d$ [142, 143].

Our theorem regarding the prevalent multifractal behavior in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ is the following. It depends on the existence of a wavelet satisfying a property that we call (R). We come back to this after the statement.

Theorem 5.14. *Let $\mu \in \mathscr{E}_d$, $\nu \in \mathscr{E}_{d'}$. Consider the product measure $\xi = \mu \times \nu \in \mathcal{C}([0,1]^D)$, with $D = d + d'$. Call C, s_1, s_2 the real numbers such that (5.14) is satisfied.*

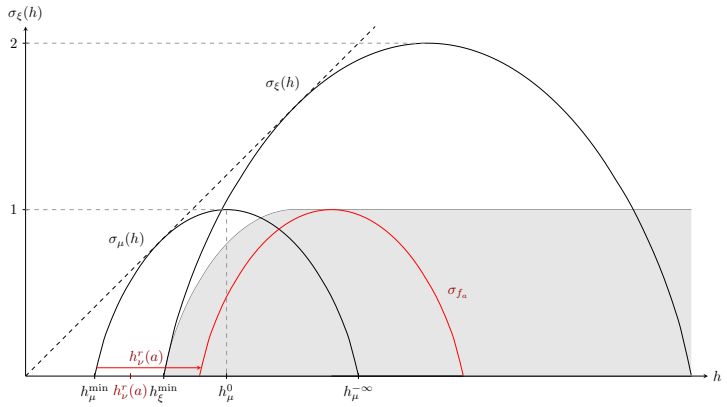


Figure 5.2: Example of complete bounds for $\sigma_{f_a}(h)$ for almost all $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ and ν_r-almost all $a \in [0,1]^{d'}$ for $\xi = \mu \times \nu$ with μ and ν have the same known singularity spectrum.

Let $n \geq \left\lfloor s_2 + \frac{D}{p} \right\rfloor + 1$. *Assume that there exists a compactly supported wavelet* $\psi \in C^n(\mathbb{R})$ *having (at least)* n *vanishing moments and satisfying the property* (R).

Let $(r_n)_{n\geq\mathbb{N}}$ *be a countable sequence of real numbers. There exists a prevalent set of function* $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ *such that for every* $n \in \mathbb{N}$, *for* ν_{r_n}*-almost all* $a \in (0,1]^{d'}$,

$$\sigma_{f_a}(h) = \begin{cases} \sigma_\mu(h - h_\nu^{r_n}) & \text{if } h \in \operatorname{supp}(\sigma_\mu) + h_\nu^{r_n} \\ -\infty & \text{else.} \end{cases}$$

Recalling that for $r = 0$ the auxiliary measure ν_r is the Lebesgue measure $\mathcal{L}^{d'}$, Theorem 5.14 gives that for $\mathcal{L}^{d'}$-almost every a, the singularity spectrum of f_a is $\sigma(\cdot - h_\nu^0)$. This is exactly similar to the result of [10] for standard Besov spaces. Here, in addition to what happens in standard Besov spaces, the traces do not all exhibit the same multifractal properties, and we can compute the singularity spectrum of many other traces: r being fixed, for ν_r-almost every a, the singularity spectrum σ_{f_a} of the trace f_a coincides with that of μ shifted by h_ν^r. This is somehow expected since the environment ξ is not spatially homogeneous, but it remains surprising nonetheless to be able to precisely describe this variety of regularity. Our work follows in the tradition of standard theorems in geometric measure theory, such as those of Marstrand–Mattila [134], where one studies the dimension of the slices of a set $E \subset \mathbb{R}^D$ of dimension α along affine subsets of dimension *strictly less than* D. In most of these results, there is a single almost-sure behaviour, and the set of exceptional behaviours is often small, or even of dimension 0. Here, by focusing on traces, we undertake a comparable analysis, but the inhomogeneity of the initial function implies that the regularity of the traces is highly dependent on the specific trace considered.

It is not entirely satisfactory that the result is proved only for a countable sequence $(r_n)_{n\in\mathbb{N}}$, which can be dense in $\mathbb{R}$. We conjecture that Theorem 5.14 holds for every $r \in \mathbb{R}$ simultaneously on a prevalent set. Similarly, it is a natural question to investigate the situation where $p < \infty$.

Theorem 5.14 depends on the existence of wavelets enjoying the following property.

Definition 5.15. *A function $\psi : \mathbb{R} \to \mathbb{R}$ satisfies the property (R) if:*
(R_1) *$\psi \in \mathcal{C}^1(\mathbb{R})$ and there exists $K_\psi \in \mathbb{N}^*$ such that $\overline{\mathrm{supp}(\psi)} = [0, K_\psi]$.*
(R_2) *The cardinality of the set $\mathcal{Z}(0,0) := \psi^{-1}(\{0\}) \bigcap [0, K_\psi]$ of zeros of ψ is finite.*
(R_3) *For all $x \in [0,1]$, $S(x) := \sum_{p=0}^{K_\psi - 1} |\psi(x+p)| > 0$.*

This property holds for Daubechies' wavelets up to order 45; see [153]. Since db45 $\in C^{10}(\mathbb{R})$, Theorem 5.14 applies whenever the local dimensions of ξ do not exceed 10. Proving that broad wavelet classes, including Daubechies', satisfy (R) is a relevant question, which we intend to address in future work. Property (R) will be crucial for estimating the pointwise regularity of prevalent traces, especially when proving Theorem 5.41 and the lower bounds for pointwise Hölder exponents derived therein.

Section 5.2 provides the necessary material to work with capacities that are products of Gibbs measures, as well as the notions related to wavelet decompositions. Section 5.3 contains the proof of Theorem 5.11 and compares it to those in standard Besov spaces. In Section 5.4, we prove Theorem 5.12. , the upper bounds for the singularity spectrum of functions in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. In Section 5.5, we introduce and study the regularity properties of a saturating function in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. Intuitively, such a function has the largest possible wavelet coefficients in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. This function is the cornerstone of the rest of the chapter. Finally, Theorem 5.14 is proved in section 5.6. We construct a prevalent set on which the multifractal behaviour of functions is controlled by finding suitable measures and probe sets based on the analysis of the saturating functions. One of the delicate issues consists of establishing the analyticity of the sets involved in our proofs.

5.2 Capacities, wavelets and multifractal formalism

5.2.1 Capacities and multifractal formalism

For $j \in \mathbb{N}$, recall that Λ_j^D is the set of dyadic cube of side length 2^{-j} on $[0,1]^D$, i.e. the dyadic cube $\prod_{i=1}^D [k_i 2^{-j}, (k_i+1)2^{-j})$ where $k = (k_1, \dots, k_D) \in \{0, \dots, 2^j - 1\}^D$. We denote $\Lambda^D = \bigcup_{j \in \mathbb{N}} \Lambda_j^D$ the set of dyadic cubes in $[0,1]^D$.

For $j \in \mathbb{N}$, $\lambda \in \Lambda_j^D$ and $N \in \mathbb{N}^*$, we denote $N\lambda$ the cube with the same centre as λ and radius equal to $N \cdot 2^{-j-1}$ in $(\mathbb{R}^D, \|\cdot\|_\infty)$. For instance, 3λ is the union of those $\lambda' \in \Lambda_j^D$ such that $\partial\lambda \cap \partial\lambda' \neq \emptyset$ where $\partial\lambda$ is the boundary of the cube λ.

For $x \in \mathbb{R}^d$, denote $\lambda_j^D(x)$ the dyadic cubes of side length 2^{-j} containing x.

For any Borel set $A \subset \mathbb{R}^D$ and capacity ν with $\nu(A) > 0$, we will denote $\nu_{|A} = \nu(. \cap A)$.

In addition to the lower local dimension already defined in Definition 5.3, one also needs the upper and limit local dimension of set functions.

Definition 5.16. *Let $\nu \in \mathcal{H}([0,1]^D)$. For $x \in \mathrm{supp}(\nu)$, the lower and upper local dimension of ν at x are respectively defined by*

$$\underline{h}_\nu(x) = \liminf_{r \to 0^+} \frac{\log \nu(B(x,r))}{\log(r)} \quad \text{and} \quad \overline{h}_\nu(x) = \limsup_{r \to 0^+} \frac{\log \nu(B(x,r))}{\log(r)}.$$

Whenever $\underline{h}_\nu(x) = \overline{h}_\nu(x)$*, the common limit is called* $h_\nu(x)$*. Then, for* $h \in \mathbb{R}$*,*

$$\underline{E}_\nu(h) = \{x \in \mathrm{supp}(\nu) \ : \ \underline{h}_\nu(x) = h\}, \quad \overline{E}_\nu(h) = \{x \in \mathrm{supp}(\nu) \ : \ \overline{h}_\nu(x) = h\}$$
$$\text{and} \quad E_\nu(h) = \underline{E}_\nu(h) \cap \overline{E}_\nu(\alpha).$$

Remark 5.17. Note that the local dimensions are sometimes defined as

$$\underline{h}_\nu(x) = \liminf_{j\to+\infty} \frac{\log_2 \nu(\lambda_j(x))}{-j} \quad \text{and} \quad \overline{h}_\nu(x) = \limsup_{j\to+\infty} \frac{\log_2 \nu(\lambda_j(x))}{-j},$$
$$\text{or} \quad \underline{h}_\nu(x) = \liminf_{j\to+\infty} \frac{\log_2 \nu(3\lambda_j(x))}{-j} \quad \text{and} \quad \overline{h}_\nu(x) = \limsup_{j\to+\infty} \frac{\log_2 \nu(3\lambda_j(x))}{-j}.$$

However, this chapter only considers doubling capacities for which all previous notions of local dimensions, level sets, singularity spectrum and scaling function do not depend on whether dyadic cubes or centred balls are considered.

Let us introduce the notation for $\varepsilon > 0$, $h \in \mathbb{R}$ and $I = [a, b]$ an interval

$$h \pm \varepsilon = [h - \varepsilon, h + \varepsilon] \quad \text{and} \quad I \pm \varepsilon = [a - \varepsilon, b + \varepsilon]. \tag{5.21}$$

Definition 5.18. *For* $\nu \in \mathcal{H}([0,1]^D)$*,* $I \subset \mathbb{R}$ *and* $j \in \mathbb{N}^*$*, set*

$$\mathcal{D}_\nu(j, I) = \Big\{\lambda \in \Lambda_j^D \ : \ \frac{\log_2 \nu(\lambda)}{-j} \in I\Big\}.$$

By Remark 5.17 for the almost doubling capacities, the level sets $E_\nu(h)$ can be rewritten as

$$E_\nu(h) = \bigcap_{m\in\mathbb{N}^*} \bigcup_{J\in\mathbb{N}} \bigcap_{j\geq J} \bigcup_{\lambda\in\mathcal{D}_\nu(j,h\pm1/m)} \lambda \tag{5.22}$$

which are Borel sets as, for all $m \in \mathbb{N}^*$,

$$\mathcal{D}_\nu(j, h \pm 1/m) = \Big\{\lambda \in \Lambda_j^D \ : \ \frac{\log_2 \nu(\lambda)}{-j} \in h \pm 1/m\Big\}$$

is a finite set of dyadic cubes at generation j.

A classical result on capacities states that for every $\nu \in \mathbb{C}([0,1]^D)$, one always has $\sigma_\nu(h) \leq \tau_\nu^*(h)$ (see [33, 90, 122]). Proposition 5.19 can be deduced from [15, 16, 19, 33, 122, 141].

Proposition 5.19. *Let* $\nu \in \mathcal{C}([0,1]^D)$ *with* $supp(\nu) \neq \emptyset$*, such that* ν *obeys the SMF. Then:*

(*i*) *For every* $h \in \mathbb{R}^+$*, one has* $\sigma_\nu(h) = \dim_H\big(\underline{E}_\nu(h)\big) = \dim_H\big(\overline{E}_\nu(h)\big) = \tau_\nu^*(h)$.

(*ii*) *For every* $h \leq \tau_\nu'(0^+)$, $\dim_H\big(\underline{E}_\mu^{\leq}(h)\big) = \tau_\mu^*(h)$.

(*iii*) *For every* $h \geq \tau_\nu'(0^-)$, $\dim_H\big(\overline{E}_\mu^{\geq}(h)\big) = \tau_\mu^*(h)$.

(*iv*) *For every* $\tau_\nu'(0^+) \leq h \leq \tau_\mu'(0^-)$, $\sigma_\mu(h) = D$.

(*v*) *Denote* $\mathrm{dom}(\tau_\nu^*) = \{\alpha \in \mathbb{R} \ : \ \tau_\nu^*(\alpha) \geq 0\}$*. If* $\mathrm{dom}(\tau_\nu^*)$ *is compact, then* $\mathrm{dom}(\tau_\nu^*) = [h_\nu^{+\infty}, h_\nu^{-\infty}]$ *and there exists a positive decreasing sequence* $(\varepsilon_j)_{j\geq0}$ *tending to 0 when* j *tends to infinity, such that for all* $j \in \mathbb{N}$ *and* $\lambda \in \Lambda_j^D$*,*

$$h_\nu^{+\infty} - \eta_j \leq \frac{\log_2 \nu(\lambda)}{-j} \leq h_\nu^{-\infty} + \eta_j. \tag{5.23}$$

Remark 5.20. (1) Direct computations show that for any $s > 0$, $\nu \in \mathcal{C}([0,1]^D)$, $t \in \mathbb{R}$, $\tau_{\nu^s}(t) = \tau_\nu(st)$, and by Legendre transform, for $h \in \mathbb{R}$, $\tau_{\nu^s}^*(h) = \tau_\nu^*\big(h/s\big)$.

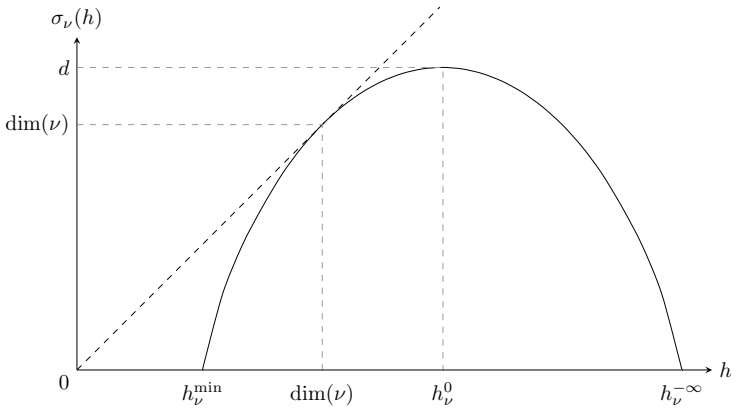


Figure 5.3: Example of singularity spectrum σ_ν for a Gibbs measure ν.

(2) Let $\nu \in \mathcal{C}([0,1]^D)$ satisfying (5.14). One can check that the set functions ν^s, $\nu^{(+s)}$ and, when s is small enough, $\nu^{(-s)}$, satisfy the same property with different exponents, and
- $\underline{E}_{\nu^s}(\alpha) = \underline{E}_\nu\big(\alpha/s\big)$, $\overline{E}_{\nu^s}(\alpha) = \overline{E}_\nu\big(\alpha/s\big)$, and $E_{\nu^s}(\alpha) = E_\nu\big(\alpha/s\big)$,
- the singularity spectrum of ν^s is $h \mapsto \sigma_\nu\big(h/s\big)$.

5.2.2 Gibbs measure and related capacity

We gather classical properties satisfied by the Gibbs capacities $\mathscr{E}_D$ (see [82, 144, 145, 146, 147, 159]).

Property 5.21. *Let ν be a Gibbs measure. Then for every $s > 0$:*

(i) The capacity $\nu^s \in \mathscr{E}_D$ is doubling (5.6) and satisfies the SMF of Definition 5.8.

(ii) The dimension of ν^s (5.9) equals $\tau'_\nu(1) = \frac{1}{s}\tau'_{\nu^s}(1/s)$. Hence, $\dim(\nu^s) = \dim(\nu)$.

(iii) The mapping τ_{ν^s} is strictly concave, real-analytic.

(iv) The mapping $\tau^*_{\nu^s}$ has $\mathrm{dom}(\tau^*_{\nu^s}) = [h_{\nu^s}^{+\infty}, h_{\nu^s}^{-\infty}] \subset \mathbb{R}_+^*$ and is strictly concave, real-analytic on $(h_\nu^{+\infty}, h_\nu^{-\infty})$.

(v) The capacity ν^s satisfies a quasi-Bernoulli property: there is a constant $C \geq 1$ such that for every $j_1, j_2 \in \mathbb{N}$, for every dyadic cube $\lambda^D_{j_1,k_1} \in \Lambda^D$ and $\lambda^D_{j_2,k_2} \in \Lambda^D$,

$$\frac{1}{C}\ \mu(\lambda^D_{j_1,k_1})\ \mu(\lambda^D_{j_2,k_2}) \leq \mu(\lambda^D_{j_1+j_2,(2^{j_2}k_1+k_2)}) \leq C\ \mu(\lambda^D_{j_1,k_1})\ \mu(\lambda^D_{j_2,k_2}) \tag{5.24}$$

In (5.24), the cube $\lambda^D_{j_1+j_2,(2^{j_2}k_1+k_2)}$ is the image of $\lambda^D_{j_2,k_2}$ by the canonical affine mapping that sends $[0,1]^D$ to $\lambda^D_{j_1,k_1}$. This quasi-Bernoulli property means that ν^s is almost multiplicative.

Figure 5.3 illustrates the typical shape of σ_ν when ν is a Gibbs measure. Observe that the smallest local dimension of ν is $h_\nu^{\min} = h_\nu^{+\infty}$ and the largest one is $h_\nu^{\max} = h_\nu^{-\infty}$. Using item (iii), one has that $h_\nu^0 = \tau'_\nu(0^+) = \tau'_\nu(0^-)$ is the unique maximum of the spectrum.

As explained earlier, auxiliary measures of Gibbs measures play a key role in our analysis. Let us regroup some important results about them, see [33, 66, 72, 78, 141].

Theorem 5.22 (Auxiliary measure)**.** *Let ν be a Gibbs measure associated to a potential φ on $\mathbb{R}^D$ and $r \in \mathbb{R}$. There exists a probability measure ν_r, called* auxiliary measure*, satisfying for a constant $C \geq 1$ and for all cubes $I \in \Lambda^D$*

$$\frac{1}{C}\, \nu(I)^r |I|^{-\tau_\nu(r)} \leq \nu_r(I) \leq C\, \nu(I)^r |I|^{-\tau_\nu(r)}. \tag{5.25}$$

The measure ν_r has the following properties (recall the definition (5.10) of h^r_ν):

(i) ν_r is a Gibbs measure associated to the potential $r\varphi$ and satisfies the SMF,

(ii) One has $h_{\nu_r} = \sigma_\nu(r^r_\nu)$ and $\nu_r\big(E_{\nu_r}(h_{\nu_r})\big) = \nu(E_\nu(h^r_\nu)) = 1$,

(iii) $\sigma_\nu(h^r_\nu) = \dim(\nu_r) = rh^r_\nu - \tau_\nu(r) \leq h^r_\nu$.

Remark 5.23. If we consider a capacity ν satisfying the SMF, one sees from the definition that $E_\nu(h^r_\nu) = E_{\nu_r}(\sigma_\nu(h^r_\nu))$. Indeed, using (5.25),

$$h_{\nu_r}(x) = \lim_{j\to+\infty} \frac{\log_2 \nu_r(\lambda_j(x))}{-j} = r \lim_{j\to+\infty} \frac{\log_2 \nu(\lambda_j(x))}{-j} - \tau_\nu(r) = rh^r_\nu - \tau_\nu(r) = \sigma_\nu(h^r_\nu).$$

Remark 5.24. Consider a Gibbs measure ν and $\mu := \nu^s \in \mathscr{E}_d$ with $s > 0$. Then using Remark 5.20, the auxiliary measure $\mu_r = \nu_{rs}$ satisfying (5.25) and the properties of Theorem 5.22:

(i) $\mu_r = \nu_{rs}$ is a Gibbs measure and satisfies the SMF,

(ii) $\mu_r\big(E_{\mu_r}(\sigma_\mu(\tau'_\mu(r)))\big) = \nu_{rs}\big(E_{\nu_{rs}}(\sigma_\nu(\tau'_\nu(rs)))\big) = 1$,

(iii) $\dim(\mu_r) = \dim(\nu_{rs}) = \sigma_\nu\big(\tau'_\nu(rs)\big) = \sigma_\mu\big(\tau'_\mu(r)\big)$.

The following result introduces quantities important for what follows. It states that, up to a restriction onto some set $A^{\nu_r}_{n,m}$ of large ν_r-measure, the ν-mass and ν_r-mass of the dyadic cubes intersecting the neighbourhood of $A^{\nu_r}_{n,m}$ can be very precisely controlled.

Proposition 5.25. *Let $\nu \in \mathscr{E}_D$, $n, m \in \mathbb{N}^*$ and $r \in \mathbb{R}$. For any fixed constant $K \in \mathbb{N}^*$, there exist an integer $J^{\nu_r}_{n,m} \in \mathbb{N}$ and a set $A^{\nu_r}_{n,m}$ satisfying the following properties :*

1. *$\nu_r(A^{\nu_r}_{n,m}) \geq 1 - 1/n$,*
2. *for every $a \in A^{\nu_r}_{n,m}$, for every $j \geq J^{\nu_r}_{n,m}$ and for $\lambda^D \in \Lambda^D_j$ such that $a \in K\lambda^D$,*

$$2^{-j(h^r_\nu + 1/m)} \leq \nu\big(\lambda^D\big) \leq 2^{-j(h^r_\nu - 1/m)}, \tag{5.26}$$

$$2^{-j(\dim(\nu_r)+1/m)} \leq \nu_r\big(\lambda^D\big) \leq 2^{-j(\dim(\nu_r)-1/m)}. \tag{5.27}$$

Proof. Consider $\nu \in \mathscr{E}_D$ and its associated auxiliary measure ν_r. Using Definition 5.18, for all $\widetilde{m} \in \mathbb{N}^*$, $\mathcal{D}_\nu(j, h \pm 1/\widetilde{m}) = \Big\{\lambda \subset [0,1]^D, \lambda \in \Lambda^D_j : \frac{\log_2 \nu(\lambda)}{-j} \in h \pm 1/\widetilde{m}\Big\}$ is a finite set of dyadic cubes at generation j. Applying (5.22), ν and ν_r being doubling, one has

$$E_\nu(h) = \bigcap_{\widetilde{m}\in\mathbb{N}^*} \bigcup_{J\in\mathbb{N}} \bigcap_{j\geq J} \bigcup_{\lambda\in\mathcal{D}_\nu(j,h\pm1/\widetilde{m})} \lambda.$$

and from Theorem 5.22, $\nu_r\big(E_{\nu_r}(\sigma_\nu(h^r_\nu))\big) = \nu_r\big(E_\nu(h^r_\nu)\big) = 1$. One deduces that for every $\widetilde{m} \in \mathbb{N}^*$,

$$\nu_r\Big(\bigcup_{J\in\mathbb{N}} \bigcap_{j\geq J} \bigcup_{\lambda\in\mathcal{D}_\nu(j,h^r_\nu\pm1/\widetilde{m})} \lambda\Big) = 1.$$

The fact that $\big(\bigcap_{j\geq J}\bigcup_{\lambda\in\mathcal{D}_\nu(j,h_\nu^r\pm 1/\widetilde{m})}\lambda\big)_{J\geq 1}$ is an increasing sequence of Borel sets implies that $\lim_{J\to+\infty}\nu_r\big(\bigcap_{j\geq J}\bigcup_{\lambda\in\mathcal{D}_\nu(j,h_\nu^r\pm 1/\widetilde{m})}\lambda\big)=1$, so there is $J_{n,\widetilde{m}}\in\mathbb{N}$ such that the Borel set

$$A_{n,\widetilde{m}}^{\nu_r}:=\bigcap_{j\geq J_{n,\widetilde{m}}}\bigcup_{\lambda\in\mathcal{D}_\nu(j,h_\nu^r\pm 1/\widetilde{m})}\lambda \tag{5.28}$$

satisfies $\nu_r(A_{n,\widetilde{m}}^{\nu_r})\geq 1-1/n$. Using item (iii) of Theorem 5.22, $\dim(\nu_r)=\sigma_\nu(h_\nu^r)\leq h_\nu^r$. From (5.28), for $n,\widetilde{m}\in\mathbb{N}^*$, there is an integer $J_{n,\widetilde{m}}$ such that for every $a\in A_{n,\widetilde{m}}^{\nu_r}$ and $j\geq J_{n,\widetilde{m}}$,

$$2^{-j(h_\nu^r+1/\widetilde{m})}\leq\nu\big(\lambda_j^D(a)\big)\leq 2^{-j(h_\nu^r-1/\widetilde{m})} \tag{5.29}$$

where we recall that $\lambda_j^D(a)$ is the dyadic cube $\lambda_j^D\in\Lambda_j^D$ such that $a\in\lambda_j^D$.

Fix $K\in\mathbb{N}^*$. For $\lambda_j^D\in\Lambda_j^D$ such that $a\in K\lambda_j^D$, call $x_{j,a}$ the centre of $\lambda_j^D(a)\in\Lambda_j^D$ and $\widetilde{x}$ the center of λ_j^D. From the definition of $K\lambda^D$, we have that $\|a-\widetilde{x}\|_\infty\leq 2^{-j-1+\log_2(K)}$, so

$$\lambda_j^D\subset B\big(x_{j,a},2^{-j+\lfloor\log_2(K)\rfloor+1}\big)\ \text{ and }\ \lambda_j^D(a)\subset B\big(\widetilde{x},2^{-j+\lfloor\log_2(K)\rfloor+1}\big).$$

Using the doubling property (5.5) of ν, one gets

$$\frac{1}{C_\nu^{\lfloor\log_2(K)\rfloor+1}}\nu(\lambda_j^D(a))\leq\nu(\lambda_j^D)\leq C_\nu^{\lfloor\log_2(K)\rfloor+1}\nu(\lambda_j^D(a))$$

and the same result holds for ν_r as it is also doubling. Hence, one has from (5.29)

$$2^{-j(h_\nu^r+1/\widetilde{m_j})}\leq\nu\big(\lambda_j^D\big)\leq 2^{-j(h_\nu^r-1/\widetilde{m_j})}$$

with $\frac{1}{\widetilde{m_j}}=\frac{1}{\widetilde{m}}+\frac{(\lfloor\log_2(K)\rfloor+2)\ \log_2(C_\nu)}{j}$. Moreover, by (5.25), for every $r\in\mathbb{R}$, there is $C_r\geq 1$ such that

$$\frac{1}{C_r}\,\nu\big(\lambda_j^D\big)^r\big|\lambda_j^D\big|^{-\tau_\nu(r)}\leq\ \nu_r\big(\lambda_j^D\big)\leq C_r\,\nu\big(\lambda_j^D\big)^r\big|\lambda_j^D\big|^{-\tau_\nu(r)}.$$

One deduces that, for every $\lambda_j^D\in\Lambda_j^D$ such that $a\in K\lambda_j^D$,

$$\begin{aligned}2^{-j(rh_\nu^r-\tau_\nu(r)+1/\widetilde{m_j}+\frac{\log_2(C_r)}{j})}&\leq\nu_r(\lambda_j^D)\leq 2^{-j(rh_\nu^r-\tau_\nu(r)-1/\widetilde{m_j}-\frac{\log_2(C_r)}{j})}\\ 2^{-j(\dim(\nu_r)+1/\widetilde{m_j}+\frac{\log_2(C_r)}{j})}&\leq\nu_r(\lambda_j^D)\leq 2^{-j(\dim(\nu_r)-1/\widetilde{m_j}-\frac{\log_2(C_r)}{j})}.\end{aligned} \tag{5.30}$$

One concludes that, for $\frac{1}{m_j}=\frac{1}{\widetilde{m}}+\frac{(\lfloor\log_2(K)\rfloor+2)\ \log_2(C_\nu)}{j}+\frac{\log_2(C_r)}{j}$, by (5.29) and (5.30),

$$\begin{aligned}2^{-j(h_\nu^r+1/m_j)}\leq\ \nu\big(\lambda_j^D\big)&\leq 2^{-j(h_\nu^r-1/m_j)}\\ 2^{-j\left(\dim(\nu_r)+1/m_j\right)}\leq\ \nu_r\big(\lambda_j^D\big)&\leq 2^{-j\left(\dim(\nu_r)-1/m_j\right)}.\end{aligned}$$

For $m\in\mathbb{N}^*$, there exists an integer $J_m\in\mathbb{N}$ large enough such that for all $j\geq J_m$,

$$\frac{(\lfloor\log_2(K)\rfloor+2)\ \log_2(C_\nu)}{j}+\frac{\log_2(C_r)}{j}\leq\frac{1}{2m}$$

and considering $J_{n,m}^{\nu_r}:=\max(J_{n,2m},J_m)$ (with $\widetilde{m}=2m$), one sees that $1/\widetilde{m_j}\leq 1/m$ and $1/m_j\leq 1/m$, inducing the second item. □

Proposition 5.25 reads as: if $K\lambda^{d'}\cap A_{n,m}^{\nu_r}\neq\emptyset$, then (5.26) and (5.27) hold true. Also, m being fixed, one sees that $(A_{n,m}^{\nu_r})_{n\in\mathbb{N}^*}$ is an increasing sequence.

5.2.3 Product capacity on $[0,1]^D$

Fix $D = d+d'$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $\widetilde{\varphi} : \mathbb{R}^{d'} \to \mathbb{R}$ be respectively a $\mathbb{Z}^d$-invariant and $\mathbb{Z}^{d'}$-invariant real valued Hölder function. Then $\varphi+\widetilde{\varphi} : (x^d, x^{d'}) \in \mathbb{R}^d \times \mathbb{R}^{d'} \mapsto \varphi(x^d)+\widetilde{\varphi}(x^{d'}) \in \mathbb{R}$ is a $\mathbb{Z}^D$-invariant real valued Hölder function. The next lemma is immediate.

Lemma 5.26. *Let μ and ν be Gibbs measures respectively on $[0,1]^d$ associated to φ and on $[0,1]^{d'}$ associated to $\widetilde{\varphi}$ as above. The product measure $\xi = \mu \times \nu$ is a Gibbs measure on $[0,1]^D$ associated to the Hölder continuous potential $\varphi + \widetilde{\varphi}$.*

The Hölder index of ξ is deduced from that of μ and ν.

Lemma 5.27. *Let $\mu \in \mathscr{E}_d$ on $[0,1]^d$ be s_μ-Hölder and $\nu \in \mathscr{E}_{d'}$ on $[0,1]^{d'}$ be s_ν-Hölder with $s_\mu + s_\nu > 0$. Then $\xi = \mu \times \nu$ is $(s_\mu + s_\nu)$-Hölder.*

Proof. Since, μ is s_μ-Hölder, there is $C_\mu > 0$ such that $\mu(E^d) \le C_\mu |E^d|^{s_\mu}$ for all $E^d \in \mathcal{B}([0,1]^d)$. Similarly, ν is s_ν-Hölder, and $\nu(E^{d'}) \le C_\nu |E^{d'}|^{s_\nu}$ for all $E^{d'} \in \mathcal{B}([0,1]^{d'})$ and some $C_\nu > 0$.

Let $E^D \in \mathcal{B}([0,1]^D)$. There exist a $x \in \mathbb{R}^d$ and $x' \in \mathbb{R}^{d'}$ such that $E^D \subset B_d(x, |E^D|) \times B_{d'}(x', |E^D|)$. One obtains that for some constant $C > 0$,

$$\begin{aligned}\xi(E^D) &\le \xi(B_d(x,|E^D|) \times B_{d'}(x',|E^D|)) = \mu(B_d(x,|E^D|)) \cdot \nu(B_{d'}(x',|E^D|)) \\ &\le C\ |E^D|^{s_\mu} \cdot |E^D|^{s_\nu},\end{aligned}$$

proving that ξ is $(s_\mu + s_\nu)$-Hölder. □

From the previous lemma and Remark 5.31, one sees that, for $s := s_\mu + s_\nu$,

$$B^\xi_{p,q}([0,1]^D) \hookrightarrow B^{s+\frac{D}{p}}_{p,q}([0,1]^D) \hookrightarrow B^{s+\frac{D}{p}-\frac{D}{p}}_{\infty,q}([0,1]^D) = C^s([0,1]^D). \tag{5.31}$$

Lemma 5.28. *Let μ and ν be doubling capacities in $\mathcal{C}([0,1]^d)$ and $\mathcal{C}([0,1]^{d'})$, respectively. The product capacity $\xi = \mu \times \nu$ is doubling.*

Proof. Let us prove (5.5) for ξ. For $x^D = (x^d, x^{d'}) \in (0,1)^d \times (0,1)^{d'} = (0,1)^D$, for all $r > 0$,

$$B_D(x^D, 2r) \subset B_d(x^d, 2r) \times B_{d'}(x^{d'}, 2r) \quad \text{and} \quad B_d(x^d, r/2) \times B_{d'}(x^{d'}, r/2) \subset B_D(x^D, r).$$

So, using (5.5) for μ and ν (with the constant C_μ and C_ν respectively),

$$\begin{aligned}\xi(B_D(x^D, 2r)) &\le \mu(B_d(x^d, 2r)) \cdot \nu(B_{d'}(x^{d'}, 2r)) \\ &\le C_\mu^2 C_\nu^2\ \mu(B_d(x^d, r/2)) \cdot \nu(B_{d'}(x^{d'}, r/2)),\end{aligned}$$

hence $\xi\big(B_D(x^D, 2r)\big) \le C_\xi\ \xi(B_D(x^D, r))$ and the lemma is proved. □

Proposition 5.29. *Let $\mu \in \mathscr{E}_d$ and $\nu \in \mathscr{E}_{d'}$. The capacity $\xi = \mu \times \nu$ verifies the SMF and*

$$\text{for all } r \in \mathbb{R}, \quad \sigma_\xi\big(h^r_\mu + h^r_\nu\big) = \sigma_\mu\big(h^r_\mu\big) + \sigma_\nu\big(h^r_\nu\big). \tag{5.32}$$

Proof. We begin with the lower bound $\sigma_\xi\big(h_\mu^r+h_\nu^r\big) \geq \sigma_\mu\big(h_\mu^r\big)+\sigma_\nu\big(h_\nu^r\big)$. Observe that for every $0 \leq \alpha \leq h$, one has $E_\mu(\alpha) \times E_\nu(h-\alpha) \subset E_\xi(h) \subset \underline{E}_\xi(h)$. Indeed, for $x^D = (x^d, x^{d'}) \in [0,1]^D$, one has $h_\mu(x^d) + h_\nu(x^{d'}) = h_\xi(x^D)$ using that

$$\lim_{j\to+\infty} \frac{\log \mu(\lambda_j^d(x^d))}{\log 2^{-j}} + \frac{\log \nu(\lambda_j^{d'}(x^{d'}))}{\log 2^{-j}} = \lim_{j\to+\infty} \frac{\log \xi(\lambda_j^D(x^D))}{\log 2^{-j}}$$

since all the limits exist. Hence $\bigcup_{0\leq\alpha\leq h} E_\mu(\alpha) \times E_\nu(h-\alpha) \subset E_\xi(h)$, from which we deduce

$$\begin{aligned}\sigma_\xi(h) = \dim_H\big(\underline{E}_\xi(h)\big) &\geq \max_{0\leq\alpha\leq h} \dim_H\big(E_\mu(\alpha) \times E_\nu(h-\alpha)\big)\\ &\geq \max_{0\leq\alpha\leq h} \dim_H\big(E_\mu(\alpha)\big) + \dim_H\big(E_\nu(h-\alpha)\big) = \max_{0\leq\alpha\leq h} \sigma_\mu(\alpha) + \sigma_\nu(h-\alpha),\end{aligned}$$

the last equality coming from the fact that μ and ν satisfy the SMF. Taking $h = h_\mu^r + h_\nu^r$ for some $r \in \mathbb{R}$ and $\alpha = h_\mu^r$, we get $\sigma_\xi\big(h_\mu^r + h_\nu^r\big) \geq \sigma_\mu\big(h_\mu^r\big) + \sigma_\nu\big(h_\nu^r\big)$.

Observe also that when r ranges in $[-\infty, +\infty]$, $h_\mu^r + h_\nu^r$ covers the whole interval $[h_\mu^{+\infty} + h_\nu^{+\infty}, h_\mu^{-\infty} + h_\nu^{+\infty}]$, which contains all possible values of the local dimensions of ξ, hence formula (5.32) describes entirely the values of σ_ξ.

We finish with the converse inequality. It is known that for every $h \in \mathbb{R}$, $\sigma_\xi(h) \leq \tau_\xi^*(h)$. Since $\mu \in \mathscr{E}_d$ and $\nu \in \mathscr{E}_{d'}$ satisfy (5.24), it is standard that the scaling function of μ and ν is a limit by Fekete's Subadditive Lemma, as used in [33]. As a Cartesian product of such objects, ξ is quasi-Bernoulli as well, and $\tau_\xi(q) = \lim_{j\to+\infty} \frac{1}{-j} \log_2 \sum_{I\in\Lambda_j^D} \xi(I)^q$. One then computes that

$$\begin{aligned}\tau_\xi(q) &= \lim_{j\to+\infty} \frac{\log \sum_{\lambda^d\times\lambda^{d'}\in\Lambda_j^D} \xi(\lambda^d \times \lambda^{d'})^q}{\log 2^{-j}}\\ &= \lim_{j\to+\infty} \frac{\log \sum_{\lambda^d\in\Lambda_j^d} \mu(\lambda^d)^q \sum_{\lambda^{d'}\in\Lambda_j^{d'}} \nu(\lambda^{d'})^q}{\log 2^{-j}}\\ &= \lim_{j\to+\infty} \frac{\log \sum_{\lambda^d\in\Lambda_j^d} \mu(\lambda^d)^q}{\log(2^{-j})} + \lim_{j\to+\infty} \frac{\log \sum_{\lambda^{d'}\in\Lambda_j^{d'}} \nu(\lambda^{d'})^q}{\log 2^{-j}} = \tau_\mu(q) + \tau_\nu(q).\end{aligned}$$

Observe then that, for a fixed $r \in \mathbb{R}$, $\tau_\xi'(r) = \tau_\mu'(r) + \tau_\nu'(r) = h_\mu^r + h_\nu^r$. Also, one knows by the properties of the Legendre transform of concave functions that the infimum $\inf_{t\in\mathbb{R}}(t\tau_\xi'(r) - \tau_\xi(t))$ is reached at $t = r$, hence, using the SMF for μ and ν,

$$\begin{aligned}\tau_\xi^*(\tau_\xi'(r)) &= r\tau_\xi'(r) - \tau_\xi(r) = rh_\mu^r - \tau_\mu(r) + h_\nu^r - \tau_\nu(r) = \tau_\mu^*(h_\mu^r) + \tau_\nu^*\big(h_\nu^r\big)\\ &= \sigma_\mu(h_\mu^r) + \sigma_\nu\big(h_\nu\big).\end{aligned}$$

One deduces that $\sigma_\xi(h_\mu^r + h_\nu^r) \leq \tau_\xi^*(h_\mu^r + h_\nu^r) = \sigma_\mu(h_\mu^r) + \sigma_\nu(h_\nu)$, hence the result. □

5.2.4 Multi-resolution wavelet analysis

Let $D = d + d' \in \mathbb{N}$ and $1 \leq p, q \leq +\infty$. A mapping $\psi : \mathbb{R}^D \to \mathbb{R}$ has $n+1$ vanishing moments when for every multi-index $\alpha \in \mathbb{N}^D$ such that $\sum_{i=1}^D \alpha_i \leq n$, $\int_{\mathbb{R}^D} x_1^{\alpha_1} \dots x_D^{\alpha_D} \psi(x) dx = 0$.

For an arbitrary integer $N \geq 1$, one can construct compactly supported functions

$\psi^0 \in C^N(\mathbb{R})$ (called the scaling function) and $\psi^1 \in C^N(\mathbb{R})$ (called the mother wavelet), with ψ^1 having at least $N+1$ vanishing moments (see [48, 50]) and such that the set of function $\psi^1_{j,k} : x \mapsto \psi^1(2^j x - k)$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ form an orthogonal basis of $L^2(\mathbb{R})$. In this case, the wavelet is said to be N-regular. We set $0^D := (0, \dots, 0)$, $1^D := (1, \dots, 1)$, $L^D = \{0,1\}^D \backslash 0^D$. An orthogonal basis of $L^2(\mathbb{R}^D)$ is obtained by tensorisation as follows (see [136, Chapters 2-3] for a general construction): for $\lambda^D = (j,k,l) \in \mathbb{Z} \times \mathbb{Z}^D \times \{0,1\}^D$, define the tensorised wavelet $\psi_{\lambda^D}(x_1, ..., x_D) = \prod_{i=1}^D \psi^{l_i}_{j,k_i}(x_i)$ with $k = (k_1, \dots, k_D)$ and $l = (l_1, \dots, l_D) \in \{0,1\}^D$.

Similar notations are used with d and d' (e.g. $L^d = \{0,1\}^d \backslash 0^d$).

For $\lambda^d = (j,k,l) \in \Lambda^d_j \times \{0,1\}^d$ and $\lambda^{d'} = (j,k',l') \in \Lambda^{d'}_j \times \{0,1\}^{d'}$, we will write $\lambda^D = \lambda^d \times \lambda^{d'}$ for $\lambda^D = (j,(k,k'),(l,l')) \in \Lambda^D_j \times \{0,1\}^D$. We will also alternatively use $c_{\lambda^d,\lambda^{d'}} = c_{\lambda^D}$ depending on the need to make the notation simpler.

Thus, with these notations, and recalling (5.13), we will consider functions written as

$$f = c_{0,0^D,0^D}\psi_{0,0^D,0^D} + \sum_{\lambda^D \in \Lambda^D \times L^D} c_{\lambda^D}\psi_{\lambda^D} = c_{0,0^D,0^D}\psi_{0,0^D,0^D} + \sum_{j \geq 0} \sum_{\lambda^D_j \in \Lambda^D_j \times L^D} c_{\lambda^D_j}\psi_{\lambda^D_j}$$

Proposition 5.30. *[83, 95] Suppose that $\gamma > 0$ and $N = \lfloor \gamma + 1 \rfloor$. Consider a wavelet $\psi \in C^N(\mathbb{R})$ having at least $N+1$-regular. Let $f : [0,1]^D \to \mathbb{R}$ be a locally bounded function with wavelet coefficients $\{c_\lambda\}_{\lambda \in \Lambda^D \times L^D}$, and let $x \in [0,1]^D$.*

If $f \in C^\gamma(x)$, then there exists a constant > 0 such that for all $\lambda = (j,k,l) \in \Lambda^D \times L^D$,

$$|c_\lambda| \leq M\ 2^{-j\gamma}\big(1 + |2^j x - k|\big)^\gamma. \tag{5.33}$$

Conversely, if (5.33) holds true and if $f \in \bigcup_{\varepsilon > 0} C^\varepsilon([0,1]^D)$, then $f \in C^{\gamma-\eta}(x)$, $\forall\ \eta > 0$.

Remark 5.31. [19, Remark 2.17] Let $\xi \in \mathcal{C}([0,1]^D)$ be s-Hölder with $s > 0$, the embedding $B^\xi_{p,q}([0,1]^D) \hookrightarrow B^{s+\frac{D}{p}}_{p,q}([0,1]^D)$ holds. When ξ satisfies (5.14) for $C > 0$ and $0 < s_1 < s_2$, one has $B^{s_2+\frac{D}{p}}_{p,q}([0,1]^D) \hookrightarrow B^\xi_{p,q}(\mathbb{R}^D) \hookrightarrow B^{s_1+\frac{D}{p}}_{p,q}([0,1]^D)$. Also, when $\alpha \geq \beta$, $B^{\xi(+\alpha)}_{p,q}([0,1]^D) \hookrightarrow B^{\xi(+\beta)}_{p,q}([0,1]^D)$ and $\widetilde{B}^{\xi(+\alpha)}_{p,q}([0,1]^D) \hookrightarrow \widetilde{B}^{\xi(+\beta)}_{p,q}([0,1]^D)$.

5.3 Traces and inhomogeneous Besov spaces

5.3.1 Wavelet characterisation of traces

Observe that, by Lemma 5.27, every function in this chapter is continuous, which implies that the trace f_a in (5.4) is well-defined for every $a \in \mathbb{R}^{d'}$. Observe that for every $a \in \mathbb{R}^{d'}$, the trace $f_a : x \in \mathbb{R}^d \mapsto f(x,a) \in \mathbb{R}$ can be explicitly written in terms of wavelet expansion (5.13) as

$$f_a(x) = c_{0,0^D,0^D}\psi_{0,0^d,0^d}(x)\psi_{0,0^{d'},0^{d'}}(a) + \sum_{\lambda^D := \lambda^d \times \lambda^{d'} \in \Lambda^D \times L^D} c_{\lambda^D}\psi_{\lambda^d}(x)\psi_{\lambda^{d'}}(a). \tag{5.34}$$

We then decompose f_a as $f_a = G_a + F_a$ where $G_a(x) = \sum_{\lambda^d \in \Lambda^d \times 0^d} d^G_{\lambda^d}(a)\psi_{\lambda^d}(x)$ with

$$d^G_{\lambda^d_j}(a) = \begin{cases} \displaystyle\sum_{\lambda^{d'} \in \Lambda_0^{d'} \times \{0,1\}^{d'}} c_{\lambda^d,\lambda^{d'}} \psi_{\lambda^{d'}}(a), & \text{for } j = 0, \\ \displaystyle\sum_{\lambda^{d'} \in \Lambda_j^{d'} \times \mathbf{L^{d'}}} c_{\lambda^d,\lambda^{d'}} \psi_{\lambda^{d'}}(a), & \text{for } j \geq 1, \end{cases} \tag{5.35}$$

and $F_a(x) = \sum_{j\geq 0} \sum_{\lambda^d_j \in \Lambda^d_j \times L^d} d^F_{\lambda^d_j}(a) \psi_{\lambda^d_j}(x)$ with

$$d^F_{\lambda^d_j}(a) = \sum_{\lambda^{d'} \in \Lambda_j^{d'} \times \{\mathbf{0},\mathbf{1}\}^{\mathbf{d'}}} c_{\lambda^d_j,\lambda^{d'}_j} \psi_{\lambda^{d'}_j}(a). \tag{5.36}$$

Formula (5.36) yields a wavelet decomposition of F_a as the wavelets $\left(\psi_{\lambda^d}\right)_{\lambda^d \in \Lambda^d \times L^d}$ form a wavelet basis of $L^2([0,1]^d)$ when completed by the scaling function $\psi_{0,0^d,0^d}$. On the contrary, the decomposition G_a in (5.35) only involves the scaling functions $\psi_{j,k,0^d}$ and not wavelets.

So, to find out whether f_a belongs to some inhomogeneous space $B^\mu_{p,q}([0,1]^d)$, one cannot directly use the definition (5.15), which involves only wavelet coefficients computed with $\psi_{\lambda^d}(a)$ for $\lambda^d \in \Lambda^d \times L^d$. Fortunately, Jaffard proved the following result for standard Besov space:

Proposition 5.32. [85, Proposition 1] *Let $s > 0$ and a function $g = \sum_{\lambda^d \in \Lambda^d \times 0^d} d_{\lambda^d} \psi_{\lambda^d}$. If $\sup_{j\geq 1} 2^{j(ps-d)} \sum_{\lambda^d \in \Lambda^d_j \times 0^d} \left|d_{\lambda^d}\right|^p < +\infty$, then $g \in B^s_{p,\infty}([0,1]^d)$.*

Proposition 5.32 is extended to inhomogeneous Besov spaces.

Proposition 5.33. *Let $1 \leq p,q \leq \infty$, $\kappa \in \mathcal{C}([0,1]^d)$ a doubling capacity and a function $g = \sum_{\lambda^d \in \Lambda^d \times 0^d} d_{\lambda^d} \psi_{\lambda^d}$.If*

$$\left\|(\widetilde{\varepsilon}^{\kappa,p}_j)_{j\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})} < \infty, \quad \textit{where} \quad \widetilde{\varepsilon}^{\kappa,p}_j = \left\| \left(\frac{d_{\lambda^d_j}}{\kappa(\lambda^d_j)} \right)_{\lambda^d_j \in \Lambda^d_j \times 0^d} \right\|_{\ell^p(\Lambda^d_j \times 0^d)}, \tag{5.37}$$

then $g \in B^\kappa_{p,q}([0,1]^d)$.

To prove that $f_a \in B^\kappa_{p,\infty}([0,1]^d)$, using (5.34), it is enough to check that

$$|f_a|_{\kappa,p,q} = \left\|(\varepsilon^{\kappa,p}_j)_{j\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})} < \infty, \text{ where } \varepsilon^{\kappa,p}_j = \left\| \left(\frac{c_\lambda}{\kappa(\lambda)} \right)_{\lambda \in \Lambda^d_j \times \{0,1\}^d} \right\|_{\ell^p(\Lambda^d_j \times \{\mathbf{0},\mathbf{1}\}^{\mathbf{d}})} \tag{5.38}$$

The difference between (5.38) and the original characterisation (5.15) is that the sum is now taken on all cubes $\Lambda^d \times \{0,1\}^d$. We keep this characterisation throughout the Section 5.3.

Proof. The idea is to decompose $\psi_{j,k,0^d}$ on the wavelets $\psi_{J,K,l}$ for $l \in L^d$ such that $j > J$.

First, let us show that $g \in L^p([0,1]^d)$. Remark that, as $\kappa \in \mathcal{C}([0,1]^d)$, κ is s-Hölder for some $s > 0$. Thus, for some constant $K > 0$, for every $\lambda^d_j \in \Lambda^d_j \times \{0,1\}^d$, $\kappa(\lambda^d_j) \leq K 2^{-js}$. Hence,

$$\|g\|_{L^p} \leq \sum_{j\geq 0} \left\| \sum_{\lambda^d_j \in \Lambda^d_j \times 0^d} d_{\lambda^d_j} \psi_{\lambda^d_j} \right\|_{L^p} \leq K \sum_{j\geq 0} 2^{-js} \left\| \sum_{\lambda^d_j \in \Lambda^d_j \times 0^d} \frac{d_{\lambda^d_j}}{\kappa(\lambda^d_j)} \psi_{\lambda^d_j} \right\|_{L^p}. \tag{5.39}$$

Each $\psi_{\lambda_j^d}$ is compactly supported, with support included in $2^{-j}[-M_\psi, M_\psi]^d + 2^{-j}k$. In addition,

$$C(\psi) := \sup_{x\in\mathbb{R}^d} \sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} |\psi_{\lambda_j^d}(x)| \le (4M_\psi + 1)^D \|\psi\|_{L^\infty}.$$

When $p = \infty$, one sees that

$$\Big| \sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \psi_{\lambda_j^d}(x) \Big| \le \sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \Big| \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \Big| |\psi_{\lambda_j^d}(x)| \le \Big\| \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \Big\|_{\ell^\infty} \sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} |\psi_{\lambda_j^d}(x)|,$$

which is smaller than $C(\psi)\, \widetilde{\varepsilon}_j^{\kappa,\infty}$. This induces that

$$\Big\| \sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \psi_{\lambda_j^d} \Big\|_{L^\infty} \le C(\psi)\, \widetilde{\varepsilon}_j^{\kappa,\infty}. \tag{5.40}$$

When $p \in [1,\infty)$, let $q \in (1,\infty]$ be such that $1/p + 1/q = 1$. The Hölder inequality gives

$$\begin{aligned} \Big| \sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \psi_{\lambda_j^d}(x) \Big| &\le \Big(\sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \Big| \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \Big|^p |\psi_{\lambda_j^d}(x)| \Big)^{1/p} \Big(\sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} |\psi_{\lambda_j^d}(x)| \Big)^{1/q} \\ &\le C(\psi)^{1/q} \Big(\sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \Big| \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \Big|^p |\psi_{\lambda_j^d}(x)| \Big)^{1/p}. \end{aligned}$$

One deduces that

$$\begin{aligned} \Big\| \sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \psi_{\lambda_j^d} \Big\|_{L^p} &\le C(\psi)^{1/q} \Big(\sum_{\lambda_j^d \in \Lambda_j^d \times 0^d} \Big| \frac{d_{\lambda_j^d}}{\kappa(\lambda_j^d)} \Big|^p \int_{\mathbb{R}^d} |\psi_{\lambda_j^d}(x)| dx \Big)^{1/p} \\ &\le C(\psi)^{1/q} \|\psi\|_{L^1}^{1/p}\, \widetilde{\varepsilon}_j^{\kappa,p}. \end{aligned} \tag{5.41}$$

Since $|\widetilde{\varepsilon}_j^{\kappa,p}| \le \big\|\widetilde{\varepsilon}_j^{\kappa,p}\big\|_{\ell^q(\mathbb{N})}$, by combining (5.40), (5.41) and (5.39), it gives that $\|g\|_{L^p} \le K \sum_{j\ge 0} 2^{-js} C \widetilde{\varepsilon}_j^{\kappa,p} \le K \sum_{j\ge 0} 2^{-js} C \big\|\widetilde{\varepsilon}_j^{\kappa,p}\big\|_{\ell^q(\mathbb{N})} \le C' \big\|\widetilde{\varepsilon}_j^{\kappa,p}\big\|_{\ell^q(\mathbb{N})} < \infty$ and $g \in L^p([0,1]^d)$.

We show that the seminorm (5.15) of g is finite. Applying (5.13) to $g \in L^p([0,1]^d)$,

$$g = c_{0,0^d,0^d} \psi_{0,0^d,0^d} + \sum_{\lambda^d \in \Lambda^d \times L^d} c_{\lambda^d} \psi_{\lambda^d}, \tag{5.42}$$

where for $(J,K,l) \in \Lambda^d \times L^d$

$$\begin{aligned} c_{0,0^d,0^d} &= \int_{\mathbb{R}^D} g(x) \psi_{(0,0^d,0^d)}(x) dx \\ &= \int_{\mathbb{R}^D} \sum_{j\ge 0} \sum_{k\in\{0,1,\dots,2^j-1\}^d} d_{j,k,0^d} \psi_{(j,k,0^d)}(x) \psi_{(0,0^d,0^d)}(x) dx, \\ \text{and} \quad c_{J,K,l} &= 2^{dJ} \int_{\mathbb{R}^D} g(x) \psi_{(J,K,l)}(x) dx \\ &= 2^{dJ} \int_{\mathbb{R}^D} \sum_{j>J} \sum_{k\in\{0,1,\dots,2^j-1\}^d} d_{j,k,0^d} \psi_{(j,k,0^d)}(x) \psi_{(J,K,l)}(x) dx. \end{aligned}$$

The sum goes over $j > J$ since $\psi_{(J,K,l)}$ is orthogonal to all $\psi_{(j,k,0^d)}$ with $j \le J$. Also,

only wavelet coefficients indexed with dyadic cubes in $[0,1]^d$ are considered, hence $k \in \{0,1,\dots,2^j-1\}^d$.

One sees that for any fixed j and J, there is a finite number of $k \in \{0,1,\dots,2^j-1\}^d$, such that the support of $\psi_{(j,k,0^d)}$ is included in the support of $\psi_{(J,K,l)}$. Since ψ^0 and ψ^1 are compactly supported, their support is included in an interval $[-M_\psi, M_\psi]$. Denote, for every j, J, K,

$$S_{j,J,K} = \big\{k \in \{0,1,\dots,2^j-1\}^d \ : \ \mathrm{supp}(\psi_{(j,k,0^d)}) \ \cap \ \mathrm{supp}(\psi_{(J,K,l)}) \neq \emptyset\big\}.$$

This set contains the dyadic cubes $\lambda_j^d \in \Lambda_j^d \times 0^d$ such that the associated $d_{\lambda_j^d}$ appears in the computation of $c_{\lambda_J^d}$ for $\lambda_J^d \in \Lambda_J^d \times L^d$ above. Observe that for every $j, J \in \mathbb{N}$,

$$\#S_{j,J,K} \leq (2M_\psi+1)^d \ 2^{d(j-J)}. \tag{5.43}$$

To establish that $g \in B^\kappa_{p,q}([0,1]^d)$, we use (5.15). First, one has

$$\left|\frac{c_{J,K,l}}{\kappa(\lambda_{J,K}^d)}\right| \leq 2^{dJ} \sum_{j>J} \sum_{k\in\{0,1,\dots,2^j-1\}^d} \left|\frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)}\right| \left|\frac{\kappa(\lambda_{j,k}^d)}{\kappa(\lambda_{J,K}^d)}\right| \int_{\mathbb{R}^D} \left|\psi_{(j,k,0^d)}(x)\psi_{(J,K,l)}(x)\right| dx.$$

Hence, using the definition of $S_{j,J,K}$, for some constant C_ψ depending on ψ only,

$$\begin{aligned}\left|\frac{c_{J,K,l}}{\kappa(\lambda_{J,K}^d)}\right| &\leq 2^{dJ} \sum_{j>J} \sum_{k\in S_{j,J,K}} \left|\frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)}\right| \left|\frac{\kappa(\lambda_{j,k}^d)}{\kappa(\lambda_{J,K}^d)}\right| \|\psi\|_{L^\infty}^d \int_{\mathbb{R}^D} \left|\psi_{(j,k,0^d)}(x)\right| dx \\ &\leq C_\psi 2^{dJ} \sum_{j>J} 2^{-jd} \sum_{k\in S_{j,J,K}} \left|\frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)}\right| \left|\frac{\kappa(\lambda_{j,k}^d)}{\kappa(\lambda_{J,K}^d)}\right|.\end{aligned}$$

The Hölder inequality gives, for $\frac{1}{p}+\frac{1}{r}=1$,

$$\left|\frac{c_{J,K,l}}{\kappa(\lambda_{J,K}^d)}\right| \leq C_\psi 2^{dJ} \sum_{j>J} 2^{-jd} \left(\sum_{k\in S_{j,J,K}} \left|\frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)}\right|^p\right)^{1/p} \left(\sum_{k\in S_{j,J,K}} \left|\frac{\kappa(\lambda_{j,k}^d)}{\kappa(\lambda_{J,K}^d)}\right|^r\right)^{1/r}.$$

Observe that for all $k \in S_{j,J,K}$, $\lambda_{j,k}^d \subset (4M_\psi+1)\,\lambda_{J,K}^d$. Thus by the doubling inequality (5.5) for κ with the constant $C_\kappa > 0$, one has, for all $k \in S_{j,J,K}$,

$$\frac{\kappa(\lambda_{j,k}^d)}{\kappa(\lambda_{J,K}^d)} = \frac{\kappa(\lambda_{j,k}^d)}{\kappa((4M_\psi+1)\,\lambda_{J,K}^d)} \frac{\kappa((4M_\psi+1)\,\lambda_{J,K}^d)}{\kappa(\lambda_{J,K}^d)} \leq C_\kappa^{\lfloor \log_2(4M_\psi+1)\rfloor+1}. \tag{5.44}$$

Applying (5.43) and (5.44) to the second sum above, one gets

$$\begin{aligned}\left|\frac{c_{J,K,l}}{\kappa(\lambda_{J,K}^d)}\right| &\leq C_{\psi,\kappa} 2^{dJ} \sum_{j>J} 2^{-jd} \left(\sum_{k\in S_{j,J,K}} \left|\frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)}\right|^p\right)^{1/p} 2^{\frac{d}{r}(j-J)} \\ &= C_{\psi,\kappa} \sum_{j>J} 2^{d(j-J)(-1/p)} \left(\sum_{k\in S_{j,J,K}} \left|\frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)}\right|^p\right)^{1/p}.\end{aligned}$$

It is immediate that $\varepsilon_J^{\kappa,p} = \Big\| \Big(\Big| \frac{c_{J,K,l}}{\kappa(\lambda_{J,K}^d)} \Big| \Big)_{(K,l)\in\{0,\dots,2^J-1\}^d\times L^d} \Big\|_{\ell^p}$ satisfies

$$\varepsilon_J^{\kappa,p} \le \left\| \left(C \sum_{j>J} 2^{-\frac{d}{p}(j-J)} \Big(\sum_{k\in S_{j,J,K}} \left| \frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)} \right|^p \Big)^{1/p} \right)_{(K,l)\in\{0,\dots,2^J-1\}^d\times L^d} \right\|_{\ell^p}$$

In the last norm, the elements do not depend on $l \in L^d$, hence

$$\varepsilon_J^{\kappa,p} \le C_{\psi,\kappa,d} \sum_{j>J} 2^{-\frac{d}{p}(j-J)} \left\| \Big(\sum_{k\in S_{j,J,K}} \left| \frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)} \right|^p \Big)^{1/p}_{K\in\{0,\dots,2^J-1\}^d} \right\|_{\ell^p}. \tag{5.45}$$

Recalling that ψ^0 and ψ^1 have their support in $[-M_\psi, M_\psi]$, for a fixed $J \in \mathbb{N}$ and $l \in L^d$, a point $x \in [0,1]^d$ is included in the support of (at most) $(2M_\psi+1)^d$ wavelets $\psi_{(J,K,l)}$. Thus, for $j > J$, a scaling function $\psi_{(j,k,0^d)}$ has a support intersecting that of $\psi_{(J,K,l)}$ for at most $(4M_\psi+1)^d$ multi-integer K. One deduces that, for j, J fixed with $j > J$, a $k \in \{0,\dots,2^j-1\}^d$ is in at most $(4M_\psi+1)^d$ sets $S_{j,J,K}$ for $K \in \{0,\dots,2^J-1\}^d$. One sees that

$$\sum_{K\in\{0,\dots,2^J-1\}^d} \sum_{k\in S_{j,J,K}} \left| \frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)} \right|^p \le C_{\psi,d} \sum_{k\in\{0,\dots,2^j-1\}^d} \left| \frac{d_{j,k,0^d}}{\kappa(\lambda_{j,k}^d)} \right|^p = C_{\psi,d} (\widetilde{\varepsilon}_j^{\kappa,p})^p.$$

So $\varepsilon_J^{\kappa,p} \le C_{\psi,\kappa,d} \sum_{j>J} 2^{-\frac{d}{p}(j-J)} \widetilde{\varepsilon}_j^{\kappa,p}$, and by the Hölder inequality, for a constant C that changes along the computations,

$$|\varepsilon_J^{\kappa,p}| \le C \Big(\sum_{j>J} 2^{-\frac{d}{p}(j-J)} |\widetilde{\varepsilon}_j^{\kappa,p}|^q \Big)^{1/q} \Big(\sum_{j>J} 2^{-\frac{d}{p}(j-J)} \Big)^{1/(1-1/q)} \le C \Big(\sum_{j>J} 2^{-\frac{d}{p}(j-J)} |\widetilde{\varepsilon}_j^{\kappa,p}|^q \Big)^{1/q}.$$

Hence, for C depending on ψ, κ, d and q,

$$\sum_{J\in\mathbb{N}} |\varepsilon_J^{\kappa,p}|^q \le C \sum_{J\in\mathbb{N}} \sum_{j>J} 2^{-\frac{d}{p}(j-J)} |\widetilde{\varepsilon}_j^{\kappa,p}|^q \le C \sum_{j\in\mathbb{N}} |\widetilde{\varepsilon}_j^{\kappa,p}|^q \sum_{J<j} 2^{-\frac{d}{p}(j-J)} \le C \sum_{j\in\mathbb{N}} |\widetilde{\varepsilon}_j^{\kappa,p}|^q.$$

One concludes by (5.37) that $g \in B_{p,q}^{\kappa}([0,1]^d)$. □

5.3.2 Proof of Theorem 5.11

The product capacity $\xi = \mu \times \nu$ is fixed. Recall that $\Gamma_{\nu,r}^p = h_\nu^r + \frac{\dim(\nu_r)}{p}$, with $\Gamma_{\nu,r}^\infty = h_\nu^r$.

Proposition 5.34. *Let $p \in [1,+\infty]$, $\Gamma < \Gamma_{\nu,r}^p$, $f \in \widetilde{B}_{p,q}^{\xi}([0,1]^D)$ and $r \in \mathbb{R}$. Then for ν_r-almost all $a \in [0,1]^{d'}$, $f_a \in B_{p,1}^{\mu^{(+\Gamma)}}([0,1]^d)$.*

Proof. Since f is continuous, each trace f_a (5.4) is also continuous and $f_a \in L^p([0,1]^d)$.

We start with the case $p < +\infty$. Consider $\Gamma < \Gamma_{\nu,r}^p$, $f \in \widetilde{B}_{p,q}^{\xi}([0,1]^D)$ and $r \in \mathbb{R}$. To prove Proposition 5.34, one needs to show that $|f_a|_{\mu^{(+\Gamma)},p,1} < \infty$. Recalling (5.38), and since μ, ν and f are fixed, we introduce a more convenient notation for the quantity $\varepsilon_j^{\mu^{(+\Gamma)},p}$ associated with f_a: we rewrite

$$S_j^p(a,\Gamma) := \sum_{\lambda^d\in\Lambda_j^d\times\{0,1\}^d} \left| \frac{d_{\lambda^d}(a)}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p \quad (= \varepsilon_j^{\mu^{(+\Gamma)},p}). \tag{5.46}$$

In $S_j^p(a,\Gamma)$, the coefficients $d_{\lambda^d}(a)$ are either $d_{\lambda^d}^G(a)$ if $\lambda^d \in \Lambda^d \times 0^d$ or $d_{\lambda^d}^F(a)$ if $\lambda^d \in \Lambda^d \times L^d$.

Let us show an upper bound for the $S_j^p(a,\Gamma)$.

First, for $j \geq 1$, for the wavelet coefficient (5.35) of G_a

$$\sum_{\lambda^d \in \Lambda_j^d \times 0^d} \left| \frac{d^G_{\lambda^d}(a)}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p = \sum_{\lambda^d \in \Lambda_j^d \times 0^d} \left| \frac{1}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p \cdot \Big| \sum_{\lambda^{d'} \in \Lambda_j^{d'} \times L^{d'}} c_{\lambda^d, \lambda^{d'}} \psi_{\lambda^{d'}}(a) \Big|^p .$$

By the compactness of the supports of ψ^0 and ψ^1, for each a, there is a finite number (bounded independently of j or a) of non-zero terms $\psi_{\lambda^{d'}}(a)$ in the sum on $\Lambda_j^{d'} \times L^{d'}$. Hence, for some constant $C_{p,\psi} > 0$ depending on p, ψ^0 and ψ^1,

$$\begin{aligned} \Big| \sum_{\lambda^{d'} \in \Lambda_j^{d'} \times L^{d'}} c_{\lambda^d, \lambda^{d'}} \psi_{\lambda^{d'}}(a) \Big|^p &\leq C_{p,\psi} \sum_{\lambda^{d'} \in \Lambda_j^{d'} \times L^{d'}} \left| c_{\lambda^d, \lambda^{d'}} \psi_{\lambda^{d'}}(a) \right|^p \\ &\leq C_{p,\psi} \sum_{\lambda^{d'} \in \Lambda_j^{d'} \times \{0,1\}^{d'}} \left| c_{\lambda^d, \lambda^{d'}} \psi_{\lambda^{d'}}(a) \right|^p . \end{aligned}$$

Using that $\xi(\lambda^d \times \lambda^{d'}) = \mu(\lambda^d) \cdot \nu(\lambda^{d'})$, one deduces that

$$\begin{aligned} \sum_{\lambda^d \in \Lambda_j^d \times 0^d} \left| \frac{d^G_{\lambda^d}(a)}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p &\leq C_{p,\psi} \sum_{\lambda^d \in \Lambda_j^d \times 0^d} \left| \frac{1}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p \sum_{\lambda^{d'} \in \Lambda_j^{d'} \times \{0,1\}^{d'}} \left| c_{\lambda^d, \lambda^{d'}} \psi_{\lambda^{d'}}(a) \right|^p \\ &= C_{p,\psi} \sum_{\lambda^D \in \Lambda_j^D \times (0^d \times \{0,1\}^{d'})} \left| \frac{1}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p \left| c_{\lambda^D} \psi_{\lambda^{d'}}(a) \right|^p \\ &= C_{p,\psi} \sum_{\lambda^D \in \Lambda_j^D \times (0^d \times \{0,1\}^{d'})} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \left| \psi_{\lambda^{d'}}(a) \right|^p . \end{aligned} \tag{5.47}$$

When $j = 0$, using that

$$\Big| \sum_{\lambda^{d'} \in \Lambda_0^{d'} \times \{0,1\}^{d'}} c_{\lambda^d, \lambda^{d'}} \psi_{\lambda^{d'}}(a) \Big|^p \leq C_{p,\psi} \sum_{\lambda^{d'} \in \Lambda_0^{d'} \times \{0,1\}^{d'}} \left| c_{\lambda^d, \lambda^{d'}} \psi_{\lambda^{d'}}(a) \right|^p ,$$

The same computations give that

$$\sum_{\lambda^d \in \Lambda_0^d \times 0^d} \left| \frac{d^G_{\lambda^d}(a)}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p \leq C_{p,\psi} \sum_{\lambda^D \in \Lambda_0^D \times (0^d \times \{0,1\}^{d'})} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \left| \psi_{\lambda^{d'}}(a) \right|^p . \tag{5.48}$$

The same computations hold for the wavelet coefficients (5.36) of F_a and, for every $j \in \mathbb{N}$,

$$\sum_{\lambda^d \in \Lambda_j^d \times L^d} \left| \frac{d^F_{\lambda^d}(a)}{\mu(\lambda^d) 2^{-j\Gamma}} \right|^p \leq C_{p,\psi} \sum_{\lambda^D \in \Lambda_j^D \times (L^d \times \{0,1\}^{d'})} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \left| \psi_{\lambda^{d'}}(a) \right|^p . \tag{5.49}$$

Noticing that $L^d \cup 0^d = \{0,1\}^d$, we obtain from (5.47), (5.48) and (5.49) that

$$S_j^p(a,\Gamma) \leq C_{p,\psi} \sum_{\lambda^D \in \Lambda_j^D \times \{0,1\}^D} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \left| \psi_{\lambda^{d'}}(a) \right|^p . \tag{5.50}$$

Recall that as a Gibbs capacity, ξ satisfies (5.14) for some $s_1 > 0$. Let us choose M_{ν_r} large enough that for every $m \geq M_{\nu_r}$,

$$m \geq \lfloor 1/s_1 \rfloor + 1 \quad \text{and} \quad \Gamma < \Gamma^p_{\nu,r} - (2p+2)/pm. \tag{5.51}$$

Let $n \in \mathbb{N}^*$, $m \geq M_{\nu_r}$ and $A^{\nu_r}_{n,m}$ be the set and $J_{n,m}$ the integer from Proposition 5.25. Using (5.50) and integrating $S^p_j(a,\Gamma)$ over $A^{\nu_r}_{n,m}$ yields

$$\begin{aligned}
&\int_{A^{\nu_r}_{n,m}} S^p_j(a,\Gamma) d\nu_r(a) = \int_{[0,1]^{d'}} S^p_j(a,\Gamma) d\nu_{r|_{A^{\nu_r}_{n,m}}}(a) \\
&\leq C_{p,\psi} \sum_{\lambda^D \in \Lambda^D_j \times \{0,1\}^D} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \int_{[0,1]^{d'}} \left| \psi_{\lambda^{d'}}(a) \right|^p d\nu_{r|_{A^{\nu_r}_{n,m}}}(a) \\
&\leq C_{p,\psi} \sum_{\lambda^D \in \Lambda^D_j \times \{0,1\}^D} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \int_{[0,1]^{d'}} \|\psi\|^{d'p}_\infty \mathbb{1}_{\widetilde{K}\lambda^{d'}}(a) d\nu_{r|_{A^{\nu_r}_{n,m}}}(a)
\end{aligned}$$

with $\|\psi\|_\infty = \max\{\|\psi^0\|_\infty, \|\psi^1\|_\infty\}$. Hence there is $C > 0$, depending also on d, such that

$$\begin{aligned}
&\int_{A^{\nu_r}_{n,m}} S^p_j(a,\Gamma) d\nu_r(a) \\
&\leq C \sum_{\lambda^D \in \Lambda^D_j \times \{0,1\}^D} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \nu_r\big(A^{\nu_r}_{n,m} \cap \widetilde{K}\lambda^{d'}\big) \\
&\leq C \sum_{\lambda^D \in \Lambda^D_j \times \{0,1\}^D} \left| \frac{c_{\lambda^D}}{\xi(\lambda^D)} \right|^p \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \nu_r\big(\widetilde{K}\lambda^{d'}\big) \mathbb{1}_{A^{\nu_r}_{n,m} \cap \widetilde{K}\lambda^{d'} \neq \emptyset} \\
&\leq C \sum_{\substack{\lambda^{d'} \in \Lambda^{d'}_j \times \{0,1\}^{d'} \\ A^{\nu_r}_{n,m} \cap \widetilde{K}\lambda^{d'} \neq \emptyset}} \left| \frac{\nu(\lambda^{d'})}{2^{-j\Gamma}} \right|^p \nu_r\big(\widetilde{K}\lambda^{d'}\big) \sum_{\lambda^d \in \Lambda^d_j \times \{0,1\}^d} \left| \frac{c_{\lambda^d,\lambda^{d'}}}{\xi(\lambda^d \times \lambda^{d'})} \right|^p
\end{aligned} \tag{5.52}$$

as $A^{\nu_r}_{n,m} \cap \widetilde{K}\lambda^{d'} \subset \widetilde{K}\lambda^{d'}$. Applying the doubling property (5.5) of ν_r to pass from $\nu_r(\widetilde{K}\lambda^{d'})$ to $\nu_r(\lambda^{d'})$, the last inequality gives that, for $C > 0$ depending on ν and r, that

$$\int_{A^{\nu_r}_{n,m}} S^p_j(a,\Gamma)\, d\nu_r(a) \leq C \sum_{\substack{\lambda^{d'} \in \Lambda^{d'}_j \times \{0,1\}^{d'} \\ \widetilde{K}\lambda^{d'} \cap\, A^{\nu_r}_{n,m} \neq \emptyset}} \frac{\nu(\lambda^{d'})^p\ \nu_r(\lambda^{d'})}{2^{-jp\Gamma}} \sum_{\lambda^d \in \Lambda^d_j \times \{0,1\}^d} \left| \frac{c_{\lambda^d,\lambda^{d'}}}{\xi(\lambda^d \times \lambda^{d'})} \right|^p.$$

Recalling that $\Gamma^p_{\nu,r} = h^r_\nu + \frac{\dim(\nu_r)}{p}$, (5.26) and (5.27) give

$$\begin{aligned}
&\int_{A^{\nu_r}_{n,m}} S^p_j(a,\Gamma)\, d\nu_r(a) \\
&\leq C \sum_{\substack{\lambda^{d'} \in \Lambda^{d'}_j \times \{0,1\}^{d'} \\ \widetilde{K}\lambda^{d'} \cap\, A^{\nu_r}_{n,m} \neq \emptyset}} 2^{-j(p\Gamma_{\nu,r} - (p+1)/m - p\Gamma)} \sum_{\lambda^d \in \Lambda^d_j \times \{0,1\}^d} \left| \frac{c_{\lambda^d,\lambda^{d'}}}{\xi(\lambda^d \times \lambda^{d'})} \right|^p \\
&\leq C\ 2^{-j(p\Gamma_{\nu,r} - (2p+1)/m - p\Gamma)} \sum_{\substack{\lambda^{d'} \in \Lambda^{d'}_j \times \{0,1\}^{d'} \\ \widetilde{K}\lambda^{d'} \cap\, A^{\nu_r}_{n,m} \neq \emptyset}} \sum_{\lambda^d \in \Lambda^d_j \times \{0,1\}^d} \left| \frac{c_{\lambda^d,\lambda^{d'}}}{\xi^{(-1/m)}(\lambda^d \times \lambda^{d'})} \right|^p.
\end{aligned}$$

By (5.13), observe that:

- for $j \geq 1$, f has no wavelet coefficient associated with the index $l^D = 0^D$, hence the

sum of λ^D over $\Lambda_j^D \times \{0,1\}^D$ is the same as the sum over $\Lambda_j^D \times L^D$ and, by (5.15),

$$\sum_{\lambda^{d'}\in\Lambda_j^{d'}\times\{0,1\}^{d'}} \sum_{\lambda^d\in\Lambda_j^d\times\{0,1\}^d} \left|\frac{c_{\lambda^d,\lambda^{d'}}}{\xi^{(-1/m)}(\lambda^d\times\lambda^{d'})}\right|^p = \sum_{\lambda^D\in\Lambda_j^D\times L^D} \left|\frac{c_{\lambda^D}}{\xi^{(-1/m)}(\lambda^D)}\right|^p,$$

- for $j=0$, since $|c_{(0,0^D,0^D)}|<\infty$,

$$\sum_{\lambda^D\in\Lambda_0^D\times\{0,1\}^D} \left|\frac{c_{\lambda^D}}{\xi^{(-1/m)}(\lambda^D)}\right|^p = \frac{1}{\xi^{(-1/m)}([0,1]^D)}\left(\left|c_{(0,0^D,0^D)}\right|^p + \sum_{\lambda^D\in\Lambda_0^D\times L^D}\left|c_{\lambda^D}\right|^p\right)$$
$$\le C\left(\left|c_{(0,0^D,0^D)}\right|^p + \left(\varepsilon_0^{\xi^{(-1/m)}}\right)^p\right).$$

The choice (5.51) for m ensures that $1/m<s_1$ and $f\in\widetilde{B}_{p,q}^{\xi^{(-1/m)}}([0,1]^D)$. This implies that $\left(\varepsilon_j^{\xi^{(-1/m)}}\right)_{j\in\mathbb{N}}\in l^q(\mathbb{N})$, and in particular that this sequence is bounded. Hence

$$\int_{A_{n,m}^{\nu_r}} S_j^p(a,\Gamma)\, d\nu_r(a) \le C\, 2^{-j(p\Gamma_{\nu,r}-(2p+1)/m-p\Gamma)}. \tag{5.53}$$

Let

$$A_{n,m,j}^{\nu_r,p} := \left\{a\in A_{n,m}^{\nu_r} \;:\; S_j^p(a,\Gamma) > C\, j^2\, 2^{-j(p\Gamma_{\nu,r}-(2p+1)/m-p\Gamma)}\right\}. \tag{5.54}$$

Markov's inequality using (5.53) gives that $\nu_r(A_{n,m,j}^{\nu_r,p}) \le \frac{2^{-j(p\Gamma_{\nu,r}-(2p+1)/m-p\Gamma)}}{j^2\, 2^{-j(p\Gamma_{\nu,r}-(2p+1)/m-p\Gamma)}} = j^{-2}$. So $\sum_{j\in\mathbb{N}}\nu_r(A_{n,m,j}^{\nu_r,p})$ converges and the Borel-Cantelli lemma yields that

$$\nu_r(\limsup_{j\to\infty} A_{n,m,j}^{\nu_r,p}) = 0. \tag{5.55}$$

So, for every $a\in\widetilde{A}_{n,m}^{\nu_r,p} := A_{n,m}^{\nu_r}\setminus \limsup_{j\to\infty} A_{n,m,j}^{\nu_r,p}$, there is $J'_{n,m}\in\mathbb{N}$ such that for $j\ge J'_{n,m}$

$$S_j^p(a,\Gamma) \le Cj^2 2^{-j(p\Gamma_{\nu,r}-(2p+1)/m-p\Gamma)} = C2^{-j\left(p\Gamma_{\nu,r}-\frac{(2p+1)}{m}-2\frac{\log_2(j)}{j}-p\Gamma\right)}$$
$$\le C2^{-j\left(p\Gamma_{\nu,r}-\frac{(2p+2)}{m}-p\Gamma\right)},$$

that last inequality being true for large integers j. Remark that for all $j\le J'_{n,m}$, $S_j^p(a,\Gamma)$ is bounded. By our choice (5.51) for m, one has $\Gamma<\Gamma_{\nu,r}-\frac{2p+2}{pm}$, implying that the sum $\sum_{j\ge1} S_j^p(a,\Gamma)$ converges and $f_a\in B_{p,1}^{\mu^{(+\Gamma)}}([0,1]^d)$ by the characterisation (5.38).

For every $n\in\mathbb{N}^*$, one has $\nu_r(\widetilde{A}_{n,m}^{\nu_r,p}) = 1-\frac{1}{n}$ by (5.55). So $\nu_r(\bigcup_{n\in\mathbb{N}^*}\widetilde{A}_{n,m}^{\nu_r,p}) = 1$ and $\bigcup_{n\in\mathbb{N}^*}\widetilde{A}_{n,m}^{\nu_r,p}$ has full ν_r-measure. So, for every $a\in\bigcup_{n\in\mathbb{N}^*}\widetilde{A}_{n,m}^{\nu_r,p}$, $f_a\in B_{p,1}^{\mu^{(+\Gamma)}}([0,1]^d)$, hence the conclusion.

Let us now explain how to adapt the proof to the case $p=\infty$. Consider $\Gamma<\Gamma_{\nu,r}^{\infty}=h_\nu^r$, $f\in\widetilde{B}_{p,q}^{\xi}([0,1]^D)$ and $r\in\mathbb{R}$. The inequality (5.50) with the infinity norm gives $S_j^\infty(a,\Gamma) := \sup_{\lambda^d\in\Lambda_j^d\times\{0,1\}^d}\left|\frac{d_{\lambda^d}(a)}{\mu(\lambda^d)2^{-j\Gamma}}\right|$ is bounded above by $C_\psi \sup_{\lambda^D\in\Lambda_j^D\times\{0,1\}^D}\left|\frac{c_{\lambda^D}}{\xi(\lambda^D)}\right|\left|\frac{\nu(\lambda^{d'})}{2^{-j\Gamma}}\right|$. By (5.26), $S_j^\infty(a,\Gamma)\le C_\psi\, 2^{-j(h_\nu^r-2/m-\Gamma)}\sup_{\substack{\lambda^D\in\Lambda_j^D\times\{0,1\}^D\\ \widetilde{K}\lambda^{d'}\cap A_{n,m}^{\nu_r}\neq\emptyset}}\left|\frac{c_{\lambda^D}}{\xi^{(-1/m)}(\lambda^D)}\right|$ for every $a\in A_{n,m}^{\nu_r}$. Then (5.53) is replaced by $S_j^\infty(a,\Gamma)\le C\, 2^{-j(h_\nu^r-2/m-\Gamma)}$, which in turn changes (5.54) into $A_{n,m,j}^{\nu_r,\infty} := \left\{a\in A_{n,m}^{\nu_r} \;:\; S_j^p(a,\Gamma) > C\, j^2\, 2^{-j(h_\nu^r-2/m-\Gamma)}\right\}$. Using $A_{n,m,j}^{\nu_r,\infty}$ with m large enough that $\Gamma<h_\nu^r-2/m$, the same Markov and Borel-Cantelli argument shows that for ν_r-almost all $a\in[0,1]^{d'}$, $f_a\in B_{\infty,1}^{\mu^{(+\Gamma)}}([0,1]^d)$. □

We explain how Proposition 5.34 implies Theorem 5.11. Theorem 5.11 can be reworded as follows: for $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$ and $r \in \mathbb{R}$ fixed, there is a set $A_r(f)$ of full ν_r-measure such that for every $a \in A_r(f)$,

$$\text{for all } 0 \le \Gamma < \Gamma^p_{\nu,r}, \quad f_a \in B^{\mu^{(+\Gamma)}}_{p,q}([0,1]^d).$$

Proof of Theorem 5.11. Fix $p \in [1,+\infty]$ and $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$. For every $r \in \mathbb{R}$ and $n \in \mathbb{N}^*$, by Proposition 5.34 for $\Gamma^p_{\nu,r} - 1/n$, call $A_{r,n} \subset [0,1]^{d'}$ a set of full ν_r-measure such that for every $a \in A_{r,n}$, $f_a \in B^{\mu^{(+\Gamma^p_{\nu,r}-1/n)}}_{p,1}([0,1]^d)$. Since the sets $B^{\mu^{(+\Gamma)}}_{p,1}([0,1]^d)$ are decreasing with Γ, taking the countable intersection $\bigcap_{n\in\mathbb{N}^*} A_{r,n}$ of full ν_r-measure gives that for ν_r-almost all $a \in [0,1]^{d'}$,

$$f_a \in \bigcap_{0\le\Gamma<\Gamma^p_{\nu,r}} B^{\mu^{(+\Gamma)}}_{p,1}([0,1]^d) = \widetilde{B}^{\mu^{(+\Gamma^p_{\nu,r})}}_{p,1}([0,1]^d) \subset \widetilde{B}^{\mu^{(+\Gamma^p_{\nu,r})}}_{p,q}([0,1]^d) \subset \widetilde{B}^{\mu^{(+\Gamma^p_{\nu,r})}}_{p,\infty}([0,1]^d).$$

□

As a consequence of the previous proof, one gets the following proposition.

Proposition 5.35. *For every $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$, every $a \in [0,1]^{d'}$, $f_a \in \widetilde{B}^{\mu^{(+h^{\min}_\nu)}}_{p,1}([0,1]^d)$.*

Proof. Let $m > \lfloor 2/h^{\min}_\nu \rfloor$. Using (5.23) and (5.50), there exists a positive sequence $(\eta_j)_{j\mathbb{N}}$ tending to 0 such that

$$\begin{aligned} S^p_j(a, h^{\min}_\nu - 2/m) &\le C_{p,\psi} \sum_{\lambda^D \in \Lambda^D_j \times \{0,1\}^D} \left| \frac{c_{\lambda^D}}{\xi^{(-1/m)}(\lambda^D)} \right|^p \left|2^{-j(1/m-\eta_j)}\right|^p \left|\psi_{\lambda^{d'}}(a)\right|^p \\ &\le C\ 2^{-jp(1/m-\eta_j)} (\varepsilon^{\xi^{(-1/m)}}_j)^p. \end{aligned}$$

Let J_m be such that for all $j \ge J_m$, $\eta_j < 1/m$. As $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$, the sequence $\left(\varepsilon^{\xi^{(-1/m)}}_j\right)_j$ belongs to $l^q(\mathbb{N})$ and is bounded. Hence $(S_j(a, h^{\min}_\nu - 2/m)^{1/p})_j \in l^1(\mathbb{N})$. One deduces that $f_a \in B^{\mu^{(+h^{\min}_\nu - 1/m)}}_{p,1}([0,1]^d)$. This holds for all m, so $f_a \in \widetilde{B}^{\mu^{(+h^{\min}_\nu)}}_{p,1}([0,1]^d)$. □

As a consequence, for every $f \in \widetilde{B}^{\xi}_{p,q}([0,1]^D)$ every $a \in [0,1]^{d'}$, f_a can be written

$$f_a = c_{0,0^d,0^d}(a)\psi_{0,0^d,0^d} + \sum_{j\ge 0} \sum_{\lambda^d_j \in \Lambda^d_j \times L^d} c_{\lambda^d_j}(a)\psi_{\lambda^d_j}. \tag{5.56}$$

Remark 5.36 (Link to the standard Besov spaces)**.** Let $\mathcal{L}^d$ be the d-dimensional Lebesgue measure. Let $D = d + d'$, $0 < p, s < \infty$, with $s > D/p$, and $q \in (0,p)$. Consider $\mu = \left(\mathcal{L}^d\right)^{s/D-1/p}$ and $\nu = \left(\mathcal{L}^{d'}\right)^{s/D-1/p}$. and $\xi = \mu \times \nu = \left(\mathcal{L}^D\right)^{s/D-1/p}$. One immediately checks that $B^{\xi}_{p,q} = B^s_{p,q}$. We recover the result of [10] by applying Theorem 5.11 to these capacities, i.e.:

Let $D \in \mathbb{N}^*$, $0 < p, s < \infty$, with $s - D/p > 0$, and let $q \in (0,\infty]$. If $f \in B^s_{p,q}([0,1]^D)$, then for Lebesgue-almost all $a \in [0,1]^{d'}$, $f_a \in \widetilde{B}^s_{p,1}([0,1]^d) = \bigcap_{0<\varepsilon<d/p} B^{s-\varepsilon}_{p,1}([0,1]^d)$.

Indeed, for all $r \in \mathbb{N}$, $\nu_r = \mathcal{L}^{d'}$ and $\Gamma_{\nu,r} = d'\left(s/D - 1/p\right) + d'/p$. One can check that $\mu^{(+\Gamma_{\nu,r})} = \left(\mathcal{L}^d\right)^{s/d-1/p}$ giving $\bigcap_{0<\Gamma<\Gamma_{\nu,r}} B^{\mu^{(+\Gamma)}}_{p,1}([0,1]^d) = \bigcap_{0<\varepsilon<d/p} B^{s-\varepsilon}_{p,1}([0,1]^d)$.

5.4 Upper bound for the spectrum of traces of $\widetilde{B}^{\xi}_{\infty,q}$ functions

Let $\mu \in \mathscr{E}_d$ on $[0,1]^d$ and $\nu \in \mathscr{E}_{d'}$ on $[0,1]^{d'}$ with $D = d + d'$. Let ξ be the product capacity on $[0,1]^D$, $\xi := \mu \times \nu$. The local behavior of elements of $B^{\xi}_{p,q}([0,1]^D)$ and $\widetilde{B}^{\xi}_{p,q}([0,1]^D)$ is described via their pointwise Hölder exponent, characterised by wavelet leaders [95].

Definition 5.37 (Wavelet Leaders)**.** *Let $(\psi^l)_{l\in L^D}$ be a family of wavelets that form an orthogonal basis of $L^2(\mathbb{R}^D)$. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^D)$ with $p \in [1,+\infty]$, call $(c_{\lambda^D})_{\lambda^D\in\Lambda^D\times L^D}$ the wavelet coefficients of f. The wavelet leader of f associated with $\lambda^D \in \Lambda^D$ is defined as*

$$L^f_{\lambda^D} = \sup\{|c_{\lambda'}| : \lambda' = (j,k,l) \in \Lambda^D \times L^D, \lambda^D_{j,k} \subset 3\lambda^D\}. \tag{5.57}$$

Notation. Consider $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ and, for $a \in [0,1]^{d'}$, the associated trace f_a with wavelet coefficients denoted by $(c_{\lambda^d}(a))_{\lambda^d\in\Lambda^d\times L^d}$. Following Definition 5.37, we denote the wavelet leader of f_a, the trace of f in a, associated with $\lambda^d \in \Lambda^d$ by (notice the dependance in a)

$$L^f_{\lambda^d}(a) = \sup\{|d_\lambda(a)| : \lambda = (j,k,l) \in \Lambda^d \times L^d, \lambda^d_{j,k} \subset 3\lambda^d\}. \tag{5.58}$$

The mention of f on leaders is omitted when the function is obvious from the context.

The pointwise Hölder exponent (5.1) is characterised by wavelet leaders [95, Prop. 4].

Proposition 5.38. *Let $N \in \mathbb{N}^*$ and assume that the wavelets belong to $C^N(\mathbb{R}^D)$ with at least $N+1$ vanishing moments. If $f \in \mathcal{C}^{\varepsilon}(R^D)$ for some $\varepsilon > 0$, then for every $x_0 \in \mathbb{R}^D$, $h_f(x_0) < N$ if and only if $\liminf_{j\to+\infty} \frac{\log L^f_{\lambda^D_j(x_0)}}{\log(2^{-j})} < N$, and in this case*

$$h_f(x_0) = \liminf_{j\to+\infty} \frac{\log L^f_{\lambda^D_j(x_0)}}{\log(2^{-j})}.$$

Let us start with an intermediate result key for proving Theorem 5.12 in Sections 5.5 and 5.6.

Lemma 5.39. *For all $f \in \widetilde{B}^{\xi}_{\infty,q}$, for every $x \in [0,1]^d$ and $a \in [0,1]^{d'}$, $h_{f_a}(x) \geq \underline{h}_\mu(x) + h^{\min}_\nu$.*

Proof. Let $f \in \widetilde{B}^{\xi}_{\infty,q}$ and $a \in [0,1]^{d'}$. Consider the wavelet decomposition (5.56) of f_a

$$f_a = c_{0,0^d,0^d}(a)\psi_{0,0^d,0^d} + \sum_{\lambda^d\in\Lambda^d\times L^d} c_{\lambda^d}(a)\psi_{\lambda^d}. \tag{5.59}$$

Pay attention that $c_{\lambda^d}(a)$ does not have an explicit formula like for $d^G_{\lambda^d}(a)$ in (5.35) and $d^F_{\lambda^d}(a)$ in (5.36). Using Proposition 5.35, one has $f_a \in \widetilde{B}^{\mu(+h^{\min}_\nu)}_{\infty,1}([0,1]^d)$. Then, for every $m \geq \lfloor 1/h^{\min}_\nu \rfloor$, there is $C > 0$ such that $\sup_{j\in\mathbb{N}} \sup_{\lambda^d\in\Lambda^d_j\times L^d} \left|\frac{c_{\lambda^d}(a)}{\mu^{(+h^{\min}_\nu - 1/m)}(\lambda^d)}\right| \leq C$. As a consequence, for every $x \in [0,1]^d$,

$$\begin{aligned} L^f_{\lambda^d_j(x)}(a) &= \sup_{\lambda\subset 3\lambda^d_j(x)} \big|c_\lambda(a)\big| \leq C \sup_{\lambda_{j'}\subset 3\lambda^d_j(x)} \big|\mu(\lambda_{j'})\ 2^{-j'(h^{\min}_\nu - \frac{1}{m})}\big| \\ &\leq C\big|\mu(3\lambda^d_j(x))\ 2^{-j(h^{\min}_\nu - \frac{1}{m})}\big|. \end{aligned}$$

The doubling property (5.5) of μ yields, for a constant $C > 0$,

$$L^f_{\lambda^d_j(x)}(a) \leq C\big|\mu(\lambda^d_j(x))\ 2^{-j(h^{\min}_\nu - \frac{1}{m})}\big|. \tag{5.60}$$

Hence, for all $j \in \mathbb{N}$, $\log L^f_{\lambda^d_j(x)}(a) \leq \log C + \log \mu(\lambda^d_j(x)) + \log\big(2^{-j(h^{\min}_\nu - 1/m)}\big)$. Dividing by $\log(2^{-j})$ and taking the lim inf on both sides gives $h_{f_a}(x) \geq \underline{h}_\mu(x) + h^{\min}_\nu - 1/m$. This holds for every large m, hence the conclusion. □

Lemma 5.40. *Let $r \in \mathbb{R}$. For all $f \in \widetilde{B}^{\xi}_{\infty,q}$, for every $x \in [0,1]^d$ and ν_r-almost all $a \in [0,1]^{d'}$, $h_{f_a}(x) \geq \underline{h}_\mu(x) + h^r_\nu$.*

Proof. The arguments are identical to those of Lemma 5.39. Theorem 5.11 gives for ν_r-almost all $a \in [0,1]^{d'}$, $f_a \in \widetilde{B}^{\mu^{(+h^r_\nu)}}_{\infty,1}([0,1]^d)$. Then, for every $m \geq \lfloor 1/h^r_\nu \rfloor$, there exists a constant $C > 0$ such that

$$\sup_{j\in\mathbb{N}} \sup_{\lambda^d \in \Lambda^d_j \times L^d} \left| \frac{c_{\lambda^d}(a)}{\mu^{(+(h^r_\nu - 1/m))}(\lambda^d)} \right| \leq C.$$

Thus, for all $x \in [0,1]^d$, as in (5.60),

$$L^f_{\lambda^d_j(x)}(a) = \sup_{\lambda \subset 3\lambda^d_j(x)} \big|c_\lambda(a)\big| \leq C \sup_{\lambda_{j'} \subset 3\lambda^d_j(x)} \big|\mu(\lambda_{j'}) 2^{-j'(h^r_\nu - \frac{1}{m})}\big| \leq C\big|\mu(\lambda^d_j(x)) 2^{-j(h^r_\nu - \frac{1}{m}))}\big|.$$

As previously, one concludes that for ν_r-almost all $a \in [0,1]^{d'}$, $h_{f_a}(x) \geq \underline{h}_\mu(x) + h^r_\nu$. □

We are now ready to prove Theorem 5.12.

Proof of Theorem 5.12. Considering a function in $\widetilde{B}^{\xi}_{\infty,q}$ without any further hypothesis, one can only get an upper bound of the spectrum of the trace of such a function. Having both Lemma 5.39 and 5.40, we now prove the upper bound for σ_{f_a}.

Let us start by (5.18). Using Lemma 5.39, one has that, for $a \in [0,1]^{d'}$ and $h \in \mathbb{R}^+$, $\{x \in \mathbb{R}^d : h_{f_a}(x) \leq h\} \subset \{x \in \mathbb{R}^d : \underline{h}_\mu(x) \leq h - h^{\min}_\nu\}$. Hence

$$E_{f_a}(h) \subset \{x \in \mathbb{R}^d : h_{f_a}(x) \leq h\} \subset \{x \in \mathbb{R}^d : \underline{h}_\mu(x) \leq h - h^{\min}_\nu\} = \underline{E}^{\leq}_\mu(h - h^{\min}_\nu).$$

When $h \leq h^0_\nu + h^{\min}_\nu$, item *(ii)* of Proposition 5.19 gives $\dim_H\big(\underline{E}^{\leq}_\mu(h - h^{\min}_\nu)\big) = \tau^*_\mu(h - h^{\min}_\nu)$. As μ verifies the SMF, for every $h \leq h^0_\nu + h^{\min}_\nu$, one has $\tau^*_\mu(h - h^{\min}_\nu) = \sigma_\mu(h - h^{\min}_\nu)$. Moreover, for every $h > h^0_\nu + h^{\min}_\nu$, $\{x \in \mathbb{R}^d : \underline{h}_\mu(x) \leq h - h^{\min}_\nu\}$ has dimension d. This induces the result as $\sigma_{f_a}(h) = \dim_H(E_{f_a}(h))$.

Let us finish with the proof of (5.19). Let $f \in \widetilde{B}^{\xi}_{\infty,q}$ and $r \in \mathbb{R}$. By Lemma 5.40, for ν_r-almost all $a \in [0,1]^{d'}$, one has, for all $h \in \mathbb{R}^+$,

$$E_{f_a}(h) \subset \{x \in \mathbb{R}^d : h_{f_a}(x) \leq h\} \subset \{x \in \mathbb{R}^d : \underline{h}_\mu(x) \leq h - h^r_\nu\} = \underline{E}^{\leq}_\mu(h - h^r_\nu).$$

Then one concludes in the same way as for (5.18). □

5.5 A saturating function in $\widetilde{B}^{\mu}_{\infty,q}$

From now on, we focus on $p = +\infty$. For the rest of the chapter, the wavelet family considered will be constructed as in Section 5.2.4 with ψ^1 verifying the property (R) of Definition 5.15.

To compute the singularity spectrum of $\mathscr{G}_a$, this property (R) of the wavelet ψ is necessary. Indeed, in [153], the following theorem is proved.

Theorem 5.41. *Let $\psi : \mathbb{R} \to \mathbb{R}$ satisfy property (R), with $\mathrm{supp}(\psi) = [0, K_\psi]$. Let*

$$G : x \in \mathbb{R} \mapsto \sum_{n\in\mathbb{Z}} \psi(x - nK_\psi). \tag{5.61}$$

There exists $\alpha > 0$ such that , for every $d' \in \mathbb{N}^*$, there exists an integer $N(d')$ and a sequence of multi-integer $(\overline{p_j} = (p_{j,1}, \dots, p_{j,d'}))_{j\in\mathbb{N}} \in (\{0, \dots, K_\psi - 1\}^{d'})^{\mathbb{N}}$ enjoying the following property: for every $J \equiv 0 \pmod{N(d')}$ and $x \in \mathbb{R}^{d'}$, there is an integer $\ell \in \{0, \dots, N(d') - 1\}$ that verifies $\prod_{i=1}^{d'} |G(2^{J+\ell}x_i - p_{J+\ell,i})| \geq \alpha^{d'}$.

In other words, (R) ensures that dilated and translated transforms of tensor products of periodised versions of ψ cannot all vanish at the same point, and even more, they are non-vanishing very regularly. This is key to proving that the traces of well-chosen *saturating* functions cannot be zero too often, and is key to our proof of the prevalence Theorem 5.14.

Consider the integer $N(d')$ and the sequence $(\overline{p_j} = (p_{j,1}, \dots, p_{j,d'}))_{j\in\mathbb{N}}$ given by Theorem 5.41.

Definition 5.42. *The map $G_j^{d'} : \mathbb{R}^{d'} \to \mathbb{R}$ is given by*

$$\text{for every } x = (x_1, ..., x_{d'}) \in \mathbb{R}^{d'}, \quad G_j^{d'}(x) = \prod_{i=1}^{d'} G(2^j x_i - p_{j,i}). \tag{5.62}$$

To establish generic or prevalent properties of traces of function in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$, a classical approach consists in perturbing a set of functions by a *saturating function* in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ whose multifractal structure exhibits the property claimed to be generic.

5.5.1 Construction of the saturating function $\mathscr{G}_q$

Definition 5.43. *Let us define, for every $\lambda_0^D \in \Lambda_0^D \times \{0,1\}^D$, $g_{\lambda_0^D} = 0$, and for every $\lambda_j^D = (j, (k, k'), (l, l')) \in \Lambda_j^D \times L^D$ with $j \geq 1$,*

$$g_{\lambda_j^D} = \begin{cases} j^{-\frac{2}{q}}\, \xi(\lambda_{j,k}^d \times \lambda_{j,k'}^{d'}) \displaystyle\sum_{n\in\mathbb{Z}^{d'}} \delta^{k'}_{p_j + nK_\psi} & \text{if } l \neq 0^d \text{ and } l' = 1^{d'}, \\ 0 & \text{if } l = 0^d \text{ or } l' \neq 1^{d'}, \end{cases} \tag{5.63}$$

with the convention $2/\infty = 0$. The saturating function is $\mathscr{G}_q := \sum_{\lambda^D \in \Lambda^D \times L^D} g_{\lambda^D} \psi_{\lambda^D}$.

The idea is that $\mathscr{G}_q$ has the largest possible wavelet coefficients to maximise $\varepsilon_j^{\xi,\infty}$ in (5.15), while ensuring that the non-zero wavelet coefficients are sparse and well-separated, to easily compute the wavelet coefficients of its traces.

Proposition 5.44. *The function $\mathscr{G}_q$ belongs to $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$.*

Proof. First, since ξ is s-Hölder, one has

$$\|\mathscr{G}_q\|_{L^\infty} \leq \sum_{j\geq 1} \Big\| \sum_{\lambda_j^d \in \Lambda_j^D \times L^D} g_{\lambda_j^D} \psi_{\lambda_j^D} \Big\|_{L^\infty} \leq C \sum_{j\geq 1} 2^{-js} \Big\| \Big(\frac{g_{\lambda_j^D}}{\xi(\lambda_j^D)} \Big) \Big\|_{\ell^\infty} \leq C \sum_{j\geq 1} 2^{-js} \varepsilon_j^{\xi,\infty}.$$

Hence $\mathscr{G}_q$ is a bounded function. Next, recall (5.15). As for every $j \geq 1$ and $\lambda_j^D \in \Lambda_j^D \times L^D$, $g_{\lambda_j^D} \leq j^{-2/q} \xi(\lambda^D)$, one has $\varepsilon_j^{\xi,\infty} = \Big\| \Big(\frac{g_{\lambda^D}}{\xi(\lambda^D)} \Big)_{\lambda^D} \Big\|_{\ell^\infty(\Lambda_j^D \times L^D)} \leq j^{-2/q}$. This implies that $\|\mathscr{G}_q\|_{B^{\xi}_{\infty,q}} < \infty$ and $\mathscr{G}_q \in B^{\xi}_{\infty,q}([0,1]^D) \subset \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. □

For $a = (a_1, \dots, a_{d'}) \in [0,1]^{d'}$, the trace $\mathscr{G}_{q,a} : x \in [0,1]^d \mapsto \mathscr{G}_q(x,a) \in \mathbb{R}$ has a convenient wavelet decomposition. Indeed, because of our choice (5.63), for the trace $\mathscr{G}_{q,a}$, all the coefficients (5.35) are zero since $g_{\lambda_j^D} = 0$ when $\lambda_j^D = \lambda_j^d \times \lambda_j^{d'}$ with $\lambda_j^d \in \Lambda_j^d \times \mathbf{0}^d$. Hence, only the coefficients (5.36) remain. Writing then

$$\mathscr{G}_{q,a}(x) = \sum_{j\geq 0} \sum_{\lambda_j^d \in \Lambda_j^d \times L^d} e_{\lambda_j^d}(a) \psi_{\lambda_j^d}(x),$$

the coefficients of (5.36) can be written as

$$e_{\lambda_j^d}(a) = \sum_{\lambda^{d'} \in \Lambda_j^{d'} \times \mathbf{1}^{\mathbf{d'}}} g_{\lambda_j^d \times \lambda_j^{d'}} \psi_{\lambda_j^{d'}}(a). \tag{5.64}$$

In addition, these remaining coefficients have a fairly simple expression. To see this, for every $\lambda_j^d = (j, k_d, l_d) \in \Lambda_j^d \times L^d$, consider the integer $k_j(a)$ defined by

$$k_j(a) := \Big(p_{j,1} + \Big\lfloor \frac{2^j a_1 - p_{j,1}}{K_\psi} \Big\rfloor K_\psi, \dots, p_{j,d'} + \Big\lfloor \frac{2^j a_{d'} - p_{j,d'}}{K_\psi} \Big\rfloor K_\psi \Big). \tag{5.65}$$

Since $\psi_{\lambda_j^{d'}}$ has a support of length $N_\psi 2^{-j}$, and since the coefficients $g_{\lambda_{j,k}^d \times \lambda_{j,k'}^{d'}}$ are zero except when $k' = p_j \mod N_\psi$, the only non-zero wavelet coefficient $g_{\lambda_j^D}$ in the sum (5.64) is the one at $\lambda_j^D = (j, (k_d, k_j(a)), (l_d, 1^{d'}))$. Observe that as soon as j is so large that $N_\psi 2^{-j} \leq a/2$, then $k_j(a)$ has strictly positive entries. One concludes that $a \in (0,1]^{d'}$ being fixed, when j is large, one has

$$e_{\lambda_j^d}(a) := j^{-2/q} \mu(\lambda_j^d) \nu\big(\lambda_{j,k_j(a)}^{d'}\big) G_j^{d'}(a). \tag{5.66}$$

This formula is key for the rest of the chapter, and justifies our choice for the saturation function $\mathcal{G}_q$. One can now understand the relevance of Theorem 5.41, which allows us to ensure that the coefficients $|e_{\lambda_j^d}(a)|$ are not too small - this will be used to bound by above the pointwise Hölder exponent of the traces.

Regarding the multifractal properties, Theorem 5.11 applied to $\mathscr{G}_q$ gives:

Corollary 5.45. *For every $r \in \mathbb{R}$, for ν_r-almost all $a \in [0,1]^{d'}$, $\mathscr{G}_{q,a} \in \widetilde{B}^{\mu^{(+h_\nu^r)}}_{\infty,q}([0,1]^d)$.*

5.5.2 Singularity spectrum of the traces $\mathscr{G}_{q,a}$ of the saturating function

Proposition 5.46. *For every $r \in \mathbb{R}$, for every $x \in [0,1]^d$ and for ν_r-almost all $a \in (0,1]^{d'}$,*

$$h_{\mathscr{G}_a}(x) = \underline{h}_\mu(x) + h^r_\nu = \underline{h}_{\mu^{(+h^r_\nu)}}(x). \tag{5.67}$$

Hence

$$\sigma_{\mathscr{G}_a} = \tau^*_{\mu^{(+h^r_\nu)},\infty} = \tau^*_\mu(\cdot - h^r_\nu). \tag{5.68}$$

Proof. Let $m, n \in \mathbb{N}^*$ be such that $m \geq M_{\nu_r} := \lfloor 1/h^r_\nu \rfloor + 1$. Let $A^{\nu_r}_{n,m}$ and $J_{n,m}$ be the set and integer obtained in Proposition 5.25. Let $a \in A^{\nu_r}_{n,m} \cap (0,1]^{d'}$. By (5.26), one has $2^{-j(h^r_\nu+1/m)} \leq \nu(\lambda^{d'}_j) \leq 2^{-j(h^r_\nu-1/m)}$ for all $\lambda^{d'}_j \in \Lambda^{d'}_j$ such that $a \in (2K_\psi)\lambda^{d'}_j$ and $j \geq J_{n,m}$.

We first bound above $L_{\lambda^d}(a)$ (5.58) of $\mathscr{G}_{q,a}$. Consider $j \equiv 0 \pmod{N(d')}$. Using (5.66), for $\lambda^d_j = (j,k) \in \Lambda^d_j$, and that $|G^{d'}_{j'}(a)| \leq \max\{\|\psi\|^{d'}_{L^\infty}, 1\}$, one has

$$\begin{aligned}
L_{\lambda^d_j}(a) &= \sup_{j' \geq j} \sup_{\Lambda^d_{j'} \ni \lambda^d_{j'} \subset 3\lambda^d_j} \left| \frac{1}{(j')^{2/q}} \mu(\lambda^d_{j'}) \nu\big(\lambda^{d'}_{j',k_{j'}(a)}\big) G^{d'}_{j'}(a) \right| \\
&\leq \sup_{j' \geq j} \sup_{\Lambda^d_{j'} \ni \lambda^d_{j'} \subset 3\lambda^d_j} \left| \mu(\lambda^d_{j'}) 2^{-j'(h^r_\nu - 1/m)} \right| \max\{\|\psi\|^{d'}_{L^\infty}, 1\} \\
&\leq C 2^{-j(h^r_\nu - 1/m)} \left| \mu(3\lambda^d_j) \right|.
\end{aligned}$$

For every $x \in \mathbb{R}^d$, since $3\lambda^d_j(x) \subset B(x, 2 \times 2^{-j})$, one has that $L_{\lambda^d_j(x)}(a) \leq C 2^{-j(h^r_\nu - 1/m)} \mu(B(x, 2 \times 2^{-j}))$. The doubling property (5.5) of μ then yields for some $C > 0$ depending on μ, ψ and d'

$$L_{\lambda^d_j(x)}(a) \leq C 2^{-j(h^r_\nu - 1/m)} \mu(B(x, 2^{-j})). \tag{5.69}$$

We now bound $L_{\lambda^d}(a)$ from below. Consider j so large that $k_j(a)$ contains only positive integers. Let $x \in [0,1]^d$. By Theorem 5.41, for every $j \equiv 0 \mod N(d')$, there is $J \in \{0, \dots, N(d') - 1\}$ such that $\left|G^{d'}_{j+J}(x)\right| \geq \alpha^{d'}$. By (5.66) and (5.26) one deduces that

$$\begin{aligned}
L_{\lambda^d_j(x)}(a) &\geq \frac{1}{(j+J)^{2/q}} \mu(\lambda^d_{j+J}(x)) \nu(\lambda^{d'}_{j+J,k_{j+J}(a)}) \left| G^{d'}_{j+J}(a) \right| \\
&\geq 2^{-2\log_2(j+J)/q} \alpha^{d'} 2^{-(j+J)(h^r_\nu + 1/m)} \mu(\lambda^d_{j+J}(x)).
\end{aligned}$$

Denoting by y_{λ_j} the center of the cube λ_j, observe that $\lambda^d_{j+J}(x) = B(y_{\lambda^d_{j+J}(x)}, 2^{-(j+J+1)})$. Hence, using the constant C_μ appearing in (5.5), one gets

$$\begin{aligned}
L_{\lambda^d_j(x)}(a) &\geq 2^{-2\log_2(j+N(d'))/q} \alpha^{d'} 2^{-N(d')(h^r_\nu + 1/m)} 2^{-j(h^r_\nu + 1/m)} \mu\big(B(y_{\lambda_{j+J}(x)}, 2^{-(j+J+1)})\big) \\
&\geq C 2^{-2\log_2(j+N(d'))/q} 2^{-j(h^r_\nu + 1/m)} \frac{1}{C^{J+2}_\mu} \mu\big(B(y_{\lambda_{j+J}(x)}, 2^{-j+1})\big) \\
&\geq C 2^{-j\left(h^r_\nu + \frac{2}{q}\frac{\log_2(j+N(d'))}{j} + 1/m\right)} \mu\big(B(x, 2^{-j})\big).
\end{aligned} \tag{5.70}$$

Last equation (5.70) holds only when $j = 0 \mod N(d')$. Since the wavelet leaders $L_{\lambda^d_j(x)}(a)$ are decreasing in j, the liminf of the above quantities when $j \to +\infty$ equals the liminf restricted to $j = 0 \mod N(d')$. Hence, Proposition 5.38 combined with (5.69), (5.70) and the

lower local dimension $\underline{h}_\mu(x)$ together yield that

$$(h_\nu^r - 1/m) + \underline{h}_\mu(x) \leq h_{\mathscr{G}_a}(x) \leq (h_\nu^r + 1/m) + \underline{h}_\mu(x).$$

These inequalities hold for $a \in A_{n,m}^{\nu_r} \cap (0,1]^{d'}$ and remain true for every $a \in \bigcup_{n\in\mathbb{N}^*} A_{n,m}^{\nu_r} \cap (0,1]^{d'}$ and $m \geq M_{\nu_r}$ as they do not depend on n. Using that the sets $\bigcup_{n\in\mathbb{N}^*} A_{n,m}^{\nu_r}$ are of full ν_r-measure for all m, $\left(\bigcap_{m\geq M_{\nu_r}} \bigcup_{n\in\mathbb{N}^*} A_{n,m}^{\nu_r}\right) \cap (0,1]^{d'}$ is of full ν_r-measure as a countable intersection of full ν_r-measure sets. Hence for every $a \in \left(\bigcap_{m\geq M_{\nu_r}} \bigcup_{n\in\mathbb{N}^*} A_{n,m}^{\nu_r}\right) \cap (0,1]^{d'}$,

$$h_{\mathscr{G}_a}(x) = \underline{h}_\mu(x) + h_\nu^r. \tag{5.71}$$

Then by definition of $\mu^{(+h_\nu^r)}$, $\underline{h}_{\mu^{(+h_\nu^r)}}(x) = \underline{h}_\mu(x) + h_\nu^r$ giving (5.67). Since μ satisfies the SMF, one concludes that for ν_r-almost all $a \in (0,1]^{d'}$, $\sigma_{\mathscr{G}_a} = \sigma_{\mu^{(+h_\nu^r)}} = \tau_\mu^*(\cdot - h_\nu^r)$. □

5.6 Prevalent singularity spectrum of traces

5.6.1 Prevalence property of an auxiliary set

The key result to obtain the prevalent value of the singularity spectrum of functions in $\widetilde{B}_{\infty,q}^{\xi}([0,1]^D)$ in Theorem 5.14 is the following:

Proposition 5.47. *Let $r \in \mathbb{R}$. The set*

$$\mathcal{F}^r = \big\{ f \in \widetilde{B}_{\infty,q}^{\xi}([0,1]^D) : \exists A(f) \text{ of full } \nu_r\text{-measure in } (0,1]^{d'} \\ \text{such that } a \in A(f) \implies \forall x \in [0,1]^d,\ h_{f_a}(x) \leq \overline{h}_\mu(x) + h_\nu^r \big\} \tag{5.72}$$

is prevalent in $\widetilde{B}_{\infty,q}^{\xi}([0,1]^D)$.

The proof of Proposition 5.47 is postponed to Sections 5.6.3 and 5.6.4. For the moment, we admit it and explain how to deduce Theorem 5.14.

Proof of Theorem 5.14. Using Lemma 5.40 and Proposition 5.47, for all $f \in \mathcal{F}^r$, there is a set $A(f)$ of full ν_r-measure in $(0,1]^{d'}$ such that for every $x \in [0,1]^d$,

$$\underline{h}_\mu(x) + h_\nu^r \leq h_{f_a}(x) \leq \overline{h}_\mu(x) + h_\nu^r. \tag{5.73}$$

Let us begin with the upper bound $\sigma_{f_a} \leq \sigma_\mu(\cdot - h_\nu^r)$. For all $h \in \mathbb{R}_+$,

$$E_{f_a}(h) \subset \{x \in \mathbb{R}^d \ :\ h_{f_a}(x) \leq h\} \subset \{x \in \mathbb{R}^d \ :\ \underline{h}_\mu(x) + h_\nu^r \leq h\} = \underline{E}_\mu^{\leq}(h - h_\nu^r),$$
$$E_{f_a}(h) \subset \{x \in \mathbb{R}^d \ :\ h_{f_a}(x) \geq h\} \subset \{x \in \mathbb{R}^d \ :\ \overline{h}_\mu(x) + h_\nu^r \geq h\} = \overline{E}_\mu^{\geq}(h - h_\nu^r).$$

Items *(ii)* and *(iii)* of Proposition 5.19 give:

- for every $h \leq \tau_\mu'(0) + h_\nu^r$, $\dim_H\big(\underline{E}_\mu^{\leq}(h - h_\nu^r)\big) = \tau_\mu^*(h - h_\nu^r)$,
- for every $h \geq \tau_\mu'(0) + h_\nu^r$, $\dim_H\big(\overline{E}_\mu^{\geq}(h - h_\nu^r)\big) = \tau_\mu^*(h - h_\nu^r)$.

Additionally, μ verifies the SMF, so for all $h \in \mathbb{R}_+$, $\tau_\mu^*(h) = \sigma_\mu(h)$. Hence, $\sigma_{f_a} \leq \sigma_\mu(\cdot - h_\nu^r)$.

To prove the lower bound $\sigma_{f_a} \geq \sigma_\mu(\cdot - h_\nu^r)$, consider $h \in \mathbb{R}_+$ and $x \in E_\mu(h)$. Recalling Definition 5.16, $\underline{h}_\mu(x) = \overline{h}_\mu(x) = h_\mu(x)$, and (5.73) implies that $h_\mu(x) = h_{f_a}(x) - h_\nu^r$. So,

for every $h \in \mathbb{R}_+$, $E_\mu(h) \subset E_{f_a}(h + h_\nu^r)$ and item (*i*) of Proposition 5.19 gives $\sigma_{f_a}(h) = \dim_H(E_{f_a}(h)) \geq \dim_H(E_\mu(h - h_\nu^r)) = \sigma_\mu(h - h_\nu^r)$.

This holds for a given $r \in \mathbb{R}$. Now, given any countable sequence $(r_n)_{n\in\mathbb{N}}$, the set $\mathcal{F}^{r_n}$ is prevalent in $\widetilde{B}^\xi_{\infty,q}([0,1]^D)$, and so is their intersection $\bigcap_{n\in\mathbb{N}} \mathcal{F}^{r_n}$. This concludes the proof of Theorem 5.14, since for every $n \in \mathbb{N}$, for ν_{r_n}-a.e $a \in (0,1]^{d'}$, $\sigma_{f_a}(h) = \sigma_\mu(h - h_\nu^{r_n})$. □

5.6.2 A useful decomposition of $\mathcal{F}^r$

The objective is to show that, r being fixed, $(\mathcal{F}^r)^\complement = \widetilde{B}^\xi_{\infty,q}([0,1]^d)\backslash\mathcal{F}^r$ is shy. For this, we include $(\mathcal{F}^r)^\complement$ in a countable union of simpler ancillary sets. For $n, m, p, M \in \mathbb{N}^*$ and $\gamma > 0$, let

$$\mathcal{O}_{n,m}^{p,\gamma,M} = \big\{ f \in \widetilde{B}^\mu_{\infty,q}([0,1]^d) \ : \ \nu_r\big(\mathcal{A}_{n,m}^{p,\gamma,M}(f)\big) > 0 \big\}.$$

with

$$\mathcal{A}_{n,m}^{p,\gamma,M}(f) = \left\{ a \in A_{n,m}^{\nu_r} \cap (0,1]^{d'} : \begin{array}{c} \exists x \in [0,1]^d, \forall \lambda_j^d = (j,k,l) \in \Lambda^d \times L^d, \\ \big|c_{\lambda_j^d}^f(a)\big| \leq M_1\ 2^{-j\gamma}\big(1 + |2^j x - k|\big)^\gamma \\ \forall j > M_2,\ \mu(\lambda_j(x)) \geq 2^{-j(\gamma - h_\nu^r - 1/p)} \end{array} \right\}. \tag{5.74}$$

The sets $\mathcal{O}_{n,m}^{p,\gamma,M}$ contains those functions $f \in \widetilde{B}^\xi_{\infty,q}([0,1]^D)$ such that, for a set of $a \in (0,1]^{d'}$ of ν_r-positive measure, for a given $x \in [0,1]^d$, $f_a \in C^\gamma(x)$ and simultaneously the $\mu^{(+h_\nu^r)}$-mass of all small dyadic cubes $\lambda_j^d(x)$ is bounded below. These two properties will allow us to control the ratio $\frac{|c_{\lambda^d}(a)|}{\mu^{(+h_\nu^r)}(\lambda^d(x))}$.

Proposition 5.48. *We have* $(\mathcal{F}^r)^\complement \subset \bigcup_{p\in\mathbb{N}^*} \bigcap_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}^*} \bigcup_{\gamma\in\mathbb{Q}_+} \bigcup_{M\in(\mathbb{N}^*)^2} \mathcal{O}_{n,m}^{p,\gamma,M}$.

Proof. We begin by observing that $(\mathcal{F}^r)^\complement$ can be written as

$$(\mathcal{F}^r)^\complement = \big\{ f \in \widetilde{B}^\xi_{\infty,q}([0,1]^D) : \forall A(f) \text{ of full } \nu_r\text{-measure}, \\ \exists a \in A(f), \exists x \in [0,1]^d, h_{f_a}(x) > \overline{h}_\mu(x) + h_\nu^r \big\}.$$

We write $(\mathcal{F}^r)^\complement = \bigcup_{p\in\mathbb{N}^*} (\mathcal{F}_p^r)^\complement$, where

$$\mathcal{F}_p^r = \big\{ f \in \widetilde{B}^\xi_{\infty,q}([0,1]^D) : \exists A(f) \text{ of full } \nu_r\text{-measure}, \\ \forall a \in A(f), \forall x \in [0,1]^d, h_{f_a}(x) - \big(\overline{h}_\mu(x) + h_\nu^r\big) \leq 2/p \big\}.$$

Suppose that $f \in (\mathcal{F}^r)^\complement$. There exists $p \in \mathbb{N}^*$ such that $f \notin \mathcal{F}_p^r$. The set $\mathcal{N} = \{a \in [0,1]^{d'} : \exists x \in [0,1]^d,\ h_{f_a}(x) - \big(\overline{h}_\mu(x) + h_\nu^r\big) > \frac{2}{p}\}$ has positive ν_r-measure. Recall from Proposition 5.25 that $\bigcap_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}^*} A_{n,m}^{\nu_r}$ has full ν_r-measure in $(0,1]^{d'}$. Considering

$$\mathcal{N}_{n,m} := \{a \in A_{n,m}^{\nu_r} \cap (0,1]^{d'} : \exists x \in [0,1]^d, h_{f_a}(x) - \big(\overline{h}_\mu(x) + h_\nu^r\big) > 2/p\},$$

One has the following decomposition

$$\Big(\bigcap_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}^*} \mathcal{N}_{n,m} \Big)^\complement = \Big(\bigcap_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}^*} A_{n,m}^{\nu_r} \Big)^\complement \\ \bigcup \Big\{ a \in \bigcap_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}^*} A_{n,m}^{\nu_r} \cap (0,1]^{d'} : \forall x \in [0,1]^d, h_{f_a}(x) - \big(\overline{h}_\mu(x) + h_\nu^r\big) \leq 2/p \Big\}.$$

Since $(\bigcap_{m\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*} A^{\nu_r}_{n,m})^{\complement}$ has zero ν_r-measure, $f\notin\mathcal{F}^r_p$ implies that the set

$$\Big\{a\in\bigcap_{m\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*} A^{\nu_r}_{n,m}\cap(0,1]^{d'} : \forall x\in[0,1]^d,\ h_{f_a}(x)-\big(\overline{h}_\mu(x)+h^r_\nu\big)\leq 2/p\Big\}$$

cannot have full ν_r-measure. Hence, if $f\notin\mathcal{F}^r_p$, then $\bigcap_{m\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*}\mathcal{N}_{n,m}$ must have positive ν_r-measure. Thus, for every $m\in\mathbb{N}^*$, $\bigcup_{n\in\mathbb{N}^*}\mathcal{N}_{n,m}$ must have positive ν_r-measure, and there exists $n_m\in\mathbb{N}^*$ such that $\mathcal{N}_{n_m,m}$ has positive ν_r-measure.

Suppose $a\in\mathcal{N}_{n_m,m}$, there exists $\gamma\in\mathbb{Q}_+$ and $x\in[0,1]^d$ such that $h_{f_a}(x)>\gamma$ and $\overline{h}_\mu(x)+h^r_\nu\leq\gamma-2/p$. Using (5.33) of Proposition 5.30, if $h_{f_a}(x)>\gamma$, then $f_a\in C^\gamma(x)$, which implies that

$$\exists M_1\in\mathbb{N},\forall\lambda_j=(j,k,l)\in\Lambda\times L^d,\big|c^f_{\lambda^d_j}(a)\big|\leq M_1 2^{-j\gamma}\big(1+|2^jx-k|\big)^\gamma.$$

Using Definition 5.16, $\overline{h}_\mu(x)+h^r_\nu\leq\gamma-2/p$ yields that $\sup_{j>M_2}\frac{\log\mu(\lambda_j(x))}{\log 2^{-j}}\leq\gamma-h^r_\nu-1/p$ for some integer M_2 large enough. This implies that

$$\exists M_2\in\mathbb{N},\forall j>M_2,\mu(\lambda_j(x))\geq 2^{-j(\gamma-h^r_\nu-1/p)}.$$

This precisely means that a belongs to the set $\mathcal{A}^{p,\gamma,M}_{n_m,m}(f)$ given in (5.74) with $M=(M_1,M_2)$. So $\mathcal{N}_{n_m,m}\subset\bigcup_{\gamma\in\mathbb{Q}_+}\bigcup_{M\in(\mathbb{N}^*)^2}\mathcal{A}^{p,\gamma,M}_{n_m,m}(f)$ and since $\nu_r(\mathcal{N}_{n,m})>0$, one of the $\mathcal{A}^{p,\gamma,M}_{n_m,m}(f)$ must have positive ν_r-measure. One concludes that $f\in\mathcal{O}^{p,\gamma,M}_{n_m,m}$, for every integer m.

To conclude, every $f\in(\mathcal{F}^r)^{\complement}$ belongs to $\bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}^*}\bigcup_{\gamma\in\mathbb{Q}_+}\bigcup_{M\in(\mathbb{N}^*)^2}\mathcal{O}^{p,\gamma,M}_{n,m}$ for some $p>0$. This proves the claim. □

5.6.3 Analyticity of $\mathcal{O}^{p,\gamma,M}_{n,m}$

To show that $\mathcal{F}^r$ is prevalent, it remains to prove the universal measurability of $\mathcal{O}^{p,\gamma,M}_{n,m}$ and that these sets are all shy. This is the purpose of the rest of the chapter.

We will actually prove that $\mathcal{O}^{p,\gamma,M}_{n,m}$ is an analytic set, which implies its universal measurability. Analytic sets in Polish spaces are defined as continuous images of Borel sets. Unfortunately, as we work with $\widetilde{B}^\xi_{\infty,q}([0,1]^d)$ which is not separable, we cannot use the Polish spaces characterisation of analytic sets and have to consider a more general definition valid for any Hausdorff topological space X endowed with a Borel σ-algebra $\mathcal{B}(X)$. More precisely, Choquet's capacitability theorem [43, Theorem 30.1] with the following definition of analyticity [43, Theorem 5.1] offers the right setting for our situation.

Definition 5.49. *For a compact topological space K, the collection of its closed subsets is written $\mathcal{K}$. Let $(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$ be the collection of countable intersections of countable unions of sets that are Cartesian products of a Borel set in X and a closed set in K.*

Call $\pi:X\times K\to X$ the canonical projection map $\pi(x,y)=x$. A set $A\subset X$ is analytic when there is a compact space K and $\mathcal{T}\in(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$ such that $A=\pi(\mathcal{T})$.

The famous Choquet's capacitability theorem is the following.

Theorem 5.50. *An analytic set of a Hausdorff topological space is universally measurable.*

We finally prove the analyticity of $\mathcal{O}_{n,m}^{p,\gamma,M}$, and so its universal measurability.

Proposition 5.51. *Let $n, m, p \in \mathbb{N}^*$, $M \in (\mathbb{N}^*)^2$ and $\gamma \in \mathbb{Q}_+$. The set $\mathcal{O}_{n,m}^{p,\gamma,M}$ is universally measurable in $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$.*

Proof. We are going to prove the analyticity of $\mathcal{O}_{n,m}^{p,\gamma,M}$ by taking the right sets X, K and $\mathcal{T}$ in Definition 5.49, and then writing it as $\pi(\mathcal{T})$. Denote $X := (\widetilde{B}^{\xi}_{\infty,q}([0,1]^D) \times (0,1]^{d'}, \mathscr{D}(\cdot,\cdot) + \|\cdot\|_\infty)$ with $\mathscr{D}$ defined in (5.17) and $K := ([0,1]^d, \|\cdot\|_\infty)$. The proof follows some ideas of [10].

Step 1: For every $\lambda_j^d = (j,k,l) \in \Lambda^d \times L^d$, every $c^f_{\lambda_j^d}(a)$ (5.56), let us introduce the maps $\phi_{\lambda_j^d}, \phi : X \times K \to \mathbb{R}$ defined as

$$\phi_{\lambda_j^d} : (f,a,x) \mapsto M_1 2^{-j\gamma}\big(1 + |2^j x - k|\big)^\gamma - \big|c^f_{\lambda_j^d}(a)\big|$$
$$\phi : (f,a,x) \mapsto \inf_{\lambda \in \Lambda^d \times L^d} \phi_\lambda(f,a,x).$$

We introduce the set of dyadic cubes

$$\mathcal{D}^d_\mu(j, \gamma - h^r_\nu - 1/p) := \big\{\lambda_j \in \Lambda^d_j \ : \ \frac{\log \mu(\lambda_j)}{\log 2^{-j}} \le \gamma - h^r_\nu - 1/p\big\}$$
$$\text{and} \quad \mathcal{D}^{\ge}_\mu(\gamma - h^r_\nu - 1/p) := \bigcap_{j \ge M_2} \bigcup_{\lambda \in \mathcal{D}^d_\mu(j, \gamma - h^r_\nu - 1/p)} \lambda. \tag{5.75}$$

Also, let

$$\widetilde{\phi} : f \mapsto \nu_r\big(\big\{a \in A^{\nu_r}_{n,m} \cap (0,1]^{d'} : \exists x \in \mathcal{D}^{\ge}_\mu(\gamma - h^r_\nu - 1/p),\ \phi(f,a,x) \ge 0\big\}\big).$$

With these definitions, one sees that $\mathcal{O}_{n,m}^{p,\gamma,M} = \widetilde{\phi}^{-1}\big((0,+\infty)\big)$.

Lemma 5.52. *The map $(f,a) \in X \longmapsto f_a \in (C^0([0,1]^d), \|\cdot\|_{L^\infty})$ is continuous.*

Proof. Let $\varepsilon > 0$, and fix $f, g \in X$ such that $\mathscr{D}(f,g) < \varepsilon$. By (5.17), for $n := \lfloor \max(1, s_1^{-1}) \rfloor + 1$, one has $2^{-n} \frac{\|f-g\|_{B^{\xi}_{\infty,q}(-1/n)}}{1+\|f-g\|_{B^{\xi}_{\infty,q}(-1/n)}} < \varepsilon$. This impies $\|f-g\|_{B^{\xi}_{\infty,q}(-1/n)} < 2^n\varepsilon$, hence $\|f-g\|_{L^\infty} < 2^n\varepsilon$.

Then, the uniform continuity of f on $[0,1]^D$ shows the existence of $\delta > 0$ such that if $a, b \in [0,1]^{d'}$ verify $\|a-b\|_\infty \le \delta$, then $\|f(\cdot,a) - f(\cdot,b)\|_{L^\infty} \le \varepsilon$. One concludes by the triangular inequality that $\|f(\cdot,a) - g(\cdot,b)\|_{L^\infty} \le \|f(\cdot,a) - f(\cdot,b)\|_{L^\infty} + \|f-g\|_{L^\infty} \le (C2^n + 1)\varepsilon$. □

Lemma 5.53. *Each $\phi_{\lambda_j^d}$ is continuous on $X \times K$.*

Proof. Indeed, we deal with continuous functions f and Lemma 5.52 shows that $(f,a) \mapsto f_a$ is continuous and $f_a \mapsto c^f_{\lambda_j^d}(a)$ is continuous as a scalar product between f_a and a fixed wavelet $\psi_{\lambda_j^d}$. So, as a composition of continuous functions, the mapping $(f,a,x) \mapsto c^f_{\lambda_j^d}(a)$ is continuous, and so is $\phi_{\lambda_j^d}$ as a sum of continuous functions. □

Step 2: Let us introduce

$$\mathcal{T} := \phi^{-1}([0,+\infty)) \cap \big(X \times \mathcal{D}_{\mu}^{\geq}(\gamma - h_{\nu}^{r} - 1/p)\big). \tag{5.76}$$

Proposition 5.54. *The set $\mathcal{T}$ belongs to $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$.*

Proof. • One first shows that the set $\phi^{-1}([0,+\infty)) \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$. Observe that, as a composition of continuous mappings, the mapping

$$\begin{array}{ccccc} X & \longrightarrow & (C^0([0,1]^d), \|\cdot\|_{L^\infty}) & \longrightarrow & \mathbb{R} \\ \big(f,a\big) & \longmapsto & f_a & \longmapsto & c_{\lambda_j^d}(a) \end{array}$$

is continuous. So, for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, the set

$$F_{\lambda_j^d}(n,m) := \big\{(f,a) \in X : c^f_{\lambda_j^d}(a) \in 2^{-n}[m,m+1]\big\}.$$

belongs to $\mathcal{B}(X)$ as the inverse of a Borel set by a continuous mapping. Let us then define

$$X_{\lambda_j^d}(n,m) := \big\{x \in [0,1]^d : \sup_{(f,a)\in F_{\lambda_j^d}(n,m)} \phi_{\lambda_j^d}\big(f,a,x\big) \geq 0\big\}.$$

Observe that when f ranges over $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ and a in $(0,1]^{d'}$, all possible values of $c^f_{\lambda_j^d}(a) \in 2^{-n}[m,m+1]$ are reached. Hence, one can rewrite $X_{\lambda_j^d}(n,m)$ as

$$\begin{aligned} X_{\lambda_j^d}(n,m) &= \Big\{x \in [0,1]^d : \max_{c\in 2^{-n}[m,m+1]} M_1\, 2^{-j\gamma}\big(1+|2^j x - k|\big)^\gamma - |c| \geq 0\Big\} \\ &= \big\{x \in [0,1]^d : \exists\, c \in 2^{-n}[m,m+1],\ M_1\, 2^{-j\gamma}\big(1+|2^j x - k|\big)^\gamma - |c| \geq 0\big\}. \end{aligned}$$

Using the mappings $v : (x,c) \in \mathbb{R}^d \times \mathbb{R} \mapsto M_1\, 2^{-j\gamma}\big(1+|2^j x - k|\big)^\gamma - |c| \in \mathbb{R}$ and the projection on the first coordinate $\widetilde{\pi}(x,c) = x$, one sees that

$$X_{\lambda_j^d}(n,m) = \widetilde{\pi}\big(v^{-1}(\mathbb{R}_+) \cap ([0,1]^d \times 2^{-n}[m,m+1])\big) \cap [0,1]^d.$$

Since v and $\widetilde{\pi}$ are continuous, $X_{\lambda_j^d}(n,m)$ is thus compact.

Lemma 5.55. *One has $\phi^{-1}_{\lambda_j^d}([0,+\infty)) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{Z}} F_{\lambda_j^d}(n,m) \times X_{\lambda_j^d}(n,m)$.*

Proof. First, let us prove that $\phi^{-1}_{\lambda_j^d}([0,+\infty)) \subset \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{Z}} F_{\lambda_j^d}(n,m) \times X_{\lambda_j^d}(n,m)$.

Let $(f,a,x) \in \phi^{-1}_{\lambda_j^d}([0,+\infty))$. Then for every $n \in \mathbb{N}$, there is an integer $m \in \mathbb{Z}$ such that $c^f_{\lambda_j^d}(a) \in 2^{-n}[m,m+1]$ and $(f,a) \in F_{\lambda_j^d}(n,m)$. In addition, $\phi_{\lambda_j^d}\big(f,a,x\big) \geq 0$, so we have $\sup_{(g,b)\in F_{\lambda_j^d}(n,m)} \phi_{\lambda_j^d}\big(g,b,x\big) \geq 0$ and $x \in X_{\lambda_j^d}(n,m)$. Hence $(f,a,x) \in F_{\lambda_j^d}(n,m) \times X_{\lambda_j^d}(n,m)$.

Next, we show the inclusion $\phi^{-1}_{\lambda_j^d}([0,+\infty)) \supset \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{Z}} F_{\lambda_j^d}(n,m) \times X_{\lambda_j^d}(n,m)$.

Let $(f,a,x) \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{Z}} F_{\lambda_j^d}(n,m) \times X_{\lambda_j^d}(n,m)$. For a fixed $n \in \mathbb{N}$, there is $m \in \mathbb{Z}$ such that $(f,a) \in F_{\lambda_j^d}(n,m)$ and $x \in X_{\lambda_j^d}(n,m)$, meaning that $c^f_{\lambda_j^d}(a) \in 2^{-n}[m,m+1]$ and

$\sup_{(g,b)\in F_{\lambda_j^d}(n,m)} \phi_{\lambda_j^d}(g,b,x) \geq 0$. One deduces that

$$\begin{aligned}\phi_{\lambda_j^d}(f,a,x) &\geq \sup_{(g,b)\in F_{\lambda_j^d}(n,m)} \phi_{\lambda_j^d}(g,b,x) - \Big| \sup_{(g,b)\in F_{\lambda_j^d}(n,m)} \phi_{\lambda_j^d}(g,b,x) - \phi_{\lambda_j^d}(f,a,x)\Big| \\ &\geq - \sup_{(g,b)\in F_{\lambda_j^d}(n,m)} \big|\phi_{\lambda_j^d}(g,b,x) - \phi_{\lambda_j^d}(f,a,x)\big| \\ &\geq - \sup_{(g,b)\in F_{\lambda_j^d}(n,m)} \big|\big|c^g_{\lambda_j^d}(b)\big| - \big|c^f_{\lambda_j^d}(a)\big|\big| \geq -2^{-n}.\end{aligned}$$

Since this holds for all $n \in \mathbb{N}$, one deduces that $\phi_{\lambda_j^d}(f,a,x) \geq 0$, hence the result. □

Lemma 5.55 ensures that $\phi^{-1}_{\lambda_j^d}([0,+\infty)) \in (\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$, which implies that

$$\phi^{-1}([0,+\infty)) = \bigcap_{\lambda^d\in\Lambda^d\times L^d} \phi^{-1}_{\lambda^d}([0,+\infty)) \in (\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}.$$

• To prove Proposition 5.54, it remains to show that in (5.76), the second term defining $\mathcal{T}$ also belongs to $(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$. By (5.75), the set $\mathcal{D}^{\geq}_{\mu}(\gamma - h^r_\nu - 1/p)$ is closed in $[0,1]^d$ as countable intersection of closed sets. Thus $X \times \mathcal{D}^{\geq}_{\mu}(\gamma - h^r_\nu - 1/p)$ is a countable intersection of countable unions of closed sets, and so belongs to $(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$.

• Finally, $\mathcal{T}$ is the intersection of two sets belonging to $(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$, hence the conclusion. □

Step 3: By Definition 5.49, the canonical projection map $\pi : (x,y) \in X\times K \to x \in X$ gives that

$$\pi(\mathcal{T}) := \big\{(f,a)\in X : \exists x \in \mathcal{D}^{\geq}_{\mu}(\gamma - h^r_\nu - 1/p), (f,a,x) \in \phi^{-1}([0,+\infty))\big\}$$

is analytic in the space $(X, \mathcal{B}(X))$.

Step 4: To conclude, notice that $\widetilde{\phi} : \widetilde{B}^{\xi}_{\infty,q}([0,1]^D) \longrightarrow [0,1]$ can be rewritten as

$$\begin{aligned}\widetilde{\phi}(f) &= \nu_r\big(\big\{a \in A^{\nu_r}_{n,m} \cap (0,1]^{d'} : \exists x \in \mathcal{D}^{\geq}_{\mu}(\gamma - h^r_\nu - 1/p), \phi(f,a,x) \geq 0\big\}\big) \\ &= \int_{A^{\nu_r}_{n,m}} \mathbb{1}_{\pi(\mathcal{T})}(f,a) d\nu_r(a).\end{aligned}$$

Since $\pi(\mathcal{T})$ is then $\beta\otimes\nu_r$-measurable, Fubini's theorem gives that $\widetilde{\phi}$ is β-measurable, for any complete Borel measure β on $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. This yields that $\mathcal{O}^{p,\gamma,M}_{n,m} = \widetilde{\phi}^{-1}\big((0,+\infty)\big)$ is also β-measurable, for any complete Borel measure β on $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. □

5.6.4 Shyness of $\mathcal{O}^{p,\gamma,M}_{n,m}$

The standard method to prove the shyness of a set is to focus on a finite-dimensional *probe space* $\mathcal{P} \subset \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ and the Lebesgue measure restricted to $\mathcal{P}$ as a transverse measure.

Recall the integer $N(d')$ and the sequence of multi-integer $(\overline{p_j} = (p_{j,1},\dots,p_{j,d'}))_{j\in\mathbb{N}}$

obtained in Theorem 5.41. Let $m \geq p(d+2)$ and d_1 be the integer given by

$$d_1 = p(d+1). \tag{5.77}$$

Definition 5.56. *The probe space $\mathcal{P}$ is the subspace of $\widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$ spanned by the functions $\mathscr{G}^{(i)} := \sum_{\lambda^D \in \Lambda^D \times L^D} g^{(i)}_{\lambda^D} \psi_{\lambda^D}$, for $i \in \{1,\dots,d_1\}$, whose wavelet coefficients $g^{(i)}_{\lambda^D}$ are defined in the following way: for every $\lambda^D = (j,k,l) \in \Lambda^D \times \{0,1\}^D$,*

$$g^{(i)}_{\lambda^D} = \begin{cases} g_{\lambda^D} & \text{if } j \pmod{d_1 N(d')} \in \{0,\dots,N(d')-1\} + (i-1)\times N(d'), \\ 0 & \text{else.} \end{cases}$$

Observe that $\sum_{i=1}^{d_1} \mathscr{G}^{(i)} = \mathscr{G}_q$. The intuition is that $\mathscr{G}_q$ is sampled into $d_1 - 1$ functions $\mathscr{G}^{(i)}$ whose wavelet coefficients are non-zero only on a regular subsequence of generation j. More precisely, for $a \in [0,1]^{d'}$, by (5.63) and (5.66), the trace $\mathscr{G}^{(i)}_a$ can be expressed as

$$\mathscr{G}^{(i)}_a(x) = \sum_{j \geq 0} \sum_{\lambda^d_j \in \Lambda^d_j \times L^d} e^{(i)}_{\lambda^d_j}(a) \psi_{\lambda^d_j}(x) \tag{5.78}$$

where, recalling (5.66) , for every $\lambda^d = (j,k,l) \in \Lambda^d$,

$$e^{(i)}_{\lambda^d_j}(a) := \begin{cases} e_{\lambda^d_j}(a) & \text{if } j \mod d_1 N(d') \in \{0,\dots,N(d')-1\} + (i-1)N(d'), \\ 0 & \text{else.} \end{cases} \tag{5.79}$$

For any arbitrary $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$, define

$$\begin{array}{ccc} \mathbb{R}^{d_1} & \to & f + \mathcal{P} \\ \beta & \mapsto & f^\beta = f + \sum_{i=1}^{d_1} \beta_i \cdot \mathscr{G}^{(i)}. \end{array} \tag{5.80}$$

The trace of f^β is denoted f^β_a, the wavelet coefficients of the trace and the associated wavelet leaders are respectively $c^\beta_{\lambda^d}(a)$ and $L^\beta_{\lambda^d}(a)$. Similarly, $c_{\lambda^d}(a)$ and $L_{\lambda^d}(a)$ are the wavelet coefficient and the associated leader of f_a. For $a \in (0,1]^{d'}$, let also

$$\mathcal{B}_a := \{\beta \in \mathbb{R}^{d_1} : \mathcal{A}^{p,\gamma,M}_{n,m}(f^\beta)\}$$

Lemma 5.57. *The mapping* $\left\{ \begin{array}{ccc} [0,1]^{d'} \times \mathbb{R}^{d_1} & \longrightarrow & \mathbb{R} \\ (a,\beta) & \longmapsto & \mathbb{1}_{\mathcal{B}_a}(\beta) \end{array} \right.$ *is measurable.*

Proof. Let $\Phi : (a,\beta,x) \mapsto \inf_{\lambda^d_j \in \Lambda^d \times L^d} \phi_{\lambda^d_j}(f^\beta, a, x)$. The map Φ is Borel on $[0,1]^{d'} \times \mathbb{R}^{d_1} \times [0,1]^d$ as an infimum of countably many continuous functions. Writing explicitly $\mathcal{B}_a$ under the form

$$\mathcal{B}_a := \left\{ \beta \in \mathbb{R}^{d_1} : \begin{array}{c} \exists x \in [0,1]^d, \forall \lambda^d_j = (j,k,l) \in \Lambda^d \times L^d, \\ \left|c^\beta_{\lambda^d_j}(a)\right| \leq M_1\, 2^{-j\gamma}\left(1 + |2^j x - k|\right)^\gamma \\ \text{and } \forall j > M_2,\ \mu(\lambda^d_j(x)) \geq 2^{-j(\gamma - h^r_\nu - 1/p)} \end{array} \right\},$$

one sees that $\mathbb{1}_{\mathcal{B}_a}(\beta) = \mathbb{1}_{\mathcal{G}}(a,\beta)$, where

$$\mathcal{G} = \left\{(a,\beta) \in (0,1]^{d'} \times \mathbb{R}^{d_1} \ : \ \exists x \in \mathcal{D}^{\geq}_\mu(\gamma - h^r_\nu - 1/p),\ \Phi(a,\beta,x) \geq 0\right\}.$$

This set can be rewritten as

$$\mathcal{G} = \widetilde{\pi}\big(\Phi^{-1}([0,+\infty)) \bigcap \big((0,1]^{d'} \times \mathbb{R}^{d_1} \times \mathcal{D}_{\mu}^{\geq}(\gamma - h_\nu^r - 1/p)\big)\big), \tag{5.81}$$

where $\widetilde{\pi}(a,\beta,x) = (a,\beta)$ is the canonical projection on the first two coordinates. The set between brackets in (5.81) being a Borel set, $\mathcal{G}$ is analytic by Definition 5.49. By Theorem 5.50, it is also Lebesgue-measurable. Since $\mathbb{1}_{\mathcal{B}_a}(\beta) = \mathbb{1}_{\mathcal{G}}(a,\beta)$, the conclusion follows. □

Lemma 5.58. *For $a \in A_{n,m}^{\nu_r} \cap (0,1]^{d'}$, the d_1-dimensional Lebesgue measure $\mathcal{L}_{d_1}$ of $\mathcal{B}_a$ is zero.*

Proof. For any $\lambda_0 := (j_0,k_0,l_0) \in \Lambda^d$, denoting

$$\mathcal{B}_{a,\lambda_0} := \left\{ \beta \in \mathbb{R}^{d_1} : \begin{array}{c} \exists x \in \lambda_0, \forall \lambda_j^d = (j,k,l) \in \Lambda^d \times L^d, \\ \big|c_{\lambda_j^d}^{\beta}(a)\big| \leq M_1\ 2^{-j\gamma}\big(1+|2^j x - k|\big)^{\gamma} \\ \text{and } \forall j > M_2,\ \mu(\lambda_j^d(x)) \geq 2^{-j(\gamma - h_\nu^r - 1/p)} \end{array} \right\},$$

one has

$$\mathcal{B}_a = \limsup_{j_0 \to \infty} \bigcup_{(j_0,k_0,l_0)\in\Lambda_{j_0}^d} \mathcal{B}_{a,\lambda_0} = \bigcap_{J\in\mathbb{N}^*} \bigcup_{j_0 \geq J} \bigcup_{(j_0,k_0,l_0)\in\Lambda_{j_0}^d} \mathcal{B}_{a,\lambda_0}. \tag{5.82}$$

We show that $\mathcal{L}_{d_1}(\mathcal{B}_a) = 0$ by bounding above each $\mathcal{L}_{d_1}\big(\mathcal{B}_{a,\lambda_0}\big)$, and conclude with the Borel-Cantelli lemma.

One can ensure that j_0 is large enough that $j_0 \geq M_2 + d_1 N(d')$, $j_0 - d_1 N(d') \equiv 0 \pmod{d_1 N(d')}$ and $k_j(a) > 0$ (recall (5.65)). Suppose β and $\widetilde{\beta}$ belong to a same $\mathcal{B}_{a,\lambda_0}$. There exist $x_\beta, x_{\widetilde{\beta}} \in \lambda_0 \subset \mathbb{R}^d$ such that for every $j \in \mathbb{N}^*$, if $k_{j,\beta}$ and $k_{j,\widetilde{\beta}}$ is the multi-integer associated to $\lambda_j^d(x_\beta)$,

$$\big|c_{\lambda_j(x_\beta)}^{\beta}(a)\big| \leq M_1\ 2^{-j\gamma}\big(1+|2^j x_\beta - k_{j,\beta}|\big)^{\gamma} \tag{5.83}$$

and similarly for $c_{\lambda_j(x_{\widetilde{\beta}})}^{\widetilde{\beta}}(a)$ with its associated $k_{j,\widetilde{\beta}}$.

For every $j \in \{j_0 - d_1 N(d'), \dots, j_0\}$, one has $\lambda_j^d(x_\beta) = \lambda_j^d(x_{\widetilde{\beta}})$: call $\widetilde{\lambda}_j$ this cube, for short. For every $i \in \{1,\dots,d_1\}$, set $j^{(i)} \equiv (i-1)N(d') \pmod{d_1 N(d')}$ with $j^{(i)} \in \{j_0 - d_1 N(d'), \dots, j_0\}$ (remark that $j^{(1)} = j_0 - d_1 N(d')$). We make use of Theorem 5.41. For every $i \in \{1,\dots,d_1\}$, for every $a \in \mathbb{R}^{d'}$, there exists an integer $J^{(i)} \in \{0,\dots,N(d')-1\}$ such that $G_{j^{(i)}+J^{(i)}}^{d'}(a) \geq \alpha^{d'} > 0$. One sees that $j^{(i)} + J^{(i)} \in \{j_0 - d_1 N(d'), \dots, j_0\}$. Using $x_\beta, x_{\widetilde{\beta}} \in \lambda_0$ and (5.83), one deduces that there is a constant $C > 0$ depending on $M_1, j_0, d_1, \psi, \gamma$ and D such that, for every $j \in \{j_0 - d_1 N(d'), \dots, j_0\}$, $\big|c_{\widetilde{\lambda}_j}^{\beta}(a)\big| \leq C 2^{-j\gamma}$ and $\big|c_{\widetilde{\lambda}_j}^{\widetilde{\beta}}(a)\big| \leq C 2^{-j\gamma}$. From all this, one deduces that

$$\begin{aligned} &\big|(\beta_i - \widetilde{\beta}_i)\ e_{\widetilde{\lambda}_{j^{(i)}+J^{(i)}}}(a)\big| = \big|c_{\widetilde{\lambda}_{j^{(i)}+J^{(i)}}}(a) + \beta_i\ e_{\widetilde{\lambda}_{j^{(i)}+J^{(i)}}}(a) - \big(c_{\widetilde{\lambda}_{j^{(i)}+J^{(i)}}}(a) + \widetilde{\beta}_i\ e_{\widetilde{\lambda}_{j^{(i)}+J^{(i)}}}(a)\big)\big| \\ &\leq \big|c_{\widetilde{\lambda}_{j^{(i)}+J^{(i)}}}^{\beta}(a)\big| + \big|c_{\widetilde{\lambda}_{j^{(i)}+J^{(i)}}}^{\widetilde{\beta}}(a)\big| \leq 2C \cdot 2^{-(j_0 - d_1 N(d'))\gamma}. \end{aligned}$$

So, recalling (5.66), one has

$$\big|\beta_i - \widetilde{\beta}_i\big| \leq 2C j^{2/q} \frac{2^{-(j_0 - d_1 N(d'))\gamma}}{\mu\big(\widetilde{\lambda}_{j^{(i)}+J^{(i)}}\big)\nu\big(\lambda_{j^{(i)}+J^{(i)},k_{j^{(i)}+J^{(i)}}(a)}^{d'}\big)G_{j^{(i)}+J^{(i)}}^{d'}(a)}.$$

When $j \in \{j_0 - d_1 N(d'), \dots, j_0\}$, $j > M_2$, so $\mu(\widetilde{\lambda}_j) \geq 2^{-j(\gamma - h_\nu^r - 1/p)}$, from which we deduce that $\mu(\widetilde{\lambda}_j) \geq 2^{-j_0(\gamma - h_\nu^r - 1/p)}$. As $a \in A_{n,m}^{\nu_r}$, this gives

$$\nu\big(\lambda^{d'}_{j^{(i)}+J^{(i)},k_{j^{(i)}+J^{(i)}}(a)}\big) \geq 2^{-(j^{(i)}+J^{(i)})(h_\nu^r+1/m)} \geq 2^{-j_0(h_\nu^r+1/m)}.$$

So for every $i \in \{1, \dots, d_1\}$, recall that $G^{d'}_{j^{(i)}+J^{(i)}}(a) \geq \alpha^{d'}$, and one has

$$\big|\beta_i - \widetilde{\beta}_i\big| \leq 2C j^{2/q} \frac{2^{-(j_0 - d_1 N(d'))\gamma}}{2^{-j_0(\gamma - h_\nu^r - 1/p)}\ 2^{-j_0(h_\nu^r+1/m)}\alpha^{d'}} \leq C j_0^{2/q} 2^{-j_0(1/p - 1/m)}.$$

The distance between two elements of $\mathcal{B}_{a,\lambda_0}$ is at most $j_0^{2/q} 2^{-j_0(1/p-1/m)}$ up to a constant. Hence $\mathcal{L}_{d_1}(\mathcal{B}_{a,\lambda_0}) \leq C\ j_0^{2d/q} 2^{-j_0 \cdot d_1(1/p-1/m)}$. Summing over all the 2^{dj_0} dyadic cubes for $k_0 \in \mathbb{Z}^d_{j_0}$ at scale j_0 gives $\mathcal{L}_{d_1}\big(\bigcup_{k_0 \in \mathbb{Z}^d_{j_0}} \mathcal{B}_{a,\lambda_0}\big) \leq C j_0^{2d/q} 2^{j_0(d + d_1/m - d_1/p)}$. Because of our choice (5.77) and $d_1/m \leq \frac{p+1}{p+2} \leq 1$, $d + d_1/m - d_1/p < 0$. The Borel-Cantelli lemma implies $\mathcal{L}_{d_1}(\mathcal{B}_a) = \mathcal{L}_{d_1}\Big(\limsup_{j_0 \to \infty} \bigcup_{k_0 \in \mathbb{Z}^d_{j_0}} \mathcal{B}_{a,\lambda_0}\Big) = 0$, which proves the claim. □

Proposition 5.59. *For any $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$, the set $\{\beta \in \mathbb{R}^{d_1} : f^\beta \in \mathcal{O}^{p,\gamma,M}_{n,m}\}$ has $\mathcal{L}_{d_1}$-Lebesgue measure 0.*

Proof. By Lemma 5.58, $\int_{a \in A^{\nu_r}_{n,m} \cap (0,1]^{d'}} \int_{\beta \in \mathbb{R}^{d_1}} \mathbb{1}_{\mathcal{B}_a}(\beta)\, d\beta d\nu_r(a) = 0$. Applying Fubini's theorem (measurability being guaranteed by Lemma 5.57),

$$\int_{\beta \in \mathbb{R}^{d_1}} \int_{a \in A^{\nu_r}_{n,m} \cap (0,1]^{d'}} \mathbb{1}_{\mathcal{B}_a}(\beta)\ d\nu_r(a) d\beta = 0.$$

In other words, for Lebesgue-almost any $\beta \in \mathbb{R}^{d_1}$, there exists a set $A(f^\beta) \subset A^{\nu_r}_{n,m}$ with same ν_r-measure as $A^{\nu_r}_{n,m}$, such that $A(f^\beta) \cap \mathcal{A}^{p,\gamma,M}_{n,m}(f) = \emptyset$. Recalling that $\mathcal{A}^{p,\gamma,M}_{n,m}(f) \subset A^{\nu_r}_{n,m}$, this implies that $\nu_r(\mathcal{A}^{p,\gamma,M}_{n,m}(f^\beta)) = 0$, and $f^\beta \notin \mathcal{O}^{p,\gamma,M}_{n,m}$ which yields the result by taking complement. □

Finally, we are now ready to prove the shyness of the sets $\mathcal{O}^{p,\gamma,M}_{n,m}$.

Proposition 5.60. *For every $n, m \in \mathbb{N}$ with $m \geq p(d+2)$, $M = (M_1, M_2) \in (\mathbb{N}^*)^2$ and $\gamma > h + h_\nu^r$, $\mathcal{O}^{p,\gamma,M}_{n,m}$ is shy.*

Proof. Let $\vartheta = G_\# \mathcal{L}_{d_1}$ be the push-forward measure of the $\mathcal{L}_{d_1}$ onto $\mathcal{P}$ by

$$G : \left\{ \begin{array}{ccc} [0,1]^{d_1} & \to & \mathcal{P} \\ \beta & \mapsto & \sum_{i=1}^{d_1} \beta_i \cdot \mathscr{G}^{(i)}. \end{array} \right. \tag{5.84}$$

This measure is supported by a compact in $\mathcal{P} \subset \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. Fix any $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$. Using Proposition 5.59, for ϑ-almost all $F \in \mathcal{P}$, $f + F \notin \mathcal{O}^{p,\gamma,M}_{n,m}$. Hence for any $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$, the set $\{f + \mathcal{O}^{p,\gamma,M}_{n,m}\}$ has ϑ-measure equal to 0, i.e., $\vartheta(\{f + \mathcal{O}^{p,\gamma,M}_{n,m}\}) = 0$. Since it is true for any $f \in \widetilde{B}^{\xi}_{\infty,q}([0,1]^D)$, the set $\mathcal{O}^{p,\gamma,M}_{n,m}$ is shy. □

Using Proposition 5.48, the set $(\mathcal{F}^r)^\complement$ is shy as it is a subset of countable unions and intersections of shy sets. This yields that $\mathcal{F}^r$ is prevalent, proving Proposition 5.47.

5.7 Further investigations

A first natural extension consists in investigating the case where $p < \infty$, for which new and interesting phenomena may arise, as was observed in [19]. One would then like to explore situations where ξ is not a product of measures, or, as a preliminary step, to consider capacities that are not Gibbs capacities associated with $\mathscr{E}_d$. For instance, a natural extension consists in taking for ξ either 2- or 3-dimensional Mandelbrot cascades or Castaing cascade models, and to see whether the same conclusion holds. However, this scenario will be significantly more intricate and will likely depend on the behaviour of the marginals of ξ on $\mathbb{R}^d$ and $\mathbb{R}^{d'}$.

Finally, in all cases, it would be valuable to study traces not only on affine subspaces but also on general submanifolds.

Chapter 6

Anisotropy in Besov-type Spaces

6.1 Introduction

Anisotropy, which happens when physics does not act the same in different directions, arises in a wide range of applications such as geophysics, hydrology [148], medical imaging (osteoporosis, muscular tissues, mammographies,...) [27, 29]. A first issue lies in introducing global and local notions of anisotropic regularity, which encompass and extend (isotropic) regularity spaces, such as Sobolev or Besov spaces, and the classical notion of pointwise Hölder regularity. To model anisotropy, stochastic processes such as anisotropic self-similar random fields have been introduced and studied [22, 27, 45, 55], and some of the properties characterising anisotropy can be analysed with functional spaces well-adapted to anisotropy, such as anisotropic Besov spaces or Besov with dominating mixed smoothness. These spaces have an extensive literature, see for example [25, 31, 56, 158, 165, 168, 169].

Performing signal analysis to extract important information out of a signal, such as its regularity, can be done with a wavelet analysis, as the information is coded in the wavelet coefficients, which are easy to compute and handle. Although wavelet coefficients of a signal contain rather complete information about it, classical wavelets are not well-suited for the analysis of anisotropic signals due to their construction principle based on isotropic dilation. For example, a signal that is relatively smooth in one direction but rough in the orthogonal direction, such as layers or stripes in a texture, cannot be properly analysed by a classical multi-resolution method.

Many authors addressed this problem with different solutions depending on the definition of anisotropic space and the anisotropic basis used, and wavelets have been adapted in many different ways to "detect" and resolve anisotropy, such as, for instance, ridgelets and curvelets studied by Candes, Donoho and Demanet [36, 37, 38, 39, 40], bandelets [119, 127], aniset [94], shearlets developped in [71, 117, 118], and anisotropic wavelets adressed by many authors, such as Ben Slimane, Ben Braiek, Bownik, Farkas, Garrigos, Hochmuth, Haroske, Liu, Tabacco, Tamási, Triebel and Yang [24, 32, 57, 69, 75, 79, 123, 139, 170, 171]. However, these wavelets require knowing the anisotropy in advance, and if anisotropy parameters are

not provided by the context of the signal considered, characterising it cannot be done without "trying out" several anisotropic bases, which is hardly implementable in practice.

In [3], it has been shown that any anisotropic Besov space on $\mathbb{R}^2$ can "almost" be characterised with the help of a so-called hyperbolic wavelet transform. This enables the detection and characterisation of the anisotropy of a signal via a specific weight applied to the wavelet coefficients. This has led to an efficient algorithm for image classification and anisotropy detection applied to both synthetic and real textures (see [6, 155]). Such a characterisation was later generalised to higher dimension and for Triebel-Lizorkin and Sobolev spaces by Schäfer-Ullrich-Vedel [157].

Vybiral [172] showed that Besov spaces with dominating mixed smoothness can also be characterised with a hyperbolic wavelet transform, and Esser-Loosveldt-Vedel [55] used this hyperbolic wavelet basis to introduce the weighted tensorised Besov spaces, which, with the help of a parameter r decreasing from 1 to 0, correspond to mixed dominating smoothness Besov spaces when $r = 0$ and isotropic hyperbolic Besov spaces when $r = 1$.

This characterisation through a hyperbolic wavelet basis is tied to consideration in nonlinear approximation of functions in PDE [73, 103] and practical applications such as analysis of anisotropic textures [155]. In [73], Harbrecht and Von Rickenbach studied the approximation of functions in Bessel potential spaces $H^s([0,1]^d)$ by their partial sum of the N largest wavelet coefficients in an isotropic and hyperbolic wavelet decomposition. They showed that a hyperbolic wavelet decomposition allows for approximating functions in $H^s([0,1]^d)$ at a fixed approximation rate with lower regularity requirement on the function to approximate than isotropic wavelet decomposition; and they proved Jackson and Bernstein inequalities in a hybrid regularity setting, linking weighted tensorised Besov spaces to Bessel potential spaces.

Our aim in the present work is to introduce an anisotropic version of the weighted tensorised Besov spaces, as well as to explain how they can be considered from the viewpoint of anisotropic hyperbolic Besov spaces and mixed dominating smoothness Besov spaces. These spaces have an anisotropic structure similar to anisotropic hyperbolic Besov and Besov with dominating mixed smoothness, with a parameter that controls the reticulated effect present in mixed dominating smoothness Besov spaces and not in anisotropic Besov spaces. Moreover, they generalise the isotropic weighted tensorised Besov spaces of Esser-Loosveldt-Vedel. We also present how the regularity of a function in an anisotropic weighted tensorised Besov space changes with the consideration of different anisotropy directions. This chapter is a starting point that aims at providing a theoretical background to the method for estimating anisotropic parameters of textures as done in [155].

Throughout this chapter, we will use the following notation. Put $\mathbb{N}_{-1} := \mathbb{N} \cup \{-1\}$. For a $(x_1, x_2) \in \mathbb{R}^2$, we shall denote by $(x_1, x_2) > 0$ that each coordinate is positive and similarly for non-negative coordinates. For $0 < p \leq \infty$ and $\overline{x} \in \mathbb{R}^2$, we define $\|\overline{x}\|_p := (|x_1|^p + |x_2|^p)^{1/p}$, with the usual modification in the case $p = \infty$. Multivariate indices are typeset with a bar, such as $\overline{k}, \overline{j}$ or $\overline{\alpha}$. We also put $x_+ := \max\{x, 0\}$ for $x \in \mathbb{R}$ and $\overline{x}_+ := ((x_1)_+, (x_2)_+)$ in the multidimensional case. In the rest of this chapter, the multi-index $\overline{\alpha} = (\alpha_1, \alpha_2) > 0$ stands

for an anisotropy index and is such that $\alpha_1 + \alpha_2 = 2$. The notation $\overline{j}/\overline{\alpha}$ shall always stand for $(j_1/\alpha_1, j_2/\alpha_2)$ and thus $\overline{j}_+/\overline{\alpha}$ shall always stand for $((j_1)_+/\alpha_1, (j_2)_+/\alpha_2)$. For $r \in [0,1]$, consider the function

$$\theta_r(\overline{j}_+/\overline{\alpha}) := (1+r)\max(\overline{j}_+/\overline{\alpha}) + (1-r)\min(\overline{j}_+/\overline{\alpha}). \tag{6.1}$$

We recall the definition of a hyperbolic wavelet basis as tensorial products of two unidimensional wavelet bases, see [51].

Definition 6.1. *Let ψ denote a Lemarié-Meyer wavelet and φ the associated scaling function. The hyperbolic wavelet basis is defined as the system $\{\psi_{\overline{j},\overline{k}} : \overline{j} \in \mathbb{N}^2_{-1}, \overline{k} \in \mathbb{Z}^2\}$ where*

- *for $\overline{j} \geq 0$, $\psi_{\overline{j},\overline{k}}(x_1,x_2) = \psi(2^{j_1}x_1 - k_1)\ \psi(2^{j_2}x_2 - k_2)$,*
- *if for $i \in \{1,2\}$, $j_i = -1$, $\psi(2^{j_i}x_i - k_i)$ is replaced by $\varphi(x_i - k_i)$.*

For any $f \in \mathcal{S}'(\mathbb{R}^2)$, one then defines its hyperbolic wavelet coefficients by

$$\forall \overline{j} \in \mathbb{N}^2_{-1}, \overline{k} \in \mathbb{Z}^2, \quad c_{\overline{j},\overline{k}} = 2^{\|\overline{j}_+\|_1} \left\langle f, \psi_{\overline{j},\overline{k}} \right\rangle .$$

This wavelet basis allows the study of regularity and oscillation of a function with different behaviour across the two orthogonal directions, which cannot be done with classical wavelet methods having isotropic dilation.

Let us first consider the anisotropic hyperbolic Besov space defined through the hyperbolic wavelet basis. These spaces have equivalent definitions in terms of what is usually called a hyperbolic Littlewood-Paley analysis. This equivalence is proved in [3] for dimension 2 and in [157] for higher dimensions.

Definition 6.2. *Let $1 \leq p,q \leq \infty$, $s \in \mathbb{R}$ and anisotropy index $\overline{\alpha} \in \mathbb{R}^2$. Consider the hyperbolic wavelet basis of Definition 6.1.*

A function $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to the anisotropic hyperbolic Besov space $\widetilde{B}^{s,\overline{\alpha}}_{p,q}(\mathbb{R}^2)$ if $\|f\|_{\widetilde{B}^{s,\overline{\alpha}}_{p,q}} < +\infty$, where

$$\|f\|_{\widetilde{B}^{s,\overline{\alpha}}_{p,q}} := \left\| (\varepsilon^{s,\overline{\alpha},p}_{\overline{j}})_{\overline{j}\in\mathbb{N}^2_{-1}} \right\|_{\ell q}, \quad \text{with} \quad \varepsilon^{s,\overline{\alpha},p}_{\overline{j}} = \left\| \left(2^{s\|\overline{j}_+/\overline{\alpha}\|_\infty}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2} \right\|_{\ell p}. \tag{6.2}$$

These spaces can be seen as sequence spaces using hyperbolic wavelet coefficients [157] as

$$\widetilde{b}^{s,\overline{\alpha}}_{p,q} := \left\{ c^f = (c^f_{\overline{j},\overline{k}}) \in \mathbb{R}^{\mathbb{N}^2_{-1}\times\mathbb{Z}^2} : \|c^f\|_{\widetilde{b}^{s,\overline{\alpha}}_{p,q}} := \left\| (\varepsilon^{s,\overline{\alpha},p}_{\overline{j}})_{\overline{j}\in\mathbb{N}^2_{-1}} \right\|_{\ell q} < \infty \right\}. \tag{6.3}$$

Similarly to classical Besov spaces, the spaces $\widetilde{B}^{s,\overline{\alpha}}_{p,q}(\mathbb{R}^2)$ are spaces of functions with controlled oscillations over scales with regularity in one direction controlled by s/α_1 and regularity in the orthogonal direction controlled by s/α_2.

Besov with dominating mixed smoothness $S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2)$ can also be characterised with the hyperbolic wavelet transform [172] giving that a function belong to $S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2)$ if (6.2) is verified with the slight modification that $\|\overline{j}_+/\overline{\alpha}\|_\infty$ is replaced by $\|\overline{j}_+/\overline{\alpha}\|_1$, with the corresponding space $s^{s,\overline{\alpha}}_{p,q}b$ using (6.3). The functions in these spaces have a regularity behaviour similar to the anisotropic hyperbolic Besov above, with a structure similar to reticulated textures such as textiles.

To link these two spaces to each other and describe a family of anisotropic Gaussian processes, Esser-Loosveldt-Vedel [55] introduced the weighted tensorised Besov spaces. Such spaces have an anisotropic structure similar to anisotropic hyperbolic Besov and Besov with dominating mixed smoothness, with a parameter that controls the reticulated effect. Let us now give an anisotropic version of these last spaces.

Definition 6.3. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$ and an anisotropy index $\overline{\alpha} \in \mathbb{R}^2$. Consider the hyperbolic wavelet basis.*

A function $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to the anisotropic weighted tensorised Besov space $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ if $\|f\|_{TB^{s,\overline{\alpha},r}_{p,q}} < +\infty$, where

$$\|f\|_{TB^{s,\overline{\alpha},r}_{p,q}} := \left\| (\varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}})_{\overline{j}\in\mathbb{N}^2_{-1}} \right\|_{\ell^q}, \quad \text{with} \quad \varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}} = \left\| \left(2^{\theta_r(\overline{j}_+/\overline{\alpha})s}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2} \right\|_{\ell^p} \tag{6.4}$$

with the convention $\frac{\|\overline{j}_+\|_1}{\infty} = 0$.

Remark 6.4. Remark that when $s_1 \leq s_2$, $TB^{s_2,\overline{\alpha},r}_{p,q}(\mathbb{R}^2) \hookrightarrow TB^{s_1,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$.

Such spaces can be linked back to anisotropic hyperbolic Besov spaces and Besov spaces with dominating mixed smoothness as follows. Directly from the definitions, one can see that, for $r = 0$, the anisotropic weighted tensorised Besov space corresponds to the Besov spaces with dominating mixed smoothness

$$S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \left\| \left(\left\| \left(2^{\|\overline{j}_+/\overline{\alpha}\|_1 s}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2} \right\|_{\ell^p} \right)_{\overline{j}\in\mathbb{N}^2_{-1}} \right\|_{\ell^q} < \infty \right\},$$

and, for $r = 1$, it corresponds to anisotropic hyperbolic Besov spaces (with regularity 2s)

$$\widetilde{B}^{2s,\overline{\alpha}}_{p,q}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \left\| \left(\left\| \left(2^{\|\overline{j}_+/\overline{\alpha}\|_\infty s}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2} \right\|_{\ell^p} \right)_{\overline{j}\in\mathbb{N}^2_{-1}} \right\|_{\ell^q} < \infty \right\}.$$

Proposition 6.5. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$ and an anisotropy index $\overline{\alpha} \in \mathbb{R}^2$. One has $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2) \subset \widetilde{B}^{(1+r)s,\overline{\alpha}}_{p,q}(\mathbb{R}^2)$ and $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2) \subset S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2)$.*

Proof. Let us first remark that for every $\overline{j} \in \mathbb{N}^2_{-1}$, one has $\theta_r(\overline{j}_+/\overline{\alpha}) \geq (1+r)\|\overline{j}_+/\overline{\alpha}\|_\infty$. One deduces that $2^{\theta_r(\overline{j}_+/\overline{\alpha})s} \geq 2^{s(1+r)\|\overline{j}_+/\overline{\alpha}\|_\infty}$, and then for any function f, $\|f\|_{TB^{s,\overline{\alpha},r}_{p,q}} \geq \|f\|_{\widetilde{B}^{(1+r)s,\overline{\alpha}}_{p,q}}$. This prove that $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2) \subset \widetilde{B}^{(1+r)s,\overline{\alpha}}_{p,q}(\mathbb{R}^2)$. Similarly, for every $\overline{j} \in \mathbb{N}^2_{-1}$, one has $\theta_r(\overline{j}_+/\overline{\alpha}) \geq \|\overline{j}_+/\overline{\alpha}\|_1$, inducing that $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2) \subset S^{(1+r)s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2)$. □

Similarly to (6.3), an interesting consideration is a sequential characterisation of the spacees $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ as follows.

Definition 6.6. *For $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$ and an anisotropy index $\overline{\alpha} \in \mathbb{R}^2$.*

The anisotropic weighted tensorised sequential Besov space $tb^{s,\overline{\alpha},r}_{p,q}$ is the set of sequences $c^f = (c^f_{\overline{j},\overline{k}}) \in \mathbb{R}^{\mathbb{N}^2_{-1}\times\mathbb{Z}^2}$ such that $\|c^f\|_{tb^{s,\overline{\alpha},r}_{p,q}} < \infty$ where

$$\|c^f\|_{tb^{s,\overline{\alpha},r}_{p,q}} := \left\| (\varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}})_{\overline{j}\in\mathbb{N}^2_{-1}} \right\|_{\ell^q}, \quad \text{with} \quad \varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}} = \left\| \left(2^{\theta_r(\overline{j}_+/\overline{\alpha})s}\ 2^{-\|\overline{j}_+\|_1/p}\ c^f_{\overline{j},\overline{k}}\right)_{\overline{k}\in\mathbb{Z}^2} \right\|_{\ell^p} \tag{6.5}$$

with the convention $\frac{\|\overline{j}_+\|_1}{\infty} = 0$.

Using the proof of Proposition 6.5, one has $tb^{s,\overline{\alpha},r}_{p,q} \subset \widetilde{b}^{(1+r)s,\overline{\alpha}}_{p,q}$ and $tb^{s,\overline{\alpha},r}_{p,q} \subset s^{s,\overline{\alpha}}_{p,q}b$. In particular, for any sequence $(c_{\overline{j},\overline{k}}) \in tb^{s,\overline{\alpha},r}_{p,q}$, the function defined as $f = \sum_{\overline{j}\in\mathbb{N}^2_{-1}} \sum_{\overline{k}\in\mathbb{Z}^2} c_{\overline{j},\overline{k}}\psi_{\overline{j},\overline{k}}$ belongs to $S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2)$, see [157, 172], and thus to $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$. Thus $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ is isometric to $tb^{s,\overline{\alpha},r}_{p,q}$ through the hyperbolic wavelet transform.

An important property of $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ is that these spaces are Banach spaces.

Proposition 6.7. *Let $1 \leq p,q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$ and an anisotropy index $\overline{\alpha} \in \mathbb{R}^2$. The normed spaces $(tb^{s,\overline{\alpha},r}_{p,q}, \|\cdot\|_{tb^{s,\overline{\alpha},r}_{p,q}})$ and $(TB^{s,\overline{\alpha},r}_{p,q}, \|\cdot\|_{TB^{s,\overline{\alpha},r}_{p,q}})$ are Banach spaces.*

We investigate how the regularity of a function in an anisotropic weighted tensorised Besov space behaves in comparison to other anisotropy directions.

Definition 6.8. *Let $1 \leq p,q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, and an anisotropy index $\overline{\alpha} = (\alpha_1,\alpha_2) \in \mathbb{R}^2$. For $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, define the function*

$$\forall x \in (0,2), \quad \rho_{f,r}(x) := \sup\{s' \in \mathbb{R} : f \in TB^{s',(x,2-x),r}_{p,q}(\mathbb{R}^2)\}.$$

Considering a function $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, typically, this function shows its largest regularity when studied considering an anisotropy index $\overline{\alpha}$ and its regularity only decreases if we consider other anisotropy indices.

Theorem 6.9. *Let $1 \leq p,q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, and an anisotropy index $\overline{\alpha} = (\alpha_1,\alpha_2) \in \mathbb{R}^2$ such that $\|\overline{\alpha}\|_1 = 2$.*

For any $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, we have

$$\rho_{f,r}(x) \geq \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{6.6}$$

For any f in a Baire second category set in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, we have

$$\rho_{f,r}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{6.7}$$

This can be interpreted that if one computes the regularity of a function for different anisotropy indices, one can deduce its anisotropy parameter as the one where the function has the largest regularity. This method was used in [155] to evaluate the anisotropy parameter of texture simulated from anisotropic Gaussian processes and the corresponding regularity. Indeed, for anisotropic textures simulated from anisotropic Gaussian processes with particular regularity and anisotropy index, they estimated the regularity $s(\overline{\alpha})$ for multiple anisotropy indices $\overline{\alpha}$ and recovered the anisotropy and regularity used for the simulation. The estimated regularity for multiple anisotropy index forms a triangular-shaped function similar to Figure 6.1.

In particular, in the case of anisotropic hyperbolic Besov spaces, if for a function f, the maximum of $\rho_{f,1}$ is known, then one can completely determine $\rho_{f,1}$ as follows.

Theorem 6.10. *Let $r = 1$ and $\rho_{f,1}(x) = \sup\{s' \in \mathbb{R} : f \in \widetilde{B}^{2s',(x,2-x)}_{p,q}(\mathbb{R}^2)\}$. Consider a function f such that there exists an anisotropy index $\overline{\alpha}$ for which $\rho_{f,1}(\alpha_1) = s = \max_{y\in(0,2)} \rho_{f,1}(y)$*

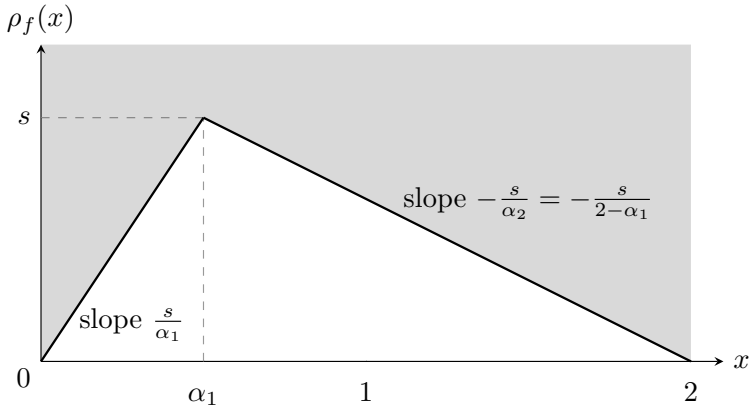


Figure 6.1: The regularity function $\rho_{f,r}$ depending on the anisotropy $(x, 2-x)$. The graph of the function $\rho_{f,r}$ is included in the grey part for all functions f and, for typical functions in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, $\rho_{f,r}$ is represented in black.

for $s \geq 0$. *Then*

$$\rho_{f,1}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}. \tag{6.8}$$

Both Theorems 6.9 and 6.10 imply that if a function f satisfies $\rho_{f,1}(\alpha_1) = s = \max_{y\in(0,2)} \rho_{f,1}(y)$ for an anisotropy index $\overline{\alpha}$ and a $s \geq 0$, then $\rho_{f,1}(x) = \min\left\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\right\}$. Thus, if one considers the setting of anisotropic hyperbolic Besov spaces, one simply has to obtain (numerically) two points that aren't aligned with either $(0,0)$ or $(2,0)$ in the plane $(x, \rho_{f,1}(x))$ as in Figure 6.1. This will give the maximum regularity by linear regression, which means that not all the anisotropies need to be tested, simplifying the method used in [155].

6.1.1 Extension

This is but a starting point, as multiple questions arise. First, anisotropic hyperbolic Besov spaces $\widetilde{B}^{s,\overline{\alpha}}_{p,q}(\mathbb{R}^2)$ and Besov with dominating mixed smoothness $S^{s,\overline{\alpha}}_{p,q}B(\mathbb{R}^2)$ have equivalent definitions in terms of hyperbolic Littlewood-Paley analysis [3, 157, 172]. In the isotropic case $\overline{\alpha} = (1,1)$, Esser-Loosveldt-Vedel [55] showed that the space $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ also has equivalent definitions with an hyperbolic Littlewood-Paley analysis. Thus, a natural question that arises is whether the anisotropic spaces $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ have an equivalent formulation in terms of hyperbolic Littlewood-Paley analysis in all generality.

Let us recall the definition of a hyperbolic Littlewood-Paley analysis. Let $\theta_0 \in \mathcal{S}(\mathbb{R})$ be a non-negative function supported on $[-2,2]$ with $\theta_0 = 1$ on $[-1,1]$. For any $j \in \mathbb{N}$, let us further define

$$\theta_j = \theta_0(2^{-j}\cdot) - \theta_0(2^{-(j-1)}\cdot)$$

such that $(\theta_j)_{j\in\mathbb{N}}$ forms a univariate resolution of unity, i.e., $\sum_{j\in\mathbb{N}} \theta_j = 1$. Observe that, for any $j \in \mathbb{N}$,

$$\operatorname{supp}(\theta_j) \subset \{\xi \in \mathbb{R} \ : \ 2^{j-1} \leq |\xi| \leq 2^{j+1}\} \quad \text{and} \quad \theta_j = \theta_1(2^{-(j-1)}\cdot).$$

The function θ_0 can be chosen with an arbitrary compact support [55, 157]. This allows us to choose θ_0 as the Fourier transform of a Meyer scaling function.

Definition 6.11. *1. For any $\overline{j} = (j_1, j_2) \in \mathbb{N}^2$ and any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ set*

$$\theta_{\overline{j}}(\xi) := \theta_{j_1}(\xi_1)\ \theta_{j_2}(\xi_2)\,.$$

The function $\theta_{\overline{j}}$ belongs to $\mathcal{S}(\mathbb{R}^2)$ for all $\overline{j} \in \mathbb{N}^2$ and is compactly supported on a dyadic rectangle. Further $\sum_{\overline{j}\in\mathbb{N}^2} \theta_{\overline{j}} = 1$ and $(\theta_{\overline{j}})_{\overline{j}\in\mathbb{N}^2}$ is called a hyperbolic resolution of unity.

2. For $f \in \mathcal{S}'(\mathbb{R}^2)$ and $\overline{j} \in \mathbb{N}^2$ set

$$\Delta_{\overline{j}} f := \mathcal{F}^{-1}(\theta_{\overline{j}} \mathcal{F} f)\,.$$

The sequence $(\Delta_{\overline{j}} f)_{\overline{j}\in\mathbb{N}^2}$ is called a hyperbolic Littlewood-Paley analysis of f.

We conjecture that the anisotropic weighted tensorised Besov space $TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$ can be characterised through a hyperbolic Littlewood-Paley analysis as follows.

Definition 6.12. *Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $0 \le r \le 1$ and an anisotropy index $\overline{\alpha} \in \mathbb{R}^2$ such that $\|\overline{\alpha}\|_1 = 2$. The anisotropic weighted tensorised Besov space $\widetilde{TB}_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$ is defined by*

$$\widetilde{TB}_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \Big(\sum_{\overline{j}\in\mathbb{N}^2} 2^{sq\theta_r(\overline{j}/\overline{\alpha})} \|\Delta_{\overline{j}} f\|_{L^p}^q \Big)^{1/q} < \infty \right\}.$$

The quantity

$$\|f\|_{\widetilde{TB}_{p,q}^{s,\overline{\alpha},r}} := \Big(\sum_{\overline{j}\in\mathbb{N}^2} 2^{sq\theta_r(\overline{j}/\overline{\alpha})} \|\Delta_{\overline{j}} f\|_{L^p}^q \Big)^{1/q}$$

is a norm on $\widetilde{TB}_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$ for $1 \le p, q \le \infty$.

Our conjecture can then be summarised as follows.

Conjecture 6.13. *Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $0 \le r \le 1$ and an anisotropy index $\overline{\alpha} \in \mathbb{R}^2$ such that $\|\overline{\alpha}\|_1 = 2$. Let $(c_{\overline{j},\overline{k}}^f)$ denote the sequence of the Meyer wavelet coefficients of a function $f \in \mathcal{S}'([0,1]^2)$. The following assertions are equivalent:*

1. *$f \in \widetilde{TB}_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$*
2. *$\big(c_{\overline{j},\overline{k}}^f\big) \in tb_{p,q}^{s,\overline{\alpha},r}$*

Moreover, there exist two constants $C_1, C_2 > 0$ such that

$$C_1 \big\|\big(c_{\overline{j},\overline{k}}^f\big)\big\|_{tb_{p,q}^{s,\overline{\alpha},r}} \le \|f\|_{\widetilde{TB}_{p,q}^{s,\overline{\alpha},r}} \le C_2 \big\|\big(c_{\overline{j},\overline{k}}^f\big)\big\|_{tb_{p,q}^{s,\overline{\alpha},r}}.$$

Another interrogation is to determine the analogue of Theorem 6.10 in the case of $r \in [0,1)$. At first glance, these cases have a less rigid setting and, if one knows the maximum of $\rho_{f,r}$ for a function f, the associated $\rho_{f,r}$ could not be a triangular-shaped function. This also asks the question of, for a function f fixed, whether the maximum of $\rho_{f,r}$ is reached at the same (x,s) for every $r \in [0,1]$ or if (x,s) depend on $r \in [0,1]$ and is intrinsically linked to the structure of the spaces in relation to r.

Another direction would be to prove that (6.7) of Theorem 6.9 is true for a prevalent set of functions in $TB_{p,q}^{s,\overline{\alpha},r}(\mathbb{R}^2)$.

Finally, one could be interested in extreme cases with anisotropy index $\overline{\alpha} = (0, 2)$ or $\overline{\alpha} = (2, 0)$ with, for example, a function $f(\overline{x}) = f_1(x_1)f_2(x_2)$ where $f_1 \in B^s_{p,q}(\mathbb{R})$ and $f_2 \in C^\infty(\mathbb{R})$ and obtain a function similar to $\rho_{f,r}$.

6.1.2 Organisation of the chapter

Section 6.2 contains the proof of Proposition 6.7. In Section 6.3, we prove the inequality (6.6) of Theorem 6.9. In Section 6.4, we introduce a saturating function in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ and study the regularity properties of this function with different anisotropies. Intuitively, such a function has the largest possible wavelet coefficients in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ and forms the cornerstone of the proof of the Baire typical result (6.7) that follows. Section 6.5 provides a proof of Theorem 6.10.

6.2 Completeness of $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$: Proof of Proposition 6.7

Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, and an anisotropy index $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that $\|\overline{\alpha}\|_1 = 2$.

We will only show that the normed space $(tb^{s,\overline{\alpha},r}_{p,q}, \|\cdot\|_{tb^{s,\overline{\alpha},r}_{p,q}})$ is a Banach space through its completeness. Indeed, the hyperbolic wavelet transform is an isometric isomorphism and induces the completeness of the normed space $(TB^{s,\overline{\alpha},r}_{p,q}, \|\cdot\|_{TB^{s,\overline{\alpha},r}_{p,q}})$.

Let $(c^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $tb^{s,\overline{\alpha},r}_{p,q}$ where, for every $n \in \mathbb{N}$, we write $c^{(n)} = \left(c^{(n)}_{\overline{j},\overline{k}}\right)_{(\overline{j},\overline{k})\in\mathbb{N}^2_{-1}\times\mathbb{Z}^2}$. We will show that this sequence converges in $tb^{s,\overline{\alpha},r}_{p,q}$.

First, we prove that, for every $(\overline{j}, \overline{k}) \in \mathbb{N}^2_{-1} \times \mathbb{Z}^2$, $c_{\overline{j},\overline{k}} := \lim_{n\in\mathbb{N}} c^{(n)}_{\overline{j},\overline{k}}$ exists in $\mathbb{R}$. Remark from (6.4), that, for every $(\overline{j}, \overline{k}) \in \mathbb{N}^2_{-1} \times \mathbb{Z}^2$, for every $n, m \in \mathbb{N}$, one has

$$2^{\theta_r(\overline{j}_+/\overline{\alpha})s}2^{-\|\overline{j}_+\|_1/p}\left|c^{(n)}_{\overline{j},\overline{k}} - c^{(m)}_{\overline{j},\overline{k}}\right| \leq \left\|c^{(n)} - c^{(m)}\right\|_{tb^{s,\overline{\alpha},r}_{p,q}}. \tag{6.9}$$

For a fixed $(\overline{j}, \overline{k}) \in \mathbb{N}^2_{-1} \times \mathbb{Z}^2$, $2^{\theta_r(\overline{j}_+/\overline{\alpha})s}2^{-\|\overline{j}_+\|_1/p}$ is a constant that does not depend of n and m. Thus (6.9) gives that, for a constant $C > 0$ and for every $n, m \in \mathbb{N}$, $\left|c^{(n)}_{\overline{j},\overline{k}} - c^{(m)}_{\overline{j},\overline{k}}\right| \leq C\left\|c^{(n)} - c^{(m)}\right\|_{tb^{s,\overline{\alpha},r}_{p,q}}$. This induces that, for every $(\overline{j}, \overline{k}) \in \mathbb{N}^2_{-1} \times \mathbb{Z}^2$, $\left(c^{(n)}_{\overline{j},\overline{k}}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and $c_{\overline{j},\overline{k}} := \lim_{n\in\mathbb{N}} c^{(n)}_{\overline{j},\overline{k}}$ exists.

Now let us show that the sequence $c = \left(c_{\overline{j},\overline{k}}\right)_{(\overline{j},\overline{k})\in\mathbb{N}^2_{-1}\times\mathbb{Z}^2}$ belongs to $tb^{s,\overline{\alpha},r}_{p,q}$ and the convergence occurs in $tb^{s,\overline{\alpha},r}_{p,q}$. The sequence $(c^{(n)})_{n\in\mathbb{N}}$ is bounded in $tb^{s,\overline{\alpha},r}_{p,q}$ so there exists $C > 0$ such that, for every $n \in \mathbb{N}$, $\left\|c^{(n)}\right\|_{tb^{s,\overline{\alpha},r}_{p,q}} \leq C$. Thus, for any fixed $N \in \mathbb{N}$, one has

$$\begin{aligned}
&\sum_{\overline{j}\in\{-1,\dots,N\}}\Bigg(\sum_{\overline{k}\in\{-N,\dots N\}} 2^{\theta_r(\overline{j}_+/\overline{\alpha})sp-\|\overline{j}_+\|_1}\left|c_{\overline{j},\overline{k}}\right|^p\Bigg)^{q/p}\\
&= \lim_{n\to\infty}\sum_{\overline{j}\in\{-1,\dots,N\}}\Bigg(\sum_{\overline{k}\in\{-N,\dots N\}} 2^{\theta_r(\overline{j}_+/\overline{\alpha})sp-\|\overline{j}_+\|_1}\left|c^{(n)}_{\overline{j},\overline{k}}\right|^p\Bigg)^{q/p} \leq C^q.
\end{aligned}$$

Since this holds for any N, we get that $\|c\|_{tb^{s,\overline{\alpha},r}_{p,q}} \leq C$ and $c \in tb^{s,\overline{\alpha},r}_{p,q}$.

Finally, for any given $\varepsilon > 0$, there exists a $n_\varepsilon \in \mathbb{N}$ such that, for every $n, m > n_\varepsilon$, $\left\|c^{(n)} - c^{(m)}\right\|_{tb^{s,\overline{\alpha},r}_{p,q}} < \varepsilon$. Applying the same method, one has, for any fixed $N \in \mathbb{N}$ and any $m > n_\varepsilon$,

$$\sum_{\overline{j}\in\{-1,\dots,N\}} \Big(\sum_{\overline{k}\in\{-N,\dots N\}} 2^{\theta_r(\overline{j}_+/\overline{\alpha})sp - \|\overline{j}_+\|_1} \left| c_{\overline{j},\overline{k}} - c^{(m)}_{\overline{j},\overline{k}} \right|^p \Big)^{q/p}$$
$$= \lim_{n\to\infty} \sum_{\overline{j}\in\{-1,\dots,N\}} \Big(\sum_{\overline{k}\in\{-N,\dots N\}} 2^{\theta_r(\overline{j}_+/\overline{\alpha})sp - \|\overline{j}_+\|_1} \left| c^{(n)}_{\overline{j},\overline{k}} - c^{(m)}_{\overline{j},\overline{k}} \right|^p \Big)^{q/p} \leq \varepsilon^q.$$

Since this holds for any N, we get that, for any $m > n_\varepsilon$, $\|c - c^{(m)}\|_{tb^{s,\overline{\alpha},r}_{p,q}} \leq \varepsilon$ and $c^{(m)}$ converges to c in $tb^{s,\overline{\alpha},r}_{p,q}$.

This proves that any Cauchy sequence $(c^{(n)})_{n\in\mathbb{N}}$ converges in $tb^{s,\overline{\alpha},r}_{p,q}$ and that the normed space $(tb^{s,\overline{\alpha},r}_{p,q}, \|\cdot\|_{tb^{s,\overline{\alpha},r}_{p,q}})$ is complete and thus Banach.

6.3 Lower-bound (6.6) in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$

Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$ and an anisotropy index $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. For any function $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, we will prove that, for other anisotropy $\overline{\beta} = (\beta_1, \beta_2) > 0$ with $\|\overline{\beta}\|_1 = 2$, f belongs to $TB^{s',\overline{\beta},r}_{p,q}(\mathbb{R}^2)$ for $s' = \min\{\frac{s}{\alpha_1}\beta_1, \frac{s}{\alpha_2}\beta_2\}$ giving (6.6) through the inclusion of Remark 6.4.

Let $f \in TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$. Consider an anisotropy index $\overline{\beta} = (\beta_1, \beta_2) > 0$ with $\|\overline{\beta}\|_1 = 2$. For $s' \in \mathbb{R}$, one gets that $f \in TB^{s',\overline{\beta},r}_{p,q}(\mathbb{R}^2)$ if $\|f\|_{TB^{s',\overline{\beta},r}_{p,q}(\mathbb{R}^2)} < \infty$. Notice then that for $s' \in \mathbb{R}$ such that $s'\theta_r(\overline{j}_+/\overline{\beta}) \leq s\theta_r(\overline{j}_+/\overline{\alpha})$ for every $\overline{j} \in \mathbb{N}^2_{-1}$, one has $\|f\|_{TB^{s',\overline{\beta},r}_{p,q}(\mathbb{R}^2)} \leq \|f\|_{TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)} < \infty$

We now show that, for $s' = \min\{\frac{s}{\alpha_1}\beta_1, \frac{s}{\alpha_2}\beta_2\}$, one has, for every $\overline{j} \in \mathbb{N}^2_{-1}$, $s'\theta_r(\overline{j}_+/\overline{\beta}) \leq s\theta_r(\overline{j}_+/\overline{\alpha})$. Recall from (6.1) that $\theta_r(\overline{j}_+/\overline{\alpha}) = (1+r)\max(\overline{j}_+/\overline{\alpha}) + (1-r)\min(\overline{j}_+/\overline{\alpha})$.

One has that $s'/\beta_1 \leq s/\alpha_1$ and $s'/\beta_2 \leq s/\alpha_2$ giving that $(j_1)_+ s'/\beta_1 \leq (j_1)_+ s/\alpha_1$ and $(j_2)_+ s'/\beta_2 \leq (j_2)_+ s/\alpha_2$. One deduces that $s'\max(\overline{j}_+/\overline{\beta}) \leq s\max(\overline{j}_+/\overline{\alpha})$ and $s'\min(\overline{j}_+/\overline{\beta}) \leq s\min(\overline{j}_+/\overline{\alpha})$. Hence, recalling (6.1), $s'\theta_r(\overline{j}_+/\overline{\beta}) \leq s\theta_r(\overline{j}_+/\overline{\alpha})$.

One concludes that, for $s' = \min\{\frac{s}{\alpha_1}\beta_1, \frac{s}{\alpha_2}\beta_2\}$, $\|f\|_{TB^{s',\overline{\beta},r}_{p,q}(\mathbb{R}^2)} \leq \|f\|_{TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)} < \infty$ meaning that $\rho_{f,r}(\beta_1) \geq \min\{\frac{s}{\alpha_1}\beta_1, \frac{s}{\alpha_2}(2-\beta_1)\}$. This concludes the proof of (6.6).

6.4 Typical equality (6.7) in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$

Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq r \leq 1$, and an anisotropy index $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$.

The idea of the proof begins with the construction of a function for which (6.7) holds. This function is called a *saturating function* because it "saturates" the norm (6.4) of the spaces $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$. Then we consider a sequence of dense sets in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$, which we slightly perturb using the saturating function. We will show that the functions in these perturbed dense sets also verify (6.7) and their intersection thus forms a dense G_δ set.

As $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ is isomorphic to $tb^{s,\overline{\alpha},r}_{p,q}$, every function f in $TB^{s,\overline{\alpha},r}_{p,q}(\mathbb{R}^2)$ can be identified

to a sequence c^f in $tb^{s,\overline{\alpha},r}_{p,q}$. Thus we will prove (6.7) by considering $tb^{s,\overline{\alpha},r}_{p,q}$.

For $\overline{j}=(j_1,j_2)\in\mathbb{N}^2_{-1}$, denote $\Lambda_{\overline{j}}=\Lambda_{j_1,j_2}:=\{-2^{(j_1)_+},\dots,2^{(j_1)_+}\}\times\{-2^{(j_2)_+},\dots,2^{(j_2)_+}\}$ and $\Lambda=\bigcup_{\overline{j}\in\mathbb{N}^2_{-1}}\Lambda_{\overline{j}}$. For $N\in\mathbb{N}^*$, we will write $N\Lambda_{\overline{j}}=\{-N2^{(j_1)_+},\dots,N2^{(j_1)_+}\}\times\{-N2^{(j_2)_+},\dots,N2^{(j_2)_+}\}$ with the corresponding $N\Lambda=\bigcup_{\overline{j}\in\mathbb{N}^2_{-1}}N\Lambda_{\overline{j}}$.

Proving this result on $tb^{s,\overline{\alpha},r}_{p,q}$ comes to showing it for $\bigcap_{N\in\mathbb{N}^*}tb^{s,\overline{\alpha},r}_{p,q}(N)$ with

$$tb^{s,\overline{\alpha},r}_{p,q}(\mathbf{N}):=\Big\{c^f=(c^f_{\overline{j},\overline{k}})\in\mathbb{R}^{\mathbb{N}^2_{-1}\times\mathbf{N}\Lambda}:\\ \Big\|\Big(\Big\|\big(2^{\theta_r(\overline{j}_+/\overline{\alpha})s-\|\overline{j}_+\|_1/p}\,c^f_{\overline{j},\overline{k}}\big)_{\overline{k}\in\mathbf{N}\Lambda_{\overline{j}}}\Big\|_{\ell^p}\Big)_{\overline{j}\in\mathbb{N}^2_{-1}}\Big\|_{\ell^q}<\infty\Big\}\quad(6.10)$$

and we will obtain (6.7) for a countable intersection of dense G_δ sets, hence on a G_δ set. The set $tb^{s,\overline{\alpha},r}_{p,q}(N)$ can be understood as $TB^{s,\overline{\alpha},r}_{p,q}([-N,N]^2)$. Thus, from now on, we work on $tb^{s,\overline{\alpha},r}_{p,q}(1)$.

We denote the norm in (6.10) as

$$\|c^f\|_{tb^{s,\overline{\alpha},r}_{p,q}(N)}:=\big\|(\varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}})_{\overline{j}\in\mathbb{N}^2_{-1}}\big\|_{\ell^q},\ \text{with}\ \varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}}=\Big\|\big(2^{\theta_r(\overline{j}_+/\overline{\alpha})s}\,2^{-\|\overline{j}_+\|_1/p}\,c^f_{\overline{j},\overline{k}}\big)_{\overline{k}\in N\Lambda_{\overline{j}}}\Big\|_{\ell^p}.\quad(6.11)$$

6.4.1 Saturating function

Definition 6.14. *Let us define, for every $\overline{j}=(j_1,j_2)\in\mathbb{N}^2_{-1}$ and $\overline{k}=(k_1,k_2)\in\Lambda_{\overline{j}}$,*

$$g^r_{\overline{j},\overline{k}}=\frac{1}{(((j_1)_+1)((j_2)_+1))^{\frac{2}{q}+1}}2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}$$

with the convention $\frac{2}{\infty}=0$. The saturating function is the sequence $\mathscr{G}_r:=\big(g^r_{\overline{j},\overline{k}}\big)_{(\overline{j},\overline{k})\in\mathbb{N}^2_{-1}\times\Lambda}$.

The idea is that $\mathscr{G}_r$ has the largest wavelet coefficients to maximise $\varepsilon^{s,\overline{\alpha},r,p}_{\overline{j}}$ in (6.11).

Proposition 6.15. *The saturating function $\mathscr{G}_r$ belongs to $tb^{s,\overline{\alpha},r}_{p,q}(1)$.*

Proof. By (6.11) for $N=1$, one has

$$\begin{aligned}\|\mathscr{G}_r\|_{tb^{s,\overline{\alpha},r}_{p,q}(1)}&=\Big(\sum_{\overline{j}\in\mathbb{N}^2_{-1}}\Big(2^{-\|\overline{j}_+\|_1}\sum_{\overline{k}\in\Lambda_{\overline{j}}}\Big|2^{s\,\theta_r(\overline{j}_+/\overline{\alpha})}g^r_{\overline{j},\overline{k}}\Big|^p\Big)^{q/p}\Big)^{1/q}\\&\le\Big(\sum_{\overline{j}\in\mathbb{N}^2_{-1}}\frac{\big(2^{-\|\overline{j}_+\|_1}(3\times2^{\|\overline{j}_+\|_1})\big)^{q/p}}{(((j_1)_+1)((j_2)_+1))^{2+q}}\Big)^{1/q}\\&\le 3^{1/p}\Big(1+\sum_{j_1\in\mathbb{N}}\frac{1}{(j_1+1)^2}\Big)^{1/q}\Big(1+\sum_{j_2\in\mathbb{N}}\frac{1}{(j_2+1)^2}\Big)^{1/q}\le C.\end{aligned}$$

This proves that $\mathscr{G}_r$ belongs to $tb^{s,\overline{\alpha},r}_{p,q}(1)$. □

Applying (6.6) of Theorem 6.9 to $\mathscr{G}_r$ gives that, for every $x\in(0,2)$, $\rho_{\mathscr{G}_r}(x)\ge\min\{\frac{s}{\alpha_1}x,\frac{s}{\alpha_2}(2-x)\}$. We now show that $\mathscr{G}_r$ verifies (6.7).

Proposition 6.16. *For every $x\in(0,2)$, $\rho_{\mathscr{G}_r}(x)=\min\{\frac{s}{\alpha_1}x,\frac{s}{\alpha_2}(2-x)\}$.*

Proof. First, remark that, for $s' \in \mathbb{R}$ and an anisotropy index $\overline{\beta}$, $tb^{s',\overline{\beta},r}_{p,q}(1) \subset t^{s',\overline{\beta},r}_{\infty,\infty} b(1)$. Let $x \in (0,2)$ and consider $s' = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$. Let us prove that for every $\varepsilon > 0$, $\|\mathscr{G}_r\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} = +\infty$. Recall that

$$\|\mathscr{G}_r\|_{tb^{s',(x,2-x),r}_{\infty,\infty}(1)} = \sup_{(j_1,j_2)\in\mathbb{N}^2_{-1}} \frac{1}{(((j_1)_+ + 1)((j_2)_+ + 1))^{\frac{2}{q}+1}} 2^{s'\,\theta_r(\overline{j}_+/(x,2-x)) - s\,\theta_r(\overline{j}_+/\overline{\alpha})}.$$

Let $\varepsilon > 0$. Consider first $s \geq 0$ and thus $s' \geq 0$. We begin with the case $x \leq \alpha_1$, for which $s' = \frac{s}{\alpha_1}x$. Considering $j_1 \in \mathbb{N}$ and $j_2 = 0$, one has $(s'+\varepsilon)\,\theta_r(\overline{j}/(x,2-x)) - s\,\theta_r(\overline{j}/\overline{\alpha}) = (s'+\varepsilon)(1+r)\frac{j_1}{x} - s(1+r)\frac{j_1}{\alpha_1} = (1+r)\varepsilon\frac{j_1}{x}$ and deduces that

$$\|\mathscr{G}_r\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} \geq \sup_{j_1\in\mathbb{N}} \frac{1}{(j_1+1)^{\frac{2}{q}+1}} 2^{(1+r)\varepsilon\frac{j_1}{x}} \geq \sup_{j_1\in\mathbb{N}} \frac{1}{(j_1+1)^3} 2^{(1+r)\varepsilon\frac{j_1}{x}} = +\infty. \tag{6.12}$$

Now considering the case $x \geq \alpha_1$, one gets that $s' = \frac{s}{\alpha_2}(2-x)$. Considering $j_1 = 0$ and $j_2 \in \mathbb{N}$, one has $(s'+\varepsilon)\,\theta_r(\overline{j}/(x,2-x)) - s\,\theta_r(\overline{j}/\overline{\alpha}) = (s'+\varepsilon)(1+r)\frac{j_2}{2-x} - s(1+r)\frac{j_2}{\alpha_2} = (1+r)\varepsilon\frac{j_2}{2-x}$ and gets

$$\|\mathscr{G}_r\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} \geq \sup_{j_2\in\mathbb{N}} \frac{1}{(j_2+1)^{\frac{2}{q}+1}} 2^{(1+r)\varepsilon\frac{j_2}{2-x}} \geq \sup_{j_2\in\mathbb{N}} \frac{1}{(j_2+1)^3} 2^{(1+r)\varepsilon\frac{j_2}{2-x}} = +\infty. \tag{6.13}$$

Consider now $s < 0$, and thus $s' < 0$. The case $x \leq \alpha_1$ corresponds to $s' = \frac{s}{\alpha_2}(2-x)$ and one gets (6.13) with the same consideration as above. Conversely, the case $x \geq \alpha_1$ corresponds to $s' = \frac{s}{\alpha_1}x$ and one gets (6.12).

One has that for every $x \in (0,2)$ and $\varepsilon > 0$, $\|\mathscr{G}_r\|_{tb^{s'+\varepsilon,(x,2-x),r}_{p,q}(1)} \geq \|\mathscr{G}_r\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} = +\infty$. Hence, for every $x \in (0,2)$ and every $\varepsilon > 0$, $\rho_{\mathscr{G}_r}(x) < \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\} + \varepsilon$. One concludes that for every $x \in (0,2)$, $\rho_{\mathscr{G}_r}(x) = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$. □

6.4.2 Baire typical result for $p, q < \infty$

We suppose first that $p \neq \infty$ and $q \neq \infty$. In that case, the space $tb^{s,\overline{\alpha},r}_{p,q}(1)$ is separable and we can pick a dense sequence $(f_n)_{n\in\mathbb{N}}$ in $tb^{s,\overline{\alpha},r}_{p,q}(1)$. We consider the redundant sequence

$$(\widetilde{f}_n)_{n\in\mathbb{N}} = (f_1, f_1, f_2, f_1, f_2, f_3, ..., f_1, ..., f_k, f_1, ..., f_{k+1}, ...).$$

We denote by g_n the following modification of $\widetilde{f}_n$:

- If $j_1 < n$ and $j_2 < n$, the wavelet coefficients of g_n are the same as those of $\widetilde{f}_n$.
- If $j_1 \geq n$ or $j_2 \geq n$, the wavelet coefficients $c_{\overline{j},\overline{k}}$ of g_n are the same as those of the saturating function $\mathscr{G}_r$ constructed in the previous subsection.

It clearly follows from our choice of the wavelet coefficients of the sequence (g_n) that, for any $k \in \mathbb{N}$, there is a subsequence $(g_{\phi(n)})_{n\in\mathbb{N}}$ such that $\|f_k - g_{\phi(n)}\|_{tb^{s,\overline{\alpha},r}_{p,q}(1)}$ tends to 0 when n tends to ∞ as the tail of a convergent series. Thus the sequence (g_n) is also dense in $tb^{s,\overline{\alpha},r}_{p,q}(1)$. The G_δ dense set that we will consider is:

$$\mathcal{A} = \bigcap_{m\in\mathbb{N}} \bigcup_{n>m} B(g_n, r_n)$$

with $r_n = \frac{1}{2(n+1)^{\frac{2}{q}+1}} 2^{-\frac{n}{p}}$. $B(g_n, r_n)$ denotes the open ball of center g_n and radius r_n using the norm (6.11). $\mathcal{A}$ is clearly a countable intersection of dense open sets.

First, observe that at each step, if a function belongs to the ball $B(g_n, r_n)$, then the following result holds for its wavelet coefficients.

Proposition 6.17. *For $n \in \mathbb{N}$, let $f \in B(g_n, r_n)$ and consider $\overline{j} = (j_1, j_2) = (n, 0)$ or $\overline{j} = (0, n)$, and $\overline{k} = (k_1, k_2) \in \Lambda_{\overline{j}}$. One has*

$$g^r_{\overline{j},\overline{k}}/2 \leq |c^f_{\overline{j},\overline{k}}| \leq 2\, g^r_{\overline{j},\overline{k}} \tag{6.14}$$

Proof. Consider $\overline{j} = (j_1, j_2) = (n, 0)$ or $\overline{j} = (0, n)$. One has

$$\left||c^f_{\overline{j},\overline{k}}| - g^r_{\overline{j},\overline{k}}\right| \leq \left|c^f_{\overline{j},\overline{k}} - g^r_{\overline{j},\overline{k}}\right| \leq r_n\, 2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha}) + \|\overline{j}_+\|_1/p} \leq \frac{2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}}{2(n+1)^{\frac{2}{q}+1}}. \tag{6.15}$$

Hence, $\left|c^f_{\overline{j},\overline{k}}\right|$ can be bounded above by

$$\left|c^f_{\overline{j},\overline{k}}\right| \leq g^r_{\overline{j},\overline{k}} + \frac{2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}}{2(n+1)^{\frac{2}{q}+1}} \leq \left(\frac{1}{(n+1)^{\frac{2}{q}+1}} + \frac{1}{2(n+1)^{\frac{2}{q}+1}}\right) 2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}.$$

Noticing that $\frac{1}{(n+1)^{\frac{2}{q}+1}} + \frac{1}{2\,(n+1)^{\frac{2}{q}+1}} \leq 2\,\frac{1}{(n+1)^{\frac{2}{q}+1}}$, one deduces the upper bound $\left|c^f_{\overline{j},\overline{k}}\right| \leq 2\, g^r_{\overline{j},\overline{k}}$.

By (6.15), the lower bound is given by

$$\begin{aligned} |c^f_{\overline{j},\overline{k}}| &\geq g^r_{\overline{j},\overline{k}} - \frac{2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}}{2(n+1)^{\frac{2}{q}+1}} \geq \left(\frac{1}{(n+1)^{\frac{2}{q}+1}} - \frac{1}{2(n+1)^{\frac{2}{q}+1}}\right) 2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})} \\ &\geq \frac{2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}}{2\,(n+1)^{\frac{2}{q}+1}} = \frac{g^r_{\overline{j},\overline{k}}}{2}. \end{aligned}$$

□

Now we can conclude that, for a function f in $\mathcal{A}$, f belongs to a subsequence of $B(g_{n_l}, r_{n_l})$ for which one has the bound (6.14) on its wavelet coefficient. This allows us to compute the norm of f in $tb^{s',\overline{\beta},r}_{p,q}(1)$ and deduce the convergence or divergence of the norm in relation to s' for different anisotropy indices $\overline{\beta}$.

Proposition 6.18. *Let $f \in \mathcal{A}$. For every $x \in (0,2)$, $\rho_{f,r}(x) = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$.*

Proof. Let $f \in \mathcal{A}$. There exists a sequence (n_l) such that, for every $l \in \mathbb{N}$, $f \in B(g_{n_l}, r_{n_l})$. Let $x \in (0,2)$ and consider $s' = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$. Let us prove that for every $\varepsilon > 0$, $\|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} = +\infty$. Using that $\|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} \leq \|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{p,q}(1)}$, one can conclude, using the same method as in the proof of Proposition 6.16, that $\rho_{f,r}(x) < s' + \varepsilon$.

Let $\varepsilon > 0$. Consider first $s \geq 0$ and thus $s' \geq 0$. We begin with the case $x \leq \alpha_1$, for which $s' = \frac{s}{\alpha_1}x$. For $\overline{j} = (j_1, j_2)$ of the form $(n_l, 0)$, one has $(s'+\varepsilon)\,\theta_r(\overline{j}/(x, 2-x)) - s\,\theta_r(\overline{j}/\overline{\alpha}) = (s'+\varepsilon)(1+r)\frac{j_1}{x} - s(1+r)\frac{j_1}{\alpha_1} = (1+r)\varepsilon\frac{j_1}{x}$. One deduces that

$$\|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} \geq \sup_{n_l} \sup_{\overline{k} \in \Lambda_{n_l,0}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{x}} \left|c^f_{(n_l,0),\overline{k}}\right|$$

$$\geq \sup_{n_l} \sup_{\overline{k}\in\Lambda_{n_l,0}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{x}} \left|\frac{1}{2}\, g^r_{(n_l,0),\overline{k}}\right| \geq \sup_{n_l} \frac{2^{\varepsilon(1+r)\frac{n_l}{x}}}{2\ (n_l+1)^{\frac{2}{q}+1}} = +\infty. \quad (6.16)$$

Now considering the case $x \geq \alpha_1$,one gets that $s' = \frac{s}{\alpha_2}(2-x)$. For $\overline{j} = (j_1, j_2)$ of the form $(0, n_l)$, one has $(s'+\varepsilon)\,\theta_r(\overline{j}/(x, 2-x)) - s\,\theta_r(\overline{j}/\overline{\alpha}) = (s'+\varepsilon)(1+r)\frac{j_2}{2-x} - s(1+r)\frac{j_2}{\alpha_2} = (1+r)\varepsilon\frac{j_2}{2-x}$. One gets

$$\|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} \geq \sup_{n_l} \sup_{\overline{k}\in\Lambda_{0,n_l}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{2-x}} \left|c^f_{(0,n_l),\overline{k}}\right|$$

$$\geq \sup_{n_l} \sup_{\overline{k}\in\Lambda_{0,n_l}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{2-x}} \left|\frac{1}{2}\, g^r_{(0,n_l),\overline{k}}\right| \geq \sup_{n_l} \frac{2^{\varepsilon(1+r)\frac{n_l}{2-x}}}{2\ (n_l+1)^{\frac{2}{q}+1}} = +\infty. \quad (6.17)$$

Consider now $s < 0$, and thus $s' < 0$. The case $x \leq \alpha_1$ corresponds to $s' = \frac{s}{\alpha_2}(2-x)$ and one gets (6.17) with the same consideration as above. Conversely, the case $x \geq \alpha_1$ corresponds to $s' = \frac{s}{\alpha_1}x$ and one gets (6.16).

One has that for every $x \in (0,2)$ and $\varepsilon > 0$, $\|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{p,q}(1)} \geq \|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} = +\infty$. Hence, for every $x \in (0,2)$ and every $\varepsilon > 0$, $\rho_{f,r}(x) < \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\} + \varepsilon$. One concludes that for every $x \in (0,2)$, $\rho_{f,r}(x) = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$. □

This gives us that (6.7) is verified in a set of second category in $tb^{s,\overline{\alpha},r}_{p,q}(1)$. Similarly, one has that (6.7) is verified in a G_δ-set in $tb^{s,\overline{\alpha},r}_{p,q}(N)$ for every $N > 1$ and thus in a a G_δ-set in $tb^{s,\overline{\alpha},r}_{p,q}$ as a countable intersection of a G_δ-set. This proves the result in the case $p, q < \infty$.

6.4.3 Baire genericity for $p = \infty$ or $q = \infty$

We suppose that $p = \infty$ or $q = \infty$. In that case, the space $tb^{s,\overline{\alpha},r}_{p,q}(1)$ is not separable anymore. For every $m > 0$, define

$$E_m = \left\{ f \in B^{s_0,\alpha_0,r}_{p,q}([0,1]^2) : \forall(\overline{j},\overline{k}) \in \mathbb{N}^2_{-1} \times \Lambda,\ \frac{|c^f_{\overline{j},\overline{k}}|}{g^r_{\overline{j},\overline{k}}} \in \frac{1}{m}\mathbb{N}^* \right\}$$

We set, with $r_m = \frac{1}{2(m+1)^{\frac{2}{q}+2}} 2^{-\frac{m}{p}}$,

$$A_m = E_m + B(0, r_m) \quad \text{and} \quad \mathcal{A} = \bigcap_{M\in\mathbb{N}} \bigcup_{m\geq M} A_m.$$

First, we show that, as in the separable case, $\mathcal{A}$ is a countable intersection of dense sets.

Proposition 6.19. *For every $M \in \mathbb{N}$, $\cup_{m\geq M} E_m$ is dense in $tb^{s,\overline{\alpha},r}_{p,q}(1)$, and $\mathcal{A}$ is a G_δ set.*

Proof. Let $f \in tb^{s,\overline{\alpha},r}_{p,q}(1)$. For every $(\overline{j},\overline{k}) \in \mathbb{N}^2_{-1} \times \Lambda$, let us denote

$$N^f_{\overline{j},\overline{k}} := \begin{cases} \left\lfloor m\ \frac{c^f_{\overline{j},\overline{k}}}{g^r_{\overline{j},\overline{k}}} \right\rfloor & \text{if } c^f_{\overline{j},\overline{k}} < 0, \\ \left\lfloor m\ \frac{c^f_{\overline{j},\overline{k}}}{g^r_{\overline{j},\overline{k}}} \right\rfloor + 1 & \text{if } c^f_{\overline{j},\overline{k}} \geq 0. \end{cases}$$

Define the function $h \in tb^{s,\overline{\alpha},r}_{p,q}(1)$ such that for every $(\overline{j},\overline{k}) \in \mathbb{N}^2_{-1} \times \Lambda$, $c^h_{\overline{j},\overline{k}} = \frac{N^f_{\overline{j},\overline{k}}}{m} g^r_{\overline{j},\overline{k}}$. One can see that $h \in E_m$ as $N^f_{\overline{j},\overline{k}} > 0$. One has, for every $(\overline{j},\overline{k}) \in \mathbb{N}^2_{-1}\times\Lambda$, $\left|c^f_{\overline{j},\overline{k}} - c^h_{\overline{j},\overline{k}}\right| \leq \frac{1}{m}\, g^r_{\overline{j},\overline{k}}$.

Hence

$$\|f-h\|_{tb^{s,\overline{\alpha},r}_{p,q}(1)} \leq \frac{1}{m}\ \|g^r\|_{tb^{s,\overline{\alpha},r}_{p,q}(1)} \leq \frac{C}{m}$$

This implies that for every $m \in \mathbb{N}$, there exists $f_m \in E_m$ such that $\|f-f_m\|_{tb^{s,\overline{\alpha},r}_{p,q}(1)} \leq \frac{1}{m}$. One concludes that for every $f \in tb^{s,\overline{\alpha},r}_{p,q}(1)$, there exists a sequence $(f_m) \in \cup_{m\geq M} E_m$ converging to f in $tb^{s,\overline{\alpha},r}_{p,q}(1)$. □

Similarly to the separable case, the wavelet coefficients of a function in A_m are comparable to those of the saturating function.

Proposition 6.20. *For $m \in \mathbb{N}$, let $f \in A_m$ and consider $\overline{j} = (j_1, j_2) = (m, 0)$ or $\overline{j} = (0, m)$, and $\overline{k} \in \Lambda_{\overline{j}}$. One has*

$$\frac{g^r_{\overline{j},\overline{k}}}{2(m+1)} \leq |c^f_{\overline{j},\overline{k}}| \tag{6.18}$$

Proof. Consider $\overline{j} = (j_1, j_2) = (m, 0)$ or $\overline{j} = (0, m)$. For every $h \in E_m$, one has, for every $(\overline{j}, \overline{k}) \in \mathbb{N}^2_{-1} \times \Lambda$, $|c^h_{\overline{j},\overline{k}}| \geq \frac{1}{m}\ g^r_{\overline{j},\overline{k}}$.

Let $f \in A_m = E_m + B(0, r_m)$, there exists $h \in E_m$ such that, for every $(\overline{j}, \overline{k}) \in \mathbb{N}^2_{-1} \times \Lambda$,

$$\big||c^f_{\overline{j},\overline{k}}| - |c^h_{\overline{j},\overline{k}}|\big| \leq |c^f_{\overline{j},\overline{k}} - c^h_{\overline{j},\overline{k}}| \leq r_m\ 2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})+\|\overline{j}_+\|_1/p} \leq \frac{2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}}{2(m+1)^{\frac{2}{q}+2}}.$$

One deduces that

$$\begin{aligned}|c^f_{\overline{j},\overline{k}}| &\geq \frac{1}{m}\ g^r_{\overline{j},\overline{k}} - \frac{2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}}{2(m+1)^{\frac{2}{q}+2}} \geq \left(\frac{1}{m+1}\ \frac{1}{(m+1)^{\frac{2}{q}+1}} - \frac{1}{2(m+1)^{\frac{2}{q}+2}}\right) 2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})} \\ &\geq \frac{2^{-s\,\theta_r(\overline{j}_+/\overline{\alpha})}}{2(m+1)^{\frac{2}{q}+2}} = \frac{g^r_{\overline{j},\overline{k}}}{2(m+1)}.\end{aligned}$$

□

We have the counterpart of Proposition 6.18 in the non-separable case with a slight modification of the proof.

Proposition 6.21. *Let $f \in \mathcal{A}$. For every $x \in (0,2)$, $\rho_{f,r}(x) = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$.*

Proof. This result is proved in the same way as Proposition 6.18 with the slight modification that (6.14) is replaced by (6.18) in (6.16) and (6.17). This modification gives, when $s \geq 0$:

1. For $x \leq \alpha_1$,

$$\begin{aligned}\|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} &\geq \sup_{n_l} \sup_{\overline{k}\in\Lambda_{n_l,0}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{x}}\ \big|c^f_{(n_l,0),\overline{k}}\big| \\ &\geq \sup_{n_l} \sup_{\overline{k}\in\Lambda_{n_l,0}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{x}}\ \Big|\frac{1}{2(n_l+1)}\ g^r_{(n_l,0),\overline{k}}\Big| \\ &\geq \sup_{n_l} \frac{2^{\varepsilon(1+r)\frac{n_l}{x}}}{2\ (n_l+1)^{\frac{2}{q}+2}} = +\infty.\end{aligned} \tag{6.19}$$

2. For $x \geq \alpha_1$,

$$\begin{aligned}\|f\|_{tb^{s'+\varepsilon,(x,2-x),r}_{\infty,\infty}(1)} &\geq \sup_{n_l} \sup_{\overline{k}\in\Lambda_{0,n_l}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{2-x}} \left|c^f_{(0,n_l),\overline{k}}\right| \\ &\geq \sup_{n_l} \sup_{\overline{k}\in\Lambda_{0,n_l}} 2^{(s+\varepsilon)(1+r)\frac{n_l}{2-x}} \left|\frac{1}{2(n_l+1)}\, g^r_{(0,n_l),\overline{k}}\right| \\ &\geq \sup_{n_l} \frac{2^{\varepsilon(1+r)\frac{n_l}{2-x}}}{2\,(n_l+1)^{\frac{2}{q}+2}} = +\infty. \end{aligned} \tag{6.20}$$

When $s < 0$, one has the same correspondence as in the proof of Proposition 6.18: for $x \leq \alpha_1$, one gets (6.20) and, for $x \geq \alpha_1$, one gets (6.19). One concludes in the same way that, for every $x \in (0,2)$, $\rho_{f,r}(x) = \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$. □

This gives us that (6.7) is verified in a set of second category in $tb^{s,\overline{\alpha},r}_{p,q}(1)$, and similarly, one has that (6.7) is verified in a G_δ-set in $tb^{s,\overline{\alpha},r}_{p,q}(N)$ for every $N > 1$, and thus in a a G_δ-set in $tb^{s,\overline{\alpha},r}_{p,q}$ as a countable intersection of a G_δ-set. This proves the result in the case $p = \infty$ or $q = \infty$.

6.5 Determination of $\rho_{f,1}$ from its maximal point: Proof of Theorem 6.10

Let $r = 1$ and consider a function f such that, there exists an anisotropy index $\overline{\alpha}$ and $s \geq 0$ for which $\rho_{f,1}(\alpha_1) = s = \max_{y\in(0,2)} \rho_{f,1}(y)$. As $\rho_{f,1}(\alpha_1) = s$, one has that

$$f \in \bigcap_{\varepsilon>0} \widetilde{B}^{2s-\varepsilon,\overline{\alpha}}_{p,q}(\mathbb{R}^2) \quad \text{and} \quad f \notin \bigcup_{\varepsilon>0} \widetilde{B}^{2s+\varepsilon,\overline{\alpha}}_{p,q}(\mathbb{R}^2). \tag{6.21}$$

From (6.6) of Theorem 6.9, one has, for every $x \in (0,2)$, that $\rho_{f,1}(x) \geq \min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$. And we only need to prove that $\min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$ is also an upper bound to $\rho_{f,1}(x)$. The idea used consists in remarking that if, for a $x \in (0,2)$ with $x \neq \alpha_1$, $\min\{\frac{s}{\alpha_1}x, \frac{s}{\alpha_2}(2-x)\}$ is strictly below $\rho_{f,1}(x)$, then one can find a γ between x and α_1 for which $\rho_{f,1}(\gamma) > s$ which contradicts the hypothesis.

Recall (6.4) and one has

$$\|f\|_{\widetilde{B}^{2s,\overline{\alpha}}_{p,q}} = \Big(\sum_{\overline{j}\in\mathbb{N}^2_{-1}} 2^{2sq\|\overline{j}_+/\overline{\alpha}\|_\infty} \left|\omega^p_{\overline{j}}\right|^q\Big)^{1/q}, \quad \text{with} \quad \omega^p_{\overline{j}} = \Big\|\big(2^{-\|\overline{j}_+\|_1/p}\, c^f_{\overline{j},\overline{k}}\big)_{\overline{k}\in\mathbb{Z}^2}\Big\|_{\ell^p}. \tag{6.22}$$

We begin by considering an anisotropy index $\overline{\beta} = (\beta_1, \beta_2) > 0$ with $\|\overline{\beta}\|_1 = 2$ such that $\beta_1 < \alpha_1$. We will show that $\rho_{f,1}(\beta_1) \leq \frac{s}{\alpha_1}\beta_1 = \min\{\frac{s}{\alpha_1}\beta_1, \frac{s}{\alpha_2}(2-\beta_1)\}$ by contradiction. Suppose that $t := \rho_{f,1}(\beta_1) > \frac{s}{\alpha_1}\beta_1$. So there exists $\varepsilon > 0$ such that $t = \frac{s}{\alpha_1}\beta_1 + \varepsilon$. Recall that $t \leq s$ by hypothesis. Let $\overline{\gamma} = (\gamma_1, \gamma_2) > 0$ be an anisotropy index, so with $\|\overline{\gamma}\|_1 = 2$, such that $\gamma_1 \in (\beta_1, \alpha_1)$. One has

$$\sum_{(j_1)_+/\gamma_1<(j_2)_+/\gamma_2} 2^{2s'q\frac{(j_2)_+}{\gamma_2}} \left|\omega^p_{\overline{j}}\right|^q \leq \sum_{(j_1)_+/\alpha_1<(j_2)_+/\alpha_2} 2^{2s'q\frac{(j_2)_+}{\alpha_2}\frac{\alpha_2}{\gamma_2}} \left|\omega^p_{\overline{j}}\right|^q \tag{6.23}$$

as there are more terms in the last sum. One has (6.23) is finite if $2s'\frac{\alpha_2}{\gamma_2} < 2s$ by (6.21).

Further,

$$\sum_{(j_1)_+/\gamma_1\geq(j_2)_+/\gamma_2} 2^{2s'q\frac{(j_1)_+}{\gamma_1}}\left|\omega_{\bar{j}}^p\right|^q \leq \sum_{(j_1)_+/\beta_1\geq(j_2)_+/\beta_2} 2^{2s'q\frac{(j_1)_+}{\beta_1}\frac{\beta_1}{\gamma_1}}\left|\omega_{\bar{j}}^p\right|^q \tag{6.24}$$

as there are more terms in the last sum. Hence, (6.23) is finite if $2s'\frac{\beta_1}{\gamma_1} < 2t$ by hypothesis on t. This induces that $\|f\|_{\widetilde{B}^{2s',\overline{\gamma}}_{p,q}} < \infty$ if $s' < \min\{s\frac{\gamma_2}{\alpha_2}, t\frac{\gamma_1}{\beta_1}\}$. One deduces that $\rho_{f,1}(\gamma_1) \geq \min\{s\frac{\gamma_2}{\alpha_2}, t\frac{\gamma_1}{\beta_1}\}$. However, as $\gamma_2 > \alpha_2$, $s\frac{\gamma_2}{\alpha_2} > s$ and

$$t\frac{\gamma_1}{\beta_1} = \big(\frac{s}{\alpha_1}\beta_1 + \varepsilon\big)\frac{\gamma_1}{\beta_1} = s\frac{\gamma_1}{\alpha_1} + \varepsilon\frac{\gamma_1}{\beta_1}. \tag{6.25}$$

Denoting $\gamma_1 = \alpha_1 - \eta$ for $\eta > 0$ small enough, (6.25) gives

$$t\frac{\gamma_1}{\beta_1} = s\frac{\alpha_1-\eta}{\alpha_1} + \varepsilon\frac{\alpha_1-\eta}{\beta_1} = s + \varepsilon\frac{\alpha_1}{\beta_1} - \eta\Big(\frac{s}{\alpha_1} + \frac{\varepsilon}{\beta_1}\Big).$$

Remark that $C := \left(\frac{s}{\alpha_1} + \frac{\varepsilon}{\beta_1}\right)$ is a positive constant, and one can chose η small enough so that $\eta C < \frac{\varepsilon}{2}\frac{\alpha_1}{\beta_1}$ and $t\frac{\gamma_2}{\beta_2} > s$. One deduces that for γ_1 close enough to α_1 (so η small enough), one has $\rho_{f,1}(\gamma_1) > s$ which is a contradiction with $s = \max_{y\in(0,2)}\rho_{f,1}(y)$. One concludes that for $\beta_1 < \alpha_1$, $\rho_{f,1}(\beta_1) = \frac{s}{\alpha_1}\beta_1$.

We continue with an anisotropy index $\overline{\beta} = (\beta_1, \beta_2)$ such that $\beta_1 > \alpha_1$ and show that $\rho_{f,1}(\beta_1) \leq \frac{s}{\alpha_2}(2-\beta_1) = \min\{\frac{s}{\alpha_1}\beta_1, \frac{s}{\alpha_2}(2-\beta_1)\}$ by contradiction. Suppose that $t := \rho_{f,1}(\beta_1) > \frac{s}{\alpha_2}(2-\beta_1)$. So there exists $\widetilde{\varepsilon} > 0$ such that $t = \frac{s}{\alpha_2}(2-\beta_1) + \widetilde{\varepsilon}$. Recall that $t \leq s$ by hypothesis. Let $\overline{\gamma} = (\gamma_1, \gamma_2) > 0$ be an anisotropy index, so with $\|\overline{\gamma}\|_1 = 2$, such that $\gamma_1 \in (\alpha_1, \beta_1)$. One has

$$\sum_{(j_1)_+/\gamma_1\geq(j_2)_+/\gamma_2} 2^{2s'q\frac{(j_1)_+}{\gamma_1}}\left|\omega_{\bar{j}}^p\right|^q \leq \sum_{(j_1)_+/\alpha_1\geq(j_2)_+/\alpha_2} 2^{2s'q\frac{(j_1)_+}{\alpha_1}\frac{\alpha_1}{\gamma_1}}\left|\omega_{\bar{j}}^p\right|^q \tag{6.26}$$

as there are more terms in the last sum. One has (6.26) is finite if $2s'\frac{\alpha_1}{\gamma_1} < 2s$ by (6.21). Further,

$$\sum_{(j_1)_+/\gamma_1<(j_2)_+/\gamma_2} 2^{2s'q\frac{(j_2)_+}{\gamma_2}}\left|\omega_{\bar{j}}^p\right|^q \leq \sum_{(j_1)_+/\beta_1<(j_2)_+/\beta_2} 2^{2s'q\frac{(j_2)_+}{\beta_2}\frac{\beta_2}{\gamma_2}}\left|\omega_{\bar{j}}^p\right|^q \tag{6.27}$$

as there are more terms in the last sum. Hence, (6.27) is finite if $2s'\frac{\beta_2}{\gamma_2} < 2t$ by hypothesis on t. This induces that $\|f\|_{\widetilde{B}^{2s',\overline{\gamma}}_{p,q}} < \infty$ if $s' < \min\{s\frac{\gamma_1}{\alpha_1}, t\frac{\gamma_2}{\beta_2}\}$. One deduces that $\rho_{f,1}(\gamma_1) \geq \min\{s\frac{\gamma_1}{\alpha_1}, t\frac{\gamma_2}{\beta_2}\}$. However, as $\gamma_1 > \alpha_1$, $s\frac{\gamma_1}{\alpha_1} > s$ and

$$t\frac{\gamma_2}{\beta_2} = \big(\frac{s}{\alpha_2}\beta_2 + \widetilde{\varepsilon}\big)\frac{\gamma_2}{\beta_2} = s\frac{\gamma_2}{\alpha_2} + \widetilde{\varepsilon}\frac{\gamma_2}{\beta_2}. \tag{6.28}$$

Denoting $\gamma_1 = \alpha_1 + \eta$ for $\eta > 0$ small enough, one has also $\gamma_2 = \alpha_2 - \eta$ and (6.28) gives

$$t\frac{\gamma_2}{\beta_2} = s\frac{\alpha_2-\eta}{\alpha_2} + \widetilde{\varepsilon}\frac{\alpha_2-\eta}{\beta_2} = s + \widetilde{\varepsilon}\frac{\alpha_2}{\beta_2} - \eta\Big(\frac{s}{\alpha_2} + \frac{\widetilde{\varepsilon}}{\beta_2}\Big).$$

Remark that $\widetilde{C} := \left(\frac{s}{\alpha_2} + \frac{\widetilde{\varepsilon}}{\beta_2}\right)$ is a positive constant, and one can chose η small enough so that $\eta\widetilde{C} < \frac{\widetilde{\varepsilon}}{2}\frac{\alpha_2}{\beta_2}$ and $t\frac{\gamma_2}{\beta_2} > s$. One deduces that for γ_1 close enough to α_1 (so η small enough), one has $\rho_{f,1}(\gamma_1) > s$ which is a contradiction with $s = \max_{y\in(0,2)}\rho_{f,1}(y)$. One concludes that for $\beta_1 > \alpha_1$, $\rho_{f,1}(\beta_1) = \frac{s}{\alpha_2}(2-\beta_1)$. This concludes the proof that for any $\beta_1 \in (0,2)$, $\rho_{f,1}(\beta_1) = \min\{\frac{s}{\alpha_1}\beta_1, \frac{s}{\alpha_2}(2-\beta_1)\}$.

Chapter 7

Riemann Fourier series

7.1 Introduction

The determination of the regularity of trigonometric series has been a topic of major importance during the last two centuries. These series have supplied examples and counterexamples of functions having specific properties and the theoretical advances made on this subject have had major implications in mathematical analysis, see e.g. Jaming-Kellay-Saba-Wang, Kahane-Lemarie, Korner and Zygmund [100, 101, 104, 113, 179]. A typical example is supplied by *Weierstrass functions*

$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(\pi b^n x) \qquad \text{where} \quad 0 < a < 1 \text{ and } b > 1$$

which settled the question of determining if there exist continuous nowhere differentiable functions. Weierstrass proved that it is the case if b in an integer and $ab > 1 + 3\pi/2$ [173], and Hardy obtained the sharp condition $b > 1$ and $ab > 1$ [74]. Another key example was proposed by Riemann: he conjectured that the function

$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2} \tag{7.1}$$

is continuous and nowhere differentiable. This function proved much harder to be dealt with than the Weierstrass functions: In [74] Hardy proved that it is nowhere differentiable except perhaps at the rationals of the form

$$r = \frac{2p+1}{2q+1},$$

and Hardy's conjecture was disproved only in 1969 by J. Gerver [70] who showed that Riemann's function is differentiable at those rationals. Its full regularity properties were completely derived only in the 1990s, first with the determination in [86] of its exact pointwise regularity at every point, see Definition 7.1, and then by an asymptotic expansion at the differentiability points, see [97], which put in evidence its *chirp behavior*, see (7.7). An important byproduct is that Riemann's function is an example of *multifractal function*: the sets of

points that share the same regularity exponent are fractal sets with various Hausdorff dimensions, see Definition 7.6. Such regularity issues are not completely settled for several variants and generalisations of the seminal Riemann's function (7.1). Let us consider the *Schrödinger series*

$$\mathcal{S}_{b,s}(x,t) = \sum_{n=1}^{\infty} \frac{e^{i\pi(n^b x + nt)}}{n^s}, \tag{7.2}$$

where $b \geq 2$ is an integer and $s \in \mathbb{R}$ (Riemann's function is the imaginary part of $\mathcal{S}_{2,2}(\cdot, 0)$). Various instances of these functions have been studied: the case $b = 2$ is met in explicit solutions of several linear PDEs, such as Schrödingers's equation (hence the notation that we use, see Section 7.3.1), so that their analysis gives insight on the regularity properties of these PDEs; furthermore, they play an important role in fluid mechanics because of their occurrence of in the dynamics of the binormal flow : In [14], it is shown that they describe the temporal trajectory of a vortex filament.

In [150], Hervé Queffelec investigated the regularity properties of the functions

$$\sum \frac{\sin(\pi P(n)x)}{P(n)} \tag{7.3}$$

where P is a polynomial function with integer coefficients. We will first specify his results in the particular case of the *Queffelec series*

$$\mathcal{Q}_{b,s}(x) = \mathrm{Im}(\mathcal{S}_{b,s}(x,0)) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^b x)}{n^s} \tag{7.4}$$

when $b = s$ and is an integer larger than 1. These results were complemented by D. Boichu in [28] and are summarised in Theorem 7.2 below. In order to state them, we will need the following notions concerning pointwise and uniform regularity.

The *uniform Hölder regularity exponent* of a periodic function f defined on $\mathbb{T} = \mathbb{R}/2\mathbb{Z}$ is

$$h_f^{\min} = \sup\{\alpha \in \mathbb{R} : f \in C^\alpha(\mathbb{T})\}. \tag{7.5}$$

For locally bounded functions, the pointwise regularity notion most often used is the Hölder exponent.

Definition 7.1. *Let $f \in L^\infty_{loc}(\mathbb{R})$, $x_0 \in \mathbb{R}$ and $\alpha \geq 0$. The function f belongs to $C^\alpha(x_0)$ when there exists a polynomial P_{x_0} of degree less than α and $C, r > 0$ such that*

$$\forall x \in (x_0 - r, x_0 + r), \quad |f(x) - P_{x_0}(x - x_0)| \leq C|x - x_0|^\alpha.$$

The Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha \in \mathbb{R} : f \in C^\alpha(x_0)\}$.

Recall that, informally, a chirp at x_0 of exponents (α, β) is a function f such that

$$h_f(x_0) = \alpha \qquad \text{and} \quad \forall n \geq 1, \quad h_{f^{(-n)}}(x_0) = \alpha + n(1 + \beta), \tag{7.6}$$

where $f^{(-n)}$ is the nth-order primitive with zero mean. Typical examples of chirps are

$$\mathcal{C}^{\alpha,\beta}(x) = |x|^\alpha \sin\left(\frac{1}{|x|^\beta}\right) \qquad \text{for} \quad \alpha, \beta > 0 \tag{7.7}$$

(see [97] for a precise definition and its wavelet characterization).

Theorem 7.2 (Queffelec [150], Boichu [28]). *Let $s \geq 2$; then we have*

$$h^{\min}_{\mathcal{Q}_{s,s}} = \frac{s-1}{s}.$$

$\mathcal{Q}_{s,s}$ is differentiable at some rational points only where its chirp exponents are

$$(\alpha, \beta) = \left(\frac{2s-1}{2s-2}, \frac{1}{s-1} \right). \tag{7.8}$$

Remark 7.3. It follows from the definition of $h^{\min}_f$ (or from (7.23) below) that, see [28, Proposition 5] and [20],

$$h^{min}_{\mathcal{Q}_{b,s}} = \frac{s-1}{b}.$$

If $1/2 < s < 1$, then the functions $Q_{b,s}$ are not bounded, but only belong to L^2; therefore, the usual notion of pointwise regularity supplied by the Hölder exponent can no more be used, and partial results using a weaker notion of pointwise regularity (see Definition 7.4) have been obtained by S. Seuret and A. Ubis when $b = 2$, [160]. Nonetheless, our understanding of the regularity properties of this family of functions is partial; their multifractal spectra (see Definition 7.6) are only partially known, and in many instances, only loose upper and lower bounds are known, see Section 7.7. One way to make precise guesses on their exact values is through the application of a *multifractal formalism*, see Section 7.2, which yields a relationship between the multifractal spectrum and numerically computable quantities. Up to now, these techniques have mainly been used for classification and model selection; in particular, wavelet methods allowed many breakthroughs in signal and image processing [175, 176], brain activity [177] and other scientific fields, so that they now stand as the state of the art in this field. Though the precise range of validity of these formulas in not known, they have been proved to hold for a large number of mathematical models, so that working out the multifractal formalism in a numerically precise way would supply "guesses" of the correct values of their multifractal spectrum. Up to now, the numerical determination of Legendre spectra of mathematical functions was performed only in cases where their multifractal spectra were theoretically known, see e.g. [5], with the aim of testing its numerical accuracy, often leading to improvements or variants of the method; therefore its purpose lies at the cross-road of applied mathematics and signal processing. Our purpose here is different: it is to supply mathematicians with reliable conjectures concerning their multifractal spectra, and therefore falls in the field of *experimental mathematics*.

Functions whose pointwise regularity varies erratically appear naturally in many areas of mathematics. The Riemann Fourier series $\mathcal{R}$ was retrospectively the first deterministic example of this type; more recently, the Brjuno function [96], was shown to share regularity properties with the Riemann function: in both cases their pointwise regularity exponent is directly related with Diophantine irrationality exponents. Stochastic examples are supplied by cascade models, see [17] and references therein, or Lévy processes, see [88, 149]. Such models, whose theoretical properties are fully understood, yield important case studies on

which new numerical algorithms can be tested, and their accuracy can be challenged. They are used in several fields of applications such as turbulence [41, 65] or finance [11].

Riemann's functions are of particular interest, because of the presence of oscillating singularities in their graphs. Such functions present strong oscillations near the singularity points such as the chirps (7.7). Such behaviors are met in various physical phenomena see [60]. For signals or function without such singularities, the Hölder or the p-exponents (see Definition 7.4 below) provide all the information of the regularity of the function as well as the regularity of its fractional integrals. However, in the presence of such singularities, studying the p-exponents of a function and its primitive yields informations on the type of oscillatory behaviour that it displays.

In Section 7.2 the necessary notions of pointwise regularity are recalled as well as the wavelet-based multifractal formalism used in the chapter. Section 7.3 gives some result on the regularity of Fourier series and their fractional derivatives and primitives as well as an explicit formula for wavelet coefficient with the Meyer wavelet. Its use will allow to study the regularity properties of the Riemann functions when they are no more locally bounded but only belong to L^p. In Section 7.4, we provide the known result on the regularity of $\mathcal{S}_{2,s}$ for $s > 1/2$ when $t = 0$ as well as some conjectures previously formulated. These results and conjectures are then compared to numerical estimation of the regularity quantities in Section 7.5 for $s = 2$ and in Section 7.6 for $1/2 < s < 1$ (we will pick $s = 0.7$). Section 7.7 recalls some theoretical result for $\mathcal{Q}_{b,s}$, with $b = 3$ and $b = 4$, compares them to numerically estimated results. Section 7.8 gives the numerical study of $\mathcal{S}_{2,2}(x,t)$ when t is non zero in order to provides some conjecture of its regularity behaviour. Finally, in the conclusion, the pertinence of previous conjectures concerning Riemann functions and their generalizations, are discussed, sharpened and extended.

7.2 Regularity exponents and multifractal formalisms

The definition of the pointwise Hölder regularity explicitly assumes that f is locally bounded. Our study will not be limited to such functions, but we will consider functions which belong to spaces L^p only for a limited range of values of p. For instance, we will see in Section 7.4 that if $1/2 \leq s \leq 1$, then the functions $\mathcal{R}_s$ defined by (7.27) belongs to L^p for every $p < 2/(1-s)$ and this result is sharp.

Calderón and Zygmund introduced in [35] a notion of pointwise regularity adapted to functions in $L^p_{loc}(\mathbb{R})$.

Definition 7.4. *Let $f \in L^p_{loc}(\mathbb{R})$ with $p \geq 1$, $x_0 \in \mathbb{R}$ and $\alpha \geq -1/p$. The function f belongs to $T^p_\alpha(x_0)$ when there exists a polynomial P_{x_0} of degree less than α and $C, R > 0$ such that*

$$\forall r \in (0,R), \quad \left(\frac{1}{r}\int_{x_0-r}^{x_0+r} |f(x) - P_{x_0}(x-x_0)|^p dx\right)^{\frac{1}{p}} \leq Cr^\alpha. \tag{7.9}$$

The p-exponent of f at x_0 is $h^p_f(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}$.

Remark 7.5. In the two previous definitions, the constant $C > 0$ may depend on x_0 and α. Furthermore, the polynomial P_{x_0} is unique and, if $\alpha \leq 0$, then it vanishes. If $p = +\infty$, then $h_f^p(x_0) = h_f(x_0)$. Furthermore, $h_f^p(x_0)$ can take values down to $-1/p$, thus supplying mathematical frameworks for the notion of singularities of negative exponent.

The purpose of multifractal analysis is to study functions, processes or signals whose pointwise regularity exponent widely varies from one point to another. When dealing with experimental data, it is not possible to calculate a pointwise regularity exponent point by point and then determine the Hausdorff dimension of its level sets. Indeed, for classical models used in applications, such as multiplicative cascades, multifractal random walks, random wavelet series, or random pulse series, see [89, 97, 156], the Hölder exponent is an extremely irregular, nowhere continuous, function which, over any interval of positive length, takes all values included in an interval $[h^{\min}, h^{\max}]$ (with $h^{\min} < h^{\max}$). One way to bypass this numerical obstruction is not to focus on the precise values taken by h_f^p at each point, but on a more qualitative information: the sizes of the sets of points that share the same p-exponent. These sets are fractals for the models mentioned above, and the mathematical notion which allows to discriminate their sizes is provided by the notion of *Hausdorff dimension*.

Definition 7.6. *Let $f \in L^p_{loc}(\mathbb{R})$ with $p \geq 1$. The p-isohölder sets of f are*

$$\forall h \in \mathbb{R}^+, \qquad E_f^{(p)}(h) = \{x \in \mathbb{R} : h_f^p(x) = h\} \tag{7.10}$$

and its multifractal p-spectrum is

$$D_f^{(p)} : \begin{cases} [-\frac{1}{p}, +\infty[& \longrightarrow & [0,1] \cup \{-\infty\} \\ h & \longmapsto & \dim_H(E_f^{(p)}(h)). \end{cases} \tag{7.11}$$

For $p = +\infty$, the ∞-isohölder sets are simply referred to as the *isohölder sets* and denoted by E_f, and the multifractal ∞-spectrum is referred to as the *multifractal spectrum* and is denoted by D_f.

The computation of the multifractal spectrum of a signal is not feasible by a direct application of its definition since the estimation of the Hausdorff dimension of a set requires the consideration of all its possible coverings. One rather has recourse to *multifractal formalisms* which relate a multifractal spectrum associated with a pointwise exponent to quantities that can be calculated numerically from experimental data, see Definition 7.9 below for the wavelet based technique that we will apply.

We will use the setting supplied by one-dimensional periodised wavelets, which we now recall.

Definition 7.7. *A mapping $\psi : \mathbb{R} \to \mathbb{R}$ has $r+1$ vanishing moments when for every integer $\alpha \leq r$, $\int_{\mathbb{R}} x^\alpha \psi(x) dx = 0$.*

A "mother wavelet" is a function $\psi \in C^r(\mathbb{R})$, such that ψ and its r first derivatives have fast decay, and the family composed of the functions $\mathbb{1}_{\mathbb{T}}$ and the

$$\widetilde{\psi}_{j,k}(x) = \sum_{\ell \in \mathbb{Z}} 2^{j/2} \psi(2^j x - k - \ell) \quad \text{for} \quad (j,k) \in \mathbb{N} \times \mathbb{Z}/2^j\mathbb{Z}$$

form an orthonormal basis of $L^2(\mathbb{T})$.

The wavelet coefficients of a 1-periodic function $f : \mathbb{T} \to \mathbb{R}$ *are*

$$c_{j,k} = 2^{j/2} \int_{\mathbb{R}/\mathbb{Z}} f(x)\widetilde{\psi}_{j,k}(x)dx = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)dx, \quad (j,k) \in \mathbb{N} \times \mathbb{Z}/2^j\mathbb{Z}. \tag{7.12}$$

For all (j,k), we denote by $\lambda_{j,k}$ the dyadic interval defined by $\lambda_{j,k} = [k2^{-j}, (k+1)2^{-j}]$. We use the notation $3\lambda_{j,k}$ for the union of $\lambda_{j,k}$ and the 2 intervals adjacent to $\lambda_{j,k}$:

$$3\lambda_{j,k} = [(k-1)2^{-j}, (k+2)2^{-j}].$$

Finally, for all $j \in \mathbb{N}$ and $x_0 \in \mathbb{R}$, we denote by $\lambda_j(x_0)$ the unique dyadic interval $\lambda_{j,k}$ such that $x_0 \in \lambda_{j,k}$. This allows to define the notion of wavelet leaders and p-leaders.

Definition 7.8. *Let* $p \in \mathbb{R}_+^*$ *and* $f \in L^p_{loc}(\mathbb{R}/\mathbb{Z})$. *The* p*-leaders are*

$$\forall (j,k) \in \mathbb{N} \times \mathbb{Z}/2^j\mathbb{Z}, \quad l^{(p)}_{j,k} = \Big(\sum_{\lambda_{j',k'} \subset 3\lambda_{j,k}} 2^{-(j'-j)} |c_{j',k'}|^p \Big)^{\frac{1}{p}}. \tag{7.13}$$

If $p = +\infty$, *the wavelet leaders of* f *are*

$$l_{j,k} = \sup_{\lambda_{j',k'} \subset 3\lambda_{j,k}} |c_{j',k'}|.$$

The following notion of spectrum yields a numerically computable quantity, which can be derived from log-log plot regression algorithms, see [92].

Definition 7.9. *Let* $p > 0$ *and* $f \in L^p(\mathbb{T})$. *The* p*-leader scaling function of* f *is*

$$\forall q \in \mathbb{R}, \quad \zeta^{(p)}_f(q) = \liminf_{j \to +\infty} \frac{\log \Big| 2^{-j} \sum_{k=0}^{2^j - 1} |l^{(p)}_{j,k}|^q \Big|}{\log(2^{-j})}$$

and the p*-Legendre spectrum of* f *is*

$$\forall h \in \mathbb{R}, \quad \mathcal{L}^{(p)}_f(h) = \inf_{q \in \mathbb{R}} (qh - \zeta^{(p)}_f(q) + 1). \tag{7.14}$$

If $p = +\infty$, we recover the notion of the wavelet leader scaling function ζ_f and of Legendre spectrum $\mathcal{L}_f$.

If f belongs to $L^p(\mathbb{T})$, then the Legendre spectra satisfy, see [93],

$$\forall h \in \mathbb{R}, \quad D^{(p)}_f(h) \leq \mathcal{L}^{(p)}_f(h), \tag{7.15}$$

When this inequality is an equality, the *multifractal formalism* holds. While the multifractal formalism holds for a wide range of functions, such as certain self-similar functions [23] or generic functions in functional spaces [19, 90], there are many counter-examples to this formalism.

Note that, by construction, the right-hand side of (7.15) is a concave function, so that the multifractal formalism fails for functions that do not have a concave multifractal spectrum, which often is the case. A typical example is supplied by the Riemann's function, see Theorem 7.16. In such situations, one rather expects the Legendre spectrum to yield the

concave hull of the multifractal spectrum; when such is the case, we will say that the *weak multifractal formalism* holds. This formulation has a wider range of validity than the classical multifractal formalism and, in practice, we will use it in order to conjecture the concave hull of several generalizations of the Riemann functions. However, in the Section 7.9, we will show an example of numerical method which is taylored to the estimation of non-concave multifractal spectra, and we will apply it to the estimation of the decreasing part of the multifractal spectrum of $\mathcal{R}$.

Other numerical methods have been devised in order to estimate multifractal Legendre spectra, such as the *multifractal detrended fluctuation analysis* (MFDFA), see [106, 107] or the *wavelet-transform modulus-maxima* (WTMM), see [140]. However, we will not use them because they present the following limitations:

- They are not backed by mathematical results such as the upper bound supplied by (7.15).
- They cannot be taylored to specific regularity exponents. Actually, MFDFA was shown to be fitted to the 2-exponent, see [121], and the WTMM was shown to be fitted to the *weak-scaling exponent* (another example of pointwise regularity exponent that we will not consider in this chapter), see [137].
- They do not yield methods adapted to detect the existence of oscillating singularities.

In order to use a multifractal analysis based on the Hölder exponent, one has to determine whether f is locally bounded; this can be numerically achieved by estimating the the value of uniform regularity exponent (7.5) which can be numerically derived form the the wavelet coefficients of f by

$$h_f^{\min} = \liminf_{j\to+\infty} \frac{\log\left(\sup_{k\in\{0,\dots 2^j-1\}} |c_{j,k}|\right)}{\log(2^{-j})}. \tag{7.16}$$

If $h_f^{\min} > 0$, then the Legendre spectrum based on wavelet leaders is appropriate to estimate the multifractal spectrum, and additionally,

$$\forall x, \qquad h_f(x) \geq h_f^{\min}.$$

On other hand, if $h^{\min} \leq 0$, then a possible alternative is to use p-leaders, in which case, one has to determine if $f \in L^p$. This is performed numerically through the evaluation of the *wavelet scaling function* defined by

$$\forall p > 0, \quad \eta_f(p) = \sup\{s \in \mathbb{R} : f \in B^{s/p}_{p,p}\}, \tag{7.17}$$

see e.g. [136] for the wavelet characterizations of these spaces, from which the following numerical procedure for the estimation of η_f is derived

$$\forall p > 0, \quad \eta_f(p) = \liminf_{j\to+\infty} \frac{\log\left(2^{-j} \sum_{k\in\{0,\dots 2^j-1\}} |c_{j,k}|^p\right)}{\log(2^{-j})}. \tag{7.18}$$

In particular, it follows from the classical embeddings between Besov and Lebesgue spaces

that, if $\eta_f(p) > 0$, then f belongs to $L^p_{loc}(\mathbb{R})$, and a multifractal analysis based on the p-exponent can be performed.

Finally, we will use the following result, which states cases where the wavelet scaling function and the wavelet leader scaling function coincide; indeed, it will imply that several Legendre spectra coincide.

Proposition 7.10. *Let $f : \mathbb{T} \to \mathbb{R}$ be a bounded function, and let $p > 0$.*

$$\text{if } \eta_f(p) > 1, \quad \text{then} \quad \zeta_f(p) = \eta_f(p).$$

This results immediately follows from the embeddings between oscillation spaces and Besov spaces, see Proposition 2 of [91].

7.3 Regularity of Fourier series

Our purpose in this section is not to survey the (huge) topic of the regularity of Fourier series but only to supply some remarks that will be relevant in the following. We start with a short overview describing one of the contexts in which the regularity of the trigonometric series studied in this chapter show up.

7.3.1 Trigonometric series in PDEs

Several solutions of classical linear PDEs can be written as trigonometric series with similar frequencies as the ones we considered. A first example is supplied by the one-dimensional periodic Schrödinger equation without potential

$$\begin{cases} i\dfrac{\partial \phi}{\partial t} = -\dfrac{\partial^2 \phi}{\partial x^2}, & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R} \\ \\ \phi(x,0) = \phi_0(x) = \displaystyle\sum_{n \in \mathbb{Z}} a_n e^{i\pi n x}. \end{cases} \tag{7.19}$$

Formally, the general solution of (7.19) can be written

$$\phi(x,t) = \sum_{n \in \mathbb{Z}} a_n e^{inx} e^{-in^2 t}.$$

In particular, let us consider the case where the initial condition $\phi_0(x)$ is the 2-periodic sawtooth function. Recall that

$$\{x\} = x - [x] - \frac{1}{2},$$

which satisfies

$$-\pi \left\{ \frac{x}{2} \right\} = \sum \frac{\sin(\pi n x)}{n}.$$

Therefore (exchanging the roles of the space and the time variables), the solution of (7.19) with initial value

$$\phi_0(x) = -\pi \left\{ \frac{x}{2} \right\} \tag{7.20}$$

is

$$\phi_0(x,t) = \mathrm{Im}(\mathcal{S}_{2,1}(t,x)).$$

Another choice is to take for initial condition the primitive of (7.20): in that case, the solution of (7.19) is

$$\phi_0(x,t) = -\,\mathrm{Re}(\mathcal{S}_{2,2}(t,x));$$

one can also take fractional primitives or derivatives of the sawtooh function, obtaining various cases of the functions $\mathcal{S}_{2,s}(t,x)$. These examples therefore show that understanding the regularity properties of the functions $\mathcal{S}_{b,s}(t,x)$ gives information of the possible developments of singularities of the solutions of the Schrödinger equation in the case where the initial solution is smooth except a one point with a cusp singularity whose order can be tuned. In [94] everywhere irregularity results for the two variables function $\phi(x,t)$ were derived when the initial condition is not smooth. Let us give a different example of irregularity, which also is a direct consequence of the techniques developed in [94].

Proposition 7.11. *Let $\phi(x,t)$ be a solution of (7.19) such that the Fourier coefficients of the initial condition $\phi_0(x)$ satisfy*

$$\exists \alpha > 1, \exists C, C' > 0 : \qquad \frac{C}{n^\alpha} \leq |a_n| \leq \frac{C'}{n^\alpha}.$$

Let x be fixed and let p be such that the function ϕ_x defined by $t \to \phi(x,t)$ belongs to L^p. Then the p-exponent of ϕ_x satisfies

$$\forall t, \qquad \frac{\alpha - 1}{2} \leq h^p_{\phi_x}(t) \leq \alpha.$$

If the first inequality is clearly sharp, the optimality of the second one is an open problem: in the case of the Riemann function, the largest Hölder exponent is $3/2$, and it is not clear what can be the largest possible Hölder exponent of a trigonometric series of the form

$$\sum a_n e^{inx} \qquad \text{with} \quad |a_n| \sim \frac{C}{n^2};$$

note however that the Gabor-wavelet techniques developed in [94] yields that it is bounded by 2.

Another example where similar considerations can be addressed is supplied by the two-dimensional supported plate equation, in a rectangular domain $\Omega = [0,A] \times [0,B]$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \Delta^2 u = 0, \\ u = \Delta u = 0 \ \text{ on } \ \partial\Omega, \\ u(x,y,0) = u_0(x,y) \ \text{ and } \ \dfrac{\partial u}{\partial t}(x,y,0) = u_1(x,y). \end{cases} \tag{7.21}$$

The general solution can formally be written

$$u(x,y,t) = \sum \sin\left(\frac{nx}{A}\right) \sin\left(\frac{my}{B}\right) \left[a_{n,m} \cos\left(\left(\frac{n^2}{A^2} + \frac{m^2}{B^2}\right) t\right) + b_{n,m} \sin\left(\left(\frac{n^2}{A^2} + \frac{m^2}{B^2}\right) t\right)\right].$$

Finally, a closely related equation is supplied by the vibrations of simply supported

beams:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^4 u}{\partial x^4} = 0, & \text{for } (x,t) \in [0,\pi] \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \text{ and } \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in [0,\pi], \\ u(0,t) = u(\pi,t) = 0 & \text{for } x = 0 \text{ and } x = \pi. \end{cases}$$

The general solution can formally be written

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n^2 t) + b_n \sin(n^2 t)) \sin(nx),$$

where the a_n and b_n are defined by

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{and } u_1(x) = \sum_{n=1}^{\infty} n^2 b_n \sin(nx).$$

Results similar to Proposition 7.11 can be derived for these equations. Additional examples of occurences of trigonometric series in PDEs can be found in [112].

7.3.2 Regularity properties of sine and cosine series

The trigonometric series that we consider are often written under different forms:

$$F(x) = \sum_{n\in\mathbb{N}} a_n e^{i\pi nx}, \quad G(x) = \sum_{n\in\mathbb{N}} a_n \sin(\pi nx), \quad \text{or} \quad H(x) = \sum_{n\in\mathbb{N}} a_n \cos(\pi nx), \tag{7.22}$$

where $x \in \mathbb{T} = \mathbb{R}/2\mathbb{Z}$ (in the following, we will always assume that $a_0 = 0$, which is of no consequence as regards to pointwise regularity). The following results shows that the global and local regularities of these functions coincide (see e.g. [136] for the definitions of these spaces).

Proposition 7.12. *The functions F, G and H defined in (7.22) belong to the same:*

- *Bessel potential spaces $H^{s,p}(\mathbb{T})$ for $p \in (1,+\infty)$,*
- *Besov spaces $B^s_{p,q}(\mathbb{T})$ for $p,q \in (0,+\infty]$.*

If there exists $\varepsilon > 0$ such that $F \in C^\varepsilon(\mathbb{T})$, then

$$\forall x \in \mathbb{T}, \qquad h_F(x) = h_G(x) = h_H(x).$$

If $F \in L^p(\mathbb{T})$, then

$$\forall p > 0, \quad \forall x \in \mathbb{T}, \qquad h^p_F(x) = h^p_G(x) = h^p_H(x).$$

In particular, this statement implies that F, G and H share the same wavelet scaling function and the same p-spectra.

Proof. First note that, since $G = \mathrm{Re}(F)$ and $H = \mathrm{Im}(F)$, it follows that the statement that F belongs to a given function space (either global, such as Besov or Bessel potential spaces, or local, such as the $C^\alpha(x_0)$ or $T^p_\alpha(x_0)$ spaces) implies the same statement for both G and H. Therefore, it is sufficient to show that G and H share the same local (or global) regularity.

For this, let us recall the key notion of *Hilbert transform* $\mathcal{H}$. Let

$$A = \mathbb{1}_{\mathbb{R}^+} - \mathbb{1}_{\mathbb{R}^-}.$$

If $f \in L^2(\mathbb{R})$, then its Hilbert transform is defined in the Fourier domain by

$$\widehat{\mathcal{H}(f)} = A\widehat{f}.$$

Recall that $\mathcal{S}_0$ is the space of the functions in the Schwartz class such that all their moments vanish. Then A clearly is continuous on $\mathcal{S}_0$ and therefore can be extended to $\mathcal{S}_0'$ by duality. In the case of Fourier series, the vanishing moments condition boils down to $a_0 = 0$, and the Hilbert transform simply is the mapping defined by

$$\text{if} \quad f(x) = \sum_{n\neq 0} a_n e^{i\pi nx}, \quad \text{then} \quad \mathcal{H}(f)(x) = \sum_{n\neq 0} \operatorname{sgn}(n)\, a_n e^{i\pi nx}.$$

Therefore, the results of Proposition 7.12 are a consequence of the following classical results:

- The Hilbert transform is continuous on the Bessel and Besov spaces mentioned in the statement of Proposition 7.12, see [136].
- The Hilbert transform is continuous on the $T^p_\alpha(x)$ spaces, if $p < \infty$, see [136].
- The Hilbert transform is continuous on the $T^p_\alpha(x_0)$ spaces, see [35] and on the two-microlocal $C^{s,s'}(x_0)$ spaces, see [84]. This last result implies that, if f has some uniform Hölder regularity, then

$$\forall x, \qquad h_f(x) = h_{\mathcal{H}(f)}(x).$$

□

As a consequence of these results, in the following, we will consider the regularity properties of functions written indifferently as series of complex exponentials with positive frequencies, or of the corresponding sines or cosines series.

7.3.3 Fractional integration and derivation

In this section, we recall the regularity properties associated with fractional integration and derivation. These operators extend to non-integer orders the natural notions of primitives and derivatives, which are naturally defined for integer order. In the case of functions defined on $\mathbb{R}$, there exists several variants of these definitions, see e.g. [77] for detailed considerations of these topicss; however these variants usually present the same regularity properties so that the one used has little implications. The situation actually is simpler in the case of trigonometric series of the form

$$f(x) = \sum_{n>0} a_n e^{i\pi nx}.$$

Indeed, note that their derivative of order k (in the sense of distributions) is

$$f^{(k)}(x) = \sum_{n>0} a_n (in\pi)^k e^{i\pi nx},$$

and their primitive of order k of vanishing mean is

$$f^{(-k)}(x) = \sum_{n>0} a_n (in\pi)^{-k} e^{i\pi nx}.$$

It follows that a natural definition (up to a constant multiplicative term) for the fractional integral of order $\gamma \in \mathbb{R}$ of a trigonometric series is

$$\mathcal{I}^\gamma f(x) = \sum_{n>0} \frac{a_n}{n^\gamma} e^{i\pi nx},$$

the case $\gamma < 0$ corresponding to a fractional derivative; it is usually referred to as the *Weyl integral*.

The regularity properties of the fractional integration are well known in the case of Bessel potential and Besov spaces, and are recalled in the following statement, see [136].

Proposition 7.13. *The fractional integration operator $\mathcal{I}^\gamma$ satisfies the following continuity properties:*

- *Let $f \in H^{t,p}(\mathbb{T})$ for $p \in (1, +\infty)$; then $\forall \gamma \in \mathbb{R}$, $\mathcal{I}^\gamma f \in H^{t+\gamma,p}(\mathbb{T})$.*
- *Let $f \in B^t_{p,q}(\mathbb{T})$ for $p, q \in (0, +\infty]$; then $\forall \gamma \in \mathbb{R}$, $\mathcal{I}^\gamma f \in B^{t+\gamma}_{p,q}(\mathbb{T})$.*
- *Let $f \in T^p_\alpha(x_0)$ for $p \in (0, +\infty]$; then $\forall \gamma > 0$, $\mathcal{I}^\gamma f \in T^p_{\gamma+\alpha}(x_0)$.*

The two first statements imply the following result for wavelet scaling functions:

$$\forall \gamma \in \mathbb{R},\ \forall p > 0, \qquad \eta_{\mathcal{I}^\gamma f}(p) = p\gamma + \eta_f(p).$$

Additionally, taking $p = q = +\infty$ and using that $B^s_{\infty,\infty} = C^s$, it follows that

$$h^{\min}_{\mathcal{I}^\gamma f} = h^{\min}_f + \gamma. \tag{7.23}$$

The last statement of the proposition implies that

$$\forall \gamma > 0,\ \forall p > 0, \qquad h^p_{\mathcal{I}^\gamma f}(x_0) \geq h^p_f(x_0).$$

Note that the last inequality can be strict, as shown by the examples of chirps as explained at the end of Section 7.2; such examples will be considered in Section 7.5.

Note that

$$\mathcal{I}^\gamma \mathcal{Q}_{b,s} = \mathcal{Q}_{b,s+b\gamma}, \quad \text{and} \quad \mathcal{I}^\gamma \mathcal{S}_{b,s} = \mathcal{S}_{k,s+b\gamma}, \tag{7.24}$$

where $\mathcal{I}^\gamma$ denotes the fractional integral with respect to the variable x. From (7.24), in order to analyze different functions $\mathcal{Q}_{b,s}$, we need not compute their wavelet coefficients for different values of s, as a consequence of the following remarks. Recall that the fractional integral of order γ is the operator which is diagonal in the Fourier system of complex exponentials, and amounts to multiply $e^{i\pi nx}$ by $1/n^\gamma$. In a similar way, we note that the derivative of order n of a wavelet $\psi(2^j x - k)$ is a function of the form $2^{nj}\widetilde{\psi}(2^j x - k)$, which constitutes a system of functions that no longer forms an orthogonal basis of L^2 but share similar properties (in particular, they are Riesz basis of $L^2(\mathbb{R})$, see [136]. This remark motivates to consider the mapping $\widetilde{\mathcal{I}}^\gamma$ which is diagonal on a given wavelet basis, and consists in multiplying the wavelet $\psi_{j,k}$ by $2^{-\gamma j}$. This mapping is referred to as the *pseudo-fractional integral of order* γ, and it

has been shown to share the same regularity properties as the fractional integral, see [176]. The following statement summarises these results.

Proposition 7.14. *Let f be a trigonometric series. Then*

- *For every $\gamma \in \mathbb{R}$, the wavelet scaling functions of $\mathcal{I}^\gamma f$ and $\widetilde{\mathcal{I}}^\gamma f$ coincide.*
- *If $f \in L^p$, then for every $\gamma > 0$, the p-leader scaling function and the p-spectrum of $\mathcal{I}^\gamma f$ and $\widetilde{\mathcal{I}}^\gamma f$ coincide.*

Remark 7.15. Recalling (7.24), this means that, in order to analyze the functions $\mathcal{Q}_{b,s}$ for different values of s, one can compute these coefficients $c_{j,k}$ for one value of s, and then, when considering another function $\mathcal{Q}_{b,s'}$, replace in the computations of the various multifractality parameters (scaling functions, h_f^{min}, h_f^{max}, ...) the wavelet coefficients of $\mathcal{Q}_{b,s'}$ by the other ones obtained as the wavelet coefficients $c^\gamma_{j,k} = 2^{-\gamma j} c_{j,k}$, where $\gamma = (s'-s)/b$. Stated otherwise, as regards to such computations, everything happens as if the wavelets $\psi_{j,k}$ were eigenfunctions of the operator $\mathcal{I}^\gamma$ with eigenvalue $2^{-\gamma j}$.

If the pointwise singularities of f are of *cusp* type, then the multifractal spectrum of $\widetilde{I}^\gamma f$ is deduced from the one of f by a translation to the right of γ. However, the singularities of a function are not always of cusp type, but can be oscillating singularities, such as the chirps (7.7). If the function has oscillating singularities, after integration of order γ, the regularity exponent which corresponds to these singularities does not translate by γ but by $\gamma(1+\beta)$ where the β is characteristic of the oscillation, in accordance with (7.6).

7.3.4 Explicit computation of wavelet coefficients

We now discuss an important numerical issue concerning the advantages of computing the wavelet coefficients of trigonometric series using Meyer's wavelet (see [120] for its construction, and the precise formulas which define its Fourier transform). First, recall that it belongs to the Schwartz class and its Fourier transform is compactly supported; it follows that the wavelet coefficients of a trigonometric series $f(x) = \sum a_n e^{i\pi n x}$ can be directly derived from its Fourier coefficients as follows: Since

$$supp(\widehat{\psi}) = \left[-\frac{8\pi}{3}, -\frac{2\pi}{3},\right] \bigcup \left[\frac{2\pi}{3}, \frac{8\pi}{3},\right],$$

in order to determine the value of the $c_{j,k}$, the Fourier coefficients a_n of the series for $n \geq A_j := \frac{8\pi}{3} 2^j$ are not involved in the computation, and a straightforward computation yields

$$c_{j,k} = \sum_{|n|<A_j} a_n e^{i\pi n k 2^{-j}} \widehat{\psi}\left(-\frac{n\pi}{2^j}\right) \quad \text{where} \quad A_j = \frac{2^{j+3}}{3}. \tag{7.25}$$

Using this formula yields a shortcut in the procedure that would consist first of estimation the sum of the trigonometric series, and then computing its wavelet coefficients. Alternatively, this formula also shows that, if one prefers this procedure, then, one can stop the simulation of the function at the frequency A_j in order to compute the coefficients $c_{j,k}$. This remark is of particular importance when one deals with trigonometric series which are not locally

bounded, which is the case for $\mathcal{S}_{2,s}$ with $1/2 < s \leq 1$ as in Section 7.6. Indeed, in such cases, the Fourier series may diverge at the points of computation, and one cannot derive the wavelet coefficients of f from numerically unstable estimates of its pointwise values. In such situations, a way to estimate the wavelet coefficients in a numerically stable way consists in using directly (7.25); the other option being to use the remark following Proposition 7.14 which suggests to compute the wavelet coefficients of the function $\mathcal{S}_{2,s}$ for a large enough value of s, so that it is continuous, and then apply a pseudofractional derivation in order to perform the multifractal analysis in the cases where the function is no more bounded.

Another advantage of using Meyer's wavelet is the following: when using a wavelet of limited regularity, one has to assume that the pointwise regularity exponents met in the data are smaller than the regularity of the wavelet, so that the theorems of wavelet characterization of regularity are valid, see e.g. [84, 87]; this limitation usually is of no consequence in applications on real-life data the regularity of which being usually limited, but it could prove more delicate for mathematical functions for which we have no a priori knowledge of their pointwise regularity (note however that this remark does not apply here because the lacunarity of the frequencies of the trigonometric series we consider, which increase polynomially, imply a bound on the largest Hölder or p-exponent met in the function, see [94].

In the particular case of $\mathcal{R}_s$ (see (7.27) below) (7.25) boils down to

$$c_{j,k} = \sum_{n=1}^{\sqrt{A_j}} \frac{1}{n^s} \exp\left(-i\pi n^2 \frac{2k-1}{2^{j+1}}\right) \sqrt{F\left(\frac{\pi n^2}{2^j}\right)}. \tag{7.26}$$

where F is the function supported on $[-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]$ represented below (see [136] for the precise formula yielding $\widehat{\psi}$).

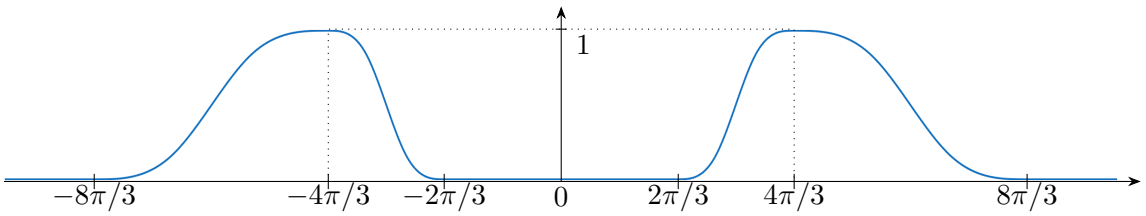


Figure 7.1: Function F used in the construction of the Meyer wavelet, and which leads to a compactly supported function $\widehat{\psi}$.

7.4 Theoretical results concerning the multifractal spectrum of the Riemann series

Recall if $b = 2$, then (7.4) boils down to

$$\mathcal{R}_s(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^s}, \tag{7.27}$$

which can be interpreted as a fractional integral or derivative of the seminal Riemann function (7.1). The following result is proved in [86].

Theorem 7.16. *The multifractal spectrum of $\mathcal{R}_s$ for $s > 1$ is*

$$\mathcal{D}_{\mathcal{R}_s}(h) = \begin{cases} 4h - 2s + 2 & \text{if } h \in [\frac{s-1}{2}, \frac{2s-1}{4}] \\ 0 & \text{if } h = s - \frac{1}{2} \\ -\infty & \text{else.} \end{cases} \tag{7.28}$$

For R_2, the point $h = s - \frac{1}{2} = 3/2$ corresponds to the differentiability points $\frac{2p+1}{2q+1}$ as proved by Gerver [70]. At such a point, the function behaves like a chirp, see Figure 7.2.

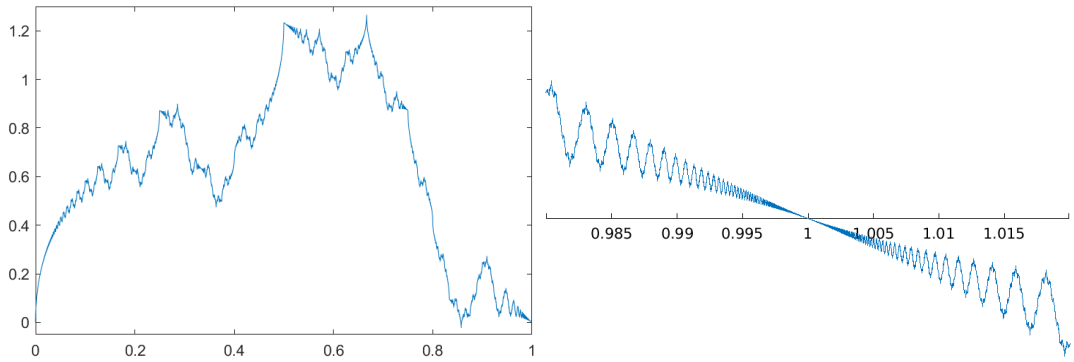


Figure 7.2: (Left) The graph of the original Riemann Fourier series $\mathcal{R}_2 = \mathcal{R}$ and (Right) a zoom around $x = 1$ where the function is differentiable and has a chirp expansion, which is typical of the behaviour near the rational points of the form $\frac{2p+1}{2q+1}$.

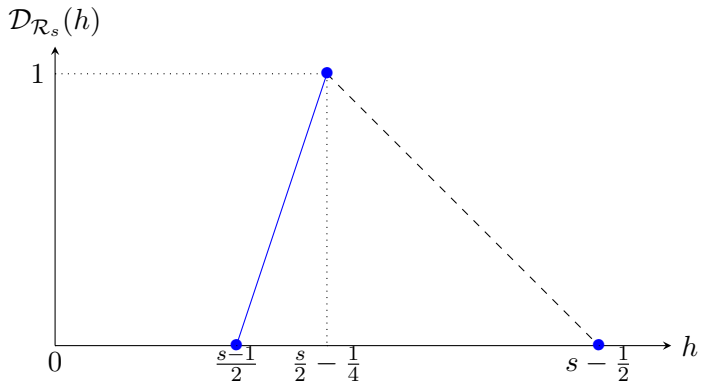


Figure 7.3: Multifractal spectrum of the Riemann Fourier series $\mathcal{R}_s$ for $s > 1$ in blue with its concave hull in dashed.

S. Seuret and A. Ubis in [160] established that the 2-spectrum of $\mathcal{R}_s$ for $s \in (1/2, 1]$ satisfies

$$\forall h \in \left[0, \frac{s}{2} - \frac{1}{4}\right], \qquad \mathcal{D}^{(2)}_{\mathcal{R}_s}(h) = 4h + 2 - 2s. \tag{7.29}$$

They conjectured that the 2-spectrum is $\mathcal{D}^{(2)}_{\mathcal{R}_s}(h) = 4h + 2 - 2s$ for $h \in [\frac{s}{2} - \frac{1}{2}, \frac{s}{2} - \frac{1}{4}]$. Their result leaves open the determination of the values of this 2-spectrum outside of the interval

$[0, \frac{s}{2} - \frac{1}{4}]$, and the values of the p-spectrum for $p \neq 2$. One of our purposes in this chapter is to establish precise conjectures based on the numerical determination of the Legendre p-spectrum in these cases.

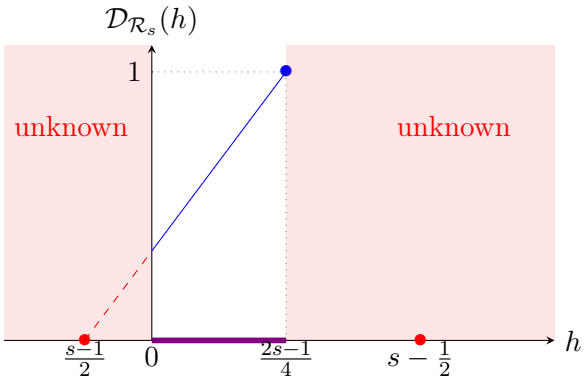


Figure 7.4: Partial 2-spectrum of the generalised Riemann's series $\mathcal{R}_s$ for $s \in \left(\frac{1}{2}, 1\right]$ with the established 2-spectrum as a blue line (with support in violet), the conjectured part is the red dashed line and and the spectrum is unknown outside of the interval $h \in [0, (2s-1)/4]$.

As a consequence of L^p estimates on partial sums of trigonometric series of frequencies n^2 obtained by Z. Zalcwasser in [178], the wavelet scaling function of $\mathcal{R}$ was shown to satisfy

$$\eta_{\mathcal{R}}(p) = \begin{cases} 3p/4 & \text{if } 0 < p \leq 4, \\ 1 + p/2 & \text{if } 4 \leq p, \end{cases} \tag{7.30}$$

see [86]. This implies in particular that the Riemann function satisfies the multifractal formalism as initially proposed by U. Frisch and G. Parisi in [66]: in this seminal article, the following notion of multifractal spectrum is introduced; let f be a bounded function; then

$$\widetilde{\mathcal{D}}_f(h) = dim_H(\{x \in \mathbb{T} : f \notin C^h(x)\}.$$

U. Frisch and G. Parisi derived the initial formulation of the multifractal formalism which can be stated as follows:

$$\widetilde{\mathcal{D}}_f(h) = \inf_{p>0}(hp - \eta_f(p) + 1), \tag{7.31}$$

where η_f is the the Kolmogorov scaling function function, which was shown to coincide with the wavelet scaling function, see [87]. It follows from [86] that the following (slight) variant of Theorem 7.16 also holds:

$$\widetilde{\mathcal{D}}_{\mathcal{R}_s}(h) = \begin{cases} 4h - 2s + 2 & \text{if } h \in [\frac{s-1}{2}, \frac{2s-1}{4}] \\ 1 & \text{if } h \geq \frac{2s-1}{4}; \end{cases} \tag{7.32}$$

taking (7.28) into account, we obtain that (7.31) holds for the functions $\mathcal{R}_s$ when $s > 1$, and in particular for the Riemann series $\mathcal{R}$.

By simulating $\mathcal{R}$ on $[0, 1]$ (whose graph is shown in Figure 7.2), we derive a numerical estimation of $\eta_{\mathcal{R}}(p)$, in Figure 7.5. The estimation is less accurate for large values of p, a limitation which has been noticed for many other examples.

For any $s \geq 0$, the wavelet scaling function of $\mathcal{R}_s$ can be deduced from the one of $\mathcal{R}$; it follows from (7.30) that

$$\eta_{\mathcal{R}_s}(p) = \begin{cases} p\left(\frac{s}{2} - \frac{1}{4}\right) & \text{if } 0 < p \leq 4, \\ 1 + p\left(\frac{s}{2} - \frac{1}{2}\right) & \text{if } 4 \leq p. \end{cases} \tag{7.33}$$

Therefore, in order to perform the multifractal analysis of $\mathcal{R}_s$, three cases appear:

- If $s > 1$, then the series is normally convergent, so that either the Hölder exponent or any p-exponent can be used,
- if $1/2 \leq s \leq 1$, then $\eta_{\mathcal{R}_s}(p) \geq 0$ for every $p < 2/(1-s)$, $\mathcal{R}_s \in L^p$ and the corresponding p-exponent can be used.
- if $0 \leq s \leq 1/2$, then $\eta_{\mathcal{R}_s}(p) \leq 0$ for every p, so that $\mathcal{R}_s$ does not belong to any L^p_{loc} and no p-exponent can be used.

In the second case, as suggested in [160], one can expect that the value of p yielding the largest value of the scaling function will give the numerically most precise results, thus leading to the choice $p = 4$ for the determination of the p-Legendre spectrum, see Figure 7.12 for a numerical illustration.

In the last case, a possibility consists in performing a multifractal analysis based on the *weak-scaling exponent*, a notion of pointwise regularity introduced by Yves Meyer in [137], and which is well defined in the very broad context of tempered distributions, see [2] for the numerical implementation of the corresponding Legendre spectrum.

The following results follows from Proposition 7.10.

Proposition 7.17. *Let $s > 1$. Then the scaling function of the Riemann functions $\mathcal{R}_s$ satisfies*

$$\text{if } p > \frac{1}{\frac{s}{2} - \frac{1}{4}} \quad \text{then} \quad \eta_{\mathcal{R}_s}(p) = \zeta_{\mathcal{R}_s}(p),$$

and

$$\forall h \leq \frac{2s-1}{4}, \qquad \mathcal{D}_{\mathcal{R}_s}(h) = \widetilde{\mathcal{D}}_{\mathcal{R}_s}(h).$$

Proof. The first result holds because it follows from (7.33) that the wavelet scaling function of $\mathcal{R}_s$ is larger than 1 for $p > \frac{1}{\frac{s}{2} - \frac{1}{4}}$, so that, as a consequence of Proposition 7.10, both scaling functions coincide in this range.

The second result holds because, in the increasing part of the Legendre spectrum, i.e. when the infimum in (7.14) (for $\mathcal{L}_f^{(+\infty)} = \mathcal{L}_f$) is reached for $q > 0$, it is actually reached for $q = 4$ and, in the range of values of s considered here, $\eta_{\mathcal{R}_s}(4) > 1$. □

7.5 Numerical estimation of the Legendre spectrum of $\mathcal{R}$

The series defining the Riemann function $\mathcal{R}$ is normally convergent, thus defining a continuous function, so that an approximation of $\mathcal{R}$ can be computed, and its wavelet leader scaling function $\eta_{\mathcal{R}}$ and its Legendre spectrum $\mathcal{L}_{\mathcal{R}}$ can be estimated. We use two types of numerical methods to estimate $\mathcal{L}_{\mathcal{R}}$:

(i) We compute an approximation of $\mathcal{R}$ numerically as $\mathcal{R}^{\text{approx}} := \sum_{n=1}^{700} \frac{\sin(\pi n^2 x)}{n^2}$ for 2 millions points in $[0,1]$ and then compute the wavelet coefficient using the Daubechies wavelet, with order at least 3, or the Meyer wavelet. From this, we compute its Legendre spectrum using Definition 7.9 through the MATLAB toolbox, see [121, 174, 176], available at: https://www.irit.fr/~Herwig.Wendt/software.html.

(ii) We compute the Legendre spectrum of $\mathcal{R}$ using the explicit formula (7.26) with the Meyer wavelet with sum for $n \in \{1, ..., 700\}$ so that the condition $n < \sqrt{A_j}$ is satisfied.

Figure 7.5 shows the estimation of the wavelet scaling function of $\mathcal{R}$ using the log-log plot regression formula (7.18).

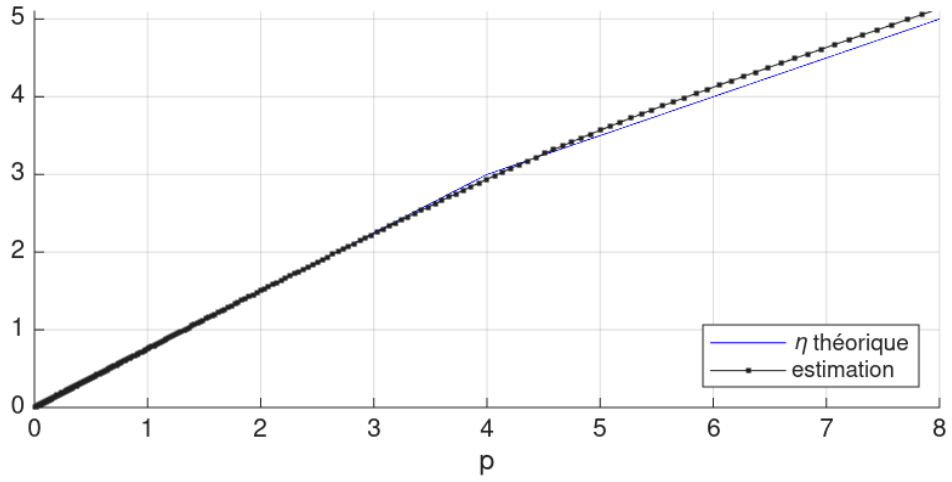


Figure 7.5: Estimated $\eta_{\mathcal{R}}$ in black with the theoretical function $\eta_{\mathcal{R}}$ in blue given in (7.30), where the *Daubechies wavelet 'db3'* is used.

Figure 7.6 shows the estimation of the Legendre spectrum $\mathcal{L}_{\mathcal{R}}$ first using the method (i) with the Daubechies wavelet 'db3' and the Meyer wavelet, then with the method (ii) using a direct computation of the wavelet coefficients. These numerical estimations are compared with the theoretical multifractal spectrum $\mathcal{D}_{\mathcal{R}}$ of (7.28). Both method yield a sharp estimation of the concave hull of $\mathcal{D}_{\mathcal{R}}$. However, the estimated $h_{\mathcal{R}}^{\max}$ does not coincïde with the theoretical maximal value of the multifractal spectrum at $3/2$ or the end of the estimated Legendre spectrum, overshooting both. This negative result questions the methods of estimation of the multifractality parameter $h_f^{\max}$.

Figures 7.7, 7.8 and 7.9 shows the estimations of $h_{\mathcal{R}}^{\min}$ and $c_1(\mathcal{R})$ in red in Figure 7.6 through a log-log regression such as (7.16) provided by the MATLAB toolbox.

7.5.1 Numerical estimation of the Legendre spectrum of $\mathcal{R}$ compared with the one of its fractionnal integration

We now turn to the functions $\mathcal{R}_s$ for $s > 1$ which are interpreted as fractional integrals or derivatives of $\mathcal{R}$, of order $\gamma = \frac{s}{2} - 1$. It follows from (7.28) that the increasing affine part

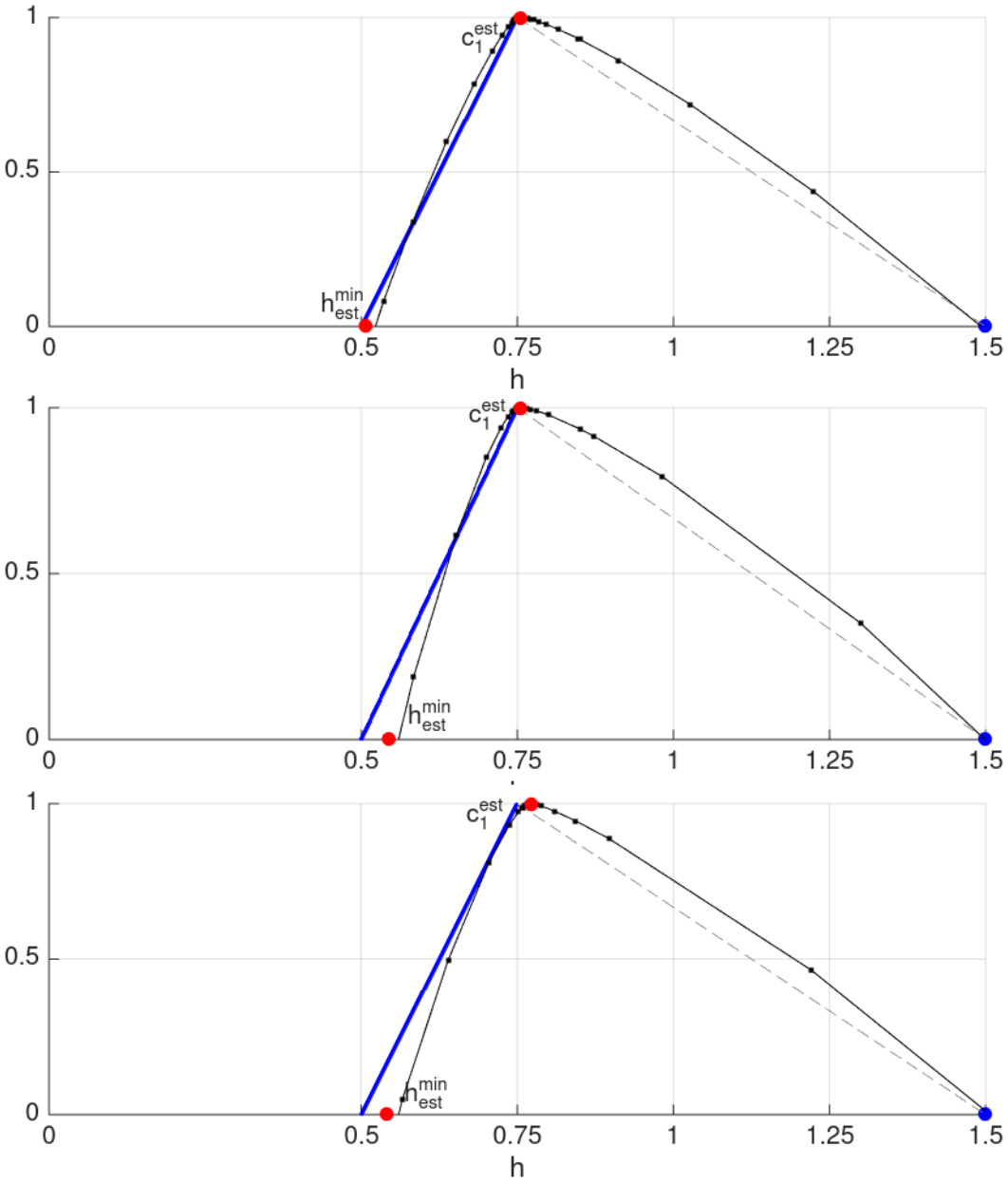


Figure 7.6: Estimated Legendre spectra of $\mathcal{R}$ in black compared with the theoretical spectrum of $\mathcal{R}$ in blue and its concave hull in dashed. The estimated $h_{\mathcal{R}}^{\min}$ and $c_1(\mathcal{R})$ are in red. (Up) The Legendre spectrum is estimated using the method (i) by computing the wavelet coefficients of $\mathcal{R}^{\text{approx}}$ with the Daubechies wavelet 'db3'. The estimated $h_{\mathcal{R}}^{\max} \approx 1.82$ are outside of the figure. (Middle) The Legendre spectrum is estimated using the method (i) by computing the wavelet coefficients of $\mathcal{R}^{\text{approx}}$ with the Meyer wavelet. The estimated $h_{\mathcal{R}}^{\max} \approx 1.85$ are outside of the figure. (Down) The Legendre spectrum is computed through the method (ii) with direct computation of the wavelet coefficients with the Meyer wavelet. The estimated $h_{\mathcal{R}}^{\max} \approx 1.70$ are outside of the figure.

of the multifractal spectrum of $\mathcal{I}^\gamma\mathcal{R}$ is the one of $\mathcal{R}$ shifted by γ so that

$$\mathcal{D}_{\mathcal{I}^\gamma\mathcal{R}}(h) = 4(h-\gamma) - 2 \quad \text{for } h \in [1/2+\gamma, 3/4+\gamma].$$

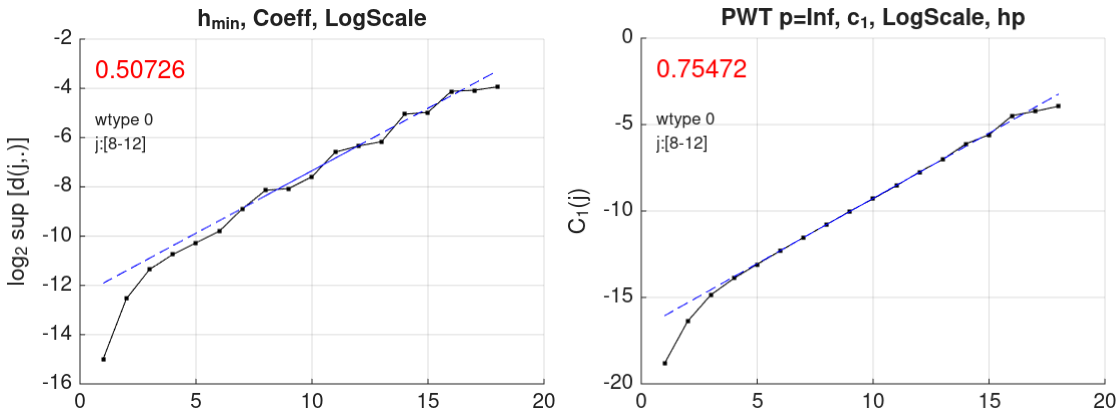


Figure 7.7: Estimated $h_{\mathcal{R}}^{\min}$ (left) and $c_1(\mathcal{R})$ (right) by log-log regressions using the method (i) by computing the wavelet coefficients of $\mathcal{R}^{\text{approx}}$ with the Daubechies wavelet 'db3'.

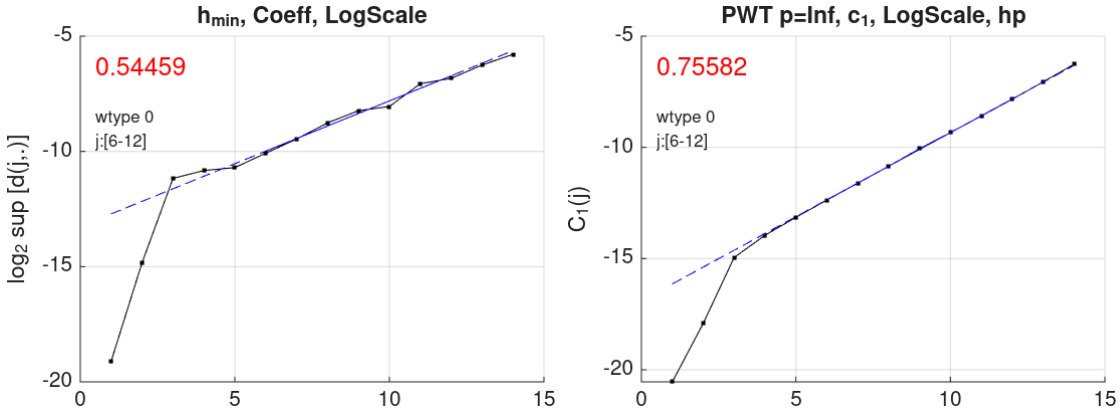


Figure 7.8: Estimated $h_{\mathcal{R}}^{\min}$ (left) and $c_1(\mathcal{R})$ (right) by log-log regressions using the method (i) by computing the wavelet coefficients of $\mathcal{R}^{\text{approx}}$ with the Meyer wavelet.

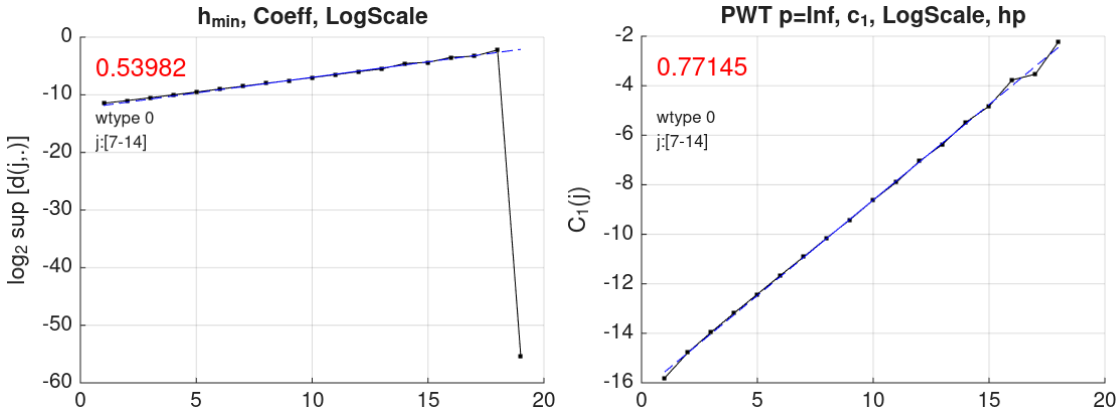


Figure 7.9: Estimated $h_{\mathcal{R}}^{\min}$ (left) and $c_1(\mathcal{R})$ (right) by log-log regressions computed through the method (ii) with direct computation of the wavelet coefficients with the Meyer wavelet.

However, the point at $h = 3/2$ in the multifractal spectrum of $\mathcal{R}$ representing the oscillating singularities at points $x = \frac{2p+1}{2q+1}$ is shifted by 2γ in $\mathcal{D}_{\mathcal{I}^\gamma \mathcal{R}}$ of $\mathcal{I}^\gamma \mathcal{R}$ and is at $h = 3/2 + 2\gamma$.

Apart from the exponent $h_f^{\min}$ which was already considered, the multifractality parameters that we will estimate have the following interpretation: $h_f^{\max}$ yields an upper bound for the largest value taken by the pointwise exponent and $c_1(f)$ (usually denoted by c_1 when there is no ambiguity on the function under study) represents the value of h for which the maximum of the spectrum is attained, and corresponds to the almost everywhere regularity.

In the case of the Riemann functions $\mathcal{R}_s$ for $s > 1$, the graph of $\mathcal{D}_{\mathcal{R}_s}(h)$ is composed of an affine function between $[h_{\mathcal{R}_s}^{\min}, c_1]$ and a point at $h_{\mathcal{R}_s}^{\max}$.

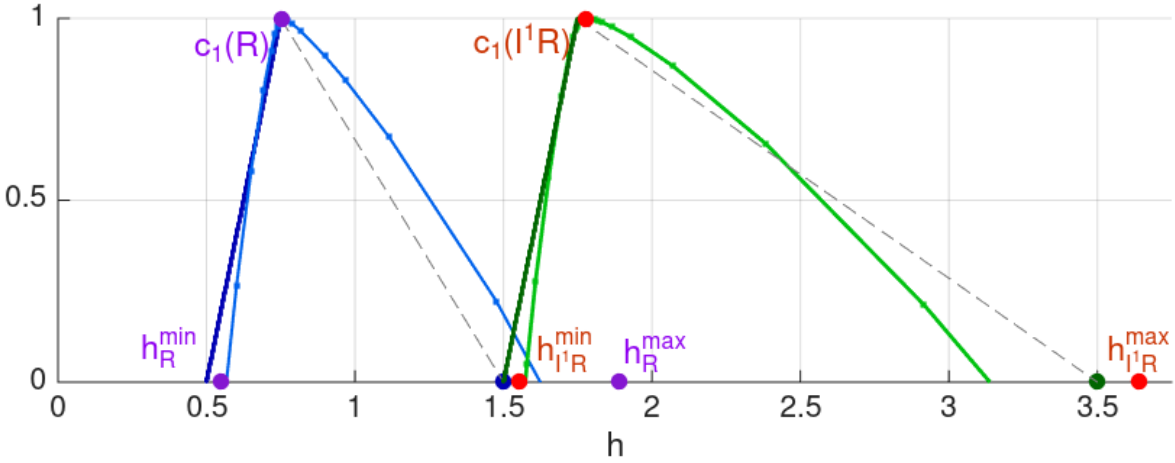


Figure 7.10: The spectra of $\mathcal{R}$ and its primitive are compared:
- Multifractal spectrum of $\mathcal{R}$ in blue,
- Legendre spectrum of $\mathcal{R}$ in light blue,
- Estimated $h_{\mathcal{R}}^{\min}$, $c_1(\mathcal{R})$ and $h_{\mathcal{R}}^{\max}$ in violet.
- Multifractal spectrum of $\mathcal{D}_{I^1\mathcal{R}}$ in dark green,
- Legendre spectrum of $\mathcal{I}^1\mathcal{R}$ in green.
- Estimated $h_{\mathcal{I}^1\mathcal{R}}^{\min}$, $c_1(\mathcal{I}^1\mathcal{R})$ and $h_{\mathcal{I}^1\mathcal{R}}^{\max}$ in red.

The Legendre spectra are estimated using the method (i) by computing the wavelet coefficients of an approximation of $\mathcal{R}$ with the fractional integration described in Section 7.2. The estimated $h_{\mathcal{R}}^{\max}$ and $h_{\mathcal{I}^1\mathcal{R}}^{\max}$ give a very poor estimate for the theoretical maximal value of the multifractal spectrum or the end of the estimated Legendre spectrum.

Figure 7.10 compares the estimated Legendre spectra of $\mathcal{R}$ and its primitive with their theoretical multifractal spectra. A first observation is that the decreasing part of $\mathcal{L}_{\mathcal{I}^1\mathcal{R}}$ is wider than the one of $\mathcal{L}_{\mathcal{R}}$ thus putting in evidence the presence of oscillating singularities. Figure 7.10 also shows that the two estimated $h_{\mathcal{R}}^{\max}$ and $h_{\mathcal{I}^1\mathcal{R}}^{\max}$ fail to estimate the theoretical maximal value of the multifractal spectrum or the end of the estimated Legendre spectrum.

Figure 7.11 explores this issue in more details, and shows the widening of the decreasing part of $\mathcal{L}_{\mathcal{I}^\gamma\mathcal{R}}$ inducing the detection of oscillating singularities. An important precision is that in Figure 7.11,the values of $h_{\mathcal{I}^\gamma\mathcal{R}}^{\max}$ correspond to the end of the Legendre spectrum estimated and not the $h_{\mathcal{I}^\gamma\mathcal{R}}^{\max}$ given by the MATLAB toolbox, as Figures 7.6 and 7.10 highlighted that the estimated $h_{\mathcal{I}^\gamma\mathcal{R}}^{\max}$ cannot be relied on.

The distance $h^{\min}$-c_1 remains constant, while the distance c_1-$h^{\max}$ increases linearly with the integration order, as expected when the highest regularity points are chirps. However, $h_{\mathcal{I}^\gamma\mathcal{R}}^{\max}$ increases at a rate 1.5γ while it should increase at rate 2γ. We conclude that such an analysis allows to put in evidence the presence of chirps in the data but is not sharp enough

to yield their oscillation exponent, thus only giving a qualitative result and not an quantitive estimation. This imprecise estimate of the regularity of the chirps may be due to the fact that, after a large order of fractional integration, the difference between their regularity and the almost everywhere regularity increases, making their detection harder and thus less precise.

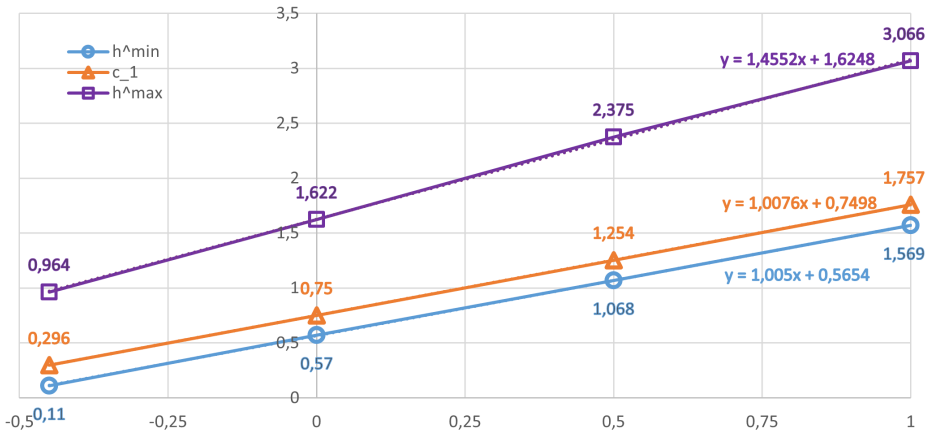


Figure 7.11: Characteristic values of $h^{\min}_{\mathcal{I}^\gamma\mathcal{R}}$, c_1 and $h^{\max}_{\mathcal{I}^\gamma\mathcal{R}}$ of estimated Legendre spectra of $\mathcal{I}^\gamma\mathcal{R}$ plotted with fractional integration parameters $\gamma \in \{-0.45, 0, 0.5, 1\}$. This analysis puts in evidence the widening of the decreasing part of $\mathcal{L}_{\mathcal{I}^\gamma\mathcal{R}}$ inducing the detection of oscillating singularities. On other hand, the lines of $h^{\min}$ and c_1 remain parallel, as expected, since they correspond to non-oscillating singularities. In this figure, the value for $h^{\max}_{\mathcal{I}^\gamma\mathcal{R}}$ corresponds to the end of the Legendre spectrum estimated and not the $h^{\max}_{\mathcal{I}^\gamma\mathcal{R}}$ given by the MATLAB toolbox.

7.6 Numerical estimate of the Legendre spectrum of $\mathcal{R}_s$

In this part, we consider the functions $\mathcal{R}_s$ defined in (7.27). The same analysis as the one for $\mathcal{R}$ can be performed for $\mathcal{R}_{0.7}$ but using p-exponents instead of Hölder exponents, since $\mathcal{R}_{0.7} \notin L^\infty_{loc}(\mathbb{R})$; here we will use wavelet coefficient estimated using (7.26). We will also discuss the presence of chirps which we expect to meet at the rational points of the form $\frac{2p+1}{2q+1}$. In practice, we consider the approximation

$$\mathcal{R}^{\text{approx}}_{0.7} := \sum_{n=1}^{1300} \frac{\sin(\pi n^2 x)}{n^{0.7}}$$

and will use the same numerical methods as in Section 7.5, with the slight modification in (ii) that $n \in \{1, ..., 1300\}$.

First, one can estimate $\eta_{\mathcal{R}_{0.7}}$ using (7.33) (which is numerically confirmed by Figure 7.12), and we conclude that one can use p-exponents for any $p \in (0, 6]$. Figure 7.13 shows the estimation of $\mathcal{L}_{\mathcal{R}_{0.7}}$ using :

- the method (i) with the Meyer wavelet,
- the method (ii) with direct computation of the wavelet coefficient,
- the theoretical multifractal spectrum $\mathcal{D}^{(2)}_{\mathcal{R}_{0.7}}$ of 7.29,

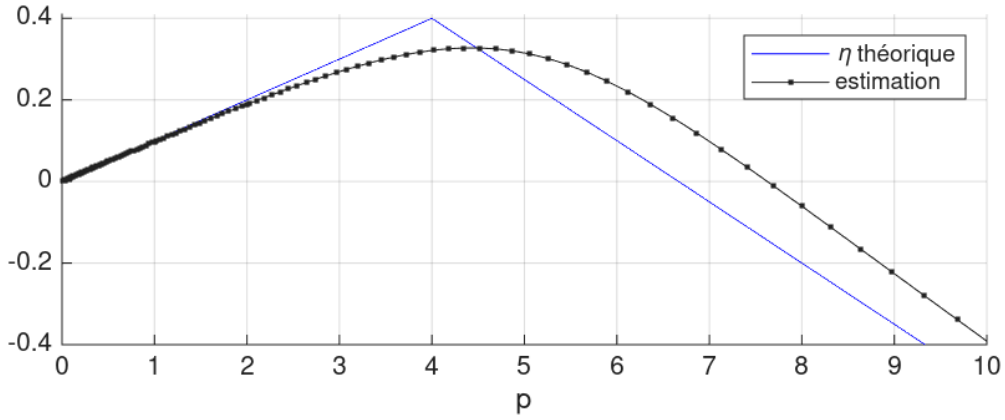


Figure 7.12: Estimated $\eta_{\mathcal{R}_{0.7}}$ in black with the theoretical function $\eta_{\mathcal{R}}$ in blue given in (7.33).

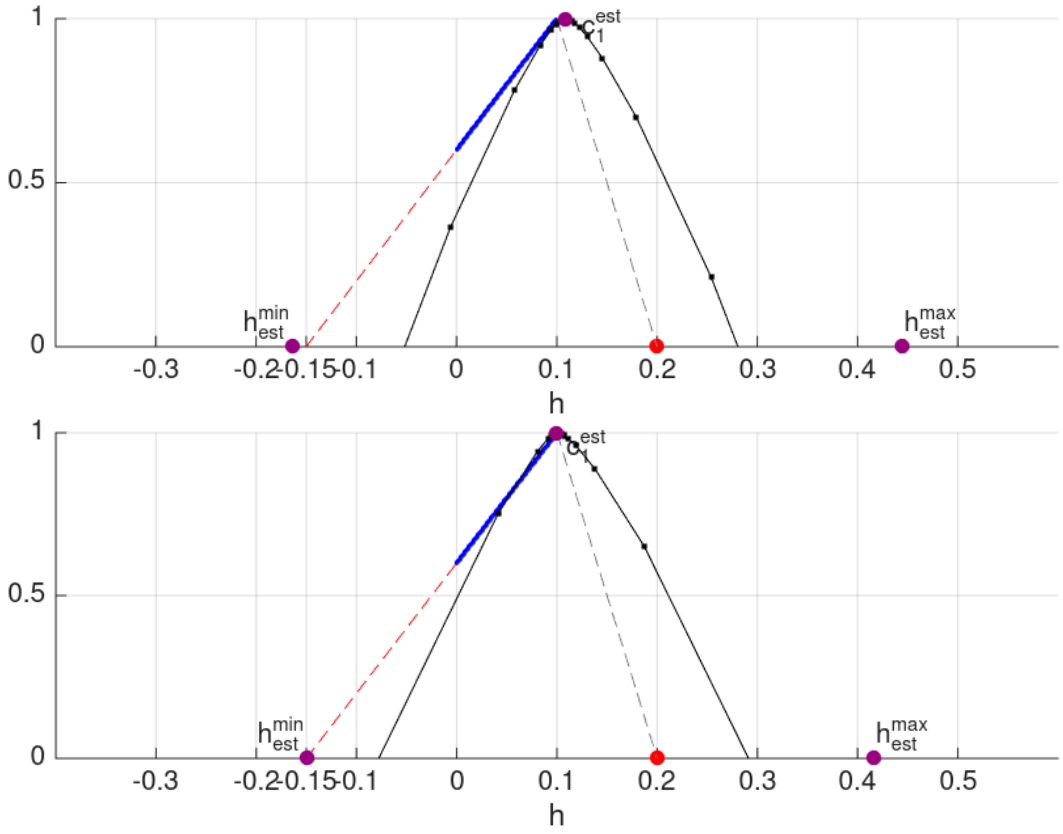


Figure 7.13: 2-Legendre spectrum of $\mathcal{R}_{0.7}$ in black compared with the multifractal 2-spectrum $\mathcal{D}^{(2)}_{\mathcal{R}_{0.7}}$ of (7.29) in blue with the additional conjecture in red of Figure 7.4. The estimated $h^{\min}_{\mathcal{R}}$ and $c_1(\mathcal{R})$ are in violet. (Up) The Legendre spectrum is estimated using the method (i) by computing the wavelet coefficients of $\mathcal{R}^{\text{approx}}_{0.7}$ with the Meyer wavelet. (Down) The Legendre spectrum is computed through the method (ii) by a direct computation of the wavelet coefficients with the Meyer wavelet. This method also allows to estimate $h^{\min}$ and gives $h^{\min} \approx -0.15$.

- the addition of the conjecture represented in Figure 7.4.

Figure 7.13 shows that the numerical estimates of $\mathcal{L}^{(2)}_{\mathcal{R}_{0.7}}$ through both method are in accordance with the part of the 2-spectrum $\mathcal{D}^{(2)}_{\mathcal{R}_{0.7}}$ of (7.29) determined by Ubis and Seuret, and they confirm the conjectures shown in Figure 7.4.

Another interesting issue is to estimate the p-Legendre spectra $\mathcal{L}^{(p)}_{R_{0.7}}$ for several values of p (it follows from (7.30) that one can use the p-exponents for p between 1 and 6).

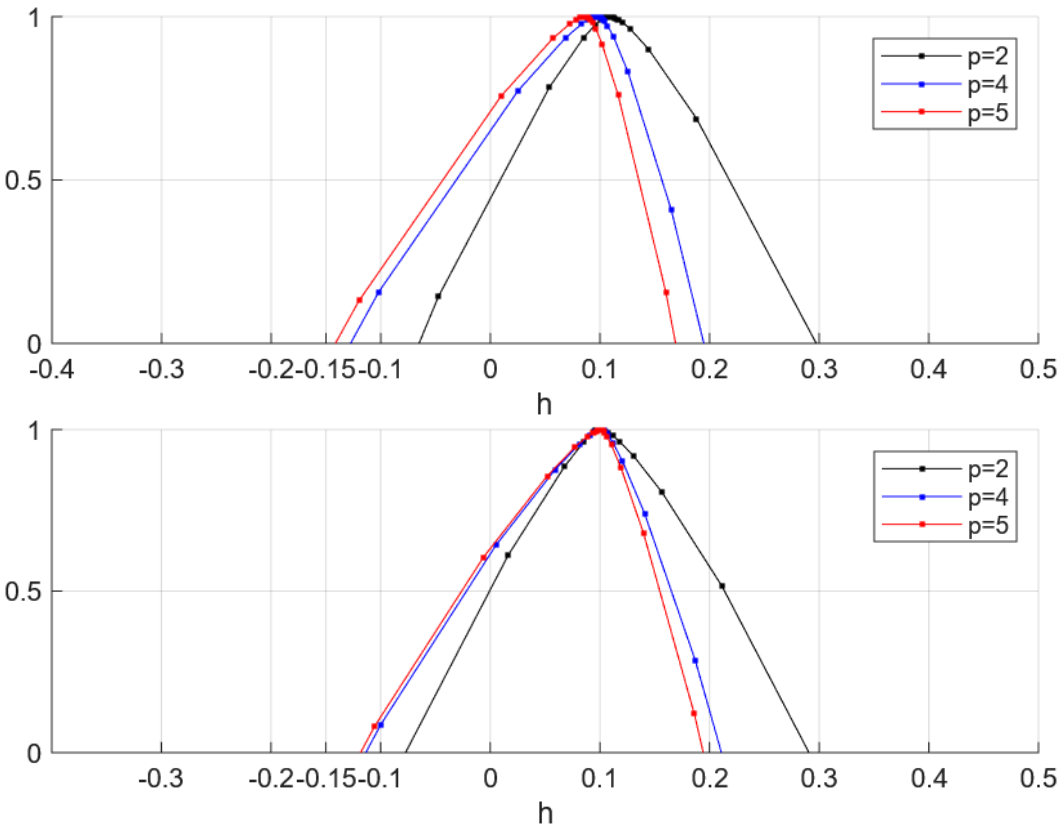


Figure 7.14: Estimated p-Legendre spectra $\mathcal{L}^{(p)}_{R_{0.7}}$ for multiple value of p, with $p = 2$ in black, $p = 4$ in blue and $p = 5$ in red. (Up) The Legendre spectrum is estimated using the method (i) by computing the wavelet coefficients of $\mathcal{R}^{\text{approx}}_{0.7}$ with the Meyer wavelet. (Down) The Legendre spectrum is computed through the method (ii) with direct computation of the wavelet coefficient using the Meyer wavelet.

Figure 7.14 represents estimated p-Legendre spectra of $\mathcal{R}_{0.7}$ for $p = 2, 4$ and 5. They are close, and their slight differences can be explained following the rule of thumb indicated above: $\eta_{\mathcal{R}_{0.7}}(4)$ and $\eta_{\mathcal{R}_{0.7}}(5)$ are larger than $\eta_{\mathcal{R}_{0.7}}(2)$ so that the 4-spectrum and 5-spectrum are better estimated. Recall also that we are only interested here in the weak formulation of the multifractal formalism: all these estimates correspond to a concave hull of the theoretical $\mathcal{D}^{(2)}_{\mathcal{R}_{0.7}}$ (partly shown in Figure 7.4). These results can be interpreted as giving a more precise information concerning the nature of the pointwise singularities, see [4].

7.7 Numerical estimate of the Legendre spectrum of $\mathcal{Q}_{b,s}$ for $b = 3, 4$

We now consider the functions $\mathcal{Q}_{b,s}$ defined in (7.4) In that case, the exact multifractal spectra are not known; in [42], Chamizo and Ubis proved the following bounds:

$$\forall h \in \left[\frac{s-1}{b}, \frac{s}{b} - \frac{1}{2b}\right), \qquad (b+1)(h-(s-1)/b) \leq \mathcal{D}_{\mathcal{Q}_{b,s}}(h) \leq \omega(h-(s-1)/b)$$

$$\text{with} \quad \omega(x) = \begin{cases} \frac{2x}{2^{-b}+x} & \text{if } 0 \leq x < \frac{1}{b2^b}, \\ \\ \frac{3}{2} - \sqrt{\frac{b+4}{4b} - 2x} & \text{if } \frac{1}{b2^b} \leq x < \frac{1}{2b}. \end{cases} \tag{7.34}$$

It follows from Theorem 7.2 and Remark 7.3 that

$$h^{\min}_{\mathcal{Q}_{b,s}} = \frac{s-1}{b}.$$

There are no known result outside of the interval $[\frac{s-1}{b}, \frac{s}{b} - \frac{1}{2b})$. When $s = b$, the point at $\frac{2b-1}{2b-2}$ is also known from Theorem 7.2.

Nonetheless, let us furthermore assume that the chirps still yield the most regular points; since the regularity of a chirps is increased by $(1+\beta)\gamma$ under a fractional integral of order γ (where β is the oscillation exponent of the chirp), then we can conjecture that

$$h^{\max}_{\mathcal{Q}_{b,s}} = \frac{2s-1}{2b-2}.$$

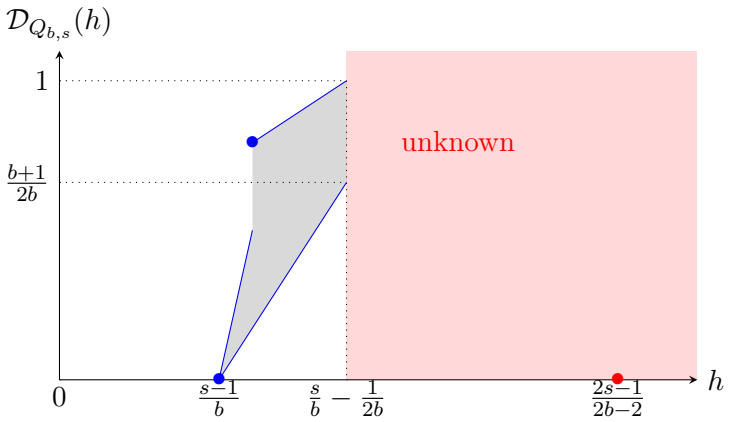


Figure 7.15: Theoretical bounds of the spectrum of $\mathcal{Q}_{b,s}(x)$ with $b \geq 3$ and $1 + b/2 < s < b$ from [42] and [28, 150]; (7.34) shows that while the upper bound seems piecewise affine, it is not the case.

7.7.1 Multifractal Legendre spectrum of $\mathcal{Q}_{3,s}$

Figures 7.16 and 7.17 show the estimation of $\mathcal{L}_{\mathcal{Q}_{3,s}}$ for $s = 2.75$ and $s = 3$ using the same methods as in Section 7.5. The numerical results display:

- the method (i) with the Daubechies wavelet of order 8 'db8',

- the method (ii) with direct computation of the wavelet coefficient.

We also show on the same pictures the theoretical $h^{\min}_{\mathcal{Q}_{3,s}}$ in blue, the bounds of the increasing part of the multifractal spectrum $\mathcal{D}_{\mathcal{Q}_{3,s}}$ from [42] in blue and green, and the conjectured value for $h^{\max}_{\mathcal{Q}_{3,s}}$. We observe that the estimated Legendre spectra seem to correspond to a spectra in two affine part represented in violet. We conjecture that the multifractal spectrum of $\mathcal{Q}_{3,s}$ in violet is

$$\mathcal{D}_{\mathcal{Q}_{3,s}}(h) = \begin{cases} 6h - 2s + 2 & \text{if } h \in [\frac{s-1}{3}, \frac{2s-1}{6}] \\ 0 & \text{if } h = \frac{2s-1}{4} \\ -\infty & \text{else,} \end{cases} \tag{7.35}$$

simply obtained as an affine part from 0 at $\frac{s-1}{3}$ to 1 at $\frac{2s-1}{6}$ and the point at $\frac{2s-1}{4}$ similarly to the Riemann functions $\mathcal{R}_s$ of Section 7.4.

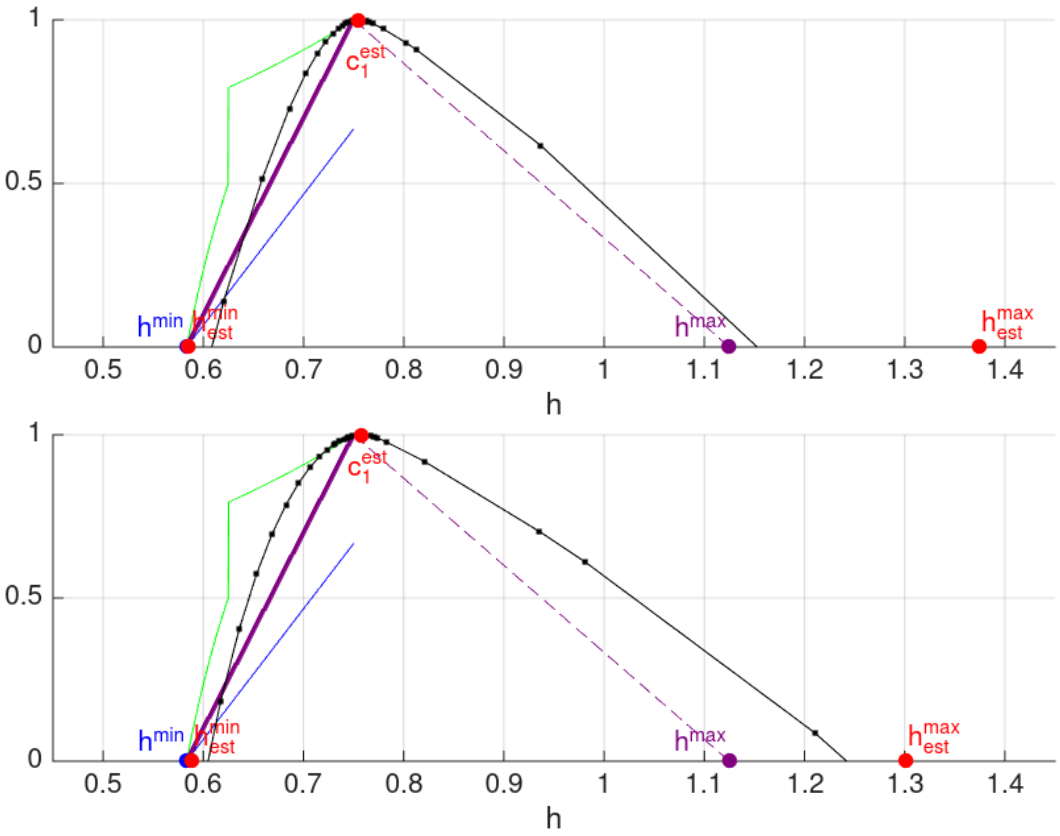


Figure 7.16: Estimated Legendre spectra of $\mathcal{Q}_{3,2.75}$ in black with estimated $h^{\min}_{\mathcal{Q}_{3,2.75}}$ and $h^{\max}_{\mathcal{Q}_{3,2.75}}$ in red. The upper bound of the theoretical spectrum is in green, the lower bound and theoretical $h^{\min}_{\mathcal{Q}_{3,2.75}}$ are in blue and the conjectured $h^{\max}_{\mathcal{Q}_{3,2.75}}$ and mutlifractal spectrum are in violet. (Up) The Legendre spectrum is estimated using the method (i) of approximating the graph of $\mathcal{Q}_{3,2.75}$ and computing its wavelet coefficients. We find that $h^{\min} \approx 0,585$. (Down) The Legendre spectrum is computed through the method (ii) with direct computation of the wavelet coefficient with the Meyer wavelet. We find that $h^{\min} \approx 0,588$.

As expected, the estimated Legendre spectrum lies in the bounds found by Chamizo-Ubis. Further, for both methods, the estimates of $h^{\min}_{\mathcal{Q}_{3,s}}$ are close to the theoretical value

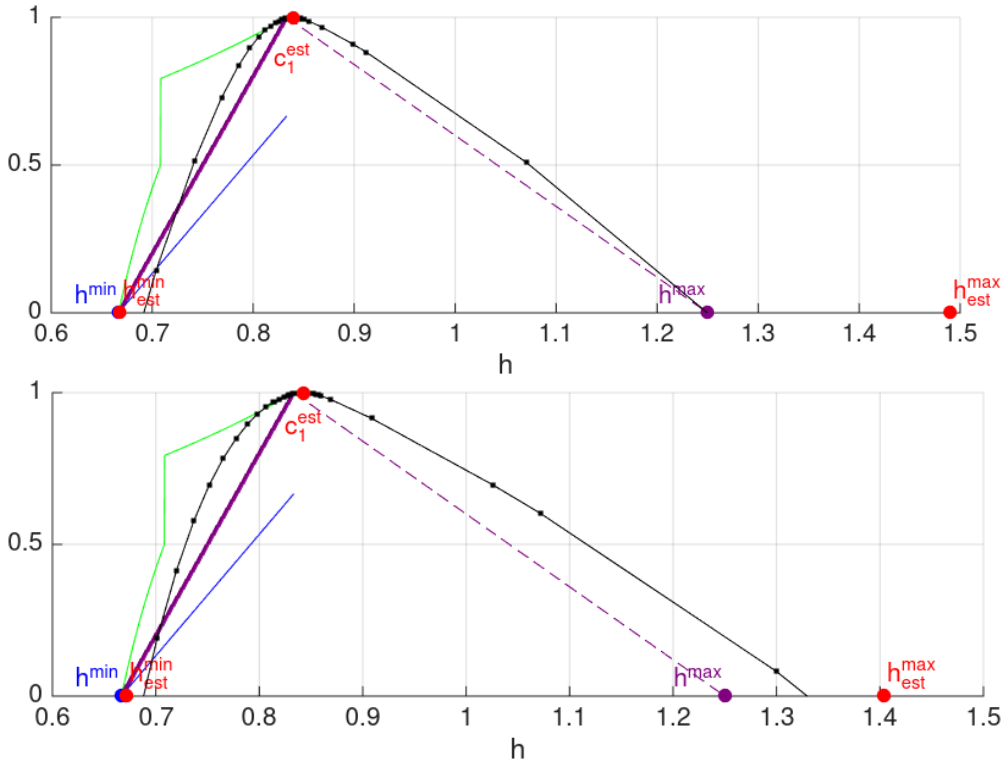


Figure 7.17: Estimated Legendre spectra of $\mathcal{Q}_{3,3}$ in black with estimated $h^{\min}_{\mathcal{Q}_{3,3}}$ and $h^{\max}_{\mathcal{Q}_{3,3}}$ in red. The upper bound of the theoretical spectrum is in green, the lower bound and theoretical $h^{\min}_{\mathcal{Q}_{3,3}}$ are in blue and the conjectured $h^{\max}_{\mathcal{Q}_{3,3}}$ and mutlifractal spectrum are in violet. (Up) The Legendre spectrum is estimated using the method (i) of approximating the graph of $\mathcal{Q}_{3,3}$ and computing its wavelet coefficients. We have that $h^{\min} \approx 0,668$. (Down) The Legendre spectrum is computed through the method (ii) with direct computation of the wavelet coefficient with the Meyer wavelet. We have that $h^{\min} \approx 0,672$.

and almost coincides with the beginning of the support of the estimated Legendre spectrum; nonetheless, the estimate of $h^{\max}_{\mathcal{Q}_{3,s}}$ is not close to either the theoretical value of $h^{\max}_{\mathcal{Q}_{3,s}}$ or the end of the support of the Legendre spectrum. This function therefore yields another example where the the numerical estimate of the exponent $h^{\max}_{\mathcal{Q}_{3,s}}$ cannot be relied on.

7.7.2 Multifractal Legendre spectrum of $\mathcal{Q}_{4,s}$

Similarly to the numerical study of $\mathcal{Q}_{3,s}$, Figures 7.18, 7.19 and 7.20 show the estimation of $\mathcal{L}_{\mathcal{Q}_{4,s}}$ for $s = 3$, $s = 3.5$ and $s = 4$ using :

- the method (i) worked out with the Daubechies wavelet of order 8 'db8',
- the method (ii) worked out with a direct computation of the wavelet coefficient.

In addition, we also show:

- the theoretical $h^{\min}_{\mathcal{Q}_{4,s}}$ and the bounds of the increasing part of the multifractal spectrum $\mathcal{D}_{\mathcal{Q}_{4,s}}$ from [42]
- the conjectured value of $h^{\max}_{\mathcal{Q}_{4,s}}$.

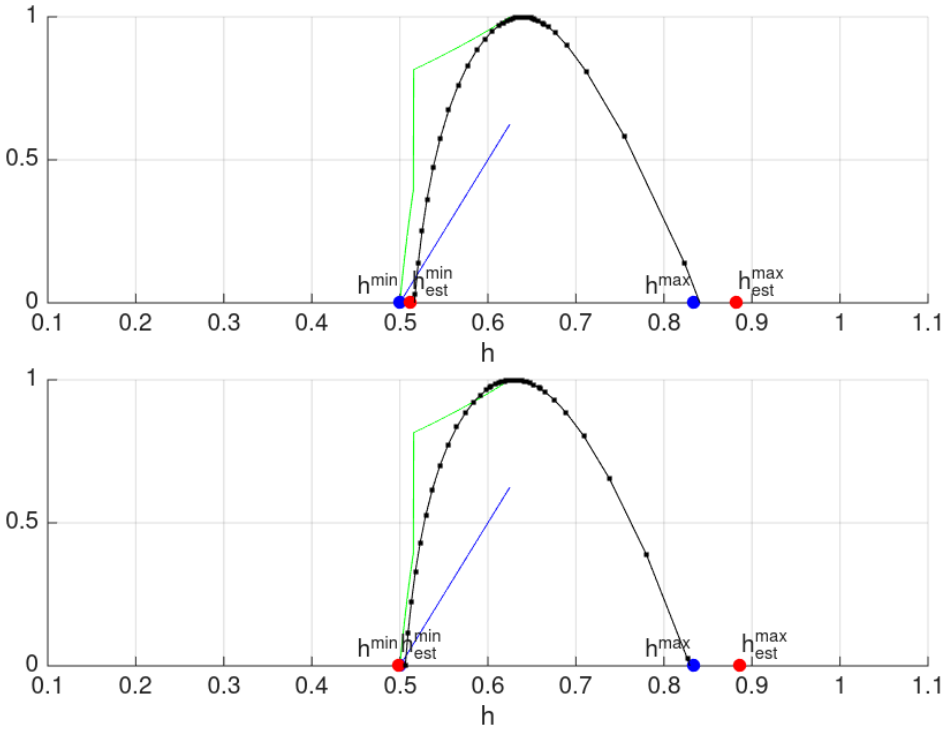


Figure 7.18: Estimated Legendre spectrum of $\mathcal{Q}_{4,3}$ in black with estimated $h^{\min}_{\mathcal{Q}_{4,3}}$ and $h^{\max}_{\mathcal{Q}_{4,3}}$ in red. The upper bound of theoretical spectra is in green and lower bound and theoretical $h^{\min}_{\mathcal{Q}_{4,3}}$ and $h^{\max}_{\mathcal{Q}_{4,3}}$ are in blue. (Up) The Legendre spectrum is estimated using the method (i) of approximating the graph of $\mathcal{Q}_{4,3}$ and computing its wavelet coefficients. We find that $h^{\min} \approx 0,511$. (Down) The Legendre spectrum is computed through the method (ii) with direct computation of the wavelet coefficient with the Meyer wavelet. We find that $h^{\min} \approx 0,498$. Note that the estimated spectrum falls inside the bounds of [42].

In these figures, both methods provide an estimate of the Legendre spectrum where the increasing part lies between the theoretical bounds of Chamizo-Ubis and suggests that neither of these bounds is optimal. The decreasing part of the estimated Legendre spectrum however fails to catch the conjectured $h^{\max}_{\mathcal{Q}_{4,s}}$.

Similarly to $\mathcal{Q}_{3,s}$, the estimate of $h^{\min}_{\mathcal{Q}_{4,s}}$ corresponds to the theoretical value and the beginning of the support of the Legendre spectrum, and the estimate of $h^{\max}_{\mathcal{Q}_{4,s}}$ lies far away from the end of the Legendre spectrum and seems, once again, to yields a very poor estimate of the conjectured $h^{\max}_{\mathcal{Q}_{4,s}}$.

7.8 Multifractal analysis of the Schrödinger series $\mathcal{S}_{2,2}$

We now consider the function $\mathcal{S}_{2,2}(\cdot,t)$, which is a function that comes up in explicit solutions of PDEs and in fluid mechanics, in the dynamics of the binormal flow [14],

$$\mathcal{S}_{2,2}(x,t)=\sum \frac{e^{i\pi(n^2x+nt)}}{n^2},$$

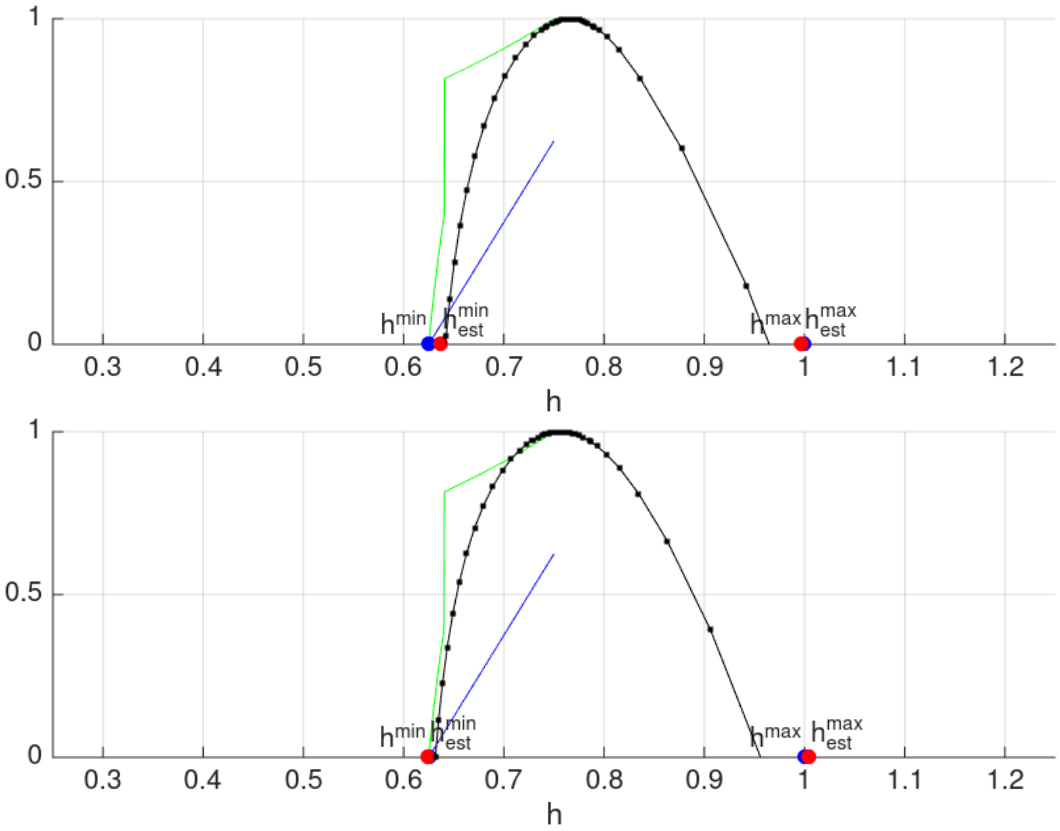


Figure 7.19: Estimated Legendre spectra of $\mathcal{Q}_{4,3.5}$ in black with estimated $h^{\min}_{\mathcal{Q}_{4,3.5}}$ and $h^{\max}_{\mathcal{Q}_{4,3.5}}$ in red. The upper bound of theoretical spectrum is in green and lower bound and theoretical $h^{\min}_{\mathcal{Q}_{4,3.5}}$ and $h^{\max}_{\mathcal{Q}_{4,3.5}}$ are in blue. (Up) The Legendre spectrum is estimated using the method (i) of approximating the graph of $\mathcal{Q}_{4,3.5}$ and computing its wavelet coefficients. We have that $h^{\min} \approx 0,637$. (Down) The Legendre spectrum is computed through the method (ii) with direct computation of the wavelet coefficient with the Meyer wavelet. We have that $h^{\min} \approx 0,624$. Note that the estimated spectrum falls inside the bounds of [42].

with a nonvanshing phase t; these functions not only appear as solutions of PDEs such as Schrödinger's equation, but also in fluid mechanics in the dynamics of the binormal flow [14]. We can consider this function as a variant of the Riemann function $\mathcal{R}$ where a phase $e^{i\pi nt}$ is added.

Figures 7.22, 7.24 and 7.26 show that, if the phase is either a rational number or an irrational number "badly approximable by rationals" such as $\sqrt{2}$, the estimated Legendre spectra of $\mathcal{S}_{2,2}$ are close to the Legendre spectrum of $\mathcal{R}$ estimated in Figure 7.6. Therefore, one can conjecture from these observations that the multifractal spectrum of $\mathcal{S}_{2,2}$ is the same one as for $\mathcal{R}$ and the addition of a phase does not impact the multifractality of the function. These observations confirm the preliminary theoretical results of Carmel Aboua (personal communication) in the case where the phase is rational, see also [14].

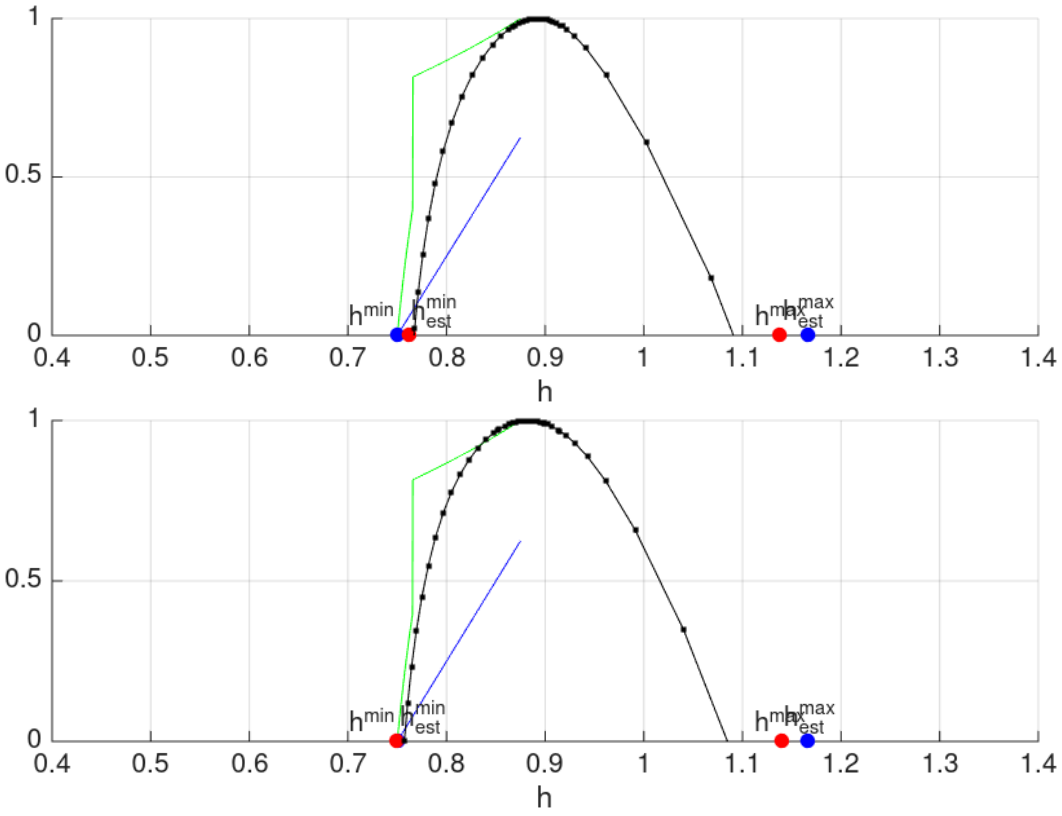


Figure 7.20: Estimated Legendre spectra of $\mathcal{Q}_{4,4}$ in black with estimated $h^{\min}_{\mathcal{Q}_{4,4}}$ and $h^{\max}_{\mathcal{Q}_{4,4}}$ in red. The upper bound of theoretical spectrum is in green and lower bound and theoretical $h^{\min}_{\mathcal{Q}_{4,4}}$ and $h^{\max}_{\mathcal{Q}_{4,4}}$ are in blue. (Up) The Legendre spectrum is estimated using the method (i) of approximating the graph of $\mathcal{Q}_{4,4}$ and computing its wavelet coefficients. We have that $h^{\min} \approx 0,762$. (Down) The Legendre spectrum is computed through the method (ii) with direct computation of the wavelet coefficient with the Meyer wavelet. We have that $h^{\min} \approx 0,750$. Note that the estimated spectrum falls inside the bounds of [42].

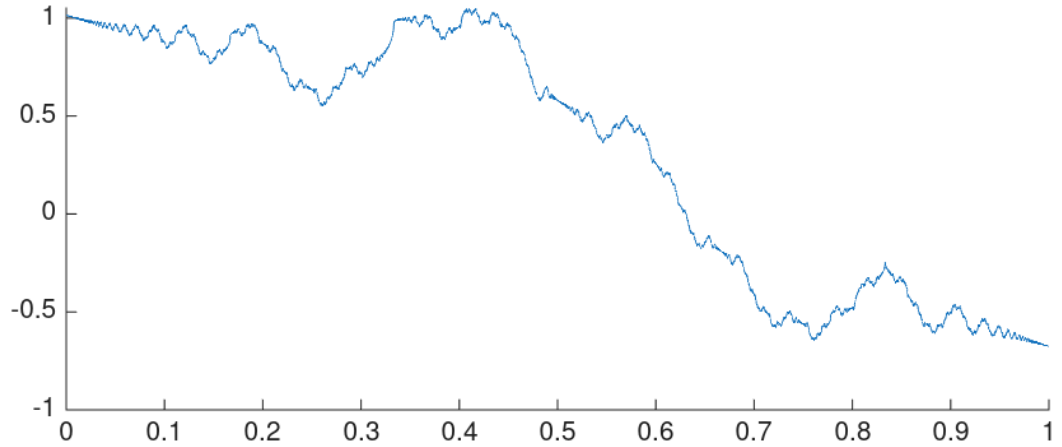


Figure 7.21: Graph of $\mathcal{S}_{2,2}(\cdot, 1/3)$.

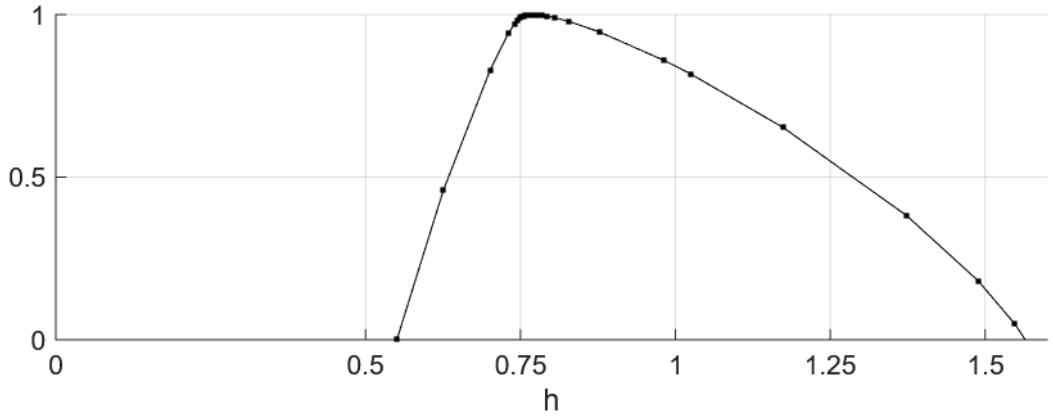


Figure 7.22: Legendre spectrum of $\mathcal{S}_{2,2}(\cdot, 1/3)$. The spectrum is estimated from the approximation of the sum of the 700 first term from which we compute its wavelet coefficient with the db3.

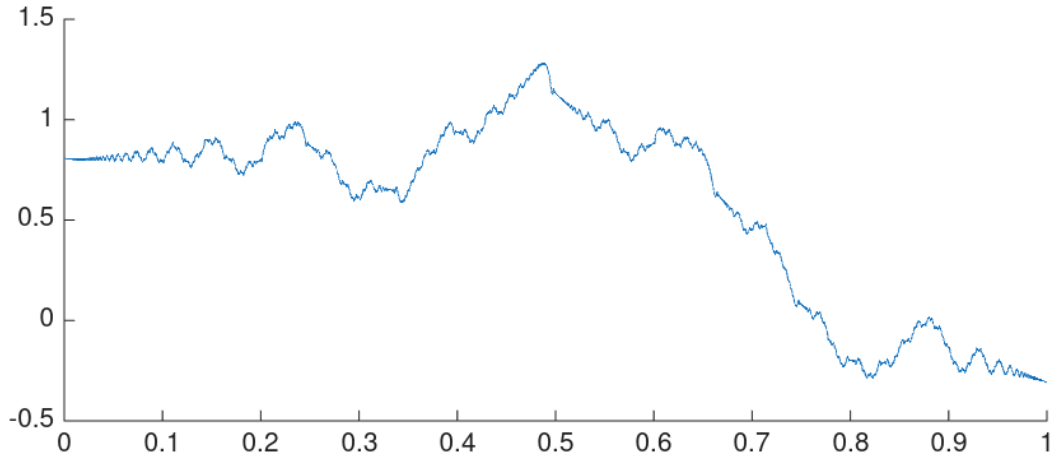


Figure 7.23: Graph of $\mathcal{S}_{2,2}(\cdot, 1/7)$.

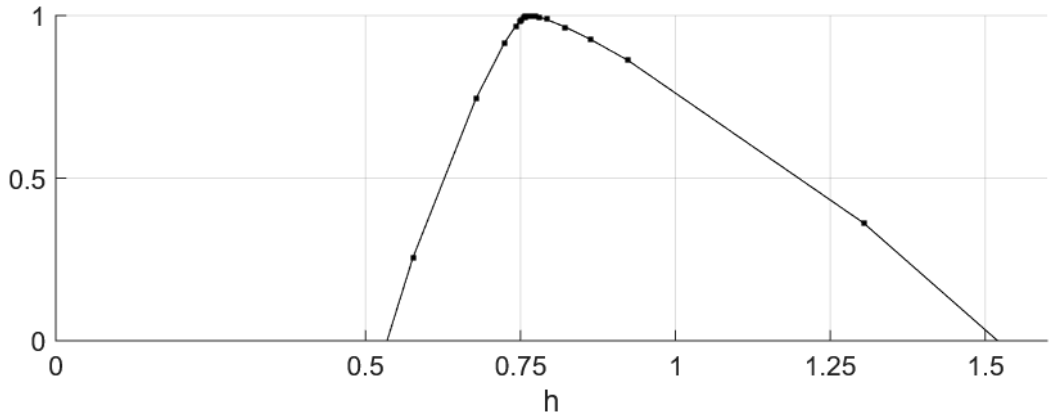


Figure 7.24: Legendre spectrum of $\mathcal{S}_{2,2}(\cdot, 1/7)$. The spectrum is estimated from the approximation of the sum of the 700 first term from which we compute its wavelet coefficient with the db3.

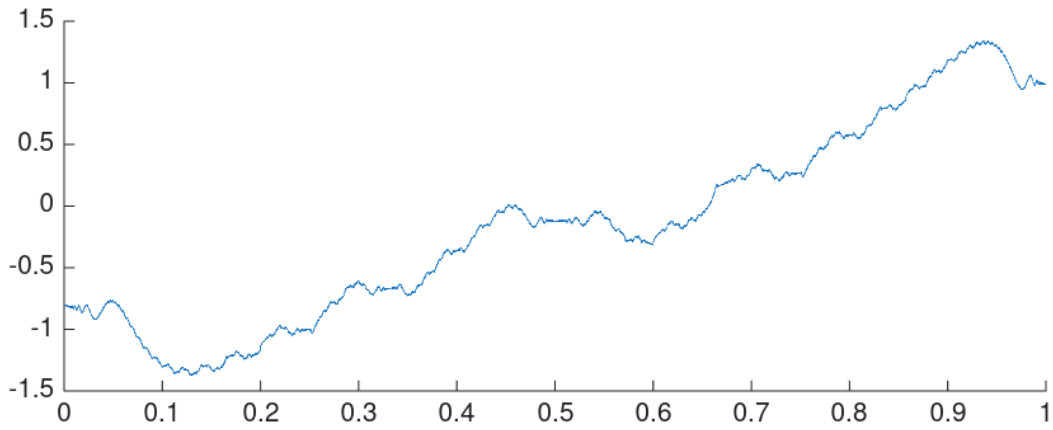


Figure 7.25: Graph of $\mathcal{S}_{2,2}(\cdot, \sqrt{2})$.

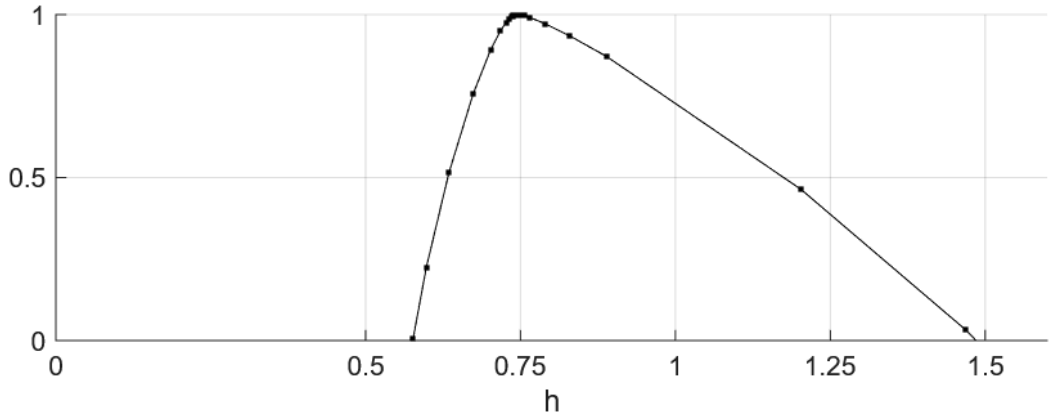


Figure 7.26: Legendre spectrum of $\mathcal{S}_{2,2}(\cdot, \sqrt{2})$. The spectrum is estimated from the approximation of the sum of the 700 first term from which we compute its wavelet coefficient with the db3.

7.9 Conclusion

In this chapter, we have applied the classical wavelet-based multifractal analysis techniques in order to numerically estimate various Legendre spectra of trigonometric series. First, we did it on function for which the multifractal spectrum is either completely or partially known. This allowed us to verify that, in these cases, the numerical results are close to the true theoretical values, which validates our approach. The second issue was to estimate Legendre spectra in cases where upper and lower bounds are known, such as the Queffelec series (7.4).

Concerning the first issue, Figure 7.6 showed the bad estimation of $h_{\mathcal{R}}^{\max}$. An important issue is to determine if this is a specific problem which might be due to the particular nature of these functions (such as the presence of chirps at the points with the largest Hölder exponent); it would therefore be interesting to verify if this numerical overshoothing is confirmed for other types of mathematical functions or for classical models in signal processing. If it is the case,

it might lead to question the way that this exponent is estimated. These results concerning the exponent $h_f^{\max}$ are in sharp contrast with those observed for $h_{\mathcal{Q}_{b,s}}^{\min}$, which coincide with the begining of the Legendre and mutlifractal spectrum of $\mathcal{Q}_{b,s}$.

Through our use of the numerical methods, we saw that current method of computation of the $h_f^{\max}$ does not provide a reliant value of either the end of the estimated Legendre spectrum or a maximal regularity exponent.

Concerning the second issue, we can draw several conclusions from our results:

- the numerical values obtained fall indeed between these bounds as expected,
- they do not seem close either to the upper nor the lower bound derived in [42], suggesting that these bounds are not sharp;
- for $\mathcal{Q}_{3,s}$, the Legendre spectrum obtained seems piecewise affine similarly to the one of the Riemann functions; this allows to conjecture that the multifractality of $\mathcal{Q}_{3,s}$ is obtained by a mechanism similar to the one shown in the case of the Riemann function, namely that the Hölder exponent is derived form Diophantine approximation conditions;
- for $\mathcal{Q}_{4,s}$, the bell-shaped spectrum obtained strongly contrast with the one obtained for the Riemann functions; and we can consider that this is not a numerical artefact, since the Legendre spectrum obtained for the Riemman functions are close to yield the characteristic piecewise affine shape which is expected; these bell-shaped spectrum suggest that the pointwise regularity of these functions are not related in a simple way with the Diophantine exponent (in which cases the spectrum obtained are affine, as shown by the Riemann functions).

Note that our conclusions are of qualitative type, and, except in the case of $\mathcal{Q}_{3,s}$, we do not pretend to conjecture precise values for the spectrum of the Queffelec functions in all generality. Indeed, spectra are estimated with some error, as shown by the results obtained for the Riemann series. The case of $\mathcal{Q}_{3,s}$ is slightly different, since the conjecture can be backed by several partial theoretical results.

As regards the Schrödinger functions $\mathcal{S}_{2,2}$, we already mentioned that the multifractal spectrum do not seem to depend on the value of t which is chosen, a conjecture which is backed by several theoretical results in the case where t is a rational number, and which seems to hold for other values of t.

We have also shown that multifractal methods allow to put in evidence the existence of chirps in the data, which is made clear by comparing the multifractal spectrum of a function f and its primitive. This technique could have consequences in the fields where the existence of chirps in the data is anticipated by some models, and contradicted by others: The present chapter shows that this method could allow to validate these models.

Finally, we address a question which finds a particular relevance in the case of the Riemann functions: their Legendre spectrum is composed of:

- an affine increasing part which coincides with the multifractal spectrum,
- an affine decreasing part which is the concave hull on its two extreme points.

Quantile methods were used in order to draw differences between these two situations: They

are taylored to estimate the *quantile spectrum* $\mathcal{D}^q(h)$ which can be (loosely) defined as follows: Let $d_{j,k}$ be a multiscale quantity corresponding to a pointwise exponent $h_f(x)$. Then

$$\mathrm{Card}(\{k \in \{0, ..., 2^j - 1\} : d_{j,k} \sim 2^{-hj}\}) \sim 2^{\mathcal{D}^q(h)j},$$

(see [5] for a precise definition) which yields a sharper upper estimates of the multifractal spectrum than the one supplied by the Legendre spectrum. The quantile spectrum of the Riemann functions are, see [5],

$$\mathcal{D}^q_{\mathcal{R}_s}(h) = \begin{cases} 4h - 2s + 2 & \text{if } h \in [\frac{s-1}{2}, \frac{2s-1}{4}] \\ 0 & \text{if } h \in [\frac{2s-1}{4}, s - \frac{1}{2}]; \\ -\infty & \text{if } h > s - \frac{1}{2}; \end{cases} \tag{7.36}$$

Figure 7.27 puts in light an important difference between the increasing and the decreasing parts of the Legendre spectrum of the Riemann functions, putting in evidence that the straight line in the increasing part is not the result of a concave hull, but instead the points in the line belong to the spectrum; this is in sharp contrast with the decreasing part, which is shown to be a consequence to taking a concave hull between the summit of the spectrum and the chirps, and is strongly modified by the computation of the quantile spectrum. Figure 7.28 shows a similar phenomenon for the function $\mathcal{Q}_{3,2.75}$ which allows to conjecture that, for this function too, there exists an "empty room" in the decreasing part of spectrum corresponding to regularity exponents that lie between the a.e. regularity (i.e. the summit of the multifractal spectrum) and the chirps.

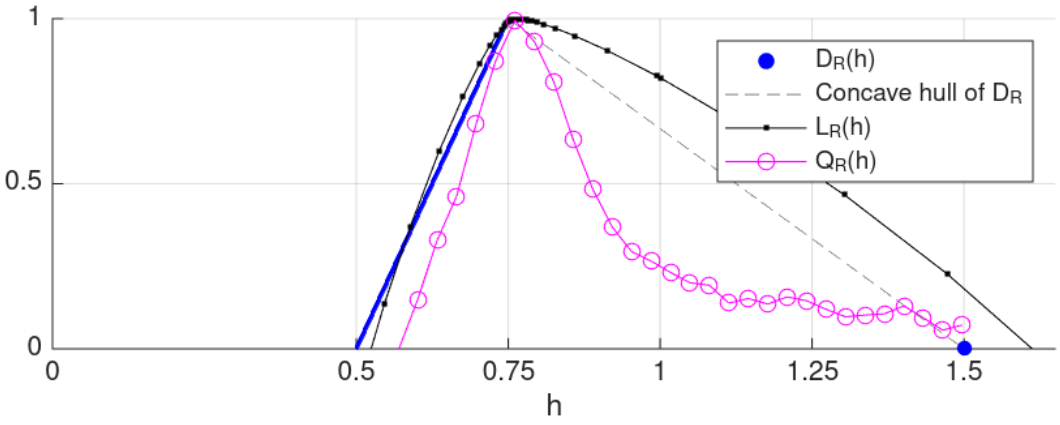


Figure 7.27: Estimated quantile leader spectrum $\mathcal{Q}_\mathcal{R}$ of $\mathcal{R}$ in violet, estimated Legendre spectrum $\mathcal{L}_\mathcal{R}$ of $\mathcal{R}$ in black with the theoretical spectrum of $\mathcal{R}$ in blue and its concave hull in dashed. The estimated spectra are computed by computing the wavelet coefficients of $\mathcal{R}^{\text{approx}}$ with the Daubechies wavelet 'db3'.

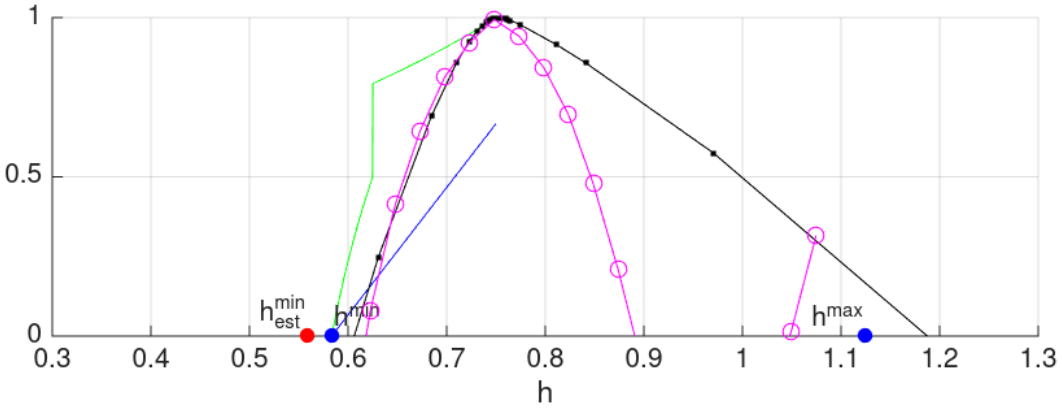


Figure 7.28: Estimated quantile leader spectrum $\mathcal{Q}_{\mathcal{Q}_{3,2.75}}$ of $\mathcal{Q}_{3,2.75}$ in violet, estimated Legendre spectrum $\mathcal{L}_{\mathcal{Q}_{3,2.75}}$ of $\mathcal{Q}_{3,2.75}$ in black and estimated $h^{\min}_{\mathcal{Q}_{3,2.75}}$ in red. The upper bound of theoretical spectrum is in green and lower bound and theoretical $h^{\min}_{\mathcal{Q}_{3,2.75}}$ and $h^{\max}_{\mathcal{Q}_{3,2.75}}$ are in blue. The estimated spectra are computed by computing the wavelet coefficients of the approximation of $\mathcal{Q}_{3,2.75}$ with the Daubechies wavelet 'db8'.

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