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JIN YAN

Composition du Jury :

M. Romain Bourdais Full Professor, CentraleSupélec	Président/Examinateur
Mme Isabelle Queinnec Directrice de recherche, LAAS-CNRS	Rapporteur
M. Pedro Rodriguez-Ayerbe Full Professor, CentraleSupélec	Rapporteur
M. Anh-Tu Nguyen Associate Professor, INSA Hauts-de-France, Université Polytechnique Hauts-de-France	Examinateur
M. Nel Samama Full Professor, Télécom SudParis, Institut Polytechnique de Paris	Directeur de thèse
M. Hoai-Nam Nguyen Associate Professor, Télécom SudParis, Institut Polytechnique de Paris	Co-encadrant de thèse

Declaration

I hereby declare that this thesis represents my own work which has been done after registration for the degree of PhD at Institut Polytechnique de Paris, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

I have read the University's current research ethics guidelines, and accept responsibility for the conduct of the procedures in accordance with the University's Committee on the Use of Human and Animal Subjects in Teaching and Research (HASC). I have attempted to identify all the risks related to this research that may arise in conducting this research, obtained the relevant ethical and/or safety approval (where applicable), and acknowledged my obligations and the rights of the participants.

Jin YAN

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Résumé

Cette thèse porte sur le problème de l'estimation du domaine d'attraction (DoA) et de la conception de lois de commande pour les systèmes linéaires à paramètres variables (LPV) soumis à des contraintes sur l'état et l'entrée. Plus précisément, les travaux de recherche abordent les deux problématiques suivantes :

- Estimation du DoA : déterminer l'ensemble des conditions initiales à partir desquelles les trajectoires du système LPV convergent vers un point d'équilibre souhaité tout en respectant les contraintes imposées.
- Conception de lois de commande : synthétiser des contrôleurs stabilisants permettant d'atteindre des objectifs de performance prescrits tout en garantissant le respect des contraintes malgré les variations paramétriques.

Dans la première partie, des conditions sous forme d'inégalités matricielles linéaires (LMI) sont développées afin de calculer des ensembles invariants ellipsoïdaux ainsi que des ensembles invariants ellipsoïdaux dépendant des paramètres pour des systèmes en boucle fermée soumis à une loi de retour d'état à gain programmé. Une contribution majeure de cette partie réside dans la dérivation, pour la première fois, de conditions LMI permettant de construire des ellipsoïdes invariants dépendant des paramètres pour des systèmes LPV soumis à une saturation de l'entrée. De plus, des fonctions de Lyapunov dépendant des paramètres appropriées sont introduites afin de réduire le conservatisme des approches LMI dépendant des paramètres en évitant la formulation à double sommation.

La deuxième partie est consacrée à l'étude de stratégies de commande prédictive (MPC) pour les systèmes LPV contraints. Deux nouveaux cadres MPC sont proposés. Tout d'abord, une formulation MPC quasi-min–max est développée sur la base d'une fonction de Lyapunov dépendant des paramètres combinée à une loi de commande par retour d'état non linéaire. L'approche proposée permet de réduire le conservatisme par rapport aux méthodes existantes. Ensuite, un cadre MPC sans double sommation basé sur des fonctions de Lyapunov dépendant des paramètres est introduit. Cette approche permet d'obtenir des résultats moins conservateurs en éliminant la formulation à double sommation. Enfin, ce cadre MPC sans double sommation est étendu au cas des systèmes soumis à une saturation de l'entrée afin d'améliorer les performances de commande tout en tenant explicitement compte des limitations des actionneurs.

La troisième partie traite du problème de la fusion de contrôleurs à faible et à fort gain dans un contexte de commande à gain programmé. Une stratégie systématique de fusion est développée afin d'obtenir une loi de commande continue combinant les avantages des contrôleurs à faible et à fort gain. Le contrôleur obtenu préserve les performances locales optimales en retrouvant le contrôleur à fort gain au voisinage du point d'équilibre, tout en garantissant un domaine d'attraction plus large que celui obtenu avec un contrôleur à faible gain. Cette approche peut également être étendue aux systèmes soumis à des contraintes de saturation de l'entrée, permettant une exploitation efficace des capacités des actionneurs et une amélioration des performances en boucle fermée.

Abstract

This thesis focuses on the problem of estimating the domain of attraction (DoA) and designing a control law for linear parameter-varying (LPV) systems subject to input and state constraints. More specially, the research addresses:

- DoA estimation: determining the set of initial conditions from which the LPV system trajectories converge to a desired equilibrium point while satisfying constraints.
- Control law design: synthesize stabilizing controllers that achieve prescribed performance objectives and guaranty constraint satisfaction despite parameter variations.

In the first part, LMI conditions are developed for computing ellipsoidal and parameter-dependent ellipsoidal invariant sets of closed-loop systems under gain-scheduling state feedback. A particular contribution is for the first time, LMI conditions for constructing parameter-dependent invariant ellipsoids under input saturation. Furthermore, appropriate parameter-dependent Lyapunov functions are introduced to reduce conservatism by avoiding the double-sum formulation in parameter-dependent LMI approaches.

The second part investigates MPC strategies. Two novel MPC frameworks are proposed. first, we develop a quasi-min–max MPC formulation based on parameter-dependent lyapunov function together with a nonlinear state-feedback law, our approach reduced conservatism compared with existing methods. then we proposed another approach introduces a double-sum-free MPC framework based on parameter-dependent Lyapunov functions. This approach give us a less conservative result by avoiding the double-sum formulation. Finally we extended this double-sum-free MPC to the input saturation case to improve the control performance.

The third part addresses the problem of merging low-gain and high-gain gain-scheduling controllers. A systematic merging strategy is developed to obtain a continuous controller that combines the advantages of both low-gain and high-gain controllers. The resulting controller preserves local optimal performance by recovering the high-gain controller near the equilibrium, while get a large domain of attraction than low-gain controller. This approach is also possible to extended to saturated input systems, enabling efficient utilization of actuator capabilities to improve the control performance.

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List of Symbols and Acronyms

$(*)$ Symmetric blocks in matrix expressions

$\mathbb{R}$ The set of real numbers

$\mathbb{R}^{n\times m}$ The set of real matrices of dimension $n \times m$

$\mathbb{S}^n$ The set of $n \times n$ symmetric positive definite matrices

$\overline{1, s}$ The set $\{1, 2, \ldots, s\}$

$\succ 0$ Symmetric positive definite

$\succeq 0$ Symmetric positive semi-definite

cov Convex hull

CLF Control Lyapunov function

DoA Domain of attraction

LDI Linear differential inclusion

LMI Linear matrix inequality

LPV Linear parameter varying

MPC Model predictive control

PDLF Parameter-dependent Lyapunov function

QP Quadratic program

SDP Semi-definite program

SNA Sector nonlinearity approach

SOS Sum of squares

Chapter 1

Introduction

Industrial processes are inherently nonlinear, and their dynamics are often inadequately represented by a single linear model. As a result, controllers designed based on such linear approximations may experience significant performance degradation when applied outside a narrow operating region. In practice, industrial systems rarely operate under fixed conditions. Changes in economic objectives, production demands, or environmental disturbances frequently lead to shifts in the operating point, which in turn modify the process dynamics [8–10].

During the 1990s, gain-scheduling control emerged as a practical tool for handling nonlinear and time-varying dynamics. However, classical gain-scheduling relies on open-loop variation of controller parameters and lacks a rigorous closed-loop stability guaranty [11, 12]. To address these limitations, [13] introduced the LPV framework, which provides a systematic way to represent nonlinear systems using a linear structure with parameter-dependent dynamics [14]. More importantly, LPV methods enable explicit stability and performance analysis, overcoming the theoretical shortcomings of traditional gain-scheduling approaches [15].

Following the construction of a LPV model, invariant set theory provides a geometric and computationally tractable framework for analyzing stability and performance [16–18]. For polytopic LPV systems, verifying invariance at all vertices ensures the invariance over the entire parameter domain [19, 20]. From a controller synthesis viewpoint, invariant set serves not only as analysis tools but also as direct design objectives. By jointly optimizing the state-feedback gains and the shape of the invariant set, one can construct LPV controllers that guarantee closed-loop stability, satisfy state/input constraints, and achieve specified performance levels. Such invariant-set–based control synthesis methods have been widely adopted in LPV control, robust MPC, and nonlinear constrained control [21–23].

MPC is widely used in industry as an effective method for dealing with constrained multivariable control problems [24–27]. In essence, this control scheme consists of the online optimization technique: at each time instant, an open-loop optimization problem is solved online using the current available measurements. The first element of the resulting optimal control sequence is applied to the system. At the next time instant, the entire procedure is repeated: the optimization

problem is refreshed with the new measurements and re-solved. The issues of feasibility of the on-line optimization problem, and of stability of the closed-loop system are largely understood for linear discrete-time systems with input and state constraints.

In the last three decades, various research efforts have been made to synthesize an MPC controller for nonlinear systems, see e.g., [28–31]. In this context, LPV system modeling framework offers a powerful tool [32]. Frequently used MPC approaches for the control of LPV systems with input and state constraints are based on the online design of closed-loop linear feedback strategies, e.g., [31, 33, 34]. The optimization is done over a feedback law. In [2], a quasi-min-max MPC algorithm was proposed. The MPC algorithm is called quasi since the first stage cost can be calculated without any uncertainty. At time instant k, the first predicted control move $u(k)$ is separated from the rest of the predicted control moves $u(k+t), \forall t \geq 1$ governed by a feedback control law. At each time instant, a solution of a convex SDP is required. It was shown that the quasi-min-max MPC control law can improve the performance and reduce the conservativeness with respect to other min-max MPC techniques, e.g., [23]. However, the feedback control law for $u(k+t), \forall t \geq 1$ in [2] is a linear function of the scheduling parameters.

Although the input-output relationships in LPV models remain linear, this linearity explicitly depends on scheduling variables-parameters that influence the system dynamics. These variables are assumed to be measurable in real time and are constrained within a predefined scheduling region, typically represented as a polytopic set. A common approach to applying MPC to LPV systems with input and state constraints is to assume that the scheduling parameters remain constant, or equivalently frozen, over the prediction horizon and equal to their currently measured values. This assumption reduces the online optimization to a QP [35], but at the expense of increased conservatism and potential sub-optimality. To alleviate this conservatism while retaining the computational advantages of the QP formulation, tube-based MPC can be employed. In this framework, a scheduling tube is introduced to bound the deviation between the actual and nominal trajectories, as shown for example in [36, 37].

Another classical strategy is built around the notion of quadratic stability, where the controller is designed so that a single state-feedback gain and a single Lyapunov matrix guarantee stability for every system that can arise from the LPV model [23]. Because this formulation does not exploit the real-time availability of the scheduling parameters, it often yields overly conservative control laws.

A more refined way to reduce conservatism is to incorporate the scheduling parameter directly into the controller design. Building on early works such as [38] and [39], less conservative LMI conditions for robust analysis and controller synthesis have been developed through the use of PDLFs. The key idea is to allow the Lyapunov function to vary explicitly with the scheduling parameter, rather than relying on a fixed worst-case function. This paradigm has led to significant improvements in closed-loop performance. When integrated into an MPC framework, as in [4], PDLFs substantially reduce conservatism compared to traditional quadratic Lyapunov-based approaches such as [23].

Over the last two decades, the application of the LMI technique in fuzzy control has inspired new ideas for designing MPC strategies for LPV systems. In [40], a nonlinear gain-scheduling state feedback control law and an associated nonlinear PDLF were proposed for the stability and

stabilization problems of discrete-time Takagi-Sugeno fuzzy systems. This approach is shown to reduce conservatism compared to traditional methods. Based on the advancements in [40], the work in [3] extended these ideas by developing an LMI-based MPC strategy that utilizes the same class of PDLFs and control laws. More recently, the work in [41] refines previous approaches by introducing a simplified version of the nonlinearly PDLF, striking a balance between computational efficiency and closed-loop performance.

A weakness of the PDLF based methods is that it leads to a bilinear structure, often referred to as the double sum positivity problem. This problem arises from the multiplication of system matrices and the Lyapunov matrices. This multiplication results in quadratic terms that involve double summations over the indices of the scheduling parameters, creating a non-convex optimization problem. It is well known [42] that for the double sum problem, the SOS approach offers an asymptotically exact convexification, but at the cost of increased computational complexity. As the degree of the polynomials of the SOS approach involved increases, the SOS relaxation becomes tighter and eventually provides the exact solution to the original problem. However, the computational burden of solving the resulting SDP also grows rapidly, making the method computationally expensive. Simpler convexification procedures, such as those found in [40] and [3], offer a more computationally tractable alternative, but they come at the cost of some performance loss.

Gain-scheduling control offers a computationally efficient alternative to MPC for LPV systems in terms of online complexity. However, a fundamental limitation arises when attempting to simultaneously address constraints and optimality. Typically, a single gain-scheduling control law can effectively achieve one of the two objectives but often performs poorly with respect to the other. For example, a low-gain, DoA-oriented controller generally ensures a relatively large safe set — that is, the set of initial conditions from which the system can evolve without violating constraints. Nevertheless, this conservative control design usually results in slower convergence to the origin. Conversely, a high-gain, performance-oriented controller optimized for a specific cost function can achieve faster regulation but typically yields a significantly smaller safe set. A simple and straightforward solution to this problem is the use of a control-switching strategy. In this approach, two gain-scheduling controllers are designed, with their respective DoA typically (though not necessarily) nested. The control system switches from the "external" low-gain controller to the locally optimal high-gain one once the system state enters the smaller region of attraction [43], [44].

A key drawback of this switching scheme is the potential for abrupt changes in the control input when transitioning between controllers. Such discontinuities can induce undesirable jumps in both the state and the control signal, which may compromise system performance or even destabilize the closed-loop behavior in real applications. Furthermore, measurement noise can cause unintended and repeated switching, potentially resulting in persistent oscillations. To mitigate these issues, it is desirable to design a time-varying control law that depends continuously on the system state and smoothly blends the two controllers within a transition region. In this way, the control input evolves gradually as the system state moves from the outer to the inner region, ensuring continuity and improved robustness.

It should be emphasized that the idea of merging control laws—particularly through the framework of CLFs —is well established in nonlinear control theory [45]. In [46], a method was proposed

for continuous-time systems to unite two controllers by constructing a composite control law that enables a smooth transition between them. This concept was later extended in [47] to the merging of CLFs. Their approach relies on the so-called control-sharing property, which assumes the existence of a domain where the two CLFs admit a common control. In [48], for constrained linear differential inclusion systems, convex computational tests were introduced to verify the control-sharing property for various classes of functions, including polyhedral, quadratic, piecewise quadratic, truncated ellipsoidal, and hybrid combinations thereof.

1.1 Scope

Motivated by the aforementioned considerations, this thesis investigates advanced control strategies for constrained LPVsystems. The research focuses on two fundamental problems:

- Stability analysis of the equilibrium point of the closed-loop system using invariant set theory.
- Control synthesis, where a stabilizing state-feedback law is designed according to given performance criteria, including the maximization of the DoA, optimization of the MPC cost function.

1.2 Main Contributions

Below the structure of the work and the main contributions are described in more detail.

Chapter 2 studies the stability of constrained LPV systems based on set invariance theory. The LPV representation derived from nonlinear dynamics via the SNA is first introduced. By employing convex and ellipsoidal set theory, sufficient LMI conditions are established for computing ellipsoidal and parameter-dependent ellipsoidal invariant sets of the closed-loop system. The analysis is further extended to the case of input saturation under gain-scheduled state feedback. The main contributions of this chapter are twofold. First, tractable LMI conditions are derived for constructing parameter-dependent ellipsoidal invariant sets for LPV systems with gain-scheduled input saturation, providing a unified convex framework for stability analysis of constrained LPV systems. Second, through appropriate modeling of the controlled LPV system and a suitable selection of PDLFs, a novel design method for parameter-dependent ellipsoidal invariant sets is proposed. The proposed formulation avoids the double-sum problem commonly encountered in LMI-based designs, thereby reducing computational complexity. Moreover, the method is naturally extended to systems subject to saturation constraints. Several examples are provided to illustrate the effectiveness of the proposed approach.

Chapter 3 aims to develop parameter-dependent Lyapunov function-based MPC framework for constrained LPV systems. We propose a novel quasi–min–max MPC algorithms that utilize a class of nonlinearly parameterized state-feedback control laws. Compared with existing works [3, 40], our nonlinearly parameterized Lyapunov function is significantly simpler. We also provide two new MPC strategies. In the first strategy, linear gain-scheduling feedback is adopted, which offers

efficient online computation. However, as the control law is purely linear, it often fails to fully exploit the available control range, and the plant's state may take longer than necessary to reach the origin. To address this limitation, the second strategy employs nonlinearly saturated gain-scheduling feedback, allowing the controller to apply larger control actions when the system is far from the desired state and thereby achieving faster convergence.

Chapter 4 investigates the problem of merging two gain-scheduling control laws for constrained discrete-time LPV systems. The primary objective is to establish a unified and computationally tractable framework that enables the systematic design and merging of low-gain and high-gain controllers without relying on the conventional control-sharing property. A central theoretical result of this chapter demonstrates that, for the considered class of constrained LPV systems, any two asymptotically stabilizing gain-scheduling controllers can be merged while preserving stability and desirable performance characteristics.

Building upon the convex synthesis framework introduced in **Chapter 2** for synthesizing DoA-oriented low-gain controllers and performance-oriented high-gain controllers, a novel merging strategy for two asymptotically stabilizing linear gain-scheduling control laws is then developed. The resulting merged controller is continuous with respect to the system state, guarantees asymptotic stability, preserves local optimality by reducing to the high-gain controller whenever feasible, and achieves a domain of attraction comparable to that of the low-gain controller. This establishes a general and systematic solution to the controller merging problem for constrained LPV systems.

To further enhance practical applicability, the merged controller is extended to saturated input case. This extension allows the full exploitation of the available control input range while respecting actuator constraints. The merged saturated controller retains all key desirable properties, including state continuity, preservation of local performance. As a result, the proposed approach provides a unified solution for both linear and saturated gain-scheduling control laws under input constraints.

1.3 Related publications

This section lists the publications written in the framework of this dissertation.

- J. Yan, H. -N. Nguyen and N. Samama. Improved Quasi-Min-Max MPC for Constrained LPV Systems via Nonlinearly Parameterized State Feedback Control. Proceedings of *2024 IEEE 63rd Conference on Decision and Control (CDC)*, Milan, Italy, 2024, pp. 5546-5551, doi: 10.1109/CDC56724.2024.10886262.

- J. Yan, H. -N. Nguyen and N. Samama. Model Predictive Control for Linear Parameter Varying Systems: A Double-Sum-Free Parameter-Dependent Lyapunov Function Approach. Accepted to *International Journal of Control*, DOI: 10.1080/00207179.2026.2700002.

- J. Yan, H. -N. Nguyen and N. Samama. Smooth Merging Low- and High-Gain Controllers for Linear Parameter-Varying Systems. Submitted to *IEEE Transaction on Automaic Control* (Second round).

- J. Yan, H. -N. Nguyen and N. Samama. H_∞ and H_2 Control Design for Discrete-time LPV Systems subject to Bounded Parameter Variation. Accepted to *The 12th 2026 International Conference on Control, Decision and Information Technologies (CoDIT 2026)*.

Chapter 2

Modeling, Invariant Sets, and Stability Analysis for Constrained LPV Systems

In this chapter, Section 2.1 presents the formulation of the constrained LPV system. Section 2.2 presents the modeling approach of the LPV system derived from nonlinear dynamics. Section 2.3 introduces the basic definitions of convex sets and ellipsoidal set. Based on these definitions, Section 2.4 investigates ellipsoidal invariant sets under linear state feedback and saturated feedback and derives the corresponding LMI conditions. Section 2.5 investigates parameter-dependent ellipsoidal invariant sets under linear state feedback and saturated feedback and derives the corresponding LMI conditions. Section 2.6 proposes DoA-oriented low-gain and performance-oriented high-gain controller design methods for LPV systems.

The main contributions of this chapter are summarized as follows:

- **LMI-Based Conditions for Invariant Set Construction:** We derive, for the first time to the best of the author's knowledge, a set of LMI conditions for constructing invariant sets of LPV systems under gain-scheduled input saturation.
- **LMI-Based Conditions for Low-Gain and High-Gain Control Design:** We present a minor contribution to the design of DoA-oriented low-gain and performance-oriented high-gain controllers for LPV systems. To the best of the author's knowledge, existing PDLF formulations for gain design typically involve non-convex double-sum structures. In this chapter, we show that by appropriately reformulating the PDLF framework, direct LMI conditions for controller gain computation can be derived without approximation.

2.1 Problem Formulation

In this chapter, consider the following discrete-time LPV system

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the measured state, $u \in \mathbb{R}^m$ is the control input. The matrices $[A(k)\ B(k)] \in \Omega$ with

$$\Omega = \text{cov}\left([A_1\ B_1], [A_2\ B_2], \ldots, [A_L\ B_L]\right) \tag{2.2}$$

where $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ are constant matrices, $\forall i \in \overline{1,L}$, L is the number of subsystems, and cov$(\cdot)$ denotes the convex hull operator. It is well known that the condition $[A(k)\ B(k)] \in \Omega$ holds if and only if $\exists p_i(k), \forall i \in \overline{1,L}$ such that

$$A(k) = \sum_{i=1}^{L} p_i(k)A_i, B(k) = \sum_{i=1}^{L} p_i(k)B_i \tag{2.3}$$

where $p(k) = [p_1(k)\ p_2(k)\ \ldots p_L(k)]^T$ satisfies

$$p(k) \in \Sigma_L := \left\{ p \in \mathbb{R}^L \mid \sum_{i=1}^{L} p_i = 1,\ p_i \geq 0 \right\} \tag{2.4}$$

LPV hypothesis: $p(k)$ is available in each time instant k.

We consider the following constraints on the state $x(k)$, and on the input $u(k)$

$$\begin{cases} x(k) \in \mathscr{X} := \left\{x \in \mathbb{R}^n \mid |f_l^T x| \leq 1, \forall l \in \overline{1,n_x}\right\} \\ u(k) \in \mathscr{U} := \left\{u \in \mathbb{R}^m \mid |u_l| \leq 1, \forall l \in \overline{1,m}\right\} \end{cases} \tag{2.5}$$

where $f_l \in \mathbb{R}^n, l \in \overline{1,n_x}$ are known and fixed, u_l is the $l-$th component of the vector u, $l \in \overline{1,m}$.

2.2 LPV System Modeling for Nonlinear Dynamics

The concept of sector nonlinearity in fuzzy model construction was first proposed in [49], and it can be further extended to the modeling of LPV systems. SNA is based on the following idea [1,50,51].

Consider a nonlinear system

$$x(k+1) = f(x(k)) \tag{2.6}$$

where $f(0) = 0$. The aim is to find the global sector such that $x(k+1) = f(x(k)) \in [a_1\ a_2]x(k)$. Fig.2.1 illustrates the sector nonlinearity approach. This approach guarantees an exact LPV model construction. However, it is sometimes difficult to find global sectors for general nonlinear systems. In this case, we can consider local sector nonlinearity. This is reasonable as variables of physical systems are always bounded.

The following example illustrates the concrete steps to construct LPV models.

Example: Consider the following nonlinear system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} (1-T)x_1(k) + Tx_1(k)x_2^3(k) \\ (1-T)x_2(k) + T(3+x_2(k))x_1^3(k) \end{bmatrix} \tag{2.7}$$

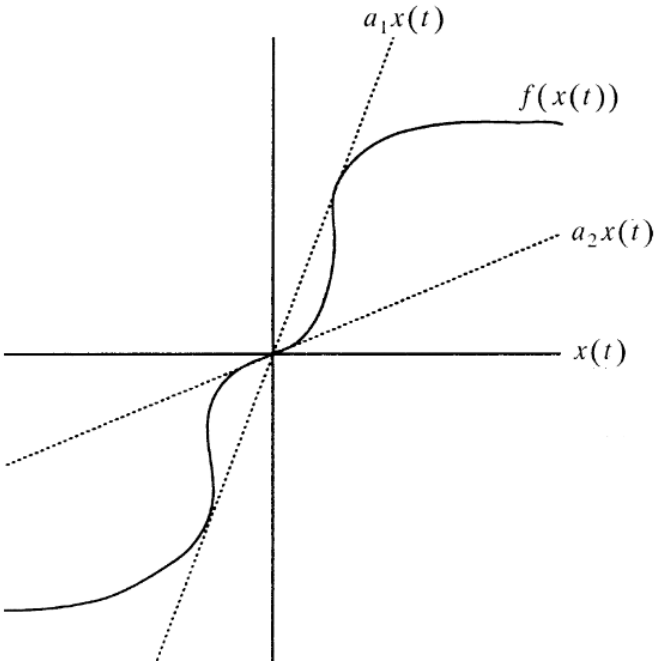


Figure 2.1: Global sector nonlinearity [1]

where T is the sampling period. We assume that $x_1 \in [-1, 1]$, $x_2 \in [-1, 1]$. (2.7) can be written as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1-T & Tx_1x_2^2(k) \\ T(3+x_2(k))x_1^2(k) & 1-T \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{2.8}$$

Note that $x_1x_2^2(k)$, $(3 + x_2(k))x_1^2(k)$ are nonlinear terms. Define $z_1(k) = x_1x_2^2(k)$, $z_2(k) = (3 + x_2(k))x_1^2(k)$. Then we have

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1-T & Tz_1(k) \\ Tz_2(k) & 1-T \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{2.9}$$

Next, calculate the minimum and maximum values of $z_1(k)$ and $z_2(k)$ under $x_1(k) \in [-1, 1]$, $x_2(k) \in [-1, 1]$. One gets

$$\begin{aligned} \max_{x_1(k),x_2(k)} z_1(k) = 1, \quad \min_{x_1(k),x_2(k)} z_1(k) = -1 \\ \max_{x_1(k),x_2(k)} z_2(k) = 4, \quad \min_{x_1(k),x_2(k)} z_2(k) = 0 \end{aligned} \tag{2.10}$$

Therefore, the four corresponding vertices of the LPV system are given by

$$\begin{aligned} A_1 &= \begin{bmatrix} 1-T & T \\ 4T & 1-T \end{bmatrix}, \; A_2 = \begin{bmatrix} 1-T & T \\ 0 & 1-T \end{bmatrix} \\ A_3 &= \begin{bmatrix} 1-T & -T \\ 4T & 1-T \end{bmatrix}, \; A_4 = \begin{bmatrix} 1-T & -T \\ 0 & 1-T \end{bmatrix} \end{aligned} \tag{2.11}$$

From the maximum and minimum values, $z_1(k)$ and $z_2(k)$ can be represented by

$$\begin{aligned} z_1(k) &= x_1(k)x_2^2(k) = \mu_1^{(1)}(k) \times 1 + \mu_1^{(2)}(k) \times (-1) \\ z_2(k) &= (3 + x_2(k))x_1^2(k) = \mu_2^{(1)}(k) \times 4 + \mu_2^{(2)}(k) \times 0 \end{aligned} \tag{2.12}$$

where $\mu_1^{(1)}(k) + \mu_1^{(2)}(k) = 1$, $\mu_2^{(1)}(k) + \mu_2^{(2)}(k) = 1$. Therefore $\mu_1^{(1)}(k)$, $\mu_1^{(2)}(k)$, $\mu_2^{(1)}(k)$, $\mu_2^{(2)}(k)$ can be calculated as

$$\begin{aligned} \mu_1^{(1)}(k) &= \frac{z_1(k)+1}{2}, \; \mu_1^{(2)}(k) = \frac{1-z_1(k)}{2} \\ \mu_2^{(1)}(k) &= \frac{z_2(k)}{4}, \; \mu_2^{(2)}(k) = \frac{4-z_2(k)}{4} \end{aligned} \tag{2.13}$$

The discrete-time LPV system is carried out as

$$x(k+1) = \sum_{i=1}^{4} p_i(k) A_i x(k) \tag{2.14}$$

where $x(k) = [x_1(k) \; x_2(k)]^T$ and

$$\begin{aligned} p_1 &= \mu_1^{(1)}(k)\mu_2^{(1)}(k) \\ p_2 &= \mu_1^{(1)}(k)\mu_2^{(2)}(k) \\ p_3 &= \mu_1^{(2)}(k)\mu_2^{(1)}(k) \\ p_4 &= \mu_1^{(2)}(k)\mu_2^{(2)}(k) \end{aligned} \tag{2.15}$$

This LPV model exactly represents the nonlinear system in the region $[-1, 1] \times [-1, 1]$ on the x_1-x_2 space.

Remark: One of major shortcomings when using the classical SNA is that the corresponding number L of vertex models exponentially grows with respect to the number α of non-constant terms (which generally represent the system nonlinearities or time-varying parameters) in the nonlinear system, i.e., $L = 2^\alpha$. Therefore, when the system contains numerous nonlinearities, the number of vertex models increases exponentially. [52] presented a noteworthy approach to reducing the number of vertices in the polytopic representation.

2.3 Set terminology

Convex sets are of central importance in optimization theory and stability analysis within control systems [16, 18]. We will first present several fundamental definitions related to convex sets, which will be instrumental for the developments that follow [53, 54].

Definition 2.3.1. ***(Closed set)*** *A set S is closed if it contains its own boundary. In other words, any point outside S has a neighborhood disjoint from S.*

Definition 2.3.2. ***(Bounded set)*** *A set $S \subset \mathbb{R}^n$ is bounded if it is contained in some ball $B_r = \{x \in \mathbb{R}^n : ||x||_2 \leq r\}$ of finite radius $r > 0$.*

Definition 2.3.3. ***(Compact set)*** *A set $S \subset \mathbb{R}^n$ is compact if it is closed and bounded.*

Definition 2.3.4. ***(Convex set)*** *A set $S \subset \mathbb{R}^n$ is said to be convex if and only if for any $x_1 \in S$ and $x_2 \in S$, it holds that*

$$\alpha x_1 + (1-\alpha)x_2 \in S, \quad \forall\, \alpha \in [0,1] \tag{2.16}$$

The point

$$x = \alpha x_1 + (1-\alpha)x_2 \tag{2.17}$$

is called a convex combination of the pair (x_1, x_2). The set of all such points forms the line segment connecting x_1 and x_2.

Equivalently, the set S is convex if and only if the line segment between any two points in S lies entirely in S.

Definition 2.3.5. ***(Convex function)*** *A function $f : S \to \mathbb{R}$, with $S \subseteq \mathbb{R}^n$, is said to be convex if and only if the set S is convex and*

$$f\big(\alpha x_1 + (1-\alpha)x_2\big) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \tag{2.18}$$

for all $x_1 \in S$, $x_2 \in S$, and for all $\alpha \in [0,1]$.

Definition 2.3.6. ***(Convex hull)*** *The convex hull of a set $S \subset \mathbb{R}^n$ is defined as the smallest convex set that contains S.*

It is well known [54] that for any finite set

$$S = \{s_1, s_2, \ldots, s_r\}, \quad r \in \mathbb{N}$$

the convex hull of S is given by

$$\text{cov}(S) = \left\{ s \in \mathbb{R}^n \;\middle|\; s = \sum_{i=1}^{r} \alpha_i s_i,\ \sum_{i=1}^{r} \alpha_i = 1,\ \alpha_i \geq 0 \right\}. \tag{2.19}$$

2.3.1 Ellipsoidal set

Ellipsoidal set, which constitutes a family of convex sets characterized by quadratic forms, plays a pivotal role in control theory. Ellipsoids have been widely utilized in stability analysis, robust control, state estimation, and constraint handling. Their inherent compatibility with Lyapunov-based analysis and linear matrix inequality (LMI) frameworks renders ellipsoidal sets a powerful and systematic tool for representing uncertainties and analyzing the stability properties of dynamical systems [55–57].

Definition 2.3.7. ***(Ellipsoidal set)*** *Let $P \in \mathbb{S}^{n\times n}$ be a positive semi-definite matrix and $c_0 \in \mathbb{R}^n$. The* ellipsoidal set *centered at c_0 with shape matrix P is defined by*

$$\mathscr{E}(P, c_0) = \{x \in \mathbb{R}^n \mid (x - c_0)^T P(x - c_0) \leq 1\} \tag{2.20}$$

If the ellipsoid is centered at the origin, it reduces to

$$\mathscr{E}(P) = \{x \in \mathbb{R}^n \mid x^T P x \leq 1\} \tag{2.21}$$

The use of ellipsoidal sets allows most optimization problems encountered in control problems to be reformulated as the optimization of a linear objective subject to LMI constraints. The optimization of a linear function subject to LMI constraints is referred to as SDP [58,59], which can be viewed as a generalization of linear programming. A key advantage of using LMIs is that efficient polynomial-time algorithms have been developed for solving SDP problems. These algorithms are implemented in widely available LMI solvers, such as MOSEK, SeDuMi.

A key tool in deriving LMI conditions is the Schur complement. The Schur complement states that the nonlinear conditions of the special forms:

$$M(x) - N(x)Y(x)^{-1}N(x)^T \succ 0, \quad \text{with } Y(x) \succ 0 \tag{2.22}$$

$$\text{or} \quad Y(x) - N(x)^T M(x)^{-1}N(x) \succ 0, \quad \text{with } M(x) \succ 0 \tag{2.23}$$

can be equivalently expressed as the linear matrix inequality

$$\begin{bmatrix} M(x) & N(x) \\ N(x)^T & Y(x) \end{bmatrix} \succ 0 \tag{2.24}$$

This equivalence allows nonlinear matrix inequalities to be reformulated in standard LMI form, which is convex and amenable to semi-definite programming.

The following lemmas will be used extensively in the PhD thesis.

Lemma 2.1. *[16] The volume of the ellipsoid $\mathscr{E}(P)$ is proportional to* $\det\left(P^{-1}\right)$.

Lemma 2.2. *[60]Consider the ellipsoid (2.21). Let c be a vector of appropriate dimension. Then, for all $x \in \mathscr{E}(P)$, the maximum value of $c^T x$ is given by*

$$\max_{x\in\mathscr{E}(P)} c^T x = \sqrt{c^T P^{-1} c} \tag{2.25}$$

Proof.

$$\begin{aligned}\max_{x\in\mathscr{E}(P)} c^T x &= \max_{x\in\mathscr{E}(P)} \left|c^T x\right| = \max_{x\in\mathscr{E}(P)} |c^T P^{-\frac{1}{2}} P^{\frac{1}{2}} x| \\ &\leq \max_{x\in\mathscr{E}(P)} \|c^T P^{-\frac{1}{2}}\| \|P^{\frac{1}{2}} x\| \leq \|c^T P^{-\frac{1}{2}}\| = \sqrt{c^T P^{-1} c}\end{aligned} \tag{2.26}$$

□

Lemma 2.3. *[16] Consider two ellipsoidal sets $\mathscr{E}(P_1)$ and $\mathscr{E}(P_2)$. Then, $\mathscr{E}(P_2) \subseteq \mathscr{E}(P_1)$ if and only if $P_1 \preceq P_2$.*

Proof. The inclusion $\mathscr{E}(P_2) \subseteq \mathscr{E}(P_1)$ holds if and only if, for any x satisfying $x^T P_1 x = 1$, there exists a scalar $0 < \alpha \leq 1$ such that

$$\alpha^2 x^T P_2 x = 1 \tag{2.27}$$

This condition is equivalent to

$$x^T (P_2 - P_1) x = \frac{1}{\alpha^2} - 1 \geq 0 \tag{2.28}$$

for all $x \neq 0$, which implies that

$$P_1 \preceq P_2 \tag{2.29}$$

□

2.4 Ellipsoidal invariant set

Invariant set plays a fundamental role in constrained system analysis and controller design.

Consider the following discrete-time nonlinear system

$$x(k+1) = f(x(k)) \tag{2.30}$$

and the system with control input

$$x(k+1) = f(x(k), u(k)) \tag{2.31}$$

The state $x(k)$ and the input $u(k)$ are subject to the following constraints for all $k \geq 0$:

$$\begin{cases} x(k) \in \mathscr{X} \\ u(k) \in \mathscr{U} \end{cases} \tag{2.32}$$

where the sets $\mathscr{X} \subset \mathbb{R}^n$ and $\mathscr{U} \subset \mathbb{R}^m$ are assumed to be closed and bounded. In the following, we provide the related definitions of invariant sets.

Definition 2.4.1. ***(Invariant set)*** *The set $\Omega \subseteq \mathscr{X}$ is said to be invariant for the system (2.30) if and only if*

$$f\big(x(k)\big) \in \Omega, \quad \forall x(k) \in \Omega \tag{2.33}$$

Definition 2.4.2. ***(Controlled invariant set)*** *The set $\mathscr{C} \subseteq \mathscr{X}$ is said to be a controlled invariant set for the system (2.31) if and only if, for all $x(k) \in \mathscr{C}$, there exists a control input $u(k) \in \mathscr{U}$ such that*

$$f\big(x(k), u(k)\big) \in \mathscr{C} \tag{2.34}$$

Ellipsoidal sets are commonly employed in the characterization of invariant sets for constrained systems, due to the fact their convenient representation via LMIs and the fact that their dimension is fixed respect to the system dimension.

2.4.1 Ellipsoidal invariant set for linear feedback

For the constrained LPV system (2.1), (2.5), consider the ellipsoid

$$\mathscr{E}(P^{-1}) = \{x \in \mathbb{R}^n \mid x^T P^{-1} x \leq 1\} \tag{2.35}$$

and the following state feeback

$$u(k) = Fx(k) \tag{2.36}$$

where $F \in \mathbb{R}^{m \times n}$ is constant control matrix. According to the controlled invariant set definition (2.4.2), the ellipsoid $\mathscr{E}(P^{-1})$ is controlled invariant set if and only if $\forall x(k) \in \mathscr{E}(P^{-1})$

$$x^T (A(k) + B(k)F)^T P^{-1} (A(k) + B(k)F) x \leq 1 \tag{2.37}$$

For system (2.1), it suffices to verify condition (2.37) on the boundary of the ellipsoid $\mathscr{E}(P^{-1})$, that is, for all x satisfying $x^T P^{-1} x = 1$. Consequently, condition (2.37) can be reformulated as

$$x^T (A(k) + B(k)F)^T P^{-1} (A(k) + B(k)F) x \leq x^T P^{-1} x \tag{2.38}$$

(2.38) is satisfied if and only if

$$P^{-1} - (A_i + B_i F)^T P^{-1} (A_i + B_i F) \succeq 0, \; i \in \overline{1, L} \tag{2.39}$$

Using Schur complement, this condition can be rewritten as

$$\begin{bmatrix} P^{-1} & (A_i + B_i F)^T \\ A_i + B_i F & P \end{bmatrix} \succeq 0, \; i \in \overline{1, L} \tag{2.40}$$

Since this condition involves the nonlinear term P^{-1} when P is considered as a decision variable, pre- and post-multiplying it by

$$\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \tag{2.41}$$

yields

$$\begin{bmatrix} P & PA_i^T + PF^T B_i^T \\ A_i P + B_i F P & P \end{bmatrix} \succeq 0, \; i \in \overline{1, L} \tag{2.42}$$

As the above condition involves the bilinear term FP, a variable substitution is employed to convexify the formulation. By defining $Y = FP$, we have

$$\begin{bmatrix} P & PA_i^T + Y^T B_i^T \\ A_i P + B_i Y & P \end{bmatrix} \succeq 0,\ i \in \overline{1, L} \tag{2.43}$$

Condition (2.43) provides a necessary and sufficient condition for the ellipsoid $\mathscr{E}(P^{-1})$ to be invariant when the linear state feedback controller $u = Fx$ is applied. It should be noted that, although the ellipsoid $\mathscr{E}(P^{-1})$ satisfying condition (2.43) is invariant, it is not necessarily contractive. Ensuring that the ellipsoid $\mathscr{E}(P^{-1})$ is contractive implies that $\lim_{k\to\infty} x(k) = 0$. Therefore, for all x on the boundary of $\mathscr{E}(P^{-1})$, i.e., for all x satisfying $x^T P^{-1} x = 1$, we require

$$x^T (A(k) + B(k)F)^T P^{-1} (A(k) + B(k)F)x < 1 \tag{2.44}$$

which is equivalent to the following LMI:

$$\begin{bmatrix} P & PA_i^T + Y^T B_i^T \\ A_i P + B_i Y & P \end{bmatrix} \succ 0,\ i \in \overline{1, L} \tag{2.45}$$

Concerning the constraint satisfaction (2.5), based on lemma (2.2), it is obvious that :

- The state constraints are satisfied if and only if

$$\begin{bmatrix} 1 & f_l^T P \\ P f_l & P \end{bmatrix} \succeq 0,\ l \in \overline{1, n_x} \tag{2.46}$$

- The input constraints are satisfied if and only if

$$\begin{bmatrix} 1 & Y_l \\ Y_l^T & P \end{bmatrix} \succeq 0,\ l \in \overline{1, m} \tag{2.47}$$

  where Y_l denotes the l-th row of the matrix Y.

Among all ellipsoids that satisfy (2.45), (2.46), and (2.47), the ellipsoid of maximum size can be determined by maximizing the volume of $\mathscr{E}(P^{-1})$, according to Lemma (2.1). Since the determinant of P is proportional to the volume of the ellipsoid $\mathscr{E}(P^{-1})$, maximizing $\log\det(P)$ is equivalent to maximizing the ellipsoid's volume. The problem of computing the largest invariant ellipsoid can be formulated as

$$\begin{aligned} &\max_{P,Y} \{\log\det(P)\}, \\ &\text{s.t. } (2.43), (2.46), (2.47) \end{aligned} \tag{2.48}$$

This problem can be interpreted as the search for the largest ellipsoid that provides an estimate of the DoA of system (2.1). It is noticed that if B_i are the same, it is possible to employ a gain-scheduling

feedback of the form $u(k) = F(k)x(k)$, $F(k) = \sum_{i=1}^{L} p_i(k)F_i$. In this case, (2.43) can be formulate as

$$\begin{bmatrix} P & PA_i^T + Y_i^T B^T \\ A_i P + BY_i & P \end{bmatrix} \succeq 0, \ i \in \overline{1, L} \tag{2.49}$$

where $Y_i = F_i P$.

The concept of ellipsoidal invariant sets provides a convenient tool for analyzing the stability of LPV systems. By constructing a Lyapunov function of the form

$$V(k) = x^T P^{-1} x, \tag{2.50}$$

the associated sub-level sets define ellipsoids in the state space that are invariant under the system dynamics. If the derivative of the Lyapunov function along system trajectories satisfies

$$V(x+1) - V(x) \leq 0 \tag{2.51}$$

these ellipsoids constitute invariant sets.

Since $V(k+1) - V(k) = 0$ holds if and only if $x = 0$, LaSalle's invariance principle guarantees the asymptotic stability of the system. Therefore, the invariant set condition (2.49) also serves as a Lyapunov stability condition.

2.4.2 Ellipsoidal invariant set for saturated feedback

Although linear gain-scheduling feedback laws are computationally efficient, the resulting control inputs are often conservative and remain unsaturated. As a consequence, the state trajectories may converge to the origin more slowly than necessary [61–64]. To overcome this challenge and to fully utilize the input range power, we parameterize the control input as

$$u(k) = \text{sat}\left(g(x(k))\right) \tag{2.52}$$

where $g(x(k))$ is a state-feedback control law. The saturation function $\text{sat}(u) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$\begin{aligned} \text{sat}(u) &= [\text{sat}(u_1) \ \text{sat}(u_2) \ \dots \ \text{sat}(u_m)]^T \\ \text{sat}(u_l) &= \begin{cases} -1 & \text{if } u_l \leq -1, \\ u_l & \text{if } -1 \leq u_l \leq 1, \\ 1 & \text{if } u_l \geq 1 \end{cases} \end{aligned} \tag{2.53}$$

It is clear that using (2.52), (2.53), the input constraints are automatically fulfilled.

In the following, we briefly review the LDI modeling framework, as proposed in [60, 65]. We will employ this framework to represent the saturation nonlinearity within our system model. Let $\mathscr{D}$ be the set of all $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathscr{D}$. For example, if $m = 2$, the set $\mathscr{D}$ contains four matrices

$$\mathscr{D} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Suppose that each element of $\mathscr{D}$ is labeled as $D_q, q \in \overline{1, 2^m}$. Then, $\mathscr{D} = \{D_q : q \in \overline{1, 2^m}\}$. Define $D_q^- = I - D_q$. It is clear that $D_q^- \in \mathscr{D}$.

Lemma 2.4. *[60] Given matrices $F, H \in \mathbb{R}^{m\times n}$. For any x such that $|H_l x| \leq 1, l \in \overline{1, m}$ where H_l denotes the $l-$th row of H, one has*

$$sat(Fx) \in cov\{D_q Fx + D_q^- Hx\}, q \in \overline{1, 2^m} \tag{2.54}$$

Although the use of a gain-scheduling feedback law can improve system performance to some extent, the ellipsoidal invariant set-based approach for system analysis remains inherently conservative. The primary source of this conservatism lies in the ellipsoidal shape itself: while an ellipsoidal invariant set guarantees quadratic stability of the system, not all stable systems are necessarily quadratically stable.

Assume that, by exploiting results in section 2.4.1, one can determine a feedback gain $F \in \mathbb{R}^{m\times n}$ such that the control law $u(k) = Fx(k)$ quadratically stabilizes the system (2.1). The aim is to get ellipsoid invariant set condition for the closed-loop system

$$x(k+1) = A(k)x(k) + B(k)\,\text{sat}(Fx(k))$$

subject to the state and input constraints given in (2.5). Consider the ellipsoid

$$\mathscr{E}(S^{-1}) = \{x \in \mathbb{R}^n \mid x^T S^{-1} x \leq 1\} \tag{2.55}$$

Theorem 2.5. *If there exist matrices $S \in \mathbb{S}^n, \tilde{Y}_i \in \mathbb{R}^{m\times n}, i \in \overline{1, L}$ such that the following LMI conditions hold:*

$$\begin{bmatrix} S & * \\ A_i S + B_i D_j F S + B_i D_j^- \tilde{Y} & S \end{bmatrix} \succeq 0, \forall i \in \overline{1, L}, j \in \overline{1, 2^m} \tag{2.56}$$

$$\begin{bmatrix} 1 & * \\ \tilde{Y}_l^T & S \end{bmatrix} \succeq 0,\ l \in \overline{1, m} \tag{2.57}$$

$$\begin{bmatrix} 1 & * \\ Sf_l & S \end{bmatrix} \succeq 0,\ \forall l \in \overline{1, n_x} \tag{2.58}$$

where $\tilde{Y} = HS$, and $\tilde{Y}_l$ is the $l-$row of the matrix $\tilde{Y}$, $l \in \overline{1, m}$, then the ellipsoid $\mathscr{E}(S^{-1})$ is controlled invariant and constraint-admissible for (2.1), (2.5) with $u(k) = sat(Fx(k))$.

Proof. For the invariance proof, we need to show that $x(k+1) \in \mathscr{E}(S^{-1})$, for any $x(k) \in \mathscr{E}(S^{-1})$. Pre and post-multiplying (2.56) by

$$\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{2.59}$$

one gets

$$\begin{bmatrix} S^{-1} & * \\ A_i + B_i D_j F + B_i D_j^- H & S \end{bmatrix} \succeq 0, \forall i \in \overline{1, L}, j \in \overline{1, 2^m} \tag{2.60}$$

thus, for each i, j, multiply the corresponding by $p_i(k)$,$\alpha_j(k)$, $i \in \overline{1,L}$, $j \in \overline{1,2^m}$ and sum to get

$$\begin{bmatrix} \sum_{i=1}^{L}\sum_{j=1}^{2^m} p_i(k)\alpha_j(k)S^{-1} & * \\ \sum_{i=1}^{L}\sum_{j=1}^{2^m} p_i(k)\alpha_j(k)(A_i + B_iD_jF + B_iD_j^-H) & \sum_{i=1}^{L}\sum_{j=1}^{2^m} p_i(k)\alpha_j(k)S \end{bmatrix} \succeq 0, \forall i \in \overline{1,L},, j \in \overline{1,2^m} \tag{2.61}$$

Based on lemma 2.4

$$sat(Fx) = \sum_{j=1}^{2^m} \alpha_j(k)(D_jFx + D_j^-Hx) \tag{2.62}$$

for all x such that such that $|H_ix(k)| \leq 1, \forall i \in \overline{1,L}$, thus

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{L} p_i(k)(A_i + B_i\sum_{j=1}^{2^m} \alpha_j(k)(D_jF + D_j^-H))x(k) \\ &= \sum_{i=1}^{L}\sum_{j=1}^{2^m} p_i(k)\alpha_j(k)(A_i + B_i(D_jF + D_j^-H))x(k) \end{aligned} \tag{2.63}$$

Note that

$$\sum_{i=1}^{L}\sum_{j=1}^{2^m} p_i(k)\alpha_j(k) = 1 \tag{2.64}$$

Combing (2.63), using Schur complement for (2.61),one gets

$$x(k)^TS^{-1}x(k) - x(k+1)^TS^{-1}x(k+1) \geq 0.$$

Note $|Hx(k)| \leq 1$, $\forall i \in \overline{1,L}, \forall l \in \overline{1,m}$. based on lemma (2.2), the conditions $|Hx(k)| \leq 1$ for $x \in \mathscr{E}(P^{-1})$ are satisfied if and only if

$$\begin{bmatrix} 1 & * \\ \tilde{Y}^T & S \end{bmatrix} \succeq 0, l \in \overline{1,m}$$

where $\tilde{Y} = HS$. Consider the case where $u(k)$ is a vector. Note that conditions (2.75) impose the constraint on each component of $Hx(k)$.

It remains to prove $u(k) \in \mathscr{U}$ and $x(k) \in \mathscr{X}$. As we employ (2.52), it is clear $u(k) \in \mathscr{U}$. For the state constraint, it is omitted here, as the proof is similar to that of section (2.4.1).

Therefore, for any $x(k)^TS^{-1}x(k) \leq 1$ one has $x(k+1)^TS^{-1}x(k+1) \leq 1$. Hence, $\mathscr{E}(S^{-1})$ is controlled invariant and constraint-admissible for (2.1), (2.5) with $u(k) = sat(Fx(k))$. □

It is noticed that if B_i are the same, it is possible to employ a gain-scheduling saturated feedback of the form $u(k) = sat(F(k)x(k))$, $F(k) = \sum_{i=1}^{L} p_i(k)F_i$.

The problem of computing the largest invariant ellipsoid can be formulated as

$$\begin{array}{l}\max\limits_{S,\tilde{Y}_i}\left\{\sum\limits_{i=1}^{L}\log\det(S)\right\},\\ \text{s.t. } (2.56),(2.57),(2.58)\end{array} \tag{2.65}$$

This is a SDP. We can solve it efficiently using free available LMI parser such as CVX [66] , or YALMIP [67].

2.5 Parameter-dependent ellipsoid invariant set

To address the conservatism associated with a single ellipsoidal invariant set, PDLF approaches have been applied to LPV systems since the early 1990s. The sub-level set of quadratic-form PDLF is naturally associated with a corresponding parameter-dependent ellipsoidal set. Hence, the search for a parameter-dependent ellipsoidal invariant set is of fundamental importance for deriving less conservative stability conditions and for controller design [68]. In this section, we present LMI conditions for the computation of parameter-dependent ellipsoidal invariant sets for constrained LPV system.

Definition 2.5.1. ***(Parameter-dependent controlled invariant set)*** *The set $\Omega(p)\subset\mathbb{R}^n$ is said to be parameter-dependent controlled invariant with respect to the dynamics (2.1) if and only if*

$$\forall x(k)\in\Omega(p(k)),\exists u(k)=U(x(k),p(k)) \text{ such that } x(k+1)\in\Omega(p(k+1)).$$

If, in addition, $\Omega(p(k))\subseteq\mathscr{X}$ and $U(x(k),p(k))\in\mathscr{U}$, then $\Omega(p(k))$ is constraint-admissible with respect to (2.5).

We will deal several times with the following double sum positivity problem of the form

$$x^T\left(\sum_{i,j=1}^{L}p_ip_jM_{ij}\right)x\geq 0 \tag{2.66}$$

where $x\in\mathbb{R}^n$, $M_{ij}\in\mathbb{R}^{n\times n}$. The parameters p_i satisfy

$$\sum_{i=1}^{L}p_i=1,p_i\geq 0,\forall i=\overline{1,L} \tag{2.67}$$

Lemma 2.6. *[42] Inequality (2.66) holds $\forall x\in\mathbb{R}^n$, $\forall p_i$ satisfying (2.67) if*

$$\begin{cases}M_{ii}\succeq 0,\forall i\in\overline{1,L}\\ \frac{2}{L-1}M_{ii}+M_{ij}+M_{ji}\succeq 0,1\leq i\neq j\leq L\end{cases} \tag{2.68}$$

We will use the following lemma to construct a parameter dependent ellipsoid invariant set.

Lemma 2.7. *[38] Given $S \in \mathbb{S}^{n\times n}$, $G \in \mathbb{R}^{n\times n}$. The following relation holds*

$$G^T S^{-1} G \succeq G + G^T - S \tag{2.69}$$

Remark: In the absence of the control input $u(k)$, i.e., when $u(k) = 0$, the dynamics (2.1) reduce to

$$x(k+1) = A(k)x(k). \tag{2.70}$$

In this case, if the same invariance property holds, i.e., $x(k+1) \in \Omega(p(k+1))$ for any $x(k) \in \Omega(p(k))$, then $\Omega(p(k))$ is called a parameter-dependent invariant set.

2.5.1 Parameter-dependent ellipsoidal invariant set with nonlinearly parameterized feedback

Consider the following gain-scheduling nonlinear control law

$$u(k) = F(k)G(k)^{-1}x(k) \tag{2.71}$$

where $F(k) = \sum_{i=1}^{L} p_i(k)F_i, F_i \in \mathbb{R}^{m\times n}$, $G(k) = \sum_{i=1}^{L} p_i(k)G_i, G_i \in \mathbb{R}^{n\times n}$ and the following parameter-dependent ellipsoid

$$\mathscr{E}(P(k)^{-1}) = \left\{x \in \mathbb{R}^n : x^T P(k)^{-1} x \leq 1\right\} \tag{2.72}$$

where $P(k) = \sum_{i=1}^{L} p_i(k)P_i$ and $P_i \in \mathbb{S}^n$.

Theorem 2.8. *If there exist matrices $P_i \in \mathbb{S}^n, F_i \in \mathbb{R}^{m\times n}, G_i \in \mathbb{R}^{n\times n}, i \in \overline{1,L}$ such that the following LMI conditions hold:*

$$X_{iis} \succeq 0, i \in \overline{1,L}, s \in \overline{1,L} \tag{2.73}$$

$$\frac{2}{L-1}X_{iis} + X_{ijs} + X_{jis} \succeq 0, 1 \leq i \neq j \leq L, s \in \overline{1,L} \tag{2.74}$$

$$\begin{bmatrix} 1 & F_{i,l} \\ F_{i,l}^T & G_i^T + G_i - P_i \end{bmatrix} \succeq 0, i \in \overline{1,L}, l \in \overline{1,m} \tag{2.75}$$

$$\begin{bmatrix} 1 & f_l^T P_i \\ P_i f_l & P_i \end{bmatrix} \succeq 0, i \in \overline{1,L}, l \in \overline{1,n_x} \tag{2.76}$$

where $F_{i,l}$ is the $l-$row of the matrix F_i, $i \in \overline{1,L}, l \in \overline{1,m}$

$$X_{ijs} = \begin{bmatrix} G_i^T + G_i - P_i & G_j^T A_i^T + F_j^T B_i^T \\ A_i G_j + B_i F_j & P_s \end{bmatrix}, i \in \overline{1,L}, j \in \overline{1,L}, s \in \overline{1,L}$$

then the ellipsoid $\mathscr{E}(P(k)^{-1})$ is parameter-dependent controlled invariant and constraint-admissible for (2.1), (2.5).

Proof. By using Lemma 2.6, (2.73) and (2.74) hold, which implies that the following inequality is satisfied:

$$\sum_{i,j=1}^{L} p_i(k)p_j(k)X_{ijs} \succeq 0 \tag{2.77}$$

The inequality can be equivalently rewritten as

$$\begin{bmatrix} G(k)^T + G(k) - P(k) & G(k)^T A(k)^T + F(k)^T B(k)^T \\ A(k)G(k) + B(k)F(k) & P_s \end{bmatrix} \succeq 0 \tag{2.78}$$

Multiplying both sides by $p_s(k+1)$ and summing over all s, we obtain

$$\begin{bmatrix} G(k)^T + G(k) - P(k) & G(k)^T A(k)^T + F(k)^T B(k)^T \\ A(k)G(k) + B(k)F(k) & P(k+1) \end{bmatrix} \succeq 0$$

Using Lemma 2.7, one gets,

$$\begin{bmatrix} G(k)^T P(k)^{-1} G(k) & G(k)^T A(k)^T + F(k)^T B(k)^T \\ A(k)G(k) + B(k)F(k) & P(k+1) \end{bmatrix} \succeq 0$$

By pre- and post-multiplying with $\mathrm{diag}(G(k)^{-T}, I)$ and $\mathrm{diag}(G(k)^{-1}, I)$, one obtains

$$\begin{bmatrix} P(k)^{-1} & A(k)^T + G(k)^{-T} F(k)^T B(k)^T \\ A(k) + B(k)F(k)G(k)^{-1} & P(k+1) \end{bmatrix} \succeq 0$$

By using Schur complement, one gets

$$P(k)^{-1} - \big(A(k) + B(k)F(k)G(k)^{-1}\big)^T P(k+1)^{-1} \big(A(k) + B(k)F(k)G(k)^{-1}\big) \succeq 0$$

It follows that

$$x(k)P(k)^{-1}x(k) - x(k)^T (A(k) + BF(k)G(k)^{-1})^T P(k+1)^{-1} (A(k) + BF(k)G(k)^{-1})x(k) \geq 0,$$

or equivalently,

$$x(k)P(k)^{-1}x(k) \geq x(k+1)P(k+1)^{-1}x(k+1).$$

Hence, for any $x(k)$ such that $x(k)P(k)^{-1}x(k) \leq 1$ one has

$$x(k+1)P(k+1)^{-1}x(k+1) \leq 1.$$

Therefore, $x(k+1) \in \mathscr{E}(P(k+1)^{-1})$. It follows that $\mathscr{E}(P(k)^{-1})$ is controlled invariant.

It remains to prove the constraint admissibility, i.e., $\forall x(k) \in \mathscr{E}(P(k)^{-1})$, one has $x(k) \in \mathscr{X}$ and $u(k) = F(k)G(k)^{-1}x(k) \in \mathscr{U}$.

Consider first the case where $u(k)$ is a scalar, define

$$\begin{aligned} &v_u(t) = \max_{x(k)} \left\{ F(k)G(k)^{-1}x(k) \right\}, \\ &\text{s.t. } x(k)^T P(k)^{-1} x(k) \leq 1 \end{aligned} \tag{2.79}$$

Using the method of Lagrange multipliers, it can be shown that

$$v_u^*(t) = \sqrt{F(k)G(k)^{-1}P(k)G(k)^{-T}F(k)^T}$$

As the set $\mathscr{E}(P(k)^{-1})$ is symmetric, one has

$$-1 \leq F(k)G(k)^{-1}x(k) \leq 1$$

$\forall x(k) \in \mathscr{E}(P(k)^{-1})$. Hence

$$1 - F(k)G(k)^{-1}P(k)G(k)^{-T}F(k)^T \geq 0$$

Thus, using Schur complements

$$\begin{bmatrix} 1 & * \\ F(k)^T & G(k)^T P(k)^{-1} G(k) \end{bmatrix} \succeq 0 \tag{2.80}$$

Using Lemma 2, it follows that if (2.75) holds, then (2.80) is satisfied $\forall t \geq 0$.

Now, consider the case where $u(k)$ is a vector. Note that conditions (2.75) impose the constraint on each component of $u(k)$. (2.76) is for the state constraint satisfaction. The proof is omitted here, as it follows closely the proof of the input constraints. □

2.5.2 Parameter-dependent ellipsoidal invariant set with linear feedback – A double-sum-free approach

Although Theorem 2.8 provides LMI conditions for the parameter-dependent ellipsoidal invariant set, it is worth noting that it leads to a double-sum formulation. The relaxation techniques required to handle the double-sum terms increase both the computational complexity and the conservatism of the resulting algorithm.

In this section, a design method for the parameter-dependent ellipsoidal invariant set is proposed to avoid the double-sum formulation.

Consider the following discrete-time LPV system

$$x(k+1) = A(k)x(k) + Bu(k) \tag{2.81}$$

It is emphasized that all local models in (2.81) share the same input matrix B. As noted in [19] that this does not impose a significant restriction. The condition is naturally satisfied if actuator dynamics are included in the models. Otherwise, it can be ensured by filtering the input.

Consider the following gain-scheduling control law

$$u(k) = \left(\sum_{i=1}^{L} p_i(k) F_i \right) x(k) \tag{2.82}$$

and the following parameter-dependent ellipsoid

$$\mathscr{E}(\mathbf{P}(k)) = \left\{x \in \mathbb{R}^n : x^T \mathbf{P}(k) x \leq 1\right\} \tag{2.83}$$

where $\mathbf{P}(k) = \sum_{i=1}^{L} p_i(k)\mathbf{P}_i^{-1}$, and $\mathbf{P}_i \in \mathbb{S}^n$.

Theorem 2.9. *If there exist matrices $\mathbf{P}_i \in \mathbb{S}^n, \mathbf{Y}_i \in \mathbb{R}^{m\times n}, i \in \overline{1,L}$ such that the following LMI conditions hold:*

$$\begin{bmatrix} \mathbf{P}_i & * \\ A_i\mathbf{P}_i + B\mathbf{Y}_i & \mathbf{P}_j \end{bmatrix} \succeq 0, \quad \forall i \in \overline{1,L},\ \forall j \in \overline{1,L}. \tag{2.84}$$

$$\begin{bmatrix} 1 & * \\ \mathbf{Y}_{i,l} & \mathbf{P}_i \end{bmatrix} \succeq 0,\ i \in \overline{1,L}, l \in \overline{1,m} \tag{2.85}$$

$$\begin{bmatrix} 1 & * \\ \mathbf{P}_i f_l & \mathbf{P}_i \end{bmatrix} \succeq 0,\ i \in \overline{1,L}, l \in \overline{1,n_x} \tag{2.86}$$

where $\mathbf{Y}_i = F_i\mathbf{P}_i, \forall i \in \overline{1,s}$, and $\mathbf{Y}_{i,l}$ is the $l-$row of the matrix $\mathbf{Y}_i$ then the ellipsoid $\mathscr{E}(\mathbf{P}(k))$ is parameter-dependent controlled invariant and constraint-admissible for (2.81)*,* (2.5)*.*

Proof. First we will prove that $\mathscr{E}(\mathbf{P}(k))$ is controlled invariant, then we will show that $\mathscr{E}(\mathbf{P}(k))$ is constrained-admissible.

For the invariance proof, we need to show that for any $x(k) \in \mathscr{E}(\mathbf{P}(k))$, there exists $u(k)$ such that $x(k+1) \in \mathscr{E}(\mathbf{P}(k+1))$. For each i, pre- and post-multiplying (2.84) by

$$\begin{bmatrix} \mathbf{P}_i^{-1} & * \\ 0 & I \end{bmatrix}$$

one gets

$$\begin{bmatrix} \mathbf{P}_i^{-1} & * \\ A_i + B\mathbf{Y}_i\mathbf{P}_i^{-1} & \mathbf{P}_j \end{bmatrix} \succeq 0, i \in \overline{1,L}, j \in \overline{1,L}.$$

Recall that $F_i = \mathbf{Y}_i\mathbf{P}_i^{-1}$. For each $i \in \overline{1,L}$, multiply the corresponding by $p_i(k)$ and sum to get

$$\begin{bmatrix} \mathbf{P}(k) & * \\ A(k) + BF(k) & \mathbf{P}_j \end{bmatrix} \succeq 0, j \in \overline{1,L}.$$

Using Schur complement, one obtains

$$\mathbf{P}(k) - (A(k) + BF(k))^T \mathbf{P}_j^{-1} (A(k) + BF(k)) \succeq 0,$$

thus, for each $j \in \overline{1,L}$, multiply the corresponding by $p_j(k+1)$ and sum to get

$$\mathbf{P}(k) - (A(k) + BF(k))^T \mathbf{P}(k+1)(A(k) + BF(k)) \succeq 0.$$

It follows that

$$x(k)^T\mathbf{P}(k)x(k) - x(k)^T(A(k)+BF(k))^T\mathbf{P}(k+1)(A(k)+BF(k))x(k) \geq 0,$$

or equivalently,

$$x(k)^T\mathbf{P}(k)x(k) \geq x(k+1)^T\mathbf{P}(k+1)x(k+1).$$

Hence, for any $x(k)$ such that $x(k)\mathbf{P}(k)x(k) \leq 1$ one has

$$x(k+1)^T\mathbf{P}(k+1)x(k+1) \leq 1.$$

Therefore, $x(k+1) \in \mathscr{E}(\mathbf{P}(k+1))$. It follows that $\mathscr{E}(\mathbf{P}(k))$ is controlled invariant.

It remains to prove the constraint admissibility, i.e., $\forall x(k) \in \mathscr{E}(\mathbf{P}(k))$, one has $x(k) \in \mathscr{X}$ and $u(k) = \left(\sum_{i=1}^{L} p_i(k)F_i\right)x(k) \in \mathscr{U}$.

For the state constraints, we need to show that $\mathscr{E}(\mathbf{P}(k)) \subseteq \mathscr{X}$. Using (2.82), one has $\mathscr{E}(\mathbf{P}(k)) \subseteq \mathscr{X}$ if and only if $\mathscr{E}(\mathbf{P}_i^{-1}) \subseteq \mathscr{X}, i \in \overline{1,L}$. Using lemma (2.2), this condition is satisfied if and only if

$$\begin{bmatrix} 1 & * \\ \mathbf{P}_i f_l & \mathbf{P}_i \end{bmatrix} \succeq 0,\ i \in \overline{1,L}, l \in \overline{1,n_x}$$

This is the condition (2.86).

We omit the proof for the input constraints, as it is analogous to that of the proof for the state constraints. □

To ensure that the set $\mathscr{E}(\mathbf{P}(k))$ is a parameter-dependent invariant and contractive set, condition (2.84) can be formulated as

$$\begin{bmatrix} \mathbf{P}_i & * \\ A_i\mathbf{P}_i + B\mathbf{Y}_i & \mathbf{P}_j \end{bmatrix} \succ 0, \quad \forall i \in \overline{1,L},\ \forall j \in \overline{1,L}. \tag{2.87}$$

Moreover, the ellipsoid $\mathscr{E}(\mathbf{P}^{-1})$ can be interpreted as a sublevel set of the parameter-dependent Lyapunov function

$$V(x) = x^T\mathbf{P}(k)x, \quad \mathbf{P}(k) = \sum_{i=1}^{L} p_i(k)\mathbf{P}_i^{-1} \tag{2.88}$$

Since $V(k+1) - V(k) = 0$ holds if and only if $x = 0$, LaSalle's invariance principle guarantees the asymptotic stability of the system. Therefore, the invariant set condition (2.84) also serves as a Lyapunov stability condition.

2.5.3 Parameter-dependent invariant ellipsoidal for saturated feedback

Based on the results in Section 2.5, assume that a feedback gain $F(k) \in \mathbb{R}^{m\times n}$, $G(k) \in \mathbb{R}^{n\times n}$ has been obtained such that the control law $u(k) = F(k)G(k)^{-1}x(k)$ ensures stability of (2.1).

The goal is to establish parameter-dependent ellipsoidal invariant set conditions for the resulting closed-loop system

$$x(k+1) = A(k)x(k) + B(k)\,\mathrm{sat}\big(F(k)G(k)^{-1}x(k)\big)$$

under the constraints specified in (2.5).

Consider the following parameter-dependent ellipsoid

$$\mathscr{E}(S(k)^{-1}) = \big\{x \in \mathbb{R}^n : x^T S(k)^{-1} x \le 1\big\} \tag{2.89}$$

where $S(k) = \sum_{i=1}^{L} p_i(k) S_i$ and $S_i \in \mathbb{S}^n$.

Based on Lemma 2.4, we have the following proposition.

Proposition 1. *Given matrices $F(k), H(k) \in \mathbb{R}^{m\times n}$, $G(k) \in \mathbb{R}^{n\times n}$ with $H(k) = \sum_{i=1}^{L} p_i(k) H_i$. For any x such that $|H(k)G(k)^{-1}x| \le 1$, one has*

$$sat(F(k)G(k)^{-1}x) \in cov\{D_q F(k)G(k)^{-1}x + D_q^- H(k)G(k)^{-1}x\}, q \in \overline{1, 2^m} \tag{2.90}$$

Theorem 2.10. *If their exist matrices $S_i \in \mathbb{S}^n$, $F_i \in \mathbb{R}^{m\times n}$, $H_i \in \mathbb{R}^{m\times n}$, $G_i \in \mathbb{R}^{n\times n}$ such that the following LMI conditions hold:*

$$X_{iisq} \succeq 0, i \in \overline{1, L},\ s \in \overline{1, L},\ q \in \overline{1, 2^m} \tag{2.91}$$

$$\frac{2}{L-1} X_{iisq} + X_{ijsq} + X_{jisq} \succeq 0, 1 \le i \ne j \le L, s \in \overline{1, L}, q \in \overline{1, 2^m} \tag{2.92}$$

$$\begin{bmatrix} 1 & H_{i,l} \\ H_{i,l}^T & G_i^T + G_i - S_i \end{bmatrix} \succeq 0, i \in \overline{1, L}, l \in \overline{1, m} \tag{2.93}$$

$$\begin{bmatrix} 1 & f_l^T S_i \\ S_i f_l & S_i \end{bmatrix} \succeq 0, i \in \overline{1, L}, l \in \overline{1, n_x} \tag{2.94}$$

where $H_{i,l}$ is the l-row of the matrix H_i, $i \in \overline{1, L}, l \in \overline{1, m}$

$$X_{ijqs} = \begin{bmatrix} G_i^T + G_i - S_i & * \\ A_i G_j + B_i(D_q F_j + D_q^- H_j) & S_s \end{bmatrix}, i \in \overline{1, L}, j \in \overline{1, L}, s \in \overline{1, L}, q \in \overline{1, 2^m} \tag{2.95}$$

then the ellipsoid $\mathscr{E}(S(k)^{-1})$ is parameter-dependent controlled invariant and constraint admissible for (2.1), (2.5).

Proof. Based on Lemma 2.6, (2.91), (2.92) imply the following inequality is satisfied

$$\sum_{i,j=1}^{L} p_i(k)p_j(k)X_{ijqs} \succeq 0 \tag{2.96}$$

and can be rewritten as

$$\begin{bmatrix} G(k)^T + G(k) - S(k) & * \\ A(k)G(k) + B(k)(D_qF(k) + D_q^- H(k)) & S_s \end{bmatrix} \succeq 0, s \in \overline{1,L}, q \in \overline{1,2^m} \tag{2.97}$$

Multiplying both sides by $p_s(k+1)$ and summing over all s, we obtain

$$\begin{bmatrix} G(k)^T + G(k) - S(k) & * \\ A(k)G(k) + B(k)(D_qF(k) + D_q^- H(k)) & S(k+1) \end{bmatrix} \succeq 0, q \in \overline{1,2^m}$$

Using Lemma 2.7, the above inequality holds which implies that the following inequality is satisfied

$$\begin{bmatrix} G(k)^T S(k)^{-1} G(k) & * \\ A(k)G(k) + B(k)(D_qF(k) + D_q^- H(k)) & S(k+1) \end{bmatrix} \succeq 0, q \in \overline{1,2^m} \tag{2.98}$$

Pre- and post-multiplying with diag($G(k)^{-T}$,I) and diag($G(k)^{-1}$,I), one obtains

$$\begin{bmatrix} S(k)^{-1} & * \\ A(k) + B(k)\big(D_qF(k)G(k)^{-1} + D_q^- H(k)G(k)^{-1}\big) & S(k+1) \end{bmatrix} \succeq 0, q \in \overline{1,2^m}$$

By using Schur complement, and based on the Proposition 1, one gets

$$\begin{aligned} x(k)^T S^{-1} x(k) - \big(A(k)x(k) + B(k)sat(F(k)G(k)^{-1}x(k))\big)^T \\ \times S(k+1)^{-1}\big(A(k)x(k) + B(k)sat(F(k)G(k)^{-1}x(k))\big) \geq 0 \end{aligned}$$

or equivalently,

$$x(k)^T S(k)^{-1} x(k) \geq x(k+1)^T S(k+1)^{-1} x(k+1)$$

Hence, for any $x(k)$ such that $x(k)S(k)^{-1}x(k) \leq 1$, one has

$$x(k+1)S(k+1)^{-1}x(k+1) \leq 1.$$

Therefore, $x(k+1) \in \mathscr{E}(S(k+1)^{-1})$. It follows that $\mathscr{E}(S(k)^{-1})$ is controlled invariant. Condition (2.93) guarantees that, for any $x(k) \in \mathscr{E}(S(k)^{-1})$ such that $|H(k)G(k)^{-1}x(k)| \leq 1$. Condition (2.94) characterizes the state constraint. The proof proceeds along the same lines as that of Theorem 2.8. □

Remark: Similar to the LMI conditions proposed in Theorem 2.8 for computing the parameter-dependent ellipsoid, Theorem 2.10 also introduces a double-sum problem, which implies that the resulting algorithm still exhibits a certain degree of conservatism. In addition, regarding the LDI modeling approach for the nonlinear input saturation $sat(F(k)G(k)^{-1}x)$, the auxiliary matrix $H(k)G(k)^{-1}$ shares the same $G(k)^{-1}$ term as $F(k)G(k)^{-1}$ in our formulation. This structural constraint also introduces additional conservatism to some extent. These issues will be specifically addressed in the next section.

2.5.3.1 Double-sum free approach

Assume that, by exploiting results in section 2.5.2, one can determine a feedback gain $F(k) \in \mathbb{R}^{m\times n}$ such that the control law $u(k) = F(k)x(k)$ quadratically stabilizes the system (2.81). The aim is to get the parameter-dependent ellipsoid invariant set condition for the closed-loop system

$$x(k+1) = A(k)x(k) + B\,\mathrm{sat}(F(k)x(k))$$

subject to the state and input constraints given in (2.5).

Consider the following ellipsoid

$$\mathscr{E}(\mathbf{S}(k)) = \{x \in \mathbb{R}^n : x^T\mathbf{S}(k)x \leq 1\}$$

where $\mathbf{S}(k) = \sum_{i=1}^{L} p_i(k)\mathbf{S}_i^{-1}$, and $\mathbf{S}_i \in \mathbb{S}^n$.

The following lemma provides a slight generalization of the result in Lemma 2.4 [60].

Lemma 2.11. *Given matrices $F(k), H(k) \in \mathbb{R}^{m\times n}$ with $H(k) = \sum_{i=1}^{L} p_i(k)H_i$. For any x such that $|H_{i,l}x| \leq 1, i \in \overline{1,L}, l \in \overline{1,n_u}$ where $H_{i,l}$ denotes the $l-$th row of H_i, one has*

$$sat(F(k)x) \in cov\{D_qF(k)x + D_q^-H(k)x\}, q \in \overline{1,2^m} \tag{2.99}$$

Based on Lemma 2.11, for LPV system (2.81)

$$x(k+1) \in A(k)x(k) + B\mathrm{cov}\{D_qF(k)x(k) + D_q^-H(k)x(k)\}, q \in \overline{1,2^m}$$

or equivalently

$$x(k+1) \in \mathrm{cov}\left\{\left(A(k) + BD_qF(k) + BD_q^-H(k)\right)x(k)\right\}, q \in \overline{1,2^m} \tag{2.100}$$

Define

$$\mathscr{A}_{i,q} = A_i + BD_qF_i + BD_q^-H_i, i \in \overline{1,L}, q \in \overline{1,2^m} \tag{2.101}$$

Using (2.100) one obtains

$$x(k+1) \in \mathrm{cov}\left\{\mathscr{A}_q(k)x(k)\right\}, q \in \overline{1,2^m} \tag{2.102}$$

where $\mathscr{A}_q(k) = \sum_{i=1}^{L} p_i(k)\mathscr{A}_{i,q}, q \in \overline{1,2^m}$. It is emphasized that (2.102) is valid only for $x(k)$ such that $|H_{i,l}x(k)| \leq 1, \forall i \in \overline{1,L}, \forall l \in \overline{1,m}$.

Theorem 2.12. *If there exist matrices $\mathbf{S}_i \in \mathbb{S}^n$, $\hat{\mathbf{Y}}_i \in \mathbb{R}^{m\times n}$, $\forall i \in \overline{1,s}$ such that the following LMI conditions hold*

$$\begin{bmatrix} \mathbf{S}_i & * \\ (A_i + BD_qF_i)\mathbf{S}_i + BD_q^-\hat{\mathbf{Y}}_i & \mathbf{S}_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m} \tag{2.103}$$

$$\begin{bmatrix} 1 & * \\ \hat{\mathbf{Y}}_{i,l}^T & \mathbf{S}_i \end{bmatrix} \succeq 0,\ \forall i \in \overline{1,L}, \forall l \in \overline{1,m} \tag{2.104}$$

$$\begin{bmatrix} 1 & * \\ \mathbf{S}_i f_l & \mathbf{S}_i \end{bmatrix} \succeq 0,\ \forall i \in \overline{1,L}, \forall l \in \overline{1,n_x} \tag{2.105}$$

where $\hat{\mathbf{Y}}_i = H_i\mathbf{S}_i, \forall i \in \overline{1,L}$*, and* $\hat{\mathbf{Y}}_{i,l}$ *is the* $l-$*row of the matrix* $\hat{\mathbf{Y}}_i$*, then the ellipsoid* $\mathscr{E}(\mathbf{S}(k))$ *is parameter-dependent invariant for* (2.81) *and constraint-admissible with respect to* (2.5) *under* $u(k) = sat(F(k)x(k))$.

Proof. We omit the proof of constraint-admissibility as it follows the same lines as the proof of Theorem 2.9.

For the parameter-dependent invariance proof, we need to show that $x(k+1) \in \mathscr{E}(\mathbf{S}(k+1))$, for any $x(k) \in \mathscr{E}(\mathbf{S}(k))$. Pre- and post-multiplying (2.103) by

$$\begin{bmatrix} \mathbf{S}_i^{-1} & * \\ \mathbf{0} & I \end{bmatrix}$$

one gets

$$\begin{bmatrix} \mathbf{S}_i^{-1} & * \\ A_i + BD_qF_i + BD_q^-H_i & \mathbf{S}_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}$$

thus, for each i, multiply the corresponding by $p_i(k)$, $i \in \overline{1,L}$ and sum to get

$$\begin{bmatrix} \mathbf{S}(k) & * \\ A(k) + B(D_qF(k) + D_q^-H(k)) & \mathbf{S}_j \end{bmatrix} \succeq 0, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}$$

Using the Schur complements, one obtains

$$\mathbf{S}(k) - \mathscr{A}_q(k)^T\mathbf{S}_j^{-1}\mathscr{A}_q(k) \succeq 0, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}$$

For each j, multiply the corresponding by $p_j(k+1)$ and sum to get

$$\mathbf{S}(k) - \mathscr{A}_q(k)^T\mathbf{S}(k+1)\mathscr{A}_q(k) \succeq 0, \forall q \in \overline{1,2^m}$$

It follows that

$$x(k)^T\mathbf{S}(k)x(k) - x(k)^T\mathscr{A}_q(k)^T\mathbf{S}(k+1)\mathscr{A}_q(k)x(k) \geq 0, \forall q \in \overline{1,2^m}$$

Using (2.102), one gets

$$x(k)^T\mathbf{S}(k)x(k) - x(k+1)^T\mathbf{S}(k+1)x(k+1) \geq 0$$

Therefore, for any $x(k)^T\mathbf{S}(k)x(k) \leq 1$ one has $x(k+1)^T\mathbf{S}(k+1)x(k+1) \leq 1$. Hence, $\mathscr{E}(\mathbf{S}(k))$ is parameter-dependent invariant. □

Once the invariance and constraint-admissibility conditions are obtained, our next step is to maximize the size of $\mathscr{E}(\mathbf{S}(k))$. It can be done by maximizing the volume of the associated ellipsoids $\mathscr{E}(\mathbf{S}_i^{-1}), i \in \overline{1,L}$. The problem of computing the largest invariant ellipsoid can be formulated as

$$\begin{aligned} &\max_{\mathbf{S}_i, \hat{Y}_i} \left\{ \sum_{i=1}^{L} \log\det(\mathbf{S}_i) \right\} \\ &\text{s.t. (2.103), (2.104), (2.105)} \end{aligned} \tag{2.106}$$

This is a SDP. We can solve it efficiently using free available LMI parser such as CVX , or YALMIP.

Problem (2.106) admits two equivalent interpretations. For a given set of feedback gains F_i, it amounts to computing the largest parameter-dependent invariant ellipsoid under the saturated control law $u(k) = \text{sat}(F(k)x(k))$. Alternatively, it can be viewed as a controller synthesis problem, where the feedback gains F_i are optimized so as to maximize the volume of the invariant ellipsoid $\mathscr{E}(\mathbf{S}(k))$. For the controller synthesis problem, (2.103) can be formulated as

$$\begin{bmatrix} \mathbf{S}_i & * \\ A_i + B(D_q Z_i + D_q^- \hat{\mathbf{Y}}_i) & S_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m} \tag{2.107}$$

where $Z_i = F_i \mathbf{S}_i$.

2.6 Low and high gain controller design

2.6.1 Low-gain controller design

Theorem 2.9 establishes a necessary and sufficient condition for the invariance and constraint admissibility of the parameter-dependent ellipsoid $\mathscr{E}(\mathbf{P}(k))$. Once the invariance and constraint-admissibility conditions are obtained, our next step is to maximize the size of $\mathscr{E}(\mathbf{P}(k))$. It can be done by maximizing the volume of the associated ellipsoids $\mathscr{E}(\mathbf{P}_i^{-1}), i \in \overline{1,L}$. The problem of computing the largest invariant ellipsoid can be formulated as

$$\begin{aligned} &\max_{\mathbf{P}_i, \mathbf{Y}_i} \left\{ \sum_{i=1}^{L} \log\det(\mathbf{P}_i) \right\} \\ &\text{s.t. (2.84), (2.85), (2.86)} \end{aligned} \tag{2.108}$$

This is a SDP. We can solve it efficiently using free available LMI parser such as CVX, YALMIP. In the following, we provide a way to take into account the information about the initial state $x(0)$ and the parameter $p(0)$. Our problem is to optimize the set $\mathscr{E}(\mathbf{P}(0))$ such that $\alpha d \in \mathscr{E}(\mathbf{P}(0))$ where $d = x(0)$ and $\alpha \geq 0$ is a scaling factor. One has

$$\alpha^2 d^T (\sum_{i=1}^{L} p_i(0) \mathbf{P}_i^{-1}) d \leq 1 \tag{2.109}$$

This constraint is non-convex due to the multiplication between α and $\mathbf{P}_i, i \in \overline{1, L}$. We can convexify it by introducing auxiliary variables t_i as

$$\alpha^2 d^T \mathbf{P}_i^{-1} d \leq t_i, \forall i \in \overline{1, L},$$

thus, using the Schur complement,

$$\begin{bmatrix} t_i & * \\ \alpha d & \mathbf{P}_i \end{bmatrix} \succeq 0, \forall i \in \overline{1, L}. \tag{2.110}$$

Using t_i, the constraint (2.109) becomes

$$\sum_{i=1}^{L} p_i(0) t_i \leq 1. \tag{2.111}$$

Thus, the problem of maximizing the size of $\mathscr{E}(S(k))$ can be formulated as

$$\begin{aligned} &\max_{\mathbf{P}_i, \mathbf{Y}_i, t_i, \alpha} \{\alpha\}, \\ &\text{s.t. (2.84), (2.85), (2.86), (2.110), (2.111).} \end{aligned} \tag{2.112}$$

This is a SDP.

2.6.2 High-gain controller design

The main advantage of the control law in Section 2.6.1 is its potentially large DoA. The trade-off, however, is in performance. Because the low-gain control law is generally de-tuned, the time required to regulate the plant to the origin is typically longer than necessary.

The objective of this section is to develop a method for designing a performance-oriented controller. For this purpose, we consider the problem of constructing a gain-scheduling control law

$$u(k) = \left(\sum_{i=1}^{L} p_i(k) K_i \right) x(k) \tag{2.113}$$

that minimizes the following performance cost

$$E \left(\max_{A(p(k)) \in \Psi} \left\{ \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k) \right\} \right) \tag{2.114}$$

subject to a random initial condition $x(0)$ with zero mean, and with unit covariance, i.e., $E(x(0)x(0)^T) = I$. Note that this is an LQR problem [69].

Denote $\sqrt{Q} \in \mathbb{R}^{n \times n}$, $\sqrt{R} \in \mathbb{R}^{m \times m}$, respectively, as a square root of Q and R, i.e., $\sqrt{Q}^T \sqrt{Q} = Q$, $\sqrt{R}^T \sqrt{R} = R$.

Theorem 2.13. *Assume that there exist matrices $\mathscr{S}_i \in \mathbb{S}^{n\times n}$, $Q_i \in \mathbb{S}^n$, and $T_i \in \mathbb{R}^{m\times n}$ such that*

$$\begin{bmatrix} \mathscr{S}_i & * & * & * \\ A_i\mathscr{S}_i + BT_i & \mathscr{S}_j & * & * \\ \sqrt{Q}\mathscr{S}_i & 0 & I & * \\ \sqrt{R}T_i & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L} \tag{2.115}$$

$$\begin{bmatrix} Q_i & * \\ I & \mathscr{S}_i \end{bmatrix} \succeq 0, \forall i \in \overline{1,L} \tag{2.116}$$

where $T_i = K_i\mathscr{S}_i, \forall i \in \overline{1,L}$. Then it holds that

$$E\left(\max_{A(p(k))\in\Psi}\left\{\sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)\right\}\right) \leq \sum_{i=1}^{L} p_i(0)\mathit{trace}(Q_i).$$

Proof. Consider the quadratic function $V(k) = x(k)^T\mathscr{S}(k)x(k)$, where $\mathscr{S}(k) = \left(\sum_{i=1}^{L} p_i(k)\mathscr{S}_i^{-1}\right)$. Pre- and post-multiplying (2.115) by $\mathrm{diag}(\mathscr{S}_i^{-1}, I, I, I)$ and using the fact that $K_i = T_i\mathscr{S}_i^{-1}$, one gets

$$\begin{bmatrix} \mathscr{S}_i^{-1} & * & * & * \\ A_i + BK_i & \mathscr{S}_j & * & * \\ \sqrt{Q} & 0 & I & * \\ \sqrt{R}K_i & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}$$

For each $i \in \overline{1,L}$, multiply the corresponding by $p_i(k)$ and sum to get

$$\begin{bmatrix} \mathscr{S}(k) & * & * & * \\ A(k) + BK(k) & \mathscr{S}_j & * & * \\ \sqrt{Q} & 0 & I & * \\ \sqrt{R}K(k) & 0 & 0 & I \end{bmatrix} \succeq 0, \forall j \in \overline{1,L}$$

thus, using the Schur complement, $\forall j \in \overline{1,L}$

$$\mathscr{S}(k) - \begin{bmatrix} (A(k)+BK(k))^T & \sqrt{Q}^T & K(k)^T\sqrt{R}^T \end{bmatrix} \begin{bmatrix} \mathscr{S}_j^{-1} & * & * \\ 0 & I & * \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} (A(k)+BK(k)) \\ \sqrt{Q} \\ \sqrt{R}K(k) \end{bmatrix} \succeq 0.$$

It follows that, $\forall j \in \overline{1,L}$

$$\begin{aligned} &x(k)\mathscr{S}(k)x(k) - x(k)(A(k)+BK(k))^T\mathscr{S}_j^{-1}(A(k)+BK(k))x(k) \\ &\quad -x(k)^T Q x(k) - x(k)^T K(k)^T R K(k) x(k) \geq 0. \end{aligned}$$

For each $j \in \overline{1, L}$, multiply the corresponding by $p_j(k+1)$ and sum to get

$$x(k)\mathscr{S}(k)x(k) - x(k+1)\mathscr{S}(k+1)x(k+1) \geq x(k)^T Q x(k) + u(k)^T R u(k). \tag{2.117}$$

Since $K(k)$ is an asymptotically stabilizing control law, one has $\lim_{k\to\infty} x(k)\mathscr{S}(k)x(k) = 0$. Summing (2.117) from $k = 0$ to $k = \infty$, one obtains

$$x(0)^T \mathscr{S}(0)x(0) \geq \sum_{k=0}^{\infty} \left(x(k)^T Q x(k) + u(k)^T R u(k) \right). \tag{2.118}$$

One has

$$E\left(x(0)^T \mathscr{S}(0)x(0)\right) = E\left(\text{trace}(\mathscr{S}(0)x(0)x(0)^T)\right).$$

Note that $x(0)^T \mathscr{S}(0)x(0)$ is a random variable with $E(x(0)^T x(0)) = I$. Therefore

$$E\left(x(0)^T S(0)x(0)\right) = \text{trace}(S(0)) = \sum_{i=1}^{L} p_i(0)\text{trace}(\mathscr{S}_i^{-1}).$$

Using the Schur complement for (2.116), one gets

$$Q_i \succeq \mathscr{S}_i^{-1}.$$

It follows that

$$\sum_{i=1}^{L} p_i(0)\text{trace}(Q_i) \geq \sum_{i=1}^{L} p_i(0)\text{trace}(\mathscr{S}_i^{-1}). \tag{2.119}$$

Combining (2.118), (2.119), one obtains

$$\sum_{i=1}^{L} p_i(0)\text{trace}(Q_i) \geq E\left(\sum_{k=0}^{\infty} \left(x(k)^T Q x(k) + u(k)^T R u(k) \right) \right). \tag{2.120}$$

The proof is complete. □

Using theorem 2.13, the problem of designing an optimal gain-scheduling control law for system (2.81) can be formulated as

$$\begin{aligned} &\min_{\mathscr{S}_i, Q_i, T_i} \left\{ \sum_{i=1}^{L} \text{trace}(Q_i) \right\} \\ &\text{s.t. (2.115), (2.116).} \end{aligned} \tag{2.121}$$

Note that in the formulation (2.121), we do not take into account the information about the scheduling parameter in the cost function.

2.7 Numerical Examples

2.7.1 Example 1

This example is adapted from Problem 4 of the benchmark problems described in [70]. The system considered is a two-mass-spring setup. By applying Euler's first-order approximation for the derivative with a sampling time of 0.1 units, the discrete-time LPV system given in (2.81) is obtained with $B = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$ and

$$A_1 = \begin{bmatrix} 1.0 & 0 & 0.1 & 0 \\ 0 & 1.0 & 0 & 0.1 \\ -0.1 & 0.1 & 1.0 & 0 \\ 0.1 & -0.1 & 0 & 1.0 \end{bmatrix}, A_2 = \begin{bmatrix} 1.0 & 0 & 0.1 & 0 \\ 0 & 1.0 & 0 & 0.1 \\ -0.3 & 0.3 & 1.0 & 0 \\ 0.3 & -0.3 & 0 & 1.0 \end{bmatrix}.$$

We impose the following input constraints $-1 \leq u \leq 1$.

To compute the largest single controlled ellipsoidal invariant set, problem (2.48) is solved, yielding

$$P = \begin{bmatrix} 2.2161 & 2.1908 & -0.5433 & -0.5744 \\ 2.1908 & 2.2100 & -0.4589 & -0.5293 \\ -0.5433 & -0.4589 & 0.4643 & 0.2551 \\ -0.5744 & -0.5293 & 0.2551 & 0.3091 \end{bmatrix} \tag{2.122}$$

with the corresponding state feedback gain

$$F = \begin{bmatrix} -1.7226 & 1.4929 & -2.2468 & 1.2714 \end{bmatrix}. \tag{2.123}$$

To obtain the largest parameter-dependent controlled invariant set $\mathscr{E}(P(k))$, problem (2.108) is solved, resulting in the following matrices:

$$\mathbf{P}_1 = \begin{bmatrix} 2.2753 & 2.2410 & -0.5834 & -0.6276 \\ 2.2410 & 2.2827 & -0.4555 & -0.5662 \\ -0.5834 & -0.4555 & 0.5875 & 0.2751 \\ -0.6276 & -0.5662 & 0.2751 & 0.3791 \end{bmatrix} \tag{2.124}$$

$$\mathbf{P}_2 = \begin{bmatrix} 2.2735 & 2.2433 & -0.5738 & -0.6280 \\ 2.2433 & 2.2801 & -0.4680 & -0.5635 \\ -0.5738 & -0.4680 & 0.5361 & 0.2736 \\ -0.6280 & -0.5635 & 0.2736 & 0.4051 \end{bmatrix} \tag{2.125}$$

with the associated feedback gains

$$F_1 = \begin{bmatrix} -2.1141 & 1.8227 & -1.9453 & 0.8394 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.8621 & 0.5277 & -1.8199 & 0.5628 \end{bmatrix}. \tag{2.126}$$

The determinants of matrices P, $\mathbf{P}_1$, and $\mathbf{P}_2$ are 0.0022, 0.0071, and 0.0067, respectively. It can be observed that the volumes of the ellipsoids $\mathscr{E}(\mathbf{P}_1^{-1})$ and $\mathscr{E}(\mathbf{P}_2^{-1})$ are both larger than that of the

ellipsoid $\mathscr{E}(P^{-1})$. Consequently, the parameter-dependent ellipsoidal set $\mathscr{E}(\mathbf{P}(k))$ admits a larger volume than the single ellipsoid case. This indicates that the use of parameter-dependent ellipsoids provides the potential to obtain a larger feasible region.

For the initial condition $x(0) = [-2.4149 \ -2.4335 \ -0.0534 \ 0.0903]^T$, Fig. 2.2 and Fig. 2.3 shows the state and input trajectories as functions of time for parameter-dependent ellipsoid case (solid blue), for single ellipsoid case (solid red).

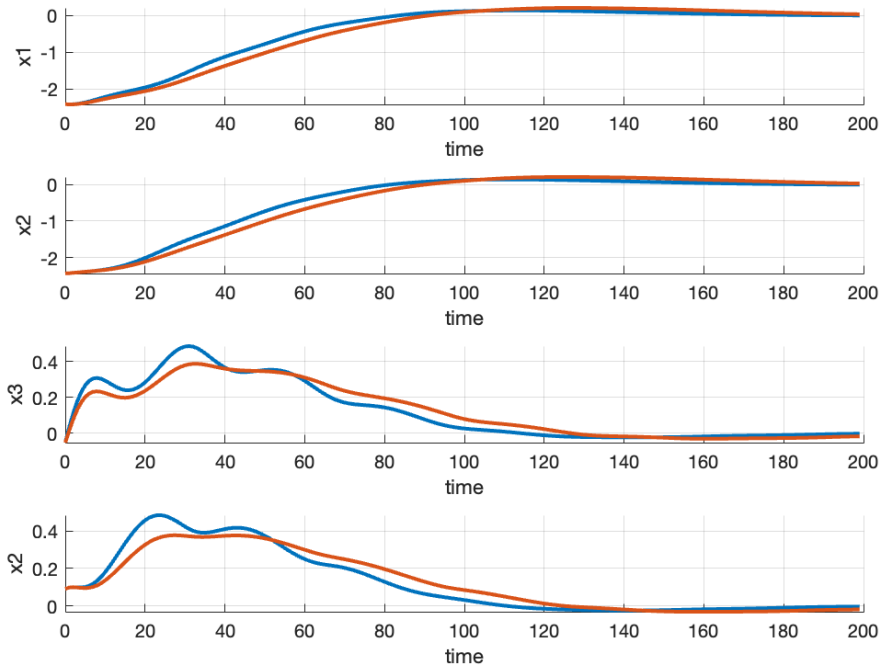


Figure 2.2: State trajectory for parameter-dependent ellipsoid-based approach and for ellipoisd-based approach for example 1

2.7.2 Example 2

We consider the following Van der Pol oscillator system in [37]

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + 2(1 - x_1^2)x_2 + u. \end{cases} \tag{2.127}$$

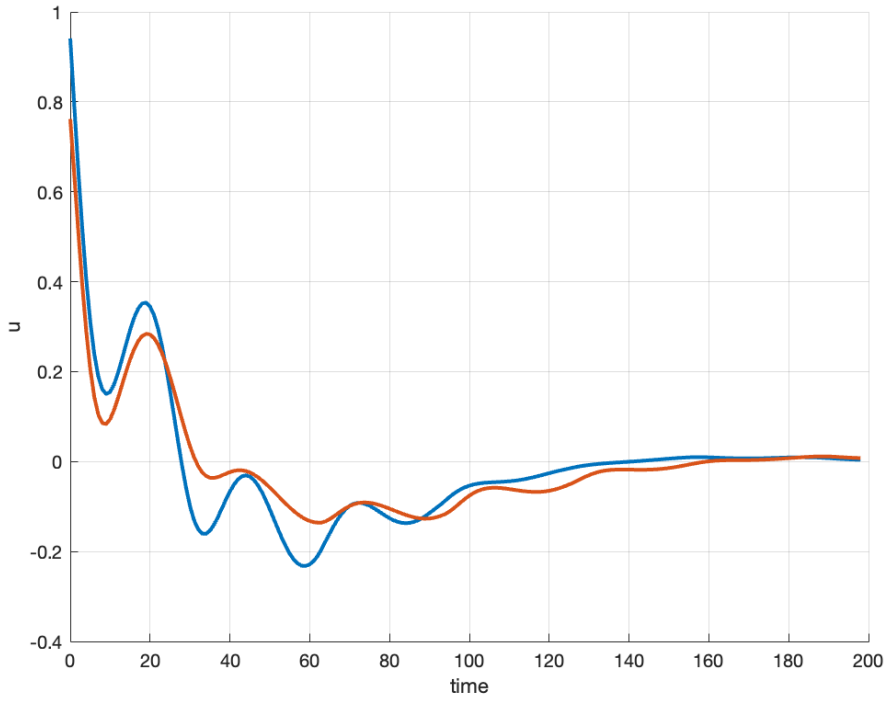


Figure 2.3: Input trajectory for parameter-dependent ellipsoid-based approach and for ellipoisd-based approach for example 1

The input and state constraints are

$$-1 \leq u \leq 1,\ -1 \leq x_1 \leq 1. \tag{2.128}$$

To obtain the discrete time dynamics, we discretize the system using the forward Euler's method with the sampling time 0.1 units. As a result

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 1+0.2\nu(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(k), \tag{2.129}$$

where $\nu(k) = 1 - x_1(k)^2$, one has $0 \leq \nu(k) \leq 1$. Further, we rewrite the obtained discrete time system in the LPV form with scheduling parameters $p_1(k) = \nu(k)$ and $p_2(k) = 1 - p_1(k)$. In this case, we obtain an LPV representation (2.81) with

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 1.2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

The robustly stabilizing feedback matrix gain is calculated by Theorem. 2.13

$$F_1 = \begin{bmatrix} 0.1706 & -3.8215 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.1702 & -3.1887 \end{bmatrix} \tag{2.130}$$

In this example, we compare the following feasible sets:

- the parameter-dependent ellipsoidal feasible set under the gain-scheduling linear feedback law $u(k) = F(k)x(k)$;
- the single invariant ellipsoid under the saturated control input $u(k) = \text{sat}(F(k)x(k))$;
- the parameter-dependent invariant ellipsoidal set under the saturated control input $u(k) = \text{sat}(F(k)x(k))$.

This comparison highlights the impact of input saturation and parameter dependence on the size of the admissible domain.

By solving (2.108), P_i is obtained

$$\mathbf{P}_1 = \begin{bmatrix} 0.3361 & -0.0630 \\ -0.0630 & 0.0622 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 0.3394 & -0.0492 \\ -0.0492 & 0.0921 \end{bmatrix} \tag{2.131}$$

Based on Theorem 2.12, S_i is obtained by solving (2.106)

$$\mathbf{S}_1 = \begin{bmatrix} 0.4072 & -0.0250 \\ -0.0250 & 0.1922 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 0.4078 & -0.0174 \\ -0.0174 & 0.2823 \end{bmatrix} \tag{2.132}$$

Fig. 2.4 presents feasible region based on the different ellipsoid-based invariant set: the parameter-dependent ellipsoidal feasible set under the gain-scheduling linear feedback law $u(k) = F(k)x(k)$(solid yellow for $\mathscr{E}(\mathbf{P}_1^{-1})$, solid purple for $\mathscr{E}(\mathbf{P}_2^{-1})$); the single invariant ellipsoid under the saturated control input $u(k) = \text{sat}(F(k)x(k))$(solid red); the parameter-dependent invariant ellipsoidal set under the saturated control input $u(k) = \text{sat}(F(k)x(k))$(solid blue for $\mathscr{E}(\mathbf{S}_1^{-1})$, solid green for $\mathscr{E}(\mathbf{S}_2^{-1})$).

It can be observed that the ellipsoid $\mathscr{E}(\mathbf{S}(k))$ attains the largest volume among the considered invariant sets, which consequently yields a larger feasible region for the system and reduces the conservatism in the stability analysis.

2.7.3 Example 3

Consider the following LPV system, i.e.

$$A_1 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 0.99 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

The input constraint is $|u(k)| \leq 1$, state constraints are $|x_1(k)| \leq 10$, $|x_2(k)| \leq 10$.

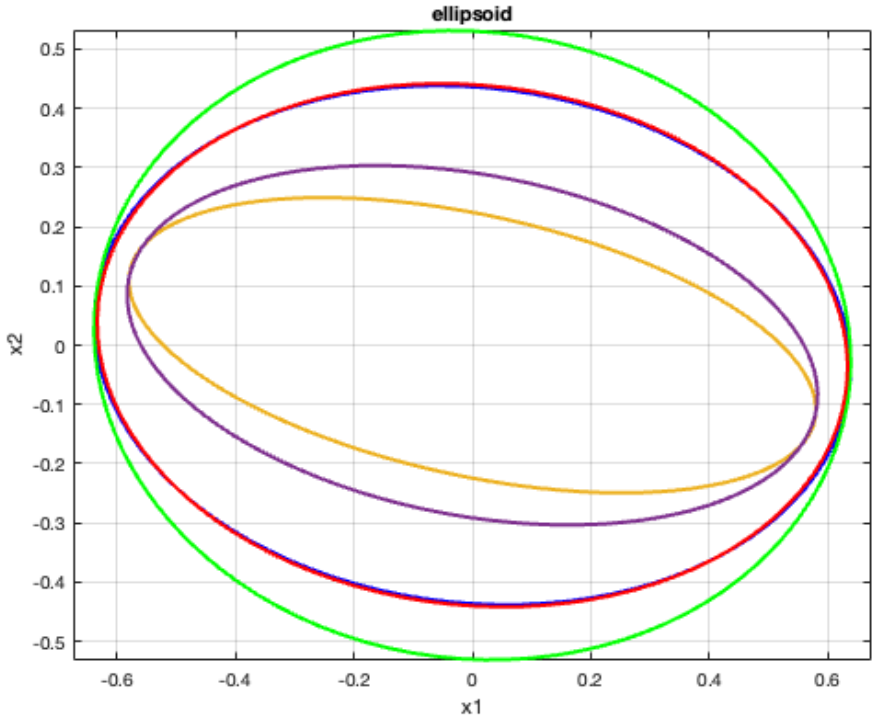


Figure 2.4: Feasible region based on the different ellipsoid-based invariant set: the parameter-dependent ellipsoidal feasible set under the gain-scheduling linear feedback law $u(k) = F(k)x(k)$(solid yellow for $\mathscr{E}(\mathbf{P}_1^{-1})$, solid purple for $\mathscr{E}(\mathbf{P}_2^{-1})$); the single invariant ellipsoid under the saturated control input $u(k) = \text{sat}(F(k)x(k))$(solid red); the parameter-dependent invariant ellipsoidal set under the saturated control input $u(k) = \text{sat}(F(k)x(k))$(solid blue for $\mathscr{E}(\mathbf{S}_1^{-1})$, solid green for $\mathscr{E}(\mathbf{S}_2^{-1})$) for example 2

Based on Theorem 2.12, $\mathbf{S}_i$ and F_i are obtained by solving (2.106)

$$\begin{aligned} \mathbf{S}_1 &= \begin{bmatrix} 5.1589 & -4.1258 \\ -4.1258 & 14.9161 \end{bmatrix} \quad \mathbf{S}_2 = \begin{bmatrix} 5.2162 & -9.4979 \\ -9.4979 & 98.9973 \end{bmatrix} \\ F_1 &= \begin{bmatrix} -1.9417 & -2.1537 \end{bmatrix} \quad F_2 = \begin{bmatrix} -1.8947 & -0.1895 \end{bmatrix} \end{aligned} \tag{2.133}$$

with auxiliary matrices

$$H_1 = \begin{bmatrix} -0.3260 & -0.2862 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.4816 & -0.0424 \end{bmatrix} \tag{2.134}$$

To compare the approach based on the theorem 2.5, solving the optimization problem (2.65), the matrices S, F_i are obtained

$$S = \begin{bmatrix} 4.5815 & -2.2855 \\ -2.2855 & 12.4859 \end{bmatrix}, F_1 = \begin{bmatrix} -0.9938 & -2.0624 \end{bmatrix}, F_2 = \begin{bmatrix} -0.9936 & -0.0915 \end{bmatrix} \quad (2.135)$$

with auxiliary matrices

$$H_1 = \begin{bmatrix} -0.3942 & -0.2403 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.4830 & -0.0404 \end{bmatrix} \quad (2.136)$$

Similarly, for comparison purpose, the matrices P, F are obtained by solving (2.48)

$$P = \begin{bmatrix} 4.1006 & -1.8029 \\ -1.8029 & 11.2236 \end{bmatrix}, \quad F = \begin{bmatrix} -0.4607 & -0.2045 \end{bmatrix} \quad (2.137)$$

Fig. 2.5 presents ellipsoid invariant sets with different approach: the set $\mathscr{E}(S(k))$ is obtained by solving problem (2.106), The set $\mathscr{E}(S^{-1})$ is obtained by solving problem (2.65) and the set $\mathscr{E}(P^{-1})$ is obtained by solving (2.48).

For the initial condition $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$, Fig. 2.6 and Fig. 2.7 shows the state and input trajectories as functions of time.

It can be observed that the time to regulate the plant to the origin by using the linear controller is longer than the time to regulate the plant to the origin by using the saturated controller. Because when using the controller $u(k) = Fx(k)$, the control action is saturated only at some points of the boundary of the ellipsoid $\mathscr{E}(P^{-1})$, when using the controller $u(k) = sat(F(k)x(k))$ the control action is saturated not only on the boundary of the set $\mathscr{E}(S^{-1})$ (or $\mathscr{E}(S(k))$, the saturation being active also inside the set $\mathscr{E}(S^{-1})$(or $\mathscr{E}(S(k))$.

2.8 Conclusion

This chapter investigates the stability of constrained LPV systems based on set invariance theory. Tractable LMI conditions are derived for LPV systems with gain-scheduled inputs to construct parameter-dependent ellipsoidal invariant sets, thereby providing a unified convex optimization framework for the stability analysis of constrained LPV systems. Furthermore, through appropriate modeling of the controlled LPV system and a suitable selection of PDLF, a novel design method for parameter-dependent ellipsoidal invariant sets is proposed. The proposed method avoids the double-sum problem commonly encountered in LMI-based designs, thereby reducing computational complexity. The proposed parameter-dependent ellipsoidal invariant set framework is further extended to LPV systems with input saturation under gain-scheduled state feedback. In addition, low-gain and high-gain controller design methods are developed, which lay the foundation for the merging controllers design approach proposed in Chapter 4.

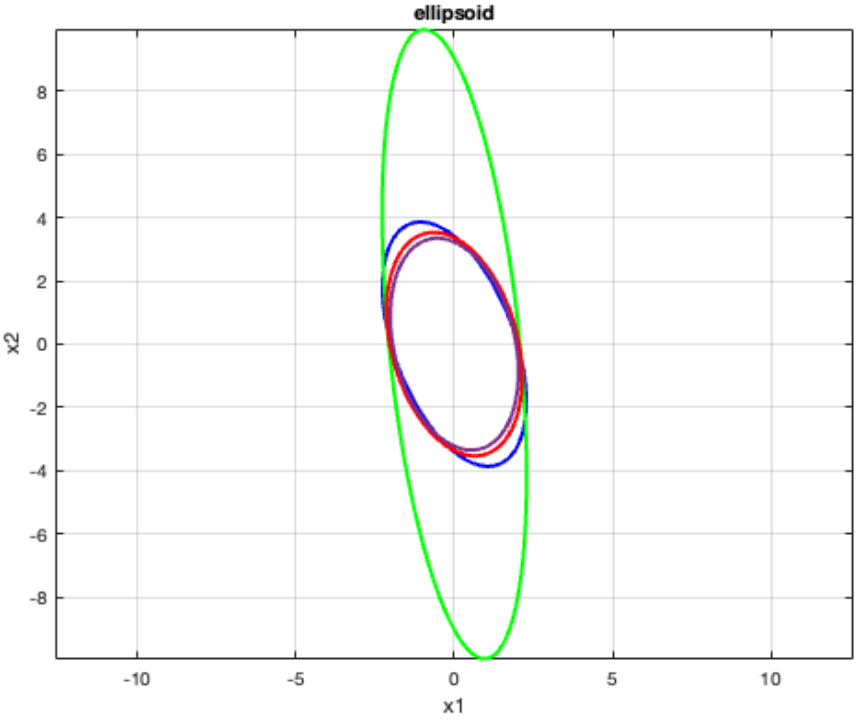


Figure 2.5: Ellipsoid invariant sets with different approach: $\mathscr{E}(\mathbf{S}(k))$ is obtained by solving problem (2.106)(solid blue for $\mathscr{E}(\mathbf{S}_1^{-1})$, solid green for $\mathscr{E}(\mathbf{S}_2^{-1})$), $\mathscr{E}(S^{-1})$ is obtained by solving problem (2.65)(solid red for $\mathscr{E}(S^{-1})$), $\mathscr{E}(P^{-1})$ is obtained by solving (2.48)(solid purple for $\mathscr{E}(P^{-1})$) for example 3

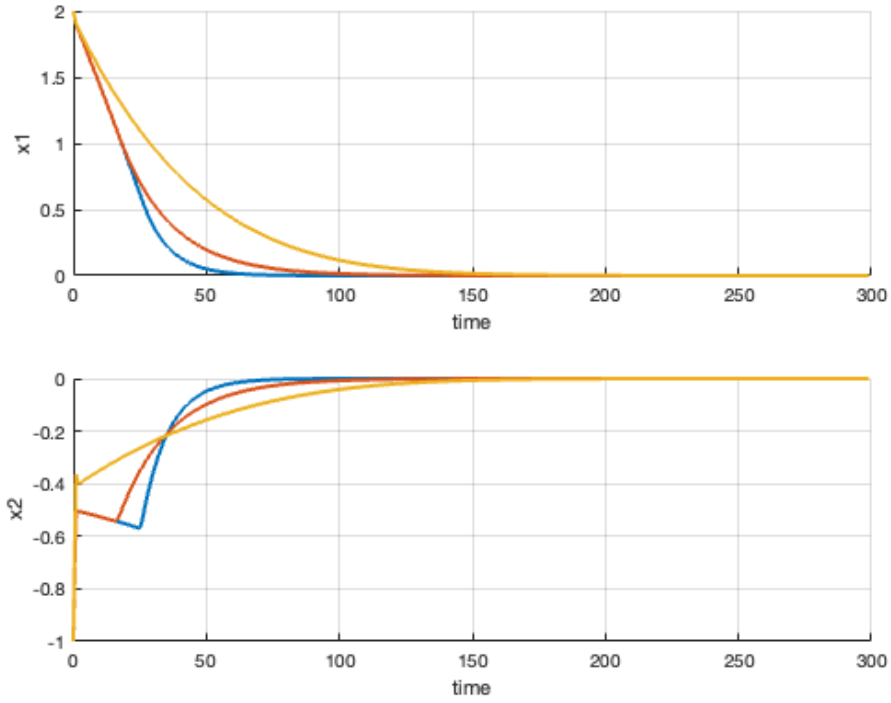


Figure 2.6: State trajectories as functions of time, $u(k) = sat(F(k)x(k))$ is obtained by solving problem (2.106) (solid blue), $u(k) = sat(F(k)x(k))$ is obtained by solving problem (2.65) (solid red), $u(k) = Fx(k)$ is obtained by solving problem (2.48) (solid yellow) for example 3.

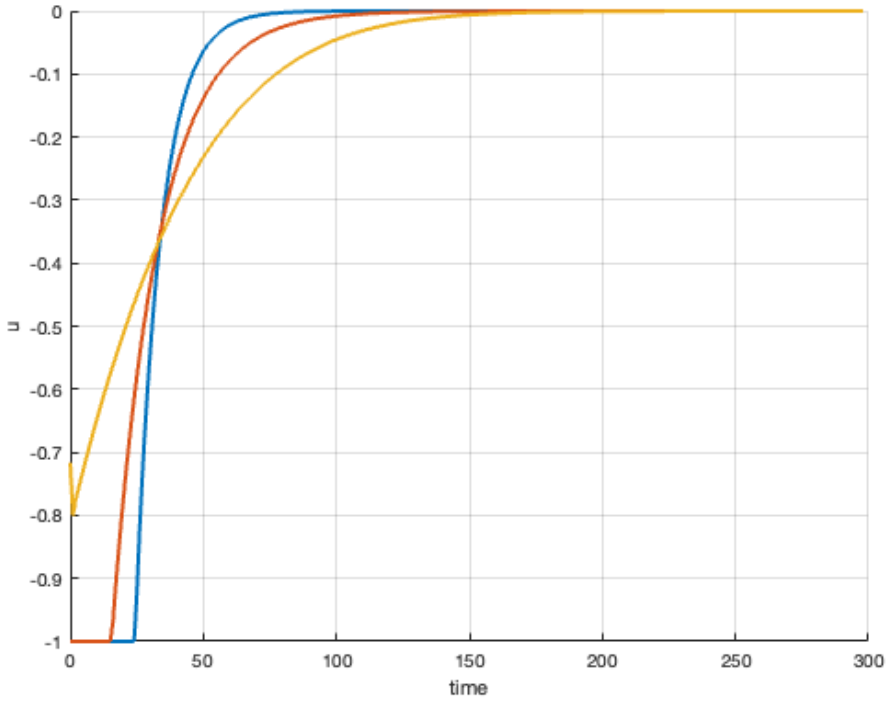


Figure 2.7: Input trajectories as functions of time, $u(k) = sat(F(k)x(k))$ is obtained by solving problem (2.106) (solid blue), $u(k) = sat(F(k)x(k))$ is obtained by solving problem (2.65) (solid red), $u(k) = Fx(k)$ is obtained by solving problem (2.48) (solid yellow) for example 3.

Chapter 3

Model Predictive Control for Constrained LPV Systems

This chapter aims to develop a parameter-dependent Lyapunov function-based MPC framework for constrained LPV systems. The main contributions of the chapter are

- We propose novel quasi-min–max MPC algorithms that employs a class of nonlinearly parameterized state-feedback control laws to generate the future control inputs $u(k+t)$, $t \geq 1$. Our nonlinearly parameterized Lyapunov function is simpler than the one in [40], [3]. We show in Section 3.3 that the use of the complex Lyapunov function in [40], [3] does not bring any advantages compared to ours.
- We provide new LMI conditions for MPC problem that fully exploit the PDLF without encountering the double sum positivity problem. Two new double-sum-free MPC methods are proposed. In the first method, linear gain-scheduling feedback is considered. The main advantage of the first method is the efficiency of online computation. However, as the control law is purely linear, the full control range is rarely exploited. The control inputs might not be saturated even when the plant's state is far from the origin. Hence the time to take the plant's state to the origin is often longer than necessary. To improve the performance, in the second MPC method, we propose a nonlinearly saturated gain-scheduling feedback. The saturation allows the controller to apply larger control actions when the system is far from the desired state, leading to faster convergence.
- We show how to reformulate the online optimization problem as a convex SDP program.

This chapter is organized as follows. Section 3.1 presents the problem formulation. In Section 3.3, we develop quasi-min–max MPC algorithms based on nonlinearly parameterized state-feedback control laws. The performance of the proposed methods is demonstrated through two simulation examples and compared with previously reported approaches. Section 3.4 is dedicated to the MPC control law with linear feedback, with saturated feedback based on the double-sum free approach.

Finally, two simulated examples with comparison to earlier solutions from the literature are evaluated.

3.1 Problem Formulation

Consider the following discrete-time LPV system

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the measured state, $u \in \mathbb{R}^m$ is the control input. The matrices $[A(k), B(k)] \in \Omega$ with

$$A(k) = \sum_{i=1}^{L} p_i(k)A_i, B(k) = \sum_{i=1}^{L} p_i(k)B_i \tag{3.2}$$

where $p(k) = [p_1(k)\ p_2(k)\ \ldots p_L(k)]^T$ satisfies

$$p(k) \in \Sigma_L := \left\{ p \in \mathbb{R}^L \mid \sum_{i=1}^{L} p_i = 1,\ p_i \geq 0 \right\} \tag{3.3}$$

The vector $p(k)$ may represent time-varying lumped parameters or state space components.

LPV hypothesis: $p(k)$ is available in each time instant k.

The state $x(k)$ and the input $u(k)$ are subject to the constraints, $\forall k \geq 0$

$$\begin{cases} x(k) \in \mathscr{X} := \left\{ x \in \mathbb{R}^n \mid |f_l^T x| \leq 1, \forall l \in \overline{1, n_x} \right\} \\ u(k) \in \mathscr{U} := \left\{ u \in \mathbb{R}^m \mid |u_l| \leq 1, \forall l \in \overline{1, m} \right\} \end{cases} \tag{3.4}$$

We denote $x(k+t), u(k+t), t = 0, 1, \ldots$, respectively, as the predicted states and the predicted control inputs at time $k+t$ from time k. Consider the following infinite horizon objective function

$$J_0^\infty(k) = \sum_{t=0}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \tag{3.5}$$

where $Q \in \mathbb{S}^n, R \in \mathbb{S}^m$ are weighting matrices.

At a generic time k, the objective of this chapter is to design a control law

$$u(k+t) = g(x(k+t), p(k+t)), t = 0, 1, \ldots \tag{3.6}$$

that asymptotically stabilizes the system (3.1). Furthermore, $u(k+t) = g(x(k+t), p(k+t))$ should solve the following min-max problem

$$\begin{aligned} &\min_{U_0^\infty(k)} \max_{[A(k+t), B(k+t)] \in \Omega} J_0^\infty(k) \\ &\text{s.t. } (3.1), (3.4) \end{aligned} \tag{3.7}$$

where

$$U_0^\infty(k) = [u(k)^T\ u(k+1)^T\ \ldots]^T \tag{3.8}$$

3.2 Background

In model predictive control for LPV systems, system uncertainties leads naturally to a min–max optimization problem, where the control inputs are designed to minimize a performance index under the worst-case realization of uncertainties [23, 71]. Formally, this problem can be expressed as the minimization of a cost function subject to state and input constraints that must hold for all admissible uncertainties [72].

Under a state-feedback control law, the Lyapunov decrease condition leads to a set of matrix inequalities that must be satisfied for all vertices of the uncertainty polytope. Through standard variable substitutions, these conditions can be reformulated as linear matrix inequalities. Consequently, the original min–max robust MPC problem is transformed into a convex SDP, which can be efficiently solved online.

An LMI-based approach to robust MPC design for LPV systems, explicitly considering plant model uncertainty, is developed in [23].

Consider the quadratic function

$$V_l(k) = x(k)^T P x(k), P \succ 0 \tag{3.9}$$

with $V_l(k) = 0$ if and only if $x(k) = 0$. At sampling time k, suppose V_l satisfies the following inequality for all $x(k+t)$, $u(k+t)$, $t \geq 0$ for any $[A(k+t)\ B(k+t)] \in \Omega$

$$V_l(k+t+1) - V_l(k+t) \leq -\big(x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t)\big) \tag{3.10}$$

For the robust performance objective function (3.5), suppose $x(\infty) = 0$ and hence $V_l(\infty) = 0$. Summing (3.10) from $t = 0$ to $t = \infty$, one gets

$$-V_l(k) \leq -J_0^\infty(k) \tag{3.11}$$

Thus

$$\max_{[A(k+t),B(k+t)]\in\Omega} J_0^\infty(k) \leq V_l(k) \tag{3.12}$$

This gives an upper bound on the performance objective.

Consider the following constant state feedback control law

$$u(k+t) = Fx(k+t), t \geq 0 \tag{3.13}$$

In [23], the following result is established.

Theorem 3.1. *[23] Let $x(k)$ denote the state of the uncertain system (3.1) at sampling time k. The scheduling parameter $p(k)$ is assumed to be unknown but bounded within a polytope Ω. Then, the state-feedback gain F in the control law (3.13), which minimizes an upper bound $V(k)$ on the robust performance objective at time k, is given by $F = Y\tilde{Q}^{-1}$, where $\tilde{Q} \succeq 0$ and Y are obtained by solving the following linear objective minimization problem:*

$$\min_{\gamma,\tilde{Q},Y} \gamma \tag{3.14}$$

$$\text{s.t.} \quad \begin{bmatrix} 1 & x(k)^T \\ x(k) & \tilde{Q} \end{bmatrix} \succeq 0 \tag{3.15}$$

$$\begin{bmatrix} \tilde{Q} & \tilde{Q}A_i^T + Y^T B_i^T & \tilde{Q}Q_c^T & Y^T R_c^T \\ A_i\tilde{Q} + B_i Y & \tilde{Q} & 0 & 0 \\ Q_c\tilde{Q} & 0 & \gamma I & 0 \\ R_c Y & 0 & 0 & \gamma I \end{bmatrix} \succeq 0, \; i \in \overline{1, L} \tag{3.16}$$

$$\begin{bmatrix} 1 & Y_l \\ Y_l^T & \tilde{Q} \end{bmatrix} \succeq 0, \; l \in \overline{1, m} \tag{3.17}$$

$$\begin{bmatrix} 1 & f_l^T\tilde{Q} \\ \tilde{Q}f_l & \tilde{Q} \end{bmatrix} \succeq 0, \; l \in \overline{1, n_x} \tag{3.18}$$

where $\tilde{Q} = \gamma P^{-1}$, $Q_c^T Q_c = Q$, $R_c^T R_c = R$, Y_l *is the l-row of the matrix* Y.

Proof. The detailed proof is omitted. It can be found in [23]. □

Proposition 2. *The results in [23] establish recursive feasibility of the optimization problem in Theorem 3.1, meaning that feasibility at time* k *ensures feasibility for all* $t > k$. *In addition, the corresponding receding horizon state-feedback law guarantees robust asymptotic stability of the closed-loop system.*

Proof. The detailed proof is omitted. It can be found in [23]. □

Note that [23] presented a min-max MPC design technique for constrained LPV systems. However, in [23], the parameters are not available in real-time; therefore, the controller is robust but not scheduling. Assume that parameters is available in each time instant k, [2] proposed a quasi–min–max MPC algorithm that exhibits reduced conservatism compared to the approach in [23]. [2] split the cost function $J_0^\infty(k)$ into two parts

$$J_0^\infty(k) = x(k)^T Q x(k) + u(k)^T R u(k) + J_1^\infty(k) \tag{3.19}$$

with

$$J_1^\infty(k) = \sum_{t=1}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \tag{3.20}$$

Using (3.19), (3.20), and the principle of optimality, problem (3.7) is equivalent to

$$\begin{aligned} &\min_{u(k)} \left(u(k)^T R u(k) + \min_{U_1^\infty(k)} \max_{[A(p(k+t)),B(p(k+t))]\in\Omega} J_1^\infty(k) \right) \\ &\text{s.t. (3.1), (3.4)} \end{aligned} \tag{3.21}$$

with $U_1^\infty(k) = [u(k+1)^T \; u(k+2)^T \; \ldots]^T$. Note that the term $x(k)^T Q x(k)$ is removed from the cost (3.19) as it does not influence the optimal argument.

In the quasi-min-max MPC, the first control move $u(k)$ is a decision variable in the problem (3.21). The rest of the future control moves is parameterized as [2]

$$u(k+t) = Fx(k+t), \forall t = 1, 2, \dots \tag{3.22}$$

where $F \in \mathbb{R}^{m \times n}$ is a decision variable in (3.21).

The following theorem is established in [2].

Theorem 3.2. *[2] At each time instant, the optimization problem (3.21) with the control law given by (3.22) can be solved by the following semi-definite programming:*

$$\min_{\gamma, \tilde{Q}, Y} \ \gamma \tag{3.23}$$

$$\text{s.t.} \quad \begin{bmatrix} 1 & \big(A(k)x(k) + B(k)u(k)\big)^T & x(k)^T Q_c^T & u(k)^T R_c^T \\ (A(k)x(k) + B(k)u(k) & \tilde{Q} & 0 & 0 \\ Q_c x(k) & 0 & \gamma I & 0 \\ R_c u(k) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0 \tag{3.24}$$

$$\begin{bmatrix} \tilde{Q} & \big(A_i\tilde{Q} + B_i Y\big)^T & \tilde{Q}Q_c^T & Y^T R_c^T \\ A_i\tilde{Q} + B_i Y & \tilde{Q} & 0 & 0 \\ Q_c\tilde{Q} & 0 & \gamma I & 0 \\ R_c Y & 0 & 0 & \gamma I \end{bmatrix} \succeq 0, \ i \in \overline{1, L} \tag{3.25}$$

$$|u_l(k)| \leq 1, l \in \overline{1, m} \tag{3.26}$$

$$\begin{bmatrix} 1 & Y_l \\ Y_l^T & \tilde{Q} \end{bmatrix} \succeq 0, l \in \overline{1, m} \tag{3.27}$$

$$\begin{bmatrix} 1 & f_l^T\tilde{Q} \\ \tilde{Q}f_l & \tilde{Q} \end{bmatrix} \succeq 0, \ l \in \overline{1, n_x} \tag{3.28}$$

where $Y = F\tilde{Q}$, $\tilde{Q} = \gamma P^{-1}$, $Q_c^T Q_c = Q$, $R_c^T R_c = R$ and Y_l is the l-row of the matrix Y.

Proof. The detailed proof is omitted for brevity and can be found in [2]. □

If we assume that $B(k)$ is constant, in [2], [33], [34], less conservative results can be achieved where linearly scheduled state feedback control laws of the form, $\forall t = 1, 2, \dots$

$$u(k+t) = \left(\sum_{i=1}^{L} p_i(k+t) F_i \right) x(k+t) \tag{3.29}$$

are employed, where $F_i \in \mathbb{R}^{m \times n}, \forall i \in \overline{1, L}$ are decision variables in (3.21).

In the above linear matrix inequalities proposed in Theorem 3.2 , $p(k)$and $x(k)$ are the measured parameter and state, respectively. Optimization determines the "first move" $u(k)$ that gets

implemented, the matrix P, and the feedback gain F which guarantees that the future states evolve inside an invariant ellipsoid $\mathscr{E}(P,\gamma)$.

It is worth noting that, since the current model $\big(A(k),\ B(k)\big)$ and the state $x(k)$ are assumed to be known, the successor state $x(k+1)$ can be computed exactly. This implies that inequality (3.10) guarantees that only the future states $x(k+t+1)$, $t \geq 1$, are confined within an invariant ellipsoid [2]. By contrast, in [23], the invariant ellipsoid also includes the state $x(k+1)$, which is uncertain due to model uncertainty.

3.3 Quasi-Min-Max MPC via Nonlinearly Parameterized State Feedback Control

In order to achieve less conservative results, this section considers the following class of nonlinearly parameterized state-feedback control laws,$\forall t \geq 1$

$$u(k+t) = F(k+t)G(k+t)^{-1}x(k+t) \tag{3.30}$$

where $F(k+t) = \sum_{i=1}^{L} p_i(k+t)F_i$, $G(k+t) = \sum_{i=1}^{L} p_i(k+t)G_i$ with $F_i \in \mathbb{R}^{m\times n}, G_i \in \mathbb{R}^{n\times n}, \forall i \in \overline{1,L}$, are decision variables in (3.21). Clearly if $G_i = I, \forall i \in \overline{1,L}$ then (3.30) coincides with (3.29).

It should be stressed that the use of (3.30) is not new. In [40], the authors employed (3.30) for the stabilization problem of unconstrained discrete-time Takagi-Sugeno fuzzy systems. The control law (3.30) was also used in the context of MPC [3] for constrained LPV systems, but the MPC strategy is not the quasi-min-max one, as the cost function does not split into two parts, and the control input is parameterized by (3.30) for all $t \geq 0$.

We will first provide a way to calculate the upper bound of $J_1^{\infty}(k)$ using the control law (3.30). Substituting (3.30) into (3.1), one obtains, $\forall t \geq 1$

$$x(k+t+1) = A_c(k+t)x(k+t) \tag{3.31}$$

where $A_c(k+t) = A(k+t) + B(k+t)F(k+t)G(k+t)^{-1}$. We will provide a way to design $F_i, G_i, \forall i \in \overline{1,L}$ such that the closed-loop system (3.31) is asymptotically stable. For the moment, it is assumed that $\lim_{t\to\infty} x(k+t) = 0$ for any admissible state $x(k)$.

Consider the following parameterized quadratic function, $\forall t \geq 1$

$$V(k+t) = x(k+t)^T \left(\sum_{i=1}^{L} p_i(k+t)S_i\right)^{-1} x(k+t) \tag{3.32}$$

where $S_i \in \mathbb{S}^{n\times n}, \forall i \in \overline{1,L}$ are chosen to satisfy, $\forall t \geq 1$.

$$x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \leq V(k+t) - V(k+t+1) \tag{3.33}$$

As $\lim_{t\to\infty} x(k+t) = 0$, one has $\lim_{t\to\infty} V(k+t) = 0$. Summing (3.33) from $t = 1$ to $t = \infty$, one obtains

$$\sum_{t=1}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \leq V(k+1)$$

or equivalently, $\forall k \geq 0$

$$J_1^{\infty}(k) \leq V(k+1) \tag{3.34}$$

Hence $V(k+1)$ is an upper bound of the cost $J_1^{\infty}(k)$.

Theorem 3.3. *At time k, for a given state $x(k)$, the problem (3.21) can be solved via the following SDP:*

$$\min_{u(k),F_j,G_j,\tilde{S}_j,\gamma} \{\gamma\} \tag{3.35}$$

subject to

$$\begin{bmatrix} 1 & * & * \\ A(k)x(k) + B(k)u(k) & \widetilde{S}_i & * \\ R_c u(k) & 0 & \gamma I \end{bmatrix} \succeq 0, i \in \overline{1,L} \tag{3.36}$$

$$M_{iis} \succeq 0,\ i \in \overline{1,L}, s \in \overline{1,L} \tag{3.37}$$

$$\frac{2}{L-1} M_{iis} + M_{ijs} + M_{jis} \succeq 0, 1 \leq i \neq j \leq L, s \in \overline{1,L} \tag{3.38}$$

$$|u_l(k)| \leq 1, \forall l \in \overline{1,m} \tag{3.39}$$

$$\begin{bmatrix} 1 & * \\ F_{i,l}^T & G_i^T + G_i - \widetilde{S}_i \end{bmatrix} \succeq 0,\ i \in \overline{1,L}, l \in \overline{1,m} \tag{3.40}$$

$$\begin{bmatrix} 1 & * \\ \widetilde{S}_i f_l & \widetilde{S}_i \end{bmatrix} \succeq 0\ i \in \overline{1,L}, l \in \overline{1,n_x} \tag{3.41}$$

where $F_{i,l}$ is the l-row of the matrix F_i, $\tilde{S}_i = \gamma S_i, \forall i \in \overline{1,L}$, and, $\forall i, \forall j, \forall s = \overline{1,L}$

$$M_{ijs} = \begin{bmatrix} G_i^T + G_i - \widetilde{S}_i & * & * & * \\ A_i G_j + B_i F_j & \widetilde{S}_s & * & * \\ Q_c G_i & 0 & \gamma I & * \\ R_c F_i & 0 & 0 & \gamma I \end{bmatrix} \tag{3.42}$$

Proof. We divide the proof into three main parts concerning the cost function, the upper bound of J_1^{∞}, the input and the state constraints.

Cost Function: Using (3.19), (3.32), the problem of minimizing J_0^{∞} can be rewritten as

$$\big(A(k)x(k) + B(k)u(k)\big)^T S(k+1)^{-1} \big(A(k)x(k) + B(k)u(k)\big) + u(k)^T R u(k) \leq \gamma \tag{3.43}$$

Thus, using Schur complements

$$\begin{bmatrix} 1 & \big(A(k)x(k)+B(k)u(k)\big)^T & u(k)^T R_c \\ A(k)x(k)+B(k)u(k) & \gamma\sum_{i=1}^{L} p_i(k+1)S_i & 0 \\ R_c u(k) & 0 & \gamma I \end{bmatrix} \succeq 0 \tag{3.44}$$

As $\tilde{S}_j = \gamma S_j$, it follows that (3.44) holds $\forall p(k+1) \in \Sigma_L$ if and only if (3.36) holds.

Upper Bound of $J_1^{\infty}(k)$: Concerning condition (3.33) it is clear that it suffices to verify (3.33) with $t = 1$, i.e

$$x^T(k+1)Qx(k+1) + u(k+1)^T Ru(k+1) \leq V(k+1) - V(k+2) \tag{3.45}$$

Using (3.31),(3.30), it follows that (3.45) holds if and only if

$$\begin{aligned} S(k+1)^{-1} - A_c(k+1)^T S(k+2)^{-1} A_c(k+1) - Q \\ - \big(F(k+1)G(k+1)^{-1}\big)^T R\big(F(k+1)G(k+1)^{-1}\big) \succeq 0 \end{aligned} \tag{3.46}$$

By pre- and post-multiplying with $\sqrt{\frac{1}{\gamma}}G(k+1)^T$ and $\sqrt{\frac{1}{\gamma}}G(k+1)$, one obtains

$$\begin{aligned} S_g(k+1) - A_g(k+1)^T \times \tilde{S}(k+2)^{-1} A_g(k+1) \\ - G(k+1)^T \frac{Q}{\gamma} G(k+1) - F(k+1)^T \frac{R}{\gamma} F(k+1) \succeq 0 \end{aligned}$$

where

$$\begin{aligned} S_g(k+1) &= G(k+1)^T \tilde{S}(k+1)^{-1} G(k+1), \\ A_g(k+1) &= A(k+1)G(k+1) + B(k+1)F(k+1) \end{aligned}$$

thus, by using Schur complements

$$\begin{bmatrix} S_g(k+1) & * & * & * \\ A_g(k+1) & \tilde{S}(k+2) & * & * \\ Q_c G(k+1) & 0 & \gamma I & 0 \\ R_c F(k+1) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0 \tag{3.47}$$

Recall that $\tilde{S}(k+2) = \sum_{s=1}^{L} p_s(k+2)\tilde{S}_s$. Hence (3.47) is satisfied, $\forall p(k+2) \in \Sigma_L$ if and only if

$$\begin{bmatrix} S_g(k+1) & * & * & * \\ A_g(k+1) & \tilde{S}_s & * & * \\ Q_c G(k+1) & 0 & \gamma I & 0 \\ R_c F(k+1) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0, \forall s \in \overline{1, L} \tag{3.48}$$

Using Lemma 2.7, one has

$$S_g(k+1) \preceq G(k+1)^T + G(k+1) - \tilde{S}(k+1)$$

Therefore, if the following condition

$$\begin{bmatrix} G(k+1)^T + G(k+1) - \tilde{S}(k+1) & * & * & * \\ A_g(k+1) & \tilde{S}_s & * & * \\ Q_c G(k+1) & 0 & \gamma I & 0 \\ R_c F(k+1) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0 \tag{3.49}$$

holds $\forall s \in \overline{1, L}$, then (3.48) holds. Note that (3.49) can be written in the following double sum form, $\forall s = \overline{1, L}$

$$\sum_{i=1}^{L} \sum_{j=1}^{L} p_i(k+1) p_j(k+1) M_{ijs} \succeq 0 \tag{3.50}$$

with M_{ijs} being defined in (3.42). Using Lemma 2.6, if (3.37), (3.38) hold then (3.50) holds.

Input and state constraints: Consider first the case where $u(k)$ is a scalar. In quasi-min-max MPC, $u(k)$ is a decision variable. Hence (3.39) imposes directly the constraint on $u(k)$. For the remaining inputs U_1^∞, one has, $\forall t \geq 1$

$$|u(k+t)| = |F(k+t)G(k+t)^{-1}x(k+t)| \leq 1 \tag{3.51}$$

Using (3.43), one obtains

$$u(k)^T R u(k) + x(k+1)^T S(k+1)^{-1} x(k+1) \leq \gamma$$

Hence

$$x(k+1)^T \tilde{S}(k+1)^{-1} x(k+1) \leq 1$$

Hence, $x(k+1)$ belongs to the parameterized ellipsoid $\mathscr{E}(\sum_{j=1}^{L} p_j(k+1)\tilde{S}_j)$. Using the feasibility proof of Theorem 3.1, it follows that, $\forall t \geq 1$

$$x(k+t) \in \mathscr{E}(\sum_{j=1}^{L} p_j(k+t)\tilde{S}_j) \tag{3.52}$$

Consequently, for the input constraint, condition (3.51) should hold $\forall x(k+t)$ satisfying (3.52).

Define, $\forall t \geq 1$

$$\begin{aligned} v_u(t) = & \max_{x(k+t)} \left\{ F(k+t)G(k+t)^{-1}x(k+t) \right\}, \\ \text{s.t. } & x(k+t)^T \tilde{S}(k+t)^{-1} x(k+t) \leq 1 \end{aligned} \tag{3.53}$$

Using the method of Lagrange multipliers, it can be shown that

$$v_u(t) = \sqrt{F(k+t)G(k+t)^{-1}\tilde{S}(k+t)G(k+t)^{-T}F(k+t)^T}$$

As the set $\mathscr{E}(\sum_{j=1}^{L} p_j(k+t)\tilde{S}_j)$ is symmetric, one has

$$-v_u(t) \leq F(k+t)G(k+t)^{-1}x(k+t) \leq v_u(t)$$

$\forall x(k+1) \in \mathscr{E}(\sum_{j=1}^{L} p_j(k+t)\tilde{S}_j)$. Hence (3.51) hold if and only if

$$1 - F(k+t)G(k+t)^{-1}\tilde{S}(k+t)G(k+t)^{-T}F(k+t)^T \geq 0$$

Thus, using Schur complements

$$\begin{bmatrix} 1 & * \\ F(k+t)^T & G(k+t)^T\widetilde{S}(k+t)^{-1}G(k+t) \end{bmatrix} \succeq 0 \tag{3.54}$$

Using Lemma 2, it follows that if (3.40) holds, then (3.54) is satisfied $\forall t \geq 1$.

Now, consider the case where $u(k)$ is a vector. Note that conditions (3.39), (3.40) impose the constraint on each component of $u(k)$ and of $u(k+t)$, $t \geq 1$.

Condition (3.41) is for the state constraint satisfaction. The proof is omitted here, as it follows closely the proof of the input constraints. □

It is worth noting that in both [40] and [3], the following nonlinearly parameterized Lyapunov function

$$V(x,p) = x^T \left(\sum_{i=1}^{L} p_i G_i\right)^{-T} \left(\sum_{i=1}^{L} p_i P_i\right) \left(\sum_{i=1}^{L} p_i G_i\right)^{-1} x \tag{3.55}$$

is employed in conjunction with the control law (3.30).

In contrast, the nonlinearly parameterized Lyapunov function proposed in (3.32) has a simpler structure, as it does not depend on $G(k)$. Furthermore, as will be illustrated in the numerical examples, the use of (3.55) does not provide any performance advantage compared to the proposed simpler Lyapunov function.

Based on Theorem 3.3, we propose the following quasi-min-max MPC algorithm

Theorem 3.4. *Assuming feasibility at the initial condition $x(0)$, then algorithm 3.1 guarantees recursive feasibility, and the closed-loop system with the MPC control law yields is asymptotically stable.*

Proof. **Feasibility Proof:** The basic idea of the feasibility proof is to show that the optimal solution at time k is a feasible solution at time $k+1$. As condition (3.36) is for the performance, one needs only to check conditions (3.37), (3.38), (3.39), (3.40), (3.41).

Algorithm 3.1 Quasi-min-max MPC

1: Set $k \leftarrow 0$
2: Measure and/or estimate $x(k)$ and $p(k)$.
3: Solve the optimization problem (3.35), (3.36), (3.37), (3.38), (3.39), (3.40), (3.41).
4: Feed the plant with $u^*(k)$.
5: Set $k \leftarrow k+1$ and go to step 2.

Using the proof of theorem 3.3, one has that conditions (3.37)), (3.38) do not contain the parameters $p(k)$, and are for the upper bound of the cost $J_1^\infty(k)$. Using (3.33), it is clear that if $J_1^\infty(k) \leq V(k+1)$, then $J_1^\infty(k+1) \leq V(k+2)$. Hence conditions (3.37), (3.38) are satisfied at time $k+1$.

Using (3.40), one has $|u_l(k+1)| \leq 1, \forall l \in \overline{1,L}$ with

$$u(k+1) = F^*(k+1)G^*(k+1)^{-1}x(k+1)$$

Hence, condition (3.39) is satisfied. It remains to show (3.40), (3.41), which are for the constraint admissibility. As (3.40), (3.41) do not contain the parameters $p(k)$, hence if it holds at time k, then it holds at time $k+1$.

Stability Proof: Consider the following parameter-dependent Lyapunov candidate function

$$\begin{aligned}\phi(k) = x(k)^T Q x(k) + u^*(k)^T R u^*(k) \\ + x(k+1)^T S^*(k+1)^{-1} x(k+1)\end{aligned} \tag{3.56}$$

where $u^*(k), S_j^*, \forall j \in \overline{1,L}$ are the outputs of Algorithm 1 at time k. Note that $\phi(k) > 0, \forall x(k) \neq 0$. Note also that in Algorithm 1, if $x(k) = 0$, then $u(k) = 0$ is the solution. In this case $\phi(k) = 0$.

As $u(k+1) = F^*(k+1)G^*(k+1)^T x(k+1)$ and $S^*(k+1)$ is a feasible solution at time $k+1$, one obtains

$$\begin{aligned}&x(k+1)^T Q x(k+1) + u(k+1)^T R u(k+1) \\ &+ x(k+2)^T S(k+2)^{-1} x(k+2) \\ &\leq x(k+1) S^*(k+1)^{-1} x(k+1)\end{aligned} \tag{3.57}$$

Using (3.56), (3.57), one gets

$$\begin{aligned}\phi(k) \geq x(k)^T Q x(k) + u^*(k)^T R u^*(k) \\ + x(k+1)^T Q x(k+1) + u(k+1)^T R u(k+1) \\ + x(k+2)^T S(k+2)^{-1} x(k+2)\end{aligned} \tag{3.58}$$

At time instant $k+1$, one has

$$\begin{aligned}\phi(k+1) &= x(k+1)^T Q x(k+1) + u^*(k+1)^T R u^*(k+1) \\ &\quad + x(k+2)^T S^*(k+2)^{-1} x(k+2) \\ &\leq x(k+1)^T Q x(k+1) + u(k+1)^T R u(k+1) \\ &\quad + x(k+2)^T S(k+2)^{-1} x(k+2)\end{aligned} \tag{3.59}$$

Combining (3.58), (3.59), one obtains

$$\phi(k) \geq \phi(k+1) + x(k)^T Q x(k) + u^*(k)^T R u^*(k)$$

Hence, $\phi(k)$ is a Lyapunov function for the closed-loop system with the control law $u(k) = u^*(k)$. In the other words, asymptotic stability is guaranteed. □

Algorithm 3.1 guarantees recursive feasibility and the closed-loop system is asymptotically stable.

In the following we provide a second quasi-min-max MPC scheme that yields an asymptotically stable closed-loop system. The motivation for the second algorithm is the observation that requiring the satisfaction of the input and state constraints for the entire infinite horizon might be too restrictive. Therefore, for the second algorithm we impose the constraints only on $u(k)$ and $x(k+1)$. To guarantee the stability of the closed-loop system, we require that the objective function decreases monotonically. This is done by explicitly incorporating a linear Lyapunov constraint in the online optimization problem. Consider the cost function

$$\varphi(k) = x(k)^T Q x(k) + u(k)^T R u(k) + x(k+1)^T S(k+1)^{-1} x(k+1) \tag{3.60}$$

We include the following Lyapunov constraint in the optimization problem

$$\varphi(k) < \varphi(k-1) \tag{3.61}$$

Using (3.60), (3.61), and the proof of theorem 3.3, we obtain the following result.

Theorem 3.5. *For a given state $x(k)$ at time k, a solution to the problem (3.21) with the constraint (3.61) is obtained by solving the following SDP problem*

$$\min_{u(k), F_j, G_j, S_j, \gamma} \quad \{\gamma\} \tag{3.62}$$

subject to

$$\begin{bmatrix} \gamma & * & * & * \\ A(k)x(k) + B(k)u(k) & S_i & * & * \\ Q_c x(k) & 0 & I & * \\ R_c u(k) & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i \in \overline{1, L} \tag{3.63}$$

$$\gamma \leq \varphi(k-1) \tag{3.64}$$

$$N_{iis} \succeq 0, \forall i, \forall s \in \overline{1,L} \tag{3.65}$$

$$\frac{2}{L-1} N_{iis} + N_{ijs} + N_{jis} \succeq 0, 1 \leq i \neq j \leq L, \forall s \in \overline{1,L} \tag{3.66}$$

$$|u_l(k)| \leq 1, l \in \overline{1,m} \tag{3.67}$$

$$f_l |A(k)x(k) + B(k)u(k)| \leq 1, l \in \overline{1,n_x} \tag{3.68}$$

where

$$N_{ijs} = \begin{bmatrix} G_i^T + G_i - S_i & * & * & * \\ A_i G_j + B_i F_j & S_s & * & * \\ Q_c G_i & 0 & I & * \\ R_c F_i & 0 & 0 & I \end{bmatrix} \tag{3.69}$$

Proof. Following the proof of Proposition 1, it is clear that conditions (3.65), (3.66) are for the upper bound of $J_1^{\infty}(k)$ in the sense that if there exist F_i, G_i, S_i satisfying (3.65), (3.66), then

$$J_1^{\infty}(k) \leq x(k+1)^T \left(\sum_{i=1}^{L} p_i(k+1) S_i \right)^{-1} x(k+1)$$

In this case, the cost function of (3.21) can be rewritten as

$$\min_{u(k), F_i, G_i, S_i, \gamma} \quad \{\gamma\}$$

subject to

$$x(k)^T Q x(k) + u(k)^T R u(k) + x(k+1)^T S(k+1)^{-1} x(k+1) \leq \gamma$$

Thus, using Schur complements, one obtains (3.63).

Note that γ provides an upper bound of $\varphi(k)$, i.e., $\varphi(k) \leq \gamma$, and there exists a particular set of parameters $p_i^o(k+1)$ such that

$$\begin{aligned} \gamma = {} & x(k)^T Q x(k) + u(k)^T R u(k) \\ & + x(k+1)^T \left(\sum_{i=1}^{L} p_i^o(k+1) S_j \right)^{-1} x(k+1) \end{aligned} \tag{3.70}$$

Hence condition (3.64) is for satisfying the Lyapunov constraint (3.61). Finally, conditions (3.67), (3.68) are, respectively for the input constraint $u(k)$ and for the state constraint $x(k+1)$. □

Denote the solution of (3.62),(3.63),(3.64),(3.65), (3.66),(3.67),(3.68) as $u^*(k), F_i^*, G_i^*, S_i^*$, $\forall i \in \overline{1,L}$. Using theorem 3.5, we propose the following quasi-min-max MPC algorithm.

Algorithm 3.2 Quasi-min-max MPC

1: Set $k \leftarrow 0$.
2: Measure and/or estimate $x(k)$ and $p(k)$.
3: **IF** k=0
4: Solve (3.62), (3.63), (3.65), (3.66), (3.67), (3.68).
5: **ELSE**
6: Solve (3.62), (3.63), (3.64), (3.65), (3.66), (3.101), (3.68).
7: **ENDIF**
8: Feed the plant with $u(k) = u^*(k)$.
9: Set $k \leftarrow k + 1$ and go to step 2.

Theorem 3.6. *If Algorithm 3.2 is feasible at time k, then it yields an asymptotically stable closed-loop system.*

Proof. The main idea of the proof is to show that $\varphi(k)$ is a Lyapunov function of the closed-loop system. The proof is omitted as it follows closely to the proof of Theorem 3.4. □

It is noticed that the recursive feasibility is not guaranteed with Algorithm 3.2. Note also that feasibility of the optimization problem of Algorithm 3.1 implies feasibility of the optimization problem of Algorithm 3.2, but the reverse does not generally hold.

3.3.1 Numerical Examples

Two example systems are shown in this section. The CVX toolbox [66] was used to solve the SDP optimization problems.

3.3.1.1 Example 4

This example is taken from [73]. Consider system (3.1) with

$$A_1 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 0.99 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 & 0.0787 \end{bmatrix}^T, B_2 = B_1$$

Note that the open-loop system is unstable. The input constraint is $|u(k)| \leq 1$. The weighting matrices are $Q = I, R = 1$.

For the initial condition $x(0) = [2 \; 1]^T$, Fig. 3.1 and Fig. 3.2 present the state and the input trajectories as functions of time for Algorithm 3.1 (solid blue), for Algorithm 3.2 (dash dot violet) for [2] (dashed red). Using Fig. 3.1, Fig. 3.2, the superior performance of both Algorithm 3.1 and Algorithm 3.2 compared to [2] is clearly observed.

Finally, Fig. 3.3 presents the accumulated cost $\mathscr{V}(k)$, which is computed as, $\mathscr{V}(-1) = 0$

$$\mathscr{V}(k) = \mathscr{V}(k-1) + x(k)^T Q x(k) + u(k)^T R u(k)$$

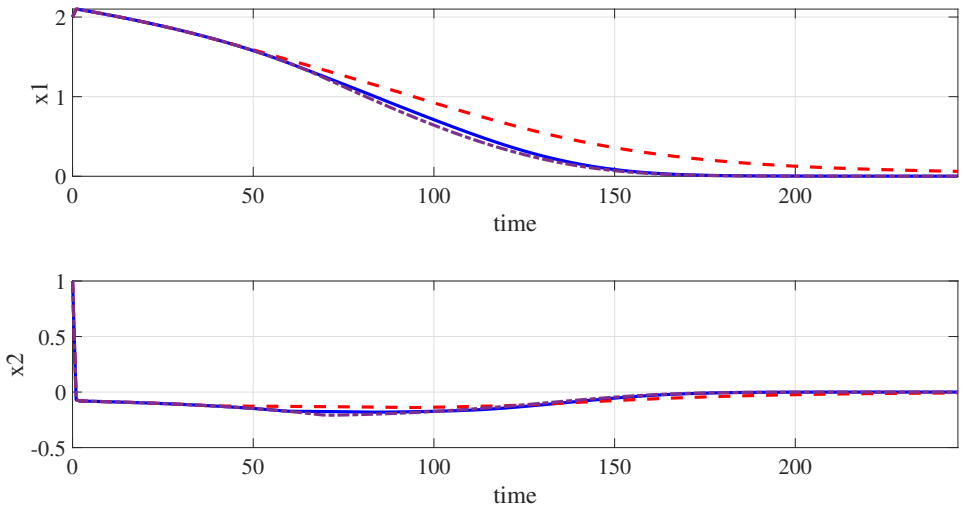


Figure 3.1: State trajectories for Algorithm 3.1 (solid blue), for Algorithm 2 (dash dot violet), for [2] (dashed red) for example 4.

for Algorithm 3.1 (solid blue), for Algorithm 3.2 (dashed-dot violet), and for [2] (dashed red). Fig. 3.3 also presents the p_1 realization as a function of time.

3.3.1.2 Example 5

Consider system (3.1) with

$$A_1 = \begin{bmatrix} 1.0 & -1.4 \\ -1.0 & -0.8 \end{bmatrix}, A_2 = \begin{bmatrix} 1.0 & 1.4 \\ -1.0 & -0.8 \end{bmatrix} \\ B_1 = \begin{bmatrix} 5.9 & 2.8 \end{bmatrix}^T, B_2 = \begin{bmatrix} 3.1 & -2.8 \end{bmatrix}^T \tag{3.71}$$

Note that system (3.71) is open loop unstable. It can also be verified that it is not quadratically stabilizable. Since $B_1 \neq B_2$, the linear parameterized control law in [2] is not applicable.

The input constraints are $|u(k)| \leq 1$. There are no state constraints. The weighting matrices are $Q = I, R = 0.01$.

For the initial condition $x(0) = [2.78\ 2]^T$, Fig. 3.4 and Fig. 3.5 show the state and the input trajectories as function of time for Algorithm 3.1 (solid blue), and for [3] (dashed red). Note that for this example both Algorithm 3.1 and Algorithm 3.2 have the identical performance.

Finally, Fig. 3.6 shows the accumulated cost for Algorithm 3.1 (solid blue), for [3] (dashed red). Fig. 3.6 also shows the p_1-realization as a function of time.

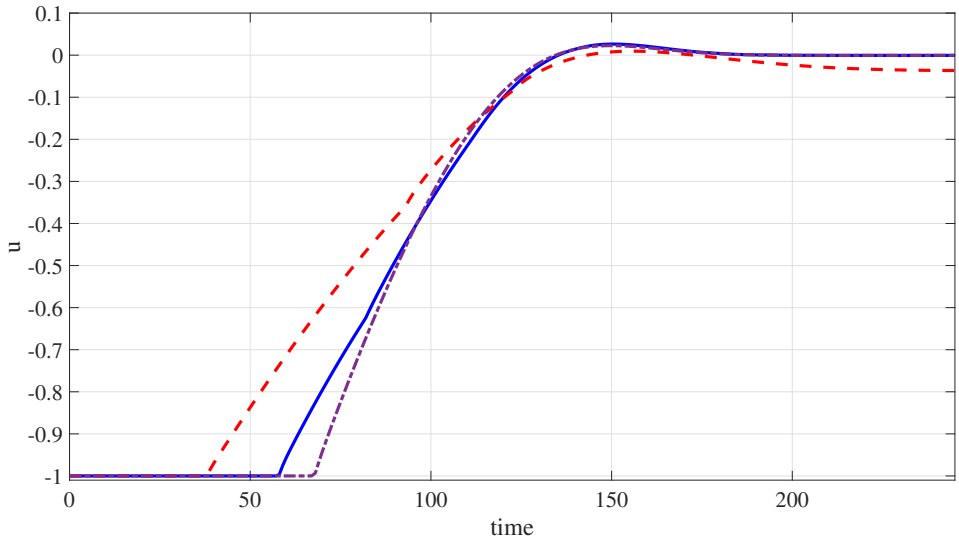


Figure 3.2: Input trajectories for Algorithm 3.1 (solid blue), for Algorithm 3.2 (dash dot violet), for [2] (dashed red) for example 4.

3.4 MPC for LPV system: A Double-Sum-Free PDLF Approach

Note that Algorithm 3.1 and Algorithm 3.2 lead to a double-sum formulation. During the algorithm design process, lemma 2.6 is employed to derive the corresponding LMI conditions. The computational complexity of solving such a set of LMIs at each iteration increases exponentially with the number of system vertices, which imposes a significant computational burden on the solver. Moreover, the relaxation technique provided by lemma 2.6 in previous chapter also introduces a certain degree of conservatism. In this section, we introduce a model predictive control framework for LPV systems based on a Double-Sum-Free Parameter-Dependent Lyapunov Function approach.

Consider the following discrete-time LPV system

$$x(k+1) = A(k)x(k) + Bu(k) \tag{3.72}$$

It is emphasized that all local models in (3.72) share the same input matrix B. As noted in [19] that this does not impose a significant restriction. The condition is naturally satisfied if actuator dynamics are included in the models. Otherwise, it can be ensured by filtering the input.

3.4.1 MPC with Linear Gain-Scheduling Feedback

For the optimization problem (3.7), we employ the following linear gain-scheduling feedback

$$u(k+t) = F(k+t)x(k+t), \forall t = 0, 1, \dots \tag{3.73}$$

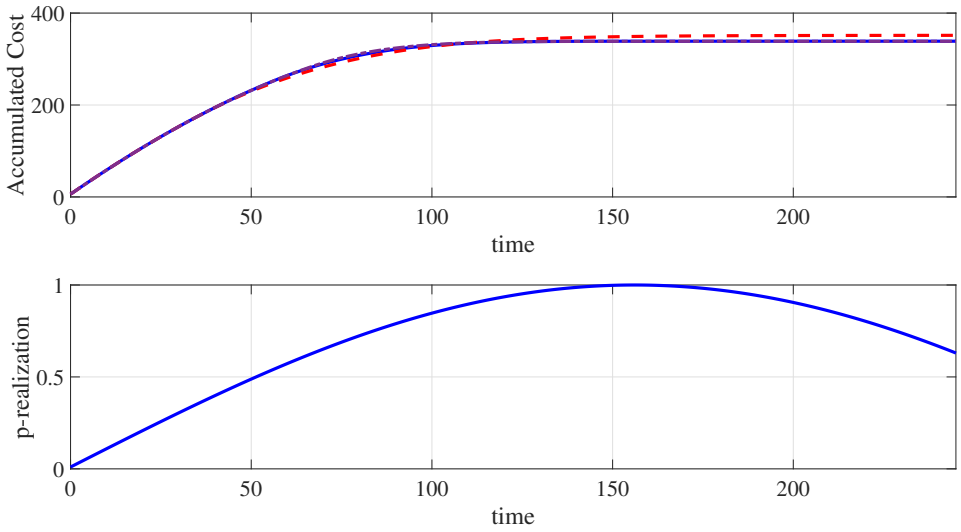


Figure 3.3: Accumulated cost for Algorithm 3.1 (solid blue), for Algorithm 3.2 (dashed-dot violet), for [2] (dashed red), and p_1-realization for example 4.

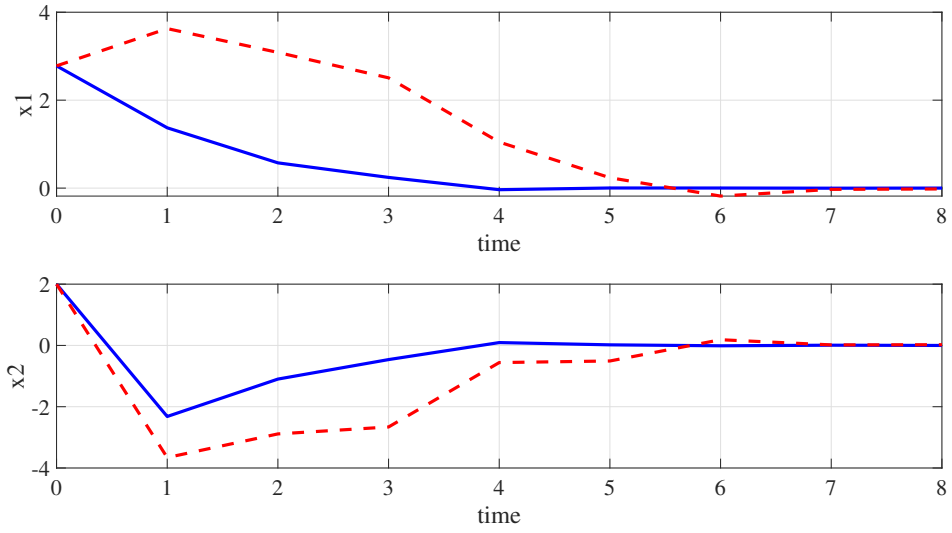


Figure 3.4: State trajectories for Algorithm 3.1 (solid blue), for [3] (dashed red) for example 5.

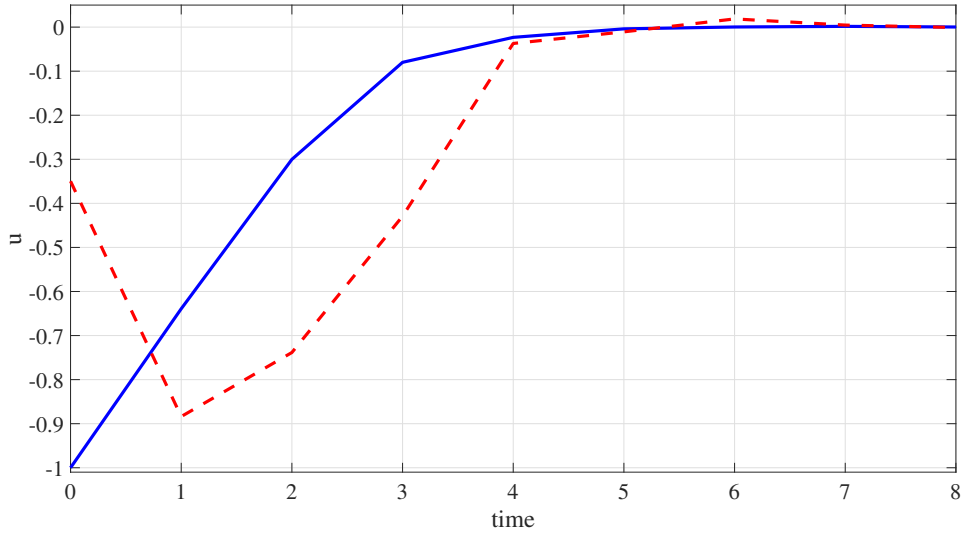


Figure 3.5: Input trajectories for Algorithm 3.1 (solid blue), for [3] (dashed red) for example 5.

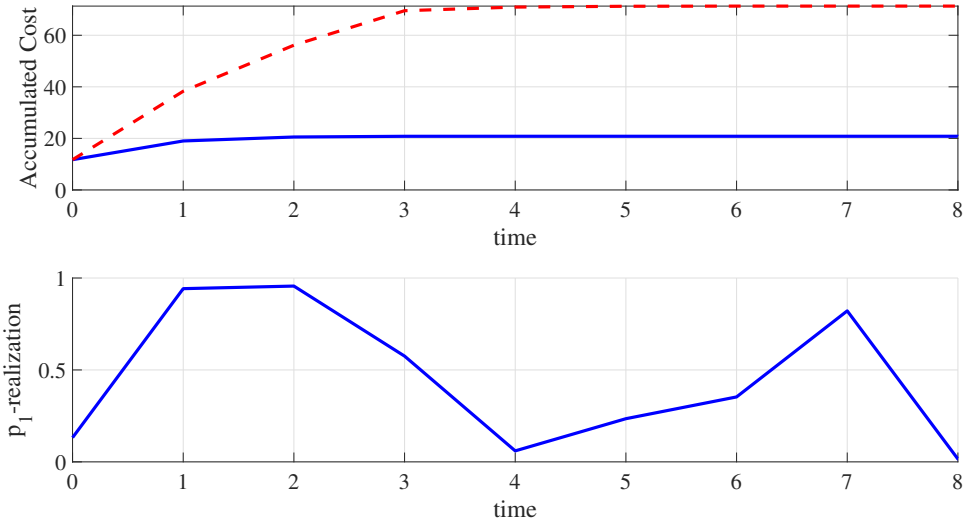


Figure 3.6: Accumulated cost for Algorithm 3.1 (solid blue), for [3] (dashed red), and p_1-realization for example 5.

to determine $u(k+t)$, where $F(k+t) = \sum_{i=1}^{L} p_i(k+t)F_i$, with $F_i \in \mathbb{R}^{m\times n}, \forall i \in \overline{1,L}$ being decision variables.

The following theorem provides a way to solve the problem (3.7) using the control law (3.73).

Theorem 3.7. *Consider the problem (3.7). Assume that there exist matrices Y_i, P_i, and positive scalars α_i, t_i such that*

$$\begin{bmatrix} \alpha_i & * \\ 1 & t_i \end{bmatrix} \succeq 0, \forall i \in \overline{1,L} \tag{3.74}$$

$$\begin{bmatrix} t_i & * \\ t_i x(k) & P_i \end{bmatrix} \succeq 0, \forall i \in \overline{1,L} \tag{3.75}$$

$$\begin{bmatrix} P_i & * & * & * \\ A_iP_i + BY_i & P_j & * & * \\ Q_cP_i & 0 & I & * \\ R_cY_i & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i, \forall j \in \overline{1,L} \tag{3.76}$$

$$\begin{bmatrix} t_i & * \\ Y_{i,l}^T & P_i \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall l \in \overline{1,m} \tag{3.77}$$

$$\begin{bmatrix} t_i & * \\ (A_iP_i + BY_i)^T f_l & P_i \end{bmatrix} \succeq 0 \ \forall i \in \overline{1,L}, \forall l \in \overline{1,n_x} \tag{3.78}$$

where $Y_i = F_iP_i, \forall i \in \overline{1,L}$ and $Y_{i,l}$ denotes the $l-$th column of the matrix Y_i, $l \in \overline{1,m}$. Then $u(k+t) \in \mathscr{U}$, $x(k+t) \in \mathscr{X}$, $\forall t \geq 0$, and the following inequality holds

$$J_\infty(k) \leq \sum_{i=1}^{L} p_i(k)\alpha_i \tag{3.79}$$

Proof. We first show that the inequality (3.79) holds. By using $Y_i = F_iP_i$ for (3.76), and pre- and post- multiplying it with $\text{diag}(P_i^{-1}, I, I, I)$, one obtains

$$\begin{bmatrix} P_i^{-1} & * & * & * \\ A_i + BF_i & P_j & * & * \\ Q_c & 0 & I & * \\ R_cF_i & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i, \forall j \in \overline{1,L} \tag{3.80}$$

By multiplying (3.80) by $p_i(k+t)$ and summing them up for $i \in \overline{1,L}$, one gets

$$\begin{bmatrix} P(k+t) & * & * & * \\ A_c(k+t) & P_j & * & * \\ Q_c & 0 & I & * \\ R_cF(k+t) & 0 & 0 & I \end{bmatrix} \succeq 0, \forall j \in \overline{1,L}$$

where $P(k+t) = \sum_{i=1}^{L} p_i(k+t)P_i^{-1}$, $A_c(k+t) = \sum_{i=1}^{L} p_i(k+t)(A_i + BF_i)$. Thus, using the Schur complement, one obtains

$$P(k+t) - \begin{bmatrix} A_c(k+t)^T & Q_c^T & F(k+t)^T R_c^T \end{bmatrix} \begin{bmatrix} P_j^{-1} & * & * \\ 0 & I & * \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_c(k+t) \\ Q_c \\ R_c F(k+t) \end{bmatrix} \succeq 0, \forall j \in \overline{1, L} \tag{3.81}$$

By multiplying (3.81) by $p_j(k+t+1)$ and summing them up for $j \in \overline{1, L}$, one gets

$$P(k+t) - \begin{bmatrix} A_c(k+t)^T & Q_c^T & F(k+t)^T R_c^T \end{bmatrix} \begin{bmatrix} P(t+k+1) & * & * \\ 0 & I & * \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_c(k+t) \\ Q_c \\ R_c F(k+t) \end{bmatrix} \succeq 0$$

or equivalently

$$P(k+t) - A_c(k+t)^T P(t+k+1) A_c(k+t) - Q - F(k+t)^T R F(k+t) \succeq 0$$

thus, by pre- and post- multiplying with $x(k+t)^T$, one obtains

$$\begin{aligned} x(k+t)^T P(k+t)x(k+t) - x(k+t)^T A_c(k+t)^T P(t+k+1) A_c(k+t)x(k+t) \\ \geq x(k+t)^T Q x(k+t) + x(k+t)^T F(k+t)^T R F(k+t)x(k+t) \end{aligned}$$

Recall that $x(k+t+1) = A_c(k+t)x(k+t)$, and $u(k+t) = F(k+t)x(k+t)$. As a consequence

$$\begin{aligned} & x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \\ & \leq x(k+t)^T P(k+t)x(k+t) - x(k+t+1)^T P(k+t+1)x(k+t+1) \end{aligned} \tag{3.82}$$

A direct conclusion of (3.82) is that the closed-loop system with the control law (3.73) is asymptotically stable. Therefore $\lim_{t \to +\infty} x(k+t) = 0$, and hence $\lim_{t \to +\infty} x(k+t)^T P(k+t)x(k+t) = 0$. Summing (3.82) from $t = 0$ to $t = \infty$, one obtains

$$\sum_{t=0}^{\infty} x^T(k+t)Qx(k+t) + u(k+t)^T R u(k+t) \leq x(k)^T P(k)x(k)$$

or equivalently, $\forall k \geq 0$

$$J_\infty(k) \leq x(k)^T P(k)x(k) \tag{3.83}$$

Hence $x(k)^T P(k)x(k)$ provides an upper bound of the cost $J_\infty(k)$.

Using the Schur complement for (3.75), one gets

$$\frac{1}{t_i} \geq x(k)^T P_i^{-1} x(k) \tag{3.84}$$

By multiplying (3.84) by $p_i(k)$ and summing them up for $i \in \overline{1, L}$, one obtains

$$\sum_{i=1}^{L} \frac{p_i(k)}{t_i} \geq x(k)^T P(k) x(k) \tag{3.85}$$

Using the Schur complement for (3.74), one gets

$$\alpha_i \geq \frac{1}{t_i}$$

thus, combining with (3.85), one obtains

$$\sum_{i=1}^{L} p_i(k)\alpha_i \geq x(k)^T P(k) x(k) \tag{3.86}$$

Using (3.83), (3.86), one gets

$$J_\infty(k) \leq \sum_{i=1}^{L} p_i(k)\alpha_i \tag{3.87}$$

Hence, the inequality (3.79) holds.

It remains to show that $u(k+t) \in \mathscr{U}$ and $x(k+t) \in \mathscr{X}$, $\forall t \geq 0$. For the input constraints, using (3.73) one needs to prove that

$$|f_{u,l}^T F(k+t) x(k+t)| \leq 1, \forall t \geq 0 \tag{3.88}$$

where $f_{u,l}$ is the $l-$th element of the standard basic in $\mathbb{R}^m$, $\forall l \in \overline{1, m}$, i.e.,

$$f_{u,l} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}^T$$

Using (3.82), (3.84), it follows that

$$x(k+t)^T P(k+t) x(k+t) \leq \sum_{i=1}^{L} \frac{p_i(k)}{t_i}, \forall i \in \overline{1, L} \tag{3.89}$$

It is clear that if (3.88) holds for $t = 0$, then (3.88) holds for any $t > 0$. Therefore, it suffices to show (3.88) for $t = 0$. Consider the following optimization problem

$$\begin{aligned} &\min_{x(k)} \left\{ f_{u,l}^T F_i x(k) \right\} \\ &\text{s.t. } x(k)^T P_i^{-1} x(k) \leq \tfrac{1}{t_i} \end{aligned} \tag{3.90}$$

The Lagrangian is

$$\mathscr{L}(x(k), \lambda) = f_{u,l}^T F_i x(k) + \frac{\lambda}{2} \left(x(k)^T P_i^{-1} x(k) - \frac{1}{t_i} \right)$$

where $\lambda \geq 0$ is the Lagrange multiplier. Since the cost function (3.90) is linear, the constraint (3.90) is always active. It follows that $\lambda > 0$. One has

$$\frac{\partial \mathscr{L}}{\partial x(k)} = F_i^T f_{u,l} + \lambda^* P_i^{-1} x^*(k) = 0$$

Consequently,

$$x^*(k) = -\frac{1}{\lambda^*} P_i F_i^T f_{u,l}$$

As the constraint (3.90) is active, one gets

$$\frac{1}{(\lambda^*)^2} f_{u,l}^T F_i P_i F_i^T f_{u,l} = \frac{1}{t_i}$$

Therefore

$$\lambda^* = \sqrt{t_i f_{u,l}^T F_i P_i F_i^T f_{u,l}}$$

It follows that

$$x^*(k) = -\frac{1}{\sqrt{t_i f_{u,l}^T F_i P_i F_i^T f_{u,l}}} P_i F_i^T f_{u,l}$$

Consequently

$$\min_{x(k)} \left\{ f_{u,l}^T F_i x(k) \right\} = -\sqrt{\frac{f_{u,l}^T F_i P_i F_i^T f_{u,l}}{t_i}}$$

Recall that $F_i = Y_i P_i^{-1}$, and $f_{u,l}^T Y_i = Y_{i,l}$. One gets

$$\min_{x(k)} \left\{ f_{u,l}^T F_i x(k) \right\} = -\sqrt{\frac{Y_{i,l} P_i^{-1} Y_{i,l}^T}{t_i}}$$

As the set (3.89) is symmetric, it is clear that to guarantee (3.88), it suffices

$$-\sqrt{\frac{Y_{i,l} P_i^{-1} Y_{i,l}^T}{t_i}} \geq -1, \forall i \in \overline{1, L}, \forall l \in \overline{1, m}$$

or equivalently

$$Y_{i,l} P_i^{-1} Y_{i,l}^T \leq t_i, \forall i \in \overline{1, L}, \forall l \in \overline{1, m}$$

Using the Schur complement, this condition becomes (3.77). Hence the input constraints are satisfied.

For the state constraints, similar arguments to those used for the input constraints apply. The only exception is that, since $x(k)$ is already available, one needs to ensure that $x(k+t) \in \mathscr{X}$ for all $t \geq 1$. The proof for the state constraints is omitted here. □

Algorithm 3.3 MPC Based on Linear Gain-Scheduling Control Law

1. Set $k \leftarrow 0$
2. Measure and/or estimate $x(k)$ and $p(k)$.
3. Solve the following optimization problem

$$\begin{aligned} &\min_{\alpha_i, t_i, P_i, Y_i} \left\{ \sum_{i=1}^{L} p_i(k)\alpha_i \right\}, \\ &\text{s.t. (3.74), (3.75), (3.76), (3.77), (3.78).} \end{aligned} \tag{3.91}$$

4. Calculate $F_i = Y_i P_i^{-1}, \forall i \in \overline{1, L}$.
5. Apply the control law $u(k) = \left(\sum_{i=1}^{L} p_i(k) F_i \right) x(k)$ to the plant.
6. Set $k \leftarrow k + 1$ and go to step 2.

Using Theorem 3.7, we propose the following MPC algorithm.

Define the optimal solution of (3.91) at time k as $\alpha_i^*(k), t_i^*(k) P_i^*(k), F_i^*(k)$. The following theorem holds.

Theorem 3.8. *Assuming feasibility at $x(k)$ for a generic time k. Then, Algorithm 3.3 guarantees recursive feasibility. Moreover, the closed-loop system under the MPC control law is asymptotically stable.*

Proof. Recursive Feasibility Proof: The basic idea of the feasibility proof is to show that the optimal solution at time k remains a feasible solution at time $k + 1$. Note that (3.74), (3.75), (3.76), (3.77), (3.78) do not contain the scheduling parameters $p(k)$. Therefore if it holds at time k, then it also holds at time $k + 1$.

Using (3.77), (3.78) it is clear that $x(k + t) \in \mathscr{X}, u(k + t) \in \mathscr{U}, \forall t \geq 1$. Hence the state and input constraints are satisfied.

By (3.74), (3.75), (3.76), it is clear that if $J_\infty(k) \leq \sum_{i=1}^{L} p_i(k)\alpha_i^*$, then

$$J_\infty(k + 1) \leq \sum_{i=1}^{L} p_i(k + 1)\alpha_i^*$$

thus (3.75) is satisfied at time instant $k + 1$. We conclude that the optimal solution at time k is a feasible solution at time $k + 1$. Hence, recursive feasibility is guaranteed.

Stability Proof. Consider the following candidate Lyapunov function

$$V(x) = x(k)^T P^*(k) x(k) \tag{3.92}$$

where $P(k) = \sum_{i=1}^{L} p_i(k)(P_i^*)^{-1}(k)$. It is clear that $V(x) > 0, \forall x \neq 0$ and $V(x) = 0$ if and only if $x = 0$.

Using the two facts that

- the optimal solution at time k is a feasible solution at time $k+1$;
- we minimize $\sum_{i=1}^{L} p_i(k)\alpha_i^*$ at each time instant;

it follows that

$$x(k+1)^T P^*(k+1)x(k+1) \leq x(k+1)^T P(k+1)x(k+1) \tag{3.93}$$

where $P(k+1) = \sum_{i=1}^{L} p_i(k+1)(P_i^*)^{-1}(k)$.

Using (3.82), one has

$$x(k+1)^T P(k+1)x(k+1) \leq x(k)^T P^*(k)x(k) - x(k)^T Qx(k) - u(k)^T Ru(k)$$

thus, using (3.93), one obtains

$$x(k+1)^T P^*(k+1)x(k+1) \leq x(k)^T P^*(k)x(k) - x(k)^T Qx(k) - u(k)^T Ru(k)$$

or equivalently

$$V(x(k+1)) \leq V(x(k)) - x(k)^T Qx(k) - u(k)^T Ru(k)$$

If follows that $V(k)$ is a Lyapunov function for the closed-loop system. In the other words, asymptotic stability is guaranteed. □

It is worth noting that, despite employing a gain-scheduling control law (3.73) and a PDLF (3.92), our approach does not result in a double summation in the obtained conditions. This reduces computational complexity and decreases conservatism.

3.4.2 MPC with saturated gain-scheduling feedback

It is observed that while the MPC method with linear gain-scheduling feedback allows for an efficient online computation, the resulting control action is often unsaturated. Consequently, the state trajectories can take longer than necessary to converge to the origin. To overcome this challenge and to fully utilize the input range power, we parameterize the control input as

$$u(k+t) = \text{sat}\left(F(k+t)x(k+t)\right), \forall t \geq 0 \tag{3.94}$$

where $F(k+t) = \sum_{i=1}^{L} p_i(k+t)F_i$. The saturation function $\text{sat}(u) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as (2.53).

Theorem 3.9. *Consider the control law (3.94). Assume that there exist matrices Y_i, Z_i, P_i and positive scalars α_i, t_i such that*

$$\begin{bmatrix} \alpha_i & * \\ 1 & t_i \end{bmatrix} \succeq 0, \forall i \in \overline{1, L} \tag{3.95}$$

$$\begin{bmatrix} t_i & * \\ t_i x(k) & P_i \end{bmatrix} \succeq 0, \forall i \in \overline{1, L} \tag{3.96}$$

$$\begin{bmatrix} P_i & * & * & * \\ A_i P_i + B(D_p Y_i + D_p^- Z_i) & P_j & * & * \\ Q_c P_i & 0 & I & * \\ R_c(D_p Y_i + D_p^- Z_i) & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i, \forall j \in \overline{1, L}, \forall q \in \overline{1, 2^m} \tag{3.97}$$

$$\begin{bmatrix} t_i & * \\ Z_{i,l}^T & P_i \end{bmatrix} \succeq 0, \ \forall i \in \overline{1, L}, \forall l \in \overline{1, m} \tag{3.98}$$

$$\begin{bmatrix} t_i & * \\ \left(A_i P_i + B D_q Y_i + B D_q^- Z_i\right)^T f_l & P_i \end{bmatrix} \succeq 0, \ \forall i \in \overline{1, L}, \forall l \in \overline{1, n_x}, \forall p \in \overline{1, 2^m} \tag{3.99}$$

where $Y_i = F_i P_i, Z_i = H_i P_i, \forall i \in \overline{1, L}$. Then $u(k+t) \in \mathscr{U}$, $x(k+t) \in \mathscr{X}$, $\forall t \geq 0$, and the following inequality holds

$$J_\infty(k) \leq \sum_{i=1}^{L} p_i(k) \alpha_i \tag{3.100}$$

Proof. Similar to the proof of Theorem 3.7, we will first show that the inequality (3.100) holds. Using $Y_i = F_i P_i, Z_i = H_i P_i$, and pre- and post- multiplying (3.97) with $\text{diag}(P_i^{-1}, I, I, I)$, one gets

$$\begin{bmatrix} P_i^{-1} & * & * & * \\ \mathscr{A}_{i,q} & P_j & * & * \\ Q_c & 0 & I & * \\ R_c(D_q F_i + D_q^- H_i) & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i, \forall j \in \overline{1, L}, \forall q \in \overline{1, 2^m} \tag{3.101}$$

where $\mathscr{A}_{i,q} = A_i + B D_q F_i + B D_q^- H_i, i \in \overline{1, L}, q \in \overline{1, 2^m}$. Multiplying (3.101) by $p_i(k+t)$ and summing them up for $i \in \overline{1, L}$, one obtains

$$\begin{bmatrix} P(k+t) & * & * & * \\ \mathscr{A}_q(k+t) & P_j & * & * \\ Q_c & 0 & I & * \\ R_c\left(D_q F(k+t) + D_q^- H(k+t)\right) & 0 & 0 & I \end{bmatrix} \succeq 0, \forall j \in \overline{1, L}, \forall q \in \overline{1, 2^m}$$

thus, using the Schur complement, one gets $\forall j \in \overline{1, L}, \forall q \in \overline{1, 2^m}$

$$\begin{aligned} &P(k+t) - \mathscr{A}_q(k+t)^T P_j^{-1} \mathscr{A}_q(k+t) - Q \\ &- \left(D_q F(k+t) + D_q^- H(k+t)\right)^T R \left(D_q F(k+t) + D_q^- H(k+t)\right) \succeq 0 \end{aligned} \tag{3.102}$$

Multiplying (3.102) by $p_i(t+k+1)$ and summing them up for $j \in \overline{1,L}$, one obtains

$$\begin{aligned} &P(k+t) - \mathscr{A}_q(k+t)^T P(k+t+1)\mathscr{A}_q(k+t) - Q \\ &- \left(D_q F(k+t) + D_q^- H(k+t)\right)^T R \left(D_q F(k+t) + D_q^- H(k+t)\right) \succeq 0 \end{aligned} \tag{3.103}$$

Pre- and post- multiplying (3.103) with $x(k+t)$, and using Lemma 2.11, one gets

$$x(k+t)^T P(k+t)x(k+k) - x(k+t+1)^T P(k+t)x(k+t+1) - x(k+t)^T Q x(k+t) - u(k+t)^T R u(k+t) \geq 0 \tag{3.104}$$

It follows that the closed loop system with the saturated control law (3.94) is asymptotically stable. Hence $\lim\limits_{t\to+\infty} x(k+t) = 0$, and $\lim\limits_{t\to+\infty} x(k+t)^T P(k+t)x(k+t) = 0$.

Summing (3.104) from $t = 0$ to $t = \infty$, one gets

$$\sum_{t=0}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \leq x(k)^T P(k) x(k)$$

It follows that

$$J_\infty(k) \leq x(k)^T P(k) x(k) \tag{3.105}$$

Therefore, $x(k)^T P(k)x(k)$ provides an upper bound of $J_\infty(k)$.

Following the proof of Theorem 3.7, the LMIs (3.95), (3.96) imply that

$$x(k)^T P(k) x(k) \leq \sum_{i=1}^{L} \frac{p_i(k)}{t_i} \leq \sum_{i=1}^{L} p_i(k)\alpha_i \tag{3.106}$$

Using (3.105), (3.106), one obtains (3.100).

Note that we have (3.105), (3.106) or consequently (3.100) only for $x(k)$ such that $|H_{i,l}x(k)| \leq 1$, $\forall i \in \overline{1,L}, \forall l \in \overline{1,m}$. Similar to the proof of Theorem 3.7, the conditions $|H_{i,l}x(k)| \leq 1$ are satisfied if and only if

$$\begin{bmatrix} t_i & * \\ Z_{i,l}^T & P_i \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall l \in \overline{1,m}$$

or equivalently (3.98) holds.

It remains to prove $u(k+t) \in \mathscr{U}$ and $x(k+t) \in \mathscr{X}$. As we employ (3.94), it is clear $u(k+t) \in \mathscr{U}$. For the state constraint, it is omitted here, as the proof is similar to that of Theorem 3.7. □

Using Theorem 3.9, we propose the following MPC algorithm.

It is noticed that both $F(k)$ and $H(k)$ are decision variables in Algorithm 3.4. If $H(k) = F(k)$, then Algorithm 3.4 becomes Algorithm 3.3. Hence Algorithm 3.4 provides more degrees of freedom than Algorithm 3.3, which generally leads to improved performance at the cost of increased computational complexity.

Algorithm 3.4 MPC Based on Saturated Gain-Scheduling Control Law

1. Set $k \leftarrow 0$
2. Measure and/estimate $x(k)$ and $p(k)$.
3. Solve the following optimization problem

$$\begin{aligned} &\min_{\alpha_i, t_i, P_i, Y_i, Z_i} \left\{ \sum_{i=1}^{L} p_i(k)\alpha_i \right\}, \\ &\text{s.t. } (3.95), (3.96), (3.97), (3.98), (3.99). \end{aligned} \tag{3.107}$$

4. Compute $F_i = Y_i P_i^{-1}$.
5. Apply the control law $u(k) = \text{sat}\left(\left(\sum_{i=1}^{L} p_i(k) F_i \right) x(k) \right)$ to the plant.
6. Set $k \leftarrow k+1$ and go to step 2.

Theorem 3.10. *Assuming feasibility at $x(k)$ for a generic time k. Then, Algorithm 3.4 guarantees recursive feasibility. Moreover the closed-loop system with the MPC control law is asymptotically stable.*

Proof. The main idea of the proof is to show that $V(k)$ is a Lyapunov function of the closed-loop system. The proof is omitted as it follows closely to the proof of Theorem 3.8. □

Remark: Note that the distinction between $F(k)$ and $H(k)$ in Algorithm 3.4 - where $F(k)$ represents the actual feedback law and $H(k)$ serves as an auxiliary law - allows for differential weighting, which can further enhance closed-loop performance. To illustrate this more clearly, consider a Lyapunov function $V(k+t) = x(k+t)^T P(k+t) x(k+t)$. In this case

$$\begin{aligned} \Delta V(k+t) &= V(k+t) - V(k+t+1) \\ &= x(k+t)^T P(k+t) x(k+t) - x(k+t+1)^T P(k+t+1) x(k+t+1) \end{aligned}$$

Without loss of generality, assume $D_1 = I$. To upper bound the cost $J_0^\infty(k)$, one can select $P(k)$ such that

$$\begin{cases} \Delta V(k+t) \geq \frac{1}{\beta} (x(k+t)^T \left(Q + F(k+t)^T R F(k+t) \right) x(k+t) \\ \Delta V(k+t) \geq \\ x(k+t)^T \left(Q + \left(D_q F(k+t) + D_q^- H(k+t) \right)^T R \left(D_q F(k+t) + D_q^- H(k+t) \right) \right) x(k+t), \\ \forall q \in \overline{2, 2^m} \end{cases} \tag{3.108}$$

where $\beta \in (0, 1]$. Note that the tunning additional parameter β assigns a heavier weight to the actual feedback law than to the auxiliary feedback laws. Similar to Theorem 3.9, we can show that

(3.108) can be equivalently rewritten as

$$\begin{bmatrix} P_i & * & * & * \\ A_iP_i + BY_i & P_j & * & * \\ Q_cP_i & 0 & \beta I & * \\ R_cY_i & 0 & 0 & \beta I \end{bmatrix} \succeq 0, \forall i, \forall j \in \overline{1, L}$$
$$\begin{bmatrix} P_i & * & * & * \\ A_iP_i + B(D_qY_i + D_q^- Z_i) & P_j & * & * \\ Q_cP_i & 0 & I & * \\ R_c(D_qY_i + D_q^- Z_i) & 0 & 0 & I \end{bmatrix} \succeq 0, \forall i, \forall j \in \overline{1, L}, \forall q \in \overline{2, 2^m} \tag{3.109}$$

3.4.3 Numerical Examples

Two example systems are shown in this section. The CVX toolbox [66] was used to solve the SDP optimization problems.

3.4.3.1 Example 6

Consider system (3.72) with

$$A_1 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 0.99 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

The input constraint is $|u(k)| \leq 1$. There are no state constraints. The weighting matrices are $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = 0.01$.

For the initial condition $x(0) = [3 \; 0]^T$, Fig. 3.7 and Fig. 3.8 present, respectively, the state and the input trajectories as functions of time for Algorithm 3.3 (solid blue lines), for Algorithm 3.4 (solid red lines). For comparison purposes, Fig. 3.7 and Fig. 3.8 also present the state and the input trajectories for the method in [3] (dash-dot green lines), and for [4] (dashed violet lines). The superior performance of the proposed algorithms compared to existing ones is clearly observed.

As highlighted by [74], the computational complexity of an SDP problem depends on the number of decision variables and the number of LMI constraints. Table 3.1 provides these parameters for Algorithm 3.3, for Algorithm 3.4, for the two existing methods in [3], and in [4].

Table 3.1: Computational complexity for example 6

	Algorithm 3.3	**Algorithm 3.4**	**Garone et al. [3]**	**Waza et al. [4]**
No. of decision variables	14	18	19	19
No. of LMI constraints	10	12	10	8

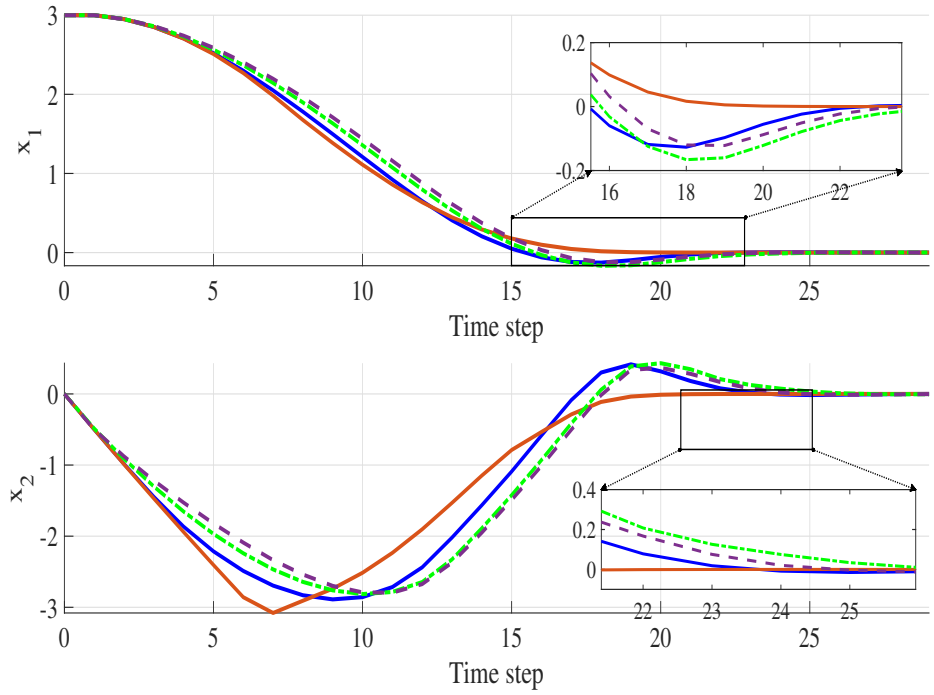


Figure 3.7: State trajectories for Algorithm 3.3 (solid blue), for Algorithm 3.4 (solid red), for [3] (dash-dot green), and for [4] (dashed violet) for example 6.

3.4.3.2 Example 7

In this example, we consider system (3.72) with

$$A_1 = \begin{bmatrix} 1.0 & 0.3 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.0 & 0.8 & 0 \\ 0 & 1 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1.0 & 0.6 & 0 \\ 0 & 1 & 0.7 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

The input and state constraints are

$$|u_1(k)| \leq 1, |u_2(k)| \leq 0.5, |x_1(k)| \leq 10$$

The weighting matrices are $Q = I, R = 10^{-2}I$.

For the initial condition $x(0) = [2 \quad 1.7 \quad 0]^T$, Fig. 3.9 and Fig. 3.10 show, respectively, the state and the input trajectories as functions of time for Algorithm 3.3 (solid blue lines), for Algorithm 3.4 (solid red lines), for [3] (dash-dot green lines), and for [4] (dashed violet lines). It can be observed that the proposed algorithms outperform existing results. In particular, for Algorithm 3.4,

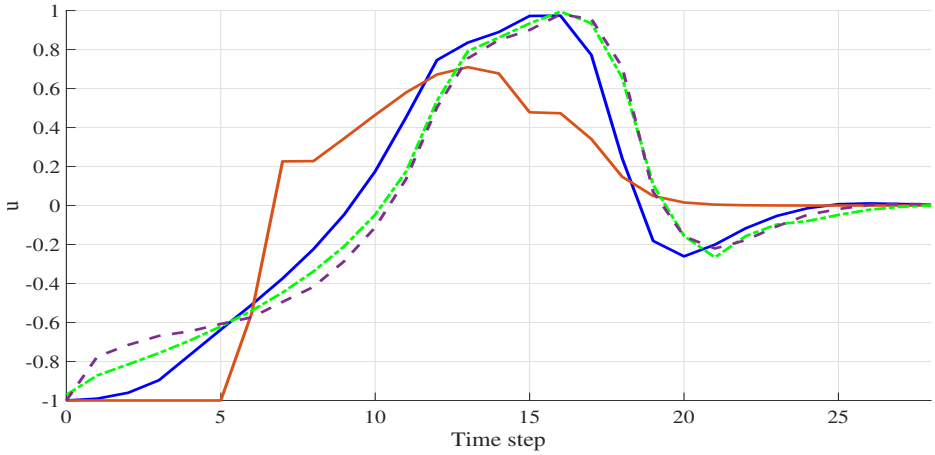


Figure 3.8: Input trajectories for Algorithm 3.3 (solid blue), for Algorithm 3.4 (solid red), for [3] (dash-dot green), and for [4] (dashed violet) for example 6.

the use of the nonlinearly saturated control law enables the control action to fully utilize the available actuator capacities.

Table 3.2: Computational complexity for example 7

	Algorithm 3.3	**Algorithm 3.4**	**Garone et al. [3]**	**Waza et al. [4]**
No. of decision variables	30	36	46	46
No. of LMI constraints	21	30	36	18

Finally, Table 3.2 gives the number of decision variables as well as the number of LMI constraints for Algorithm 3.3, for Algorithm 3.4, for the two existing methods in [3], and in [4].

3.5 Conclusion

In this chapter, we first review the related work on min-max MPC and quasi-min-max MPC, analyzing the sources of conservatism in existing approaches. Motivated by these sources of conservatism, we then propose three new quasi-min-max MPC algorithms for discrete-time LPV systems with input and state constraints. The main idea is to use a class of nonlinearly parameterized state feedback control laws. In Algorithm 3.1, recursive feasibility of the online optimization problem and asymptotic stability of the closed-loop system are guaranteed. In Algorithm 3.2, a linear Lyapunov constraint is introduced to ensure stability.

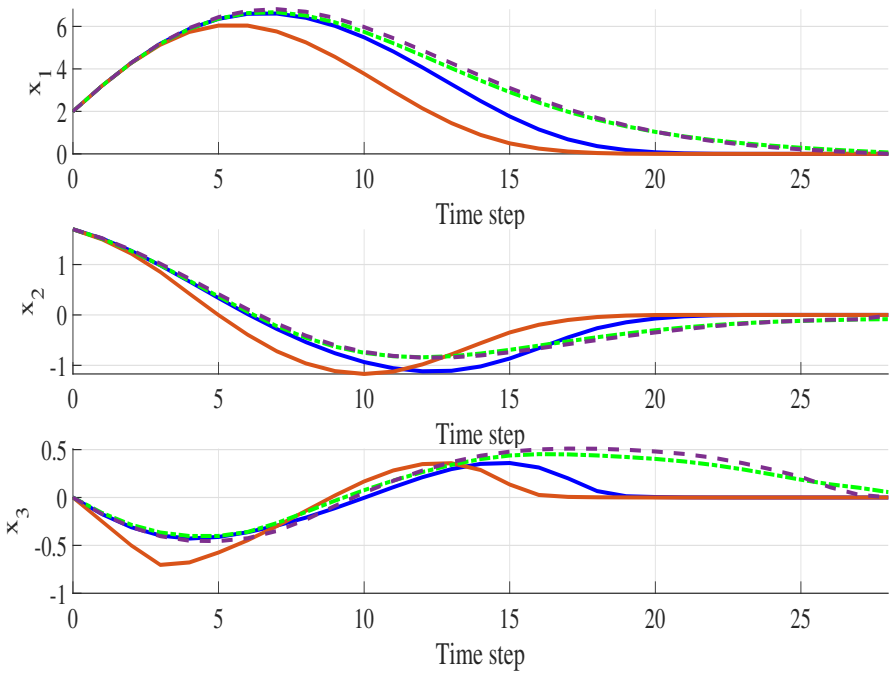


Figure 3.9: State trajectories for Algorithm 3.3 (solid blue), for Algorithm 3.4 (solid red), for [3] (dash-dot green), and for [4] (dashed violet) for example 7.

It is observed that the proposed quasi-min–max MPC algorithms lead to a double-sum formulation. To circumvent this issue, we derive new LMI conditions for the MPC problem that fully exploit the PDLF without the double-sum formulation, and further extend the results to the case of saturated feedback.

Nevertheless, the requirement of solving a convex semi-definite program at each sampling instant introduces a non-negligible computational burden, particularly for systems with high complexity or fast sampling rates. Consequently, the applicability of the proposed approach may be restricted to systems equipped with sufficient computational resources to meet real-time implementation requirements.

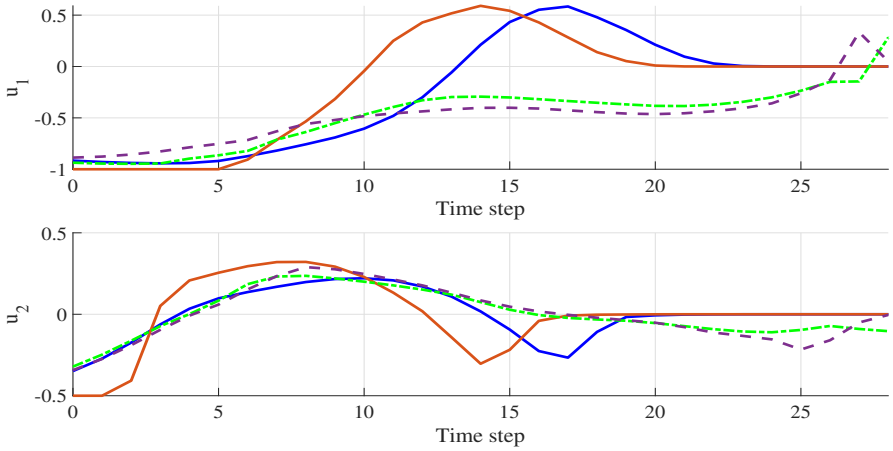


Figure 3.10: Input trajectories for Algorithm 3.3 (solid blue), for Algorithm 3.4 (solid red), for [3] (dash-dot green), and for [4] (dashed violet) for example 7.

Chapter 4

Smooth Merging Gain-Scheduling Low- and High-Gain Controllers for Constrained LPV System

In this chapter, we address the problem of merging two gain-scheduling control laws for constrained discrete-time LPV systems. A key result demonstrates that, for this class of systems, it is possible to merge any two controllers without requiring the control-sharing property [46, 47].

The main contributions of this chapter are summarized as follows:

- **Merging of Two Linear Gain-Scheduling Control Laws:** We propose a method to merge two asymptotically stabilizing gain-scheduling control laws without relying on the control-sharing property. The resulting control law is continuous with respect to the state, preserves local optimality by reducing to the high-gain controller whenever feasible, and achieves a domain of attraction comparable to that of the low-gain controller.
- **Merging of Two Nonlinear Saturated Gain-Scheduling Control Laws:** To fully exploit the available control input range, we extend the proposed framework to the merging of two asymptotically stabilizing saturated gain-scheduling control laws. The resulting control law retains all the desired properties: (i) it is continuous in the state; (ii) it guarantees local optimality; and (iii) it achieves a domain of attraction as large as that of the saturated low-gain controller.

This chapter is organized as follows. Section 4.1 covers the problem formulation. Section 4.2 and Section 4.3 are, respectively, concerned with the problem of merging two linear gain-scheduling control laws, and of merging two saturated gain-scheduling control laws. Two numerical examples with comparison to earlier solutions from the literature are evaluated in Section 4.4. Finally, Section 4.5 concludes this chapter.

4.1 Problem formulation

Consider the following discrete-time LPV system

$$x(k+1) = A(k)x(k) + Bu(k) \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the measured state, $u \in \mathbb{R}^m$ is the control input, $B \in \mathbb{R}^{n\times m}$ is known and fixed. The matrix $A(k)$ satisfies

$$A(k) = \sum_{i=1}^{L} p_i(k) A_i \tag{4.2}$$

where $A_i \in \mathbb{R}^{n\times n}, i \in \overline{1, L}$ are known and fixed, s is the number of subsystems, and the parameter $p(k) = [p_1(k)\ p_2(k)\ \ldots\ p_L(k)]^T$ satisfies $p(k) \in \Sigma_s$ with

$$\Sigma_L := \left\{ p \in \mathbb{R}^L \mid \sum_{i=1}^{L} p_i = 1,\ p_i \geq 0 \right\} \tag{4.3}$$

Equations (4.2), (4.3) implies that $A(k) \in \Psi$, where Ψ is the convex hull of $A_1, A_2, \ldots, A_L$.

LPV hypothesis: $p(k)$ is available in each time instant k.

We consider the following constraints on the state $x(k)$, and on the input $u(k)$

$$\begin{cases} x(k) \in \mathscr{X} := \left\{ x \in \mathbb{R}^n \mid |f_l^T x| \leq 1, \forall l \in \overline{1, n_x} \right\} \\ u(k) \in \mathscr{U} := \left\{ u \in \mathbb{R}^m \mid |u_l| \leq 1, \forall l \in \overline{1, m} \right\} \end{cases} \tag{4.4}$$

where $f_l \in \mathbb{R}^n, l \in \overline{1, n_x}$ are known and fixed, u_l is the $l-$th component of the vector u, $l \in \overline{1, m}$.

Control Objective: Denote $x(k+t), u(k+t), t = 0, 1, \ldots$, respectively, as the predicted states and the predicted control inputs at time $k+t$ from time k. The control design problem is to select $u(k+t), t = 0, 1, 2, ...$, to stabilize the system, while satisfying the constraints (4.4), and solving the following min-max problem

$$\begin{aligned} &\min_{U_0^\infty(k)} \max_{A(k+t)\in\Psi} \{J(k)\} \\ &\text{s.t. } (4.1), (4.4), \end{aligned} \tag{4.5}$$

where

$$U_0^\infty(k) = [u(k)^T\ u(k+1)^T\ \ldots]^T \tag{4.6}$$

and

$$J(k) = \sum_{t=0}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \tag{4.7}$$

with $Q \in \mathbb{S}^n, R \in \mathbb{S}^m$ being the state and input weighting matrices [75].

In the following, we present a method to achieve the control objective. The main idea is to blend low-gain and high-gain controllers. The low-gain control law serves to enlarge the DoA, whereas the high-gain control law ensures the desired performance.

4.2 Merging Linear Low- and High-Gain Controllers

4.2.1 The Control Law

In this section, we assume that the asymptotically stabilizing high-gain $K(k)$, and low-gain $F(k)$ control matrices are available.

As discussed in Section 2.6.2, the LQR control law (2.113) provides a solution to problem (4.5) in the absence of the constraints (4.4). A weakness of this control law is that its DoA can be rather small. This is because the control input may hit the constraint even for states close to the origin.

In the following we will provide a way to overcome this weakness. The main idea is to blend the DoA-oriented low-gain and the performance-oriented high-gain controllers [76, 77]. To this end, we consider the following control law

$$u(k) = K(k)x(k) + (F(k) - K(k))c(k) \tag{4.8}$$

where $K(k)$ is the high-gain control matrix, $F(k)$ is the low-gain control matrix, and the internal controller state $c(k) \in \mathbb{R}^n$ provides a tunable degree of freedom. By updating $c(k)$ dynamically, the control law (4.8) can simultaneously achieve the performance of the high-gain controller and the large DoA associated with the low-gain controller. In particular, when $c(k) = 0$, (4.8) reduces to the high-gain law $u(k) = K(k)x(k)$, which emphasizes the performance. Conversely, when $c(k) = x(k)$ then (4.8) becomes $u(k) = F(k)x(k)$, corresponding to the low-gain law that ensures a large DoA.

We present now a method to address the problem (4.5) using the control law (4.8). For this purpose, we assume that $c(k)$ evolves according to the dynamics

$$c(k+1) = (A(k) + BF(k))c(k) \tag{4.9}$$

Combining (4.1), (4.8), (4.9), one obtains

$$z(k+1) = \mathscr{A}(k)z(k), \tag{4.10}$$

where

$$z(k) = \begin{bmatrix} x(k) \\ c(k) \end{bmatrix}, \mathscr{A}(k) = \sum_{i=1}^{L} p_i(k)\mathscr{A}_i$$

$$\mathscr{A}_i = \begin{bmatrix} A_i + BK_i & B(F_i - K_i) \\ 0 & A_i + BF_i \end{bmatrix}, \forall i \in \overline{1, L}.$$

Consider the following quadratic function

$$\mathbf{V}(k) = z(k)^T \left(\sum_{i=1}^{L} p_i(k)\mathscr{W}_i \right) z(k), \tag{4.11}$$

where the matrices $\mathscr{W}_i \in \mathbb{S}^{2n}, i \in \overline{1,L}$ are chosen to satisfy, $\forall t = 0, 1, \ldots,$

$$x(k+t)Qx(k+t) + u(k+t)^T Ru(k+t) \leq \mathbf{V}(k+t) - \mathbf{V}(k+t+1). \tag{4.12}$$

We will discuss the conditions for the existence of $\mathscr{W}_i$ latter. For the moment, we assume that there exist $\mathscr{W}_i, i \in \overline{1,L}$ fulfilling (4.12).

Since $F(k), K(k)$ are asymptotically stabilizing control laws, it follows that system (4.10) is asymptotically stable for states near the origin, i.e., $\lim\limits_{t\to+\infty} z(k+t) = 0$. Hence $\lim\limits_{t\to+\infty} \mathbf{V}(k+t) = 0$. Summing (4.12) from $t = 0$ to $t = \infty$, one gets

$$\sum_{t=0}^{\infty} \left(x(k+t)^T Qx(k+t) + u(k+t)^T Ru(k+t)\right) \leq \mathbf{V}(k). \tag{4.13}$$

Therefore $\mathbf{V}(k)$ provides an upper bound of $J(k)$.

Define $\mathbf{K}(k) = \sum\limits_{i=1}^{L} p_i(k)\mathbf{K}_i$ with $\mathbf{K}_i = [K_i \quad F_i - K_i], i \in \overline{1,L}$. It follows that $u(k) = \mathbf{K}(k)z(k)$. Define also $\mathbf{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$.

The following result provides a necessary and sufficient condition for the existence of $\mathscr{W}_i$ in (4.12).

Theorem 4.1. *There exist matrices $\mathscr{W}_i \in \mathbb{S}^{2n}, \forall i \in \overline{1,L}$ satisfying (4.12) if and only if the following LMI conditions hold*

$$\begin{bmatrix} \mathscr{W}_i - \mathbf{Q} & * & * \\ \mathscr{W}_j\mathscr{A}_i & \mathscr{W}_j & * \\ \mathbf{K}_i\sqrt{R} & 0 & I \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}. \tag{4.14}$$

Proof. Since t is arbitrary, it suffices to verify (4.12) for $t = 0$. Using the facts that x(k) =$[I_n\ 0_n]z(k)$ and $u(k) = \mathbf{K}(k)z(k)$, we can rewrite equivalently condition (4.12) as

$$\begin{aligned} &\mathbf{Q} + \left(\sum_{i=1}^{s} p_i(k)\mathbf{K}_i\right)^T R \left(\sum_{i=1}^{s} p_i(k)\mathbf{K}_i\right) \\ &\quad \preceq \sum_{i=1}^{s} p_i(k)\mathscr{W}_i - \left(\sum_{i=1}^{s} p_i(k)\mathscr{A}_i\right)^T \left(\sum_{j=1}^{s} p_j(k+1)\mathscr{W}_j\right) \left(\sum_{i=1}^{s} p_i(k)\mathscr{A}_i\right). \end{aligned} \tag{4.15}$$

Condition (4.15) holds if and only if, $\forall j \in \overline{1,L}$

$$\begin{aligned} &\mathbf{Q} + \left(\sum_{i=1}^{s} p_i(k)\mathbf{K}_i\right)^T R \left(\sum_{i=1}^{s} p_i(k)\mathbf{K}_i\right) \\ &\quad \preceq \sum_{i=1}^{s} p_i(k)\mathscr{W}_i - \left(\sum_{i=1}^{s} p_i(k)\mathscr{A}_i\right)^T \mathscr{W}_j \left(\sum_{i=1}^{s} p_i(k)\mathscr{A}_i\right). \end{aligned}$$

Using Schur complement, one obtains

$$\begin{bmatrix} \sum_{i=1}^{s} p_i(k)\mathscr{W}_i - \mathbf{Q} & * & * \\ \mathscr{W}_j\left(\sum_{i=1}^{s} p_i(k)\mathscr{A}_i\right) & \mathscr{W}_j & * \\ \sqrt{R}\left(\sum_{i=1}^{s} p_i(k)\mathbf{K}_i\right) & 0 & I \end{bmatrix} \succeq 0, \forall j \in \overline{1,L},$$

or equivalently,

$$\begin{bmatrix} \mathscr{W}_i - \mathbf{Q} & * & * \\ \mathscr{W}_j\mathscr{A}_i & \mathscr{W}_j & * \\ \sqrt{R}\mathbf{K}_i & 0 & I \end{bmatrix} \geq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}.$$

The proof is complete. □

Since the matrices $\mathscr{A}_i$ are upper triangular, there exist $\mathscr{W}_i$ satisfying (4.14), if and only if there exist $\mathscr{W}_i$ in the following diagonal form

$$\mathscr{W}_i = \begin{bmatrix} W_{i,x} & 0_{n\times n} \\ 0_{n\times n} & W_{i,c} \end{bmatrix}, i \in \overline{1,L} \tag{4.16}$$

where $W_{i,x} \in \mathbb{S}^{n\times n}$, $W_{i,c} \in \mathbb{S}^{n\times n}$. We adopt the block-diagonal structure for $\mathscr{W}_i$ to ensure the local optimality. In particular, the control law (4.8) reduces to the high-gain control law $u(k) = K(k)x(k)$ whenever the latter is feasible. We will provide a more detailed explanation of this choice in the following section.

We use the matrices $\mathscr{W}_i$ to provide the upper bound of the cost function $J(k)$. We can find the optimal $\mathscr{W}_i$ by solving the following SDP problem

$$\begin{aligned} &\min_{\mathscr{W}_{i,x},\mathscr{W}_{i,c}} \left\{ \sum_{i=1}^{L} \left(\text{trace}(\mathscr{W}_{i,x}) + \text{trace}(\mathscr{W}_{i,c})\right) \right\} \\ &\text{s.t. (4.14).} \end{aligned} \tag{4.17}$$

Our next step is to characterize the DoA of the control law (4.8). Define $\mathbf{f}_l = [f_l^T \ 0]^T, l \in \overline{1,n_x}$. Using (4.8), the constraints (4.4) becomes $z(k) \in \mathscr{Z}$, with

$$\mathscr{Z} = \left\{ z(k) \in \mathbb{R}^{2n} : \begin{array}{l} |\mathbf{f}_l^T z(k)| \leq 1, l \in \overline{1,n_x} \\ |\mathbf{K}(k)z(k)| \leq \mathbf{1} \end{array} \right\}. \tag{4.18}$$

Consider the following parameter dependent ellipsoid $\mathscr{E}(\mathscr{P}(k)) \subset \mathbb{R}^{2n}$

$$\mathscr{E}(\mathscr{P}(k)) = \left\{ z(k) \in \mathbb{R}^{2n} : z(k)^T \mathscr{P}(k) z(k) \leq 1 \right\}$$

where $\mathscr{P}(k) = \sum_{i=1}^{L} p_i(k)\mathscr{P}_i^{-1}$ with $\mathscr{P}_i \in \mathbb{S}^{2n}, i \in \overline{1,L}$.

We have the following result.

Theorem 4.2. *If there exist matrices $\mathscr{P}_i \in \mathbb{S}^{2n}, \forall i \in \overline{1,L}$ such that the following LMI conditions hold:*

$$\begin{bmatrix} \mathscr{P}_i & * \\ \mathscr{A}_i\mathscr{S}_i & \mathscr{P}_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L} \tag{4.19}$$

$$\begin{bmatrix} 1 & * \\ \mathscr{P}_i\mathbf{K}_{i,l}^T & \mathscr{P}_i \end{bmatrix} \succeq 0, \ \forall i \in \overline{1,L}, \forall l \in \overline{1,m} \tag{4.20}$$

$$\begin{bmatrix} 1 & * \\ \mathscr{P}_i\mathbf{f}_l & \mathscr{P}_i \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall l \in \overline{1,n_x} \tag{4.21}$$

where $\mathbf{K}_{i,l}$ is the $l-$th row of $\mathbf{K}_i$, then $\mathscr{E}(\mathscr{P}(k))$ is parameter-dependent invariant and constraint-admissible for system (4.10) with constraints (4.18).

Proof. The proof is analogous to that of Theorem 2.9 and is omitted here. □

We divide the matrix $\mathscr{P}_i$ into blocks as $\mathscr{P}_i = \begin{bmatrix} \mathscr{P}_{i,11} & \mathscr{P}_{i,12} \\ \mathscr{P}_{i,12}^T & \mathscr{P}_{i,22} \end{bmatrix}$.

The orthogonal projection $\mathscr{E}_x$ of $\mathscr{E}(\mathscr{P}(k))$ onto the $x-$space represents the DoA for the system (4.1) under the control law (4.8). The set $\mathscr{E}_x$ should be maximized. It is clear that $\mathscr{E}_x$ is proportional to the volume of $\mathscr{E}(\mathscr{P}_{i,11}^{-1})$. On the other hand, it is clear that the feasible set for c is proportional to the volume of $\mathscr{E}(\mathscr{P}_{i,22}^{-1})$. This set should also be maximized for the performance. The problem of optimizing the set $\mathscr{E}(\mathscr{P}(k))$ can be formulated as

$$\begin{aligned} &\max_{\mathscr{P}_i} \left\{ \sum_{i=1}^{L} (\text{logdet}(\mathscr{P}_{i,11}) + \rho\text{logdet}(\mathscr{P}_{i,22})) \right\} \\ &\text{s.t. (4.19), (4.20), (4.21).} \end{aligned} \tag{4.22}$$

where $\rho > 0$ is a tuning variable.

Define Ω_F as invariant and constraint-admissible ellipsoid for (4.1), (4.4) under the low-gain controller $u(k) = F(k)x(k)$. The following result shows that we can optimize $\mathscr{P}_i, i \in \overline{1,L}$ in such a way that $\Omega_F \subseteq \mathscr{E}_x$.

Proposition 3. *The set $\mathscr{E}(\mathscr{P}(k))$ can be optimized in such a way that $\Omega_F \subseteq \Omega_x$.*

Proof. As Ω_F is invariant and constraint-admissible set for (4.1), (4.4) under the low gain controller $u(k) = F(k)x(k)$, it follows that for any $x(k) \in \Omega_F$, one has

$$x(k+1) = \sum_{i=1}^{L} p_i(k)(A_i + BF_i)x(k) \in \Omega_F$$

and $\Omega_F \subseteq \mathscr{X}$, and $u(k) = F(k)x(k) \in \mathscr{U}$.

Rewrite (4.10) as

$$\begin{aligned}\begin{bmatrix} x(k+1) \\ c(k+1) \end{bmatrix} &= \sum_{i=1}^{L} p_i(k) \begin{bmatrix} A_i + BK_i & B(F_i - K_i) \\ 0_{n\times n} & A_i + BF_i \end{bmatrix} \begin{bmatrix} x(k) \\ c(k) \end{bmatrix} \\ &= \begin{bmatrix} A(k) + BK(k) & B(F(k) - K(k)) \\ 0_{n\times n} & A(k) + BF(k) \end{bmatrix} \begin{bmatrix} x(k) \\ c(k) \end{bmatrix}\end{aligned} \tag{4.23}$$

For any $z(k)$ such that $c(k) = x(k)$, one has

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{L} p_i(k)((A_i + BK_i)x(k) + B(F_i - K_i)c(k)) \\ &= \sum_{i=1}^{L} p_i(k)(A_i + BF_i)x(k) \\ &= \sum_{i=1}^{L} p_i(k)(A_i + BF_i)c(k) = c(k+1) \end{aligned} \tag{4.24}$$

In this case, (4.23) consists of two identically decoupled systems. It follows that $\Omega_F \subseteq \Omega_x$. The proof is complete. □

4.2.2 Overall MPC Algorithm

In this section, we assume that the matrices $\mathscr{W}_{i,x}, \mathscr{W}_{i,c}$ as well as $\mathscr{P}_i$ are available.

We show in Section 4.2.1 that, under the control law (4.8),

- $\mathbf{V}(k)$ provides as an upper bound for the cost function (4.5), where

$$\mathbf{V}(k) = z(k)^T \left(\sum_{i=1}^{L} p_i(k) \mathscr{W}_i \right) z(k) = x(k)^T \mathscr{W}_x(k) x(k) + c(k)^T \mathscr{W}_c(k) c(k);$$

- $\mathscr{E}(\mathscr{P}(k))$ constitutes a feasible set for (4.8).

Therefore, for a given current state $x(k)$ at a generic time instant k, using the control law (4.8), the problem (4.5) becomes

$$\begin{aligned} &\min_{c(k)} \left\{ x(k)^T \mathscr{W}_x(k) x(k) + c(k)^T \mathscr{W}_c(k) c(k) \right\} \\ &\text{s.t.} \quad \begin{bmatrix} x(k)^T & c(k)^T \end{bmatrix} \left(\sum_{i=1}^{L} p_i(k) \mathscr{P}_i^{-1} \right) \begin{bmatrix} x(k) \\ c(k) \end{bmatrix} \leq 1, \end{aligned} \tag{4.25}$$

where

$$\mathscr{W}_x(k) = \sum_{i=1}^{L} p_i(k) \mathscr{W}_{x,i}, \ \mathscr{W}_c(k) = \sum_{i=1}^{L} p_i(k) \mathscr{W}_{c,i}$$

Denote $\mathscr{Q}_i = \mathscr{P}_i^{-1}$. We partition the matrix $\mathscr{Q}_i$ into block as $\mathscr{Q}_i = \begin{bmatrix} \mathscr{Q}_{i,11} & \mathscr{Q}_{i,12} \\ \mathscr{Q}_{i,21} & \mathscr{Q}_{i,22} \end{bmatrix}$. We rewrite the problem (4.25) as

$$\begin{aligned} &\min_{c(k)} \left\{ c(k)^T \mathscr{W}_c(k) c(k) \right\} \\ &\text{s.t. } \begin{bmatrix} x(k)^T & c(k)^T \end{bmatrix} \begin{bmatrix} \mathscr{Q}_{11}(k) & \mathscr{Q}_{12}(k) \\ \mathscr{Q}_{21}(k) & \mathscr{Q}_{22}(k) \end{bmatrix} \begin{bmatrix} x(k) \\ c(k) \end{bmatrix} \leq 1 \end{aligned} \tag{4.26}$$

where $\mathscr{Q}_{l_1 l_2}(k) = \sum_{i=1}^{L} p_i(k) \mathscr{Q}_{i,l_1 l_2}, l_1 = \overline{1,2}, l_2 = \overline{1,2}$

Note that we omit the term $x(k)^T \mathscr{W}_x(k) x(k)$ in the cost function (4.26), as this term does not affect the optimal solution.

Define $\mathscr{Q}_c(k) = \mathscr{Q}_{22}(k)^{-1} \mathscr{Q}_{21}(k)$ and

$$\mathscr{Q}_x(k) = \mathscr{Q}_{11}(k) - \mathscr{Q}_{12}(k) \mathscr{Q}_{22}(k)^{-1} \mathscr{Q}_{21}(k).$$

We rewrite the problem (4.26) as

$$\begin{aligned} &\min_{c(k)} \left\{ c(k)^T \mathscr{W}_c(k) c(k) \right\} \\ &\text{s.t. } \left(c(k) + \mathscr{Q}_c(k) x(k) \right)^T \mathscr{Q}_{22}(k) \left(c(k) + \mathscr{Q}_c(k) x(k) \right) \leq 1 - x(k)^T \mathscr{Q}_x(k) x(k). \end{aligned} \tag{4.27}$$

The constraint of (4.27) is a parameter-dependent ellipsoidal set, that depends on the current state $x(k)$. As $\mathscr{W}_c(k) \succ 0$, problem (4.27) is a convex quadratically constrained quadratic program (QCQP), and therefore admits a unique solution [78]. In addition, as noted in [79], [80], problem (4.27) has a very nice geometrical interpretation. Specially, if the ellipsoidal set contains the origin, then the trivial solution $c(k) = 0$ is optimal. Otherwise, provided the set is non-empty, the solution to (4.27) corresponds to the point on the ellipsoid that provides the shortest distance to the origin, measured in the $\mathscr{W}_c(k)$ norm. The solution can be obtained using the ADMM algorithm [78], or alternatively via the Newton–Raphson procedure, which guarantees a quadratic convergence.

4.2.2.1 Newton-Raphson Procedure

As mentioned above, problem (4.27) has a very elegant geometric interpretation: if $c(k) = 0$ makes problem (4.27) feasible, then its optimal solution is $c^* = 0$; otherwise, the solution to problem (4.27) corresponds to the point on the ellipsoid that is closest to the origin. In other words, the vector c will lie on the boundary of $\mathscr{E}(\mathscr{P}(k))$ and be perpendicular to the hyperplane tangent to the boundary $\mathscr{E}(\mathscr{P}(k))$, that is,

$$\begin{aligned} 2\mathscr{W}_c c(k) &= -2\mu(k) \mathscr{Q}_{22} (c(k) + \mathscr{Q}_c x(k)), \\ \left(c(k) + \mathscr{Q}_c(k) x(k) \right)^T \mathscr{Q}_{22}(k) \left(c(k) + \mathscr{Q}_c(k) x(k) \right) &= 1 - x(k)^T \mathscr{Q}_x(k) x(k) \end{aligned} \tag{4.28}$$

one has

$$c(k) = -\mu(k)\left(\mathscr{W}_c + \mu\mathscr{Q}_{22}\right)^{-1}\mathscr{Q}_c x(k), \tag{4.29}$$

For the sake of simplicity, we omit the index k, define $\mathscr{W}_\mu = \left(\mathscr{W}_c + \mu\mathscr{Q}_{22}\right)^{-1}$, combining (4.28), one has

$$2\mu x^T\mathscr{Q}_{12}\mathscr{W}_\mu\mathscr{Q}_{12}^T x + \mu^2 x^T\mathscr{Q}_{12}\mathscr{W}_\mu\mathscr{Q}_{22}\mathscr{W}_\mu\mathscr{Q}_{12}^T x = 1 - x^T\mathscr{Q}_{11}x \tag{4.30}$$

We can use the Newton–Raphson procedure to find the numerical solution of equation (4.30) via an iterative method.

Define

$$f_\mu = x^T\mathscr{Q}_{11}x + 2\mu x^T\mathscr{Q}_{12}\mathscr{W}_\mu\mathscr{Q}_{12}^T x + \mu^2 x^T\mathscr{Q}_{12}\mathscr{W}_\mu\mathscr{Q}_{22}\mathscr{W}_\mu\mathscr{Q}_{12}^T x - 1 \tag{4.31}$$

we have

$$\frac{df}{d\mu} = 2x^T\mathscr{Q}_{12}\frac{dc}{d\mu} + 2(\mu\mathscr{W}_\mu\mathscr{Q}_{12}^T x)^T\mathscr{Q}_{22}\frac{dc}{d\mu} \tag{4.32}$$

where

$$\frac{dc}{d\mu} = \mathscr{W}_\mu\mathscr{Q}_{12}^T x + \mu\mathscr{W}_\mu\mathscr{Q}_{22}\mathscr{W}_\mu\mathscr{Q}_{12}^T x \tag{4.33}$$

A superscript is used to denote the iteration index one has

$$\mu^{(t+1)} = \mu^{(t)} - \left(\frac{df_\mu^{(t)}}{d\mu^{(t)}}\right)^{-1} f_\mu^{(t)} \tag{4.34}$$

Once μ^* is found, the optimal value c^* can be expressed in terms of (4.29) as

$$c^* = \mu^* W_\mu^*\mathscr{Q}_c^T x \tag{4.35}$$

where $W_\mu^* = \left(\mathscr{W}_c + \mu^*\mathscr{Q}_{22}\right)^{-1}$.

Denote $c^*(k)$ as the optimal solution of (4.27). The control action at time k is computed as

$$u(k) = K(k)x(k) + (F(k) - K(k))c^*(k). \tag{4.36}$$

In summary, the proposed method consists of two stages: offline and online.

Algorithm 4.1 Improved MPC algorithm based on merging low and high gain controllers – Offline Stage

1: Obtain the low-gain control matrices F_i, $\forall i \in \overline{1, L}$ by solving (2.112).
2: Obtain the high-gain control matrices K_i, $\forall i \in \overline{1, L}$ by solving (2.121).
3: Obtain $\mathscr{W}_{i,c}$, $\forall i \in \overline{1, L}$ by solving (4.17).
4: Obtain $\mathscr{P}_i$, $\forall i \in \overline{1, L}$ by solving (4.22).

Algorithm 4.2 Improved MPC algorithm based on merging low and high gain controllers – Online Stage - At time k

1: Measure or estimate $x(k)$ and $p(k)$.
2: Using $p(k)$ to update $F(k), K(k), \mathscr{W}_c(k), \mathscr{P}(k)$.
3: Obtain $c^*(k)$ by solving (4.27).
4: Apply the control law (4.36) to the system (4.1).

Theorem 4.3. *Assuming feasibility at the initial state, the control law* (4.36) *guarantees satisfaction of the constraints* (4.4) *at all times, and ensures asymptotic stability of the closed-loop system* (4.1)*.*

Proof. We divide the proof into two parts:

Constraint satisfaction: We need to show that $u(k) \in \mathscr{U}$, and $x(k+1) \in \mathscr{X}$.

As (4.27) is feasible at time k, it follows that for a given $x(k)$, there exists $c(k)$ such that $z(k) = [x(k)^T \; c(k)^T]^T \in \mathscr{E}(\mathscr{Q}(k))$, where $\mathscr{E}(\mathscr{Q}(k)) = \sum_{i=1}^{L} p_i(k)\mathscr{E}(\mathscr{Q}_i)$.

Recall that $\mathscr{E}(\mathscr{P}(k))$ is invariant and constraint-admissible for (4.1), (4.4) under the control law (4.8). It follows that

$$u(k) = K(k)x(k) + (F(k) - K(k))c^*(k) \in \mathscr{U},$$

and $[x(k+1)^T c(k+1)^T]^T \in \mathscr{E}(\mathscr{P}(k+1))$. As $\mathscr{E}(\mathscr{P}(k+1))$ is constraint-admissible with respect to $\mathscr{X}$, we conclude that $x(k+1) \in \mathscr{X}$.

Asymptotic stability: With a slight abuse of notation, consider the following candidate Lyapunov function

$$V^*(k) = x(k)^T \mathscr{W}_x(k)x(k) + c^*(k)^T \mathscr{W}_c(k)c^*(k).$$

Note that if $x(k) = 0$, then (4.27) has the trivial solution $c^*(k) = 0$. In this case $V^*(k) = 0$.

Using (4.12), one gets

$$V(k+1) \geq V^*(k) - x(k)^T Q x(k) - u(k)^T R u(k)$$

where $V(k+1) = x(k+1)^T \mathscr{W}_x(k+1)x(k+1) + c(k+1)^T \mathscr{W}_c(k+1)c(k+1)$.

At time $k+1$, one has

$$V(k+1) \geq x(k+1)^T \mathscr{W}_x(k+1)x(k+1) + c^*(k+1)^T \mathscr{W}_c(k+1)c^*(k+1).$$

It follows that

$$V^*(k+1) \geq V^*(k) - x(k)^T Q x(k) - u(k)^T R u(k).$$

Therefore, $V^*(k)$ is a Lyapunov function for the closed-loop system (4.1) under the control law (4.36). The proof is complete. □

From now on, we will omit the time index k whenever it is clear from context. We have the following result.

Theorem 4.4. *The control law* (4.36) *can be represented as a continuous piecewise linear function of the state.*

Proof. The problem (4.27) is a QCQP multiparametric programming. It is well known [81] that the solution c^* is a continuous function of the state $x(k)$. Hence $u(k)$ is a continuous function of $x(k)$.

It remains to show that the control law (4.36) can be represented as a piecewise linear function of $x(k)$. To this end, we form the Lagrange for the problem (4.27) as

$$\mathscr{L}(c,\lambda) = c^T \mathscr{W}_c c + \lambda \left((c + \mathscr{Q}_c x)^T \mathscr{Q}_{22} (c + \mathscr{Q}_c x) - 1 + x^T \mathscr{Q}_x x \right).$$

For the optimal solution c^*, λ^*, one has

$$\frac{\partial}{\partial c}\mathscr{L}(c,\lambda) = 2\mathscr{W}_c c^* + 2\lambda^* \mathscr{Q}_{22} (c + \mathscr{Q}_c x) = 0,$$

or equivalently,

$$c^* = -\lambda^* (\mathscr{W}_c + \lambda^* \mathscr{Q}_{22})^{-1} \mathscr{Q}_c x,$$

thus, using (4.36), one obtains

$$u(k) = \left(K(k) - \lambda^*(k)(F(k) - K(k)) (\mathscr{W}_c(k) + \lambda^*(k)\mathscr{Q}_{22}(k))^{-1} \mathscr{Q}_c(k) \right) x(k).$$

We conclude that the control input is a piecewise linear continuous function of the state. □

Consider the ellipsoid $\mathscr{E}(\mathscr{Q}(k))$. It is clear that the cut of $\mathscr{E}(\mathscr{Q}(k))$ through $c = 0$ is the ellipsoid $\mathscr{E}(\mathscr{Q}_{11}(k))$, i.e., for any $x(k)$ such that $[x(k)^T \ 0^T]^T \in \mathscr{E}(\mathscr{Q}(k))$ if and only if $x(k) \in \mathscr{E}(\mathscr{Q}_{11}(k))$.

The following result concerns the local optimality of the control law (4.36).

Theorem 4.5. *Consider the ellipsoid $\mathscr{E}(\mathscr{Q}_{11}(k))$. The following properties hold:*

a) For any $x(k) \in \mathscr{E}(\mathscr{Q}_{11}(k))$, the control law (4.36) *reduces to $u(k) = K(k)x(k)$, that is, the performance-oriented high-gain control law.*

b) The state trajectory $x(k)$ reaches $\mathscr{E}(\mathscr{Q}_{11}(k))$ in finite time. Moreover, once $x(k)$ enters $\mathscr{E}(\mathscr{Q}_{11}(k))$, it remains in $\mathscr{E}(\mathscr{Q}_{11}(k))$ for all future times. In other words, $\mathscr{E}(\mathscr{Q}_{11}(k))$ is invariant and constraint-admissible with respect to (4.1), (4.4).

Proof. **Proof of 4.5a)** As $\mathscr{E}(\mathscr{Q}_{11}(k))$ is the cut of $\mathscr{E}(\mathscr{Q}(k))$ through $c = 0$, it is clear that $c = 0$ is a feasible solution for (4.27). It is indeed the optimal solution for (4.27) because the cost function is zero. In this case, the control law (4.36) reduces to $u(k) = K(k)x(k)$.

Proof of 4.5b) Recall from Theorem 4.5 that the closed-loop system (4.1) under the control law (4.36) is asymptotically stable. Hence, $x(k)$ enters any neighborhood of the origin in finite time. Since $\mathscr{E}(\mathscr{Q}_{11}(k))$ containing the origin, it follows that $x(k)$ reaches $\mathscr{E}(\mathscr{Q}_{11}(k))$ in finite time.

Furthermore, for any $x(k) \in \mathscr{E}(\mathscr{Q}_{11}(k))$, one has $c^* = 0$ is the optimal solution to (4.27). Using (4.10), one gets

$$\begin{bmatrix} x(k+1) \\ c(k+1) \end{bmatrix} = \begin{bmatrix} A(k)+BK(k) & B(F(k)-K(k)) \\ 0 & A(k)+BF(k) \end{bmatrix} \begin{bmatrix} x(k) \\ c^*(k) \end{bmatrix}.$$

Hence $c(k+1) = 0$ is a feasible solution, and then the optimal solution to (4.27) at time $k+1$. Therefore, $x(k+1) \in \mathscr{E}(\mathscr{Q}_{11}(k+1))$. We conclude that $\mathscr{E}(\mathscr{Q}_{11}(k+1))$ is invariant and constraint-admissible. □

4.3 Merging Saturated Low- and High-Gain Controllers

As shown in Section 4.2, the merged low- and high-gain control law yields a smooth feedback law that simultaneously enlarges the DoA and guarantees local optimality. However, as proved in Theorem 4.4, the resulting control law remains linear, and thus the control action typically operates below the input bounds, i.e., it is often unsaturated.

To overcome this limitation and fully exploit the available input range, we propose a method that blends two nonlinear saturated control laws. Specifically, instead of the linear feedback in (4.8), we consider the following saturated control law:

$$u(k) = \text{sat}\left(K(k)x(k) + (F(k) - K(k))c(k)\right) \tag{4.37}$$

where $K(k)$ and $F(k)$ are respectively, the performance-oriented high-gain, and the DoA-oriented low-gain control matrices. The saturation function $\text{sat}(u) : \mathbb{R}^m \to \mathbb{R}^m$ is defined component-wise as (2.53).

4.3.1 Auxiliary Gain Matrix Computation

The aim of this section is to provide a way to calculate an auxiliary gain matrix for the control law (4.37). To this end, we consider the closed-loop discrete-time LPV system (4.1) under the saturated low-gain control law $u(k) = \text{sat}(F(k)x(k))$

$$x(k+1) = A(k)x(k) + B\text{sat}(F(k)x(k)) \tag{4.38}$$

Using proposition 1, one obtains

$$x(k+1) \in A(k)x(k) + B\text{cov}\{D_qF(k)x(k) + D_q^-\tilde{F}(k)x(k)\}, q \in \overline{1, 2^m} \tag{4.39}$$

for any $x(k)$ such that $|\tilde{F}(k)x(k)| \leq \mathbf{1}$.

Because the nonlinear saturation function $\text{sat}(F(k)x)$ is not linear in x, it prevents the direct application of a quadratic Lyapunov framework. The introduction of the auxiliary gain matrix $\tilde{F}(k)$ allows us to express the saturation function as an element of the convex hull of the vectors $D_qF(k)x + D_q^-\tilde{F}(k)x$, $q \in \overline{1, 2^m}$. Note that each term $D_qF(k)x + D_q^-\tilde{F}(k)x$ is linear in x. This

convex reformulation enables the use of LMIs for stability and performance analysis. We use the auxiliary gain matrix $\tilde{F}(k)$ to maximize the DoA of the system (4.38).

Using (2.101), one gets

$$x(k+1) \in \text{cov}\left\{\overline{\mathbf{A}}_q(k)x(k)\right\}, q \in \overline{1,2^m} \tag{4.40}$$

where

$$\overline{\mathbf{A}}_q(k) = A(k) + BD_qF(k) + BD_q^-\tilde{F}(k)$$

Define, $i \in \overline{1,L}, q \in \overline{1,2^m}$

$$\overline{\mathbf{A}}_{i,q} = A_i + BD_qF_i + BD_q^-\tilde{F}_i$$

It follows that $\overline{\mathbf{A}}_q(k) = \sum_{i=1}^{L} p_i(k)\overline{\mathbf{A}}_{i,q}$.

Consider the following ellipsoid

$$\mathscr{E}(\overline{\mathbf{S}}(k)) = \{x \in \mathbb{R}^n : x^T\overline{\mathbf{S}}(k)x \leq 1\}$$

where $\overline{\mathbf{S}}(k) = \sum_{i=1}^{L} p_i(k)\overline{\mathbf{S}}_i^{-1}$, and $\overline{\mathbf{S}}_i \in \mathbb{S}^n$.

Theorem 4.6. *If there exist matrices $\overline{\boldsymbol{S}}_i \in \mathbb{S}^n$, $\overline{\boldsymbol{Y}}_i \in \mathbb{R}^{m\times n}$, $\forall i \in \overline{1,L}$ such that the following LMI conditions hold*

$$\begin{bmatrix} \overline{\boldsymbol{S}}_i & * \\ (A_i + BD_qF_i)\overline{\boldsymbol{S}}_i + D_q^-\overline{\boldsymbol{Y}}_i & \overline{\boldsymbol{S}}_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m} \tag{4.41}$$

$$\begin{bmatrix} 1 & * \\ \overline{\boldsymbol{Y}}_{i,l}^T & \overline{\boldsymbol{S}}_i \end{bmatrix} \succeq 0, \ \forall i \in \overline{1,L}, \forall l \in \overline{1,m} \tag{4.42}$$

$$\begin{bmatrix} 1 & * \\ \overline{\boldsymbol{S}}_i f_l & \overline{\boldsymbol{S}}_i \end{bmatrix} \succeq 0, \ \forall i \in \overline{1,L}, \forall l \in \overline{1,n_x} \tag{4.43}$$

where $\overline{\boldsymbol{Y}}_i = \tilde{F}_i\overline{\boldsymbol{S}}_i, \forall i \in \overline{1,L}$, and $\overline{\boldsymbol{Y}}_{i,l}$ is the $l-$row of the matrix $\overline{\boldsymbol{Y}}_i$, then the ellipsoid $\mathscr{E}(\overline{\boldsymbol{S}}(k))$ is parameter-dependent invariant for (4.38) *and constraint-admissible with respect to* (4.4).

Proof. We omit the proof of constraint-admissibility as it follows the same lines as the proof of theorem 2.9. For the parameter-dependent invariance proof, we need to show that $x(k+1) \in \mathscr{E}(\overline{\mathbf{S}}(k+1))$, for any $x(k) \in \mathscr{E}(\overline{\mathbf{S}}(k))$. Pre- and post-multiplying (4.41) by

$$\begin{bmatrix} \overline{\mathbf{S}}_i^{-1} & * \\ 0 & I \end{bmatrix}$$

one gets

$$\begin{bmatrix} \overline{\mathbf{S}}_i^{-1} & * \\ A_i + BD_qF_i + D_q^-\tilde{F}_i & \overline{\mathbf{S}}_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}$$

thus, for each i, multiply the corresponding by $p_i(k)$, $i \in \overline{1,L}$ and sum to get

$$\begin{bmatrix} \overline{\mathbf{S}}(k) & * \\ A(k) + B(D_q F(k) + D_q^- \tilde{F}(k)) & \overline{\mathbf{S}}_j \end{bmatrix} \succeq 0, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}$$

Using the Schur complements, one obtains

$$\overline{\mathbf{S}}(k) - \overline{\mathbf{A}}_q(k)^T \overline{\mathbf{S}}_j^{-1} \overline{\mathbf{A}}_q(k) \succeq 0, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}$$

For each j, multiply the corresponding by $p_j(k+1)$ and sum to get

$$\overline{\mathbf{S}}(k) - \overline{\mathbf{A}}_q(k)^T \overline{\mathbf{S}}(k+1) \overline{\mathbf{A}}_q(k) \succeq 0, \forall q \in \overline{1,2^m}$$

It follows that

$$x(k)^T \overline{\mathbf{S}}(k) x(k) - x(k)^T \overline{\mathbf{A}}_q(k)^T \overline{\mathbf{S}}(k+1) \overline{\mathbf{A}}_q(k) x(k) \geq 0, \forall q \in \overline{1,2^m}$$

Using (4.40), one gets

$$x(k)^T \overline{\mathbf{S}}(k) x(k) - x(k)^T \overline{\mathbf{S}}(k+1) x(k) \geq 0$$

Therefore, for any $x(k)^T \overline{\mathbf{S}}(k) x(k) \leq 1$ one has $x(k+1)^T \overline{\mathbf{S}}(k+1) x(k+1) \leq 1$. Hence, $\mathscr{E}(\overline{\mathbf{S}}(k))$ is parameter-dependent invariant. □

Once the invariance and constraint-admissibility conditions for $\mathscr{E}(\overline{\mathbf{S}}(k))$ is obtained, our next step is to optimize the size of $\mathscr{E}(\overline{\mathbf{S}}(k))$, where $\tilde{F}_i, i \in \overline{1,L}$ serve as decision variables. For instance, one can maximize the volume of the associated ellipsoids $\mathscr{E}(\overline{\mathbf{S}}_i)$, or alternatively optimize the size of $\mathscr{E}(\overline{\mathbf{S}}(k))$ along specific directions of interest. The details of this optimization procedure are omitted here, as they closely follow the developments in Section 2.6.1.

Concerning the auxiliary gain matrix $\tilde{K}(k) = \sum_{i=1}^{L} p_i(k) \tilde{K}_i$ of the saturated high-gain control law $u(k) = \text{sat}(K(k)x(k))$, we can establish a result analogous to theorem 4.6, stated as follows.

Proposition 4. *Consider the ellipsoid* $\mathscr{E}(\underline{\mathbf{S}}(k)) = \{x \in \mathbb{R}^n : x^T \underline{\mathbf{S}}(k) x \leq 1\}$ *with* $\underline{\mathbf{S}}(k) = \sum_{i=1}^{L} p_i(k) \underline{\mathbf{S}}_i^{-1}$. *If there exist matrices* $\underline{\mathbf{S}}(k)_i \in \mathbb{S}^n$, $\underline{\mathbf{Y}}_i \in \mathbb{R}^{m \times n}$, $\forall i \in \overline{1,L}$ *such that the following LMI conditions hold*

$$\begin{bmatrix} \underline{\mathbf{S}}_i & * \\ (A_i + B D_q K_i) \overline{\mathbf{S}}_i + D_q^- \underline{\mathbf{Y}}_i & \underline{\mathbf{S}}_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}$$

$$\begin{bmatrix} 1 & * \\ \underline{\mathbf{Y}}_{i,l}^T & \underline{\mathbf{S}}_i \end{bmatrix} \succeq 0, \ \forall i \in \overline{1,L}, \forall l \in \overline{1,m}$$

$$\begin{bmatrix} 1 & * \\ \underline{\mathbf{S}}_i f_l & \underline{\mathbf{S}}_i \end{bmatrix} \succeq 0, \ \forall i \in \overline{1,L}, \forall l \in \overline{1,n_x}$$

where $\underline{\mathbf{Y}}_i = \tilde{K}_i \underline{\mathbf{S}}_i, \forall i \in \overline{1,L}$, *and* $\underline{\mathbf{Y}}_{i,l}$ *is the* $l-$*row of the matrix* $\underline{\mathbf{Y}}_i$, *then the ellipsoid* $\mathscr{E}(\underline{\mathbf{S}}(k))$ *is parameter-dependent invariant for the system*

$$x(k+1) = A(k)x(k) + B\,\text{sat}(K(k)x(k))$$

and constraint-admissible with respect to (4.4).

4.3.2 Upper Bound of The Cost and DoA

In this section, we assume that $F(k)$, $K(k)$ as well as $\tilde{F}(k)$, and $\tilde{K}(k)$ are available. Our aim here is to employ these matrices to compute an upper bound for the cost function (4.5) and to characterize DoA associated with the control law (4.37). For this purpose, define

$$L(k) = [K(k)\ F(k) - K(k)],\ \tilde{L}(k) = [\tilde{K}(k)\ \tilde{F}(k) - \tilde{K}(k)]$$

In this case $L(k) = \sum_{i=1}^{L} p_i(k) L_i$, and $\tilde{L}(k) = \sum_{i=1}^{L} p_i(k) \tilde{L}_i$, where

$$L_i = [K_i\ F_i - K_i],\ \tilde{L}_i = [\tilde{K}_i\ \tilde{F}_i - \tilde{K}_i], \forall i \in \overline{1,L}.$$

We rewrite the control law (4.37) as

$$u(k) = \text{sat}(L(k)z(k))$$

thus, using (4.1)

$$x(k+1) = A(k)x(k) + B\,\text{sat}(L(k)z(k))$$

Using Proposition 1, one obtains

$$x(k+1) \in A(k)x(k) + B\text{cov}\left\{D_q L(k)z(k) + D_q^- \tilde{L}(k)z(k)\right\}, q \in \overline{1,2^m} \tag{4.44}$$

for any $z(k)$ such that $|\tilde{L}(k)z(k)| \leq \mathbf{1}$.

(4.44) holds if and only if there exit $\gamma_q(k) \geq 0, \forall q \in \overline{1,2^m}$ such that $\sum_{q=1}^{2^m} \gamma_q(k) = 1$, and

$$x(k+1) = A(k)x(k) + B\sum_{q=1}^{2^m} \gamma_q(k) \left\{D_q L(k)z(k) + D_q^- \tilde{L}(k)z(k)\right\}$$

or equivalently

$$\begin{aligned} x(k+1) \quad &= \sum_{q=1}^{2^m} \gamma_q(k) \left(A(k) + BD_q K(k) + D_q^- \tilde{K}(k)\right) x(k) \\ &+B \sum_{q=1}^{2^m} \gamma_q(k) \left(D_q(F(k) - K(k)) + D_q^-(\tilde{F}(k) - \tilde{K}(k))\right) c(k) \end{aligned} \tag{4.45}$$

Now we assume that $c(k)$ follows the dynamics

$$c(k+1) = \sum_{q=1}^{2^m} \gamma_q(k) \left(A(k) + BD_qF(k) + D_q^- \tilde{F}(k) \right) c(k) \tag{4.46}$$

Using (4.46), one obtains

$$z(k+1) = \tilde{\mathscr{A}}(k) z(k) \tag{4.47}$$

where $\tilde{\mathscr{A}}(k) = \sum_{i=1}^{L} \sum_{q=1}^{2^m} p(k)\gamma_q(k)\tilde{\mathscr{A}}_{i,q}$, and

$$\tilde{\mathscr{A}}_{i,q} = \begin{bmatrix} A_i + B(D_qK_i + D_q^- \tilde{K}_i) & B(D_q(F_i - K_i) + D_q^-(\tilde{F}_i - \tilde{K}_i)) \\ 0 & A_i + B(D_qF_i + D_q^- \tilde{F}_i) \end{bmatrix}, i \in \overline{1, L}, q \in \overline{1, 2^m}$$

Note that (4.47) evolves according to the dynamics (4.47) only for $z(k)$ such that $|\tilde{L}(k)z(k)| \leq \mathbf{1}$. Now we provide a way to compute an upper bound of the cost (4.5) under the control law (4.37). Consider the following quadratic function

$$V_a(k) = z(k)^T \Gamma(k) z(k) \tag{4.48}$$

where $\Gamma(k) = \sum_{i=1}^{L} p_i(k)\Gamma_i$, and $\Gamma_i \in \mathbb{S}^{2n}, i \in \overline{1, L}$ are chosen to satisfy, $\forall t = 0, 1, \ldots$

$$\begin{aligned} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \leq \\ V_a(k+t, z(k+t)) - V_a(k+t+1, z(k+t+1)) \end{aligned} \tag{4.49}$$

We will provide LMI conditions for Γ_i to satisfy (4.49) latter. For the moment, let us assume that there exist Γ_i such that (4.49) holds.

Since the saturated control laws $u(k) = \text{sat}(K(k)x(k))$ and $u(k) = \text{sat}(F(k)x(k))$ are asymptotically stabilizing for states near the origin, one has $\lim_{t\to\infty} z(k+t) = 0$. Summing (4.49) from $t = 0$ to $t = \infty$, one obtains

$$\sum_{t=0}^{\infty} (x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t)) \leq V_a(k)$$

Hence $V_a(k, z(k))$ provides an upper bound of the cost (4.5).

Concerning the LMI conditions for Γ_i to satisfy (4.49), we have the following result.

Proposition 5. *There exist matrices $\Gamma_i \in \mathbb{S}^{2n}$, $\forall i \in \overline{1, L}$ satisfying (4.49) if and only if following LMI conditions hold*

$$\begin{bmatrix} \Gamma_i - \mathbf{Q} & * & * \\ \Gamma_j \mathscr{A}_{i,q} & \Gamma_j & * \\ \sqrt{R}\mathbf{X}_{i,q} & 0 & I \end{bmatrix} \succeq 0, i \in \overline{1, L}, j \in \overline{1, L}, q \in \overline{1, 2^m} \tag{4.50}$$

where

$$\mathbf{X}_{i,q} = \begin{bmatrix} D_qK_i + D_q^- \tilde{K}_i & D_q(F_i - K_i) + D_q^-(\tilde{F}_i - \tilde{K}_i) \end{bmatrix}.$$

Proof. The proof is omitted here, as it follows the same lines of the proof of Theorem 4.1. □

Since $\mathscr{A}_{i,q}$ are upper triangular matrices, there exist Γ_i satisfying (4.50), if and only if there exist Γ_i in the following diagonal form

$$\Gamma_i = \begin{bmatrix} \Gamma_{i,x} & 0 \\ 0 & \Gamma_{i,c} \end{bmatrix}, \forall i \in \overline{1,L}$$

where $\Gamma_{i,x} \in \mathbb{S}^n, \Gamma_{i,c} \in \mathbb{S}^n, \forall i \in \overline{1,L}$.

The matrices Γ_i are used to provide an upper bound of the cost (4.5). The optimal Γ_i can be found by solving the following SDP problem

$$\begin{aligned} &\min_{\Gamma_{i,x},\Gamma_{i,c}} \left\{ \sum_{i=1}^{L} (\text{trace}(\Gamma_{i,x}) + \text{trace}(\Gamma_{i,c})) \right\}, \\ &\text{s.t. (4.50)}. \end{aligned} \tag{4.51}$$

Our next step is to provide a method to estimate the DoA of (4.47). Consider the following parameter-dependent ellipsoid

$$\mathscr{E}(\mathscr{X}(k)) = \{z(k) \in \mathbb{R}^{2n} : z(k)^T \mathscr{N}(k) z(k) \leq 1\} \tag{4.52}$$

where $\mathscr{N}(k) = \sum_{i=1}^{L} p_i(k) \mathscr{N}_i^{-1}$, and $\mathscr{N}_i \in \mathbb{S}^{2n}$.

Theorem 4.7. *If there exist matrices $\mathscr{N}_i \in \mathbb{S}^{2n}, \forall i \in \overline{1,L}$ such that the following LMI conditions hold:*

$$\begin{bmatrix} \mathscr{N}_i & * \\ \mathscr{A}_{i,q}\mathscr{N}_i & \mathscr{N}_j \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall j \in \overline{1,L}, \forall q \in \overline{1,2^m}; \tag{4.53}$$

$$\begin{bmatrix} 1 & * \\ \mathscr{N}_i \tilde{L}_{i,l}^T & \mathscr{N}_i \end{bmatrix} \succeq 0, \ i \in \overline{1,L}, l \in \overline{1,m} \tag{4.54}$$

$$\begin{bmatrix} 1 & * \\ \mathscr{N}_i \mathbf{f}_l & \mathscr{N}_i \end{bmatrix} \succeq 0, \forall i \in \overline{1,L}, \forall l \in \overline{1,n_x} \tag{4.55}$$

where $\tilde{L}_{i,l}$ is the $l-$th row of the matrix $\tilde{L}_i$, then the ellipsoid $\mathscr{E}(\mathscr{N}(k))$ is parameter-dependent invariant for (4.47), and constraint-admissible with respect to (4.4).

Proof. The proof is omitted for brevity, as it reproduces the arguments used in the proof of Theorem 4.2. □

We divide the matrices $\mathscr{N}_i$ into blocks as $\mathscr{N}_i = \begin{bmatrix} \mathscr{N}_{i,11} & \mathscr{N}_{i,12} \\ \mathscr{N}_{i,12}^T & \mathscr{N}_{i,22} \end{bmatrix}, \forall i \in \overline{1,L}$.

Following the same idea of the linear feedback case, the problem of optimizing the parameter-dependent invariant and constraint-admissible ellipsoid $\mathscr{E}(\mathscr{N}(k))$ can be formulated as

$$\begin{aligned} &\max_{\mathscr{N}_i} \left\{ \sum_{i=1}^{L} \left(\text{logdet}(\mathscr{N}_{i,11}) + \tilde{\rho}\,\text{logdet}(\mathscr{N}_{i,22})\right) \right\} \\ &\text{s.t. } (4.53), (4.54), (4.55). \end{aligned} \tag{4.56}$$

where $\tilde{\rho} > 0$ is a tuning variable.

Define $\tilde{\Omega}_x$ as the projection of $\mathscr{E}(\mathscr{N}(k))$ onto the $x-$space. Define also $\tilde{\Omega}_F$ as parameter-dependent invariant and constraint-admissible ellipsoid for (4.1), (4.4) under the saturated low-gain control law $u(k) = \text{sat}(F(k)x(k))$. The following result shows that it is possible to optimize $\mathscr{N}_i$ in such a way that $\tilde{\Omega}_F \subseteq \tilde{\Omega}_x$.

Theorem 4.8. *We can optimize $\mathscr{E}(\mathscr{N}(k))$ in such a way that $\tilde{\Omega}_F \subseteq \tilde{\Omega}_x$.*

4.3.3 Overall Algorithm

Once Γ_i, $\mathscr{N}_i$ are available from the offline stage, we can obtain the optimal $c^*(k)$ at a generic time k by solving the following optimization problem

$$\begin{aligned} &\min_{c(k)} \left\{ x(k)^T \Gamma_x(k) x(k) + c(k)^T \Gamma_c(k) c(k) \right\} \\ &\text{s.t. } \begin{bmatrix} x(k)^T & c(k)^T \end{bmatrix} \left(\sum_{i=1}^{L} p_i(k) \mathscr{N}_i^{-1} \right) \begin{bmatrix} x(k) \\ c(k) \end{bmatrix} \leq 1 \end{aligned} \tag{4.57}$$

where $\Gamma_x(k) = \sum_{i=1}^{L} p_i(k)\Gamma_{i,x}$, $\Gamma_c(k) = \sum_{i=1}^{L} p_i(k)\Gamma_{i,c}$.

Define $\mathscr{Y}_i = \mathscr{N}_i^{-1}$. We divide $\mathscr{Y}_i$ into blocks as $\mathscr{Y}_i = \begin{bmatrix} \mathscr{Y}_{i,11} & \mathscr{Y}_{i,12} \\ \mathscr{Y}_{i,12}^T & \mathscr{Y}_{i,22} \end{bmatrix}$. We can rewrite the optimization problem (4.57) as

$$\begin{aligned} &\min_{c(k)} \left\{ c(k)^T \Gamma_c(k) c(k) \right\} \\ &\text{s.t. } \begin{bmatrix} x(k)^T & c(k)^T \end{bmatrix} \begin{bmatrix} \mathscr{Y}_{11}(k) & \mathscr{Y}_{12}(k) \\ \mathscr{Y}_{12}(k)^T & \mathscr{Y}_{22}(k) \end{bmatrix} \begin{bmatrix} x(k) \\ c(k) \end{bmatrix} \leq 1 \end{aligned} \tag{4.58}$$

where $\mathscr{Y}_{11}(k) = \sum_{i=1}^{L} p_i(k)\mathscr{Y}_{i,11}$, $\mathscr{Y}_{12}(k) = \sum_{i=1}^{L} p_i(k)\mathscr{Y}_{i,12}$, and $\mathscr{Y}_{22}(k) = \sum_{i=1}^{L} p_i(k)\mathscr{Y}_{i,22}$.

Note that it is a convex QCQP problem.

Once $c(k)^*$ is available, the control action at time k is given as

$$u(k) = \text{sat}(K(k)x(k) + (F(k) - K(k))c^*(k)) \tag{4.59}$$

In summary, the overall algorithm with the actuator saturation consists of two stages: offline and online.

Algorithm 4.3 Improved MPC algorithm based on merging saturated low and high gain controllers– Offline Stage

1: Obtain the auxiliary matrices $\tilde{K}_i$, $\tilde{F}_i$, $\forall i \in \overline{1, L}$.
2: Obtain $\Gamma_{i,c}$, $\forall i \in \overline{1, L}$ by solving (4.51).
3: Obtain $\mathscr{N}_i$, $\forall i \in \overline{1, L}$ by solving (4.56).

Algorithm 4.4 Improved MPC algorithm based on merging saturated low and high gain controllers– Online Stage - At time k

1: Measure or estimate $x(k)$ and $\gamma(k)$.
2: Using $p(k)$ to $K(k), F(k), \Gamma_c(k), \mathscr{N}(k)$.
3: Obtain $c^*(k)$ by solving (4.58).
4: Apply the control law (4.59) to the system (4.1).

Proposition 6. *Assuming feasibility at the initial state, the control law (4.59) guarantees satisfaction of the constraints (4.4) at all times, and ensures asymptotic stability of the closed-loop system (4.1).*

Proposition 7. *The control law (4.59) can be represented as a continuous saturated function of the state.*

Proposition 8. *The following properties hold:*

a) For any $x(k) \in \mathscr{E}(\mathscr{Y}_{11}(k))$, the control law (4.59) reduces to $u(k) = \text{sat}(K(k)x(k))$, i.e., the performance-oriented saturated high-gain control law.

b) The state trajectory $x(k)$ reaches $\mathscr{E}(\mathscr{Y}_{11}(k))$ in finite time. In addition, once $x(k)$ enters $\mathscr{E}(\mathscr{Y}_{11}(k))$, it remains in $\mathscr{E}(\mathscr{Y}_{11}(k))$ for all future times. In other words, $\mathscr{E}(\mathscr{Y}_{11}(k))$ is parameter-dependent invariant and constraint-admissible with respect to (4.1), (4.4).

4.4 Numerical Examples

This section illustrates the potential benefit of the new approaches by simulations of two examples system. We used the CVX toolbox to solve SDP optimization problems. In section 4.2 with linear feedback, in section 4.3 with saturated feedback, the high-gain control law and the low-gain control law as MergedLin, MergedSat, HighGain, and LowGain, respectively.

4.4.1 Example 8

One important application of the proposed methods lies in the control of nonlinear systems subject to both input and state constraints. We consider the nonlinear system which mentioned in example 2.

The weighting matrices are $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 0.1$. The high-gain controller is obtained by solving (2.121). In this case,

$$K_1 = [-1.6889 \ \ -4.7618], \ K_2 = [-1.7209 \ \ -4.0693].$$

The low-gain controller is chosen as

$$F_1 = [0.5005 \ \ -2.3535], \ F_2 = [0.4882 \ \ -1.9616].$$

Using Theorem 4.6 and Proposition 4, we obtain the following auxiliary matrix gains

$$\begin{cases} \tilde{K}_1 = [-0.0519 \ \ -2.3224], \ \tilde{K}_2 = [-0.5980 \ \ -1.9905], \\ \tilde{F}_1 = [0.5518 \ \ -2.0525], \ \tilde{F}_2 = [0.4519 \ \ -1.7545]. \end{cases}$$

For the tunning parameter $\rho = 0.01$, Fig. 4.1 presents the feasible ellipsoids of MergedLin (dashed lines), of MergedSat (solid lines), and of HighGain (dash-dot lines) for the two extreme cases $p_1 = 1, p_2 = 0$ and $p_1 = 0, p_2 = 1$. As seen in the figure, MergedSat provides the largest DoA. MergedLin achieves a slightly smaller DoA, while HighGain exhibits a much more limited one.

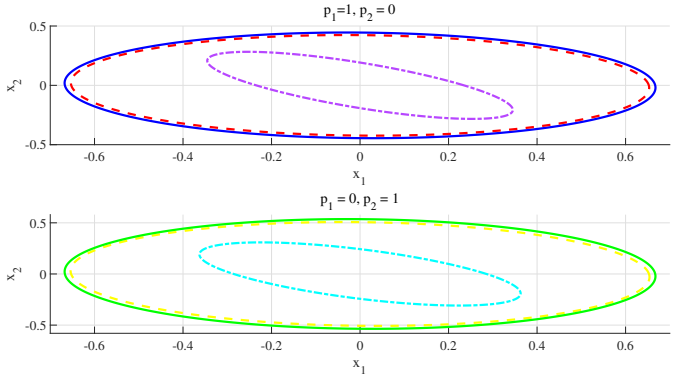
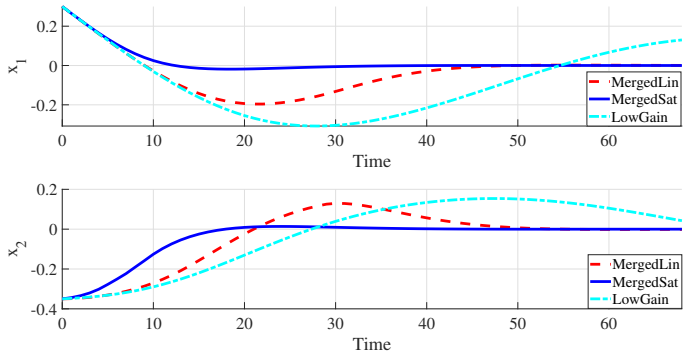
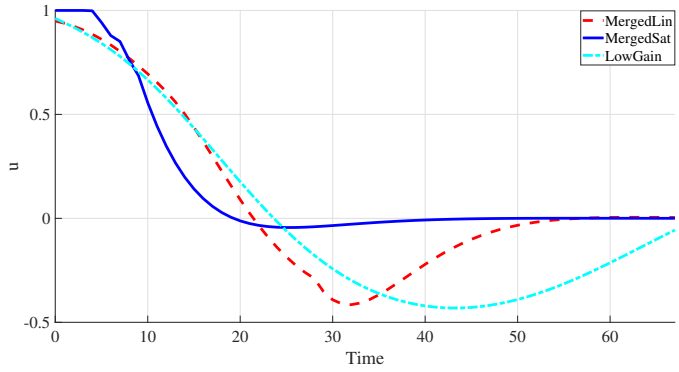


Figure 4.1: Feasible sets for MergedLin (dashed lines), for MergedSat (solid lines), and for High-Gain (dash-dot lines) for example 8.

For the initial condition $x(0) = [0.3 \ \ -0.35]^T$, Fig. 4.2 shows the state trajectories and input trajectories as functions of time for MergedLin (dashed red lines), for MergedSat (solid blue lines), and of LowGain (dash-dot cyan lines). Using Fig. 4.2, we can observe that MergedSat achieves the fastest convergence to the origin, followed by MergedLin. Both significantly outperform LowGain.



(a) State trajectories.



(b) Input trajectories

Figure 4.2: (a) State trajectories; (b) Input trajectories for MergedLin (dashed red line), for MergedSat (solid blue line), for LowGain (dash-dot cyan line) for example 8.

4.4.2 Example 9

We consider the same LPV system presented in example 1.

We impose the following input constraints $-1 \leq u \leq 1$. The weighting matrices are $Q = I, R = 1$. In this case, the high-gain control law is

$$\begin{cases} K_1 = [-5.8803 \ 4.3306 \ -4.0051 \ -0.5361], \\ K_2 = [-5.5518 \ 3.9433 \ -4.0063 \ -0.6821]. \end{cases}$$

The low-gain control law is chosen as

$$\begin{cases} F_1 = [-3.6780 \ 2.9382 \ -3.3505 \ 0.6869], \\ F_2 = [-3.3350 \ 2.5418 \ -3.3083 \ 0.4879]. \end{cases}$$

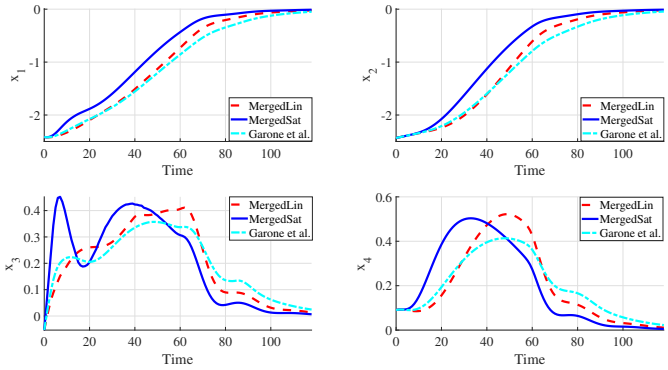
Using Theorem 4.6 and Proposition 4, we obtain the following auxiliary matrix gains

$$\begin{cases} \tilde{K}_1 = [-2.2168\ 1.8096\ -1.7252\ 0.2756], \\ \tilde{K}_2 = [-1.2914\ 0.8963\ -1.6612\ 0.1957], \\ \tilde{F}_1 = [-1.7496\ 1.5373\ -1.5132\ 0.5436], \\ \tilde{F}_2 = [-0.6731\ 0.4566\ -1.4239\ 0.4071]. \end{cases}$$

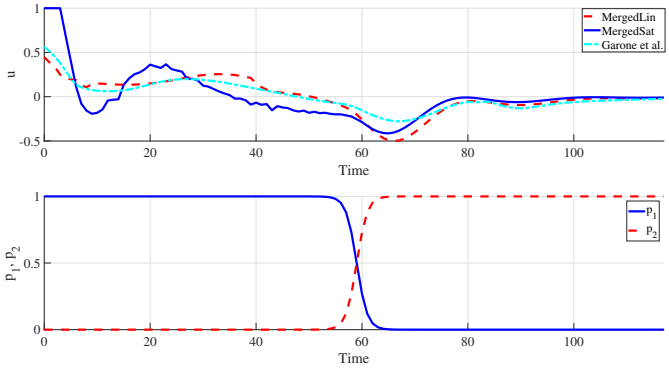
For the initial condition $x(0) = [-2.4149\ -2.4335\ -0.0534\ 0.0903]^T$, Fig. 4.3 shows the state and input trajectories as functions of time for MergedLin (dashed red lines), for MergedSat (solid blue lines). For comparison purposes, the trajectories obtained from [3] are also included (dash-dot cyan lines). Note that the approach in [3] requires solving an SDP problem at each time instant. In addition, Fig. 4.3(b) also presents the realization of the scheduling parameters p_1, p_2. Using Fig. 4.3, we can observe that MergedSat has the best performance, followed by MergedLin. Both of them outperform the one proposed in [3].

4.5 Conclusion

In this chapter, we considered a discrete-time linear parameter-varying system subject to input and state constraints. For this class of systems, we proposed two new control schemes that merge two gain-scheduling control laws with different and potentially conflicting objectives. The first controller features a large DoA but offers limited performance, while the second achieves improved performance for small states near the origin but has a smaller DoA. The proposed merging control law combines the advantages of both approaches: (i) it preserves local optimality by reducing to the high-gain controller whenever the latter is feasible, and (ii) it maintains a large DoA guaranteed by the low-gain controller. Moreover, we proved that the resulting control law is a continuous function of the state. The proposed design procedures do not rely on the control-sharing property. Two numerical examples, including comparisons with existing approaches from the literature, illustrated the effectiveness of the proposed methods.



(a) State trajectories.



(b) Input trajectories

Figure 4.3: (a) State trajectories; (b) Input trajectories for MergedLin (dashed red line), for MergedSat (solid blue line), for [3] (dash-dot cyan line) for example 9.

Chapter 5

Conclusions and Perspective

This thesis investigates several fundamental challenges in the analysis and control of constrained discrete-time LPV systems. The research focuses primarily on two core issues: estimation of the DoA, and the problem of controller synthesis, specifically the design of stabilizing state-feedback control laws based on given performance criteria.

In Chapter 2, we propose, for the first time, LMI conditions for invariant set construction of LPV systems with gain-scheduled input saturation using parameter-dependent ellipsoids. It is noticed that existing LPV controller design methods based on PDLFs introduce a double-sum problem, which is typically solved using sum-of-squares techniques—though their computational complexity increases exponentially with the system order—or through certain relaxation techniques, at the cost of increased algorithm conservativeness. To address this issue, we propose a novel PDLF reconstruction method that eliminates the classic double-sum problem commonly encountered in LPV control design.

Chapter 3 investigates MPC for constrained LPV systems and presents several significant results. First, based on a nonlinear parameterized state feedback control law, we construct a quasi-min–max MPC algorithm. Our results show that the new method achieves better performance than current state-of-the-art methods. However, it is worth noting that the quasi-min-max method we proposed still has some drawbacks: the construction of LMI constraints introduces a double-sum problem, which increases the conservativeness of the algorithm. To address this issue, building on the results in Chapter 2, we extend the double-sum-free PDLF method to the MPC framework. This method effectively eliminates the positive-definiteness issue associated with double-summing, significantly reducing conservativeness while maintaining convexity. We propose two new double-sum-free MPC strategies. The first strategy employs linear gain-scheduling feedback, which enables efficient online computation. However, due to the purely linear nature of the control law, it may not fully utilize the available control authority, causing the system state to converge to the origin more slowly than desired. To overcome this drawback, the second strategy introduces nonlinearly saturated gain-scheduling feedback, which permits larger control inputs when the system is far from the equilibrium point, thereby improving the convergence speed.

To enhance the computational efficiency of the algorithm, we propose a sub-optimal algorithm in Chapter 4. This algorithm is based on an offline-online solution strategy. Building upon the low-gain and high-gain controllers proposed in Chapter 2, we present a general framework that integrates low- and high-gain scheduling control laws. During the offline phase, the maximum invariant set and the upper bound of the objective function are solved. For the online phase, the optimization problem can be solved efficiently by the Newton Raphson algorithm. We prove that for constrained discrete-time LPV systems, a systematic combination of stable controllers can be achieved while maintaining continuity, local optimality, and good attractor region properties. We also extend this framework to saturated-gain scheduling controllers, thereby more effectively utilizing actuator constraints while ensuring stability and performance.

This thesis investigates the stability analysis and controller design for constrained discrete-time LPV systems; however, some issues have not yet been fully addressed and require further investigation. First, in many practical applications, scheduling parameters are bounded not only in magnitude but also in their rates of variation due to physical or operational constraints. Ignoring such rate information can lead to conservative designs and degraded performance [82]; therefore, the case of bounded parameter variation warrants consideration. Some ongoing work in this direction is included in the Appendix. Secondly, This thesis mainly addresses the state-feedback MPC problem. For practical applications, however, output-feedback must be considered since only partial state measurements are typically available. One idea is to integrate the design of the observer gain and control gain, and then solve the resulting optimization problem online. Thirdly, this thesis assume that the exact values of the scheduling parameter are available. However, in practical systems, this assumption is generally difficult to satisfy due to uncertainties arising from measurement errors, noise, or estimation inaccuracies. Therefore, the design of controllers for discrete-time LPV systems under parameter uncertainty is of significant practical relevance and will constitute one of our future research directions.

Chapter 6

Appendix

The research presented in this appendix is still ongoing. The results reported below are preliminary and should be regarded as indicative rather than conclusive.

6.1 Introduction

In many practical applications, scheduling parameters are bounded not only in magnitude but also in their rates of variation due to physical or operational constraints. Ignoring such rate information can lead to conservative designs and degraded performance [82]. As a result, LPV control methods that explicitly account for bounded parameter variation rates have received increasing attention [33, 83–85]. For instance, [5] proposed a gain-scheduled approach for discrete-time LPV systems that incorporates rate bounds via a polytopic representation and difference inequalities, yielding improved performance over rate-independent designs. However, the approach significantly increases the number of LMI constraints and decision variables, and thus the computational complexity [86] .

To overcome this limitation, [87] introduced a novel modeling framework for discrete-time LPV systems with bounded parameter variation rates. By embedding the correlation between the scheduling parameters and their rates of variation into an augmented space, the resulting polytopic representation maintains a fixed number of vertices, independent of the rate bounds. This approach enables effective exploitation of rate information while preserving the computational tractability of the LMI conditions. The method has since become a cornerstone in the control design of LPV systems with bounded parameter variations [84].

In this appendix, we present the design methodologies for H_∞ and H_2 control and the improved quasi-min–max MPC for LPV systems subject to bounded parameter variations.

6.2 H_∞ and H_2 Control Design for Discrete-time LPV Systems with Bounded Parameter Variation

The objective of this section is to propose a design methodology for H_∞ and H_2 controllers for discrete-time LPV systems with bounded parameter variations. The main contributions of this chapter are summarized as follows:

- We propose a class of nonlinearly parameterized state-feedback control laws for H_∞ and H_2 control design.
- To the best of our knowledge, this is the first method that enables the design of continuous state-dependent H_∞ and H_2 control laws for LPV systems while explicitly accounting for bounded parameter variations.

6.2.1 Problem Formulation

We consider the following LPV system

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_w(k)w(k) + B(k)u(k) \\ y(k) &= C(k)x(k) + D(k)u(k) \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the measured state, $y \in \mathbb{R}^{n_y}$ is the performance output, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^{n_w}$ is the disturbance. $[A(k)\ B_w(k)\ B(k)\ C(k)\ D(k)]$ with

$$\begin{aligned} A(k) &= \sum_{i=1}^{L} p_i(k)A_i, B_w(k) = \sum_{i=1}^{L} p_i(k)B_{wi} \\ B(k) &= \sum_{i=1}^{L} p_i(k)B_i, C(k) = \sum_{i=1}^{L} p_i(k)C_i \\ D(k) &= \sum_{i=1}^{L} p_i(k)D_i \end{aligned} \tag{6.2}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $B_{wi} \in \mathbb{R}^{n\times n_w}$, $C_i \in \mathbb{R}^{n_y\times n}$, $D_i \in \mathbb{R}^{n_y\times n_w}$ are constant matrices, $\forall i \in \overline{1, L}$, L is the number of subsystems.

$p(k) = [p_1(k)\ p_2(k)\ \dots p_L(k)]^T$ satisfies

$$p(k) \in \Sigma_L := \left\{ p \in \mathbb{R}^L \mid \sum_{i=1}^{L} p_i = 1,\ p_i \geq 0 \right\} \tag{6.3}$$

The vector $p(k)$ may represent time-varying lumped parameters or state space components.

LPV hypothesis: $p(k)$ is available in each time instant k.

The rate of parameter variation $\Delta_i(k) = p_i(k+1) - p_i(k)$ is bounded and satisfies the following condition:

$$-b \leq \Delta_i(k) \leq b, \forall i \in \overline{1, L} \tag{6.4}$$

where $b \in \mathbb{R}$, $b \in [0, 1]$ is known.

Since $\sum_{i=1}^{L} p_i(k) = 1$, it follows that

$$\sum_{i=1}^{L} \Delta_i(k) = 0 \tag{6.5}$$

we recall the modeling framework in [87] that takes into account the bounded rate of parameter variations. Using Fig. 6.1, any pair (p_i, Δ_i) belongs to the polytope Γ_i, $i \in \overline{1, L}$ given by

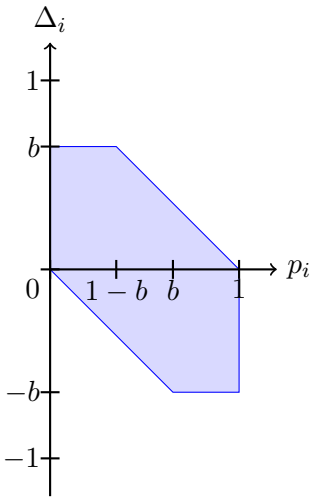


Figure 6.1: Region on the plane $\Delta_i \times p_i$ where Δ_i can assume values as a function of p_i (dark region).

$$\Gamma_i \triangleq \left\{ \delta \in \mathbb{R}^2 \mid \delta = \sum_{i=1}^{6} \lambda_i h_i, \lambda \in \Sigma_6 \right\} \tag{6.6}$$

where

$$\begin{bmatrix} h_1, \ldots, h_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1-b & 1 & 1 & b \\ 0 & b & b & 0 & -b & -b \end{bmatrix} \tag{6.7}$$

i.e. Γ_i is the convex hull of the vertices of the feasible region.

To model the space of (p, Δ), we take the Cartesian product of all Γ_i for $i \in \overline{1, L}$, and then remove any vertices that violate (6.3) and (6.5). The resulting polytope is denoted by Γ, i.e.

$$\Gamma \triangleq \left\{ \delta \in \mathbb{R}^{2L} \mid \delta = \sum_{i=1}^{s} \gamma_i h_i, \gamma \in \Sigma_s \right\} \tag{6.8}$$

where $h_i \in \mathbb{R}^{2L}$ are given vectors, and the number of vertices s grows as a function of L. A key feature of this modeling approach is that both the parameter vector p and its variation Δ are jointly embedded in a higher-dimensional space—referred to as the γ-*space*—of dimension $2L$.

When solving a control problem that depends simultaneously on both p and Δ, we first lift p and Δ to the γ-space. This step can be achieved using the relationship given in (6.8)

$$\begin{bmatrix} p \\ \Delta \end{bmatrix} = H\gamma \tag{6.9}$$

where $H = \begin{bmatrix} h_1 & \dots & h_m \end{bmatrix}$, $\gamma \in \Sigma_m$. For example, for $L = 2$, the polytope Γ is constructed by $\Gamma_1 \times \Gamma_2$, vertices h_i are given by

$$\begin{bmatrix} h_1, \dots, h_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1-b & b \\ 0 & 0 & 1 & 1 & b & 1-b \\ -b & 0 & 0 & b & b & -b \\ b & 0 & 0 & -b & -b & b \end{bmatrix} \tag{6.10}$$

It follows that

$$\begin{aligned} p_1 &= \gamma_1 + \gamma_2 + (1-b)\gamma_5 + b\gamma_6 \\ p_2 &= \gamma_3 + \gamma_4 + b\gamma_5 + (1-b)\gamma_6 \\ \Delta_1 &= -b\gamma_1 + b\gamma_4 + b\gamma_5 - b\gamma_6 \\ \Delta_2 &= b\gamma_1 - b\gamma_4 - b\gamma_5 + b\gamma_6 \end{aligned} \tag{6.11}$$

Thus

$$\begin{aligned} A(k) &= \hat{A}(\gamma) = \sum_{i=1}^{s} \gamma_i \hat{A}_i \\ A(k+1) &= \tilde{A}(\gamma) = \sum_{i=1}^{s} \gamma_i \tilde{A}_i \end{aligned} \tag{6.12}$$

where $\hat{A}_1 = A_1$, $\hat{A}_2 = A_1$, $\hat{A}_3 = A_2$, $\hat{A}_4 = A_2$, $\hat{A}_5 = (1-b)A_1 + bA_2$, $\hat{A}_6 = bA_1 + (1-b)A_2$, $\tilde{A}_1 = (1-b)A_1 + bA_2$, $\tilde{A}_2 = A_1$, $\tilde{A}_3 = A_2$, $\tilde{A}_4 = bA_1 + (1-b)A_2$, $\tilde{A}_5 = A_1$, $\tilde{A}_6 = A_2$.

6.2.2 H_∞ Non-linear Gain-scheduling Control Design

In this section, we address the problem of H_∞ control synthesis for LPV systems subject to bounded parameter variations.

Consider the following parameter dependent Lyapunov function candidate

$$V(k) = x(k)^T P(k)^{-1} x(k) \tag{6.13}$$

where $P(k) = \left(\sum_{i=1}^{L} p_i(k) P_i \right)$, $P_i \in \mathbb{S}^{n\times n}$, $\forall i \in \overline{1, L}$.

It is worth noting that the matrices $A(k)$, $B_w(k)$, $B(k)$, $C(k)$, $D_1(k)$, as well as the Lyapunov matrix $P(k)$ are functions of $p(k)$. As such they are all subject to the bounded parameter variation assumption (6.4). Consequently, the method [87] can be employed to lift the original parameter space (p, Δ) into the γ-space.

Consider the following nonlinear gain-scheduling state feedback

$$u(k) = F(k)G(k)^{-1} x(k) \tag{6.14}$$

where $F(k) = \sum_{i=1}^{L} p_i(k) F_i$, $G(k) = \sum_{i=1}^{L} p_i(k) G_i$, with $F_i \in \mathbb{R}^{n\times m}$, $G_i \in \mathbb{R}^{n\times n}$.

Our aim in this section is to find the matrix gains F_i, G_i such that the closed-loop system

$$\begin{aligned} x(k+1) &= A_c x(k) + B_w(k) w(k) \\ y(k) &= C_c x(k) \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} A_c(k) &= A(k) + B(k)F(k)G(k)^{-1} \\ C_c(k) &= C(k) + D(k)F(k)G(k)^{-1} \end{aligned}$$

is robustly asymptotically stable for all $(p(k), \Delta(k)) \in \Gamma$ and has the optimal H_∞ norm bound on the $w - y$ channel.

Recall that m is the number of vertices in the lifted $\gamma-$space. For given matrices G_i, F_i, P_i, A_i, B_i, B_{wi}, C_i, D_i, we denote $\hat{G}_i$, $\hat{F}_i$,$\hat{P}_i$, $\hat{A}_i$, $\hat{B}_i$, $\hat{B}_{wi}$, $\hat{C}_i$, $\hat{D}_i$, $\tilde{P}_i$ as the coefficients of the matrices $\hat{G}(\gamma)$, $\hat{F}(\gamma)$, $\hat{P}(\gamma)$, $\hat{A}(\gamma)$, $\hat{B}(\gamma)$, $\hat{B}_w(\gamma)$, $\hat{D}(\gamma)$, $\tilde{P}(\gamma)$ obtained with change of variable (6.9) for $G(k)$,$F(k)$,$P(k)$,$A(k)$,$B(k)$,$B_w(k)$, $C(k)$, $D(k)$,$P(k+1)$, respectively.

We have the following theorem.

Theorem 6.1. *Consider the LPV system (6.1). If there exist matrices $G_i \in \mathbb{R}^{n\times n}$, $F_i \in \mathbb{R}^{m\times n}$,$P_i \in \mathbb{S}^{n\times n}$, $i \in \overline{1, L}$ such that the following LMI conditions hold:*

$$X_{jj} \succeq 0,\ j \in \overline{1, s} \tag{6.16}$$

$$\frac{2}{L-1} X_{jj} + X_{jl} + X_{lj} \succeq 0, 1 \le j \ne l \le s, \tag{6.17}$$

$$X_{jl} = \begin{bmatrix} \hat{G}_j^T + \hat{G}_j - \hat{P}_j & 0 & * & * \\ 0 & I & * & * \\ \hat{A}_j \hat{G}_l + \hat{B}_j \hat{F}_l & \hat{B}_{wj} & \tilde{P}_j & 0 \\ \hat{C}_j \hat{G}_l + \hat{D}_j \hat{F}_l & 0 & 0 & \beta I \end{bmatrix} \tag{6.18}$$

then the system (6.1) *is robustly asymptotically stable under the bounded parameter variation* (6.4), (6.5) *with the gain-scheduling state feedback* (6.14) *and has a* H_∞ *norm bound on the* $w-y$ *channel less than* β*, that is,* $\sup_{w\in L_2} \frac{||y||_2}{||w||_2} \leq \beta$.

Proof. Using Lemma 2, the following inequality holds when (6.16), (6.17) are satisfied

$$\left(\sum_{j,l=1}^{m} \gamma_j \gamma_l X_{jl}\right) \succeq 0$$

Or equivalently

$$\begin{bmatrix} \hat{G}(\gamma)^T + \hat{G}(\gamma) - \hat{P}(\gamma) & 0 & * & * \\ 0 & I & * & * \\ \hat{A}(\gamma)\hat{G}(\gamma) + \hat{B}(\gamma)\hat{F}(\gamma) & \hat{B}_w(\gamma) & \tilde{P}(\gamma) & 0 \\ \hat{C}(\gamma)\hat{G}(\gamma) + \hat{D}(\gamma)\hat{F}(\gamma) & 0 & 0 & \beta I \end{bmatrix} \succeq 0 \tag{6.19}$$

Using Lemma 1, (6.19) holds if

$$\begin{bmatrix} \hat{G}(\gamma)^T \hat{P}(\gamma)^{-1} \hat{G}(\gamma) & 0 & * & * \\ 0 & I & * & * \\ \hat{A}(\gamma)\hat{G}(\gamma) + \hat{B}(\gamma)\hat{F}(\gamma) & \hat{B}_w(\gamma) & \tilde{P}(\gamma) & 0 \\ \hat{C}(\gamma)\hat{G}(\gamma) + \hat{D}(\gamma)\hat{F}(\gamma) & 0 & 0 & \beta I \end{bmatrix} \succeq 0 \tag{6.20}$$

Pre- and post-multiplying (6.20) by diag($G^{-1}(k)^T$,0,0,0) and diag($G^{-1}(k)$,0,0,0), one obtains

$$\begin{bmatrix} \hat{P}(\gamma)^{-1} & 0 & * & * \\ 0 & I & * & * \\ \hat{A}(\gamma) + \hat{B}(\gamma)\hat{F}(\gamma)\hat{G}(\gamma)^{-1} & \hat{B}_w(\gamma) & \tilde{P}(\gamma) & 0 \\ \hat{C}(\gamma) + \hat{D}(\gamma)\hat{F}(\gamma)\hat{G}(\gamma)^{-1} & 0 & 0 & \beta I \end{bmatrix} \succeq 0 \tag{6.21}$$

By converting back from the γ-space to the $(p\ \ \Delta)$ space, (6.21) is equivalent to the following inequality

$$\begin{bmatrix} P(k)^{-1} & 0 & * & * \\ 0 & I & * & * \\ A_c(k) & B_w(k) & P(k+1) & 0 \\ C_c(k) & 0 & 0 & \beta I \end{bmatrix} \succeq 0 \tag{6.22}$$

Using Shur complement, (6.22) is satisfied if and only if

$$\begin{aligned} &\begin{bmatrix} P(k)^{-1} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_c(k) & B_w(k) \\ C_c(k) & 0 \end{bmatrix}^T \\ &\times \begin{bmatrix} P(k+1)^{-1} & 0 \\ 0 & \frac{1}{\beta} I \end{bmatrix} \begin{bmatrix} A_c(k) & B_w(k) \\ C_c(k) & 0 \end{bmatrix} \succeq 0 \end{aligned} \tag{6.23}$$

Pre- and post-multiplying (6.23) by [x(k)^T w(k)^T] and its transpose, one gets

$$\begin{aligned} &x(k)^T P(k)^{-1} x(k) - x(k+1)^T P(k+1)^{-1} x(k+1) \\ &+w(k)^T w(k) - \tfrac{1}{\beta} y(k)^T y(k) \geq 0 \end{aligned}$$

Clearly, this condition implies that the system is asymptotically stable. Hence $\lim_{k\to\infty} x(k) = 0$. It follows that

$$\frac{1}{\beta}\sum_{k=0}^{\infty} y(k)^T y(k) - x(0)^T P(0)^{-1} x(0) \leq \sum_{k=0}^{\infty} w(k)^T w(k)$$

Under the zero initial condition, i.e., $x(0) = 0$, one gets

$$\sum_{k=0}^{\infty} y(k)^T y(k) \leq \beta \sum_{k=0}^{\infty} w(k)^T w(k)$$

Hence the H_∞ norm is less than β. The proof is complete. □

It is worth noticing that the problem of designing an H_∞ optimal control law for LPV systems with bounded parameter variations was already addressed in [5]. Compared with [5], the proposed approach offers two main advantages. First, when the parameter variation rate is small, the modeling strategy adopted in [5] results in a large number of LMI conditions, which significantly increases the computational complexity of the associated algorithms. Second, [5] employs a discontinuous control law with respect to the state, whereas the proposed approach ensures continuity by construction.

Using Theorem 1, the problem of designing an H_∞ optimal gain-scheduling control law for system (6.1) can be formulated as

$$\begin{aligned} &\min_{G_i, F_i, P_i, \beta} \beta, \\ &\text{s.t. (6.16), (6.17), (6.18)} \end{aligned} \tag{6.24}$$

This is a convex semidefinite program, for which several free solvers are available.

6.2.3 H_2 Non-linear Gain-scheduling Control Design

We first introduce the following lemma to support the method to be developed.

Lemma 6.2. *[88] Consider the following discrete-time LTI syetem*

$$\begin{aligned} x(k+1) &= Ax(k) + B_w w(k) + Bu(k) \\ u(k) &= Kx(k) \\ z(k) &= Cx(k) + Du(k) \end{aligned} \tag{6.25}$$

where matrices A, B, B_w, C, D are of appropriate dimensions.

Let the closed-loop matrices of the system be denoted as $A_{cl} = A + BK$ and $C_{cl} = C + DK$. Suppose that a state-feedback gain K is designed such that A_{cl} is asymptotically stable. Denote by L_o the observability Gramian of (A_{cl}, C_{cl}), i.e., L_o is the solution of the following discrete-time Lyapunov equation

$$A_{cl}^T L_o A_{cl} - L_o + C_{cl}^T C_{cl} = 0. \tag{6.26}$$

The H_2 norm of the closed-loop system is given by

$$\|H\|_2 = \text{Tr}(B_w^T L_o B_w). \tag{6.27}$$

In the following we provide a way to extend the result in [88] for the LPV system (4.1). For this purpose, consider the non-linear gain scheduling state feedback controller (6.14), and the following parameter-dependent Lyapunov functions

$$V_c(k) = x(k)^T P(k)^{-1} x(k) \tag{6.28}$$

where $P(k) = \left(\sum_{i=1}^{L} p_i(k) P_i \right)$, $P_i \in \mathbb{S}^{n \times n}$, $\forall i \in \overline{1, L}$.

The following result holds.

Theorem 6.3. *Consider LPV system (6.1). If there exist matrices $G_i \in \mathbb{R}^{n \times n}$, $F_i \in \mathbb{R}^{m \times n}$, $Z_i \in \mathbb{R}^{n_w \times n_w}$, $P_i \in \mathbb{S}^{n \times n}$, $\forall i \in \overline{1, L}$ such that the following LMI conditions hold:*

$$Y_{jj} \succeq 0, \ j \in \overline{1, s} \tag{6.29}$$

$$\frac{2}{m-1} Y_{jj} + Y_{jl} + Y_{lj} \succeq 0, 1 \leq j \neq l \leq s \tag{6.30}$$

$$Y_{jl} = \begin{bmatrix} \hat{G}_j^T + \hat{G}_j - \hat{P}_j & * & * \\ \hat{A}_j \hat{G}_l + \hat{B}_j \hat{F}_l & \tilde{P}_j & * \\ \hat{C}_j \hat{G}_l + \hat{D}_j \hat{F}_l & 0 & I \end{bmatrix} \tag{6.31}$$

$$\begin{bmatrix} \hat{Z}_j & \hat{B}_w^T \\ \hat{B}_w & \hat{P}_j \end{bmatrix} \succeq 0 \tag{6.32}$$

$$Tr(\hat{Z}_j) \leq \eta \tag{6.33}$$

then the system (6.1) is robustly asymptotically stable under the bounded parameter variations (6.4), (6.5) with the gain-scheduling state feedback (6.14) and has a H_2 norm bound on the $w - y$ channel less than η.

Proof. Following a similar argument as in the proof of Theorem 6.1, it can be shown that inequalities (6.29)–(6.31) imply

$$A_{cl}^T(k) P(k+1) A_{cl}(k) - P(k) + C_c^T(k) C_c(k) \preceq 0,$$

Hence the system is asymptotically stable, and $P(k)$ plays a role of the controllabity gramian for the closed-loop LPV system.

Note that inequalities (6.32)–(6.33) imply

$$\|H\|_2 = \text{Tr}(B_w^T(k)P(k)B_w) \leq \text{Tr}(Z(k)) \leq \eta.$$

The proof is complete. □

Using Theorem 6.3, the problem of designing an optimal gain-scheduling control law for system (6.1) can be formulated as

$$\begin{aligned} &\min_{G_i,F_i,P_i,\eta} \eta, \\ &\text{s.t. (6.29), (6.30), (6.31), (6.32), (6.33)} \end{aligned} \tag{6.34}$$

6.2.4 Numerical Examples

This section illustrates the potential benefit of the new approaches by simulations of two example systems. We used the CVX toolbox to solve SDP optimization problems. For comparison purpose, we denote the our proposed algorithm, i.e. the H_∞ control law with parameter bounded variation as Hinfpvb, without parameter bounded variation as Hinf, the H_2 control law with parameter bounded variation as H2pbv, without parameter bounded variation as H2.

6.2.4.1 Example 10

This example is taken from [89], and is used to investigate the performance of the H_∞ controller. For this system, the proposed H_∞ control approach is compared with the method in [5] as well as a conventional H_∞ design that does not consider bounded parameter variations.

Consider system (6.1) with

$$\begin{aligned} A_1 &= \alpha \begin{bmatrix} 1.0 & 0 & -2 \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \; A_2 = \alpha \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix} \\ B_1 &= B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_{w1} = B_{w2} = \begin{bmatrix} 0.1 \\ -0.15 \\ 0.16 \end{bmatrix} \\ C_1 &= C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D_1 = D_2 = 1 \end{aligned}$$

The parameter α is 0.66, with a bounded rate of variation $b = 0.25$, i.e.,

$$|p_i(k+1) - p_i(k)| \leq 0.25, \forall i = 1, 2$$

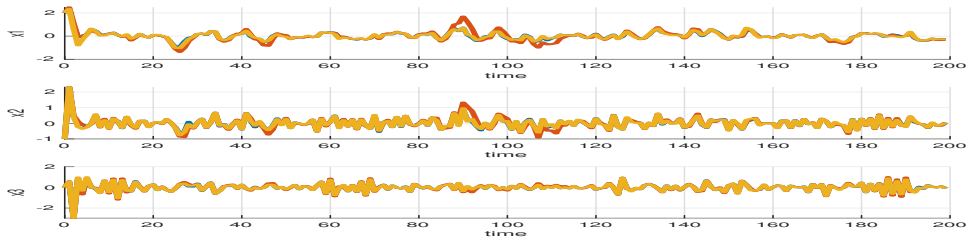


Figure 6.2: State trajectories for Hinfpvb (solid blue), Hinf (solid red), and the method in [5] (solid yellow) for Example 10.

By solving the optimization problem (6.24), an upper bound of the H_∞ norm is obtained as $\beta = 0.733$, with

$$\begin{aligned} F_1 &= \begin{bmatrix} -0.3691 & -0.2784 & 0.6915 \end{bmatrix} \\ F_2 &= \begin{bmatrix} -0.2817 & -0.2580 & 0.5617 \end{bmatrix} \\ G_1 &= \begin{bmatrix} 0.6605 & 0.3947 & -0.6813 \\ 0.4323 & 0.3840 & -0.3832 \\ -0.2124 & -0.1058 & 0.4278 \end{bmatrix} \\ G_2 &= \begin{bmatrix} 0.5442 & 0.3452 & -0.4034 \\ 0.4819 & 0.3909 & -0.4711 \\ -0.5406 & -0.3722 & 0.7446 \end{bmatrix} \end{aligned} \tag{6.35}$$

Using the approach proposed in [5], an upper bound of the H_∞ norm is obtained as $\beta = 1.18$. Without considering bounded parameter variations ($b = 1$), the computed upper bound of the H_∞ norm is $\beta = 3.85$.

For the initial condition $x(0) = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T$, Fig. 6.2 shows the state trajectories as functions of time for the proposed H_∞ controller with parameter bounded variation (Hinfpvb, solid blue), the conventional (without parameter bounded variation) H_∞ controller (Hinf, solid red), and the method in [5] (solid purple) under the Gaussian white noise. Fig. 6.3 shows the corresponding accumulated output energy.

It can be observed that the proposed method exhibits superior disturbance attenuation capability. It is worth noting that the number of decision variables is the same for our approach and the rate-independent methods. As a result, the proposed method has lower computational complexity compared with the method in [5].

6.2.4.2 Example 11

The second example is adopted from [6] and is used to evaluate the performance of the proposed H_2 controller.

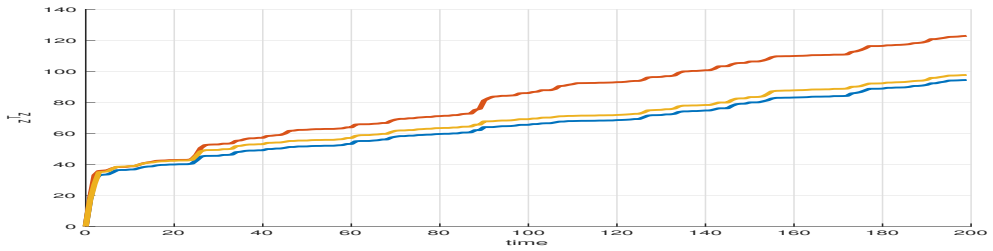


Figure 6.3: Accumulated output energy trajectories for Hinfpvb (solid blue), Hinf (solid red), and the method in [5] (solid yellow) for Example 10.

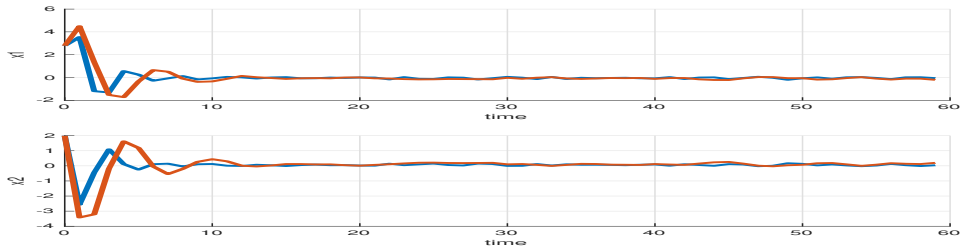


Figure 6.4: State trajectories for H2pvb (solid blue), H2 (solid red) for Example 11.

Consider system (4.1) with

$$A_1 = \begin{bmatrix} 1.0 & -1.4 \\ -1.0 & -0.8 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.0 & 1.4 \\ -1.0 & -0.8 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 5.9 \\ 2.8 \end{bmatrix}, B_2 = \begin{bmatrix} 3.1 \\ -2.8 \end{bmatrix}, B_{w1} = B_{w2} = \begin{bmatrix} -0.2 \\ 0.16 \end{bmatrix}$$
$$C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We consider the bound rate of variation $b = 0.18$.

By solving the optimization problem (6.34), we get $\eta = 0.059$. If we do not take into account the bounded rate of parameter variations information, we get $\eta = 0.113$. Therefore, by fully exploiting the information on the bounded rate of parameter variation, the proposed method achieves a smaller H_2 norm.

Fig. 6.4 and Fig. 6.5 show the state and output trajectories as functions of time for the proposed H_2 controller with parameter bounded variation (H2pvb, solid blue), the conventional(without parameter bounded variation) H_2 controller (H2, solid red), Fig. 6.6 shows the corresponding input trajectories.

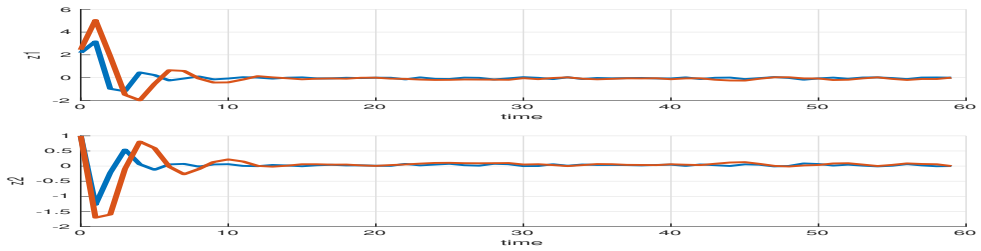


Figure 6.5: Output trajectories for H2pvb (solid blue), H2 (solid red) for Example 11.

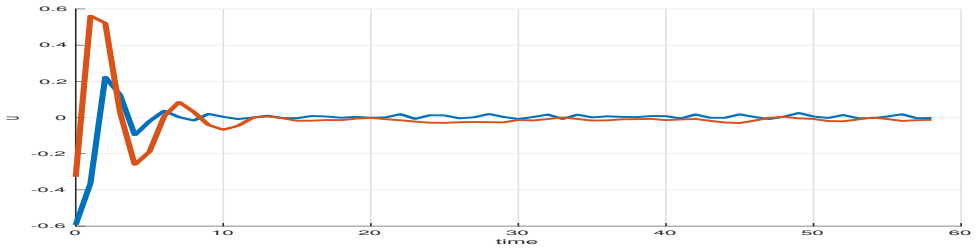


Figure 6.6: Input trajectories for H2pvb (solid blue), H2 (solid red) for Example 11.

6.2.5 Conclusion

This section presents a systematic design methodology for H_∞ and H_2 controllers for discrete-time LPV systems with bounded parameter variations. By introducing parameter-dependent Lyapunov functions and employing nonlinear gain-scheduling control laws, the proposed approach explicitly incorporates bounds on parameter variation rates into the controller synthesis process. Theoretical and simulation results show that the proposed approach outperforms state-of-the-art methods.

6.3 Improved Quasi-Min-Max MPC for LPV system subject to parameter bounded variation

6.3.1 Problem Formulation

Consider the following discrete-time linear parameter varying (LPV) system

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{6.36}$$

where $x \in \mathbb{R}^n$ is the measured state, $u \in \mathbb{R}^m$ is the control input. The matrices $[A(k)\ B(k)] \in \Omega$ with

$$\Omega = \text{Co}\left([A_1\ B_1], [A_2\ B_2], \ldots, [A_L\ B_L]\right) \tag{6.37}$$

where $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ are constant matrices, $\forall i \in \overline{1,L}$, L is the number of subsystems, and $\mathrm{Co}(\cdot)$ denotes the convex hull operator. It is well known [7] that the condition $[A(k)\ B(k)] \in \Omega$ holds if and only if $\exists p_i(k), \forall i = \overline{1,L}$ such that

$$A(k) = \sum_{i=1}^{L} p_i(k)A_i, B(k) = \sum_{i=1}^{L} p_i(k)B_i \tag{6.38}$$

where $p(k) = [p_1(k)\ p_2(k)\ \ldots p_L(k)]^T$ satisfies

$$p(k) \in \Sigma_L := \left\{ p \in \mathbb{R}^L \mid \sum_{i=1}^{L} p_i = 1,\ p_i \geq 0 \right\} \tag{6.39}$$

The vector $p(k)$ may represent time-varying lumped parameters or state space components.

The rate of parameter variation $\Delta p_i(k) = p_i(k+1) - p_i(k)$ is bounded and satisfies the following condition:

$$-b \leq \Delta p_i(k) \leq b, \forall i \in \overline{1,L} \tag{6.40}$$

where $b \in \mathbb{R}$, $b \in [0,1]$ is known.

Since $\sum_{i=1}^{L} p_i(k) = 1$, it follows that

$$\sum_{i=1}^{L} \Delta p_i(k) = 0 \tag{6.41}$$

LPV hypothesis: $p(k)$ is available in each time instant k.

The state $x(k)$ and the input $u(k)$ are subject to the constraints, $\forall k \geq 0$

$$\begin{cases} x(k) \in \mathscr{X} := \left\{ x \in \mathbb{R}^n \mid |f_l^T x| \leq 1, \forall l \in \overline{1,n_x} \right\} \\ u(k) \in \mathscr{U} := \left\{ u \in \mathbb{R}^m \mid |u_l| \leq 1, \forall l \in \overline{1,m} \right\} \end{cases} \tag{6.42}$$

We denote $x(k+t), u(k+t), t = 0, 1, \ldots$, respectively, as the predicted states and the predicted control inputs at time $k+t$ from time k.

Consider the following infinite horizon objective function

$$J_0^\infty(k) = \sum_{t=0}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \tag{6.43}$$

where $Q \in \mathbb{S}^{n_x}, R \in \mathbb{S}^{n_u}$ are weighting matrices.

At a generic time k, the objective of the paper is to design a control law $u(k+t), t = 0, 1, \ldots$ that asymptotically stabilizes the system (6.36). Furthermore, $u(k+t)$ should solve the following min-max problem

$$\begin{aligned} &\min_{U_0^\infty(k)} \max_{[A(k+t),B(k+t)]\in\Omega} J_0^\infty(k) \\ &\text{s.t. } (6.36), (6.42) \end{aligned} \tag{6.44}$$

where

$$U_0^{\infty}(k) = [u(k)^T \; u(k+1)^T \; \ldots]^T \tag{6.45}$$

One way to solve the problem (6.44) is to apply the quasi-min-max MPC control strategy [2]. Using this method, we split the cost function $J_0^{\infty}(k)$ into two parts

$$J_0^{\infty}(k) = x(k)^T Q x(k) + u(k)^T R u(k) + J_1^{\infty}(k) \tag{6.46}$$

with

$$J_1^{\infty}(k) = \sum_{t=1}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \tag{6.47}$$

Using (6.46), (6.47), and the principle of optimality [90], problem (6.44) is equivalent to

$$\begin{aligned} &\min_{u(k)} \left(u(k)^T R u(k) + \min_{U_1^{\infty}(k)} \max_{[A(k+t),B(k+t)]\in\Omega} J_1^{\infty}(k) \right) \\ &\text{s.t. } (6.36), (6.42) \end{aligned} \tag{6.48}$$

with $U_1^{\infty}(k) = [u(k+1)^T \; u(k+2)^T \; \ldots]^T$. Note that the term $x(k)^T Q x(k)$ is removed from the cost (6.48) as it does not influence the optimal argument.

In the quasi-min-max MPC, the first control move $u(k)$ is a decision variable in the problem (6.48). The rest of the future control moves is parameterized as [2]

$$u(k+t) = F x(k+t), \forall t = 1, 2, \ldots \tag{6.49}$$

where $F \in \mathbb{R}^{n_u \times n_x}$ is a decision variable in (6.49). In [2], [33], [34], less conservative results can be achieved where linearly scheduled state feedback control laws of the form, $\forall t = 1, 2, \ldots$

$$u(k+t) = \left(\sum_{i=1}^{L} p_i(k+t) F_i \right) x(k+t) \tag{6.50}$$

are employed, where $F_i \in \mathbb{R}^{m \times n}, \forall i = \overline{1, L}$ are decision variables in (6.44).

We recall the modeling framework in [87] that takes into account the bounded rate of parameter variations. Using Fig. 6.7, any pair (p_i, Δ_i) belongs to the polytope Γ_i, $i \in \overline{1, L}$ given by

$$\Gamma_i \triangleq \left\{ \delta \in \mathbb{R}^2 \mid \delta = \sum_{i=1}^{6} \lambda_i v_i, \lambda \in \Sigma_6 \right\} \tag{6.51}$$

where

$$\begin{bmatrix} v_1, \ldots, v_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1-b & 1 & 1 & b \\ 0 & b & b & 0 & -b & -b \end{bmatrix} \tag{6.52}$$

i.e. Γ_i is the convex hull of the vertices of the feasible region.

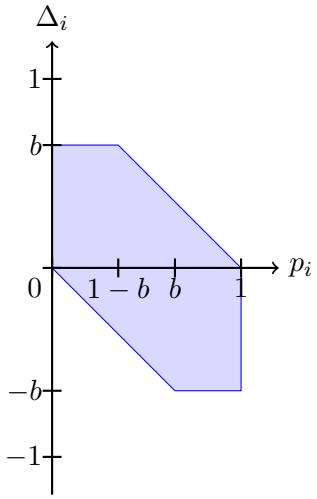


Figure 6.7: Region on the plane $\Delta_i \times p_i$ where Δ_i can assume values as a function of p_i (dark region).

To model the space of (p, Δ), we take the Cartesian product of all Γ_i for $i \in \overline{1, L}$, and then remove any vertices that violate (6.40) and (6.41). The resulting polytope is denoted by Γ, i.e.

$$\Gamma \triangleq \left\{ \delta \in \mathbb{R}^{2L} \mid \delta = \sum_{i=1}^{n_\eta} \eta_i h_i, \eta \in \Sigma_{n_\eta} \right\} \tag{6.53}$$

where $h_i \in \mathbb{R}^{2L}$ are given vectors, and the number of vertices n_η grows as a function of L. A key feature of this modeling approach is that both the parameter vector p and its variation Δ are jointly embedded in a higher-dimensional space—referred to as the η-*space*—of dimension $2L$.

When solving a control problem that depends simultaneously on both $p(k+t)$ and $\Delta(k+t)$ $t \geq 0$, we first lift $p(k+t)$ and $\Delta(k+t)$ to the η-space. This step can be achieved using the relationship given in (6.53)

$$\begin{bmatrix} p(k+t) \\ \Delta(k+t) \end{bmatrix} = H\eta \tag{6.54}$$

where $H = \begin{bmatrix} h_1 & \dots & h_\eta \end{bmatrix}$, $\eta \in \Sigma_{n_\eta}$, $p(k+t)$ and $\Delta(k+t)$ are column vectors composed of $p_i(k+t)$ and $\Delta_i(k+t)$, respectively. For example, for $L = 2$, the polytope Γ is constructed by $\Gamma_1 \times \Gamma_2$, vertices h_i are given by

$$\begin{bmatrix} h_1, \dots, h_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1-b & b \\ 0 & 0 & 1 & 1 & b & 1-b \\ -b & 0 & 0 & b & b & -b \\ b & 0 & 0 & -b & -b & b \end{bmatrix} \tag{6.55}$$

Thus $\forall t \geq 0$,

$$\begin{aligned} A(k+t) &= \hat{A}(\eta) = \sum_{i=1}^{n_\eta} \eta_i \hat{A}_i \\ A(k+t+1) &= \tilde{A}(\eta) = \sum_{i=1}^{n_\eta} \eta_i \tilde{A}_i \end{aligned} \tag{6.56}$$

where $\hat{A}_1 = A_1$, $\hat{A}_2 = A_1$, $\hat{A}_3 = A_2$, $\hat{A}_4 = A_2$, $\hat{A}_5 = (1-b)A_1 + bA_2$. $\hat{A}_6 = bA_1 + (1-b)A_2$, $\tilde{A}_1 = (1-b)A_1 + bA_2$, $\tilde{A}_2 = A_1$, $\tilde{A}_3 = A_2$, $\tilde{A}_4 = bA_1 + (1-b)A_2$, $\tilde{A}_5 = A_1$, $\tilde{A}_6 = A_2$.

Similarly, the input matrix $B(k)$ can also be lifted to the η-space using the above approach.

6.3.2 Main Results

At sampling time k, the next state matrix $A(k+1)$ can be written as

$$\begin{aligned} A(k+1) &= \sum_{i=1}^{L} p_i(k+1)A_i = \sum_{i=1}^{L} (p_i(k) + \Delta p_i)A_i \\ &= A(k) + \sum_{i=1}^{L} (\Delta p_i(k))A_i \end{aligned} \tag{6.57}$$

It is worth noting that $\Delta p_i(k)$ depends not only on the variation bound b but also on the current value $p_i(k)$, in order to guarantee that $p_i(k+1) \in [0, 1]$. Therefore, we have

$$\Delta p_i(k) \in \begin{cases} [-p_i(k),\, b], & \text{if } p_i(k) \in [0,\, b], \\ [p_i(k) - b,\, p_i(k) + b], & \text{if } p_i(k) \in [b,\, 1-b], \\ [-b,\, 1 - p_i(k)], & \text{if } p_i(k) \in [1-b,\, 1]. \end{cases} \tag{6.58}$$

Consider the parameter vector $p(k+1) = [p_1(k+1), \ldots, p_i(k+1)]^T, i \in \overline{1, L}$ satisfying $p_i(k+1) \in [\underline{p_i}, \overline{p}_i]$

The feasible set for $p(k+1)$ is defined as

$$\mathscr{P} = \left\{ p \in \mathbb{R}^L \,\middle|\, \underline{p_i} \leq p_i \leq \overline{p_i},\ \sum_{i=1}^{L} p_i = 1 \right\}, \tag{6.59}$$

which is a convex polytope.

The vertices of $\mathscr{P}$ are obtained by fixing $(L-1)$ components of p at their bounds (i.e., $\underline{p_i}$ or $\overline{p}_i$), while the remaining component is determined from the equality constraint

$$p_w = 1 - \sum_{i \neq w} p_i, \tag{6.60}$$

where $w \in \overline{1, L}$, provided that $p_w \in [\underline{p_w}, \overline{p_w}]$.

Therefore, all admissible vertices are constructed by enumerating all 2^{L-1} boundary combinations of $(L-1)$ variables and retaining only those that satisfy the interval constraint of the remaining variable.

Define s is number of vertices for $\mathscr{P}$, thus $[A(k+1), B(k+1)]$ belongs to a polytope $\Omega^o(k)$:

$$\Omega^o(k) = \text{Co}\{[A_1^o, B_1^o], \dots [A_s^o, B_s^o]\} \tag{6.61}$$

where $A_l^o(k), B_l^o(k)$, $l \in \overline{1, s}$ represent vertices of $A(k+1)$ and $B(k+1)$ and both are constant matrices. One has

$$A(k+1) = \sum_{l=1}^{s} p_l^o A_l^o, \; B(k+1) = \sum_{l=1}^{s} p_l^o B_l^o \tag{6.62}$$

where p_l^o are scheduling parameters. Define $p^o = [p_1^o \; p_2^o \; \dots p_s^o]^T$ satisfies $p^o \in \Sigma_s$.

Similarly, the admissible variation $\Delta p_i(k+1)$ depends on $p_i(k+1)$, the feasible set of $p(k+2)$ is determined in a state-dependent manner. When $p(k+1)$ lies within a given subregion, the admissible range of $\Delta p(k+1)$ is fixed accordingly. Therefore, the vertices of the feasible set for $p(k+2)$ are obtained by combining the vertices of $p(k+1)$ with the extreme values of $\Delta p(k+1)$.

Similarly, for each vertex $[A_l^o(k), B_l^o(k)]$ associated with the prediction at time $k+1$, we have

$$[A_l^o(k), B_l^o(k)] \in \mathscr{X}_l, \tag{6.63}$$

where $\mathscr{X}_l$ is defined as a convex polytope:

$$\mathscr{X}_l = \text{Co}\left\{[A_{l1}^o, B_{l1}^o], \dots, [A_{ls}^o, B_{ls}^o]\right\}. \tag{6.64}$$

where $A_l^o(k), B_l^o(k)$ are constant vertices.

Thus $[A(k+2), B(k+2)]$ belongs to the following polytope:

$$[A(k+2), B(k+2)] \in \text{Co}\left\{\mathscr{X}_1, \mathscr{X}_2 \dots \mathscr{X}_s\right\} \tag{6.65}$$

i.e.

$$\begin{aligned} A(k+2) &= \sum_{l=1}^{s}\sum_{q=1}^{s} p_l^o \iota_q^o A_{lq}^o \\ B(k+2) &= \sum_{l=1}^{s}\sum_{q=1}^{s} p_l^o \iota_q^o B_{lq}^o \end{aligned} \tag{6.66}$$

Define $\iota^o = [\iota_1^o \; \iota_2^o \; \dots \iota_s^o]^T$ satisfies $\iota^o \in \Sigma_s$. $\sum_{l=1}^{s}\sum_{q=1}^{s} p_l^o \iota_q^o = 1$.

Remark 1: It is noticed that the new polytope is smaller than the original one in the sense that the range of $p(k+1)$ defined in 6.59 is smaller than the range of 0 and 1. By updating the polytope, the feasibility properties of quasi-min-max are improved. □

After updating the polytope $\Omega^o(k)$ that characterizes the next state matrix, the second-stage cost can be separated from the infinite-horizon objective function $J_2^\infty(k)$, along with the first-stage cost.

$$\begin{aligned} J_0^\infty(k) =& x(k)^T Q x(k) + u(k)^T R u(k) \\ &+ x(k+1)^T Q x(k+1) + u(k+1)^T R u(k+1) \\ &+ J_2^\infty(k) \end{aligned} \tag{6.67}$$

with

$$J_2^\infty(k) = \sum_{t=2}^{\infty} x(k+t)^T Q x(k+t) + u(k+t)^T R u(k+t) \tag{6.68}$$

Then the following holds

$$x(k+2) = \sum_{l=1}^{s} p_l^o (A_l^o x(k+1) + B_l^o u(k+1)) \tag{6.69}$$

where $x(k+1) = A(k)x(k) + B(k)u(k)$. thus problem (6.44) is equivalent to

$$\begin{aligned} \min_{u(k),u(k+1)} \Big\{ & u(k)^T R u(k) + x(k+1)^T Q x(k+1) \\ & + u(k+1)^T R u(k+1) \\ & + \min_{U_2^\infty(k)} \max_{[A(k+t),B(k+t)]\in\Omega} J_2^\infty(k) \Big\} \\ \text{s.t. } & (6.36), (6.42) \end{aligned} \tag{6.70}$$

the first and the second control move $u(k)$, $u(k+1)$ are decision variables in the problem (6.70).

With the aim of obtaining even less conservative results, we adopt the following class of non-linearly parameterized state feedback control laws for U_2^∞. $\forall t = 1, 2, \ldots$

$$u(k+t) = F(k+t)G(k+t)^{-1} x(k+t) \tag{6.71}$$

where $F(k+t) = \sum_{i=1}^{L} p_i(k+t)F_i$, $G(k+t) = \sum_{i=1}^{L} p_i(k+t)G_i$ with $F_i \in \mathbb{R}^{m\times n}, G_j \in \mathbb{R}^{n\times n}, \forall i = \overline{1, L}$, are decision variables in (6.70). Clearly if $G_i = I, \forall i = \overline{1, L}$ then (6.71) coincides with (6.50).

Now we will provide a way to calculate the upper bound of $J_1^\infty(k)$ using the control law (6.71). Substituting (6.71) into (6.36), one obtains, $\forall t \geq 2$

$$x(k+t+1) = A_c(k+t)x(k+t) \tag{6.72}$$

where $A_c(k+t) = A(k+t) + B(k+t)F(k+t)G(k+t)^{-1}$. We will provide a way to design $F_i, G_i, \forall i \in \overline{1, L}$ such that the closed-loop system (6.72) is asymptotically stable. For the moment, it is assumed that $\lim_{t\to\infty} x(k+t) = 0$ for any admissible state $x(k)$.

Consider the following parameterized quadratic function, $\forall t \geq 2$

$$V(k+t) = x(k+t)^T S(k+t)^{-1} x(k+t) \tag{6.73}$$

where $S(k+t) = \sum_{i=1}^{L} p_i(k+t) S_i$, $S_i \in \mathbb{S}^{n\times n}, \forall i = \overline{1, L}$ are chosen to satisfy,

$$\begin{aligned} x^T(k+t)Qx(k+t) + u(k+t)^T Ru(k+t) \leq \\ V(k+t) - V(k+t+1) \end{aligned} \tag{6.74}$$

As $\lim_{t\to\infty} x(k+t) = 0$, one has $\lim_{t\to\infty} V(k+t) = 0$. Summing (6.74) from $t = 2$ to $t = \infty$, one obtains

$$\sum_{t=2}^{\infty} x^T(k+t)Qx(k+t) + u(k+t)^T Ru(k+t) \leq V(k+2)$$

or equivalently, $\forall k \geq 0$

$$J_2^{\infty}(k) \leq V(k+2) = x(k+2)^T S(k+2)^{-1} x(k+2)$$

Hence $V(k+2)$ is an upper bound of the cost $J_2^{\infty}(k)$.

Similar to (6.66), $S(k+2)$ can be expressed as

$$S(k+2) = \sum_{l=1}^{s}\sum_{q=1}^{s} p_l^o \iota_q^o S_{lq}^o \tag{6.75}$$

where S_{lq} is the linear function of S_i.

Define $R_c \in \mathbb{R}^{n_u\times n_u}, Q_c \in \mathbb{R}^{n_x\times n_x}$, respectively, as a squared root of R, Q, i.e., $R_c^T R_c = R$, $Q_c^T Q_c = Q$. Recall that n_η is the number of vertices in the lifted $\eta-$space. For given matrices G_i, F_i, S_i, A_i, B_i, we denote $\hat{G}_i$, $\hat{F}_i$, $\hat{S}_i$, $\hat{A}_i$, $\hat{B}_i$, $\tilde{S}_i$ as the coefficients of the matrices $\hat{G}(\eta)$, $\hat{F}(\eta)$, $\hat{S}(\eta)$, $\hat{A}(\eta)$, $\hat{B}(\eta)$, $\tilde{S}(\eta)$ obtained with change of variable (6.54) for $G(k+t)$,$F(k+t)$,$S(k+t)$,$A(k+t)$,$B(k+t)$,$S(k+t+1)$ $\forall t \geq 2$, respectively.

Theorem 6.4. *For a given $x(k)$ at a generic time instant k, a solution to the problem (6.70) may be found by solving the following SDP program*

$$\min_{u(k),u(k+1)F_i,G_i,S_i,\gamma} \{\gamma\} \tag{6.76}$$

subject to

$$\begin{bmatrix} 1 & * & * & * & * \\ A_l^o\big(A(k)x(k) + B(k)u(k)\big) + B_l^o u(k+1) & Q_{lq}^o & * & * & * \\ R_c u(k) & 0 & \gamma I & * & * \\ Q_c\big(A(k)x(k) + B(k)u(k)\big) & 0 & 0 & \gamma I & * \\ R_c u(k+1) & 0 & 0 & 0 & \gamma I \end{bmatrix} \succeq 0 \tag{6.77}$$

$l \in \overline{1,s},\ q \in \overline{1,s}$

$$X_{ii} \succeq 0,\ i \in \overline{1,n_\eta} \tag{6.78}$$

$$\frac{2}{L-1}X_{ii} + X_{ij} + X_{ji} \succeq 0, 1 \leq i \neq j \leq n_\eta \tag{6.79}$$

$$|u_l(k)| \leq 1, \forall l \in \overline{1,m} \tag{6.80}$$

$$|u_l(k+1)| \leq 1, \forall l \in \overline{1,m} \tag{6.81}$$

$$\begin{bmatrix} 1 & * \\ \hat{F}_{i,l}^T & \hat{G}_i^T + \hat{G}_i - \hat{Q}_i \end{bmatrix} \succeq 0,\ i \in \overline{1,n_\eta},\ l \in \overline{1,m} \tag{6.82}$$

$$\begin{bmatrix} 1 & * \\ \hat{Q}_i f_l & \hat{Q}_i \end{bmatrix} \succeq 0\ i \in \overline{1,n_\eta}, l \in \overline{1,n_x} \tag{6.83}$$

where

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{6.84}$$

$$X_{ij} = \begin{bmatrix} \hat{G}_i^T + \hat{G}_i - \hat{Q}_i & * & * & * \\ \hat{A}_i\hat{G}_j + \hat{B}_i\hat{F}_j & \tilde{Q}_i & * & * \\ Q_c\hat{G}_i & 0 & \gamma I & * \\ R_c\hat{F}_i & 0 & 0 & \gamma I \end{bmatrix} \tag{6.85}$$

where $\hat{F}_{i,l}$ is the l-row of the matrix $\hat{F}_i$, $Q_{lq}^o = \gamma S_{lq}^o$ $\hat{Q}_i = \gamma\hat{S}_i, \tilde{Q}_i = \gamma\tilde{S}_i$, $\forall i \in \overline{1,n_\eta}$.

Based on Theorem 6.4, we propose the following quasi-min-max MPC algorithm subject to parameter bounded variation

Algorithm 6.1 Improved quasi-min-max MPC subject to parameter bounded variation

1: Set $k \leftarrow 0$
2: Measure and/or estimate $x(k)$ and $p(k)$.
3: Solve the optimization problem (6.76), (6.77), (6.78), (6.79), (6.80),(6.81), (6.82), (6.83).
4: Feed the plant with $u^*(k)$.
5: Set $k \leftarrow k+1$ and go to step 2.

The following theorem holds.

Theorem 6.5. *Assuming feasibility at the initial condition $x(0)$. Then, Algorithm 6.1 guarantees recursive feasibility, and the closed-loop system with the MPC control law yields is asymptotically stable.*

Proof. **Feasibility Proof:** The basic idea of the feasibility proof is to show that the optimal solution at time k is a feasible solution at time $k+1$.

As condition (6.77) is for the performance, one needs only to check conditions (6.78), (6.79), (6.80), (6.82), (6.83).

Using the proof of Theorem 1, one has that conditions (6.78), (6.79) do not contain the parameters η, and are for the upper bound of the cost $J_2^\infty(k)$. Using (6.74), it is clear that if $J_2^\infty(k) \leq V(k+2)$, then $J_2^\infty(k+1) \leq V(k+3)$. Hence conditions (6.78), (6.79) are satisfied at time $k+1$.

Using (6.82), one has $|u_l(k+2)| \leq u_{l,max}, \forall l = \overline{1,L}$ Hence, condition (6.80) is satisfied. It remains to show (6.82), (6.83), which are for the constraint admissibility. As (6.82), (6.83) do not contain the parameters $p(k)$, hence if it holds at time k, then it holds at time $k+1$.

Stability Proof: Consider the following parameter-dependent Lyapunov candidate function

$$\begin{aligned}\phi(k) = x(k)^T Q x(k) + u^*(k)^T R u^*(k) \\ + x(k+1)^T S^*(k+1)^{-1} x(k+1)\end{aligned} \tag{6.86}$$

where $u^*(k), S_i^*, \forall i = \overline{1,L}$ are the outputs of Algorithm 1 at time k. Note that $\phi(k) > 0, \forall x(k) \neq 0$. Note also that in Algorithm 1, if $x(k) = 0$, then $u(k) = 0$ is the solution. In this case $\phi(k) = 0$.

As $u*(k+1)$ and $S^*(k+1)$ is a feasible solution at time $k+1$, one obtains

$$\begin{aligned}&x(k+1)^T Q x(k+1) + u(k+1)^T R u(k+1) \\ &+ x(k+2)^T S(k+2)^{-1} x(k+2) \\ &\leq x(k+1) S^*(k+1)^{-1} x(k+1)\end{aligned} \tag{6.87}$$

Using (6.86), (6.87), one gets

$$\begin{aligned}\phi(k) \geq x(k)^T Q x(k) + u^*(k)^T R u^*(k) \\ + x(k+1)^T Q x(k+1) + u(k+1)^T R u(k+1) \\ + x(k+2)^T S(k+2)^{-1} x(k+2)\end{aligned} \tag{6.88}$$

At time instant $k+1$, one has

$$\begin{aligned}\phi(k+1) &= x(k+1)^T Q x(k+1) + u^*(k+1)^T R u^*(k+1) \\ &\quad + x(k+2)^T S^*(k+2)^{-1} x(k+2) \\ &\leq x(k+1)^T Q x(k+1) + u(k+1)^T R u(k+1) \\ &\quad + x(k+2)^T S(k+2)^{-1} x(k+2)\end{aligned} \tag{6.89}$$

Combining (6.88), (6.89), one obtains

$$\phi(k) \geq \phi(k+1) + x(k)^T Q x(k) + u^*(k)^T R u^*(k)$$

Hence, $\phi(k)$ is a Lyapunov function for the closed-loop system with the control law $u(k) = u^*(k)$. In the other words, asymptotic stability is guaranteed. This complete the proof. □

6.3.3 Numerical Examples

This section illustrates the potential benefit of the new approaches by simulations of two examples system. We used the CVX toolbox to solve SDP optimization problems. For comparison purpose, we denote the control strategies in [6], [2], [3], as Quasi_Yan, Quasi_Lu, MPC_Garone, respectively.

6.3.3.1 Example 12

This example is taken from [6]. Consider system (6.36) with

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.0 & -1.4 \\ -1.0 & -0.8 \end{bmatrix}, A_2 = \begin{bmatrix} 1.0 & 1.4 \\ -1.0 & -0.8 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 5.9 & 2.8 \end{bmatrix}^T, B_2 = \begin{bmatrix} 3.1 & -2.8 \end{bmatrix}^T \end{aligned} \tag{6.90}$$

The input constraints are $|u(k)| \leq 1$. There are no state constraints. The weighting matrices are $Q = I, R = 0.1$. Consider the bounded rate of variation $b = 0.15$, i.e.,

$$|p_i(k+1) - p_i(k)| \leq 0.15, \forall i = 1, 2$$

For the initial condition $x(0) = [2.78\ 2]^T$.

Note that system (6.90) is open loop unstable. It can also be verified that it is not quadratically stabilizable. Thus Quasi_Lu strategy is not applicable.

The initial parameter $p_1(k)$ is chosen as 0.4, in this case, $[A(k+1), B(k+1), S(k+1)] \in$ Co$([A_1^o, B_1^o, S_1^o],\ [A_2^o, B_2^o, S_2^o])$ where $A_1^o = A(k) + bA_1 - bA_2$, $A_2^o = A(k) - bA_1 + bA_2$, B_l^o,S_l^o, $l \in \overline{1, s}$ are constructed in the same manner . For prediction $k+2$ from time instant k, $[S(k+2)] \in$ Co$(S_{11}^o, S_{12}^o, S_{21}^o, S_{22}^o,)$. where $S_{11}^o = S(k) + 2bS_1 - 2bS_2$, $S_{12}^o = S(k)$, $S_{21}^o = S(k)$, $S_{22}^o = S(k) - 2bS_1 + 2bS_2$.

Fig. 6.8 and Fig. 6.9 show the state and the input trajectories as function of time for Algorithm 6.1 (solid blue), and for [6] (solid red).

We can observe that by using Algorithm 6.1, the system converges to the origin faster compared to [6]. This is because Algorithm 6.1 makes more effective use of the information of the parameter p, specifically its bounded rate of variation b.

Finally, Fig. 6.10 shows the cost for Algorithm 6.1 (solid blue), for [6] (solid red). Fig. 6.11 shows the p_1-realization as a function of time.

6.3.4 Example 13

The second example is adapted from Problem 4 of the benchmark problems described in [70]. The system considered is a two-mass-spring setup. By applying Euler's first-order approximation for the derivative with a sampling time of 0.1 units, the discrete-time LPV system given in (6.36) is

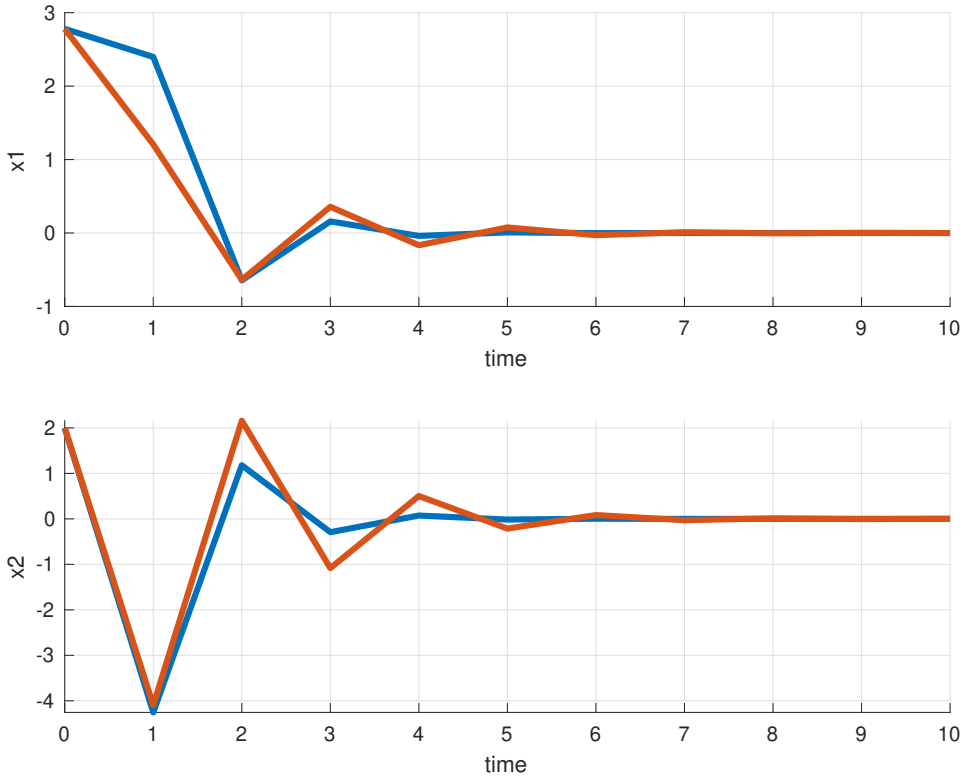


Figure 6.8: State trajectories for Algorithm 6.1 (solid blue), for Quasi_Yan [6] (solid red) for example 12.

obtained with

$$A_1 = \begin{bmatrix} 1.0 & 0 & 0.1 & 0 \\ 0 & 1.0 & 0 & 0.1 \\ -0.1 & 0.1 & 1.0 & 0 \\ 0.1 & -0.1 & 0 & 1.0 \end{bmatrix}$$

,

$$A_2 = \begin{bmatrix} 1.0 & 0 & 0.1 & 0 \\ 0 & 1.0 & 0 & 0.1 \\ -0.4 & 0.4 & 1.0 & 0 \\ 0.4 & -0.4 & 0 & 1.0 \end{bmatrix}.$$

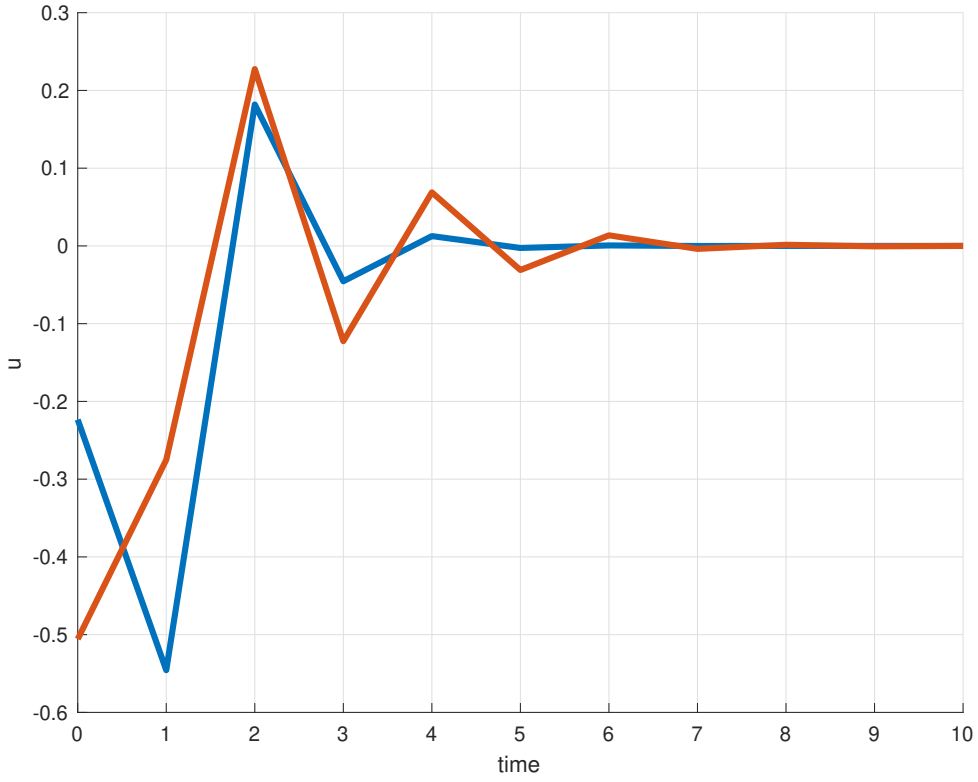


Figure 6.9: input trajectories for Algorithm 6.1 (solid blue), for Quasi_Yan [6] (solid red) for example 12.

and $B = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$ The input constraints are $|u(k)| \leq 1$. There are no state constraints. The weighting matrices are $Q = I, R = 0.1$. Consider the bounded rate of variation $b = 0.1$, i.e.,

$$|p_i(k+1) - p_i(k)| \leq 0.1, \forall i = 1, 2$$

For the initial condition $x(0) = [-2.4149 \ -2.4335 \ -0.0534 \ 0.0903]^T$, Fig. 6.12 and Fig. 6.13 shows the state and input trajectories as functions of time for Algorithm 1 (dashed red lines), For comparison purposes, the trajectories obtained from Quasi_Yan [7], MPC_Garone [3], Quasi_Lu [2] are also included. Using Fig. 6.12, Fig. 6.13 we can observe that our proposed algorithm has the best performance, and outperform Quasi_Yan [7], MPC_Garone [3], Quasi_Lu [2].

In addition, Fig. 6.14 presents the realization of the scheduling parameter p_1,.

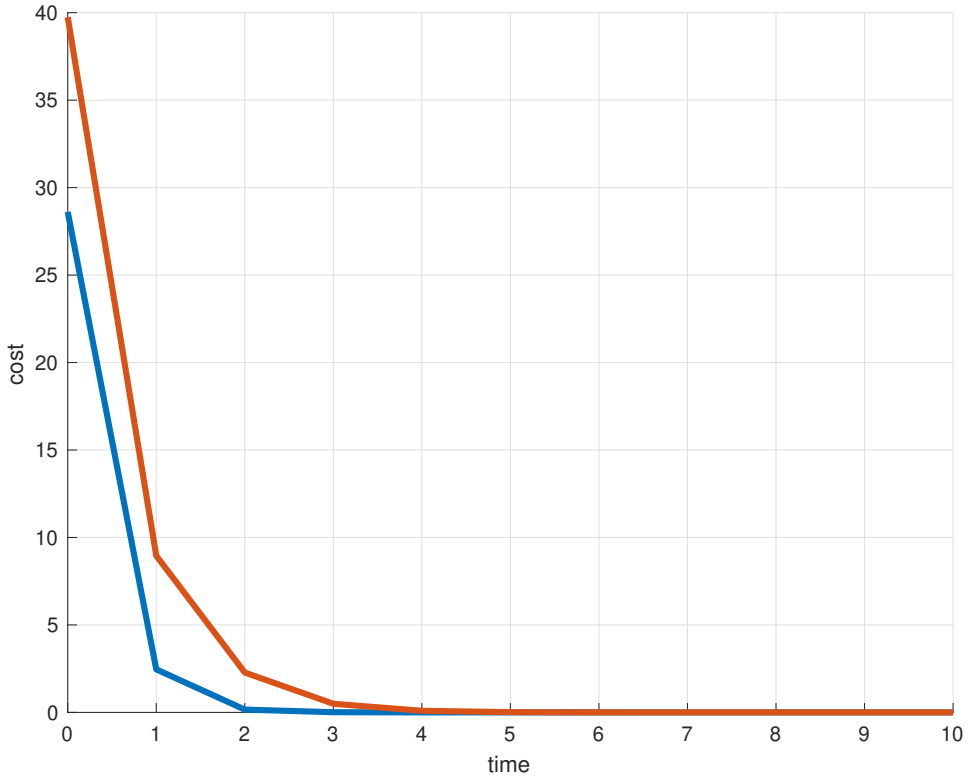


Figure 6.10: Cost for for Algorithm 6.1 (solid blue), for [6](solid red) for example 12.

6.3.5 Conclusion

In this section, we propose a new quasi-min-max MPC algorithm for discrete-time LPV systems with input and state constraints, subject to bounded parameter variations. The main idea is to exploit upper and lower bounds on the parameter variation rates to update the parameter polytope over one prediction step, which allows the second-stage cost to be decoupled from the infinite-horizon cost function. For $u(k+t), \forall t \geq 2$, a class of nonlinearly parameterized state feedback control laws is employed. The proposed algorithm guarantees recursive feasibility of the online optimization problem as well as asymptotic stability of the closed-loop system. Two numerical examples, along with comparisons to existing methods in the literature, demonstrate the effectiveness of the proposed approach.

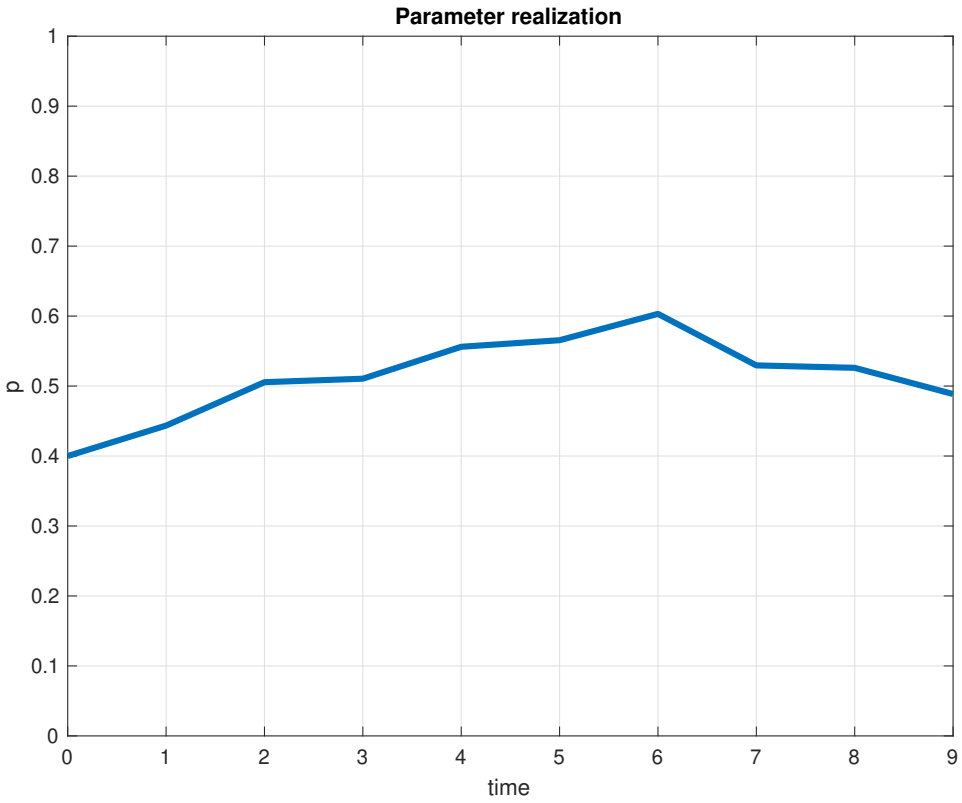


Figure 6.11: p_1-realization for example 12.

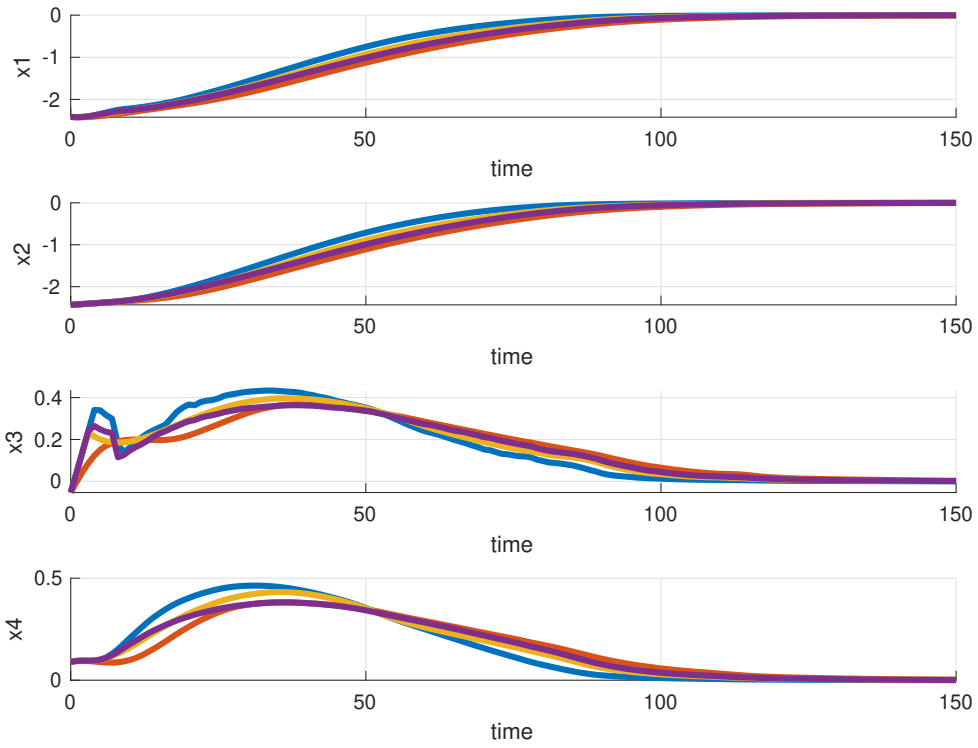


Figure 6.12: State trajectories for Algorithm 6.1 (solid blue), Quasi_Yan [7](solid yellow), MPC_Garone [3](solid red), Quasi_Lu [2] (solid purple) for example 13

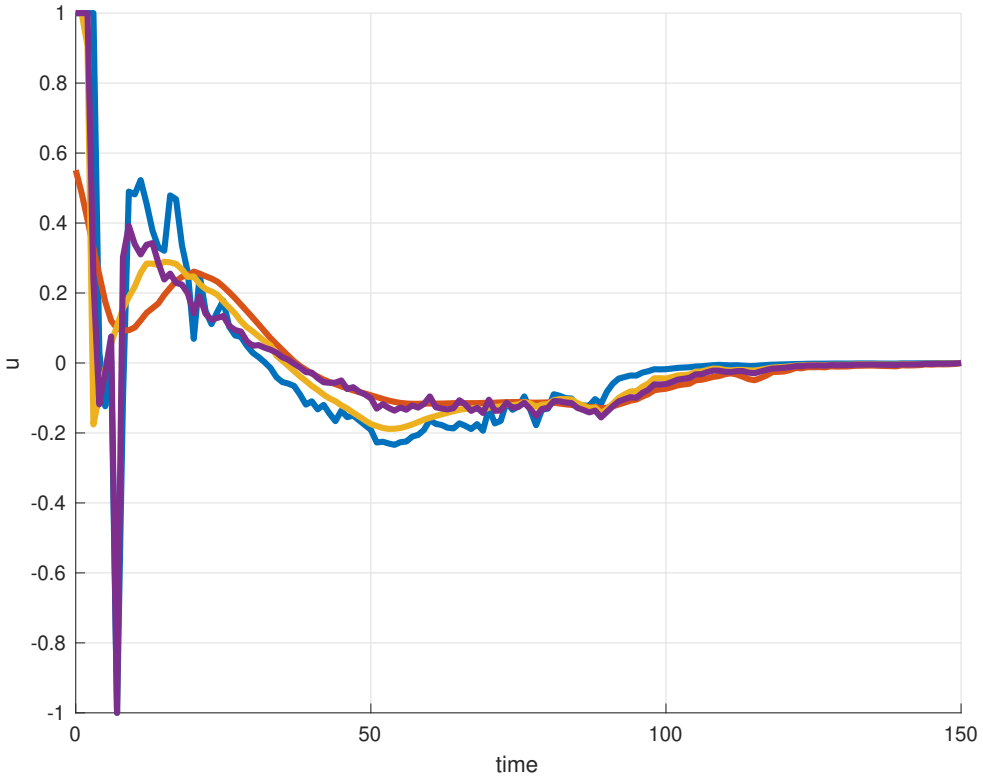


Figure 6.13: input trajectories for Algorithm 6.1 (solid blue), Quasi_Yan [7](solid yellow), MPC_Garone [3](solid red), Quasi_Lu [2] (solid purple) for example 13

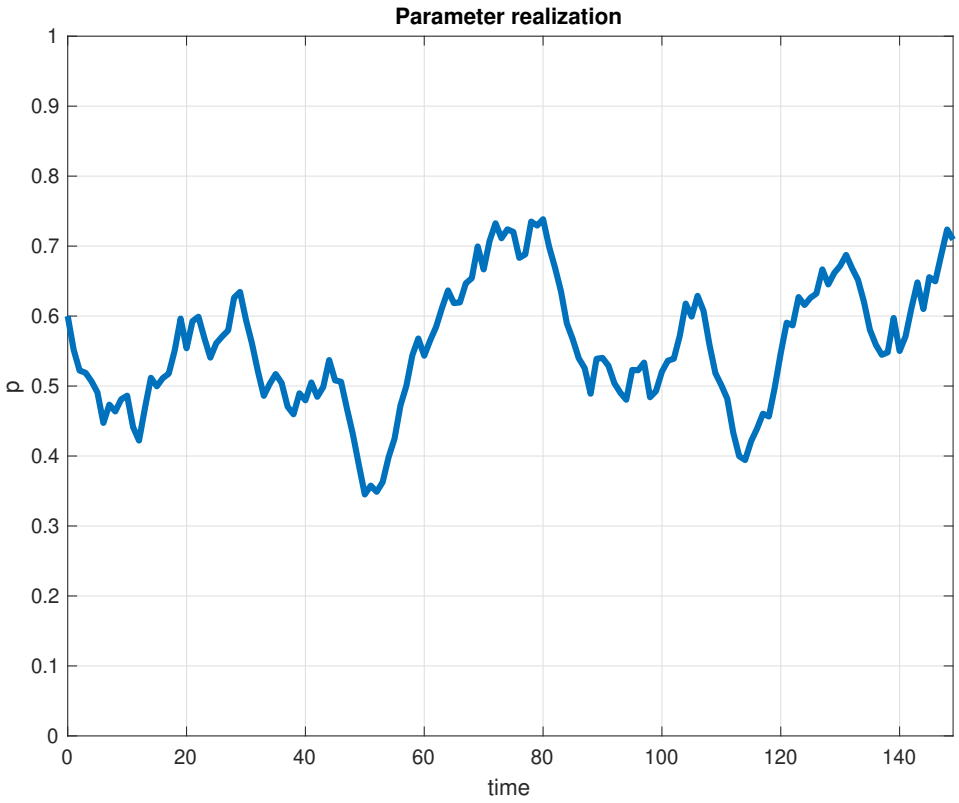


Figure 6.14: p_1-realization for example 13.

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DOCTORAL SCHOOL

Titre : Commande contrainte des systèmes linéaires à paramètres variables

Mots clés : Commande prédictive ; Ensemble invariant ; Systèmes linéaires à paramètres variables

Résumé : Cette thèse porte sur le problème de l'estimation du domaine d'attraction (DoA) et de la synthèse d'une loi de commande pour des systèmes linéaires à paramètres variables (LPV) soumis à des contraintes sur l'état et l'entrée. Plus précisément, les travaux abordent :
i) Estimation du DoA : déterminer l'ensemble des conditions initiales à partir desquelles les trajectoires du système LPV convergent vers un point d'équilibre donné tout en respectant les contraintes ;
ii) Synthèse de lois de commande : concevoir des contrôleurs stabilisants permettant d'atteindre des performances prescrites tout en garantissant le respect des contraintes malgré les variations paramétriques.
Dans la première partie, des conditions sous forme d'inégalités matricielles linéaires (LMI) sont établies afin de calculer des ensembles invariants ellipsoïdaux ainsi que des ensembles ellipsoïdaux dépendant des paramètres pour des systèmes LPV en boucle fermée sous retour d'état à gain programmé. Une contribution majeure réside dans la dérivation, pour la première fois, de conditions LMI permettant le calcul d'ellipsoïdes invariants par rapport aux paramètres pour des systèmes LPV soumis à des contraintes de saturation de l'entrée, fournissant ainsi des estimations constructives du DoA en présence de saturation.
La deuxième partie propose deux nouvelles approches de commande prédictive (MPC) pour les systèmes LPV contraints : i) un algorithme MPC quasi min–max basé sur une loi de commande par retour d'état à paramétrisation non linéaire ; ii) un cadre MPC sans double sommation basé sur une fonction de Lyapunov dépendant des paramètres. Enfin, ce dernier cadre est étendu pour intégrer un retour à gain programmé avec saturation, en tenant explicitement compte des limites des actionneurs afin d'améliorer les performances en boucle fermée.
La troisième partie traite du problème de la fusion progressive de lois de commande à faible et à fort gain dans un cadre de gain scheduling. Une stratégie systématique de fusion est développée afin de garantir que la loi de commande résultante : i) est continue ; ii) préserve les performances locales autour du point d'équilibre ; iii) assure un large domaine d'attraction.
Globalement, cette thèse propose des outils unifiés basés sur les LMI pour le calcul d'ensembles invariants, la conception de MPC à conservatisme réduit, l'intégration du gain scheduling, ainsi que la commande optimale de systèmes LPV contraints.

Title : Constrained Control of Linear Parameter-Varying Systems

Keywords : Model predictive control ; Invariant set ; LPV system

Abstract : This thesis focuses on the problem of estimating the domain of attraction (DoA) and designing a control law for linear parameter-varying (LPV) systems subject to input and state constraints. More specially, the research addresses :
i)DoA estimation :determining the set of initial conditions from which the LPV system trajectories converge to a desired equilibrium point while satisfying constraints.
ii)Control law design : synthesizing stabilizing controllers that achieve prescribed performance objectives and guarantee constraint satisfaction despite parameter variations.
In the first part, linear matrix inequality (LMI) conditions are derived to compute ellipsoidal and parameter-dependent ellipsoidal invariant sets for closed-loop LPV systems under gain-scheduled feedback. A main contribution is the derivation, for the first time, of LMI conditions for computing parameter-invariant ellipsoids for LPV systems subject to saturated input constraints, providing constructive DoA estimates under saturation.
The second part proposes two new Model Predictive Control (MPC) schemes for constrained LPV systems : i) a quasi–min–max MPC algorithm based on a nonlinearly parameterized state-feedback control law ; ii) a double-sum-free MPC framework based on a parameter-dependent Lyapunov function. Finally, the latter framework is extended to incorporate saturated gain-scheduling feedback, explicitly accounting for actuator saturation to improve the closed loop performance.
The third part addresses the problem of smoothly merging low- and high-gain gain-scheduling controllers. A systematic merging strategy is developed to guarantee that the resulting control law : i) is continuous ; ii) preserves local performance around the equilibrium point ; iii) achieves a large domain of attraction.
Overall, the thesis provides unified LMI-based tools for invariant-set computation, reduced conservatism MPC design, gain-scheduling integration, and optimal control design for constrained LPV systems.

Institut Polytechnique de Paris
91120 Palaiseau, France