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Stabilisation de systèmes décrivant le mouvement de solides rigides en interaction avec un fluide.

Stabilization of systems describing the motion of rigid bodies interacting with a fluid.

Sous la direction de : **Marius Tucsnak**

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Résumé

Cette thèse est consacrée à l'analyse et au contrôle de systèmes décrivant le mouvement de corps rigides dans un fluide. La principale difficulté mathématique réside dans la présence d'interfaces mobiles : le domaine occupé par le fluide dépend du mouvement inconnu du solide et doit donc être déterminé comme partie intégrante de la solution. Les modèles considérés se répartissent en deux classes: les systèmes avec corps flottants, dans lesquels le solide interagit avec la surface libre du fluide, et les systèmes avec corps immergés, dans lesquels le solide est entièrement entouré par le fluide.

La première partie est consacrée à un système linéarisé d'interaction fluide–structure décrivant le couplage entre un écoulement visqueux en eaux peu profondes dans un domaine non borné et le mouvement vertical d'un corps flottant. Nous montrons que ce système couplé est bien posé dans un espace de Hilbert approprié. Nous considérons ensuite le problème de contrôle linéaire–quadratique à horizon infini associé. Bien que le caractère non borné du domaine spatial exclue la stabilité exponentielle du semi-groupe non contrôlé, nous établissons le caractère bien posé du système de contrôle et obtenons une caractérisation du contrôle optimal par retour d'état de type Riccati, en nous appuyant sur la théorie LQR pour les systèmes fortement stables.

Dans la deuxième partie, nous étudions un système bidimensionnel d'interaction fluide–corps rigide dans lequel un disque rigide se déplace à l'intérieur d'une cavité circulaire remplie d'un fluide visqueux incompressible. La dynamique couplée est gouvernée par les équations de Navier–Stokes incompressibles et la deuxième loi de Newton. Nous prouvons qu'aucun contact entre le disque intérieur et la frontière extérieure ne peut se produire en temps fini, sans imposer a priori d'hypothèse de verticalité sur la trajectoire du disque intérieur. Il en résulte l'existence globale et l'unicité de solutions fortes.

La troisième partie est consacrée à un modèle unidimensionnel fluide–particule, qui constitue un système à frontière libre réduit conservant les caractéristiques essentielles des modèles d'interaction fluide–solide visqueux. Nous proposons une loi de feedback proportionnel agissant sur la particule et nous montrons que le système correspondant en boucle fermé est bien posé, globalement en temps. De plus, nous établissons que ce feedback assure la stabilisation exponentielle globale de la particule vers toute position d'équilibre prescrite à l'intérieur du domaine fluide. Un ingrédient essentiel de l'analyse est l'existence d'une borne inférieure strictement positive pour la distance entre la par-

ticule et la frontière, ce qui exclut tout contact et garantit un contrôle global en temps de l'interface mobile. Enfin, nous établissons un résultat de robustesse, au sens d'une propriété de type stabilité entrée–état (*input-to-state stability*, ISS), ce qui permet de quantifier l'effet de perturbations extérieures sur le système.

Mots-clés

Les systèmes d'interaction fluide–structure; problèmes à frontière libre; les équations de Saint-Venant; équations de Navier–Stokes; dynamique des corps rigides; contrôle linéaire–quadratique; bien posé; stabilisation exponentielle globale; stabilité entrée–état; non-contact.

Abstract

This thesis is devoted to the analysis and control of systems describing the motion of a rigid body in a viscous fluid. The main mathematical difficulty lies in the presence of moving interfaces: the fluid domain depends on the unknown motion of the rigid body and must therefore be determined as part of the solution. The models under consideration fall into two classes: floating-body systems, in which the solid interacts with the free surface of the fluid, and immersed-body systems, in which the solid is completely surrounded by the fluid.

Part I is devoted to a linearized fluid–solid interaction system describing the coupling between a viscous shallow-water flow in an unbounded domain and the vertical motion of a floating body. We prove that this coupled system is well-posed in an appropriate Hilbert space. We then consider the associated infinite-horizon linear–quadratic control problem. Even though the unbounded spatial domain rules out exponential stability of the uncontrolled semigroup, we establish the well-posedness of the control system and derive a Riccati-based feedback characterization of the optimal control by relying on LQR theory for strongly stable systems.

In Part II, we consider a two-dimensional fluid–rigid body interaction system, in which a rigid disk moves inside a circular cavity filled with a viscous incompressible fluid. The coupled dynamics are governed by the incompressible Navier–Stokes equations and Newton's second law. We prove that collision between the inner disk and the outer boundary cannot occur in finite time, without imposing any apriori verticality assumption on the trajectory of the inner disk. As a consequence, we obtain the global existence and uniqueness of strong solutions.

Part III is devoted to a one-dimensional fluid–particle model, which serves as a reduced free-boundary system retaining the essential features of viscous fluid–solid interaction models. We design a proportional feedback law acting on the particle and prove the global well-posedness of the corresponding closed-loop system. Moreover, we show that the closed-loop system achieves global exponential stabilization of the particle to any prescribed equilibrium position in the interior of the fluid domain. A key ingredient is the existence of a uniform positive lower bound on the distance between the particle and the boundary, which excludes collision and ensures global-in-time control of the moving interface. We further establish a robustness result, in the sense of a type of

input-to-state stability (ISS), thereby quantifying the effect of external disturbances on the system.

Keywords

fluid–rigid-body interaction; free-boundary problems; viscous shallow-water equations; Navier–Stokes equations; rigid-body dynamics; linear–quadratic control; Well-posedness; global exponential stabilization; input-to-state stability; non-collision.

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Publications

1. M. Tucsnak and Z. Xu, *LQR Control for a System Describing the Interaction Between a Floating Solid and the Surrounding Fluid*, *Mathematical Control and Related Fields* **14** (2024), no. 4, pp. 1477–1500 (24 pp.).

2. M. Tucsnak and Z. Xu, *Global Exponential Stabilization for a Simplified Fluid–Particle Interaction System*, *Journal of Differential Equations* **442** (2025), 16 pp.

3. Z. Xu, *Input-to-State Type Stability for a Simplified Fluid–Particle Interaction System*, *IMA Journal of Mathematical Control and Information*, accepted (to appear), 29 pp.

4. Z. Xu, *Global Well-Posedness of Fluid–Solid Interaction System*, preprint, HAL: `hal-05484267`, 33 pp.

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Chapter 1 General Introduction

This chapter is devoted to the scientific context and the mathematical motivations of the thesis.

Fluid–solid interaction systems play a central role in a wide range of problems arising in engineering, physics, and mathematics. Among the many configurations studied in the literature, two are of particular relevance here. The first is the floating-body configuration, where the solid interacts with the free surface of the fluid. The second is the immersed-body configuration, where the solid is entirely surrounded by the fluid.

The mathematical study of such systems leads to coupled PDE–ODE models. On the fluid side, the dynamics are governed by evolution equations, such as the Navier–Stokes equations for viscous incompressible flows or the Saint–Venant equations in shallow-water regimes. On the solid side, the motion of the rigid body is described by Newton's second law. One of the main analytical difficulties is that the fluid region is itself unknown: it depends on the position of the solid and evolves in time together with it. The resulting problem is therefore a free-boundary problem involving a strong coupling between fluid and solid dynamics.

The purpose of this thesis is to investigate several classes of fluid–solid interaction models with an emphasis on well-posedness, optimal control, stabilization and robustness. More specifically, we first establish the well-posedness results for coupled systems in moving domains. We then design feedback and optimal-control mechanisms that yield quantitative stability and robustness properties for the interaction systems.

The chapter concludes with a concise overview of the main contributions of the thesis, which are presented in detail in the subsequent chapters.

Part I. *LQR control for a system describing the interaction between a floating solid and the surrounding fluid*

The paper corresponding to this part of the thesis was published in:

LQR control for a system describing the interaction between a floating solid and the surrounding fluid
Z. Xu and M. Tucsnak
Mathematical Control and Related Fields.
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In this part, we consider a fluid–solid interaction system describing the motion of a rigid body floating on the free surface of a viscous fluid and undergoing purely vertical displacements. The body remains partially submerged and never becomes fully immersed in the fluid, see Figure 1.1.

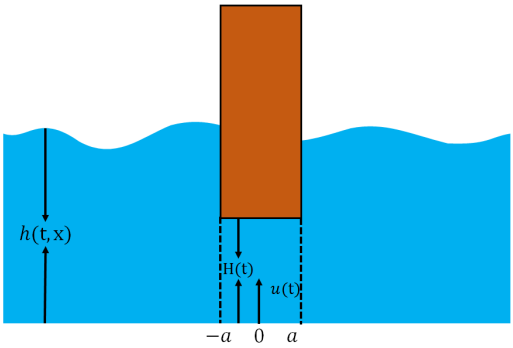


Figure 1.1: A floating body interacting with free-surface waves.

The fluid is assumed to be incompressible and to occupy an unbounded horizontal domain. The flow is described by a viscous version of the Saint–Venant (shallow-water) equations. The rigid body has vertical side walls and is constrained to move only in the vertical direction; in particular, its horizontal footprint on the flat bottom remains fixed in time. This setting corresponds to an open-sea configuration rather than a bounded container.

The coupled system can be viewed as a PDE–ODE system: the fluid is governed in the exterior and interior regions by equations coupled through transmission conditions at the contact points $x = \pm a$, while the motion of the rigid body is governed by Newton's law under the action of the resultant force exerted by the fluid.

The mathematical study of floating-body problems goes back to the classical works of John [Joh49, Joh50], who considered an inviscid fluid and adopted simplifying assumptions, notably by neglecting nonlinear terms and certain contributions related to the wetted surface. Subsequently, motivated in part by the development of numerical methods and engineering applications, several works incorporated nonlinear effects (see, e.g., [GSS89, HS09]). While these models are more accurate, they also raise delicate issues concerning the rigorous derivation and closure of the rigid-body translation

equation.

To address these modeling difficulties, Lannes [Lan17a] proposed a unified formulation of the floating-body problem based on a standard water-wave system. This approach provides a systematic way to couple the fluid dynamics with the body motion. Lannes also developed further analytical results and higher-dimensional extensions; see, for instance, [Lan17b, Lan13, Lan20].

The above contributions are formulated in the inviscid setting. In contrast, Maity, Takahashi, Martín and Tucsnak [MSMTT19] introduced a viscous model for a floating body in the shallow-water regime. Using a Hamiltonian-type approach to derive the coupled equations, they established a well-posedness theory for the resulting PDE–ODE system.

In this part, we focus on the optimal control of the linearized floating-body model introduced in [MSMTT19], but we work in an *unbounded* horizontal domain, that is, the fluid extends over the whole real line, as is natural for open-sea configurations. The aim of this part is to formulate the control problem and to design a feedback law that ensures a suitable compromise between the attenuation of the body motion and the control effort. A key motivation for introducing optimal control is that, in practical wave-energy or motion-regulation settings, one seeks a systematic trade-off between reducing the body oscillations (or shaping a desired output) and limiting the actuation effort. This viewpoint leads naturally to an infinite-horizon linear–quadratic regulator (LQR) formulation, where a quadratic cost encodes performance objectives and control energy.

The main objectives of this part are twofold. First, we prove the well-posedness of the linearized floating-body system, namely, the existence and uniqueness of strong solutions in the appropriate functional framework. Second, we study the corresponding infinite-horizon linear–quadratic optimal control problem, and show that it admits a unique optimal control, which can be expressed in feedback form. A distinctive feature of this problem is that the fluid domain is unbounded, so that the analysis falls outside the classical exponentially stabilizable LQR framework.

Part II. *Global well-posedness for a fluid–solid interaction system*

The contributions in this part of the thesis are contained in the preprint:

> *Global well-posedness of fluid-solid interaction system*
> Z. Xu
> https://hal.science/hal-05484267

In many engineering and physical applications, the rigid body is not floating on the free surface, but is instead fully immersed and moves within the surrounding fluid (see Figure 1.2), where the blue region represents the fluid and the brown part represents the inner solid. The coupling is bidirectional: the fluid motion exerts forces on the rigid body and thereby affects its dynamics, while the body motion, in turn, modifies

the surrounding flow. For this reason, such models are referred to as fluid–solid (fluid–structure) interaction systems.

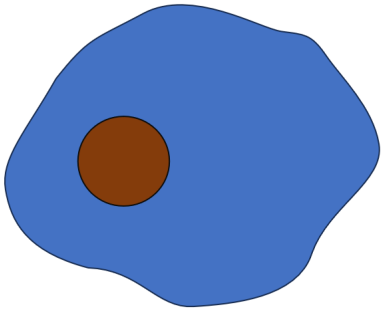
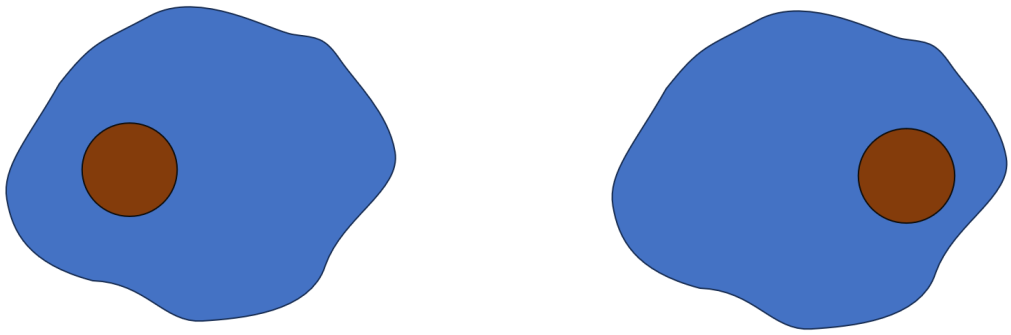


Figure 1.2: A fluid–solid interaction system

From a mathematical viewpoint, such immersed-body configurations lead to coupled PDE–ODE models: the fluid dynamics is governed by incompressible fluid equations (typically the Euler or the Navier–Stokes equations), whereas the rigid-body motion follows Newton's second law. The coupling is enforced through interface conditions on the fluid–solid boundary, expressing that the fluid velocity coincides with the solid velocity at the interface. Since the rigid body moves, the fluid domain depends on time; see Figure 1.3. In particular, the geometry of the fluid region evolves with the body motion, and the resulting coupled problem naturally takes the form of a moving-boundary (free-boundary) system.



(a) $t = 0$ (b) $t = t_0 \neq 0$

Figure 1.3: Time evolution of the fluid domain induced by the rigid-body motion.

The analysis of fluid–solid interaction systems has developed along several comple-

mentary directions. A first line of research is devoted to well-posedness issues, including existence, uniqueness and continuation of solutions, both in weak and in strong solution frameworks. A second line concerns control questions: one introduces additional inputs (or controls) into the coupled dynamics in order to steer the system toward a desired behavior, for instance stabilization or controllability.

Across these different questions, a common and fundamental difficulty is the well-posedness of the coupled interaction system. In bounded domains, the maximal existence time is often characterized by a continuation criterion: as long as the rigid body does not collide with the boundary, the solution can be extended in time. This viewpoint is emphasized, for example, in the works of Glass and Sueur [GS15] and Takahashi [Tak03], where the global well-posedness of this interaction system is linked to the exclusion of boundary contact. In other words, ruling out collision in finite time is a key step toward global well-posedness.

The collision problem has been studied from several viewpoints in the literature. On the one hand, in no-slip viscous settings one may observe "no-collision" mechanisms driven by smooth effects: for instance, Hillairet [Hil07] proved that, in two dimensions, if a rigid disk moves in the upper half-plane in $\mathbb{R}^2$, then the disk cannot reach the container boundary $y = 0$ in any finite time, and then Hillairet–Takahashi [HT09] obtained an analogous no-contact result for a rigid ball above a plane in three dimensions. Sabbagh [Sab19] proved that, for a system of several rigid disks moving in the whole space $\mathbb{R}^2$, the disks never come into contact with one another in finite time. On the other hand, collisions may occur and can be accompanied by singular behavior of strong solutions: see Hillairet–Takahashi [HT10] for finite-time contact together with blow-up in a viscous cavity problem under symmetry assumptions. Moreover, the possibility of contact crucially depends on boundary conditions: Gérard-Varet–Hillairet–Wang [GVHW15] show that introducing slip may circumvent the classical no-collision paradox and Hillairet–Takahashi [HT21] prove finite-time contact for the model with Tresca-type boundary conditions.

For the disk-in-disk geometry, Hesla [Hes04] introduced a fruitful variational viewpoint on the contact problem. The guiding principle is that, if one can represent the physical gap—more precisely, an "effective" or "equivalent" distance—between the rigid body and the fluid boundary by an explicit scalar quantity, then one can build a *special test function* by minimizing an appropriate energy functional among admissible fields. Thanks to the specific structure of the minimizer, the viscous dissipation term can be related—up to controlled error terms—to the time variation of this distance. Using this minimizer as a test function in the weak formulation of the coupled system yields a differential inequality that governs the evolution of the minimal distance between the inner disk and the outer boundary.

Within this approach, Hesla proved a no-collision result for purely vertical motions of the inner disk. The key ingredient is a specially constructed test function yielding an estimate on the distance between the inner disk and the outer boundary. However, the

case of *general trajectories* in the disk-in-disk setting remains open.

This thesis addresses this problem by studying the motion of an inner rigid disk without imposing additional geometric restrictions (i.e., without a priori assumptions on the trajectory). We prove that any local strong solution issued from admissible initial data remains collision-free for all finite times. The main difficulty lies not in constructing a test function in the spirit of [Hes04], but in deriving suitable estimates for it when no restriction is imposed on the trajectory of the inner disk.

Part III. *One-dimensional simplified fluid-particle interaction system*

The work presented in this part is published in the following papers:

> *Global Exponential Stabilization for a Simplified Fluid-Particle Interaction System*
> Z. Xu and M. Tucsnak
> Journal of Differential Equations
> https://doi.org/10.1016/j.jde.2025.113490

> *Input-to-state type Stability for Simplified Fluid-Particle Interaction System*
> Z. Xu
> IMA Journal of Mathematical Control and Information
> https://doi.org/10.1093/imamci/dnag004

In Part II, we have already introduced the general higher-dimensional fluid–solid interaction models, where an immersed rigid body moves inside a viscous incompressible fluid and the coupled dynamics is posed on a time-dependent domain. We now explain why, for the purpose of studying control and robustness, it is natural to start from a simplified one-dimensional model.

In genuinely higher-dimensional settings, viscous fluid–solid interaction leads to coupled PDE–ODE systems of Navier–Stokes type with moving boundaries. The analytical and geometric complexity of these models makes it difficult to investigate fine control objectives in a fully rigorous way directly at this level. For this reason, we first consider a reduced one-dimensional fluid–particle model. Although the fluid is not governed by the Navier–Stokes equations and the solid is reduced to a point mass, the model retains several essential mechanisms that also appear in higher-dimensional viscous FSI: two-way coupling between the fluid and the solid, a moving (free) boundary, and the possibility of boundary contact, which may obstruct global continuation of solutions. Studying stabilization and robustness in this reduced framework helps us identify the core mechanisms and provides a stepping stone toward the analysis of more realistic higher-dimensional coupled systems.

This toy model can be described as follows. A point particle moves in a bounded viscous fluid. The fluid dynamics is governed by the viscous Burgers equation, while

the particle motion is described by Newton's second law. The coupling is imposed through a kinematic condition: the particle velocity coincides with the fluid velocity evaluated at the particle position, see Figure 1.4. Here, we take the fluid domain to be $(-1, h(t)) \cup (h(t), 1)$, and the particle position is $h(t)$.

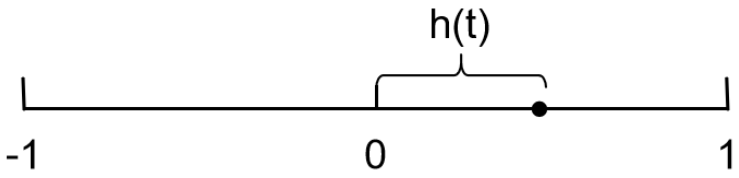


Figure 1.4: A one-dimensional fluid–particle configuration.

The one-dimensional model was first introduced by [VZ03], [VZ06], where the existence, uniqueness and large-time behavior of the solutions were first investigated, in the case of a fluid filling the entire space. Subsequently, Doubova and Fernandez-Cara [DFC05] addressed the boundary controllability problem for this system, proving local null controllability for the nonlinear system when control acts on both boundaries. Then, specifically regarding the establishment of null controllability when control is applied at only one end of the domain, Liu, Takahashi, and Tucsnak [LTT13] solved this problem by employing spectral methods and introducing the so-called source point method. In 2015, Cîndea, Micu, Roventa and Tucsnak [CMRT15] considered a model in which control is applied directly to the particle. Their analysis focused on a closed-loop system with initial data close to equilibrium, and they examined the long-time behavior of the solutions. Using a combination of spectral and fixed-point techniques, they established local null controllability for the nonlinear interaction system.

Within this framework, we address two practical questions that are central in applications. First, can a simple feedback action steer the particle to a prescribed target position and ensure that it remains away from the boundary for all times? Second, how sensitive is this feedback-stabilized behavior to external perturbations, such as modeling errors, environmental disturbances, or actuation imperfections? Our first paper provides a global stabilization result: the coupled dynamics is well defined for all times and the particle converges to the desired location while staying uniformly separated from the boundary. Our second paper focuses on robustness: we quantify how disturbances affect the long-time behavior and prove an input-to-state-type stability estimate. A major difficulty is that the energy estimates yield only a coarse bound on the particle trajectory, namely, that the particle stays in a bounded region. Such information is too weak for our purposes, since both well-posedness and stability with respect to perturbations require more precise information on the particle position.

The remainder of the thesis is organized as follows. Chapter 2 gives an overview of the main contributions and highlights the principal mathematical difficulties arising

in each part. Chapter 3 is devoted to the LQR problem for the floating-body model. Chapter 4 establishes the global well-posedness of a two-dimensional fluid–solid interaction system. Chapters 5 and 6 deal with the one-dimensional fluid–particle model, focusing respectively on global stabilization and robustness. Finally, Chapter 7 gathers the background material from functional analysis and control theory used throughout the thesis.

Chapter 2 Overview of the Main Contributions

This chapter provides a concise overview of the mathematical problems introduced in Chapter 1 and of the main results obtained in this thesis. We present the three models under consideration, highlight the main analytical difficulties and summarize the corresponding contributions.

The chapter is organized around the following three problems:

- Part I studies an infinite-horizon linear–quadratic regulator problem for a floating body interacting with a viscous shallow-water fluid;
- Part II establishes the global-in-time strong well-posedness of a two-dimensional fluid-solid interaction system through a no-collision result;
- Part III addresses the global stabilization and robustness of a one-dimensional simplified fluid–particle model.

Detailed notation and background material on Bochner spaces, semigroup theory, infinite-horizon LQR problems for strongly stable systems, and input-to-state stability for linear systems are collected in Chapter 7, and nonlinear ISS notions used in Part III are recalled later in Section 6.2.

2.1 Part I: LQR control for a floating body in a viscous shallow-water fluid

In the first part of this thesis, we investigate a linearized fluid–rigid body interaction model in the shallow-water regime. The system describes the vertical motion of a floating rigid body coupled with a viscous flow in an unbounded one-dimensional horizontal domain.

More precisely, the rigid body is assumed to have vertical sidewalls and to undergo purely vertical motions; see Figure 1.1. In particular, its projection onto the flat bottom

remains fixed in time. We denote by

$$\mathcal{I} = [-a, a], \qquad a > 0,$$

the projection of the body onto the bottom, and we set

$$\mathcal{E} = \mathbb{R} \setminus [-a, a].$$

Moreover, $t \geq 0$ denotes time, $x \in \mathbb{R}$ is the horizontal spatial variable, and $\mu > 0$ is the viscosity coefficient.

The model under consideration is the following:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad (t > 0,\ x \in \mathbb{R}), \tag{2.1.1}$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0 \qquad (t > 0,\ x \in \mathbb{R}), \tag{2.1.2}$$

$$H(t) = h(t, x) \qquad (t > 0,\ x \in \mathcal{I}), \tag{2.1.3}$$

$$p(t, x) = 0 \qquad (t > 0,\ x \in \mathcal{E}), \tag{2.1.4}$$

$$\begin{aligned} &h\left(t, -a^-\right) - \mu \frac{\partial q}{\partial x}(t, -a^-) \\ &\qquad = p\left(t, -a^+\right) + H(t) - \mu \frac{\partial q}{\partial x}\left(t, -a^+\right) \qquad (t > 0), \end{aligned} \tag{2.1.5}$$

$$\begin{aligned} &h\left(t, a^+\right) - \mu \frac{\partial q}{\partial x}\left(t, a^+\right) \\ &\qquad = p\left(t, a^-\right) + H(t) - \mu \frac{\partial q}{\partial x}(t, a^-) \qquad (t > 0), \end{aligned} \tag{2.1.6}$$

$$\ddot{H}(t) = \int_{-a}^{a} p(t, x)\, \mathrm{d}x + u(t) \qquad (t > 0), \tag{2.1.7}$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \qquad (t > 0,\ x \in \mathcal{I}), \tag{2.1.8}$$

$$q(t, -a^-) = q\left(t, -a^+\right) = q(t, -a) \qquad (t > 0), \tag{2.1.9}$$

$$q\left(t, a^+\right) = q(t, a^-) = q(t, a) \qquad (t > 0), \tag{2.1.10}$$

$$\lim_{|x| \to \infty} q(t, x) = 0 \qquad (t > 0), \tag{2.1.11}$$

$$H(0) = H_0, \qquad \dot{H}(0) = G_0, \tag{2.1.12}$$

$$h(0, x) = h_0(x) \qquad (x \in \mathcal{E}), \tag{2.1.13}$$

$$q(0, x) = q_0(x) \qquad (x \in \mathbb{R}). \tag{2.1.14}$$

Here

$$f(a^+) = \lim_{\substack{x \to a \\ x > a}} f(x), \qquad f(a^-) = \lim_{\substack{x \to a \\ x < a}} f(x).$$

The unknowns of the system are:

- $q(t,x)$, the horizontal fluid flux;
- $h(t,x)$, the free-surface elevation;
- $H(t)$, the distance between the rigid body and the bottom of the fluid domain;
- $p(t,x)$, the pressure underneath the rigid body,

and $u(t)$ denotes the external control input applied to the rigid body.

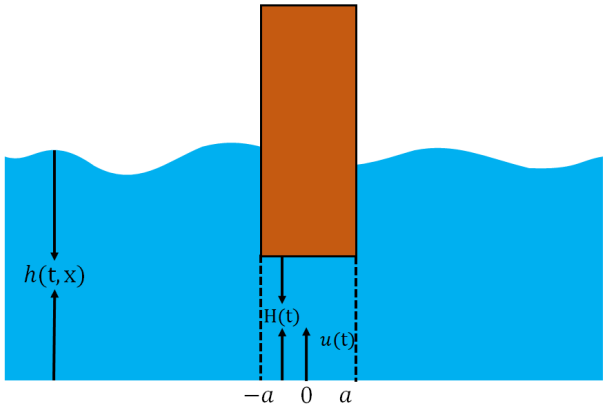


Figure 2.1: A floating body interacting with free-surface waves.

The first objective of Part I is to establish the well-posedness of this coupled PDE–ODE system.

Theorem 2.1.1 (Well-posedness of the floating-body system)**.** *Let $T > 0$ and $u \in L^2(0,T)$. For all initial data*

$$(H_0, G_0, h_0, q_0) \in \mathbb{R}^2 \times H^1(\mathcal{E}) \times H^1(\mathbb{R})$$

satisfying the compatibility condition

$$q_0(x) = -G_0 x + \frac{q_0(a) + q_0(-a)}{2}, \qquad x \in \mathcal{I},$$

there exists a unique strong solution

$$(H, h, q, p)$$

to system (2.1.1)–(2.1.14) such that

$$H \in H^2(0,T), \qquad h \in H^1\big((0,T); H^1(\mathcal{E})\big), \tag{2.1.15}$$

$$p \in L^2\big((0,T); \mathcal{P}_2(\mathcal{I})\big), \tag{2.1.16}$$

$$q \in H^1\big((0,T); L^2(\mathbb{R})\big) \cap C\big([0,T]; H^1(\mathbb{R})\big), \tag{2.1.17}$$

$$q|_{\mathcal{E}} \in L^2\big((0,T); H^2(\mathcal{E})\big), \qquad q|_{\mathcal{I}} \in H^1\big((0,T); \mathcal{P}_1(\mathcal{I})\big). \tag{2.1.18}$$

Here $\mathcal{P}_1(\mathcal{I})$ and $\mathcal{P}_2(\mathcal{I})$ denote the spaces of polynomials of degree at most 1 and 2 on $\mathcal{I}$, respectively, while the definition of the Bochner spaces is given in Subsection 7.2. Moreover, the solution depends continuously on the initial data and on the control input u.

We next consider the infinite-horizon linear–quadratic regulator problem associated with system (2.1.1)–(2.1.14). For given admissible initial data as in Theorem 2.1.1 and a control $u \in L^2(0,\infty)$, we define

$$J(u) = \int_0^\infty \Big(|u(t)|^2 + |\dot{H}(t)|^2 \Big)\, \mathrm{d}t, \qquad u \in L^2(0,\infty). \tag{2.1.19}$$

The control objective is to minimize J over all controls $u \in L^2(0,\infty)$.

Our second main result establishes the existence of an optimal control and shows that it admits a feedback representation.

Theorem 2.1.2 (Existence of an optimal feedback law)**.** *For every admissible initial data that satisfies the assumptions of Theorem 2.1.1, there exists a unique optimal control $\overline{u} \in L^2(0,\infty)$ minimizing J. Moreover, this control can be written in feedback form.*

We now briefly explain why the analysis carried out in this part is not standard. The main novelty is that the fluid domain is unbounded. This setting is natural from the physical viewpoint, since it corresponds to the motion of a floating body in an open-sea configuration, but it also changes the mathematical nature of the problem. In most of the existing literature on viscous shallow-water fluid–structure interaction, the fluid is confined to a bounded domain. By contrast, the unbounded geometry considered here leads to a substantially different asymptotic behavior and requires new arguments. Moreover, while earlier works mainly addressed well-posedness and stability issues, the present analysis also includes an infinite-horizon optimal control problem and a feedback characterization of the corresponding minimizer.

Several difficulties arise from this setting. First, the unboundedness of the fluid domain destroys the exponential stability properties that are typically available in bounded geometries. As a consequence, the standard theory of infinite-dimensional LQR problems recalled in Section 7.5 cannot be applied directly. Secondly, the system couples a partial differential equation for the fluid with an ordinary differential equation for the rigid body, and this interaction must be handled in a functional framework that is suitable both for well-posedness and for control. Finally, the lack of exponential stabilizability requires a more delicate analysis of open-loop dynamics.

The strategy adopted in this part is as follows. We first rewrite the coupled model as an abstract evolution equation on a suitable Hilbert space. This operator-theoretic formulation allows us to analyze the resolvent equation, derive resolvent estimates, and, using the maximal L^2-regularity theory for analytic semigroups recalled in Section 7.4, establish the well-posedness of the system in a suitable regularity class. We then exploit

the natural energy structure of the model to construct a suitable feedback law. We prove that the corresponding closed-loop system is strongly stable and output stable. This makes it possible to apply an LQR framework tailored to strongly stable systems and thereby to obtain the existence, uniqueness and feedback representation of the optimal control.

To summarize, the contribution of Part I is twofold. On the one hand, it provides a well-posedness and stability analysis for a fluid–solid interaction model in an unbounded domain. On the other hand, it shows that an infinite-horizon LQR problem can still be solved in this setting, despite the failure of the classical exponential-stability framework. The results of this part therefore illustrate how the difficulties created by the unbounded geometry can be overcome by combining semigroup methods, energy estimates, and an optimal control theory tailored to strongly stable systems.

2.2 Part II: A two-dimensional disk-in-disk fluid –solid interaction system

In Part II, we study the motion of a rigid disk immersed in a two-dimensional viscous incompressible fluid. The fluid occupies the region between a fixed circular container and an inner rigid disk, see Figure 2.2. The fluid dynamics is governed by the Navier–Stokes equations, while the motion of the rigid body is described by the balance equations for linear and angular momentum (Newton's laws), with no-slip on the fluid–solid interface. More precisely, we consider the following system:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad x \in \Omega(t),\ t \in [0,T], \tag{2.2.1}$$

$$\operatorname{div} u = 0, \qquad x \in \Omega(t),\ t \in [0,T], \tag{2.2.2}$$

$$u = 0, \qquad x \in \partial B(0,R)\ , t \in [0,T], \tag{2.2.3}$$

$$u = \dot h(t) + \omega(t)(x-h(t))^{\perp}, \qquad x \in \partial S(t),\ t \in [0,T], \tag{2.2.4}$$

$$M\ddot h(t) = -\int_{\partial S(t)} \sigma n\, d\Gamma, \qquad t \in [0,T], \tag{2.2.5}$$

$$J\dot\omega(t) = -\int_{\partial S(t)} (x-h(t))^{\perp}\cdot \sigma n\, d\Gamma, \quad t \in [0,T], \tag{2.2.6}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega(0), \tag{2.2.7}$$

$$h(0) = h_0, \quad \dot h(0) = g_0 \in \mathbb{R}^2, \quad \omega(0) = \omega_0 \in \mathbb{R}. \tag{2.2.8}$$

Here

$$\Omega(t) = B(0,R) \setminus S(t),$$

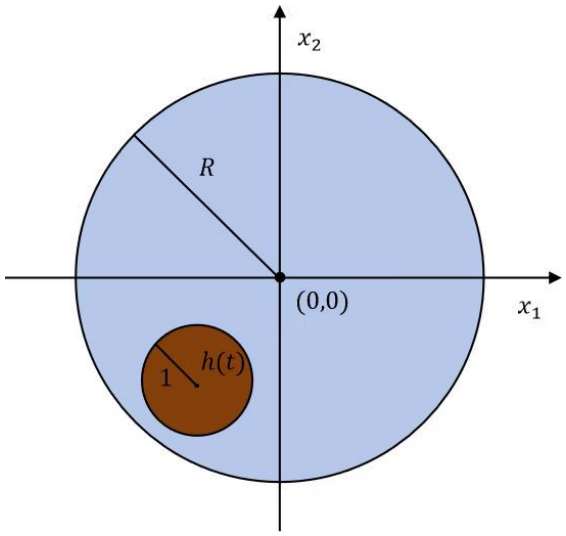


Figure 2.2: A two-dimensional disk-in-disk fluid–solid interaction system

where $B(0,R)$ denotes the disk of radius $R>1$ centered at the origin, and $S(t)$ denotes the unit disk centered at $h(t)$. The unknowns are $u(x,t)=\begin{pmatrix}u_1(x,t)\\u_2(x,t)\end{pmatrix}$ (the velocity field of the fluid), $p(x,t)$ (the pressure of the fluid), $h(t)=\begin{pmatrix}h_1(t)\\h_2(t)\end{pmatrix}$ (the position of the mass center of the rigid body) and $\omega(t)$ (the angular velocity of the rigid body), see Figure 2.2.

The constants M and J are the mass and the moment of inertia of the inner rigid body, respectively.

For all $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2$, we denote by $x^\perp$ the vector

$$x^\perp=\begin{pmatrix}x_2\\-x_1\end{pmatrix}.$$

We denote by $\dot{w}$ and $\ddot{w}$ the first and second derivatives of a function w depending only on time t.

If $x,y\in\mathbb{R}^2$, then $x\cdot y$ stands for the Euclidean inner product of x and y, and $|x|$ stands for the corresponding Euclidean norm.

Moreover, we denote by $\partial S(t)$ the boundary of the inner rigid body at time t, and by $n(x,t)$ the unit normal to $\partial S(t)$ at the point x, directed toward the interior of the rigid body. The positive constant ν denotes the viscosity of the fluid. The Cauchy stress tensor is defined by

$$\sigma(x,t)=-p(x,t)I_d+2\nu D(u), \tag{2.2.9}$$

where I_d is the identity matrix in $\mathbb{R}^2$ and

$$D(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^\top\right) \tag{2.2.10}$$

is the symmetric part of the velocity gradient. Equivalently, for all $i, j \in \{1, 2\}$,

$$D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In fact, by the continuation criterion in [Tak03], the global-in-time well-posedness of strong solutions is equivalent to excluding collision: a local strong solution can be continued as long as $d(t) := \operatorname{dist}(S(t), \partial B(0, R)) > 0$. Since the fluid domain depends on time, stating a complete global well-posedness result requires introducing function spaces on moving domains, together with the corresponding notation. To keep the present overview concise, we focus here on the no-collision property, and we postpone the detailed well-posedness framework to Chapter 4, Section 4.2.

Theorem 2.2.1 (No collision in finite time)**.** *Assume that the initial data* $(u_0, h_0, g_0, \omega_0)$ *satisfy*

$$\begin{cases} u_0 \in H^1(\Omega(0)), \quad g_0 \in \mathbb{R}^2, \quad \omega_0 \in \mathbb{R}, \\ \operatorname{div} u_0 = 0 \quad \text{in } \Omega(0), \\ u_0 = 0 \quad \text{on } \partial B(0, R), \\ u_0(x) = g_0 + \omega_0 (x - h_0)^\perp \quad \text{on } \partial S(0), \\ d(0) = \operatorname{dist}(S(0), \partial B(0, R)) > 0. \end{cases}$$

Let (u, p, h, ω) *be the corresponding maximal local strong solution of system* (2.2.1)-(2.2.8) *in the sense of [Tak03, Theorem 2.2], defined on* $[0, T)$*. Then for all* $T > 0$

$$d(t) := \operatorname{dist}(S(t), \partial B(0, R)) > 0, \qquad t \in [0, T).$$

In particular, by the continuation criterion of [Tak03, Theorem 2.2], the above local strong solution extends globally in time.

Theorem 2.2.1 extends the non-collision analysis for the disk-in-disk fluid–solid interaction system beyond the special class of purely vertical motions considered in [Hes04]. The main novelty of this part is that any local strong solution issued from admissible initial data remains collision-free for all times. The main difficulty comes from the fact that, in the general case, the thin fluid region between the moving disk and the outer boundary no longer enjoys the simple structure available in the vertical-motion setting studied by Hesla. In that work, the assumption that the inner disk moves only vertically plays a decisive role. It allows one to introduce a test function defined through a stationary Stokes problem and to extract, from the viscous dissipation term, a crucial inequality governing the distance between the inner disk and the boundary of the container. This special geometry also permits an explicit construction of the test function

without generating additional complicated terms, and moreover the analysis of the test function reduces to controlling its L^2 and L^∞ norms.

By contrast, when the inner disk is no longer restricted to a prescribed trajectory, the situation becomes much more delicate. The motion of the disk modifies the orientation of the boundary condition imposed on the solid. As a result, if one attempts to adapt Hesla's argument directly, the explicit form of the test function becomes considerably more complicated and one is naturally led to establish further estimates, such as L^∞ bounds for its gradient and other related quantities. These new terms and estimates are a direct consequence of the fact that the motion is no longer restricted to a fixed vertical direction. More details can be found in [Hes04, Part III].

The strategy adopted in this part is inspired by the ideas developed in [Sab19] and [Hil07]. A central point in these works is that the test function is not introduced through a stationary Stokes problem, but rather by means of an explicit construction depending directly on the geometry of the fluid domain and of the rigid body. This approach avoids the need to derive and analyze an explicit solution of an auxiliary Stokes system, while preserving the possibility of obtaining the fundamental inequality for the distance between the inner disk and the outer boundary. Its main advantage is that each term arising in the variational formulation can be computed and estimated in a direct and transparent manner.

A further source of inspiration is the change of variables introduced in [Sab19]. The essential point is that this transformation preserves the structure of the interaction between the dissipative term and the test function. Following this idea, we introduce a time-dependent rotating frame in order to fix the relevant boundary direction. In the transformed coordinates, the geometry of the narrow gap becomes closer in spirit to the one encountered in the vertical-motion setting. This makes it possible to construct a divergence-free test function, in the spirit of [Hil07] and [Sab19], adapted to the singular geometry near contact.

Once this test function has been transported back to the original variables, we derive the estimates required to control the distance between the two boundaries. This leads to the exclusion of collision in finite time and, combined with the continuation criterion, yields the global well-posedness of the system.

2.3 Part III: A one-dimensional simplified fluid–particle model

The third part of the thesis concerns a one-dimensional simplified fluid–particle interaction model. The fluid velocity is governed by the viscous Burgers equation, while the particle motion is described by Newton's second law. Since the particle position is itself unknown, the problem has a free-boundary character.

2.3.1 Global stabilization

The background and main questions of the one-dimensional simplified fluid–particle model were introduced in Chapter 1, Part III. We now recall the system under consideration.

$$\begin{aligned}
&\dot{v}(t,y) + v(t,y)v_y(t,y) - v_{yy}(t,y) = 0 \\
&\qquad t \in (0,T), \quad y \in (-1,1), \quad y \neq h(t), && && (2.3.1)\\
&v(t,-1) = v(t,1) = 0 && t \in (0,T), && (2.3.2)\\
&\dot{h}(t) = v(t,h(t)) && t \in (0,T), && (2.3.3)\\
&\ddot{h}(t) = [v_y](t,h(t)) + u(t) && t \in (0,T), && (2.3.4)\\
&v(0,y) = v_0(y) && y \in (-1,1), && (2.3.5)\\
&h(0) = h_0, \qquad \dot{h}(0) = g_0. && && (2.3.6)
\end{aligned}$$

where

$$[v_y](t,h(t)) = v_y\left(t,h(t)^+\right) - v_y\left(t,h(t)^-\right) \qquad (t \geq 0). \tag{2.3.7}$$

Here $v = v(t,y)$ denotes the fluid velocity, governed by the viscous Burgers equation, while $h(t)$ is the particle position and $u(t)$ is the scalar input acting on the particle.

We observe that the system defined by (2.3.1)- (2.3.6) constitutes a free boundary value problem since the position $h(t)$ of the particle, thus the domain filled by the fluid, is one of the unknowns of the problem. In this respect, the system (2.3.1)- (2.3.6) is similar to Stefan-type problems, see Koga and Krstic [KK22] for the control theoretic properties of the corresponding systems. However, in our case, the presence of a second derivative of h in (2.3.4), which is the main difference with respect to Stefan type systems (where a single time derivative appears), leads to more regular state trajectories. The presence of a free boundary necessitates a precise definition of the concept of a finite-energy solution. This notion was introduced in [CMRT15, Definition 1.1].

Definition 2.3.1. Given $T > 0$, $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, $h_0 \in (-1,1)$ and $u \in L^2(0,T)$, we say that

$$\begin{aligned}
\begin{bmatrix} v \\ g \\ h \end{bmatrix} \in &\left(C([0,T];L^2(-1,1)) \cap L^2((0,T);H_0^1(-1,1))\right) \\
&\times L^2(0,T) \times H^1(0,T).
\end{aligned} \tag{2.3.8}$$

is a finite-energy solution of system (2.3.1)-(2.3.6) on $[0,T]$ if $h(0) = h_0$, $\dot{h}(t) = g(t) =$

$v\,(t,h(t))$, $h(t)\in(-1,1)$ for almost every $t\in[0,T]$ and

$$\begin{aligned}&\int_{-1}^{1} v(t,y)\psi(t,y)\,\mathrm{d}y-\int_{-1}^{1} v_0(y)\psi(0,y)\,\mathrm{d}y+g(t)l(t)-g_0l(0)\\&-\int_0^t\int_{-1}^{1} v(\sigma,y)\dot{\psi}(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma+\int_0^t\int_{-1}^{1} v_y(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma\\&-\int_0^t g(\sigma)\dot{l}(\sigma)\,\mathrm{d}\sigma-\frac{1}{2}\int_0^t\int_{-1}^{1} v^2(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma\\=&\int_0^t u(\sigma)l(\sigma)\,\mathrm{d}\sigma,\end{aligned}\tag{2.3.9}$$

for every $t\in[0,T]$ and for every

$$\begin{bmatrix}\psi\\ l\end{bmatrix}\in\Big(H^1\big((0,T);L^2(-1,1)\big)\cap L^2\big((0,T);H_0^1(-1,1)\big)\Big)\times H^1(0,T),\tag{2.3.10}$$

$$l(t)=\psi(t,h(t)),\qquad t\in[0,T].\tag{2.3.11}$$

A triple $\begin{bmatrix}v\\ g\\ h\end{bmatrix}$ which is a finite-energy solution on $[0,T]$ for every $T>0$ is called a global finite-energy solution.

Note that the test functions used above depend on the solution and more precisely on its component h. Here and in the rest of this paper, for each $m\in\mathbb{N}$, H^m and H_0^m denote the classical Sobolev spaces and H^{-m} stands for the dual of H_0^m with respect to the pivot space L^2.

One of the contributions of our work is that we prove the existence and uniqueness of a global finite-energy solution of (2.3.1)-(2.3.6) when u is given by a feedback law of the form

$$u(t)=\mathcal{K}(h_1-h(t))\qquad(\mathcal{K}\geq 0,\qquad t>0),\tag{2.3.12}$$

with a given $h_1\in(-1,1)$ and $\mathcal{K}\geqslant 0$. This feedback may be regarded as a proportional (P) controller, as is often used in control engineering. Another interpretation is that the particle is actuated by a force which is generated by a spring (with elastic constant $\mathcal{K}$ and fixed at h_1). Unlike in [CMRT15], in Theorem 2.3.2 below we do not assume that the initial velocity field v_0 and the initial data g_0 and h_1-h_0 are small, in a suitable sense.

The first main result of this part is:

Theorem 2.3.2. *Let $v_0\in L^2(-1,1)$, $g_0\in\mathbb{R}$, $h_0,\ h_1\in(-1,1)$ and $\mathcal{K}\geqslant 0$. Then system (2.3.1)-(2.3.6), with u given by the feedback law (2.3.12), admits a unique global finite-energy solution. Moreover, there exists $\alpha>0$ depending on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1 and*

$\mathcal{K}$, such that the trajectory h of the particle satisfies

$$-1+\alpha \le h(t) \le 1-\alpha \qquad (t \geqslant 0). \tag{2.3.13}$$

Finally, for every $T>0$, the map defined in

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto \begin{bmatrix} v \\ g \\ h \end{bmatrix}, \tag{2.3.14}$$

is continuous from $L^2(-1,1)\times\mathbb{R}\times(-1,1)$ to $[C([0,T];L^2(-1,1))\cap L^2((0,T);H^1_0(-1,1))]\times L^2(0,T)\times H^1(0,T)$.

Remark 2.3.3. The fact that $h(t)\in(-1,1)$ for every $t\geqslant 0$ is consistent with earlier results obtained in a related setting for smoother initial data; see [VZ06]. The novelty of Theorem 2.3.2 is the uniform lower bound on the distance from $h(t)$ to the boundary, expressed solely in terms of the initial data and the parameters of the problem.

Our second main result describes the large time behavior of the solutions obtained in Theorem 2.3.2.

Theorem 2.3.4. *With the notation and assumptions of Theorem 2.3.2 we have:*

- *If $\mathcal{K}=0$ the finite-energy solutions of (2.3.1)-(2.3.6) satisfy*

$$\begin{aligned} &\|v(t,\cdot)\|^2_{L^2(-1,1)} + g^2(t) \\ &\le \exp\left(-\frac{1}{4}t\right)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2\right) \qquad (t>0). \end{aligned} \tag{2.3.15}$$

  Moreover, there exists $h^*\in(-1,1)$ such that

$$|h(t)-h^*|^2 \le \exp\left(-\frac{1}{4}t\right)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2\right). \tag{2.3.16}$$

- *If $\mathcal{K}>0$ there exists a constant $\eta_2>0$, depending on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1 and $\mathcal{K}$, such that*

$$\begin{aligned} &\|v(t,\cdot)\|^2_{L^2(-1,1)} + g^2(t) + \mathcal{K}(h(t)-h_1)^2 \\ &\le 16\exp\left(-\eta_2 t\right)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_0-h_1)^2\right) \qquad (t>0). \end{aligned} \tag{2.3.17}$$

The aforementioned theorems show, in particular, that when the feedback control (2.3.12) is active (that is $\mathcal{K}>0$) the position of the particle tends to the prescribed "target" h_1, whereas when $\mathcal{K}=0$ it tends to an "undetermined" position h^*. By "undetermined" we mean that we are unable to describe h^* in terms of the initial data.

The analysis faces two main difficulties. First, the low regularity of finite-energy solutions prevents a direct use of the arguments developed in [VZ03, VZ06], which rely on smoother trajectories. Second, since neither the initial data nor the feedback gain is assumed to be small, the exclusion of collisions with the boundary cannot be obtained through a perturbative argument, unlike in [CMRT15]. To overcome this difficulty, we exploit a special test function introduced by Hesla [Hes04]. The same idea also plays a central role in the construction of a perturbed Lyapunov functional yielding the decay estimates in Theorem 5.1.3.

An important novelty in Theorem 2.3.4 is that, unlike in [CMRT15, Section 6], where it has been shown that $\lim_{t\to\infty} h(t) = h_1$, we obtain an explicit decay estimate for $h_1 - h(t)$. An important ingredient in the proof of Theorem 2.3.4 is again the special test function mentioned above, which is used to construct a perturbed Lyapunov function. We will give the proof of the above two theorems in Chapter 5.

2.3.2 Robustness and ISS-type estimates for the simplified system

In the previous subsection, we introduced a proportional (P) state-feedback law acting on the particle, of the form

$$u(t) = \mathcal{K}(h_1 - h(t)), \tag{2.3.18}$$

where $h_1 \in (-1, 1)$ is a prescribed target position and $\mathcal{K} > 0$ is the feedback gain.

In this subsection, the proportional feedback law $\mathcal{K}(h_1 - h(t))$ is incorporated into the particle equation, while the symbol $u = u(t)$ denotes an additional external disturbance. Our goal is to quantify the effect of this disturbance on the closed-loop behavior. More precisely, we consider the following system:

$$\begin{aligned}
&\dot{v}(t,y) + v(t,y)v_y(t,y) - v_{yy}(t,y) = 0 \\
&\qquad t \in (0,T), \quad y \in (-1,1), \quad y \neq h(t), && && (2.3.19)\\
&v(t,-1) = v(t,1) = 0 && t \in (0,T), && (2.3.20)\\
&\dot{h}(t) = v(t,h(t)) && t \in (0,T), && (2.3.21)\\
&\ddot{h}(t) = [v_y](t,h(t)) + \mathcal{K}(h_1 - h(t)) + u(t) && t \in (0,T), && (2.3.22)\\
&v(0,y) = v_0(y) && y \in (-1,1), && (2.3.23)\\
&h(0) = h_0, \qquad \dot{h}(0) = g_0. && && (2.3.24)
\end{aligned}$$

where $\mathcal{K} \geq 0$ is a given constant and

$$[v_y]\,(t,h(t)) = v_y\left(t,h(t)^+\right) - v_y\left(t,h(t)^-\right) \qquad (t>0), \tag{2.3.25}$$

with

$$v_y(t,h(t)^+) := \lim_{\substack{y\to h(t)\\ y>h(t)}} v_y(t,y),$$

$$v_y(t,h(t)^-) := \lim_{\substack{y\to h(t)\\ y<h(t)}} v_y(t,y).$$

Here, $v = v(t,y)$ denotes the velocity field of a viscous fluid, governed by the Burgers equation. The motion of the immersed particle is described by Newton's second law, see (2.3.22). The particle position is denoted by $h(t)$ and its velocity by $\dot{h}(t)$; both depend only on time. We fix a target point $h_1 \in (-1,1)$.

System (2.3.19)–(2.3.24) constitutes a free boundary value problem because the particle's position $h(t)$, which influences the domain of the fluid, is an unknown of the system. This fact leads us to adopt the notion of a finite-energy solution, as defined in [CMRT15, Definition 1.1]:

Definition 2.3.5. Given $T > 0$, $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, h_0, $h_1 \in (-1,1)$, $\mathcal{K} \geq 0$ and $u \in L^2(0,T)$, we say that

$$\begin{bmatrix} v \\ g \\ h \end{bmatrix} \in \Big(C([0,T];L^2(-1,1)) \cap L^2((0,T);H_0^1(-1,1))\Big) \times L^2(0,T) \times H^1(0,T). \tag{2.3.26}$$

is a finite-energy solution of system (2.3.19)–(2.3.24) on $[0,T]$ if $h(0) = h_0 \in (-1,1)$, $\dot{h}(t) = g(t) = v\,(t,h(t))$, and $h(t) \in (-1,1)$ for almost every $t \in [0,T)$, and if for all $t \in [0,T)$ the following equation holds:

$$\begin{aligned} &\int_{-1}^{1} v(t,y)\psi(t,y)\,\mathrm{d}y - \int_{-1}^{1} v_0(y)\psi(0,y)\,\mathrm{d}y + g(t)l(t) - g_0 l(0) \\ &- \int_0^t \int_{-1}^{1} v(\sigma,y)\dot{\psi}(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma + \int_0^t \int_{-1}^{1} v_y(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ &- \int_0^t g(\sigma)\dot{l}(\sigma)\,\mathrm{d}\sigma - \frac{1}{2}\int_0^t \int_{-1}^{1} v^2(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ =& \int_0^t u(\sigma)l(\sigma)\,\mathrm{d}\sigma + \int_0^t \mathcal{K}\,(h_1 - h(\sigma))\,l(\sigma)\,\mathrm{d}\sigma, \end{aligned} \tag{2.3.27}$$

for every test pair $\begin{pmatrix} \psi \\ l \end{pmatrix}$ that satisfies the condition (2.3.10)-(2.3.11). A triple $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ which is a finite-energy solution on $[0,T]$ for every $T > 0$ is called a global finite-energy solution.

The first theorem establishes the global existence and uniqueness of finite-energy solutions for system (2.3.19)-(2.3.24):

Theorem 2.3.6. *Let $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, h_0, $h_1 \in (-1,1)$, $\mathcal{K} \geq 0$ and $u \in L^2(0,\infty)$, then system (2.3.19)–(2.3.24) admits a unique global finite-energy solution as defined above. Moreover, for every $t \geq 0$ the following energy equality holds:*

$$\begin{aligned}&\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y - \int_{-1}^{1} v_0^2(y)\,\mathrm{d}y + g^2(t) - g_0^2 + \mathcal{K}\left(h_1 - h(t)\right)^2 \\ &- \mathcal{K}\left(h_1 - h_0\right)^2 + 2\int_0^t \int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma = \int_0^t u(\sigma)g(\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{2.3.28}$$

The second result implies the boundedness of the particle trajectory under additional assumptions on the input u.

Theorem 2.3.7. *With the same initial conditions as in Theorem 2.3.6, if the disturbance satisfies $u \in L^2 \cap L^1(0,\infty)$. Then system (2.3.19)–(2.3.24) admits a unique global finite-energy solution. Moreover, there exists $\alpha > 0$, depending on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1, $\mathcal{K}$, $\|u\|_{L^2(0,\infty)}$, and $\|u\|_{L^1(0,\infty)}$, such that*

$$-1 + \alpha \leq h(t) \leq 1 - \alpha. \tag{2.3.29}$$

Remark 2.3.8. The global well-posedness and the energy identity (2.3.28) of system (2.3.19)-(2.3.24) hold for any input $u \in L^2(0,\infty)$. The uniform gap estimate (2.3.29) requires, in addition, $u \in L^1(0,\infty)$ to control the cumulative impulse acting on the particle; without this, inputs in $L^2(0,\infty)\setminus L^1(0,\infty)$ can still generate unbounded impulse and may drive $h(t)$ toward the boundary. This split—L^2 for energy, L^1 for impulse—is natural from a physical viewpoint and is easy to enforce in applications where the external input is integrable.

Background on ISS is provided later in Section 6.2 for nonlinear systems and in Section 7.6 for linear systems in the semigroup framework. We now state the main stability result for (2.3.19)-(2.3.24):

Theorem 2.3.9. *With the notation and assumptions of Theorem 2.3.7, system (2.3.19)–(2.3.24) exhibits the following behavior:*

- *If $\mathcal{K} = 0$, we have*

$$\begin{aligned}&\|v(t,\cdot)\|_{L^2(-1,1)}^2 + g^2(t) \\ \leq& 16\exp\left(-\frac{1}{4}t\right)\left(\|v_0\|_{L^2(-1,1)}^2 + g_0^2\right) + 16\|u\|_{L^2(0,\infty)}^2 \qquad (t>0). \end{aligned} \tag{2.3.30}$$

- *If $\mathcal{K} > 0$, then there exist constants $\eta > 0$ depending on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1, $\|u\|_{L^2(0,\infty)}$, $\|u\|_{L^1(0,\infty)}$, and $\mathcal{K}$, such that*

$$\begin{aligned}&\|v(t,\cdot)\|_{L^2(-1,1)}^2 + g^2(t) + \mathcal{K}(h(t)-h_1)^2 \\ \leq& 16\exp\left(-\eta t\right)\left(\|v_0\|_{L^2(-1,1)}^2 + g_0^2 + \mathcal{K}(h_0-h_1)^2\right) + \frac{3}{2}\|u\|_{L^2(0,\infty)}^2 \qquad (t>0). \end{aligned} \tag{2.3.31}$$

Remark 2.3.10. By Theorem 1.4 of [TX25], in the case without P–control and without perturbations the particle converges to some (a priori unknown) limit position h^* inside the fluid domain. In contrast, our estimate (2.3.30) controls only the velocity: since the assumption $u \in L^1(0,\infty) \cap L^2(0,\infty)$ does not imply $u(t) \to 0$ as $t \to \infty$, the bound (2.3.30) yields no information about the particle's asymptotic position and velocity. This is why no explicit $h(t)$–term appears in (2.3.30).

Concerning robustness, the estimate (2.3.31) is *not* uniform ISS in the sense of Definition 6.2.3 which will be introduced in Section 6.2, because the decay rate η may depend on a norm of the input u. On the other hand, although (2.3.30) exhibits a uniform decay rate, it does not involve the position component of the state and therefore cannot establish standard ISS either. Finally, as illustrated in Section 6.6, numerical experiments with different inputs u exhibit different decay rates, further confirming the lack of a uniform (input-independent) rate. Both estimates show that small external disturbances produce only small effects on the fluid and particle velocities. Moreover, the numerical experiments indicate that, under P feedback and for disturbances satisfying the hypotheses of Theorem 2.3.9, we also have $h(t) \to h_1$ as $t \to \infty$. Taken together, these facts demonstrate the robustness of the feedback-controlled interaction system.

Chapter 3 LQR control for a system describing the interaction between a floating solid and the surrounding fluid

Chapter 2 provides an overview of the fluid–structure interaction problems investigated in this thesis. The present chapter is devoted to the first of these problems (see Part I of Chapter 1 and Section 2.1), namely the control of a floating rigid body interacting with a viscous shallow-water flow in an unbounded one-dimensional horizontal domain. This chapter is based on the work of Tucsnak and Xu [TX24].

3.1 Introduction

We consider a linearized coupled PDE–ODE system describing the vertical motion of a rigid body with vertical sidewalls floating on a viscous free-surface fluid. The body is assumed to move only in the vertical direction, while the fluid domain is unbounded in the horizontal variable, see Figure 3.1.

The general background of the model considered in this chapter, together with the main differences between our approach and those in the existing literature, was presented in Part I of Chapter 1 and in Section 2.1. For the convenience of the reader, we now briefly recall the model under consideration, as well as the two main results of this chapter.

We denote by

$$\mathcal{I} = [-a, a], \qquad a > 0,$$

the projection of the rigid body onto the flat bottom, and we set

$$\mathcal{E} = \mathbb{R} \setminus [-a, a].$$

Here $t \geq 0$ denotes time, $x \in \mathbb{R}$ is the horizontal spatial variable, and $\mu > 0$ is the viscosity coefficient.

The dynamics are governed by

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad (t > 0,\ x \in \mathbb{R}), \tag{3.1.1}$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0 \qquad (t > 0,\ x \in \mathbb{R}), \tag{3.1.2}$$

$$H(t) = h(t, x) \qquad (t > 0,\ x \in \mathcal{I}), \tag{3.1.3}$$

$$p(t, x) = 0 \qquad (t > 0,\ x \in \mathcal{E}), \tag{3.1.4}$$

$$\begin{aligned} &h\left(t, -a^-\right) - \mu \frac{\partial q}{\partial x}(t, -a^-) \\ &\qquad = p\left(t, -a^+\right) + H(t) - \mu \frac{\partial q}{\partial x}\left(t, -a^+\right) \qquad (t > 0), \end{aligned} \tag{3.1.5}$$

$$\begin{aligned} &h\left(t, a^+\right) - \mu \frac{\partial q}{\partial x}\left(t, a^+\right) \\ &\qquad = p\left(t, a^-\right) + H(t) - \mu \frac{\partial q}{\partial x}(t, a^-) \qquad (t > 0), \end{aligned} \tag{3.1.6}$$

$$\ddot{H}(t) = \int_{-a}^{a} p(t, x)\, \mathrm{d}x + u(t) \qquad (t > 0), \tag{3.1.7}$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \qquad (t > 0,\ x \in \mathcal{I}), \tag{3.1.8}$$

$$q(t, -a^-) = q\left(t, -a^+\right) = q(t, -a) \qquad (t > 0), \tag{3.1.9}$$

$$q\left(t, a^+\right) = q(t, a^-) = q(t, a) \qquad (t > 0), \tag{3.1.10}$$

$$\lim_{|x| \to \infty} q(t, x) = 0 \qquad (t > 0), \tag{3.1.11}$$

$$H(0) = H_0, \qquad \dot{H}(0) = G_0, \tag{3.1.12}$$

$$h(0, x) = h_0(x) \qquad (x \in \mathcal{E}), \tag{3.1.13}$$

$$q(0, x) = q_0(x) \qquad (x \in \mathbb{R}). \tag{3.1.14}$$

For any function f admitting one-sided traces at the contact points, we use the notation

$$f(a^+) = \lim_{\substack{x \to a \\ x > a}} f(x), \qquad f(a^-) = \lim_{\substack{x \to a \\ x < a}} f(x),$$

and similarly at $x = -a$.

The unknowns of the above system are as follows:

- $q(t, x)$, the horizontal fluid flux;
- $h(t, x)$, the free-surface elevation;

- $H(t)$, the distance between the rigid body and the bottom of the fluid domain;
- $p(t,x)$, the pressure underneath the rigid body,

The state variables are $q(t,x)$, $h(t,x)$, $H(t)$, $p(t,x)$, while $u(t)$ denotes the external control input applied to the rigid body.

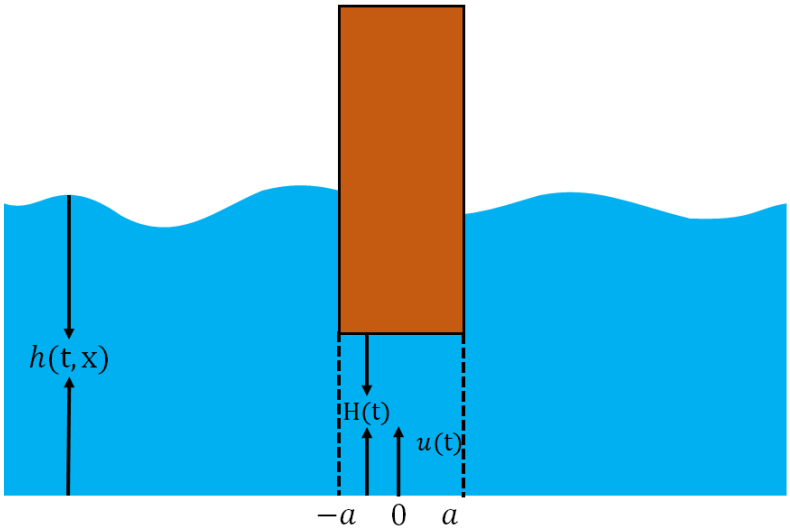


Figure 3.1: A rigid body floating in an unbounded viscous shallow-water fluid.

We say that initial data

$$(H_0, G_0, h_0, q_0) \in \mathbb{R}^2 \times H^1(\mathcal{E}) \times H^1(\mathbb{R})$$

is admissible if it satisfies the compatibility condition

$$q_0(x) = -G_0 x + \frac{q_0(a) + q_0(-a)}{2}, \qquad x \in \mathcal{I}. \tag{3.1.15}$$

The first main result of this chapter is the well-posedness of system (3.1.1)–(3.1.14).

Theorem 3.1.1 (Well-posedness). *Let $T > 0$, let $u \in L^2(0,T)$, and let*

$$(H_0, G_0, h_0, q_0) \in \mathbb{R}^2 \times H^1(\mathcal{E}) \times H^1(\mathbb{R})$$

be an admissible initial data. Then there exists a unique strong solution

$$(H, h, q, p)$$

of system (3.1.1)–(3.1.14) *such that:*

$$H \in H^2(0,T), \tag{3.1.16}$$
$$h \in H^1\big((0,T);H^1(\mathcal{E})\big), \tag{3.1.17}$$
$$p \in L^2\big((0,T);\mathcal{P}_2(\mathcal{I})\big), \tag{3.1.18}$$
$$q \in H^1\big((0,T);L^2(\mathbb{R})\big) \cap C\big([0,T];H^1(\mathbb{R})\big), \tag{3.1.19}$$
$$q|_{\mathcal{E}} \in L^2\big((0,T);H^2(\mathcal{E})\big), \tag{3.1.20}$$
$$q|_{\mathcal{I}} \in H^1\big((0,T);\mathcal{P}_1(\mathcal{I})\big). \tag{3.1.21}$$

Here $\mathcal{P}_1(\mathcal{I})$ and $\mathcal{P}_2(\mathcal{I})$ denote the spaces of polynomials of degree at most 1 and 2 on $\mathcal{I}$, respectively. Moreover, the solution depends continuously on the initial datum and on the control input u.

We next consider the infinite-horizon linear–quadratic regulator problem associated with system (3.1.1)–(3.1.14). For a given admissible initial datum and a control $u \in L^2(0,\infty)$, we define the cost functional

$$J(u) = \int_0^\infty \Big(|u(t)|^2 + |\dot{H}(t)|^2\Big)\,\mathrm{d}t. \tag{3.1.22}$$

The control objective is to minimize J over all controls $u \in L^2(0,\infty)$.

The second main result of this chapter is the following.

Theorem 3.1.2 (Existence of an optimal feedback law)**.** *For every admissible initial datum, there exists a unique control $\bar{u} \in L^2(0,\infty)$ such that*

$$J(\bar{u}) \le J(u) \qquad \text{for every } u \in L^2(0,\infty),$$

with equality if and only if $u = \bar{u}$. Moreover, $\bar{u}$ can be written in feedback form.

A more detailed version of the above result, including the construction of the feedback operator, will be given later in Chapter 3, Theorem 3.5.7.

The main difficulty comes from the unboundedness of the fluid domain. In contrast with bounded geometries, the semigroup generated by the uncontrolled system is neither exponentially stable nor exponentially stabilizable. As a consequence, the standard theory of infinite-dimensional LQR problems cannot be applied directly. Our analysis combines an operator-theoretic reformulation of the model with resolvent estimates and energy arguments. In particular, we prove that the system is strongly stable and output stabilizable, which allows us to use the framework developed by Curtain and Oostveen [CO99].

To prove Theorems 3.1.1 and 3.1.2, we proceed in several steps. Firstly, we reformulate the floating-body model as an abstract control system on a suitable Hilbert space.

We then study the resolvent of the state operator and show that it generates an analytic semigroup on this space. Using the maximal regularity theorem for analytic semigroups, we establish the well-posedness of system (3.1.1)–(3.1.14). Next, we prove that the associated semigroup is neither exponentially stable nor exponentially stabilizable. Then, the standard infinite-dimensional Riccati theory cannot be applied directly. We therefore recall in the next section an abstract LQR result for strongly stable systems. Finally, by introducing a bounded feedback control, motivated by the energy identity, acting on the solid, we show that the cost functional (3.2.4) can be optimized. We then verify that the resulting closed-loop system satisfies the assumptions of the abstract result.

The remaining part of this work is organized as follows. In Section 3.2 we provide some necessary background on LQR problems for infinite dimensional systems. In Sections 3.3 and 3.4 we give the proof of Theorem 3.1.1. Finally, Section 3.5 is devoted to the proof of Theorem 3.1.2.

3.2 A LQR criterion for strongly stable systems

In this section, we recall the abstract LQR result for strongly stable systems that will be used in the proof of Theorem 3.1.2. A more detailed introduction to the LQR problem is given in Section 7.5.

Let X, U and Y be Hilbert spaces and

$$A : \mathcal{D}(A) \subset X \to X$$

be the generator of a C_0-semigroup

$$(\mathbb{T}(t))_{t\geq 0}$$

on X, and let

$$B \in \mathcal{L}(U, X), \qquad C \in \mathcal{L}(X, Y).$$

We consider the linear control system $\Sigma(A, B, C)$, with state space X, input space U, and output space Y, given by

$$\dot{z}(t) = Az(t) + Bu(t), \qquad t > 0, \tag{3.2.1}$$
$$z(0) = z_0, \tag{3.2.2}$$
$$y(t) = Cz(t), \qquad t > 0, \tag{3.2.3}$$

where $z_0 \in X$ is arbitrary.

For a control $u \in L^2(0, \infty; U)$, the associated infinite-horizon cost functional is defined by

$$J(u; z_0) = \int_0^\infty \left(\|Cz(t)\|_Y^2 + \|u(t)\|_U^2 \right) \mathrm{d}t. \tag{3.2.4}$$

The corresponding LQR problem consists in minimizing $J(u; z_0)$ over $L^2(0,\infty;U)$.

We recall that the system $\Sigma(A,B,C)$ is said to be *optimizable* if for every $z_0 \in X$ there exists $u \in L^2(0,\infty;U)$ such that $J(u;z_0) < \infty$. Under the classical assumptions of the standard LQR theory, optimizability, together with exponential stabilizability, yields the existence of a unique optimal control and of an associated Riccati operator; see again [CZ95, Theorem 6.2.4 and Theorem 6.2.7]. Since these assumptions fail for the floating-body system, we shall use a different criterion based on strong stability.

Recall that a C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ on X is said to be *strongly stable* if

$$\lim_{t\to\infty} \|\mathbb{T}(t)x\|_X = 0 \qquad \text{for every } x \in X.$$

Definition 3.2.1. For every $t > 0$, the output map

$$\Psi_t \in \mathcal{L}(X, L^2(0,\infty;Y))$$

is defined by

$$(\Psi_t z_0)(\sigma) = \begin{cases} C\mathbb{T}(\sigma)z_0, & 0 < \sigma < t, \\ 0, & \sigma \geq t, \end{cases} \qquad z_0 \in X.$$

We say that the system (3.2.1)–(3.2.3) is *output stable* if there exists

$$\Psi \in \mathcal{L}(X, L^2(0,\infty;Y))$$

such that

$$\lim_{t\to\infty} \|\Psi_t - \Psi\|_{\mathcal{L}(X,L^2(0,\infty;Y))} = 0.$$

Remark 3.2.2. In the terminology of Tucsnak and Weiss [TW09, Section 4.6], output stability means that C is an infinite-time admissible observation operator for the semigroup $(\mathbb{T}(t))_{t\geq 0}$, and Ψ is the corresponding extended output map.

The following proposition is the main abstract result used in the proof of Theorem 3.1.2.

Proposition 3.2.3. *Assume that there exists $F \in \mathcal{L}(X,U)$ such that the operator $A+BF$ generates a strongly stable C_0-semigroup*

$$(\mathbb{T}^F(t))_{t\geq 0}$$

on X, and that the system

$$\Sigma\left(A+BF,\ 0,\ \begin{bmatrix} C \\ F \end{bmatrix}\right)$$

with state space X and output space $Y \times U$ is output stable. Then the system $\Sigma(A,B,C)$ is optimizable and, for every $z_0 \in X$, the LQR problem associated with (3.2.4) admits a unique optimal control $\bar{u} \in L^2(0,\infty;U)$.

Moreover, there exists a minimal self-adjoint nonnegative operator $P_0 \in \mathcal{L}(X)$ such that the optimal control is given by

$$\bar{u}(t) = -B^* P_0 z(t), \qquad t \geq 0,$$

where z denotes the corresponding optimal trajectory, and P_0 satisfies the algebraic Riccati equation

$$A^* P_0 z + P_0 A z - P_0 B B^* P_0 z + C^* C z = 0, \qquad z \in \mathcal{D}(A). \tag{3.2.5}$$

Finally, the minimal cost is given by

$$J(\bar{u}; z_0) = \langle P_0 z_0, z_0 \rangle_X. \tag{3.2.6}$$

Remark 3.2.4. The precise sense in which P_0 satisfies (3.2.5) is described in [BDPDM07, Part IV, Section 3.1 and Part V]. In particular, one has $P_0 z \in \mathcal{D}(A^*)$ for every $z \in \mathcal{D}(A)$.

In the forthcoming sections, we rewrite the floating-body model in the form $\Sigma(A, B, C)$ and verify the assumptions of Proposition 3.2.3. This will yield the existence of an optimal control and its feedback representation.

3.3 Operator form of the governing equations and semigroup generator

We now rewrite the floating-body model (3.1.1)-(3.1.14) as a linear control system on a Hilbert space in Section 3.2.

$$\dot{z}(t) = \mathcal{A} z(t) + \mathcal{B} u(t) \qquad (t > 0), \tag{3.3.1}$$

$$z(0) = z_0, \tag{3.3.2}$$

where $\mathcal{A}$ is the generator of an analytic semigroup on a Hilbert space $\mathcal{X}$ and $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, where $\mathcal{U}$ is another Hilbert space. To write system (3.1.1)-(3.1.14) in the form (3.3.1)-(3.3.2) we follow the procedure proposed in [MSMTT19], where the corresponding system, with the fluid confined in a bounded container, has been investigated.

We begin by setting

$$q_-(t) = q(t, -a), \qquad q_+(t) = q(t, a) \qquad (t > 0). \tag{3.3.3}$$

Next we remark that the methodology developed in [MSMTT19] to eliminate the pressure term p can be applied with no major modification to our case. In this way we obtain

the following system, equivalent to the original governing equations (3.1.1)-(3.1.14):

$$\dot{H}(t) = -\frac{q_+(t) - q_-(t)}{2a} \qquad (t > 0), \tag{3.3.4}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial x} \qquad (t > 0,\ x \in \mathcal{E}), \tag{3.3.5}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial h}{\partial x} + \mu \frac{\partial^2 q}{\partial x^2} \qquad (t > 0,\ x \in \mathcal{E}), \tag{3.3.6}$$

$$\begin{bmatrix} \dot{q}_-(t) \\ \dot{q}_+(t) \end{bmatrix} = 2aM \begin{bmatrix} u(t) \\ -u(t) \end{bmatrix} + 4a^2 M \begin{bmatrix} \mu \left(\dfrac{q_+(t) - q_-(t)}{2a} \right) + \left(h(t, -a^-) - H(t) \right) - \mu \dfrac{\partial q}{\partial x}(t, -a^-) \\ -\mu \left(\dfrac{q_+(t) - q_-(t)}{2a} \right) - \left(h(t, a^+) - H(t) \right) + \mu \dfrac{\partial q}{\partial x}(t, a^+) \end{bmatrix} \qquad (t > 0), \tag{3.3.7}$$

$$\lim_{|x| \to \infty} q(t, x) = 0 \qquad (t > 0), \tag{3.3.8}$$

$$H(0) = H_0, \tag{3.3.9}$$

$$h(0, x) = h_0(x) \qquad (x \in \mathcal{E}), \tag{3.3.10}$$

$$q(0, x) = q_0(x) \qquad (x \in \mathcal{E}), \tag{3.3.11}$$

$$q_-(0) = q_{-,0}, \qquad q_+(0) = q_{+,0}, \tag{3.3.12}$$

where the matrix M is defined by

$$M = \frac{1}{8a^3 \left(1 + \frac{2a^3}{3} \right)} \begin{bmatrix} 1 + \frac{8a^3}{3} & 1 - \frac{4a^3}{3} \\ 1 - \frac{4a^3}{3} & 1 + \frac{8a^3}{3} \end{bmatrix}. \tag{3.3.13}$$

At this stage we introduce more notation, which will be used in the remainder of this chapter. Firstly, we define the spaces

$$\mathcal{X} = \left\{ [H \ h \ q \ q_- \ q_+]^\top \in \mathbb{C} \times H^1(\mathcal{E}) \times L^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C} \right\}, \tag{3.3.14}$$

$$\mathcal{Z} = \left\{ [H \ h \ q \ q_- \ q_+]^\top \in \mathbb{C} \times H^1(\mathcal{E}) \times H^2(\mathcal{E}) \times \mathbb{C}^2 \;\middle|\; \begin{array}{r} q(-a) = q_-, \\ q(a) = q_+ \end{array} \right\}, \tag{3.3.15}$$

$$\mathcal{W} = \left\{ [H \ h \ q \ q_- \ q_+]^\top \in \mathbb{C} \times H^1(\mathcal{E}) \times H^1(\mathcal{E}) \times \mathbb{C}^2 \;\middle|\; \begin{array}{r} q(-a) = q_-, \\ q(a) = q_+ \end{array} \right\}, \tag{3.3.16}$$

$$\mathcal{U} = \mathbb{C}. \tag{3.3.17}$$

Moreover, denoting by $z = [H \ h \ q \ q_- \ q_+]^\top$ a generic element of $\mathcal{X}$, we also introduce the operators $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ and $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ defined by

$$\mathcal{D}(\mathcal{A}) = \mathcal{Z}, \tag{3.3.18}$$

$$\mathcal{A}z = \begin{bmatrix} -\dfrac{q_+ - q_-}{2a} \\ -\dfrac{\mathrm{d}q}{\mathrm{d}x} \\ -\dfrac{\mathrm{d}h}{\mathrm{d}x} + \mu\dfrac{\mathrm{d}^2 q}{\mathrm{d}x^2} \\ R_1 z \\ R_2 z \end{bmatrix}, \qquad \mathcal{B}u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{au}{(1+\frac{2a^3}{3})} \\ -\dfrac{au}{(1+\frac{2a^3}{3})} \end{bmatrix} \qquad (z \in \mathcal{D}(\mathcal{A}),\ u \in \mathcal{U}), \tag{3.3.19}$$

where

$$\begin{bmatrix} R_1 z \\ R_2 z \end{bmatrix} = 4a^2 M \begin{bmatrix} \mu\left(\dfrac{q_+ - q_-}{2a}\right) + (h(-a^-) - H) - \mu\dfrac{\mathrm{d}q}{\mathrm{d}x}(-a^-) \\ -\mu\left(\dfrac{q_+ - q_-}{2a}\right) - \Big(h(a^+) - H\Big) + \mu\dfrac{\mathrm{d}q}{\mathrm{d}x}(a^+) \end{bmatrix}. \tag{3.3.20}$$

It is easy to check that $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X})$.

With the above notation, following step by step the procedure in [MSMTT19], we have:

Proposition 3.3.1. *Let $T > 0$ and let $\begin{bmatrix} H_0 \ G_0 \ h_0 \ q_0 \end{bmatrix}^\top$ satisfy the assumptions in Theorem 3.1.1. Let $\begin{bmatrix} H \ h \ q \ p \end{bmatrix}^\top$ be a solution of (3.1.1)-(3.1.14) satisfying (3.1.16)-(3.1.21). Then $z = [H \ h \ q \ q_- \ q_+]^\top$, with q_+ and q_- defined in (3.3.3), lies in*

$$C([0,T]; \mathcal{W}) \cap L^2((0,T); \mathcal{Z}) \cap H^1((0,T); \mathcal{X}), \tag{3.3.21}$$

and it satisfies

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t) \qquad (t > 0), \tag{3.3.22}$$

$$z(0) = z_0. \tag{3.3.23}$$

Conversely, assume that z satisfies (3.3.21)-(3.3.23). Then there exists an extension of q (still denoted by q) and p such that $\begin{bmatrix} H \ h \ q \ p \end{bmatrix}^\top$ is a solution of (3.1.1)-(3.1.14) satisfying (3.1.16)-(3.1.21).

In this section and the following one we prove that operator $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$ and we give the first consequences of this result on the well-posedness

of system (3.3.4)-(3.3.12). The remaining part of this section is devoted to the study of the spectrum and of the resolvents of the operator $\mathcal{A}$ defined above.

As usual, we denote by $\sigma(\mathcal{A})$ and $\rho(\mathcal{A})$ the spectrum and the resolvent set of the operator $\mathcal{A}$, respectively. We also set

$$\mathbb{C}_- = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}, \qquad \mathbb{R}_- = \mathbb{C}_- \cap \mathbb{R}. \tag{3.3.24}$$

In order to state the main theorem of this section, we need four technical lemmas. Although based on elementary facts, the proofs of these four lemmas require a certain computational effort, so that they are postponed to the appendix in subsection 3.6.1. The first one of these lemmas is:

Lemma 3.3.2. *Assume that $\mu > 0$ and $\lambda \in \mathbb{C}\backslash\{0\}$, $\lambda \neq -\frac{1}{\mu}$. Then $\frac{\lambda^2}{1+\mu\lambda} \in \mathbb{R}_-$ if and only if*

$$\lambda \in \left(-\infty, -\frac{1}{\mu}\right) \quad \text{or} \quad \left|\lambda + \frac{1}{\mu}\right| = \frac{1}{\mu}\,. \tag{3.3.25}$$

The second auxiliary result to be proved in the Appendix is:

Lemma 3.3.3. *Let $a > 0$ and $\mu > 0$. We assume that $\lambda \in \mathbb{C}$ satisfies*

$$\lambda \notin \left(-\infty, -\frac{1}{\mu}\right] \quad \text{and} \quad \left|\lambda + \frac{1}{\mu}\right| \neq \frac{1}{\mu}\,. \tag{3.3.26}$$

Let

$$\omega_\lambda := \sqrt{\frac{\lambda^2}{1+\mu\lambda}}\,, \tag{3.3.27}$$

where $\sqrt{\cdot}$ stands for the principal determination of the square root function. (Note that this definition is correct due to Lemma 3.3.2.)

Let $\mathcal{M}_\lambda$ be defined by

$$M_\lambda = \begin{bmatrix} \alpha_\lambda & \beta_\lambda \\ \beta_\lambda & \alpha_\lambda \end{bmatrix}, \tag{3.3.28}$$

where

$$\alpha_\lambda := \lambda\left(1 + \frac{8a^3}{3}\right) + 2a\left(\mu + \frac{1}{\lambda}\right) + \frac{4a^2}{\omega_\lambda}, \tag{3.3.29}$$

$$\beta_\lambda := -\lambda\left(1 - \frac{4a^3}{3}\right) - 2a\left(\mu + \frac{1}{\lambda}\right). \tag{3.3.30}$$

Then, there exists a set $\mathcal{S} \subset \mathbb{C}_-$, having at most four elements, such that for every λ satisfying (3.3.26) and not lying in $\mathcal{S}$ we have that the matrix $\mathcal{M}_\lambda$ is invertible.

An important role in proving our resolvent estimates is played by the following result:

Lemma 3.3.4. *Let $\omega \in \mathbb{C}$, $\operatorname{Re}\omega > 0$ and $a > 0$. We define the following operators:*

$$(D_-(\omega)\gamma_-)(x) = \gamma_- \exp[\omega(a+x)] \qquad (\gamma_- \in \mathbb{C}, \quad x \le -a), \tag{3.3.31}$$

$$(D_+(\omega)\gamma_+)(x) = \gamma_+ \exp[\omega(a-x)] \qquad (\gamma_+ \in \mathbb{C}, \quad x \ge a), \tag{3.3.32}$$

$$\begin{aligned}
&(R_-(\omega)\varphi)(x)\\
&= \frac{1}{2\omega}\left(\int_x^{-a} \exp(-\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(\omega x)\\
&\quad + \frac{1}{2\omega}\left(\int_{-\infty}^{x} \exp(\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(-\omega x)\\
&\quad - \frac{1}{2\omega}\left(\int_{-\infty}^{-a} \exp(\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(2\omega a)\exp(\omega x)\\
&\qquad\qquad (\varphi \in L^2(-\infty,-a),\ x \le -a),
\end{aligned} \tag{3.3.33}$$

$$\begin{aligned}
&(R_+(\omega)\varphi)(x)\\
&= \frac{1}{2\omega}\left(\int_a^{x} \exp(\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(-\omega x)\\
&\quad + \frac{1}{2\omega}\left(\int_{x}^{\infty} \exp(-\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(\omega x)\\
&\quad - \frac{1}{2\omega}\left(\int_{a}^{\infty} \exp(-\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(2\omega a)\exp(-\omega x)\\
&\qquad\qquad (\varphi \in L^2(a,\infty),\ x \ge a).
\end{aligned} \tag{3.3.34}$$

Then we have

$$D_-(\omega) \in \mathcal{L}(\mathbb{C},\ L^2(-\infty,-a)), \qquad D_+(\omega) \in \mathcal{L}(\mathbb{C},\ L^2(a,\infty)),$$
$$R_-(\omega) \in \mathcal{L}(L^2(-\infty,-a)), \qquad R_+(\omega) \in \mathcal{L}(L^2(a,\infty)),$$

with the norm estimates

$$\|D_-(\omega)\|_{\mathcal{L}(\mathbb{C},\ L^2(-\infty,-a))} \le \left(\frac{1}{2\operatorname{Re}\omega}\right)^{\frac{1}{2}}, \quad \|D_+(\omega)\|_{\mathcal{L}(\mathbb{C},\ L^2(a,\infty))} \le \left(\frac{1}{2\operatorname{Re}\omega}\right)^{\frac{1}{2}}, \tag{3.3.35}$$

$$\|R_-(\omega)\|_{\mathcal{L}(L^2(-\infty,-a))} \le \frac{3}{2|\omega|\operatorname{Re}\omega}, \qquad \|R_+(\omega)\|_{\mathcal{L}(L^2(a,\infty))} \le \frac{3}{2|\omega|\operatorname{Re}\omega}. \tag{3.3.36}$$

Finally, the structure of the resolvent of our semigroup generator will be shown to be a consequence of the following lemma:

Lemma 3.3.5. *Let $a > 0$ and $\mu > 0$ and assume that $\lambda \in \mathbb{C}$ satisfies condition (3.3.26). Then for every $\varphi \in L^2(\mathcal{E})$ and every γ_-, $\gamma_+ \in \mathbb{C}$, there exists a unique $q_\lambda \in H^2(\mathcal{E})$ such that*

$$\begin{cases} -\dfrac{\mathrm{d}^2 q_\lambda}{\mathrm{d}x^2} + \omega_\lambda^2 q_\lambda = \varphi(x) \quad (x \in \mathcal{E}), \\ q_\lambda(-a) = \gamma_-, \quad q_\lambda(a) = \gamma_+, \end{cases} \tag{3.3.37}$$

where ω_λ is defined in (3.3.27). Moreover, we have

$$q_\lambda(x) = \begin{cases} [D_-(\omega_\lambda)\gamma_-](x) + [R_-(\omega_\lambda)\varphi](x) & (x \le -a), \\ [D_+(\omega_\lambda)\gamma_+](x) + [R_+(\omega_\lambda)\varphi](x) & (x \ge a), \end{cases} \tag{3.3.38}$$

where the operators $D_+(\omega)$, $D_-(\omega)$, $R_+(\omega)$ and $R_-(\omega)$ have been defined in Lemma 3.3.4.

The main result in this section is:

Theorem 3.3.6. *For every $a > 0$ and $\mu > 0$, there exists a set $\mathcal{S} \subset \mathbb{C}_-$, having at most four elements, such that the spectrum of the operator $\mathcal{A}$ defined in (3.3.14)-(3.3.20) satisfies*

$$\sigma(\mathcal{A}) \subset E, \tag{3.3.39}$$

where E is defined by

$$E = \{0\} \cup \mathcal{S} \cup \left(-\infty, -\frac{1}{\mu}\right] \cup \left\{\lambda \in \mathbb{C}_- \quad \middle| \quad \left|\lambda + \frac{1}{\mu}\right| = \frac{1}{\mu}\right\}.$$

Moreover, for every $\lambda \notin E$, $\mathcal{F} = [f_1 \ f_2 \ f_3 \ f_4 \ f_5]^\top \in \mathcal{X}$ we denote

$$(\lambda \mathbb{I} - \mathcal{A})^{-1} F = z_\lambda = [H_\lambda \ h_\lambda \ q_\lambda \ q_{-,\lambda} \ q_{+,\lambda}]^\top \in \mathcal{D}(\mathcal{A}). \tag{3.3.40}$$

Then we have

$$\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda} \end{bmatrix} = \mathcal{M}_\lambda^{-1} \left(M^{-1} \begin{bmatrix} f_4 \\ f_5 \end{bmatrix} + \frac{4a^2}{\lambda} \begin{bmatrix} f_2(-a^-) - f_1 \\ f_1 - f_2(a^+) \end{bmatrix} + \frac{4a^2\lambda}{\omega_\lambda^2} \begin{bmatrix} -\int_{-\infty}^{0} \exp(\omega_\lambda \xi)\varphi_\lambda(\xi - a)\,\mathrm{d}\xi \\ \int_{0}^{\infty} \exp(-\omega_\lambda \xi)\varphi_\lambda(\xi + a)\,\mathrm{d}\xi \end{bmatrix} \right). \tag{3.3.41}$$

where

$$\varphi_\lambda = \frac{\lambda}{1 + \mu\lambda} f_3(x) - \frac{1}{1 + \mu\lambda} \frac{\mathrm{d}f_2}{\mathrm{d}x} \quad (x \in \mathcal{E}),$$

and ω_λ, $\mathcal{M}_\lambda$ have been defined in (3.3.27) and (3.3.28), respectively. In addition, we have

$$q_\lambda(x) = \begin{cases} [D_-(\omega_\lambda)q_{-,\lambda}](x) + [R_-(\omega_\lambda)\varphi_\lambda](x) & (x \le -a), \\ [D_+(\omega_\lambda)q_{+,\lambda}](x) + [R_+(\omega_\lambda)\varphi_\lambda](x) & (x \ge a), \end{cases} \tag{3.3.42}$$

$$h_\lambda(x) = \frac{1}{\lambda}\left(f_2 - \frac{\mathrm{d}q_\lambda(x)}{\mathrm{d}x}\right) \quad (x \in \mathcal{E}), \tag{3.3.43}$$

$$H_\lambda = \frac{1}{\lambda}\left(f_1 - \frac{q_{+,\lambda} - q_{-,\lambda}}{2a}\right), \tag{3.3.44}$$

where the operators $D_-(\omega)$, $D_+(\omega)$, $R_-(\omega)$ and $R_+(\omega)$ have been defined in Lemma 3.3.4.

Proof. We firstly remark that the resolvent equation (3.3.40) can be written as:

$$\lambda H_\lambda + \frac{q_{+,\lambda} - q_{-,\lambda}}{2a} = f_1, \tag{3.3.45}$$

$$\lambda h_\lambda(x) + \frac{\mathrm{d}q_\lambda}{\mathrm{d}x}(x) = f_2(x) \qquad (x \in \mathcal{E}), \tag{3.3.46}$$

$$\lambda q_\lambda(x) + \frac{\mathrm{d}h_\lambda}{\mathrm{d}x}(x) - \mu \frac{\mathrm{d}^2 q_\lambda}{\mathrm{d}x^2}(x) = f_3(x) \qquad (x \in \mathcal{E}), \tag{3.3.47}$$

$$q_\lambda(-a) = q_{-,\lambda}, \quad q_\lambda(a) = q_{+,\lambda}, \tag{3.3.48}$$

$$\lambda \begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda} \end{bmatrix} - 4a^2 M \begin{bmatrix} \mu \dfrac{q_{+,\lambda} - q_{-,\lambda}}{2a} + (h_\lambda(-a^-) - H_\lambda) - \mu \dfrac{\mathrm{d}q_\lambda}{\mathrm{d}x}(-a^-) \\ -\mu \dfrac{q_{+,\lambda} - q_{-,\lambda}}{2a} - \Big(h_\lambda(a^+) - H_\lambda\Big) + \mu \dfrac{\mathrm{d}q_\lambda}{\mathrm{d}x}(a^+) \end{bmatrix} = \begin{bmatrix} f_4 \\ f_5 \end{bmatrix}, \tag{3.3.49}$$

where M is a matrix defined in (3.3.13). From equations (3.3.46)-(3.3.48), it follows that

$$\begin{cases} -\dfrac{\mathrm{d}^2 q_\lambda}{\mathrm{d}x^2}(x) + \omega_\lambda^2 q_\lambda(x) = \dfrac{\lambda}{1+\mu\lambda} f_3(x) - \dfrac{1}{1+\mu\lambda} \dfrac{\mathrm{d}f_2}{\mathrm{d}x}(x) \quad (x \in \mathcal{E}), \\ q_\lambda(-a) = q_{-,\lambda}, \quad q_\lambda(a) = q_{+,\lambda}, \end{cases} \tag{3.3.50}$$

This implies, using Lemma 3.3.5, that (3.3.42) holds true. Formulas (3.3.43) and (3.3.44) are then obtained as direct consequences of (3.3.45) and (3.3.47).

We still have to prove (3.3.41). To this aim, we use Lemma 3.3.5 in order to reduce equations (3.3.49) to a linear algebraic system of unknowns $q_{-,\lambda}$ and $q_{+,\lambda}$. We note that from the first equation in (3.3.42), it follows that for every $x \geqslant a$, we have

$$\begin{aligned} \frac{\mathrm{d}q_\lambda}{\mathrm{d}x}(x) = & -\frac{1}{2}\left(\int_a^x \exp(\omega_\lambda \xi)\varphi_\lambda(\xi)\,\mathrm{d}\xi\right)\exp(-\omega_\lambda x) \\ & + \frac{1}{2}\left(\int_x^\infty \exp(-\omega_\lambda \xi)\varphi_\lambda(\xi)\,\mathrm{d}\xi\right)\exp(\omega_\lambda x) \\ & - \omega_\lambda q_{+,\lambda}\exp(\omega_\lambda(a-x)) \\ & + \frac{1}{2}\left(\int_a^\infty \exp(-\omega_\lambda \xi)\varphi_\lambda(\xi)\,\mathrm{d}\xi\right)\exp(2\omega_\lambda a)\exp(-\omega_\lambda x). \end{aligned} \tag{3.3.51}$$

Taking the limit when $x \to a^+$ in the above equation, it follows that

$$\frac{\mathrm{d}q_\lambda}{\mathrm{d}x}(a^+) = \left(\int_a^\infty \exp(-\omega_\lambda \xi)\varphi_\lambda(\xi)\mathrm{d}\xi\right)\exp(\omega_\lambda a) - \omega_\lambda q_{+,\lambda}. \tag{3.3.52}$$

In a similar manner, we have

$$\frac{\mathrm{d}q_\lambda}{\mathrm{d}x}(-a^-) = -\left(\int_{-\infty}^{-a} \exp(\omega_\lambda \xi)\varphi_\lambda(\xi)\mathrm{d}\xi\right)\exp(\omega_\lambda a) + \omega_\lambda q_{-,\lambda}. \tag{3.3.53}$$

Inserting the above two formulas in (3.3.49) and using the fact that

$$\frac{1+\mu\lambda}{\lambda} = \frac{\lambda}{\omega_\lambda^2},$$

we obtain that

$$\begin{aligned}
&\lambda M^{-1}\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix} - 4a^2\begin{bmatrix} -\left(\frac{\mu}{2a}+\frac{\lambda}{\omega_\lambda}+\frac{1}{2a\lambda}\right)q_{-,\lambda} + \left(\frac{\mu}{2a}+\frac{1}{2a\lambda}\right)q_{+,\lambda} \\ \left(\frac{\mu}{2a}+\frac{1}{2a\lambda}\right)q_{-,\lambda} - \left(\frac{\mu}{2a}+\frac{\lambda}{\omega_\lambda}+\frac{1}{2a\lambda}\right)q_{+,\lambda}\end{bmatrix} \\
=&M^{-1}\begin{bmatrix} f_4 \\ f_5\end{bmatrix} + \frac{4a^2}{\lambda}\begin{bmatrix} f_2(-a^-)-f_1 \\ f_1 - f_2(a^+)\end{bmatrix} + \frac{4a^2\lambda}{\omega_\lambda^2}\begin{bmatrix} -\int_{-\infty}^0 \exp(\omega_\lambda\xi)\varphi_\lambda(\xi-a)\mathrm{d}\xi \\ \int_0^\infty \exp(-\omega_\lambda\xi)\varphi_\lambda(\xi+a)\mathrm{d}\xi\end{bmatrix}.
\end{aligned}$$

The above equation can be rewritten

$$\mathcal{M}_\lambda\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix} = M^{-1}\begin{bmatrix} f_4 \\ f_5\end{bmatrix} + \frac{4a^2}{\lambda}\begin{bmatrix} f_2(-a^-)-f_1 \\ f_1 - f_2(a^+)\end{bmatrix} + \frac{4a^2\lambda}{\omega_\lambda^2}\begin{bmatrix} -\int_{-\infty}^0 \exp(\omega_\lambda\xi)\varphi_\lambda(\xi-a)\mathrm{d}\xi \\ \int_0^\infty \exp(-\omega_\lambda\xi)\varphi_\lambda(\xi+a)\mathrm{d}\xi\end{bmatrix},$$

where the matrices M and $\mathcal{M}_\lambda$ have been defined in(3.3.13), (3.3.28), respectively. By combining the above formula and Lemma 3.3.3, we obtain (3.3.41) and thus the conclusion of Theorem 3.3.6. □

3.4 Semigroup generation and well-posedness

In this section we continue to use the notation introduced in the previous one, in particular for the space $\mathcal{X}$ defined in (3.3.14) and for the operator $\mathcal{A}$ defined in (3.3.18)-(3.3.20). We prove below that $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$ and we give the proof of Theorem 3.1.1.

The proof that $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$ is partially based on two technical results given below. Although based on elementary facts, the proofs of these two lemmas are provided, for completeness, in the Appendix in subsection 3.6.2. The first of these lemmas is:

Lemma 3.4.1. *Let $\mu > 0$ and $\theta \in [0, \frac{\pi}{2})$. Then*

$$\frac{\lambda^2}{1+\mu\lambda} \notin \mathbb{R}_- \qquad \left(\lambda \in \Sigma_\theta,\ |\lambda| \geq \frac{4}{\mu(1-\sin(\theta))}\right), \tag{3.4.1}$$

and

$$\operatorname{Re}\left(\sqrt{\frac{\lambda^2}{1+\mu\lambda}}\right) \geq \frac{1}{4}\sqrt{\frac{|\lambda|(1-\sin(\theta))}{\mu}} \qquad \left(\lambda \in \Sigma_\theta,\ |\lambda| \geq \frac{4}{\mu(1-\sin(\theta))}\right), \tag{3.4.2}$$

where

$$\Sigma_\theta = \left\{\lambda = \rho\exp(i\phi) \in \mathbb{C} \;\middle|\; \arg\phi \in \left(-\frac{\pi}{2}-\theta, \frac{\pi}{2}+\theta\right)\right\}. \tag{3.4.3}$$

The second technical result, which is also proved in the Appendix in Section 3.6 is:

Lemma 3.4.2. *Let $a,\ \mu > 0$ and $\theta \in \left[0, \frac{\pi}{2}\right)$. Then there exists a constant $R = R(a,\ \mu,\ \theta)$, such that for every*

$$\lambda \in \Sigma_\theta, \quad |\lambda| \geq R(a,\ \mu,\ \theta), \tag{3.4.4}$$

the matrix $\mathcal{M}_\lambda$ defined in (3.3.28) is invertible and there is a constant $K(\theta,\ \mu) > 0$, such that

$$\left\|\lambda\mathcal{M}_\lambda^{-1}\right\| \leq K(\theta,\mu) \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a,\ \mu,\ \theta)). \tag{3.4.5}$$

An important ingredient of the proof of the fact that $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$ is the following result:

Proposition 3.4.3. *With the notation in Theorem 3.3.6, for every $\lambda \in \Sigma_\theta$, $|\lambda| \geq \frac{4}{\mu(1-\sin\theta)}$ we have*

$$\|q_\lambda\|_{L^2(\mathcal{E})} \leq \tilde{c}\left(\frac{1}{|\omega_\lambda|\operatorname{Re}\omega_\lambda}\|\varphi_\lambda\|_{L^2(\mathcal{E})} + \frac{1}{(\operatorname{Re}\omega_\lambda)^{\frac{1}{2}}}\left\|\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix}\right\|_{\mathbb{C}^2}\right), \tag{3.4.6}$$

and

$$\left\|\frac{\mathrm{d}q_\lambda}{\mathrm{d}x}\right\|_{L^2(\mathcal{E})} \leq \tilde{c}\left(\frac{1}{\operatorname{Re}\omega_\lambda}\|\varphi_\lambda\|_{L^2(\mathcal{E})} + \left\|\omega_\lambda\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix}\right\|_{\mathbb{C}^2}\right), \tag{3.4.7}$$

where $\tilde{c} > 0$ is a constant depending only on a, θ and μ.

Proof. Firstly, by combining (3.3.42) and Lemma 3.3.4, we have

$$\|q_\lambda\|_{L^2(\mathcal{E})} \leq \tilde{c}\left(\frac{1}{|\omega_\lambda|\operatorname{Re}\omega_\lambda}\|\varphi_\lambda\|_{L^2(\mathcal{E})} + \frac{1}{(\operatorname{Re}\omega_\lambda)^{\frac{1}{2}}}\left\|\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix}\right\|_{\mathbb{C}^2}\right) \qquad (\lambda \in \sigma(\mathcal{A})).$$

Similarly, we obtain

$$\left\|\frac{\mathrm{d}q_\lambda}{\mathrm{d}x}\right\|_{L^2(\mathcal{E})} \leq \tilde{c}\left(\frac{1}{\operatorname{Re}\omega_\lambda}\|\varphi_\lambda\|_{L^2(\mathcal{E})} + \left\|\frac{\omega_\lambda}{(\operatorname{Re}\omega_\lambda)^{\frac{1}{2}}}\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix}\right\|_{\mathbb{C}^2}\right),$$

for $\lambda \in \Sigma_\theta$, $|\lambda| \geq \frac{4}{\mu(1-\sin\theta)}$. The last estimate and the fact that

$$\frac{1}{(\operatorname{Re}\omega_\lambda)^{\frac{1}{2}}} \geq 1.$$

imply the conclusion (3.4.7). □

We are now in a position to prove that $\mathcal{A}$ is a sectorial operator on $\mathcal{X}$.

Theorem 3.4.4. *Let $a,\ \mu > 0$ and $\theta \in [0, \frac{\pi}{2})$, then there exists a positive constant $c = c(a,\ \theta,\ \mu)$ such that*

$$\left\|\lambda(\lambda\mathbb{I} - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{X})} \leq c \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a,\ \mu,\ \theta))\,, \tag{3.4.8}$$

so that the operator $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$.

Proof. From Theorem 3.3.6, it follows that if $R(a,\ \mu,\ \theta)$ is large enough then

$$\{\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a,\ \mu,\ \theta)\} \subset \rho(\mathcal{A}).$$

Using again Theorem 3.3.6, we infer that for every λ satisfying (3.4.4), and for every $\mathcal{F} = [f_1\ \ f_2\ \ f_3\ \ f_4\ \ f_5]^\top \in \mathcal{X}$, one can set

$$(\lambda\mathbb{I} - \mathcal{A})^{-1}\mathcal{F} = z_\lambda = [H_\lambda\ \ h_\lambda\ \ q_\lambda\ \ q_{-,\lambda}\ \ q_{+,\lambda}]^\top,$$

with H_λ, h_λ, q_λ, $q_{-,\lambda}$ and $q_{+,\lambda}$ satisfying (3.3.41)-(3.3.44). Combining this with (3.4.5), we get

$$\begin{aligned}\left\|\lambda\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix}\right\|_{\mathbb{C}^2} \leq& \tilde{c}\left(\left|\int_{-\infty}^{0}\exp(\omega_\lambda\xi)\varphi_\lambda(\xi - a)\mathrm{d}\xi\right| + \left|\int_{0}^{\infty}\exp(-\omega_\lambda\xi)\varphi_\lambda(\xi + a)\mathrm{d}\xi\right|\right)\\ &+ \tilde{c}\left(|f_1| + \|f_2\|_{H^1(\mathcal{E})} + |f_4| + |f_5|\right) \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a,\ \mu,\ \theta))\,,\end{aligned}$$

where

$$\varphi_\lambda(x) = \frac{\lambda}{1+\mu\lambda}f_3(x) - \frac{1}{1+\mu\lambda}\frac{\mathrm{d}f_2(x)}{\mathrm{d}x} \qquad (x \in \mathcal{E}),$$

and $\tilde{c} = \tilde{c}(\theta,\ \mu\ , a)$. Noting that

$$\|\varphi_\lambda\|_{L^2(\mathcal{E})} \leq \tilde{c}\left(\|f_3\|_{L^2(\mathcal{E})} + \frac{1}{|\lambda|}\|f_2\|_{H^1(\mathcal{E})}\right) \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a,\ \mu,\ \theta))\,,$$

the above inequalities, combined with Hölder's inequality, imply that

$$\begin{aligned}&\left\|\lambda\begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda}\end{bmatrix}\right\|_{\mathbb{C}^2}\\ \leq& \tilde{c}(|f_1| + \|f_2\|_{H^1(\mathcal{E})} + \|f_3\|_{L^2(\mathcal{E})} + |f_4| + |f_5|) \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a,\ \mu,\ \theta))\,.\end{aligned} \tag{3.4.9}$$

On the other hand, combining Lemma 3.3.4, Lemma 3.4.1 and (3.3.42), for every $\lambda \in \Sigma_\theta$ with $|\lambda| \geq R(a, \mu, \theta)$, we have

$$\begin{aligned}\|\lambda q_\lambda\|_{L^2(\mathcal{E})} \leq \tilde{c} \left\| \lambda \begin{bmatrix} q_{-,\lambda} \\ q_{+,\lambda} \end{bmatrix} \right\|_{\mathbb{C}^2} \\ + \frac{|\lambda|}{2|\omega_\lambda| \operatorname{Re} \omega_\lambda} \left(\|f_3\|_{L^2(\mathcal{E})} + \frac{1}{|\lambda|} \|f_2\|_{H^1(\mathcal{E})} \right),\end{aligned} \tag{3.4.10}$$

On the other hand, since

$$\operatorname{Re} \sqrt{z} = \frac{1}{\sqrt{2}} \sqrt{|z| + \operatorname{Re} z} \qquad (z \in \mathbb{C} \backslash \mathbb{R}_-), \tag{3.4.11}$$

for $|\lambda|$ large enough, we have

$$\frac{|\lambda|}{2|\omega_\lambda| \operatorname{Re} \omega_\lambda} \leq c, \tag{3.4.12}$$

where c is a constant depending only on a, μ and θ.

Estimates (3.4.10) and (3.4.12), combined with (3.4.2) and (3.4.5) yield that

$$\begin{aligned}&\|\lambda q_\lambda\|_{L^2(\mathcal{E})} \\ \leq &\tilde{c} \left(|f_1| + \|f_2\|_{H^1(\mathcal{E})} + \|f_3\|_{L^2(\mathcal{E})} + |f_4| + |f_5| \right) \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a, \mu, \theta)).\end{aligned} \tag{3.4.13}$$

Using Theorem 3.3.6 and Remark 3.4.3, for every $\lambda \in \Sigma_\theta$ with $|\lambda| \geq R(a, \mu, \theta)$, we derive:

$$\begin{aligned}&\|\lambda h_\lambda\|_{H^1(\mathcal{E})} \\ = &\left\| f_2 - \frac{\mathrm{d} q_\lambda}{\mathrm{d} x} \right\|_{H^1(\mathcal{E})} \\ \leq &\tilde{c} \|f_2\|_{H^1(\mathcal{E})} + \left\| \frac{\mathrm{d} q_\lambda}{\mathrm{d} x} \right\|_{L^2(\mathcal{E})} + \left\| \frac{\mathrm{d}^2 q_\lambda}{\mathrm{d} x^2} \right\|_{L^2(\mathcal{E})} \\ \leq &\tilde{c} \left(|f_1| + \|f_2\|_{H^1(\mathcal{E})} + \|f_3\|_{L^2(\mathcal{E})} + |f_4| + |f_5| \right),\end{aligned} \tag{3.4.14}$$

and

$$\begin{aligned}&\|\lambda H_\lambda\|_{\mathbb{C}} \\ = &\left\| f_1 - \frac{q_{+,\lambda} - q_{-,\lambda}}{2a} \right\|_{\mathbb{C}} \\ \leq &\tilde{c} \left(|f_1| + \|f_2\|_{H^1(\mathcal{E})} + \|f_3\|_{L^2(\mathcal{E})} + |f_4| + |f_5| \right) \quad (\lambda \in \Sigma_\theta,\ |\lambda| \geq R(a, \mu, \theta)).\end{aligned} \tag{3.4.15}$$

Putting together (3.4.9), (3.4.13), (3.4.14) and (3.4.15) yields the conclusion (3.4.8). □

We are now in a position to prove our first main result.

Proof of Theorem 3.1.1. The proof is based on Theorem 3.4.4 and on an interpolation argument. Following Lunardi [Lun18, Chapter 1], given two Banach spaces X, Y, where $Y \subset X$ with continuous embedding, we denote by $(X, Y)_{\frac{1}{2},2}$ the real interpolation space of order $\frac{1}{2}$ between Y and X.

Recall the operators $\mathcal{A}$ and $\mathcal{B}$ defined in (3.3.18) and (3.3.19). It is not difficult to check that the real interpolation space $(\mathcal{X}, \mathcal{D}(\mathcal{A}))_{\frac{1}{2},2}$ coincides with the space $\mathcal{W}$ defined in (3.3.16). Moreover, we know from Theorem 3.4.4 that $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$ and we also know that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathcal{X})$. According to a classical result on analytic semigroups (see, for instance, [BDPDM07, Theorem 2.1, p.207]), for every $z_0 \in (\mathcal{X}, \mathcal{D}(\mathcal{A}))_{\frac{1}{2},2} = \mathcal{W}$, $T > 0$ and $\mathcal{B}u \in L^2([0,T]; \mathcal{X})$, the system (3.3.1), (3.3.2) admits a unique solution

$$z \in C([0,T]; \mathcal{W}) \cap L^2((0,T); \mathcal{Z}) \cap H^1((0,T); \mathcal{X}).$$

The desired conclusion follows now by applying Proposition 3.3.1. □

3.5 The optimal control problem

We give below the proof of our main result in Theorem 3.1.2 and we describe the construction of the feedback control by applying Proposition 3.2.3. To apply this result we have to prove the existence of a feedback operator F and of an observation operator C such that our system satisfies all the assumptions of Proposition 3.2.3. The definition of C is easy to find, whereas the construction of F is slightly more involved. Therefore, in the first subsection below we derive some energy estimates which will suggest the choice of this feedback operator.

3.5.1 Feedback design

We begin by deriving the following energy estimate:

Proposition 3.5.1. *Let* $\begin{bmatrix} H \\ h \\ q \\ p \end{bmatrix}$ *be the solution of system* (3.1.1)-(3.1.14) *whose existence and uniqueness has been proved in Theorem 3.1.1 and let*

$$E(t) = \int_{\mathbb{R}} \frac{1}{2} \left[q^2(t,x) + h^2(t,x) \right] \mathrm{d}x + \frac{1}{2} \dot{H}^2(t) \qquad (t > 0). \tag{3.5.1}$$

Then we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\mu \int_{\mathbb{R}} \left(\frac{\partial q}{\partial x} \right)^2 (t,x) \mathrm{d}x + u(t) \dot{H}(t) \qquad (t > 0). \tag{3.5.2}$$

Proof. From (3.5.1), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \int_{\mathbb{R}} \left[q(t,x)\frac{\partial q}{\partial t}(t,x) + h(t,x)\frac{\partial h}{\partial t}(t,x) \right] \mathrm{d}x + \dot{H}(t)\ddot{H}(t) \quad (t > 0 \ a.e.).$$

Combining the above formula with (3.1.1), (3.1.2), (3.1.3) and (3.1.4), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) = &- \int_{\mathcal{E}} \frac{\partial (hq)}{\partial x}(t,x)\mathrm{d}x + \mu \int_{\mathcal{E}} q(t,x)\frac{\partial^2 q}{\partial x^2}(t,x)\mathrm{d}x \\ &+ \mu \int_{\mathcal{I}} q(t,x)\frac{\partial^2 q}{\partial x^2}(t,x)\mathrm{d}x - H(t)\int_{\mathcal{I}} \frac{\partial q}{\partial x}(t,x)\mathrm{d}x \\ &- \int_{\mathcal{I}} q(t,x)\frac{\partial p}{\partial x}(t,x)\mathrm{d}x + \dot{H}(t)\ddot{H}(t) \qquad (t > 0 \ a.e.).\end{aligned}$$

Integrating by parts and combining (3.1.5), (3.1.6), (3.1.8) and (3.1.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\mu \int_{\mathbb{R}} \left(\frac{\partial q}{\partial x}\right)^2 (t,x)\mathrm{d}x - \dot{H}(t)\int_{\mathcal{I}} p(t,x)\mathrm{d}x + \dot{H}(t)\ddot{H}(t) \qquad (t > 0 \ a.e.).$$

Using (3.1.7) and the above equality, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\mu \int_{\mathbb{R}} \left(\frac{\partial q}{\partial x}\right)^2 (t,x)\mathrm{d}x + u(t)\dot{H}(t) \qquad (t > 0),$$

which concludes the proof of this lemma. □

The above lemma suggests that $\tilde{u}_\alpha(t) = -\alpha\dot{H}(t)$, with $\alpha \geqslant 0$, is a feedback control for which the energy of the system is not increasing. Moreover, if $\alpha > 0$ it follows from (3.5.2) that $J(\tilde{u}_\alpha)$ is finite. In order to express this feedback law in terms of z introduced in Section 3.3, we note that from (3.3.4) this feedback can be rewritten

$$\tilde{u}_\alpha(t) = \alpha\left(\frac{q_+(t) - q_-(t)}{2a}\right) \qquad (t > 0, \ \alpha \geq 0).$$

In the next subsection we show how the above formula allows us to construct a feedback operator F and operators A, B and C such that all the assumptions in Proposition 3.2.3 are satisfied.

3.5.2 Proof of Theorem 3.1.2

In this subsection we show that equations (3.1.1)-(3.1.11), together with the output law $y(t) = \dot{H}(t)$ for $t > 0$ define a system satisfying all the assumptions in Proposition 3.2.3. To this aim, we define the following spaces and operators:

- The state space is $X = \mathcal{W}$, where $\mathcal{W}$ has been introduced in (3.3.16);

- A is the part in $\mathcal{W}$ of the operator $\mathcal{A}$ introduced in (3.3.18)-(3.3.20);
- $Y = \mathbb{C}$ and $C \in \mathcal{L}(X, Y)$ is defined by

$$Cz = -\frac{q_+ - q_-}{2a} \qquad \left(z = [H \;\; h \;\; q \;\; q_- \;\; q_+]^\top \in X\right). \tag{3.5.3}$$

- $U = \mathbb{C}$ and $B = \mathcal{B} \in \mathcal{L}(U, X)$, where $\mathcal{B}$ has been defined in (3.3.19);
- $F \in \mathcal{L}(X, U)$ is defined by

$$Fz = \frac{q_+ - q_-}{2a} \qquad (z \in X). \tag{3.5.4}$$

Remark 3.5.2. The motivation behind the above choice of the feedback operator has been explained above, in the comments following the proof of Proposition 3.5.1.

Proposition 3.5.3. *With the above notation, the operators A and $A_F = A + BF$ generate analytic semigroups on X.*

Proof. As already mentioned in the Proof of Theorem 3.1.1, $X = \mathcal{W}$ coincides with the real interpolation space $(\mathcal{X}, \mathcal{D}(\mathcal{A}))_{\frac{1}{2},2}$, where $\mathcal{A}$ has been introduced in (3.3.18)-(3.3.20). On the other hand, we have seen in Theorem 3.4.4 that $\mathcal{A}$ is a sectorial operator on the space $\mathcal{X}$ defined in (3.3.14). The two above facts, combined with [Lun18, Theorem 6.5, p.165] imply that A is the part of $\mathcal{A}$ in X and A is sectorial on X, so that A generates an analytic semigroup on X. This, the fact that $BF \in \mathcal{L}(X)$ and Corollary 2.2 in Pazy [Paz12, Corollary 2.2, p.81] imply that A_F generates an analytic semigroup on X. □

The result below provides some useful spectral properties of the operator A_F introduced above.

Proposition 3.5.4. *With the notation in Proposition 3.5.3, there exists a set $\mathcal{S} \subset \mathbb{C}_-$ (recall (3.3.24) for the definition of $\mathbb{C}_-$), having at most four elements, such that the spectrum of A_F satisfies*

$$\sigma(A_F) \subset \{0\} \cup \mathcal{S} \cup \left(-\infty, -\frac{1}{\mu}\right] \cup \left\{\lambda \in \mathbb{C}_- \;\middle|\; \left|\lambda + \frac{1}{\mu}\right| = \frac{1}{\mu}\right\}. \tag{3.5.5}$$

Proof. The proof follows that of Theorem 3.3.6 line by line. The only difference is that we need to replace the matrix $\mathcal{M}_\lambda$ defined in (3.3.28) by

$$\tilde{\mathcal{M}}_\lambda = \begin{bmatrix} \lambda\left(1 + \dfrac{8a^3}{3}\right) + 2a\left(\mu + \dfrac{1}{2a} + \dfrac{1}{\lambda}\right) + \dfrac{4a^2\lambda}{\omega_\lambda} & -\lambda\left(1 - \dfrac{4a^3}{3}\right) - 2a\left(\mu + \dfrac{1}{\lambda}\right) \\ -\lambda\left(1 - \dfrac{4a^3}{3}\right) - 2a\left(\mu + \dfrac{1}{\lambda}\right) & \lambda\left(1 + \dfrac{8a^3}{3}\right) + 2a\left(\mu + \dfrac{1}{2a} + \dfrac{1}{\lambda}\right) + \dfrac{4a^2\lambda}{\omega_\lambda} \end{bmatrix}.$$

where ω_λ is defined in (3.3.27). □

We also need the result below, which is easy to check, so we skip the proof.

Proposition 3.5.5. *With the notation in Proposition 3.5.3, we have that* 0 *is not an eigenvalue of* A_F*. Moreover, for every* $G \in \mathcal{L}(X, \mathbb{C})$*, we have that* $0 \in \sigma(A+BG)$*, which implies that the system* $\Sigma(A, B, C)$ *is not exponentially stabilizable.*

The result below shows that the system $\Sigma\left(A + BF,\, 0,\, \begin{bmatrix} C \\ F \end{bmatrix}\right)$ satisfies one of the main assumptions in Proposition 3.2.3.

Proposition 3.5.6. *With the notation in Proposition 3.5.3, the semigroup* $\mathbb{T}^F$ *generated by* A_F *is strongly stable in* X*.*

Proof. We first remark that, since $\mathbb{T}^F$ is analytic, its growth bound $\omega_0\left(\mathbb{T}^F\right)$ is given ([BDPDM07, Proposition 2.9, p.120-121]) by

$$\omega_0\left(\mathbb{T}^F\right) = \sup_{\lambda \in \sigma(A_F)} \operatorname{Re} \lambda.$$

This fact, combined with Proposition 3.5.4 and 3.5.5, implies that $\mathbb{T}^F$ is a bounded semigroup.

On the other hand, from Proposition 3.5.4 and 3.5.5 it follows that A_F has no eigenvalues on the imaginary axis and that $\sigma\left(A_F\right) \cap i\mathbb{R} = \{0\}$. Using the above properties and applying the Arendt-Batty theorem (see Arendt and Batty [AB88]), we obtain the announced conclusion. □

We are now in a position to prove the main result of this section, which can be seen as a reinforced version of Theorem 3.1.2, already announced in Section 3.1. Recall that the operators A, B and C appearing in the theorem below are those introduced at the beginning of this subsection.

Theorem 3.5.7. *For every initial data satisfying the assumptions in Theorem 3.1.1, there exists a unique optimal control* $\bar{u} \in L^2(0, \infty)$ *minimizing the cost functional* $J(u)$ *defined in (3.1.22). Moreover,* $\bar{u}$ *can be expressed in the feedback form* $\bar{u}(t) = -B^* P_0 z(t)$*, where* $P_0 \in \mathcal{L}(X)$ *is the minimal self-adjoint nonnegative solution to the algebraic Riccati equation*

$$A^* P z + P A z - P B B^* P z + C^* C z = 0 \qquad (z \in \mathcal{D}(A)).$$

Finally, the corresponding minimal cost functional is given by

$$J(\bar{u},\ z_0) = \langle P_0 z_0,\ z_0 \rangle_X .$$

Proof. As stated in Proposition 3.3.1, the system (3.1.1)-(3.1.14) is equivalent to system (3.3.4)-(3.3.12). Hence, finding an optimal control for the latter system implies that the resulting optimal control is also optimal for the original system.

To apply Proposition 3.2.3 we first remark that, according to Proposition 3.5.6, the semigroup generated by $A + BF$ is strongly stable. Moreover, from Proposition 3.5.1 it follows that

$$\int_0^\infty \left|\dot{H}(t)\right|^2 \mathrm{d}t \leq 2\|z_0\|_X^2.$$

Combining the last estimate with (3.3.4) and with the definitions (3.5.3), (3.5.4) of the operators C and F, it follows that the system $\Sigma\left(A + BF,\, 0,\, \begin{bmatrix} C \\ F \end{bmatrix}\right)$ is output stable. We can thus apply Proposition 3.2.3 to obtain the announced conclusions. □

3.6 Appendix

3.6.1 Appendix A

This paragraph is devoted to the proof of Lemmas 3.3.2-3.3.5.

Proof of Lemma 3.3.2. Since

$$\frac{\lambda^2}{1+\mu\lambda} = \frac{1}{\mu^2}\frac{(\mu\lambda)^2}{\mu\lambda+1},$$

it clearly suffices to prove (3.4.1) and (3.4.2) for $\mu = 1$.

We take $\lambda \neq -1$ and write $\lambda = \rho\exp(i\phi)$ with $\rho > 0$. Since

$$\frac{\lambda^2}{1+\lambda} = \frac{\rho^2\left(\cos(2\phi) + \rho\cos(\phi) + i(\sin(2\phi) + \rho\sin(\phi)\right)}{|1+\lambda|^2},$$

it follows that $\frac{\lambda^2}{1+\lambda} \in \mathbb{R}_-$ if and only if

$$\cos(2\phi) + \rho\cos(\phi) < 0 \quad \text{and} \quad \sin(2\phi) + \rho\sin(\phi) = 0.$$

The above condition implies that

$$\phi = \pi \quad \text{and} \quad \rho > 1 \quad \text{or} \quad 2\cos(\phi) + \rho = 0. \tag{3.6.1}$$

The first condition in (3.6.1) writes $\lambda \in (-\infty, -1)$. If the second condition in (3.6.1)

holds then

$$\begin{aligned}
&|1+\lambda|^2\\
=&(1+\rho\cos(\phi)+i\rho\sin(\phi))\,(1+\rho\cos(\phi)-i\rho\sin(\phi))\\
=&\rho^2+2\rho\cos(\phi)+1\\
=&\rho(2\cos(\phi)+\rho)+1\\
=&1.
\end{aligned}$$

So that the second condition is equivalent to $|\lambda+1|=1$.
We have thus proved the conclusion for $\mu=1$ and, consequently, for any positive μ. □

Proof of Lemma 3.3.3. We first note that

$$\det\mathcal{M}_\lambda = 4a^2\lambda\left(a+\frac{1}{\omega_\lambda}\right)\times\left[\lambda\left(2+\frac{4a^3}{3}\right)+4a\left(\mu+\frac{1}{\lambda}\right)+\frac{4a^2\lambda}{\omega_\lambda}\right].$$

In order to obtain the conclusion of Lemma 3.3.3, it suffices to check that the equation

$$4a^2\lambda\left[a+\frac{1}{\omega_\lambda}\right]\times\left[\lambda\left(2+\frac{4a^3}{3}\right)+4a\left(\mu+\frac{1}{\lambda}\right)+\frac{4a^2\lambda}{\omega_\lambda}\right]=0, \tag{3.6.2}$$

admits at most four solutions satisfying (3.3.26) and lying in $\mathbb{C}_-$. To this aim, we begin by remarking that for any λ satisfying (3.3.26) we have

$$\mathrm{Re}\left(a+\frac{1}{\omega_\lambda}\right)\geq a\,.$$

Consequently, the complex λ satisfying (3.3.26) is a solution of $\det\mathcal{M}_\lambda=0$, if and only if

$$\lambda\left(2+\frac{4a^3}{3}\right)+4a\left(\mu+\frac{1}{\lambda}\right)+\frac{4a^2\lambda}{\omega_\lambda}=0. \tag{3.6.3}$$

Let now $\lambda\in\mathbb{C}^*$ with $\mathrm{Re}\,\lambda\geqslant 0$. Then $\lambda=\rho\exp(i\phi)$ with $|\phi|\leq\frac{\pi}{2}$ and $\rho>0$. Then there exists an angle ψ with $|\psi|\leq\frac{\pi}{2}$ such that

$$1+\mu\lambda=1+\mu\rho\cos(\phi)+i\mu\rho\sin(\phi)=|1+\mu\lambda|\exp(i\psi).$$

Moreover, it is easy to see that $\psi\geq 0$ if and only if $\phi\geq 0$. Thus $\left|\phi-\frac{\psi}{2}\right|\leq\frac{\pi}{2}$. Combining this estimate with the fact, following from the definition (3.3.27) of ω_λ, that

$$\omega_\lambda=\frac{\rho}{|1+\mu\lambda|^{\frac{1}{2}}}\exp\left(i\left(\phi-\frac{\psi}{2}\right)\right),$$

it follows that

$$\mathrm{Re}\left(\frac{\omega_\lambda}{\lambda}\right)\geq 0\qquad\qquad(\lambda\in\mathbb{C}^*,\ \ \mathrm{Re}\,\lambda\geqslant 0).$$

The last inequality implies that for every $\lambda \in \mathbb{C}^*$ with $\operatorname{Re}\lambda \geqslant 0$ we have

$$\operatorname{Re}\left(\lambda\left(2+\frac{4a^3}{3}\right)+4a\left(\mu+\frac{1}{\lambda}\right)+\frac{4a^2\lambda}{\omega_\lambda}\right) \geq 4a\mu > 0.$$

We have thus shown that any the solution of (3.6.3) either vanishes or it lies in the left half plane.
Moreover, if λ satisfying (3.3.26) is a solution of $\det \mathcal{M}_\lambda = 0$, then we have

$$\left[\lambda^2\left(2+\frac{4a^3}{3}\right)+4a\left(\mu\lambda+1\right)\right]^2 = 16a^4\lambda^2(1+\mu\lambda),$$

so that $\det \mathcal{M}_\lambda$ can vanish for at most four values of λ satisfying (3.3.26). □

Proof of Lemma 3.3.4. Firstly, by the definition of $D_+(\omega)$ (3.3.32), for every $\gamma_+ \in \mathbb{C}$, we have

$$\int_a^\infty |(D_+(\omega)\gamma_+)(x)|^2 \mathrm{d}x \leq \exp(2(\operatorname{Re}\omega)a)|\gamma_+|^2 \int_a^\infty \exp\left(-2(\operatorname{Re}\omega)x\right)\mathrm{d}x = \frac{|\gamma_+|^2}{2\operatorname{Re}\,\omega},$$

which implies the second equality in (3.3.35).

On the other hand, to estimate the norm of operator $R_+(\omega)$, defined in (3.3.34), we remark that

$$\begin{aligned}
&\left|\frac{1}{2\omega}\left(\int_a^x \exp(\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(-\omega x)\right| \\
\leq&\frac{1}{2|\omega|}\left(\int_a^x \exp\left((\operatorname{Re}\omega)\,\xi\right)|\varphi(\xi)|\mathrm{d}\xi\right)\exp\left(-(\operatorname{Re}\omega)x\right) \\
\leq&\frac{\|\varphi\|_{L^\infty(a,\infty)}}{2|\omega|}\left(\int_a^x \exp((\operatorname{Re}\omega)\xi)\mathrm{d}\xi\right)\exp(-(\operatorname{Re}\omega)x) \\
=&\frac{\|\varphi\|_{L^\infty(a,\infty)}}{2|\omega|}\frac{\exp((\operatorname{Re}\omega)x)-\exp((\operatorname{Re}\omega)a)}{\operatorname{Re}\omega}\exp(-(\operatorname{Re}\omega)x) \\
\leq&\frac{\|\varphi\|_{L^\infty(a,\infty)}}{2|\omega|\operatorname{Re}\omega} \qquad (\varphi \in L^\infty(a,\infty),\ x \in (a,\infty)).
\end{aligned} \tag{3.6.4}$$

With a similar calculation for the other two terms of $R_+(\omega)$, we obtain

$$\left|\frac{1}{2\omega}\left(\int_x^\infty \exp(-\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(\omega x)\right| \leq \frac{\|\varphi\|_{L^\infty(a,\infty)}}{2|\omega|\operatorname{Re}\omega}, \tag{3.6.5}$$

and

$$\left|\frac{1}{2\omega}\left(\int_a^\infty \exp(-\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(2\omega a)\exp(-\omega x)\right| \leq \frac{\|\varphi\|_{L^\infty(a,\infty)}}{2|\omega|\operatorname{Re}\omega}, \tag{3.6.6}$$

where

$$\varphi \in L^\infty(a,\infty) \qquad \text{and} \qquad x \in (a,\infty).$$

Combining the defintion (3.3.34) of $R_+(\omega)$ with the (3.6.4)-(3.6.6), it follows that

$$\|R_+(\omega)\|_{\mathcal{L}(L^\infty(a,\infty))} \leq \frac{3}{2|\omega|\mathrm{Re}\,\omega} \qquad (\mathrm{Re}\,\omega > 0,\ \varphi \in L^\infty(a,\infty)). \tag{3.6.7}$$

We next estimate the norm in $\mathcal{L}(L^1(a,\infty))$ of the operator $R_+(\omega)$ defined in (3.3.34). For the first term in the right hand side of (3.3.34) we can use Tonelli's theorem to get

$$\begin{aligned}
&\int_a^\infty \left|\frac{1}{2\omega}\left(\int_a^x \exp(\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(-\omega x)\right|\mathrm{d}x\\
\leq&\frac{1}{2|\omega|}\int_a^\infty\left(\int_a^x \exp((\mathrm{Re}\,\omega)\xi)|\varphi(\xi)|\mathrm{d}\xi\right)\exp(-(\mathrm{Re}\,\omega)x)\mathrm{d}x\\
=&\frac{1}{2|\omega|}\int_a^\infty\left(\int_\xi^\infty \exp(-(\mathrm{Re}\,\omega)x)\mathrm{d}x\right)\exp((\mathrm{Re}\,\omega)\xi)|\varphi(\xi)|\mathrm{d}\xi\\
\leq&\frac{\|\varphi\|_{L^1(a,\infty)}}{2|\omega|\mathrm{Re}\,\omega} \qquad (\varphi \in L^1(a,\infty)).
\end{aligned} \tag{3.6.8}$$

In a fully similar manner it can be checked that, for every $\varphi \in L^1(a,\infty)$, the other terms in the right hand side of (3.3.34) satisfy the estimates,

$$\int_a^\infty \left|\frac{1}{2\omega}\left(\int_x^\infty \exp(-\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(\omega x)\right|\mathrm{d}x \leq \frac{\|\varphi\|_{L^1(a,\infty)}}{2|\omega|\mathrm{Re}\,\omega}, \tag{3.6.9}$$

and

$$\int_a^\infty \left|\frac{1}{2\omega}\left(\int_a^\infty \exp(-\omega\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(2\omega a)\exp(-\omega x)\right|\mathrm{d}x \leq \frac{\|\varphi\|_{L^1(a,\infty)}}{2|\omega|\mathrm{Re}\,\omega}. \tag{3.6.10}$$

Combining the definition (3.3.34) of $R_+(\omega)$ with the (3.6.8)-(3.6.10), we obtain that

$$\|R_+(\omega)\|_{\mathcal{L}(L^1(a,\infty))} \leq \frac{3}{2|\omega|\mathrm{Re}\,\omega} \qquad (\mathrm{Re}\,\omega > 0). \tag{3.6.11}$$

Using the (3.6.7) and (3.6.11), combined with the Riesz-Thorin theorem (see, for instance, [Kra19, p.318]), we find the second inequality in (3.3.36) also holds.

The first inequalities in (3.3.35) and (3.3.36) can be proved in a fully similar manner, so we skip the corresponding details. □

Proof of Lemma 3.3.5. After a standard caculation, we find that the general solution of equation (3.3.37) is

$$\begin{aligned}
q_\lambda(x) =&\frac{1}{2\omega_\lambda}\left(\int_a^x \exp(\omega_\lambda\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(-\omega_\lambda x) - \frac{1}{2\omega_\lambda}\left(\int_a^x \exp(-\omega_\lambda\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(\omega_\lambda x)\\
&+ K_1\exp(-\omega_\lambda x) + K_2\exp(\omega_\lambda x) \qquad (x \geq a),
\end{aligned}$$

where K_1 and K_2 are complex constants.

Combined Lemma 3.3.2 and 3.3.3, we get $\operatorname{Re}\omega_\lambda > 0$, where ω_λ is defined by (3.3.27). In order to have $q_\lambda \in L^2(\mathcal{E})$ we necessarily choose

$$K_2 = \frac{1}{2\omega_\lambda}\left(\int_a^\infty \exp(-\omega_\lambda\xi)\varphi(\xi)\mathrm{d}\xi\right),$$

so that

$$\begin{aligned} q_\lambda(x) =& \frac{1}{2\omega_\lambda}\left(\int_a^x \exp(\omega_\lambda\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(-\omega_\lambda x) \\ &+ \frac{1}{2\omega_\lambda}\left(\int_x^\infty \exp(-\omega_\lambda\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(\omega_\lambda x) + K_1\exp(-\omega_\lambda x) \qquad (x \geq a). \end{aligned}$$

Using next the boundary condition in (3.3.37) we obtain that

$$\gamma_+ = \frac{1}{2\omega_\lambda}\left(\int_a^\infty \exp(-\omega_\lambda\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(\omega_\lambda a) + K_1\exp(-\omega_\lambda a),$$

which yields

$$K_1 = \gamma_+\exp(\omega_\lambda a) - \frac{1}{2\omega_\lambda}\left(\int_a^\infty \exp(-\omega_\lambda\xi)\varphi(\xi)\mathrm{d}\xi\right)\exp(2\omega_\lambda a). \tag{3.6.12}$$

With the above obtained values of K_1 and K_2 we see that if $q_\lambda \in H^2(\mathcal{E})$ satisfies (3.3.37) then q_λ satisfies the second equation in (3.3.38). By a fully similar calculation we can check that if $q_\lambda \in H^2(\mathcal{E})$ satisfies (3.3.37) then q_λ satisfies the first equation in (3.3.38). Thus the unique possible solution $q_\lambda \in H^2(\mathcal{E})$ of (3.3.37) is given by (3.3.38).

On the other hand, we know from Lemma 3.3.4 (respectively from the above calculations) that q_λ defined by (3.3.38) lies in $L^2(\mathcal{E})$ (respectively that it satisfies (3.3.37)). This implies that indeed q_λ defined by (3.3.38) lies in $H^2(\mathcal{E})$ and it is the unique solution in this space of (3.3.37). □

3.6.2 Appendix B

This subsection is devoted to the proofs of the lemmas in Section 3.4.

Proof of Lemma 3.4.1. By obvious homogeneity reasons (see the proof of Lemma 3.3.2), it suffices to prove the conclusion of the present lemma for $\mu = 1$. In this case we have

$$|1+\lambda|^2 = 1 + \rho^2 + 2\rho\cos(\phi),$$

so that

$$\begin{aligned} \frac{\lambda^2}{1+\lambda} &= \frac{\left(\rho^2\cos(2\phi) + i\rho\sin(2\phi)\right)\left(1+\rho\cos(\phi) - i\rho\sin(\phi)\right)}{1+\rho^2+2\rho\cos(\phi} \\ &= \frac{\rho^2\cos(2\phi) + \rho^3\cos(\phi) + i(\rho^2\sin(2\phi) + \rho^3\sin(\phi))}{1+\rho^2+2\rho\cos(\phi)} \qquad (\lambda \in \Sigma_\theta). \end{aligned} \tag{3.6.13}$$

Moreover, we have that

$$\rho^2 \sin(2\phi) + \rho^3 \sin(\phi) = \rho^2 \sin(\phi)(2\cos(\phi) + \rho) \neq 0,$$

where

$$\left(\phi \in \left(-\frac{\pi}{2} - \theta,\ \frac{\pi}{2} + \theta\right) \backslash \{0\},\ \rho \geq \frac{4}{1 - \sin(\theta)}\right).$$

The above three formulas imply that the imaginary part of $\frac{\lambda^2}{1+\lambda}$, with $\lambda = \rho\exp(i\phi)$ and $\rho \geq \frac{4}{1-\sin(\theta)}$ vanishes iff $\phi = 0$. On the other hand, formula (3.6.13) clearly yields that $\operatorname{Re} \frac{\lambda^2}{1+\lambda} > 0$. We have thus proved that (3.4.1) holds.

We still prove that formula (3.4.2) holds for $\mu = 1$. To this aim, we first recall the classical formula

$$\operatorname{Re} \sqrt{z} = \frac{1}{\sqrt{2}} \sqrt{|z| + \operatorname{Re} z} \qquad (z \in \mathbb{C} \backslash \mathbb{R}_-). \tag{3.6.14}$$

We next apply the above formula for $z = \frac{\lambda^2}{1+\lambda}$ and $\lambda = \rho\exp(i\phi)$, with

$$\rho \geq \frac{4}{1 - \sin(\theta)} \qquad \text{and} \qquad \phi \in \left(-\frac{\pi}{2} - \theta,\ \frac{\pi}{2} + \theta\right). \tag{3.6.15}$$

Combining (3.6.14), (3.6.15) and (3.6.13), we obtain that

$$\begin{aligned}
&2\left[\operatorname{Re}\left(\sqrt{\frac{\lambda^2}{1+\lambda}}\right)\right]^2 \\
&= \frac{\rho^2\sqrt{1 + \rho^2 + 2\rho\cos(\phi)}}{1 + \rho^2 + 2\rho\cos(\phi)} + \frac{\rho^2\cos(2\phi) + \rho^3\cos(\phi)}{1 + \rho^2 + 2\rho\cos(\phi)} \\
&= \frac{\rho^2}{1 + \rho^2 + 2\rho\cos(\phi)} \left(\sqrt{1 + \rho^2 + 2\rho\cos(\phi)} + \cos(2\phi) + \rho\cos(\phi)\right),
\end{aligned} \tag{3.6.16}$$

for every $\lambda = \rho\exp(i\phi)$ with ρ and ϕ satisfying (3.6.15). Moreover, since $\rho > 1$, we have

$$\begin{aligned}
&\sqrt{1 + \rho^2 + 2\rho\cos(\phi)} + \cos(2\phi) + \rho\cos(\phi) \\
\geq &\rho - 1 + \cos(2\phi) + \rho\cos(\phi) \\
\geq &\rho(1 - \sin(\theta)) - 2 \\
\geq &\frac{1}{2}\rho(1 - \sin(\theta)).
\end{aligned}$$

Using the above equation and (3.6.16), for every $\lambda = \rho\exp(i\phi)$ satisfying (3.6.15), it follows that

$$2\left[\operatorname{Re}\left(\sqrt{\frac{\lambda^2}{1+\lambda}}\right)\right]^2 \geqslant \frac{\rho^3(1 - \sin(\theta))}{2(1 + \rho^2 + 2\rho\cos(\phi))} \geq \frac{\rho^3(1 - \sin(\theta))}{2(1+\rho)^2} \geq \frac{\rho(1 - \sin(\theta))}{8},$$

which ends the proof of Lemma 3.4.1. □

Proof of Lemma 3.4.2. From Lemma 3.3.3, we find for every λ satisfying (3.4.4), the matrix $\mathcal{M}_\lambda$ defined in (3.3.28) is invertible. Moreover, from the definition (3.3.28) of $\mathcal{M}_\lambda$, we have

$$\lambda^{-1}\mathcal{M}_\lambda = \begin{bmatrix} 1+\dfrac{8a^3}{3} & -1+\dfrac{4a^3}{3} \\ -1+\dfrac{4a^3}{3} & 1+\dfrac{8a^3}{3} \end{bmatrix} + 2a\left(\frac{\mu}{\lambda}+\frac{1}{\lambda^2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{4a^2}{\omega_\lambda}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The above formula can be rewrite as,

$$\lambda^{-1}\mathcal{M}_\lambda = \begin{bmatrix} 1+\dfrac{8a^3}{3} & -1+\dfrac{4a^3}{3} \\ -1+\dfrac{4a^3}{3} & 1+\dfrac{8a^3}{3} \end{bmatrix} + \mathcal{P}_\lambda,$$

where

$$\mathcal{P}_\lambda = 2a\left(\frac{\mu}{\lambda}+\frac{1}{\lambda^2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{4a^2}{\omega_\lambda}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using next Lemma 3.4.1 we see that there exists a constant $\tilde{K}(\theta,\,\mu) > 0$ such that

$$\|\mathcal{P}_\lambda\| \le \frac{\tilde{K}(\theta,\,\mu)}{\sqrt{|\lambda|}} \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \ge R(a,\,\mu,\,\theta))\,.$$

Moreover, after a simple calculation, we see that the inverse of the matrix M defined in (3.3.13) is

$$M^{-1} = \begin{bmatrix} 1+\dfrac{8a^3}{3} & -\left(1-\dfrac{4a^3}{3}\right) \\ -\left(1-\dfrac{4a^3}{3}\right) & 1+\dfrac{8a^3}{3} \end{bmatrix},$$

Consequently,

$$\begin{aligned} &\|\lambda\mathcal{M}_\lambda^{-1}\| \\ &= \left\|(M^{-1}+\mathcal{P}_\lambda)^{-1}\right\| \\ &\le \left\|M^{-1}\right\| \left\|\left(\mathbb{I}+\frac{\mathcal{P}_\lambda}{M^{-1}}\right)\right\| \\ &\le \left\|M^{-1}\right\| \left\|\frac{1}{1-\left\|\frac{\mathcal{P}_\lambda}{\|M^{-1}\|}\right\|}\right\| \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \ge R(a,\,\mu,\,\theta))\,. \end{aligned} \tag{3.6.17}$$

The above formula implies the inequality (3.4.5). □

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Chapter 4 Global well-posedness for a two-dimensional fluid–solid interaction system

The present chapter is devoted to the second of these problems (see Part II of Chapter 1 and Section 2.2), namely global well-posedness of the fluid–solid interaction system describing a rigid body moving in a two-dimensional viscous flow in a disk. This chapter is based on the preprint Xu [Xu26a].

4.1 Introduction

We study a two-dimensional fluid–rigid body system consisting of a viscous incompressible fluid and a rigid disk moving inside a fixed circular cavity. Since the proof depends crucially on the geometry of the configuration, we begin by recalling the model and the notation used throughout this chapter.

Let $R > 1$ be fixed. For $a \in \mathbb{R}^2$ and $r > 0$, we denote by

$$B(a, r) := \{x \in \mathbb{R}^2 : |x - a| < r\}$$

the open disk of centre a and radius r. The outer container is the disk $B(0, R)$. At time t, the rigid body occupies the unit disk

$$S(t) := B(h(t), 1),$$

where $h(t) \in \mathbb{R}^2$ denotes the centre of mass of the rigid body. The fluid occupies the time-dependent domain

$$\Omega(t) := B(0, R) \setminus S(t).$$

The unknowns are the fluid velocity

$$u = u(x, t) \in \mathbb{R}^2,$$

the fluid pressure

$$p = p(x,t) \in \mathbb{R},$$

the center-of-mass position

$$h = h(t) \in \mathbb{R}^2,$$

and the angular velocity

$$\omega = \omega(t) \in \mathbb{R}.$$

For a vector $x = (x_1, x_2)^\top \in \mathbb{R}^2$, we write

$$x^\perp := (x_2, -x_1)^\top.$$

A dot denotes differentiation with respect to time. We denote by $M > 0$ and $J > 0$ the mass and the moment of inertia of the rigid disk, respectively, and by $\nu > 0$ the viscosity of the fluid. For $x \in \partial S(t)$, we let $n(x,t)$ be the unit normal to $\partial S(t)$ pointing into the rigid body. The Cauchy stress tensor is defined by

$$\sigma(x,t) := -p(x,t)I_2 + 2\nu D(u)(x,t), \qquad D(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top),$$

where I_2 denotes the identity matrix in $\mathbb{R}^2$.

With this notation, the coupled fluid–solid system reads

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad x \in \Omega(t),\ t \in [0,T], \tag{4.1.1}$$

$$\operatorname{div} u = 0, \qquad x \in \Omega(t),\ t \in [0,T], \tag{4.1.2}$$

$$u = 0, \qquad x \in \partial B(0,R)\ , t \in [0,T], \tag{4.1.3}$$

$$u = \dot h(t) + \omega(t)(x - h(t))^\perp, \qquad x \in \partial S(t),\ t \in [0,T], \tag{4.1.4}$$

$$M\ddot h(t) = -\int_{\partial S(t)} \sigma n\, d\Gamma, \qquad t \in [0,T], \tag{4.1.5}$$

$$J\dot\omega(t) = -\int_{\partial S(t)} (x - h(t))^\perp \cdot \sigma n\, d\Gamma, \quad t \in [0,T], \tag{4.1.6}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega(0), \tag{4.1.7}$$

$$h(0) = h_0, \quad \dot h(0) = g_0 \in \mathbb{R}^2, \quad \omega(0) = \omega_0 \in \mathbb{R}. \tag{4.1.8}$$

A central question for system (4.1.1)–(4.1.8) is whether the rigid disk can touch the cavity wall in finite time. For compatible initial data, Takahashi [Tak03] proved local-in-time existence and uniqueness of strong solutions, together with a continuation criterion showing that the only possible obstruction to global continuation is collision with the outer boundary.

The main result of this chapter is the following no-collision theorem.

Theorem 4.1.1. *Let*

$$\Omega_0 := \Omega(0) = B(0,R) \setminus \overline{S(0)}.$$

Assume that the initial data $(u_0, h_0, g_0, \omega_0)$ *satisfy*

$$\begin{cases} u_0 \in H^1(\Omega_0), \quad g_0 \in \mathbb{R}^2, \quad \omega_0 \in \mathbb{R}, \\ \operatorname{div} u_0 = 0 \quad \text{in } \Omega_0, \\ u_0 = 0 \quad \text{on } \partial B(0,R), \\ u_0(x) = g_0 + \omega_0 (x - h_0)^\perp \quad \text{for } x \in \partial S(0), \\ d(0) = \operatorname{dist}(S(0), \partial B(0,R)) > 0. \end{cases}$$

Let (u, p, h, ω) *be the corresponding maximal local strong solution of system* (4.1.1)-(4.1.8) *provided by [Tak03, Theorem 2.2]. Then*

$$d(t) = \operatorname{dist}(S(t), \partial B(0,R)) > 0 \qquad \text{for all } t \in [0, T_{\max}).$$

In particular, $T_{\max} = +\infty$.

In the disk-in-disk geometry, the previously understood no-collision result was obtained under a symmetry assumption: the center of the inner disk was required to move along a fixed diameter of the cavity (equivalently, after a suitable choice of coordinates, along a vertical line); see [Hes04]. In that special setting, the relevant boundary direction remains constant in time, and one can exploit this fact to construct a special test function through an auxiliary stationary Stokes problem. This leads to a differential inequality controlling the gap $d(t)$.

The purpose of this chapter is to remove that symmetry assumption and to prove Theorem 4.1.1 for arbitrary admissible trajectories of the inner disk. The main difficulty is that, for a general motion, the distinguished direction between the centers is no longer fixed. Near a possible contact one necessarily has $h(t) \neq 0$, and the unit vector

$$\mathbf{e}(t) := \frac{h(t)}{|h(t)|}$$

depends on time. Consequently, the boundary datum associated with the test function rotates with the motion of the rigid body, and the explicit construction available in the symmetric case can no longer be used directly.

Our approach is to freeze this direction by working in a time-dependent rotating frame. More precisely, we introduce a rotation $Q(t) \in SO(2)$ such that

$$Q(t)e(t) = -e_2, \qquad e_2 = (0,1)^\top.$$

In the rotated coordinates, the thin fluid region near a possible contact has a fixed orientation. This allows us to construct an explicit divergence-free test function adapted

to the singular geometry, in the spirit of [Hil07, Sab19]. We then transport this test function back to the original variables by using the invariance of the coupled equations under rigid rotations. When inserted into the weak formulation, this test function yields the key differential inequality that prevents $d(t)$ from vanishing in finite time. Combined with Takahashi's continuation criterion, this proves Theorem 4.1.1 and therefore the global-in-time existence of strong solutions.

The broader literature on fluid–solid interaction and collision phenomena has already been reviewed in Chapter 2, Section 2.2.

To prove Theorem 4.1.1, we proceed as follows. We first construct a divergence-free test function reflecting the geometry of the gap between the inner disk and the outer boundary. After deriving suitable L^p -bounds for this test function, we insert it into the weak formulation of the coupled system. This allows us to identify a dissipative term that captures the behavior of the distance between the inner disk and the outer boundary. We then show that all the other terms remain bounded, while the dissipative term blows up if this distance tends to zero. Arguing by contradiction, we conclude that no collision can occur in finite time. The conclusion then follows from [Tak03, Theorem 2.2], which implies that the local strong solution extends globally in time.

4.2 Notation and Main Result

In this section we adopt the notation of [Tak03, Section 2] and specify the functional framework used to state the global well-posedness result for system (4.1.1)–(4.1.8). We denote by

$$\Omega_0 := \Omega(0) = B(0,R) \setminus S(0)$$

the initial fluid domain. For $t \geq 0$, we denote by $\Omega(t)$ and $S(t)$ the fluid and solid domains at time t, respectively.

Throughout this chapter, Sobolev spaces on moving domains are defined by means of a smooth diffeomorphism

$$\psi : \Omega_0 \times [0,T] \to \bigcup_{t\in[0,T]} \Omega(t) \times \{t\}, \qquad (y,t) \mapsto \psi(y,t),$$

such that, for each $t \in [0,T]$, the map $\psi(\cdot,t)$ is a C^∞ diffeomorphism from Ω_0 onto $\Omega(t)$ and

$$\frac{\partial^{i+\alpha_1+\alpha_2}\psi}{\partial t^i\, \partial y_1^{\alpha_1}\, \partial y_2^{\alpha_2}} \quad (i \leqslant 1,\ \alpha_1, \alpha_2 \geqslant 0)$$

exist and are continuous. More details can be found in [Tak03, Section 3].

For all functions $u(\cdot,t) : \Omega(t) \to \mathbb{R}^2$, we denote by

$$U(y,t) = u(\psi(y,t),t), \qquad y \in \Omega_0,$$

their pullback to the reference domain.

With this identification, the spaces above can be equivalently defined on Ω by

$$\begin{aligned}
L^2(0,T;H^2(\Omega(t))) &= \{u : U \in L^2(0,T;H^2(\Omega))\},\\
H^1(0,T;L^2(\Omega(t))) &= \{u : U \in H^1(0,T;L^2(\Omega))\},\\
C([0,T];H^1(\Omega(t))) &= \{u : U \in C([0,T];H^1(\Omega))\},\\
L^2(0,T;H^1(\Omega(t))) &= \{u : U \in L^2(0,T;H^1(\Omega))\}.
\end{aligned}$$

For any domain $\mathcal{O}$, we denote by $H^m(\mathcal{O})$ the Sobolev space of functions with weak derivatives up to order m in $L^2(\mathcal{O})$. Finally, we introduce the space

$$\mathcal{U}(0,T;\Omega(t)) = L^2(0,T;H^2(\Omega(t))) \cap H^1(0,T;L^2(\Omega(t))) \cap C([0,T];H^1(\Omega(t))). \quad (4.2.1)$$

Definition 4.2.1 (Strong solution)**.** Suppose that $T > 0$. A quadruple (u,p,h,ω) is called a strong solution of system (4.1.1)–(4.1.8) on $[0,T]$ if

$$(u,p,h,\omega) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))) \times H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R}),$$

and if the distance between $S(t)$ and $\partial B(0,R)$ remains strictly positive for all $t \in [0,T]$, while the equations (4.1.1)–(4.1.8) are satisfied almost everywhere in $(0,T)\times\Omega(t)$ (or in the trace sense on the boundaries).

We recall the local in time well-posedness of system (4.1.1)–(4.1.8).

Proposition 4.2.2 (Theorem 2.2, [Tak03])**.** *Let the initial data (u_0,h_0,g_0,ω_0) satisfy*

$$\begin{cases}
u_0 \in H^1(\Omega(0)),\ g_0 \in \mathbb{R}^2,\ \omega_0 \in \mathbb{R}, & \\
\operatorname{div} u_0 = 0 & \text{in } \Omega_(0),\\
u_0(x) = 0 & \text{on } \partial B(0,R),\\
u_0(x) = g_0 + \omega_0(x-h_0)^\perp & \text{on } \partial S(0),\\
\operatorname{dist}(S(0),\partial B(0,R)) > 0. &
\end{cases}$$

Then there exists a maximal time $T_0 > 0$ (T_0 may be infinite) such that system (4.1.1)–(4.1.8) admits a unique strong solution

$$(u,p,h,\omega) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))) \times H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R}),$$

for all $T \in (0,T_0)$ with the following energy estimate

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega(t)} |u(x,t)|^2\,\mathrm{d}x + \frac{M}{2}|\dot h(t)|^2 + \frac{J}{2}|\omega(t)|^2 + 2\nu\int_0^t\int_{\Omega(s)} D(u) : D(u)(x,s)\,\mathrm{d}x\,\mathrm{d}s\\
=&\frac{1}{2}\int_{\Omega(0)} |u(x,0)|^2\,\mathrm{d}x + \frac{M}{2}|\dot h(0)|^2 + \frac{J}{2}|\omega(0)|^2. \quad (4.2.2)
\end{aligned}$$

Moreover, one of the following alternatives holds:

1. *$T_0 = +\infty$, i.e., the solution is global;*

2. *There exists a finite T_0, such that* $\lim_{t \to T_0} \operatorname{dist}(S(t), \partial B(0,R)) = 0$.

We can now state the main result of this chapter.

Theorem 4.2.3. *Under the assumptions of Proposition 4.2.2, system (4.1.1)–(4.1.8) admits a unique global strong solution. In particular, for every $t > 0$, we have*

$$d(t) = \operatorname{dist}(S(t), \partial B(0,R)) > 0. \tag{4.2.3}$$

In view of Proposition 4.2.2, the question of global well-posedness reduces to *excluding the collision scenario*. To this end, we will exploit the weak formulation of the coupled system (4.1.1)–(4.1.8), testing it with a carefully chosen time-dependent test function.

To make this strategy precise, we first recall the weak formulation of the fluid–solid interaction system, which serves as the starting point for all subsequent estimates. Since the problem involves a moving boundary (the surface of the rigid disk), the class of admissible test functions must respect the coupling conditions between the fluid and the solid (see, for instance, [Hes04, Chapter 5, Page 52]). The formulation is expressed in terms of test triples (φ, ℓ, r), where φ is a divergence-free vector field defined in the fluid domain and (ℓ, r) are functions of time that represent, respectively, a translational and a rotational velocity imposed on the disk.

Definition 4.2.4 (Admissible test functions)**.** Let $T > 0$. We say that (φ, ℓ, r) is an admissible test triple of system (4.1.1)–(4.1.8) on $(0,T)$ if

- $\ell \in H^1((0,T);\mathbb{R}^2)$ and $r \in H^1((0,T);\mathbb{R})$,
- $\varphi \in H^1\big((0,T); L^2(B(0,R))\big) \cap L^2\big((0,T); H^1(B(0,R))\big)$,
- for a.e. $t \in (0,T)$,

$$\operatorname{div} \varphi(t,\cdot) = 0 \text{ in } \Omega(t), \qquad \varphi(t,\cdot) = 0 \text{ on } \partial B(0,R),$$

  and φ matches a rigid velocity on the disk:

$$\varphi(t,x) = \ell(t) + r(t)\,(x - h(t))^{\perp} \qquad \text{for a.e. } x \in S(t).$$

The result below can be obtained by simple integrations by parts, so we state it without proof.

Proposition 4.2.5 (Weak formulation)**.** *The strong solution of system* (4.1.1)–(4.1.8) *satisfies the following weak identity for every admissible test triple* (φ, ℓ, r)*:*

$$\begin{aligned}
&\int_{\Omega(T)} u(T,x)\cdot\varphi(T,x)\,\mathrm{d}x - \int_{\Omega(0)} u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x \\
&\quad + M\Big(\dot h(T)\cdot\ell(T) - \dot h_0\cdot\ell(0)\Big) + J\Big(\omega(T)\,r(T) - \omega_0\,r(0)\Big) \\
&\quad + \int_0^T\!\!\int_{\Omega(t)} \Big(- u\cdot\partial_t\varphi - (u\otimes u) : \nabla\varphi + 2\nu\, D(u) : D(\varphi)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \Big(M\,\dot h(t)\cdot\dot\ell(t) + J\,\omega(t)\,\dot r(t)\Big)\,\mathrm{d}t = 0.
\end{aligned} \tag{4.2.4}$$

The remaining parts of this chapter are organized as follows. In Section 4.3 we present the rotational change of variables, while the construction of the test function is carried out in Section 4.4. The necessary estimates of the test function are established in Sections 4.5 and 4.6, and the proof of the main theorem is completed in the last Section.

4.3 Change of Variables and Invariance Properties

In this section we recall the invariance properties of the relevant quantities under rigid orthogonal changes of coordinates; see [Sab19, Lemma 3.2].

For any given point $\tau = (\tau_1, \tau_2) \in B(0,R)$ and $\tau \neq 0$, we introduce an orthonormal basis (e_1, e_2) of $\mathbb{R}^2$ defined by

$$e_2 = -\frac{\tau}{|\tau|}, \qquad e_1 = -e_2^{\perp}. \tag{4.3.1}$$

Then (e_1, e_2) is a positively oriented orthonormal basis of $\mathbb{R}^2$. We then define the rotation map Y_τ by

$$Y_\tau(x) = (y_1(x), y_2(x)) = (x\cdot e_1,\ x\cdot e_2) = \left(x\cdot\frac{\tau^{\perp}}{|\tau|},\ -x\cdot\frac{\tau}{|\tau|}\right). \tag{4.3.2}$$

Moreover, the inverse mapping of Y_τ is given by

$$Y_\tau^{-1}(y) = y_1\ e_1 + y_2 e_2 = y_1\frac{\tau^{\perp}}{|\tau|} - y_2\frac{\tau}{|\tau|}. \tag{4.3.3}$$

For every function $u \in H^1(B(0,R))$, we define its pullback under the transformation Y_τ by

$$\bar u(y) := J_y Y_\tau^{-1}(y)\, u(Y_\tau^{-1}(y)), \qquad y \in B(0,R). \tag{4.3.4}$$

Similarly, for $\bar{v} \in H^1(B(0,R))$ we set

$$v(x) = J_x Y_\tau(x)\, \bar{v}(Y_\tau(x)), \qquad x \in B(0,R). \tag{4.3.5}$$

Here $J_x Y_\tau(x)$ and $J_y Y_\tau^{-1}(y)$ denote the Jacobian matrices of Y_τ and Y_τ^{-1}, respectively. In explicit coordinates, writing $\tau = (\tau_1, \tau_2)$, we have

$$J_y Y_\tau^{-1}(y) = \frac{1}{|\tau|}\begin{pmatrix} -\tau_2 & -\tau_1 \\ \tau_1 & -\tau_2 \end{pmatrix}, \qquad J_x Y_\tau(x) = \frac{1}{|\tau|}\begin{pmatrix} -\tau_2 & \tau_1 \\ -\tau_1 & -\tau_2 \end{pmatrix}. \tag{4.3.6}$$

In this way, $\bar{u}$ represents the same vector field as u, but expressed in the rotated coordinate system associated with the basis (e_1, e_2).

For the transformation Y_τ defined in (4.3.2), we recall the following invariance properties proved in [Sab19, Lemma 3.2].

Proposition 4.3.1. *Let $u, v \in H^1(B(0,R))$. For every $p \in [1,\infty)$,*

$$\|u\|_{L^p(B(0,R))} = \|\bar{u}\|_{L^p(B(0,R))}, \tag{4.3.7}$$

and

$$D(u) : D(v) = D(\bar{u}) : D(\bar{v}). \tag{4.3.8}$$

Moreover, if $\operatorname{div}_x u = 0$ in $B(0,R)$, then also $\operatorname{div}_y \bar{u} = 0$ in $B(0,R)$.

Here

$$D(w) := \frac{1}{2}(\nabla w + \nabla w^\top)$$

denotes the symmetric part of the gradient of a vector field w, and $\bar{u}, \bar{v}$ are defined in (4.3.4). For two matrices $A, B \in \mathbb{R}^{2\times 2}$, their Frobenius inner product is

$$A : B := \sum_{i,j=1}^{2} A_{ij} B_{ij}. \tag{4.3.9}$$

4.4 Construction of the Test Function

In this section we construct an admissible test function $v(t,\cdot)$ for system (4.1.1)–(4.1.8). It is required to satisfy

1. $\operatorname{div} v(t,\cdot) = 0$ in $\Omega(t)$;
2. $v(t,x) = e_{h(t)}$ for $x \in \partial S(t)$, where $e_{h(t)} := h(t)/|h(t)|$ is the unit vector pointing from the origin towards the disk centre;
3. $v(t,x) = 0$ for $x \in \partial B(0,R)$.

Main idea The main difficulty comes from the narrow fluid gap when the inner disk approaches the outer boundary. To handle this geometry, we follow the classical strategy of [Hil07, Sab19]:

- we first rotate the coordinates so that the disk centres lie on the vertical axis;
- in the rotated frame, the gap becomes a simple strip-like region described by two graphs;
- we then build a scalar profile which varies smoothly from the outer boundary to the disk boundary;
- to enforce incompressibility, we introduce an auxiliary scalar potential $\bar{g}_d$ and set $\bar{v}_d = \nabla^\perp \bar{g}_d$;
- finally we rotate the vector field back to the original variables.

We define the (time-dependent) distance between the inner disk and the outer boundary by

$$d(t) := dist\big(S(t), \partial B(0,R)\big) = R - |h(t)| - 1, \tag{4.4.1}$$

where $S(t) = B(h(t), 1)$ denotes the position of the inner disk at time t, $B(0,R)$ is the fluid domain, and $h(t)$ is the disk centre. Throughout this section we fix a time t such that $d(t) > 0$, and for simplicity we write

$$d := d(t).$$

Step 1. Rotating the coordinate system. For each time t, we apply the rotation map $Y_{h(t)}$ introduced in (4.3.2), which sends the direction of $h(t)$ to the negative y_2-axis:

$$Y_{h(t)}(x) = (y_1(t,x), y_2(t,x)) = \Big(x \cdot \frac{h(t)^\perp}{|h(t)|},\ -x \cdot \frac{h(t)}{|h(t)|}\Big). \tag{4.4.2}$$

In these rotated coordinates, the centers of the two disks are aligned along the y_2-axis, and the gap is symmetric with respect to the vertical axis (see Figure 4.1). In this frame, the inner disk is defined by

$$S_d := B(y_C, 1), \qquad y_C = (0, -R + d + 1)^\top.$$

Step 2. The gap region in rotated coordinates.

In the rotated frame, the minimal distance between the disk and the outer boundary is attained near $y_1 = 0$, and it equals d. Moreover, the gap opens up quadratically as $|y_1|$ increases. Therefore, the most delicate part of the construction is concentrated in a small strip around $y_1 = 0$.

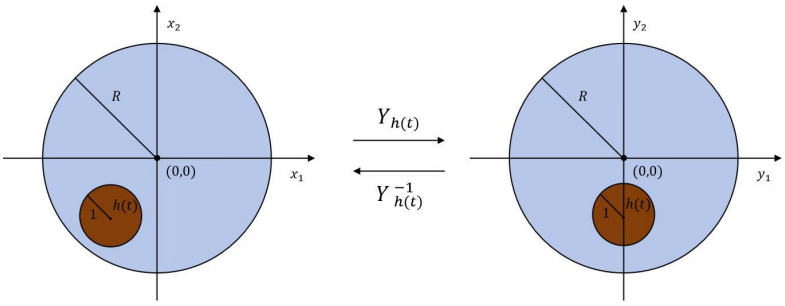


Figure 4.1: Change of coordinates. Left: original configuration. Right: rotated frame where the disk centres are aligned vertically.

We fix a small constant $\delta > 0$ depending only on R:

$$\delta = \min\left\{\frac{1}{8},\ \frac{1}{2}\sqrt{\frac{1}{2}R^3\Big(1-\frac{1}{R}\Big)}\right\}, \tag{4.4.3}$$

and we focus on the strip $|y_1| \le 2\delta$. Inside this strip, both boundaries can be represented as graphs $y_2 = \psi(y_1)$.

Boundary graphs. For $|y_1| \le 2\delta$, the outer boundary (lower graph) is

$$\psi_{\text{low}}(y_1) = -\sqrt{R^2 - y_1^2} \qquad \text{(outer circle)}, \tag{4.4.4}$$

and the part of the disk boundary facing the outer boundary (upper graph) is

$$\psi_{\text{up}}(y_1) = -\sqrt{1 - y_1^2} - R + d + 1 \qquad \text{(inner disk)}. \tag{4.4.5}$$

Note that $\psi_{\text{up}}(0) - \psi_{\text{low}}(0) = d$, so d is indeed the minimal gap width.

Gap subregions. We define the strip-like gap region

$$A_{\delta,d} = \{(y_1, y_2) : |y_1| \le 2\delta,\ \psi_{\text{low}}(y_1) \le y_2 \le \psi_{\text{up}}(y_1)\}, \tag{4.4.6}$$

$$A_{0,d} = \{(y_1, y_2) : |y_1| \le \delta,\ \psi_{\text{low}}(y_1) \le y_2 \le \psi_{\text{up}}(y_1)\}. \tag{4.4.7}$$

The core region $A_{0,d}$ is where the gap is the narrowest, and it is where we will build the main profile explicitly.

Local gap width. For each y_1, we denote the vertical thickness of the gap by

$$\mathcal{G}(y_1) := \psi_{\text{up}}(y_1) - \psi_{\text{low}}(y_1). \tag{4.4.8}$$

Lemma 4.4.1 (Gap estimate)**.** *Let $R > 1$ and $d > 0$. Then for every $|y_1| \le 2\delta$, the width $\mathcal{G}(y_1)$ satisfies*

$$d + \Big(\frac{1}{4} - \frac{1}{4R}\Big)y_1^2 \ \le\ \mathcal{G}(y_1)\ \le\ d + \Big(1 - \frac{1}{R}\Big)y_1^2. \tag{4.4.9}$$

A complete proof of the above is provided in Appendix 4.8.

Step 3. Constructing a scalar profile across the gap.

We now build a scalar function $\bar{\varphi}_d$ satisfying:

$$\bar{\varphi}_d = 0 \text{ on the outer boundary}, \qquad \bar{\varphi}_d = 1 \text{ on the disk boundary},$$

with a smooth transition inside the fluid region. This scalar profile will later be converted into a divergence-free vector field.

(a) Core region $A_{0,d}$.

We introduce the normalized vertical coordinates.

$$F(y) = \frac{y_2 - \psi_{\text{low}}(y_1)}{\mathcal{G}(y_1)}, \tag{4.4.10}$$

so that $F = 0$ on ψ_{low} and $F = 1$ on ψ_{up}. We set

$$\bar{\varphi}_d(y) = 3F(y)^2 - 2F(y)^3 \qquad (y \in A_{0,d}). \tag{4.4.11}$$

The polynomial $p(s) = 3s^2 - 2s^3$ is chosen because

$$p(0) = 0, \qquad p(1) = 1, \qquad p'(0) = p'(1) = 0,$$

which yields a C^1-matching at the two boundaries of the gap.

(b) Region away from the gap. Outside the strip $A_{\delta,d}$, we use a smooth cut-off function $\phi \in C^\infty(\overline{B(0,R)})$ such that

$$\phi(y) = \begin{cases} 1, & y \in B\left(y_C,\, 1 + \left(\frac{1}{8} - \frac{1}{8R}\right)\delta\right) \setminus A_{0,d}, \\ 0, & y \in \partial B(0,R) \setminus \left(B\left(y_C,\, 1 + \left(\frac{1}{4} - \frac{1}{4R}\right)\delta\right) \cup A_{0,d}\right), \end{cases} \tag{4.4.12}$$

here $y_C = (0, -R + d + 1)$. For $y \in B(0,R) \setminus A_{\delta,d}$, we define $\bar{\varphi}_d(y) := \phi(y)$.

(c) Transition region $A_{\delta,d} \setminus A_{0,d}$. We glue the two definitions using a cut-off depending only on y_1. Let $\mathcal{X} \in C^\infty(\mathbb{R})$ satisfy

$$\mathcal{X}(y_1) = \begin{cases} 1, & |y_1| \le \delta, \\ 0, & |y_1| \ge 2\delta. \end{cases} \tag{4.4.13}$$

Then for $y \in A_{\delta,d} \setminus A_{0,d}$ we define

$$\bar{\varphi}_d(y) = \mathcal{X}(y_1)(3F^2(y) - 2F^3(y)) + (1 - \mathcal{X}(y_1))\,\phi(y).$$

Remark 4.4.2. For $\delta \leq |y_1| \leq 2\delta$, Lemma 4.4.1 guarantees $\mathcal{G}(y_1) \geq c_\delta > 0$, so the transition region has a thickness bounded uniformly with respect to the gap size d.

Step 4. From the scalar profile to a divergence-free vector field.

To enforce incompressibility, we introduce an auxiliary scalar potential $\bar{g}_d$ and define the vector field through the perpendicular gradient (which automatically yields zero divergence). More precisely, we set

$$\bar{g}_d(y) = y_1 \, \bar{\varphi}_d(y), \tag{4.4.14}$$

and define

$$\bar{v}_d(y) = \nabla^\perp \bar{g}_d(y) = \begin{pmatrix} -\partial_{y_2} \bar{g}_d \\ \partial_{y_1} \bar{g}_d \end{pmatrix}. \tag{4.4.15}$$

Proposition 4.4.3 (Properties of $\bar{v}_d$)**.** *For each $d > 0$, the vector field $\bar{v}_d$ satisfies:*

1. $\operatorname{div} \bar{v}_d = 0$;
2. $\bar{v}_d(y) = e_2$ *for every* $y \in \partial S(t)$, *where* $e_2 = (0,1)^\top$;
3. $\bar{v}_d(y) = 0$ *for every* $y \in \partial B(0,R)$.

A detailed proof of these properties can be found in [Hil07].

Step 5. Rotating back to the original coordinates.

Finally, we return to the physical variables by applying the inverse rotation. Let $J_x Y_{h(t)}$ be the Jacobian matrix of $Y_{h(t)}$ (defined in (4.3.6)). We define the test function in the original (x_1, x_2)-coordinates by

$$v(t,x) = J_x Y_{h(t)} \; \bar{v}_d(Y_{h(t)}(x)). \tag{4.4.16}$$

Proposition 4.4.4 (Properties of v)**.** *For every $t \in (0,T)$ with $d(t) > 0$, the vector field $v(t,\cdot)$ defined by (4.4.16) satisfies:*

1. $\operatorname{div} v(t,\cdot) = 0$ *in* $\Omega(t)$;
2. $v(t,x) = e_{h(t)}$, *with* $e_{h(t)} = \left(\frac{h_1(t)}{|h(t)|}, \frac{h_2(t)}{|h(t)|}\right)$ *for* $x \in \partial S(t)$;
3. $v(t,x) = 0$ *for* $x \in \partial B(0,R)$,

where $T > 0$ is the maximal existence time provided by Proposition 4.2.2 and $h(t) = (h_1(t), h_2(t))^\top$.

Proof. These properties follow directly from Proposition 4.4.3 and the fact that $Y_{h(t)}$ is an isometry (see Proposition 4.3.1). □

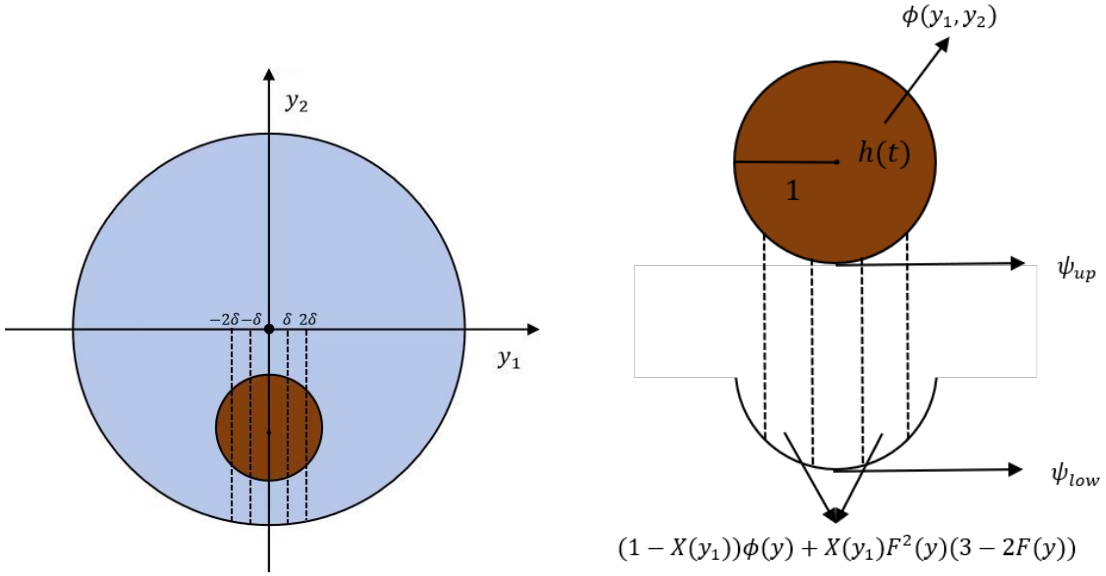


Figure 4.2: Test function.

4.5 Some estimates of the test function

In this section, we derive several preliminary estimates that will be used throughout the remainder of the paper. These estimates were proved in [Sab19]. However, since our geometric setting is different, we include the argument for completeness, indicating the minor modifications needed.

We begin by recalling the definition of $\bar{v}_d(y)$ from (4.4.15) and we establish L^p estimates.

Lemma 4.5.1. *Let $d \in (0, R-1]$. Then, for every $p \in [2, 3)$, there exists a constant $C = C(R) > 0$, depending only on R, such that*

$$\|\bar{v}_d\|_{L^p(B(0,R))} \leq C(R). \tag{4.5.1}$$

Proof. We work first on the core strip $A_{0,d}$ from (4.4.7). By the definition of $\bar{v}_d$ in (4.4.15), it follows that:

$$\bar{v}_d = \nabla^\perp(y_1\,\bar{\varphi}_{\text{core}}) = (\nabla^\perp y_1)\,\bar{\varphi}_{\text{core}} \;+\; y_1\,\nabla^\perp\bar{\varphi}_{\text{core}} = \bar{\varphi}_{\text{core}}\,e_2 \;+\; y_1\,\nabla^\perp\bar{\varphi}_{\text{core}} \quad \text{in } A_{0,d}.$$

Here $\bar{\varphi}_{\text{core}}(y) = F^2(y)(3 - 2F(y))$ and $F(y)$ is defined in (4.4.10), hence $0 \leq \bar{\varphi}_{\text{core}} \leq 2$ and

$$\partial_{y_i}\bar{\varphi}_{\text{core}} = 6F(y)(1 - F(y))\,\partial_{y_i}F(y), \qquad i = 1, 2.$$

Since

$$F(y) = \frac{y_2 - \psi_{\text{low}}(y_1)}{\mathcal{G}(y_1)}, \qquad \mathcal{G}(y_1) = \psi_{\text{up}}(y_1) - \psi_{\text{low}}(y_1),$$

we have $\partial_{y_2} F(y) = \frac{1}{\mathcal{G}(y_1)}$. Using the facts $|y_2 - \psi_{\text{low}(y_1)}| \le \mathcal{G}(y_1)$ together with

$$|\psi'_{\text{up}}(y_1)| + |\psi'_{\text{low}}(y_1)| \le C(R)\,|y_1| \qquad (|y_1| \le 2\delta), \tag{4.5.2}$$

$$|\partial_{y_1} F(y)| = \frac{|-\psi'_{\text{low}}\mathcal{G}(y_1) - (y_2 - \psi_{\text{low}})(y_1)(\psi'_{\text{up}} - \psi'_{\text{low}})(y_1)|}{\mathcal{G}^2(y_1)} \;\le\; \frac{C(R)\,|y_1|}{\mathcal{G}(y_1)}.$$

Consequently, for $y \in A_{0,d}$,

$$|\bar{v}_d(y)| \;\le\; C(R) \;+\; C(R)\,\frac{|y_1|}{\mathcal{G}(y_1)} \;+\; C(R)\,\frac{|y_1|^2}{\mathcal{G}(y_1)},$$

and hence, for any $p \in [1, \infty)$,

$$|\bar{v}_d(y)|^p \;\le\; C(R) \;+\; C(R)\,\frac{|y_1|^p}{\mathcal{G}(y_1)^p} \;+\; C(R)\,\frac{|y_1|^{2p}}{\mathcal{G}(y_1)^p}.$$

Integrating over $A_{0,d}$ and using Tonelli's theorem, we have

$$\begin{aligned}\int_{A_{0,d}} |\bar{v}_d|^p dy \le& C(R) + C(R)\int_{\psi_{low}(y_1)}^{\psi_{up}(y_1)}\int_{-\delta}^{\delta} \frac{|y_1|^p}{\mathcal{G}(y_1)^p} + \frac{|y_1|^{2p}}{\mathcal{G}(y_1)^p} \mathrm{d}y_1 \mathrm{d}y_2 \\ \le& C(R) + C(R)\int_{-\delta}^{\delta}\left(\frac{|y_1|^p}{\mathcal{G}(y_1)^{p-1}} + \frac{|y_1|^{2p}}{\mathcal{G}(y_1)^{p-1}}\right)\mathrm{d}y_1.\end{aligned}$$

By the lower bound in Lemma 4.4.1, we have

$$\frac{1}{\mathcal{G}(y_1)^{p-1}} \;\le\; \frac{C(R)}{|y_1|^{2(p-1)}} \qquad (|y_1| \le 2\delta),$$

and therefore

$$\begin{aligned}\int_{A_{0,d}} |\bar{v}_d|^p \mathrm{d}y &\le C(R) + C(R)\int_{-\delta}^{\delta}\left(|y_1|^{p-2(p-1)} + |y_1|^{2p-2(p-1)}\right)\mathrm{d}y_1 \\ &= C(R) + C(R)\int_{-\delta}^{\delta}\left(|y_1|^{2-p} + |y_1|^2\right)\mathrm{d}y_1.\end{aligned}$$

The second integral is always finite, and the first one is finite provided $p < 3$. Hence,

$$\int_{A_{0,d}} |\bar{v}_d|^p \mathrm{d}y \;\le\; C(R) \qquad \text{for any } p \in [1, 3). \tag{4.5.3}$$

On the exterior region $B(0,R)\setminus A_{\delta,d}$ we have, by definition, $\bar\varphi_d=\phi$, where ϕ defined in (4.4.12) is a fixed radial cut-off independent of d. Hence

$$\bar v_d=\nabla^\perp(y_1\,\phi)$$

and all derivatives of ϕ are only depending on R; thus, for any $p\in[1,\infty)$,

$$\int_{B(0,R)\setminus A_{\delta,d}}|\bar v_d|^p\,\mathrm{d}y\ \le\ C(R). \tag{4.5.4}$$

On the transition strip $A_{\delta,d}\setminus A_{0,d}$, the definition

$$\bar\varphi_d=(1-\mathcal{X}(y_1))\,\phi+\mathcal{X}(y_1)\,F^2(y)(3-F(y))$$

matches the core and exterior expressions. Using that $\mathcal{X}$ defined in (4.4.13) depends only on R, that $\mathcal{G}(y_1)\ge c_R\delta^2$ on this strip, and the same computations as above, we obtain a bound uniform in d:

$$\int_{A_{\delta,d}\setminus A_{0,d}}|\bar v_d|^p\,\mathrm{d}y\ \le\ C(R)\qquad\text{for all } p\in[1,\infty). \tag{4.5.5}$$

Combining (4.5.4), (4.5.5), and the estimate on the core strip $A_{0,d}$, we conclude that

$$\int_{B(0,R)}|\bar v_d(y)|^p\,\mathrm{d}y\ \le\ C(R)\qquad\text{for all } p\in[1,3),$$

with a constant $C(R)$ independent of d.

□

Lemma 4.5.2. *Let $d\in(0,R-1]$. Then there exists a constant $C=C(R)>0$, depending only on R, such that*

$$\left\|\frac{\partial\bar g_d}{\partial d}\right\|_{L^2(A_{\delta,d})}\le C(R), \tag{4.5.6}$$

where $\bar g_d$ has been defined in (4.4.14).

Proof. Recall that

$$\bar g_d(y)=y_1\Big((1-\mathcal{X}(y_1))\,\phi(y)+\mathcal{X}(y_1)F^2(y)\,(3-2F(y))\Big)\qquad(y\in A_{\delta,d}), \tag{4.5.7}$$

where $F(y)$ and $\mathcal{X}(y_1)$ are defined by (4.4.10), (4.4.13), respectively.

Since only ψ_{up} in (4.4.5) depends on d (while ψ_{low}, ϕ, and $\mathcal{X}$ do not), differentiating $F(y)$ with respect to d yields:

$$\frac{\partial F(y,d)}{\partial d}=-\,\frac{y_2-\psi_{\mathrm{low}}(y_1)}{\mathcal{G}(y_1)^2},$$

where $\mathcal{G}$ is the gap defined in (4.4.8).

With a simple calculation, we get

$$\frac{\partial \bar{g}_d}{\partial d}(y) = y_1\,\mathcal{X}(y_1)\left(-\frac{6\,(y_2-\psi_{\mathrm{low}}(y_1))^2}{\mathcal{G}(y_1)^3}+\frac{6\,(y_2-\psi_{\mathrm{low}}(y_1))^3}{\mathcal{G}(y_1)^4}\right), \qquad y\in A_{\delta,d}.$$

Using $0\le y_2-\psi_{\mathrm{low}}(y_1)\le \mathcal{G}(y_1)$, we get the pointwise bound

$$\left|\frac{\partial \bar{g}_d}{\partial d}(y)\right| \;\le\; C(R)\,\frac{|y_1|}{\mathcal{G}(y_1)} \qquad (y\in A_{\delta,d}), \tag{4.5.8}$$

with a constant $C(R)$ independent of d.

Integrating (4.5.8) over $A_{\delta,d}$ and using Tonelli's theorem, we have

$$\int_{A_{\delta,d}}\left|\frac{\partial \bar{g}_d}{\partial d}\right|^2 \mathrm{d}y \;\le\; C\int_{-2\delta}^{2\delta}\frac{y_1^2}{\mathcal{G}(y_1)^2}\left(\int_{\psi_{\mathrm{low}}(y_1)}^{\psi_{\mathrm{up}}(y_1)}\mathrm{d}y_2\right)\mathrm{d}y_1 \;=\; C(R)\int_{-2\delta}^{2\delta}\frac{y_1^2}{\mathcal{G}(y_1)}\,\mathrm{d}y_1.$$

By Lemma 4.4.1, there is a $c_R>0$, such that

$$\mathcal{G}(y_1)\ge d+c_R y_1^2,$$

with the fact $d\ge 0$, it follows that

$$\int_{-2\delta}^{2\delta}\frac{y_1^2}{\mathcal{G}(y_1)}\,\mathrm{d}y_1 \;\le\; \int_{-2\delta}^{2\delta}\frac{y_1^2}{d+c_R y_1^2}\,\mathrm{d}y_1 \;\le\; C(R).$$

Therefore

$$\left\|\frac{\partial \bar{g}_d}{\partial d}\right\|_{L^2(A_{\delta,d})} \;\le\; C(R).$$

This completes the proof. □

We now derive some bounds on the derivatives of $\bar{v}$.

Lemma 4.5.3. *Let $d\in(0,R-1]$. Then there exists a constant $C=C_i(R)>0$, depending only on R, such that*

$$\|\partial_1\bar{v}_{d,2}\|_{L^2(A_{\delta,d})}=\|\partial_{11}\bar{g}_d\|_{L^2(A_{\delta,d})}\le C(R) \tag{4.5.9}$$

$$\frac{C_1(R)}{d^{3/4}}\le\|\partial_2\bar{v}_{d,1}\|_{L^2(A_{\delta,d})}\le\frac{C_2(R)}{d^{3/4}}, \tag{4.5.10}$$

where $\bar{v}_d$ is defined in (4.4.15).

The proof involves routine but lengthy computations; we put it in Appendix 4.8.

4.6 The main estimate for the test function

In this section we establish the key estimates needed for the proof of Theorem 4.2.3. The argument follows [Sab19]; we adapt it to our geometric setting and provide full details for the reader's convenience.

Standing notation. To simplify the computations, we assume throughout this section and the next that the viscosity coefficient satisfies $\nu = 1$. Fix $t \in (0,T)$ such that $d(t) > 0$, and set

$$d := d(t) \in (0, R-1).$$

All objects in this section are written in the rotated variables $y = (y_1, y_2)$. Recall that $\bar{g}_d$ and $\bar{v}_d = \nabla^\perp \bar{g}_d = (\bar{v}_{d,1}, \bar{v}_{d,2})^\top$ are defined in (4.4.14)–(4.4.15). We denote by $A_{\delta,d}$ the gap region defined in (4.4.6), bounded by the graphs $y_2 = \psi_{\mathrm{low}}(y_1)$ and $y_2 = \psi_{\mathrm{up}}(y_1)$ introduced in (4.4.4)–(4.4.5). We also define the inner disk in the rotated frame by

$$S_d := B(y_C, 1), \qquad y_C = (0, -R + d + 1)^\top,$$

and we write $n = (n_1, n_2)^\top$ for the outward unit normal to $\partial A_{\delta,d}$. Finally, $\bar{e}_2 = (0,1)^\top$, and $C = C(R) > 0$ denotes a generic constant depending only on R. For brevity, we write

$$\partial_i := \frac{\partial}{\partial y_i} \qquad (i = 1, 2).$$

The goal of this section is to prove Propositions 4.6.7 and 4.6.8. More precisely, we will first establish a comparison estimate between the viscous term $\int_{A_{\delta,d}} 2D(\bar{v}_d) : D(\bar{u})$ and a scalar boundary quantity involving $\tilde{m}(d)$. We then show that $\tilde{m}(d)$ blows up as $d \to 0$, which is the key mechanism leading to the non-collision argument. Since the proofs require additional notation and preparatory results, we begin by introducing an auxiliary pressure and a boundary coefficient.

4.6.1 Auxiliary pressure and boundary coefficient

We introduce an auxiliary pressure q_d and a scalar coefficient $\tilde{m}(d)$, which will capture the boundary contribution on ∂S_d.

Definition 4.6.1. We define $q_d : A_{\delta,d} \to \mathbb{R}$ by

$$q_d(y) = \partial_{12} \bar{g}_d(y) + \int_{-2\delta}^{y_1} \frac{12\, s\, \mathcal{X}(s)}{\big(\psi_{\mathrm{up}}(s) - \psi_{\mathrm{low}}(s)\big)^3}\, \mathrm{d}s, \tag{4.6.1}$$

where δ is defined in (4.4.3), $\mathcal{X}$ is defined in (4.4.13), and $\psi_{\mathrm{up}}, \psi_{\mathrm{low}}$ are given in (4.4.5)–(4.4.4) (with parameter d). This choice of q_d is tailored so that the defect $-\Delta \bar{v}_d + \nabla q_d$ admits a convenient explicit form and enjoys good symmetry properties.

We also define

$$\tilde{m}(d) := \int_{\partial A_{\delta,d} \cap \partial S_d} (2D(\bar{v}_d) - q_d I) n \cdot \bar{e}_2 \, \mathrm{d}\Gamma. \tag{4.6.2}$$

Comment. The pressure q_d is only used as an auxiliary quantity in the thin region $A_{\delta,d}$. It is introduced so that the defect $-\Delta \bar{v}_d + \nabla q_d$ can be written explicitly and, crucially, so that it satisfies convenient parity properties with respect to the reflection $y_1 \mapsto -y_1$. Moreover, $\tilde{m}(d)$ depends only on the gap parameter d and will later provide the effective "dissipation coefficient" in the contradiction argument.

Throughout the rest of this section, we keep the above notation and do not repeat it in each statement.

4.6.2 A decomposition formula

The next lemma is the basic integration-by-parts identity used in the proof of Proposition 4.6.7.

Lemma 4.6.2 (Integration by parts identity)**.** *Let $\bar{u} \in H_0^1(B(0,R))$ be divergence-free. Then*

$$\begin{aligned} &\int_{A_{\delta,d}} (\Delta \bar{v}_d - \nabla q_d) \cdot \bar{u} \, \mathrm{d}y \\ &= -\int_{A_{\delta,d}} 2D(\bar{v}_d) : D(\bar{u}) \, \mathrm{d}y + \int_{\partial A_{\delta,d}} (2D(\bar{v}_d) - q_d I) n \cdot \bar{u} \, \mathrm{d}\Gamma. \end{aligned} \tag{4.6.3}$$

Proof. This follows from integration by parts, the definition of $D(\cdot)$ in (2.2.10), and the incompressibility $\operatorname{div} \bar{u} = 0$. □

4.6.3 Estimate of the left term in (4.6.3)

We first compute an explicit expression for the defect $-\Delta \bar{v}_d + \nabla q_d$ in $A_{\delta,d}$, and then bound its action on divergence-free test fields.

Lemma 4.6.3 (Expression of the defect)**.** *From the definitions of $\bar{v}_d$ and $\bar{g}_d$ in (4.4.14)–(4.4.15), we have*

$$\Delta \bar{v}_d(y) - \nabla q_d(y) = \begin{pmatrix} -2\,\partial_{112}\bar{g}_d(y) - y_1(1 - \mathcal{X}(y_1))\,\partial_{222}\varphi(y) \\ \partial_{111}\bar{g}_d(y) \end{pmatrix}, \qquad y \in A_{\delta,d}. \tag{4.6.4}$$

and,

$$\begin{aligned} \int_{A_{\delta,d}} (-\Delta \bar{v}_d + \nabla q_d) \cdot \bar{u} \, \mathrm{d}y = &\int_{A_{\delta,d}} (2\partial_{112}\bar{g}_d \, \bar{u}_1 - \partial_{111}\bar{g}_d \, \bar{u}_2) \, \mathrm{d}y \\ &+ \int_{A_{\delta,d}} y_1(1 - \mathcal{X}(y_1))\,\partial_{222}\varphi(y)\,\bar{u}_1 \, \mathrm{d}y. \end{aligned} \tag{4.6.5}$$

Here $\mathcal{X}$ is defined in (4.4.13) and φ is defined in (4.4.12).

Proof. The formula follows from a direct differentiation of $\bar{v}_d = \nabla^\perp \bar{g}_d$ and from the definition of q_d in (4.6.1). □

Lemma 4.6.4. *There exists $C(R) > 0$ such that for any divergence-free $\bar{u} \in H^1_0(B(0,R))$, we have*

$$\left| \int_{A_{\delta,d}} (-\Delta \bar{v}_d + \nabla q_d) \cdot \bar{u} \, \mathrm{d}y \right| \leq C(R) \Big(\|\bar{u}\|_{L^2(B(0,R))} + \|\nabla \bar{u}\|_{L^2(B(0,R))} + \|\bar{u}\|_{L^\infty(\partial S_d)} \Big).$$

Proof. Starting from (4.6.5), we first estimate

$$\int_{A_{\delta,d}} (2\partial_{112}\bar{g}_d \, \bar{u}_1 - \partial_{111}\bar{g}_d \, \bar{u}_2) \, \mathrm{d}y.$$

Integrating by parts gives

$$\begin{aligned} \int_{A_{\delta,d}} (2\partial_{112}\bar{g}_d \, \bar{u}_1 - \partial_{111}\bar{g}_d \, \bar{u}_2) \, \mathrm{d}y = &- \int_{A_{\delta,d}} \partial_{11}\bar{g}_d \, (2\partial_2 \bar{u}_1 - \partial_1 \bar{u}_2) \, \mathrm{d}y \\ &+ \int_{\partial A_{\delta,d}} \partial_{11}\bar{g}_d (2\bar{u}_1 n_2 - \bar{u}_2 n_1) \, \mathrm{d}\Gamma. \end{aligned} \tag{4.6.6}$$

Using (4.5.9) and Hölder's inequality,

$$\left| \int_{A_{\delta,d}} \partial_{11}\bar{g}_d \, (2\partial_2 \bar{u}_1 - \partial_1 \bar{u}_2) \, \mathrm{d}y \right| \leq C(R) \Big(\|\partial_2 \bar{u}_1\|_{L^2(A_{\delta,d})} + \|\partial_1 \bar{u}_2\|_{L^2(A_{\delta,d})} \Big). \tag{4.6.7}$$

Boundary decomposition. We now estimate the boundary term in (4.6.6). We split $\partial A_{\delta,d}$ into the inner boundary, the two vertical segments $|y_1| = 2\delta$, and the outer circle. This allows us to use (i) $\bar{u} = 0$ on $\partial B(0,R)$, (ii) uniform smoothness on the vertical parts, and (iii) symmetry properties on ∂S_d. We decompose

$$\partial A_{\delta,d} = (\partial A_{\delta,d} \cap \partial S_d) \ \cup \ (\partial A_{\delta,d} \cap \{|y_1| = 2\delta\}) \ \cup \ (\partial A_{\delta,d} \cap \partial B(0,R)).$$

Since $\bar{u} = 0$ on $\partial B(0,R)$, the outer boundary does not contribute. Using the symmetry in y_1 and boundary parametrizations, we obtain

$$\begin{aligned} \left| \int_{\partial A_{\delta,d}} \partial_{11}\bar{g}_d \, \bar{u}_2 n_1 \, \mathrm{d}\Gamma \right| \leq C(R) \Bigg(&\|\bar{u}\|_{L^\infty(\partial S_d)} \int_{-2\delta}^{2\delta} \frac{|y_1| \, |\partial_1 \psi_{\mathrm{up}}(y_1)|^3}{\big(\psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1)\big)^2} \, \mathrm{d}y_1 \\ &+ \left| \int_{\partial A_{\delta,d} \cap \{|y_1| = 2\delta\}} \partial_{11}\bar{g}_d \, \bar{u}_2 n_1 \, \mathrm{d}\Gamma \right| \Bigg). \end{aligned} \tag{4.6.8}$$

For the first term on the right-hand side, using (4.4.9) and (4.5.2) gives

$$\|\bar{u}\|_{L^\infty(\partial S_d)} \int_{-2\delta}^{2\delta} \frac{|y_1| \, |\partial_1 \psi_{\mathrm{up}}(y_1)|^3}{\big(\psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1)\big)^2} \, \mathrm{d}y_1 \leq C(R) \, \|\bar{u}\|_{L^\infty(\partial S_d)}.$$

For the vertical parts, we use the parametrization and the fact that $\partial_{11}\bar{g}_d$ is odd in y_1 to write

$$\begin{aligned}
&\int_{\partial A_{\delta,d}\cap\{|y_1|=2\delta\}} \partial_{11}\bar{g}_d\,\bar{u}_2 n_1\,\mathrm{d}\Gamma \\
&= \int_{\psi_{\mathrm{low}}(2\delta)}^{\psi_{\mathrm{up}}(2\delta)} \partial_{11}\bar{g}_d(2\delta,y_2)\,(\bar{u}_2(2\delta,y_2)-\bar{u}_2(-2\delta,y_2))\,\mathrm{d}y_2 \\
&= \int_{\psi_{\mathrm{low}}(2\delta)}^{\psi_{\mathrm{up}}(2\delta)} \partial_{11}\bar{g}_d(2\delta,y_2)\int_{-2\delta}^{2\delta}\partial_1\bar{u}_2(s,y_2)\,\mathrm{d}s\,\mathrm{d}y_2. \qquad (4.6.9)
\end{aligned}$$

Applying (4.5.9) and Hölder's inequality yields

$$\left|\int_{\partial A_{\delta,d}\cap\{|y_1|=2\delta\}} \partial_{11}\bar{g}_d\,\bar{u}_2 n_1\,\mathrm{d}\Gamma\right| \le C(R)\,\|\partial_1\bar{u}_2\|_{L^2([-2\delta,2\delta]\times[\psi_{\mathrm{low}}(2\delta),\psi_{\mathrm{up}}(2\delta)])}. \qquad (4.6.10)$$

Combining the above bounds gives

$$\left|\int_{\partial A_{\delta,d}} \partial_{11}\bar{g}_d\,\bar{u}_2 n_1\,\mathrm{d}\Gamma\right| \le C(R)\Big(\|\bar{u}\|_{L^\infty(\partial S_d)}+\|\partial_1\bar{u}_2\|_{L^2([-2\delta,2\delta]\times[\psi_{\mathrm{low}}(2\delta),\psi_{\mathrm{up}}(2\delta)])}\Big). \qquad (4.6.11)$$

Cancellation on the top boundary. On the upper boundary $y=(y_1,\psi_{\mathrm{up}}(y_1))$, the integration is taken over a symmetric interval $[-2\delta,2\delta]$. Since $\partial_{11}\bar{g}_d$ is odd in y_1 while ψ_{up} is even, the corresponding boundary integrand becomes odd and the contribution cancels.

On the upper boundary $y=(y_1,\psi_{\mathrm{up}}(y_1))$, we have $n_2\,\mathrm{d}\Gamma=\mathrm{d}y_1$, hence

$$\int_{\partial A_{\delta,d}} \partial_{11}\bar{g}_d\,\bar{u}_2\,n_2\,\mathrm{d}\Gamma = \int_{-2\delta}^{2\delta} \partial_{11}\bar{g}_d(y_1,\psi_{\mathrm{up}}(y_1))\;\bar{u}_2(y_1,\psi_{\mathrm{up}}(y_1))\;\mathrm{d}y_1. \qquad (4.6.12)$$

Since $\partial_{11}\bar{g}_d$ is odd in y_1 while ψ_{up} is even, the integrand is odd, and therefore

$$\int_{\partial A_{\delta,d}} \partial_{11}\bar{g}_d\,\bar{u}_2\,n_2\,\mathrm{d}\Gamma = 0. \qquad (4.6.13)$$

Finally, since $y_1(1-\mathcal{X}(y_1))$ and $\partial_{222}\varphi$ are uniformly bounded on $A_{\delta,d}$, we have

$$\left|\int_{A_{\delta,d}} y_1(1-\mathcal{X}(y_1))\,\partial_{222}\varphi(y)\,\bar{u}_1\,\mathrm{d}y\right| \le C(R)\|\bar{u}\|_{L^2(A_{\delta,d})}.$$

Collecting the above bounds together with (4.6.7), (4.6.11), and (4.6.13) completes the proof. □

4.6.4 Boundary reduction by symmetry

We now control the boundary contributions appearing in Lemma 4.6.2. The vertical parts $|y_1| = 2\delta$ are estimated by standard trace inequalities, and the part on ∂S_d is reduced to $\tilde{m}(d)$ by symmetry.

Lemma 4.6.5 (Vertical boundary contribution)**.** *There exists $C(R) > 0$ such that*

$$\left| \int_{\partial A_{\delta,d} \cap \{|y_1| = 2\delta\}} (2D(\bar{v}_d) - q_d I) n \cdot \bar{u} \, \mathrm{d}\Gamma \right| \leq C(R) \, \|\bar{u}\|_{H^1(A_{\delta,d})}. \tag{4.6.14}$$

Proof. Since $\nabla \bar{v}_d$ and q_d are smooth up to the boundary, they are uniformly bounded on $\partial A_{\delta,d} \cap \{|y_1| = 2\delta\}$. Moreover, $\delta > 0$ depends only on R, and

$$\psi_{\mathrm{up}}(2\delta) - \psi_{\mathrm{low}}(2\delta) \leq C(R).$$

Hence

$$|(2D(\bar{v}_d) - q_d I) n| \leq C(R) \qquad \text{for } y \in \partial A_{\delta,d} \cap \{|y_1| = 2\delta\}. \tag{4.6.15}$$

The claim follows from the trace theorem and Hölder's inequality. □

Key step. The next lemma shows that the boundary integral on ∂S_d depends only on the translational component $V(\bar{u})$ of the rigid motion inside S_d, and it reduces to the scalar coefficient $\tilde{m}(d)$. This reduction relies on parity cancellations with respect to $y_1 \mapsto -y_1$.

Lemma 4.6.6 (Reduction on ∂S_d by symmetry)**.** *Assume in addition that $\bar{u}$ coincides with a rigid motion in S_d, namely*

$$\bar{u}(y) = V(\bar{u}) + \omega \, (y - y_C)^{\perp}, \qquad y \in S_d, \tag{4.6.16}$$

where $V(\bar{u}) \in \mathbb{R}^2$, $\omega \in \mathbb{R}$, and $y_C = (0, -R + d + 1)^{\top}$. Then

$$\int_{\partial A_{\delta,d} \cap \partial S_d} (2D(\bar{v}_d) - q_d I) n \cdot \bar{u} \, \mathrm{d}\Gamma = \tilde{m}(d) \, (V(\bar{u}) \cdot \bar{e}_2).$$

Proof. Using the explicit formulas for $(\bar{v}_d, q_d)$ and the reflection $y_1 \mapsto -y_1$, one checks that

$$q_d, \quad \partial_{y_1} \bar{v}_{d,1}, \quad \partial_{y_2} \bar{v}_{d,2} \text{ are even in } y_1, \qquad \partial_{y_2} \bar{v}_{d,1}, \quad \partial_{y_1} \bar{v}_{d,2} \text{ are odd in } y_1. \tag{4.6.17}$$

Therefore, when integrating over the symmetric boundary piece $\partial A_{\delta,d} \cap \partial S_d$, all odd contributions cancel.

Reduction to translation. Since the boundary piece $\partial A_{\delta,d} \cap \partial S_d$ is symmetric with respect to $y_1 \mapsto -y_1$, the rotational part $\omega \, (y - y_C)^{\perp}$ yields only odd contributions and thus vanishes after integration. Consequently, only the vertical component of $V(\bar{u})$ survives.

Using the rigid-motion trace (4.6.16), the rotational part does not contribute by symmetry, and only the vertical component of $V(\bar{u})$ remains. This gives the desired identity. □

4.6.5 Main estimate

We now combine the estimate (4.6.4) and the boundary reductions to prove the main inequality.

Proposition 4.6.7 (Main inequality for the test function)**.** *Let $d \in (0, R-1)$ and let $\bar{u} \in H_0^1(B(0,R))$ be divergence-free. Assume moreover that $\bar{u}$ satisfies the rigid-motion trace condition (4.6.16) in S_d. Then there exists a constant $C(R) > 0$ such that*

$$\begin{aligned}&\left|\int_{A_{\delta,d}} 2D(\bar{v}_d) : D(\bar{u})\,\mathrm{d}y - \tilde{m}(d)\,(V(\bar{u})\cdot \bar{e}_2)\right| \\ \leq & C(R)\Big(\|\bar{u}\|_{L^2(B(0,R))} + \|\bar{u}\|_{L^\infty(\partial S_d)} + \|\nabla \bar{u}\|_{L^2(B(0,R))}\Big).\end{aligned} \tag{4.6.18}$$

Proof. We start from the integration-by-parts identity in Lemma 4.6.2. The right term of (4.6.2) is controlled by Lemma 4.6.4, the vertical boundary terms by Lemma 4.6.5, and the inner boundary term on ∂S_d is reduced to $\tilde{m}(d)$ by Lemma 4.6.6.

Starting from Lemma 4.6.2, we rewrite (4.6.3) as

$$\int_{A_{\delta,d}} 2D(\bar{v}_d) : D(\bar{u})\,\mathrm{d}y = -\int_{A_{\delta,d}} (\Delta \bar{v}_d - \nabla q_d)\cdot \bar{u}\,\mathrm{d}y + \int_{\partial A_{\delta,d}} (2D(\bar{v}_d) - q_d I)n \cdot \bar{u}\,\mathrm{d}\Gamma.$$

The first term in the right equality is estimated by Lemma 4.6.4. For the boundary term, we decompose

$$\partial A_{\delta,d} = (\partial A_{\delta,d} \cap \{|y_1| = 2\delta\}) \cup (\partial A_{\delta,d} \cap \partial B(0,R)) \cup (\partial A_{\delta,d} \cap \partial S_d).$$

Since $\bar{u} = 0$ on $\partial B(0,R)$, the outer boundary contribution vanishes. The vertical contribution is controlled by Lemma 4.6.5. Finally, the inner boundary term is reduced by Lemma 4.6.6, yielding

$$\int_{\partial A_{\delta,d}\cap \partial S_d} (2D(\bar{v}_d) - q_d I)n \cdot \bar{u}\,\mathrm{d}\Gamma = \tilde{m}(d)\,(V(\bar{u})\cdot \bar{e}_2).$$

Combining the above estimates proves (4.6.18). □

4.6.6 Lower bound on $\tilde{m}(d)$

We finally prove that $\tilde{m}(d)$ blows up as $d \to 0$, which is crucial for the dissipation mechanism.

Proposition 4.6.8 (Lower bound on $\tilde{m}(d)$)**.** *There exists $\eta = \eta(R) > 0$, independent of d, such that for all $0 < d \leq \eta$,*

$$\tilde{m}(d) \;\geq\; \frac{C(R)}{d^{3/2}}. \tag{4.6.19}$$

Proof. Main mechanism. The blow-up of $\tilde{m}(d)$ as $d \to 0$ comes from the dominant shear inside the thin gap, captured by $\partial_2 \bar{v}_{d,1}$, whose L^2-norm scales like $d^{-3/4}$. This yields the scaling $\tilde{m}(d) \gtrsim \|\partial_2 \bar{v}_{d,1}\|_{L^2}^2 \sim d^{-3/2}$.

Taking $\bar{u} = \bar{v}_d$ in Proposition 4.6.7, we obtain

$$\begin{aligned}&\left|\int_{A_{\delta,d}} 2D(\bar{v}_d) : D(\bar{v}_d)\,\mathrm{d}y - \tilde{m}(d)\,(V(\bar{v}_d)\cdot \bar{e}_2)\right| \\ \le& C(R)\Big(\|\bar{v}_d\|_{L^2(B(0,R))} + \|\bar{v}_d\|_{L^\infty(\partial S_d)} + \|\nabla \bar{v}_d\|_{L^2(B(0,R))}\Big). \end{aligned} \tag{4.6.20}$$

Since $\bar{v}_d = \bar{e}_2$ in S_d (by Proposition 4.4.3), we have $V(\bar{v}_d) = \bar{e}_2$ and hence $V(\bar{v}_d)\cdot \bar{e}_2 = 1$. Therefore, (4.6.20) implies

$$\begin{aligned}\tilde{m}(d) \ge& \int_{A_{\delta,d}} 2D(\bar{v}_d) : D(\bar{v}_d)\,\mathrm{d}y - C(R)\Big(\|\bar{v}_d\|_{L^2(A_{\delta,d})} + \|\bar{v}_d\|_{L^\infty(S_d)} \\ &+ \|\partial_{y_2}\bar{v}_{d,1}\|_{L^2(A_{\delta,d})} + \|\partial_{y_1}\bar{v}_{d,2}\|_{L^2(A_{\delta,d})}\Big). \end{aligned} \tag{4.6.21}$$

From (4.5.9) we have

$$\|\partial_1 \bar{v}_{d,2}\|_{L^2(A_{\delta,d})} = \|\partial_{11}\bar{g}_d\|_{L^2(A_{\delta,d})} \le C(R), \qquad \|\partial_2 \bar{v}_{d,1}\|_{L^2(A_{\delta,d})} \ge \frac{C(R)}{d^{\frac{3}{4}}}. \tag{4.6.22}$$

Using the elementary inequality

$$(a+b)^2 \ge \frac{1}{2}a^2 - b^2 \qquad (a, b \in \mathbb{R}),$$

together with the explicit structure of $D(\bar{v}_d)$, we obtain (for d small enough)

$$\begin{aligned}\int_{A_{\delta,d}} 2D(\bar{v}_d) : D(\bar{v}_d)\,\mathrm{d}y \ge& \frac{1}{2}\|\partial_2 \bar{v}_{d,1} + \partial_1 \bar{v}_{d,2}\|_{L^2(A_{\delta,d})}^2 \\ \ge& \frac{1}{4}\|\partial_2 \bar{v}_{d,1}\|_{L^2(A_{\delta,d})}^2 - \frac{1}{2}\|\partial_1 \bar{v}_{d,2}\|_{L^2(A_{\delta,d})}^2 \ \ge\ \frac{C(R)}{d^{3/2}}. \end{aligned} \tag{4.6.23}$$

This proves (4.6.19) for all $0 < d \le \eta(R)$. □

4.7 Proof of Theorem 4.2.3

In this section, we prove Theorem 4.2.3 by a contradiction argument. The strategy is the same as in [Sab19, Section 3.2]; for completeness, we provide the details.

To begin with, we first restrict to the case where the disk stays away from the origin. This is only for technical convenience: the rotation map Y_τ introduced in (4.3.2) involves the normalization $\tau/|\tau|$ and becomes singular as $\tau \to 0$. More precisely, we assume that

$$|h(t)| > \frac{1}{8}(R-1) \qquad \text{for } t \in (0, T^*). \tag{4.7.1}$$

Here, T^* denotes the maximal existence time of system (4.1.1)-(4.1.8) in Proposition 4.2.2; it may be finite or equal to $+\infty$. (We will explain at the end of the proof why this assumption does not restrict the contradiction argument.)

We justify that the test function constructed in Section 4.4 is admissible in the weak formulation in Definition 4.2.4.

Lemma 4.7.1 (Regularity of the test function)**.** *Assume* (4.7.1)*. Let v be the test function defined by* (4.4.16)*. Then the associated rigid-body coefficients can be chosen as*

$$\ell(t) := e_{h(t)} = \frac{h(t)}{|h(t)|}, \qquad r(t) := 0,$$

and they satisfy

$$\ell \in H^1((0,T);\mathbb{R}^2), \qquad r \in H^1((0,T);\mathbb{R}).$$

Moreover,

$$v \in H^1((0,T);L^2(B(0,R))) \cap L^2((0,T);H^1(B(0,R))).$$

Proof sketch. **Step 1: Regularity of ℓ and r.** By (4.7.1) there exists $c_0 > 0$ such that $|h(t)| \geq c_0$ for all $t \in (0,T)$. Since the map $z \mapsto z/|z|$ is smooth on $\{z \in \mathbb{R}^2 : \ |z| \geq c_0\}$, the chain rule yields

$$\dot{\ell}(t) = \Big(\frac{1}{|h(t)|} I - \frac{h(t) \otimes h(t)}{|h(t)|^3}\Big)\dot{h}(t),$$

hence $|\dot{\ell}(t)| \leq C(c_0)\,|\dot{h}(t)|$. Therefore $\ell \in H^1((0,T);\mathbb{R}^2)$. Clearly $r(t) \equiv 0 \in H^1((0,T);\mathbb{R})$.

Step 2: Regularity of the gap function. Recall that $d(t) = R - 1 - |h(t)|$. Differentiating gives

$$\dot{d}(t) = -\frac{h(t)}{|h(t)|} \cdot \dot{h}(t) = \dot{h}(t) \cdot e_2(t), \qquad e_2(t) := -\frac{h(t)}{|h(t)|},$$

and therefore $|\dot{d}(t)| \leq |\dot{h}(t)|$, so $d \in H^1(0,T)$.

Step 3: Regularity of v before contact. Fix $T_0 \in (0,T^*)$ such that the disk remains at a positive distance from the outer boundary on $[0,T_0]$, namely

$$d_0 := \min_{t\in[0,T_0]} d(t) > 0. \tag{4.7.2}$$

We claim that on $(0,T_0)$ the test function v defined by (4.4.16) is admissible, i.e.

$$v \in L^2((0,T_0);H^1(B(0,R))) \cap H^1((0,T_0);L^2(B(0,R))).$$

Recall that

$$v(t,x) = J_x Y_{h(t)}\, \bar{v}_{d(t)}(Y_{h(t)}(x)),$$

where $Y_{h(t)}$ is a rotation, hence an isometry. In particular, the change of variables $y = Y_{h(t)}(x)$ preserves Lebesgue measure and L^2-norms. Therefore,

$$\|\nabla v(t)\|_{L^2(B(0,R))} = \|\nabla \bar{v}_{d(t)}\|_{L^2(B(0,R))}.$$

Spatial regularity. We split the domain into the thin region $A_{\delta,d(t)}$ and its complement. Outside $A_{\delta,d(t)}$, the field $\bar{v}_{d(t)}$ is built using smooth cut-off functions independent of $d(t)$, hence

$$\|\nabla \bar{v}_{d(t)}\|_{L^2(B(0,R)\setminus A_{\delta,d(t)})} \le C(R).$$

Inside $A_{\delta,d(t)}$, Lemma 4.5.3 yields

$$\|\partial_1 \bar{v}_{d(t),2}\|_{L^2(A_{\delta,d(t)})} \le C(R), \qquad \|\partial_2 \bar{v}_{d(t),1}\|_{L^2(A_{\delta,d(t)})} \le \frac{C(R)}{d(t)^{3/4}} \le \frac{C(R)}{d_0^{3/4}}.$$

Combining these bounds, we obtain for a.e. $t \in (0, T_0)$,

$$\|\nabla v(t)\|^2_{L^2(B(0,R))} = \|\nabla \bar{v}_{d(t)}\|^2_{L^2(B(0,R))} \le C(R) + \frac{C(R)}{d_0^{3/2}}.$$

Hence $v \in L^2((0,T_0); H^1(B(0,R)))$.

Time regularity. Differentiating $v(t,x)$ with respect to t and using the chain rule, we obtain

$$\begin{aligned}\partial_t v(t,x) = {} & (\partial_t J_x Y_{h(t)})\, \bar{v}_{d(t)}(Y_{h(t)}(x)) \\ & + J_x Y_{h(t)} \Big(\dot{d}(t)\, \partial_d \bar{v}_{d(t)}(Y_{h(t)}(x)) + (\nabla \bar{v}_{d(t)})(Y_{h(t)}(x))\, \partial_t Y_{h(t)}(x) \Big).\end{aligned}$$

Using the isometry of $Y_{h(t)}$, the Cauchy–Schwarz inequality, and the bound

$$\|\partial_t Y_{h(t)}\|_{L^\infty(B(0,R))} \le C(R)\, |\dot{h}(t)|,$$

we infer that for a.e. $t \in (0, T_0)$,

$$\|\partial_t v(t)\|_{L^2(B(0,R))} \le C(R)(|\dot{h}(t)| + |\dot{d}(t)|) \left(\|\bar{v}_{d(t)}\|_{H^1(B(0,R))} + \|\partial_d \bar{v}_{d(t)}\|_{L^2(B(0,R))} \right). \quad (4.7.3)$$

Moreover, since $d(t) = R - 1 - |h(t)|$, we have $|\dot{d}(t)| \le |\dot{h}(t)|$. Using again the uniform lower bound $d(t) \ge d_0$ on $[0, T_0]$ and the estimates collected in Lemmas 4.5.1, 4.5.2, and 4.5.3, we obtain

$$\|\partial_t v(t)\|_{L^2(B(0,R))} \le \frac{C(R)}{d_0^{3/4}}\, |\dot{h}(t)| \qquad \text{for a.e. } t \in (0, T_0). \quad (4.7.4)$$

Finally, the energy inequality (4.2.2) implies $\dot{h} \in L^2(0,T_0)$. Therefore (4.7.4) yields $\partial_t v \in L^2((0,T_0);L^2(B(0,R)))$, and hence $v \in H^1((0,T_0);L^2(B(0,R)))$.

Remark. Estimate (4.7.4) shows explicitly that the admissibility of v relies on the strict positivity of the gap. If $d(t) \to 0$, then the factor $d_0^{-3/4}$ may blow up, so the bounds on $\partial_t v$ (and similarly on ∇v) degenerate. This is precisely why the test function is only used on time intervals where contact has not yet occurred. □

Proof of Theorem 4.2.3. **Step 1. Extension of the velocity.** For later use, we extend the velocity to the whole ball $B(0,R)$ by

$$\widetilde{u}(t,x) = \begin{cases} u(t,x), & x \in \Omega(t), \\ \dot{h}(t) + \omega(t)(x-h(t))^{\perp}, & x \in S(t), \end{cases} \tag{4.7.5}$$

where u is the fluid velocity, $\dot{h}$ is the translational velocity of the disk center, and ω is the angular velocity. By the no-slip condition (4.1.4) and Proposition 4.2.2,

$$\widetilde{u} \in H^1(0,T;L^2(B(0,R))) \cap C([0,T];H^1(B(0,R))). \tag{4.7.6}$$

Step 2. Weak formulation with the test function. We choose the test function v defined in (4.4.16) and integrate from 0 to t. Then, for a.e. $t \in (0,T)$,

$$\begin{aligned} &\int_{B(0,R)} \widetilde{u}(t,x)\cdot v(t,x)\,\mathrm{d}x - \int_{B(0,R)} \widetilde{u}(0,x)\cdot v(0,x)\,\mathrm{d}x \\ &= \int_0^t \int_{B(0,R)} \Big(\widetilde{u}\cdot\partial_t v(s,x) + (\widetilde{u}\otimes\widetilde{u}) : D[v](s,x) - 2\,D[\widetilde{u}] : D[v](s,x) \Big)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.7.7}$$

Insertion of the coefficient term. Since $\tilde{m}(d(t))\,V(\bar{\widetilde{u}})\cdot\bar{e}_2$ is a scalar quantity depending only on time, we may add its time antiderivative to both sides of (4.7.7) without altering the equality. Thus we obtain

$$P(t) - P(0) - \int_0^t A_1(s)\,\mathrm{d}s + \int_0^t A_2(s)\,\mathrm{d}s + \int_0^t A_3(s)\,\mathrm{d}s = \int_0^t \tilde{m}(d(s))\,V(\bar{\widetilde{u}})\cdot\bar{e}_2\,\mathrm{d}s, \tag{4.7.8}$$

where

$$P(t) = \int_{B(0,R)} \widetilde{u}(t,x)\cdot v(t,x)\,\mathrm{d}x, \tag{4.7.9}$$

$$A_1(t) = \int_{B(0,R)} \widetilde{u}(t,x)\cdot\partial_t v(t,x)\,\mathrm{d}x, \tag{4.7.10}$$

$$A_2(t) = \int_{B(0,R)} (\widetilde{u}(t,x)\otimes\widetilde{u}(t,x)) : D[v](t,x)\,\mathrm{d}x, \tag{4.7.11}$$

$$A_3(t) = \int_{B(0,R)} 2\,D[\widetilde{u}](t,x) : D[v](t,x)\,\mathrm{d}x - \tilde{m}(d(t))\,V(\bar{\widetilde{u}})\cdot\bar{e}_2. \tag{4.7.12}$$

Outline of the estimates. We next show that $P(t)$ is uniformly bounded, that $\int_0^t A_1$, $\int_0^t A_2$ and $\int_0^t A_3 \mathrm{d}s$ are controlled by the initial energy. These bounds yield a global upper bound on the right-hand side of (4.7.8).

Preparatory estimates

To carry out the estimates below, we introduce the energy associated with the system (4.1.1)–(4.1.8):

$$E(t) = \frac{1}{2}\int_{\Omega(t)} |u(x,t)|^2 \,\mathrm{d}x + \frac{M}{2}\,|\dot{h}(t)|^2 + \frac{J}{2}\,|\omega(t)|^2. \tag{4.7.13}$$

Lemma 4.7.2 (Bound for the momentum term)**.** *There exists $C(R) > 0$ such that for all $t \in (0,T)$,*

$$|P(t)| \le C(R)\sqrt{E(0)}.$$

Proof. By Proposition 4.3.1 and Lemma 4.5.1,

$$\|v(t)\|_{L^2(B(0,R))} = \|\bar{v}_{d(t)}\|_{L^2(B(0,R))} \le C(R).$$

Hence,

$$|P(t)| \le \|\widetilde{u}(t)\|_{L^2(B(0,R))}\,\|v(t)\|_{L^2(B(0,R))} \le C(R)\,\|\widetilde{u}(t)\|_{L^2(B(0,R))}.$$

Using the energy estimate (4.2.2) yields $\|\widetilde{u}(t)\|^2_{L^2(B(0,R))} \le C\,E(0)$, which proves the claim. □

Lemma 4.7.3 (Estimate of the time-derivative term)**.** *There exists $C(R) > 0$ such that for a.e. $t \in (0,T)$,*

$$|A_1(t)| \le C(R)\,\|\nabla\widetilde{u}(t)\|_{L^2(B(0,R))}\,|\dot{h}(t)| \le C(R)\,\|\nabla\widetilde{u}(t)\|^2_{L^2(B(0,R))} + C(R)\,|\dot{h}(t)|^2.$$

Consequently, for all $t \in (0,T)$,

$$\left|\int_0^t A_1(s)\,\mathrm{d}s\right| \le C(R)\,E(0) + C(R)\,E(0)\,t.$$

Proof. Recall that

$$v(t,x) = \nabla_x^{\perp}(\bar{g}_{d(t)}(Y_{h(t)}(x))).$$

Differentiating with respect to t gives

$$\partial_t v(t,x) = \nabla_x^{\perp}\Big(\dot{d}(t)\,\partial_d \bar{g}_{d(t)}(Y_{h(t)}(x)) + \nabla_y \bar{g}_{d(t)}(Y_{h(t)}(x)) \cdot \partial_t Y_{h(t)}(x)\Big). \tag{4.7.14}$$

As a consequence, integration by parts in the x-variables yields

$$\begin{aligned}
&\int_{B(0,R)} \widetilde{u}(x,t)\cdot\partial_t v(x,t)\,\mathrm{d}x\\
&= \int_{B(0,R)} \left(\partial_{x_1}\widetilde{u}_2 - \partial_{x_2}\widetilde{u}_1\right)\partial_t(\bar{g}_{d(t)}(Y(x,t)))\,\mathrm{d}x\\
&= \int_{B(0,R)} \left(\partial_{x_1}\widetilde{u}_2 - \partial_{x_2}\widetilde{u}_1\right)\dot{d}(t)\,\partial_d\bar{g}_{d(t)}(Y(x,t))\mathrm{d}x \qquad (4.7.15)\\
&\quad + \int_{B(0,R)} \left(\partial_{x_1}\widetilde{u}_2 - \partial_{x_2}\widetilde{u}_1\right)\Big(\nabla_y\bar{g}_{d(t)}(Y(x,t))\cdot\partial_t Y(x,t)\Big)\mathrm{d}x. \qquad (4.7.16)
\end{aligned}$$

Hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\Big|\int_{B(0,R)} \widetilde{u}\cdot\partial_t v\,dx\Big|\\
\le&\|\nabla\widetilde{u}\|_{L^2(B(0,R))}|\dot{d}(t)|\,\|\partial_d\bar{g}_{d(t)}(Y)\|_{L^2(B(0,R))}\\
&+\|\nabla\widetilde{u}\|_{L^2(B(0,R))}\|\nabla_y\bar{g}_{d(t)}(Y)\|_{L^2(B(0,R))}\,\|\partial_t Y\|_{L^\infty(B(0,R))}. \qquad (4.7.17)
\end{aligned}$$

Moreover, since $d(t) = R - 1 - |h(t)|$, we have

$$\dot{d}(t) = -\dot{h}(t)\cdot e_2(t), \qquad e_2(t) = -\frac{h(t)}{|h(t)|},$$

and thus $|\dot{d}(t)| \le |\dot{h}(t)|$. Under (4.7.1), one also has $\|\partial_t Y_{h(t)}\|_{L^\infty(B(0,R))} \le C(R)\,|\dot{h}(t)|$. Finally, Lemma 4.5.2 yields

$$\|\partial_d\bar{g}_{d(t)}\|_{L^2(B(0,R))} + \|\nabla_y\bar{g}_{d(t)}\|_{L^2(B(0,R))} \le C(R).$$

Therefore,

$$\Big\|\partial_t(\bar{g}_{d(t)}(Y_{h(t)}(\cdot)))\Big\|_{L^2(B(0,R))} \le C(R)\,|\dot{h}(t)|.$$

This gives

$$|A_1(t)| \le C(R)\,\|\nabla\widetilde{u}(t)\|_{L^2(B(0,R))}\,|\dot{h}(t)|.$$

The quadratic bound follows from Young's inequality.

Integrating in time and using (4.2.2), we obtain

$$\int_0^t \|\nabla\widetilde{u}(s)\|^2_{L^2(B(0,R))}\,\mathrm{d}s \le C(R)\,E(0), \qquad \int_0^t |\dot{h}(s)|^2\,\mathrm{d}s \le C\,E(0)\,t,$$

which yields the integrated estimate. □

Lemma 4.7.4 (Estimate of the nonlinear term)**.** *There exists $C(R) > 0$ such that for a.e. $t \in (0,T)$,*

$$|A_2(t)| \le C(R)\,\|\nabla\widetilde{u}(t)\|^2_{L^2(B(0,R))}.$$

Consequently,

$$\left|\int_0^t A_2(s)\,\mathrm{d}s\right| \le C(R)\,E(0) \qquad \textit{for all } t \in (0,T).$$

Proof. We split the domain into the thin gap region $\bar{A}_{\delta,d}$ and its complement:

$$A_2(t) = \int_{B(0,R)\setminus\bar{A}_{\delta,d}} (\widetilde{u}\otimes\widetilde{u}) : D[v]\,\mathrm{d}x + \int_{\bar{A}_{\delta,d}} (\widetilde{u}\otimes\widetilde{u}) : D[v]\,\mathrm{d}x,$$

where $\bar{A}_{\delta,d} := \{x \in B(0,R) : Y_{h(t)}(x) \in A_{\delta,d}\}$ and Y_τ defined in (4.4.2) with $h(t)$ is the center mass of the inner disk.

(i) Outside the gap. On $B(0,R)\setminus\bar{A}_{\delta,d}$, the construction of v involves only smooth cut-offs independent of d, hence $\|D[v]\|_{L^\infty(B(0,R)\setminus\bar{A}_{\delta,d})} \le C(R)$. Therefore,

$$\left|\int_{B(0,R)\setminus\bar{A}_{\delta,d}} (\widetilde{u}\otimes\widetilde{u}) : D[v]\,\mathrm{d}x\right| \le C(R)\,\|\widetilde{u}(t)\|^2_{L^2(B(0,R))} \le C(R)\,E(0).$$

(ii) Inside the gap. Using incompressibility and integrating by parts, we write

$$\int_{\bar{A}_{\delta,d}} (\widetilde{u}\otimes\widetilde{u}) : D[v]\,\mathrm{d}x = -\int_{\bar{A}_{\delta,d}} (u\cdot\nabla u)\cdot v\,\mathrm{d}x + \int_{\partial\bar{A}_{\delta,d}} (u\cdot v)(u\cdot n)\,\mathrm{d}\Gamma.$$

For the first term on the right-hand side of this equality, Hölder's inequality and the 2D Gagliardo–Nirenberg inequality yield

$$\left|\int_{\bar{A}_{\delta,d}} (u\cdot\nabla u)\cdot v\,\mathrm{d}x\right| \le \|\nabla u\|_{L^2}\,\|u\|_{L^{10}}\,\|v\|_{L^{5/2}} \le C(R)\,\|\nabla\widetilde{u}(t)\|^2_{L^2(B(0,R))},$$

where we used the uniform bound $\|v\|_{L^{5/2}(B(0,R))} \le C(R)$ (Lemma 4.5.1). For the boundary term, using $\|v\|_{L^\infty(\partial\bar{A}_{\delta,d})} \le C(R)$ and the trace theorem,

$$\left|\int_{\partial\bar{A}_{\delta,d}} (u\cdot v)(u\cdot n)\,\mathrm{d}\Gamma\right| \le C(R)\,\|u\|^2_{L^2(\partial\bar{A}_{\delta,d})} \le C(R)\,\|\nabla\widetilde{u}(t)\|^2_{L^2(B(0,R))}.$$

Combining the above bounds yields the desired pointwise estimate for $A_2(t)$. Integrating in time and using (4.2.2) gives the integrated bound. □

Lemma 4.7.5 (Estimate of the viscous term)**.** *There exists $C(R) > 0$ such that for all $t \in (0,T)$,*

$$\int_0^t A_3(s)\,\mathrm{d}s \le C(R)\,E(0) + C(R)\,E(0)\,t.$$

Proof. By Proposition 4.6.7 applied with $\bar{u} = \bar{\widetilde{u}}$, we have for a.e. $t \in (0,T)$,

$$\begin{aligned}&\left|\int_{B(0,R)} 2\,D[\widetilde{u}](t,x) : D[v](t,x)\,\mathrm{d}x - \tilde{m}(d(t))\,V(\bar{\widetilde{u}})\cdot\bar{e}_2\right| \\ \le& C(R)\Big(\|\widetilde{u}\|_{L^2} + \|\widetilde{u}\|_{L^\infty(\partial S(t))} + \|\nabla\widetilde{u}\|_{L^2}\Big).\end{aligned}$$

Recalling the definition of $A_3(t)$, we obtain

$$|A_3(t)| \le C(R)\Big(\|\widetilde{u}(t)\|_{L^2(B(0,R))} + \|\widetilde{u}(t)\|_{L^\infty(\partial S(t))} + \|\nabla\widetilde{u}(t)\|_{L^2(B(0,R))}\Big).$$

Using Young's inequality and the fact that $\widetilde{u}$ is a rigid motion on $S(t)$, $\|\widetilde{u}(t)\|_{L^\infty(\partial S(t))} \le |\dot{h}(t)| + |\omega(t)|$, together with the energy bound (4.2.2), we obtain

$$\begin{aligned}\int_0^t |A_3(s)|\,\mathrm{d}s \le& C(R)\int_0^t \|\nabla\widetilde{u}(s)\|_{L^2}^2\,\mathrm{d}s + C(R)\int_0^t \|\widetilde{u}(s)\|_{L^2}^2\,\mathrm{d}s \\ &+ C(R)\,E(0)\,t \le C(R)\,E(0) + C(R)\,E(0)\,t. \end{aligned} \tag{4.7.18}$$

□

Completion of the proof

Step 4. Derivation of the key inequality. Combining (4.7.8) with Lemmas 4.7.2, 4.7.3, 4.7.4, 4.7.5, we obtain that for every $t \in (0,T)$,

$$\int_0^t \tilde{m}(d(s))\,V(\bar{\widetilde{u}})\cdot e_2\,\mathrm{d}s \le C(R)E(0) + C(R)\sqrt{E(0)} + C(R)E(0)\,t. \tag{4.7.19}$$

Step 5. Contradiction and exclusion of contact. Assume by contradiction that there exists $0 < T^* < \infty$ such that

$$\lim_{t\to T^*} d(t) = 0.$$

Then there exists $T_1 \in (0,T^*)$ such that for all $t \in (T_1,T^*)$,

$$0 < d(t) < \eta(R), \tag{4.7.20}$$

where $\eta(R)$ is the constant in Proposition 4.6.8.

Moreover, By using (4.7.19) and the fact that $V(\bar{\widetilde{u}})\cdot e_2 = -\dot{d}(t)$, we obtain for all $t \in (T_1,T^*)$,

$$-\int_{T_1}^t \tilde{m}(d(s))\,\dot{d}(s)\,\mathrm{d}s \le C(R,T^*,E(0)).$$

Writing $H(s) = d(s)$, this yields

$$-\int_{d(T_1)}^{d(t)} \tilde{m}(H)\,\mathrm{d}H \le C(R,T^*,E(0)).$$

Using Proposition 4.6.8, namely $\tilde{m}(H) \ge C(R)H^{-3/2}$ for $H \in (0,\eta]$, we deduce

$$-\int_{d(T_1)}^{d(t)} \frac{C(R)}{H^{3/2}}\,\mathrm{d}H \le C(R,T^*,E(0)),$$

which implies

$$\frac{1}{d^{1/2}(t)} \leq \frac{1}{d^{1/2}(T_1)} + C(R, T^*, E(0)).$$

In particular $d(t)$ is bounded below by a strictly positive constant on (T_1, T^*), contradicting $\lim_{t\to T^*} d(t) = 0$. Therefore, no finite-time contact can occur.

Remark 4.7.6 (On the assumption (4.7.1))**.** The lower bound (4.7.1) is imposed only to avoid the singularity of the rotation map Y_τ, which involves the normalization $\tau/|\tau|$. It is not an essential restriction for the non-collision argument. Indeed, in the contradiction proof we assume $d(t) \to 0$ as $t \to T^*$. Since $d(t) = R-1-|h(t)|$, this implies $|h(t)| \to R-1$ as $t \to T^*$. Hence there exists $T_0 < T^*$ such that (4.7.1) holds for all $t \in (T_0, T^*)$. Therefore, all estimates involving the rotated coordinates can be carried out on (T_0, T^*), and the proof does not require any global assumption on $|h(t)|$.

Finally, by Proposition 4.2.2, the strong solution is global in time. □

4.8 Appendix

In this section, we prove Lemma 4.4.1 and Lemma 4.5.3.

Reminder of notation. For convenience, we briefly recall the main objects used in this appendix and indicate where they are defined:

- The parameter $R > 1$ is the radius of the outer disk, and $d \in (0, R-1]$ denotes the (instantaneous) gap parameter.
- The boundary profiles ψ_{up} and ψ_{low} of the gap region are introduced in (4.4.5)–(4.4.4).
- The (small) constant $\delta > 0$ is defined in (4.4.3), and the gap region $A_{\delta,d}$ is defined in (4.4.6). The core region $A_{0,d}$ is defined in (4.4.7).
- The cut-off functions $\mathcal{X}$ and ϕ are defined in (4.4.13) and (4.4.12).
- The stream function $\bar{g}_d$ and the associated divergence-free field $\bar{v}_d = \nabla^\perp \bar{g}_d$ are defined in (4.4.14)–(4.4.15).

Throughout the appendix, all computations are performed in the rotated variables $y = (y_1, y_2)$, and we use the shorthand $\partial_i = \frac{\partial}{\partial y_i}$.

A note on the piecewise structure of $\bar{g}_d$ and $\bar{v}_d$. Recall from (4.4.14) that the construction involves the cut-off $\mathcal{X}(y_1)$, hence $\bar{g}_d$ (and therefore $\bar{v}_d = \nabla^\perp \bar{g}_d$) should be viewed as a *piecewise-defined* function:

- On the *core region* $A_{0,d}$ one has $\mathcal{X}(y_1) \equiv 1$, so the expression of $\bar{g}_d$ reduces to the thin-gap polynomial profile depending on $F(y)$.

- On the *transition strip* $A_{\delta,d} \setminus A_{0,d}$ one has $0 \le \mathcal{X}(y_1) \le 1$, and the definition of $\bar{g}_d$ includes the additional smooth term involving φ.

This is why, in the estimates below, we systematically split the domain into these two parts.

Proof of Lemma 4.4.1. The purpose of this lemma is to provide a two-sided estimate of the gap thickness $\psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1)$ for $|y_1| \le 2\delta$.

Step 1: Exact expression of the gap. From the definitions of ψ_{up} and ψ_{low} we obtain

$$\psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1) = d + \left(1 - \sqrt{1 - y_1^2}\right) - \left(R - \sqrt{R^2 - y_1^2}\right), \tag{4.8.1}$$

valid for

$$-2\delta \le y_1 \le 2\delta. \tag{4.8.2}$$

Step 2: Elementary bounds for the square roots. Using $|y_1| \le 1$, we have the standard estimates

$$1 - \frac{y_1^2}{2} - \frac{y_1^4}{2} \;\le\; \sqrt{1 - y_1^2} \;\le\; 1 - \frac{y_1^2}{2}. \tag{4.8.3}$$

Similarly, for $\sqrt{R^2 - y_1^2}$, we obtain

$$R - \frac{y_1^2}{2R} - \frac{y_1^4}{2R^3} \le \sqrt{R^2 - y_1^2} \le R - \frac{y_1^2}{2R}. \tag{4.8.4}$$

Step 3: Two-sided estimate of the gap. Combining the above estimates with (4.8.1), it follows that

$$d + \left(\frac{1}{2} - \frac{1}{2R}\right) y_1^2 - \frac{1}{2R^3} y_1^4 \;\le\; \psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1) \;\le\; d + \left(\frac{1}{2} - \frac{1}{2R}\right) y_1^2 + \frac{1}{2} y_1^4. \tag{4.8.5}$$

Step 4: Simplifying the lower bound. We use the fact

$$-\frac{1}{2R^3} y_1^4 \ge -\left(\tfrac{1}{4} - \tfrac{1}{R}\right) y_1^2 \qquad\qquad y_1^2 \le \tfrac{R^3}{2}\left(\tfrac{1}{4} - \tfrac{1}{4R}\right),$$

so that the negative quartic term can be absorbed into the quadratic term. Hence we get the left-hand side of (4.4.9):

$$\psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1) \;\ge\; d + \left(\tfrac{1}{4} - \tfrac{1}{4R}\right) y_1^2. \tag{4.8.6}$$

Step 5: Upper bound. From the right-hand side of (4.8.5) and using $|y_1| \le 1$, it is easy to check that

$$\psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1) \;\le\; d + \left(1 - \tfrac{1}{R}\right) y_1^2. \tag{4.8.7}$$

Conclusion. Combining (4.8.6)–(4.8.7), for all $|y_1| \le 2\delta$ we finally obtain

$$d + \left(\tfrac{1}{4} - \tfrac{1}{4R}\right) y_1^2 \;\le\; \psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1) \;\le\; d + \left(1 - \tfrac{1}{R}\right) y_1^2. \tag{4.8.8}$$

This completes the proof. □

Proof of Lemma 4.5.3. The goal is to estimate $\|\partial_1 \bar{v}_{d,2}\|_{L^2(A_{\delta,d})}$ and $\|\partial_2 \bar{v}_{d,1}\|_{L^2(A_{\delta,d})}$ uniformly in d (up to the expected singular scaling in d).

Step 1: Estimate of $\partial_1 \bar{v}_{d,2}$. From (4.4.14)–(4.4.15), we have

$$\partial_1 \bar{v}_{d,2} = \partial_{11} \bar{g}_d.$$

As explained above, we split the gap region into the core part $A_{0,d}$ and the transition strip $A_{\delta,d} \setminus A_{0,d}$, since $\bar{g}_d$ (hence $\bar{v}_d$) has different expressions on these two sets. We first estimate $\partial_{11} \bar{g}_d$ on $A_{0,d}$, where the cut-off satisfies $\mathcal{X}(y_1) = 1$.

On $A_{0,d}$ we have $\mathcal{X}(y_1) = 1$ by (4.4.13), and a direct calculation gives

$$\partial_{11} \bar{g}_d(y_1) = 3\,\partial_1 F(y) \;-\; 6\,y_1 (\partial_1 F(y))^2, \tag{4.8.9}$$

where $F(y)$ is given by (4.4.10).

Moreover,

$$\begin{cases} \partial_1 F(y) = \dfrac{-\psi'_{\text{low}}(y_1)\,\mathcal{G}(y_1) - (y_2 - \psi_{\text{low}}(y_1))\,\mathcal{G}'(y_1)}{\mathcal{G}(y_1)^2}, \\ \mathcal{G}'(y_1) = \psi'_{\text{up}}(y_1) - \psi'_{\text{low}}(y_1) \end{cases} \tag{4.8.10}$$

Using (4.5.2), we obtain, for $y \in A_{0,d}$,

$$|\partial_1 F(y)| \;\le\; C(R) \left(\frac{|y_1|}{\mathcal{G}(y_1)} + \frac{|y_1|^3}{\mathcal{G}(y_1)^3} \right). \tag{4.8.11}$$

Therefore, arguing as in the calculation leading to (4.5.3), we conclude that

$$\|\partial_1 F\|_{L^2(A_{0,d})} \;\le\; C(R). \tag{4.8.12}$$

Combining the above inequality with (4.8.9), we obtain

$$\|\partial_1 \bar{v}_{d,2}\|_{L^2(A_{0,d})} \;\le\; C(R).$$

For the remaining part on $A_{\delta,d} \setminus A_{0,d}$, we proceed exactly as in the proof of Lemma 4.5.1: on this transition strip, $0 \le \mathcal{X} \le 1$ and the definition of $\bar{g}_d$ involves only smooth cut-off functions (namely $\mathcal{X}$ and φ), whose derivatives are uniformly bounded. Moreover, the gap thickness is uniformly bounded from below away from zero on this strip. Hence the corresponding contributions are uniformly bounded by $C(R)$. This completes the proof of the estimate (4.5.9).

Step 2: Estimate of $\partial_2 \bar{v}_{d,1}$ (upper bound).

Recall that $\bar{v}_d = \nabla^\perp \bar{g}_d$, hence

$$\bar{v}_{d,1} = -\partial_{y_2} \bar{g}_d.$$

Differentiating once more with respect to y_2 gives

$$\partial_{y_2}\bar{v}_{d,1} = -\partial_{y_2y_2}\bar{g}_d.$$

From the explicit formula of $\bar{g}_d$ in (4.4.14), for every $y \in A_{\delta,d}$, we have we obtain the following expression:

$$\begin{aligned}\partial_2\bar{v}_{d,1}(y) = &- y_1(1-\mathcal{X}(y_1))\,\partial_{22}\varphi(y)\\ &- 6y_1\mathcal{X}(y_1)\left(\frac{1}{\left(\psi_{\mathrm{up}}(y_1)-\psi_{\mathrm{low}}(y_1)\right)^2} - \frac{2\left(y_2-\psi_{\mathrm{low}}(y_1)\right)}{\left(\psi_{\mathrm{up}}(y_1)-\psi_{\mathrm{low}}(y_1)\right)^3}\right). \qquad (4.8.13)\end{aligned}$$

Step 2.1: Pointwise bound. We estimate each term in (4.8.13) separately. Note that on the core region $A_{0,d}$ we have $\mathcal{X} \equiv 1$, so the first term in (4.8.13) vanishes there; it only contributes on the transition strip.

(i) The cut-off part. Since $\mathcal{X} \in C^\infty(\mathbb{R})$ satisfies $0 \le \mathcal{X} \le 1$ and $\varphi \in C^\infty(\overline{B(0,R)})$ is fixed (independent of d), we have

$$|\partial_{22}\varphi(y)| \le C(R) \qquad \text{for all } y \in B(0,R).$$

Moreover, $|y_1| \le 2\delta$, hence

$$\left|-y_1(1-\mathcal{X}(y_1))\partial_{22}\varphi(y)\right| \le |y_1|\cdot 1\cdot C(R) \le C(R).$$

(ii) The thin-gap part. Set for simplicity

$$\mathcal{G}(y_1) := \psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1).$$

Then the second line in (4.8.13) becomes

$$-6y_1\mathcal{X}(y_1)\left(\frac{1}{\mathcal{G}(y_1)^2} - \frac{2\left(y_2-\psi_{\mathrm{low}}(y_1)\right)}{\mathcal{G}(y_1)^3}\right).$$

Taking absolute values and using $0 \le \mathcal{X} \le 1$, we get

$$\begin{aligned}&\left|-6y_1\mathcal{X}(y_1)\left(\frac{1}{\mathcal{G}(y_1)^2} - \frac{2\left(y_2-\psi_{\mathrm{low}}(y_1)\right)}{\mathcal{G}(y_1)^3}\right)\right|\\ &\qquad \le 6|y_1|\left(\frac{1}{\mathcal{G}(y_1)^2} + \frac{2|y_2-\psi_{\mathrm{low}}(y_1)|}{\mathcal{G}(y_1)^3}\right).\end{aligned}$$

Since $y \in A_{\delta,d}$ implies

$$\psi_{\mathrm{low}}(y_1) \le y_2 \le \psi_{\mathrm{up}}(y_1) \quad\Longrightarrow\quad 0 \le y_2 - \psi_{\mathrm{low}}(y_1) \le \mathcal{G}(y_1),$$

we obtain

$$\frac{|y_2 - \psi_{\text{low}}(y_1)|}{\mathcal{G}(y_1)^3} \leq \frac{\mathcal{G}(y_1)}{\mathcal{G}(y_1)^3} = \frac{1}{\mathcal{G}(y_1)^2}.$$

Therefore,

$$\left| -6y_1 \mathcal{X}(y_1) \left(\frac{1}{\mathcal{G}(y_1)^2} - \frac{2\,(y_2 - \psi_{\text{low}}(y_1))}{\mathcal{G}(y_1)^3} \right) \right| \leq C(R)\, \frac{|y_1|}{\mathcal{G}(y_1)^2}.$$

(iii) Conclusion of the pointwise bound. Combining (i) and (ii), we obtain that for every $y \in A_{\delta,d}$,

$$|\partial_2 \bar{v}_{d,1}(y)| \leq C(R) \left(1 + \frac{|y_1|}{\mathcal{G}(y_1)^2} \right). \tag{4.8.14}$$

Step 2.2: From the pointwise bound to an L^2 bound. Squaring (4.8.14) and using $(a+b)^2 \leq 2a^2 + 2b^2$, we get

$$|\partial_2 \bar{v}_{d,1}(y)|^2 \leq C(R) \left(1 + \frac{y_1^2}{\mathcal{G}(y_1)^4} \right), \qquad y \in A_{\delta,d}.$$

Integrating over $A_{\delta,d}$ gives

$$\begin{aligned} \|\partial_2 \bar{v}_{d,1}\|^2_{L^2(A_{\delta,d})} &= \int_{A_{\delta,d}} |\partial_2 \bar{v}_{d,1}(y)|^2 \, \mathrm{d}y \\ &\leq C(R) \int_{A_{\delta,d}} 1 \, \mathrm{d}y + C(R) \int_{A_{\delta,d}} \frac{y_1^2}{\mathcal{G}(y_1)^4} \, \mathrm{d}y. \end{aligned} \tag{4.8.15}$$

The first integral is uniformly bounded because $A_{\delta,d} \subset B(0,R)$, hence

$$\int_{A_{\delta,d}} 1 \, \mathrm{d}y \leq |B(0,R)| \leq C(R).$$

We now estimate the second integral in (4.8.15). Using the definition of $A_{\delta,d}$, we apply Tonelli's theorem:

$$\int_{A_{\delta,d}} \frac{y_1^2}{\mathcal{G}(y_1)^4} \, \mathrm{d}y = \int_{-2\delta}^{2\delta} \int_{\psi_{\text{low}}(y_1)}^{\psi_{\text{up}}(y_1)} \frac{y_1^2}{\mathcal{G}(y_1)^4} \, \mathrm{d}y_2 \, \mathrm{d}y_1.$$

Since the integrand does not depend on y_2, the inner integral equals the thickness $\psi_{\text{up}}(y_1) - \psi_{\text{low}}(y_1) = \mathcal{G}(y_1)$:

$$\begin{aligned} \int_{\psi_{\text{low}}(y_1)}^{\psi_{\text{up}}(y_1)} \frac{y_1^2}{\mathcal{G}(y_1)^4} \, \mathrm{d}y_2 &= \frac{y_1^2}{\mathcal{G}(y_1)^4} \int_{\psi_{\text{low}}(y_1)}^{\psi_{\text{up}}(y_1)} 1 \, \mathrm{d}y_2 \\ &= \frac{y_1^2}{\mathcal{G}(y_1)^4} \, \mathcal{G}(y_1) = \frac{y_1^2}{\mathcal{G}(y_1)^3}. \end{aligned}$$

Hence

$$\int_{A_{\delta,d}} \frac{y_1^2}{\mathcal{G}(y_1)^4}\,\mathrm{d}y = \int_{-2\delta}^{2\delta} \frac{y_1^2}{\mathcal{G}(y_1)^3}\,\mathrm{d}y_1. \tag{4.8.16}$$

Step 2.3: Using the gap lower bound. By Lemma 4.4.1, there exists $c_R > 0$ such that for all $|y_1| \le 2\delta$,

$$\mathcal{G}(y_1) \ge d + c_R y_1^2.$$

Therefore,

$$\frac{1}{\mathcal{G}(y_1)^3} \le \frac{1}{(d + c_R y_1^2)^3},$$

and hence (4.8.16) implies

$$\int_{A_{\delta,d}} \frac{y_1^2}{\mathcal{G}(y_1)^4}\,\mathrm{d}y \le \int_{-2\delta}^{2\delta} \frac{y_1^2}{(d + c_R y_1^2)^3}\,\mathrm{d}y_1. \tag{4.8.17}$$

We now estimate the right-hand side by a scaling argument. Using symmetry, it is enough to consider $y_1 \ge 0$ and set $y_1 = \sqrt{d}\,z$:

$$\begin{aligned}\int_0^{2\delta} \frac{y_1^2}{(d + c_R y_1^2)^3}\,\mathrm{d}y_1 &= \int_0^{2\delta/\sqrt{d}} \frac{(dz^2)}{\left(d + c_R d z^2\right)^3}\,\sqrt{d}\,\mathrm{d}z \\ &= \int_0^{2\delta/\sqrt{d}} \frac{d^{3/2} z^2}{d^3 (1 + c_R z^2)^3}\,\mathrm{d}z = d^{-3/2} \int_0^{2\delta/\sqrt{d}} \frac{z^2}{(1 + c_R z^2)^3}\,\mathrm{d}z.\end{aligned}$$

Since $\frac{z^2}{(1+c_R z^2)^3} \sim z^{-4}$ as $z \to \infty$, the integral $\int_0^{+\infty} \frac{z^2}{(1+c_R z^2)^3}\,\mathrm{d}z$ is finite, and therefore

$$\int_0^{2\delta/\sqrt{d}} \frac{z^2}{(1 + c_R z^2)^3}\,\mathrm{d}z \le C(R).$$

We conclude that

$$\int_{-2\delta}^{2\delta} \frac{y_1^2}{(d + c_R y_1^2)^3}\,\mathrm{d}y_1 \le \frac{C(R)}{d^{3/2}}. \tag{4.8.18}$$

Step 2.4: Conclusion. Plugging (4.8.18) into (4.8.15), we obtain

$$\left\|\partial_2 \bar{v}_{d,1}\right\|^2_{L^2(A_{\delta,d})} \le C(R) + \frac{C(R)}{d^{3/2}}.$$

Taking the square root yields

$$\left\|\partial_2 \bar{v}_{d,1}\right\|_{L^2(A_{\delta,d})} \le C(R) + \frac{C_2(R)}{d^{3/4}},$$

which is the desired upper bound.

Step 3: Estimate of $\partial_2 \bar{v}_{d,1}$ (lower bound). Fix constants $0 < \eta_1 < \eta_2$ independent of d. In this step we work in the *core region* $A_{0,d}$, where the cut-off satisfies $\mathcal{X}(y_1) \equiv 1$ and the field $\bar{v}_d$ has the explicit thin-gap form. We assume $d > 0$ is small enough such that $\eta_2\sqrt{d} \le \delta$. Then $y_1 \in [\eta_1\sqrt{d}, \eta_2\sqrt{d}] \subset [-\delta, \delta]$ and therefore $\mathcal{X}(y_1) \equiv 1$.

Recall that on $A_{0,d}$ we have

$$\bar{g}_d(y) = y_1(3F(y)^2 - 2F(y)^3), \qquad F(y) = \frac{y_2 - \psi_{\mathrm{low}}(y_1)}{\mathcal{G}(y_1)}, \qquad \mathcal{G}(y_1) = \psi_{\mathrm{up}}(y_1) - \psi_{\mathrm{low}}(y_1).$$

Moreover,

$$\bar{v}_{d,1} = -\partial_2 \bar{g}_d.$$

Since $\partial_2 F = 1/\mathcal{G}(y_1)$ and $\partial_{22} F = 0$, we compute

$$\partial_2(3F^2 - 2F^3) = (6F - 6F^2)\,\partial_2 F = \frac{6F(1-F)}{\mathcal{G}(y_1)},$$

and hence

$$\bar{v}_{d,1}(y) = -y_1 \frac{6F(1-F)}{\mathcal{G}(y_1)}.$$

Differentiating once more with respect to y_2, we obtain

$$\partial_2 \bar{v}_{d,1}(y) = -y_1\,\partial_2\left(\frac{6F(1-F)}{\mathcal{G}(y_1)}\right) = -y_1\,\frac{6(1-2F)}{\mathcal{G}(y_1)^2}.$$

Therefore

$$|\partial_2 \bar{v}_{d,1}(y)| = \frac{6|y_1|\,|1-2F(y)|}{\mathcal{G}(y_1)^2}. \tag{4.8.19}$$

Choice of a subregion where $|1-2F|$ is uniformly positive. Define the subregion

$$A := \Big\{(y_1, y_2) \in A_{\delta,d} :\ \eta_1\sqrt{d} \le y_1 \le \eta_2\sqrt{d}, \quad \psi_{\mathrm{low}}(y_1) \le y_2 \le \psi_{\mathrm{low}}(y_1) + \tfrac{1}{3}\mathcal{G}(y_1)\Big\}. \tag{4.8.20}$$

For every $y \in A$, we have

$$0 \le F(y) = \frac{y_2 - \psi_{\mathrm{low}}(y_1)}{\mathcal{G}(y_1)} \le \frac{1}{3},$$

hence $1 - 2F(y) \ge 1/3$. Plugging this into (4.8.19) yields

$$|\partial_{y_2} \bar{v}_{d,1}(y)| \ge \frac{2\,y_1}{\mathcal{G}(y_1)^2}, \qquad y \in A. \tag{4.8.21}$$

L^2 lower bound. Using (4.8.21) and integrating first in y_2 over an interval of length $\mathcal{G}(y_1)/3$, we start from the definition of the L^2 norm on the set A:

$$\|\partial_{y_2} \bar{v}_{d,1}\|^2_{L^2(A)} = \int_A |\partial_{y_2} \bar{v}_{d,1}(y_1, y_2)|^2 \,\mathrm{d}y_1\,\mathrm{d}y_2.$$

By the definition of A in (4.8.20), this integral can be written as an iterated integral:

$$\|\partial_2 \bar{v}_{d,1}\|_{L^2(A)}^2 = \int_{\eta_1\sqrt{d}}^{\eta_2\sqrt{d}} \int_{\psi_{\mathrm{low}}(y_1)}^{\psi_{\mathrm{low}}(y_1)+\mathcal{G}(y_1)/3} |\partial_2 \bar{v}_{d,1}(y_1,y_2)|^2 \,\mathrm{d}y_2\,\mathrm{d}y_1.$$

Moreover, for every $(y_1, y_2) \in A$, we proved in (4.8.21) that

$$|\partial_2 \bar{v}_{d,1}(y_1,y_2)| \geq \frac{2y_1}{\mathcal{G}(y_1)^2}.$$

Squaring both sides yields

$$|\partial_2 \bar{v}_{d,1}(y_1,y_2)|^2 \geq \left(\frac{2y_1}{\mathcal{G}(y_1)^2}\right)^2 = \frac{4y_1^2}{\mathcal{G}(y_1)^4}, \qquad (y_1,y_2) \in A.$$

Therefore, inserting this pointwise lower bound into the integral, we obtain

$$\begin{aligned}\|\partial_2 \bar{v}_{d,1}\|_{L^2(A)}^2 &= \int_{\eta_1\sqrt{d}}^{\eta_2\sqrt{d}} \int_{\psi_{\mathrm{low}}(y_1)}^{\psi_{\mathrm{low}}(y_1)+\mathcal{G}(y_1)/3} |\partial_2 \bar{v}_{d,1}(y_1,y_2)|^2 \,\mathrm{d}y_2\,\mathrm{d}y_1 \\ &\geq \int_{\eta_1\sqrt{d}}^{\eta_2\sqrt{d}} \int_{\psi_{\mathrm{low}}(y_1)}^{\psi_{\mathrm{low}}(y_1)+\mathcal{G}(y_1)/3} \frac{4y_1^2}{\mathcal{G}(y_1)^4} \,\mathrm{d}y_2\,\mathrm{d}y_1.\end{aligned}$$

Now we compute the inner integral with respect to y_2. Since the integrand $\frac{4y_1^2}{\mathcal{G}(y_1)^4}$ does not depend on y_2, it factors out:

$$\begin{aligned}\int_{\psi_{\mathrm{low}}(y_1)}^{\psi_{\mathrm{low}}(y_1)+\mathcal{G}(y_1)/3} \frac{4y_1^2}{\mathcal{G}(y_1)^4} \,\mathrm{d}y_2 &= \frac{4y_1^2}{\mathcal{G}(y_1)^4} \int_{\psi_{\mathrm{low}}(y_1)}^{\psi_{\mathrm{low}}(y_1)+\mathcal{G}(y_1)/3} 1 \,\mathrm{d}y_2 \\ &= \frac{4y_1^2}{\mathcal{G}(y_1)^4} \left(\psi_{\mathrm{low}}(y_1) + \frac{\mathcal{G}(y_1)}{3} - \psi_{\mathrm{low}}(y_1)\right) \\ &= \frac{4y_1^2}{\mathcal{G}(y_1)^4} \cdot \frac{\mathcal{G}(y_1)}{3} = \frac{4}{3}\,\frac{y_1^2}{\mathcal{G}(y_1)^3}.\end{aligned}$$

Plugging this back into the outer integral, we conclude

$$\|\partial_{y_2} \bar{v}_{d,1}\|_{L^2(A)}^2 \geq \frac{4}{3} \int_{\eta_1\sqrt{d}}^{\eta_2\sqrt{d}} \frac{y_1^2}{\mathcal{G}(y_1)^3} \,\mathrm{d}y_1. \tag{4.8.22}$$

Now we use the *upper bound* of Lemma 4.4.1, namely

$$\mathcal{G}(y_1) \leq d + \Big(1 - \frac{1}{R}\Big) y_1^2.$$

For $y_1 \in [\eta_1\sqrt{d}, \eta_2\sqrt{d}]$ we have $y_1^2 \leq \eta_2^2 d$, hence

$$\mathcal{G}(y_1) \leq \Big(1 + \Big(1 - \frac{1}{R}\Big)\eta_2^2\Big) d =: C_0\, d.$$

Moreover $y_1^2 \geq \eta_1^2 d$ on the same interval. Therefore (4.8.22) implies

$$\|\partial_{y_2}\bar{v}_{d,1}\|_{L^2(A)}^2 \geq \frac{4}{3}\int_{\eta_1\sqrt{d}}^{\eta_2\sqrt{d}} \frac{\eta_1^2 d}{(C_0 d)^3}\,\mathrm{d}y_1 = \frac{4}{3}\cdot\frac{\eta_1^2}{C_0^3}\cdot(\eta_2-\eta_1)\,d^{-3/2}.$$

Taking the square root gives

$$\|\partial_{y_2}\bar{v}_{d,1}\|_{L^2(A)} \geq C_1(R)\,d^{-3/4},$$

for some constant $C_1(R) > 0$ (depending only on R and the fixed η_1, η_2). Finally, since $A \subset A_{\delta,d}$, we obtain

$$\|\partial_{y_2}\bar{v}_{d,1}\|_{L^2(A_{\delta,d})} \geq \|\partial_{y_2}\bar{v}_{d,1}\|_{L^2(A)} \geq C_1(R)\,d^{-3/4}.$$

This concludes the proof of the lower bound.

Step 4: Conclusion. Combining the upper and lower bounds, we obtain that there exist constants $C_1(R), C_2(R) > 0$ such that

$$\frac{C_1(R)}{d^{3/4}} \leq \|\partial_2\bar{v}_{d,1}\|_{L^2(A_{\delta,d})} \leq C(R) + \frac{C_2(R)}{d^{3/4}}. \tag{4.8.23}$$

Since $d \in (0, R-1]$, the additive constant $C(R)$ can be absorbed into the singular term, and we may write

$$\frac{C_1(R)}{d^{3/4}} \leq \|\partial_2\bar{v}_{d,1}\|_{L^2(A_{\delta,d})} \leq \frac{C_2(R)}{d^{3/4}}. \tag{4.8.24}$$

This finishes the proof of inequality (4.5.10). □

Chapter 5 Global Stabilization of a One-Dimensional Fluid–Particle Interaction System

The present chapter is devoted to the third of the problems outlined earlier (see Part III of Chapter 1 and the first subsection of Section 2.3), namely the global stabilization of a fluid–particle interaction system describing the motion of a particle in a one-dimensional viscous fluid. This chapter is based on the work of Tucsnak and Xu [TX25].

5.1 Introduction

The general background and motivation of the one-dimensional fluid–particle model considered in this chapter, together with the main differences between our setting and related references, were already described in Part III of Chapter 1 and in Section 2.3 of Chapter 2. In the present section, we restrict ourselves to the mathematical analysis of the corresponding closed-loop system and recall only the ingredients needed for the proofs. The system under consideration is the following:

$$\dot{v}(t,y) + v(t,y)v_y(t,y) - v_{yy}(t,y) = 0 \quad t \in (0,T), \quad y \in (-1,1), \quad y \neq h(t), \tag{5.1.1}$$

$$v(t,-1) = v(t,1) = 0 \quad t \in (0,T), \tag{5.1.2}$$

$$\dot{h}(t) = v(t,h(t)) \quad t \in (0,T), \tag{5.1.3}$$

$$\ddot{h}(t) = [v_y](t,h(t)) + u(t) \quad t \in (0,T), \tag{5.1.4}$$

$$v(0,y) = v_0(y) \quad y \in (-1,1), \tag{5.1.5}$$

$$h(0) = h_0, \qquad \dot{h}(0) = g_0. \tag{5.1.6}$$

Here v denotes the fluid velocity, h the particle position, and u the force acting on the particle. Moreover,

$$[v_y](t,h(t)) = v_y(t,h(t)^+) - v_y(t,h(t)^-), \qquad t \geq 0. \tag{5.1.7}$$

Since the interface position $h(t)$ is itself unknown, the system has a free-boundary character. Cîndea, Micu, Roventa and Tucsnak [CMRT15] considered the case where the control acts directly on the particle and proved local well-posedness for small initial data. Two important issues, however, remained open: the global well-posedness of the closed-loop system for arbitrary finite-energy initial data, and the derivation of quantitative decay rates under the feedback law. The main objective of this chapter is to resolve both of them. We study system (5.1.1)–(5.1.6) under the feedback law

$$u(t) = \mathcal{K}(h_1 - h(t)), \qquad \mathcal{K} \geq 0, \quad t > 0, \tag{5.1.8}$$

where $h_1 \in (-1, 1)$ is a prescribed target position. This feedback may be interpreted as a spring-type restoring force anchored at h_1, with stiffness $\mathcal{K}$. Throughout this chapter, we work in the framework of finite-energy solutions introduced in [CMRT15, Definition 1.1]; the precise definition is recalled below.

Definition 5.1.1. Given $T > 0$, $v_0 \in L^2(-1, 1)$, $g_0 \in \mathbb{R}$, $h_0 \in (-1, 1)$ and $u \in L^2(0, T)$, we say that

$$\begin{aligned} \begin{bmatrix} v \\ g \\ h \end{bmatrix} \in \Big(C([0,T]; L^2(-1,1)) \cap L^2((0,T); H_0^1(-1,1)) \Big) \\ \times L^2(0,T) \times H^1(0,T). \end{aligned} \tag{5.1.9}$$

is a finite energy solution of system (5.1.1)-(5.1.6) on $[0, T]$ if $h(0) = h_0$, $\dot{h}(t) = g(t) = v\,(t, h(t))$, $h(t) \in (-1, 1)$ for almost every $t \in [0, T]$ and

$$\begin{aligned} &\int_{-1}^{1} v(t,y)\psi(t,y)\,\mathrm{d}y - \int_{-1}^{1} v_0(y)\psi(0,y)\,\mathrm{d}y + g(t)l(t) - g_0 l(0) \\ &- \int_0^t \int_{-1}^{1} v(\sigma,y)\dot{\psi}(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma + \int_0^t \int_{-1}^{1} v_y(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ &- \int_0^t g(\sigma)\dot{l}(\sigma)\,\mathrm{d}\sigma - \frac{1}{2}\int_0^t \int_{-1}^{1} v^2(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ &= \int_0^t u(\sigma) l(\sigma)\,\mathrm{d}\sigma, \end{aligned} \tag{5.1.10}$$

for every $t \in [0, T]$ and for every

$$\begin{aligned} \begin{bmatrix} \psi \\ l \end{bmatrix} \in \Big(H^1((0,T); L^2(-1,1)) \cap L^2((0,T); H_0^1(-1,1)) \Big) \\ \times H^1(0,T), \end{aligned} \tag{5.1.11}$$

$$l(t) = \psi(t, h(t)), \qquad t \in [0, T]. \tag{5.1.12}$$

A triple $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ which is a finite energy solution on $[0,T]$ for every $T>0$ is called a global finite energy solution.

Note that the test functions used above depend on the solution, more precisely on its component h. Here and in the rest of this paper, for each $m \in \mathbb{N}$, H^m and H_0^m denote the classical Sobolev spaces and H^{-m} stands for the dual of H_0^m with respect to the pivot space L^2. Our first main result establishes global well-posedness together with a uniform non-collision property.

Theorem 5.1.2. *Let $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, $h_0, h_1 \in (-1,1)$, and $\mathcal{K} \geq 0$. Then system (5.1.1)–(5.1.6), with u given by the feedback law (5.1.8), admits a unique global finite-energy solution. Moreover, there exists $\alpha > 0$, depending only on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1, and $\mathcal{K}$, such that*

$$-1+\alpha \leq h(t) \leq 1-\alpha \qquad (t \geq 0). \tag{5.1.13}$$

Finally, for every $T>0$, the solution map

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto \begin{bmatrix} v \\ g \\ h \end{bmatrix}$$

is continuous from

$$L^2(-1,1) \times \mathbb{R} \times (-1,1)$$

to

$$\Big(C([0,T];L^2(-1,1)) \cap L^2((0,T);H_0^1(-1,1))\Big) \times L^2(0,T) \times H^1(0,T).$$

Our second main result concerns the large-time behavior of the solutions constructed in Theorem 5.1.2.

Theorem 5.1.3. *With the notation and assumptions of Theorem 5.1.2, the following assertions hold.*

- *If $\mathcal{K}=0$, then the finite-energy solutions of (5.1.1)–(5.1.6) satisfy*

$$\|v(t,\cdot)\|^2_{L^2(-1,1)} + g^2(t) \leq \exp\left(-\frac{1}{4}t\right)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2\right) \qquad (t>0). \tag{5.1.14}$$

  Moreover, there exists $h^* \in (-1,1)$ such that

$$|h(t)-h^*|^2 \leq \exp\left(-\frac{1}{4}t\right)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2\right) \qquad (t>0).$$

- *If $\mathcal{K} > 0$, then there exists a constant $\eta_2 > 0$, depending only on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1, and $\mathcal{K}$, such that*

$$\begin{aligned}&\|v(t,\cdot)\|^2_{L^2(-1,1)} + g^2(t) + \mathcal{K}(h(t)-h_1)^2\\ \leq& 16\exp(-\eta_2 t)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_0-h_1)^2\right) \qquad (t>0). \end{aligned} \tag{5.1.15}$$

Theorems 5.1.2 and 5.1.3 show, in particular, that when the feedback is active, that is, when $\mathcal{K} > 0$, the particle position converges exponentially to the prescribed target h_1. By contrast, when $\mathcal{K} = 0$, the particle converges to a limit position $h^* \in (-1, 1)$ that is not explicitly characterized in terms of the initial data.

The analysis faces two main difficulties. First, the low regularity of finite-energy solutions prevents a direct use of the arguments developed in [VZ03, VZ06], which rely on smoother trajectories. Second, since neither the initial data nor the feedback gain is assumed to be small, the exclusion of collisions with the boundary cannot be obtained through a perturbative argument, unlike in [CMRT15]. To overcome this difficulty, we exploit a special test function introduced by Hesla [Hes04]. The same idea also plays a central role in the construction of a perturbed Lyapunov functional yielding the decay estimates in Theorem 5.1.3.

We next briefly describe the strategy of the proofs of the above two theorems. We first construct a test function adapted to the geometry of the narrow gap between the inner disk and the outer boundary. When inserted into the weak formulation, this test function gives rise to a dissipative term that captures the behavior of thde distance to the boundary. The crucial observation is that this term has a logarithmic singularity of the form

$$\log(1+h(t)) \qquad \text{and} \qquad \log(1-h(t)),$$

as the disk approaches the boundary, whereas all the other terms remain uniformly bounded. This contradiction rules out finite-time contact and yields the desired result.

The chapter is organized as follows. In Section 5.2, we prove Theorem 5.1.2. Section 5.3 is devoted to the proof of Theorem 5.1.3.

5.2 Global Well-Posedness of the Closed-Loop System

We begin by recalling a local in time existence and uniqueness result which is proved in [CMRT15, Sections 2, 3 and 4]

Proposition 5.2.1. *For every $0 < \varepsilon \leq \frac{1}{4}$ and $\kappa > 0$, we denote by $\mathcal{B}_{\varepsilon,\kappa}$ the set of*

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \in L^2(-1,1) \times \mathbb{R} \times (-1,1)$$

satisfying

$$\|v_0\|_{L^2(-1,1)} + g_0 \leq \kappa, \tag{5.2.1}$$

$$-1 + 4\varepsilon \leq h_0 \leq 1 - 4\varepsilon. \tag{5.2.2}$$

Then there exists $T > 0$, depending only on ε and κ, such that for every $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \in \mathcal{B}_{\varepsilon,\kappa}$, *system (5.1.1)-(5.1.6) with feedback law (5.1.8) admits a unique finite energy solution (in the sense of Definition 5.1.1) on the time interval $[0,T]$. Moreover, for every $t \in [0,T]$, we have the following energy estimate:*

$$\begin{aligned} &\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}\left(h_1 - h(t)\right)^2 + 2\int_0^t \int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ &= \int_{-1}^{1} v_0^2(y)\,\mathrm{d}y + g_0^2 + \mathcal{K}\left(h_1 - h_0\right)^2, \end{aligned} \tag{5.2.3}$$

Remark 5.2.2. Let $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ be the finite energy solution of system (5.1.1)-(5.1.6) on $[0,T]$, then the map

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto \begin{bmatrix} v \\ g \\ h \end{bmatrix}, \tag{5.2.4}$$

is continuous from $\mathcal{B}_{\kappa,\varepsilon}$ to $\{\mathcal{C}([0,T];L^2(-1,1)) \cap L^2((0,T);\mathcal{H}_0^1(-1,1))\} \times L^2(0,T) \times \mathcal{H}^1(0,T)$.

As a consequence of the above result we have:

Corollary 5.2.3. *Assume that for every $\tau > 0$, a finite energy solution of (5.1.1)-(5.1.6) defined on $[0,\tau)$ satisfies*

$$\sup_{t\in[0,\tau)} |h(t)| < 1. \tag{5.2.5}$$

Then the considered finite energy solution is global.

Proposition below is the essential ingredient of our global well-posedness result:

Proposition 5.2.4. *Let $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, h_0, $h_1 \in (-1,1)$. Moreover, let $\tau > 0$ and let* $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ *a finite energy solution of (5.1.1)-(5.1.6) defined on $[0,\tau)$. Then for every*

$t \in [0,\tau)$, the position of the particle of system (5.1.1)-(5.1.6) with feedback law in (5.1.8) statisfies

$$-1+\kappa_1(t) \le h(t) \le 1-\kappa_2(t) \qquad (t\in[0,\tau)), \tag{5.2.6}$$

where

$$\kappa_1(t) = \frac{2}{1+\max\left\{2,\frac{1-h_0}{1+h_0}\right\}\exp\left(C+2\mathcal{K}t\right)} \qquad (t\in[0,\tau)),$$
$$\kappa_2(t) = \frac{2}{1+\max\left\{2,\frac{1+h_0}{1-h_0}\right\}\exp\left((C+2\mathcal{K}t)\right)} \qquad (t\in[0,\tau)),$$
$$C = 10\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1-h_0)^2 + \sqrt{\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1-h_0)^2}\right).$$

Proof. Following the approach proposed in [Hes04, Section II], we begin by choosing the test function $\begin{bmatrix}\psi\\ l\end{bmatrix} = \begin{bmatrix}\varphi\\ 1\end{bmatrix}$ in the Definition 5.1.1, where φ is defined by:

$$\varphi(t,y) = \begin{cases} \frac{1}{1+h(t)}(y+1) & (y\in(-1,h(t))),\\ \frac{1}{1-h(t)}(1-y) & (y\in(h(t),1)).\end{cases} \tag{5.2.7}$$

It follows that

$$\begin{aligned}&\int_{-1}^{1} v(t,y)\varphi(t,y)\,\mathrm{d}y - \int_{-1}^{1} v_0(y)\varphi(0,y)\,\mathrm{d}y - \int_0^t\int_{-1}^{1} v(\sigma,y)\frac{\partial\varphi}{\partial\sigma}(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma\\ &+\int_0^t\int_{-1}^{1} v(\sigma,y)v_y(\sigma,y)\varphi(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma + g(t) - g_0 + \int_0^t\int_{-1}^{1} v_y(\sigma,y)\varphi_y(\sigma,y)\\ =&\int_0^t \mathcal{K}(h_1-h(\sigma))\,\mathrm{d}\sigma,\end{aligned} \tag{5.2.8}$$

Moreover, using the defintion of φ in (5.2.7) and the fact that $v(\sigma,h(\sigma)) = \dot{h}(\sigma)$ for almost every $t\in[0,\tau)$, $\sigma\in[0,\tau)$, we have

$$\begin{aligned}&\int_{-1}^{1} v_y(\sigma,y)\varphi_y(\sigma,y)\,\mathrm{d}y\\ =&\int_{-1}^{h(\sigma)} v_y(\sigma,y)\varphi_y(\sigma,y)\,\mathrm{d}y + \int_{h(\sigma)}^{1} v_y(\sigma,y)\varphi_y(\sigma,y)\,\mathrm{d}y\\ =&\frac{\dot{h}(\sigma)}{1+h(\sigma)} + \frac{\dot{h}(\sigma)}{1-h(\sigma)}\\ =&\frac{\mathrm{d}}{\mathrm{d}\sigma}(\ln(1+h(\sigma))) - \frac{\mathrm{d}}{\mathrm{d}\sigma}(\ln(1-h(\sigma))).\end{aligned} \tag{5.2.9}$$

Using (5.2.11), (5.2.12), (5.2.13) and (5.2.9), we can rewrite (5.2.8) as

$$\begin{aligned}&\ln\frac{1+h(t)}{1+h_0}-\ln\frac{1-h(t)}{1-h_0}\\&=\int_0^t\mathcal{K}(h_1-h(\sigma))\,\mathrm{d}\sigma+P(0)-P(t)+\int_0^t A_1(\sigma)\mathrm{d}\sigma-\int_0^t A_2(\sigma)\,\mathrm{d}\sigma,\end{aligned}\tag{5.2.10}$$

where, for almost every $t\in[0,\tau)$, $\sigma\in[0,t]$, we have set

$$P(\sigma)=\int_{-1}^1\varphi(\sigma,y)v(\sigma,y)\,\mathrm{d}y+g(\sigma),\tag{5.2.11}$$

$$A_1(\sigma)=\int_{-1}^1 v(\sigma,y)\frac{\partial\varphi}{\partial\sigma}(\sigma,y)\,\mathrm{d}y,\tag{5.2.12}$$

$$A_2(\sigma)=\int_{-1}^1 v(\sigma,y)v_y(\sigma,y)\varphi(\sigma,y)\,\mathrm{d}y.\tag{5.2.13}$$

We use next the energy estimate (5.2.3) to estimate the four last terms on the right hand side of (5.2.10).

Firstly, we remark that for P defined in (5.2.11) we can use (5.2.3), the Cauchy-Schwarz inequality and the fact that $|\varphi|$ is uniformly bounded by 1 to get that

$$|P(\sigma)|^2\le 4\left(\|v_0\|^2_{L^2(-1,1)}+g_0^2+\mathcal{K}\,(h_1-h_0)^2\right)\qquad(t\in[0,\tau),\ \sigma\in[0,t]).\tag{5.2.14}$$

To estimate A_1, we note that from (5.2.12) and the definition of φ in (5.2.7), for $t\in[0,\tau))$ and almost every $\sigma\in[0,t]$ we have:

$$\begin{aligned}A_1(\sigma)=&-\frac{g(\sigma)}{(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v(\sigma,y)(1+y)\,\mathrm{d}y\\&+\frac{g(\sigma)}{(1-h(\sigma))^2}\int_{h(\sigma)}^{1}v(\sigma,y)(1-y)\,\mathrm{d}y.\end{aligned}\tag{5.2.15}$$

To estimate the first term on the right hand side of the above formula, we firstly integrate by parts, then for $t\in[0,\tau)$ and $\sigma\in[0,t]$ a.e., we have

$$\begin{aligned}&\frac{g(\sigma)}{(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v(\sigma,y)(1+y)\,\mathrm{d}y\\&=\frac{g(\sigma)}{2(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v(\sigma,y)\frac{\mathrm{d}}{\mathrm{d}y}\left[(1+y)^2\right]\,\mathrm{d}y\\&=-\frac{g(\sigma)}{2(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v_y(\sigma,y)(1+y)^2\,\mathrm{d}y+\frac{1}{2}g^2(\sigma).\end{aligned}\tag{5.2.16}$$

Moreover, by combining the Cauchy-Schwarz inequality, a trace theorem, the facts $|1+h(\sigma)| \leqslant 2$ and $v(t,h(t)) = g(t)$, for $t \in [0,\tau)$ and $\sigma \in [0,t]$ a.e., we have

$$\begin{aligned}
&\left| \frac{g(\sigma)}{2(1+h(\sigma))^2} \int_{-1}^{h(\sigma)} v_y(\sigma,y)(1+y)^2 \,\mathrm{d}y - \frac{1}{2}g^2(\sigma) \right| \\
&\leq \frac{|g(\sigma)|}{2(1+h(\sigma))^2} \left(\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{-1}^{h(\sigma)} (1+y)^4 \,\mathrm{d}y \right)^{\frac{1}{2}} + \frac{1}{2}g^2(\sigma) \\
&\leq \frac{1}{\sqrt{5}} |g(\sigma)| (1+h(\sigma))^{\frac{1}{2}} \left(\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \right)^{\frac{1}{2}} + \frac{1}{2}g^2(\sigma) \\
&\leq g(\sigma) \left(\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \right)^{\frac{1}{2}} + \frac{1}{2}g^2(\sigma) \\
&\leq 2g^2(\sigma) + 2\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \\
&\leq 6\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \qquad\qquad (t\in[0,\tau),\ \sigma\in[0,t] \text{ a.e.}).
\end{aligned}$$

Combining the above formula and (5.2.17) it follows that

$$\begin{aligned}
&\left| \frac{g(\sigma)}{(1+h(\sigma))^2} \int_{-1}^{h(\sigma)} v(\sigma,y)(1+y)\,\mathrm{d}y \right| \\
\leqslant& 6\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \qquad\qquad (t\in[0,\tau),\ \sigma\in[0,t] \text{ a.e.}).
\end{aligned} \tag{5.2.17}$$

In a completely similar manner we can estimate the second term in the right hand side of left term of (5.2.15) to get

$$\left| \frac{g(\sigma)}{(1-h(\sigma))^2} \int_{h(\sigma)}^{1} v(\sigma,y)(1-y)\,\mathrm{d}y \right| \leqslant 6\int_{h(\sigma)}^{1} v_y^2(\sigma,y)\,\mathrm{d}y \qquad (t\in[0,\tau),\ \sigma\in[0,t] \text{ a.e.}).$$

Putting together the last two inequalities and using (5.2.3), it follows that

$$\begin{aligned}
\left| \int_0^t A_1(\sigma)\mathrm{d}\sigma \right| &\leq \int_0^t |A_1(\sigma)|\mathrm{d}\sigma \leq 6\int_0^t \int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\
&\leq 6\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}\left(h_1-h_0\right)^2 \right) \qquad (t\in[0,\tau)).
\end{aligned} \tag{5.2.18}$$

We next estimate of A_2 defined in (5.2.13). Since φ defined by (5.2.7) is uniformly bounded by 1, using the Cauchy-Schwarz and Poincaré inequalities, it follows that for

every $t \in [0,\tau)$ and almost every $\sigma \in [0,t]$, we have

$$\begin{aligned}|A_2(\sigma)| &\leq \int_{-1}^{1} |v(\sigma,y)v_y(\sigma,y)\phi(\sigma,y)|\,\mathrm{d}y \\ &\leq \left(\int_{-1}^{1} v^2(\sigma,y)\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\right)^{\frac{1}{2}} \leq 4\int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y. \end{aligned} \tag{5.2.19}$$

From the above estimate and the energy estimate (5.2.3) we obtain:

$$\begin{aligned}\left|\int_0^t A_2(\sigma)\mathrm{d}\sigma\right| &\leq 4\int_0^t \int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ &\leq 4\left(\|v_0\|_{L^2(-1,1)} + g_0^2 + \mathcal{K}\left(h_1 - h_0\right)^2\right) \qquad (t \in [0,\tau)). \end{aligned} \tag{5.2.20}$$

Moreover, using the fact $|h(\sigma) - h_1| \leq 2$, for every $t \in (0,\tau)$, $\sigma \in [0,t]$, it follows that

$$\left|\int_0^t \mathcal{K}(h_1 - h(\sigma))\mathrm{d}\sigma\right| \leq 2\mathcal{K}t. \tag{5.2.21}$$

By combing (5.2.14), (5.2.18), (5.2.20), (5.2.22) and (5.2.10), we have

$$\ln\frac{1+h(t)}{1+h_0} - \ln\frac{1-h(t)}{1-h_0} \leq C + 2\mathcal{K}t \qquad (t \in [0,\tau)), \tag{5.2.22}$$

where $C = 10\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1-h_0)^2 + \sqrt{\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1-h_0)^2}\right)$.

Since for every $t \geq 0$, we have

$$1 + \frac{1+h_0}{1-h_0}\exp\left(C + 2\mathcal{K}t\right) \leq 1 + \max\left\{2, \frac{1+h_0}{1-h_0}\right\}\exp\left(C + 2\mathcal{K}t\right),$$

then, after a simple calculation, the above inequality (5.2.22) can be rephrased to

$$h(t) \leq 1 - \frac{2}{1 + \max\left\{2, \frac{1+h_0}{1-h_0}\right\}\exp\left(C + 2\mathcal{K}t\right)} \qquad (t \in [0,\tau)). \tag{5.2.23}$$

Choosing next the test function $\begin{bmatrix}\psi \\ l\end{bmatrix} = \begin{bmatrix}-\varphi \\ 1\end{bmatrix}$ in Defintion 5.1.1 and following step by step the procedure used to prove (5.2.23), it follows that:

$$h(t) \geq -1 + \frac{2}{1 + \max\left\{2, \frac{1-h_0}{1+h_0}\right\}\exp\left(C + 2\mathcal{K}t\right)} \qquad (t \in [0,\tau)). \tag{5.2.24}$$

□

By combining Corollary 5.2.3 and Proposition 5.2.4 it follows that

Corollary 5.2.5. *Let $\mathcal{K} \geqslant 0$. Then for every $v_0 \in L^2(-1,1)$, $g \in \mathbb{R}$ and $h_0,\ h_1 \in (-1,1)$ the system (5.1.1)-(5.1.6) with feedback law (5.1.8) admits unique global finite energy solution.*

We also need the result below, which asserts that for $\mathcal{K} > 0$ the equilibrium state $\begin{bmatrix} 0 \\ 0 \\ h_1 \end{bmatrix}$ is asymptotically stable.

Theorem 5.2.6. *Let $\mathcal{K} > 0$. Then for every $v_0 \in L^2(-1,1)$, $g \in \mathbb{R}$ and $h_0,\ h_1 \in (-1,1)$, the unique global finite energy solution of (5.1.1)-(5.1.6) with feedback law (5.1.8) satisfies*

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{L^2(-1,1)} = 0, \qquad \lim_{t\to\infty} g(t) = 0, \qquad \lim_{t\to\infty} h(t) = h_1. \tag{5.2.25}$$

The proof of the above result can be done by closely following the procedure in [CMRT15, Section 6]. To be more precise, the result in [CMRT15, Section 6] required some smallness assumptions on the initial data, but those assumptions were used there only to be sure that the solutions are global. Since now this fact is known from Corollary 5.2.5, the conclusion of Theorem 5.2.6 holds with no smallness assumption on the initial data. However, for the sake of completeness, we provide a complete proof of Theorem 5.2.6 in the Appendix in Section 5.4.

We are now in a position to prove the main result of this section.

Proof of Theorem 5.1.2. The existence and uniqueness of finite energy solutions has already been proved in Corollary 5.2.5. We only have to prove (5.1.13). To this aim, we remark that

1. If $\mathcal{K} = 0$, from (5.2.23) and (5.2.24), we have

$$-1 + \frac{2}{1 + \max\left\{2, \frac{1-h_0}{1+h_0}\right\}\exp(C)} \leq h(t) \leq 1 - \frac{2}{1 + \max\left\{2, \frac{1+h_0}{1-h_0}\right\}\exp(C)}, \tag{5.2.26}$$

   which directly implies estimate (5.1.13).

2. If $\mathcal{K} > 0$, from Theorem 5.2.6, we have

$$\lim_{t\to\infty} h(t) = h_1,$$

   so that there exists $T = T(h_1) > 0$ with

$$|h_1 - h(t)| \leq \frac{1}{8}(1 - h_1), \tag{5.2.27}$$

   which implies that for every $t > T(h_1)$,

$$1 - h(t) \geq \frac{7}{8}(1 - h_1). \tag{5.2.28}$$

For $t \in (0, T(h_1))$, from (5.2.23), we have

$$h(t) \leq 1 - \frac{2}{1 + \max\left\{2, \frac{1+h_0}{1-h_0}\right\} \exp(C + 2\mathcal{K}T(h_1))}. \tag{5.2.29}$$

Therefore,

$$h(t) \leq 1 - \alpha, \tag{5.2.30}$$

where

$$\begin{aligned}
&C = 10E(0) + 10\sqrt{E(0)},\\
&E(0) = \|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1 - h_0)^2,\\
&\alpha = \min\left\{\frac{7}{8}(1 - h_1),\ \frac{2}{1 + \max\left\{2, \frac{1+h_0}{1-h_0}\right\} \exp(C + 2\mathcal{K}T(h_1))}\right\}.
\end{aligned}$$

The lower bound of $h(t)$ can be obtained similarly, completing the proof.

□

5.3 Global exponential stability

In this section, we give the proof of Theorem 5.1.3.

Proof of Theorem 5.1.3. To begin let ε be such that

$$0 \leq \varepsilon \leq \min\left\{\frac{1}{8}, \frac{\mathcal{K}}{8}\right\}, \tag{5.3.1}$$

and we define

$$\begin{aligned}
V_\varepsilon(t) = &\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}\left(h(t) - h_1\right)^2\\
&- \varepsilon\left(h_1 - h(t)\right) P(t) \qquad\qquad (t \geqslant 0),
\end{aligned} \tag{5.3.2}$$

where $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ is the global finite energy solution of system (5.1.1)-(5.1.6) with feedback (5.1.8), which has been defined in Definition 5.1.1 and P has been defined in (5.2.11).

Meanwhile, for every $t \geq 0$, we also define

$$E(t) = \int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}\left(h(t) - h_1\right)^2 \qquad (\mathcal{K} \geq 0). \tag{5.3.3}$$

By combining the Cauchy-Schwarz inequality and the fact $|\varphi|$ is uniform bounded by 1, it is straightforward to verify that,

$$\frac{1}{4}E(t) \leq V_\varepsilon(t) \leq 2E(t), \qquad \frac{1}{4}V_\varepsilon(t) \leq E(t) \leq 2V_\varepsilon(t) \qquad (t \geq 0). \tag{5.3.4}$$

We next remark that $V_\varepsilon(t)$ is differentiable with respect to t. Indeed, from the energy estimate given by (5.2.3) and Theorem 5.1.2, we deduce that the mapping

$$t \mapsto \frac{1}{2}\left(\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}(h(t)-h_1)^2\right) \tag{5.3.5}$$

is absolutely continuous and differentiable for almost every $t \in [0,\infty)$. Consequently, by taking the derivative of (5.3.3) with respect to t, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -2\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y \qquad (t \geq 0 \text{ a.e.}). \tag{5.3.6}$$

On the other hand, from (5.2.10) it follows that P is absolutely continuous on every bounded interval contained in $[0,\infty)$, thus differentiable almost everywhere on $[0,\infty)$. Moreover, from (5.2.11) it also follows that for almost every $t \geqslant 0$ we have

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t}(t) &= A_1(t) - A_2(t) - \int_{-1}^{1} v_y(t,y)\varphi_y(t,y)\,\mathrm{d}y + \mathcal{K}(h_1 - h(t)) \\ &= A_1(t) - A_2(t) + \mathcal{K}(h_1 - h(t)) - \left(\frac{g(t)}{1+h(t)} + \frac{g(t)}{1-h(t)}\right),\end{aligned} \tag{5.3.7}$$

where A_1 and A_2 have been defined in (5.2.12) and (5.2.13), respectively.

From (5.3.6) and (5.3.7) we get that $V_\varepsilon(t)$ defined by (5.3.2) is absolutely continuous, thus differentiable for almost every $t \in [0,\infty)$, with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \\ =& -2\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y \\ &+ \varepsilon g(t)\left(g(t) + \int_{-1}^{1}\varphi(t,y)v(t,y)\,\mathrm{d}y\right) \\ &- \varepsilon\,(h_1 - h(t))\left(A_1(t) - A_2(t) + \mathcal{K}(h_1 - h(t)) - \left(\frac{g(t)}{1+h(t)} + \frac{g(t)}{1-h(t)}\right)\right).\end{aligned} \tag{5.3.8}$$

Moreover, we know from (5.1.13) that there exists a constant $\alpha \in (0,1]$, such that

$$\min\{1-h(t),\ 1+h(t)\} \geq \alpha \qquad (t \geq 0).$$

The above estimate, combined with an elementary inequality, shows that for every $t \geq 0$ we have

$$(h_1 - h(t))\left(\frac{g(t)}{1+h(t)} + \frac{g(t)}{1-h(t)}\right) \leq \frac{g^2(t)}{\mathcal{K}\alpha^2} + \frac{\mathcal{K}}{4}(h_1 - h(t))^2 \qquad (\mathcal{K} > 0).$$

From the above inequality it follows that for every ε satisfying (5.3.1) and almosts every $t \geqslant 0$ and $\mathcal{K} > 0$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq &\left(-\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y - \varepsilon(h_1 - h(t))A_1(t) + \varepsilon(h_1 - h(t))A_2(t)\right)\\ &+ \left(5 + \frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t) - \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2.\end{aligned}$$

Using the inequality above and the fact that $|h(t) - h_1| \leq 2$, along with the estimates of A_1 and A_2 in (5.2.17), (5.2.20), respectively, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq (-2 + 20\varepsilon)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y + \left(5 + \frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t) - \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2. \quad (5.3.9)$$

Using the estimates

$$\begin{aligned}g(t) = v(t,h(t)) = \int_{-1}^{h(t)} v_y(t,y)\,\mathrm{d}y \leq \left(\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{-1}^{1} 1\,\mathrm{d}y\right)^{\frac{1}{2}}\\ \leq \sqrt{2}\left(\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y\right)^{\frac{1}{2}}, \qquad (5.3.10)\end{aligned}$$

inequality (5.3.9) implies that for almost every $t \geqslant 0$ we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq (-2 + 20\varepsilon)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y + \left(5 + \frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t) - \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2\\ =&(-2 + 20\varepsilon)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y + \left(7 + \frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t) - 2\varepsilon g^2(t) - \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2\\ \leq&\left(-1 + 20\varepsilon + 2\left(7 + \frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon\right)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y - 2\varepsilon g^2(t) - \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2. \quad (5.3.11)\end{aligned}$$

We choose $\varepsilon = \frac{1}{16\left(34 + \frac{2}{\mathcal{K}\alpha^2}\right)}$, which satisfies (5.3.1). Combining this with the Poincaré inequality, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq &-\frac{1}{4}\left[1 - \left(34 + \frac{2}{\mathcal{K}\alpha^2}\right)\varepsilon\right]\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y\\ &- 2\varepsilon g^2(t) - \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2. \qquad (5.3.12)\end{aligned}$$

Denote

$$\eta = \frac{1}{4}\min\left\{\frac{1}{34+\frac{2}{\mathcal{K}\alpha^2}},\ \frac{3\mathcal{K}\varepsilon}{4}\right\}.$$

Using the above inequality and (5.3.4), for almost every $t \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq -4\eta E(t) \leq -\eta V_\varepsilon(t), \tag{5.3.13}$$

where $E(t)$ is defined by (5.3.3).

By Gronwall's inequality, for $\mathcal{K} > 0$, we obtain

$$V_\varepsilon(t) \leq \exp(-\eta t)V_\varepsilon(0). \tag{5.3.14}$$

Using (5.3.4) again, we conclude that if $\mathcal{K} > 0$ then

$$E(t) \leq 16\exp(-\eta t)E(0) \qquad (t \geqslant 0). \tag{5.3.15}$$

For $\mathcal{K} = 0$, using (5.3.6) and (5.3.10) for almost every $t \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -2\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y \leq -\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y - \frac{1}{2}g^2(t). \tag{5.3.16}$$

Applying the Poincaré inequality, the above estimate implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\frac{1}{4}E(t) \qquad (t \geq 0) \tag{5.3.17}$$

By Gronwall's inequality, it follows that

$$E(t) \leq \exp\left(-\frac{1}{4}t\right)E(0), \qquad (t \geq 0).$$

The above estimate implies that

$$|g(t)| \leq \exp\left(-\frac{1}{8}t\right)\sqrt{E(0)}. \tag{5.3.18}$$

Next for any $\varepsilon > 0$, we choose $T(\varepsilon) = 16\ln\frac{\sqrt{E_0}}{4\varepsilon}$, then, for every $t_2 > t_1 \geq T(\varepsilon)$, using Newton-Leibniz formula we have

$$\begin{aligned}
&|h(t_2) - h(t_1)| = \left|\int_{t_2}^{t_1} g(s)\,\mathrm{d}s\right| \leq \int_{t_2}^{t_1} |g(s)|\,\mathrm{d}s \\
\leq &\int_{t_2}^{t_1} \exp\left(-\frac{1}{8}t\right)\sqrt{E(0)}\,\mathrm{d}s \leq \frac{\sqrt{E_0}}{8}\left(\exp\left(-\frac{1}{8}t_1\right) - \exp\left(-\frac{1}{8}t_2\right)\right) \\
\leq &\frac{\sqrt{E_0}}{4}\exp\left(-\frac{1}{8}T(\varepsilon)\right) < \varepsilon,
\end{aligned}$$

by combining the above with (5.2.26), there exists $h^* \in (-1,1)$, such that

$$\lim_{t\to\infty} h(t) = h^*. \tag{5.3.19}$$

Moreover, using (5.3.18) and above equality, we have

$$|h(t) - h^*| \leq \int_t^\infty |g(s)|\,\mathrm{d}s \leq \frac{\sqrt{E_0}}{8}\exp\left(-\frac{1}{8}t\right).$$

(5.3.15), (5.3.16) and the above inequality allow us to end the proof of Theorem 5.1.3. □

5.4 Appendix

This appendix is devoted to the proof of Theorem 5.2.6. This proof is essentially the same as the one proposed in [CMRT15] for a weaker version of this result, where it was assumed that the initial data satisfy some smallness conditions. Nevertheless, in order to convince the reader that the result holds without these assumptions, we give the proof below.

We first introduce the functions $W_1,\ W_2 : L^2(-1,1)\times\mathbb{R}\times(-1,1) \to [0,\infty)$ defined by

$$W_1\begin{bmatrix}\varphi\\ g\\ h\end{bmatrix} = \frac{1}{2}\left(\int_{-1}^1 \varphi^2\,\mathrm{d}y + |g|^2\right) \qquad (\varphi\in L^2(-1,1),\ g\in\mathbb{R},\ h\in(-1,1)), \tag{5.4.1}$$

$$W_2\begin{bmatrix}\varphi\\ g\\ h\end{bmatrix} = \frac{\mathcal{K}}{2}|h-h_1|^2 \qquad (\varphi\in L^2(-1,1),\ g\in\mathbb{R},\ h\in(-1,1)). \tag{5.4.2}$$

We also consider $D : H_0^1(-1,1)\times\mathbb{R}\times(-1,1)\to[0,\infty)$ defined by

$$D\begin{bmatrix}\varphi\\ g\\ h\end{bmatrix} = \int_{-1}^1 \varphi_y^2(y)\,\mathrm{d}y \qquad (\varphi\in H_0^1(-1,1),\ g\in\mathbb{R},\ h\in(-1,1)). \tag{5.4.3}$$

Moreover, for $v_0\in L^2(-1,1)$, $g_0\in\mathbb{R}$ and $h_0\in(-1,1)$ we set

$$S(t)\begin{bmatrix}v_0\\ g_0\\ h_0\end{bmatrix} = \begin{bmatrix}v(t,\cdot)\\ g(t)\\ h(t)\end{bmatrix} \qquad (t\geqslant 0), \tag{5.4.4}$$

where $\begin{bmatrix}v\\ g\\ h\end{bmatrix}$ is the corresponding finite energy solution of (5.1.1)-(5.1.6) constructed in Theorem 5.1.2. From Theorem 5.1.2 it follows that to say that

$$S(t)\in C\left(L^2(-1,1)\times\mathbb{R}\times(-1,1);\ L^2(-1,1)\times\mathbb{R}\times(-1,1)\right) \qquad (t\geqslant 0), \tag{5.4.5}$$

and that, for every $T > 0$, the map

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto D\left(S(\cdot) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) \tag{5.4.6}$$

is continuous from $L^2(-1,1) \times \mathbb{R} \times (-1,1)$ to $L^1(0,T)$.

With the above notation, the energy estimate (5.2.3) can be written:

$$\begin{aligned} &W_1 \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} + W_2 \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} - W_1 \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) - W_2 \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) \\ &= \int_0^t D\left(S(\sigma) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) \mathrm{d}\sigma \qquad\qquad (t \geqslant 0). \end{aligned} \tag{5.4.7}$$

Proposition 5.4.1. *Under the assumptions of Theorem 5.1.2 and with the above notation, for every $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$ and $h_0 \in (-1,1)$ we have*

$$\lim_{t\to\infty} W_1 \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) = 0.$$

Proof. Let

$$W_k \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) = W_k(t) \qquad (k \in \{1,2\}, \quad t \geqslant 0).$$

We use a contradiction argument.

If W_1 does not converge to zero when $t \to \infty$. then there exist $\varepsilon > 0$ and a sequence $(t_n)_{n\geq 0}$ of positive numbers with $t_n \to \infty$ and

$$W_1(t_n) \geqslant \varepsilon \qquad (n \in \mathbb{N}).$$

Denote

$$\delta_n = \max\left\{ \delta > 0 \;\mid\; W_1(t_n - \delta) \geqslant \frac{\varepsilon}{2} \right\} \qquad\qquad (n \in \mathbb{N}).$$

Since, according to (5.4.7) and to Poincaré's inequality and a trace theorem, we have that $W_1 \in L^1[0,\infty)$, it follows that

$$\sum_{n\in\mathbb{N}} \delta_n < \infty,$$

so that

$$\lim_{n\to\infty} \delta_n = 0.$$

On the other hand, we know from (5.4.7) that $W_1 + W_2$ is nonincreasing, so that

$$\begin{aligned}\frac{\varepsilon}{2} + \frac{1}{2}|h(t_n - \delta_n) - h_1|^2 &= W_1(t_n - \delta_n) + W_2(t_n - \delta_n)\\ \geqslant W_1(t_n) + W_2(t_n) &\geqslant \varepsilon + \frac{1}{2}|h(t_n) - h_1|^2 \qquad (n \in \mathbb{N}).\end{aligned}$$

From the above estimate it follows that

$$|h(t_n - \delta_n) - h_1|^2 - |h(t_n) - h_1|^2 \geqslant \varepsilon \qquad (n \in \mathbb{N}).$$

By combining the mean value theorem with the obvious inequality

$$|h(t) - h_1| \leqslant 2 \qquad (t \geqslant 0),$$

we obtain that for every $n \in \mathbb{N}$ there exist $\alpha_n \in (0,1)$ with

$$|\dot{h}(t_n - \alpha_n \delta_n)| \geqslant \frac{\varepsilon}{4\delta_n} \to \infty. \tag{5.4.8}$$

On the other hand, according to (5.4.7) we have that

$$W_1(t) \leqslant W_1(0) + W_2(0) \qquad (t \geqslant 0).$$

The above estimate and (5.4.1) clearly contradict (5.4.8), which ends the proof. □

We are now in a position to give the main proof of this section.

Proof of Theorem 5.2.6. We first remark that Proposition 5.4.1 implies that

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{L^2(-1,1)} = 0, \qquad \lim_{t\to\infty} g(t) = 0. \tag{5.4.9}$$

On the other hand, the fact that $h(t) \in (-1,1)$ for every $t \geqslant 0$ implies that the set $(h(t))_{t\geqslant 0}$ is relatively compact in $\mathbb{R}$. Consequently, there exists a sequence $(t_n)_{n\geqslant 0}$ of positive numbers with

$$t_n \to \infty, \qquad \lim_{n\to\infty} h(t_n) = h^* \in [-1,1]. \tag{5.4.10}$$

Moreover, (5.4.7) implies that the map $t \mapsto D\left(S(t)\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right)$ is in $L^1[0,\infty)$. From this fact it follows that for all $T > 0$ we have

$$\lim_{n\to\infty} \int_{t_n}^{T+t_n} D\left(S(t)\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right) \mathrm{d}t = 0.$$

The above estimate can be combined with the semigroup property of the family $(S(t))_{t\geqslant 0}$ to imply that

$$\lim_{n\to\infty} \int_0^T D\left(S(s)S(t_n) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) \mathrm{d}s = 0.$$

On the other hand, we know from Theorem 5.1.2 that $S(s)$ is continuous on $L^2(-1,1)\times\mathbb{R}\times(-1,1)$, so that we can use (5.4.9) and (5.4.10) to obtain

$$\lim_{n\to\infty} S(t_n) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h^* \end{bmatrix}. \tag{5.4.11}$$

From the last two formulae and the fact that the map defined in (5.4.6) is continuous we obtain that

$$D\left(S(s) \begin{bmatrix} 0 \\ 0 \\ h^* \end{bmatrix} \right) = 0 \qquad (s\in[0,T]).$$

To end the proof, we define

$$\begin{bmatrix} \widetilde{v}(t,\cdot) \\ \widetilde{g}(t) \\ \widetilde{h}(t) \end{bmatrix} = S(t) \begin{bmatrix} 0 \\ 0 \\ h^* \end{bmatrix} \qquad (t\geqslant 0).$$

Then, from (5.4.11), it follows that $\widetilde{v}_y(s,\cdot)=0$ in $L^2(-1,1)$ for almost every $s\in[0,T]$. Moreover, since $\widetilde{v}$ vanishes for $y=\pm 1$, it follows that $\widetilde{v}(s,\cdot)=0$ in $H_0^1(-1,1)$ for almost every $s\in[0,T]$. This implies, in particular, that $g(s)=0$ for almost every $s\in[0,T]$. Finally, using (5.1.4) and (5.1.8) (with $\widetilde{v}$ instead of v and h^* instead of h_0) we obtain that $h^*=h_1$, which concludes the proof. □

Chapter 6 Input-to-State Type Stability for a One-Dimensional Fluid -Particle Interaction System

The present chapter is devoted to the third of these problems (see Part III of Chapter 1 and the second subsection of Section 2.3), namely an input-to-state stability analysis of a fluid–particle interaction system describing the motion of a particle in a one-dimensional viscous fluid. This chapter is based on the work of Xu [Xu26b].

6.1 Introduction

In the previous Chapter 5, we introduced a proportional (P) state-feedback law acting on the particle, of the form

$$u(t) = \mathcal{K}(h_1 - h(t)), \tag{6.1.1}$$

where $h_1 \in (-1, 1)$ is a prescribed target position and $\mathcal{K} > 0$ is the feedback gain.

In this subsection, the proportional feedback law $\mathcal{K}(h_1 - h(t))$ is incorporated into the particle equation, while the symbol $u = u(t)$ denotes an additional external disturbance. Our goal is to quantify the effect of this disturbance on the closed-loop behavior. More precisely, we consider the following system:

$$\dot{v}(t,y) + v(t,y)v_y(t,y) - v_{yy}(t,y) = 0 \qquad t \in (0,T), \quad y \in (-1,1), \quad y \neq h(t), \tag{6.1.2}$$

$$v(t,-1) = v(t,1) = 0 \qquad t \in (0,T), \tag{6.1.3}$$

$$\dot{h}(t) = v(t,h(t)) \qquad t \in (0,T), \tag{6.1.4}$$

$$\ddot{h}(t) = [v_y](t,h(t)) + \mathcal{K}(h_1 - h(t)) + u(t) \qquad t \in (0,T), \tag{6.1.5}$$

$$v(0,y) = v_0(y) \qquad y \in (-1,1), \tag{6.1.6}$$

$$h(0) = h_0, \qquad \dot{h}(0) = g_0. \tag{6.1.7}$$

Here $\mathcal{K} \geq 0$ is the feedback gain and

$$[v_y](t,h(t)) = v_y(t,h(t)^+) - v_y(t,h(t)^-), \qquad t > 0, \tag{6.1.8}$$

with

$$v_y(t,h(t)^+) := \lim_{\substack{y\to h(t)\\ y>h(t)}} v_y(t,y),$$

$$v_y(t,h(t)^-) := \lim_{\substack{y\to h(t)\\ y<h(t)}} v_y(t,y).$$

Here $v = v(t,y)$ denotes the fluid velocity, $h(t)$ the particle position, and $g(t) = \dot{h}(t)$ the particle velocity. The signal $u = u(t)$ is not a control to be designed; rather, it is an external disturbance acting on the particle dynamics. As in the previous chapter, the problem is a nonlinear free-boundary problem, since the unknown position $h(t)$ determines the fluid domain.

System (6.1.2)–(6.1.7) constitutes a free boundary value problem because the particle's position $h(t)$, which influences the domain of the fluid, is an unknown of the system. This fact leads us to adopt the notion of a finite energy solution, as defined in [CMRT15, Definition 1.1]:

Definition 6.1.1. Given $T > 0$, $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, h_0, $h_1 \in (-1,1)$, $\mathcal{K} \geq 0$ and $u \in L^2(0,T)$, we say that

$$\begin{bmatrix} v \\ g \\ h \end{bmatrix} \in \left[C\left([0,T];\ L^2(-1,1)\right) \cap L^2\left((0,T);\ H_0^1(-1,1)\right)\right] \times L^2(0,T) \times H^1(0,T), \quad (6.1.9)$$

is a finite energy solution of system (6.1.2)–(6.1.7) on $[0,T]$ if $h(0) = h_0 \in (-1,1)$, $\dot{h}(t) = g(t) = v\left(t,h(t)\right)$, and $h(t) \in (-1,1)$ for almost every $t \in [0,T)$, and if for all $t \in [0,T)$ the following equation holds:

$$\begin{aligned} &\int_{-1}^{1} v(t,y)\psi(t,y)\,\mathrm{d}y - \int_{-1}^{1} v_0(y)\psi(0,y)\,\mathrm{d}y + g(t)l(t) - g_0 l(0) - \int_0^t g(\sigma)\dot{l}(\sigma)\,\mathrm{d}\sigma \\ &- \int_0^t \int_{-1}^{1} v(\sigma,y)\dot{\psi}(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma + \int_0^t \int_{-1}^{1} v_y(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ &- \frac{1}{2}\int_0^t \int_{-1}^{1} v^2(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma = \int_0^t u(\sigma)l(\sigma)\,\mathrm{d}\sigma + \int_0^t \mathcal{K}\left(h_1 - h(\sigma)\right) l(\sigma)\,\mathrm{d}\sigma, \end{aligned} \quad (6.1.10)$$

for every test function

$$\begin{bmatrix} \psi \\ l \end{bmatrix} \in \left\{H^1\left((0,T);\ L^2(-1,1)\right) \cap L^2\left((0,T);\ H_0^1(-1,1)\right)\right\} \times H^1(0,T), \quad (6.1.11)$$

$$l(t) = \psi(t,h(t)). \quad (6.1.12)$$

A triple $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ which is a finite energy solution on $[0,T]$ for every $T>0$ is called a global finite energy solution.

Note that the test functions depend explicitly on the solution, through its component $h(t)$. Throughout the paper, for $m \in \mathbb{N}$, we denote by H^m and H_0^m the usual Sobolev spaces, and by H^{-m} the dual of H_0^m with respect to the pivot space L^2.

We now present the main results of the paper. The first theorem ensures the global existence and uniqueness result of system (6.1.2)-(6.1.7):

Theorem 6.1.2. *Let $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, h_0, $h_1 \in (-1,1)$, $\mathcal{K} \geq 0$ and $u \in L^2(0,\infty)$, then system (6.1.2)–(6.1.7) admits a unique global finite energy solution as defined above. Moreover, for every $t \geq 0$ the following energy equality holds:*

$$\begin{aligned} &\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y - \int_{-1}^{1} v_0^2(y)\,\mathrm{d}y + g^2(t) - g_0^2 + \mathcal{K}\left(h_1 - h(t)\right)^2 - \mathcal{K}\left(h_1 - h_0\right)^2 \\ &+ 2\int_0^t \int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma = \int_0^t u(\sigma)g(\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{6.1.13}$$

The next result shows that, under the additional assumption $u \in L^1(0,\infty) \cap L^2(0,\infty)$, the particle remains uniformly separated from the boundary for all times.

Theorem 6.1.3. *With the same initial conditions as in Theorem 6.1.2, if the open control $u \in L^2 \cap L^1(0,\infty)$, then system (6.1.2)–(6.1.7) admits a unique global finite energy solution. Moreover, there exists $\alpha > 0$, depending on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1, $\mathcal{K}$, $\|u\|_{L^2(0,\infty)}$, and $\|u\|_{L^1(0,\infty)}$, such that*

$$-1 + \alpha \leq h(t) \leq 1 - \alpha. \tag{6.1.14}$$

Before stating the main ISS-type result, we briefly mention that a formal discussion of the ISS will be provided in Section 6.2. We now present the main stability theorem of system (6.1.2)-(6.1.7):

Theorem 6.1.4. *With the notation and assumptions of Theorem 6.1.3, system (6.1.2)–(6.1.7) exhibits the following behavior:*

- *If $\mathcal{K} = 0$, then we have*

$$\begin{aligned} &\|v(t,\cdot)\|_{L^2(-1,1)}^2 + g^2(t) \\ \leq &16\exp\left(-\frac{1}{4}t\right)\left(\|v_0\|_{L^2(-1,1)}^2 + g_0^2\right) + 16\|u\|_{L^2(0,\infty)}^2 \qquad (t>0). \end{aligned} \tag{6.1.15}$$

- *If $\mathcal{K} > 0$, then there exist constants $\eta > 0$ depending on $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1, $\|u\|_{L^2(0,\infty)}$, $\|u\|_{L^1(0,\infty)}$, and $\mathcal{K}$, such that*

$$\begin{aligned} &\|v(t,\cdot)\|^2_{L^2(-1,1)} + g^2(t) + \mathcal{K}(h(t)-h_1)^2 \\ &\leq 16\exp(-\eta t)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_0-h_1)^2\right) \\ &\quad + \frac{3}{2}\|u\|^2_{L^2(0,\infty)}, \qquad t > 0. \end{aligned} \tag{6.1.16}$$

In connection with the above three theorems, the reader may wonder why the notion considered here is not the standard ISS property, and why an additional L^1-assumption on the input is imposed. These points were already discussed in Subsection 2.3.2 and in Remarks 2.3.8 and 2.3.10. The proof strategy is also close to that of Chapter 5. We first establish a non-collision estimate and showing that the particle remains uniformly away from the boundary; this is the key step toward global well-posedness for the disturbed system. Finally, we adapt the Lyapunov functional introduced in the previous chapter 5 to derive ISS-type estimates for the open-loop dynamics.

This chapter is organized as follows. In Section 6.3 we adapt the approach of [TX25, Section 2] to prove Theorem 6.1.2; the main idea is unchanged, with the only additional ingredient being the external input u. Section 6.4 establishes Theorem 6.1.3 via a comparison argument for ODEs. In Section 6.5 we employ Lyapunov-based techniques, inspired by [TX25, Section 3], to derive the ISS-type estimate (6.1.16) stated in Theorem 6.1.4. Finally, Section 6.6 concludes with numerical simulations illustrating both the stabilization mechanism from the previous chapter and the robustness properties established here.

6.2 Background on Input-to-State Stability

In this section, we briefly recall some basic notions of input-to-state stability (ISS) for nonlinear systems. For a more detailed discussion of ISS in the linear setting, we refer the reader to Appendix 7.6. We begin with the general definition of a control system. Our presentation follows closely the framework developed by Mironchenko and Prieur [MP20].

Definition 6.2.1. A control system is defined by the triple $\Sigma = (X, \mathcal{U}, \phi)$, where:

(i) X is a normed vector space, called the *state space*, equipped with the norm $\|\cdot\|_X$.

(ii) $\mathcal{U}$ is a normed vector space of inputs, where each $u \in \mathcal{U}$ is a function $u : \mathbb{R}_+ \to U$ and U is a normed vector space of input values. The space $\mathcal{U}$ satisfies the following two properties:

  i For all $u \in \mathcal{U}$ and $\tau \geq 0$, the time-shifted function $u(\cdot + \tau)$ also belongs to $\mathcal{U}$, and $\|u\|_{\mathcal{U}} \geq \|u(\cdot + \tau)\|_{\mathcal{U}}$.

ii For all $u_1, u_2 \in \mathcal{U}$ and $t > 0$, the concatenation

$$(u_1 \diamondsuit_t u_2)(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0, t], \\ u_2(\tau - t), & \text{otherwise,} \end{cases}$$

also belongs to $\mathcal{U}$.

(iii) $\phi : D_\phi \to X$ is a transition map, where $D_\phi \subset \mathbb{R}_+ \times X \times \mathcal{U}$. For every initial state $x \in X$ and input $u \in \mathcal{U}$, there exists a maximal existence time $t_m = t_m(x, u) \in (0, +\infty]$ such that

$$D_\phi \cap (\mathbb{R}_+ \times \{(x, u)\}) = [0, t_m) \times \{(x, u)\} \subset D_\phi.$$

The interval $[0, t_m)$ is called the *maximal domain of definition* of the map $t \mapsto \phi(t, x, u)$.

The triple Σ is called a *(control) system* if the following conditions hold:

(Σ1) For all $(x, u) \in X \times \mathcal{U}$, we have $\phi(0, x, u) = x$.

(Σ2) If u and $\tilde{u}$ agree on $[0, t]$, then $\phi(t, x, u) = \phi(t, x, \tilde{u})$.

(Σ3) For each fixed $(x, u) \in X \times \mathcal{U}$, the map $t \mapsto \phi(t, x, u)$ is continuous on its domain.

(Σ4) For all $x \in X$, $u \in \mathcal{U}$, and $t, h \geq 0$ such that $[0, t + h] \times \{(x, u)\} \subset D_\phi$, it holds that

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

This abstract definition covers many important types of systems, such as ordinary differential equations, partial differential equations, time-delay systems, and even more complex systems like interconnected or hybrid systems. A similar definition of nonlinear systems can be found in Section 7 of Tucsnak and Weiss [TW14]. They explain that a nonlinear system can be regarded as a linear system with a nonlinear feedback term.

Next, we introduce a basic well-posedness condition.

Definition 6.2.2. A control system (as defined above) is said to be *forward complete* if

$$D_\phi = \mathbb{R}_+ \times X \times \mathcal{U},$$

i.e., for every $(x, u) \in X \times \mathcal{U}$ and for all $t \geq 0$, the value $\phi(t, x, u) \in X$ is well-defined.

To define input-to-state stability, we use several standard classes of comparison functions:

$$\begin{aligned}
\mathcal{P} &:= \{\gamma \in C(\mathbb{R}_+) : \gamma(0) = 0,\ \gamma(r) > 0 \text{ for } r > 0\}, \\
\mathcal{K} &:= \{\gamma \in \mathcal{P} : \gamma \text{ is strictly increasing}\}, \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} : \gamma \text{ is unbounded}\}, \\
\mathcal{L} &:= \left\{\gamma \in C(\mathbb{R}_+) : \gamma \text{ is strictly decreasing and } \lim_{t\to\infty} \gamma(t) = 0\right\}, \\
\mathcal{KL} &:= \left\{\beta \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) : \begin{array}{l} \beta(\cdot, t) \in \mathcal{K} \text{ for all } t \geq 0, \\ \beta(r, \cdot) \in \mathcal{L} \text{ for all } r > 0 \end{array}\right\}.
\end{aligned}$$

Functions in class $\mathcal{P}$ are also called *positive definite functions*.

Now we introduce the main concept of this section: input-to-state stability.

Definition 6.2.3. System $\Sigma = (X, \mathcal{U}, \phi)$ is said to be *(uniformly) input-to-state stable (ISS)* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $x \in X$, $u \in \mathcal{U}$, and $t \geq 0$, we have

$$\|\phi(t, x, u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{6.2.1}$$

This inequality describes how the state of the system behaves over time. Specifically, the first term $\beta(\|x\|_X, t)$ shows that the effect of the initial condition decreases as time increases. The second term $\gamma(\|u\|_{\mathcal{U}})$ shows how the input signal influences the state in the long run.

6.3 Proof of Theorem 6.1.2

In this section, we establish the global existence and uniqueness of finite energy solutions as defined in Definition (6.1.1). To begin with, we introduce the following theorem, which concerns the local (in time) well-posedness of the system (6.1.2)–(6.1.7). The theorem is stated as follows:

Theorem 6.3.1. *For every $0 < \varepsilon \leq \frac{1}{4}$, $\kappa > 0$ and $h_1 \in (-1, 1)$, we denote by $\mathcal{B}_{\varepsilon,\kappa}$ the set* $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \\ u \end{bmatrix} \in L^2(-1,1) \times \mathbb{R} \times \mathbb{R} \times L^2(0,\infty)$ *satisfying*

$$\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \|u\|_{L^2(0,\infty)} + (h_1 - h_0)^2 \leq \kappa, \tag{6.3.1}$$

$$-1 + 4\varepsilon \leq h_0 \leq 1 - 4\varepsilon. \tag{6.3.2}$$

Then there exists $T > 0$, depending on ε and κ, such that for every $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \\ u \end{bmatrix} \in \mathcal{B}_{\varepsilon,\kappa}$, *system* (6.1.2)-(6.1.7) *admits a unique finite energy solution defined by Definition 6.1.1 in the time interval $[0,T]$. Moreover, for every $t \in [0,T]$ we have the following energy estimate:*

$$\begin{aligned} &\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}\left(h_1 - h(t)\right)^2 + \int_0^t \int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\mathrm{d}\sigma \\ \leq &\int_0^t u^2(\sigma)\,\mathrm{d}\sigma + \int_{-1}^{1} v^2(0,y)\,\mathrm{d}y + g^2(0) + \mathcal{K}\left(h_1 - h_0\right)^2, \end{aligned} \tag{6.3.3}$$

Here, we provide the proof of the above theorem in the Appendix 6.7.
As a consequence of the above results, we have:

Corollary 6.3.2. *Assume that for every $\tau > 0$, a finite energy solution in Definition 6.1.1 of system* (6.1.2)-(6.1.7) *defined on $[0,\tau)$ satisfies*

$$\sup_{t\in[0,\tau)} |h(t)| < 1. \tag{6.3.4}$$

Then the considered finite energy solution is global.

Our main result in this section can be stated as follows.

Proposition 6.3.3. *Let $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, h_0, $h_1 \in (-1,1)$ and $u \in L^2(0,\infty)$. Moreover, let* $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ *be the finite energy solution defined by Definition 6.1.1 on $(0,\tau)$. Then for every $t \in (0,\tau)$, the position of the particle $h(t)$ satisfies*

$$-1 + c_2(t) \leq h(t) \leq 1 - c_1(t) \qquad (t \in (0,\tau)) \tag{6.3.5}$$

where

$$\begin{aligned} c_1(t) =& \frac{2}{1 + \max\{2, \frac{(1+h_0)}{1-h_0}\} \exp\left(C + t^{\frac{1}{2}} \left(\int_0^t u^2(s)\mathrm{d}s\right)^{\frac{1}{2}}\right)}, \\ c_2(t) =& \frac{2}{1 + \max\{2, \frac{(1-h_0)}{1+h_0}\} \exp\left(\left(C + t^{\frac{1}{2}} \left(\int_0^t u^2(s)\mathrm{d}s\right)^{\frac{1}{2}}\right)\right)}, \\ C =& 10\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1 - h_0)^2 \right. \\ &\left. + \|u\|^2_{L^2(0,\infty)} + \sqrt{\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1 - h_0)^2 + \|u\|^2_{L^2(0,\infty)}}\right). \end{aligned} \tag{6.3.6}$$

Proof. Following the approach proposed in [Hes04, Section II], we begin by choosing the test function $\begin{bmatrix}\psi\\ l\end{bmatrix} = \begin{bmatrix}\varphi\\ 1\end{bmatrix}$ in the Definition 6.1.1, where φ is defined by:

$$\varphi(t,y) = \begin{cases} \frac{1}{1+h(t)}(y+1) & (y \in (-1, h(t))), \\ \frac{1}{1-h(t)}(1-y) & (y \in (h(t), 1)). \end{cases} \tag{6.3.8}$$

Here, φ is the minimizer of the following Functional:

$$J(w) = \int_{-1}^{1} w_y^2(y)\, \mathrm{d}y, \tag{6.3.9}$$

where

$$w \in \left\{ w \in H_0^1(-1,1), \quad w(h) = 1 \right\}. \tag{6.3.10}$$

By [Hes04, Sec. 7], the test function with dissipative term in (6.1.2) can provide quantitative information on the separation between the particle and the fluid boundary. More precisely, using the definition of φ in (6.3.8) and the fact that $v(\sigma, h(\sigma)) = \dot{h}(\sigma)$ for almost every $t \in [0,\tau)$, $\sigma \in [0,\tau)$, we have

$$\begin{aligned} &\int_{-1}^{1} v_y(\sigma,y)\varphi_y(\sigma,y)\, \mathrm{d}y \\ =&\int_{-1}^{h(\sigma)} v_y(\sigma,y)\varphi_y(\sigma,y)\, \mathrm{d}y + \int_{h(\sigma)}^{1} v_y(\sigma,y)\varphi_y(\sigma,y)\, \mathrm{d}y \\ =&\frac{\dot{h}(\sigma)}{1+h(\sigma)} + \frac{\dot{h}(\sigma)}{1-h(\sigma)} \\ =&\frac{\mathrm{d}}{\mathrm{d}\sigma}(\ln(1+h(\sigma))) - \frac{\mathrm{d}}{\mathrm{d}\sigma}(\ln(1-h(\sigma))). \end{aligned} \tag{6.3.11}$$

Then we choose the test function $\begin{bmatrix}\psi\\ l\end{bmatrix} = \begin{bmatrix}\varphi\\ 1\end{bmatrix}$ in the Definition 6.1.1, it follows that

$$\begin{aligned} &\int_{-1}^{1} v(t,y)\varphi(t,y)\, \mathrm{d}y - \int_{-1}^{1} v_0(y)\varphi(0,y)\, \mathrm{d}y - \int_0^t \int_{-1}^{1} v(\sigma,y)\frac{\partial\varphi}{\partial\sigma}(\sigma,y)\, \mathrm{d}y\, \mathrm{d}\sigma \\ &+ \int_0^t \int_{-1}^{1} v(\sigma,y)v_y(\sigma,y)\varphi(\sigma,y)\, \mathrm{d}y\, \mathrm{d}\sigma + g(t) - g_0 + \int_0^t \int_{-1}^{1} v_y(\sigma,y)\varphi_y(\sigma,y) \\ =&\int_0^t \mathcal{K}(h_1 - h(\sigma))\, \mathrm{d}\sigma + \int_0^t u(\sigma)\, \mathrm{d}\sigma, \end{aligned} \tag{6.3.12}$$

Using (6.3.14), (6.3.15), (6.3.16) and (6.3.11), we can rewrite (6.3.12) as

$$\begin{aligned}&\ln\frac{1+h(t)}{1+h_0}-\ln\frac{1-h(t)}{1-h_0}\\ =&\int_0^t\mathcal{K}(h_1-h(\sigma))\,\mathrm{d}\sigma+\int_0^t u(\sigma)\,\mathrm{d}\sigma\\ &+P(0)-P(t)+\int_0^t A_1(\sigma)\mathrm{d}\sigma-\int_0^t A_2(\sigma)\,\mathrm{d}\sigma.\end{aligned}\tag{6.3.13}$$

where, for almost every $t\in[0,\tau)$, $\sigma\in[0,t]$, we have set

$$P(\sigma)=\int_{-1}^1\varphi(\sigma,y)v(\sigma,y)\,\mathrm{d}y+g(\sigma),\tag{6.3.14}$$

$$A_1(\sigma)=\int_{-1}^1 v(\sigma,y)\frac{\partial\varphi}{\partial\sigma}(\sigma,y)\,\mathrm{d}y,\tag{6.3.15}$$

$$A_2(\sigma)=\int_{-1}^1 v(\sigma,y)v_y(\sigma,y)\varphi(\sigma,y)\,\mathrm{d}y.\tag{6.3.16}$$

Next, we use the energy estimate (6.3.3) to estimate the four last terms in the right hand side of (6.3.13).

First, we note that, for P defined in (6.3.14), by combining (6.3.3), the Cauchy–Schwarz inequality, and the uniform bound $|\varphi|\le 1$, we obtain, for every $t\in[0,\tau)$ and almost every $\sigma\in[0,t]$, that

$$|P(\sigma)|^2\le 4\left(\|v_0\|^2_{L^2(-1,1)}+g_0^2+\mathcal{K}(h_1-h_0)^2+\|u\|^2_{L^2(0,\infty)}\right).\tag{6.3.17}$$

To estimate A_1, we note that from (6.3.15) and the definition of φ in (6.3.8), for $t\in[0,\tau)$ and almost every $\sigma\in[0,t]$ we have:

$$\begin{aligned}A_1(\sigma)=&-\frac{g(\sigma)}{(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v(\sigma,y)(1+y)\,\mathrm{d}y\\ &+\frac{g(\sigma)}{(1-h(\sigma))^2}\int_{h(\sigma)}^{1}v(\sigma,y)(1-y)\,\mathrm{d}y.\end{aligned}\tag{6.3.18}$$

In order to estimate the first term on the right-hand side of the above formula, we begin by integrating by parts. Thus, for $t\in[0,\tau)$ and for a.e. $\sigma\in[0,t]$, we get

$$\begin{aligned}&\frac{g(\sigma)}{(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v(\sigma,y)(1+y)\,\mathrm{d}y\\ =&\frac{g(\sigma)}{2(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v(\sigma,y)\frac{\mathrm{d}}{\mathrm{d}y}\left[(1+y)^2\right]\,\mathrm{d}y\\ =&-\frac{g(\sigma)}{2(1+h(\sigma))^2}\int_{-1}^{h(\sigma)}v_y(\sigma,y)(1+y)^2\,\mathrm{d}y+\frac{1}{2}g^2(\sigma).\end{aligned}\tag{6.3.19}$$

Moreover, by combining the Cauchy-Schwarz inequality, a trace theorem, the facts $|1+h(\sigma)| \leqslant 2$ and $v(t,h(t)) = g(t)$, we have

$$\begin{aligned}
&\left| \frac{g(\sigma)}{2(1+h(\sigma))^2} \int_{-1}^{h(\sigma)} v_y(\sigma,y)(1+y)^2 \,\mathrm{d}y - \frac{1}{2}g^2(\sigma) \right| \\
&\leq \frac{|g(\sigma)|}{2(1+h(\sigma))^2} \left(\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{-1}^{h(\sigma)} (1+y)^4 \,\mathrm{d}y \right)^{\frac{1}{2}} + \frac{1}{2}g^2(\sigma) \\
&\leq \frac{1}{\sqrt{5}} |g(\sigma)| (1+h(\sigma))^{\frac{1}{2}} \left(\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \right)^{\frac{1}{2}} + \frac{1}{2}g^2(\sigma) \\
&\leq g(\sigma) \left(\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \right)^{\frac{1}{2}} + \frac{1}{2}g^2(\sigma) \\
&\leq 2g^2(\sigma) + 2\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \\
&\leq 6\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \qquad\qquad (t\in[0,\tau),\ \sigma\in[0,t] \text{ a.e.}).
\end{aligned}$$

Combining the above formula and (6.3.20) it follows that

$$\begin{aligned}
&\left| \frac{g(\sigma)}{(1+h(\sigma))^2} \int_{-1}^{h(\sigma)} v(\sigma,y)(1+y)\,\mathrm{d}y \right| \\
\leqslant& 6\int_{-1}^{h(\sigma)} v_y^2(\sigma,y)\,\mathrm{d}y \qquad\qquad (t\in[0,\tau),\ \sigma\in[0,t] \text{ a.e.}). \qquad (6.3.20)
\end{aligned}$$

In a completely similar manner we can estimate the second term the right hand side of left term of (6.3.18) to get

$$\left| \frac{g(\sigma)}{(1-h(\sigma))^2} \int_{h(\sigma)}^{1} v(\sigma,y)(1-y)\,\mathrm{d}y \right| \leqslant 6\int_{h(\sigma)}^{1} v_y^2(\sigma,y)\,\mathrm{d}y \qquad (t\in[0,\tau),\ \sigma\in[0,t] \text{ a.e.}).$$

Putting together the last two inequalities and using (6.3.3), it follows that

$$\begin{aligned}
&\left| \int_0^t A_1(\sigma)\mathrm{d}\sigma \right| \leq \int_0^t |A_1(\sigma)|\mathrm{d}\sigma \leq 6\int_0^t \int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\
\leq& 6\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}\left(h_1-h_0\right)^2 + \|u\|^2_{L^2(0,\infty)} \right) \qquad (t\in[0,\tau)). \qquad (6.3.21)
\end{aligned}$$

We next estimate of A_2 defined in (6.3.16). Since φ defined by (6.3.8) is uniformly bounded by 1, using the Cauchy-Schwarz and Poincaré inequalities, it follows that for

every $t \in [0,\tau)$ and almost every $\sigma \in [0,t]$, we have

$$\begin{aligned}|A_2(\sigma)| &\le \int_{-1}^{1} |v(\sigma,y)v_y(\sigma,y)\phi(\sigma,y)|\,\mathrm{d}y \\ &\le \left(\int_{-1}^{1} v^2(\sigma,y)\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\right)^{\frac{1}{2}} \le 4\int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y. \end{aligned} \tag{6.3.22}$$

From the above estimate and the energy estimate (6.3.3) we obtain:

$$\begin{aligned}&\left|\int_0^t A_2(\sigma)\mathrm{d}\sigma\right| \le 4\int_0^t\int_{-1}^{1} v_y^2(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ \le& 4\left(\|v_0\|_{L^2(-1,1)} + g_0^2 + \mathcal{K}\,(h_1-h_0)^2 + \|u\|^2_{L^2(0,\infty)}\right) \qquad (t\in[0,\tau)). \end{aligned} \tag{6.3.23}$$

Moreover, using the fact $|h(\sigma)-h_1| \le 2$, for every $t \in (0,\tau)$, $\sigma \in [0,t]$, it follows that

$$\left|\int_0^t \mathcal{K}(h_1-h(\sigma))\mathrm{d}\sigma\right| \le 2\mathcal{K}t \qquad \left|\int_0^t u(\sigma)\mathrm{d}\sigma\right| \le t^{\frac{1}{2}}\|u\|_{L^2(0,\infty)}. \tag{6.3.24}$$

By combing (6.3.17), (6.3.21), (6.3.23), (6.3.24) and (6.3.13), we have

$$\ln\frac{1+h(t)}{1+h_0} - \ln\frac{1-h(t)}{1-h_0} \le C + 2\mathcal{K}t + t^{\frac{1}{2}}\|u\|_{L^2(0,\infty)} \qquad (t\in[0,\tau)), \tag{6.3.25}$$

where

$$\begin{aligned}C =& 10\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1-h_0)^2\right. \\ &\left. + \|u\|^2_{L^2(0,\infty)} + \sqrt{\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1-h_0)^2 + \|u\|^2_{L^2(0,\infty)}}\right). \end{aligned} \tag{6.3.26}$$

Since for every $t > 0$, we have

$$\begin{aligned}&\frac{1+h_0}{1-h_0}\exp\left(C + 2\mathcal{K}t + t^{\frac{1}{2}}\|u\|_{L^2(0,\infty)}\right) \\ \le& \max\left\{2, \frac{1+h_0}{1-h_0}\right\}\exp\left(C+2\mathcal{K}t\right) \times \exp\left(t^{\frac{1}{2}}\|u\|_{L^2(0,\infty)}\right). \end{aligned}$$

then after a simple calculation, the above inequality (6.3.25) can be rephrased to

$$h(t) \le 1 - \frac{2}{1+\max\left\{2, \frac{1+h_0}{1-h_0}\right\}\exp\left(C + 2\mathcal{K}t + t^{\frac{1}{2}}\|u\|_{L^2(0,\infty)}\right)} \qquad (t\in[0,\tau)). \tag{6.3.27}$$

Choosing next the test function $\begin{bmatrix}\psi\\ l\end{bmatrix} = \begin{bmatrix}-\varphi\\ 1\end{bmatrix}$ in Definition 6.1.1 and following step by step the procedure used to prove (6.3.27), it follows that:

$$h(t) \ge -1 + \frac{2}{1+\max\left\{2, \frac{1-h_0}{1+h_0}\right\}\exp\left(C + 2\mathcal{K}t + t^{\frac{1}{2}}\|u\|_{L^2(0,\infty)}\right)} \qquad (t\in[0,\tau)). \tag{6.3.28}$$

By combining Corollary 6.3.2 and Proposition 6.3.3 , we finish the proof of Theorem 6.1.2. □

6.4 Proof of Theorem 6.1.3

This section is devoted to the proof of one of our main results.

Proof of Theorem 6.1.3. Let $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ be the finite energy solution defined by (6.1.1) to system (6.1.2)–(6.1.7). Then the energy inequality (6.3.3) holds for every $t \geq 0$.

From this inequality we know that

$$\dot{h} = g \in L^{\infty}(0, \infty),$$

together with the estimate (6.3.5) and the Sobolev embedding theorem, we obtain

$$h \in W^{1,\infty}(0, \infty) \hookrightarrow C[0, \infty),$$

which implies that h is Lipschitz continuous on $[0, \infty)$.

We now prove Theorem 6.1.3. Since global existence and uniqueness have already been established, it remains to prove the uniform bound (6.1.14) on the particle's position $h(t)$.

By the continuity of h and the local sign-preserving property of continuous functions, it suffices to verify that $h(t)$ stays uniformly away from the value 1. This can be done by considering three possible scenarios:

(1) For all $t > 0$,
$$h(t) \leq h_1.$$

(2) If there exists $T^* > 0$ such that
$$h(T^*) = h_1,$$
and for all $t > T^*$,
$$h(t) > h_1.$$
Then there exists a constant $\alpha > 0$ such that
$$h(t) \leq 1 - \alpha, \quad \forall t > 0.$$

(3) If there exists a sequence $\{T_i^*\}_{i=1}^N$ (N could be infinite) such that

$$h(T_i^*) = h_1,$$

and for each $t \in (T_i^*, T_{i+1}^*)$,

$$h(t) > h_1.$$

Then again, there exists a uniform constant $\alpha > 0$, independent of i, such that

$$h(t) \le 1 - \alpha, \quad \forall t \in (T_i^*, T_{i+1}^*).$$

Case 1. The bound is already satisfied, so there is nothing to prove.

Case 2. From equation (6.3.13), we have:

$$\ln \frac{1+h(t)}{1+h_0} - \ln \frac{1-h(t)}{1-h_0} = \int_0^t \mathcal{K}(h_1 - h(\sigma))\, d\sigma + \int_0^t u(\sigma)\, d\sigma + P(0) - P(t) + \int_0^t A_1(\sigma)\, d\sigma - \int_0^t A_2(\sigma)\, d\sigma.$$

Since $h(t) > h_1$ for $t > T^*$, we have $\mathcal{K}(h_1 - h(t)) \le 0$ on (T^*, ∞).
Then for $t > T^*$, we rewrite the above equation as:

$$\begin{aligned}\ln \frac{1+h(t)}{1+h_0} - \ln \frac{1-h(t)}{1-h_0} =& \ln \frac{1+h_1}{1+h_0} - \ln \frac{1-h_1}{1-h_0} \\ &+ \int_{T^*}^t \mathcal{K}(h_1 - h(\sigma))\, d\sigma + \int_{T^*}^t u(\sigma)\, d\sigma \\ &+ P(T^*) - P(t) + \int_{T^*}^t A_1(\sigma)\, d\sigma - \int_{T^*}^t A_2(\sigma)\, d\sigma.\end{aligned}$$

Since $\mathcal{K}(h_1 - h(\sigma)) \le 0$, from the above equality, we get:

$$\begin{aligned}\ln \frac{1+h(t)}{1+h_0} - \ln \frac{1-h(t)}{1-h_0} \le& \ln \frac{1+h_1}{1+h_0} - \ln \frac{1-h_1}{1-h_0} \\ &+ \int_{T^*}^t u(\sigma)\, d\sigma + P(T^*) - P(t) \\ &+ \int_{T^*}^t A_1(\sigma)\, d\sigma - \int_{T^*}^t A_2(\sigma)\, d\sigma.\end{aligned}$$

By assumption $u \in L^1(0,\infty)$ and previous estimates (6.3.17), (6.3.21), (6.3.23), all terms on the right-hand side are bounded by $\|v_0\|_{L^2(-1,1)}$, g_0, h_0, h_1, $\mathcal{K}$, $\|u\|_{L^2(0,\infty)}$, and $\|u\|_{L^1(0,\infty)}$. Therefore,

$$\ln \frac{1+h(t)}{1+h_0} - \ln \frac{1-h(t)}{1-h_0} \le C,$$

for some constant $C > 0$. This implies

$$h(t) \leq 1 - \alpha,$$

with

$$\alpha = \frac{2}{1+C} \tag{6.4.1}$$

where

$$C = \frac{1+h_0}{1-h_0} \exp\left(\ln \frac{1+h_1}{1+h_0} - \ln \frac{1-h_1}{1-h_0} + 10(E_0 + \sqrt{E_0}) + \|u\|_{L^1(0,\infty)} \right), \tag{6.4.2}$$

with

$$E_0 := \|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_1 - h_0)^2 + \|u\|^2_{L^2(0,\infty)}. \tag{6.4.3}$$

Case 3. The same procedures as in Case 2 can be applied on each interval (T_i^*, T_{i+1}^*), since $h(T_i^*) = \beta$ and $h(t) > \beta$ inside the interval. Therefore, the same bound holds on each sub-interval, with a uniform constant $\alpha > 0$ which has been defined in (6.4.1) and (6.4.2).

Combining the three cases, we conclude that there exists a constant $\alpha > 0$ such that

$$h(t) \leq 1 - \alpha, \quad \forall t > 0. \tag{6.4.4}$$

Moreover, since h is almost continuous, we can follow the same steps as in the proof of (6.4.4) to show that there exists a constant α such that the position trajectory h of the system (6.1.2)–(6.1.7) satisfies the desired estimate.

$$-1 + \alpha \leq h(t), \quad \forall t > 0. \tag{6.4.5}$$

Finally, combining (6.4.4), (6.4.5) and Theorem 6.1.2, we complete the proof of Theorem 6.1.3. □

6.5 Proof of Theorem 6.1.4

We first give the proof of inequality (6.1.16).

Proof of Theorem 6.1.4. Let ε be such that

$$0 \leq \varepsilon \leq \min\left\{ \frac{1}{8}, \frac{\mathcal{K}}{8} \right\} \qquad (\mathcal{K} > 0), \tag{6.5.1}$$

and we define

$$V_\varepsilon(t) = \int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}\left(h(t)-h_1\right)^2 - \varepsilon\left(h_1 - h(t)\right)P(t) \qquad (t \geqslant 0)\,, \tag{6.5.2}$$

where $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ is the global finite energy solution of system (6.1.2)-(6.1.7), which has been defined in Definition 6.1.1 and P has been defined in (6.3.14).

Meanwhile, for every $t \geq 0$, we also define

$$E(t) = \int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}\left(h(t)-h_1\right)^2 \qquad (\mathcal{K}>0). \tag{6.5.3}$$

By combining the Cauchy-Schwarz inequality and the fact $|\varphi|$ is uniform bounded by 1, it is straightforward to verify that,

$$\frac{1}{4}E(t) \leq V_\varepsilon(t) \leq 2E(t), \qquad \frac{1}{4}V_\varepsilon(t) \leq E(t) \leq 2V_\varepsilon(t) \qquad (t \geq 0). \tag{6.5.4}$$

We next remark that $V_\varepsilon(t)$ is differentiable with respect to t. Indeed, from the energy estimate given by (6.1.13) and Theorem 6.1.2, we deduce that the mapping

$$t \mapsto \frac{1}{2}\left(\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}(h(t)-h_1)^2\right) \tag{6.5.5}$$

is absolutely continuous and differentiable for almost every $t \in [0,\infty)$. Consequently, by taking the derivative of (6.5.3) with respect to t, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -2\int_{-1}^{1} v_y^2(t,y)\mathrm{d}y + g(t)u(t) \qquad (t \geq 0 \text{ a.e.}). \tag{6.5.6}$$

On the other hand, from (6.3.13) it follows that P is absolutely continuous on every bounded interval contained in $[0,\infty)$, thus differentiable almost everywhere on $[0,\infty)$. Moreover, from (6.3.14) it also follows that for almost every $t \geqslant 0$ we have

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t}(t) =& A_1(t) - A_2(t) - \int_{-1}^{1} v_y(t,y)\varphi_y(t,y)\,\mathrm{d}y + \mathcal{K}(h_1 - h(t)) \\ =& A_1(t) - A_2(t) + \mathcal{K}(h_1 - h(t)) - \left(\frac{g(t)}{1+h(t)} + \frac{g(t)}{1-h(t)}\right) + u(t),\end{aligned} \tag{6.5.7}$$

where A_1 and A_2 have been defined in (6.3.15) and (6.3.16), respectively.

From (6.5.6) and (6.5.7) we get that $V_\varepsilon(t)$ defined by (6.5.2) is absolutely continuous, thus differentiable for almost every $t \in [0, \infty)$, with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t)\\ =&-2\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y + g(t)u(t) + \varepsilon g(t)\left(g(t) + \int_0^1 \varphi(t,y)v(t,y)\,\mathrm{d}y\right) - \varepsilon\,(h_1 - h(t))\\ &\left(A_1(t) - A_2(t) + \mathcal{K}(h_1 - h(t)) - \left(\frac{g(t)}{1+h(t)} + \frac{g(t)}{1-h(t)}\right) + u(t)\right). \end{aligned} \tag{6.5.8}$$

By using Cauchy-Schwarz inequality, for every $\mathcal{K} > 0$ we have

$$\begin{cases} g(t)u(t) \le \dfrac{1}{2}g^2(t) + \dfrac{1}{2}u^2(t), \\ (h_1 - h(t))u(t) \le \dfrac{\mathcal{K}}{4}(h_1 - h(t))^2 + \dfrac{1}{\mathcal{K}}u^2(t), \end{cases} \qquad t \ge 0. \tag{6.5.9}$$

Moreover, we know from (6.3.5) that there exists a constant $\alpha \in (0, 1]$, such that

$$\min\{1 - h(t),\ 1 + h(t)\} \ge \alpha \qquad (t \ge 0).$$

The above estimate, combined with an elementary inequality, shows that for every $t \ge 0$ we have

$$(h_1 - h(t))\left(\frac{g(t)}{1+h(t)} + \frac{g(t)}{1-h(t)}\right) \le \frac{g^2(t)}{\mathcal{K}\alpha^2} + \frac{\mathcal{K}}{4}(h_1 - h(t))^2 \qquad (\mathcal{K} > 0). \tag{6.5.10}$$

From the above inequality it follows that for every ε satisfying (6.5.1) and almost every $t \geqslant 0$ and $\mathcal{K} > 0$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \le &\left(-\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y - \varepsilon(h_1 - h(t))A_1(t) + \varepsilon(h_1 - h(t))A_2(t)\right)\\ &+ \frac{1}{2}g^2(t) + \left(5 + \frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t) - \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2 + \left(\frac{1}{2} + \frac{\varepsilon}{\mathcal{K}}\right)u^2(t).\end{aligned}$$

Using the inequality above and the fact that $|h(t) - h_1| \le 2$, along with the estimates of A_1 and A_2 in (6.3.20), (6.3.23), respectively, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \le &(-2 + 20\varepsilon)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y + \frac{1}{2}g^2(t) + \left(5 + \frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t)\\ &- \frac{3\mathcal{K}\varepsilon}{4}(h_1 - h(t))^2 + \left(\frac{1}{2} + \frac{\varepsilon}{\mathcal{K}}\right)u^2(t)\end{aligned} \tag{6.5.11}$$

Using the estimates

$$\begin{aligned} g(t) = v(t,h(t)) &= \int_{-1}^{h(t)} v_y(t,y)\,\mathrm{d}y \\ \leq \left(\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y\right)^{\frac{1}{2}} &\left(\int_{-1}^{1} 1\,\mathrm{d}y\right)^{\frac{1}{2}} \leq \sqrt{2}\left(\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y\right)^{\frac{1}{2}}, \end{aligned} \tag{6.5.12}$$

inequality (6.5.11) implies that for almost every $t \geqslant 0$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq{}& (-1+20\varepsilon)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y + \left(5+\frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t) \\ &- \frac{3\mathcal{K}\varepsilon}{4}(h_1-h(t))^2 + \left(\frac{1}{2}+\frac{\varepsilon}{\mathcal{K}}\right)u^2(t) \\ ={}& (-1+20\varepsilon)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y + \left(7+\frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon g^2(t) - 2\varepsilon g^2(t) \\ &- \frac{3\mathcal{K}\varepsilon}{4}(h_1-h(t))^2 + \left(\frac{1}{2}+\frac{\varepsilon}{\mathcal{K}}\right)u^2(t) \\ \leq{}& \left(-1+20\varepsilon+2\left(7+\frac{1}{\mathcal{K}\alpha^2}\right)\varepsilon\right)\int_{-1}^{1} v_y^2(t,y)\,\mathrm{d}y - 2\varepsilon g^2(t) \\ &- \frac{3\mathcal{K}\varepsilon}{4}(h_1-h(t))^2 + \left(\frac{1}{2}+\frac{\varepsilon}{\mathcal{K}}\right)u^2(t). \end{aligned} \tag{6.5.13}$$

We choose $\varepsilon = \frac{1}{16\left(34+\frac{2}{\mathcal{K}\alpha^2}\right)}$, which satisfies (6.5.1). Combining this with the Poincaré inequality, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq{}& -\frac{1}{4}\left[1-\left(34+\frac{2}{\mathcal{K}\alpha^2}\right)\varepsilon\right]\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y - 2\varepsilon g^2(t) \\ &- \frac{3\mathcal{K}\varepsilon}{4}(h_1-h(t))^2 + \left(\frac{1}{2}+\frac{\varepsilon}{\mathcal{K}}\right)u^2(t), \end{aligned} \tag{6.5.14}$$

since $\alpha \leq 1$, then we have

$$\frac{\varepsilon}{\mathcal{K}} = \frac{1}{544\mathcal{K}+\frac{2}{\alpha^2}} \leq 1$$

Denote

$$\eta = \frac{1}{4}\min\left\{\frac{1}{34+\frac{2}{\mathcal{K}\alpha^2}},\ \frac{3\mathcal{K}\varepsilon}{4}\right\}. \tag{6.5.15}$$

Using the above inequality and (6.5.4), for almost every $t \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(t) \leq -4\eta E(t) + \frac{3}{2}u^2(t) \leq -\eta V_\varepsilon(t) + \frac{3}{2}u^2(t), \tag{6.5.16}$$

where $E(t)$ is defined by (6.5.3).

By Gronwall's inequality, for $\mathcal{K} > 0$, we obtain

$$V_\varepsilon(t) \leq \exp(-\eta t) V_\varepsilon(0) + \frac{3}{2}\|u\|^2_{L^2(0,\infty)}. \tag{6.5.17}$$

Using (6.5.4) again, we conclude that if $\mathcal{K} > 0$ then

$$E(t) \leq 16\exp(-\eta t)E(0) + \frac{3}{2}\|u\|^2_{L^2(0,\infty)} \qquad (t \geqslant 0). \tag{6.5.18}$$

Here, we finish the proof of inequality (6.1.16).

Next, for $\mathcal{K} = 0$, using (6.5.6), (6.5.12), and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|^2_{L^2(-1,1)} + g^2(t)\right)\\
\leq& -2\int_{-1}^{1} |v_y(t,y)|^2\,\mathrm{d}y + \frac{1}{2}g^2(t) + \frac{1}{2}u^2(t)\\
\leq& -2\int_{-1}^{1} |v_y(t,y)|^2\,\mathrm{d}y + \frac{3}{4}g^2(t) - \frac{1}{4}g^2(t) + \frac{1}{2}u^2(t)\\
\leq& -\frac{1}{2}\int_{-1}^{1} |v_y(t,y)|^2\,\mathrm{d}y - \frac{1}{4}g^2(t) + \frac{1}{2}u^2(t).
\end{aligned} \tag{6.5.19}$$

Applying the Poincaré inequality on $(-1,1)$, we have

$$\int_{-1}^{1} |v(t,y)|^2\,\mathrm{d}y \leq C_P \int_{-1}^{1} |v_y(t,y)|^2\,\mathrm{d}y, \tag{6.5.20}$$

where $C_P = \left(\frac{1}{\pi/2}\right)^2 = \frac{4}{\pi^2}$, see for Payne and Weinberger [PW60]. For simplicity, we absorb the constant into the inequality and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t,\cdot)\|^2_{L^2(-1,1)} + g^2(t)\right) \leq -\frac{1}{4}\|v(t,\cdot)\|^2_{L^2(-1,1)} - \frac{1}{4}g^2(t) + \frac{1}{2}u^2(t). \tag{6.5.21}$$

Now, applying Gronwall's inequality, we deduce that

$$\|v(t,\cdot)\|^2_{L^2(-1,1)} + g^2(t) \leq \exp\left(-\frac{1}{4}t\right)\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2\right) + \frac{1}{2}\int_0^t u^2(\sigma)\,\mathrm{d}\sigma.$$

Combining the estimate above with (6.5.18), we complete the proof of Theorem 6.1.4. □

Finally, we state a standard local input-to-state stability result for system (6.1.2)-(6.1.7).

Proposition 6.5.1 (local eISS). *Assume that* $v_0 \in L^2(-1,1)$, $g_0 \in \mathbb{R}$, $h_0, h_1 \in (-1,1)$, *and* $u \in L^2(0,\infty)$ *satisfy:*

$$0 < |h_0 - h_1| < \frac{1}{2\sqrt{2}} \min\{1 - h_1, 1 + h_1\}, \tag{6.5.22}$$

$$\mathcal{K} > \frac{\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \|u\|^2_{L^2(0,\infty)}}{|h_0 - h_1|^2}. \tag{6.5.23}$$

Then system (6.1.2)–(6.1.7) *is exponentially input-to-state stable. That is, there exists constants*
$C_1(h_1) > 0$ *such that for all* $t \geq 0$, *we have*

$$\begin{aligned} &\|v(t)\|^2_{L^2(-1,1)} + g^2(t) + \mathcal{K}(h(t) - h_1)^2 \\ \leq &16e^{-C_1 t}\left(\|v_0\|^2_{L^2(-1,1)} + g_0^2 + \mathcal{K}(h_0 - h_1)^2\right) + \frac{3}{2}\|u\|^2_{L^2(0,\infty)}. \end{aligned} \tag{6.5.24}$$

Proof. From the energy estimate and assumption (6.5.23), we have:

$$\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y + g^2(t) + \mathcal{K}(h(t) - h_1)^2 \leq 2\mathcal{K}|h_0 - h_1|^2.$$

Then,

$$|h(t) - h_1| \leq \sqrt{2}|h_0 - h_1| < \frac{1}{2}\min(1 - h_1, 1 + h_1),$$

which implies

$$\min(1 - h(t), 1 + h(t)) \geq \frac{1}{2}\min(1 - h_1, 1 + h_1).$$

Taking $\alpha = \frac{1}{2}\min(1-h_1, 1+h_1)$, we can apply the same analysis from (6.5.8)-(6.5.18) to get (6.5.24). Then combining the estimate (6.5.24) with the definition of ISS in Section 6.2, we get the local ISS property of system (6.1.2)-(6.1.7). □

Remark 6.5.2 (On the choice of the Lyapunov functional). From the energy identity (6.5.6), the right–hand side does not provide a direct bound on $|h_1 - h(t)|$. Consequently, Grönwall's inequality cannot be applied *directly* to obtain the ISS estimate in Theorem 6.1.4.

To recover information on the position error, we select a test function tailored to the kinematic relation and, using (6.3.13), introduce the auxiliary quantity $P(t)$ defined in (6.3.14). Differentiating $P(t)$ yields a term that is comparable to $|h_1 - h(t)|$. Moreover, thanks to (6.3.21), (6.3.23), and Theorem 6.1.3, The remaining contributions A_1, A_2 and (6.3.11) can then be bounded uniformly by the dissipative quantity

$$\int_{-1}^{1} v_y^2(y)\,\mathrm{d}y.$$

This is precisely why we work with the *perturbed Lyapunov functional* in (6.5.2): it couples fluid dissipation with the position error and leads to the desired ISS type conclusion.

6.6 Numerical Results and Discussion

In this section we describe the numerical method used to simulate the coupled fluid–particle system (6.1.2)–(6.1.7). The purpose of this section is not to establish a convergence result for the numerical scheme, but rather to illustrate the qualitative behavior predicted by the analytical results proved above.

For the interaction system, we formally define the total energy by

$$E(t) \;=\; \frac{1}{2}\int_{-1}^{1} v^2(t,y)\,\mathrm{d}y \;+\; \frac{1}{2}\,\dot{h}^2(t) \;+\; \frac{\mathcal{K}}{2}\,|h(t)-h_1|^2, \tag{6.6.1}$$

where $\mathcal{K} \geq 0$ denotes the feedback gain in (6.1.5). At the continuous level, viscosity dissipates kinetic energy, while the external input $u \in L^1(0,\infty) \cap L^2(0,\infty)$ injects only a finite amount of energy into the system. This suggests, at least formally, that $E(t)$ remains bounded on $[0,\infty)$.

To approximate (6.1.2)–(6.1.7) numerically, we proceed in two steps. First, we discretize the spatial variable y by standard continuous piecewise affine (P_1) finite elements on a uniform mesh of the interval $(-1,1)$. More precisely, let

$$-1 = y_0 < y_1 < \cdots < y_{N+1} = 1$$

be a uniform partition, and let $\{\phi_i\}_{i=1}^{N}$ be the usual hat basis of $H_0^1(-1,1)$, that is, ϕ_i is affine on each interval $[y_j, y_{j+1}]$, satisfies $\phi_i(y_j) = \delta_{ij}$ for $j = 1,\dots,N$, and vanishes at the boundary $y = \pm 1$. We approximate the fluid velocity by

$$v_h(t,y) \;=\; \sum_{i=1}^{N} V_i(t)\,\phi_i(y),$$

where $V(t) = (V_1(t),\dots,V_N(t))^{\top} \in \mathbb{R}^N$ is the vector of nodal values.

Using this ansatz in the weak formulation of (6.1.2)–(6.1.7) leads to a coupled system of ordinary differential equations for $V(t)$ and $h(t)$. To write this system explicitly, we introduce the standard mass and stiffness matrices

$$M_{ij} \;=\; \int_{-1}^{1} \phi_i(y)\phi_j(y)\,\mathrm{d}y, \qquad K_{ij} \;=\; \int_{-1}^{1} \phi_i'(y)\phi_j'(y)\,\mathrm{d}y, \qquad 1 \leq i,j \leq N,$$

which correspond respectively to the L^2 inner product and to the viscous term.

We also need a discrete approximation of the nonlinear convection term $v\,v_y$. Using integration by parts together with the homogeneous Dirichlet boundary conditions, we obtain

$$\int_{-1}^{1} v\,v_y\,\psi\,\mathrm{d}y \;=\; -\frac{1}{2}\int_{-1}^{1} v^2\,\psi_y\,\mathrm{d}y.$$

This motivates the definition of the nonlinear vector $F(V) \in \mathbb{R}^N$ by

$$F_j(V) \;=\; \frac{1}{2}\int_{-1}^{1}\Big(\sum_{i=1}^{N} V_i\phi_i(y)\Big)^2\phi_j'(y)\,\mathrm{d}y, \qquad j=1,\dots,N.$$

In addition, we must evaluate the fluid velocity at the time-dependent particle position $y = h(t)$. For this purpose we introduce the interpolation row

$$I_{h(t)} \;=\; \big(\phi_1(h(t)),\dots,\phi_N(h(t))\big),$$

so that

$$v_h(t,h(t)) = I_{h(t)}V(t).$$

In the weak formulation, the coupling with the particle produces terms involving the value of the test function at $y = h(t)$; after spatial discretization, this leads to coupling terms of the form $I_{h(t)}^{\top}\ddot{h}(t)$.

By testing the semi-discrete weak form against each basis function ϕ_j, $j = 1,\dots,N$, we obtain the coupled ODE system

$$\dot{h}(t) = I_{h(t)}V(t), \tag{6.6.2}$$

$$M\dot{V}(t) \;+\; KV(t) \;-\; F(V(t)) \;+\; I_{h(t)}^{\top}\ddot{h}(t) \;=\; \mathcal{K}(h_1-h(t))I_{h(t)}^{\top} \;+\; u(t)\,I_{h(t)}^{\top}. \tag{6.6.3}$$

Equation (6.6.2) is the discrete counterpart of the kinematic condition $\dot{h} = v(t,h(t))$, whereas (6.6.3) is the finite-element version of the Burgers equation coupled with the particle dynamics. In particular, the viscosity is represented by $KV(t)$, the nonlinear convection by $F(V(t))$, and the fluid–particle interaction appears through the coupling terms involving $I_{h(t)}$.

We next discretize the system in time by a first-order semi-implicit scheme with time step $\Delta t > 0$. Denoting $t^n = n\Delta t$, $h^n \approx h(t^n)$, and $V^n \approx V(t^n)$, we first update the particle position explicitly by

$$h^{n+1} \;=\; h^n \;+\; \Delta t\, I_{h^n}V^n. \tag{6.6.4}$$

This is a forward Euler discretization of the equation $\dot{h} = I_hV$.

We then update the fluid velocity by solving a linear system for V^{n+1}. The viscous term is treated implicitly, which improves stability, whereas the nonlinear convection is treated explicitly, which avoids solving a nonlinear system at each time step. The restoring force is evaluated at the updated particle location h^{n+1}. The resulting discrete equation is

$$\Big(\frac{M}{\Delta t}+K\Big)V^{n+1} \;=\; \frac{M}{\Delta t}V^n \;+\; F(V^n) \;+\; \mathcal{K}(h_1-h^{n+1})I_{h^{n+1}}^{\top} \;+\; u(t^n)\,I_{h^{n+1}}^{\top} \;+\; \mathcal{C}^n, \tag{6.6.5}$$

where $\mathcal{C}^n$ denotes an explicit finite-difference approximation of the coupling contribution $I_{h(t)}^{\top}\ddot{h}(t)$. Hence, at each time step, one only has to solve a sparse linear system with coefficient matrix $(M/\Delta t + K)$.

The fully discrete scheme (6.6.4)–(6.6.5) is first-order accurate in time. Since the nonlinear convection term is treated explicitly, we choose the time step Δt proportional to Δy^2, where Δy denotes the spatial mesh size. This is consistent with a diffusion-type CFL restriction and prevents numerical instabilities generated by the explicit part of the scheme.

To monitor the long-time behavior of the numerical solution, we introduce the discrete energy

$$E^n \;=\; \frac{1}{2}(V^n)^{\top} M V^n \;+\; \frac{1}{2}\left|I_{h^n} V^n\right|^2 \;+\; \frac{\mathcal{K}}{2}\,|h^n - h_1|^2. \tag{6.6.6}$$

The first term is the discrete fluid kinetic energy, the second term plays the role of the particle kinetic energy, and the last term is the elastic potential associated with the feedback.

Because the scheme is only semi-implicit and does not preserve the continuous energy identity exactly at the fully discrete level, the sequence E^n is not expected to be strictly decreasing. In the simulations we typically observe a short transient phase during which the energy may increase slightly. After this transient, the viscosity dominates and the discrete energy decreases rapidly. This initial overshoot is a numerical artifact caused by the explicit treatment of the nonlinear convection and of part of the coupling. More implicit variants of the scheme can reduce this effect, but they require the solution of additional linear systems at each time step.

Unless stated otherwise, the simulations use the fluid initial profile

$$v_0(y) = 2\sin(\pi y), \qquad h_1 = 0.5.$$

The initial particle position and the external input are specified separately in each test case.

We present three numerical experiments to illustrate the behavior of the fluid–particle system under different control configurations.

We report three groups of simulations. Each figure contains three panels: the particle position, the discrete energy, and the logarithmic energy profile $\log(E^n/E^0)$.

Group I: zero-input simulations. Figures (6.1) and (6.2) illustrate the open-loop case $u(t) = 0$ with two different initial particle positions. They show that, in the absence of feedback, the velocity decays while the long-time particle position depends on the initial condition.

Figures (6.3) and (6.4) correspond to the closed-loop case with $u(t) = 0$ and two different feedback gains. These simulations illustrate the effect of the proportional feedback on the convergence of the particle toward the target position h_1.

Group II: disturbed open-loop simulations. Figures (6.5) and (6.6) compare two decaying inputs in the open-loop case $\mathcal{K} = 0$. They show that different input

amplitudes may lead to different long-time particle positions, even though the velocity component still decays.

Group III: disturbed closed-loop simulations. Figures (6.7) and (6.8) compare two feedback gains for the same decaying input. They illustrate how the gain affects the transient behavior and the speed at which the particle is driven back toward the target position.

Figures (6.9) and (6.10) compare two input amplitudes for a fixed feedback gain. They show that the observed decay profile depends on the size of the perturbation, even when the same feedback law is used.

- With out any feedback control and disturbance , but with the different initial position of the particle, we could see the following

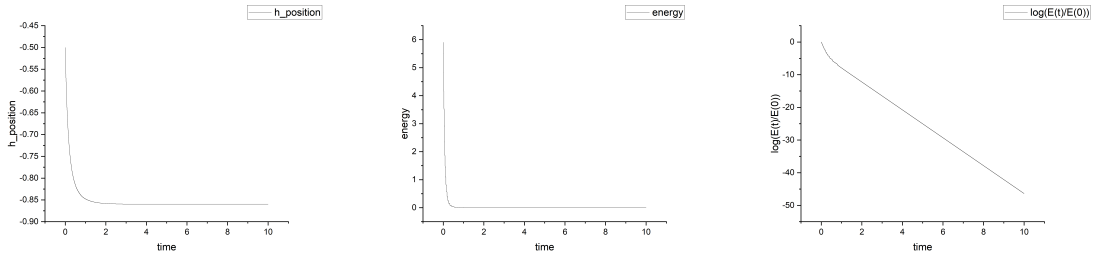


Figure 6.1: $\mathcal{K} = 0$, $u(t) = 0$ and $h_0 = -0.5$

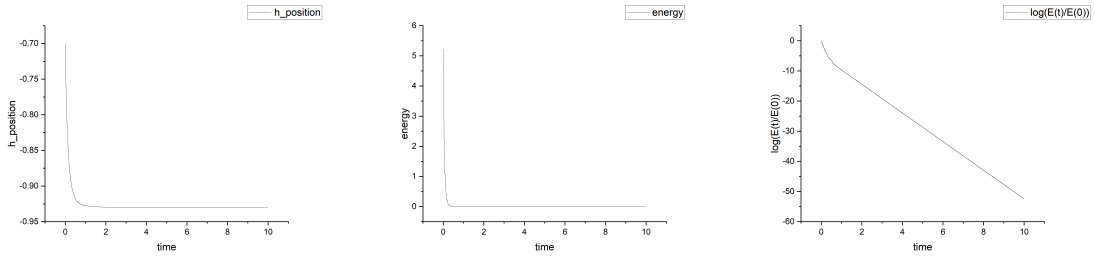


Figure 6.2: $\mathcal{K} = 0$, $u(t) = 0$ and $h_0 = 0.7$

- Feedback only (no external input). We fix $u(t) = 0$ and consider $\mathcal{K} \in \{20, 100\}$ with the same initial position $h_0 = -0.5$. This highlights the effect of the feedback gain without external input.

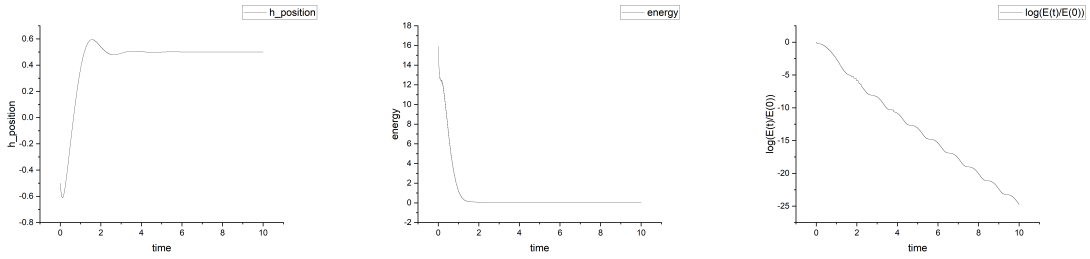


Figure 6.3: $\mathcal{K} = 20$, $u(t) = 0$ and $h_0 = -0.5$

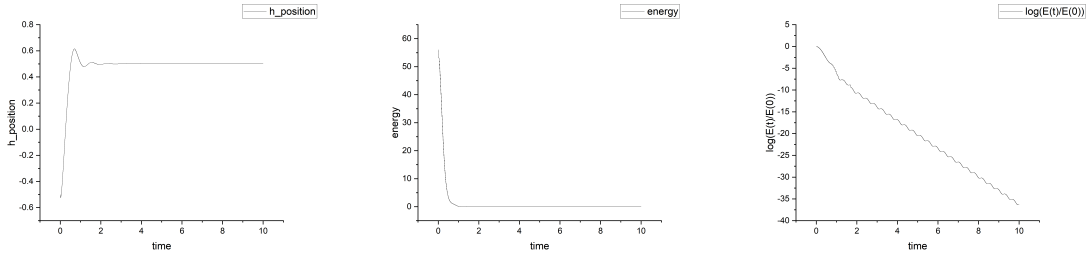


Figure 6.4: $\mathcal{K} = 100$, $u(t) = 0$ and $h_0 = -0.5$

- Without feedback, that is, with $\mathcal{K} = 0$ and and with different perturbations, we fix $h_0 = -0.5$, and compare $u_1(t) = \frac{1}{1+t^2}$ versus $u_2(t) = \frac{100}{1+t^2}$, we have that

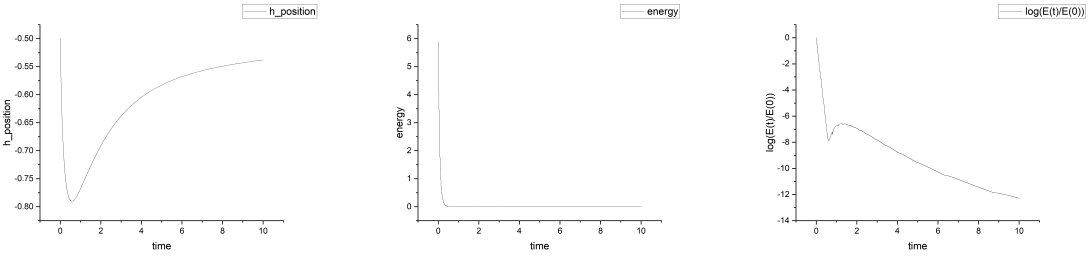


Figure 6.5: $\mathcal{K} = 0$, $u(t) = \frac{1}{1+t^2}$ and $h_0 = -0.5$

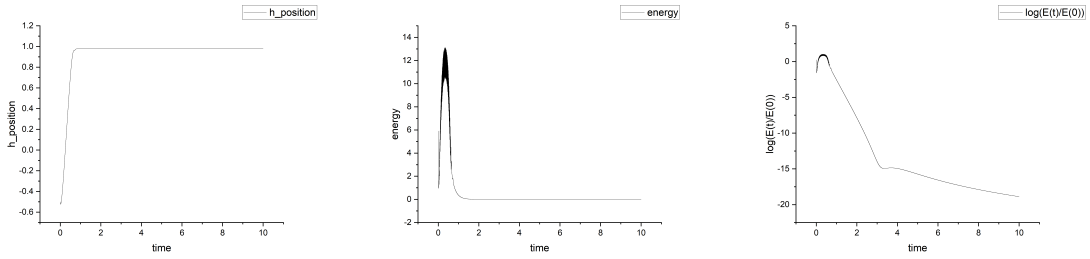


Figure 6.6: $\mathcal{K} = 0$, $u(t) = \frac{100}{1+t^2}$ and $h_0 = -0.5$

- Same perturbation and initial position, different feedback gains. We fix $h_0 = -0.5$ and $u(t) = \frac{1}{1+t^2}$, then compare the responses for $\mathcal{K} = 20$ and $\mathcal{K} = 100$. This shows how stronger feedback affects robustness and convergence speed.

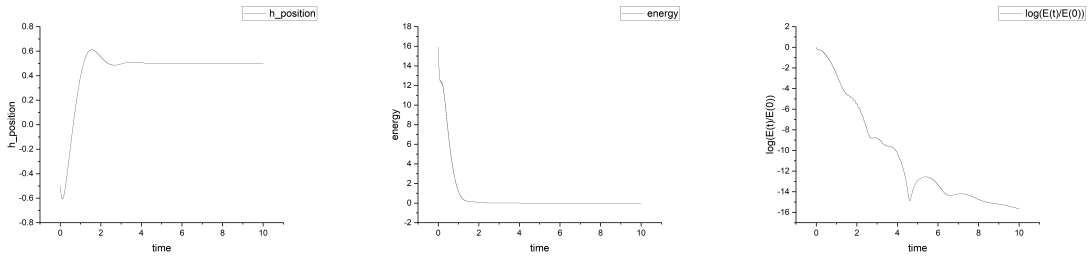


Figure 6.7: $\mathcal{K} = 20$, $u(t) = \frac{1}{1+t^2}$ and $h_0 = -0.5$

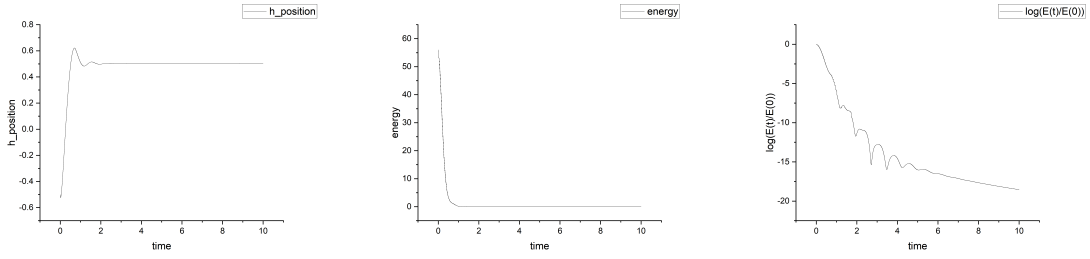


Figure 6.8: $\mathcal{K} = 100$, $u(t) = \frac{1}{1+t^2}$ and $h_0 = -0.5$

- Different perturbations, same initial position and feedback gain. We fix $h_0 = -0.5$ and $\mathcal{K} = 20$, and compare $u_1(t) = \frac{1}{1+t^2}$ versus $u_2(t) = \frac{100}{1+t^2}$. This quantifies the sensitivity to the input level under a fixed feedback gain.

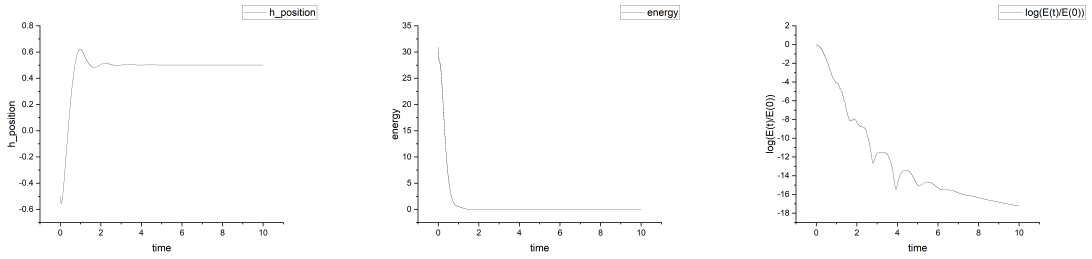


Figure 6.9: $\mathcal{K} = 50$, $u(t) = \frac{1}{1+t^2}$ and $h_0 = -0.5$

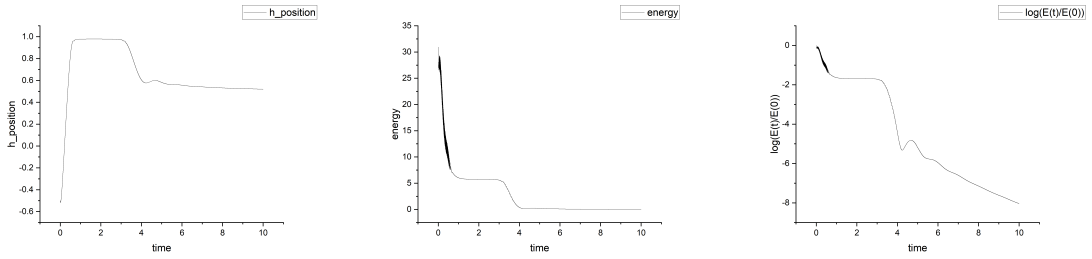


Figure 6.10: $\mathcal{K} = 50$, $u(t) = \frac{100}{1+t^2}$ and $h_0 = -0.5$

Numerical observations. The simulations lead to the following observations.

- In the absence of feedback ($\mathcal{K} = 0$), with or without an external input, the long-time particle position is not prescribed a priori. In particular, different initial conditions or different inputs may lead to different asymptotic positions.

- When the proportional feedback is activated and $u \equiv 0$, the particle is numerically steered toward the target position h_1. After a short transient phase, the logarithmic energy plots suggest an approximately exponential decay, which is consistent with Theorem 1.4 in [TX25].

- In the disturbed open-loop regime, Figures (6.5) and (6.6) show that the asymptotic particle position depends on the input. This is consistent with the fact that the estimate (6.1.15) controls only the velocity variables and does not provide an ISS estimate for the full state.

- In the disturbed closed-loop regime, Figures (6.7), (6.8), (6.9), and (6.10) show that, for the decaying inputs considered here, the particle remains bounded and is driven back toward the target position h_1. These simulations are consistent with the ISS-type estimate (6.1.16).

- Finally, Figures (6.9) and (6.10) indicate that the observed decay rate depends on the amplitude of the input. This is consistent with the fact that the closed-loop system satisfies an ISS-type estimate rather than a standard ISS estimate with an input-independent exponential decay rate.

6.7 Appendix

In this appendix, we prove Theorem 6.3.1. The argument closely follows [CMRT15, Sections 2–4]. The proof has three steps: first, we straighten the moving boundary by a change of variables, obtaining an equivalent problem on a fixed spatial domain; second, we linearize the transformed system and derive the required a priori estimates for the linear problem; third, we reformulate the nonlinear problem as a fixed-point map and apply Banach's contraction principle to obtain local existence and uniqueness.

6.7.1 Change of variables and transformed equations

In this section we introduce a change of variables that rewrites system (6.1.2)–(6.1.7) in an equivalent form on a fixed spatial domain. The standard approach is to associate to each $p \in (-1,1)$ a strictly increasing homeomorphism $\Psi(\cdot,p)$ of $[-1,1]$ such that $\Psi(p,p) = h_0$; see, e.g., [DFC05, LTT13]. In [DFC05], $\Psi(\cdot,p)$ is chosen piecewise affine on $[-1,p]$ and $[p,1]$. Because we work with a weaker notion of solution than in [Tak03, CMRT15], we require a smoother transformation with $\Psi_y(y,p) = 1$ in a neighborhood of p. We therefore adapt the construction of [Tak03] for rigid bodies in a Navier–Stokes flow. We now introduce the change of variables used in [CMRT15, Section 2].

Fix $\varepsilon > 0$ and $p, h_0 \in [-1+2\varepsilon, 1-2\varepsilon]$. Let

$$\Lambda(y,p) = (p-h_0)\,\vartheta(y), \qquad y \in [-1,1],$$

where $\vartheta \in \mathcal{D}(-1,1)$ satisfies $\vartheta(y) = 1$ for $y \in [-1+\varepsilon, 1-\varepsilon]$ and $\vartheta(y) = 0$ for $y \notin [-1+\varepsilon/2,\, 1-\varepsilon/2]$. Consider the initial value problem

$$\begin{cases} \widetilde{\Psi}_s(s;y,p) = \Lambda(\widetilde{\Psi}(s;y,p),p), \quad s \in [0,1], \\ \widetilde{\Psi}(0;y,p) = y. \end{cases} \tag{6.7.1}$$

Since $y \mapsto (p-h_0)\vartheta(y)$ is C^∞, problem (6.7.1) admits, for every $y \in [-1,1]$, a unique solution $\widetilde{\Psi}(\cdot;y,p)$ on $[0,\varrho]$ for some $\varrho > 0$. Because ϑ vanishes on $[-1,1] \setminus [-1+\varepsilon/2, 1-\varepsilon/2]$, we have $\widetilde{\Psi}(s;y,p) \in [-1,1]$ for all $s \in [0,\varrho]$, $y \in [-1,1]$ and $p \in [-1+2\varepsilon, 1-2\varepsilon]$. Hence $\widetilde{\Psi}$ extends to all $s \geq 0$. We then define

$$\Psi(y,p) := \widetilde{\Psi}(1;y,p).$$

Lemma 6.7.1. *We have $\Psi \in C^\infty([-1,1]\times(-1+\varepsilon,1-\varepsilon))$. For every $p \in [-1+2\varepsilon, 1-2\varepsilon]$, the map $y \mapsto \Psi(y,p)$ is a C^∞ diffeomorphism from $[-1,1]$ onto itself, from $[-1,p]$ onto $[-1,h_0]$, and from $[p,1]$ onto $[h_0,1]$. Moreover,*

$$\Psi(y,p) = y - p + h_0 \qquad y \in [-1+\varepsilon,\ 1-\varepsilon], \tag{6.7.2}$$

$$\Psi(y,p) = y \qquad y \notin [-1+\tfrac{\varepsilon}{2},\ 1-\tfrac{\varepsilon}{2}]. \tag{6.7.3}$$

For every $p \in [-1+\varepsilon, 1-\varepsilon]$, the inverse $x \mapsto \Phi(x,p)$ belongs to $C^\infty([-1,1]\times(-1+\varepsilon,1-\varepsilon))$ and

$$\Phi_x(x,p) \geq e^{-|p-h_0|K_1(\varepsilon)} \qquad (x \in [-1,1],\ p \in [-1+\varepsilon, 1-\varepsilon]), \tag{6.7.4}$$

$$\Phi_x(x,h_0) = 1, \qquad \Phi_{xx}(x,h_0) = 0 \qquad (x \in [-1,1]), \tag{6.7.5}$$

where $K_1(\varepsilon) = \|\vartheta_x\|_{\mathcal{C}[-1,1]}$.

Let $\varepsilon > 0$ and let $h \in H^1(0,T)$ satisfy $h(t) \in [-1+2\varepsilon, 1-2\varepsilon]$ for all $t \in [0,T]$. We introduce the change of variables

$$\begin{cases} x = \Psi(y, h(t)), \\ y = \Phi(x, h(t)). \end{cases} \tag{6.7.6}$$

Remark 6.7.2. By Lemma 6.7.1 and $h \in H^1(0,T)$, the map

$$\begin{aligned} \mathcal{T} : H^1(0,T;L^2(-1,1)) \cap L^2(0,T;H_0^1(-1,1)) \\ \longrightarrow H^1(0,T;L^2(-1,1)) \cap L^2(0,T;H_0^1(-1,1)), \\ \psi \ \longmapsto \mathcal{T}(\psi), \\ \mathcal{T}(\psi)(t,x) \ = \psi(t, \Phi(x,h(t))). \end{aligned}$$

is a well-defined bijection with inverse $\mathcal{T}^{-1}(\varphi)(t,y) = \varphi(t,\Psi(y,h(t)))$.

We can now state the definition of a finite-energy solution for the fixed-domain formulation of system 6.1.2- 6.1.7in the fixed-domain.

Proposition 6.7.3. *Let $T > 0$, $v_0 \in L^2(-1,1)$, $h_0 \in (-1,1)$, $g_0 \in \mathbb{R}$, $u \in L^2(0,T)$, and*

$$\begin{bmatrix} v \\ g \\ h \end{bmatrix} \in \left\{ \mathcal{C}([0,T];L^2(-1,1)) \cap L^2(0,T;H_0^1(-1,1)) \right\} \times L^2(0,T) \times H^1(0,T).$$

Then $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ is a finite-energy solution of (6.1.2)–(6.1.7) on $[0,T]$ if and only if the triplet $\begin{bmatrix} w \\ g \\ h \end{bmatrix}$, where $w(t,x) = v(t,\Phi(x,h(t)))$, $h(0) = h_0$, $\dot{h}(t) = g(t) = w(t,h_0)$ and $h(t) \in$

$[-1,1]$ *for a.e.* $t\in[0,T]$, *satisfies for every* $t\in[0,T]$ *and every*

$$\begin{cases}\begin{bmatrix}\varphi\\ l\end{bmatrix}\in\Big(H^1(0,T;L^2(-1,1))\cap L^2(0,T;H^1_0(-1,1))\Big)\times H^1(0,T),\\ l(t)=\varphi(t,h_0).\end{cases}\tag{6.7.7}$$

the following equality holds

$$\begin{aligned}&\int_{-1}^1 w(t,x)\,\varphi(t,x)\,\Phi_x(x,h(t))\,\mathrm{d}x\;-\;\int_{-1}^1 v_0(x)\,\varphi(0,x)\,\mathrm{d}x\;+\;g(t)\,l(t)\;-\;g_0\,l(0)\\ &-\int_0^t\!\!\int_{-1}^1 w(\sigma,x)\,\dot\varphi(\sigma,x)\,\Phi_x(x,h(\sigma))\,\mathrm{d}x\,\mathrm{d}\sigma+\int_0^t g(\sigma)\int_{-1}^1 w(\sigma,x)\,\varphi_x(\sigma,x)\,\Phi_p(x,h(\sigma))\,\mathrm{d}x\,\mathrm{d}\sigma\\ &\qquad-\int_0^t g(\sigma)\,\dot l(\sigma)\,\mathrm{d}\sigma+\int_0^t\!\!\int_{-1}^1\frac{1}{\Phi_x(x,h(\sigma))}\,w_x(\sigma,x)\,\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma\\ &\quad-\frac12\int_0^t\!\!\int_{-1}^1 w^2(\sigma,x)\,\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma\;=\;\int_0^t\big(u(\sigma)+\mathcal{K}(h_1-h(\sigma))\big)\,l(\sigma)\,\mathrm{d}\sigma.\end{aligned}\tag{6.7.8}$$

Proof. The proof follows, step by step, the arguments of [CMRT15, Proposition 2.3] and is omitted.

The previous proposition, together with $\Phi_x(x,h_0)=1$ for all $x\in[-1,1]$, yields the following reformulation.

Corollary 6.7.4. *Let* $T>0$, $v_0\in L^2(-1,1)$, $g_0\in\mathbb{R}$, $h_0\in(-1,1)$, $u\in L^2(0,T)$, *and*

$$\begin{bmatrix}v\\ g\\ h\end{bmatrix}\in\Big\{\mathcal{C}\big([0,T];L^2(-1,1)\big)\cap L^2(0,T;H^1_0(-1,1))\Big\}\times L^2(0,T)\times H^1(0,T).$$

Define z *by*

$$z(t,x)=\Phi_x(x,h(t))\,v(t,\Phi(x,h(t))),\qquad t\in[0,T].$$

Then $h(0)=h_0$, $\dot h(t)=g(t)=z(t,h_0)$, $h(t)\in[-1,1]$ *for a.e.* $t\in[0,T]$, *and for every test pair* (6.7.7) *we have*

$$\begin{aligned}&\int_{-1}^1 z(t,x)\,\varphi(t,x)\,\mathrm{d}x\;-\;\int_{-1}^1 z_0(x)\,\varphi(0,x)\,\mathrm{d}x\;+\;g(t)\,l(t)\;-\;g_0\,l(0)\\ &-\int_0^t\!\!\int_{-1}^1 z(\sigma,x)\,\dot\varphi(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma+\int_0^t g(\sigma)\int_{-1}^1 z(\sigma,x)\,\varphi_x(\sigma,x)\,\frac{\Phi_p(x,h(\sigma))}{\Phi_x(x,h(\sigma))}\,\mathrm{d}x\,\mathrm{d}\sigma\\ &-\int_0^t g(\sigma)\,\dot l(\sigma)\,\mathrm{d}\sigma+\int_0^t\!\!\int_{-1}^1\frac{z_x(\sigma,x)}{\Phi_x(x,h(\sigma))^2}\,\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma\\ &-\int_0^t\!\!\int_{-1}^1\frac{z(\sigma,x)\,\Phi_{xx}(x,h(\sigma))}{\Phi_x(x,h(\sigma))^3}\,\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma-\frac12\int_0^t\!\!\int_{-1}^1\frac{z(\sigma,x)^2}{\Phi_x(x,h(\sigma))^2}\,\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma\\ &=\int_0^t u(\sigma)\,l(\sigma)\,\mathrm{d}\sigma+\int_0^t\mathcal{K}\,(h_1-h(t))\;l(\sigma)\,\mathrm{d}\sigma.\end{aligned}\tag{6.7.9}$$

6.7.2 Study of a linear operator

We next recall a self-adjoint operator in [CMRT15, Section 3] that plays a central role in the sequel . On the Hilbert space

$$H = L^2(-1,1) \times \mathbb{R},$$

endowed with

$$\left\langle \begin{bmatrix} \varphi_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} \varphi_2 \\ p_2 \end{bmatrix} \right\rangle = \int_{-1}^{1} \varphi_1(x)\,\varphi_2(x)\,\mathrm{d}x + p_1 p_2, \tag{6.7.10}$$

we define the linear operator $A_0 : \mathcal{D}(A_0) \to H$ by

$$\mathcal{D}(A_0) = \left\{ \begin{bmatrix} \varphi \\ p \end{bmatrix} \in H_0^1(-1,1) \times \mathbb{R} \;\middle|\; \begin{array}{l} \varphi|_{(-1,h_0)} \in H^2(-1,h_0), \\ \varphi|_{(h_0,1)} \in H^2(h_0,1), \\ \varphi(h_0) = p \end{array} \right\}. \tag{6.7.11}$$

$$A_0 \begin{bmatrix} \varphi \\ p \end{bmatrix} = \begin{bmatrix} -\{\varphi_{xx}\}_{h_0} \\ -[\varphi_x]_{h_0} \end{bmatrix} \qquad \Big(\begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{D}(A_0) \Big). \tag{6.7.12}$$

Here $\{\varphi_{xx}\}_{h_0} \in L^2(-1,1)$ is defined a.e. by

$$\{\varphi_{xx}\}_{h_0}(x) = \begin{cases} \psi_1(x), & x \in [-1,h_0], \\ \psi_2(x), & x \in [h_0,1], \end{cases}$$

where ψ_1 (resp. ψ_2) is the second distributional derivative of φ on $(-1,h_0)$ (resp. $(h_0,1)$). Then

$$\varphi_{xx} = \{\varphi_{xx}\}_{h_0} + [\varphi_x]_{h_0}\,\delta_{h_0} \qquad (\varphi \in \mathcal{D}(A_0)). \tag{6.7.13}$$

Proposition 6.7.5. *The operator A_0 is self-adjoint and strictly positive on H. Consequently, $-A_0$ generates a contraction C^0-semigroup $\mathbb{T}$ on H. Moreover,*

$$H_{1/2} := \mathcal{D}\left(A_0^{1/2}\right) = \left\{ \begin{bmatrix} \varphi \\ p \end{bmatrix} \in H_0^1(-1,1) \times \mathbb{R} \;\middle|\; \varphi(h_0) = p \right\}, \tag{6.7.14}$$

endowed with the inner product

$$\left\langle \begin{bmatrix} \varphi_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} \varphi_2 \\ p_2 \end{bmatrix} \right\rangle_{1/2} = \int_{-1}^{1} \varphi_{1,x}(x)\,\varphi_{2,x}(x)\,\mathrm{d}x. \tag{6.7.15}$$

The dual space $H_{-1/2}$ (with pivot H) is isometric to the quotient $W = \big(H^{-1}(-1,1) \times \mathbb{R}\big)/\mathrm{span}\{\begin{bmatrix} \delta_{h_0} \\ -1 \end{bmatrix}\}$ via

$$\left\| \widehat{\begin{bmatrix} \psi \\ \alpha \end{bmatrix}} \right\|_{-\frac{1}{2}} = \|\psi + \alpha\,\delta_{h_0}\|_{H^{-1}(-1,1)}, \tag{6.7.16}$$

with duality

$$\left\langle \widehat{\begin{bmatrix}\psi\\ \alpha\end{bmatrix}}, \begin{bmatrix}\varphi\\ l\end{bmatrix} \right\rangle_{-1/2,1/2} = \langle \psi, \varphi\rangle_{H^{-1},H_0^1} + \alpha\, l. \tag{6.7.17}$$

Remark 6.7.6. For brevity we write $\left\langle \begin{bmatrix}\psi\\ \alpha\end{bmatrix}, \begin{bmatrix}\varphi\\ l\end{bmatrix} \right\rangle_{-1/2,1/2}$ for the duality in (6.7.17).

We recall the following well-posedness result, which will be used repeatedly in the next subsection.

Proposition 6.7.7. *Let $T > 0$. For any $\begin{bmatrix}z_0\\ g_0\end{bmatrix} \in H$, $\begin{bmatrix}f_1\\ f_2\end{bmatrix} \in L^2(0,T;H_{-1/2})$ and $u \in L^2(0,T)$, there exists a unique*

$$\begin{bmatrix}z\\ g\end{bmatrix} \in \mathcal{C}([0,T];H) \cap L^2(0,T;H_{1/2}) \cap H^1(0,T;H_{-1/2}) \tag{6.7.18}$$

such that $z(0) = z_0$, $g(0) = g_0$ and, for a.e. $t \in [0,T]$ and all $\begin{bmatrix}\varphi\\ l\end{bmatrix} \in L^2(0,T;H_{1/2}) \cap H^1(0,T;H_{-1/2})$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} z(t,x)\,\varphi(t,x)\,\mathrm{d}x - \langle \dot\varphi(t,\cdot), z(t,\cdot)\rangle_{H^{-1},H_0^1} + \frac{\mathrm{d}}{\mathrm{d}t}\big(g(t)\,l(t)\big) - g(t)\,\dot l(t)\\ &+ \int_{-1}^{1} z_x(t,x)\,\varphi_x(t,x)\,\mathrm{d}x = f_2(t)\,l(t) + u(t)\,l(t) + \langle f_1(t), \varphi(t,\cdot)\rangle_{H^{-1},H_0^1}.\end{aligned} \tag{6.7.19}$$

Moreover, for any $t \in [0,T]$,

$$\begin{aligned}&\|z(t,\cdot)\|^2_{L^2(-1,1)} + |g(t)|^2 + \int_0^t \|z_x(\sigma,\cdot)\|^2_{L^2(-1,1)}\,\mathrm{d}\sigma\\ \le&\|z_0\|^2_{L^2(-1,1)} + |g_0|^2 + \|u\|^2_{L^2(0,T)} + \left\| \begin{bmatrix}f_1\\ f_2\end{bmatrix} \right\|^2_{L^2(0,T;H_{-1/2})}.\end{aligned} \tag{6.7.20}$$

Proof. Standard variational theory for parabolic problems (see, e.g., [Zei13]) applied to A_0 yields existence, uniqueness and the energy estimate.

6.7.3 Local well-posedness of the open-loop problem

Formally, if v and h are smooth and satisfy (6.1.11)–(6.1.12), multiplying the fluid equation by v and integrating on $(-1, h(t))$ and $(h(t), 1)$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v^2\,\mathrm{d}y = -\int_{-1}^{1} v_y^2\,\mathrm{d}y - [v_y](t,h(t))\,v(t,h(t)).$$

Multiplying (6.1.5) by $\dot{h}$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{h}(t)^2) = [v_y](t,h(t))\,\dot{h}(t) + u(t)\,\dot{h}(t) + \mathcal{K}\,(h_1 - h(t))\,h(t).$$

Summing up,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v^2\,\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{h}(t)^2) + \frac{\mathcal{K}}{2}\frac{\mathrm{d}}{\mathrm{d}t}|h(t)-h_1|^2 = -\int_{-1}^{1} v_y^2\,\mathrm{d}y + u(t)\,\dot{h}(t). \tag{6.7.21}$$

By Hölder's inequality and the trace theorem, for every $t>0$, the total energy is finite.

The main local well-posedness result is as follows.

Theorem 6.7.8. *Let $\kappa>0$ and $\varepsilon>0$. Set*

$$\mathcal{B}_{\kappa,\varepsilon} := \left\{ \begin{bmatrix} v_0 \\ g_0 \\ h_0 \\ u \end{bmatrix} \in H\times(-1,1)\times L^2(0,\infty) \;\middle|\; \begin{array}{r} \|v_0\|^2_{L^2(-1,1)} + |g_0|^2 + \|u\|^2_{L^2(0,\infty)} < \kappa^2, \\ |h_0| \le 1-4\varepsilon \end{array} \right\}.$$

Then there exists $T>0$, depending only on κ and ε, such that for every $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \\ u \end{bmatrix} \in \mathcal{B}_{\kappa,\varepsilon}$, *system (6.1.2)–(6.1.7) admits a unique solution* $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$, *in the sense of Definition 6.1.1, on* $[0,T]$.

An essential ingredient in the proof is the study of the operators $\mathcal{G}_k$ $(k=1,2,3,4)$ suggested by (6.7.9). Given $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2(0,T;H_{-1/2})$, let $\begin{bmatrix} z \\ g \end{bmatrix}$ be the unique solution of (6.7.18)–(6.7.19) provided by Proposition 6.7.7. Define, for $\begin{bmatrix} \varphi \\ l \end{bmatrix} \in H_{1/2}$,

$$\left\langle \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-1/2,1/2} = \int_{-1}^{1} z_x(t,x)\left(1 - \frac{1}{\Phi_x(x,h(t))^2}\right)\varphi_x(x)\,\mathrm{d}x, \tag{6.7.22}$$

$$\begin{aligned} \left\langle \mathcal{G}_2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-1/2,1/2} &= \int_{-1}^{1} z(t,x)\left(\frac{\Phi_{xx}(x,h(t))}{\Phi_x(x,h(t))^3} - \frac{\Phi_p(x,h(t))}{\Phi_x(x,h(t))}\,g(t)\right)\varphi_x(x)\,\mathrm{d}x \\ &= \int_{-1}^{1} z(t,x)\left[\frac{\Phi_{xx}(x,h(t))}{\Phi_x(x,h(t))^3} - \frac{\Phi_p(x,h(t))}{\Phi_x(x,h(t))}\,g(t)\right]\varphi_x(x)\,\mathrm{d}x. \end{aligned} \tag{6.7.23}$$

$$\left\langle \mathcal{G}_3 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-1/2,1/2} = \frac{1}{2}\int_{-1}^{1} z(t,x)^2 \, \frac{1}{\Phi_x(x,h(t))^2}\, \varphi_x(x)\,\mathrm{d}x, \tag{6.7.24}$$

$$\left\langle \mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-1/2,1/2} = \mathcal{K}\,(h_1 - h(t))\, l. \tag{6.7.25}$$

Notice that $\begin{bmatrix} z \\ g \end{bmatrix} \in \mathcal{C}([0,T];H) \cap L^2(0,T;H_{1/2})$ solves (6.7.9) iff it solves (6.7.19) with right-hand side $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = (\mathcal{G}_1+\mathcal{G}_2+\mathcal{G}_3+\mathcal{G}_4)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, i.e., $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a fixed point of $\mathcal{G}_1+\mathcal{G}_2+\mathcal{G}_3+\mathcal{G}_4$.

Lemma 6.7.9. *For any $T > 0$, the operators $\mathcal{G}_k$ $(1 \le k \le 4)$ define bounded maps $L^2(0,T;H_{-1/2}) \to L^2(0,T;H_{-1/2})$. Moreover, assume there exist $\kappa, \varepsilon > 0$ such that*

$$\left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2(0,T;H_{-1/2})} \le \kappa, \qquad \left\| \begin{bmatrix} z_0 \\ g_0 \end{bmatrix} \right\| \le \kappa, \qquad \|u\|_{L^2(0,\infty)} \le \kappa, \qquad 1 - |h_0| \ge 4\varepsilon. \tag{6.7.26}$$

Then there exists $K(\varepsilon) > 0$ such that, for all $T \le \min\{\frac{\sqrt{2}\,\varepsilon}{\kappa}, 1\}$,

$$1 - |h(t)| \ge 2\varepsilon \qquad (t \in [0,T]), \tag{6.7.27}$$

$$\|\mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\|_{L^2(0,T;H_{-1/2})} \le K(\varepsilon)\, T^{1/4}\, \kappa^2 \qquad (k = 1,2,3), \tag{6.7.28}$$

$$\|\mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\|_{L^2(0,T;H_{-1/2})} \le K(\varepsilon)\, T^{1/2}\, (|h_1 - h_0| + \kappa). \tag{6.7.29}$$

The estimates could be proved by following along the lines of [CMRT15, Lemma 4.2] and we omit here.

Lemma 6.7.10. *Under the assumption of Lemma 6.7.9, let $\begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \in L^2(0,T;H_{-1/2})$ with $\|\begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}\|_{L^2(0,T;H_{-1/2})} \le \kappa$. Then there exists $K(\varepsilon) > 0$ such that, for all $T \le \min\left\{\frac{\sqrt{2}\,\varepsilon}{\kappa}, 1\right\}$,*

$$\begin{aligned} &\left\| \mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_k \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{L^2(0,T;H_{-1/2})} \\ &\le K(\varepsilon)\, T^{1/4}\, \kappa(\kappa+1) \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{L^2(0,T;H_{-1/2})} \qquad (k = 1,2,3), \end{aligned} \tag{6.7.30}$$

and

$$\|\mathcal{G}_4\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_4\begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}\|_{L^2(0,T;H_{-1/2})} \leq K(\varepsilon)\, T^{1/2} \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{L^2(0,T;H_{-1/2})}. \tag{6.7.31}$$

Proof of Theorem 6.7.8. For $T > 0$ and $\kappa, \varepsilon > 0$, let

$$\mathcal{X}_{T,\kappa} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2(0,T;H_{-1/2}) \;\middle|\; \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2(0,T;H_{-1/2})} \leq \kappa \right\}.$$

Define $\mathcal{N} : L^2(0,T;H_{-1/2}) \times \mathcal{B}_{\kappa,\varepsilon} \to L^2(0,T;H_{-1/2})$ by

$$\mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v_0 \\ g_0 \\ h_0 \\ u \end{bmatrix} \right) = \sum_{k=1}^{4} \mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

By Lemma 6.7.9, for all $T \leq \min\{1, \frac{\sqrt{2}\,\varepsilon}{\kappa}\}$ and all $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \\ u \end{bmatrix} \in \mathcal{B}_{\kappa,\varepsilon}$,

$$\left\| \mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v_0 \\ g_0 \\ h_0 \\ u \end{bmatrix} \right) \right\|_{L^2(0,T;H_{-1/2})} \leq K(\varepsilon)\, T^{1/4}\,(\kappa^2+1) \qquad \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{X}_{T,\kappa} \right),$$

which shows that $\mathcal{N}$ leaves $\mathcal{X}_{T,\kappa}$ invariant provided

$$T \leq \min\left\{ 1, \frac{\sqrt{2}\,\varepsilon}{\kappa}, \left(\frac{K(\varepsilon)(\kappa^2+1)}{\kappa} \right)^{-4} \right\}. \tag{6.7.32}$$

Moreover, Lemma 6.7.10 implies that for such T,

$$\begin{aligned} &\left\| \mathcal{N}\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \cdot \right) - \mathcal{N}\left(\begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{pmatrix}, \cdot \right) \right\|_{L^2(0,T;H^{-1/2})} \\ &\leq K(\varepsilon)\, T^{1/4}\,(1+\kappa^2) \left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{pmatrix} \right\|_{L^2(0,T;H^{-1/2})}. \end{aligned}$$

Thus $\mathcal{N}$ is a strict contraction on $\mathcal{X}_{T,\kappa}$ provided

$$T \leq \min\left\{ 1, \frac{\sqrt{2}\,\varepsilon}{\kappa}, \frac{1}{16}\left[K(\varepsilon)(1+\kappa^2)\right]^{-4} \right\}. \tag{6.7.33}$$

By Banach's fixed point theorem, $\mathcal{N}$ admits a unique fixed point $\begin{bmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{bmatrix}$, which in turn yields the desired solution. □

Chapter 7 Appendix: Some Background on Functional Analysis

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This chapter recalls the basic notions from functional analysis, LQR control, and input-to-state stability (ISS) that will be used throughout the thesis. Since all evolution problems considered here are formulated in Hilbert spaces, we work exclusively in the Hilbert-space setting. We first introduce weak derivatives and Sobolev spaces in the

L^2-framework, then recall the notion of measurability for Hilbert-valued functions, and finally define the corresponding Bochner spaces. For general background, we refer, for example, to [Eva22, Chapter 5], [TW09, Appendix II], [BDPDM07] and [Lun18].

7.1 Weak derivatives and Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ be a nonempty open set. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we set

$$|\alpha| := \alpha_1 + \cdots + \alpha_d, \qquad \partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}.$$

Let $u \in L^1_{\mathrm{loc}}(\Omega)$. A function $v \in L^1_{\mathrm{loc}}(\Omega)$ is called the *weak derivative* of u of order α if

$$\int_\Omega u(x)\, \partial^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v(x)\, \varphi(x)\, dx \qquad \forall\, \varphi \in C_c^\infty(\Omega).$$

In this case, we write $\partial^\alpha u = v$.

For $m \in \mathbb{N}$, we define the Sobolev space

$$H^m(\Omega) := \left\{ u \in L^2(\Omega)\,;\, \partial^\alpha u \in L^2(\Omega) \text{ for every multi-index } \alpha \text{ with } |\alpha| \le m \right\}.$$

Equipped with its usual norm, $H^m(\Omega)$ is a Hilbert space. In particular,

$$H^1(\Omega) = \left\{ u \in L^2(\Omega)\,;\, \nabla u \in L^2(\Omega)^d \right\}.$$

Let $\Omega \subset \mathbb{R}^d$ be an open set. We denote by $C_c^\infty(\Omega)$ the space of all infinitely differentiable functions with compact support contained in Ω, that is,

$$C_c^\infty(\Omega) := \{\, \varphi \in C^\infty(\Omega)\ ;\ \mathrm{supp}(\varphi) \subset\subset \Omega \,\}.$$

Here $\mathrm{supp}(\varphi)$ denotes the support of φ. We also denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

If Ω has Lipschitz boundary, then the trace operator is well defined on $H^1(\Omega)$, and $H_0^1(\Omega)$ may be characterized as the subspace of functions in $H^1(\Omega)$ having vanishing trace on $\partial\Omega$.

7.2 Sobolev spaces with respect to time

In this thesis, the unknowns in the evolution problems depend on time and take values in Hilbert spaces. For this reason, before introducing Sobolev regularity with respect to the time variable, we briefly recall the notions of measurability and integrability for Hilbert space-valued mappings.

Let $T > 0$ be fixed, and let H be a real Hilbert space, endowed with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. We denote by H^* the topological dual of H, and by $\langle\cdot,\cdot\rangle_{H^*,H}$ the duality pairing between H^* and H. If $E \subset [0,T]$ is Lebesgue measurable, then χ_E denotes its characteristic function and $|E|$ its Lebesgue measure. Moreover, $C_c^\infty(0,T)$ stands for the space of all infinitely differentiable real-valued functions with compact support in $(0,T)$.

7.2.1 Hilbert space-valued measurable and integrable functions

A function $u : [0,T] \to H$ is said to be *simple* if there exist $m \in \mathbb{N}$, measurable sets $E_1, \dots, E_m \subset [0,T]$, and vectors $x_1, \dots, x_m \in H$ such that

$$u(t) = \sum_{i=1}^{m} \chi_{E_i}(t)\, x_i, \qquad t \in [0,T].$$

A mapping $u : [0,T] \to H$ is called *strongly measurable* if there exists a sequence of simple functions $(u_n)_{n\in\mathbb{N}}$ such that

$$u_n(t) \to u(t) \qquad \text{for a.e. } t \in [0,T].$$

Here and in the sequel, "a.e." means "almost everywhere" with respect to the Lebesgue measure on $[0,T]$.

We say that $u : [0,T] \to H$ is *weakly measurable* if, for every $x^* \in H^*$, the scalar-valued function

$$t \longmapsto \langle x^*, u(t)\rangle_{H^*,H}$$

is Lebesgue measurable on $[0,T]$.

Finally, u is said to be *almost separably valued* if there exists a measurable set $N \subset [0,T]$ with $|N| = 0$ such that the set

$$\{u(t)\ ;\ t \in [0,T] \setminus N\}$$

is contained in a separable subspace of H.

A standard result, usually referred to as Pettis' theorem, asserts that a mapping $u : [0,T] \to H$ is strongly measurable if and only if it is weakly measurable and almost separably valued.

A strongly measurable function $u : [0,T] \to H$ is said to be *Bochner integrable* on $[0,T]$ if

$$\int_0^T \|u(t)\|_H \, dt < \infty.$$

In this case, its Bochner integral is denoted by

$$\int_0^T u(t)\,dt \in H,$$

and it satisfies

$$\left\| \int_0^T u(t)\,dt \right\|_H \leq \int_0^T \|u(t)\|_H\,dt.$$

7.2.2 The space $L^2(0,T;H)$

We define

$$L^2(0,T;H) = \left\{ u : [0,T] \to H \ ;\ u \text{ is strongly measurable and } \int_0^T \|u(t)\|_H^2\,dt < \infty \right\}.$$

For $u, v \in L^2(0,T;H)$, we equip this space with the inner product

$$(u,v)_{L^2(0,T;H)} := \int_0^T (u(t), v(t))_H\,dt,$$

and the associated norm

$$\|u\|_{L^2(0,T;H)} := \left(\int_0^T \|u(t)\|_H^2\,dt \right)^{1/2}.$$

Since H is a Hilbert space, $L^2(0,T;H)$ is itself a Hilbert space.

7.2.3 Weak derivative with respect to time

Let $u, v \in L^1(0,T;H)$. We say that v is the *weak derivative* of u with respect to the time variable if, for every test function $\varphi \in C_c^\infty(0,T)$, one has

$$\int_0^T u(t)\,\varphi'(t)\,dt = -\int_0^T v(t)\,\varphi(t)\,dt \qquad \text{in } H.$$

In this case, we write

$$\partial_t u = v \qquad \text{or simply} \qquad u_t = v.$$

Equivalently, since the previous identity holds in H, it is equivalent to require that for every $x^* \in H^*$ and every $\varphi \in C_c^\infty(0,T)$,

$$\int_0^T \langle x^*, u(t)\rangle_{H^*,H}\,\varphi'(t)\,dt = -\int_0^T \langle x^*, v(t)\rangle_{H^*,H}\,\varphi(t)\,dt.$$

Thus, the scalar function $t \mapsto \langle x^*, u(t)\rangle_{H^*,H}$ has distributional derivative $t \mapsto \langle x^*, v(t)\rangle_{H^*,H}$ for every $x^* \in H^*$.

As usual, if the weak derivative exists, then it is unique up to sets of measure zero.

7.2.4 Sobolev spaces in time

We can now introduce the Sobolev space in time associated with the Hilbert space H. We define

$$H^1(0,T;H) := \left\{ u \in L^2(0,T;H) \ ; \ \partial_t u \in L^2(0,T;H) \right\}.$$

This space is equipped with the norm

$$\|u\|_{H^1(0,T;H)} := \left(\|u\|^2_{L^2(0,T;H)} + \|\partial_t u\|^2_{L^2(0,T;H)} \right)^{1/2}.$$

More generally, for $m \in \mathbb{N}$, one defines

$$H^m(0,T;H) := \left\{ u \in L^2(0,T;H) \ ; \ \partial_t^k u \in L^2(0,T;H) \text{ for all } k = 1, \dots, m \right\},$$

where $\partial_t^k u$ denotes the k-th weak derivative of u with respect to time.

A basic fact, which is frequently used in the study of evolution equations, is that every function $u \in H^1(0,T;H)$ admits a representative, still denoted by u, which belongs to $C([0,T];H)$. Moreover, for every $0 \le s \le t \le T$,

$$u(t) = u(s) + \int_s^t \partial_\tau u(\tau)\, d\tau \qquad \text{in } H,$$

where the integral on the right-hand side is understood in the Bochner sense. In particular, the values $u(0)$ and $u(T)$ are well defined for functions in $H^1(0,T;H)$.

Therefore, the space $H^1(0,T;H)$ is the natural functional framework for describing time-dependent unknowns which are square integrable in time together with their first weak time derivative.

7.3 Strongly continuous semigroups

In this section, we recall some standard facts on strongly continuous semigroups that will be used in the sequel. We work throughout on a complex Hilbert space H.

7.3.1 Definition and infinitesimal generator

Definition 7.3.1 (C_0-semigroup)**.** A family $\mathbb{T} = (\mathbb{T}(t))_{t \ge 0} \subset \mathcal{L}(H)$ is called a *strongly continuous semigroup* (or *C_0-semigroup*) on H if

1. $\mathbb{T}(0) = I$;
2. $\mathbb{T}(t+s) = \mathbb{T}(t)\mathbb{T}(s)$ for every $t, s \ge 0$;
3. for every $x \in H$, the map $t \mapsto \mathbb{T}(t)x$ is continuous from $[0,\infty)$ into H.

The notion of infinitesimal generator plays a central role in the semigroup approach to evolution equations.

Definition 7.3.2 (Infinitesimal generator)**.** Let $\mathbb{T} = (\mathbb{T}(t))_{t\geq 0}$ be a C_0-semigroup on H. Its *infinitesimal generator* is the operator $A : \mathcal{D}(A) \subset H \to H$ defined by

$$\mathcal{D}(A) := \left\{ x \in H \; ; \; \lim_{h\to 0} \frac{\mathbb{T}(h)x - x}{h} \text{ exists in } H \right\},$$

and

$$Ax := \lim_{h\to 0} \frac{\mathbb{T}(h)x - x}{h}, \qquad x \in \mathcal{D}(A).$$

The generator of a C_0-semigroup is in general an unbounded operator, and therefore its domain $\mathcal{D}(A)$ must always be specified explicitly. This point is crucial in the infinite-dimensional setting.

The following basic facts will be used repeatedly: the operator A is densely defined and closed; for every $x \in \mathcal{D}(A)$ one has $\mathbb{T}(t)x \in \mathcal{D}(A)$ for all $t \geq 0$, and

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbb{T}(t)x = A\mathbb{T}(t)x = \mathbb{T}(t)Ax, \qquad t \geq 0.$$

7.3.2 The spaces H_1 and H_{-1}

Associated with the generator A, we introduce the space

$$H_1 := \mathcal{D}(A),$$

endowed with the graph norm

$$\|x\|_{H_1} := \|x\|_H + \|Ax\|_H, \qquad x \in \mathcal{D}(A).$$

Thus, H_1 is a Hilbert space continuously and densely embedded in H.

Fix $\beta \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. The extrapolation space H_{-1} associated with A is defined as the completion of H with respect to the norm

$$\|x\|_{H_{-1}} := \|(\beta I - A)^{-1}x\|_H, \qquad x \in H.$$

The definition is independent, up to equivalent norms, of the particular choice of β. Moreover, the semigroup $\mathbb{T}$ extends uniquely to a C_0-semigroup $\mathbb{T}_{-1}$ on H_{-1}.

These spaces are particularly useful when the forcing term, or the control operator, naturally takes values in H_{-1} rather than in H.

7.3.3 Generation results

A general characterization of generators is provided by the Hille–Yosida theorem.

Theorem 7.3.3 (Hille–Yosida theorem)**.** *Let $A : \mathcal{D}(A) \subset H \to H$ be a densely defined linear operator. Then A generates a C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ on H satisfying*

$$\|\mathbb{T}(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}, \qquad t \geq 0,$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$, if and only if

$$(\omega, \infty) \subset \rho(A)$$

and

$$\|(\lambda I - A)^{-n}\|_{\mathcal{L}(H)} \leq \frac{M}{(\lambda - \omega)^n}, \qquad \lambda > \omega, \quad n \in \mathbb{N}.$$

In the Hilbert space setting, an especially useful criterion is the Lumer–Phillips theorem.

Definition 7.3.4. An operator $A : \mathcal{D}(A) \subset H \to H$ is said to be *dissipative* if

$$\operatorname{Re} \langle Ax, x\rangle_H \leq 0, \qquad \forall x \in \mathcal{D}(A).$$

It is said to be *maximal dissipative*, or *m-dissipative*, if it is dissipative and

$$\operatorname{Ran}(\lambda I - A) = H \quad \text{for some (equivalently, for every) } \lambda > 0.$$

Theorem 7.3.5 (Lumer–Phillips)**.** *Let $A : \mathcal{D}(A) \subset H \to H$ be a densely defined linear operator. Then A generates a strongly continuous contraction semigroup on H if and only if A is m-dissipative.*

Having introduced the semigroup framework, we now turn to its application to linear evolution equations. More precisely, we explain how the properties of the generator can be used to establish the well-posedness and regularity of the corresponding abstract Cauchy problem.

Let $A : \mathcal{D}(A) \subset H \to H$ be the generator of a C_0-semigroup $\mathbb{T} = (\mathbb{T}(t))_{t\geq 0}$. We consider the inhomogeneous abstract Cauchy problem

$$\dot{z}(t) = Az(t) + f(t), \qquad t > 0, \tag{7.3.1}$$

$$z(0) = z_0. \tag{7.3.2}$$

If $z_0 \in H$ and $f \in L^1_{\mathrm{loc}}([0,\infty); H)$, then a natural notion of solution is the *mild solution*, defined by the variation-of-constants formula

$$z(t) = \mathbb{T}(t)z_0 + \int_0^t \mathbb{T}(t-s)f(s)\,ds, \qquad t \geq 0.$$

In this case, $z \in C([0,\infty); H)$.

If, in addition, $z_0 \in \mathcal{D}(A)$ and f is sufficiently regular, then the solution is more regular, typically

$$z \in C([0,\infty); \mathcal{D}(A)) \cap C^1([0,\infty); H),$$

and (7.3.1) holds in H for every $t \geq 0$.

When the source term only belongs to H_{-1}, the problem is interpreted in the extrapolation space. More precisely, if $z_0 \in H_{-1}$ and

$$f \in L^1_{\mathrm{loc}}([0,\infty); H_{-1}),$$

then one defines a mild solution in H_{-1} by

$$z(t) = \mathbb{T}_{-1}(t) z_0 + \int_0^t \mathbb{T}_{-1}(t-s) f(s)\, ds, \qquad t \geq 0.$$

Thus, the natural state space for the abstract equation depends on the regularity of the data: for data in H one obtains solutions in H, while for rougher data in H_{-1} one first works in the extrapolation space H_{-1}.

7.4 Analytic semigroups

In this section, we briefly recall the notion of analytic semigroup, a standard generation criterion, and the interpolation spaces naturally appearing in maximal L^2-regularity for abstract evolution equations.

7.4.1 Analytic semigroups and their generators

Let H be a Hilbert space.

Definition 7.4.1 (Analytic semigroup)**.** Let $\mathbb{T} = (\mathbb{T}(t))_{t\geq 0}$ be a C_0-semigroup on H. We say that $\mathbb{T}$ is an *analytic semigroup* if there exists $\theta \in (0, \pi/2)$ such that $\mathbb{T}$ admits an analytic extension, still denoted by $\mathbb{T}$, to the sector

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} \ ; \ |\arg z| < \theta\},$$

and for every $\theta' \in (0, \theta)$ there exists a constant $M_{\theta'} > 0$ such that

$$\sup_{z \in \overline{\Sigma_{\theta'}}} \|\mathbb{T}(z)\|_{\mathcal{L}(H)} \leq M_{\theta'}.$$

A standard characterization of analytic semigroups is given by the following sectorial resolvent criterion.

Theorem 7.4.2 (Generation of analytic semigroups)**.** *Let $A : \mathcal{D}(A) \subset H \to H$ be a densely defined linear operator. Assume that there exist $\omega \in \mathbb{R}$, $\theta_0 \in (\pi/2, \pi)$ and $M > 0$ such that*

$$S_{\omega,\theta_0} := \{\lambda \in \mathbb{C} \,;\, \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta_0\} \subset \rho(A),$$

and

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{M}{|\lambda - \omega|}, \qquad \forall \lambda \in S_{\omega,\theta_0}.$$

Then A generates an analytic C_0-semigroup on H.

We next introduce a scale of Hilbert spaces lying between $\mathcal{D}(A)$ and H. These spaces arise naturally in connection with the maximal L^2-regularity result stated below, since they provide the appropriate trace spaces for the corresponding solutions.

More generally, let H and Y be Hilbert spaces such that H is continuously and densely embedded in Y. For $\theta \in (0, 1)$, we denote by

$$[H, Y]_\theta$$

the real interpolation space between H and Y; for further details, we refer to [Lun18].

In this thesis, we only need the particular interpolation space

$$[\mathcal{D}(A), H]_{1/2},$$

where $\mathcal{D}(A)$ is equipped with its graph norm. This space is an intermediate Hilbert space between $\mathcal{D}(A)$ and H, and it will serve as the natural trace space in the maximal L^2-regularity framework used below.

7.4.2 Maximal L^2 regularity

In this subsection, we present the main results that will be used in this thesis, which can be found in [BDPDM07].

Let $A : \mathcal{D}(A) \subset H \to H$ generate an analytic C_0-semigroup on H, and let $T > 0$. We define

$$W(0, T; \mathcal{D}(A), H) := \Big\{ y \in L^2(0, T; \mathcal{D}(A)) \,;\, y' \in L^2(0, T; H) \Big\},$$

endowed with the norm

$$\|y\|_{W(0,T;\mathcal{D}(A),H)} := \|y\|_{L^2(0,T;\mathcal{D}(A))} + \|y'\|_{L^2(0,T;H)}.$$

We consider the abstract Cauchy problem

$$y'(t) - Ay(t) = f(t), \qquad \text{for a.e. } t \in (0, T), \tag{7.4.1}$$

$$y(0) = y_0. \tag{7.4.2}$$

Theorem 7.4.3 (Maximal L^2-regularity)**.** *For every $T > 0$, the linear map*

$$\begin{aligned} D:\ W(0,T;\mathcal{D}(A),H) &\longrightarrow L^2(0,T;H)\times[\mathcal{D}(A),H]_{1/2},\\ y &\longmapsto (y'-Ay,\ y(0)), \end{aligned}$$

is an isomorphism.

In particular, for every

$$f\in L^2(0,T;H),\qquad y_0\in[\mathcal{D}(A),H]_{1/2},$$

there exists a unique solution

$$y\in W(0,T;\mathcal{D}(A),H)$$

to (7.4.1)–(7.4.2). Moreover, there exists a constant $C_T>0$ such that

$$\|y\|_{L^2(0,T;\mathcal{D}(A))}+\|y'\|_{L^2(0,T;H)}\le C_T\Big(\|f\|_{L^2(0,T;H)}+\|y_0\|_{[\mathcal{D}(A),H]_{1/2}}\Big).$$

7.5 Background on infinite-horizon LQR problems for strongly stable systems

In this subsection, we briefly recall some background on infinite-horizon linear–quadratic regulator (LQR) problems in Hilbert spaces. Our purpose is not to present the general theory in its full scope, but rather to introduce the notions and results that will be used later for the floating-body system studied in Part I.

For exponentially stabilizable systems, the infinite-horizon LQR problem is by now classical and can be treated within the standard Riccati framework; see, for instance, Curtain and Zwart [CZ95, Section 6.2]. If the system is not stabilizable, we shall instead use the LQR theory developed for strongly stable systems in Curtain and Oostveen [CO99]; see also Bensoussan, Da Prato, Delfour and Mitter [BDPDM07], and Oostveen, Curtain and Ito [OCI00].

Let H, U and Y be Hilbert spaces. We consider the abstract linear system

$$\dot z(t)=Az(t)+Bu(t),\qquad t>0, \tag{7.5.1}$$

$$z(0)=z_0, \tag{7.5.2}$$

$$y(t)=Cz(t),\qquad t>0, \tag{7.5.3}$$

where $A:\mathcal{D}(A)\to H$ is the generator of a C^0-semigroup $\mathbb{T}=(\mathbb{T}(t))_{t\ge0}$ on H, while

$$B\in\mathcal{L}(U,H),\qquad C\in\mathcal{L}(H,Y).$$

For every initial condition $z_0 \in H$ and every input $u \in L^2((0,\infty);U)$, the corresponding state is understood in the mild sense defined in Section 7.3, namely

$$z(t) = \mathbb{T}(t)z_0 + \int_0^t \mathbb{T}(t-s)Bu(s)\,\mathrm{d}s, \qquad t \geq 0. \tag{7.5.4}$$

The associated infinite-horizon quadratic cost is

$$J(z_0;u) = \int_0^\infty \left(\|Cz(t)\|_Y^2 + \|u(t)\|_U^2\right)\,\mathrm{d}t, \qquad u \in L^2((0,\infty);U). \tag{7.5.5}$$

The corresponding LQR problem consists in minimizing (7.5.5) over all controls $u \in L^2((0,\infty);U)$.

A natural condition ensuring that the above problem is meaningful is the following.

Definition 7.5.1. The system $\Sigma(A,B,C)$ is said to be *optimizable* if, for every $z_0 \in H$, there exists a control

$$\overline{u} \in L^2((0,\infty);U)$$

such that the corresponding trajectory satisfies

$$J(z_0;\overline{u}) < \infty.$$

In the standard infinite-dimensional LQR theory, optimizability is often derived from stabilizability assumptions, typically in an exponentially stable framework. Since this is not the case for the systems considered in this thesis, we need a notion adapted to strong stability.

Definition 7.5.2. For every $t > 0$, the *truncated output map*

$$\Psi_t : H \to L^2((0,\infty);Y)$$

is defined by

$$(\Psi_t z_0)(\sigma) = \begin{cases} C\mathbb{T}(\sigma)z_0, & \sigma \in (0,t], \\ 0, & \sigma > t. \end{cases}$$

The system (7.5.1)–(7.5.3) is called *output stable* if there exists an operator

$$\Psi \in \mathcal{L}(H, L^2((0,\infty);Y))$$

such that

$$\lim_{t\to\infty} \|\Psi_t - \Psi\|_{\mathcal{L}(H,L^2((0,\infty);Y))} = 0.$$

Remark 7.5.3. In the terminology of Tucsnak and Weiss [TW09, Section 4.6], output stability means that C is an infinite-time admissible observation operator for the semigroup $\mathbb{T}$, and the operator Ψ is the associated extended output map.

The following result, taken from Curtain and Oostveen [CO99], is the main abstract tool used later in this thesis.

Proposition 7.5.4. *Assume that there exists an operator $F \in \mathcal{L}(H,U)$ such that the operator $A+BF$ generates a strongly stable semigroup $\mathbb{T}^F$ on H, and that the system*

$$\Sigma\left(A+BF,\ 0,\ \begin{bmatrix} C \\ F \end{bmatrix}\right)$$

is output stable, with state space H and output space $Y \times U$. Then, for every $z_0 \in H$, the cost functional (7.5.5) admits a unique minimizer

$$\overline{u} \in L^2((0,\infty);U).$$

Moreover, there exists a minimal self-adjoint nonnegative operator $P_0 \in \mathcal{L}(H)$ such that the optimal control is given in feedback form by

$$\overline{u}(t) = -B^* P_0 z(t),$$

where z is the corresponding optimal trajectory, and the minimal cost satisfies

$$J(z_0;\overline{u}) = \langle P_0 z_0, z_0\rangle_H.$$

The operator P_0 is the minimal self-adjoint nonnegative solution of the algebraic Riccati equation

$$A^*Pz + PAz - PBB^*Pz + C^*Cz = 0, \qquad z \in \mathcal{D}(A).$$

Remark 7.5.5. The precise sense in which the Riccati equation is satisfied is described in Bensoussan, Da Prato, Delfour and Mitter [BDPDM07, Part IV, Section 3.1 and Part V]. In particular, for every $z \in \mathcal{D}(A)$, one has $P_0 z \in \mathcal{D}(A^*)$.

Remark 7.5.6 (On unbounded control operators)**.** The above presentation is restricted to the case where the control operator

$$B \in \mathcal{L}(U,H)$$

is bounded. In many PDE control problems, however, and in particular for boundary control systems, the natural control operator is unbounded on the state space H. A standard way to handle this situation is to introduce the extrapolation space H_{-1} associated with A.

More precisely, let $\beta \in \rho(A)$ and define H_{-1} as the completion of H with respect to the norm

$$\|z\|_{H_{-1}} = \|(\beta I - A)^{-1} z\|_H.$$

Then the semigroup $\mathbb{T}$ extends uniquely to a C^0-semigroup $\mathbb{T}_{-1}$ on H_{-1}, with generator denoted by A_{-1}. In this more general setting, one considers control operators

$$B \in \mathcal{L}(U, H_{-1}),$$

and the corresponding mild solution is written as

$$z(t) = \mathbb{T}(t)z_0 + \int_0^t \mathbb{T}_{-1}(t-s)Bu(s)\,\mathrm{d}s, \qquad t \geq 0, \tag{7.5.6}$$

where the integral is a priori understood in H_{-1}.

To ensure that the trajectory actually takes values in H, one imposes an admissibility condition on B. More precisely, for every $t > 0$, define

$$\Phi_t u = \int_0^t \mathbb{T}_{-1}(t-s)Bu(s)\,\mathrm{d}s, \qquad u \in L^2((0,t);U).$$

The operator B is called an *admissible control operator* for $\mathbb{T}$ if, for some $t > 0$,

$$\Phi_t(L^2((0,t);U)) \subset H.$$

In that case, for every $z_0 \in H$ and every $u \in L^2_{\mathrm{loc}}((0,\infty);U)$, the solution given by (7.5.6) belongs to

$$C([0,\infty);H) \cap H^1_{\mathrm{loc}}((0,\infty);H_{-1}).$$

Observe that if $B \in \mathcal{L}(U,H)$, then B is automatically admissible.

The infinite-horizon LQR problem with unbounded control operators is usually formulated in the framework of well-posed (or regular) linear systems in the sense of Salamon and Weiss. Under suitable stabilizability and detectability assumptions, one can still obtain existence and uniqueness of optimal controls, together with a feedback representation. However, compared with the bounded case, the corresponding Riccati theory is more delicate: the feedback operator may itself be unbounded, and even in the regular case the algebraic Riccati equation may take a nonstandard form. Since the control operator arising in Part I is bounded on the chosen state space, we shall not need this more general framework here; we only mention it in order to place the above presentation in a broader context.

The theory of optimal control for infinite-dimensional systems with unbounded operators has been developed along a natural line from the abstract representation of such systems to the analysis of admissibility and, ultimately, to Riccati-based feedback synthesis. In the PDE setting, the generator A is typically unbounded, while the main additional difficulty arises when the control or observation operators are themselves unbounded, as is the case for boundary or point control problems. A functional-analytic framework for linear systems with unbounded control and observation operators was established by Salamon, and the corresponding linear-quadratic problem was subsequently

investigated by Pritchard and Salamon [Sal87, PS87]. A crucial role in this theory is played by the notion of admissibility of unbounded control operators, introduced and systematically studied by Weiss, since it guarantees that inputs acting through extrapolation spaces still generate meaningful state trajectories and allows one to formulate the cost functional in a rigorous way [Wei89].

Building on this foundation, the theory was further extended to well-posed linear systems and to infinite-horizon quadratic optimization by Staffans [Sta97] and [Sta98]. In parallel, Weiss and Rebarber clarified the notions of optimizability and estimatability, which are particularly relevant when the classical exponentially stabilizable framework is not directly applicable [WR00]. These developments, in turn, provided the basis for the study of Riccati equations and feedback synthesis for stable well-posed systems, as shown by Curtain [Cur03], and were later extended to more complex coupled PDE models with boundary control by Acquistapace, Bucci, and Lasiecka [ABL05]. Altogether, these works show that the optimal control of systems with unbounded control actions can now be treated within a rather general framework that combines admissibility, well-posedness, finite-cost conditions, Riccati theory, and feedback stabilization.

7.6 Background on Input-to-State Stability

In this section, we briefly review the notion of input-to-state stability (ISS) for linear infinite-dimensional systems in the semigroup framework., a robustness property introduced by Sontag [Son89]. ISS provides a natural framework for describing how external inputs influence the state of a dynamical system. In the finite-dimensional setting, the theory was significantly developed by Sontag and Wang [SW95], who established fundamental Lyapunov characterizations of ISS. Since then, the ISS framework has been extended to infinite-dimensional and abstract systems; see, for example, [JNPS16, JNPS18, JMS22]. More recently, ISS results have also been obtained for nonlinear infinite-dimensional systems [HJST24] and for nonlinear parabolic partial differential equations with boundary disturbances [KK16a, KK16b, KK17, MKK19, KK21]. We begin by recalling the standard classes of comparison functions used in the definition of ISS. We restrict our attention to the linear system (7.5.1)–(7.5.2), whose well-posedness has already been established in Section 7.3 and Section 7.5. For the convenience of the reader, we summarize below several notions and results related to ISS that will be used later; see [CZ95, MP20].

7.6.1 Linear systems in the semigroup setting

Let H and U be Hilbert spaces. We assume that

$$A : D(A) \subset H \to H$$

is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on H, and that

$$B \in \mathcal{L}(U, H)$$

is a bounded linear operator. Throughout this section, the input space is taken as

$$\mathcal{U} := L^\infty(\mathbb{R}_+; U),$$

equipped with the norm

$$\|u\|_{\mathcal{U}} := \|u\|_{L^\infty(\mathbb{R}_+;U)} = \operatorname*{ess\,sup}_{t\geq 0} \|u(t)\|_U.$$

Since A generates a C_0-semigroup, the homogeneous equation

$$\dot{y}(t) = Ay(t), \qquad y(0) = y_0,$$

admits the solution $y(t) = T(t)y_0$. For the inhomogeneous system (7.5.1)–(7.5.2), the natural solution concept is the following.

Because B is bounded and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+; U)$, the integral in (7.5.4) is well defined in the Bochner sense. Moreover, for every $y_0 \in H$ and every $u \in \mathcal{U}$, the formula (7.5.4) defines a unique global mild solution. Hence the system is forward complete.

Definition 7.6.1 (Forward completeness)**.** System (7.5.1)-(7.5.2) is said to be *forward complete* if, for every initial state $y_0 \in H$ and every input $u \in \mathcal{U}$, the corresponding mild solution is defined for all $t \geq 0$.

7.6.2 Exponential stability of the semigroup

The asymptotic behaviour of (7.5.1)–(7.5.2) is closely related to the stability properties of the semigroup generated by A.

Definition 7.6.2 (Exponential stability)**.** The semigroup $(T(t))_{t\geq 0}$ is said to be *exponentially stable* if there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\|T(t)\|_{\mathcal{L}(H)} \leq Me^{-\omega t}, \qquad \forall t \geq 0. \tag{7.6.1}$$

If (7.6.1) holds, then the influence of the initial condition decays exponentially fast along the homogeneous dynamics.

7.6.3 Input-to-state stability

Denote by $\mathbb{R}_+ = [0, \infty)$. We set

$$\begin{aligned}
\mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,;\, \gamma \text{ is continuous, strictly increasing, and } \gamma(0) = 0\},\\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \,;\, \gamma(r) \to \infty \text{ as } r \to \infty\},\\
\mathcal{L} &:= \{\ell : \mathbb{R}_+ \to \mathbb{R}_+ \,;\, \ell \text{ is continuous, strictly decreasing, and } \ell(t) \to 0 \text{ as } t \to \infty\},\\
\mathcal{KL} &:= \Big\{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \,;\, \beta(\cdot, t) \in \mathcal{K} \text{ for every } t \geq 0,\ \beta(r, \cdot) \in \mathcal{L} \text{ for every } r > 0\Big\}.
\end{aligned}$$

Definition 7.6.3 (Input-to-state stability)**.** The linear system (7.5.1)-(7.5.2) is said to be *input-to-state stable* (ISS) with respect to $\mathcal{U} = L^\infty(\mathbb{R}_+; U)$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, for every $y_0 \in H$, every $u \in \mathcal{U}$, and every $t \geq 0$, the mild solution satisfies

$$\|y(t)\|_H \leq \beta(\|y_0\|_H, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{7.6.2}$$

Estimate (7.6.2) expresses two fundamental features of the system: the effect of the initial condition decreases with time, while the effect of the input is bounded uniformly in time by a function of the input size.

For linear systems, it is natural to consider the stronger estimate

$$\|y(t)\|_H \leq Me^{-\omega t}\|y_0\|_H + C\|u\|_{\mathcal{U}}, \qquad \forall t \geq 0, \tag{7.6.3}$$

for some constants $M \geq 1$, $\omega > 0$, and $C > 0$. In this case one speaks of *exponential ISS*.

7.6.4 ISS and exponential stability for bounded input operators

For the class of systems considered here, ISS is in fact equivalent to the exponential stability of the semigroup generated by A.

Proposition 7.6.4. *Assume that $(T(t))_{t\geq 0}$ is exponentially stable. Then system (7.5.1)-(7.5.2) is ISS with respect to $\mathcal{U} = L^\infty(\mathbb{R}_+; U)$. More precisely, if (7.6.1) holds, then every mild solution satisfies*

$$\|y(t)\|_H \leq Me^{-\omega t}\|y_0\|_H + \frac{M\|B\|_{\mathcal{L}(U,H)}}{\omega}\|u\|_{\mathcal{U}}, \qquad \forall t \geq 0. \tag{7.6.4}$$

The converse is also true: for bounded input operators, ISS is equivalent to exponential stability of the semigroup. Therefore, for linear systems of the form (7.5.1)-(7.5.3), the ISS analysis reduces to a semigroup stability problem.

Theorem 7.6.5. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on H, and let $B \in \mathcal{L}(U, H)$. Then the following assertions are equivalent:*

(i) system (7.5.1)-(7.5.2) is ISS with respect to $\mathcal{U} = L^\infty(\mathbb{R}_+; U)$;

(ii) system (7.5.1)-(7.5.2) is exponentially ISS, i.e., (7.6.3) holds for some constants $M \geq 1$, $\omega > 0$, and $C > 0$;

(iii) the semigroup $(T(t))_{t\geq 0}$ is exponentially stable.

Remark 7.6.6. The previous theorem is specific to the case of bounded input operators. In infinite dimensions, strong stability of the semigroup is in general strictly weaker than exponential stability, and therefore it is not sufficient for ISS.

Remark 7.6.7. For systems with unbounded input operators, which typically arise in boundary control problems, the same formula (7.5.4) remains meaningful only after introducing the extrapolation space H_{-1} and suitable admissibility assumptions on B. In that setting, ISS is characterized by exponential stability of the semigroup together with infinite-time admissibility of the input operator. Since in the present work we only need the bounded-operator framework, we do not pursue this more general setting here.

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