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Contents

Remerciements i

Acknowledgements iii

Table of contents iv

List of symbols and notations vii

Acronyms viii

1 Introduction 1
- 1.1 Managing uncertainty in finance . . . 3
  - 1.1.1 Robust finance and super replication . . . 3
  - 1.1.2 From robust finance to martingale optimal transport . . . 5
  - 1.1.3 Mean field games in financial modeling . . . 5
- 1.2 Martingale optimal transport . . . 6
  - 1.2.1 Optimal transport . . . 6
    - 1.2.1.1 Definition . . . 6
    - 1.2.1.2 Duality and cyclical monotonicity . . . 8
    - 1.2.1.3 Wasserstein spaces and quantile functions . . . 9
  - 1.2.2 Martingale optimal transport . . . 11
    - 1.2.2.1 Definition . . . 11
    - 1.2.2.2 Duality . . . 12
  - 1.2.3 Weak optimal transport . . . 13
  - 1.2.4 Numerical methods for XOT . . . 15
    - 1.2.4.1 Numerical methods for OT . . . 16
    - 1.2.4.2 Numerical methods for MOT . . . 17
    - 1.2.4.3 Numerical methods under the weak framework . . . 19
  - 1.2.5 Our contributions . . . 19
    - 1.2.5.1 Chapter 3 Maximal Martingale Wasserstein Inequality . . . 19
    - 1.2.5.2 Chapter 4 Non-decreasing martingale couplings . . . 21
    - 1.2.5.3 Chapter 5 WOT . . . 24
- 1.3 Graphon mean field games . . . 25
  - 1.3.1 Finite horizon mean field games . . . 25
  - 1.3.2 From finite games to graphon mean field games . . . 27
  - 1.3.3 Our contribution : extended graphon mean field games . . . 30

2 Introduction en français 33

2.1 Gestion de l'incertitude en finance . . . 35
2.1.1 Finance robuste et sur-réplication . . . 35
2.1.2 De la finance robuste au transport optimal martingale . . . 37
2.1.3 Jeux à champ moyen en modélisation financière . . . 38
2.2 Transport optimal martingale . . . 39
2.2.1 Transport optimal . . . 39
2.2.1.1 Définition . . . 39
2.2.1.2 Dualité et monotonicité cyclique . . . 40
2.2.1.3 Espaces de Wasserstein et fonctions quantiles . . . 42
2.2.2 Transport optimal martingale . . . 44
2.2.2.1 Définition . . . 44
2.2.2.2 Dualité . . . 45
2.2.3 Transport optimal faible . . . 46
2.2.4 Méthodes numériques pour XOT . . . 48
2.2.4.1 Méthodes numériques pour OT . . . 48
2.2.4.2 Méthodes numériques pour MOT . . . 50
2.2.4.3 Méthodes numériques dans le cadre faible . . . 51
2.2.5 Nos contributions . . . 52
2.2.5.1 Chapitre 3 Inégalité de Wasserstein martingale maximale . . . 52
2.2.5.2 Chapitre 4 Couplages martingales croissants . . . 54
2.2.5.3 Chapitre 5 WOT . . . 57
2.3 Jeux à champ moyen sur graphons . . . 58
2.3.1 Jeux à champ moyen en horizon fini . . . 58
2.3.2 Des jeux finis aux jeux à champ moyen sur graphons . . . 60
2.3.3 Notre contribution : jeux à champ moyen sur graphon étendu . . . 63

3 Maximal Martingale Wasserstein inequality 67
3.1 Introduction . . . 68
3.2 Proof . . . 70

4 Non-decreasing martingale couplings 77
4.1 Introduction . . . 78
4.2 Statement of the results . . . 81
4.2.1 Numerical experiments . . . 81
4.2.2 Infimum and supremum of $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ for the convex order . 83
4.2.3 Definition and optimality of non-decreasing and non-increasing martingale couplings . . . 83
4.2.4 Main result : directionally decomposed non-decreasing martingale couplings 86
4.2.5 Do $\pi^{\uparrow}$ and $\pi^{\downarrow}$ optimize the cost function $|x-y|^{\rho}$ for $\rho > 1$ under the sandwich condition ? . . . 88
4.2.5.1 Preservation of the optimality . . . 88
4.2.5.2 Non-preservation of the optimality . . . 90
4.3 Proofs . . . 91
4.3.1 Proof of Lemma 4.2.2 . . . 92
4.3.2 Proofs of the optimality results in Section 4.2.3 apart from Proposition 4.2.8 93
4.3.3 Proof of the results in Section 4.2.4 and Proposition 4.2.8 . . . 98
4.3.4 Proofs of the results in Section 4.2.5 concerning optimality of $\pi^{\uparrow}$ and $\pi^{\downarrow}$ for the cost function $|x-y|^{\rho}$ with $\rho > 1$ under the sandwich condition . . 117

5 An optimization approach to the numerical approximation of the weak optimal transport problem **127**
5.1 Introduction . . . 128
5.2 Numerical approximation of the problem . . . 130
5.3 Frank Wolfe Algorithm . . . 132
5.4 Applications . . . 134
5.4.1 Wasserstein projection (WP) in dimension one . . . 134
5.4.1.1 WP with $\rho = 1$: reduction to LP . . . 135
5.4.1.2 WP with $\rho = 2$: reduction to a quadratic problem . . . 137
5.4.1.3 WP solved by Frank Wolfe algorithm . . . 140
5.4.2 Stretched Brownian motion . . . 141

6 Extended graphon mean field games in discrete time **146**
6.1 Introduction . . . 147
6.2 Graphon mean field game model . . . 149
6.2.1 Setup and preliminaries . . . 149
6.2.2 Graphons . . . 149
6.2.3 Model . . . 150
6.2.4 Assumptions . . . 151
6.3 Link between Graphon Nash Equilibrium (GNE) and forward-backward system . 152
6.4 Existence of equilibrium . . . 153
6.4.1 Equilibrium condition and existence . . . 153
6.4.2 Proofs . . . 154
6.4.2.1 Useful lemmas . . . 154
6.4.2.2 Proof of Theorem 6.4.1 . . . 163
6.5 Uniqueness Results . . . 168
6.5.1 Uniqueness by monotonicity . . . 168
6.5.2 Uniqueness by contraction . . . 170
6.6 Approximate Equilibrium in Finite Player Games . . . 172
6.6.1 From GNE to N-player Nash equilibrium . . . 173
6.6.2 Useful lemmas . . . 175
6.6.3 Proofs of the lemmas in Section 6.6 . . . 176
6.7 Application in finance - optimal investment with relative performance criterion . 185
6.7.1 Model . . . 185
6.7.2 Numerical results . . . 191

List of publications **198**

Bibliography **199**

List of symbols and notations

$\mathbb{N}^*$	the set of all positive integers
$\|\cdot\|$	the Euclidean norm
$\|\cdot\|_p$	the L_p norm
$\|\cdot\|_\square$	the cut norm
$\|\cdot\|_{\infty\to 1}$	the operator norm as an operator $L_\infty \to L_1$
$\mathbb{R}_+$	non-negative real numbers, i.e. $[0, +\infty)$
A^B	the set of functions from set B to set A
$\mathcal{P}(\cdot)$	the set of probability measures (on the relevant space)
$\mathcal{P}_\rho(\cdot)$	the set of probability measures with finite ρ-th moment
δ_x	Dirac mass concentrated at point x
$\mathbb{1}_A$	indicator function of a set A
$\nabla, \nabla\cdot$	the gradients and the divergence
$D^2 u$	Hessian of the scalar function u
$\wedge$	the min operator, i.e. $a \wedge b = \min\{a, b\}$
$\vee$	the max operator, i.e. $a \vee b = \max\{a, b\}$
$T\#\mu$	pushforward measure of μ through the map T
$\pi^\uparrow$	non-decreasing martingale coupling
$\pi^\downarrow$	non-increasing martingale coupling
π^{HN}	the Hobson and Neuberger martingale coupling
π^{HK}	the Hobson and Klimmek martingale coupling
$\Pi(\mu, \nu)$	the set of probability couplings between μ and ν
$\Pi_M(\mu, \nu)$	the set of martingale couplings between μ and ν
ess sup	the essential supremum
$\lfloor x \rfloor$	the floor function (greatest integer less than or equal to x)
argmax_a	the arguments of the maxima
$\lim, \liminf, \limsup$	limit, inferior limit, superior limit
$\mathcal{T} := \{0, 1, \ldots, T-1\}$	the set of finite time steps without the terminal time T
$\overline{\mathcal{T}} := \mathcal{T} \cup \{T\}$	the set of finite time steps with the terminal time T
$I := [0, 1]$	unit interval equipped with the Euclidean distance
$\leq_{cx}$	dominance in the convex order
$\leq_{st}$	dominance in the stochastic order
F_η	cumulative distribution function of probability measure η
F_η^{-1}	quantile function of probability measure η

Acronyms

COT *Causal Optimal Transport*

ELMM *Equivalent local martingale measure*

GMFG *Graphon Mean Field Games*

GNE *Graphon Nash equilibrium*

MFG *Mean Field Games*

MFNE *Mean Field Nash equilibrium*

MOT *Martingale Optimal Transport*

OT *Optimal Transport*

SEP *Skorokhod Embedding Problem*

WMOT *Weak Martingale Optimal Transport*

WOT *Weak Optimal Transport*

Chapter 1

Introduction

Abstract

This thesis investigates two distinct directions: *Martingale Optimal Transport* (MOT) and *Graphon Mean Field Games* (GMFG), both inspired by the overarching principles of mathematical finance. While these two areas are treated separately, they share a common motivation: managing uncertainty in financial markets and constructing strategies that perform reliably across a wide range of conditions.

Contents

1.1 Managing uncertainty in finance . . . **3**
1.1.1 Robust finance and super replication . . . 3
1.1.2 From robust finance to martingale optimal transport . . . 5
1.1.3 Mean field games in financial modeling . . . 5
1.2 Martingale optimal transport . . . **6**
1.2.1 Optimal transport . . . 6
1.2.2 Martingale optimal transport . . . 11
1.2.3 Weak optimal transport . . . 13
1.2.4 Numerical methods for XOT . . . 15
1.2.5 Our contributions . . . 19
1.3 Graphon mean field games . . . **25**
1.3.1 Finite horizon mean field games . . . 25
1.3.2 From finite games to graphon mean field games . . . 27
1.3.3 Our contribution: extended graphon mean field games . . . 30

In financial markets, uncertainty and incomplete information are intrinsic challenges that investors have to continuously manage. Market participants face unpredictability in asset prices, volatility in returns, and uncertainty in the actions and behaviors of others. These factors complicate the decision-making process. Traditional financial models often assume idealized conditions or rely on precise knowledge of market dynamics, which are rarely accurate in practice. To address these issues, the field of *Robust Finance* has emerged and focused on constructing a framework that remains effective across a wide range of possible market environments. Rather than depending on highly specific assumptions, robust finance framework builds around strategies that hedge against risk and optimize performance with lower dependency on specific models. This approach has become more popular in model calibration, risk control, and portfolio management etc.

There are two powerful mathematical frameworks that have developed within the domain of robust finance: *Martingale Optimal Transport* (MOT) and *Mean Field Games* (MFG). These frameworks approach uncertainty from different angles, offering complementary insights. The MOT problem, derived from the theory of optimal transport by adding an additional martingale constraint, provides a way to evaluate upper and lower bounds for the price of financial instruments without any dependence of specific model. On the other hand, the MFG problem, introduced to model strategic interactions within large populations, addresses uncertainty in a dynamic environment, and aims to search an equilibrium solution where individuals optimize strategies relative to the collective behavior. These frameworks, although developed independently, both derive from the fundamental idea of financial decision-making in uncertain environments. Beyond MOT and MFG, robust finance encompasses other areas not covered by this thesis, such as backward stochastic differential equations (BSDEs), robust portfolio optimization, model uncertainty and control theory, which have a vast literature.

Even though MOT and MFG have little direct overlap, there are several lines of research connecting classical *Optimal Transport* (OT) and MFG. For instance, Acciaio, Backhoff-Veraguas, and Carmona [1] investigated extended mean field control problems using a stochastic maximum principle and a transport perspective. Further work has combined Cournot-Nash equilibrium concepts with OT, for instance, see Acciaio et al. [7, 8]. Notably, entropic OT and MFG have been studied by Benamou, Carlier, Di Marino, and Nenna [44], who proposed an entropy minimization approach to second-order variational mean field games under a quadratic Hamiltonian, and by Backhoff, Conforti, Gentil, Ivan, and Léonard [21], who investigated the mean field Schrödinger problem and its intrinsic link to entropic OT. Ren, Tan, Touzi, and Yang [199] further examined entropic optimal planning for path-dependent mean field games. Finally, machine learning approaches to OT and MFG have also emerged in works such as [30, 67].

In this thesis, although there is no direct link established between MOT and MFG, both frameworks address critical aspects of decision-making under uncertainty and competition in financial markets. The thesis, therefore, contributes to the growing body of research on robust finance by offering new insights into how MOT and MFG can be leveraged independently to tackle complex financial problems involving risk, competition, and market dynamics.

1.1 Managing uncertainty in finance

1.1.1 Robust finance and super replication

Let us consider a financial market with $d+1$ assets that can be traded at times $t \geq 0$, where $d \in \mathbb{N}^*$. We denote by S_t^i the price of asset i at time t. The price process $(S_t^i)_{t\geq 0}$ evolve on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the historical (real-world) probability measure under which we model the market. The asset with index $i = 0$ is considered the risk-free asset, and we set $S_0^0 = 1$ by convention. The dynamics of the risk-free asset with a constant interest rate $r \in \mathbb{R}$ are given by

$$dS_t^0 = r\, S_t^0\, dt, \quad \text{so that} \quad S_t^0 = S_0^0\, e^{rt}.$$

Since modeling interest rates is not the main focus of this thesis, we assume $r = 0$ for simplicity, which implies $S_t^0 = 1$ for all $t \geq 0$.

For the risky assets $i \in \{1, \dots, d\}$, their price dynamics are often modeled using stochastic differential equations (SDEs), such as the *Geometric Brownian Motion* (GBM):

$$dS_t^i = \mu^i\, S_t^i\, dt + \sigma^i\, S_t^i\, dW_t^{i,\mathbb{P}},$$

where μ^i is the drift (expected return), σ^i is the volatility, and $W_t^{i,\mathbb{P}}$ is a one-dimensional standard Brownian motion under $\mathbb{P}$.

Self-financing portfolios Let Δ_t^i denote the quantity of asset i held at time t, then, the portfolio at time t is given by

$$V_t = \sum_{i=0}^{d} \Delta_t^i S_t^i.$$

Note that $\Delta_t^i < 0$ corresponds to a debt in asset i. This is a possible situation as long as the investors are able to repay and have $V_t \geq 0$ for all t. We impose the *self-financing* condition: we can readjust the allocations Δ_t^i, but no cash can ever be injected into or removed from the portfolio, which means between t and $t + dt$, the variation in the portfolio value is only due to the variation of the values of the assets, i.e.

$$dV_t = \sum_{i=1}^{d} \Delta_t^i\, dS_t^i,$$

Let us introduce two concepts: admissible portfolio and arbitrage.

Admissible portfolio $\mathcal{D} := (\Delta_t, 0 \leq t \leq T)$ is called an **admissible portfolio** if V_t is bounded from below for all t $\mathbb{P}$-a.s., i.e. there exist $L \in \mathbb{R}$ such that

$$\mathbb{P}(V_t \geq L) = 1, \qquad \forall\, t \in [0, T].$$

Arbitrage A self-financing admissible portfolio is called an **arbitrage** if the corresponding value process V_t satisfies $V_0 = 0$ and

$$V_T \geq 0 \quad \mathbb{P} - a.s. \qquad \text{and} \qquad \mathbb{P}(V_T > 0) \neq 0.$$

We denote the equivalence of two probability measures $\mathbb{Q}$ and $\mathbb{P}$ by $\mathbb{Q} \sim \mathbb{P}$, i.e. for all $A \in \mathcal{F}_T$, $\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0$. Then, let us introduce a very important concept called *Equivalent local martingale measure*, which is defined as following:

Definition 1.1.1 (ELMM)**.** *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ is called an Equivalent local martingale measure (ELMM) if $\mathbb{Q} \sim \mathbb{P}$ and the discounted process*

$$\left(\frac{S_t^1}{S_t^0}, \ldots, \frac{S_t^d}{S_t^0}\right)_{t\in[0,T]}$$

is local martingale with respective to $\mathbb{Q}$. An ELMM is also called a risk-neutral measure.

Complete market To define a complete market, we first introduce the notion of an *attainable payoff*. Suppose we buy and delta-hedge a European option written on the $d+1$ assets with maturity T and payoff ξ_T. A payoff ξ_T is said to be *attainable* at time 0 if there exists an admissible portfolio $\Delta \in \mathcal{D}$ and a price $x_0 \in \mathbb{R}$ such that

$$x_0 + V_T(\Delta) = \xi_T, \quad \mathbb{P} - \text{a.s.}, \quad \text{where} \quad V_T(\Delta) = \sum_{i=1}^{d} \int_0^T \Delta_s^i \, dS_s^i = \int_0^T \Delta_s \, dS_s,$$

and Δ is a self-financing portfolio that is a true martingale under some equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$.

A market is said to be *complete* if every payoff is attainable at time 0.

The following characterization of an arbitrage-free and complete market is known as the *Fundamental Theorem of Asset Pricing*; see, for instance, [53].

Theorem 1.1.2 (Fundamental Theorem of Asset Pricing)**.** *The market is arbitrage-free if and only if there exists an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ under which discounted asset prices are martingales. Furthermore, the market is complete if and only if $\mathbb{Q}$ is unique.*

Since we assume $r = 0$ implying $S_t^0 = 1$ for all $t \in [0, T]$, it follows that the asset price processes $(S_t^1, , \ldots, S_t^d)_{t\in[0,T]}$ is a martingale under $\mathbb{Q}$. Additionally, the self-financing portfolio $(V_t(\Delta))_{t\in[0,T]}$ is also a martingale under $\mathbb{Q}$.

However, real financial markets are subject to constraints that prevent perfect replication and, consequently, completeness. A well-known example of an incomplete market was first studied by Harrison and Kreps [130] and further developed by El Karoui and Quenez [101]. In incomplete markets, we consider the *super-replication price*, representing the minimal initial cost required to ensure that a hedging strategy exists such that the final portfolio value exceeds the contingent claim in all possible scenarios.

Theorem 1.1.3 (Super-replication price [101]). *The super-replication price is given by*

$$\sup_{\mathbb{Q}\in ELMM} \mathbb{E}^{\mathbb{Q}}[\xi_T] = \inf\left\{x \in \mathbb{R} : \exists\, \Delta \in \mathcal{D} \ s.t.\ x + V_T(\Delta) \geq \xi_T,\ \mathbb{P} - a.s.\right\}.$$

This concept of super-replication is closely related to the development of robust finance after the financial crisis in the post-2008 era, where model uncertainty is addressed by considering a family of models. Instead of relying on a single probability measure, we consider a set of probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$. A model-free version of the fundamental theorem of asset pricing and the super-replication theorem was proposed by Acciaio et al. [4]. The robust pricing problems have also garnered significant attention; see, for example [54, 10]. The details of these connections are elaborated in the next subsection.

1.1.2 From robust finance to martingale optimal transport

Recall that we assumed a zero interest rate in the section above, hence, the basket of assets $(S_t = (S_t^1, \ldots, S_t^d))_{t\in[0,T]}$ is a martingale under $\mathbb{Q}$ as well as the process $(V_t)_{t\in[0,T]}$. Now, let us consider a European option with maturity T whose payoff depends on the value of the financial assets at times $0 < T_1 < T_2 < T$, and the option pays $c(S_{T_1}, S_{T_2})$, where $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ and S_{T_1}, S_{T_2} are the stock prices at T_1, T_2 respectively. To replicates the option with our portfolio, we need to have $V_T = c(S_{T_1}, S_{T_2})$. Since $(V_t)_{t\in[0,T]}$ is a martingale, we have

$$V_0 = \mathbb{E}^{\mathbb{Q}}[V_T] = \mathbb{E}^{\mathbb{Q}}[c(S_{T_1}, S_{T_2})],$$

and in the complete market, $\mathbb{E}^{\mathbb{Q}}[c(S_{T_1}, S_{T_2})]$ will be the price of this option.

For $d = 1$, in the work of [58], it is observed that the marginal distributions of the underlying asset $S_{T_1} \sim \mu$ and $S_{T_2} \sim \nu$ can be retrieved uniquely from the prices of calls and puts for all strike prices K at maturities T_1 and T_2.

For $d > 1$, the laws of $S_{T_1} \sim \mu$ and $S_{T_2} \sim \nu$ are determined from the prices of options with payoff $(\lambda \cdot S_{T_1} - K)^+$ and $(\lambda \cdot S_{T_2} - K)^+$ for all strike prices K with $\lambda \in \mathbb{R}^d$ such that $|\lambda| = 1$. In addition, Theorem 1.1.2 implies that (S_{T_1}, S_{T_2}) is a martingale that $\mathbb{E}^{\mathbb{Q}}[S_{T_2} \mid S_{T_1}] = S_{T_1}$ almost surely. We deduce that the price x of the option with payoff $c(S_{T_1}, S_{T_2})$ satisfies

$$\inf_{\substack{X\sim\mu, Y\sim\nu \\ \mathbb{E}[Y|X]=X\ a.s.}} \mathbb{E}[c(X,Y)] \leq x \leq \sup_{\substack{X\sim\mu, Y\sim\nu \\ \mathbb{E}[Y|X]=X\ a.s.}} \mathbb{E}[c(X,Y)].$$

The calculation of the lower bound constitutes the MOT problem, which are introduced with more details in Section 1.2.2 below.

1.1.3 Mean field games in financial modeling

Mean Field Games (MFG) is another area with significant applications under the mathematical finance framework, particularly in scenarios where strategic interactions between agents must be taken into account. Traditional portfolio optimization or trading problems often consider an

investor's decisions in isolation, based solely on risk preferences or market forecasts. However, in realistic settings, the actions of other participants such as competitors, market makers, or institutional investors can significantly affect outcomes. Therefore, strategies must be robust not only to market uncertainty but also to the unpredictability of competitors' actions.

Mean Field Games, introduced independently by Lasry and Lions [166] and Huang, Malhamé, and Caines [140], have gained significant attention in financial modeling due to their ability to handle systems with many interacting agents. In finance, MFG models scenarios where agents (such as traders, investors, or institutions) make decisions based on their state and the distribution of others. These agents often face uncertainty not only in the price evolution of assets but also in the actions of their competitors, which makes MFG particularly relevant for studying large markets where agents' actions influence market dynamics and vice versa.

The motivation for MFG in finance arises from its capacity to model the uncertainty in others' behavior. Robust models within the mathematical finance framework aim to develop strategies that perform well under a variety of possible scenarios, and MFG facilitates the formulation of equilibrium strategies where agents consider both their own risks and those stemming from others' actions. Two popular directions in MFG applications are trading and systemic risk. In optimal trading, agents aim to minimize costs or maximize profits while considering the aggregate impact of other participants, leading to competitive or cooperative behavior. Applications of MFG to systemic risk focus on managing the collective risk posed by the financial system and how individual agents might manage their exposure to such risks within an MFG framework. For a detailed study of the MFG applications in finance, one can refer to [71].

An extension of the MFG problem that has gained attentions is the *Graphon Mean Field Games* (GMFG), which models interactions between heterogeneous agents across networks or graphs. In the context of finance, this allows for the representation of agents with different characteristics, such as varying risk preferences or investment strategies, interacting in a non-homogeneous manner. By allowing heterogeneous interactions, GMFG provides a more realistic model for financial markets where players may have different information, preferences, or connectivity.

1.2 Martingale optimal transport

Section 1.1.1 and Section 1.1.2 provide the motivation for our investigation into the MOT problem under the robust finance framework. In the following of this section, the foundation of Chapter 3, Chapter 4, and Chapter 5 will be presented.

1.2.1 Optimal transport

1.2.1.1 Definition

Monge and Kantorovich problem Introduced by Monge in 1781 in his work "Théorie des déblais et des remblais", the *Optimal Transport* (OT) problem is concerned with transferring mass from one location to another in such a way as to minimize an integrated cost function c.

Let E and F be two Polish spaces, that is, complete and separable metric spaces. For $\mu \in \mathcal{P}(E)$ and $\nu \in \mathcal{P}(F)$, where $\mathcal{P}(\cdot)$ denotes the set of probability measures on the corresponding space. Monge was looking for measurable transport maps $T : E \times F$ such that the image $T\#\mu$ of μ by T is equal to ν, where $\#$ denotes the pushforward operation of measures, i.e. for $B \in F$, $T\#\mu(B) = \mu\left(T^{-1}(B)\right) = \nu(B)$. The *Monge Problem* (MP) is defined as

$$\mathrm{MP}(\mu, \nu) := \inf_{T : T\#\mu = \nu} \int_E c(x, T(x))\mu(dx). \tag{MP}$$

This optimization problem is the first historical formulation of an OT problem, however, Monge problem does not support mass spliting, which leads to non-existence of transport maps.

Later on after World War II, Kantorovich proposed a very fruitful relaxation by considering probability couplings (or plans) which are probability measures on the product space $E \times F$ with marginals given by μ and ν :

$$\mathrm{KP}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{E \times F} c(x, y)\pi(dx, dy) \tag{KP}$$

where

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}(E \times F) \mid \pi(dx, F) = \mu(dx) \text{ and } \pi(E, dy) = \nu(dy)\}.$$

Theorem 1.2.1 (Existence)**.** *Let $\mu \in \mathcal{P}(E)$ and $\nu \in \mathcal{P}(F)$, and let $c : E \times F \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and bounded from below, then* (KP) *admits a solution.*

A student of Kantorovich named Svetlozar Rachev [194] provided a good overview of applications for the Monge-Kantorovich problem across numerous field: differential geometry [224, 217], functional analysis [153, 173], infinite-dimensional linear programming [221, 203, 197, 155], probability theory [104, 106, 99, 236], statistics [237, 142], information theory [220, 123, 122], statistical physics [98, 76], dynamical systems theory [184], and matrix theory [109, 176, 183].

Motivated by the theory of fluid mechanics, Yann Brenier formulated in a famous polar decomposition theorem [59, 60] that under fairly general assumptions: there exists a unique solution to the Monge-Kantorovich problem in $E = F = \mathbb{R}^d$ for the quadratic cost $c : (x, y) \mapsto |y - x|^2$ of the form $(X, \nabla\varphi(X))$, where φ is a uniquely determined convex function up to the addition of a constant, and this solution coincides with the minimizer of the Monge problem. Furthermore, the function φ is also the solution of a Monge-Ampère type equation, which we can refer to [89, 85, 88, 87, 86].

In the recent years, OT has become a popular tool for machine learning studies such as shape analysis [9, 193, 214, 111], domain adaptation [82], multi-label classification [228], and generative adversarial networks [18, 118]. For an in-depth studies of the OT, one can refer to [195, 196, 222, 223, 14, 15, 163, 205].

1.2.1.2 Duality and cyclical monotonicity

Kantorovich dual formulation Let $\mu \in \mathcal{P}(E)$ and $\nu \in \mathcal{P}(F)$, and let $c : E \times F \to [0, +\infty)$ be a measurable cost function. The Kantorovich dual problem is formulated as

$$(\text{DP}) := \sup \left\{ \int_E \varphi(x)\, \mu(dx) + \int_F \psi(y)\, \nu(dy) \,\middle|\, (\varphi, \psi) \in L^1(\mu) \times L^1(\nu), \varphi \oplus \psi \leq c \right\}, \qquad (\text{DP})$$

where $\varphi \oplus \psi$ denotes the function $(\varphi \oplus \psi)(x, y) := \varphi(x) + \psi(y)$ for all $(x, y) \in E \times F$. If E and F are compact and c is continuous, then there exists a solution (φ, ψ) to the (DP) problem.

Theorem 1.2.2 (Kantorovich Duality)**.** *Let E and F be two Polish spaces, and let $c : E \times F \to \mathbb{R}$ be a uniformly continuous and bounded function. Then, the dual problem* (DP) *admits a solution and we have*

$$\max \textit{ (DP)} = \min \textit{ (KP)}.$$

This theorem establishes a fundamental equivalence between the primal problem (KP) and its dual (DP), ensuring that the optimal values of both problems coincide. Note that the proof of Theorem 1.2.2 can be found in [205]. This duality allows us to study optimal transport plans through the properties of the dual pairs (φ, ψ).

An immediate question arises: how does this duality connect to the Monge problem (MP)? Understanding this connection is crucial because it provides insights into the structure of optimal solutions.

Consider the Monge problem (MP), where we aim to transport goods from place A to place B in an optimal way. Suppose that, in the optimal solution, we transport at least one unit from location x_i in A to location y_i in B for $i \in \{1, \ldots, N\}$, where $N \geq 2$. Now, imagine we alter this transportation plan by redirecting the unit originally sent from x_i to y_i to instead go from x_i to y_{i+1} with the convention that $y_{N+1} = y_1$, i.e. we send

$$x_1 \to y_2, \qquad x_2 \to y_3, \qquad \ldots, \qquad x_{N-1} \to y_N, \qquad x_N \to y_1.$$

The change in cost for each redirected unit is $c(x_i, y_{i+1}) - c(x_i, y_i)$. If the original transport plan is indeed optimal, then the total cost of this new arrangement must be greater than or equal to the original cost. This leads to the inequality:

$$\sum_{i=1}^{N} (c(x_i, y_{i+1}) - c(x_i, y_i)) \geq 0 \quad \Longrightarrow \quad \sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}).$$

Cyclically monotone The observation above motivates the following definition, which characterizes optimal transport plans. Consider a cost function $c : E \times F \to \mathbb{R} \cup \{+\infty\}$. A measurable set $\Gamma \subset E \times F$ is called *c-cyclically monotone* (c-CM in short) if for all $N \in \mathbb{N}^*$ and $N \geq 2$,

$$\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}), \qquad \forall\, (x_1, y_1), \ldots, (x_N, y_N) \in \Gamma.$$

The law of a pair of random variables (X, Y) taking values in $E \times F$ is said to be c-cyclically

monotone if there exists a c-cyclically monotone set Γ such that $P((X,Y)\in\Gamma)=1$. Intuitively, a transport plan is c-CM if no cyclic rearrangement of destinations can reduce the total transport cost. Building on this concept, we present the following theorem, which characterizes optimal transport plans in terms of cyclical monotonicity (see [223] for a detailed proof).

Theorem 1.2.3. *Let $\mu\in\mathcal{P}(E)$ and $\nu\in\mathcal{P}(F)$, and let $c: E\times F\to\mathbb{R}_+$ be a lower semicontinuous cost function such that the value of the Kantorovich problem (KP) is finite. Then, the following two statements are equivalent:*

(i) $\pi\in\Pi(\mu,\nu)$ is optimal for problem (KP);

(ii) The support of π is a c-CM set.

The characterization of optimal transport plans via cyclical monotonicity has significant implications for the stability of optimal transport with respect to its marginals.

Stability For simplicity, we assume the cost function $c: E\times F\to\mathbb{R}$ is continuous and bounded. For every $n\in\mathbb{N}$, let μ_n be a probability measure on E converging weakly to μ, and let ν_n be a probability measure on F converging weakly to ν. For each n, let π_n being an optimal distribution for $\mathrm{KP}(\mu_n,\nu_n)$. Then, up to extracting a subsequence, $(\pi_n)_{n\in\mathbb{N}}$ converges weakly to a distribution $\pi\in\mathcal{P}(E\times F)$. On one hand, by Portmanteau's theorem,

$$\mathrm{KP}(\mu_n,\nu_n)=\int_{X\times Y}c(x,y)\pi_n(dx,dy)\to\int_{X\times Y}c(x,y)\pi(dx,dy).$$

On the other hand, since each π_n is supported on a c-CM set due to the optimality, and cyclical monotonicity is preserved under weak convergence so that π is also c-CM. By Theorem 1.2.3, π is an optimal transport plan for $\mathrm{KP}(\mu,\nu)$. Consequently, we conclude that:

$$\mathrm{KP}(\mu,\nu)=\lim_{n\to\infty}\mathrm{KP}(\mu_n,\nu_n).$$

For the proofs of the c-CM and stability results, one can refer to [205].

1.2.1.3 Wasserstein spaces and quantile functions

Wasserstein distances For $\rho\in[1,+\infty)$, let $\mathcal{P}_\rho(E)$ denote the set of all probability measures on a metric space (E,d) with finite ρ-th moment

$$\mathcal{P}_\rho(E)=\left\{\eta\in\mathcal{P}(E)\colon\ \exists\, x_0\in E\quad\text{s.t.}\quad\int_E d(x,x_0)^\rho\,\eta(dx)<+\infty\right\}.$$

The ρ-Wasserstein distance between two probability measures $\mu,\nu\in\mathcal{P}_\rho(E)$ is defined by

$$W_\rho(\mu,\nu)=\left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{E\times E}d(x,y)^\rho\,d\pi(x,y)\right)^{1/\rho}.$$

When $\rho=1$, the Wasserstein distance W_1 corresponds to a a particular case of the Monge-Kantorovich problem, we have a dual formulation as in Theorem 1.2.2. If (φ,ψ) is optimal for

the dual problem (DP), then any pair (X, Y) optimal for $W_1(\mu, \nu)$ is concentrated on the set of $(x, y) \in E \times E$ satisfying $\varphi(x) + \psi(y) = d(x, y)$, and $\varphi(x) = \psi(y)$ if $x = y$. In addition, since the mappings $x \mapsto d(x, y)$ (for fixed y) and $y \mapsto d(x, y)$ (for fixed x) are 1-Lipschitz, an optimal pair for (DP) can be reduced to a pair of 1-Lipschitz function $(f, -f)$ with $f : E \to \mathbb{R}$. This observation leads to the Kantorovich-Rubinstein duality formula for the W_1 distance:

$$W_1(\mu, \nu) = \sup_{f:E\to\mathbb{R},\ 1\text{-Lipschitz}} \left(\int_E f(y)\nu(dy) - \int_E f(x)\mu(dx) \right).$$

For an in-depth study of Wasserstein distances and their applications, we refer to the works on classical optimal transport theory [57, 198, 202, 204, 233].

Quantile function In the previous sections, the Monge-Kantorovich problem is considered in a general Polish space. However, for our purposes, it is sufficient to restrict our attention to the real line $\mathbb{R}$. In dimension one, we have specialized tools that provide us deeper insights into the problems.

For a probability measure $\eta \in \mathcal{P}(\mathbb{R})$, we denote $F_\eta : x \mapsto \eta((-\infty, x])$ the cumulative distribution function and F_η^{-1} the quantile function of η, where

$$F_\eta^{-1}(u) = \inf\{x \in \mathbb{R} \mid F_\eta(x) \geq u\}, \qquad \forall\, u \in (0, 1).$$

Note that F_η^{-1} is the pseudo-inverse of F_η. It is well-know that the following properties hold:

(a) F_η is right-continuous with left-hand limits, F_η^{-1} is left-continuous with right-hand limits;

(b) For all $u \in (0, 1)$ and $x \in \mathbb{R}$,

$$F_\eta^{-1}(u) \leq x \quad \Longleftrightarrow \quad u \leq F_\eta(x), \tag{1.1}$$

denoting $F_\eta(y-)$ the left-hand limit of F_η at $y \in \mathbb{R}$,

$$F_\eta(x-) < u \leq F_\eta(x) \implies x = F_\eta^{-1}(u) \quad \text{and} \quad F_\eta(F_\eta^{-1}(u)-) \leq u \leq F_\eta(F_\eta^{-1}(u)); \tag{1.2}$$

(c) For $\eta(dx)$-almost every $x \in \mathbb{R}$,

$$F_\eta(x) > 0,\ F_\eta(x-) < 1, \quad \text{and} \quad F_\eta^{-1}(F_\eta(x)) = x; \tag{1.3}$$

(d) By the inverse transform sampling, for $f : \mathbb{R} \to \mathbb{R}$ measurable and bounded,

$$\int_{\mathbb{R}} f(x)\, \eta(dx) = \int_0^1 f(F_\eta^{-1}(v))\, dv \tag{1.4}$$

These properties facilitate computations involving probability measures on the real line, particularly in the context of the following.

Hoeffding-Fréchet coupling Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ and $X \sim \mu$ and $Y \sim \nu$ be two random variables. By the properties of quantile function above, we can find two random variables U and V uniformly on $(0,1)$ almost surely that

$$(X, Y) = \left(F_\mu^{-1}(U), F_\nu^{-1}(V)\right).$$

According to Sklar's theorem [210], the joint distribution can also be represented as the product of the copula and the marginals, i.e. the joint distribution of (X, Y) is equivalent to the joint distribution of (U, V). The famous *Hoeffding-Fréchet coupling*, also known as the *co-monotonic coupling*, is defined by taking $U = V$, i.e.

$$(X, Y) = \left(F_\mu^{-1}(U), F_\nu^{-1}(U)\right).$$

where U is uniformly distributed on $(0,1)$.

This coupling aligns the quantile functions of μ and ν, and it has intrinsic connections to Wasserstein distances for $d = 1$. In fact, the Hoeffding-Fréchet coupling is optimal for $W_\rho(\mu, \nu)$, independently of ρ:

$$W_\rho(\mu, \nu) = \left(\int_0^1 \left|F_\nu^{-1}(u) - F_\mu^{-1}(u)\right|^\rho du\right)^{1/\rho}, \qquad \forall\, \rho \in [1, +\infty).$$

When $\rho > 1$, the Hoeffding-Fréchet coupling is the unique optimal coupling for $W_\rho(\mu, \nu)$, however, for $\rho = 1$, uniqueness does not hold. In this case, $W_1(\mu, \nu)$ corresponds to the area between the graphs of F_μ^{-1} and F_ν^{-1}, and therefore between the graphs on F_μ and F_ν:

$$W_1(\mu, \nu) = \int_{\mathbb{R}} \left| F_\nu(x) - F_\mu(x) \right| dx.$$

1.2.2 Martingale optimal transport

1.2.2.1 Definition

The *Martingale Optimal Transport* (MOT) problem was first introduced by Beiglböck, Henry-Labordère, and Penkner [38] in the discrete case, later on Galichon, Henry-Labordère, and Touzi [110] presented in continuous time setting. It plays an essential role in robust finance, where it is used to establish upper and lower bounds for derivative prices without assuming a specific underlying model. Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$, the problem consist with

$$\mathrm{MOT}(\mu, \nu) := \inf_{\pi \in \Pi_M(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y)\, \pi(dx, dy), \tag{MOT}$$

where Π_M denotes the set of martingale couplings between μ and ν :

$$\Pi_M(\mu, \nu) = \left\{\pi(dx, dy) = \mu(dx)\pi_x(dy) \in \Pi(\mu, \nu) \mid \mu(dx)\text{-a.e.}, \int_{y \in \mathbb{R}^d} y\, \pi_x(dy) = x\right\}.$$

By Strassen's theorem [216],

$$\Pi_M(\mu,\nu) \neq \emptyset \quad \Longleftrightarrow \quad \mu \leq_{cx} \nu,$$

where $\mu \leq_{cx} \nu$ means that μ is smaller than ν in the convex order :

$$\forall\, \varphi : \mathbb{R} \to \mathbb{R} \text{ convex}, \qquad \int_{\mathbb{R}} \varphi(x)\, \mu(dx) \leq \int_{\mathbb{R}} \varphi(y)\, \nu(dy).$$

The continuous time setting was first introduced by Henry-Labordère and Touzi [110] to solve an super-hedging problem, which was classically solved through a Skorokhod embedding. The latter, formulated by Skorokhod in 1961 [211, 212], consisted in determining for a centered probability measure μ of finite second-order moment and a Brownian motion B, an integrable stopping time τ such that $B_\tau \sim \mu$. Obłój [180] provided good overview of the development this problem of up to 2004. The link between the two problems is explained by Guo, Tan, and Touzi [124, 125, 126] and Beiglböck et al. [39, 37].

The stability question for MOT was first raised by Alfonsi, Corbetta, and Jourdain [11] and later on by Guo and Obłój [108]. Nicolas Juillet [152] has established the "left-curtain" couplings. In 2019, Backhoff-Veraguas and Pammer [22] confirmed stability in dimension one under fairly general conditions, which was closely followed by an independent work by Wiesel [227]. Backhoff-Veraguas and Pammer demonstrated stability for continuous costs with at most linear growth by leveraging the equivalence between martingale monotonicity and optimality, a martingale analogue to a result previously established by Beiglböck and Juillet [35]. On the other hand, Johannes Wiesel [227] tackled the primal formulation directly, introducing the concept of martingale rearrangement coupling. This concept was further studied by Jourdain and Margheriti [147]. The structure of martingale optimal transport couplings has also been investigated extensively by [35, 133, 132, 66, 145, 94, 115].

For a detailed study of the application of the martingale optimal transport problem in finance, one can refer to the book by [134].

1.2.2.2 Duality

The problem (MOT) is referred as the primal problem in martingale optimal transport. Analogous to the dual problem (DP) in the classical OT framework, we can consider a dual approach that incorporates a martingale constraint. This martingale dual problem is associated with the formulation of sub-hedging strategies in mathematical finance.

Recall the setting from Section 1.1.1: we consider a market with d financial assets, and the interest rate is assumed to be zero. Under the risk-neutral probability measure $\mathbb{Q}$, we seek a sub-hedging strategy for options with payoff $c(S_{T_1}, S_{T_2})$. Specifically, we consider Δ-hedging strategies of the form $h(S_{T_1}) \cdot (S_{T_2} - S_{T_1})$, where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function. Thus, we consider options with payoffs of the form:

$$\varphi(S_{T_1}) + \psi(S_{T_2}) + h(S_{T_1}) \cdot (S_{T_2} - S_{T_1}),$$

which provide a sub-hedge if this total payoff is always less than or equal to $c(S_{T_1}, S_{T_2})$.

According to Theorem 1.1.2, the price process (S_{T_1}, S_{T_2}) is a martingale under $\mathbb{Q}$. This implies that the Δ-hedging term $h(S_{T_1}) \cdot (S_{T_2} - S_{T_1})$ has zero cost, as it represents gains from trading the underlying assets without any net investment. However, the options with payoffs $\varphi(S_{T_1})$ and $\psi(S_{T_2})$ have prices equal to their expected values $\int \varphi(x)\,\mu(dx)$ and $\int \psi(y)\,\nu(dy)$ respectively under $\mathbb{Q}$.

Our goal is to determine a sub-hedging strategy that maximizes the total price of the hedging portfolio while ensuring that the payoff does not exceed $c(S_{T_1}, S_{T_2})$. This naturally leads to the dual formulation of the martingale optimal transport problem: given two probability measures μ and ν on $\mathbb{R}^d$ (with $d \in \mathbb{N}$) and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

$$\mathrm{DPM}(\mu,\nu) := \sup_{(\varphi,\psi,h)} \left\{ \int_{\mathbb{R}^d} \varphi(x)\,\mu(dx) + \int_{\mathbb{R}^d} \psi(y)\,\nu(dy) \,\middle|\, \varphi \oplus \psi(x,y) + h(x)\cdot(y-x) \le c(x,y) \right\}, \tag{DPM}$$

where $(\varphi, \psi, h) \in L^1(\mu) \times L^1(\nu) \times C_b^0(\mathbb{R}^d; \mathbb{R}^d)$, and $C_b^0(\cdot)$ denotes the set of bounded continuous functions. [38] showed that the duality gap in dimension 1 is zero.

Theorem 1.2.4. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$ be such that $\mu \le_{cx} \nu$ and let $c : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that there exists a constant $K \in \mathbb{R}$ satisfying*

$$\max\{c(x,y), 0\} \le K(1 + x + y), \qquad \forall\,(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Then, (MOT)=(DPM).

In the framework of Theorem 1.2.4, the supremum of (DPM) is not necessarily attained, there is a counterexample by [38]; however, it can be achieved with the quasi-sure formulation by [42].

1.2.3 Weak optimal transport

The optimal transport problem with weak costs was first introduced by Gozlan, Roberto, Samson, and Tetali [121] and has since been developed further by various researchers (see [119, 27, 120, 24, 13, 207]).

For a coupling $\pi \in \Pi(\mu,\nu)$, let $(\pi_x)_{x\in E}$ be a regular disintegration of π with respect to the first marginal μ. Given $\pi \in \Pi(\mu,\nu)$ (resp. $\pi \in \Pi_M(\mu,\nu)$), it is well known (see [95] pages 78–80 or [192] page 117) that there exists a $\mu(dx)$-a.e. unique Markov kernel (resp. martingale Markov kernel) π_x such that $\pi(dx,dy) = \mu(dx)\pi_x(dy)$.

We consider cost functionals of the form:

$$C : E \times \mathcal{P}(F) \to \mathbb{R} \cup \{+\infty\},$$

where C is lower bounded and lower semi-continuous, and for every $x \in E$, the mapping $C(x,\cdot)$ is convex on $\mathcal{P}(F)$. Then, the *Weak Optimal Transport* (WOT) problem is to determine

$$V_C(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_E C(x,\pi_x)\mu(dx). \tag{WOT}$$

Note that the classical optimal transport problem is a special case of the weak transport problem when the cost functional is linear in the second variable and given by $C(x,p) = \int c(x,y)\,p(dy)$ for some measurable cost $c : E \times F \to \mathbb{R}\cup\{+\infty\}$. Fundamental results in classical optimal transport theory such as the existence of optimal plans, duality principles, and the cyclical monotonicity of optimizers (for instance, see [205]), have been extended to the weak transport setting through various contributions [24, 27]. From the perspective of financial applications, Acciaio, Beiglböck, and Pammer [3] show that model-independent pricing problems in fixed-income markets lead to a WOT problem.

Motivated by problems in finance that naturally involve weak formulations of the martingale optimal transport problem (MOT), Backhoff-Veraguas and Pammer [22] incorporate martingale constraints into the weak optimal transport framework. For a given cost function $C : \mathbb{R}\times\mathcal{P}(\mathbb{R}) \to \mathbb{R}_+$, the *Weak Martingale Optimal Transport* (WMOT) problem is defined as

$$V_C^M(\mu,\nu) := \inf_{\pi\in\Pi_M(\mu,\nu)} \int_{\mathbb{R}} C(x,\pi_x)\,\mu(dx). \tag{WMOT}$$

The WMOT problem can be viewed as a special case of the WOT problem where the cost function $C(x,p)$ is set to $+\infty$ whenever the barycenter (mean) of p does not equal x, thereby enforcing the martingale constraint within the weak transport framework. Recent contributions have advanced the theoretical understanding and practical applications of WMOT; see [34, 40, 149].

Benamou-Brenier formulation and its martingale counterparts Given two probability measures $\mu,\nu \in \mathcal{P}(\mathbb{R}^d)$ with finite second moment, the Benamou-Brenier formulation [144] expresses the squared Wasserstein-2 distance as the minimal kinetic energy required to transport μ to ν over time:

$$T_2(\mu,\nu) := \inf_{\substack{X_t = X_0 + \int_0^t v_s\,ds \\ X_0\sim\mu,\, X_1\sim\nu}} \mathbb{E}\left[\int_0^1 |v_t|^2 dt\right], \tag{BB}$$

where $(X_t)_{t\in[0,1]}$ represents a stochastic process transporting μ to ν, and v_t is the velocity field. According to [59], if μ is absolutely continuous with respect to the Lebesgue measure, the problem (BB) has a unique optimizer $(X^*)_{t\in[0,1]}$.

Similarly, when $\mu \le_{cx} \nu$, (i.e., ν dominates μ in convex order), the martingale version of the Benamou-Brenier formulation can be considered. Denoting by B a Brownian motion on $\mathbb{R}^d$, the martingale Benamou-Brenier problem is defined as

$$MT(\mu,\nu) := \sup_{\substack{M_t = M_0 + \int_0^t \sigma_s\,ds \\ M_0\sim\mu,\, M_1\sim\nu}} \mathbb{E}\left[\int_0^1 \mathrm{tr}(\sigma_t) dt\right], \tag{MBB}$$

where σ is an $\mathbb{R}^{d\times d}$-valued $\mathcal{F}$-progressive process over a filtered probability spaces $(\Omega,\mathcal{F},\mathbb{P})$

and B a d-dimensional $\mathcal{F}$-Brownian motion, ensuring that M is a martingale. When ν has finite second moments, the (MBB) problem has a unique maximizer $(M^*)_{t\in[0,1]}$, known as the stretched Brownian motion from μ to ν [23, Theorem 1.5], since it is the martingale satisfying $M_0^* \sim \mu$ and $M_1^* \sim \nu$ which correlates the most with the Brownian motion.

At first glance, the optimization problems (BB) and (MBB) appear quite different. However, they can be related through appropriate formulations. In Section 6 of [23], it has been established that

$$X^* = \arg \min_{X_0\sim\mu,\, X_1\sim\nu} W^2(X, \text{constant-speed particle}),$$
$$M^* = \arg \min_{M_0\sim\mu,\, M_1\sim\nu} W_c^2(M, \text{constant-volatility martingale}),$$

where W^2 denotes the Wasserstein-2 distance with respect to the squared Cameron-Martin norm, and W_c^2 denotes an adapted or causal analogue (in the terminology of [167]). This connection highlights how weak transport methods can be employed to relate classical optimal transport (as in Benamou-Brenier) with its martingale counterparts, by considering adapted processes and causal transport plans.

Let us now consider a specific cost function $C_2 : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ defined by $C_2(x,p) = W_2^2(p, \mathcal{N}(0,1))$ for all $(x,p) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, where $\mathcal{N}(0,1)$ denotes the uni-dimensional standard normal distribution. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$, and let $V_{C_2}^M(\mu,\nu)$ be the value function given by (WMOT) with the cost function C_2.

Let $\pi^* \in \Pi_M(\mu,\nu)$ be optimal for $V_{C_2}^M(\mu,\nu)$, and let M^* be the stretched Brownian motion from μ to ν. Then, by Remark 2.1, Theorem 2.2, and Remark 2.3 from [23], the following results illustrate the connections between the (WMOT) problem and the (MBB) formulation:

(*a*) $MT(\mu,\nu) = \frac{1}{2}\left(1 + \int_{\mathbb{R}} |y|^2 \nu(dy) - V_{C_2}^M(\mu,\nu)\right)$

(*b*) π^* is the joint probability distribution of (M_0^*, M_1^*), and conversely

$$M_t^* = \mathbb{E}\left[F_{\pi_X^*}^{-1}\left(F_{\mathcal{N}(0,1)}(B_1)\right) \,\middle|\, X, (B_s)_{0\leq s\leq t}\right],$$

where $X \sim \mu$ is a random variable independent of the Brownian motion $(B_t)_{t\in[0,1]}$.

For financial applications, one can refer to the calibration of the Bass local volatility model studied by Conze and Henry-Labordère [81] and Acciaio, Marini, and Pammer [6].

1.2.4 Numerical methods for XOT

In this section, we present numerical methods for a variety of *Optimal Transport* (OT) problems, each distinguished by a specific characteristic or constraint. The "X" is a variable that represents a particular type of optimal transport problem. For example, MOT refers to *Martingale Optimal Transport*, WOT represents *Weak Optimal Transport*, and "EOT" stands for Entropic Optimal Transport, which is mentioned briefly. A growing area known as *Causal Optimal Transport* (COT) has also attracted considerable attention, with numerous numerical methods emerging in this field, for instance, see Acciaio et al. [2, 230, 229, 5]).

We begin by discussing numerical methods for the classical OT problem, followed by dedicated subsections addressing the specific nuances and computational approaches for MOT, and finally under the weak framework, we present WOT and WMOT.

1.2.4.1 Numerical methods for OT

Since the Monge-Kantorovich problem is stable with respect to its marginals (as established by Theorem 1.2.3), we can approximate arbitrary probability measures μ and ν by probability measures $\hat{\mu}$ and $\hat{\nu}$ with finite support. By solving the Kantorovich problem $\mathrm{KP}(\hat{\mu}, \hat{\nu})$ for these approximations, we can as approximate the solution to $\mathrm{KP}(\mu, \nu)$ in the limit as $\hat{\mu}$ and $\hat{\nu}$ converge to μ and ν respectively. Let $\hat{\mu}$ and $\hat{\nu}$ be measures with finite support

$$\hat{\mu} = \sum_{i=1}^{I} \mu_i \, \delta_{x_i}, \quad \hat{\nu} = \sum_{j=1}^{J} \nu_j \, \delta_{y_j}, \tag{1.5}$$

where $\mu_i, \nu_j \in [0,1]$ and $\sum_{i=1}^{I} \mu_i = \sum_{j=1}^{J} \nu_j = 1$. Any coupling π of $\hat{\mu}$ and $\hat{\nu}$ can be expressed as

$$\pi = \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} \, \delta_{(x_i, y_j)}, \quad \pi_{i,j} \geq 0, \quad \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} = 1,$$
$$\text{s.t.} \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i \quad \text{for } i = 1, \ldots, I, \quad \text{and} \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j \quad \text{for } j = 1, \ldots, J.$$

Adding the marginal constrains, the $\mathrm{KP}(\hat{\mu}, \hat{\nu})$ problem reduces to minimizing

$$\min_{(\pi_{i,j})} \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} \, c(x_i, y_j) \tag{1.6}$$

subject to the constraints $\pi_{i,j} \geq 0$ and the marginal conditions above. This is a linear programming problem over the variables $\pi_{i,j}$.

In 1939, Kantorovich and his student Gavurin proposed a potential method to solve the OT problem [114]. Additionally, classical algorithms from operations research, including the simplex algorithm [91], the Hungarian algorithm [160], the interior point method [154], and the auction algorithm [49, 51], can also be employed. However, these methods are often computationally prohibitive in terms of both time and memory, making them unsuitable for large-scale practical applications, such as image analysis, which involves processing millions of pixels [213, 215].

One approach to address the computational challenges of optimal transport is to introduce

an entropic regularization term to the objective function :

$$\inf_{\pi_{i,j}} \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{i,j}\, c(x_i, y_j) + \lambda \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{i,j} \ln(\pi_{i,j} - 1)$$
$$\text{s.t.} \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i \quad \text{for } i = 1, \dots, I, \quad \text{and} \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j \quad \text{for } j = 1, \dots, J,$$

where $\lambda > 0$ is a regularization parameter. Under certain conditions, it has been demonstrated that the solution of the entropy regularized problem converges to that of the original optimal transport problem[80, 172, 70]. This regularization has enabled the use of efficient algorithms, such as the Sinkhorn algorithm [208, 209], which significantly improves computational performance, especially for large-scale problems [90, 82, 43, 191].

There are some cost functions with special structures that are widely studied, such as the Wasserstein distance of order 2. When the cost function $c(x, y) = \frac{1}{2}|x - y|^2$, the optimal transport map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T_{\#}\mu = \nu$ can be expressed as the gradient of a convex potential function u, i.e. $T(x) = \nabla u(x)$. The potential function u satisfies the Monge-Ampère equation [129]:

$$\det\left(D^2 u(x)\right) = \frac{\mu(x)}{\nu(\nabla u(x))},$$

which is a degenerate and nonlinear elliptic equation. This equation can be solved using finite difference methods [107, 45, 46, 47].

Alternatively, the Benamou-Brenier formulation [144, 43] that we mentioned in Section 1.2.3 provides a dynamic characterization of optimal transport between two probability densities ρ_0 and ρ_1 on $\mathbb{R}^d$. It expresses the squared Wasserstein-2 distance as the minimal action required to transport ρ_0 to ρ_1:

$$W_2^2(\rho_0, \rho_1) = \inf_{\rho_t,\, v_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} \rho_t(x) |v_t(x)|^2 \, dx \, dt \right\},$$

subject to the continuity equation:

$$\partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0, \quad \text{for } t \in [0, 1],$$

where $\rho_0 = \mu$ and $\rho_1 = \nu$. In this formulation, $\rho_t(x)$ represents a time-dependent probability density, and $v_t(x)$ is the velocity field that transports ρ_0 to ρ_1 over the time $t \in [0, 1]$. Discretizing the Benamou–Brenier formulation yields a convex quadratic program whose structure can be efficiently exploited by modern solvers such as augmented Lagrangian, primal–dual splitting, or Sinkhorn methods etc.

1.2.4.2 Numerical methods for MOT

By adding the martingale constraint to (1.6), solving the MOT problem between two measures with finite support, $\hat{\mu} = \sum_{i=1}^{I} \mu_i \delta_{x_i}$ and $\hat{\nu} = \sum_{j=1}^{J} \nu_j \delta_{y_j}$ reduces to the following:

$$\inf_{\pi_{i,j}} \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{i,j}\, c(x_i, y_j),$$
$$\text{s.t. } \pi_{i,j} \geq 0, \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i, \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j, \quad \sum_{j=1}^{J} \pi_{i,j}\, y_j = \mu_i\, x_i.$$

Here, the last constraint $\sum_{j=1}^{J} \pi_{i,j}\, y_j = \mu_i\, x_i$ enforces the martingale property. Note that this formulation is a linear programming problem.

However, introducing the martingale constraint poses a significant challenge in sampling while preserving convex order. In practice, empirical measures obtained from independent and identically distributed (i.i.d.) samples often do not share the same mean and therefore generally fail to preserve convex order. Alfonsi, Corbetta, and Jourdain addressed this issue in [11, 12].

Sampling of one-dimensional probability measures in the convex order Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. We consider $(X_i)_{1\leq i\leq I}$ i.i.d with distribution μ and $(Y_j)_{1\leq j\leq J}$ i.i.d with distribution ν. The empirical means $\bar{X}_I = \frac{1}{I}\sum_{i=1}^{I} X_i$ and $\bar{Y}_J = \frac{1}{J}\sum_{j=1}^{J} Y_j$ are distinct in general and $\bar{X}_I \leq_{cx} \bar{Y}_J$ is not preserved. However, $\bar{X}_I$ and $\bar{Y}_J$ converge almost surely to $\int_{\mathbb{R}} x\,\mu(dx)$ and $\int_{\mathbb{R}} y\,\nu(dy)$ respectively as $I, J \to +\infty$. Following [11, Section 3], we can slightly modify μ_I and ν_J such that

$$\hat{\mu}_I = \frac{1}{I}\sum_{i=1}^{I} \delta_{X_i - \bar{X}_I + \int_{\mathbb{R}} x\,\mu(dx)} \quad \text{and} \quad \hat{\nu}_J = \frac{1}{J}\sum_{j=1}^{J} \delta_{Y_j - \bar{Y}_J + \int_{\mathbb{R}} y\,\nu(dy)},$$

then, $\hat{\mu}_I$ and $\hat{\nu}_J$ have the same mean. We denote $\mu \vee \nu$ (resp. $\mu \wedge \nu$) as the supremum (resp. infimum) of μ and ν for the convex order. This way, $\mu \vee \nu$ (resp. $\mu \wedge \nu$) is greater than μ (resp. smaller than ν) in the convex order.

It is shown in [11] that when $\mu \leq_{cx} \nu$, after adjusting the mean, $\hat{\mu}_I \vee \hat{\nu}_J$ and $\hat{\mu}_I \wedge \hat{\nu}_J$ converge weakly to ν and μ respectively as $I, J \to +\infty$ and $\hat{\mu}_I \leq_{cx} (\hat{\mu}_I \vee \hat{\nu}_J)$. Then, applying [11, Algorithm 1], we compute $\hat{\mu}_I \vee \hat{\nu}_J$. This technique is also applied to the approximation of the (WMOT) problem.

Several groups of researchers have contributed to the study of numerical methods for the MOT problem. Hadrien De March [93] proposed an algorithm based on the Newton method combined with entropic regularization. More recently, Eckstein and Kupper [100] introduced a unified neural network-based method to solve a broad class of problems, including OT and MOT. However, their algorithm can be unstable for large regularization parameters, potentially failing to converge, as shown by Pierre Henry-Labordère [131], who also introduced a neural network method for a primal-dual algorithm with cost functions satisfying the Spence-Mirrlees condition. Additionally, Tan and Touzi [218] addressed a continuous semi-martingale version of the OT problem.

1.2.4.3 Numerical methods under the weak framework

We retake the discretization of $\hat{\mu}$ and $\hat{\nu}$ as in (1.5) above, denote $p_{j|i} = \frac{\pi_{i,j}}{\mu_i}$ the conditional probability of y_j given x_i. Then the (WOT) problem becomes:

$$\begin{aligned} \min \quad & \sum_{i=1}^{I} \mu_i \, C\left(x_i, \sum_{j=1}^{J} p_{j|i} \, \delta_{y_j}\right) \\ \text{s.t.} \quad & p_{j|i} \geq 0, \quad \sum_{j=1}^{J} p_{j|i} = \mathbf{1}, \quad \sum_{i=1}^{I} \mu_i \, p_{j|i} = \nu_j. \end{aligned}$$

Similarly, the discretized version of the (WMOT) problem becomes:

$$\begin{aligned} \min \quad & \sum_{i=1}^{I} \mu_i \, C\left(x_i, \sum_{j=1}^{J} p_{j|i} \, \delta_{y_j}\right) \\ \text{s.t.} \quad & p_{j|i} \geq 0, \quad \sum_{j=1}^{J} p_{j|i} = \mathbf{1}, \quad \text{and} \quad \sum_{i=1}^{I} \mu_i \, p_{j|i} = \nu_j, \quad \sum_{j=1}^{J} p_{j|i} \, y_j = x_i. \end{aligned}$$

Recent advancements in the numerical methods for WOT have a significant focus on the barycenter problem. There are several algorithms developed to compute weak barycenter such as mirror descent [187], stochastic gradient descent in Wasserstein space [26], and adversarial learning objectives [158], with applications in image analysis [77]. Beyond the barycenter problem, [161] studied convex risk measures with WOT penalties, applying neural network-based computational methods for financial applications. Meanwhile, [157] focused on statistical testing and implemented an entropic Frank-Wolfe algorithm for computation for Wasserstein projections.

Simultaneously and independently, we develop a parallel approach by applying the Frank-Wolfe algorithm to solve the WOT and WMOT problems within a more general framework, without restricting ourselves to specific cost functions. A summary of our methodology is presented in our contribution; see Section 1.2.5.3. For a comprehensive exposition, we direct the interested reader to Chapter 5.

1.2.5 Our contributions

In the first part of this thesis, we focus on three distinct projects within the MOT framework: the Maximal Martingale Wasserstein Inequality in Chapter 3, Non-decreasing martingale couplings in Chapter 4, and numerical methods of weak optimal transport in Chapter 5.

1.2.5.1 Chapter 3 Maximal Martingale Wasserstein Inequality

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. Define

$$\underline{\mathcal{M}}_{\rho}^{\rho}(\mu, \nu) = \inf_{M \in \Pi^{\mathrm{M}}(\mu,\nu)} \int_{\mathbb{R}^{2d}} |x - y|^{\rho} M(dx, dy).$$

For $d = 1$, Jourdain and Margheriti [146, Proposition 1] established the Martingale Wasserstein inequality for $\rho \in [1, +\infty)$,

$$\exists\, \underline{C}_{(\rho,\rho),1} < \infty,\ \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \ \text{ with } \ \mu \leq_{cx} \nu,\ \underline{\mathcal{M}}^\rho_\rho(\mu,\nu) \leq \underline{C}_{(\rho,\rho),1} \mathcal{W}_\rho(\mu,\nu)\sigma^{\rho-1}_\rho(\nu), \tag{1.7}$$

where the central moment $\sigma_\rho(\nu)$ of ν is defined by

$$\sigma_\rho(\nu) = \inf_{c\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |y-c|^\rho \, \nu(dy) \right)^{1/\rho} \ \text{ when } \rho \in [1, +\infty).$$

This provides a quantitative stability estimation for the MOT problem with the cost function $|x-y|^\rho$ in the limit $\mathcal{W}_\rho(\mu,\nu) \to 0$. Note that the stability of MOT problem with respect to the marginal distributions μ and ν is established in dimension $d = 1$ in [22, 227], while it fails in higher dimension according to [62].

The generalization of (1.7) to higher dimensions d is also investigated in [146] where it is proved that for any $d \geq 2$,

$$\underline{C}_{(\rho,\rho),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_\rho(\mathbb{R}^d)\\ \mu\leq_{cx}\nu, \mu\neq\nu}} \frac{\underline{\mathcal{M}}^\rho_\rho(\mu,\nu)}{\mathcal{W}_\rho(\mu,\nu)\sigma^{\rho-1}_\rho(\nu)}$$

is infinite when $\rho \in [1, \frac{1+\sqrt{5}}{2})$ by considering the counterexample to stablility of the MOT problem by Juillet and Brückerhoff [62] when $d \geq 2$. Chapter 3 answers the question of the finiteness of $\underline{C}_{(\rho,\rho),d}$ when $\rho \in [\frac{1+\sqrt{5}}{2}, +\infty)$ and $d \geq 2$, which remained open.

It turns out that $\underline{C}_{(\rho,\rho),d} = +\infty$ for $d \geq 2$ when $\rho \in [1, 2)$ while for $\rho \in [2, +\infty)$ the inequality (1.7) generalizes in any dimension d into a Maximal Martingale Wasserstein inequality with the left-hand side $\underline{\mathcal{M}}^\rho_\rho(\mu,\nu)$ replaced by the larger $\overline{\mathcal{M}}^\rho_\rho(\mu,\nu)$ defined by

$$\overline{\mathcal{M}}^\rho_\rho(\mu,\nu) = \sup_{M\in\Pi^{\mathrm{M}}(\mu,\nu)} \int_{\mathbb{R}^{2d}} |x-y|^\rho \, M(dx,dy).$$

Let us define

$$\underline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d)\\ \mu\leq_{cx}\nu, \mu\neq\nu}} \frac{\underline{\mathcal{M}}^\rho_\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)}$$

and

$$\overline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d)\\ \mu\leq_{cx}\nu, \mu\neq\nu}} \frac{\overline{\mathcal{M}}^\rho_\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)},$$

with

$$\mathcal{W}_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \pi - \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} |x-y|,$$

and

$$\sigma_\infty(\nu) = \inf_{c\in\mathbb{R}^d} \nu - \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |y-c|.$$

Since $\underline{\mathcal{M}}_\rho \leq \overline{\mathcal{M}}_\rho$, one has $\underline{C}_{(\rho,q),d} \leq \overline{C}_{(\rho,q),d}$.

Theorem 1.2.5. *(i) Let $\rho \in [1,2)$. For $q \in [1, \frac{1}{2-\rho}]$ (and even $q \in [1,+\infty]$ when $\rho = 1$), one has $\underline{C}_{(\rho,q),1} \leq K_\rho < +\infty$ where the constant K_ρ is studied in [146, Proposition 1] while, for $q \in [1,+\infty]$, $\overline{C}_{(\rho,q),1} = +\infty$ and $\underline{C}_{(\rho,q),d} = +\infty$ for $d \geq 2$.*

(ii) Let $\rho \in [2,+\infty)$ and $q \in [1,+\infty]$. One has $\overline{C}_{(\rho,q),d} < +\infty$ whatever d. Moreover, when each vector space $\mathbb{R}^d$ is endowed with the Euclidean norm, $\overline{C}_{(2,q),d} = 2$ and $\sup_{d\geq 1} \overline{C}_{(\rho,q),d} < +\infty$.

1.2.5.2 Chapter 4 Non-decreasing martingale couplings

In Chapter 4, we study the optimization problems defining $\underline{\mathcal{M}}_\rho$ when $\rho > 2$ and $\overline{\mathcal{M}}_\rho$ when $\rho \in (0,2)$ and investigate the structure of the optimal couplings.

The couplings attaining $\overline{\mathcal{M}}_1(\mu,\nu)$ were first investigated in [136]. Hobson and Neuberger [136, Theorem 8.2] state that for the cost function $|x-y|$, there exists a maximizing martingale coupling with the form

$$\pi^{\mathrm{HN}} = \int_0^1 \left(\frac{r(u)-q(u)}{r(u)-p(u)} \delta_{(q(u),p(u))} + \frac{q(u)-p(u)}{r(u)-p(u)} \delta_{(q(u),r(u))} \right) du$$

with p,q,r non-decreasing on $(0,1)$ and such that $p \leq q \leq r$. When μ is continuous, according to [35],

$$\pi^{\mathrm{HN}} = \int_{\mathbb{R}} \left(\frac{g(x)-x}{g(x)-f(x)} \delta_{(x,f(x))} + \frac{x-f(x)}{g(x)-f(x)} \delta_{(x,g(x))} \right) \mu(dx)$$

for non-decreasing functions f and g such that $\forall x \in \mathbb{R}$, $f(x) \leq x \leq g(x)$ and, according to Theorem 7.3 [35], uniqueness of maximizing martingale couplings holds.

This π^{HN} provides an example of a martingale coupling non-decreasing in the sense of the following definition:

Definition 1.2.6. *Let $\mu,\nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. A martingale coupling $\pi(dx,dy) \in \Pi_M(\mu,\nu)$ is called **non-decreasing** if there exists a Borel set $\Gamma \subseteq \mathbb{R}\times\mathbb{R}$ such that $\pi(\Gamma) = 1$:*

(a) if $(x_-,y_-),(x_+,y_+) \in \Gamma$ with $y_- \leq x_-$, $y_+ \leq x_+$, and $x_- < x_+$, then $y_- \leq y_+$

(b) if $(x_-,z_-),(x_+,z_+) \in \Gamma$ with $x_- \leq z_-$, $x_+ \leq z_+$, and $x_- < x_+$, then $z_- \leq z_+$.

In Section 4.2.1 of Chapter 4, for many choices of $\mu,\nu \in \mathcal{P}_{\rho\vee 1}(\mathbb{R})$ such that $\mu \leq_{cx} \nu$, we observed numerically that the coupling π^{HN} maximizing when $\rho = 1$, is still very close to achieving $\overline{\mathcal{M}}_\rho$ when $\rho \in (0,2)$ and $\underline{\mathcal{M}}_\rho$ when $\rho > 2$. It is not surprising that $\rho = 2$ appears as a threshold, since the martingale property ensures that when $\mu,\nu \in \mathcal{P}_2(\mathbb{R})$,

$$\forall \pi \in \Pi_M(\mu,\nu),\ \int_{\mathbb{R}^2} (x-y)^2 \pi(dx,dy) = \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx) = \underline{\mathcal{M}}_2^2(\mu,\nu) = \overline{\mathcal{M}}_2^2(\mu,\nu).$$

Our contributions in Chapter 4 are divided into two main aspects: we investigate the optimality of non-decreasing martingale couplings and we introduce directionally decomposed non-decreasing martingale couplings. Note that, in this context, the optimality of non-decreasing martingale couplings means that they attain $\overline{\mathcal{M}}_\rho$ when $\rho \in (0,2)$ and $\underline{\mathcal{M}}_\rho$ when $\rho > 2$.

First, we study the preservation of optimality of π^{HN} and exhibit rather restrictive sufficient conditions. We also provide examples of pairs (μ,ν) where π^{HN} is not optimal. Second, we establish a one-to-one correspondence between certain decompositions of ν into three parts and martingale couplings that are non-decreasing in a generalized sense.

The optimality results of non-decreasing martingale couplings:

With the complete lattice structure of $\{\eta \in \mathcal{P}_1(\mathbb{R}) : \int_{\mathbb{R}} z\eta(dz) = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)\}$ for the convex order stated in [156], we deduce

Proposition 1.2.7. *Let $\mu,\nu \in \mathcal{P}_2(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. Then, the set*

$$\left\{|x-y|^2\#\pi : \pi \in \Pi_M(\mu,\nu)\right\}$$

admits an infimum for the convex order.

When there exists $\underline{\pi} \in \Pi_M(\mu,\nu)$ such that $|x-y|^2\#\underline{\pi}$ is equal to the infimum, then $\int_{\mathbb{R}^2} |x-y|^\rho \underline{\pi}(dx,dy)$ is equal to $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho \in (0,2)$ and to $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho > 2$. In particular, when

$$\nu(dy) = \int_{x\in\mathbb{R}} \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx)$$

for some $a \in \mathbb{R}$, the infimum is equal to δ_{a^2} and is attained with

$$\underline{\pi}(dx,dy) = \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx).$$

When μ is continous, the uniqueness of the maximizing and minimizing couplings for $\rho = 1$ stated in [35, Theorems 7.3] implies that π^{HN} is the only possible value for $\underline{\pi}$.

Proposition 1.2.8. *Let $\mu,\nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be some increasing concave function. If $\tilde{\pi} \in \Pi_M(\mu,\nu)$ maximizes $\int \varphi(|x-y|)\pi(dx,dy)$ over $\pi \in \Pi_M(\mu,\nu)$, then $\tilde{\pi}$ is non-decreasing and, when φ is continuous, there exists such a maximizing coupling.*

Remark 1.2.9. *For $\rho \in (0,1]$, since $\mathbb{R}_+ \ni z \mapsto z^\rho$ is concave and increasing, a martingale coupling that maximizes $\int |x-y|^\rho \pi(dx,dy)$ is non-decreasing.*

Unfortunately, even if we are able to exhibit in Proposition 4.2.16 a simple situation where the infimum is attained by $|x-y|^2\#\underline{\pi}$, we also provide examples where they are not attained (see Proposition 4.2.10 and Proposition 4.2.18 where some of the above consequences of attainment do not hold).

The following proposition exhibits that the optimality of π^{HN} holds under rather restrictive sufficient conditions:

Proposition 1.2.10. *Let $\mu \leq_{cx} \nu$ be such that there exist $\underline{y} < \overline{y} < \underline{x} < \overline{x} < \underline{z} < \overline{z}$ with $\mu([\underline{x},\overline{x}]) = 1$ and $\nu\left([\underline{y},\overline{y}] \cup [\underline{z},\overline{z}]\right) = 1$. If*

$$\underline{x} - \overline{y} \geq \alpha_\rho(\overline{z} - \overline{y}),$$
$$\underline{z} - \overline{x} \geq \alpha_\rho(\underline{z} - \underline{y}),$$

where $\alpha_\rho \in (0, \frac{1}{2})$ *is the unique solution of* $\psi_\rho(\alpha) = 0$ *with* $\psi_\rho : (0,1] \to \mathbb{R}$ *defined by*

$$\psi_\rho(\alpha) = \alpha + \alpha^{2-\rho}(1-\alpha)^{\rho-1} + 1 - \rho.$$

Then, the unique non-decreasing coupling $\pi^\uparrow$ *in* $\Pi_M(\mu,\nu)$ *is the unique optimal coupling in* $\Pi_M(\mu,\nu)$ *that attains* $\overline{\mathcal{M}}_\rho(\mu,\nu)$ *when* $\rho \in (1,2)$ *and* $\underline{\mathcal{M}}_\rho(\mu,\nu)$ *when* $\rho > 2$.

The properties of the parameter α_ρ introduced in Proposition 1.2.10 are further analyzed in Lemma 4.3.9. Moreover, Fig. 1.1 illustrates the dependence of α_ρ on ρ:

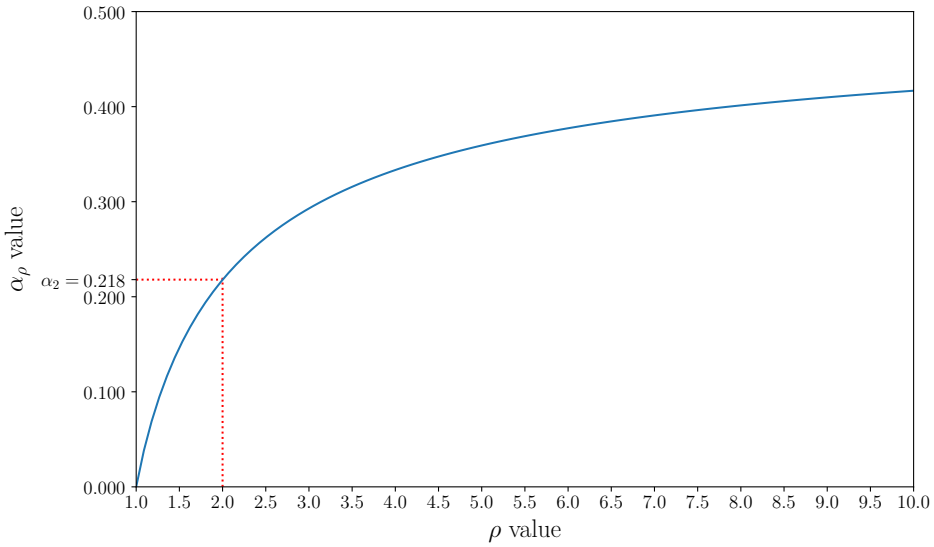


Figure 1.1: The function $\rho \mapsto \alpha_\rho$ (with α_ρ computed by a root finding algorithm).

Directionally decomposed non-decreasing martingale couplings:

Since we believe that non-attainment of the infimum for the convex order is more generic, we directly study couplings that attain $\overline{\mathcal{M}}^\rho_\rho(\mu,\nu)$ and $\underline{\mathcal{M}}^\rho_\rho(\mu,\nu)$. We introduce the following decomposition of ν into three parts:

$$\begin{aligned}
\nu_l^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y<x\}}\pi(dx,dy),\\
\nu_0^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y=x\}}\pi(dx,dy),\\
\nu_r^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y>x\}}\pi(dx,dy).
\end{aligned}$$

Note that $\nu_l^\pi + \nu_0^\pi + \nu_r^\pi = \nu$. For (ν_l, ν_r) a couple of non-negative measures such that $\nu_l + \nu_r \leq \nu$, we denote

$$\Pi_M(\mu,\nu,\nu_l,\nu_r) = \{\pi \in \Pi_M(\mu,\nu) : (\nu_l^\pi, \nu_0^\pi, \nu_r^\pi) = (\nu_l, \nu - \nu_l - \nu_r, \nu_r)\}.$$

Theorem 1.2.11. *Let* $\mu,\nu \in \mathcal{P}_1(\mathbb{R})$ *be such that* $\mu \leq_{cx} \nu$ *and* $\mu \neq \nu$. *Let* (ν_l,ν_r) *a couple of*

non-negative measures such that $\nu_l + \nu_r = \nu$. *We have*

$$\Pi_M(\mu,\nu,\nu_l,\nu_r) \neq \emptyset \Longleftrightarrow \exists!\, \pi^{\uparrow} \in \Pi_M(\mu,\nu,\nu_l,\nu_r) \quad \textit{non-decreasing},$$

and then, for each continuous increasing concave function $\varphi : \mathbb{R}_+ \to \mathbb{R}$,

$$\forall \pi \in \Pi_M(\mu,\nu,\nu_l,\nu_r) \setminus \{\pi^{\uparrow}\},\ \int_{\mathbb{R}^2} \varphi(|y-x|)\pi(dx,dy) < \int_{\mathbb{R}^2} \varphi(|y-x|)\pi^{\uparrow}(dx,dy). \tag{1.8}$$

Corollary 1.2.12. *For* (ν_l,ν_r) *a couple of non-negative measures such that* $\nu_l+\nu_r \leq \nu$, *we have*

$$\Pi_M(\mu,\nu,\nu_l,\nu_r) \neq \emptyset \Longleftrightarrow \exists!\, \pi^{\uparrow} \in \Pi_M(\mu,\nu,\nu_l,\nu_r)$$

such that $\dfrac{\pi^{\uparrow}(dx,dy) + (\nu_l+\nu_r-\nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}$ *is non-decreasing and equation* (1.8) *above still holds.*

1.2.5.3 Chapter 5 WOT

In Chapter 5, we apply the Frank-Wolfe algorithm, also known as the conditional gradient method, to approximate the WOT and WMOT problems formulated in Section 1.2.4.3 from an optimization perspective. We provide the following two examples to illustrate this approach and demonstrate its effectiveness.

Wasserstein projection in dimension one For $\mu,\nu \in \mathcal{P}_\rho(\mathbb{R}^d)$, we consider

$$V^{\rho}_{\mathrm{WP},\rho}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y\, \pi_x(dy) \right|^{\rho} \mu(dx),$$

Alfonsi, Corbetta and Jourdain [12] introduced, for $\rho \in [1,+\infty)$, the Wasserstein projections in the convex order:

$$\mathcal{I}_\rho(\mu,\nu) := \operatorname{argmin}\left\{\mathcal{W}_\rho(\mu,\eta) : \eta \leq_{cx} \nu\right\}.$$

The quantile functions of $\mathcal{I}(\mu,\nu)$ is obtained for all $u \in (0,1)$ by:

$$F^{-1}_{\mathcal{I}(\mu,\nu)} = F^{-1}_{\mu} - \partial_- \mathrm{co}(G)(u),$$

where $G(u) := \int_0^u (F^{-1}_\mu - F^{-1}_\nu)(v)dv$, co denotes the convex hull, and ∂_- the left-hand derivative. According to [12, 24], the values of $V_{\mathrm{WP},\rho}(\mu,\nu)$ and $\mathcal{W}_\rho(\mu,\mathcal{I}(\mu,\nu))$ coincide. In the numerical experiment, we applied the Frank-Wolfe algorithm to solve the following

$$\min_{p_{j|i}} \ \sum_{i=1}^{I} \mu_i \left| x_i - \sum_{j=1}^{J} p_{j|i}\, \delta_{y_j} \right|^{\rho}$$
$$\text{s.t.} \quad p_{j|i} \geq 0, \quad \sum_{j=1}^{J} p_{j|i} = \mathbf{1}, \quad \text{and} \quad \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j, \quad \sum_{j=1}^{J} p_{j|i}\, y_j = x_i.$$

Then, we find the optimal $\pi^* = \sum_{i=1}^{I}\sum_{j=1}^{J} \mu_i\, p_{j|i}\, \delta_{(x_i,y_j)}$ that attains $V_{\mathrm{WP}}^{\rho}\left(\hat{\mu},\hat{\nu}\right)$ and make a comparison between the value of $\mathcal{I}_{\rho}\left(\hat{\mu},\hat{\nu}\right)$ and the image of μ by $x \mapsto \int_{\mathbb{R}^d} y\, \pi_x^*(dy)$, which coincide as expected.

Stretched Brownian motion Let $\mu,\nu \in \mathcal{P}_2(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. Recall from Section 1.2.3 that we define the cost function $C_2 : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ as $C_2(x,p) = W_2^2(p,\mathcal{N}(0,1))$ for all $(x,p) \in \mathbb{R}\times\mathcal{P}_2(\mathbb{R})$, where $\mathcal{N}(0,1)$ denotes the standard normal distribution. Utilizing the Hoeffding-Fréchet coupling introduced in Section 1.2.1.3, which is optimal for the Wasserstein distance $W_\rho(\mu,\nu)$, we can express $C_2(x,p)$ explicitly in terms of the respective quantile functions F_p^{-1} and F_γ^{-1} of p and γ:

$$C_2(x,p) = \mathcal{W}_2^2(p,\gamma) = \int_0^1 \left(F_p^{-1}(u) - F_\gamma^{-1}(u)\right)^2 du,$$

where $F_p^{-1}(u)$ and F_γ^{-1} are the quantile functions of p and $\mathcal{N}(0,1)$, respectively.

According to [23, Remark 2.1], there exists a direct link between the martingale Benamou-Brenier formulation $MT(\mu,\nu)$ and the weak formulation $V_{C_2}^M(\mu,\nu)$ given by:

$$V_{C_2}^M(\mu,\nu) = 1 + \int_{\mathbb{R}} |y|^2\nu(dy) - 2\, MT(\mu,\nu).$$

We applied the Frank-Wolfe algorithm to demonstrate the optimal coupling π^* that attains $V_{C_2}^M$, illustrating our results with a 3D plot in Chapter 5.

1.3 Graphon mean field games

1.3.1 Finite horizon mean field games

Mean Field Games (MFGs) have been introduced to study differential games with an infinite number of players, under indistinguishability and symmetrical interaction assumptions. This theory has been introduced by J.-M. Lasry and P-L. Lions [166], and independently by Caines, Huang, and Malhamé [140]. While MFGs correspond to noncooperative games and focus on Nash equilibria, the cooperative counterpart has been studied under the name mean field control (MFC) or optimal control of McKean-Vlasov dynamics [73, 48, 97]. The equilibrium of the MFG system can also be characterized using PDE methods [28]. For more details, we refer to Lions' lectures at Collège de France [175] and the notes by Cardaliaguet [69], as well as the books by Bensoussan, Frehse and Yam [48], Gomes, Pimentel and Voskanyan [116], Carmona and Delarue [74], and the surveys by Gomes and Saúde [117] and Başar and Zaccour [29].

A mean field game is a temporally extended decision making problem involving an infinite number of identical and anonymous players. It can be solved by focusing on the optimal policy of a representative player in response to the behavior of the entire population. Even if MFGs were introduced in continuous time and continuous state and action sets, in this thesis we will focus

on models with discrete time, discrete state and action spaces. Therefore, we start introducing MFG in the following form.

Let $\mathcal{X}$ and $\mathcal{A}$ are state finite space and action space respectively. The representative player starts the game in state $x_0 \in \mathcal{X}$ according to an initial distribution μ_0 over $\mathcal{X}$. We focus on discrete time setting with finite horizon $T > 0$, denote by $\mathcal{T} := \{0, 1, \dots, T-1\}$ and $\overline{\mathcal{T}} := \mathcal{T} \cup \{T\} = \{0, 1, \dots, T\}$. Let $P : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be a one-step transition probability kernel and let $r : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ be a reward function.

Intuitively speaking, at each time step $t \in \overline{\mathcal{T}}$, a representative player arrives at state x_{t+1} from x_t by taking an action a_t according to a policy $\pi_t(a_t|x_t)$. The sequences of population's state and policy are defined as $\mu = (\mu_t)_{t\in\overline{\mathcal{T}}}$ and $\pi = (\pi_t)_{t\in\overline{\mathcal{T}}}$ respectively. Between each time step, a representative player moves to the next state x_{t+1} based on a transition kernel $P(\cdot|x_t, a_t, \mu_t) \in \mathcal{P}(\mathcal{X})$, and receives a reward $r(x_t, a_t, \mu_t)$.

Total reward The cumulative reward function of a representative player using policy π when the mean field is μ is given by:

$$J(\pi; \mu) := \mathbb{E}\left[\sum_{t\in\mathcal{T}} r(x_t, a_t, \mu_t)\right], \tag{1.9}$$

subject to

$$x_0 \sim \mu_0, \qquad a_t \sim \pi_t(\cdot \,|\, x_t), \qquad x_{t+1} \sim P(\cdot \,|\, x_t, a_t, \mu_t), \qquad \forall\, t \in \mathcal{T}.$$

Note that in this simple case, there is no reward at time T (or in other words, the reward is 0).

Dynamics of state distributions The sequence of state distributions induced by π is defined as

$$\begin{cases} \mu^\pi_{t+1}(x) = \displaystyle\sum_{x'\in\mathcal{X}} \mu^\pi_t(x') \sum_{a\in\mathcal{A}} \pi_t(a \mid x') P(x \mid x', a, \mu^\pi_t), \\ \mu^\pi_0(x) = \mu_0(x). \end{cases}$$

This is interpreted as the evolution of the population distribution when everyone uses policy π.

Mean field and best response maps First, let us introduce the mean field map, which gives the population behavior associated to a policy, by:

$$\mathrm{MF} : \mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}} \to \mathcal{P}(\mathcal{X})^{\overline{\mathcal{T}}}, \quad \pi \mapsto \mathrm{MF}(\pi) := \mu^\pi,$$

On the other hand, given a population state distribution, the goal for a representative agent is to find the policy that maximizes their total reward. To formalize this, we define the (set-valued) best response map as:

$$\mathrm{BR} : \mathcal{P}(\mathcal{X})^{\overline{\mathcal{T}}} \to 2^{\mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}}}, \qquad \mu \mapsto \mathrm{BR}(\mu) := \operatorname*{argmax}_{\pi\in\Pi} J(\pi; \mu) \subseteq \mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}},$$

where $2^{\mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}}}$ denotes the set of all subsets of $\mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}}$. We are looking for a Nash equilibrium of the game, defined as follows.

Definition 1.3.1 (MFG Nash Equilibrium)**.** *A pair* $(\hat{\pi},\hat{\mu}) \in \mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}} \times \mathcal{P}(\mathcal{X})^{\overline{\mathcal{T}}}$ *is a Mean Field Nash equilibrium (MFNE) if it satisfies the following two conditions:*

- *Optimality of the policy:* $\hat{\pi} \in \mathrm{BR}(\hat{\mu})$*;*
- *Consistency of the mean field sequence:* $\hat{\mu} = \mathrm{MF}(\hat{\pi})$*.*

Value function Given μ, we define the value function associated to π as:

$$V_t^{\pi,\mu}(x) = \mathbb{E}_{a\sim\pi(\cdot|x)}\left[\sum_{s=t}^{T} r(x_s,a_s,\mu_s)\,\middle|\, x_t = x, a_t = a, x_{s+1}\sim P(\cdot\mid x_s,a_s,\mu_s), a_s\sim\pi_s(\cdot\mid x_s)\right],$$

and the optimal value function as:

$$V_t^{*,\mu}(x) = \max_{\pi}\ \mathbb{E}\left[\sum_{s=t}^{T} r(x_s,a_s,\mu_s)\,\middle|\, x_t = x, a_t = a, x_{s+1}\sim P(\cdot\mid x_s,a_s,\mu_s), a_s\sim\pi_s(\cdot\mid x_s)\right].$$

It satisfies the Bellman equation for all x:

$$\begin{cases} V_T^{*,\mu}(x) = 0, \\ V_t^{*,\mu}(x) = \max\limits_{a}\left\{ r(x,a,\mu_t) + \sum\limits_{x'\in\mathcal{X}} P(x'\mid x,a,\mu_t) V_{t+1}^{*,\mu}(x')\right\}, \quad \forall t = T-1, T-2,\dots,0. \end{cases}$$

Exploitability One way to measure how far a policy is from being a Nash equilibrium is the exploitability. It is a function $\mathcal{E}:\Pi\to\mathbb{R}$ defined by:

$$\mathcal{E}(\pi) := \max_{\tilde{\pi}} J(\tilde{\pi};\mu) - J(\pi;\mu). \tag{1.10}$$

Remark 1.3.2. $\mathcal{E}(\hat{\pi}) = 0$ *if and only if* $(\hat{\pi},\mathrm{MF}(\hat{\pi}))$ *is a MFNE.*

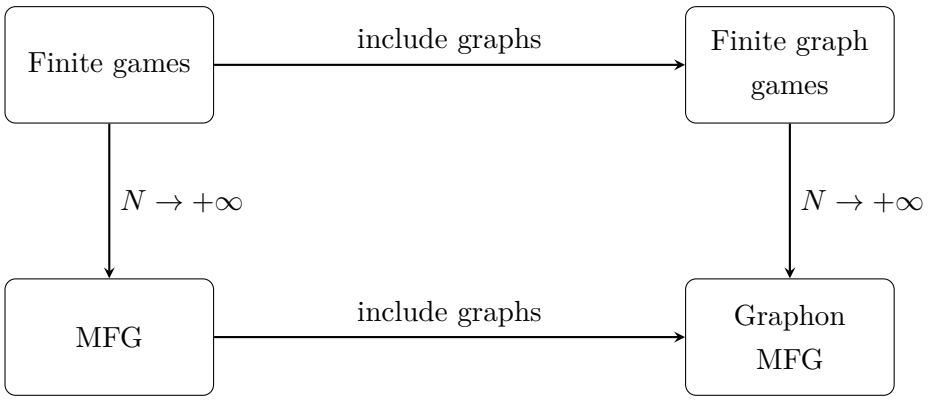
1.3.2 From finite games to graphon mean field games

The diagram below illustrates the progression from classical finite games (or N-player games) to mean field games, and from finite graph games to graphon mean field games.

- *Finite games or N-player* involve a fixed number of players, each making decisions to optimize individual objectives. When we allow a large population limit $N\to+\infty$, these finite games give rise to mean field games, where each agent interacts with the aggregate effect of the entire population.
- *Finite graph games* extend finite games by placing the players on a graph structure, so that each agent's interactions are restricted to its neighbors. In the large-population limit, the

resulting system may be described by a *graphon mean field game*, where the underlying graph is replaced by a continuous limit object known as a *graphon*.

- *Mean field games (MFGs)* can further incorporate graph structures to capture heterogeneous interactions. Transitioning to *Graphon Mean Field Games* (GMFG) involves extending MFGs by including a continuous description of the underlying network.



Thus, moving horizontally in the diagram (*"include graphs"*) adds a network structure to the problem, while moving vertically (*"$N \to +\infty$"*) converts the finite game or finite graph game into its mean field counterpart in the infinite-population limit.

Graphons Let $I := [0,1]$. A graphon is defined as a symmetric measurable function $G : I \times I \to I$, which is regarded as the limits of edge matrices of weighted graphs when the size of graphs (number of vertices) tends to infinity. We denote $\mathcal{W}$ the space of graphons.

We can also view a graphon as an operator from $L^\infty(I)$ to $L^1(I)$, associating any $\phi \in L^1(I)$ with:

$$G\phi(u) := \int_I G(u,v)\phi(v)dv.$$

If $G \in L^p(I \times I)$, we denote by $\|G\|_p = \left(\int_{I\times I} G(u,v)^p du dv\right)^{1/p}$ its $L^p(I \times I)$ norm.

The cut norm of a graphon is defined by

$$\|G\|_\square := \sup_{A,B\in\mathcal{B}(I)} \left|\int_{A\times B} G(u,v)\, du\, dv\right|.$$

For any simple graph $\mathbb{G} = (V,E)$ with sets of vertices and edges $V = \{1,2,\dots,N\}$ and $E \subseteq V \times V$ respectively, we define its corresponding step-graphon as:

$$G_N(u,v) = \sum_{i,j\in V} \mathbb{1}_{(i,j)\in E}\, \mathbb{1}_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]}\, \mathbb{1}_{v\in\left(\frac{j-1}{N},\frac{j}{N}\right]}.$$

By Lovász [177, Lemma 8.11], the resulting operator norm turns out to be equivalent to the cut norm

$$\|G\|_\square \le \|G\|_{\infty\to 1} \le 4\|G\|_\square, \quad \text{with} \quad \|G\|_{\infty\to 1} := \sup_{|\phi|\le 1} \|G\phi\|_{L^1}.$$

Finite agent graph game Let G_N be a step graphon associated with some simple graph $\mathbb{G}_N = (V_N, E_N)$. Let $\zeta_{ij}^N := \mathbb{1}_{(i,j)\in E_N}$ be a $N \times N$ interaction matrix generated by G_N which describes the interaction between players i and j. In particular, $\zeta_{ij}^N = G_N(\frac{i}{N}, \frac{j}{N})$.

We adapt the notation in Section 1.3.1. Consider finite state and action spaces $\mathcal{X}$, $\mathcal{A}$ as well as $\mathcal{T}$ and $\overline{\mathcal{T}}$. For $N \in \mathbb{N}^*$, we denote $[N] := \{1, \dots, N\}$. Let us define the set of policies for one player as $\mathbf{\Pi} := \mathcal{A}^{\mathcal{T}\times\mathcal{X}}$. A policy $\pi \in \mathbf{\Pi}$, $(t, x) \mapsto \pi_t(x) := \pi(t, x) \in \mathcal{A}$ provides an action given each state x at time t.

In the finite graph game setting, the empirical mean with graph interaction is defined as for player $i \in [N]$:

$$Z_t^i := \frac{1}{N} \sum_{j\in V_N} \zeta_{i,j} \delta_{X_t^j},$$

where the player i evolves with the following dynamic:

$$X_0^i \sim \mu_0, \quad a_t^i \sim \pi_t^i(\cdot \mid X_t^i), \quad X_{t+1}^i \sim P(\cdot \mid X_t^i, a_t^i, Z_t^i), \quad \forall\, t \in \overline{\mathcal{T}}, i \in V_N.$$

The total reward of player i in the finite agent graph game setting is defined as: for $\pi^i \in \mathbf{\Pi}$ and $\pi^{-i} \in \mathbf{\Pi}^{N-1}$,

$$\mathcal{J}^{N,i}(\pi^i; \pi^{-i}) := \mathbb{E}\left[\sum_{t\in\mathcal{T}} r(X_t^i, a_t^i, Z_t^i)\right].$$

Definition 1.3.3 (ϵ-Nash)**.** *For any $\epsilon > 0$, a tuple of policies $\hat{\boldsymbol{\pi}}^N = (\hat{\pi}^1, \dots, \hat{\pi}^N) \in \mathbf{\Pi}^N$ is an ϵ-Nash equilibrium if there exists $N' \in \mathbb{N}$ such that for all $N > N'$,*

$$\sup_{\tilde{\pi}\in\mathbf{\Pi}} \mathcal{J}^{N,i}(\tilde{\pi}; \hat{\pi}^{-i}) - \mathcal{J}^{N,i}(\hat{\boldsymbol{\pi}}^N) \le \epsilon, \qquad \forall\, i \in [N].$$

A Nash equilibrium is an ϵ-Nash equilibrium with $\epsilon = 0$.

Graphon mean field game As $N \to \infty$ limit of the N-agent game constitutes its *Graphon Mean Field Games* (GMFG). Let us first introduce the following notations:

- $\mathcal{M} := L^2(I; \mathcal{P}(\mathcal{X}))^{\overline{\mathcal{T}}}$ denotes the set of graphon mean fields.
- $\Pi := L^2(I; \mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}}$ denotes the set of graphon policies.
- $\mathcal{Z} := L^2(I)^{\overline{\mathcal{T}}}$ denotes the set of graphon aggregates.

In the GMFG setting, we consider infinitely many agents $u \in I$ instead of finitely many $i \in V_N$. A mean field $\mu \in \mathcal{P}(\mathcal{X})$, $(u, t, x) \mapsto \mu_t^u(x) := \mu(u, t, x)$ gives a probability distribution over states. A policy $\pi \in \Pi$, $(u, t, x) \mapsto \pi_t^u(x) := \pi(u, t, x) \in \mathcal{A}$ provides an action given each state x at time t and $u \in I$. The graphon mean field aggregate $Z \in \mathcal{Z}$ is defined by

$$Z_t^u := \int_{v\in I} G(u, v)\mu_t^v \, dv, \qquad \forall\, u \in I, t \in \mathcal{T}.$$

Then, the total reward of player $u \in I$ is

$$J(\pi^u; Z^u) = \mathbb{E}\left[\sum_{t\in\mathcal{T}} r(X_t^u, a_t^u, Z_t^u)\right],$$

subject to

$$X_0^u \sim \mu_0, \quad a_t^u \sim \pi_t^u(\cdot \mid X_t^u), \quad X_{t+1}^u \sim P(\cdot \mid X_t^u, a_t^u, Z_t^u), \quad \forall\, t \in \mathcal{T}$$

Similar to the mean field and best response maps in Section 1.3.1, we introduce the graphon mean field map with pure policy that for all $x \in \mathcal{X}$, $\mathrm{MF} : \Pi \to \mathcal{M},\ \pi \mapsto \mathrm{MF}(\pi) := \mu^\pi$

$$\mu_0^{\pi,u} = \mu_0, \quad \mu_{t+1}^{\pi,u}(x) = \sum_{x'\in\mathcal{X}} \mu_t^{\pi,u}(x')\, P(x \mid x', \pi_t^u(x'), Z_t^u), \qquad \forall\, t \in \mathcal{T}, u \in I$$

And the graphon best response map is defined as :

$$\mathrm{BR} : \mathcal{M} \to 2^\Pi, \qquad \mu \mapsto \mathrm{BR}(\mu) := \operatorname*{argmax}_{\pi\in\Pi} J(\pi; \mu) \subseteq \Pi.$$

1.3.3 Our contribution: extended graphon mean field games

Definition 1.3.4. *A finite-horizon, discrete time extended Graphon Mean Field Game (GMFG) is defined by a tuple* $(\overline{\mathcal{T}}, G, \mathcal{X}, \mathcal{A}, p, r, \varphi, \mu_0)$ *where*

- $\overline{\mathcal{T}} = \{0, \dots, T\}$ *is a set of discrete time steps,*
- G *is a graphon,*
- $\mathcal{X}$ *and* $\mathcal{A}$ *are state space and action space respectively, where* $\mathcal{X}$ *is finite and* $\mathcal{A}$ *is a compact subset of* $\mathbb{R}^k$ *for some integer* $k > 0$*; we endow* $\mathcal{A}$ *with the Euclidean norm, i.e.* $d_{\mathcal{A}} := \|\cdot\|$.
- $P : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathcal{P}(\mathcal{X})$ *is a one-step transition probability kernel,*
- $r : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ *is a reward function,*
- $\varphi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ *is a measurable function for the interactions,*
- $\mu_0^u := \mu_0 \in \mathcal{P}(\mathcal{X})$ *is an initial distribution for all* $u \in I$.

Given a policy $\pi \in \Pi$ and $\mu \in \mathcal{M}$, we define the aggregate $Z^u \in \mathcal{Z}$ for player $u \in I$ as :

$$Z_t^u = \int_{v\in I} G(u, v) \sum_{x\in\mathcal{X}} \varphi\left(x, \pi_t^v(x)\right)\, \mu_t^v(x)\, dv, \qquad \forall\, t \in \overline{\mathcal{T}}.$$

The total reward of player $u \in I$ in the GMFG setting is defined as:

$$J(\pi^u; Z^u) = \mathbb{E}\left[\sum_{t\in\mathcal{T}} r(X_t^u, a_t^u, Z_t^u)\right],$$

subject to

$$X_0^u \sim \mu_0, \qquad a_t^u = \pi_t^u(X_t^u), \qquad X_{t+1}^u \sim P(\cdot \mid X_t^u, a_t^u, Z_t^u), \qquad \forall\, t \in \mathcal{T}.$$

Definition 1.3.5 (GMFG Nash Equilibrium)**.** *A policy profile $\hat{\pi} \in \Pi$ is a Graphon Nash equilibrium (GNE) if, for every $u \in I$, $\hat{\pi}^u$ maximizes $J(\,\cdot\,; Z^{\hat{\pi},u})$. Alternatively, a pair $(\hat{\pi}, \hat{Z}) \in \Pi \times \mathcal{Z}$ of policy and aggregate is a (graphon MFG) Nash equilibrium if the following two conditions are satisfied:*

$$\hat{Z}^u = Z^{\hat{\pi},u} \quad \text{and} \quad J(\hat{\pi}^u; \hat{Z}^u) \geq J(\pi^u; \hat{Z}^u), \qquad \forall\, u \in I, \pi^u \in \Pi.$$

Link between Graphon Nash Equilibrium (GNE) and forward-backward system

Theorem 1.3.6. *$(\hat{\pi}, \hat{Z}) \in \Pi \times \mathcal{Z}$ is a graphon MFG Nash equilibrium if and only if it satisfies the forward-backward system in Theorem 1.3.7.*

Existence of equilibrium The following result shows that the forward-backward system admits a solution.

Theorem 1.3.7. *Let μ_0^u be a probability distribution on $\mathcal{X}$. Suppose that the Assumptions in Section 6.2.4 are satisfied. The system*

$$\begin{cases} \mu_{t+1}^u(x) = \displaystyle\sum_{x' \in \mathcal{X}} \mu_t^u(x') P(x \mid x', \pi_t^u(x'), Z_t^u), & \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T} \\ V_t^u(x) = \max_a \left\{ r(x, a, Z_t^u) + \displaystyle\sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z_t^u)\, V_{t+1}^u(x') \right\}, & \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T} \\ Z_t^u = \displaystyle\int_I G(u, v) \sum_{x \in \mathcal{X}} \varphi\left(x, \pi_t^v(x)\right)\, \mu_t^v(x)\, dv, & \forall u \in I, t \in \mathcal{T} \\ \pi_t^u(x) = \operatorname*{argmax}_a \left\{ r(x, a, Z_t^u) + \displaystyle\sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z_t^u)\, V_{t+1}^u(x') \right\}, & \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T} \\ \mu_0^u(x) = \mu_0(x),\ V_T^u(x) = 0 & \forall x \in \mathcal{X}, u \in I. \end{cases}$$

admits a bounded solution (V, μ) in $L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ such that for each $u \in I$, μ_0 is a probability distribution function on $\mathcal{X}$.

Epsilon-Nash We show that any GNE provides an ϵ-NE for the N-player graph game, and $\epsilon \to 0$ as $N \to +\infty$. Let us first introduce the model.

Definition 1.3.8. *A N-player graph model is defined by a tuple $(N, \overline{\mathcal{T}}, \mathcal{X}, \mathcal{A}, p, r, \varphi, \mu_0)$ where*

- *$N \in \mathbb{N}$ is number of players,*
- *$\overline{\mathcal{T}} = \{0, \ldots, T\}$ is a set of discrete time steps,*
- *$\mathcal{X}$ and $\mathcal{A}$ are state space and action space respectively, same as in the GMFG model,*

- $P : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathcal{P}(\mathcal{X})$ *is a one-step transition probability kernel,*
- $r : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ *is a reward function,*
- $\varphi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ *is a measurable function for the interactions,*
- $\mu_0^i := \mu_0 \in \mathcal{P}(\mathcal{X})$ *is an initial distribution for all* $i \in [N]$.

In the finite graph game setting, given a tuple of N-player policies $\boldsymbol{\pi}^N = (\pi^1, \dots, \pi^N) \in \boldsymbol{\Pi}^N$, we define the finite graph aggregate for player $i \in [N]$ as:

$$Z_t^{N,\boldsymbol{\pi}^N,i} := \sum_{j \in V_N} \zeta_{ij}^N \, \varphi(X_t^j, \pi_t^j(X_t^j)), \qquad \forall\, i \in V_N, t \in \overline{\mathcal{T}},$$

where the player i evolves with the following dynamics:

$$X_0^i \sim \mu_0, \qquad a_t^i = \pi_t^i(X_t^i), \qquad X_{t+1}^i \sim P(\cdot \,|\, X_t^i, a_t^i, Z_t^{N,\boldsymbol{\pi}^N,i}), \qquad \forall\, i \in V_N, t \in \mathcal{T}. \tag{1.11}$$

Note that the evolution of the players' states is coupled through the aggregate.

To build a connection between N-player graph game and GMFG settings, for N-player tuple $\boldsymbol{\pi}^N = (\pi^1, \dots, \pi^N) \in \boldsymbol{\Pi}^N$, we define the step policy as:

$$\pi_t^{N,\boldsymbol{\pi}^N,u}(x) := \sum_{i \in V_N} \mathbb{1}_{u \in \left(\frac{i-1}{N}, \frac{i}{N}\right]} \pi_t^i(x)\,, \qquad \forall\, x \in \mathcal{X}, u \in I, t \in \overline{\mathcal{T}}.$$

Given $\boldsymbol{\pi}^N$, let us define the distribution:

$$\mu_t^{N,\boldsymbol{\pi}^N,u} := \sum_{i \in V_N} \mathbb{1}_{u \in \left(\frac{i-1}{N}, \frac{i}{N}\right]} \delta_{X_t^i}, \qquad \forall\, i \in V_N, t \in \overline{\mathcal{T}},$$

where X^i follows the evolution in (1.11). Let us define the map $\Lambda : \Pi \to \boldsymbol{\Pi}^N$ such that for any $\pi \in \Pi$,

$$\Lambda(\pi) = (\pi^{\frac{1}{N}}, \pi^{\frac{2}{N}}, \dots, \pi^{\frac{N}{N}}).$$

Theorem 1.3.9. *Let* $\hat{\pi} \in \Pi$ *be a GNE. Suppose that the assumptions in Section 6.2.4 hold. For any* $\epsilon > 0$*, there exists* $N' \in \mathbb{N}$ *such that for all* $N > N'$*,* $\Lambda(\hat{\pi})$ *is an* ϵ*-Nash equilibrium to the* N*-player game.*

Lemma 1.3.10. *Given* $\pi \in \Pi$ *and* $\tilde{\pi} \in \boldsymbol{\Pi}$*. For* $i \in [N]$*, let us consider the state distribution sequences*

$$\mu^N = (\mu_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),u})_{u \in I, t \in \overline{\mathcal{T}}} \qquad \text{and} \qquad \mu = (\mu_t^{\pi,u})_{u \in I, t \in \overline{\mathcal{T}}}.$$

For any measurable and bounded function $\mathcal{X} \times I \ni (x,u) \mapsto f^u(x) := f(x,u) \in \mathbb{R}$*, we have*

$$\mathbb{E}[|\mu_t^N(f) - \mu_t(f)|] \to 0 \text{ as } N \to +\infty, \qquad \forall\, t \in \overline{\mathcal{T}}.$$

Chapter 2

Introduction en français

Abstract

Cette thèse s'intéresse à deux sujets distincts : le *Transport Optimal Martingale* (MOT) et la théorie des *Jeux à champ moyen* (MFG), tous deux inspirés par les principes fondamentaux de la finance mathématique. Bien que ces deux domaines soient traités séparément, ils partagent une motivation commune : la gestion de l'incertitude sur les marchés financiers et la construction de stratégies qui fonctionnent de manière fiable dans un large éventail de scenarii.

Contents

2.1 Gestion de l'incertitude en finance **35**
2.1.1 Finance robuste et sur-réplication 35
2.1.2 De la finance robuste au transport optimal martingale 37
2.1.3 Jeux à champ moyen en modélisation financière 38
2.2 Transport optimal martingale **39**
2.2.1 Transport optimal 39
2.2.2 Transport optimal martingale 44
2.2.3 Transport optimal faible 46
2.2.4 Méthodes numériques pour XOT 48
2.2.5 Nos contributions 52
2.3 Jeux à champ moyen sur graphons **58**
2.3.1 Jeux à champ moyen en horizon fini 58
2.3.2 Des jeux finis aux jeux à champ moyen sur graphons 60
2.3.3 Notre contribution : jeux à champ moyen sur graphon étendu 63

Dans les marchés financiers, l'incertitude et l'information incomplète constituent des défis intrinsèques que les investisseurs doivent gérer en permanence. Les participants au marché sont confrontés à l'imprévisibilité de l'évolution des prix des actifs, de la volatilité des rendements et à l'incertitude sur les comportements des autres. Ces facteurs compliquent la prise de décision. Les modèles financiers traditionnels supposent souvent des conditions idéalisées ou reposent sur une connaissance précise des dynamiques du marché, ce qui est rarement exact en pratique. Pour répondre à ces problématiques, le domaine de la *finance robuste* a émergé en se concentrant sur la construction d'un cadre efficace dans une large gamme de conditions de marché possibles. Plutôt que de dépendre d'hypothèses hautement spécifiques, le cadre de la finance robuste s'appuie sur des stratégies qui permettent de couvrir les risques et d'optimiser les performances tout en réduisant la dépendance à des modèles précis. Cette approche connaît une popularité croissante dans des domaines tels que la calibration des modèles, le contrôle des risques et la gestion de portefeuille.

Deux cadres mathématiques puissants ont été développés dans le domaine de la finance robuste : le *Transport Optimal Martingale* (MOT) et les *Jeux à champ moyen* (MFG). Ces cadres abordent l'incertitude sous des angles différents, offrant des perspectives complémentaires. Le problème du MOT, issu de la théorie du transport optimal avec l'ajout d'une contrainte martingale, permet d'évaluer les bornes supérieures et inférieures des prix des instruments financiers sans dépendance à un modèle spécifique. D'un autre côté, le problème des MFG, introduit pour modéliser les interactions stratégiques au sein de grandes populations, traite de l'incertitude dans un environnement dynamique et vise à rechercher une solution d'équilibre où les individus optimisent leurs stratégies en fonction du comportement collectif. Ces cadres, bien que développés indépendamment, découlent tous deux de l'idée fondamentale de la prise de décision financière dans des environnements incertains. Au-delà du MOT et des MFG, la finance robuste englobe d'autres domaines qui ne sont pas couverts par cette thèse tels que les équations différentielles stochastiques rétrogrades (EDSRs), l'optimisation de portefeuille robuste, l'incertitude des modèles et la théorie du contrôle, qui disposent d'une vaste littérature.

Bien que le MOT et les MFG aient peu de points de recoupement direct, plusieurs axes de recherche relient le transport optimal (OT) classique et les MFG. Par exemple, Acciaio, Backhoff-Veraguas et Carmona [1] ont étudié des problèmes de contrôle de champ moyen étendu en utilisant un principe du maximum stochastique et des aspects de transport. D'autres travaux ont combiné les concepts d'équilibre de Cournot-Nash avec le OT, voir par exemple Acciaio et al. [7, 8]. Notamment, le OT entropique et les MFG ont été étudiées par Benamou, Carlier, Di Marino et Nenna [44], qui ont proposé une approche de minimisation d'entropie pour les jeux de champ moyen variationnels d'ordre deux sous un Hamiltonien quadratique, ainsi que par Backhoff, Conforti, Gentil, Ivan et Léonard [21], qui ont étudié le problème de Schrödinger à champ moyen et son lien intrinsèque avec le OT entropique. Ren, Tan, Touzi et Yang [199] ont par ailleurs examiné la planification optimale entropique pour les jeux à champ moyen dépendants du chemin. Enfin, des approches d'apprentissage automatique appliquées au OT et aux MFG ont également émergé dans [30, 67].

Dans cette thèse, bien qu'aucun lien direct ne soit établi entre le MOT et les MFG, les deux

cadres abordent des aspects critiques de la prise de décision sous incertitude sur les marchés financiers. Cette thèse contribue ainsi à l'avancement des recherches sur la finance robuste en offrant de nouvelles perspectives sur la manière dont le MOT et les MFG peuvent être utilisés indépendamment pour résoudre des problèmes financiers complexes impliquant le risque, la concurrence et les dynamiques de marché.

2.1 Gestion de l'incertitude en finance

2.1.1 Finance robuste et sur-réplication

Considérons un marché financier composé de $d+1$ actifs pouvant être échangés aux instants $t \geq 0$, où $d \in \mathbb{N}^*$. Nous notons S_t^i le prix de l'actif i au temps t. Le processus de prix $(S_t^i)_{t\geq 0}$ évolue sur un espace de probabilité filtré $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, où $\mathbb{P}$ est la mesure de probabilité historique sous laquelle le marché est modélisé. L'actif d'indice $i = 0$ est considéré comme l'actif sans risque, et nous fixons $S_0^0 = 1$ par convention. La dynamique de l'actif sans risque avec un taux d'intérêt constant $r \in \mathbb{R}$ est donnée par

$$dS_t^0 = r\, S_t^0\, dt, \quad \text{d'où} \quad S_t^0 = S_0^0\, e^{rt}.$$

La modélisation des taux d'intérêt n'étant pas l'objectif principal de cette thèse, nous prenons $r = 0$ pour simplifier, ce qui implique $S_t^0 = 1$ pour tout $t \geq 0$.

Pour les actifs risqués $i \in \{1, \dots, d\}$, leur dynamique de prix est souvent modélisée par des équations différentielles stochastiques (EDS), telles que le *Mouvement Brownien Géométrique* (MBG) :

$$dS_t^i = \mu^i\, S_t^i\, dt + \sigma^i\, S_t^i\, dW_t^{i,\mathbb{P}},$$

où μ^i représente la dérive (rendement espéré), σ^i la volatilité, et $W_t^{i,\mathbb{P}}$ un mouvement brownien standard unidimensionnel sous $\mathbb{P}$.

Portefeuilles autofinancés Soit Δ_t^i la quantité d'actif i détenue au temps t. La valeur du portefeuille à l'instant t est alors

$$V_t = \sum_{i=0}^{d} \Delta_t^i S_t^i.$$

Notons que $\Delta_t^i < 0$ correspond à une position vendeuse sur l'actif i. Cela est possible tant que l'investisseur peut rembourser et que $V_t \geq 0$ pour tout t. Nous imposons la condition d'*autofinancement* : il est possible de réajuster les allocations Δ_t^i, mais aucun apport ni retrait de cash n'est autorisé. Entre t et $t + dt$, la variation de la valeur du portefeuille est uniquement due à la variation de prix des actifs, c'est-à-dire

$$dV_t = \sum_{i=1}^{d} \Delta_t^i\, dS_t^i,$$

Introduisons maintenant deux notions : le portefeuille admissible et l'arbitrage.

Portefeuille admissible Un processus $\mathcal{D} := (\Delta_t, 0 \leq t \leq T)$ est appelé **portefeuille admissible** si V_t est borné inférieurement pour tout t $\mathbb{P}$-p.s., c'est-à-dire qu'il existe $L \in \mathbb{R}$ tel que

$$\mathbb{P}(V_t \geq L) = 1, \qquad \forall\, t \in [0, T].$$

Arbitrage Un portefeuille auto-financé admissible est appelé un **arbitrage** si le processus de valeur V_t associé satisfait $V_0 = 0$ et

$$V_T \geq 0 \quad \mathbb{P}\text{-p.s.} \qquad \text{et} \qquad \mathbb{P}(V_T > 0) \neq 0.$$

Nous notons l'équivalence de deux mesures de probabilité $\mathbb{Q}$ et $\mathbb{P}$ par $\mathbb{Q} \sim \mathbb{P}$, c'est-à-dire que pour tout $A \in \mathcal{F}_T$, $\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0$. Introduisons alors une notion très importante appelée « probabilité martingale locale equivalent », définie comme suit :

Définition 2.1.1 (ELMM)**.** *Une mesure de probabilité $\mathbb{Q}$ sur $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ est dite probabilité martingale locale equivalent (ELMM) si $\mathbb{Q} \sim \mathbb{P}$ et que le processus actualisé*

$$\left(\frac{S_t^1}{S_t^0}, \ldots, \frac{S_t^d}{S_t^0}\right)_{t\in[0,T]}$$

est une martingale sous $\mathbb{Q}$. Une ELMM est aussi appelée mesure risque-neutre.

Marché complet Pour définir un marché complet, introduisons d'abord la notion de *payoff répliquable*. Supposons que l'on achète et couvre (delta-hedge) une option européenne écrite sur les $d+1$ actifs, de maturité T et de payoff ξ_T. Un payoff ξ_T est dit *répliquable* à l'instant 0 s'il existe un portefeuille admissible $\Delta \in \mathcal{D}$ et un prix $x_0 \in \mathbb{R}$ tels que

$$x_0 + V_T(\Delta) = \xi_T, \quad \mathbb{P}\text{-p.s.}, \quad \text{où} \quad V_T(\Delta) = \sum_{i=1}^{d} \int_0^T \Delta_s^i \, dS_s^i = \int_0^T \Delta_s \, dS_s,$$

et Δ est un portefeuille autofinancé dont la valeur est une véritable martingale sous une certaine mesure de martingale équivalente $\mathbb{Q} \sim \mathbb{P}$.

Un marché est dit *complet* si tout payoff est répliquable à l'instant 0.

La caractérisation suivante d'un marché sans arbitrage et complet est connue sous le nom de *théorème fondamental de l'évaluation des actifs*; voir, par exemple, [53].

Théorème 2.1.2 (Théorème fondamental de l'évaluation des actifs)**.** *Le marché est sans arbitrage si et seulement s'il existe une probabilité martingale équivalente $\mathbb{Q} \sim \mathbb{P}$ sous laquelle les prix des actifs actualisés sont des martingales. De plus, le marché est complet si et seulement si $\mathbb{Q}$ est unique.*

Puisque nous avons supposé $r = 0$, impliquant $S_t^0 = 1$ pour tout $t \in [0, T]$, il s'ensuit que le processus de prix des actifs $(S_t^1, \ldots, S_t^d)_{t\in[0,T]}$ est une martingale sous $\mathbb{Q}$. De plus, la valeur du portefeuille autofinancé $(V_t(\Delta))_{t\in[0,T]}$ est également une martingale sous $\mathbb{Q}$.

Cependant, les marchés financiers réels sont soumis à des contraintes empêchant une réplication parfaite et, de ce fait, la complétude. Un exemple bien connu de marché incomplet a été étudié en premier lieu par Harrison et Kreps [130] et développé ensuite par El Karoui et Quenez [101]. Dans les marchés incomplets, on considère le *prix de sur-réplication*, qui représente le coût initial minimal nécessaire pour s'assurer qu'il existe une stratégie de couverture telle que la valeur finale du portefeuille soit supérieure au produit dérivé dans tous les scénarios possibles.

Théorème 2.1.3 (Prix de sur-réplication [101])**.** *Le prix de sur-réplication est donné par*

$$\sup_{\mathbb{Q}\in ELMM} \mathbb{E}^{\mathbb{Q}}[\xi_T] = \inf\Big\{x \in \mathbb{R} : \exists\, \Delta \in \mathcal{D} \text{ telle que } x + V_T(\Delta) \geq \xi_T,\ \mathbb{P}\text{-p.s.}\Big\}.$$

Cette notion de sur-réplication est étroitement liée au développement de la finance robuste après la crise financière à l'ère post-2008, où l'incertitude de modèle est abordée en considérant une famille de modèles. Plutôt que de s'appuyer sur une seule mesure de probabilité, on considère un ensemble de mesures de probabilité sur $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$. Une version indépendante des modèles du théorème fondamental de l'évaluation des actifs et du théorème de sur-réplication a été proposée par Acciaio et al. [4]. Les problèmes de tarification robuste ont également fait l'objet d'une attention particulière ; voir, par exemple, [54, 10]. Les détails de ces liens sont développés dans la sous-section suivante.

2.1.2 De la finance robuste au transport optimal martingale

Rappelons que nous avons supposé un taux d'intérêt nul dans la section précédente, de sorte que le panier d'actifs $(S_t = (S_t^1, \ldots, S_t^d))_{t\in[0,T]}$ est une martingale sous $\mathbb{Q}$, ainsi que le processus $(V_t)_{t\in[0,T]}$. Considérons maintenant une option européenne de maturité T dont le payoff dépend de la valeur des actifs financiers aux instants $0 < T_1 < T_2 < T$, et dont le payoff est $c(S_{T_1}, S_{T_2})$, où $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ et S_{T_1}, S_{T_2} sont les prix des actifs aux temps T_1, T_2 respectivement. Pour répliquer l'option avec notre portefeuille, nous devons avoir $V_T = c(S_{T_1}, S_{T_2})$. Puisque $(V_t)_{t\in[0,T]}$ est une martingale, il vient

$$V_0 = \mathbb{E}^{\mathbb{Q}}[V_T] = \mathbb{E}^{\mathbb{Q}}[c(S_{T_1}, S_{T_2})],$$

et dans le marché complet, $\mathbb{E}^{\mathbb{Q}}[c(S_{T_1}, S_{T_2})]$ constitue le prix de cette option.

Pour le cas $d = 1$, il a été observé dans [58] que les distributions marginales de l'actif sous-jacent $S_{T_1} \sim \mu$ et $S_{T_2} \sim \nu$ peuvent être déduites de manière unique à partir des prix des options call et put pour tous les prix d'exercice K aux maturités T_1 et T_2.

Pour le cas $d > 1$, les lois de $S_{T_1} \sim \mu$ et $S_{T_2} \sim \nu$ sont déterminées à partir des prix d'options de payoff $(\lambda \cdot S_{T_1} - K)^+$ et $(\lambda \cdot S_{T_2} - K)^+$ pour tous les prix d'exercice K et pour tout vecteur $\lambda \in \mathbb{R}^d$ tel que $|\lambda| = 1$. De plus, d'après Théorème 2.1.2, (S_{T_1}, S_{T_2}) est une martingale sous la mesure $\mathbb{Q}$, c'est-à-dire $\mathbb{E}^{\mathbb{Q}}[S_{T_2} \mid S_{T_1}] = S_{T_1}$ presque sûrement. Nous en déduisons que le prix x de l'option de payoff $c(S_{T_1}, S_{T_2})$ satisfait

$$\inf_{\substack{X\sim\mu,\, Y\sim\nu \\ \mathbb{E}[Y|X]=X\ p.s.}} \mathbb{E}[c(X,Y)] \leq x \leq \sup_{\substack{X\sim\mu,\, Y\sim\nu \\ \mathbb{E}[Y|X]=X\ p.s.}} \mathbb{E}[c(X,Y)].$$

La calcule de la borne inferier constitute le problem MOT, qui seront introduits plus en détail dans les paragraphes Section 2.2.2.

2.1.3 Jeux à champ moyen en modélisation financière

Les jeux à champ moyens MFG constituent un autre domaine ayant des applications significatives en finance mathématique, notamment dans les situations où les interactions stratégiques entre agents doivent être prises en compte. Les méthodes classiques d'optimisation de portefeuille ou de stratégies de trading considèrent souvent les décisions d'un investisseur de manière isolée, en se fondant sur ses préférences de risque ou ses anticipations de marché. Cependant, dans des contextes réalistes, les actions d'autres participants tels que des concurrents, des teneurs de marché ou des investisseurs institutionnels peuvent fortement influer sur les résultats. Il devient donc nécessaire de concevoir des stratégies robustes non seulement vis-à-vis de l'incertitude du marché, mais aussi de l'imprévisibilité des décisions de la concurrence.

La théorie des jeux à champ moyen, introduits indépendamment par Lasry et Lions [166] et Huang, Malhamé, et Caines [140], a suscité un grand intérêt en modélisation financière en raison de sa capacité à traiter des systèmes à grand nombre d'agents en interaction. En finance, un MFG modélise des situations dans lesquelles des agents (tels que des traders, investisseurs ou institutions) prennent des décisions en fonction de leur état et de la distribution des autres agents. Ces agents sont confrontés à une double incertitude : celle liée à l'évolution des prix des actifs, mais aussi celle relative aux actions de leurs concurrents. Ceci rend les MFG particulièrement pertinents pour étudier les grands marchés où les actions des agents influent sur les dynamiques de marché, et réciproquement.

L'intérêt pour les MFG en finance réside dans leur capacité à modéliser l'incertitude quant au comportement des autres. Les modèles robustes en finance mathématique visent à concevoir des stratégies performantes face à une variété de scénarios possibles, et les MFG offrent un cadre d'équilibre où les agents tiennent compte à la fois de leurs propres risques et de ceux induits par les actions d'autrui. Deux orientations majeures dans les applications des MFG sont le trading et le risque systémique. En trading optimal, les agents cherchent à minimiser leurs coûts ou à maximiser leurs profits tout en intégrant l'impact global des autres participants, débouchant ainsi sur des comportements compétitifs ou coopératifs. Dans le cadre du risque systémique, on s'intéresse à la gestion du risque collectif induit par le système financier, et à la manière dont chaque agent peut gérer son exposition à ce risque dans un contexte MFG. Pour plus de détails sur les applications des MFG en finance, on peut se référer à [71].

Une extension du problème de MFG qui a suscité un intérêt grandissant est le problème du *Graphon Mean Field Games* (GMFG), qui modélise les interactions entre des agents hétérogènes au sein de réseaux ou graphes. Dans un contexte financier, cela permet de représenter des agents aux caractéristiques variées, comme des aversions au risque ou des stratégies d'investissement différentes, interagissant de manière non homogène. En autorisant des interactions hétérogènes, le GMFG fournit un modèle plus réaliste pour les marchés financiers, où les acteurs peuvent avoir des informations, des préférences ou des niveaux de connectivité distincts.

2.2 Transport optimal martingale

Les paragraphes Section 2.1.1 et Section 2.1.2 présentent la motivation de notre étude du problème de MOT dans le cadre de la finance robuste. Dans la suite de cette section, nous établirons les fondements de Chapitre 3, Chapitre 4 et Chapitre 5.

2.2.1 Transport optimal

2.2.1.1 Définition

Problème de Monge et de Kantorovich Introduit par Monge en 1781 dans son travail intitulé « Théorie des déblais et des remblais », le problème de Transport Optimal (OT) concerne le transfert de masse d'un lieu à un autre de manière à minimiser une fonction de coût intégrée c.

Soient E et F deux espaces polonais, c'est-à-dire des espaces métriques complets et séparables. Pour $\mu \in \mathcal{P}(E)$ et $\nu \in \mathcal{P}(F)$, où $\mathcal{P}(\cdot)$ désigne l'ensemble des mesures de probabilité sur l'espace correspondant, Monge cherchait des applications de transport mesurables $T : E \times F$ telles que l'image $T\#\mu$ de μ par T soit égale à ν, où $\#$ désigne l'opération de mesure image, c'est-à-dire pour $B \in F$, $T\#\mu(B) = \mu\left(T^{-1}(B)\right) = \nu(B)$. Le *problème de Monge* (MP) se définit comme

$$\mathrm{MP}(\mu,\nu) := \inf_{T:T\#\mu=\nu} \int_E c(x,T(x))\,\mu(dx). \tag{MP}$$

Ce problème d'optimisation constitue la première formulation historique d'un problème de OT, mais il ne permet pas le « splitting » (fractionnement) de la masse, ce qui peut mener à l'inexistence de fonctions de transport.

Après la Seconde Guerre mondiale, Kantorovich proposa une relaxation très fructueuse en considérant des couplages de probabilité, c'est-à-dire des mesures de probabilité sur l'espace produit $E \times F$ dont les marginales sont respectivement μ et ν :

$$\mathrm{KP}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{E\times F} c(x,y)\,\pi(dx,dy) \tag{KP}$$

où

$$\Pi(\mu,\nu) = \{\pi \in \mathcal{P}(E\times F) \mid \pi(dx,F) = \mu(dx) \text{ et } \pi(E,dy) = \nu(dy)\}.$$

Théorème 2.2.1 (Existence)**.** *Soient $\mu \in \mathcal{P}(E)$ et $\nu \in \mathcal{P}(F)$, et soit $c : E \times F \to \mathbb{R} \cup \{+\infty\}$ une fonction semi-continue inférieurement et bornée inférieurement. Alors* (KP) *admet une solution.*

Un élève de Kantorovich, Svetlozar Rachev [194], a écrit une excellente synthèse des applications du problème de Monge-Kantorovich dans de nombreux domaines : géométrie différentielle [224, 217], analyse fonctionnelle [153, 173], programmation linéaire en dimension infinie [221, 203, 197, 155], théorie des probabilités [104, 106, 99, 236], statistiques [237, 142], théorie de l'information [220, 123, 122], physique statistique [98, 76], théorie des systèmes dynamiques [184], et théorie des matrices [109, 176, 183].

Motivé par la théorie de la mécanique des fluides, Yann Brenier a établi, dans un célèbre théorème de décomposition polaire [59, 60], que sous des hypothèses assez générales : il existe une solution unique au problème de Monge-Kantorovich dans $E = F = \mathbb{R}^d$ pour le coût quadratique $c : (x, y) \mapsto |y - x|^2$, de la forme $(X, \nabla\varphi(X))$, où φ est une fonction convexe déterminée de façon unique à une constante près, et cette solution coïncide avec le minimiseur du problème de Monge. De plus, la fonction φ est aussi solution d'une équation du type Monge-Ampère, à laquelle on peut se référer dans [89, 85, 88, 87, 86].

Ces dernières années, le problème OT est devenue un outil populaire en apprentissage automatique, notamment pour l'analyse de formes [9, 193, 214, 111], l'adaptation de domaines [82], la classification multi-étiquettes [228], et les réseaux génératifs adverses [18, 118]. Pour une étude approfondie de le OT, on peut consulter [195, 196, 222, 223, 14, 15, 163, 205].

2.2.1.2 Dualité et monotonicité cyclique

Formulation duale de Kantorovich Soient $\mu \in \mathcal{P}(E)$ et $\nu \in \mathcal{P}(F)$, et soit $c : E \times F \to [0, +\infty)$ une fonction de coût mesurable. Le problème dual de Kantorovich se formule comme suit :

$$(\text{DP}) := \sup \left\{ \int_E \varphi(x)\, \mu(dx) + \int_F \psi(y)\, \nu(dy) \;\middle|\; (\varphi, \psi) \in L^1(\mu) \times L^1(\nu), \varphi \oplus \psi \leq c \right\}, \qquad (\text{DP})$$

où $\varphi \oplus \psi$ désigne la fonction $(\varphi \oplus \psi)(x, y) := \varphi(x) + \psi(y)$ pour tout $(x, y) \in E \times F$. Si E et F sont compacts et c est continue, alors il existe une solution (φ, ψ) au problème (DP).

Théorème 2.2.2 (Dualité de Kantorovich)**.** *Soient E et F deux espaces polonais, et soit $c : E \times F \to \mathbb{R}$ une fonction uniformément continue et bornée. Alors, le problème dual* (DP) *admet une solution et on a*

$$\max(\text{DP}) = \min(\text{KP}).$$

Ce théorème établit une équivalence fondamentale entre le problème primal (KP) et son dual (DP), garantissant que les valeurs optimales des deux problèmes coïncident. La preuve du Théorème 2.2.2 peut être consultée dans [205]. Cette dualité nous permet d'étudier les plans de transport optimaux à travers les propriétés des paires duales (φ, ψ).

Une question naturelle se pose alors : comment relier cette dualité au problème de Monge (MP) ? Comprendre ce lien est crucial, car il éclaire la structure des solutions optimales.

Considérons le problème de Monge (MP), où l'on cherche à transporter de la marchandise d'un point A à un point B de manière optimale. Supposons que, dans la solution optimale, on transporte au moins une unité depuis la position x_i à A vers la position y_i à B, pour $i \in \{1, \ldots, N\}$, où $N \geq 2$. Imaginons que nous modifions ce plan de transport en redirigeant l'unité initialement envoyée de x_i à y_i vers y_{i+1}, avec la convention $y_{N+1} = y_1$, c'est-à-dire :

$$x_1 \to y_2, \qquad x_2 \to y_3, \qquad \ldots, \qquad x_{N-1} \to y_N, \qquad x_N \to y_1.$$

La variation de coût pour chaque unité redirigée est $c(x_i, y_{i+1}) - c(x_i, y_i)$. Si le plan de transport initial est effectivement optimal, alors le coût total de ce nouveau schéma doit être supérieur ou

égal au coût initial, d'où l'inégalité :

$$\sum_{i=1}^{N} (c(x_i, y_{i+1}) - c(x_i, y_i)) \geq 0 \quad \Longrightarrow \quad \sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}).$$

Monotonicité cyclique L'observation ci-dessus motive la définition suivante, qui caractérise les plans de transport optimaux. Considérons une fonction de coût $c : E \times F \to \mathbb{R} \cup \{+\infty\}$ définie sur deux espaces polonais E et F. Un ensemble mesurable $\Gamma \subset E \times F$ est appelé *c-cycliquement monotone* (abrégé c-CM) si, pour tout $N \in \mathbb{N}^*$ et $N \geq 2$,

$$\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}), \qquad \forall\, (x_1, y_1), \ldots, (x_N, y_N) \in \Gamma.$$

La loi d'une paire de variables aléatoires (X, Y) à valeurs dans $E \times F$ est dite c-cycliquement monotone s'il existe un ensemble c-CM, Γ, tel que $P((X, Y) \in \Gamma) = 1$. Intuitivement, un plan de transport est c-CM si aucune réaffectation cyclique des destinations ne peut réduire le coût total de transport.

Sur cette base, nous présentons le théorème suivant, qui caractérise les plans de transport optimaux via la monotonicité cyclique (voir [223] pour une preuve détaillée).

Théorème 2.2.3. *Soient $\mu \in \mathcal{P}(E)$ et $\nu \in \mathcal{P}(F)$, et soit $c : E \times F \to \mathbb{R}_+$ une fonction semi-continue inférieurement telle que la valeur du problème de Kantorovich (KP) soit finie. Alors, les deux propositions suivantes sont équivalentes :*

(i) $\pi \in \Pi(\mu, \nu)$ est optimale pour le problème (KP);

(ii) Le support de π est un ensemble c-CM.

Stabilité Pour simplifier, nous supposons que la fonction de coût $c : E \times F \to \mathbb{R}$ est continue et bornée. Pour tout $n \in \mathbb{N}$, soit μ_n une mesure de probabilité sur E qui converge faiblement vers μ, et ν_n une mesure de probabilité sur F qui converge faiblement vers ν. Pour chaque n, notons π_n une solution optimale pour $\mathrm{KP}(\mu_n, \nu_n)$. Alors, en extrayant si nécessaire une sous-suite, la suite $(\pi_n)_{n \in \mathbb{N}}$ converge faiblement vers une mesure $\pi \in \mathcal{P}(E \times F)$. D'une part, par le théorème de Portmanteau,

$$\mathrm{KP}(\mu_n, \nu_n) = \int_{X \times Y} c(x, y)\, \pi_n(dx, dy) \;\to\; \int_{X \times Y} c(x, y)\, \pi(dx, dy).$$

D'autre part, chaque π_n étant supporté sur un ensemble c-CM en raison de son optimalité, et la monotonicité cyclique se conservant sous convergence faible, on déduit que π est également c-CM. Par le Théorème 2.2.3, π est donc un plan de transport optimal pour $\mathrm{KP}(\mu, \nu)$. Par conséquent, on obtient :

$$\mathrm{KP}(\mu, \nu) = \lim_{n \to \infty} \mathrm{KP}(\mu_n, \nu_n).$$

Pour les preuves relatives à la c-CM et à la stabilité, on peut consulter [205].

2.2.1.3 Espaces de Wasserstein et fonctions quantiles

Distances de Wasserstein Pour $\rho \in [1, +\infty)$, notons $\mathcal{P}_\rho(E)$ l'ensemble des mesures de probabilité sur un espace métrique (E, d) ayant un moment d'ordre ρ fini :

$$\mathcal{P}_\rho(E) = \left\{ \eta \in \mathcal{P}(E) \colon \ \exists\, x_0 \in E \quad \text{tel que} \quad \int_E d(x, x_0)^\rho \, \eta(dx) < +\infty \right\}.$$

La distance de Wasserstein d'ordre ρ entre deux mesures de probabilité $\mu, \nu \in \mathcal{P}_\rho(E)$ est définie par

$$W_\rho(\mu, \nu) = \Big(\inf_{\pi \in \Pi(\mu,\nu)} \int_{E\times E} d(x, y)^\rho \, d\pi(x, y) \Big)^{1/\rho}.$$

Lorsque $\rho = 1$, la distance de Wasserstein W_1 correspond à un cas particulier du problème de Monge-Kantorovich, et possède une formulation duale analogue à celle du Théorème 2.2.2. Si (φ, ψ) est optimale pour le problème dual (DP), alors toute paire (X, Y) optimale pour $W_1(\mu, \nu)$ est concentrée sur l'ensemble des $(x, y) \in E \times E$ satisfaisant $\varphi(x) + \psi(y) = d(x, y)$, et $\varphi(x) = \psi(y)$ si $x = y$. De plus, comme les fonctions $x \mapsto d(x, y)$ (pour y fixé) et $y \mapsto d(x, y)$ (pour x fixé) sont 1-lipschitz, on peut ramener une paire optimale du problème (DP) à une paire de fonctions 1-lipschitz $(f, -f)$ avec $f : E \to \mathbb{R}$. Cela mène à la formule de dualité de Kantorovich-Rubinstein pour la distance W_1 :

$$W_1(\mu, \nu) = \sup_{f : E \to \mathbb{R},\ 1\text{-Lipschitz}} \Big(\int_E f(y) \, \nu(dy) \ - \ \int_E f(x) \, \mu(dx) \Big).$$

Pour une étude plus approfondie des distances de Wasserstein et de leurs applications, nous renvoyons aux travaux classiques sur la théorie du transport optimal [57, 198, 202, 204, 233].

Dans les sections précédentes, le problème de Monge-Kantorovich est envisagé dans un espace polonais général. Cependant, pour nos besoins, il est suffisant de nous restreindre à la droite réelle $\mathbb{R}$. En dimension un, nous disposons d'outils spécifiques qui nous offrent une compréhension plus approfondie des problèmes.

Fonction quantile Pour une mesure de probabilité $\eta \in \mathcal{P}(\mathbb{R})$, notons $F_\eta : x \mapsto \eta((-\infty, x])$ la fonction de répartition et F_η^{-1} la fonction quantile de η, où

$$F_\eta^{-1}(u) = \inf\{x \in \mathbb{R} \mid F_\eta(x) \geq u\}, \qquad \forall\, u \in (0, 1).$$

Notons que F_η^{-1} est le pseudo-inverse de F_η. Il est bien connu que les propriétés suivantes sont vérifiées :

(a) F_η est càdlàg, F_η^{-1} est càglàd;

(b) Pour tout $u \in (0, 1)$ et $x \in \mathbb{R}$,

$$F_\eta^{-1}(u) \leq x \quad \Longleftrightarrow \quad u \leq F_\eta(x), \tag{2.1}$$

en notant $F_\eta(y-)$ la limite à gauche de F_η en $y \in \mathbb{R}$,

$$F_\eta(x-) < u \leq F_\eta(x) \implies x = F_\eta^{-1}(u) \quad \text{et} \quad F_\eta(F_\eta^{-1}(u)-) \leq u \leq F_\eta(F_\eta^{-1}(u)); \tag{2.2}$$

(c) Pour $\eta(dx)$ presque tout $x \in \mathbb{R}$,

$$F_\eta(x) > 0,\ F_\eta(x-) < 1, \quad \text{et} \quad F_\eta^{-1}(F_\eta(x)) = x; \tag{2.3}$$

(d) Grâce à la méthode de simulation par inversion de la fonction de répartition, pour toute fonction mesurable et bornée $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f(x)\, \eta(dx) = \int_0^1 f(F_\eta^{-1}(v))\, dv. \tag{2.4}$$

Ces propriétés facilitent les calculs impliquant des lois de probabilité sur la droite réelle, notamment dans le contexte suivant.

Couplage de Hoeffding-Fréchet Soient $\mu, \nu \in \mathcal{P}(\mathbb{R})$ et $X \sim \mu$, $Y \sim \nu$ deux variables aléatoires. D'après ce qui précède, on peut trouver deux variables aléatoires U et V uniformes sur $]0,1[$ telles que presque sûrement,

$$(X, Y) = (F_\mu^{-1}(U), F_\nu^{-1}(V)).$$

D'après le théorème de Sklar [210], la donnée de la loi est donc équivalente à la donnée de la loi jointe de (X, Y), ou encore à la donnée d'une copule bidimensionnelle, qui représente la fonction de répartition d'un vecteur bidimensionnel ayant pour marginales des lois uniformes sur $]0,1[$. Le célèbre *couplage de Hoeffding-Fréchet*, aussi appelé *couplage co-monotone*, est défini comme la loi du vecteur

$$(X, Y) = \left(F_\mu^{-1}(U), F_\nu^{-1}(U)\right),$$

où U suit une loi uniforme sur $]0,1[$.

Ce couplage aligne les fonctions quantiles de μ et de ν, et il est intrinsèquement lié aux distances de Wasserstein pour $d = 1$. En effet, le couplage de Hoeffding-Fréchet est optimal pour $W_\rho(\mu, \nu)$, indépendamment de ρ :

$$W_\rho(\mu, \nu) = \left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^\rho\, du\right)^{1/\rho}, \qquad \forall\, \rho \in [1, +\infty).$$

Lorsque $\rho > 1$, le couplage de Hoeffding-Fréchet est l'unique couplage optimal pour $W_\rho(\mu, \nu)$. En revanche, pour $\rho = 1$, l'unicité n'est plus vérifiée. Dans ce cas, $W_1(\mu, \nu)$ correspond à l'aire entre les graphes de F_μ^{-1} et F_ν^{-1}, et donc entre ceux de F_μ et F_ν:

$$W_1(\mu, \nu) = \int_{\mathbb{R}} |\, F_\nu(x) - F_\mu(x)|\, dx.$$

2.2.2 Transport optimal martingale

2.2.2.1 Définition

Le problème de *Martingale Optimal Transport* (MOT) a été introduit pour la première fois par Beiglböck, Henry-Labordère et Penkner [38] dans le cas discret, puis par Galichon, Henry-Labordère et Touzi [110] en temps continu. Il joue un rôle essentiel en finance robuste, où il est utilisé pour obtenir des bornes supérieures et inférieures pour le prix de produits dérivés sans supposer de modèle sous-jacent spécifique. Soient $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ et une fonction de coût $c : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$. Le problème s'écrit

$$\mathrm{MOT}(\mu,\nu) := \inf_{\pi \in \Pi_{\mathrm{M}}(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x,y)\, \pi(dx,dy), \tag{MOT}$$

où Π_M désigne l'ensemble des couplages martingales entre μ et ν :

$$\Pi_M(\mu,\nu) = \{\pi(dx,dy) = \mu(dx)\pi_x(dy) \in \Pi(\mu,\nu) \mid \mu(dx)\text{-p.s.}, \int_{y\in\mathbb{R}^d} y\, \pi_x(dy) = x\}.$$

D'après le théorème de Strassen [216],

$$\Pi_M(\mu,\nu) \neq \emptyset \quad \Longleftrightarrow \quad \mu \leq_{cx} \nu,$$

où $\mu \leq_{cx} \nu$ signifie que μ est plus petite que ν au sens de l'ordre convexe :

$$\forall\, \varphi : \mathbb{R} \to \mathbb{R} \text{ convexe}, \quad \int_{\mathbb{R}} \varphi(x)\, \mu(dx) \leq \int_{\mathbb{R}} \varphi(y)\, \nu(dy).$$

Le cadre en temps continu a d'abord été introduit par Henry-Labordère et Touzi [110] pour résoudre un problème de sur-couverture, habituellement résolu par une méthode de plongement de Skorokhod. Celle-ci, formulée par Skorokhod en 1961 [211, 212], consistait à trouver, pour une mesure de probabilité centrée μ à second moment fini et un mouvement brownien B, un temps d'arrêt intégrable τ tel que $B_\tau \sim \mu$. Obłój [180] a fait une bonne synthèse des résultats sur le plongement de Skorokhod jusqu'en 2004. Le lien entre les deux problèmes est expliqué pars Guo, Tan, Touzi [124, 125, 126] et Beiglböck et al. [39, 37].

La question de la stabilité pour le MOT a été soulevée pour la première fois par Alfonsi, Corbetta, et Jourdain [11], puis par Guo et Obłój [108]. Nicolas Juillet [152] a établi les couplages « left-curtain ». En 2019, Backhoff-Veraguas et Pammer [22] ont confirmé la stabilité en dimension 1 sous des hypothèses très générales, suivis de près par des travaux indépendants de Wiesel [227]. Backhoff-Veraguas et Pammer ont démontré la stabilité pour des coûts continus à croissance au plus linéaire, en exploitant l'équivalence entre la monotonie martingale et l'optimalité, un résultat de Beiglböck et Juillet [35] analogue en version martingale entre la monotonie cyclique et l'optimalité pour le problème OT. Parallèlement, Johannes Wiesel [227] a étudié directement la formulation primale en introduisant la notion de « martingale rearrangement coupling ». Cette notion a été ultérieurement approfondie par Jourdain et Margheriti [147]. La structure des couplages optimaux en transport martingale a également fait l'objet d'études approfondies par [35,

132, 133, 66, 145, 94, 115].

Pour une étude détaillée de l'application du transport optimal martingale en finance, on peut se référer à l'ouvrage [134].

2.2.2.2 Dualité

Le problème (MOT) est appelé le problème primal du transport optimal martingale. De manière analogue au problème dual (DP) dans le cadre classique de le OT, on peut envisager une approche duale qui intègre la contrainte de martingale. Ce problème dual martingale est relié à la formulation de stratégies de sous-couverture en finance mathématique.

Rappelons le cadre de Section 2.1.1 : nous considérons un marché avec d actifs financiers et un taux d'intérêt supposé nul. Sous la mesure risque-neutre $\mathbb{Q}$, nous cherchons une stratégie de sous-couverture pour des options de payoff $c(S_{T_1}, S_{T_2})$. Concrètement, on considère des stratégies de couverture en delta de la forme $h(S_{T_1}) \cdot (S_{T_2} - S_{T_1})$, où $h : \mathbb{R}^d \to \mathbb{R}^d$ est une fonction continue et bornée. Nous considérons ainsi des portefeuilles de la valeur terminale :

$$\varphi(S_{T_1}) + \psi(S_{T_2}) + h(S_{T_1}) \cdot (S_{T_2} - S_{T_1}),$$

qui fournissent une sous-couverture si ce payoff total est toujours inférieur ou égal à $c(S_{T_1}, S_{T_2})$.

D'après le Théorème 2.1.2, le processus de prix (S_{T_1}, S_{T_2}) est une martingale sous $\mathbb{Q}$. Cela implique que le terme de couverture en delta $h(S_{T_1}) \cdot (S_{T_2} - S_{T_1})$ a un coût nul, puisqu'il représente les gains issus du trading de l'actif sous-jacent sans investissement net. En revanche, les options de payoff $\varphi(S_{T_1})$ et $\psi(S_{T_2})$ ont un prix égal à leur espérance $\int \varphi(x)\, \mu(dx)$ et $\int \psi(y)\, \nu(dy)$ respectivement sous $\mathbb{Q}$.

Nous souhaitons déterminer une stratégie de sous-couverture qui maximise le prix total du portefeuille de couverture, tout en garantissant que le payoff ne dépasse pas $c(S_{T_1}, S_{T_2})$. Cela conduit naturellement à la formulation duale du problème de transport optimal martingale : étant données deux mesures de probabilité μ et ν sur $\mathbb{R}^d$ (avec $d \in \mathbb{N}$) et une fonction de coût $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

$$\mathrm{DPM}(\mu, \nu) := \sup_{(\varphi, \psi, h)} \Big\{ \int_{\mathbb{R}^d} \varphi(x)\, \mu(dx) + \int_{\mathbb{R}^d} \psi(y)\, \nu(dy) \,\Big|\, \varphi \oplus \psi(x, y) + h(x) \cdot (y - x) \leq c(x, y) \Big\}, \tag{DPM}$$

où $(\varphi, \psi, h) \in L^1(\mu) \times L^1(\nu) \times C_b^0(\mathbb{R}^d; \mathbb{R}^d)$, et $C_b^0(\cdot)$ désigne l'ensemble des fonctions continues et bornées. Les travaux contenus dans [38] ont montré que l'écart de dualité est nul en dimension un.

Théorème 2.2.4. *Soient $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$ telles que $\mu \leq_{cx} \nu$ et soit $c : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ une fonction continue pour laquelle il existe une constante $K \in \mathbb{R}$ satisfaisant*

$$\max\{c(x, y), 0\} \leq K(1 + x + y), \qquad \forall\, (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Alors, (MOT)=(DPM).

Dans le cadre du Théorème 2.2.4, le supremum de (DPM) n'est pas nécessairement atteint,

comme l'a illustré un contre-exemple dans [38] ; cependant, il peut être réalisé dans la formulation presque-sûre étudiée par [42].

2.2.3 Transport optimal faible

Le problème de transport optimal avec coûts faibles (*weak costs*) a été introduit pour la première fois par Gozlan, Roberto, Samson et Tetali [121] et a par la suite connu de nombreux développements (voir [119, 27, 120, 24, 13, 207]).

Soient E et F deux espaces polonais. Étant données des mesures de probabilité $\mu \in \mathcal{P}(E)$ et $\nu \in \mathcal{P}(F)$, on note $\Pi(\mu,\nu)$ l'ensemble des couplages sur $E \times F$ dont les marginales sont respectivement μ et ν. Pour un couplage $\pi \in \Pi(\mu,\nu)$, désignons par $(\pi_x)_{x\in E}$ une désintégration régulière de π par rapport à la première marginale μ. Étant donné $\pi \in \Pi(\mu,\nu)$ (resp. $\pi \in \Pi_M(\mu,\nu)$), il est bien connu (voir [95] pages 78–80 ou [192] page 117) qu'il existe un noyau de Markov (resp. un noyau de Markov martingale) unique $\mu(dx)$-p.s. tel que $\pi(dx,dy) = \mu(dx)\,\pi_x(dy)$.

Nous considérons des fonctionnelles de coût de la forme

$$C : E \times \mathcal{P}(F) \to \mathbb{R} \cup \{+\infty\},$$

où C est minorée et semi-continue inférieurment, et pour chaque $x \in E$, l'application $C(x,\cdot)$ est convexe sur $\mathcal{P}(F)$. Le problème de *Weak Optimal Transport* (WOT) consiste alors à déterminer

$$V_C(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_E C(x,\pi_x)\,\mu(dx). \tag{WOT}$$

Notons que le problème classique de transport optimal est un cas particulier du problème de transport faible lorsque le fonctionnel de coût est linéaire en la seconde variable, c'est-à-dire $C(x,p) = \int c(x,y)\,p(dy)$ pour un coût mesurable $c : E \times F \to \mathbb{R} \cup \{+\infty\}$. Des résultats fondamentaux de la théorie du transport optimal classique, tels que l'existence de plans optimaux, les principes de dualité ou la monotonicité cyclique des optimiseurs (par exemple, voir [205]), ont été étendus au cadre du transport faible dans [24, 27]. Dans le contexte des applications financières, Acciaio, Beiglböck et Pammer [3] montrent que les problèmes de tarification indépendants du modèle sur les marchés à revenu fixe se ramènent à un problème de WOT.

Motivés par des problèmes financiers qui impliquent naturellement des formulations faibles du problème de transport optimal martingale (MOT), Backhoff-Veraguas et Pammer [22] intègrent des contraintes de martingale dans le cadre du transport faible. Pour une fonction de coût $C : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+$ donnée, le problème de *Weak Martingale Optimal Transport* (WMOT) se définit comme

$$V_C^M(\mu,\nu) := \inf_{\pi\in\Pi_M(\mu,\nu)} \int_{\mathbb{R}} C(x,\pi_x)\,\mu(dx), \tag{WMOT}$$

où $\Pi_M(\mu,\nu)$ désigne l'ensemble des couplages martingales entre μ et ν.

Le problème WMOT peut être vu comme un cas particulier du problème WOT, où la fonction de coût $C(x,p)$ vaut $+\infty$ dès lors que la barycentre (la moyenne) de p n'est pas égale à x, imposant ainsi la contrainte de martingale dans le cadre du transport faible. Le problème

WMOT suscite un intérêt considérable du fait de ses applications en finance robuste et en tarification indépendante de modèle. Des travaux récents ont permis des avancées théoriques et pratiques sur ce sujet; voir [34, 40, 149].

Formulation de Benamou-Brenier et variantes martingales Soient deux mesures de probabilité $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ de moment d'ordre deux fini. La formulation de Benamou-Brenier [144] exprime le carré de la distance de Wasserstein d'ordre 2 comme l'énergie cinétique minimale requise pour transporter μ vers ν au cours du temps :

$$T_2(\mu,\nu) := \inf_{\substack{X_t = X_0 + \int_0^t v_s\, ds \\ X_0 \sim \mu,\, X_1 \sim \nu}} \mathbb{E}\Big[\int_0^1 |v_t|^2\, dt\Big], \tag{BB}$$

où $(X_t)_{t\in[0,1]}$ est un processus stochastique transportant μ vers ν, et v_t le champ de vitesse. D'après [59], si μ est absolument continue par rapport à la mesure de Lebesgue, le problème (BB) admet un unique optimiseur $(X^*)_{t\in[0,1]}$.

De même, lorsque $\mu \leq_{cx} \nu$ (c'est-à-dire lorsque ν domine μ pour l'ordre convexe), on peut considérer la version martingale de la formulation de Benamou-Brenier. Soit B un mouvement brownien sur $\mathbb{R}^d$. Le problème de Benamou-Brenier martingale est défini par

$$MT(\mu,\nu) := \sup_{\substack{M_t = M_0 + \int_0^t \sigma_s\, ds \\ M_0 \sim \mu,\, M_1 \sim \nu}} \mathbb{E}\Big[\int_0^1 \mathrm{tr}(\sigma_t)\, dt\Big], \tag{MBB}$$

où σ est un processus $\mathbb{R}^{d\times d}$-valué $\mathcal{F}$-progressif sur l'ensemble des espaces de probabilités filtrés $(\Omega, \mathcal{F}, \mathbb{P})$, et B est un mouvement brownien $\mathcal{F}$-adapté à d dimensions, assurant la propriété de martingale pour $(M_t)_{t\in[0,1]}$. Lorsque ν possède un second moment fini, le problème (MBB) admet un unique maximiseur $(M^*)_{t\in[0,1]}$, appelé « Brownien étiré » (stretched Brownian motion) allant de μ à ν [23, Théorème 1.5], puisqu'il s'agit de la martingale vérifiant $M_0^* \sim \mu$ et $M_1^* \sim \nu$ qui ressemble le plus au mouvement brownien.

À première vue, les problèmes d'optimisation (BB) et (MBB) semblent très différents. Toutefois, ils peuvent être reliés par des formulations appropriées. Dans la Section 6 de [23], il a été établi que

$$\begin{aligned} X^* &= \arg\min_{X_0\sim\mu,\, X_1\sim\nu} W^2(X, \text{particule à vitesse constante}), \\ M^* &= \arg\min_{M_0\sim\mu,\, M_1\sim\nu} W_c^2(M, \text{martingale à volatilité constante}), \end{aligned}$$

où W^2 désigne la distance de Wasserstein d'ordre 2 par rapport à la norme de Cameron-Martin au carré, et W_c^2 en est une variante adaptée ou causale (dans la terminologie de [167]). Cette connexion met en évidence la manière dont les méthodes de transport faible peuvent relier le transport optimal classique (formulation de Benamou-Brenier) à ses analogues martingales, en considérant des processus adaptés et des plans de transport causaux.

Examinons à présent une fonction de coût faible particulière $C_2 : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ définie

par $C_2(x,p) = W_2^2(p, \mathcal{N}(0,1))$ pour tout $(x,p) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, où $\mathcal{N}(0,1)$ désigne la loi normale standard unidimensionnelle. Soient $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ telles que $\mu \leq_{cx} \nu$, et désignons par $V_{C_2}^M(\mu,\nu)$ la fonction de valeur associée à (WMOT) avec la fonction de coût C_2.

Soit $\pi^* \in \Pi_M(\mu,\nu)$ une solution optimale pour $V_{C_2}^M(\mu,\nu)$, et soit M^* le « Brownien étiré » allant de μ à ν. Alors, d'après la Remarque 2.1, le Théorème 2.2 et la Remarque 2.3 de [23], on a :

(a) $MT(\mu,\nu) = \frac{1}{2}\Big(1 + \int_{\mathbb{R}} |y|^2 \, \nu(dy) \; - \; V_{C_2}^M(\mu,\nu)\Big)$

(b) π^* est la loi jointe de (M_0^*, M_1^*), et réciproquement

$$M_t^* = \mathbb{E}\Big[F_{\pi_X^*}^{-1}\Big(F_{\mathcal{N}(0,1)}(B_1)\Big) \;\Big|\; X, \; (B_s)_{0\leq s\leq t}\Big],$$

où $X \sim \mu$ est une variable aléatoire indépendante du mouvement brownien $(B_t)_{t\in[0,1]}$.

Ces résultats illustrent la profondeur des liens entre le problème (WMOT) et la formulation (MBB), ainsi que la manière dont la notion de transport faible peut servir de passerelle entre les cadres de transport optimal classique et de transport optimal martingale. Pour les applications financières, on peut se référer à la calibration du modèle de volatilité locale de Bass étudiée par Conze et Henry-Labordère [81], ainsi qu'à celle réalisée par Acciaio, Marini et Pammer [6].

2.2.4 Méthodes numériques pour XOT

Dans cette section, nous présentons des méthodes numériques pour diverses variantes de *Optimal Transport* (OT), chacune étant caractérisée par une propriété ou contrainte spécifique. Afin de conserver la généralité des différentes méthodes, nous désignons la section par « Méthodes Numériques pour XOT », où « X » indique un certain type de problème de transport optimal. Par exemple, MOT fait référence au *Martingale Optimal Transport*, WOT renvoie au *Weak Optimal Transport*, et «EOT» désigne le transport optimal entropique, brièvement mentionné. Une nouvelle discipline, appelée *Causal Optimal Transport* (COT), suscite également un vif intérêt, avec de nombreuses méthodes numériques émergentes dans ce domaine ; voir par exemple Acciaio et al. [2, 230, 229, 5].

Nous débutons par les méthodes numériques pour le problème classique de OT, suivies de sous-sections dédiées aux particularités et méthodes de calcul pour le MOT, et enfin, dans le cadre faible, nous présentons le WOT et le WMOT.

2.2.4.1 Méthodes numériques pour OT

Du fait que le problème de Monge-Kantorovich est stable par rapport à ses marginales (comme l'établit le Theorem 1.2.3), on peut approcher des mesures de probabilité générales μ et ν par des mesures de probabilité à support fini $\hat{\mu}$ et $\hat{\nu}$. En résolvant le problème de Kantorovich $\mathrm{KP}(\hat{\mu},\hat{\nu})$ pour ces approximations, on approche la solution de $\mathrm{KP}(\mu,\nu)$ à la limite lorsque $\hat{\mu}$ et $\hat{\nu}$ convergent respectivement vers μ et ν.

Soient $\hat{\mu}$ et $\hat{\nu}$ deux mesures à support fini

$$\hat{\mu} = \sum_{i=1}^{I} \mu_i \, \delta_{x_i}, \quad \hat{\nu} = \sum_{j=1}^{J} \nu_j \, \delta_{y_j}, \tag{2.5}$$

où $\mu_i, \nu_j \in [0,1]$ et $\sum_{i=1}^{I} \mu_i = \sum_{j=1}^{J} \nu_j = 1$. Tout couplage π de $\hat{\mu}$ et $\hat{\nu}$ s'écrit

$$\pi = \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} \, \delta_{(x_i, y_j)}, \quad \pi_{i,j} \geq 0, \quad \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} = 1,$$
$$\text{avec} \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i \quad \text{pour } i = 1, \ldots, I, \quad \text{et} \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j \quad \text{pour } j = 1, \ldots, J.$$

En ajoutant les contraintes marginales, le problème $\mathrm{KP}(\hat{\mu}, \hat{\nu})$ se ramène à la minimisation

$$\min_{(\pi_{i,j})} \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} \, c(x_i, y_j) \tag{2.6}$$

sous les contraintes $\pi_{i,j} \geq 0$ et les conditions marginales ci-dessus. Il s'agit là d'un problème de programmation linéaire en variables $\pi_{i,j}$.

En 1939, Kantorovich et son élève Gavurin ont proposé une méthode potentielle pour résoudre le problème de OT [114]. Par ailleurs, des algorithmes classiques de recherche opérationnelle, tels que l'algorithme du simplexe [91], l'algorithme hongrois [160], la méthode de points intérieurs [154], et l'algorithme d'enchères [49, 51], peuvent aussi être utilisés. Toutefois, ces méthodes se révèlent souvent coûteuses en temps et en mémoire, ce qui les rend peu adaptées à des problèmes de grande échelle, comme l'analyse d'images comportant des millions de pixels [213, 215].

Une approche pour contourner les difficultés de calcul en transport optimal consiste à introduire un terme de régularisation entropique dans la fonction objectif, conduisant au problème de minimisation suivant :

$$\inf_{\pi_{i,j}} \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} \, c(x_i, y_j) + \lambda \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} \ln(\pi_{i,j} - 1)$$
$$\text{s.c.} \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i \quad \text{pour } i = 1, \ldots, I, \quad \text{et} \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j \quad \text{pour } j = 1, \ldots, J,$$

où $\lambda > 0$ est un paramètre de régularisation. Sous certaines conditions, il a été montré que la solution du problème régularisé par l'entropie converge vers celle du problème de transport optimal original [80, 172, 70]. Cette régularisation permet de recourir à des algorithmes efficaces, tels que l'algorithme de Sinkhorn [208, 209], qui offre un gain de performance significatif pour des problèmes de grande taille [90, 82, 43, 191].

Certaines fonctions de coût possédant des structures particulières font l'objet d'études ap-

profondies, à l'exemple de la distance de Wasserstein d'ordre 2. Lorsque $c(x,y) = \frac{1}{2}|x-y|^2$, l'application de transport optimale $T : \mathbb{R}^d \to \mathbb{R}^d$ telle que $T_{\#}\mu = \nu$ peut s'écrire comme le gradient d'un potentiel convexe u, i.e. $T(x) = \nabla u(x)$. Ce potentiel u satisfait l'équation de Monge-Ampère [129] :

$$\det(D^2 u(x)) = \frac{\mu(x)}{\nu(\nabla u(x))},$$

qui est une équation elliptique dégénérée et non linéaire. On peut la résoudre par des méthodes de différences finies [107, 45, 46, 47].

Une autre possibilité est la formulation de Benamou-Brenier [144, 43], évoquée dans la Section 1.2.3, qui fournit une caractérisation dynamique du transport optimal entre deux densités de probabilité ρ_0 et ρ_1 sur $\mathbb{R}^d$. Elle exprime le carré de la distance de Wasserstein-2 comme l'action minimale nécessaire pour transporter ρ_0 vers ρ_1 :

$$W_2^2(\rho_0,\rho_1) = \inf_{\rho_t,\, v_t} \Big\{ \int_0^1 \int_{\mathbb{R}^d} \rho_t(x)\, |v_t(x)|^2 \, dx\, dt \Big\},$$

sous la contrainte de l'équation de continuité :

$$\partial_t \rho_t + \nabla \cdot (\rho_t\, v_t) = 0, \quad \text{pour } t \in [0,1],$$

avec $\rho_0 = \mu$ et $\rho_1 = \nu$. Dans cette formulation, $\rho_t(x)$ désigne une densité de probabilité dépendant du temps, et $v_t(x)$ est le champ de vitesse qui transporte progressivement ρ_0 vers ρ_1 sur l'intervalle $t \in [0,1]$. La discrétisation de la formulation de Benamou–Brenier conduit à un programme quadratique convexe dont la structure peut être exploitée efficacement par des solveurs modernes tels que les méthodes de lagrangien augmenté, de décomposition primale-duale ou encore les méthodes de Sinkhorn.

2.2.4.2 Méthodes numériques pour MOT

En ajoutant la contrainte de martingale à (2.6), la résolution du problème de MOT entre deux mesures à support fini, $\hat{\mu} = \sum_{i=1}^{I} \mu_i \delta_{x_i}$ et $\hat{\nu} = \sum_{j=1}^{J} \nu_j \delta_{y_j}$, se ramène à :

$$\inf_{\pi_{i,j}} \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{i,j}\, c(x_i,y_j),$$
$$\text{s.c.} \quad \pi_{i,j} \geq 0, \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i, \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j, \quad \sum_{j=1}^{J} \pi_{i,j}\, y_j = \mu_i\, x_i.$$

Ici, la dernière contrainte $\sum_{j=1}^{J} \pi_{i,j}\, y_j = \mu_i\, x_i$ impose la propriété de martingale. On note que cette formulation constitue un problème de programmation linéaire.

Cependant, l'introduction de la contrainte de martingale soulève un défi majeur quant à l'échantillonnage tout en préservant l'ordre convexe. En pratique, les mesures empiriques

obtenues à partir d'échantillons indépendants et identiquement distribués (i.i.d.) n'ont pas la même moyenne et, de manière générale, ne satisfont pas l'ordre convexe. Alfonsi, Corbetta et Jourdain ont abordé ce problème dans [11, 12].

Échantillonnage de mesures de probabilité unidimensionnelles dans l'ordre convexe Soient $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ telles que $\mu \leq_{cx} \nu$. On considère $(X_i)_{1\leq i\leq I}$ des variables i.i.d. de loi μ et $(Y_j)_{1\leq j\leq J}$ des variables i.i.d. de loi ν. Les moyennes empiriques $\bar{X}_I = \frac{1}{I}\sum_{i=1}^I X_i$ et $\bar{Y}_J = \frac{1}{J}\sum_{j=1}^J Y_j$ sont distinctes en général, et la relation $\bar{X}_I \leq_{cx} \bar{Y}_J$ n'est pas préservée. Cependant, $\bar{X}_I$ et $\bar{Y}_J$ convergent presque sûrement vers $\int_{\mathbb{R}} x\,\mu(dx)$ et $\int_{\mathbb{R}} y\,\nu(dy)$ respectivement lorsque $I, J \to +\infty$. En suivant [11, Section 3], on peut ajuster légèrement μ_I et ν_J de sorte que

$$\hat{\mu}_I = \frac{1}{I}\sum_{i=1}^I \delta_{X_i - \bar{X}_I + \int_{\mathbb{R}} x\,\mu(dx)} \quad \text{et} \quad \hat{\nu}_J = \frac{1}{J}\sum_{j=1}^J \delta_{Y_j - \bar{Y}_J + \int_{\mathbb{R}} y\,\nu(dy)},$$

alors $\hat{\mu}_I$ et $\hat{\nu}_J$ partagent la même moyenne. Nous notons $\mu \vee \nu$ (resp. $\mu \wedge \nu$) le supremum (resp. l'infimum) de μ et ν pour l'ordre convexe. Ainsi, $\mu \vee \nu$ (resp. $\mu \wedge \nu$) est plus grand que μ (resp. plus petit que ν) pour l'ordre convexe.

Dans [11], il est établi que lorsque $\mu \leq_{cx} \nu$, après ajustement des moyennes, $\hat{\mu}_I \vee \hat{\nu}_J$ et $\hat{\mu}_I \wedge \hat{\nu}_J$ convergent faiblement vers ν et μ respectivement quand $I, J \to +\infty$, et $\hat{\mu}_I \leq_{cx} (\hat{\mu}_I \vee \hat{\nu}_J)$. En appliquant [11, Algorithm 1], on calcule ainsi $\hat{\mu}_I \vee \hat{\nu}_J$. Cette technique est aussi utilisée pour l'approximation du problème (WMOT).

Plusieurs équipes de chercheurs se sont intéressées aux méthodes numériques pour le problème de MOT. Hadrien De March a proposé un nouvel algorithme s'appuyant sur la méthode de Newton combinée à une régularisation entropique, améliorant les performances tout en traitant la difficulté liée à l'échantillonnage pour l'ordre convexe [93]. Plus récemment, Eckstein et Kupper ont introduit une méthode basée sur les réseaux de neurones pour résoudre une large classe de problèmes, dont le transport optimal et le transport optimal martingale [100]. Cependant, leur algorithme peut devenir instable pour de grands paramètres de régularisation et éventuellement ne pas converger, comme l'a montré Pierre Henry-Labordère [131], qui a également proposé un réseau de neurones dans un algorithme primal-dual pour des fonctions de coût satisfaisant la condition de Spence-Mirrlees. Par ailleurs, Tan et Touzi [218] ont étudié une version continue semi-martingale du problème de transport optimal.

2.2.4.3 Méthodes numériques dans le cadre faible

Nous reprenons la discrétisation de $\hat{\mu}$ et $\hat{\nu}$ comme dans (2.5), et notons $p_{j|i} = \dfrac{\pi_{i,j}}{\mu_i}$ la probabilité conditionnelle de y_j sachant x_i. Le problème (WOT) devient alors :

$$\begin{aligned} &\min \ \sum_{i=1}^I \mu_i\, C\Big(x_i,\ \sum_{j=1}^J p_{j|i}\,\delta_{y_j}\Big) \\ &\text{s.c.} \quad p_{j|i} \geq 0, \quad \sum_{j=1}^J p_{j|i} = 1, \quad \sum_{i=1}^I \mu_i\, p_{j|i} = \nu_j. \end{aligned}$$

De même, la version discrétisée du problème (WMOT) s'écrit :

$$\min \sum_{i=1}^{I} \mu_i \, C\Big(x_i, \sum_{j=1}^{J} p_{j|i} \, \delta_{y_j}\Big)$$
$$\text{t.q.} \quad p_{j|i} \geq 0, \quad \sum_{j=1}^{J} p_{j|i} = 1, \quad \text{et} \quad \sum_{i=1}^{I} \mu_i \, p_{j|i} = \nu_j, \quad \sum_{j=1}^{J} p_{j|i} \, y_j = x_i.$$

Les progrès récents dans les méthodes numériques pour le *Weak Optimal Transport* (WOT) ont porté une attention particulière au problème de barycentre. Différents algorithmes ont été développés afin de calculer le barycentre faible, notamment la descente miroir [187], la descente de gradient stochastique dans l'espace de Wasserstein [26] et des méthodes d'apprentissage adverse [158], avec des applications dans le traitement de flux de données et l'analyse d'images [77]. Au-delà du problème de barycentre, [161] étudie des mesures de risque convexes pénalisées par le WOT, en recourant à des méthodes de calcul s'appuyant sur des réseaux de neurones pour des applications financières. Parallèlement, [157] se concentrent sur des tests statistiques et mettent en œuvre un algorithme de Frank-Wolfe entropique pour le calcul des projections de Wasserstein.

En parallèle et indépendamment de ces travaux, nous proposons une approche similaire en appliquant l'algorithme de Frank-Wolfe pour résoudre les problèmes WOT et WMOT dans un cadre plus général, sans nous restreindre à des fonctions de coût particulières. Un résumé de notre méthodologie est présenté dans notre contribution ; voir Section 1.2.5.3. Pour une présentation plus complète, incluant des formulations détaillées et d'autres applications telles que la projection de Wasserstein en dimension un ou le mouvement brownien étiré, on pourra se reporter au Chapitre 5.

2.2.5 Nos contributions

Dans la première partie de cette thèse, nous nous concentrons sur trois projets distincts dans le cadre du MOT : l'inégalité de Wasserstein martingale maximale au Chapitre 3, les couplages martingales croissants au Chapitre 4, ainsi que les méthodes numériques pour le transport optimal faible au Chapitre 5.

2.2.5.1 Chapitre 3 Inégalité de Wasserstein martingale maximale

Soient $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ telles que $\mu \leq_{cx} \nu$. Posons

$$\underline{\mathcal{M}}_\rho^\rho(\mu, \nu) = \inf_{M \in \Pi^{\mathrm{M}}(\mu,\nu)} \int_{\mathbb{R}^{2d}} |x - y|^\rho \, M(dx, dy).$$

Pour $d = 1$, Jourdain et Margheriti [146, Proposition 1] ont établi l'inégalité de Wasserstein martingale pour $\rho \in [1, +\infty)$,

$$\exists \, \underline{C}_{(\rho,\rho),1} < \infty, \; \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \;\; \text{avec} \;\; \mu \leq_{cx} \nu, \; \underline{\mathcal{M}}_\rho^\rho(\mu, \nu) \leq \underline{C}_{(\rho,\rho),1} \mathcal{W}_\rho(\mu, \nu) \, \sigma_\rho^{\rho-1}(\nu), \tag{2.7}$$

où le moment central $\sigma_\rho(\nu)$ de ν est défini par

$$\sigma_\rho(\nu) = \inf_{c\in\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |y-c|^\rho\,\nu(dy)\Big)^{1/\rho} \quad \text{lorsque } \rho\in[1,+\infty).$$

Cette inégalité fournit une estimation de stabilité quantitative pour le problème de MOT dont la fonction de coût est $|x-y|^\rho$, lorsque $\mathcal{W}_\rho(\mu,\nu)\to 0$. Notons que la stabilité du problème de MOT par rapport aux distributions marginales μ et ν est établie en dimension $d=1$ dans [22, 227], tandis qu'elle échoue en dimension supérieure selon [62].

La généralisation de (2.7) aux dimensions supérieures d est également étudiée dans [146] où il est prouvé que, pour tout $d\geq 2$,

$$\underline{C}_{(\rho,\rho),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_\rho(\mathbb{R}^d)\\ \mu\leq_{cx}\nu,\mu\neq\nu}} \frac{\underline{\mathcal{M}}^\rho_\rho(\mu,\nu)}{\mathcal{W}_\rho(\mu,\nu)\,\sigma_\rho^{\rho-1}(\nu)}$$

est infini lorsque $\rho\in[1,\frac{1+\sqrt{5}}{2})$, en se basant sur l'exemple contre-exemple de stabilité du problème MOT fourni par Juillet et Brückerhoff [62] pour $d\geq 2$. Le Chapitre 3 répond à la question de la finitude de $\underline{C}_{(\rho,\rho),d}$ lorsque $\rho\in[\frac{1+\sqrt{5}}{2},+\infty)$ et $d\geq 2$, qui restait ouverte.

Il s'avère que $\underline{C}_{(\rho,\rho),d}=+\infty$ pour $d\geq 2$ dès que $\rho\in[1,2)$, tandis que pour $\rho\in[2,+\infty)$ l'inégalité (2.7) se généralise à toute dimension d via une *Maximal Martingale Wasserstein inequality* où le terme $\underline{\mathcal{M}}^\rho_\rho(\mu,\nu)$ (membre de gauche) est remplacé par

$$\overline{\mathcal{M}}^\rho_\rho(\mu,\nu) = \sup_{M\in\Pi^{\mathrm{M}}(\mu,\nu)}\int_{\mathbb{R}^{2d}} |x-y|^\rho\,M(dx,dy).$$

Définissons alors

$$\underline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d)\\ \mu\leq_{cx}\nu,\mu\neq\nu}} \frac{\underline{\mathcal{M}}^\rho_\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\,\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)}$$

et

$$\overline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d)\\ \mu\leq_{cx}\nu,\mu\neq\nu}} \frac{\overline{\mathcal{M}}^\rho_\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\,\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)},$$

avec

$$\mathcal{W}_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)}\pi\text{-}\operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}|x-y|,$$

et

$$\sigma_\infty(\nu) = \inf_{c\in\mathbb{R}^d}\nu\text{-}\operatorname*{ess\,sup}_{y\in\mathbb{R}^d}|y-c|.$$

Comme $\underline{\mathcal{M}}_\rho\leq\overline{\mathcal{M}}_\rho$, il s'ensuit que $\underline{C}_{(\rho,q),d}\leq\overline{C}_{(\rho,q),d}$.

Théorème 2.2.5. *(i) Soit $\rho\in[1,2)$. Pour $q\in[1,\frac{1}{2-\rho}]$ (et même $q\in[1,+\infty]$ lorsque $\rho=1$), on a $\underline{C}_{(\rho,q),1}\leq K_\rho<+\infty$, où la constante K_ρ est étudiée dans [146, Proposition 1], tandis que pour $q\in[1,+\infty]$, on a $\overline{C}_{(\rho,q),1}=+\infty$ et $\underline{C}_{(\rho,q),d}=+\infty$ pour $d\geq 2$.*

(*ii*) *Soit* $\rho \in [2, +\infty)$ *et* $q \in [1, +\infty]$. *Alors* $\overline{C}_{(\rho,q),d} < +\infty$ *quelle que soit la dimension* d. *De plus, lorsque chaque espace vectoriel* $\mathbb{R}^d$ *est muni de la norme euclidienne, on obtient* $\overline{C}_{(2,q),d} = 2$ *et* $\sup_{d\geq 1} \overline{C}_{(\rho,q),d} < +\infty$.

2.2.5.2 Chapitre 4 Couplages martingales croissants

Au Chapitre 4, nous étudions les problèmes d'optimisation définissant $\underline{\mathcal{M}}_\rho$ lorsque $\rho > 2$ et $\overline{\mathcal{M}}_\rho$ lorsque $\rho \in (0, 2)$, ainsi que la structure des couplages optimaux.

Les couplages atteignant $\overline{\mathcal{M}}_1(\mu, \nu)$ ont d'abord été étudiés dans [136]. Hobson et Neuberger [136, Théorème 8.2] ont montré que, pour la fonction de coût $|x - y|$, il existe un couplage martingale maximisant la forme

$$\pi^{\mathrm{HN}} = \int_0^1 \Big(\tfrac{r(u)-q(u)}{r(u)-p(u)}\, \delta_{(q(u),p(u))} + \tfrac{q(u)-p(u)}{r(u)-p(u)}\, \delta_{(q(u),r(u))} \Big)\, du,$$

où p, q, r sont des fonctions croissantes sur $(0, 1)$ satisfaisant $p \leq q \leq r$. Lorsque μ est continue, d'après [35],

$$\pi^{\mathrm{HN}} = \int_{\mathbb{R}} \Big(\tfrac{g(x)-x}{g(x)-f(x)}\, \delta_{(x,f(x))} + \tfrac{x-f(x)}{g(x)-f(x)}\, \delta_{(x,g(x))} \Big) \mu(dx),$$

pour deux fonctions croissantes f et g telles que $\forall x \in \mathbb{R}$, $f(x) \leq x \leq g(x)$. En outre, selon le Théorème 7.3 de [35], le couplage martingale maximisant est unique.

Le couplage π^{HN} fournit un exemple de couplage martingale croissant au sens de la définition suivante :

Définition 2.2.6. *Soient* $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ *telles que* $\mu \leq_{cx} \nu$. *Un couplage martingale* $\pi(dx, dy) \in \Pi_M(\mu, \nu)$ *est dit* ***croissant*** *s'il existe un ensemble borélien* $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ *tel que* $\pi(\Gamma) = 1$ *et :*

(a) si $(x_-, y_-), (x_+, y_+) \in \Gamma$ *avec* $y_- \leq x_-$, $y_+ \leq x_+$ *et* $x_- < x_+$, *alors* $y_- \leq y_+$,

(b) si $(x_-, z_-), (x_+, z_+) \in \Gamma$ *avec* $x_- \leq z_-$, $x_+ \leq z_+$ *et* $x_- < x_+$, *alors* $z_- \leq z_+$.

Dans le Section 4.2.1 du Chapitre 4, pour de nombreuses paires de mesures $\mu, \nu \in \mathcal{P}_{\rho\vee 1}(\mathbb{R})$ telles que $\mu \leq_{cx} \nu$, nos observations numériques indiquent que le couplage π^{HN}, maximisant pour $\rho = 1$, est encore très proche d'atteindre $\overline{\mathcal{M}}_\rho$ lorsque $\rho \in (0, 2)$ et $\underline{\mathcal{M}}_\rho$ lorsque $\rho > 2$. Il n'est pas surprenant que $\rho = 2$ apparaisse comme un seuil, puisque la propriété de martingale garantit que pour toute $\pi \in \Pi_M(\mu, \nu)$ avec $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$,

$$\int_{\mathbb{R}^2} (x - y)^2\, \pi(dx, dy) = \int_{\mathbb{R}} y^2\, \nu(dy) - \int_{\mathbb{R}} x^2\, \mu(dx) = \underline{\mathcal{M}}_2^2(\mu, \nu) = \overline{\mathcal{M}}_2^2(\mu, \nu).$$

Nos contributions au Chapitre 4 se répartissent en deux volets principaux : (1) nous étudions l'optimalité des couplages martingales croissants et (2) nous introduisons les couplages martingales croissants avec décomposition directionnelle. Notons qu'ici, l'*optimalité* d'un couplage martingale croissant signifie qu'il atteint $\overline{\mathcal{M}}_\rho$ pour $\rho \in (0, 2)$ et $\underline{\mathcal{M}}_\rho$ pour $\rho > 2$.

Résultats d'optimalité de couplages martingales croissants :

En exploitant la structure de treillis complet de l'ensemble $\{\eta \in \mathcal{P}_1(\mathbb{R}) : \int_{\mathbb{R}} z\,\eta(dz) = \int_{\mathbb{R}} y^2\,\nu(dy) - \int_{\mathbb{R}} x^2\,\mu(dx)\}$ pour l'ordre convexe établie dans [156], nous obtenons :

Proposition 2.2.7. *Soient $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ telles que $\mu \leq_{cx} \nu$. Alors l'ensemble*

$$\{|x-y|^2 \# \pi : \pi \in \Pi_M(\mu,\nu)\}$$

admet un infimum pour l'ordre convexe.

Lorsque existe $\underline{\pi} \in \Pi_M(\mu,\nu)$ tel que $|x-y|^2\#\underline{\pi}$ soit cet infimum, alors $\int_{\mathbb{R}^2} |x-y|^\rho\,\underline{\pi}(dx,dy)$ coïncide avec $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ pour $\rho \in (0,2)$ et avec $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ pour $\rho > 2$. Par exemple, si

$$\nu(dy) = \int_{x\in\mathbb{R}} \tfrac{1}{2}(\delta_{x-a}(dy) + \delta_{x+a}(dy))\mu(dx)$$

pour un certain $a \in \mathbb{R}$, on a $|x-y|^2\#\underline{\pi} = \delta_{a^2}$ pour

$$\underline{\pi}(dx,dy) = \tfrac{1}{2}(\delta_{x-a}(dy) + \delta_{x+a}(dy))\mu(dx).$$

Lorsque μ est continue, l'unicité des couplages maximisant ou minimisant pour $\rho = 1$ démontrée dans [35, Théorème 7.3] implique que π^{HN} est la seule valeur possible pour $\underline{\pi}$.

Proposition 2.2.8. *Soient $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ telles que $\mu \leq_{cx} \nu$, et $\varphi : \mathbb{R}_+ \to \mathbb{R}$ une fonction strictement croissante et concave. Si $\tilde{\pi} \in \Pi_M(\mu,\nu)$ maximise $\int \varphi(|x-y|)\,\pi(dx,dy)$, alors $\tilde{\pi}$ est croissant. De plus, quand φ est continue, il existe un tel couplage maximisant.*

Remarque 2.2.9. *Pour $\rho \in (0,1]$, comme la fonction $\mathbb{R}_+ \ni z \mapsto z^\rho$ est concave et strictement croissante, un couplage martingale maximisant $\int |x-y|^\rho\,\pi(dx,dy)$ est croissant.*

Malheureusement, même si nous parvenons dans la Proposition 4.2.16 à exhiber un cas simple où les bornes inférieure sont atteintes par $|x-y|^2\#\underline{\pi}$, nous présentons également des exemples (voir Proposition 4.2.10 et Proposition 4.2.18) pour lesquels elles ne le sont pas.

Le résultat suivant montre que l'optimalité de π^{HN} se vérifie sans que la borne inférieure soit nécessairement atteint mais sous des hypothèses plutôt restrictives :

Proposition 2.2.10. *Soient $\mu \leq_{cx} \nu$ telles qu'il existe $\underline{y} < \overline{y} < \underline{x} < \overline{x} < \underline{z} < \overline{z}$ avec $\mu([\underline{x},\overline{x}]) = 1$ et $\nu([\underline{y},\overline{y}] \cup [\underline{z},\overline{z}]) = 1$. Si*

$$\underline{x} - \overline{y} \geq \alpha_\rho(\overline{z} - \overline{y}),$$
$$\underline{z} - \overline{x} \geq \alpha_\rho(\underline{z} - \underline{y}),$$

où $\alpha_\rho \in (0, \frac{1}{2})$ est l'unique solution de $\psi_\rho(\alpha) = 0$, avec $\psi_\rho : (0,1] \to \mathbb{R}$ définie par

$$\psi_\rho(\alpha) = \alpha + \alpha^{2-\rho}(1-\alpha)^{\rho-1} + 1 - \rho,$$

alors le (seul) couplage croissant $\pi^\uparrow$ dans $\Pi_M(\mu,\nu)$ est l'unique couplage optimal dans $\Pi_M(\mu,\nu)$ atteignant $\overline{\mathcal{M}}_\rho(\mu,\nu)$ quand $\rho \in (1,2)$ et $\underline{\mathcal{M}}_\rho(\mu,\nu)$ quand $\rho > 2$.

Les propriétés du paramètre α_ρ de Proposition 2.2.10 sont étudiées plus en détail dans le Lemma 4.3.9, tandis que la Fig. 2.1 illustre la dépendance de α_ρ en fonction de ρ :

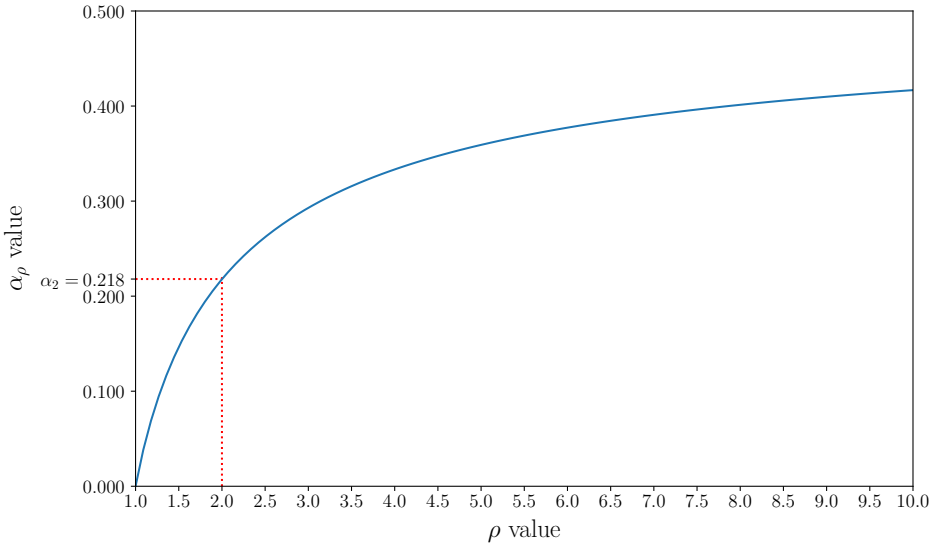


Figure 2.1: La fonction $\rho \mapsto \alpha_\rho$ (avec α_ρ calculé par un algorithme de recherche de racines).

Couplages martingales croissants avec décomposition directionnelle :

Comme nous pensons que le non-atteinte de l'infimum pour l'ordre convexe est un phénomène générique, nous étudions directement les couplages qui atteignent $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ et $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$. Nous introduisons la décomposition de ν en trois parties :

$$\begin{aligned}\nu_l^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y<x\}}\pi(dx,dy),\\ \nu_0^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y=x\}}\pi(dx,dy),\\ \nu_r^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y>x\}}\pi(dx,dy).\end{aligned}$$

Ainsi, $\nu_l^\pi + \nu_0^\pi + \nu_r^\pi = \nu$. Pour une paire (ν_l,ν_r) de mesures positifs telles que $\nu_l+\nu_r \le \nu$, posons

$$\Pi_M(\mu,\nu,\nu_l,\nu_r) = \{\pi \in \Pi_M(\mu,\nu) : (\nu_l^\pi,\nu_0^\pi,\nu_r^\pi) = (\nu_l, \nu-\nu_l-\nu_r, \nu_r)\}.$$

Théorème 2.2.11. *Soient $\mu,\nu \in \mathcal{P}_1(\mathbb{R})$ avec $\mu \le_{cx} \nu$ et $\mu \neq \nu$. Soit (ν_l,ν_r) une paire de mesures non-négatives telles que $\nu_l+\nu_r=\nu$. Alors*

$$\Pi_M(\mu,\nu,\nu_l,\nu_r) \neq \emptyset \iff \exists!\, \pi^\uparrow \in \Pi_M(\mu,\nu,\nu_l,\nu_r) \ \ \textit{croissant},$$

et pour toute fonction $\varphi : \mathbb{R}_+ \to \mathbb{R}$ continue, strictement croissante et concave,

$$\forall \pi \in \Pi_M(\mu,\nu,\nu_l,\nu_r)\setminus\{\pi^\uparrow\},\ \int_{\mathbb{R}^2} \varphi(|y-x|)\,\pi(dx,dy) < \int_{\mathbb{R}^2} \varphi(|y-x|)\,\pi^\uparrow(dx,dy). \tag{2.8}$$

Corollaire 2.2.12. *Pour (ν_l, ν_r) une paire de mesures positifs telles que $\nu_l + \nu_r \leq \nu$, on a*

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) \neq \emptyset \iff \exists! \, \pi^{\uparrow} \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$$

tel que

$$\frac{\pi^{\uparrow}(dx, dy) + (\nu_l + \nu_r - \nu)(dx)\, \delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}$$

soit croissant, et l'équation (2.8) ci-dessus reste valable.

2.2.5.3 Chapitre 5 WOT

Dans le Chapitre 5, nous appliquons l'algorithme de Frank-Wolfe, aussi appelé méthode du gradient conditionnel, pour approcher dans une perspective d'optimisation les problèmes WOT et WMOT formulés au paragraphe Section 2.2.4.3. Nous présentons ci-dessous deux exemples afin d'illustrer cette approche et de mettre en évidence son efficacité.

Projection de Wasserstein en dimension un Alfonsi, Corbetta et Jourdain [12] introduisent, pour $\rho \in [1, +\infty)$, les *projections de Wasserstein* dans l'ordre convexe :

$$\mathcal{I}_\rho(\mu, \nu) := \operatorname{argmin}\{\mathcal{W}_\rho(\mu, \eta) : \eta \leq_{cx} \nu\}.$$

La fonction quantile de $\mathcal{I}(\mu, \nu)$ est obtenue pour tout $u \in (0, 1)$ par :

$$F^{-1}_{\mathcal{I}(\mu,\nu)}(u) = F^{-1}_{\mu}(u) \; - \; \partial_{-}\mathrm{co}(G)(u),$$

où $G(u) := \int_0^u (F_\mu^{-1} - F_\nu^{-1})(v)\, dv$, co désigne l'enveloppe convexe et ∂_- la dérivée à gauche. D'après [12, 24], la valeur de $\mathcal{W}_\rho(\mu, \mathcal{I}(\mu, \nu))$ coïncide. Pour $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$, considérons

$$V^{\rho}_{\mathrm{WP},\rho}(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}} \Big| x - \int_{\mathbb{R}} y \, \pi_x(dy) \Big|^{\rho} \mu(dx).$$

Dans l'expérience numérique, nous avons appliqué l'algorithme de Frank-Wolfe pour résoudre avec

$$\min_{p_{j|i}} \; \sum_{i=1}^{I} \mu_i \left| x_i - \sum_{j=1}^{J} p_{j|i}\, \delta_{y_j} \right|^{\rho}$$
$$\text{s.c.} \quad p_{j|i} \geq 0, \quad \sum_{j=1}^{J} p_{j|i} = 1, \quad \text{et} \quad \sum_{i=1}^{I} \mu_i \, p_{j|i} = \nu_j, \quad \sum_{j=1}^{J} p_{j|i}\, y_j = x_i.$$

Ensuite, nous déduisons le couplage optimal

$$\pi^* = \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_i \, p_{j|i} \, \delta_{(x_i, y_j)}$$

qui atteint $V^{\rho}_{\mathrm{WP}}\big(\hat{\mu},\hat{\nu}\big)$, puis nous comparons $\mathcal{I}_{\rho}\big(\hat{\mu},\hat{\nu}\big)$ à l'image de μ par $x \mapsto \int y\,\pi^*_x(dy)$, lesquelles coïncident conformément à la théorie.

Brownien étiré (*Stretched Brownian motion*) Soient $\mu,\nu \in \mathcal{P}_2(\mathbb{R})$ telles que $\mu \leq_{cx} \nu$. Rappelons (voir Section 1.2.3) que nous définissons la fonction de coût $C_2 : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ par $C_2(x,p) = W_2^2(p,\mathcal{N}(0,1))$ pour tout $(x,p)\in\mathbb{R}\times\mathcal{P}_2(\mathbb{R})$, où $\mathcal{N}(0,1)$ désigne la loi normale standard. En nous appuyant sur le couplage de Hoeffding-Fréchet introduit en Section 1.2.1.3, qui est optimal pour la distance de Wasserstein $W_\rho(\mu,\nu)$, nous exprimons $C_2(x,p)$ de façon explicite via les fonctions quantiles F_p^{-1} et F_γ^{-1} associées à p et γ :

$$C_2(x,p) = \mathcal{W}_2^2(p,\gamma) = \int_0^1 \Big(F_p^{-1}(u) - F_\gamma^{-1}(u)\Big)^2\,du.$$

Selon [23, Remarque 2.1], il existe un lien direct entre la formulation de Benamou-Brenier martingale $MT(\mu,\nu)$ et la formulation faible $V^M_{C_2}(\mu,\nu)$ donnée par :

$$V^M_{C_2}(\mu,\nu) = 1 + \int_{\mathbb{R}} |y|^2\nu(dy) \;-\; 2\,MT(\mu,\nu).$$

Nous avons appliqué l'algorithme de Frank-Wolfe afin de construire le couplage optimal π^* qui atteint $V^M_{C_2}$, et illustrons nos résultats au moyen d'une représentation 3D au Chapitre 5.

2.3 Jeux à champ moyen sur graphons

2.3.1 Jeux à champ moyen en horizon fini

Les jeux à champ moyen (MFGs) ont été introduits pour étudier des jeux différentiels avec un nombre infini de joueurs, sous des hypothèses d'indistinguabilité et de symétrie. Cette théorie a été proposée par J.-M. Lasry et P.-L. Lions [166], et indépendamment par Caines, Huang et Malhamé [140]. Tandis que les MFGs correspondent à des jeux non coopératifs et se concentrent sur des équilibres de Nash, leur pendant coopératif, généralement appelé contrôle de type champ moyen (ou *mean field control*, ou encore MFC) ou contrôle optimal de dynamique de type McKean-Vlasov [73, 48, 97], a également été étudié. On trouve aussi une formulation des MFG utilisant des méthodes d'équations aux dérivées partielles (EDP) [28]. Pour plus de détails, on peut se référer aux cours de Lions au Collège de France [175] et aux notes de Cardaliaguet [69], ainsi qu'aux ouvrages de Bensoussan, Frehse et Yam [48], Gomes, Pimentel et Voskanyan [116], Carmona et Delarue [74], et aux exposés de Gomes et Saúde [117] ainsi que Başar et Zaccour [29].

Un *Mean Field Games* est un problème de décision s'étendant sur plusieurs instants, impliquant un nombre infini de joueurs identiques et anonymes. La résolution s'effectue en se concentrant sur la politique optimale d'un joueur représentatif, réagissant au comportement de la population entière. Soient $\mathcal{X}$ un espace d'états fini et $\mathcal{A}$ un espace d'actions fini. Le joueur représentatif commence la partie dans un état $x_0 \in \mathcal{X}$ selon une distribution initiale μ_0 sur $\mathcal{X}$. Nous nous plaçons dans un cadre en temps discret avec horizon fini $T > 0$, et notons $\mathcal{T} := \{0,1,\dots,T-1\}$

ainsi que $\overline{\mathcal{T}} := \mathcal{T} \cup \{T\} = \{0, 1, \dots, T\}$. Soit $P : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ un noyau de transition (une probabilité de passer d'un état à l'autre en un pas), et $r : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ une fonction de récompense.

Intuitivement, à chaque pas de temps $t \in \overline{\mathcal{T}}$, un joueur représentatif arrive à l'état x_{t+1} et choisit une action a_t selon une politique $\pi_t(a_t \mid x_t)$. Les séquences des états de la population et des politiques sont notées respectivement $\mu = (\mu_t)_{t \in \overline{\mathcal{T}}}$ et $\pi = (\pi_t)_{t \in \overline{\mathcal{T}}}$. Entre chaque pas de temps, le joueur représentatif passe à l'état suivant x_{t+1} selon le noyau de transition $P(\cdot \mid x_t, a_t, \mu_t)$ et perçoit une récompense $r(x_t, a_t, \mu_t)$.

Récompense totale La récompense cumulée d'un joueur représentatif utilisant la politique π lorsque le champ moyen est μ est

$$J(\pi; \mu) := \mathbb{E}\Big[\sum_{t \in \mathcal{T}} r(x_t, a_t, \mu_t)\Big], \tag{2.9}$$

sous les contraintes

$$x_0 \sim \mu_0, \quad a_t \sim \pi_t(\cdot \mid x_t), \quad x_{t+1} \sim P(\cdot \mid x_t, a_t, \mu_t), \quad \forall\, t \in \mathcal{T}.$$

Dans ce cadre simple, on suppose qu'il n'y a pas de récompense à l'instant T.

Dynamique des distributions d'états La séquence de distributions d'états induite par la politique π est donnée par

$$\begin{cases} \mu^{\pi}_{t+1}(x) = \displaystyle\sum_{x' \in \mathcal{X}} \mu^{\pi}_t(x') \sum_{a \in \mathcal{A}} \pi_t(a \mid x')\, P(x \mid x', a, \mu^{\pi}_t), \\ \mu^{\pi}_0(x) = \mu_0(x). \end{cases}$$

Cette dynamique décrit l'évolution de la distribution de la population lorsque chacun utilise la politique π.

Fonctions de champ moyen et de meilleure réponse Commençons par définir la fonction de champ moyen, qui associe à une politique la distribution de la population, de la façon suivante:

$$\mathrm{MF} : \mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}} \to \mathcal{P}(\mathcal{X})^{\overline{\mathcal{T}}}, \qquad \pi \;\mapsto\; \mathrm{MF}(\pi) := \mu^{\pi}.$$

D'autre part, étant donnée une distribution d'états de la population, un joueur représentatif cherche à maximiser sa récompense totale. Nous définissons donc la fonction (multivaluée) de meilleure réponse par :

$$\mathrm{BR} : \mathcal{P}(\mathcal{X})^{\overline{\mathcal{T}}} \to 2^{\mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}}}, \qquad \mu \;\mapsto\; \mathrm{BR}(\mu) := \operatorname*{argmax}_{\pi \in \Pi} J(\pi; \mu) \;\subseteq\; \mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}},$$

où $2^{\mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}}}$ désigne l'ensemble des parties de $\mathcal{P}(\mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}}$. Nous cherchons un équilibre de Nash du jeu, défini comme suit.

Définition 2.3.1 (Équilibre de Nash à champ moyen (MFNE))**.** *Une paire* $(\hat{\pi}, \hat{\mu}) \in \mathcal{P}(\mathcal{A})^{\overline{T} \times \mathcal{X}} \times \mathcal{P}(\mathcal{X})^{\overline{T}}$ *est un équilibre de Nash à champ moyen (MFNE) si elle satisfait les deux conditions suivantes :*

- *Optimalité de la politique :* $\hat{\pi} \in \mathrm{BR}(\hat{\mu})$*;*
- *Cohérence de la séquence de champ moyen :* $\hat{\mu} = \mathrm{MF}(\hat{\pi})$*.*

Fonction valeur Pour une distribution μ et une politique π, la fonction valeur (dépendant de l'état uniquement) est définie par :

$$V_t^{\pi,\mu}(x) = \mathbb{E}_{a\sim\pi(\cdot|x)}\left[\sum_{s=t}^{T} r(x_s, a_s, \mu_s) \,\middle|\, x_t = x, a_t = a, x_{s+1} \sim P(\cdot \mid x_s, a_s, \mu_s), a_s \sim \pi_s(\cdot \mid x_s)\right],$$

tandis que la fonction valeur optimale est :

$$V_t^{*,\mu}(x) = \max_{\pi} \mathbb{E}\left[\sum_{s=t}^{T} r(x_s, a_s, \mu_s) \,\middle|\, x_t = x, a_t = a, x_{s+1} \sim P(\cdot \mid x_s, a_s, \mu_s), a_s \sim \pi_s(\cdot \mid x_s)\right].$$

Elle satisfait l'équation de Bellman :

$$\begin{cases} V_T^{*,\mu}(x) = 0, \\ V_t^{*,\mu}(x) = \max_a \Big\{ r(x, a, \mu_t) + \sum_{x' \in \mathcal{X}} P(x' \mid x, a, \mu_{t+1})\, V_t^{*,\mu}(x') \Big\}, \quad \forall t = T-1, \ldots, 0. \end{cases}$$

Exploitabilité : Une façon de mesurer l'écart d'une politique à l'équilibre de Nash est l'exploitabilité, une fonction $\mathcal{E} : \Pi \to \mathbb{R}$ définie par

$$\mathcal{E}(\pi) \;:=\; \max_{\tilde{\pi}} J(\tilde{\pi}; \mu) \;-\; J(\pi; \mu).$$

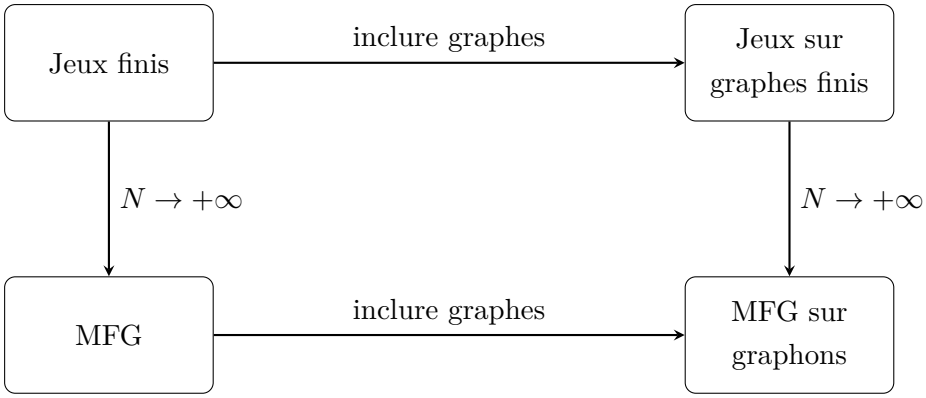
Remarque 2.3.2. *On a* $\mathcal{E}(\hat{\pi}) = 0$ *si et seulement si* $(\hat{\pi}, \mathrm{MF}(\hat{\pi}))$ *est un MFNE.*

2.3.2 Des jeux finis aux jeux à champ moyen sur graphons

Le schéma ci-dessous illustre la progression allant des jeux finis à N joueurs vers les jeux de champ moyen (MFG), et des jeux sur graphe finis vers les MFG sur graphon.

- *Jeux finis ou à N joueurs.* Ils impliquent un nombre fini de joueurs, chacun prenant des décisions afin d'optimiser son objectif individuel. Lorsque l'on fait tendre le nombre N de joueurs vers $+\infty$, ces jeux finis donnent naissance aux jeux de champ moyen, où chaque agent interagit avec l'effet agrégé de l'ensemble de la population.
- *Jeux sur graphe finis.* Ils étendent les jeux finis en plaçant les joueurs sur un graphe, de sorte que chaque agent n'interagisse qu'avec ses voisins. Dans la limite de grande population, on obtient alors ce que l'on appelle un *jeu de champ moyen sur graphon*, où le graphe sous-jacent est remplacé par un objet continu appelé *graphon*.

- *Jeux de champ moyen (MFG).* Ceux-ci peuvent également prendre en compte des structures de graphe (GMFG) pour modéliser des interactions hétérogènes. Le passage vers les MFGs sur graphons consiste à étendre les MFG classiques en introduisant une description continue du réseau sous-jacent.



Ainsi, le passage horizontal dans le schéma (« *inclure des graphes* ») ajoute une structure de réseau au problème, tandis que le passage vertical (« $N \to +\infty$ ») convertit un jeu fini ou un jeu sur graphe fini en son équivalent de champ moyen dans la limite d'une population infinie.

Graphons Soit $I := [0,1]$. Un graphon est défini comme une fonction symétrique et mesurable $G : I \times I \to I$, que l'on peut considérer comme la limite de matrices de poids (pour les arêtes) de graphes lorsque la taille (le nombre de sommets) de ces graphes tend vers l'infini. Nous notons $\mathcal{W}$ l'espace des graphons.

On peut également voir un graphon comme un opérateur de $L^\infty(I)$ vers $L^1(I)$, associant à tout $\phi \in L^1(I)$ la fonction

$$G\phi(u) := \int_I G(u,v)\,\phi(v)\,dv.$$

Lorsque $G \in L^p(I \times I)$, nous notons

$$\|G\|_p = \Big(\int_{I\times I} |G(u,v)|^p \,du\,dv \Big)^{1/p}$$

sa norme dans $L^p(I \times I)$.

La norme en coupure (cut norm) d'un graphon est définie par

$$\|G\|_\square := \sup_{A,B\in\mathcal{B}(I)} \Big| \int_{A\times B} G(u,v)\,du\,dv \Big|.$$

Pour un graphe simple $\mathbb{G} = (V,E)$ avec un ensemble de sommets $V = \{1,2,\dots,N\}$ et un ensemble d'arêtes $E \subseteq V \times V$, nous définissons le *step-graphon* correspondant par

$$G_N(u,v) = \sum_{i,j\in V} \mathbb{1}_{\{(i,j)\in E\}}\, \mathbb{1}_{\left\{u\in\left(\frac{i-1}{N},\,\frac{i}{N}\right]\right\}}\, \mathbb{1}_{\left\{v\in\left(\frac{j-1}{N},\,\frac{j}{N}\right]\right\}}. \tag{2.10}$$

D'après Lovász [177, Lemme 8.11], la norme opérateur qui en résulte s'avère équivalente à la norme en coupure :

$$\|G\|_{\square} \leq \|G\|_{\infty\to 1} \leq 4\,\|G\|_{\square}, \quad \text{où} \quad \|G\|_{\infty\to 1} := \sup_{|\phi|\leq 1} \|G\phi\|_{L^1}.$$

Jeu de graphes à N agents Soit G_N un step-graphon associé à un graphe simple $\mathbb{G}_N = (V_N, E_N)$. Notons $\zeta_{ij}^N := \mathbb{1}_{\{(i,j)\in E_N\}}$ la matrice d'interaction $N \times N$ produite par G_N, décrivant l'interaction entre les joueurs i et j. En particulier, on a $\zeta_{ij}^N = G_N(\frac{i}{N}, \frac{j}{N})$.

Nous reprenons la notation de la Section 2.3.1, en considérant des espaces finis d'états $\mathcal{X}$ et d'actions $\mathcal{A}$, ainsi que les ensembles de temps $\mathcal{T}$ et $\overline{\mathcal{T}}$. Pour $N \in \mathbb{N}^*$, on pose $[N] := \{1, \dots, N\}$. Définissons l'ensemble des politiques pour un joueur par $\mathbf{\Pi} := \mathcal{A}^{\mathcal{T}\times\mathcal{X}}$. Une politique $\pi \in \mathbf{\Pi}$, donnée par $(t, x) \mapsto \pi_t(x) := \pi(t, x) \in \mathcal{A}$, prescrit l'action à entreprendre pour chaque état x à l'instant t.

Dans le cadre du jeu avec graphe fini, la moyenne empirique tenant compte de l'interaction via le graphe est définie, pour un joueur $i \in [N]$, par

$$Z_t^i := \frac{1}{N} \sum_{j\in V_N} \zeta_{i,j}\, \delta_{X_t^j},$$

où le joueur i évolue selon la dynamique suivante :

$$X_0^i \sim \mu_0, \quad a_t^i \sim \pi_t^i(\cdot \mid X_t^i), \quad X_{t+1}^i \sim P(\cdot \mid X_t^i, a_t^i, Z_t^i), \quad \forall\, t \in \overline{\mathcal{T}},\ i \in V_N.$$

La récompense totale du joueur i dans le cadre du jeu avec graphe à N agents est alors définie, pour $\pi^i \in \mathbf{\Pi}$ et $\pi^{-i} \in \mathbf{\Pi}^{N-1}$, par

$$\mathcal{J}^{N,i}(\pi^i; \pi^{-i}) := \mathbb{E}\Big[\sum_{t\in\mathcal{T}} r(X_t^i, a_t^i, Z_t^i)\Big].$$

Définition 2.3.3 (ϵ-Nash)**.** *Pour tout $\epsilon > 0$, une famille de politiques $\hat{\boldsymbol{\pi}}^N = (\hat{\pi}^1, \dots, \hat{\pi}^N) \in \mathbf{\Pi}^N$ est un équilibre de Nash ϵ-approximé s'il existe $N' \in \mathbb{N}$ tel que, pour tout $N > N'$, on ait*

$$\sup_{\tilde{\pi}\in\mathbf{\Pi}} \mathcal{J}^{N,i}(\tilde{\pi}; \hat{\pi}^{-i}) \;-\; \mathcal{J}^{N,i}(\hat{\boldsymbol{\pi}}^N) \leq \epsilon, \qquad \forall\, i \in [N].$$

Un équilibre de Nash est un équilibre de Nash ϵ-approximé avec $\epsilon = 0$.

Jeux à champ moyen sur graphon Lorsque $N \to \infty$, on obtient le *Graphon Mean Field Games* (GMFG) comme limite du jeu à N agents. Introduisons d'abord les notations suivantes:

- $\mathcal{M} := L^2(I; \mathcal{P}(\mathcal{X}))^{\overline{\mathcal{T}}}$ désigne l'ensemble des champs moyens sur graphon.
- $\Pi := L^2(I; \mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}}$ désigne l'ensemble des politiques sur graphon.
- $\mathcal{Z} := L^2(I)^{\overline{\mathcal{T}}}$ désigne l'ensemble des agrégats sur graphon.

Dans le cadre GMFG, on considère un nombre infini d'agents $u \in I$ au lieu d'un nombre fini $i \in V_N$. Un champ moyen $\mu \in \mathcal{P}(\mathcal{X})$, décrit par $(u,t,x) \mapsto \mu_t^u(x) := \mu(u,t,x)$, fournit une distribution de probabilité sur les états. Une politique $\pi \in \Pi$, décrite par $(u,t,x) \mapsto \pi_t^u(x) := \pi(u,t,x) \in \mathcal{A}$, prescrit l'action à entreprendre pour chaque état x à l'instant t et pour tout $u \in I$. L'agrégat du champ moyen sur graphon $Z \in \mathcal{Z}$ est défini par

$$Z_t^u := \int_{v \in I} G(u,v)\mu_t^v \, dv, \qquad \forall\, u \in I, t \in \mathcal{T}.$$

La récompense totale du joueur $u \in I$ est

$$J(\pi^u; Z^u) = \mathbb{E}\Big[\sum_{t \in \mathcal{T}} r\big(X_t^u, a_t^u, Z_t^u\big)\Big],$$

sous les contraintes

$$X_0^u \sim \mu_0, \quad a_t^u \sim \pi_t^u(\cdot \mid X_t^u), \quad X_{t+1}^u \sim P(\cdot \mid X_t^u, a_t^u, Z_t^u), \quad \forall\, t \in \mathcal{T}.$$

De façon analogue à la fonction de champ moyen et à l'opérateur de meilleure réponse introduits en Section 1.3.1, nous définissons l'application de champ moyen sur graphon avec la politique pure que pour tout $x \in \mathcal{X}$, $\mathrm{MF} : \Pi \to \mathcal{M}$ par $\pi \mapsto \mathrm{MF}(\pi) := \mu^\pi$

$$\mu_0^{\pi,u} = \mu_0, \quad \mu_{t+1}^{\pi,u}(x) = \sum_{x' \in \mathcal{X}} \mu_t^{\pi,u}(x')\, P(x \mid x', \pi_t^u(x'), Z_t^u), \qquad \forall\, t \in \mathcal{T}, u \in I$$

L'opérateur de meilleure réponse sur graphon est alors

$$\mathrm{BR} : \mathcal{M} \to 2^\Pi, \quad \mu \;\mapsto\; \mathrm{BR}(\mu) \;:=\; \operatorname*{argmax}_{\pi \in \Pi} J(\pi; \mu) \;\subseteq\; \Pi.$$

2.3.3 Notre contribution : jeux à champ moyen sur graphon étendu

Définition 2.3.4. *Un Graphon Mean Field Games (GMFG) en horizon fini et temps discret est défini par un septuplet $(\overline{\mathcal{T}}, G, \mathcal{X}, \mathcal{A}, p, r, \varphi, \mu_0)$ où :*

- $\overline{\mathcal{T}} = \{0, \dots, T\}$ *est l'ensemble des instants de temps discrets,*
- G *est un graphon,*
- $\mathcal{X}$ *et* $\mathcal{A}$ *sont respectivement l'espace d'états et l'espace d'actions, où* $\mathcal{X}$ *est fini et* $\mathcal{A}$ *est un sous-ensemble compact de* $\mathbb{R}^k$ *pour un entier* $k > 0$ *; nous munissons* $\mathcal{A}$ *de la norme euclidienne, notée* $d_{\mathcal{A}} := \|\cdot\|$*,*
- $P : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathcal{P}(\mathcal{X})$ *est un noyau de transition en un pas,*
- $r : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ *est une fonction de récompense,*
- $\varphi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ *est une fonction mesurable décrivant les interactions,*
- $\mu_0^u := \mu_0 \in \mathcal{P}(\mathcal{X})$ *est une distribution initiale commune à tous les* $u \in I$*.*

Étant donnée une politique $\pi \in \Pi$ et une distribution $\mu \in \mathcal{M}$, on définit l'agrégat $Z^u \in \mathcal{Z}$ pour chaque joueur $u \in I$ par :

$$Z_t^u = \int_{v \in I} G(u,v) \sum_{x \in \mathcal{X}} \varphi\big(x, \pi_t^v(x)\big)\, \mu_t^v(x)\, dv, \qquad \forall\, t \in \overline{\mathcal{T}}.$$

La récompense totale du joueur $u \in I$ dans le cadre d'une GMFG est alors :

$$J(\pi^u; Z^u) = \mathbb{E}\Big[\sum_{t \in \mathcal{T}} r\big(X_t^u, a_t^u, Z_t^u\big)\Big],$$

sous les contraintes

$$X_0^u \sim \mu_0, \quad a_t^u = \pi_t^u(X_t^u), \quad X_{t+1}^u \sim P(\cdot \mid X_t^u, a_t^u, Z_t^u), \quad \forall\, t \in \mathcal{T}.$$

Définition 2.3.5. *Un profil de politiques $\hat{\pi} \in \Pi$ est un Graphon Nash equilibrium (GNE) si, pour chaque $u \in I$, la politique $\hat{\pi}^u$ maximise la fonction $J(\,\cdot\,; Z^{\hat{\pi},u})$. De façon équivalente, une paire $(\hat{\pi}, \hat{Z}) \in \Pi \times \mathcal{Z}$, composée d'une politique et d'un agrégat, est un équilibre de Nash (GMFG) si elle satisfait :*

$$\hat{Z}^u = Z^{\hat{\pi},u} \quad \textit{et} \quad J(\hat{\pi}^u; \hat{Z}^u) \geq J(\pi^u; \hat{Z}^u), \quad \forall\, u \in I,\ \pi^u \in \Pi.$$

Lien entre équilibre de Nash sur graphon et système progressif-rétrogressif

Théorème 2.3.6. *La paire $(\hat{\pi}, \hat{Z}) \in \Pi \times \mathcal{Z}$ est un équilibre de Nash (MFG sur graphon) si et seulement si elle satisfait le système progressif-rétrogressif dans le Theorem 6.4.1.*

Existence de l'équilibre Le résultat suivant montre que le système progressif-rétrogressif admet une solution.

Théorème 2.3.7. *Soit μ_0^u une loi de probabilité sur $\mathcal{X}$. Supposons que les hypothèses de la Section 6.2.4 sont satisfaites. Le système*

$$\begin{cases}
\mu_{t+1}^u(x) = \displaystyle\sum_{x' \in \mathcal{X}} \mu_t^u(x')\, P\big(x \mid x', \pi_t^u(x'), Z_t^u\big), \quad \forall x \in \mathcal{X},\ u \in I,\ t \in \mathcal{T}, \\
V_t^u(x) = \max_a \Big\{ r\big(x, a, Z_t^u\big) + \displaystyle\sum_{x' \in \mathcal{X}} P\big(x' \mid x, a, Z_t^u\big)\, V_{t+1}^u(x') \Big\}, \quad \forall x \in \mathcal{X},\ u \in I,\ t \in \mathcal{T}, \\
Z_t^u = \displaystyle\int_I G(u,v) \sum_{x \in \mathcal{X}} \varphi\big(x, \pi_t^v(x)\big)\, \mu_t^v(x)\, dv, \quad \forall u \in I,\ t \in \mathcal{T}, \\
\pi_t^u(x) = \operatorname*{argmax}_a \Big\{ r\big(x, a, Z_t^u\big) + \displaystyle\sum_{x' \in \mathcal{X}} P\big(x' \mid x, a, Z_t^u\big)\, V_{t+1}^u(x') \Big\}, \quad \forall x \in \mathcal{X},\ u \in I,\ t \in \mathcal{T}, \\
\mu_0^u(x) = \mu_0(x), \quad V_T^u(x) = 0, \quad \forall x \in \mathcal{X},\ u \in I.
\end{cases}$$

admet une solution bornée (V, μ) dans $L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ telle que, pour chaque $u \in I$, μ_0 soit une loi de probabilité sur $\mathcal{X}$.

Équilibre de Nash ϵ-approché Nous montrons que tout GNE fournit un équilibre de Nash ϵ-approché pour le jeu à N joueurs sur graphe et que $\epsilon \to 0$ quand $N \to +\infty$. Présentons d'abord le modèle.

Définition 2.3.8. *Un modèle à N joueurs sur graphe est défini par un tuplet $(N, \overline{\mathcal{T}}, \mathcal{X}, \mathcal{A}, p, r, \varphi, \mu_0)$ où :*

- $N \in \mathbb{N}$ *est le nombre de joueurs,*
- $\overline{\mathcal{T}} = \{0, \dots, T\}$ *l'ensemble des instants de temps discrets,*
- $\mathcal{X}$ *et* $\mathcal{A}$ *sont les espaces d'états et d'actions, identiques à ceux du modèle GMFG,*
- $P : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathcal{P}(\mathcal{X})$ *est un noyau de transition en un pas,*
- $r : \mathcal{X} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ *est une fonction de récompense,*
- $\varphi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ *est une fonction mesurable d'interactions,*
- $\mu_0^i := \mu_0 \in \mathcal{P}(\mathcal{X})$ *est une distribution initiale commune à tous les joueurs $i \in [N]$.*

Dans le cadre du jeu de graphes fini, étant donné une famille de politiques à N joueurs $\boldsymbol{\pi}^N = (\pi^1, \dots, \pi^N) \in \mathbf{\Pi}^N$, nous définissons l'agrégat fini associé au joueur $i \in [N]$ comme

$$Z_t^{N,\boldsymbol{\pi}^N,i} := \sum_{j \in V_N} \zeta_{ij}^N \, \varphi(X_t^j, \pi_t^j(X_t^j)), \qquad \forall\, i \in V_N,\ t \in \overline{\mathcal{T}},$$

où le joueur i évolue selon la dynamique

$$X_0^i \sim \mu_0, \quad a_t^i = \pi_t^i(X_t^i), \quad X_{t+1}^i \sim P(\cdot \mid X_t^i, a_t^i, Z_t^{N,\boldsymbol{\pi}^N,i}), \quad \forall\, i \in V_N,\, t \in \mathcal{T}. \tag{2.11}$$

Notons que l'évolution des états des joueurs est couplée via l'agrégat.

Pour établir un lien entre le jeu à N joueurs sur graphe et le cadre GMFG, pour toutes les politiques à N joueurs $\boldsymbol{\pi}^N = (\pi^1, \dots, \pi^N) \in \mathbf{\Pi}^N$, nous définissons la « politique en escalier » par :

$$\pi_t^{N,\boldsymbol{\pi}^N,u}(x) := \sum_{i \in V_N} \mathbb{1}_{u \in \left(\frac{i-1}{N}, \frac{i}{N}\right]} \, \pi_t^i(x), \quad \forall\, x \in \mathcal{X},\, u \in I,\, t \in \overline{\mathcal{T}}.$$

Étant donné $\boldsymbol{\pi}^N$, on définit la distribution :

$$\mu_t^{N,\boldsymbol{\pi}^N,u} := \sum_{i \in V_N} \mathbb{1}_{u \in \left(\frac{i-1}{N}, \frac{i}{N}\right]} \, \delta_{X_t^i}, \quad \forall\, i \in V_N,\, t \in \overline{\mathcal{T}},$$

où X^i suit l'évolution décrite en (2.11). Enfin, définissons l'application $\Lambda : \Pi \to \mathbf{\Pi}^N$ telle que, pour toute politique $\pi \in \Pi$,

$$\Lambda(\pi) = \left(\pi^{\frac{1}{N}},\ \pi^{\frac{2}{N}},\ \dots,\ \pi^{\frac{N}{N}}\right).$$

Théorème 2.3.9. *Soit $\hat{\pi} \in \Pi$ un GNE. Supposons que les hypothèses de la Section 6.2.4 sont vérifiées. Pour tout $\epsilon > 0$, il existe $N' \in \mathbb{N}$ tel que, pour tout $N > N'$, $\Lambda(\hat{\pi})$ constitue un équilibre*

de Nash ϵ-approché pour le jeu à N joueurs, c'est-à-dire que, pour tout $i \in V_N$,

$$\sup_{\tilde{\pi} \in \mathbf{\Pi}} \mathcal{J}^{N,i}(\tilde{\pi}; \Lambda(\hat{\pi})^{-i}) \; - \; \mathcal{J}^{N,i}(\Lambda(\hat{\pi})) \leq \epsilon.$$

Lemme 2.3.10. *Étant donnés $\pi \in \Pi$ et $\tilde{\pi} \in \mathbf{\Pi}$. Pour $i \in [N]$, considérons les séquences de distributions d'état*

$$\mu^N = (\mu_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),u})_{u \in I, t \in \overline{\mathcal{T}}} \quad et \quad \mu = (\mu_t^{\pi,u})_{u \in I, t \in \overline{\mathcal{T}}}.$$

Pour toute fonction mesurable et bornée $f : \mathcal{X} \times I \to \mathbb{R}$, $(x,u) \mapsto f^u(x) := f(x,u)$, on a

$$\mathbb{E}[\, |\mu_t^N(f) - \mu_t(f)|] \; \to \; 0 \quad lorsque \; N \to +\infty, \qquad \forall\, t \in \overline{\mathcal{T}}.$$

Chapter 3

Maximal Martingale Wasserstein inequality

The content presented in this chapter is based on a published paper:

[150] Benjamin Jourdain and Kexin Shao. "Maximal Martingale Wasserstein inequality". *Electronic Communications in Probability* 29 (2024), pp. 1–8.

Abstract

In this article, we complete the analysis of the Martingale Wasserstein Inequality started in [146] by checking that this inequality fails in dimension $d \geq 2$ when the integrability parameter ρ belongs to $[1, 2)$ while a stronger Maximal Martingale Wasserstein Inequality holds whatever the dimension d when $\rho \geq 2$.

Contents

3.1 Introduction 68
3.2 Proof 70

3.1 Introduction

The present paper elaborates on the convergence to 0 as $n \to \infty$ of

$$\inf_{M \in \Pi^{\mathrm{M}}(\mu_n, \nu_n)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho M(dx, dy)$$

when the Wassertein distance $\mathcal{W}_\rho(\mu_n, \nu_n)$ goes to 0 and for each $n \in \mathbb{N}$, μ_n and ν_n belong to the set $\mathcal{P}_\rho(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ with a finite moment of order $\rho \in [1, +\infty)$ and μ_n is smaller than ν_n in the convex order. The convex order between $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ which is denoted by $\mu \leq_{cx} \nu$ amounts to

$$\int_{\mathbb{R}^d} f(x)\, \mu(dx) \leq \int_{\mathbb{R}^d} f(y)\, \nu(dy) \text{ for each convex function } f : \mathbb{R}^d \to \mathbb{R}, \tag{3.1}$$

and, by Strassen's theorem [216], is equivalent to the non emptyness of the set of martingale couplings between μ and ν defined by

$$\Pi^{\mathrm{M}}(\mu, \nu) = \left\{ M(dx, dy) = \mu(dx) m(x, dy) \in \Pi(\mu, \nu) \mid \mu(dx)\text{-a.e., } \int_{\mathbb{R}^d} y\, m(x, dy) = x \right\}$$

where

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d) \mid \forall A \in \mathcal{B}(\mathbb{R}^d),\ \pi(A \times \mathbb{R}^d) = \mu(A) \text{ and } \pi(\mathbb{R}^d \times A) = \nu(A)\}.$$

The Wasserstein distance with index ρ is defined by

$$\mathcal{W}_\rho(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho\, \pi(dx, dy) \right)^{1/\rho}$$

and we also introduce $\underline{\mathcal{M}}_\rho(\mu, \nu)$ and $\overline{\mathcal{M}}_\rho(\mu, \nu)$ respectively defined by

$$\underline{\mathcal{M}}_\rho^\rho(\mu, \nu) = \inf_{M \in \Pi^{\mathrm{M}}(\mu, \nu)} \int_{\mathbb{R}^{2d}} |x - y|^\rho M(dx, dy), \tag{3.2}$$

$$\overline{\mathcal{M}}_\rho^\rho(\mu, \nu) = \sup_{M \in \Pi^{\mathrm{M}}(\mu, \nu)} \int_{\mathbb{R}^{2d}} |x - y|^\rho\, M(dx, dy). \tag{3.3}$$

In dimension $d = 1$, the optimization problems defining $\underline{\mathcal{M}}_\rho$ and $\overline{\mathcal{M}}_\rho$ are the respective subjects of [135] and [136] when $\rho = 1$, while the general case $\rho \in (0, +\infty)$ is studied in [151].

The question of interest is related to the stability of Martingale Optimal Transport problems with respect to the marginal distributions μ and ν established in dimension $d = 1$ in [22, 227] while it fails in higher dimension according to [62]. A quantitative answer is given in dimension $d = 1$ by the Martingale Wasserstein inequality established in [146, Proposition 1] for $\rho \in [1, +\infty)$,

$$\exists \underline{C}_{(\rho,\rho),1} < \infty,\ \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \text{ with } \mu \leq_{cx} \nu,\ \underline{\mathcal{M}}_\rho^\rho(\mu, \nu) \leq \underline{C}_{(\rho,\rho),1} \mathcal{W}_\rho(\mu, \nu) \sigma_\rho^{\rho-1}(\nu), \tag{3.4}$$

where the central moment $\sigma_\rho(\nu)$ of ν is defined by

$$\sigma_\rho(\nu) = \inf_{c\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |y-c|^\rho \, \nu(dy) \right)^{1/\rho} \text{ when } \rho \in [1, +\infty),$$

and

$$\sigma_\infty(\nu) = \inf_{c\in\mathbb{R}^d} \nu - \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |y-c|.$$

The proposition also states that $\mathcal{W}_\rho(\mu,\nu)$ and $\sigma_\rho(\nu)$ have the right exponent in this inequality in the sense that for $1 < s \le \rho$,

$$\sup_{\substack{\mu,\nu\in\mathcal{P}_\rho(\mathbb{R}) \\ \mu\le_{cx}\nu, \mu\neq\nu}} \frac{\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_\rho^s(\mu,\nu)\sigma_\rho^{\rho-s}(\nu)} = +\infty.$$

The generalization of (3.4) to higher dimensions d is also investigated in [146] where it is proved that for any $d \ge 2$,

$$\underline{C}_{(\rho,\rho),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_\rho(\mathbb{R}^d) \\ \mu\le_{cx}\nu, \mu\neq\nu}} \frac{\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_\rho(\mu,\nu)\sigma_\rho^{\rho-1}(\nu)}$$

is infinite when $\rho \in [1, \frac{1+\sqrt{5}}{2})$, while the one-dimensional constant $\underline{C}_{(\rho,\rho),1}$ is preserved when μ and ν are products of one-dimensional probability measures or when, for X distributed according to μ, the conditional expectation of X given the direction of $X - \mathbb{E}[X]$ is a.s. equal to $\mathbb{E}[X]$ and ν is the distribution of $X + \lambda(X - \mathbb{E}[X])$ for some $\lambda \ge 0$. The present paper answers the question of the finiteness of $\underline{C}_{(\rho,\rho),d}$ when $\rho \in [\frac{1+\sqrt{5}}{2}, +\infty)$ and $d \ge 2$, which remained open. It turns out that $\underline{C}_{(\rho,\rho),d} = +\infty$ for $d \ge 2$ when $\rho \in [1,2)$ while for $\rho \in [2, +\infty)$ the inequality (3.4) generalizes in any dimension d into a Maximal Martingale Wasserstein inequality with the left-hand side $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ replaced by the larger $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$. We even replace conjugate exponents ρ and $\frac{\rho}{\rho-1}$ leading to the respective indices $\rho = \rho \times 1$ and $\rho = \frac{\rho}{\rho-1} \times (\rho-1)$ in the factors $\mathcal{W}$ and σ in (3.4) by general conjugate exponents $q \in [1, +\infty]$ and $\frac{q}{q-1} \in [1, +\infty]$ leading to indices q and $\frac{q(\rho-1)}{q-1}$ (equal to $+\infty$ and $\rho - 1$ when q is respectively equal to 1 and $+\infty$) and define

$$\underline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d) \\ \mu\le_{cx}\nu, \mu\neq\nu}} \frac{\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)}$$

and

$$\overline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d) \\ \mu\le_{cx}\nu, \mu\neq\nu}} \frac{\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)},$$

with

$$\mathcal{W}_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \pi - \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} |x-y|.$$

Since $\underline{\mathcal{M}}_\rho \le \overline{\mathcal{M}}_\rho$, one has $\underline{C}_{(\rho,q),d} \le \overline{C}_{(\rho,q),d}$. These constants of course depend on the norm $|\cdot|$

on $\mathbb{R}^d$ (even if we do not make this dependence explicit), but, by equivalence of the norms, their finiteness does not. Since the Euclidean norm plays a particular role, we will denote it by $\|\cdot\|$ rather than $|\cdot|$.

Theorem 3.1.1. *(i) Let $\rho \in [1,2)$. For $q \in [1, \frac{1}{2-\rho}]$ (and even $q \in [1,+\infty]$ when $\rho = 1$), one has $\underline{C}_{(\rho,q),1} \leq K_\rho < +\infty$ where the constant K_ρ is studied in [146, Proposition 1] while, for $q \in [1,+\infty]$, $\overline{C}_{(\rho,q),1} = +\infty$ and $\underline{C}_{(\rho,q),d} = +\infty$ for $d \geq 2$.*

(ii) Let $\rho \in [2,+\infty)$ and $q \in [1,+\infty]$. One has $\overline{C}_{(\rho,q),d} < +\infty$ whatever d. Moreover, when each vector space $\mathbb{R}^d$ is endowed with the Euclidean norm, $\overline{C}_{(2,q),d} = 2$ and $\sup_{d\geq 1} \overline{C}_{(\rho,q),d} < +\infty$.

Remark 3.1.2.

- *The fact that $\rho = 2$ appears as a threshold is related to the equality*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \|y-x\|^2 M(dx,dy) = \int_{\mathbb{R}^d} \|y\|^2 \nu(dy) - \int_{\mathbb{R}^d} \|x\|^2 \mu(dx)$$

for $M \in \Pi^M(\mu,\nu)$ when $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$ are such that $\mu \leq_{cx} \nu$, which implies that when $\mathbb{R}^d$ is endowed with the Euclidean norm

$$\underline{\mathcal{M}}_2^2(\mu,\nu) = \overline{\mathcal{M}}_2^2(\mu,\nu) = \int_{\mathbb{R}^d} \|y\|^2 \nu(dy) - \int_{\mathbb{R}^d} \|x\|^2 \mu(dx).$$

- *For $\rho \in [1,2)$ and $q \in [1,+\infty]$, since $\overline{C}_{(\rho,q),d} \geq \overline{C}_{(\rho,q),1}$, one has $\overline{C}_{(\rho,q),d} = +\infty$ by Theorem 3.1.1 (i), while*

$$\sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{q}{q-1}}(\mathbb{R}^d)\\ \mu\leq_{cx}\nu,\mu\neq\nu}} \frac{\overline{\mathcal{M}}_\rho^2(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma_{\frac{q}{q-1}}(\nu)} \leq \overline{C}_{(2,q),d} < +\infty$$

since $\overline{\mathcal{M}}_\rho \leq \overline{\mathcal{M}}_2$.

3.2 Proof

The proof of Theorem 3.1.1 (*ii*) relies on the next lemma, the proof of the lemma is postponed after the proof of the theorem. In what follows, to avoid making distinctions in case $q \in \{1,+\infty\}$, we use the convention that for any probability measure γ and any measurable function f on the same probability space $\left(\left(\int |f(z)|^{\frac{q}{q-1}}\gamma(dz)\right)^{(q-1)/q}, \left(\int |f(z)|^{\frac{q(\rho-1)}{q-1}}\gamma(dz)\right)^{(q-1)/q}\right)$ (resp. $\left(\int |f(z)|^q\gamma(dz)\right)^{1/q}$) is equal to $(\gamma-\operatorname{ess\,sup}_z |f(z)|, \gamma-\operatorname{ess\,sup}_z |f(z)|^{\rho-1})$ (resp. $\gamma-\operatorname{ess\,sup}_z |f(z)|$) when $q = 1$ (resp. $q = +\infty$).

Lemma 3.2.1. *Given $\rho \in [2,+\infty)$, there exist constants $\kappa_\rho, \tilde{\kappa}_\rho \in [0,+\infty)$ such that for all $d \geq 1$ and $x,y \in \mathbb{R}^d$,*

$$\|x-y\|^\rho \leq \kappa_\rho \left((\rho-1)\|x\|^\rho + \|y\|^\rho - \rho\|x\|^{\rho-2}\langle x,y\rangle\right), \tag{3.5}$$

$$\|y\|^\rho - \|x\|^\rho \leq \tilde{\kappa}_\rho \|y-x\| \left(\|x\|^{\rho-1} + \|y\|^{\rho-1}\right). \tag{3.6}$$

Remark 3.2.2. *When $\rho = 2$, then (3.5) holds as an equality with $\kappa_\rho = 1$ while, by the Cauchy-Schwarz and the triangle inequalities,*

$$\|y\|^2 - \|x\|^2 \leq \langle y - x, y + x\rangle \leq \|y - x\| \times \|y + x\| \leq \|y - x\| \left(\|x\| + \|y\|\right)$$

so that (3.6) holds with $\tilde{\kappa}_\rho = 1$.

Proof of Theorem 3.1.1. (i) In dimension $d = 1$, one has $\underline{\mathcal{M}}_1 \leq K_1 \mathcal{W}_1$ with $K_1 = 2$ according to [146, Proposition 1] and we deduce that $\underline{C}_{(1,q),1} \leq K_1$ for $q \in [1, +\infty]$ since $\mathcal{W}_1 \leq \mathcal{W}_q$. Now, let $\rho \in (1, 2)$ and $q \in [1, \frac{1}{2-\rho}]$. One has $\frac{q(\rho-1)}{q-1} \geq 1$ since, when $q > 1$, $\frac{q}{q-1} = 1 + \frac{1}{q-1} \geq 1 + \frac{2-\rho}{\rho-1} = \frac{1}{\rho-1}$. For $\mu, \nu \in \mathcal{P}_{q \vee \frac{q(\rho-1)}{q-1}}(\mathbb{R})$ with respective quantile functions F_μ^{-1} and F_ν^{-1}, by optimality of the comonotonic coupling and Hölder's inequality, one has

$$\begin{aligned}
\mathcal{W}_\rho^\rho(\mu, \nu) &= \int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)| \times |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\rho-1} du \\
&\leq \left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^q du\right)^{\frac{1}{q}} \left(\left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}}\right)^{\rho-1}.
\end{aligned}$$

Since, by the triangle inequality and $\mu \leq_{cx} \nu$, one has for $c \in \mathbb{R}$

$$\begin{aligned}
&\left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}} \\
&\quad \leq \left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}} + \left(\int_0^1 |F_\mu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}} \\
&\quad \leq 2 \left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}},
\end{aligned}$$

we deduce by minimizing over the constant c that

$$\mathcal{W}_\rho^\rho(\mu, \nu) \leq \mathcal{W}_q(\mu, \nu) \times 2^{\rho-1} \sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu).$$

With this inequality replacing (30) in the proof of [146, Proposition 1] and the general inequality

$$\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)||F_\nu^{-1}(u) - c|^{\rho-1} du \leq \mathcal{W}_q(\mu, \nu) \left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q}},$$

replacing the special case $q = \rho$ in the second equation p840 in this proof, we deduce that $\underline{\mathcal{M}}_\rho^\rho(\mu, \nu) \leq K_\rho \mathcal{W}_q(\mu, \nu) \sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)$.

To check that $\overline{C}_{(\rho,q),1} = +\infty$ for $\rho \in [1, +\infty)$ and $q \in [1, +\infty]$, let us introduce for $n \geq 2$ and $z > 0$,

$$\mu_{n,z} = \frac{1}{2((n-1)z+1)}\left((1+z)(\delta_1+\delta_n) + 2z\sum_{i=2}^{n-1}\delta_i\right)$$
$$\text{and } \nu_{n,z} = \frac{1}{2((n-1)z+1)}\left(\delta_{1-z}+\delta_{n+z}+z(\delta_1+\delta_n) + 2z\sum_{i=2}^{n-1}\delta_i\right).$$

This example generalizes the one introduced by Brückerhoff and Juillet in [62] which corresponds to the choice $z=1$. Since

$$M_{n,z} = \frac{1}{2((n-1)z+1)}\left(\delta_{(1,1-z)} + z\delta_{(1,2)} + z\delta_{(n,n-1)} + \delta_{(n,n+z)} + z\sum_{i=2}^{n-1}\left(\delta_{(i,i-1)}+\delta_{(i,i+1)}\right)\right)$$

belongs to $\Pi^M(\mu_{n,z},\nu_{n,z})$, we have

$$\overline{\mathcal{M}}_\rho^\rho(\mu_{n,z},\nu_{n,z}) \geq \int_{\mathbb{R}\times\mathbb{R}} |y-x|^\rho M_{n,z}(dx,dy) = \frac{(n-1)z+z^\rho}{(n-1)z+1}.$$

On the other hand, by optimality of the comonotonic coupling $\mathcal{W}_\rho^\rho(\mu_{n,z},\nu_{n,z}) = \frac{z^\rho}{(n-1)z+1}$ for $\rho\in[1,+\infty)$ and $\mathcal{W}_\infty(\mu_{n,z},\nu_{n,z}) = z$. Last $\sigma_\infty(\nu_{n,z}) = \dfrac{n-1+2z}{2}$ and, when $\rho\in[1,+\infty)$,

$$\sigma_\rho^\rho(\nu_{n,z}) = \frac{1}{2^\rho((n-1)z+1)}\left((n-1+2z)^\rho + z(n-1)^\rho + 2z\sum_{i=2}^{\lfloor\frac{n+1}{2}\rfloor}(n+1-2i)^\rho\right),$$

where

$$2\sum_{i=2}^{\lfloor\frac{n+1}{2}\rfloor}(n+1-2i)^\rho \sim \frac{n^{1+\rho}}{1+\rho} \quad \text{as} \quad n\to\infty.$$

Let $\alpha\in[0,1)$. The sequence $n^{1-\alpha}$ goes to ∞ with n and for $\rho\in[1,+\infty)$ and $q\in[1,+\infty]$, we have

$$\int_{\mathbb{R}\times\mathbb{R}} |y-x|^\rho M_{n,n^{-\alpha}}(dx,dy) \to 1,\ \mathcal{W}_q(\mu_{n,n^{-\alpha}},\nu_{n,n^{-\alpha}}) \sim n^{\alpha\frac{(1-q)}{q}-\frac{1}{q}}$$

and

$$\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_{n,n^{-\alpha}}) \sim \frac{n^{\rho-1}}{2^{\rho-1}\left(1+\frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}}}$$

where $\left(1+\frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}} = 1$ by convention when $q=1$ so that

$$\frac{\int_{\mathbb{R}\times\mathbb{R}}|y-x|^\rho M_{n,n^{-\alpha}}(dx,dy)}{\mathcal{W}_q(\mu_{n,n^{-\alpha}},\nu_{n,n^{-\alpha}})\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_{n,n^{-\alpha}})} \sim 2^{\rho-1}\left(1+\frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}} n^{\frac{q-1}{q}\alpha+\frac{1}{q}+1-\rho}.$$

Let $\rho\in[1,2)$. For $q=1$, the exponent of n in the equivalent of the ratio is equal to $2-\rho>0$ so that the right-hand side goes to $+\infty$ with n. For $q\in(1,+\infty]$, we may choose $\alpha\in\left(\frac{q(\rho-1)-1}{q-1},1\right)$

(with left boundary equal to $\rho-1$ when $q=+\infty$) so that $\frac{q-1}{q}\alpha+\frac{1}{q}+1-\rho>0$ and the right-hand side still goes to $+\infty$ with n. Therefore, $\overline{C}_{(\rho,q),1}=+\infty$. To prove that $\underline{C}_{(\rho,q),d}=+\infty$ for $d\geq 2$ it is enough by [146, Lemma 1] to deal with the case $d=2$, in which we use the rotation argument in [62]. For $n\geq 2$ and $\theta\in(0,\pi)$, M_n^θ defined as $\frac{1}{2((n-1)n^{-\alpha}+1)}$ times

$$\begin{aligned}&\delta_{((1,0),(1-n^{-\alpha}\cos\theta,-n^{-\alpha}\sin\theta))}+n^{-\alpha}\delta_{((1,0),(1+\cos\theta,\sin\theta))}+n^{-\alpha}\delta_{((n,0),(n-\cos\theta,-\sin\theta))}\\&+\delta_{((n,0),(n+n^{-\alpha}\cos\theta,n^{-\alpha}\sin\theta))}+n^{-\alpha}\sum_{i=2}^{n-1}\left(\delta_{((i,0),(i-\cos\theta,-\sin\theta))}+\delta_{((i,0),(i+\cos\theta,\sin\theta))}\right)\end{aligned}$$

which is a martingale coupling between the image μ_n of $\mu_{n,n^{-\alpha}}$ by $\mathbb{R}\ni x\mapsto(x,0)\in\mathbb{R}^2$ and its second marginal ν_n^θ which, as $\theta\to 0$, converges in any $\mathcal{W}_q$ with $q\in[1,+\infty]$ to the image of $\nu_{n,n^{-\alpha}}$ by the same mapping. According to the proof of [62, Lemma 1.1], $\Pi^M(\mu_n,\nu_n^\theta)=\{M_n^\theta\}$ so that

$$\underline{\mathcal{M}}_\rho^\rho(\mu_n,\nu_n^\theta)=\int_{\mathbb{R}^2\times\mathbb{R}^2}|y-x|^\rho M_n^\theta(dx,dy)$$

and

$$\lim_{\theta\to 0}\frac{\underline{\mathcal{M}}_\rho^\rho(\mu_n,\nu_n^\theta)}{\mathcal{W}_q(\mu_n,\nu_n^\theta)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_n^\theta)}=\frac{\int_{\mathbb{R}\times\mathbb{R}}|y-x|^\rho M_{n,n^{-\alpha}}(dx,dy)}{\mathcal{W}_q(\mu_{n,n^{-\alpha}},\nu_{n,n^{-\alpha}})\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_{n,n^{-\alpha}})}.$$

With the above analysis of the asymptotic behaviour of the right-hand side as $n\to\infty$, we conclude that $\underline{C}_{(\rho,q),d}=+\infty$.

(ii) Now, let $\rho\in[2,+\infty)$ and $M\in\Pi^M(\mu,\nu)$. Applying Equation (3.5) in Lemma 3.2.1 for the inequality and then using the martingale property of M, we obtain that for $c\in\mathbb{R}^d$, we have

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d}\|x-y\|^\rho M(dx,dy)&=\int_{\mathbb{R}^d\times\mathbb{R}^d}\|(x-c)-(y-c)\|^\rho M(dx,dy)\\&\leq\kappa_\rho\int_{\mathbb{R}^d\times\mathbb{R}^d}\left((\rho-1)\|x-c\|^\rho+\|y-c\|^\rho-\rho\|x-c\|^{\rho-2}\langle x-c,y-c\rangle\right)M(dx,dy)\\&=\kappa_\rho\left(\int_{\mathbb{R}^d}\|y-c\|^\rho\nu(dy)-\int_{\mathbb{R}^d}\|x-c\|^\rho\mu(dx)\right).\end{aligned}\tag{3.7}$$

Denoting by $\pi\in\Pi(\mu,\nu)$ an optimal coupling for $\mathcal{W}_q(\mu,\nu)$, we have using Equation (3.6) in Lemma 3.2.1 for the inequality

$$\begin{aligned}\int_{\mathbb{R}^d}\|y-c\|^\rho\nu(dy)-\int_{\mathbb{R}^d}\|x-c\|^\rho\mu(dx)&=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\|y-c\|^\rho-\|x-c\|^\rho\right)\pi(dx,dy)\\&\leq\tilde{\kappa}_\rho\int_{\mathbb{R}^d\times\mathbb{R}^d}\|y-x\|\left(\|x-c\|^{\rho-1}+\|y-c\|^{\rho-1}\right)\pi(dx,dy).\end{aligned}\tag{3.8}$$

By the fact that all norms are equivalent in finite dimensional vector spaces, there exists $\lambda\in[1,\infty)$ such that for all $z\in\mathbb{R}^d$, we have

$$\frac{\|z\|}{\lambda}\leq|z|\leq\lambda\|z\|.$$

Therefore, using (3.7) and (3.8) for the second inequality, Hölder's inequality for the fourth, the

triangle inequality for the fifth and $\mu \leq_{cx} \nu$ for the sixth, we get that for $c \in \mathbb{R}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^\rho \, M(dx,dy) \leq \lambda^\rho \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\|^\rho \, M(dx,dy) \\
&\leq \kappa_\rho\tilde{\kappa}_\rho\lambda^\rho \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\| \left(\|x-c\|^{\rho-1} + \|y-c\|^{\rho-1}\right) \pi(dx,dy) \\
&\leq \kappa_\rho\tilde{\kappa}_\rho\lambda^{2\rho} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y| \left(|x-c|^{\rho-1} + |y-c|^{\rho-1}\right) \pi(dx,dy) \\
&\leq \kappa_\rho\tilde{\kappa}_\rho\lambda^{2\rho}\mathcal{W}_q(\mu,\nu) \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(|x-c|^{\rho-1} + |y-c|^{\rho-1}\right)^{\frac{q}{q-1}} \pi(dx,dy)\right)^{\frac{q-1}{q}} \\
&\leq \kappa_\rho\tilde{\kappa}_\rho\lambda^{2\rho}\mathcal{W}_q(\mu,\nu) \left(\left(\int_{\mathbb{R}^d} |x-c|^{\frac{q(\rho-1)}{q-1}} \mu(dx)\right)^{(q-1)/q} + \left(\int_{\mathbb{R}^d} |y-c|^{\frac{q(\rho-1)}{q-1}} \nu(dy)\right)^{(q-1)/q}\right) \\
&\leq 2\kappa_\rho\tilde{\kappa}_\rho\lambda^{2\rho}\mathcal{W}_q(\mu,\nu) \left(\int_{\mathbb{R}^d} |y-c|^{\frac{q(\rho-1)}{q-1}} \nu(dy)\right)^{\frac{q-1}{q}}.
\end{aligned}$$

By taking the infimum with respect to $c \in \mathbb{R}^d$, we conclude that the statement holds with $\overline{C}_{(\rho,q),d} \leq 2\kappa_\rho\tilde{\kappa}_\rho\lambda^{2\rho}$. Finally, let us suppose that $\mathbb{R}^d$ is endowed with the Euclidean norm. Then we can choose $\lambda = 1$, so that $\overline{C}_{(\rho,q),d} \leq 2\kappa_\rho\tilde{\kappa}_\rho$ with the right-hand side not depending on d according to Lemma 3.2.1. Moreover, by Remark 3.2.2, $\overline{C}_{(2,q),d} \leq 2$ and since for $\alpha \in [0,1)$,

$$\lim_{n\to\infty} \frac{\overline{\mathcal{M}}_2^2(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}})}{\sqrt{\mathcal{W}_1(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}})\sigma_\infty(\nu_{n,n^{-\alpha}})}} = 2,$$

we have $\overline{C}_{(2,q),d} = 2$. □

Proof of Lemma 3.2.1. We suppose that $\rho > 2$ since the case $\rho = 2$ has been addressed in Remark 3.2.2.

Suppose $x \neq 0$ and $y \neq x$ and set $e = \frac{x}{\|x\|}$ and $z = \frac{\langle y,x\rangle}{\|x\|^2}$. The vector $\frac{y}{\|x\|} - ze$ is orthogonal to e and can be rewritten as $\omega e^\perp$ with $\omega \geq 0$ and $e^\perp \in \mathbb{R}^d$ such that $\|e^\perp\| = 1$ and $\langle e, e^\perp\rangle = 0$. One then has $\frac{y}{\|x\|} = ze + \omega e^\perp$ and since $y \neq x$, $(z,\omega) \neq (1,0)$.

The first inequality (3.5) divided by $\|x\|^\rho$ writes:

$$\left((1-z)^2 + \omega^2\right)^{\frac{\rho}{2}} \leq \kappa_\rho \left((\rho-1) + \left(z^2+\omega^2\right)^{\frac{\rho}{2}} - \rho z\right).$$

Let us define

$$\varphi(z,\omega) = \rho - 1 + (z^2+\omega^2)^{\frac{\rho}{2}} - \rho z = -\rho(z-1) - 1 + \left(1 + 2(z-1) + (z-1)^2 + \omega^2\right)^{\frac{\rho}{2}}$$

as the second factor in the right-hand side. Applying a second order Taylor's expansion at $t = 0$ to $t \mapsto (1+t)^{\frac{\rho}{2}}$ and using that, by Young's inequality,

$$|(z-1)\omega^2| \leq \frac{|z-1|^3}{3} + \frac{2|\omega|^3}{3} = o((z-1)^2 + \omega^2),$$

we obtain

$$\begin{aligned}\varphi(z,\omega) =& -\rho(z-1) - 1 + 1 + \frac{\rho}{2}\left(2(z-1) + (z-1)^2 + \omega^2\right)\\ &+ \frac{\rho(\rho-2)}{8}(2(z-1) + (z-1)^2 + \omega^2)^2\\ &+ \mathcal{O}((2(z-1) + (z-1)^2 + \omega^2)^3)\\ =&\frac{\rho}{2}\omega^2 + \frac{\rho}{2}(\rho-1)(z-1)^2 + o((z-1)^2 + \omega^2).\end{aligned}$$

Since $\rho > 2$, we conclude that

$$\lim_{(z,\omega)\to(1,0)} \frac{((1-z)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)} = 0.$$

As $|(z,\omega)| \to +\infty$, $\varphi(z,\omega) \sim (z^2+\omega^2)^{\frac{\rho}{2}} \sim ((z-1)^2+\omega^2)^{\rho}$. Therefore,

$$\lim_{|(z,\omega)|\to+\infty} \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)} = 1.$$

The function $(z,\omega) \mapsto \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)}$ being continuous on $\mathbb{R}^2 \setminus \{(1,0)\}$, we deduce that

$$1 \leq \sup_{(z,\omega)\neq(1,0)} \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)} < +\infty.$$

Since when $x = 0$ or $y = x$, (3.5) holds with κ_ρ replaced by 1, we conclude that the optimal constant is

$$\kappa_\rho = \sup_{(z,\omega)\neq(1,0)} \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)}.$$

For the second inequality (3.6), we can apply the same approach: divided by $\|x\|^\rho$, it writes

$$\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1 \leq \tilde{\kappa}_\rho \left((z-1)^2+\omega^2\right)^{\frac{1}{2}} \left((z^2+\omega^2)^{\frac{\rho-1}{2}} + 1\right).$$

As $(z,\omega) \to (1,0)$, $\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1 = \left(1 + 2(z-1) + (z-1)^2 + \omega^2\right)^{\frac{\rho}{2}} - 1 \sim \frac{\rho}{2}\left(2(z-1) + \omega^2\right)$

$$\limsup_{(z,\omega)\to(1,0)} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1}{((z-1)^2+\omega^2)^{\frac{1}{2}}\left(1 + (z^2+\omega^2)^{\frac{\rho-1}{2}}\right)} = \limsup_{z\to1} \frac{\rho(z-1)}{2|z-1|} = \frac{\rho}{2}.$$

On the other hand,

$$\lim_{|(z,\omega)|\to+\infty} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1}{((z-1)^2+\omega^2)^{\frac{1}{2}}\left(1 + (z^2+\omega^2)^{\frac{\rho-1}{2}}\right)} = 1.$$

By continuity of the considered function over $\mathbb{R}^2 \setminus \{(1,0)\}$, we deduce that

$$\frac{\rho}{2} \vee 1 \leq \sup_{(z,\omega)\neq(1,0)} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1}{\left((z-1)^2+\omega^2\right)^{\frac{1}{2}} \left(1+\left(z^2+\omega^2\right)^{\frac{\rho-1}{2}}\right)} < +\infty.$$

Since when $x = 0$ or $y = x$, (3.6) holds with $\tilde{\kappa}_\rho$ replaced by 1, we conclude that the optimal constant is

$$\tilde{\kappa}_\rho = \sup_{(z,\omega)\neq(1,0)} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1}{\left((z-1)^2+\omega^2\right)^{\frac{1}{2}} \left(1+\left(z^2+\omega^2\right)^{\frac{\rho-1}{2}}\right)}.$$

□

Chapter 4

Non-decreasing martingale couplings

The content presented in this chapter is based on a published paper:

[151] Benjamin Jourdain and Kexin Shao. "Non-decreasing martingale couplings". *ESAIM: PS* 29 (2025), pp. 1–44.

Abstract

For many examples of couples (μ, ν) of probability measures on the real line in the convex order, we observe numerically that the Hobson and Neuberger martingale coupling, which maximizes for $\rho = 1$ the integral of $|y - x|^\rho$ with respect to any martingale coupling between μ and ν, is still very close to maximize this integral for $\rho \in (0, 2)$ and minimize it for $\rho > 2$. We investigate theoretically when it is an optimizer for $\rho \neq 1$ and exhibit rather restrictive sufficient conditions. We also exhibit couples (μ, ν) such that it is not an optimizer. The support of the Hobson and Neuberger coupling is known to satisfy some monotonicity property which we call non-decreasing. We check that the non-decreasing property is preserved for maximizers when $\rho \in (0, 1]$. In general, there exist distinct non-decreasing martingale couplings, and we find some decomposition of ν which is in one-to-one correspondence with martingale couplings non-decreasing in a generalized sense.

Contents

4.1 Introduction **78**
4.2 Statement of the results **81**
4.2.1 Numerical experiments 81
4.2.2 Infimum and supremum of $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$ for the convex order 83
4.2.3 Definition and optimality of non-decreasing and non-increasing martingale couplings . 83
4.2.4 Main result : directionally decomposed non-decreasing martingale couplings 86
4.2.5 Do $\pi^\uparrow$ and $\pi^\downarrow$ optimize the cost function $|x-y|^\rho$ for $\rho > 1$ under the sandwich condition ? 88
4.3 Proofs **91**
4.3.1 Proof of Lemma 4.2.2 92
4.3.2 Proofs of the optimality results in Section 4.2.3 apart from Proposition 4.2.8 93
4.3.3 Proof of the results in Section 4.2.4 and Proposition 4.2.8 98
4.3.4 Proofs of the results in Section 4.2.5 concerning optimality of $\pi^\uparrow$ and $\pi^\downarrow$ for the cost function $|x - y|^\rho$ with $\rho > 1$ under the sandwich condition 117

4.1 Introduction

In this paper, for μ, ν in the set $\mathcal{P}_1(\mathbb{R})$ of probability measures on the real line with a finite first order moment, we are interested in martingale couplings between μ and ν that attain $\overline{\mathcal{M}}_\rho(\mu,\nu)$ and $\underline{\mathcal{M}}_\rho(\mu,\nu)$ defined for $\rho > 0$ by

$$\overline{\mathcal{M}}_\rho^\rho(\mu,\nu) = \sup_{\pi\in\Pi_M(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^\rho \,\pi(dx,dy),$$
$$\underline{\mathcal{M}}_\rho^\rho(\mu,\nu) = \inf_{\pi\in\Pi_M(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^\rho \,\pi(dx,dy).$$

Here $\Pi_M(\mu,\nu)$ denotes the subset of

$$\Pi(\mu,\nu) = \left\{\pi \in \mathcal{P}(\mathbb{R}\times\mathbb{R}) \mid \int_{y\in\mathbb{R}} \pi(dx,dy) = \mu(dx), \int_{x\in\mathbb{R}} \pi(dx,dy) = \nu(dy)\right\}$$

consisting in martingale couplings :

$$\Pi_M(\mu,\nu) = \left\{\pi(dx,dy) = \mu(dx)\pi_x(dy) \in \Pi(\mu,\nu) \mid \mu(dx)\text{-a.e., } \int_{y\in\mathbb{R}} y\,\pi_x(dy) = x\right\}.$$

By Strassen's theorem [216],

$$\Pi_M(\mu,\nu) \neq \emptyset \quad \Longleftrightarrow \quad \mu \leq_{cx} \nu,$$

where $\mu \leq_{cx} \nu$ means that μ is smaller than ν in the convex order :

$$\forall \varphi : \mathbb{R} \to \mathbb{R} \text{ convex}, \int_{\mathbb{R}} \varphi(x)\,\mu(dx) \leq \int_{\mathbb{R}} \varphi(y)\,\nu(dy).$$

When $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ are such that $\mu \leq_{cx} \nu$, then $\underline{\mathcal{M}}_\rho(\mu,\nu) \leq \overline{\mathcal{M}}_\rho(\mu,\nu) < +\infty$ for $\rho \in (0,1]$. For $\rho > 1$, the finiteness remains true when μ, ν belong to the set $\mathcal{P}_\rho(\mathbb{R})$ of probability measures on the real line with a finite moment of order ρ. For $\rho \geq 1$, a martingale Wasserstein inequality is investigated in [146]: it is proved that there exists a finite constant $\underline{K}_\rho$ such that for $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$ with $\mu \leq_{cx} \nu$,

$$\underline{\mathcal{M}}_\rho^\rho(\mu,\nu) \leq \underline{K}_\rho \left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^\rho \,\pi(dx,dy)\right)^{\frac{1}{\rho}} \min_{z\in\mathbb{R}} \left(\int_{\mathbb{R}} |z-y|^\rho \nu(dy)\right)^{\frac{\rho-1}{\rho}}.$$

According to [150], when $\rho \geq 2$, there even exists a finite constant $\overline{K}_\rho$ such that for $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$ with $\mu \leq_{cx} \nu$,

$$\overline{\mathcal{M}}_\rho^\rho(\mu,\nu) \leq \overline{K}_\rho \left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^\rho \,\pi(dx,dy)\right)^{\frac{1}{\rho}} \min_{z\in\mathbb{R}} \left(\int_{\mathbb{R}} |z-y|^\rho \nu(dy)\right)^{\frac{\rho-1}{\rho}}.$$

The couplings attaining $\overline{\mathcal{M}}_1(\mu,\nu)$ and $\underline{\mathcal{M}}_1(\mu,\nu)$ were first investigated in the literature. Motivated by the robust pricing and hedging of forward start straddle options, Hobson and Neuberger

state in Theorem 8.2 [136] that for the cost function $|x-y|$, there exists a maximizing martingale coupling with the form

$$\pi^{\mathrm{HN}} = \int_0^1 \left(\frac{r(u)-q(u)}{r(u)-p(u)} \, \delta_{(q(u),p(u))} + \frac{q(u)-p(u)}{r(u)-p(u)} \, \delta_{(q(u),r(u))} \right) du$$

with p, q, r non-decreasing on $(0,1)$ and such that $p \leq q \leq r$ (by convention the integrand is equal to $\delta_{(q(u),q(u))}$ when $p(u) = q(u) = r(u)$). Note that $u \mapsto q(u)$ is then a quantile function of μ. This provides an example of a martingale coupling non-decreasing in the sense of Definition 4.2.3 that we give below. The study of such non-decreasing martingale couplings is one of main contributions of the present paper. The necessary optimality criterion given by Beiglböck and Juillet in Lemma 1.11 [35] ensures that any maximizing martingale coupling is non-decreasing. When μ does not weight points, we have the equality

$$\pi^{\mathrm{HN}} = \int_{\mathbb{R}} \left(\frac{g(x)-x}{g(x)-f(x)} \, \delta_{(x,f(x))} + \frac{x-f(x)}{g(x)-f(x)} \, \delta_{(x,g(x))} \right) \mu(dx)$$

for non-decreasing functions f and g such that $\forall x \in \mathbb{R}$, $f(x) \leq x \leq g(x)$ and, according to Theorem 7.3 [35], uniqueness of maximizing martingale couplings holds. Under the dispersion assumption that there exists a finite interval I such that $(\nu-\mu)^+(I) = 0$ and $(\mu-\nu)^+(I) = (\mu-\nu)^+(\mathbb{R})$, Hobson and Klimmek state in Theorem [135] that for the cost function $|x-y|$, there is a minimizing martingale coupling of the form

$$\pi^{\mathrm{HK}} = \int_{\mathbb{R}} \delta_{(z,z)} \mu \wedge \nu(dz) + (1-\mu\wedge\nu(\mathbb{R})) \int_0^1 \left(\frac{r(u)-q(u)}{r(u)-p(u)} \, \delta_{(q(u),p(u))} + \frac{q(u)-p(u)}{r(u)-p(u)} \, \delta_{(q(u),r(u))} \right) du$$

for a non-decreasing function q and non-increasing functions p, r such that $p \leq q \leq r$. When $\mu \neq \nu$ the integral over u provides an example of a coupling in $\Pi_M\left(\frac{(\mu-\nu)^+}{1-\mu\wedge\nu(\mathbb{R})}, \frac{(\nu-\mu)^+}{1-\mu\wedge\nu(\mathbb{R})}\right)$ non-increasing in the sense of Definition 4.2.3 below. Without the dispersion assumption, according to Theorem 7.4 [35], there is a unique minimizing coupling when μ does not weight points and this coupling writes

$$\int_{\mathbb{R}} \delta_{(z,z)} \mu \wedge \nu(dz) + \int_{\mathbb{R}} \left(\frac{g(x)-x}{g(x)-f(x)} \delta_{(x,f(x))} + \frac{x-f(x)}{g(x)-f(x)} \delta_{(x,g(x))} \right) (\mu-\nu)^+(dx)$$

with f and g such that $\forall x \in \mathbb{R}$, $f(x) \leq x \leq g(x)$.In [143], Huesmann and Stebegg leverage on the optimal transport viewpoint on the Skorokhod Embedding Problem provided in [36] to recover the above properties of the optimal couplings π^{HN} and π^{HK} (and martingale couplings optimal for other costs) using the monotonicity results for the *Skorokhod Embedding Problem* (SEP).

For many choices of $\mu, \nu \in \mathcal{P}_{\rho\vee 1}(\mathbb{R})$ such that $\mu \leq_{cx} \nu$, when solving the linear programming problem for the cost function $|x-y|^\rho$ obtained by approximating μ and ν by finitely supported probability measures still in the convex order, it turns out that the coupling π^{HN} maximizing when $\rho = 1$ is still very close to maximize when $\rho \in (0,2)$ and minimize when $\rho > 2$: see the

numerical results in Section 4.2.1. It is not surprising that 2 appears as a threshold for the power ρ since the martingale property ensures that when $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$,

$$\forall \pi \in \Pi_M(\mu,\nu),\ \int_{\mathbb{R}^2} (x-y)^2 \pi(dx,dy) = \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx) = \underline{\mathcal{M}}_2^2(\mu,\nu) = \overline{\mathcal{M}}_2^2(\mu,\nu).$$

Let

$$\mathrm{sq} : \mathbb{R}^2 \ni (x,y) \mapsto (y-x)^2 \in \mathbb{R} \tag{4.1}$$

denote the square cost function and for $\pi \in \Pi(\mu,\nu)$, $\mathrm{sq}\#\pi \in \mathcal{P}_1(\mathbb{R})$ denote the image of the measure π by this function. With the complete lattice structure of $\{\eta \in \mathcal{P}_1(\mathbb{R}) : \int_{\mathbb{R}} z\eta(dz) = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)\}$ for the convex order stated in [156], we deduce in Proposition 4.2.1 below that $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ admits an infimum and a supremum for this order. When there exists $\underline{\pi} \in \Pi_M(\mu,\nu)$ such that $\mathrm{sq}\#\underline{\pi}$ is equal to the infimum, then $\int_{\mathbb{R}^2} |x-y|^\rho \underline{\pi}(dx,dy)$ is equal to $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho \in (0,2)$ and to $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho > 2$. In particular, when

$$\nu(dy) = \int_{x\in\mathbb{R}} \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx)$$

for some $a \in \mathbb{R}$, the infimum is equal to δ_{a^2} and is attained with

$$\underline{\pi}(dx,dy) = \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx).$$

When there exists $\overline{\pi} \in \Pi_M(\mu,\nu)$ such that $\mathrm{sq}\#\overline{\pi}$ is equal to the supremum, then $\int_{\mathbb{R}^2} |x-y|^\rho \overline{\pi}(dx,dy)$ is equal to $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho \in (0,2)$ and to $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho > 2$. When μ does not weight points, the uniqueness of the maximizing and minimizing couplings for $\rho = 1$ stated in Theorems 7.3 and 7.4 [35] implies that π^{HN} (resp. π^{HK}) is the only possible value for $\underline{\pi}$ (resp. $\overline{\pi}$). Unfortunately, even if we are able to exhibit in Proposition 4.2.16 a simple situation where the infimum and supremum are attained, we also provide examples where they are not attained (see Propositions 4.2.10 and 4.2.18 where some of the above consequences of attainment do not hold). Giving reasonable sufficient conditions for attainment is an interesting question that we plan to address in the future. Since we believe that non-attainment is more generic, we directly study couplings that attain $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ and $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$. It turns out that, the non-decreasing property of maximizing couplings as well as their uniqueness when μ does not weight points can be extended from the specific value $\rho = 1$ to the case $\rho \in (0,1]$. But the optimizers may be distinct for distinct values of ρ. This motivates our investigation of non-decreasing martingale couplings and we find that the decomposition of ν into

$$\left(\int_{x\in\mathbb{R}} \mathbb{1}_{\{y<x\}}\pi(dx,dy), \int_{x\in\mathbb{R}} \mathbb{1}_{\{y=x\}}\pi(dx,dy), \int_{x\in\mathbb{R}} \mathbb{1}_{\{y>x\}}\pi(dx,dy)\right)$$

for some $\pi \in \Pi_M(\mu,\nu)$ is in one-to-one correspondence with martingale couplings in $\Pi_M(\mu,\nu)$ non-decreasing in a generalized sense (see Corollary 4.2.12), which is the main result of the paper. We also show that the existence of a non-increasing martingale coupling is equivalent to a restrictive sandwich condition on the supports of μ and ν, under which there is a unique

non-increasing and a unique non-decreasing martingale couplings. Even under the sandwich condition, the image $\mathrm{sq}\#\pi^{\mathrm{HN}}$ of the unique non-decreasing martingale coupling π^{HN} is not necessarily the infimum of $\{\mathrm{sq}\#\pi : \pi \in \Pi_{\mathcal{M}}(\mu,\nu)\}$ for the convex order. But under a more and more restrictive supports condition as ρ grows, this coupling π^{HN} attains $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho \in (1,2)$ and $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ when $\rho > 2$.

Our results are stated in Section 4.2 while their proofs are given in Section 4.3.

We denote the cumulative distribution function of $\eta \in \mathcal{P}(\mathbb{R})$ by $F_\eta : \mathbb{R} \ni x \mapsto \eta(-\infty, x] \in [0,1]$, and its quantile function by $F_\eta^{-1} : (0,1) \ni u \mapsto \inf\{x \in \mathbb{R} : u \leq F_\eta(x)\} \in \mathbb{R}$. The proofs rely on several properties of the quantile function, which are listed in Section 1.2.1.3.

We also denote by $u_\eta : \mathbb{R} \ni x \mapsto \int_{\mathbb{R}} |y-x|\eta(dy) \in \mathbb{R}_+$ the potential function of $\eta \in \mathcal{P}_1(\mathbb{R})$. For $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, we have

$$\mu \leq_{cx} \nu \iff \forall x \in \mathbb{R},\ u_\mu(x) \leq u_\nu(x).$$

We say that μ is smaller than ν in the stochastic order and denote $\mu \leq_{st} \nu$ if there exists $\pi \in \Pi(\mu,\nu)$ such that $\pi(\{(x,y) \in \mathbb{R}^2 : x \leq y\}) = 1$. We have $\mu \leq_{st} \nu \iff F_\mu \geq F_\nu \iff F_\mu^{-1} \leq F_\nu^{-1}$.

4.2 Statement of the results

4.2.1 Numerical experiments

For couples (μ,ν) such that μ and ν both belong to the same family of distributions (binomial, Poisson, exponential, normal, log-normal) up to recentering and $\mu \leq_{cx} \nu$, we first construct finitely supported approximations $(\hat{\mu},\hat{\nu})$ such that the convex order is preserved: $\hat{\mu} \leq_{cx} \hat{\nu}$. The computation of $\overline{\mathcal{M}}_\rho(\hat{\mu},\hat{\nu})$ (resp. $\underline{\mathcal{M}}_\rho(\hat{\mu},\hat{\nu})$) is a linear programming problem that we solve using the CVXPY package [96] in python. We call π^{HN} the obtained maximizer for the cost $|y-x|$. When $\rho \in (0,1) \cup (1,2)$ (resp. $\rho > 2$), we compare the optimal value $\overline{\mathcal{M}}_\rho(\hat{\mu},\hat{\nu})$ (resp. $\underline{\mathcal{M}}_\rho(\hat{\mu},\hat{\nu})$) and corresponding maximizer $\overline{\pi}_\rho$ (resp. minimizer $\underline{\pi}_\rho$) computed by the solver with

$$\mathcal{I}_\rho^{\pi^{\mathrm{HN}}} := \left(\int |x-y|^\rho \pi^{\mathrm{HN}}(dx,dy)\right)^{1/\rho}$$

and π^{HN} respectively. Furthermore, we also check whether the maximizing (resp. minimizing) couplings are non-decreasing in the sense of Definition 4.2.3 by computing for π equal to $\overline{\pi}_\rho$ (resp. $\underline{\pi}_\rho$)

$$\mathcal{I}^{\downarrow}(\pi) := \int \left(\mathbb{1}_{\{y_- \leq x_-\}}\mathbb{1}_{\{y_+ \leq x_+\}} + \mathbb{1}_{\{x_- \leq y_-\}}\mathbb{1}_{\{x_+ \leq y_+\}}\right)\mathbb{1}_{\{x_- < x_+\}}\mathbb{1}_{\{y_- > y_+\}}\,\pi(dx_-,dy_-)\,\pi(dx_+,dy_+).$$

It turns out that the values of $\overline{\mathcal{M}}_\rho(\hat{\mu},\hat{\nu})$ (resp. $\underline{\mathcal{M}}_\rho(\hat{\mu},\hat{\nu})$) are very close (generally with relative difference not greater than 10^{-4}) to the value of $\mathcal{I}_\rho^{\pi^{HN}}$ for $\rho \in (0,2)$ (resp. for $\rho > 2$) and, in the Poisson distributions example, the optimizing coupling is equal to π^{HN}. Moreover, $\mathcal{I}^{\downarrow}(\overline{\pi}_\rho)$ and $\mathcal{I}^{\downarrow}(\underline{\pi}_\rho)$ always vanish up to numerical accuracy, so that the optimal couplings always are non-decreasing.

For the binomial distributions which are finitely supported, we choose $(\hat{\mu}, \hat{\nu}) = (\mu, \nu)$.

Binomial distributions : $\mu = \mathcal{L}(X-5)$, $\nu = \mathcal{L}(Y-20)$ with $X \sim \mathcal{B}(10, 0.5)$, $Y \sim \mathcal{B}(40, 0.5)$									
ρ	$\overline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\lVert\overline{\pi}_\rho - \pi^{HN}\rVert$	$\mathcal{I}^{\downarrow}(\overline{\pi}_\rho)$	ρ	$\underline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\lVert\underline{\pi}_\rho - \pi^{HN}\rVert$	$\mathcal{I}^{\downarrow}(\underline{\pi}_\rho)$
0.3	4.436854	4.436854	0.000030	1.1e-09	2.1	3.977389	3.977389	0.025883	4.9e-09
0.7	2.808433	2.808433	0.000207	2.3e-09	2.3	4.513236	4.513237	0.015873	7.6e-09
1.0	2.651451	2.651451	0.0	2.2e-09	2.5	5.174434	5.174438	0.031158	1.5e-08
1.4	2.847949	2.847949	0.000025	5.5e-09	3.0	7.566087	7.566713	0.000045	-9.7e-08
1.9	3.546872	3.546872	0.000022	7.9e-08	5.0	51.444800	51.526879	0.045762	-2.8e-18

For Poisson distributions, we need to deal with the possible arbitrarily large integer values. We choose the restrictions of $\hat{\mu}$ and μ below $n_\mu = \min\{n \in \mathbb{N} : \mu((n, +\infty)) \leq 10^{-2}\}$ equal and replace in $\hat{\mu}$ the restriction of μ above n_μ by $\mathbb{P}(Z > n_\mu)\, \delta_{\mathbb{E}[Z \,|\, Z > n_\mu]}$ where $Z \sim \mu$. We use the same construction with threshold $n_\nu = \min\{n \in \mathbb{N} : \nu((n, +\infty)) \leq 10^{-2}\}$ to compute $\hat{\nu}$.

Poisson distributions : $\mu = \mathcal{L}(X-1)$, $\nu = \mathcal{L}(Y-4)$ with $X \sim \mathcal{P}(1)$, $Y \sim \mathcal{P}(4)$									
ρ	$\overline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\lVert\overline{\pi}_\rho - \pi^{HN}\rVert$	$\mathcal{I}^{\downarrow}(\overline{\pi}_\rho)$	ρ	$\underline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\lVert\underline{\pi}_\rho - \pi^{HN}\rVert$	$\mathcal{I}^{\downarrow}(\underline{\pi}_\rho)$
0.3	3.779516	3.779516	1.2e-09	2.9e-11	2.1	1.438060	1.438060	1.9e-09	8.9e-11
0.7	1.949747	1.949747	3.4e-10	8.2e-11	2.3	1.494325	1.494325	1.4e-08	8.3e-10
1.0	1.588868	1.588868	0.0	6.7e-11	2.5	1.569906	1.569906	1.2e-09	3.4e-11
1.4	1.412622	1.412622	4.6-10	5.0e-11	3.0	1.847970	1.847970	2.2e-09	2.0e-10
1.9	1.401570	1.401570	4.9-10	5.7e-11	5.0	5.171327	5.171327	1.7e-09	8.3e-12

The quantile function of an exponential distribution and its antiderivative have closed form expressions. This enables us to apply Baker's construction [92, Theorem 2.4.11] to preserve $\mu \leq_{cx} \nu$ in the following way:

$$\hat{\mu} = \frac{1}{I}\sum_{i=1}^{I} \delta_{I \int_{\frac{i-1}{I}}^{\frac{i}{I}} F_\mu^{-1}(u)\, du} \quad \text{and} \quad \hat{\nu} = \frac{1}{I}\sum_{i=1}^{I} \delta_{I \int_{\frac{i-1}{I}}^{\frac{i}{I}} F_\nu^{-1}(u)\, du} \quad \text{for } I = 100.$$

Exponential distributions : $\mu = \mathcal{L}(X-1)$, $\nu = \mathcal{L}(Y-2)$ with $X \sim \mathcal{E}(1)$ and $Y \sim \mathcal{E}(0.5)$									
ρ	$10 \times \overline{\mathcal{M}}_\rho$	$10 \times \mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\lVert\overline{\pi}_\rho - \pi^{HN}\rVert$	$\mathcal{I}^{\downarrow}(\overline{\pi}_\rho)$	ρ	$10 \times \underline{\mathcal{M}}_\rho$	$10 \times \mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\lVert\underline{\pi}_\rho - \pi^{HN}\rVert$	$\mathcal{I}^{\downarrow}(\underline{\pi}_\rho)$
0.3	29.398430	29.396399	0.038048	-3.6e-13	2.1	2.702017	2.702146	0.061036	4.3e-13
0.7	10.880587	10.880342	0.022718	-5.7e-14	2.3	2.435760	2.436259	0.066940	1.0e-13
1.0	6.941355	6.941355	0.0	-1.9e-14	2.5	2.228230	2.229254	0.072639	8.4e-13
1.4	4.490510	4.490383	0.026590	9.5e-15	3.0	1.885559	1.888544	0.080549	1.1e-12
1.9	3.046885	3.046792	0.051981	5.5e-14	5.0	1.683823	1.700810	0.086346	1.2e-12

For normal and log-normal distributions, we combine random drawings with Baker's construction. For $I = 100$ and $n = 10000$, we denote by $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(nI)}$ (resp. $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(nI)}$) the order statistics of nI random variables i.i.d. according to μ (resp. ν) and compute $\hat{\mu} = \frac{1}{I}\sum_{i=1}^{I} \delta_{\hat{X}_{(i)}}$ (resp. $\hat{\nu} = \frac{1}{I}\sum_{i=1}^{I} \delta_{\hat{Y}_{(i)}}$) where, for $1 \leq i \leq I$, $\hat{X}_{(i)} = \frac{1}{n}\sum_{j=1}^{n} X_{(n(i-1)+j)} - \frac{1}{nI}\sum_{j=1}^{nI} X_{(j)}$ (resp. $\hat{Y}_{(i)} = \frac{1}{n}\sum_{j=1}^{n} Y_{(n(i-1)+j)} - \frac{1}{nI}\sum_{j=1}^{nI} Y_{(j)}$).

Normal distributions : $\mu = \mathcal{N}(0, 0.24)$, $\nu = \mathcal{N}(0, 0.28)$									
ρ	$10 \times \overline{\mathcal{M}}_\rho$	$10 \times \mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\|\overline{\pi}_\rho - \pi^{HN}\|$	$\mathcal{I}^{\downarrow}(\overline{\pi}_\rho)$	ρ	$100 \times \underline{\mathcal{M}}_\rho$	$100 \times \mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\|\underline{\pi}_\rho - \pi^{HN}\|$	$\mathcal{I}^{\downarrow}(\underline{\pi}_\rho)$
0.3	18.616692	18.616692	0.000471	2.2e-13	2.1	0.815057	0.815058	0.001395	7.1e-13
0.7	3.672101	3.672101	0.000332	1.5e-12	2.3	0.505845	0.505846	0.001400	1.3e-12
1.0	1.437086	1.437086	0.0	1.1e-12	2.5	0.316421	0.316421	0.002014	1.3e-12
1.4	0.473121	0.473121	0.000935	2.4e-12	3.0	0.100648	0.100648	0.006310	1.4e-11
1.9	0.132566	0.132566	0.001101	2.9e-13	5.0	0.001313	0.001313	0.008049	3.3e-11

Log-normal distributions : $\mu = \exp\#\mathcal{N}(0, 0.24)$, $\nu = \exp\#\mathcal{N}(-0.0104, 0.28)$									
ρ	$10 \times \overline{\mathcal{M}}_\rho$	$10 \times \mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\|\overline{\pi}_\rho - \pi^{HN}\|$	$\mathcal{I}^{\downarrow}(\overline{\pi}_\rho)$	ρ	$100 \times \underline{\mathcal{M}}_\rho$	$100 \times \mathcal{I}_\rho^{\pi^{\mathrm{HN}}}$	$\|\underline{\pi}_\rho - \pi^{HN}\|$	$\mathcal{I}^{\downarrow}(\underline{\pi}_\rho)$
0.3	18.628224	18.628223	0.005638	1.0e-12	2.1	0.922616	0.922616	0.005181	8.2e-13
0.7	3.707914	3.707914	0.005471	1.4e-13	2.3	0.588962	0.588963	0.006438	1.2e-12
1.0	1.470487	1.470487	0.0	1.5e-12	2.5	0.380220	0.380222	0.006438	2.1e-12
1.4	0.497265	0.497265	0.001848	6.6e-12	3.0	0.132954	0.132958	0.006443	1.1e-11
1.9	0.146360	0.146360	0.005189	1.9e-12	5.0	0.003344	0.003347	0.012090	4.3e-11

4.2.2 Infimum and supremum of $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ for the convex order

We recall that $\mathrm{sq}\#\pi$ denotes the image of the probability measure π by the square cost function sq defined in (4.1).

Proposition 4.2.1. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. Then the set $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ admits an infimum $\inf_{cx}\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ and a supremum $\sup_{cx}\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ for the convex order.*

The conclusion follows from the next lemma and the complete lattice structure of the set $\{\eta \in \mathcal{P}_1(\mathbb{R}) : \int_{\mathbb{R}} z\eta(dz) = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)\}$ with minimal element $\delta_{\int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)}$ stated in [156].

Lemma 4.2.2. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. Then the set $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ is bounded from above in the convex order.*

4.2.3 Definition and optimality of non-decreasing and non-increasing martingale couplings

Definition 4.2.3. *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. A martingale coupling $\pi(dx, dy) \in \Pi_M(\mu,\nu)$ is called **non-decreasing** if there exists a Borel set $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ such that $\pi(\Gamma) = 1$ and*

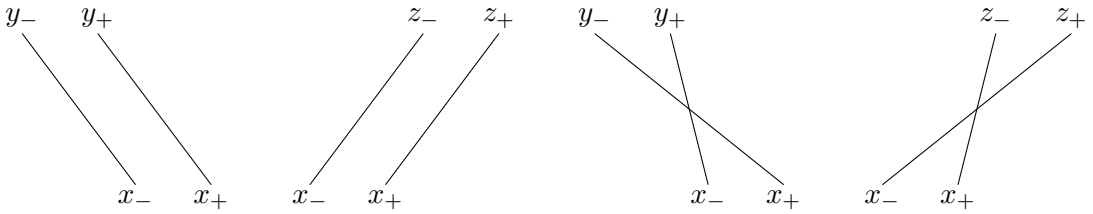
(a) if $(x_-, y_-), (x_+, y_+) \in \Gamma$ with $y_- \leq x_-$, $y_+ \leq x_+$, and $x_- < x_+$, then $y_- \leq y_+$

(b) if $(x_-, z_-), (x_+, z_+) \in \Gamma$ with $x_- \leq z_-$, $x_+ \leq z_+$, and $x_- < x_+$, then $z_- \leq z_+$.

*Respectively, $\pi(dx, dy)$ is called **non-increasing** if there exists a Borel set $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ such that $\pi(\Gamma) = 1$ and*

(c) if $(x_-, y_+), (x_+, y_-) \in \Gamma$ *with* $y_+ \leq x_-$, $y_- \leq x_+$, *and* $x_- < x_+$, *then* $y_- \leq y_+$

(d) if $(x_-, z_+), (x_+, z_-) \in \Gamma$ *with* $x_- \leq z_+$, $x_+ \leq z_-$, *and* $x_- < x_+$, *then* $z_- \leq z_+$.



(a) non-decreasing (left) (b) non-decreasing (right) (c) non-increasing (left) (d) non-increasing (right)

Figure 4.1: Non-decreasing / non-increasing coupling

Conditions $(a), (b), (c)$ and (d) are illustrated in Fig. 4.1. A different notion of monotone martingale couplings is introduced by Beiglböck and Juillet in [35] where they prove that there exists a unique martingale coupling π left-monotone (resp. right-monotone) in the sense that $\pi(\Gamma) = 1$ for some Borel subset Γ of $\mathbb{R}^2$ such that (x_-, y_-), (x_-, z_-) and $(x_+, w) \in \Gamma$ with $x_- < x_+$ and $y_- < z_-$ implies that $w \notin (y_-, z_-)$ (resp. (x_+, y_+), (x_+, z_+) and $(x_-, w) \in \Gamma$ with $x_- < x_+$ and $y_+ < z_+$ implies that $w \notin (y_+, z_+)$). According to [133], the left-monotone (resp. right-monotone) martingale coupling minimizes (resp. maximizes) over $\Pi_M(\mu, \nu)$ the integral of smooth cost functions $c(x, y)$ satisfying the martingale Spence-Mirrlees condition $\partial^3_{xyy} c \leq 0$.

In the beginning of the proof of Theorem 7.3 [35], combining the necessary optimality condition that they give in Lemma 1.11 together with some specific properties of the cost function $\mathbb{R}^2 \ni (x, y) \mapsto |x - y|$ stated in Lemma 7.5, Beiglböck and Juillet show that any coupling maximizing $\int_{\mathbb{R}^2} |x - y| \pi(dx, dy)$ over $\pi \in \Pi_M(\mu, \nu)$ is non-decreasing. Replacing Lemma 7.5 [35] by Lemma 4.3.4 below, we generalize this result to costs $\varphi(|x - y|)$ where $\mathbb{R}_+ \ni z \mapsto \varphi(z) \in \mathbb{R}$ is increasing and concave. To check that, on the other hand, any coupling minimizing $\int_{\mathbb{R}^2} \varphi(|x - y|) \pi(dx, dy)$ for such a function is non-increasing, we need μ and ν to satisfy the following sandwich condition.

Definition 4.2.4. *We say that* $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ *satisfy the sandwich condition if there exist* $-\infty < a \leq b < +\infty$ *such that* $\mu([a, b]) = 1$ *and* $\nu((a, b)) = 0$.

The sandwich condition is slightly weaker than the Reinforced Dispersion Assumption 7.1 considered by Hobson and Klimmek [135] and which amounts to the existence of some interval I such that $\mu(I) = 1$ and $\nu(I) = 0$. Indeed, under the sandwich condition, it may happen that $\mu(\{a\})\nu(\{a\}) > 0$ or $\mu(\{b\})\nu(\{b\}) > 0$.

Proposition 4.2.5. *Let* $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ *be such that* $\mu \leq_{cx} \nu$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}$ *be some increasing concave function.*

(i) If $\tilde{\pi} \in \Pi_M(\mu, \nu)$ *maximizes* $\int \varphi(|x-y|)\pi(dx, dy)$ *over* $\pi \in \Pi_M(\mu, \nu)$, *then* $\tilde{\pi}$ *is non-decreasing and, when* φ *is continuous, there exists such a maximizing coupling.*

(*ii*) *There exists* $\tilde{\pi} \in \Pi_M(\mu,\nu)$ *which minimizes* $\int \varphi(|x-y|)\pi(dx,dy)$ *over* $\pi \in \Pi_M(\mu,\nu)$ *and when* μ *and* ν *satisfy the sandwich condition, then any such coupling* $\tilde{\pi}$ *is non-increasing.*

Note that Remark 3.8 [143] exploits the links between MOT and SEP to characterize couplings minimizing $\int |x-y|^\rho \pi(dx,dy)$ over $\pi \in \Pi_M(\mu,\nu)$ for $\rho \in (0,1]$.

Remark 4.2.6. *Let* $\lambda \in [\varphi_r'(1), \varphi_l'(1)]$ *where* $\varphi_r'(1)$ *and* $\varphi_l'(1)$ *are the respective right-hand and left-hand derivatives of the increasing concave function* φ *at point* 1.

We have $\forall z \in \mathbb{R}_+,\ \varphi(0) \le \varphi(z) \le \varphi(1) + \lambda(z-1)$ *so that for* $\pi \in \Pi_M(\mu,\nu)$,

$$\begin{aligned}\varphi(0) &\le \int_{\mathbb{R}^2} \varphi(|x-y|)\pi(dx,dy) \le \varphi(1) + \lambda \int_{\mathbb{R}^2} |x-y|\pi(dx,dy) \\ &\le \varphi(1) + \lambda\left(\int_{\mathbb{R}} |x|\mu(dx) + \int_{\mathbb{R}} |y|\nu(dy)\right),\end{aligned}$$

where the right-hand side is finite for $\mu,\nu \in \mathcal{P}_1(\mathbb{R})$. *The concavity of the function* $\varphi : \mathbb{R}_+ \to \mathbb{R}$ *implies that this function is continuous on* $(0,+\infty)$. *It may have a positive jump* $\lim_{x\to 0+}\varphi(x) - \varphi(0)$ *at the origin and then* $\mathbb{R}^2 \ni (x,y) \mapsto \varphi(|x-y|)$ *is lower semi-continuous but not upper-semicontinuous.*

Remark 4.2.7. *For* $\rho \in (0,1]$, *since* $\mathbb{R}_+ \ni z \mapsto z^\rho$ *is concave and increasing, a martingale coupling that maximizes* $\int |x-y|^\rho\, \pi(dx,dy)$ *is non-decreasing.*

It turns out that the sandwich condition is also necessary for the existence of a non-increasing coupling in $\Pi_M(\mu,\nu)$ and that, under this condition there is exists a unique non-increasing coupling and also a unique non-decreasing coupling in $\Pi_M(\mu,\nu)$.

Proposition 4.2.8. *Let* $\mu,\nu \in \mathcal{P}_1(\mathbb{R})$ *be such that* $\mu \le_{cx} \nu$. *The following assertions are equivalent*

(*i*) *there exists a non-increasing martingale coupling in* $\Pi_M(\mu,\nu)$,

(*ii*) μ *and* ν *satisfy the sandwich condition :* $\exists -\infty < a \le b < +\infty$ *s.t.* $\mu([a,b]) = 1$ *and* $\nu((a,b)) = 0$,

(*iii*) *there exists a unique non-increasing martingale coupling* $\pi^\downarrow$ *in* $\Pi_M(\mu,\nu)$,

and, when they hold, $\pi^\downarrow(\{(a,a)\}) = \mu(\{a\}) \wedge \nu(\{a\})$ *and* $\pi^\downarrow(\{(b,b)\}) = \mu(\{b\}) \wedge \nu(\{b\})$. *Moreover, under the sandwich condition, there exists a unique non-decreasing martingale coupling* $\pi^\uparrow \in \Pi_M(\mu,\nu)$.

The next proposition states uniqueness of the optimal coupling in the setting of Proposition 4.2.5 (*i*) when μ does not weight points by generalizing to continuous, increasing and concave functions $\varphi : \mathbb{R}_+ \to \mathbb{R}$ the statement in Theorem 7.3 [35] which deals with the case when φ is the identity function.

Proposition 4.2.9. *Let* $\mu,\nu \in \mathcal{P}(\mathbb{R})$ *be such that* $\mu \le_{cx} \nu$ *and that* μ *does not weight points. Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}$ *be some continuous increasing concave function. There exists a unique non-decreasing martingale coupling* $\tilde{\pi}$ *that maximizes* $\int \varphi(|x-y|)\pi(dx,dy)$. *Moreover, there exist*

two non-decreasing functions $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ *such that* $T_1(x) \leq x \leq T_2(x)$ *and* $\tilde{\pi}$ *is concentrated on the graphs of these functions.*

Notice that the example in Section 7.3.1 [35] shows that for general continuous, increasing and concave functions φ, the uniqueness of maximizers may fail without the continuity assumption on μ. The next proposition illustrates that unique maximizers may be distinct for distinct power functions φ. Since those maximizers are non-decreasing couplings, this motivates the investigation of the set of non-decreasing couplings in the next subsection.

Proposition 4.2.10. *Let* $0 < \rho' < \rho \leq 1$, $y < \frac{y+z}{2} < m < z$, *and*

$$\beta \in \left(0, 2\rho\left(\frac{m-y}{z-y}(z-m)^{\rho-1} - \frac{z-m}{z-y}(m-y)^{\rho-1}\right) \wedge \frac{2m-y-z}{z-m}\right).$$

Let

$$\mu_{\varepsilon,\rho} = \frac{1}{1+(1+\beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}}\delta_{m-(1+\beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}\varepsilon} + \frac{(1+\beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}}{1+(1+\beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}}\delta_{m+\varepsilon},$$

$$\nu = \frac{1}{3}\delta_m + \frac{2(z-m)}{3(z-y)}\delta_y + \frac{2(m-y)}{3(z-y)}\delta_z.$$

For $\varepsilon < (z-m) \wedge \dfrac{m-y}{(1+\beta)^{\frac{1}{\rho}}} \wedge \dfrac{2(z-m)(m-y)}{3(z-y)} \wedge 1$, $\Pi_M(\mu_{\varepsilon,\rho}, \nu)$ *contains distinct martingale couplings and any element of* $\Pi_M(\mu_{\varepsilon,\rho}, \nu)$ *is non-decreasing. For* ε *small enough, the non-decreasing martingale coupling that attains* $\overline{\mathcal{M}}_{\rho'}(\mu_{\varepsilon,\rho}, \nu)$ *is different from the one that attains* $\overline{\mathcal{M}}_{\rho}(\mu_{\varepsilon,\rho}, \nu)$.

4.2.4 Main result : directionally decomposed non-decreasing martingale couplings

The uniqueness of non-decreasing couplings under the sandwich condition which determines the parts of the measure ν attained in the left direction and in the right direction in any martingale coupling motivates the introduction of the following decomposition. For $\pi \in \Pi_M(\mu, \nu)$, we denote by

$$\begin{aligned}
\nu_l^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y<x\}}\pi(dx, dy) = \int_{x\in\mathbb{R}} \mathbb{1}_{\{y<x\}}\pi_x(dy)\mu(dx),\\
\nu_0^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y=x\}}\pi(dx, dy) = \int_{x\in\mathbb{R}} \mathbb{1}_{\{y=x\}}\pi_x(dy)\mu(dx),\\
\nu_r^\pi(dy) &= \int_{x\in\mathbb{R}} \mathbb{1}_{\{y>x\}}\pi(dx, dy) = \int_{x\in\mathbb{R}} \mathbb{1}_{\{y>x\}}\pi_x(dy)\mu(dx).
\end{aligned}$$

Note that $\nu_l^\pi + \nu_0^\pi + \nu_r^\pi = \nu$. For (ν_l, ν_r) a couple of non-negative measures such that $\nu_l + \nu_r \leq \nu$, we denote

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) = \{\pi \in \Pi_M(\mu, \nu) : (\nu_l^\pi, \nu_0^\pi, \nu_r^\pi) = (\nu_l, \nu - \nu_l - \nu_r, \nu_r)\}.$$

Our main result, stated in the next theorem and in assertion (i) of the corollary that follows is that $\Pi_M(\mu,\nu,\nu_l,\nu_r)$ is not empty if and only if there exists a coupling in this set which is non-decreasing away from the support of $\nu-\nu_l-\nu_r$ where it goes straight.

Theorem 4.2.11. *Let $\mu,\nu\in\mathcal{P}_1(\mathbb{R})$ be such that $\mu\leq_{cx}\nu$ and $\mu\neq\nu$. Let (ν_l,ν_r) a couple of non-negative measures such that $\nu_l+\nu_r=\nu$. We have*

$$\Pi_M(\mu,\nu,\nu_l,\nu_r)\neq\emptyset\iff\exists!\,\pi^{\uparrow}\in\Pi_M(\mu,\nu,\nu_l,\nu_r)\ \ \textit{non-decreasing},$$

and then, for each continuous increasing concave function $\varphi:\mathbb{R}_+\to\mathbb{R}$,

$$\forall\pi\in\Pi_M(\mu,\nu,\nu_l,\nu_r)\setminus\{\pi^{\uparrow}\},\ \int_{\mathbb{R}^2}\varphi(|y-x|)\pi(dx,dy)<\int_{\mathbb{R}^2}\varphi(|y-x|)\pi^{\uparrow}(dx,dy).\tag{4.2}$$

Corollary 4.2.12. *Let $\mu,\nu\in\mathcal{P}_1(\mathbb{R})$ be such that $\mu\leq_{cx}\nu$ and $\mu\neq\nu$. For (ν_l,ν_r) a couple of non-negative measures such that $\nu_l+\nu_r\leq\nu$, we have that*

(i) $\Pi_M(\mu,\nu,\nu_l,\nu_r)\neq\emptyset\iff\exists!\pi^{\uparrow}\in\Pi_M(\mu,\nu,\nu_l,\nu_r)$ *s.t.* $\dfrac{\pi^{\uparrow}(dx,dy)+(\nu_l+\nu_r-\nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}$ *is non-decreasing, and then*

$$\forall\pi\in\Pi_M(\mu,\nu,\nu_l,\nu_r)\setminus\{\pi^{\uparrow}\},\ \int_{\mathbb{R}^2}\varphi(|y-x|)\pi(dx,dy)<\int_{\mathbb{R}^2}\varphi(|y-x|)\pi^{\uparrow}(dx,dy)\tag{4.3}$$

for each continuous increasing concave function $\varphi:\mathbb{R}_+\to\mathbb{R}$,

(ii) $\exists\,\pi\in\Pi_M(\mu,\nu,\nu_l,\nu_r)$ *non-decreasing*

$$\implies\nu-\nu_l-\nu_r=\mathbb{1}_{\{u_\mu=u_\nu\}}\mu+\sum_{\substack{x\in\mathbb{R}:\\ \mu\wedge\nu(\{x\})\times(u_\nu(x)-u_\mu(x))>0}}p(x)\,\delta_x\ \textit{for some}\ p(x)\in[0,\mu\wedge\nu(\{x\})],$$

(iii) There is at most one non-decreasing coupling $\pi\in\Pi_M(\mu,\nu,\nu_l,\nu_r)$.

According to the next proposition, some restrictive structure on the supports of μ and ν is needed to ensure the existence of a martingale coupling in $\Pi_M(\mu,\nu,\nu_l,\nu_r)$ non-increasing away from the support of $\nu-\nu_l-\nu_r$ where it goes straight.

Proposition 4.2.13. *Let $\mu,\nu\in\mathcal{P}_1(\mathbb{R})$ be such that $\mu\leq_{cx}\nu$ and $\mu\neq\nu$. For (ν_l,ν_r) a couple of non-negative measures such that $\nu_l+\nu_r\leq\nu$, the existence of $\pi^{\downarrow}\in\Pi_M(\mu,\nu,\nu_l,\nu_r)$ such that*

$$\frac{\pi^{\downarrow}(dx,dy)+(\nu_l+\nu_r-\nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}$$

is non-increasing is equivalent to the Dispersion Assumption 2.1 in [135] i.e. the existence of a non-empty finite interval I with ends $a\leq b$ such that $(\mu-\nu)^+(I)=(\mu-\nu)^+(\mathbb{R})$ and $(\nu-\mu)^+(I)=0$ combined with $\nu_l(dy)=\mathbb{1}_{\{y\leq a\}}(\nu-\mu)^+(dy)$ and $\nu_r(dy)=\mathbb{1}_{\{y\geq b\}}(\nu-\mu)^+(dy)$.

4.2.5 Do $\pi^{\uparrow}$ and $\pi^{\downarrow}$ optimize the cost function $|x-y|^{\rho}$ for $\rho > 1$ under the sandwich condition ?

In this section, we suppose that the probability measures μ and ν in the convex order satisfy the sandwich condition so that, by Proposition 4.2.8, there exist a unique non-decreasing coupling $\pi^{\uparrow}$ and a unique non-increasing coupling $\pi^{\downarrow}$ in $\Pi_M(\mu,\nu)$. Then, by Proposition 4.2.5, $\pi^{\uparrow}$ maximises (resp. $\pi^{\downarrow}$ minimizes)

$$\int_{\mathbb{R}^2} c(x,y)\pi(dx,dy)$$

over $\pi \in \Pi_M(\mu,\nu)$ for cost functions $c(x,y) = \varphi(|x-y|)$ with $\varphi : \mathbb{R}_+ \to \mathbb{R}$ continuous, increasing and concave and in particular for $c(x,y) = |x-y|^{\rho}$ with $\rho \in (0,1]$. We investigate whether $\pi^{\uparrow}$ (resp. $\pi^{\downarrow}$) still maximises (resp. minimises) for $c(x,y) = |x-y|^{\rho}$ with $\rho \in (1,2)$ and minimises (resp. maximises) for $c(x,y) = |x-y|^{\rho}$ with $\rho > 2$. It turns out that this is the case under a reinforced support condition with μ and ν not weighting some non empty intervals at the left of a and at the right of b. The larger ρ, the larger these gaps should be chosen. When μ only weights two points and ν weights two points to the left of the support of μ and two points to the right, we exhibit conditions ensuring that $\mathrm{sq}\#\pi^{\uparrow}$ is equal to the infimum $\inf_{cx}\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ and $\mathrm{sq}\#\pi^{\downarrow}$ is equal to the supremum $\sup_{cx}\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\nu)\}$ for the convex order. Last, we check that, still with such finite supports, optimality may fail when the gaps are too small.

4.2.5.1 Preservation of the optimality

Proposition 4.2.14. *Let $\mu \leq_{cx} \nu$ be such that there exist $\underline{y} < \overline{y} < \underline{x} < \overline{x} < \underline{z} < \overline{z}$ with $\mu([\underline{x},\overline{x}]) = 1$ and $\nu\left([\underline{y},\overline{y}] \cup [\underline{z},\overline{z}]\right) = 1$. If*

$$\underline{x} - \overline{y} \geq \alpha_\rho(\overline{z} - \overline{y}),$$
$$\underline{z} - \overline{x} \geq \alpha_\rho(\underline{z} - \underline{y}),$$

where $\alpha_\rho \in (0, \frac{1}{2})$ is the unique solution of $\psi_\rho(\alpha) = 0$ with $\psi_\rho : (0,1] \to \mathbb{R}$ defined by

$$\psi_\rho(\alpha) = \alpha + \alpha^{2-\rho}(1-\alpha)^{\rho-1} + 1 - \rho.$$

Then, the unique non-decreasing coupling $\pi^{\uparrow}$ (resp. non-increasing coupling $\pi^{\downarrow}$) in $\Pi_M(\mu,\nu)$ is the unique optimal coupling in $\Pi_M(\mu,\nu)$ that attains $\overline{\mathcal{M}}_\rho(\mu,\nu)$ (resp. $\underline{\mathcal{M}}_\rho(\mu,\nu)$) when $\rho \in (1,2)$ and $\underline{\mathcal{M}}_\rho(\mu,\nu)$ (resp. $\overline{\mathcal{M}}_\rho(\mu,\nu)$) when $\rho > 2$.

Remark 4.2.15. *Let $\rho \in (1,2) \cup (2,+\infty)$ and $\overline{y} < \underline{z}$. Since, by the fifth and seventh assertions in Lemma 4.3.9 below, $\alpha_\rho < \frac{1}{2}$, it is possible to find $\underline{x}, \overline{x} \in (\overline{y} + \alpha_\rho(\underline{z} - \overline{y}), \underline{z} - \alpha_\rho(\underline{z} - \overline{y}))$ such that $\overline{x} - \underline{x} > 0$. Then for any $\underline{y} \in \left[\frac{1}{\alpha_\rho}(\overline{x} - (1-\alpha_\rho)\underline{z}), \overline{y}\right)$ and any $\overline{z} \in \left(\underline{z}, \frac{1}{\alpha_\rho}(\underline{x} - (1-\alpha_\rho)\overline{y})\right]$, the conditions on the points $\underline{y}, \overline{y}, \underline{x}, \overline{x}, \underline{z}, \overline{z}$ in Proposition 4.2.14 are satisfied.*

Proposition 4.2.16. *Let $p \in (0,1)$ and $y_- < y_+ < x_- < x_+ < z_- < z_+$ be such that*

$$(x_- - y_-) \wedge (x_+ - y_+) \geq (z_- - x_+) \quad \text{and} \quad (z_- - x_-) \wedge (z_+ - x_+) \geq (x_- - y_+). \tag{4.4}$$

We set $\mu = p\delta_{x_-} + (1-p)\delta_{x_+}$,

$$\underline{\nu} = p\frac{z_- - x_-}{z_- - y_-}\delta_{y_-} + p\frac{x_- - y_-}{z_- - y_-}\delta_{z_-} + (1-p)\frac{z_+ - x_+}{z_+ - y_+}\delta_{y_+} + (1-p)\frac{x_+ - y_+}{z_+ - y_+}\delta_{z_+},$$
$$\pi^{\uparrow} = p\left(\frac{z_- - x_-}{z_- - y_-}\delta_{(x_-,y_-)} + \frac{x_- - y_-}{z_- - y_-}\delta_{(x_-,z_-)}\right) + (1-p)\left(\frac{z_+ - x_+}{z_+ - y_+}\delta_{(x_+,y_+)} + \frac{x_+ - y_+}{z_+ - y_+}\delta_{(x_+,z_+)}\right),$$
$$\overline{\nu} = p\frac{z_+ - x_-}{z_+ - y_+}\delta_{y_+} + p\frac{x_- - y_+}{z_+ - y_+}\delta_{z_+} + (1-p)\frac{z_- - x_+}{z_- - y_-}\delta_{y_-} + (1-p)\frac{x_+ - y_-}{z_- - y_-}\delta_{z_-},$$
$$\pi^{\downarrow} = p\left(\frac{z_+ - x_-}{z_+ - y_+}\delta_{(x_-,y_+)} + \frac{x_- - y_+}{z_+ - y_+}\delta_{(x_-,z_+)}\right) + (1-p)\left(\frac{z_- - x_+}{z_- - y_-}\delta_{(x_+,y_-)} + \frac{x_+ - y_-}{z_- - y_-}\delta_{(x_+,z_-)}\right).$$

We have $\pi^{\uparrow} \in \Pi_M(\mu,\underline{\nu})$, $\mathrm{sq}\#\pi^{\uparrow}(dx,dy) = \inf_{cx}\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu,\underline{\nu})\}$ *and* $\pi^{\downarrow} \in \Pi_M(\mu,\overline{\nu})$, $\mathrm{sq}\#\pi^{\downarrow}(dx,dy) = \sup_{cx}\{\mathrm{sq}\#\pi(dx,dy) : \pi \in \Pi_M(\mu,\overline{\nu})\}$.

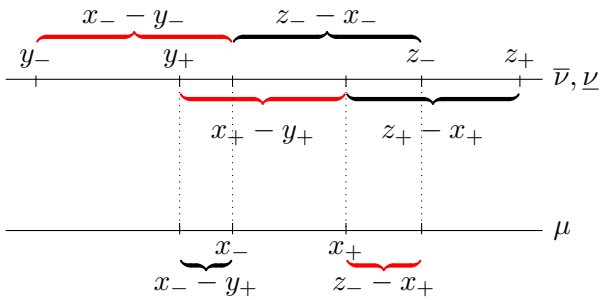


Figure 4.2: A visualization of the inequality condition (4.4), the lower red (resp. black) distance is smaller than the upper red (resp. black) distances.

Remark 4.2.17.

- *The inequality condition* (4.4) *is illustrated by Fig. 4.2. If* $y_- < y_+ < x_- < x_+ < z_- < z_+$ *and* $x_- - y_+ = z_- - x_+$, *then condition* (4.4) *is satisfied. More generally, for* $y_+ < x_- < x_+ < z_-$, *the condition* $(x_+ - y_+) \geq (z_- - x_+)$ *and* $(z_- - x_-) \geq (x_- - y_+)$ *is equivalent to the inequality* $x_- \leq \frac{y_+ + z_-}{2} \leq x_+$ *and, when they hold, we may choose* z_+ *large enough and* y_- *small enough so that* (4.4) *holds.*

- *We deduce that for any convex function* $\varphi : \mathbb{R}_+ \to \mathbb{R}$

$$\int_{\mathbb{R}^2} \varphi(|y-x|^2)\pi^{\uparrow}(dx,dy) = \inf_{\pi\in\Pi_M(\mu,\underline{\nu})} \int_{\mathbb{R}^2} \varphi(|y-x|^2)\pi(dx,dy) \text{ and}$$
$$\int_{\mathbb{R}^2} \varphi(|y-x|^2)\pi^{\downarrow}(dx,dy) = \sup_{\pi\in\Pi_M(\mu,\overline{\nu})} \int_{\mathbb{R}^2} \varphi(|y-x|^2)\pi(dx,dy).$$

Indeed the function $\varphi(z) = \mathbb{1}_{\{z=0\}} \lim_{w\to 0+} \varphi(w) + \mathbb{1}_{\{z>0\}}\varphi(z)$ *is the uniform limit of the restriction to* $\mathbb{R}_+$ *of the non-decreasing sequence* $(\varphi_n)_{n\geq 1}$ *of convex functions on* $\mathbb{R}$ *defined by*

$$\varphi_n(z) = \mathbb{1}_{\{z<\frac{1}{n}\}}(\varphi(\frac{1}{n}) + \varphi_l'(\frac{1}{n})(z - \frac{1}{n})) + \mathbb{1}_{\{z\geq\frac{1}{n}\}}\varphi(z)$$

where φ_l' *denotes the left-hand derivative of* φ. *Moreover,* $\varphi(0) - \lim_{w\to 0+}\varphi(w) \geq 0$ *and*

for probability measures $\theta, \vartheta \in \mathcal{P}_1(\mathbb{R})$ such that $\theta(\mathbb{R}_+) = 1 = \vartheta(\mathbb{R}_+)$,

$$\theta \leq_{cx} \vartheta \quad \Longrightarrow \quad \theta(\{0\}) \leq \vartheta(\{0\}).$$

In particular, since $\mathbb{R}_+ \ni z \mapsto z^{\rho/2}$ is concave for $\rho \in (0,2)$ and convex for $\rho > 2$,

$$\forall \rho > 0,\ (\rho-2)\int_{\mathbb{R}^2} |y-x|^\rho \pi^{\uparrow}(dx,dy) = \inf_{\pi \in \Pi_M(\mu,\underline{\nu})} (\rho-2)\int_{\mathbb{R}^2} |y-x|^\rho \pi(dx,dy) \text{ and}$$
$$(\rho-2)\int_{\mathbb{R}^2} |y-x|^\rho \pi^{\downarrow}(dx,dy) = \sup_{\pi \in \Pi_M(\mu,\overline{\nu})} (\rho-2)\int_{\mathbb{R}^2} |y-x|^\rho \pi(dx,dy).$$

4.2.5.2 Non-preservation of the optimality

For $p \in (0,1)$ and $y_- < y_+ < x_- < x_+ < z_- < z_+$, we set

$$\begin{aligned}
\mu &= p\delta_{x_-} + (1-p)\delta_{x_+},\\
\overline{\nu} &= p\frac{z_+ - x_-}{z_+ - y_-}\delta_{y_-} + p\frac{x_- - y_-}{z_+ - y_-}\delta_{z_+} + (1-p)\frac{z_- - x_+}{z_- - y_+}\delta_{y_+} + (1-p)\frac{x_+ - y_+}{z_- - y_+}\delta_{z_-},\\
\underline{\nu} &= p\frac{z_- - x_-}{z_- - y_+}\delta_{y_+} + p\frac{x_- - y_+}{z_- - y_+}\delta_{z_-} + (1-p)\frac{z_+ - x_+}{z_+ - y_-}\delta_{y_-} + (1-p)\frac{x_+ - y_-}{z_+ - y_-}\delta_{z_+}.
\end{aligned}$$

The couplings

$$\begin{aligned}
\overline{\pi}^\star &= p\left(\frac{z_+ - x_-}{z_+ - y_-}\delta_{(x_-,y_-)} + \frac{x_- - y_-}{z_+ - y_-}\delta_{(x_-,z_+)}\right) + (1-p)\left(\frac{z_- - x_+}{z_- - y_+}\delta_{(x_+,y_+)} + \frac{x_+ - y_+}{z_- - y_+}\delta_{(x_+,z_-)}\right),\\
\underline{\pi}^\star &= p\left(\frac{z_- - x_-}{z_- - y_+}\delta_{(x_-,y_+)} + \frac{x_- - y_+}{z_- - y_-}\delta_{(x_-,z_-)}\right) + (1-p)\left(\frac{z_+ - x_+}{z_+ - y_-}\delta_{(x_+,y_-)} + \frac{x_+ - y_-}{z_+ - y_-}\delta_{(x_+,z_+)}\right),
\end{aligned}$$

respectively belong to $\Pi_M(\mu,\overline{\nu})$ and $\Pi_M(\mu,\underline{\nu})$. Since (x_-,z_+) and (x_+,z_-) (resp. (x_-,y_+) and (x_+,y_-)) belong to any Borel subset Γ of $\mathbb{R}^2$ such that $\overline{\pi}^\star(\Gamma) = 1$ (resp. $\underline{\pi}^\star(\Gamma) = 1$), $\overline{\pi}^\star$ and $\underline{\pi}^\star$ are not non-decreasing. Since (x_-,y_-) and (x_+,y_+) (resp. (x_-,z_-) and (x_+,z_+)) belong to any Borel subset Γ of $\mathbb{R}^2$ such that $\overline{\pi}^\star(\Gamma) = 1$ (resp. $\underline{\pi}^\star(\Gamma) = 1$), $\overline{\pi}^\star$ and $\underline{\pi}^\star$ are not non-increasing.

Proposition 4.2.18. *Let $\rho \in (1,2)\cup(2,+\infty)$ and $y_- < y_+ < z_- < z_+$ be such that $z_- - y_- > z_+ - z_-$ and $z_- - y_+ \geq (\rho-1)^{\frac{1}{2-\rho}}(z_+ - z_-)$. For $\rho \in (1,2)$ (resp. $\rho > 2$), there exists $x_\rho \in (y_+, z_-)$ such that for any x_- and x_+ satisfying $x_\rho < x_- < x_+ < z_-$, $\overline{\pi}^\star$ is the unique optimal coupling that attains $\overline{\mathcal{M}}_\rho(\mu,\overline{\nu})$ (resp. $\underline{\mathcal{M}}_\rho(\mu,\overline{\nu})$) and $\underline{\pi}^\star$ is the unique optimal coupling that attains $\underline{\mathcal{M}}_\rho(\mu,\underline{\nu})$ (resp. $\overline{\mathcal{M}}_\rho(\mu,\underline{\nu})$).*

Remark 4.2.19. *For $\rho \in (1,2)\cup(2,+\infty)$, $(\rho-1)^{\frac{1}{2-\rho}} < 1$ so that when $y_- < y_+ < z_- < z_+$, it is enough that $z_- - y_+ \geq z_+ - z_-$ to ensure $z_- - y_- > z_+ - z_-$ and $z_- - y_+ \geq (\rho-1)^{\frac{1}{2-\rho}}(z_+ - z_-)$.*

Remark 4.2.20. *Let $y_- < y_+ < x_- < x_+ < z_- < z_+$ be such that $z_- - y_+ \geq z_+ - z_-$ and $\beta = \frac{z_- - y_+}{z_+ + z_- - y_+ - y_-}$. Since $\beta \in (0, \frac{1}{2})$, by Lemma 4.3.9, the equation $\alpha_\rho = \beta$ admits a unique root $\rho_\beta > 1$. Since $(1,+\infty) \ni \rho \mapsto \alpha_\rho$ is increasing by the fifth item in Lemma 4.3.9, for $\rho \in (1, \rho_\beta)$,*

$\alpha_\rho < \beta$ *and setting* $x_+ = (1-\alpha_\rho)z_- + \alpha_\rho y_-$, *we have*

$$x_+ - y_+ = z_- - y_+ + \alpha_\rho(y_- - z_-) = z_- - y_+ - \alpha_\rho(z_+ + z_- - y_+ - y_-) + \alpha_\rho(z_+ - y_+)$$
$$> z_- - y_+ - \beta(z_+ + z_- - y_+ - y_-) + \alpha_\rho(z_+ - y_+) = \alpha_\rho(z_+ - y_+).$$

We may choose $x_- < x_+$ *close to* x_+ *such that* $x_- - y_+ \geq \alpha_\rho(z_+ - y_+)$ *and then, by Proposition 4.2.14, the only martingale coupling in* $\Pi_M(\mu, \overline{\nu})$ *that maximizes (resp. minimizes)* $\int |x-y|^\rho \pi(dx,dy)$ *when* $\rho \in (1, 2 \wedge \rho_\beta)$ *(resp.* $\rho \in (2, \rho_\beta)$*) is non-decreasing. Hence for* $\rho \in (1,2) \cup (2, +\infty)$ *such that* $\rho < \rho_\beta$, *one has* $x_\rho \geq (1-\alpha_\rho)z_- + \alpha_\rho y_-$. *In particular, since* $\lim_{\rho \to 1+} \alpha_\rho = 0$ *by the sixth item in Lemma 4.3.9, we have* $\lim_{\rho \to 1+} x_\rho = z_-$.

Remark 4.2.21. *By Proposition 4.2.5 and Proposition 4.2.8, the unique non-decreasing coupling in* $\Pi_M(\mu, \overline{\nu})$ *(resp. non-increasing coupling in* $\Pi_M(\mu, \underline{\nu})$*) is the unique optimal coupling that attains* $\overline{\mathcal{M}}_\rho(\mu, \overline{\nu})$ *(resp.* $\underline{\mathcal{M}}_\rho(\mu, \underline{\nu})$*) for* $\rho \in (0,1]$. *When* $\overline{\pi}^\star$ *(resp.* $\underline{\pi}^\star$*) is the unique optimal coupling that attains* $\overline{\mathcal{M}}_\rho(\mu, \overline{\nu})$ *(resp.* $\underline{\mathcal{M}}_\rho(\mu, \underline{\nu})$*) for* $\rho \in (1,2)$ *and* $\underline{\mathcal{M}}_\rho(\mu, \overline{\nu})$ *(resp.* $\overline{\mathcal{M}}_\rho(\mu, \underline{\nu})$*) for* $\rho > 2$, *then* $\inf_{cx}\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu, \overline{\nu})\}$ *(resp.* $\sup_{cx}\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu, \underline{\nu})\}$*) does not belong to* $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu, \overline{\nu})\}$ *(resp.* $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu, \underline{\nu})\}$*).*

4.3 Proofs

To recall the necessary optimality condition stated by Beiglböck and Juillet in Lemma 1.11 [35], we first recall their definition of competitors :

Definition 4.3.1 (Competitor, taken from Definition 1.10 [35])**.** *Let* α *be a measure on* $\mathbb{R} \times \mathbb{R}$ *with finite first moment in the second variable. We say that* α', *a measure on the same space, is a competitor of* α *if* α' *has the same marginals as* α *and*

$$\int_{y \in \mathbb{R}} \alpha(dx, dy) \ a.e., \ \int y \, \alpha_x(dy) = \int y \, \alpha'_x(dy).$$

Now, we state a version of Lemma 1.11 [35] slightly reinforced by also taking into account Lemma 4.3.3 just below.

Lemma 4.3.2. *Let* $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ *be such that* $\mu \leq_{cx} \nu$ *and* $c : \mathbb{R}^2 \to \mathbb{R}$ *be a measurable cost function such that* $\forall (x,y) \in \mathbb{R}^2$, $c(x,y) \geq a(x) + b(y)$ *for measurable functions* a *and* b *integrable with respect to* μ *and* ν *respectively. When* $\pi \in \Pi_M(\mu, \nu)$ *is a minimizing coupling leading to finite cost, there exists a Borel set* $\Gamma \subset \mathbb{R}^2$ *such that* $\pi(\Gamma) = 1$,

$$\forall x \in \mathbb{R}, \ \exists y < x \ s.t. \ (x,y) \in \Gamma \iff \exists z > x \ s.t. \ (x,z) \in \Gamma, \tag{4.5}$$

and if α *is a measure finitely supported on* Γ, *then* $\int c(x,y)\alpha(dx,dy) \leq \int c(x,y)\alpha'(dx,dy)$ *for each competitor* α' *of* α.

Lemma 4.3.3. *Let* $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ *be such that* $\mu \leq_{cx} \nu$ *and* $\pi \in \Pi_M(\mu, \nu)$. *For any Borel subset*

Γ *of* $\mathbb{R}^2$ *such that* $\pi(\Gamma) = 1$*, we may find a Borel subset* $\widetilde{\Gamma}$ *of* $\mathbb{R}^2$ *such that* $\widetilde{\Gamma} \subset \Gamma$*,* $\pi(\widetilde{\Gamma}) = 1$ *and*

$$\forall x \in \mathbb{R},\ \exists y < x \ s.t.\ (x,y) \in \widetilde{\Gamma} \Longleftrightarrow \exists z > x \ s.t.\ (x,z) \in \widetilde{\Gamma}.$$

Proof of Lemma 4.3.3. For $x \in \mathbb{R}$, let $\Gamma_x = \{y \in \mathbb{R} : (x,y) \in \Gamma\}$. We set

$$A = \{x \in \mathbb{R} : \pi_x\left(\Gamma_x \cap (-\infty, x)\right) \times \pi_x\left(\Gamma_x \cap (x, +\infty)\right) = 0\},$$

and

$$\widetilde{\Gamma} = \Gamma \setminus \bigcup_{x \in A} \{x\} \times \{\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}\}.$$

Since $\mu(dx)$ almost everywhere, $\pi_x(\Gamma_x) = 1$ and $\int_{\Gamma_x} y\, \pi_x(dy) = \int_{\mathbb{R}} y\, \pi_x(dy) = x$,

$$\mu(dx) \text{ a.e., } x \in A \Rightarrow \pi_x\left(\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}\right) = 0.$$

Therefore

$$\pi\left(\bigcup_{x \in A} \{x\} \times \{\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}\}\right) = \int_{x \in A} \pi_x\left(\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}\right) \mu(dx) = 0,$$

which implies $\pi(\widetilde{\Gamma}) = \pi(\Gamma) = 1$. Let $\widetilde{\Gamma}_x = \left\{y \in \mathbb{R} : (x,y) \in \widetilde{\Gamma}\right\}$ for $x \in \mathbb{R}$. For $x \in A$, $\widetilde{\Gamma}_x \subset \{x\}$. For $x \notin A$, $\widetilde{\Gamma}_x \cap (-\infty, x) = \Gamma_x \cap (-\infty, x) \neq \emptyset$ and $\widetilde{\Gamma}_x \cap (x, +\infty) = \Gamma_x \cap (x, +\infty) \neq \emptyset$, which concludes the proof.

□

4.3.1 Proof of Lemma 4.2.2

When $\nu = \mu$, the conclusion holds since the set under consideration is $\{\delta_0\}$. We now suppose that $\nu \neq \mu$, which implies that $\int_{\mathbb{R}} y^2 \nu(dy) > \int_{\mathbb{R}} x^2 \mu(dx)$. Since $2(x^2 + y^2) \geq (y - x)^2$, we have that $(y - x)^2 > z$ implies $4x^2 > z$ or $4y^2 > z$ so that, for $\pi \in \Pi(\mu, \nu)$ and $z \geq 0$,

$$\pi\left(\left\{(x,y) : (y-x)^2 > z\right\}\right) \leq f(z) \text{ where } f(z) = \left(\mu(\{x : 4x^2 > z\}) + \nu(\{y : 4y^2 > z\})\right) \wedge 1. \tag{4.6}$$

The non-increasing function f is right-continuous on $[0, +\infty)$ and so is $[0, +\infty) \ni z \mapsto zf(z) + \int_z^{+\infty} f(w)dw$. We have

$$zf(z) + \int_z^{+\infty} f(w)dw \leq 4\left(\int \mathbb{1}_{\{x > \sqrt{z}/2\}} x^2 \mu(dx) + \int \mathbb{1}_{\{y > \sqrt{z}/2\}} y^2 \nu(dy)\right) \underset{z \to +\infty}{\longrightarrow} 0.$$

On the other hand

$$\int_0^{+\infty} f(w)dw \geq \int_0^{+\infty} \nu(\{y : 4y^2 \geq w\})dw = 4\int_{\mathbb{R}} y^2 \nu(dy) > \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx).$$

As a consequence $\overline{z} := \sup\{z > 0 : zf(z) + \int_z^{+\infty} f(w)dw \geq \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)\}$ belongs to $(0, +\infty)$ and is such that

$$\overline{z} \lim_{z\to\overline{z}-} f(z) + \int_{\overline{z}}^{+\infty} f(w)dw \geq \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx) \geq \overline{z}f(\overline{z}) + \int_{\overline{z}}^{+\infty} f(w)dw.$$

Therefore we may find $p \in [f(\overline{z}), \lim_{z\to\overline{z}-} f(z)] \subset [0, 1]$ such that

$$\overline{z}p + \int_{\overline{z}}^{+\infty} f(w)dw = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx).$$

The probability measure $\overline{\eta}$ with cumulative distribution function

$$F_{\overline{\eta}}(z) = \mathbb{1}_{\{0\leq z\leq\overline{z}\}}(z)(1-p) + \mathbb{1}_{\{z>\overline{z}\}}(1 - f(z))$$

is such that

$$\int_{\mathbb{R}} z\,\overline{\eta}(dz) = \int_0^{+\infty} (1 - F_{\overline{\eta}}(z))\,dz = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx).$$

Let $\pi \in \Pi_M(\mu, \nu)$, $\eta = \mathrm{sq}\#\pi$ and $G(z) = \int_z^{+\infty}(1-F_{\overline{\eta}}(w))dw - \int_z^{+\infty}(1-F_\eta(w))dw$ for $z \in \mathbb{R}$. By (4.6), we have $\forall z > \overline{z}$, $1 - F_\eta(z) \leq f(z) = 1 - F_{\overline{\eta}}(z)$ so that $G(z) \geq 0$ for all $z \in [\overline{z}, +\infty)$. Since $\int_0^{+\infty}(1-F_\eta(w))dw = \int_{\mathbb{R}} z\eta(dz) = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)$, $G(0) = 0$ and since $F_{\overline{\eta}}(z) = F_\eta(z) = 0$ for $z \in (-\infty, 0)$, $G(z) = 0$ for $z \in (-\infty, 0]$. The function G being concave on $[0, \overline{z}]$ by constancy of $F_{\overline{\eta}}$ on this interval, we conclude that $G(z) \geq 0$ for all $z \in \mathbb{R}$. By [206, Theorem 3.A.1 (a)], this together with the equality of means implies that $\eta \leq_{cx} \overline{\eta}$. Therefore $\{\mathrm{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$ is bounded by $\overline{\eta}$ for the convex order.

4.3.2 Proofs of the optimality results in Section 4.2.3 apart from Proposition 4.2.8

The proof of Proposition 4.2.8, which relies on the one of Theorem 4.2.11, is postponed to the next section.

Lemma 4.3.4. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be concave and increasing. Let $y < m < z$ and $f : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f(x) = \frac{z-m}{z-y}\varphi(|x-y|) + \frac{m-y}{z-y}\varphi(|z-x|) - \varphi(|x-m|).$$

Then, f is increasing on $[y, m]$ and decreasing on $[m, z]$.

Proof of Lemma 4.3.4. To check the monotonicity on $[y, m]$, we choose x_- and x_+ such that $y \leq x_- < x_+ \leq m$,

$$\begin{aligned} f(x_+) - f(x_-) = {} & \frac{z-m}{z-y}\left(\varphi(x_+ - y) - \varphi(x_- - y) - \varphi(m - x_+) + \varphi(m - x_-)\right) \\ & + \frac{m-y}{z-y}\left(\varphi(z - x_+) - \varphi(z - x_-) - \varphi(m - x_+) + \varphi(m - x_-)\right). \end{aligned}$$

Since $x_+ - y > x_- - y$ and $m - x_- > m - x_+$, we have $\varphi(x_+ - y) > \varphi(x_- - y)$ and $\varphi(m - x_-) >$

$\varphi(m-x_+)$ so that the first term in the right-hand side is positive. On the other hand, we know that

$$m - x_+ \leq (z - x_+) \wedge (m - x_-) \leq (z - x_+) \vee (m - x_-) \leq z - x_-.$$

By the concavity of φ, we have

$$\begin{aligned}
\varphi(z - x_+) &\geq \frac{z-m}{z-m+x_+-x_-}\varphi(z-x_-) + \frac{x_+-x_-}{z-m+x_+-x_-}\varphi(m-x_+),\\
\varphi(m - x_-) &\geq \frac{z-m}{z-m+x_+-x_-}\varphi(m-x_+) + \frac{x_+-x_-}{z-m+x_+-x_-}\varphi(z-x_-),
\end{aligned}$$

which implies $\varphi(z-x_+)+\varphi(m-x_-) \geq \varphi(z-x_-)+\varphi(m-x_+)$. Therefore, $f(x_+)-f(x_-)>0$ and f is increasing on $[y,m]$.

Similarly, to check the monotonicity on $[m,z]$, we choose x_- and x_+ such that $m \leq x_- < x_+ \leq z$,

$$\begin{aligned}
f(x_+) - f(x_-) = &\frac{z-m}{z-y}\left(\varphi(x_+-y) - \varphi(x_- - y) - \varphi(x_+ - m) + \varphi(x_- - m)\right)\\
&+ \frac{m-y}{z-y}\left(\varphi(z-x_+) - \varphi(z-x_-) - \varphi(x_+ - m) + \varphi(x_- - m)\right).
\end{aligned}$$

Since $z-x_+ < z-x_-$ and $x_- - m < x_+ - m$, we have $\varphi(z-x_+) < \varphi(z-x_-)$ and $\varphi(x_- - m) < \varphi(x_+ - m)$ so that the second term in the right-hand side is negative. On the other hand, we know that

$$x_- - m \leq (x_+ - m) \wedge (x_- - y) \leq (x_+ - m) \vee (x_- - y) \leq x_+ - y.$$

By the concavity of φ, we have

$$\begin{aligned}
\varphi(x_+ - m) &\geq \frac{m-y}{m-y+x_+-x_-}\varphi(x_- - m) + \frac{x_+-x_-}{m-y+x_+-x_-}\varphi(x_+ - y),\\
\varphi(x_- - y) &\geq \frac{m-y}{m-y+x_+-x_-}\varphi(x_+ - y) + \frac{x_+-x_-}{m-y+x_+-x_-}\varphi(x_- - m),
\end{aligned}$$

which implies $\varphi(x_+-y)+\varphi(x_- - m) \leq \varphi(x_- - y)+\varphi(x_+ - m)$. Therefore, $f(x_+)-f(x_-)<0$ and f is decreasing on $[m,z]$.

□

Proof of Proposition 4.2.5. The cost function $\mathbb{R}^2 \ni (x,y) \mapsto \varphi(|x-y|)$ is lower-semicontinuous and even continuous when φ is continuous. Since it satisfies the bounds in Remark 4.2.6, the existence of optimizers follows from [38, Theorem 1]. Let us establish the monotonicity properties. By Lemma 4.3.2, if $\tilde{\pi} \in \Pi_M(\mu,\nu)$ is a maximizing (resp. minimizing) martingale coupling which leads to finite costs, then there exists a Borel set $\Gamma \subseteq \mathbb{R}\times\mathbb{R}$ (resp. $\Gamma \subseteq [a,b]\times(-\infty,a]\cup[b,+\infty)$) with $\tilde{\pi}(\Gamma)=1$ such that

$$\forall x \in \mathbb{R},\ \exists y < x \text{ s.t. } (x,y)\in\Gamma \iff \exists z > x \text{ s.t. } (x,z)\in\Gamma, \tag{4.7}$$

and for a measure α which is finitely supported on Γ, we have

$$\int \varphi(|x-y|)\,\alpha'(dx,dy) \overset{\text{(resp. } \geq)}{\leq} \int \varphi(|x-y|)\,\alpha(dx,dy), \tag{4.8}$$

for every competitor α' of α. Assume that $(x_-,y_-),(x_+,y_+)\in\Gamma$ (resp. $(x_-,y_+),(x_+,y_-)\in\Gamma$) with $y_-\leq x_-$, $y_+\leq x_+$ (resp. $y_+\leq x_-$, $y_-\leq x_+$) and $x_-<x_+$. Let us assume $y_+<y_-$ and obtain a contradiction against (4.8). Then $y_+<y_-\leq x_-<x_+$ (resp. $y_+<y_-\leq a\leq x_-<x_+$). By (4.7), there exists $z>x_+$ such that $(x_+,z)\in\Gamma$ (resp. $z>x_-$ such that $(x_-,z)\in\Gamma$ and then $y_+<y_-\leq a\leq x_-<x_+\leq b\leq z$). By Lemma 4.3.4 applied with $(y,m,z)=(y_+,y_-,z)$, the function f defined by

$$f(x)=\frac{z-y_-}{z-y_+}\varphi(|x-y_+|)+\frac{y_--y_+}{z-y_+}\varphi(|z-x|)-\varphi(|x-y_-|)$$

is decreasing on $[y_-,z]$. We deduce that for

$$\begin{aligned}\beta(dx,dy)&=\delta_{(x_-,y_-)}+\frac{z-y_-}{z-y_+}\delta_{(x_+,y_+)}+\frac{y_--y_+}{z-y_+}\delta_{(x_+,z)},\\ \gamma(dx,dy)&=\delta_{(x_+,y_-)}+\frac{z-y_-}{z-y_+}\delta_{(x_-,y_+)}+\frac{y_--y_+}{z-y_+}\delta_{(x_-,z)},\end{aligned}$$

$$\int \varphi(|x-y|)\,\beta(dx,dy)-\int \varphi(|x-y|)\,\gamma(dx,dy)=f(x_+)-f(x_-)<0.$$

This contradicts inequality (4.8) since γ (resp. β) is a competitor of β (resp. γ), which is finitely supported on Γ. Therefore, we deduce that $y_-\leq y_+$.

On the other hand, assume that $(x_-,z_-),(x_+,z_+)\in\Gamma$ (resp. $(x_-,z_+),(x_+,z_-)\in\Gamma$) with $x_-\leq z_-$, $x_+\leq z_+$ (resp. $x_-\leq z_+$, $x_+\leq z_-$) and $x_-<x_+$. Let us assume that $z_+<z_-$ and obtain a contradiction against (4.8). Then $x_-<x_+\leq z_+<z_-$ (resp. $x_-<x_+\leq b\leq z_+<z_-$). By (4.7), there exists $y<x_-$ such that $(x_-,y)\in\Gamma$ (resp. $y<x_+$ such that $(x_+,y)\in\Gamma$ and then $y\leq a\leq x_-<x_+\leq b\leq z_+<z_-$). By Lemma 4.3.4 applied with $(y,m,z)=(y,z_+,z_-)$, the function g defined by

$$g(x)=\frac{z_--z_+}{z_--y}\varphi(|x-y|)+\frac{z_+-y}{z_--y}\varphi(|z_--x|)-\varphi(|x-z_+|)$$

is increasing on $[y,z_+]$. We deduce that for

$$\begin{aligned}\beta(dx,dy)&=\delta_{(x_+,z_+)}+\frac{z_--z_+}{z_--y}\delta_{(x_-,y)}+\frac{z_+-y}{z_--y}\delta_{(x_-,z_-)},\\ \gamma(dx,dy)&=\delta_{(x_-,z_+)}+\frac{z_--z_+}{z_--y}\delta_{(x_+,y)}+\frac{z_+-y}{z_--y}\delta_{(x_+,z_-)},\end{aligned}$$

$$\int |x-y|\,\beta(dx,dy)-\int |x-y|\,\gamma(dx,dy)=g(x_-)-g(x_+)<0.$$

This contradicts inequality (4.8) since γ (resp. β) is a competitor of β (resp. γ), which is finitely supported on Γ. Therefore, we deduce that $z_-\leq z_+$. □

Remark 4.3.5. *The support assumption $\mu([a,b]) = 1 = 1 - \nu((a,b))$ made in Proposition 4.2.5 to ensure that any minimizing martingale coupling for the cost $\varphi(|y-x|)$ is non-increasing is crucial to avoid in the proof the situations $(x_-, y_+), (x_+, y_-) \in \Gamma$ with $y_+ \le x_- < y_- \le x_+$ and $(x_-, z_+), (x_+, z_-) \in \Gamma$ with $x_- \le z_+ < x_+ \le z_-$ where we could not derive a contradiction.*

Proof of Proposition 4.2.9. The existence follows from Proposition 4.2.5. The proof for uniqueness is the same as the one of Theorem 7.3 [35]. Since, from Lemma 4.3.4, we only know that f is increasing on $[y,m]$ and decreasing on $[m,z]$, the forbidden cases in Equation (23) [35] should be replaced by

$$y' \le x' < x \le y^+ \text{ or } y^- \le x < x' \le y'. \tag{4.9}$$

Actually, in the case $\varphi(u) = u$ considered in Theorem 7.3 [35], f is constant on $(-\infty, y]$ and on $[z, +\infty)$ and the forbidden cases should be $y' \le x' < x \wedge y^+$ or $y^- \vee x < x' \le y'$ instead of the more general condition in Equation (23) [35]. Nevertheless, in view of Lemma 4.3.3, in the second paragraph of the proof of Theorem 7.3 [35], we can assume that $b^- < a < b^+$ so that we end up with a situation in the restricted set of forbidden cases (4.9). □

Proof of Proposition 4.2.10. Since ρ is fixed, for notational simplicity, we denote μ_ε in place of $\mu_{\varepsilon,\rho}$ in the proof. We set $x_- = m - \left(1 + \beta\varepsilon^{1-\rho}\right)^{\frac{1}{\rho}} \varepsilon$ and $x_+ = m + \varepsilon$. Let us suppose that $\varepsilon < (z-m) \wedge \frac{m-y}{(1+\beta)^{\frac{1}{\rho}}} \wedge 1$. This ensures that $y < x_-$ and $x_+ < z$. For $x \in (-\infty, x_-] \cup [x_+, +\infty)$, $u_{\mu_\varepsilon}(x) = |x-m| \le u_\nu(x)$. The function u_{μ_ε} is affine on $[x_-, x_+]$. Since $y < x_- < m < x_+ < z$, the function u_ν is affine on $[x_-, m]$ and on $[m, x_+]$. The inequality $\varepsilon < \frac{2(z-m)(m-y)}{3(z-y)}$ in our assumptions implies that $u_{\mu_\varepsilon}(m) = \frac{2\left(1+\beta\varepsilon^{1-\rho}\right)^{\frac{1}{\rho}}}{1+\left(1+\beta\varepsilon^{1-\rho}\right)^{\frac{1}{\rho}}}\varepsilon < \frac{4(z-m)(m-y)}{3(z-y)} = u_\nu(m)$. Then,

$$\forall x \in \mathbb{R}, \quad u_{\mu_\varepsilon}(x) \le u_\nu(x) \qquad \text{and} \qquad \mu_\varepsilon \le_{cx} \nu.$$

Since $y < x_- < m < x_+ < z$, the set $\Gamma = \{(x_-, y), (x_-, m), (x_-, z), (x_+, y), (x_+, m), (x_+, z)\}$ is such that $\Gamma \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le x_1\} = \{(x_-, y), (x_+, y), (x_+, m)\}$ and $\Gamma \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le x_2\} = \{(x_-, m), (x_-, z), (x_+, z)\}$ so that Γ satisfies conditions $(a)(b)$ in Definition 4.2.3. Since any martingale coupling gives full weight to Γ, it is non-decreasing.

Then, let $\pi(dx, dy) = \mu_\varepsilon(dx)\, \pi_x(dy) \in \Pi_M(\mu_\varepsilon, \nu)$. Since $u_{\mu_\varepsilon}(m) < u_\nu(m)$ and

$$\begin{aligned} u_{\mu_\varepsilon}(m) &= \mu_\varepsilon(\{x_-\})(m - x_-) + \mu_\varepsilon(\{x_+\})(x_+ - m) \\ u_\nu(m) &= \mu_\varepsilon(\{x_-\}) \int |m-w|\, \pi_{x_-}(dw) + \mu_\varepsilon(\{x_+\}) \int |m-w|\, \pi_{x_+}(dw), \end{aligned}$$

we have either $m - x_- < \int |m-w|\, \pi_{x_-}(dw)$ so that $\pi_{x_-}(\{y\})\pi_{x_-}(\{z\}) > 0$ or $x_+ - m < \int |m-w|\, \pi_{x_+}(dw)$ so that $\pi_{x_+}(\{y\})\pi_{x_+}(\{z\}) > 0$. We deduce that when $\pi_{x-}(\{m\})\pi_{x+}(\{m\}) > 0$, then either $\pi_{x_-}(\{m\})\, \pi_{x_+}(\{y\})\, \pi_{x_+}(\{z\}) > 0$ or $\pi_{x_+}(\{m\})\, \pi_{x_-}(\{y\})\, \pi_{x_-}(\{z\}) > 0$. If $\pi_{x_+}(\{m\}) = 0$ (resp. $\pi_{x_-}(\{m\}) = 0$), then $\pi_{x_+}(\{y\})\pi_{x_+}(\{z\}) > 0$ (resp. $\pi_{x_-}(\{y\})\pi_{x_-}(\{z\}) > 0$) by the martingale property, and since $\nu(\{m\}) > 0$, we have $\pi_{x_-}(\{m\}) > 0$ (resp. $\pi_{x_+}(\{m\}) > 0$).

Therefore, we always have

$$\pi_{x_-}(\{m\})\,\pi_{x_+}(\{y\})\,\pi_{x_+}(\{z\}) > 0 \quad \text{or} \quad \pi_{x_-}(\{y\})\,\pi_{x_-}(\{z\})\,\pi_{x_+}(\{m\}) > 0,$$

and there exists more than one martingale coupling between μ_ε and ν since for α positive and small enough in the first case and $-\alpha$ positive and small enough in the second case, adding

$$\alpha\left(\delta_{(x_+,m)} + \frac{z-m}{z-y}\delta_{(x_-,y)} + \frac{m-y}{z-y}\delta_{(x_-,z)} - \delta_{(x_-,m)} - \frac{z-m}{z-y}\delta_{(x_+,y)} + \frac{m-y}{z-y}\delta_{(x_+,z)}\right)$$

to π leads to a distinct coupling in $\Pi_M(\mu_\varepsilon,\nu)$. To further parametrize the coupling π, let us apply the martingale property again. For $x \in \{x_-, x_+\}$, we have

$$\begin{cases} \pi_x(\{y\}) + \pi_x(\{m\}) + \pi_x(\{z\}) = 1 \\ y\,\pi_x(\{y\}) + m\,\pi_x(\{m\}) + z\,\pi_x(\{z\}) = x \end{cases} \Longrightarrow \begin{cases} \pi_x(\{y\}) &= \dfrac{z-x-(z-m)\pi_x(\{m\})}{z-y}, \\ \pi_x(\{z\}) &= \dfrac{x-y-(m-y)\pi_x(\{m\})}{z-y}. \end{cases}$$

Since $\mu_\varepsilon(\{x_+\})\pi_{x_+}(\{m\}) = \nu(\{m\}) - \mu_\varepsilon(\{x_-\})\pi_{x_-}(\{m\})$, the coupling π can be parameterized with respect to the variable $\pi_{x_-}(\{m\})$ only as follows:

$$\begin{aligned} \pi = \gamma + \mu_\varepsilon(\{x_-\})\pi_{x_-}(\{m\}) &\left(\frac{z-m}{z-y}\delta_{(x_+,y)} + \frac{m-y}{z-y}\delta_{(x_+,z)} - \delta_{(x_+,m)}\right. \\ &\left. -\frac{z-m}{z-y}\delta_{(x_-,y)} - \frac{m-y}{z-y}\delta_{(x_-,z)} + \delta_{(x_-,m)}\right) \text{ with} \\ \gamma = \mu_\varepsilon(\{x_-\})&\left(\frac{z-x_-}{z-y}\delta_{(x_-,y)} + \frac{x_- - y}{z-y}\delta_{(x_-,z)}\right) + \mu_\varepsilon(\{x_+\})\left(\frac{z-x_+}{z-y}\delta_{(x_+,y)} + \frac{x_+-y}{z-y}\delta_{(x_+,z)}\right) \\ &+ \nu(\{m\})\left(\delta_{(x_+,m)} - \frac{z-m}{z-y}\delta_{(x_+,y)} - \frac{m-y}{z-y}\delta_{(x_+,z)}\right). \end{aligned}$$

For $\tilde\rho \in (0,1]$, defining the cost function $f_{\tilde\rho} : [y,z] \to \mathbb{R}$ by

$$f_{\tilde\rho}(x) = \frac{z-m}{z-y}(x-y)^{\tilde\rho} + \frac{m-y}{z-y}(z-x)^{\tilde\rho} - |x-m|^{\tilde\rho},$$

we have

$$\int |x-y|^{\tilde\rho}\pi(dx,dy) = \int |x-y|^{\tilde\rho}\gamma(dx,dy) + \mu_\varepsilon(\{x_-\})\pi_{x_-}(\{m\})(f_\rho(x_+) - f_\rho(x_-)).$$

To maximize $\int |x-y|^{\tilde\rho}\pi(dx,dy)$, we need to maximize $\pi_{x_-}(\{m\})$ when $f_{\tilde\rho}(x_+) > f_{\tilde\rho}(x_-)$ and minimize $\pi_{x_-}(\{m\})$ when $f_{\tilde\rho}(x_+) < f_{\tilde\rho}(x_-)$. Applying the first order Taylor expansion at $x = m$ to the first two terms of the function $f_{\tilde\rho}$ with $\tilde\rho \in (0,1]$, we get

$$\begin{aligned} f_{\tilde\rho}(x_+) - f_{\tilde\rho}(x_-) = \tilde\rho&\left(\frac{z-m}{z-y}(m-y)^{\tilde\rho-1} - \frac{m-y}{z-y}(z-m)^{\tilde\rho-1}\right)(x_+ - x_-) + o(x_+ - x_-) \\ &+ (m-x_-)^{\tilde\rho} - (x_+ - m)^{\tilde\rho}. \end{aligned} \tag{4.10}$$

Let us suppose that $\rho \in (0,1)$. We have that $(m-x_-)^\rho - (x_+-m)^\rho = \beta\varepsilon$. When $\varepsilon \to 0+$,

$(1+\beta\varepsilon^{1-\rho})^{\frac{1}{\rho}} \to 1$ so that $x_+ - x_- \sim 2\varepsilon$ and

$$f_\rho(x_+) - f_\rho(x_-) = \left(2\rho\left(\frac{z-m}{z-y}(m-y)^{\rho-1} - \frac{m-y}{z-y}(z-m)^{\rho-1}\right) + \beta\right)\varepsilon + o(\varepsilon).$$

Since $\beta \in \left(0, 2\rho\left(\frac{m-y}{z-y}(z-m)^{\rho-1} - \frac{z-m}{z-y}(m-y)^{\rho-1}\right)\right)$ by assumption, for ε positive small enough, $f_\rho(x_+) - f_\rho(x_-) < 0$. On the other hand, for $\rho' \in (0,\rho)$, we have $\left(1+\beta\varepsilon^{1-\rho}\right)^{\frac{\rho'}{\rho}} - 1 \sim \frac{\rho'}{\rho}\beta\,\varepsilon^{1-\rho}$ so that $(m-x_-)^{\rho'} - (x_+-m)^{\rho'} \sim \frac{\rho'}{\rho}\beta\,\varepsilon^{1-\rho+\rho'}$. Hence, by (4.10) for $\tilde\rho = \rho'$, $f_{\rho'}(x_+) - f_{\rho'}(x_-) \sim \frac{\rho'}{\rho}\beta\,\varepsilon^{1-\rho+\rho'} > 0$.

Finally, let us suppose that $\rho = 1$. We have

$$f_1(x_+) - f_1(x_-) = \frac{2\varepsilon(\beta(z-m) - (2m-y-z))}{z-y}.$$

Since $y < \frac{y+z}{2} < m < z$ and $\beta < \frac{2m-y-z}{z-m}$ by assumption, we have $f_1(x_+) - f_1(x_-) < 0$. For $\rho' \in (0,1)$, $(m-x_-)^{\rho'} - (x_+-m)^{\rho'} \sim \left((1+\beta)^{\rho'} - 1\right)\varepsilon^{\rho'}$ and, since $x_+ - x_- = (2+\beta)\varepsilon$, by (4.10) applied with $\tilde\rho = \rho'$, $f_{\rho'}(x_+) - f_{\rho'}(x_-) \sim \left((1+\beta)^{\rho'} - 1\right)\varepsilon^{\rho'} > 0$.

Therefore, for $0 < \rho' < \rho \le 1$ and ε positive small enough, to maximize $\int |x-y|^{\rho'}\pi(dx,dy)$, we need to maximize $\pi_{x_-}(\{m\})$ and to maximize $\int |x-y|^{\rho}\,\pi(dx,dy)$, we need to minimize $\pi_{x_-}(\{m\})$. □

4.3.3 Proof of the results in Section 4.2.4 and Proposition 4.2.8

Proof of Corollary 4.2.12. (i) As $\mu \neq \nu$, the non vacuity of $\pi \in \Pi_M(\mu,\nu,\nu_l,\nu_r)$ implies that $\nu_l + \nu_r$ is not the zero measure. Since

$$\begin{aligned}\Pi_M(\mu,\nu,\nu_l,\nu_r) \ni \pi \mapsto &\frac{\pi(dx,dy) + (\nu_l+\nu_r-\nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}\\ &\in \Pi_M\left(\frac{\mu+\nu_l+\nu_r-\nu}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}, \frac{\nu_l+\nu_r}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}, \frac{\nu_l}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}, \frac{\nu_r}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}\right)\end{aligned}$$

is a bijection, the statement follows from Theorem 4.2.11 (concerning (4.3), this relies on the fact that the corresponding mapping between the integrals is affine with a positive factor).

(ii) Let $\pi \in \Pi_M(\mu,\nu,\nu_l,\nu_r)$ be non-decreasing and let Γ denote a Borel subset of $\mathbb{R}\times\mathbb{R}$ satisfying the properties in Definition 4.2.3. Since $\pi(\{(x,x)\}) \le \mu(\{x\}) \wedge \nu(\{x\})$, we remove $\{(x,x) : \mu(\{x\}) > 0 \text{ and } \nu(\{x\}) = 0\} \cap \Gamma$ from Γ, while preserving these properties. Since $\pi(\Gamma) = 1$, we have

$$\nu_0^\pi(\mathbb{R}) = \pi(\{(x,x) : x \in \mathbb{R}\}) = \pi(\{(x,x) : x \in \mathbb{R}\} \cap \Gamma) = \nu_0^\pi(\{x \in \mathbb{R} : (x,x) \in \Gamma\}).$$

Let $x \in \mathbb{R}$ be such that $(x,x) \in \Gamma$. Then, $\Gamma \subset \{(-\infty,x]\times(-\infty,x]\} \cup \{[x,+\infty)\times[x,+\infty)\}$ and either $\mu(\{x\}) > 0$ or

$$\begin{aligned} u_\nu(x) &= \int_{\mathbb{R}^2} |z-x|\pi(dy,dz) \\ &= \int_{(y,z)\in(-\infty,x)\times\mathbb{R}} (x-z)\pi_y(dz)\mu(dy) + \int_{(y,z)\in(x,+\infty)\times\mathbb{R}} (z-x)\pi_y(dz)\mu(dy) \\ &= \int_{(-\infty,x)} (x-y)\mu(dy) + \int_{(x,+\infty)} (y-x)\mu(dy) = u_\mu(x). \end{aligned}$$

Hence,

$$\nu_0^\pi(\mathbb{R}) = \nu_0^\pi\left(\{x\in\mathbb{R} : u_\nu(x) = u_\mu(x) \text{ or } (u_\nu(x)-u_\mu(x))\times(\mu(\{x\})\wedge\nu(\{x\})) > 0\}\right).$$

The conclusion follows since $\nu_0^\pi \le \mu\wedge\nu$ by definition and $\nu_0^\pi(dx) \ge \mathbb{1}_{\{u_\mu(x)=u_\nu(x)\}}\mu(dx)$ from the decomposition in irreducible components stated in [35, Theorem A.4].

(iii) Since for any non-decreasing coupling $\pi\in\Pi_M(\mu,\nu,\nu_l,\nu_r)$,

$$\frac{\pi(dx,dy) + (\nu_l+\nu_r-\nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R})+\nu_r(\mathbb{R})}$$

is non-decreasing, the uniqueness is ensured by (i). □

The proof of Theorem 4.2.11 relies on the following lemmas.

Lemma 4.3.6. *Let $\eta\in\mathcal{P}(\mathbb{R})$ and $\hat\eta$ be a non-negative measure on the real line such that $\hat\eta\le\eta$. For $f:\mathbb{R}\to\mathbb{R}$ non-decreasing such that $\int_{\mathbb{R}}|f(x)|\eta(dx) < +\infty$, one has $\int_{\mathbb{R}}|f(x)|\hat\eta(dx) < +\infty$ and*

$$\int_0^{\hat\eta(\mathbb{R})} f(F_\eta^{-1}(u))du \le \int_{\mathbb{R}} f(x)\hat\eta(dx) \le \int_{1-\hat\eta(\mathbb{R})}^1 f(F_\eta^{-1}(u))du,$$

and if $\hat\eta(\mathbb{R})>0$, $\frac{1}{\hat\eta(\mathbb{R})}\int_0^{\hat\eta(\mathbb{R})}\delta_{F_\eta^{-1}(u)}du \le_{st} \frac{\hat\eta}{\hat\eta(\mathbb{R})} \le_{st} \frac{1}{\hat\eta(\mathbb{R})}\int_{1-\hat\eta(\mathbb{R})}^1\delta_{F_\eta^{-1}(u)}du$.

Lemma 4.3.7. *For $\mu,\nu\in\mathcal{P}(\mathbb{R})$,*

$$\exists\pi\in\Pi(\mu,\nu) \text{ s.t. } \pi\left(\{(x,y)\in\mathbb{R}^2 : x>y\}\right) = 1 \iff du \text{ a.e. on } (0,1),\ F_\mu^{-1}(u) > F_\nu^{-1}(u).$$

Proof of Lemma 4.3.6. Since the inequalities are obvious when $\hat\eta(\mathbb{R}) = 0$, we suppose that $\hat\eta(\mathbb{R})>0$ and denote $\tilde\eta = \frac{\hat\eta}{\hat\eta(\mathbb{R})}$. Since $\hat\eta\le\eta$, $\int_{\mathbb{R}}|f(x)|\hat\eta(dx) \le \int_{\mathbb{R}}|f(x)|\eta(dx) < +\infty$. Moreover, by the inverse transform sampling,

$$\int_{\mathbb{R}} f(x)\hat\eta(dx) = \hat\eta(\mathbb{R})\int_{\mathbb{R}} f(x)\tilde\eta(dx) = \hat\eta(\mathbb{R})\int_0^1 f(F_{\tilde\eta}^{-1}(u))du.$$

For $u\in(0,1)$, using (1.2) for the first inequality then $\hat\eta(\mathbb{R})\tilde\eta = \hat\eta\le\eta$ for the second, we have

$$\hat\eta(\mathbb{R})u \le \hat\eta(\mathbb{R})F_{\tilde\eta}(F_{\tilde\eta}^{-1}(u)) \le F_\eta(F_{\tilde\eta}^{-1}(u))$$

so that $F_{\tilde{\eta}}^{-1}(u) \geq F_{\eta}^{-1}(\hat{\eta}(\mathbb{R})u)$. With the monotonicity of f, we deduce that

$$\int_{\mathbb{R}} f(x)\hat{\eta}(dx) \geq \hat{\eta}(\mathbb{R}) \int_0^1 f(F_{\eta}^{-1}(\hat{\eta}(\mathbb{R})u))du = \int_0^{\hat{\eta}(\mathbb{R})} f(F_{\eta}^{-1}(v))dv.$$

On the other hand, still for $u \in (0,1)$, using $\tilde{\eta} \leq \frac{\eta}{\hat{\eta}(\mathbb{R})}$ for the first inequality and (1.2) for the second,

$$\begin{aligned} 1 - F_{\tilde{\eta}}(F_{\eta}^{-1}(1-\hat{\eta}(\mathbb{R})u)) = \tilde{\eta}((F_{\eta}^{-1}(1-\hat{\eta}(\mathbb{R})u), +\infty)) &\leq \frac{\eta((F_{\eta}^{-1}(1-\hat{\eta}(\mathbb{R})u), +\infty))}{\hat{\eta}(\mathbb{R})} \\ &= \frac{1 - F_{\eta}(F_{\eta}^{-1}(1-\hat{\eta}(\mathbb{R})u))}{\hat{\eta}(\mathbb{R})} \leq u \end{aligned}$$

so that $F_{\tilde{\eta}}^{-1}(1-u) \leq F_{\eta}^{-1}(1-\hat{\eta}(\mathbb{R})u)$. We conclude that

$$\int_{\mathbb{R}} f(x)\hat{\eta}(dx) = \hat{\eta}(\mathbb{R}) \int_0^1 f(F_{\tilde{\eta}}^{-1}(1-u))du \leq \hat{\eta}(\mathbb{R}) \int_0^1 f(F_{\eta}^{-1}(1-\hat{\eta}(\mathbb{R})u))du = \int_{1-\hat{\eta}(\mathbb{R})}^1 f(F_{\eta}^{-1}(v))dv.$$

□

Proof of Lemma 4.3.7. Since the image of the Lebesgue measure on $(0,1)$ by $(F_{\mu}^{-1}, F_{\nu}^{-1})$ belongs to $\Pi(\mu,\nu)$, the sufficient condition is obvious. To prove the necessary condition, we let (X,Y) be distributed according to $\pi \in \Pi(\mu,\nu)$ such that $\pi(\{(x,y) \in \mathbb{R}^2 : x > y\}) = 1$. We have $\mathbb{P}(X > Y) = 1$ and, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\{X > Y\} = \bigcup_{q\in\mathbb{Q}}\{X \geq q > Y\}$. Since $\mathbb{Q}$ is countable, there exists a bijection $\mathbb{N} \ni n \mapsto q_n \in \mathbb{Q}$. For $n \in \mathbb{N}$, let

$$A_n = \{X \geq q_n > Y\} \cap \left\{\bigcup_{k=0}^{n-1} \{X \geq q_k > Y\}\right\}^c.$$

The events $(A_n)_{n\in\mathbb{N}}$ are disjoint and such that $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \mathbb{P}(X > Y) = 1$. The discrete random variable $Z = \sum_{n\in\mathbb{N}} q_n \mathbb{1}_{A_n}$ is such that $\mathbb{P}(X \geq Z > Y) = 1$ so that $\nu \leq_{st} \eta \leq_{st} \mu$, where η denotes the distribution of Z. For $n \in \mathbb{N}$, we have

$$F_{\eta}(q_n) = \mathbb{P}(Z \leq q_n) = \mathbb{P}(Y < Z \leq q_n) \leq \mathbb{P}(Y < q_n) = F_{\nu}(q_n-).$$

We deduce that for all $u \in (F_{\eta}(q_n-), F_{\eta}(q_n))$, $F_{\nu}^{-1}(u) < q_n = F_{\eta}^{-1}(u)$. Since

$$\int_0^1 \mathbb{1}_{\bigcup_{n\in\mathbb{N}}(F_{\eta}(q_n-),F_{\eta}(q_n))}(u)du = \sum_{n\in\mathbb{N}} \mathbb{P}(Z = q_n) = 1,$$

and $F_{\eta}^{-1} \leq F_{\mu}^{-1}$ as $\eta \leq_{st} \mu$, the statement follows. □

Proof of Theorem 4.2.11. Since $\mu \neq \nu$, the existence of $\pi \in \Pi_M(\mu,\nu,\nu_l,\nu_r)$ implies that $\nu_l + \nu_r$ is not the zero measure. Since the equality of the means of μ and ν implies that

$$0 = \int_{\mathbb{R}^2} (y-x)\mathbb{1}_{\{y<x\}}\pi_x(dy)\mu(dx) + \int_{\mathbb{R}^2} (y-x)\mathbb{1}_{\{y>x\}}\pi_x(dy)\mu(dx),$$

then $\nu_l \neq 0$ and $\nu_r \neq 0$. Let $\tilde{\nu}_l = \frac{\nu_l}{\nu_l(\mathbb{R})}$ and $\tilde{\nu}_r = \frac{\nu_r}{\nu_r(\mathbb{R})}$. The restriction of ν_l (resp. ν_r) to its left-most part with mass $v \in [0, \nu_l(\mathbb{R})]$ (resp. $v \in [0, \nu_r(\mathbb{R})]$) writes $\int_0^v \delta_{F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)} dw$ (resp. $\int_0^v \delta_{F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)} dw$). Intuitively, to specify a non-decreasing element of $\Pi_M(\mu, \nu, \nu_l, \nu_r)$, it is enough to know the part $\phi_\uparrow(F_\mu(x))$ of the mass $F_\mu(x)$ of starting points $y \in (-\infty, x]$ sent to the left by this martingale coupling for each $x \in \mathbb{R}$ (the complementary mass $F_\mu(x) - \phi_\uparrow(F_\mu(x))$ is of course sent to the right). Indeed, the non-decreasing property ensures that $\int_0^{F_\mu(x)} \delta_{F_\mu^{-1}(w)} dw$ is sent to the left to $\int_0^{\phi_\uparrow(F_\mu(x))} \delta_{F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)} dw$ and is sent to the right to $\int_0^{F_\mu(x)-\phi_\uparrow(F_\mu(x))} \delta_{F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)} dw$. The martingale property then ensures that

$$\begin{aligned}\int_0^{F_\mu(x)} F_\mu^{-1}(w)dw &= \int_0^{\phi_\uparrow(F_\mu(x))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw + \int_0^{\frac{F_\mu(x)-\phi_\uparrow(F_\mu(x))}{\nu_r(\mathbb{R})}} F_{\tilde{\nu}_r}^{-1}(w)dw \\ &= G(F_\mu(x), \phi_\uparrow(F_\mu(x))),\end{aligned}$$

where for $u \in [0,1]$ and $v \in [(u - \nu_r(\mathbb{R}))^+, u \wedge \nu_l(\mathbb{R})]$,

$$G(u,v) = \int_0^v F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw + \int_0^{u-v} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw.$$

We are going to prove the existence of a non-decreasing function $\phi_\uparrow : [0,1] \to [0, \nu_l(\mathbb{R})]$ such that $u \mapsto u - \phi_\uparrow(u)$ is also non-decreasing and

$$\forall u \in [0,1],\ \phi_\uparrow(u) \in [(u - \nu_r(\mathbb{R}))^+, u \wedge \nu_l(\mathbb{R})] \text{ and } \int_0^u F_\mu^{-1}(w)dw = G(u, \phi_\uparrow(u)), \tag{4.11}$$

Let $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$. Since for $x < x'$,

$$0 \leq \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x'\}}\pi(dy,dz) - \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}}\pi(dy,dz) \leq \int_{\mathbb{R}^2} \mathbb{1}_{\{x<y\leq x'\}}\pi(dy,dz) = F_\mu(x') - F_\mu(x),$$

there exists a function ϕ_π such that

$$\phi_\pi(F_\mu(x)) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}}\pi(dy,dz) \in [0, \nu_l(\mathbb{R})]. \tag{4.12}$$

One has, using $\nu_0^\pi(\mathbb{R}) = 0$,

$$\begin{aligned}\nu_r(\mathbb{R}) \geq \int_{\mathbb{R}^2} \mathbb{1}_{\{y\leq x, y<z\}}\pi(dy,dz) &= \int_{\mathbb{R}^2} \mathbb{1}_{\{y\leq x\}}\pi(dy,dz) - \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}}\pi(dy,dz) \\ &= F_\mu(x) - \phi_\pi(F_\mu(x)).\end{aligned}$$

The existence of $\phi_\uparrow(F_\mu(x)) \in [(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x))]$ for $x \in \mathbb{R}$ is addressed in **Step 1** below. **Step 2** deals with the monotonicity of $x \mapsto \phi_\uparrow(F_\mu(x))$ and $x \mapsto F_\mu(x) - \phi_\uparrow(F_\mu(x))$. Both steps rely on the fact that $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ and make use of the function ϕ_π. The existence of $\phi_\uparrow(u)$ for $u \in \bigcup_{x\in\mathbb{R}}[F_\mu(x-), F_\mu(x))$, and the monotonicity of $u \mapsto \phi_\uparrow(u)$ and $u \mapsto u - \phi_\uparrow(u)$ on $(0,1)$ are addressed in **Step 3**, while (4.2) is proved in **Step 4**. When $F_\mu(x) > 0$ for each $x \in \mathbb{R}$ (resp. $F_\mu(x) < 1$ for each $x \in \mathbb{R}$), the function $\phi_\uparrow$ is continuously and monotonically extended

by $\phi_\uparrow(0) = 0$ since $\lim_{u\to 0+}(u - \nu_r(\mathbb{R}))^+ = 0 = \lim_{u\to 0+} u \wedge \nu_l(\mathbb{R})$ (resp. $\phi_\uparrow(1) = \nu_l(\mathbb{R})$ since $\lim_{u\to 1-}(u - \nu_r(\mathbb{R}))^+ = \nu_l(\mathbb{R}) = \lim_{u\to 1-} u \wedge \nu_l(\mathbb{R})$) so that (4.11) holds for $u = 0$ (resp. $u = 1$). We are now going to prove the existence of a non-decreasing coupling $\pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ such that $\phi_{\pi^\uparrow} = \phi_\uparrow$. Because of the monotonicity of $[0,1] \ni u \mapsto \phi_\uparrow(u)$ and $[0,1] \ni u \mapsto u - \phi_\uparrow(u)$, these two functions are 1-Lipschitz and therefore absolutely continuous. Up to modifying the derivative $\phi'_\uparrow$ of $\phi_\uparrow$ in the sense of distributions on a Lebesgue negligible subset of $(0,1)$, we may suppose that it is $[0,1]$-valued. The functions $[0,1] \ni u \mapsto \int_0^{\phi_\uparrow(u)} F^{-1}_{\tilde\nu_l}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw$ and $[0,1] \ni u \mapsto \int_0^{u-\phi_\uparrow(u)} F^{-1}_{\tilde\nu_r}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw$ are convex and continuous and therefore absolutely continuous. Using [178, Lemma 1.2] to differentiate (4.11), we obtain

$$F^{-1}_\mu(u) = \phi'_\uparrow(u) F^{-1}_{\tilde\nu_l}\left(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}\right) + (1 - \phi'_\uparrow(u)) F^{-1}_{\tilde\nu_r}\left(\frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})}\right), \ du \text{ a.e. on } (0,1), \tag{4.13}$$

with the convention that the first (resp. second) product in the right-hand side is 0 when $\phi'_\uparrow(u) = 0$ (resp. $\phi'_\uparrow(u) = 1$). We set

$$\pi^\uparrow(dx,dy) = \int_0^1 \Big(\phi'_\uparrow(u)\delta_{\left(F^{-1}_\mu(u), F^{-1}_{\tilde\nu_l}\left(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}\right)\right)}(dx,dy) + (1-\phi'_\uparrow(u))\delta_{\left(F^{-1}_\mu(u), F^{-1}_{\tilde\nu_r}\left(\frac{u-\phi_\uparrow(u)}{\nu_r(\mathbb{R})}\right)\right)}(dx,dy)\Big) du.$$

We have $\int_{y\in\mathbb{R}} \pi^\uparrow(dx,dy) = \int_0^1 \delta_{F^{-1}_\mu(u)}(dx)du$, so that, by the inverse transform sampling, the first marginal of $\pi^\uparrow$ is μ. Using Lemma 2.6 in [145] like in the proof of Proposition 2.3 in [145], one checks that $\pi^\uparrow$ is a martingale coupling. We are next going to check that

$$\phi'_\uparrow(u)du \text{ a.e. on } (0,1),\ F^{-1}_{\tilde\nu_l}\left(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}\right) < F^{-1}_\mu(u) \tag{4.14}$$

$$\text{and } (1-\phi'_\uparrow(u))du \text{ a.e. on } (0,1),\ F^{-1}_\mu(u) < F^{-1}_{\tilde\nu_r}\left(\frac{u-\phi_\uparrow(u)}{\nu_r(\mathbb{R})}\right). \tag{4.15}$$

With the definition of $\pi^\uparrow$, we deduce that $\nu_l^{\pi^\uparrow} = \int_0^1 \phi'_\uparrow(u)\delta_{F^{-1}_{\tilde\nu_l}\left(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}\right)} du$. For $g : [0,1] \to \mathbb{R}$ measurable and bounded, by Proposition 4.9 [201] and the remark just above applied with $f(u) = g\left(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}\right)$ and $A_u = \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}$,

$$\int_0^1 g\left(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}\right)\frac{\phi'_\uparrow(u)}{\nu_l(\mathbb{R})}du = \int_0^1 g(v)dv. \tag{4.16}$$

Hence $\nu_l^{\pi^\uparrow} = \nu_l(\mathbb{R})\int_0^1 \delta_{F^{-1}_{\tilde\nu_l}(w)}dw = \nu_l(\mathbb{R})\tilde\nu_l = \nu_l$. In the same way,

$$\nu_r^{\pi^\uparrow} = \int_0^1 (1-\phi'_\uparrow(u))\delta_{F^{-1}_{\tilde\nu_r}\left(\frac{u-\phi_\uparrow(u)}{\nu_r(\mathbb{R})}\right)}du = \nu_r,$$

so that, since $\nu = \nu_l + \nu_r$, $\pi^{\uparrow} \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$. Let

$$\Gamma = \left\{ \left\{ \left(F_{\mu}^{-1}(u), F_{\tilde{\nu}_l}^{-1}\left(\frac{\phi_{\uparrow}(u)}{\nu_l(\mathbb{R})} \right) \right) : u \in \mathbb{R} \right\} \cap \{(x,y) \in \mathbb{R}^2 : y < x\} \right\}$$
$$\cup \left\{ \left\{ \left(F_{\mu}^{-1}(u), F_{\tilde{\nu}_r}^{-1}\left(\frac{u - \phi_{\uparrow}(u)}{\nu_r(\mathbb{R})} \right) \right) : u \in \mathbb{R} \right\} \cap \{(x,y) \in \mathbb{R}^2 : x < y\} \right\}.$$

The definition of $\pi^{\uparrow}$ combined with (4.14) and (4.15) ensures that $\pi^{\uparrow}(\Gamma) = 1$. By monotonicity of $u \mapsto F_{\mu}^{-1}(u)$, $u \mapsto F_{\tilde{\nu}_l}^{-1}\left(\frac{\phi_{\uparrow}(u)}{\nu_l(\mathbb{R})} \right)$ and $u \mapsto F_{\tilde{\nu}_r}^{-1}\left(\frac{u - \phi_{\uparrow}(u)}{\nu_r(\mathbb{R})} \right)$, Γ satisfies conditions (a) and (b) in Definition 4.2.3. Hence $\pi^{\uparrow}$ is non-decreasing.

To check (4.14), we introduce

$$\tilde{\mu}_l(dy) = \frac{1}{\nu_l(\mathbb{R})} \int_{z \in \mathbb{R}} \mathbb{1}_{\{z<y\}} \pi(dy, dz).$$

As $\frac{\mathbb{1}_{\{z<y\}}}{\nu_l(\mathbb{R})} \pi(dy, dz) \in \Pi(\tilde{\mu}_l, \tilde{\nu}_l)$, Lemma 4.3.7 implies that du a.e., $F_{\tilde{\nu}_l}^{-1}(u) < F_{\tilde{\mu}_l}^{-1}(u)$. With (4.16), we deduce that

$$\phi_{\uparrow}'(u)du \text{ a.e. on } (0,1),\ F_{\tilde{\nu}_l}^{-1}\left(\frac{\phi_{\uparrow}(u)}{\nu_l(\mathbb{R})} \right) < F_{\tilde{\mu}_l}^{-1}\left(\frac{\phi_{\uparrow}(u)}{\nu_l(\mathbb{R})} \right). \tag{4.17}$$

For $x \in \mathbb{R}$, by (4.12), $F_{\tilde{\mu}_l}(x) = \dfrac{\phi_{\pi}(F_{\mu}(x))}{\nu_l(\mathbb{R})}$ and since $F_{\mu}(F_{\mu}^{-1}(F_{\mu}(x))) = F_{\mu}(x)$,

$$\phi_{\pi}\Big(F_{\mu}(F_{\mu}^{-1}(F_{\mu}(x)))\Big) = \phi_{\pi}(F_{\mu}(x)) \quad \text{and} \quad F_{\tilde{\mu}_l}^{-1}\left(\frac{\phi_{\pi}(F_{\mu}(x))}{\nu_l(\mathbb{R})} \right) \leq F_{\mu}^{-1}(F_{\mu}(x)).$$

Since, by **Step 1**, $\phi_{\uparrow}(F_{\mu}(x)) \leq \phi_{\pi}(F_{\mu}(x))$ and $F_{\tilde{\mu}_l}^{-1}$ is non-increasing, we deduce that

$$F_{\tilde{\mu}_l}^{-1}\left(\frac{\phi_{\uparrow}(F_{\mu}(x))}{\nu_l(\mathbb{R})} \right) \leq F_{\mu}^{-1}(F_{\mu}(x)).$$

As $(0,1) \cap \left\{ \bigcup_{x \in \mathbb{R}: \mu(\{x\})>0} [F_{\mu}(x-), F_{\mu}(x)) \right\}^c$ is included in the range of F_{μ}, with (4.17) we conclude that

$$\phi_{\uparrow}'(u)du \text{ a.e. on } (0,1) \cap \left\{ \bigcup_{x \in \mathbb{R}: \mu(\{x\})>0} [F_{\mu}(x-), F_{\mu}(x)) \right\}^c,\quad F_{\tilde{\nu}_l}^{-1}\left(\frac{\phi_{\uparrow}(u)}{\nu_l(\mathbb{R})} \right) < F_{\mu}^{-1}(u).$$

If $\mu(\{x\}) > 0$, then by **Step 3** below, $\phi_{\uparrow}'(u)du$ a.e. on $(F_{\mu}(x-), F_{\mu}(x))$, $F_{\tilde{\nu}_l}^{-1}\left(\frac{\phi_{\uparrow}(u)}{\nu_l(\mathbb{R})} \right) < F_{\mu}^{-1}(u)$ and therefore (4.14) holds.

In a symmetric way, since $\frac{\mathbb{1}_{\{y<z\}}}{\nu_r(\mathbb{R})} \pi(dy, dz) \in \Pi(\tilde{\mu}_r, \tilde{\nu}_r)$ where

$$\tilde{\mu}_r(dy) = \frac{1}{\nu_r(\mathbb{R})} \int_{z \in \mathbb{R}} \mathbb{1}_{\{y<z\}} \pi(dy, dz),$$

Lemma 4.3.7 implies that du a.e., $F_{\tilde{\nu}_r}^{-1}(u) > F_{\tilde{\mu}_r}^{-1}(u)$. For $x \in \mathbb{R}$, $F_{\tilde{\mu}_r}(x) = \frac{F_{\mu}(x) - \phi_{\pi}(F_{\mu}(x))}{\nu_r(\mathbb{R})}$.

Hence, for $w > \frac{F_\mu(x-)-\lim_{y\to x-}\phi_\pi(F_\mu(y))}{\nu_r(\mathbb{R})}$, $F^{-1}_{\tilde{\mu}_r}(w) \geq x$. Since $\phi_\uparrow(F_\mu(x-)) \leq \lim_{y\to x-}\phi_\pi(F_\mu(y))$, we deduce that when $\mu(\{x\}) > 0$, $F^{-1}_{\tilde{\nu}_r}(w) > x$ for $w > \frac{F_\mu(x-)-\phi_\uparrow(F_\mu(x-))}{\nu_r(\mathbb{R})}$. With this argument replacing the above reference to **Step 3** when $\mu(\{x\}) > 0$, we check that (4.15) holds.

Uniqueness : Let $\hat{\pi}^\uparrow \in \Pi_M(\mu,\nu,\nu_l,\nu_r)$ be non-decreasing. Let Γ be a Borel subset of $\mathbb{R}^2$ such that $\hat{\pi}^\uparrow(\Gamma) = 1$ and properties $(a)(b)$ in Definition 4.2.3 are satisfied. Let $x \in \mathbb{R}$ be such that $F_\mu(x) > 0$. The set $\{z \in (-\infty,x) : \exists\, y \in (z,x] \text{ s.t. } (y,z) \in \Gamma\}$ is not empty since

$$\int_{\mathbb{R}^2} \mathbb{1}_\Gamma(y,z)\mathbb{1}_{\{z<y\leq x\}}\hat{\pi}^\uparrow(dy,dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}}\hat{\pi}^\uparrow(dy,dz) = \int_{(-\infty,x]} \hat{\pi}^\uparrow_y((-\infty,y))\mu(dy) > 0,$$

as $\nu_0^{\hat{\pi}^\uparrow} = 0$ implies that $\mu(dy)$ a.e. $\hat{\pi}^\uparrow_y((-\infty,y)) > 0$. Therefore $l(x) := \sup\{z \in (-\infty,x) : \exists y \in (z,x] \text{ s.t. } (y,z) \in \Gamma\}$ belongs to $(-\infty,x]$. Since $\Gamma \cap \{(y,z) \in \mathbb{R}^2 : z < y \leq x \text{ and } z > l(x)\} = \emptyset$,

$$0 = \int_{\mathbb{R}^2} \mathbb{1}_\Gamma\,(y,z)\mathbb{1}_{\{z<y\leq x\}}\mathbb{1}_{\{z>l(x)\}}\hat{\pi}^\uparrow(dy,dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}}\mathbb{1}_{\{z>l(x)\}}\hat{\pi}^\uparrow(dy,dz).$$

On the other hand, by property (a) in Definition 4.2.3, $\Gamma \cap (x,+\infty)\times(-\infty,l(x)) = \emptyset$ so that

$$0 = \int_{\mathbb{R}^2} \mathbb{1}_\Gamma(y,z)\mathbb{1}_{\{y>x\}}\mathbb{1}_{\{z<l(x)\}}\hat{\pi}^\uparrow(dy,dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{y>x\}}\mathbb{1}_{\{z<l(x)\}}\hat{\pi}^\uparrow(dy,dz).$$

With the equality, $\frac{1}{\nu_l(\mathbb{R})}\int_{y\in\mathbb{R}} \mathbb{1}_{\{z<y\}}\hat{\pi}^\uparrow(dy,dz) = \tilde{\nu}_l(dz)$, we deduce that

$$\mathbb{1}_{\{z<l(x)\}}\tilde{\nu}_l(dz) \leq \frac{1}{\nu_l(\mathbb{R})}\int_{y\in(-\infty,x]} \mathbb{1}_{\{z<y\}}\hat{\pi}^\uparrow(dy,dz) \leq \mathbb{1}_{\{z\leq l(x)\}}\tilde{\nu}_l(dz).$$

Therefore $\frac{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))}{\nu_l(\mathbb{R})} \in [F_{\tilde{\nu}_l}(l(x)-), F_{\tilde{\nu}_l}(l(x))]$ and

$$\begin{aligned}\frac{1}{\nu_l(\mathbb{R})}\int_{y\in(-\infty,x]} \mathbb{1}_{\{z<y\}}\hat{\pi}^\uparrow(dy,dz) &= \mathbb{1}_{\{z<l(x)\}}\tilde{\nu}_l(dz) + \left(\frac{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))}{\nu_l(\mathbb{R})} - F_{\tilde{\nu}_l}(l(x)-)\right)\delta_{l(x)}(dz)\\ &= \int_0^{\frac{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))}{\nu_l(\mathbb{R})}} \delta_{F^{-1}_{\tilde{\nu}_l}(u)}(dz)du = \frac{1}{\nu_l(\mathbb{R})}\int_0^{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))} \delta_{F^{-1}_{\tilde{\nu}_l}\left(\frac{w}{\nu_l(\mathbb{R})}\right)}(dz)dw. \qquad (4.18)\end{aligned}$$

In a symmetric way, with property (b) in Definition 4.2.3, we check that

$$\int_{y\in(-\infty,x]} \mathbb{1}_{\{z>y\}}\hat{\pi}^\uparrow(dy,dz) = \int_0^{F_\mu(x)-\phi_{\hat{\pi}^\uparrow}(F_\mu(x))} \delta_{F^{-1}_{\tilde{\nu}_r}\left(\frac{w}{\nu_r(\mathbb{R})}\right)}(dz)dw. \qquad (4.19)$$

Using the martingale property of $\hat{\pi}^\uparrow$ for the second equality and $\nu_0^{\hat{\pi}^\uparrow} = 0$ for the fourth, we deduce that

$$\begin{aligned}\int_0^{F_\mu(x)} F^{-1}_\mu(u)du &= \int_{y\in(-\infty,x]} y\mu(dy) = \int_{y\in(-\infty,x]}\int_{z\in\mathbb{R}} z\hat{\pi}^\uparrow_y(dz)\mu(dy) = \int_{(-\infty,x]\times\mathbb{R}} z\hat{\pi}^\uparrow(dy,dz)\\ &= \int_{(-\infty,x]\times\mathbb{R}} z\mathbb{1}_{\{z<y\}}\hat{\pi}^\uparrow(dy,dz) + \int_{(-\infty,x]\times\mathbb{R}} z\mathbb{1}_{\{z>y\}}\hat{\pi}^\uparrow(dy,dz)\end{aligned}$$

$$= \int_0^{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw + \int_0^{F_\mu(x)-\phi_{\hat{\pi}^\uparrow}(F_\mu(x))} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw$$
$$= G(F_\mu(x), \phi_{\hat{\pi}^\uparrow}(F_\mu(x))).$$

Equations (4.18) and (4.19) also hold with $\hat{\pi}^\uparrow$ replaced by $\pi^\uparrow$ so that $G(F_\mu(x), \phi_{\pi^\uparrow}(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(u)du$. Note that the equalities

$$\int_{y\in(-\infty,x]} \mathbb{1}_{\{z<y\}}\pi^\uparrow(dy,dz) = \int_0^{\phi_{\pi^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)}(dz)dw$$
$$\int_{y\in(-\infty,x]} \mathbb{1}_{\{z>y\}}\pi^\uparrow(dy,dz) = \int_0^{F_\mu(x)-\phi_{\pi^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)}(dz)dw,$$

together with $\phi_{\pi^\uparrow}(F_\mu(x)) = \int_0^{F_\mu(x)} \phi_\uparrow'(u)du = \phi_\uparrow(F_\mu(x))$ can also be derived from the definition of $\pi^\uparrow$ combined with (4.14), (4.15), (1.1), Proposition 4.9 [201] and the remark just above applied like in the above derivation of (4.16).

By **Step 1** below applied with $\pi = \hat{\pi}^\uparrow$ and $\pi = \pi^\uparrow$, there exists a unique

$$v \in [(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_{\hat{\pi}^\uparrow}(F_\mu(x)) \vee \phi_{\pi^\uparrow}(F_\mu(x))]$$

such that $G(F_\mu(x), v) = \int_0^{F_\mu(x)} F_\mu^{-1}(w)\, dw$ so that $\phi_{\hat{\pi}^\uparrow}(F_\mu(x)) = \phi_{\pi^\uparrow}(F_\mu(x))$ for each $x \in \mathbb{R}$ and

$$\int_{y\in(-\infty,x]} \hat{\pi}^\uparrow(dy,dz) = \int_0^{\phi_{\pi^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)}(dz)dw + \int_0^{F_\mu(x)-\phi_{\pi^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)}(dz)dw$$
$$= \int_{y\in(-\infty,x]} \pi^\uparrow(dy,dz).$$

Therefore the set

$$\mathcal{M} := \left\{ A \in \mathcal{B}(\mathbb{R}) \mid \int_A \pi_y^\uparrow \mu(dy) = \int_A \hat{\pi}_y^\uparrow \mu(dy) \right\}$$

contains $\{(-\infty, x] : x \in \mathbb{R}\}$. One easily checks that if $A, B \in \mathcal{M}$ are such that $B \subset A$, then $A \setminus B \in \mathcal{M}$. Now let $(A_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\mathcal{M}$ such that $A_n \subseteq A_{n+1}$. For $B \in \mathcal{B}(\mathbb{R})$, since for all $n \in \mathbb{N}$, $\int_{A_n} \left(\hat{\pi}_y^\uparrow(B) - \pi_y^\uparrow(B)\right) \mu(dy) = 0$ and $|\hat{\pi}_y^\uparrow(B) - \pi_y^\uparrow(B)| \leq 1$, by Lebesgue's dominated convergence theorem,

$$\int_{\bigcup_{n\in\mathbb{N}} A_n} \left(\hat{\pi}_y^\uparrow(B) - \pi_y^\uparrow(B)\right) \mu(dy) = 0.$$

We deduce that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$ and $\mathcal{M}$ is a monotone class. Since $\mathcal{M}$ contains the class $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$, which is stable by finite intersections, by the monotone class theorem, we have $\sigma(\mathcal{C}) \subset \mathcal{M}$. Since $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$ and $\mathcal{M} \subseteq \mathcal{B}(\mathbb{R})$, we conclude that $\mathcal{M} = \mathcal{B}(\mathbb{R})$. In particular, for any $B \in \mathcal{B}(\mathbb{R})$,

$$\mu\left(\left\{y \in \mathbb{R} : \hat{\pi}_y^\uparrow(B) > \pi_y^\uparrow(B)\right\}\right) = 0 = \mu\left(\left\{y \in \mathbb{R} : \hat{\pi}_y^\uparrow(B) < \pi_y^\uparrow(B)\right\}\right).$$

We deduce that

$$\mu\left(\left\{y\in\mathbb{R}:\forall x\in\mathbb{Q},\ \hat{\pi}_y^{\uparrow}((-\infty,x])=\pi_y^{\uparrow}((-\infty,x])\right\}\right)=1=\mu\left(\left\{y\in\mathbb{R}:\hat{\pi}_y^{\uparrow}=\pi_y^{\uparrow}\right\}\right).$$

Step 1 : For $x\in\mathbb{R}$, we will verify the following three statements in three substeps:

Step 1.1 : $G(F_\mu(x),\phi_\pi(F_\mu(x)))\leq\int_0^{F_\mu(x)}F_\mu^{-1}(w)dw$,

Step 1.2 : $G(F_\mu(x),(F_\mu(x)-\nu_r(\mathbb{R}))^+)\geq\int_0^{F_\mu(x)}F_\mu^{-1}(w)dw$,

Step 1.3 : $v\mapsto G(F_\mu(x),v)$ is decreasing on $[(F_\mu(x)-\nu_r(\mathbb{R}))^+,\phi_\pi(F_\mu(x))]$.

Since this function is also continuous, this implies the existence of a unique $\phi_\uparrow(F_\mu(x))\in[(F_\mu(x)-\nu_r(\mathbb{R}))^+,\phi_\pi(F_\mu(x))]$ such that $G(F_\mu(x),\phi_\uparrow(F_\mu(x)))=\int_0^{F_\mu(x)}F_\mu^{-1}(v)dv$, where the notation $\phi_\uparrow(F_\mu(x))$ is justified since this point clearly only depends on x through $F_\mu(x)$.

Step 1.1 : By Lemma 4.3.6, applied with

$$\eta=\tilde{\nu}_l\quad\text{and}\quad\hat{\eta}(dz)=\frac{1}{\nu_l(\mathbb{R})}\int_{y\in(-\infty,x]}\mathbb{1}_{\{z<y\}}\pi(dy,dz)$$

such that $\hat{\eta}(\mathbb{R})=\frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}$,

$$\frac{1}{\nu_l(\mathbb{R})}\int_{\mathbb{R}^2}z\mathbb{1}_{\{z<y\leq x\}}\pi(dy,dz)\geq\int_0^{\frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}}F_{\tilde{\nu}_l}^{-1}(w)dw=\frac{1}{\nu_l(\mathbb{R})}\int_0^{\phi_\pi(F_\mu(x))}F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)dw$$

and, with

$$\eta=\tilde{\nu}_r\quad\text{and}\quad\hat{\eta}(dz)=\frac{1}{\nu_r(\mathbb{R})}\int_{y\in(-\infty,x]}\mathbb{1}_{\{z>y\}}\pi(dy,dz)\leq\tilde{\nu}_r(dz)$$

such that $\hat{\eta}(\mathbb{R})=\frac{F_\mu(x)-\phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}$,

$$\begin{aligned}\frac{1}{\nu_r(\mathbb{R})}\int_{\mathbb{R}^2}z\mathbb{1}_{\{y\leq x,y<z\}}\pi(dy,dz)&\geq\int_0^{\frac{F_\mu(x)-\phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}}F_{\tilde{\nu}_r}^{-1}(w)dw\\&=\frac{1}{\nu_r(\mathbb{R})}\int_0^{F_\mu(x)-\phi_\pi(F_\mu(x))}F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)dw.\end{aligned}$$

Using (1.1) for the first equality, the inverse transform sampling for the second and the martingale property of π, and $\nu_l^\pi+\nu_r^\pi=\nu$ for the third, we deduce that

$$\begin{aligned}\int_0^{F_\mu(x)}F_\mu^{-1}(w)dw&=\int_0^1F_\mu^{-1}(w)\mathbb{1}_{\{F_\mu^{-1}(w)\leq x\}}dw=\int_{\mathbb{R}}y\mathbb{1}_{\{y\leq x\}}\mu(dy)\\&=\int_{\mathbb{R}^2}z\mathbb{1}_{\{z<y\leq x\}}\pi(dy,dz)+\int_{\mathbb{R}^2}z\mathbb{1}_{\{y\leq x,y<z\}}\pi(dy,dz)\\&\geq G(F_\mu(x),\phi_\pi(F_\mu(x))).\end{aligned}$$

Step 1.2 : If $F_\mu(x) < \nu_r(\mathbb{R})$, setting

$$\psi(z) = \mathbb{1}_{\{z<F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\}} + \mathbb{1}_{\{z=F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\}} \mathbb{1}_{\{\tilde{\nu}_r(\{F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\})>0\}} \frac{\frac{F_\mu(x)}{\nu_r(\mathbb{R})} - \tilde{\nu}_r((-\infty, F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})))}{\tilde{\nu}_r(\{F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\})},$$

we have that

$$\begin{aligned}\int_0^{F_\mu(x)} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw &= \nu_r(\mathbb{R}) \int_{\mathbb{R}} z\psi(z)\tilde{\nu}_r(dz) = \int_{\mathbb{R}\times\mathbb{R}} z\psi(z)\mathbb{1}_{\{y<z\}}\pi(dy,dz) \\ &\geq \int_{\mathbb{R}\times\mathbb{R}} y\psi(z)\mathbb{1}_{\{y<z\}}\pi(dy,dz) = \int_{\mathbb{R}} y\hat{\mu}(dy)\end{aligned}$$

where $\hat{\mu}(dy) := \left(\int_{z\in\mathbb{R}} \psi(z)\mathbb{1}_{\{y<z\}}\pi_y(dz)\right)\mu(dy) \leq \mu(dy)$ since $\int_{z\in\mathbb{R}} \psi(z)\mathbb{1}_{\{y<z\}}\pi_y(dz) \in [0,1]$ and satisfies

$$\hat{\mu}(\mathbb{R}) = \int_{\mathbb{R}\times\mathbb{R}} \psi(z)\mathbb{1}_{\{y<z\}}\pi(dy,dz) = \nu_r(\mathbb{R}) \int_{\mathbb{R}} \psi(z)\tilde{\nu}_r(dz) = F_\mu(x).$$

By Lemma 4.3.6 applied with $(\eta,\hat{\eta}) = (\mu,\hat{\mu})$, this ensures that $\int_{\mathbb{R}} y\hat{\mu}(dy) \geq \int_0^{F_\mu(x)} F_\mu^{-1}(v)dv$ so that

$$\int_0^{F_\mu(x)} F_\mu^{-1}(v)dv \leq \int_0^{F_\mu(x)} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw = G(F_\mu(x),0).$$

If $F_\mu(x) \geq \nu_r(\mathbb{R})$, one has using $\int_0^{\nu_r(\mathbb{R})} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw = \int_{\mathbb{R}} y\nu_r(dy) = \int_{\mathbb{R}} y\mu(dy) - \int_{\mathbb{R}} y\nu_l(dy)$ and $\nu_r(\mathbb{R}) = 1 - \nu_l(\mathbb{R})$ for the second equality,

$$\begin{aligned}G(F_\mu(x), F_\mu(x) - \nu_r(\mathbb{R})) &= \int_0^{F_\mu(x)-\nu_r(\mathbb{R})} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw + \int_0^{\nu_r(\mathbb{R})} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw \\ &= \int_0^{\nu_l(\mathbb{R})-(1-F_\mu(x))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw + \int_{\mathbb{R}} y\mu(dy) - \int_{\mathbb{R}} y\nu_l(dy) \\ &= \int_0^{F_\mu(x)} F_\mu^{-1}(w)dw + \int_{F_\mu(x)}^1 F_\mu^{-1}(w)dw - \int_{\nu_l(\mathbb{R})-(1-F_\mu(x))}^{\nu_l(\mathbb{R})} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw.\end{aligned}$$

By a reasoning similar to the above derivation of $\int_0^{F_\mu(x)} F_\mu^{-1}(v)dv \leq \int_0^{F_\mu(x)} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw$, we check that the sum of the last two terms in the right-hand side is non-negative. Combining the two cases, we obtain that $G(F_\mu(x),(F_\mu(x)-\nu_r(\mathbb{R}))^+) \geq \int_0^{F_\mu(x)} F_\mu^{-1}(w)dw$.

Step 1.3 : To prove that $v \mapsto G(F_\mu(x),v)$ is decreasing on $[(F_\mu(x)-\nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x))]$, it is enough to check that

$$\forall w \in \left(0, \frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}\right),\ F_{\tilde{\nu}_l}^{-1}(w) < x \text{ and } \forall w \in \left(\frac{F_\mu(x)-\phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}, 1\right),\ x < F_{\tilde{\nu}_r}^{-1}(w). \quad (4.20)$$

Let x be such that $\phi_\pi(F_\mu(x)) > 0$. Then $F_\mu(x) > 0$ and $F_\mu(x) - \phi_\pi(F_\mu(x)) > 0$. Since, by definition of $\phi_\pi(F_\mu(x))$ and $\nu_l^\pi = \nu_l$,

$$\begin{aligned}\phi_\pi(F_\mu(x)) &= \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}}\pi(dy,dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\wedge x\}}\pi(dy,dz) - \int_{\mathbb{R}^2} \mathbb{1}_{\{z<x<y\}}\pi(dy,dz)\\ &= \nu_l(\mathbb{R})\tilde{\nu}_l((-\infty,x)) - \int_{\mathbb{R}^2} \mathbb{1}_{\{z<x<y\}}\pi(dy,dz),\end{aligned}$$

we have $\tilde{\nu}_l((-\infty,x)) \geq \frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}$, from which we deduce the first part in (4.20). In a symmetric way and since $\nu_0^\pi = \nu_0 = 0$,

$$\begin{aligned}F_\mu(x) - \phi_\pi(F_\mu(x)) &= \int_{\mathbb{R}^2} \mathbb{1}_{\{y\leq x\}}\pi(dy,dz) - \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}}\pi(dy,dz)\\ &= \int_{\mathbb{R}^2} \mathbb{1}_{\{y<z\leq x\}}\pi(dy,dz) + \int_{\mathbb{R}^2} \mathbb{1}_{\{y\leq x<z\}}\pi(dy,dz)\\ &= \nu_r(\mathbb{R})F_{\tilde{\nu}_r}(x) + \int_{\mathbb{R}^2} \mathbb{1}_{\{y\leq x<z\}}\pi(dy,dz),\end{aligned}$$

so that $F_{\tilde{\nu}_r}(x) \leq \frac{F_\mu(x)-\phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}$. With the right-continuity of $F_{\tilde{\nu}_r}$, we deduce the other part in (4.20).

Step 2 : monotonicity of $x \mapsto \phi_\uparrow(F_\mu(x))$ and $x \mapsto F_\mu(x) - \phi_\uparrow(F_\mu(x))$.

Step 2.1 : Let us first check that when $F_\mu(x) < F_\mu(x')$, $F_\mu(x) - \phi_\uparrow(F_\mu(x)) \leq F_\mu(x') - \phi_\uparrow(F_\mu(x'))$. If $\phi_\pi(F_\mu(x')) \leq \phi_\uparrow(F_\mu(x)) + F_\mu(x') - F_\mu(x)$, then we conclude with the inequality $\phi_\uparrow(F_\mu(x')) \leq \phi_\pi(F_\mu(x'))$ established in **Step 1**. Let us now suppose that $\phi_\uparrow(F_\mu(x)) + F_\mu(x') - F_\mu(x) \leq \phi_\pi(F_\mu(x'))$. Then, by the monotonicity of $F_{\tilde{\nu}_l}^{-1}$,

$$\int_{\phi_\uparrow(F_\mu(x))}^{\phi_\uparrow(F_\mu(x))+F_\mu(x')-F_\mu(x)} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)dw \leq \int_{\phi_\pi(F_\mu(x'))+F_\mu(x)-F_\mu(x')}^{\phi_\pi(F_\mu(x'))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)dw.$$

We are going to prove that

$$\int_{\phi_\pi(F_\mu(x'))+F_\mu(x)-F_\mu(x')}^{\phi_\pi(F_\mu(x'))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)dw \leq \int_{F_\mu(x)}^{F_\mu(x')} F_\mu^{-1}(w)\,dw. \tag{4.21}$$

Adding the two inequalities to $G(F_\mu(x),\phi_\uparrow(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(w)\,dw$, we obtain that

$$G(F_\mu(x'),\phi_\uparrow(F_\mu(x)) + F_\mu(x') - F_\mu(x)) \leq \int_0^{F_\mu(x')} F_\mu^{-1}(w)\,dw.$$

Since by **Step 1.3**, $v \mapsto G(F_\mu(x'),v)$ is decreasing on $[(F_\mu(x')-\nu_r(\mathbb{R}))^+,\phi_\pi(F_\mu(x'))]$, we deduce that $\phi_\uparrow(F_\mu(x')) \leq \phi_\uparrow(F_\mu(x)) + F_\mu(x') - F_\mu(x)$.

To prove (4.21), we first remark that since

$$F_\mu(x') - F_\mu(x) \leq \phi_\pi(F_\mu(x')) = \int_0^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty,F_\mu^{-1}(w)))dw,$$

where we used the inverse transform sampling and (1.1) for the equality,

$$\exists u \in [0, F_\mu(x')] \text{ s.t. } \int_u^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w)))dw = F_\mu(x') - F_\mu(x). \tag{4.22}$$

By Lemma 4.3.6, applied with $\eta = \frac{1}{F_\mu(x')} \int_0^{F_\mu(x')} \delta_{F_\mu^{-1}(w)} dw$ such that $F_\eta^{-1}(w) = F_\mu^{-1}(F_\mu(x')w)$,

$$\hat{\eta} = \frac{1}{F_\mu(x')} \int_u^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w)))\delta_{F_\mu^{-1}(w)} dw$$

with mass $\hat{\eta}(\mathbb{R}) = \frac{F_\mu(x') - F_\mu(x)}{F_\mu(x')}$, we have

$$\frac{1}{F_\mu(x')} \int_{F_\mu(x)}^{F_\mu(x')} F_\mu^{-1}(w)dw \geq \frac{1}{F_\mu(x')} \int_u^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w)))F_\mu^{-1}(w)dw. \tag{4.23}$$

For the disintegrations, $\pi(dy, dz) = \mu(dy)\pi_y(dz) = \nu(dz)\overleftarrow{\pi}_z(dy)$, let

$$\begin{aligned}
\sigma(dy) &= \frac{\mathbb{1}_{\{y \leq x'\}}}{\phi_\pi(F_\mu(x'))} \pi_y((-\infty, y))\mu(dy) \text{ and} \\
\theta(dz) &= \mathbb{1}_{\{z<x'\}} \overleftarrow{\pi}_z((z, x'])\frac{\nu(dz)}{\phi_\pi(F_\mu(x'))} = \mathbb{1}_{\{z<x'\}} \frac{\overleftarrow{\pi}_z((z, x'])}{\overleftarrow{\pi}_z((z, +\infty))} \times \frac{\nu_l(dz)}{\phi_\pi(F_\mu(x'))}.
\end{aligned}$$

The coupling $\frac{\mathbb{1}_{\{z<y\leq x'\}}}{\phi_\pi(F_\mu(x'))}\pi(dy, dz) \in \Pi(\sigma, \theta)$ giving full weight to $\{(y, z) \in \mathbb{R}^2 : y > z\}$, one has $\sigma \geq_{st} \theta$. On the other hand, Lemma 4.3.6 applied with $\eta = \tilde{\nu}_l$ and $\hat{\eta} = \frac{\phi_\pi(F_\mu(x'))}{\nu_l(\mathbb{R})}\theta$ such that $\hat{\eta}(\mathbb{R}) = \frac{\phi_\pi(F_\mu(x'))}{\nu_l(\mathbb{R})}$ implies that

$$\theta \geq_{st} \vartheta := \frac{\nu_l(\mathbb{R})}{\phi_\pi(F_\mu(x'))} \int_0^{\frac{\phi_\pi(F_\mu(x'))}{\nu_l(\mathbb{R})}} \delta_{F_{\tilde{\nu}_l}^{-1}(w)} dw.$$

Hence $\sigma \geq_{st} \vartheta$ so that $F_\sigma^{-1} \geq F_\vartheta^{-1}$ where $F_\vartheta^{-1}(w) = F_{\tilde{\nu}_l}^{-1}\left(\frac{\phi_\pi(F_\mu(x'))}{\nu_l(\mathbb{R})}w\right)$ and

$$\begin{aligned}
\int_{1-\frac{F_\mu(x')-F_\mu(x)}{\phi_\pi(F_\mu(x'))}}^1 F_\sigma^{-1}(w)dw &\geq \int_{1-\frac{F_\mu(x')-F_\mu(x)}{\phi_\pi(F_\mu(x'))}}^1 F_\vartheta^{-1}(w)dw \\
&= \frac{1}{\phi_\pi(F_\mu(x'))} \int_{\phi_\pi(F_\mu(x'))+F_\mu(x)-F_\mu(x')}^{\phi_\pi(F_\mu(x'))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw.
\end{aligned}$$

By the inverse transform sampling and (1.1),

$$\sigma = \frac{1}{\phi_\pi(F_\mu(x'))} \int_0^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w)))\delta_{F_\mu^{-1}(w)} dw$$

so that, with (4.22),

$$\int_{1-\frac{F_\mu(x')-F_\mu(x)}{\phi_\pi(F_\mu(x'))}}^{1} F_\sigma^{-1}(w)dw = \frac{1}{\phi_\pi(F_\mu(x'))}\int_u^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w)))F_\mu^{-1}(w)dw.$$

Hence,

$$\int_u^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w)))F_\mu^{-1}(w)dw \geq \int_{\phi_\pi(F_\mu(x'))+F_\mu(x)-F_\mu(x')}^{\phi_\pi(F_\mu(x'))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)dw.$$

Combining this inequality with (4.23), we obtain (4.21).

Step 2.2 : Let us now check that when $F_\mu(x) < F_\mu(x')$, $\phi_\uparrow(F_\mu(x)) \leq \phi_\uparrow(F_\mu(x'))$. If $\phi_\uparrow(F_\mu(x)) \leq F_\mu(x') - \nu_r(\mathbb{R})$, then we have $\phi_\uparrow(F_\mu(x)) \leq \phi_\uparrow(F_\mu(x'))$, since $(F_\mu(x') - \nu_r(\mathbb{R}))^+ \leq \phi_\uparrow(F_\mu(x'))$ by **Step 1**. Let us now suppose that $F_\mu(x') - \phi_\uparrow(F_\mu(x)) \leq \nu_r(\mathbb{R})$. Since, by **Step 1**, $\phi_\uparrow(F_\mu(x)) \leq \phi_\pi(F_\mu(x))$ and $F_{\tilde{\nu}_r}^{-1}$ is non-decreasing, we have, using a reasoning analogous to the above derivation of (4.21) for the second inequality,

$$\int_{F_\mu(x)-\phi_\uparrow(F_\mu(x))}^{F_\mu(x')-\phi_\uparrow(F_\mu(x))} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)dw \geq \int_{F_\mu(x)-\phi_\pi(F_\mu(x))}^{F_\mu(x')-\phi_\pi(F_\mu(x))} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)dw \geq \int_{F_\mu(x)}^{F_\mu(x')} F_\mu^{-1}(w)\,dw.$$

Adding this inequality to $G(F_\mu(x), \phi_\uparrow(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(w)dw$, we obtain that

$$G(F_\mu(x'), \phi_\uparrow(F_\mu(x))) \geq \int_0^{F_\mu(x')} F_\mu^{-1}(w)\,dw.$$

Since $\phi_\uparrow(F_\mu(x)) \leq \phi_\pi(F_\mu(x)) \leq \phi_\pi(F_\mu(x'))$ and, by **Step 1.3**, $v \mapsto G(F_\mu(x'), v)$ is decreasing on $[(F_\mu(x') - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x'))]$, we deduce that $\phi_\uparrow(F_\mu(x')) \geq \phi_\uparrow(F_\mu(x))$.

Step 3 : Let $x \in \mathbb{R}$ be such that $\mu(\{x\}) > 0$. The monotonicity proved in **Step 2** ensures the existence of the left-hand limit $\lim_{y\to x-} \phi_\uparrow(F_\mu(y))$ such that

$$F_\mu(x-) - \lim_{y\to x-} \phi_\uparrow(F_\mu(y)) \leq F_\mu(x) - \phi_\uparrow(F_\mu(x)).$$

Taking the limit $y \to x-$ in the inequality $(F_\mu(y) - \nu_r(\mathbb{R}))^+ \leq \phi_\uparrow(F_\mu(y)) \leq F_\mu(y) \wedge \nu_l(\mathbb{R})$ implies that

$$(F_\mu(x-) - \nu_r(\mathbb{R}))^+ \leq \lim_{y\to x-} \phi_\uparrow(F_\mu(y)) \leq F_\mu(x-) \wedge \nu_l(\mathbb{R}).$$

By continuity of $(u, v) \mapsto G(u, v)$ and $u \mapsto \int_0^u F_\mu^{-1}(w)dw$, taking the limit $y \to x-$ in the equality $G(F_\mu(y), \phi_\uparrow(F_\mu(y))) = \int_0^{F_\mu(y)} F_\mu^{-1}(w)dw$, we obtain that

$$G\left(F_\mu(x-), \lim_{y\to x-} \phi_\uparrow(F_\mu(y))\right) = \int_0^{F_\mu(x-)} F_\mu^{-1}(w)dw, \tag{4.24}$$

so that we can set $\phi_\uparrow(F_\mu(x-)) = \lim_{y\to x-} \phi_\uparrow(F_\mu(y))$. Substracting (4.24) from $G(F_\mu(x), \phi_\uparrow(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(w)dw$ and using that, according to (1.2), $F_\mu^{-1}(w) = x$ for $w \in (F_\mu(x-), F_\mu(x)]$, we

obtain

$$\int_{\phi_\uparrow(F_\mu(x-))}^{\phi_\uparrow(F_\mu(x))} \left(F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) - x \right) dw + \int_{F_\mu(x-)-\phi_\uparrow(F_\mu(x-))}^{F_\mu(x)-\phi_\uparrow(F_\mu(x))} \left(F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x \right) dw = 0. \quad (4.25)$$

By (4.20) and $\phi_\uparrow(F_\mu(x)) \le \phi_\pi(F_\mu(x))$, $F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) - x < 0$ for $w \in (\phi_\uparrow(F_\mu(x-)), \phi_\uparrow(F_\mu(x)))$. Since, for $y < x$, by (4.20) and $\phi_\uparrow(F_\mu(y)) \le \phi_\pi(F_\mu(y))$,

$$F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - y > 0 \quad \text{for} \quad w \in (F_\mu(y) - \phi_\uparrow(F_\mu(y)), \nu_r(\mathbb{R}))\,,$$

one has

$$F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x \ge 0 \quad \text{for} \quad w \in (F_\mu(x-) - \phi_\uparrow(F_\mu(x-)), F_\mu(x) - \phi_\uparrow(F_\mu(x))).$$

Now let $u \in (F_\mu(x-), F_\mu(x))$ and, for $v \in [\phi_\uparrow(F_\mu(x-)) \vee (u + \phi_\uparrow(F_\mu(x)) - F_\mu(x)), \phi_\uparrow(F_\mu(x)) \wedge (u + \phi_\uparrow(F_\mu(x-)) - F_\mu(x-))]$,

$$H(u,v) = \int_{\phi_\uparrow(F_\mu(x-))}^{v} \left(F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) - x \right) dw + \int_{F_\mu(x-)-\phi_\uparrow(F_\mu(x-))}^{u-v} \left(F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x \right) dw.$$

If $\phi_\uparrow(F_\mu(x-)) \ge u + \phi_\uparrow(F_\mu(x)) - F_\mu(x)$,

$$H(u, \phi_\uparrow(F_\mu(x-))) = \int_{F_\mu(x-)-\phi_\uparrow(F_\mu(x-))}^{u-\phi_\uparrow(F_\mu(x-))} \left(F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x \right) dw \ge 0$$

while if $\phi_\uparrow(F_\mu(x-)) \le u + \phi_\uparrow(F_\mu(x)) - F_\mu(x)$, using (4.25) for the equality,

$$H(u, u + \phi_\uparrow(F_\mu(x)) - F_\mu(x)) = \int_{u+\phi_\uparrow(F_\mu(x))-F_\mu(x)}^{\phi_\uparrow(F_\mu(x))} \left(x - F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) \right) dw \ge 0,$$

so that $H(u, \phi_\uparrow(F_\mu(x-)) \vee (u + \phi_\uparrow(F_\mu(x)) - F_\mu(x))) \ge 0$. In a similar way, $H(u, \phi_\uparrow(F_\mu(x)) \wedge (u + \phi_\uparrow(F_\mu(x-)) - F_\mu(x-))) \le 0$ and since $v \mapsto H(u,v)$ is decreasing on the interval, there exists a unique $\phi_\uparrow(u)$ such that $\phi_\uparrow(F_\mu(x-)) \le \phi_\uparrow(u) \le \phi_\uparrow(F_\mu(x))$,

$$F_\mu(x-) - \phi_\uparrow(F_\mu(x-)) \le u - \phi_\uparrow(u) \le F_\mu(x) - \phi_\uparrow(F_\mu(x) \quad \text{and} \quad H(u, \phi_\uparrow(u)) = 0.$$

With (4.24), this implies that $G(u, \phi_\uparrow(u)) = 0$. Repeating the above reasoning with the equality $H(u, \phi_\uparrow(u)) = 0$ replacing (4.25), we obtain for $u' \in (F_\mu(x-), u)$ the existence of a unique

$$\phi_\uparrow(u') \in [\phi_\uparrow(F_\mu(x-)) \vee (u' + \phi_\uparrow(u) - u),\ \phi_\uparrow(u) \wedge (u' + \phi_\uparrow(F_\mu(x-)) - F_\mu(x-))]$$

solving $H(u', \phi_\uparrow(u')) = 0$, which implies that $u' + \phi_\uparrow(u) - u \le \phi_\uparrow(u')$ and $\phi_\uparrow(u') \le \phi_\uparrow(u)$. Hence $\phi_\uparrow$ and $u \mapsto u - \phi_\uparrow(u)$ are non-decreasing on $(0,1)$. Moreover, $\phi'_\uparrow(u)du$ a.e. on $(F_\mu(x-), F_\mu(x))$, $\phi_\uparrow(u) < \phi_\uparrow(F_\mu(x))$ and $F_{\tilde{\nu}_l}^{-1}\left(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}\right) < x < F_{\tilde{\nu}_r}^{-1}\left(\frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})}\right)$.

Step 4 : Now, let us prove (4.2). Since $\pi^{\uparrow}$ is the only non-decreasing coupling in the set $\Pi_M(\mu,\nu,\nu_l,\nu_r)$, it is enough to check that any coupling maximizing $\int_{\mathbb{R}^2}\varphi(|x-y|)\pi(dx,dy)$ over $\pi\in\Pi_M(\mu,\nu,\nu_l,\nu_r)$ is non-decreasing. To do so, we are going to adapt the proof of Lemma 1.11 [35]. We modify the definition of the set M in this proof into $M=\bigcup_{\tau\in\mathcal{S}_3}\{M_{l,\tau}\cup M_{r,\tau}\}$, where for τ in the set $\mathcal{S}_3$ of permutations of $\{1,2,3\}$, $M_{l,\tau}$ and $M_{r,\tau}$ are the respective images of

$$\begin{aligned} M_l &= \{((x_-,y_-),(x_+,y_+),(x_+,z))\in\{\mathbb{R}^2\}^3 : y_+<y_-<x_-<x_+<z\} \\ \text{and } M_r &= \{((x_-,y),(x_-,z_-),(x_+,z_+))\in\{\mathbb{R}^2\}^3 : y<x_-<x_+<z_+<z_-\}. \end{aligned}$$

by $\{\mathbb{R}^2\}^3\ni(\chi_1,\chi_2,\chi_3)\mapsto(\chi_{\tau(1)},\chi_{\tau(2)},\chi_{\tau(3)})\in\{\mathbb{R}^2\}^3$. Applying Theorem 3.1 [35] like in the proof of Lemma 1.11 [35], we obtain in case (1) of this theorem a Borel subset $\hat{\Gamma}$ of $\mathbb{R}^2$ such that $\pi(\hat{\Gamma})=1$ and $M_l\cap\hat{\Gamma}^3=\emptyset=M_r\cap\hat{\Gamma}^3$. By Lemma 4.3.3, we may suppose that

$$\forall x\in\mathbb{R},\quad \exists\, y<x \text{ s.t. } (x,y)\in\hat{\Gamma} \iff \exists z>x \text{ s.t. } (x,z)\in\hat{\Gamma}.$$

We deduce that

$$\begin{aligned} \{((x_-,y_-),(x_+,y_+))\in\{\mathbb{R}^2\}^2 : y_+<y_-<x_-<x_+\}\cap\hat{\Gamma}^2=\emptyset \\ \text{and } \{((x_-,z_-),(x_+,z_+))\in\{\mathbb{R}^2\}^2 : x_-<x_+<z_+<z_-\}\cap\hat{\Gamma}^2=\emptyset. \end{aligned}$$

Setting $\Gamma=\hat{\Gamma}\cap\{(x,y)\in\mathbb{R}^2 : y\neq x\}$, we have

$$\begin{aligned} \{((x_-,y_-),(x_+,y_+))\in\{\mathbb{R}^2\}^2 : y_+<y_-\le x_-<x_+\}\cap\Gamma^2=\emptyset \\ \text{and } \{((x_-,z_-),(x_+,z_+))\in\{\mathbb{R}^2\}^2 : x_-<x_+\le z_+<z_-\}\cap\Gamma^2=\emptyset. \end{aligned}$$

Conditions $(a)(b)$ in Definition 4.2.3 hold and since

$$\begin{aligned} \pi(\{(x,y)\in\mathbb{R}^2 : y\neq x\}) &= \pi(\{(x,y)\in\mathbb{R}^2 : y<x\})+\pi(\{(x,y)\in\mathbb{R}^2 : y>x\}) \\ &= \nu_l(\mathbb{R})+\nu_r(\mathbb{R})=1, \end{aligned}$$

$\pi(\Gamma)=1$ so that π is non-decreasing. Like in the proof of Lemma 1.11 [35], we still need to check that case (2) in Theorem 3.1 [35] cannot occur. In the notation of the proof of the lemma, we need to construct a competitor ω' of ω such that $\pi-\omega+\omega'\in\Pi_M(\mu,\nu,\nu_l,\nu_r)$ and

$$\int_{\mathbb{R}^2}\varphi(|x-y|)\omega'(dx,dy)>\int_{\mathbb{R}^2}\varphi(|x-y|)\omega(dx,dy). \tag{4.26}$$

This is done by choosing

$$\alpha'_p=\frac{1}{3}\left(\frac{z-y_-}{z-y_+}\left(\delta_{(x_-,y_+)}+\delta_{(x_+,z)}\right)+\frac{y_--y_+}{z-y_+}\left(\delta_{(x_-,z)}+\delta_{(x_+,y_+)}\right)+\delta_{(x_+,y_-)}\right)$$

when $\alpha_p=\frac{1}{3}\left(\delta_{(x_-,y_-)}+\delta_{(x_+,y_+)}+\delta_{(x_+,z)}\right)$ for some $y_+<y_-<x_-<x_+<z$ and by choosing

$$\alpha'_p = \frac{1}{3}\left(\frac{z_- - z_+}{z_- - y}\left(\delta_{(x_-,z_-)} + \delta_{(x_+,y)}\right) + \frac{z_+ - y}{z_- - y}\left(\delta_{(x_-,y)} + \delta_{(x_+,z_-)}\right) + \delta_{(x_-,z_+)}\right)$$

when $\alpha_p = \frac{1}{3}\left(\delta_{(x_-,y)} + \delta_{(x_-,z_-)} + \delta_{(x_+,z_+)}\right)$ for some $y < x_- < x_+ < z_+ < z_-$. Note that, in the first case,

$$\int_{v\in\mathbb{R}} \mathbb{1}_{\{w<v\}}\alpha_p(dv,dw) = \frac{1}{3}\left(\delta_{y_-}(dw) + \delta_{y_+}(dw)\right) = \int_{v\in\mathbb{R}} \mathbb{1}_{\{w<v\}}\alpha'_p(dv,dw)$$

and

$$\int_{v\in\mathbb{R}} \mathbb{1}_{\{w>v\}}\alpha_p(dv,dw) = \frac{1}{3}\delta_z(dw) = \int_{v\in\mathbb{R}} \mathbb{1}_{\{w>v\}}\alpha'_p(dv,dw),$$

which contributes to ensuring that $\pi - \omega + \omega' \in \Pi_M(\mu,\nu,\nu_l,\nu_r)$. Moreover, by Lemma 4.3.4 applied with $(y,m,z) = (y_+,y_-,z)$, the function f defined by

$$f(x) = \frac{z-y_-}{z-y_+}\varphi(|x-y_+|) + \frac{y_- - y_+}{z-y_+}\varphi(|z-x|) - \varphi(|x-y_-|)$$

is decreasing on $[y_-,z]$. We deduce that, still in the first case,

$$\int_{\mathbb{R}^2} \varphi(|v-w|)\alpha'_p(dv,dw) - \int_{\mathbb{R}^2} \varphi(|v-w|)\alpha_p(dv,dw) = \frac{1}{3}\left(f(x_-) - f(x_+)\right) > 0,$$

which contributes to ensuring (4.26). One checks in the same way that the two properties still hold in the second case.

Let us now give an alternative simple argument in the particular case when φ is the identity function. For $\pi \in \Pi_M(\mu,\nu,\nu_l,\nu_r)$, we have

$$\int_{\mathbb{R}^2} (y-z)\mathbb{1}_{\{z<y\}}\pi(dy,dz) - \int_{\mathbb{R}^2} (z-y)\mathbb{1}_{\{z>y\}}\pi(dy,dz) = 0.$$

Since $\nu_l^\pi = \nu_l$, we deduce with Fubini's theorem and (4.12) that

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^2} |y-z|\pi(dy,dz) &= \int_{\mathbb{R}^2} (y-z)\mathbb{1}_{\{z<y\}}\pi(dy,dz) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}_+} \mathbb{1}_{\{y>x\}}dx\mathbb{1}_{\{z<y\}}\pi(dy,dz) - \int_{\mathbb{R}^2}\int_{\mathbb{R}_-} \mathbb{1}_{\{y\le x\}}dx\mathbb{1}_{\{z<y\}}\pi(dy,dz) - \int_{\mathbb{R}} z\nu_l(dz) \\
&= \int_{\mathbb{R}_+} (\nu_l(\mathbb{R}) - \phi_\pi(F_\mu(x)))dx - \int_{\mathbb{R}_-} \phi_\pi(F_\mu(x)))dx - \int_{\mathbb{R}} z\nu_l(dz).
\end{aligned}$$

Since, according to **Step 1**, $\forall x \in \mathbb{R}$, $\phi_\uparrow(F_\mu(x))) \le \phi_\pi(F_\mu(x))$, we conclude that

$$\int_{\mathbb{R}^2} |y-z|\pi^\uparrow(dy,dz) = \sup_{\pi\in\Pi_M(\mu,\nu,\nu_l,\nu_r)} \int_{\mathbb{R}^2} |y-z|\pi(dy,dz).$$

□

Proof of Proposition 4.2.13. Let us suppose that there exists $\pi^{\downarrow} \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ such that the coupling

$$\begin{aligned}\pi(dx, dy) &:= \frac{\pi^{\downarrow}(dx, dy) + (\nu_l + \nu_r - \nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} \\ &\in \Pi_M\left(\frac{\mu + \nu_l + \nu_r - \nu}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_l + \nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_l}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}\right)\end{aligned}$$

is non-increasing. Then, by Proposition 4.2.8, there exist $-\infty < a \leq b < +\infty$ such that $(\mu + \nu_l + \nu_r - \nu)([a, b]) = (\nu_l + \nu_r)(\mathbb{R})$, $(\nu_l + \nu_r)((a, b)) = 0$ and

$$\pi(\{(x, x)\}) = \frac{(\mu + \nu_l + \nu_r - \nu)(\{x\})}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} \wedge \frac{(\nu_l + \nu_r)(\{x\})}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}$$

for $x \in \{a, b\}$. Since $\pi(\{(x, x) : x \in \mathbb{R}\}) = 0$, we deduce that we may choose a closed or semi-open or open interval I with ends a and b such that

$$\frac{\mu + \nu_l + \nu_r - \nu}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}(I) = 1 \quad \text{and} \quad \frac{\nu_l + \nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}(I) = 0.$$

Since $\nu - (\nu_l + \nu_r) = \nu_0^{\pi^{\downarrow}} \leq \mu \wedge \nu$, we have that $\mu + \nu_l + \nu_r - \nu - (\mu - \nu)^+ = \nu_l + \nu_r - (\nu - \mu)^+$ is a non-negative measure and we deduce that $\nu_l + \nu_r = (\nu - \mu)^+$ and $\mu + \nu_l + \nu_r - \nu = (\mu - \nu)^+$. Hence the Dispersion Assumption is satisfied. Moreover, since

$$\pi(\{(-\infty, a) \cup (b, +\infty)\} \times \mathbb{R}) \leq \frac{(\mu + \nu_l + \nu_r - \nu)(\mathbb{R} \setminus I)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} = 0, \pi(\mathbb{R} \times (a, b)) \leq \frac{\nu_l + \nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}(I) = 0,$$

and $\pi(\{a, a\}) = 0$, we have

$$\begin{aligned}\nu_l(dy) = \nu_l^{\pi}(dy) &= \int_{x \in \mathbb{R}} \mathbb{1}_{\{y < x\}} \pi(dx, dy) = \int_{x \in \mathbb{R}} \mathbb{1}_{\{y < x \leq b\}} \pi(dx, dy) \\ &= \int_{x \in \mathbb{R}} \mathbb{1}_{\{y < x, y \leq a\}} \pi(dx, dy) = \int_{x \in \mathbb{R}} \mathbb{1}_{\{y \leq a\}} \pi(dx, dy) = \mathbb{1}_{\{y \leq a\}} (\nu - \mu)^+(dy).\end{aligned}$$

and, in a symmetric way, $\nu_r(dy) = \mathbb{1}_{\{y \geq b\}}(\nu - \mu)^+(dy)$ (notice that, when $a = b$, then $\mu = \delta_a$ and $(\nu - \mu)^+(\{a\}) = 0$).

In the other direction, under the Dispersion Assumption, denoting by $a \leq b$ the ends of the interval I, we have $(\mu - \nu)^+([a, b]) = (\mu - \nu)^+(\mathbb{R})$ and $(\nu - \mu)^+((a, b)) = 0$ so that Proposition 4.2.8 ensures the existence of a unique non-increasing coupling $\pi \in \Pi_M\left(\frac{(\mu-\nu)^+}{(\mu-\nu)^+(\mathbb{R})}, \frac{(\nu-\mu)^+}{(\mu-\nu)^+(\mathbb{R})}\right)$ such that

$$\pi(\{(a, a)\}) = \frac{(\mu - \nu)^+(\{a\})}{(\mu - \nu)^+(\mathbb{R})} \wedge \frac{(\nu - \mu)^+(\{a\})}{(\mu - \nu)^+(\mathbb{R})} = 0 = \pi(\{(b, b)\}).$$

Hence,

$$\int_{x \in \mathbb{R}} \mathbb{1}_{\{y < x\}} \pi(dx, dy) = \mathbb{1}_{\{y \leq a\}} \frac{(\nu - \mu)^+(dy)}{(\mu - \nu)^+(\mathbb{R})} \text{ and } \int_{x \in \mathbb{R}} \mathbb{1}_{\{y > x\}} \pi(dx, dy) = \mathbb{1}_{\{y \geq b\}} \frac{(\nu - \mu)^+(dy)}{(\mu - \nu)^+(\mathbb{R})}.$$

The coupling $(\mu-\nu)^+(\mathbb{R})\pi(dx,dy)+\mu\wedge\nu(dx)\delta_x(dy)$ belongs to

$$\Pi_M\left(\mu,\nu,\mathbb{1}_{\{y\le a\}}(\nu-\mu)^+(dy),\mathbb{1}_{\{y\ge b\}}(\nu-\mu)^+(dy)\right).$$

□

Proof of Proposition 4.2.8. Since, clearly, $(iii)\Rightarrow(i)$, to prove that $(i)\Leftrightarrow(ii)\Leftrightarrow(iii)$, it is enough to check that $(i)\Rightarrow(ii)$ and $(ii)\Rightarrow(iii)$, which we do now.

Proof of $(i)\Rightarrow(ii)$ **:** Let π be a non-increasing coupling in $\Pi_M(\mu,\nu)$ and Γ be a Borel subset of $\mathbb{R}^2$ such that $\pi(\Gamma)=1$ and conditions $(c)(d)$ in Definition 4.2.3 hold. Let for $x\in\mathbb{R}$, $\Gamma_x=\{y:(x,y)\in\Gamma\}$. We have $\mu(dx)$ a.e. $\pi_x(\Gamma_x)=1$. With the martingale property, this implies

$$\mu(dx)\text{ a.e., }\int_{\Gamma_x}y\,\pi_x(dy)=x\quad\text{so that}\quad\pi_x(\Gamma_x\cap(-\infty,x])\wedge\pi_x(\Gamma_x\cap[x,+\infty))>0. \tag{4.27}$$

Therefore $\{(x,y)\in\Gamma:y\le x\}\neq\emptyset$ and $\{(x,z)\in\Gamma:z\ge x\}\neq\emptyset$. Let

$$a=\inf\{x\in\mathbb{R}:\exists\,y\le x,\ (x,y)\in\Gamma\}\quad\text{and}\quad b=\sup\{x\in\mathbb{R}:\exists\,z\ge x,\ (x,z)\in\Gamma\}.$$

Since when $x<a$, $\Gamma_x\cap(-\infty,x]=\emptyset$ and when $x>b$, $\Gamma_x\cap[x,+\infty)=\emptyset$, (4.27) implies that

$$\mu((-\infty,a)\cup(b,+\infty))=0.$$

Let $(x_0,y_0)\in\Gamma$ be such that $y_0\le x_0$. By condition (c) in Definition 4.2.3, for any $(x,y)\in\Gamma$ with $y\le x<x_0$, we have $y_0\le y$ so that $y_0\le x$. Hence $y_0\le a$, $-\infty<a$ and $\Gamma\cap\{(x,y):y\le x\}\subset[a,+\infty)\times(-\infty,a]$. In a symmetric way, condition (d) in Definition 4.2.3 implies that $b<+\infty$ and $\Gamma\cap\{(x,z):z\ge x\}\subset(-\infty,b]\times[b,+\infty)$. Therefore $\Gamma\subset\mathbb{R}\times\{(-\infty,a]\cup[b,+\infty)\}$ and

$$\nu((-\infty,a]\cup[b,+\infty))=\pi(\mathbb{R}\times\{(-\infty,a]\cup[b,+\infty)\})\ge\pi(\Gamma)=1.$$

Proof of $(ii)\Rightarrow(iii)$ **:** By Proposition 4.2.5, there exists a coupling $\pi^\star\in\Pi_M(\mu,\nu)$ minimizing $\int_{\mathbb{R}^2}|x-y|\pi(dx,dy)$ over $\pi\in\Pi_M(\mu,\nu)$ and this coupling is non-increasing. It remains to check that $\pi^\star$ is the only non-increasing coupling in $\Pi_M(\mu,\nu)$. This is clear when $a=b$ since then $\delta_a(dx)\nu(dy)$ is the only element of $\Pi(\mu,\nu)$. We thus suppose that $a<b$. Let $\pi^\downarrow\in\Pi_M(\mu,\nu)$ be non-increasing, $x\in\mathbb{R}$ and $\psi_{\pi^\downarrow}(x)=\pi^\downarrow((-\infty,x]\times(-\infty,a])$. Since $\psi_{\pi^\downarrow}(x)\le\pi^\downarrow(\mathbb{R}\times(-\infty,a])=F_\nu(a)$ and

$$\begin{aligned}0\le F_\mu(x)-\psi_{\pi^\downarrow}(x)&=\pi^\downarrow((-\infty,x]\times\mathbb{R})-\pi^\downarrow((-\infty,x]\times(-\infty,a])\\&=\pi^\downarrow((-\infty,x]\times[b,+\infty))\le\nu([b,+\infty))=1-F_\nu(a),\end{aligned}$$

we have $\psi_{\pi^\downarrow}(x)\in[(F_\nu(a)+F_\mu(x)-1)^+,F_\nu(a)\wedge F_\mu(x)]$. Using properties $(c)(d)$ in Definition 4.2.3, we check like in the derivation of (4.18) and (4.19) in the uniqueness part of the proof of

Theorem 4.2.11 that

$$\int_{y\in(-\infty,x]} \mathbb{1}_{\{z\le a\}}\pi^{\downarrow}(dy,dz) = \int_{F_\nu(a)-\psi_{\pi^\downarrow}(x)}^{F_\nu(a)} \delta_{F_\nu^{-1}(u)}(dz)du$$
$$\int_{y\in(-\infty,x]} \mathbb{1}_{\{z\ge b\}}\pi^{\downarrow}(dy,dz) = \int_{1+\psi_{\pi^\downarrow}(x)-F_\mu(x)}^{1} \delta_{F_\nu^{-1}(u)}(dz)du.$$

With the martingale property, this ensures that

$$\begin{aligned}\int_{(-\infty,x]} y\,\mu(dy) &= \int_{(-\infty,x]}\int_{\mathbb{R}} z\,\pi_y^{\downarrow}(dz)\,\mu(dy)\\ &= \int_{(-\infty,x]\times\mathbb{R}} \mathbb{1}_{\{z\le a\}}\,z\,\pi^{\downarrow}(dz,dy) + \int_{(-\infty,x]\times\mathbb{R}} \mathbb{1}_{\{z\ge b\}}\,z\,\pi^{\downarrow}(dz,dy)\\ &= \int_{F_\nu(a)-\psi_{\pi^\downarrow}(x)}^{F_\nu(a)} F_\nu^{-1}(v)\,dv + \int_{1+\psi_{\pi^\downarrow}(x)-F_\mu(x)}^{1} F_\nu^{-1}(v)\,dv = G_x(\psi_{\pi^\downarrow}(x)),\end{aligned}$$

where

$$[(F_\nu(a)+F_\mu(x)-1)^+, F_\nu(a)\wedge F_\mu(x)] \ni u \mapsto G_x(u) = \int_{F_\nu(a)-u}^{F_\nu(a)} F_\nu^{-1}(v)dv + \int_{1+u-F_\mu(x)}^{1} F_\nu^{-1}(v)dv.$$

The three last equalities also hold with $\pi^\downarrow$ and $\psi_{\pi^\downarrow}(x)$ replaced by $\pi^\star$ (the coupling minimizing $\int_{\mathbb{R}^2}|x-y|\pi(dx,dy)$ over $\pi\in\Pi_M(\mu,\nu)$) and $\psi_{\pi^\star}(x)=\pi^\star((-\infty,x]\times(-\infty,a])$. For $(F_\nu(a)+F_\mu(x)-1)^+\le u'\le u\le F_\nu(a)\wedge F_\mu(x)$, we have

$$G_x(u)-G_x(u') = \int_{F_\nu(a)-u}^{F_\nu(a)-u'} F_\nu^{-1}(v)\,dv - \int_{1+u'-F_\mu(x)}^{1+u-F_\mu(x)} F_\nu^{-1}(v)\,dv \le (u-u')a-(u-u')b.$$

Therefore G_x is a decreasing function and the equality $G_x(\psi_{\pi^\downarrow}(x)) = \int_{(-\infty,x]} y\mu(dy) = G_x(\psi_{\pi^\star}(x))$ implies that $\psi_{\pi^\downarrow}(x)=\psi_{\pi^\star}(x)$. Hence,

$$\begin{aligned}\forall x\in\mathbb{R},\qquad &\int_{y\in(-\infty,x]} \mathbb{1}_{\{z\le a\}}\pi^{\downarrow}(dy,dz) = \int_{y\in(-\infty,x]} \mathbb{1}_{\{z\le a\}}\pi^{\star}(dy,dz)\\ &\int_{y\in(-\infty,x]} \mathbb{1}_{\{z\ge b\}}\pi^{\downarrow}(dy,dz) = \int_{y\in(-\infty,x]} \mathbb{1}_{\{z\ge b\}}\pi^{\star}(dy,dz)\\ \text{and }&\int_{y\in(-\infty,x]} \pi_y^{\downarrow}\mu(dy) = \int_{y\in(-\infty,x]} \pi_y^{\star}\mu(dy).\end{aligned}$$

We conclude that $\pi^\downarrow=\pi^\star$ like in the end of the uniqueness part of the proof of Theorem 4.2.11.

Let us now suppose that (i), (ii) and (iii) hold and check that $\pi^\star(\{(a,a)\}) = \mu(\{a\})\wedge\nu(\{a\})$, the equality $\pi^\star(\{(b,b)\}) = \mu(\{b\})\wedge\nu(\{b\})$ being derived in a symmetric way. This is clear when $a=b$ since then $\delta_a(dx)\nu(dy)$ is the only element of $\Pi(\mu,\nu)$ and we thus suppose that $a<b$. We have $F_\mu(a)=\mu(\{a\})$. When $\mu(\{a\})\le\nu(\{a\})$, $G_a(\mu(\{a\})) = a\mu(\{a\})$ so that $\psi_{\pi^\star}(a)=\mu(\{a\})$. When $\mu(\{a\})>\nu(\{a\})$, $G_a(\nu(\{a\}))\ge a\nu(\{a\})+b(\mu(\{a\})-\nu(\{a\})) > a\mu(\{a\})$ so that $\psi_{\pi^\star}(a)>\nu(\{a\})$. Since

$$\mu(\{a\})\mathbb{1}_{\{z\le a\}}\pi_a^\star(dz) = \int_{F_\nu(a)-\psi_{\pi^\star}(a)}^{F_\nu(a)} \delta_{F_\nu^{-1}(u)}(dz)du,$$

we deduce that $\pi^\star(\{(a,a)\}) = \mu(\{a\}) \wedge \nu(\{a\})$.

Let us turn to the existence of a unique non-decreasing martingale coupling. The existence is a consequence of Proposition 4.2.5 (i) for the choice φ equal to the identity function. To check the uniqueness, we suppose that $\pi \in \Pi_M(\mu,\nu)$ is non-decreasing. If $a = b$, then $\delta_a(dx)\nu(dy)$ is the only element of $\Pi(\mu,\nu)$, so now we suppose that $a < b$. Since $\mu([a,b]) = 1 = 1 - \nu((a,b))$, we have $\nu_l^\pi(dy) = \mathbb{1}_{\{y<a\}}\nu(dy) + (\mu(\{a\}) - \pi(\{(a,a)\}))\delta_a(dy)$. Moreover, since $\mu((-\infty,a)) = 0$,

$$u_\mu(a) = \int_{\mathbb{R}} x\mu(dx) - a = \int_{\mathbb{R}} y\nu(dy) - a = 2\int_{(-\infty,a)} (y-a)\nu(dy) + u_\nu(a). \tag{4.28}$$

Therefore, when $u_\mu(a) = u_\nu(a)$, $\mathbb{1}_{\{y<a\}}\nu(dy) = 0$, and, by the martingale property, $\pi(\{(a,a)\}) = \mu(\{a\})$ so that $\nu_l^\pi = 0$. Let us suppose that $u_\mu(a) < u_\nu(a)$. When $\pi(\{(a,a)\}) = 0$, we have $\nu_l^\pi(dy) = \mathbb{1}_{\{y<a\}}\nu(dy) + \mu(\{a\})\delta_a$ (this is still true when π is not monotonic). Let us now suppose that $\pi(\{(a,a)\}) > 0$. Since π is non-decreasing, one has $\pi((a,b]\times(-\infty,a)) = 0$ so that $\mu(\{a\})\pi_a((-\infty,a)) = \nu((-\infty,a))$. On the other hand, by the non-decreasing property of π,

$$\mu(\{a\})\mathbb{1}_{\{y\geq b\}}\pi_a(dy) = \int_{F_\nu(a)}^{F_\nu(a)+\mu(\{a\})\pi_a([b,+\infty))} \delta_{F_\nu^{-1}(v)}dv.$$

The probability $\pi_a([b,+\infty))$ is determined by the equality

$$\int_{F_\nu(a)}^{F_\nu(a)+\mu(\{a\})\pi_a([b,+\infty))} (F_\nu^{-1}(v) - a)dv = \int_{(-\infty,a)} (a-y)\nu(dy)$$

and so is

$$\nu_l^\pi = \mathbb{1}_{\{y<a\}}\nu(dy) + \mu(\{a\})\pi_a((-\infty,a)\cup[b,+\infty))\delta_a(dy).$$

In a symmetric way,

$$\nu_r^\pi(dy) = \mathbb{1}_{\{y>b\}}\nu(dy) + \mu(\{b\})\pi_b((-\infty,a]\cup(b,+\infty))\delta_b(dy)$$

is also determined. If $\nu_l^\pi + \nu_r^\pi = 0$, then $\pi(dx,dy) = \mu(dx)\delta_x(dy)$. Otherwise, since

$$\frac{\pi(dx,dy) + (\nu_l^\pi + \nu_r^\pi - \nu)(dx)\delta_x(dy)}{\nu_l^\pi(\mathbb{R}) + \nu_r^\pi(\mathbb{R})}$$

is non-decreasing, the uniqueness of π is ensured by Corollary 4.2.12 (i). □

4.3.4 Proofs of the results in Section 4.2.5 concerning optimality of $\pi^\uparrow$ and $\pi^\downarrow$ for the cost function $|x-y|^\rho$ with $\rho > 1$ under the sandwich condition

Proof of Proposition 4.2.14. Since, by Proposition 4.2.8 applied with $a = \underline{x}$ and $b = \overline{x}$, there exists a unique non-decreasing (resp. non-increasing) coupling in $\Pi_M(\mu,\nu)$, it is enough to check that any coupling $\pi \in \Pi_M(\mu,\nu)$ that attains $\overline{\mathcal{M}}_\rho(\mu,\nu)$ (resp. $\underline{\mathcal{M}}_\rho(\mu,\nu)$) when $\rho \in (1,2)$ or $\underline{\mathcal{M}}_\rho(\mu,\nu)$ (resp. $\overline{\mathcal{M}}_\rho(\mu,\nu)$) when $\rho > 2$ is non-decreasing (resp. non-increasing). Let Γ be a set associated with π by Lemma 4.3.2. Since $\pi([\underline{x},\overline{x}]\times\{[\underline{y},\overline{y}]\cup[\underline{z},\overline{z}]\}) = 1$, we suppose without

restriction that $\Gamma \subset [\underline{x}, \overline{x}] \times \{[\underline{y}, \overline{y}] \cup [\underline{z}, \overline{z}]\}$. Let us suppose the existence of $(x_-, y_-), (x_+, y_+) \in \Gamma$ (resp. $(x_-, y_+), (x_+, y_-) \in \Gamma$) such that $x_- < x_+$, $y_- \leq x_-$, $y_+ \leq x_+$, $y_+ < y_-$ and obtain a contradiction. By (4.5), there exists $z > x_+$ such that $(x_+, z) \in \Gamma$ (resp. $z > x_-$ such that $(x_-, z) \in \Gamma$). We have

$$\underline{y} \leq y_+ < y_- \leq \overline{y} < \underline{x} \leq x_- < x_+ \leq \overline{x} < \underline{z} \leq z \leq \overline{z}.$$

Let us define the function $f_\rho : \mathbb{R} \to \mathbb{R}$ by

$$f_\rho(x) = \frac{z - y_-}{z - y_+}|x - y_+|^\rho + \frac{y_- - y_+}{z - y_+}|z - x|^\rho - |x - y_-|^\rho.$$

Since $(1 - \alpha_\rho)y_- + \alpha_\rho z \leq (1 - \alpha_\rho)\overline{y} + \alpha_\rho \overline{z} \leq \underline{x}$, by Lemma 4.3.8 just below applied with $(y, m, z) = (y_+, y_-, z)$, f_ρ is decreasing on $[\underline{x}, \overline{x}]$ when $\rho \in (1, 2)$ and increasing on $[\underline{x}, \overline{x}]$ when $\rho > 2$. We deduce the desired contradiction like in the proof of Proposition 4.2.5.

In a symmetric way, the existence of $(x_-, z_-), (x_+, z_+) \in \Gamma$ (resp. $(x_-, z_+), (x_+, z_-) \in \Gamma$) such that $x_- < x_+$, $x_- \leq z_-$, $x_+ \leq z_+$, $z_+ < z_-$ implies the existence of $y < x_-$ such that $(x_-, y) \in \Gamma$ (resp. $y < x_+$ such that $(x_+, y) \in \Gamma$) and

$$\underline{y} \leq y \leq \overline{y} < \underline{x} \leq x_- < x_+ \leq \overline{x} < \underline{z} \leq z_+ < z_- \leq \overline{z}.$$

Since $(1 - \alpha_\rho)z_+ + \alpha_\rho y \geq (1 - \alpha_\rho)\underline{z} + \alpha_\rho \underline{y} \geq \overline{x}$, by Lemma 4.3.8 just below applied with $(y, m, z) = (y, z_+, z_-)$, the function $\mathbb{R} \ni x \mapsto \frac{z_- - z_+}{z_- - y}|x - y|^\rho + \frac{z_+ - y}{z_- - y}|z_- - x|^\rho - |z_+ - x|^\rho$ is increasing on $[\underline{x}, \overline{x}]$ when $\rho \in (1, 2)$ and decreasing on $[\underline{x}, \overline{x}]$ when $\rho > 2$, yielding again a contradiction. □

Lemma 4.3.8. *Let $y < m < z$ and $f_\rho : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f_\rho(x) = \frac{z - m}{z - y}|x - y|^\rho + \frac{m - y}{z - y}|z - x|^\rho - |x - m|^\rho.$$

When $\rho \in (1, 2)$ (resp. $\rho > 2$), f_ρ is increasing (resp. decreasing) on $[y, (1 - \alpha_\rho)m + \alpha_\rho y)]$ and decreasing (resp. increasing) on $[(1-\alpha_\rho)m+\alpha_\rho z, z]$, where $\alpha_\rho \in (0, \frac{1}{2})$ is defined in Lemma 4.3.9 just below.

On the other hand, for any $\rho \in (1, 2) \cup (2, +\infty)$ and any $\alpha \in (0, \alpha_\rho)$, we may find $y < m < z$ such that $(2 - \rho)f'_\rho((1 - \alpha)m + \alpha y) < 0$ and $y < m < z$ such that $(2 - \rho)f'_\rho((1 - \alpha)m + \alpha z) > 0$.

Lemma 4.3.9. *For any $\rho > 1$, let $\psi_\rho : (0, 1] \to \mathbb{R}$ be the function defined by*

$$\psi_\rho(\alpha) = \alpha + \alpha^{2-\rho}(1 - \alpha)^{\rho-1} + 1 - \rho, \tag{4.29}$$

the following statements hold:

- *for all $\rho \in (1, 2) \cup (2, +\infty)$, there exists a unique $\alpha_\rho \in (0, 1)$ such that $\psi_\rho(\alpha_\rho) = 0$,*
- *if $\alpha \in (\alpha_\rho, 1]$, then $\psi_\rho(\alpha) > 0$ when $\rho \in (1, 2)$ and $\psi_\rho(\alpha) < 0$ when $\rho > 2$,*

- *if* $\alpha \in (0, \alpha_\rho)$*, then* $\psi_\rho(\alpha) < 0$ *when* $\rho \in (1,2)$ *and* $\psi_\rho(\alpha) > 0$ *when* $\rho > 2$*,*
- $\rho \mapsto \alpha_\rho$ *(represented by Fig. 4.3 below) is continuously differentiable on* $(1,2) \cup (2,\infty)$ *and can be extended by continuity at* $\rho = 2$ *with the value* α_2 *which is the unique solution to* $h(\alpha) = 0$ *where* $(0,1) \ni \alpha \mapsto h(\alpha) = 1 + (1-\alpha)\ln\frac{\alpha}{1-\alpha}$*,*
- *the extended function* $\rho \in (1,\infty) \mapsto \alpha_\rho \in (0,1]$ *is increasing,*
- $\alpha_\rho < \frac{\rho-1}{2}$ *when* $\rho \in (1,2)$*,*
- $\lim_{\rho\to\infty} \alpha_\rho = \frac{1}{2}$.

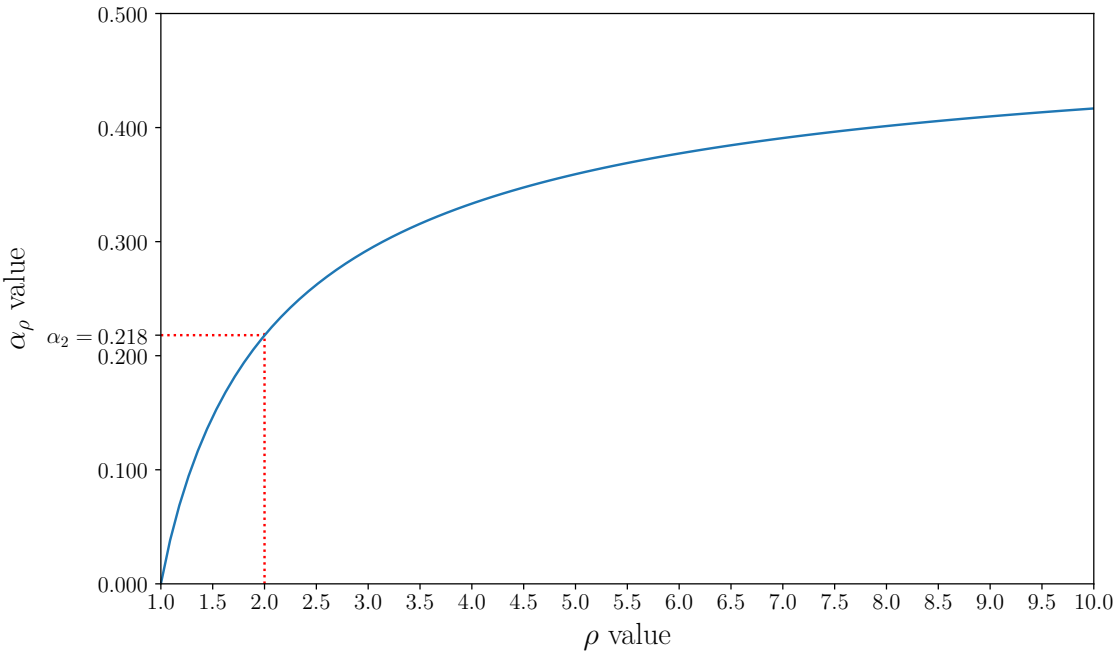


Figure 4.3: The function $\rho \mapsto \alpha_\rho$ (with α_ρ computed by a root finding algorithm).

The proof of Lemma 4.3.9 is postponed after the one of Lemma 4.3.8 which relies on the next lemma.

Lemma 4.3.10. *For* $\rho \in (1,+\infty) \setminus \{2\}$*, let us define the function* $\varphi_\rho : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ *by*

$$\varphi_\rho(\alpha, w) = (\alpha w)^\rho - \frac{w}{1+w}(1+\alpha w)^\rho - \frac{w^\rho}{1+w}(1-\alpha)^\rho .$$

Then, we have

$$\forall \alpha \in [\alpha_\rho, 1],\ \forall w > 0, \quad (2-\rho)\partial_\alpha \varphi_\rho(\alpha, w) > 0,$$
$$\forall \alpha \in [0, \alpha_\rho),\ \textit{for } w \textit{ large enough}, \quad (2-\rho)\partial_\alpha \varphi_\rho(\alpha, w) < 0.$$

where α_ρ *is defined in Lemma 4.3.9.*

The proof of Lemma 4.3.10 is postponed after the one of Lemma 4.3.8.

Proof of Lemma 4.3.8. When $x \in [y, m]$, we have

$$\begin{aligned} f_\rho(x) &= \frac{z-m}{z-y}(x-y)^\rho + \frac{m-y}{z-y}(z-x)^\rho - (m-x)^\rho \\ &= (z-m)^\rho \left[\frac{z-m}{z-y} \left(\frac{m-y}{z-m} \right)^\rho \left(1 - \frac{m-x}{m-y} \right)^\rho + \frac{m-y}{z-y} \left(1 + \frac{m-x}{z-m} \right)^\rho - \left(\frac{m-x}{z-m} \right)^\rho \right] \\ &= -(z-m)^\rho \, \varphi_\rho \left(\frac{m-x}{m-y}, \frac{m-y}{z-m} \right), \end{aligned}$$

By Lemma 4.3.10, since $x \in [y, (1-\alpha_\rho)m + \alpha_\rho y)] \Leftrightarrow \frac{m-x}{m-y} \in [\alpha_\rho, 1]$, we deduce that for $\rho \in (1,2)$ (resp. $\rho > 2$), f_ρ is increasing (resp. decreasing) on $[y, (1-\alpha_\rho)m + \alpha_\rho y)]$. Moreover, for any $\alpha \in (0, \alpha_\rho)$, when $\frac{m-y}{z-m}$ is large enough, $(2-\rho) f'_\rho((1-\alpha)m + \alpha y) < 0$.

When $x \in [m, z]$, we have

$$\begin{aligned} f_\rho(x) &= \frac{z-m}{z-y}(x-y)^\rho + \frac{m-y}{z-y}(z-x)^\rho - (x-m)^\rho \\ &= (m-y)^\rho \left[\frac{z-m}{z-y} \left(1 + \frac{x-m}{m-y} \right)^\rho + \frac{m-y}{z-y} \left(\frac{z-m}{m-y} \right)^\rho \left(1 - \frac{x-m}{z-m} \right)^\rho - \left(\frac{x-m}{m-y} \right)^\rho \right]. \\ &= -(m-y)^\rho \, \varphi_\rho \left(\frac{x-m}{z-m}, \frac{z-m}{m-y} \right). \end{aligned}$$

By Lemma 4.3.10 again, since $x \in [(1-\alpha_\rho)m + \alpha_\rho z, z] \Leftrightarrow \frac{x-m}{z-m} \in [\alpha_\rho, 1]$, we deduce that for $\rho \in (1,2)$ (resp. $\rho > 2$), f_ρ is decreasing (resp. increasing) on $[(1-\alpha_\rho)m + \alpha_\rho z, z]$. Moreover, for any $\alpha \in (0, \alpha_\rho)$, when $\frac{z-m}{m-y}$ is large enough, $(2-\rho) f'_\rho((1-\alpha)m + \alpha z) > 0$.

□

Proof of Lemma 4.3.10. Taking the derivative of $\varphi_\rho(\alpha, w)$ with respect to α, we have

$$\frac{1}{\rho} \partial_\alpha \varphi_\rho(\alpha, w) = \alpha^{\rho-1} w^\rho - \frac{w^2}{1+w} (1+\alpha w)^{\rho-1} + \frac{w^\rho}{1+w} (1-\alpha)^{\rho-1}$$

so that $\frac{1}{\rho} \partial_\alpha \varphi_\rho(0, w) = \frac{w^\rho - w^2}{1+w}$ is negative (resp. positive) when $w > 1$ and $\rho \in (1,2)$ (resp. $\rho > 2$). Let us now suppose that $\alpha \in (0,1)$. Then

$$\begin{aligned} \frac{1}{\rho} \partial_\alpha \varphi_\rho(\alpha, w) &= \frac{\alpha^{\rho-2} w^\rho}{1+w} \left(\alpha + \alpha^{2-\rho} (1-\alpha)^{\rho-1} - (\rho-1) \right) \\ &\quad + \frac{w^2}{1+w} \left((\rho-1)(\alpha w)^{\rho-2} + (\alpha w)^{\rho-1} - (1+\alpha w)^{\rho-1} \right) \\ &= \frac{\alpha^{\rho-2} w^\rho}{1+w} \psi_\rho(\alpha) + \frac{w^2(\rho-1)}{1+w} \int_{\alpha w}^{1+\alpha w} \left((\alpha w)^{\rho-2} - v^{\rho-2} \right) dv \\ &= \frac{\alpha^{\rho-2} w^\rho}{1+w} \left(\psi_\rho(\alpha) + (\rho-1) \int_{\alpha w}^{1+\alpha w} \left(1 - \left(\frac{\alpha w}{v} \right)^{2-\rho} \right) dv \right). \end{aligned}$$

Since

$$0 < \int_{\alpha w}^{1+\alpha w} \left(1 - \left(\frac{\alpha w}{v}\right)^{2-\rho}\right) dv \leq 1 - \left(\frac{\alpha w}{1+\alpha w}\right)^{2-\rho} \quad \text{when } \rho \in (1,2),$$
$$0 > \int_{\alpha w}^{1+\alpha w} \left(1 - \left(\frac{\alpha w}{v}\right)^{2-\rho}\right) dv \geq 1 - \left(\frac{1+\alpha w}{\alpha w}\right)^{\rho-2} \quad \text{when } \rho > 2,$$

by the first and second items in Lemma 4.3.9, we deduce that for $\alpha \in [\alpha_\rho, 1]$, we have $\partial_\alpha \varphi_\rho(\alpha, w) > 0$ when $\rho \in (1,2)$ and $\partial_\alpha \varphi_\rho(\alpha, w) < 0$ when $\rho > 2$.
For $\alpha \in (0, \alpha_\rho)$, since $\lim_{w\to\infty} \int_{\alpha w}^{1+\alpha w} \left(1 - (\frac{\alpha w}{v})^{2-\rho}\right) dv = 0$, by the third item in Lemma 4.3.9, we may find w large enough so that $\partial_\alpha \varphi_\rho(\alpha, w) < 0$ when $\rho \in (1,2)$ and $\partial_\alpha \varphi_\rho(\alpha, w) > 0$ when $\rho > 2$. □

Proof of Lemma 4.3.9. Taking the derivative of ψ_ρ, we get

$$\psi'_\rho(\alpha) = 1 + (2-\rho)\left(\frac{1-\alpha}{\alpha}\right)^{\rho-1} - (\rho-1)\left(\frac{\alpha}{1-\alpha}\right)^{2-\rho}.$$

It appears that

- when $1 < \rho < 2$,
  (1) ψ'_ρ is a decreasing function, so ψ_ρ is strictly concave,
  (2) $\lim_{\alpha\to 0^+} \psi_\rho(\alpha) = 1 - \rho < 0$ and $\psi_\rho(1) = 2 - \rho > 0$.
- when $\rho > 2$,
  (1) ψ'_ρ is an increasing function, and ψ_ρ is strictly convex,
  (2) $\lim_{\alpha\to 0^+} \psi_\rho(\alpha) = +\infty$ and $\psi_\rho(1) = 2 - \rho < 0$.

With the continuity of ψ_ρ, we deduce that there exists a unique $\alpha_\rho \in (0,1)$ such that $\psi_\rho(\alpha_\rho) = 0$. Moreover, for $\alpha \in (\alpha_\rho, 1]$, $\psi_\rho(\alpha) > 0$ when $\rho \in (1,2)$ and $\psi_\rho(\alpha) < 0$ when $\rho > 2$. And for $\alpha \in (0, \alpha_\rho)$, $\psi_\rho(\alpha) < 0$ when $\rho \in (1,2)$ and $\psi_\rho(\alpha) > 0$ when $\rho > 2$.

Since $\psi_\rho(\alpha_\rho) = 0$, we have $\left(\frac{1-\alpha_\rho}{\alpha_\rho}\right)^{\rho-2} = \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho}$,

$$\psi'_\rho(\alpha_\rho) = 1 + (2-\rho)\frac{\rho-1-\alpha_\rho}{\alpha_\rho} - (\rho-1)\frac{\rho-1-\alpha_\rho}{1-\alpha_\rho} = \frac{(\rho-1)(2-\rho)}{\alpha_\rho(1-\alpha_\rho)},$$

so that $\psi'_\rho(\alpha_\rho) > 0$ for $\rho \in (1,2)$ and $\psi'_\rho(\alpha_\rho) < 0$ for $\rho > 2$. Furthermore, by the implicit function theorem, as the mapping $(\rho, \alpha) \in \{(1,2) \cup (2,\infty)\} \times (0,1] \mapsto \psi_\rho(\alpha) \in \mathbb{R}$ is C^1, it follows that the function $\rho \mapsto \alpha_\rho$ is also C^1 on $(1,2) \cup (2,\infty)$ and its derivative writes

$$\partial_\rho \alpha_\rho = \frac{1 - \alpha_\rho^{2-\rho}(1-\alpha_\rho)^{\rho-1} \ln \frac{1-\alpha_\rho}{\alpha_\rho}}{\psi'_\rho(\alpha_\rho)}.$$

Since $\psi_\rho(\alpha_\rho) = 0$, we have

$$(\rho-2)\ln\frac{1-\alpha_\rho}{\alpha_\rho} = \ln\frac{\rho-1-\alpha_\rho}{1-\alpha_\rho} = (\rho-2)\int_0^1 \frac{du}{1-\alpha_\rho+(\rho-2)u}.$$

Then, since $\alpha_\rho^{2-\rho}(1-\alpha_\rho)^{\rho-1} = \rho-1-\alpha_\rho$, the numerator of the formula of $\partial_\rho\alpha_\rho$ can be written as

$$1-\alpha_\rho^{2-\rho}(1-\alpha_\rho)^{\rho-1}\ln\frac{1-\alpha_\rho}{\alpha_\rho} = 1-\int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)u}du.$$

When $\rho \in (1,2)$,

$$\psi_\rho\left(\frac{\rho-1}{2}\right) = \frac{\rho-1}{2}+\frac{\rho-1}{2}\left(\frac{3-\rho}{\rho-1}\right)^{\rho-1}+1-\rho = \frac{\rho-1}{2}\left(\left(\frac{3-\rho}{\rho-1}\right)^{\rho-1}-1\right) > 0,$$

so that $\alpha_\rho < \frac{\rho-1}{2}$, and

$$\int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)u}\,du < \int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)}\,du = 1.$$

When $\rho > 2$,

$$\int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)u}\,du > \int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)}\,du = 1.$$

Hence, we can deduce that

- $1-\alpha_\rho^{2-\rho}(1-\alpha_\rho)^{\rho-1}\ln\frac{1-\alpha_\rho}{\alpha_\rho} > 0$ when $\rho \in (1,2)$,
- $1-\alpha_\rho^{2-\rho}(1-\alpha_\rho)^{\rho-1}\ln\frac{1-\alpha_\rho}{\alpha_\rho} < 0$ when $\rho > 2$.

We conclude that $\partial_\rho\alpha_\rho > 0$ on $(1,2)\cup(2,+\infty)$.

Since $h'(\alpha) = \frac{1}{\alpha} - \ln\frac{\alpha}{1-\alpha}$ is a decreasing function, h is strictly concave. We have $\lim_{\alpha\to 0+} h(\alpha) = -\infty$ and $\lim_{\alpha\to 1-} h(\alpha) = 1$. With the continuity of h and $h(\frac{1}{2}) = 1$, we deduce that there exists a unique $\alpha_2 \in (0,\frac{1}{2})$ such that $h(\alpha_2) = 0$. Furthermore, by the fact that $e^x \geq 1+x$ with strict inequality when $x \neq 0$, we have

$$\begin{aligned}\psi_\rho(\alpha) &= \alpha+(1-\alpha)e^{(2-\rho)\ln\frac{\alpha}{1-\alpha}}+1-\rho\\ &\geq \alpha+(1-\alpha)\left(1+(2-\rho)\ln\frac{\alpha}{1-\alpha}\right)+1-\rho\\ &= (2-\rho)\left(1+(1-\alpha)\ln\frac{\alpha}{1-\alpha}\right) = (2-\rho)h(\alpha),\end{aligned}$$

with strict inequality when $\ln\frac{\alpha}{1-\alpha} \neq 0$. Hence $\psi_\rho(\alpha_2) > (2-\rho)h(\alpha_2) = 0$. We deduce that

- for $\rho \in (1,2)$, we have $\alpha_\rho < \alpha_2$ and $h(\alpha_\rho) < 0$,
- for $\rho \in (2,+\infty)$, we have $\alpha_\rho > \alpha_2$ and $h(\alpha_\rho) > 0$.

On the other hand, applying Taylor's theorem to the term $e^{(2-\rho)\ln\frac{\alpha}{1-\alpha}}$, we have

$$\psi_\rho(\alpha) = \alpha + (1-\alpha)\left(1 + (2-\rho)\ln\frac{\alpha}{1-\alpha}\right) + 1 - \rho + \mathcal{O}\left((2-\rho)^2\right) = (2-\rho)\left(h(\alpha) + \mathcal{O}(2-\rho)\right).$$

For $\alpha \in (0, \alpha_2)$, we have $h(\alpha) < 0$ and therefore for $\rho < 2$ close enough to 2, $\psi_\rho(\alpha) < 0$, which implies that $\alpha < \alpha_\rho < \alpha_2$. Hence $\lim_{\rho\to 2^-} \alpha_\rho = \alpha_2$. In the same way, we can prove that $\lim_{\rho\to 2^+} \alpha_\rho = \alpha_2$.

We have $\psi_\rho(\frac{1}{2}) = 2 - \rho$, so that, when $\rho > 2$, $\psi_\rho(\frac{1}{2}) < 0$ and $\alpha_\rho < \frac{1}{2}$. On the other hand, for $\alpha < \frac{1}{2}$, $\frac{1-\alpha}{\alpha} > 1$ and $\lim_{\rho\to+\infty} \psi_\rho(\alpha) = +\infty$ so that for large values of ρ, we have $\psi_\rho(\alpha) > 0$ and $\alpha < \alpha_\rho$. We deduce that

$$\lim_{\rho\to+\infty} \alpha_\rho = \frac{1}{2}.$$

□

Proof of Proposition 4.2.16. For $\pi \in \Pi_M(\mu, \underline{\nu})$, we have

$$\begin{aligned} \pi_{x_-}(\{y_+\}) \geq 0 = \pi^\uparrow_{x_-}(\{y_+\}), \qquad & \pi_{x_-}(\{z_+\}) \geq 0 = \pi^\uparrow_{x_-}(\{z_+\}), \\ \pi^\uparrow_{x_-}(\{y_-\}) = \frac{\nu(\{y_-\})}{\mu(\{x_-\})} \geq \pi_{x_-}(\{y_-\}), \qquad & \pi^\uparrow_{x_-}(\{z_-\}) = \frac{\nu(\{z_-\})}{\mu(\{x_-\})} \geq \pi_{x_-}(\{z_-\}). \end{aligned}$$

Now suppose that $\pi \neq \pi^\uparrow$ and define

$$\hat{\tau} = \frac{\pi_{x_-}(\{y_+\})\delta_{y_+} + \pi_{x_-}(\{z_+\})\delta_{z_+}}{\pi_{x_-}(\{y_+\}) + \pi_{x_-}(\{z_+\})},$$

$$\check{\tau} = \frac{\left(\pi^\uparrow_{x_-}(\{y_-\}) - \pi_{x_-}(\{y_-\})\right)\delta_{y_-} + \left(\pi^\uparrow_{x_-}(\{z_-\}) - \pi_{x_-}(\{z_-\})\right)\delta_{z_-}}{\pi^\uparrow_{x_-}(\{y_-\}) - \pi_{x_-}(\{y_-\}) + \pi^\uparrow_{x_-}(\{z_-\}) - \pi_{x_-}(\{z_-\})}.$$

Since

$$\pi_{x_-}(\{y_-\}) + \pi_{x_-}(\{y_+\}) + \pi_{x_-}(\{z_-\}) + \pi_{x_-}(\{z_+\}) = 1 = \pi^\uparrow_{x_-}(\{y_-\}) + \pi^\uparrow_{x_-}(\{z_-\}),$$

the denominators of the formulas defining $\hat{\tau}$ and $\check{\tau}$ are equal. Let us denote the common value by s. Clearly $s > 0$ and $\pi_{x_-} - \pi^\uparrow_{x_-} = s(\hat{\tau} - \check{\tau})$. Moreover, since $p\pi_{x_-} + (1-p)\pi_{x_+} = \nu = p\pi^\uparrow_{x_-} + (1-p)\pi^\uparrow_{x_+}$, $\pi_{x_+} - \pi^\uparrow_{x_+} = \frac{ps}{1-p}(\check{\tau} - \hat{\tau})$.

Since

$$\int y\, \pi_{x_-}(dy) = x_- = \int y\, \pi^\uparrow_{x_-}(dy),$$

$\hat{\tau}$ and $\check{\tau}$ have a common expectation m and $m \in [y_+, z_+] \cap [y_-, z_-] = [y_+, z_-]$ and we have

$$\hat{\tau} = \frac{z_+ - m}{z_+ - y_+}\delta_{y_+} + \frac{m - y_+}{z_+ - y_+}\delta_{z_+}, \qquad \check{\tau} = \frac{z_- - m}{z_- - y_-}\delta_{y_-} + \frac{m - y_-}{z_- - y_-}\delta_{z_-}. \tag{4.30}$$

The difference between π and $\pi^\uparrow$ can be expressed as $\pi - \pi^\uparrow = ps\,\delta_{x_-} \otimes (\hat{\tau} - \check{\tau}) + ps\delta_{x_+} \otimes (\check{\tau} - \hat{\tau})$.

Let us denote

$$\underline{\gamma}_m = \frac{1}{2}\left(\frac{z_- - m}{z_- - y_-}\delta_{(x_- - y_-)^2} + \frac{m - y_-}{z_- - y_-}\delta_{(z_- - x_-)^2} + \frac{z_+ - m}{z_+ - y_+}\delta_{(x_+ - y_+)^2} + \frac{m - y_+}{z_+ - y_+}\delta_{(z_+ - x_+)^2}\right),$$
$$\overline{\gamma}_m = \frac{1}{2}\left(\frac{z_+ - m}{z_+ - y_+}\delta_{(x_- - y_+)^2} + \frac{m - y_+}{z_+ - y_+}\delta_{(z_+ - x_-)^2} + \frac{z_- - m}{z_- - y_-}\delta_{(x_+ - y_-)^2} + \frac{m - y_-}{z_- - y_-}\delta_{(z_- - x_+)^2}\right).$$

We have

$$\mathrm{sq}\#\pi - \mathrm{sq}\#\pi^{\uparrow} = 2ps(\overline{\gamma}_m - \underline{\gamma}_m). \tag{4.31}$$

Let us check that for any $m \in [y_+, z_-]$, $\underline{\gamma}_m \leq_{cx} \overline{\gamma}_m$. Since $\underline{\gamma}_m$ and $\overline{\gamma}_m$ linearly depend on m, it is equivalent to check that $\underline{\gamma}_{y_+} \leq_{cx} \overline{\gamma}_{y_+}$ and $\underline{\gamma}_{z_-} \leq_{cx} \overline{\gamma}_{z_-}$. Note that $x_- - y_- \geq z_- - x_+$ implies that $x_+ - y_- \geq z_- - x_-$. Since

$$\begin{aligned}(x_- - y_+)^2 \vee (z_- - x_+)^2 &\leq (x_+ - y_+)^2 \wedge (x_- - y_-)^2 \wedge (z_- - x_-)^2 \\ &\leq (x_+ - y_+)^2 \vee (x_- - y_-)^2 \vee (z_- - x_-)^2 \leq (x_+ - y_-)^2,\end{aligned}$$

we have

$$\begin{aligned}\underline{\gamma}_{y_+} &= \frac{1}{2}\left(\delta_{(x_+ - y_+)^2} + \frac{z_- - y_+}{z_- - y_-}\delta_{(x_- - y_-)^2} + \frac{y_+ - y_-}{z_- - y_-}\delta_{(z_- - x_-)^2}\right) \\ &\leq_{cx} \frac{1}{2}\left(\delta_{(x_- - y_+)^2} + \frac{z_- - y_+}{z_- - y_-}\delta_{(x_+ - y_-)^2} + \frac{y_+ - y_-}{z_- - y_-}\delta_{(z_- - x_+)^2}\right) = \overline{\gamma}_{y_+}.\end{aligned}$$

In a symmetric way $z_+ - x_+ \geq x_- - y_+$ implies $z_+ - x_- \geq x_+ - y_+$ and since

$$\begin{aligned}(x_- - y_+)^2 \vee (z_- - x_+)^2 &\leq (x_+ - y_+)^2 \wedge (z_+ - x_+)^2 \wedge (z_- - x_-)^2 \\ &\leq (x_+ - y_+)^2 \vee (z_+ - x_+)^2 \vee (z_- - x_-)^2 \leq (z_+ - x_-)^2,\end{aligned}$$

we have $\underline{\gamma}_{z_-} \leq_{cx} \overline{\gamma}_{z_-}$. Combining the two cases together, we conclude that for any $m \in [y_+, z_-]$, $\underline{\gamma}_m \leq_{cx} \overline{\gamma}_m$, so that, by (4.31), $|y - x|^2 \# \pi^{\uparrow}(dx, dy) \leq_{cx} |y - x|^2 \# \pi(dx, dy)$.

For $\pi \in \Pi_M(\mu, \overline{\nu}) \setminus \{\pi^{\downarrow}\}$, we check in the same way that

$$\pi - \pi^{\downarrow} = ps\, \delta_{x_-} \otimes (\check{\tau} - \hat{\tau}) + ps\delta_{x_+} \otimes (\hat{\tau} - \check{\tau})$$

with $\hat{\tau}$ and $\check{\tau}$ satisfying (4.30) for some $m \in [y_+, z_-]$ and

$$s = \pi^{\downarrow}_{x_-}(\{y_+\}) - \pi_{x_-}(\{y_+\}) + \pi^{\downarrow}_{x_-}(\{z_+\}) - \pi_{x_-}(\{z_+\}) = \pi_{x_-}(\{y_-\}) + \pi_{x_-}(\{z_-\}) > 0.$$

As a consequence, $\mathrm{sq}\#\pi - \mathrm{sq}\#\pi^{\downarrow} = 2ps(\underline{\gamma}_m - \overline{\gamma}_m)$ so that $\mathrm{sq}\#\pi \leq_{cx} \mathrm{sq}\#\pi^{\downarrow}$. □

Proof of Proposition 4.2.18. Reasoning like in the beginning of the proof of Proposition 4.2.16, we check that for $\pi \in \Pi_M(\mu, \overline{\nu}) \setminus \{\overline{\pi}^{\star}\}$, $\pi = \overline{\pi}^{\star} + ps\delta_{x_-} \otimes (\hat{\tau} - \check{\tau}) + ps\delta_{x_+} \otimes (\check{\tau} - \hat{\tau})$ where

$$\hat{\tau} = \frac{z_- - m}{z_- - y_+}\delta_{y_+} + \frac{m - y_+}{z_- - y_+}\delta_{z_-}, \qquad \check{\tau} = \frac{z_+ - m}{z_+ - y_-}\delta_{y_-} + \frac{m - y_-}{z_+ - y_-}\delta_{z_+} \tag{4.32}$$
$$s = \pi_{x_-}(\{y_+\}) + \pi_{x_-}(\{z_-\}) = \overline{\pi}^{\star}_{x_-}(\{y_-\}) - \pi_{x_-}(\{y_-\}) + \overline{\pi}^{\star}_{x_-}(\{z_+\}) - \pi_{x_-}(\{z_+\}) > 0.$$

Therefore the difference between the costs of π and $\overline{\pi}^\star$ can be calculated as

$$\int |x-y|^\rho \pi(dx,dy) - \int |x-y|^\rho \overline{\pi}^\star(dx,dy) \quad = ps\left(f(x_-,m) - f(x_+,m)\right), \tag{4.33}$$

where for $x \in [y_+, z_-]$,

$$f(x,m) = \frac{z_- - m}{z_- - y_+}(x-y_+)^\rho + \frac{m-y_+}{z_- - y_+}(z_- - x)^\rho - \frac{z_+ - m}{z_+ - y_-}(x-y_-)^\rho - \frac{m-y_-}{z_+ - y_-}(z_+ - x)^\rho.$$

In the same way, we can check that any $\pi \in \Pi_M(\mu, \underline{\nu})$ distinct from $\underline{\pi}^\star$ can be expressed as

$$\pi = \underline{\pi}^\star + ps\delta_{x_-} \otimes (\check{\tau} - \hat{\tau}) + ps\delta_{x_+} \otimes (\hat{\tau} - \check{\tau}),$$

for $\hat{\tau}$ and $\check{\tau}$ satisfying (4.32) for some $m \in [y_+, z_-]$ and

$$s = \pi_{x_-}(\{y_-\}) + \pi_{x_-}(\{z_+\}) = \underline{\pi}^\star_{x_-}(\{y_+\}) - \pi_{x_-}(\{y_+\}) + \underline{\pi}^\star_{x_-}(\{z_-\}) - \pi_{x_-}(\{z_-\}) > 0.$$

Hence the difference between the costs of π and $\underline{\pi}^\star$ can be calculated as

$$\int |x-y|^\rho \pi(dx,dy) - \int |x-y|^\rho \underline{\pi}^\star(dx,dy) = ps(f(x_+,m) - f(x_-,m)).$$

Checking the optimality of $\overline{\pi}^\star$ and $\underline{\pi}^\star$ is equivalent to check when $1 < \rho < 2$ (resp. $\rho > 2$) that for every $m \in [y_+, z_-]$, $f(x_-,m) < f(x_+,m)$ (resp. $f(x_-,m) > f(x_+,m)$).

The function $m \mapsto f(x,m)$ being affine, it is also equivalent to show $f(x_-,y_+) < f(x_+,y_+)$ and $f(x_-,z_-) < f(x_+,z_-)$ for $1 < \rho < 2$ (resp. $f(x_-,y_+) > f(x_+,y_+)$ and $f(x_-,z_-) > f(x_+,z_-)$ for $\rho > 2$), which we are now going to demonstrate. We have

$$\begin{aligned}
f(x,y_+) &= (x-y_+)^\rho - \frac{z_+ - y_+}{z_+ - y_-}(x-y_-)^\rho - \frac{y_+ - y_-}{z_+ - y_-}(z_+ - x)^\rho, \\
\frac{1}{\rho}\partial_x f(x,y_+) &= (x-y_+)^{\rho-1} - \frac{z_+ - y_+}{z_+ - y_-}(x-y_-)^{\rho-1} + \frac{y_+ - y_-}{z_+ - y_-}(z_+ - x)^{\rho-1}, \\
f(x,z_-) &= (z_- - x)^\rho - \frac{z_+ - z_-}{z_+ - y_-}(x-y_-)^\rho - \frac{z_- - y_-}{z_+ - y_-}(z_+ - x)^\rho, \\
\frac{1}{\rho}\partial_x f(x,z_-) &= -(z_- - x)^{\rho-1} - \frac{z_+ - z_-}{z_+ - y_-}(x-y_-)^{\rho-1} + \frac{z_- - y_-}{z_+ - y_-}(z_+ - x)^{\rho-1}.
\end{aligned}$$

We are going to check that $(2-\rho)\partial_x f(z_-,z_-) > 0$ and $(2-\rho)\partial_x f(z_-,y_+) > 0$, so that by continuity of $x \mapsto (\partial_x f(x,z_-), \partial_x f(x,y_+))$, there exists $x_\rho \in (y_+, z_-)$ such that for all $x \in [x_\rho, z_-]$, $(2-\rho)\partial_x f(x,z_-) > 0$, $(2-\rho)\partial_x f(x,y_+) > 0$ and for all x_-, x_+ such that $x_\rho < x_- < x_+ < z_-$, $(2-\rho)\left(f(x_+,z_-) - f(x_-,z_-)\right) > 0$ and $(2-\rho)\left(f(x_+,y_+) - f(x_-,y_+)\right) > 0$.

On the one hand, we have

$$\begin{aligned}
\frac{1}{\rho}\partial_x f(z_-,z_-) &= -\frac{z_+ - z_-}{z_+ - y_-}(z_- - y_-)^{\rho-1} + \frac{z_- - y_-}{z_+ - y_-}(z_+ - z_-)^{\rho-1} \\
&= \frac{z_+ - z_-}{z_+ - y_-}(z_- - y_-)^{\rho-1}\left(\left(\frac{z_- - y_-}{z_+ - z_-}\right)^{2-\rho} - 1\right).
\end{aligned}$$

Since, by assumption $z_- - y_- > z_+ - z_-$, we deduce that $\partial_x f(z_-, z_-) > 0$ when $1 < \rho < 2$ and $\partial_x f(z_-, z_-) < 0$ when $\rho > 2$. On the other hand,

$$\begin{aligned}\frac{(z_+ - y_-)}{\rho}\partial_x f(z_-, y_+) &= (z_+ - y_-)(z_- - y_+)^{\rho-1} - (z_+ - y_+)(z_- - y_-)^{\rho-1} + (y_+ - y_-)(z_+ - z_-)^{\rho-1} \\ &= g\left(y_+ - y_-, z_- - y_+, z_+ - z_-\right),\end{aligned}$$

where, for $a, b, c \geq 0$,

$$g(a, b, c) = (a + b + c)\, b^{\rho-1} - (b + c)(a + b)^{\rho-1} + ac^{\rho-1}.$$

When $\rho \in (1, 2)$ and $b > 0$, by concavity of $\mathbb{R}_+ \ni x \mapsto x^{\rho-1}$, we have

$$\begin{aligned}g(a, b, c) &\geq (a + b + c)\, b^{\rho-1} - (b + c)(b^{\rho-1} + (\rho - 1)b^{\rho-2}a) + ac^{\rho-1} \\ &= (2 - \rho)ab^{\rho-1} + ac\left(c^{\rho-2} - ((\rho - 1)^{\frac{1}{\rho-2}}b)^{\rho-2}\right).\end{aligned}$$

and the inequality is reversed when $\rho > 2$ by convexity of $\mathbb{R}_+ \ni x \mapsto x^{\rho-1}$. We deduce that

$$\forall a, b > 0,\ \forall c \in [0, (\rho - 1)^{\frac{1}{\rho-2}}b],\ g(a, b, c) > 0 \text{ when } \rho \in (1, 2) \text{ and } g(a, b, c) < 0 \text{ when } \rho > 2. \tag{4.34}$$

Since $y_+ - y_- > 0$, $z_- - y_+ > 0$ and $z_- - y_+ \geq (\rho - 1)^{\frac{1}{2-\rho}}(z_+ - z_-)$, we have $\partial_x f(z_-, y_+) > 0$ when $\rho \in (1, 2)$ and $\partial_x f(z_-, y_+) < 0$ when $\rho > 2$. □

Chapter 5

An optimization approach to the numerical approximation of the weak optimal transport problem

Abstract

The *Weak Optimal Transport* (WOT) problem was introduced by Gozlan, Roberto, Samson, and Tetali [121], extending the classical Monge-Kantorovich framework by incorporating more generalized penalties, which are known as "weak" cost functionals. Existing numerical methods for WOT are largely confined to special cases, such as the quadratic barycentric cost. The primary challenge lies in the high computational cost of projecting under all marginal constraints while minimizing non-linear objective functions. In this work, we develop numerical approximations for both the WOT and WMOT problems from an optimization perspective by applying the Frank-Wolfe algorithm within a generalized framework.

Contents

5.1 Introduction **128**
5.2 Numerical approximation of the problem **130**
5.3 Frank Wolfe Algorithm **132**
5.4 Applications **134**
5.4.1 Wasserstein projection (WP) in dimension one 134
5.4.2 Stretched Brownian motion 141

5.1 Introduction

Let E and F be Polish spaces endowed with the compatible and complete metrics d_E and d_F, respectively. Let $c : E \times F \to \mathbb{R}_+$ be a non-negative measurable function. We denote by $\mathcal{P}(E)$ the set of probability measures on E. For probability measures $\mu \in \mathcal{P}(E)$ and $\nu \in \mathcal{P}(F)$, the classical *Optimal Transport* (OT) problem is stated as

$$OT(\mu, \nu, c) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{E\times F} c(x, y)\, \pi(dx, dy) \tag{OT}$$

where $\Pi(\mu, \nu)$ denotes the set of couplings between μ and ν, that is

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}(E \times F) \mid \pi(dx, F) = \mu(dx) \text{ and } \pi(E, dy) = \nu(dy)\}.$$

Extending this framework, consider a cost functional $C : E \times \mathcal{P}(F) \to \mathbb{R}_+$ that is measurable and convex in its second argument. The *Weak Optimal Transport* (WOT) problem is formulated as the following minimization problem:

$$V_C(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_E C(x, \pi_x)\mu(dx), \tag{WOT}$$

where $(\pi_x(dy))_{x\in E}$ is the disintegration of $\pi \in \Pi(\mu, \nu)$ with respect to the initial distribution μ, i.e. $\pi(dx, dy) = \mu(dx)\pi_x(dy)$. It is important to note that if $c : E \times F \to \mathbb{R}_+$ is a lower semi-continuous measurable function, and we define $C : (x, p) \mapsto \int_F c(x, y)\, p(dy)$, which is linear in the measure argument p, then the (WOT) problem reduces to the classical (OT) problem.

The convexity of C in its second argument is a standard assumption in weak transport theory. According to Acciaio et al. [3, Theorem 3.9], for probability measures $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ where μ is continuous, and for a continuous, totally bounded cost function $C \in \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, the following holds:

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}} C(x, \pi_x)\, \mu(dx) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}} C^*(x, \pi_x)\, \mu(dx) \tag{5.1}$$

where the function $C^*(x, \cdot)$ denotes the convex hull of the function $p \mapsto C(x, p)$.

To recall some basic results on the existence, duality, and stability of the (WOT) problem, we introduce appropriate topologies on the underlying spaces. Fix $\rho \geq 1$ and select arbitrary elements $x_0 \in E$ and $y_0 \in F$, note that their specific values will not impact our study. Let $\mathcal{P}_\rho(E)$ denote the set of all probability measures on E with finite ρ-th moment,

$$\mathcal{P}_\rho(E) = \left\{\eta \in \mathcal{P}(E) \;\middle|\; \int_E d_E^\rho(x, x_0)\, \eta(dx) < +\infty\right\}.$$

When $E = F$ and $c(x, y) = d_E^\rho(x, y)$ for some $\rho \geq 1$, the (OT) problen corresponds to the well-known Wasserstein distance with index ρ to the power ρ,

$$\mathcal{W}_\rho^\rho(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{E\times E} d_E^\rho(x, y)\, \pi(dx, dy).$$

For more details on Wasserstein distances and their properties, we refer the reader to [15, 205, 222, 223]. In analogy with the classical transport problem, it is natural to assume that the cost functional $C : E \times \mathcal{P}(F) \to \mathbb{R}_+$ is lower semi-continuous. Under this assumption, the (WOT) problem admits a minimizer, and a natural duality relation holds, see [24].

Motivated by the model-independent pricing framework, the *Martingale Optimal Transport* (MOT) problem has received considerable attention, see [38, 231, 110, 136]. For the multi-dimensional case, we refer to [94, 115, 181] and for connections to the Skorokhod embedding problem, see [36, 41, 125, 124].

For $d \in \mathbb{N}^*$, denoting by $\Pi_M(\mu, \nu)$ the set of martingale couplings between μ and ν:

$$\Pi_M(\mu,\nu) = \left\{ \pi(dx,dy) = \mu(dx)\pi_x(dy) \in \Pi(\mu,\nu) \mid \mu(dx)\text{-a.e.}, \int_{\mathbb{R}^d} y\, \pi_x(dy) = x \right\},$$

the martingale optimal transport problem is then defined as

$$MOT(\mu,\nu,c) = \inf_{\pi \in \Pi_M(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x,y)\, \pi(dx,dy). \tag{MOT}$$

The martingale constraint reflects the no-arbitrage condition in a financial market with zero interest rates. Recall that two probability measures $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ are in the convex order, denoted by $\mu \leq_{cx} \nu$, if for any convex function $\varphi : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(x)\, \mu(dx) \leq \int_{\mathbb{R}} \varphi(y)\, \nu(dy).$$

By Strassen's theorem [216], we have

$$\mu \leq_{cx} \nu \quad \iff \quad \Pi_M(\mu,\nu) \neq \emptyset.$$

Building upon this, we can incorporate martingale constraints into the weak optimal transport framework to address problems in finance that naturally involve weak formulations of the MOT problem. For a cost functional $C : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+$, the the *Weak Martingale Optimal Transport* (WMOT) problem is defined as

$$V_C^M(\mu,\nu) := \inf_{\pi \in \Pi_M(\mu,\nu)} \int_{\mathbb{R}} C(x, \pi_x)\mu(dx). \tag{WMOT}$$

Similarly, if we consider a measurable map $c : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$, and define $C : (x,p) \mapsto \int c(x,y)\, p(dy)$, which is linear in the measure argument p, the (WMOT) problem reduces to the ordinary (MOT) problem. Note that given $\pi \in \Pi(\mu,\nu)$ (resp. $\pi \in \Pi_M(\mu,\nu)$), it is well known (see [95] pages 78–80 or [192] page 117) that there exists a $\mu(dx)$-a.e. unique Markov kernel (resp. martingale Markov kernel) π_x such that $\pi(dx,dy) = \mu(dx)\pi_x(dy)$.

Numerical methods for the weak optimal transport problem have attracted considerable attention recently, especially in the context of the barycenter problem. Cazelle et al. [77] explore the relation between weak barycenter with classical barycenter problems, and develop iterative algorithms with applications on streaming data. Paty et al. [187] apply mirror descent methods

for both primal and dual formulations of the barycentric WOT problem, with applications in economic models of labor market matching. Backhoff-Veraguas et al. [26] apply stochastic gradient descent to compute 2-Wasserstein population barycenters, providing convergence analysis and conditions on the uniqueness of the solution. Kolesov et al. [158] propose a method to estimate Wasserstein barycenters by employing a bi-level adversarial learning objective applicable to general cost functions with applications of image data analysis.

Beyond the barycenter problem, Kupper et al. [161] investigated convex risk measures with WOT penalties, offering explicit representations via nonlinear transforms of loss functions and discussing computational approaches including neural network approximations, with applications to insurance and model-free pricing of European options. Kim et al. [157] focused on statistical testing within the WOT framework, introducing a test for convex order based on the Wasserstein projection distance, and implementing an efficient entropic Frank-Wolfe algorithm for computation.

Coinciding independently with [157], we developed a similar approach by applying the Frank-Wolfe algorithm to solve the (WOT) and (WMOT) problems in a more general setting without limiting ourselves to restricting cost functions, as detailed in Section 5.2. Furthermore, we present applications in Section 5.4 to the Wasserstein projection in one dimension for various values of the exponent ρ and the stretched Brownian motion.

5.2 Numerical approximation of the problem

For $I, J \in \mathbb{N}^*$, let $\hat{\mu}$ and $\hat{\nu}$ be probability measures with finite support:

$$\hat{\mu} = \sum_{i=1}^{I} \mu_i \, \delta_{x_i}, \quad \hat{\nu} = \sum_{j=1}^{J} \nu_j \, \delta_{y_j}, , \quad \mu_i, \nu_j \in [0,1] \quad \text{and} \quad \sum_{i=1}^{I} \mu_i = \sum_{j=1}^{J} \nu_j = 1,$$

where $(x_i)_{1 \le i \le I}$ and $(y_j)_{1 \le j \le J}$ are independent and identically distributed (i.i.d.) random variables with distribution $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$. Note that for empirical measures, we simply have

$$\hat{\mu} = \frac{1}{I} \sum_{i=1}^{I} \delta_{x_i}, \quad \hat{\nu} = \frac{1}{J} \sum_{j=1}^{J} \delta_{y_j}. \tag{5.2}$$

Any coupling π of $\hat{\mu}$ and $\hat{\nu}$ can be expressed as

$$\pi = \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} \, \delta_{(x_i, y_j)}, \quad \pi_{i,j} \ge 0, \quad \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_{i,j} = 1,$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i \quad \text{for } i = 1, \dots, I, \quad \text{and} \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j \quad \text{for } j = 1, \dots, J.$$

The approximation of the (OT) problem is to find $(\pi_{i,j})_{\substack{1\leq i\leq I,\\ 1\leq j\leq J}}$ that minimizes the following:

$$\min_{\pi_{i,j}} \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{i,j}\, c(x_i, y_j)$$
$$\text{s.t.} \quad \pi_{i,j} \geq 0, \quad \sum_{i=1}^{I} \pi_{i,j} = \nu_j, \quad \sum_{j=1}^{J} \pi_{i,j} = \mu_i.$$

For $\mu \leq_{cx} \nu$, the approximation of the (MOT) problem is

$$\min_{\pi_{i,j}} \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{i,j}\, c(x_i, y_j)$$
$$\text{s.t.} \quad \pi_{i,j} \geq 0, \quad \sum_{i=1}^{I} \pi_{i,j} = q_j, \quad \sum_{j=1}^{J} \pi_{i,j} = p_i, \quad \text{and} \quad \sum_{j=1}^{J} \pi_{i,j}\, y_j = p_i\, x_i.$$

Approximation of WOT and WMOT Denote $p_{j|i} = \dfrac{\pi_{i,j}}{\mu_i}$ the conditional probability of y_j given x_i. The WOT problem becomes

$$\min \sum_{i=1}^{I} \mu_i\, C\left(x_i, \sum_{j=1}^{J} p_{j|i}\, \delta_{y_j}\right)$$
$$\text{s.t.} \quad p_{j|i} \geq 0, \quad \sum_{j=1}^{J} p_{j|i} = \mathbf{1}, \quad \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j.$$

The problem is subject to constraints that enforce non-negativity and normalization of the conditional probability variables $p_{j|i}$, ensuring that each $p_{j|i}$ is non-negative and sums to 1 for every i. Additionally, the marginal consistency constraint guarantees that the weighted sums of the $p_{j|i}$ probabilities across i align with the target distribution ν_j.

Respectively, the approximation of the (WMOT) problem becomes

$$\min \sum_{i=1}^{I} \mu_i\, C\left(x_i, \sum_{j=1}^{J} p_{j|i}\, \delta_{y_j}\right)$$
$$\text{s.t.} \quad p_{j|i} \geq 0, \quad \sum_{j=1}^{J} p_{j|i} = \mathbf{1}, \quad \text{and} \quad \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j, \quad \sum_{j=1}^{J} p_{j|i}\, y_j = x_i.$$

The last constraint enforces the martingale property. Note that this formulation is also a *Linear Programming* (LP) problem.

However, introducing the martingale constraint poses a significant challenge in sampling while preserving convex order. In practice, empirical measures obtained from i.i.d. samples often do not share the same mean and therefore generally fail to preserve convex order. Alfonsi, Corbetta, and Jourdain addressed this issue in one dimension [11] and extended their approach to higher dimensions [12].

5.3 Frank Wolfe Algorithm

The Frank-Wolfe algorithm, also known as the Conditional Gradient algorithm [105, 174], is a projection-free first-order algorithm for constrained optimization. The algorithm finds the optimal solution iteratively by moving along the gradient direction of the objective function, subject to the constraints. It can be applied to a general class of constrained convex optimization problems of the following form:

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{C}, \end{aligned}$$

where $\mathcal{C}$ is a compact convex set in a vector space and $f : \mathcal{C} \to \mathbb{R}$ convex, differentiable real-valued function. Instead of directly moving all the way to the gradient direction (as in gradient descent), it takes a step towards the optimal solution by solving a linear optimization problem which involves finding the point on the constraint set that minimizes the linear approximation of the objective function.

There are only two oracles directly accessing the objective function f and the feasible region $\mathcal{C}$. The First-Order Oracle (denoted as FOO and shown in Oracle 1), gives information about the function f: when queried with a point $x \in \mathcal{C}$, FOO returns the function value $f(x)$ (i.e., zero-order information) and the gradient $\nabla f(x)$ of f at x. We applied Forward-Mode Automatic Differentiation in Julia [200] for the FOO in our program.

Oracle 1 First-Order Oracle for f (FOO)

Input: Point $x \in \mathcal{C}$

Output: $\nabla f(x)$ and $f(x)$

Secondly, there is a Linear Minimization Oracle (denoted as LMO and shown in Oracle 2), gives information about the domain $\mathcal{C}$ (note that v is not necessarily unique, we slightly abuse notation for simplicity):

Oracle 2 Linear Minimization Oracle over $\mathcal{C}$ (LMO)

Input: Linear approximation c

Output: $v = \underset{x \in \mathcal{C}}{\operatorname{argmin}} \langle c, x \rangle$

The vanilla Frank–Wolfe algorithm was first introduced by Frank and Wolfe in 1956 [105], which generates a sequence of feasible points $x_t \in \mathcal{C}$ by optimizing the first-order (linear) approximations of the objective function f.

In each iteration, the Frank-Wolfe algorithm computes a linear approximation of the objective function and solves a linear sub-problem over the feasible set, which is computationally less expensive than directly enforcing the constraints in traditional methods like projected gradient descent. This makes it highly suitable for optimization problems such as ours, where the

dimensionality of p can be large, and the constraint set is relatively simple but spans many variables.

Furthermore, the algorithm's iterative nature ensures convergence to the optimal solution, and its simplicity allows for efficient implementation even when the number of variables grows significantly. This efficiency makes Frank-Wolfe a natural choice for solving problems involving transport-like projections with convex costs and linear constraints.

Algorithm 3 (vanilla) Frank-Wolfe algorithm

Input: initial point $x_0 \in \mathcal{C}$, function $f : \mathbb{R} \to \mathbb{R}$ convex, step sizes $\gamma_t > 0$, smoothness L.

Output: Iterates $x_1, \cdots \in \mathcal{C}$

1: **for** $t \leftarrow 1$ to n **do**

2: $\quad v_t \leftarrow \underset{v \in \mathcal{C}}{\operatorname{argmin}} \langle \nabla f(x_t), v \rangle$

3: $\quad \gamma_t \leftarrow \min \left\{ \dfrac{\langle \nabla f(x_t), x_t - v_t \rangle}{L \|x_t - v_t\|^2}, 1 \right\}$

4: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$

5: **end for**

When the cost function $p \mapsto C(x, p)$ is convex for all x, the (WOT) problem and (WMOT) particularly well-suited for the Frank-Wolfe algorithm. This is due to the problems' inherent structure, which features a convex objective function and linear constraints. Specifically, the mapping

$$(p_{j|i})_{1 \le i \le I,\, 1 \le j \le J} \;\mapsto\; \sum_{i=1}^{I} \mu_i \, C\left(x_i, \sum_{j=1}^{J} p_{j|i} \delta_{y_j} \right)$$

is a composition of linear and convex functions, resulting in a convex objective function with respect to the probabilities $p_{j|i}$. The constraints over non-negativity, normalization, and marginal consistency of $p_{j|i}$, are linear, forming a convex feasible set $\mathcal{C}$ that is a polytope.

The Frank-Wolfe algorithm excels in such settings because it efficiently handles optimization over convex polytopes without the need for complex projections onto the feasible set. In this context, the LMO in step 2 of Algorithm 3 reduces to solving a linear programming problem, thanks to the linearity of the constraints. Moreover, step 3 involves a linear approximation of the convex objective function, aligning perfectly with the algorithm's framework. By avoiding computationally intensive projections and leveraging linear approximations, the Frank-Wolfe algorithm becomes highly effective for large-scale problems like (WOT) and (WMOT).

Following the suitability of the Frank-Wolfe algorithm for convex problems, we also consider its applicability when the cost function is non-convex. The result of [3, Theorem 3.9] stated in (5.1) implies that when μ is continuous, we can replace any non-convex cost function with its convex envelope without altering the minimum value of the problem. Consequently, even in the case of a non-convex cost function, the optimization can be reformulated with a convex approximation, allowing the Frank-Wolfe algorithm to be applied effectively.

Thus, for problems like (WOT) and (WMOT), we can utilize the Frank-Wolfe algorithm by first convexifying the cost function where necessary. This approach preserves the problem's

structure while leveraging the efficiency of the algorithm. In our numerical experiments, we used the Julia package FrankWolfe.jl [52] to solve the optimization problem, with the package's default parameters used to determine the step sizes and convergence criteria. This implementation provided robust and scalable performance for our problem instances.

5.4 Applications

5.4.1 Wasserstein projection (WP) in dimension one

In the recent years, the barycentric weak optimal transport problem has received special attention, we can refer to [11, 12, 25, 120, 121]. For $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$, we consider

$$V^\rho_{\mathrm{WP},\rho}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}} \left| x - \int_{\mathbb{R}} y\, \pi_x(dy) \right|^\rho \mu(dx), \tag{5.3}$$

This problem has an intrinsic connection with the problem of finding *Wasserstein Projection* (WP). Alfonsi, Corbetta and Jourdain [12] introduced, for $\rho \in [1, +\infty)$, the Wasserstein projections in the convex order:

$$\mathcal{I}_\rho(\mu,\nu) := \operatorname{argmin}\left\{\mathcal{W}_\rho(\mu,\eta) : \eta \leq_{cx} \nu\right\}, \tag{5.4}$$

$$\mathcal{J}_\rho(\mu,\nu) := \operatorname{argmin}\left\{\mathcal{W}_\rho(\eta,\nu) : \mu \leq_{cx} \eta\right\}. \tag{5.5}$$

According to [12, 24], the values of $V_{\mathrm{WP},\rho}(\mu,\nu)$ and $\mathcal{W}_\rho(\mu, \mathcal{I}_\rho(\mu,\nu))$ coincide. Moreover, if π^* is an optimizer of (5.3), then the image of its first marginal μ under the map $x \mapsto \int_{\mathbb{R}^d} y\, \pi^*_x(dy)$ is a minimizer of (5.4) and coincides with $\mathcal{I}_\rho(\mu,\nu)$ when $\rho > 1$. Therefore, when $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)$ are finitely supported, we can use (5.3) to compute the Wasserstein projection.

Wasserstein projections in the special case $d = 1$ were further studied by Alfonsi, Corbetta, and Jourdain [12] using quantile functions. We denote the cumulative distribution function of $\eta \in \mathcal{P}(\mathbb{R})$ by $F_\eta : \mathbb{R} \ni x \mapsto \eta(-\infty, x] \in [0,1]$, and its quantile function by $F^{-1}_\eta : (0,1) \ni u \mapsto \inf\{x \in \mathbb{R} : u \leq F_\eta(x)\} \in \mathbb{R}$. By [12, Theorem 2.6 and Proposition 4.2], the respective quantile functions of such optimizers $\mathcal{I}(\mu,\nu)$ and $\mathcal{J}(\mu,\nu)$ are obtained for all $u \in (0,1)$ by

$$F^{-1}_{\mathcal{I}(\mu,\nu)} = F^{-1}_\mu - \partial_-\mathrm{co}(G)(u) \quad \text{and} \quad F^{-1}_{\mathcal{J}(\mu,\nu)} = F^{-1}_\nu + \partial_-\mathrm{co}(G)(u),$$

where $G(u) := \int_0^u (F^{-1}_\mu - F^{-1}_\nu)(v)dv$, co denotes the convex hull, and ∂_- the left-hand derivative. More specifically, we have $\mathcal{I}(\mu,\nu) \leq_{cx} \nu$ and $\mu \leq_{cx} \mathcal{J}(\mu,\nu)$; if $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$,

$$W_\rho(\mu, \mathcal{I}(\mu,\nu)) = \inf_{\eta \leq_{cx} \nu} W_\rho(\mu,\eta) \quad \text{and} \quad W_\rho(\mathcal{J}(\mu,\nu),\nu) = \inf_{\mu \leq_{cx} \eta} W_\rho(\eta,\nu).$$

When $d = 1$ and $\rho \in [1, +\infty)$, according to [148, Theorem 1.1], the Wasserstein projections $\mathcal{I}$ and $\mathcal{J}$ are Lipschitz continuous. For $\mu, \nu, \tilde{\mu}, \tilde{\nu} \in \mathcal{P}_\rho(\mathbb{R})$,

$$W_\rho(\mathcal{I}(\mu,\nu),\mathcal{I}(\tilde{\mu},\tilde{\nu})) \le 2W_\rho(\mu,\tilde{\mu}) + W_\rho(\nu,\tilde{\nu}),$$
$$W_\rho(\mathcal{J}(\mu,\nu),\mathcal{J}(\tilde{\mu},\tilde{\nu})) \le W_\rho(\mu,\tilde{\mu}) + 2W_\rho(\nu,\tilde{\nu}).$$

Example 5.4.1 (Constructed by [148])**.** *Let μ,ν be the probability measures with quantile functions:*

$$F_\mu^{-1}(u) = u\mathbb{1}_{(0,\frac{1}{2}]}(u) + \frac{12+5u}{18}\mathbb{1}_{(\frac{1}{2},1)}(u), \qquad F_\nu^{-1}(u) = \frac{u}{3}\mathbb{1}_{(0,\frac{1}{2}]}(u) + \frac{u}{2}\mathbb{1}_{(\frac{1}{2},1)}(u).$$

It has been checked that

$$F_{\mathcal{I}(\mu,\nu)}^{-1}(u) = \frac{u}{3}\mathbb{1}_{(0,\frac{1}{2}]}(u) + \frac{3+5u}{18}\mathbb{1}_{(\frac{1}{2},1)}(u).$$

In the numerical experiment of the Wasserstein projection problem, we construct couples of probability measures μ,ν in the sense of Example 5.4.1 by the approximation of $\hat{\mu},\nu_I$ as (5.2). Then, we find the optimal π^* that attains $V_{\mathrm{WP}}^\rho(\mu,\nu)$ and make a comparison between the value of $\mathcal{I}_\rho(\mu,\nu)$ and the image of μ by $x \mapsto \int_{\mathbb{R}^d} y\,\pi_x^*(dy)$.

5.4.1.1 WP with $\rho = 1$: reduction to LP

When $\rho = 1$, the Wasserstein projection reduces to a linear programming problem. We introduce an auxiliary variable z_i, which bounds on the absolute differences between the original points and the projections. This approach is very commonly used to transform the absolute value into a set of linear constraints, making it easier to solve using standard LP methods.

$$\min_{p_{j|i},\,z_i} \quad \sum_{i=1}^{I} \mu_i\, z_i$$
$$\text{subject to} \quad \begin{cases} z_i \ge x_i - \sum_{j=1}^{J} p_{j|i}\, y_j, & \forall i = 1,\dots,I, \\ z_i \ge \sum_{j=1}^{J} p_{j|i}\, y_j - x_i, & \forall i = 1,\dots,I, \\ p_{j|i} \ge 0, & \forall i = 1,\dots,I, \quad \forall j = 1,\dots J, \\ \sum_{j=1}^{J} p_{j|i} = 1, & \forall i = 1,\dots,I, \\ \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j & \forall j = 1,\dots J. \end{cases}$$

When $\rho > 1$, $\mathcal{I}(\mu,\nu)$ is the unique metric projection but the uniqueness may fail when $\rho = 1$, c.f. [12, Remark 2.3]. When $\rho = 1$, $\mathcal{W}_1(\mu,\nu)$ corresponds to the area between the graphs of F_μ^{-1}

and F_ν^{-1}, and therefore between the graphs on F_μ and F_ν:

$$\mathcal{W}_1(\mu,\nu) = \int_{\mathbb{R}} |\, F_\mu(x) - F_\nu(x)\,|\, dx.$$

For $\eta \in \mathcal{P}_1(\mathbb{R})$, we say that η is smaller than μ in the stochastic order and denote $\eta \leq_{st} \mu$ if there exists $\pi \in \Pi(\eta,\mu)$ such that $\pi(\{(x,y) \in \mathbb{R}^2 : x \leq y\}) = 1$. We have

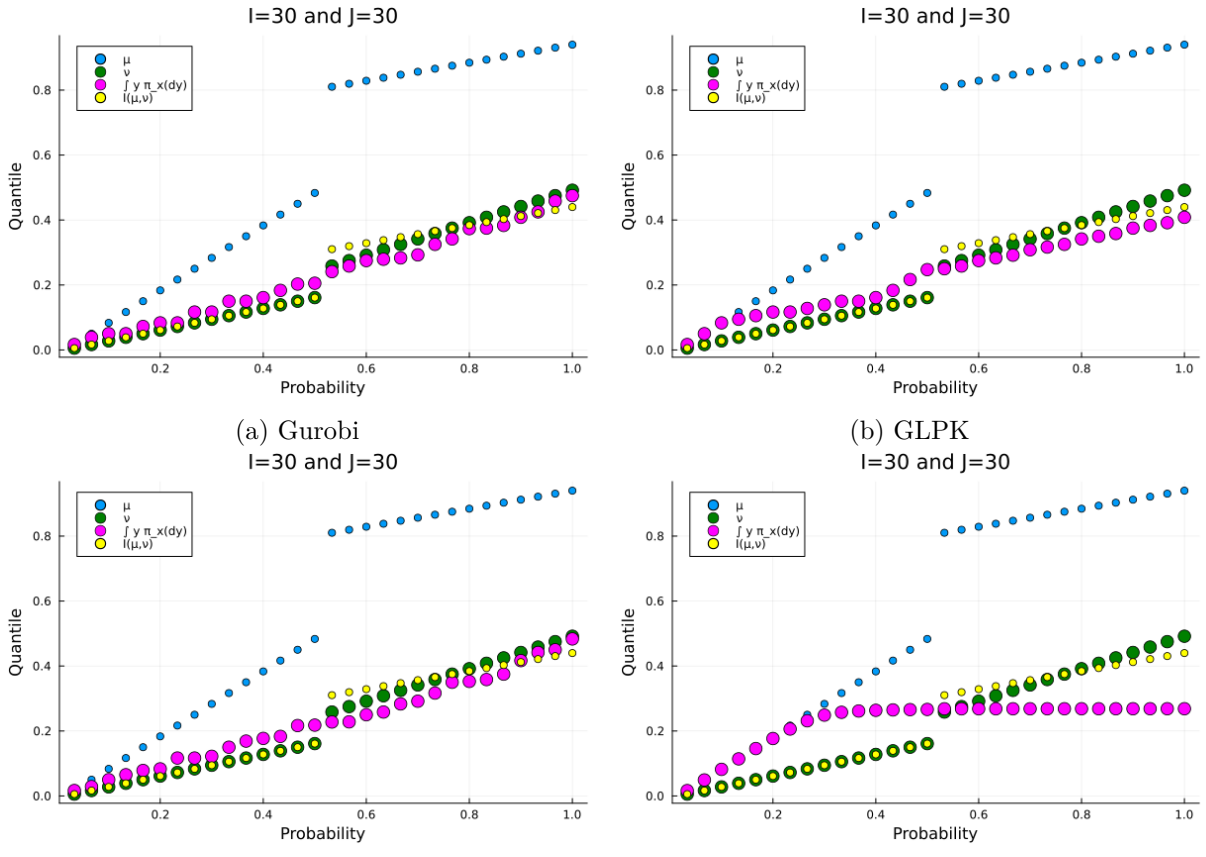
$$\eta \leq_{st} \mu \iff F_\eta \geq F_\mu \iff F_\eta^{-1} \leq F_\mu^{-1}.$$

Hence, for any $\eta \in \mathcal{P}_\infty(\mathbb{R})$ such that $\eta \leq_{cx} \nu$ and $\eta \leq_{st} \mu$, we have

$$\mathcal{W}_1(\mu, \mathcal{I}(\mu,\nu)) = \int_{\mathbb{R}} x\,\mu(dx) - \int_{\mathbb{R}} z\,\mathcal{I}(\mu,\nu)(dz) = \int_{\mathbb{R}} x\,\mu(dx) - \int_{\mathbb{R}} y\,\nu(dy) = \mathcal{W}_1(\mu,\eta).$$

Therefore, the uniqueness fails when $\rho = 1$, as confirmed by Fig. 5.1.

For $I = 30$ and $J = 30$, we demonstrated the non-uniqueness of the one-dimensional Wasserstein projection problem for $\rho = 1$. To investigate this, we applied several solvers, including Gurobi [128], GLPK [182], HiGHS [141], and Ipopt [225]. The results, presented in the scatter plots below, clearly exhibit the multiple solutions that arise in this case.



(a) Gurobi (b) GLPK

(c) HiGHS (d) Ipopt

Figure 5.1: WP with $\rho = 1$, the uniqueness fails

In each scatter plot, we displayed the quantile functions F_μ^{-1} and F_ν^{-1} in Example 5.4.1, the projection $x \mapsto \int_{\mathbb{R}} y\, \pi_x^*(dy)$, and $\mathcal{I}(\mu, \nu)$ to provide a comprehensive comparison between the solver outputs. To enhance the clarity of the visual comparison, we increased the marker sizes for F_ν^{-1} in green and $x \mapsto \int_{\mathbb{R}} y\, \pi_x^*(dy)$ in magenta, allowing for a more distinct representation of the different outcomes produced by the solvers. This approach highlights the sensitivity of the Wasserstein projection problem to solver choice when the uniqueness condition does not hold.

Although the optimal solutions π^* obtained by Gurobi, GLPK, HiGH, and Ipopt are all different, as presented in the graph above, they all yield the same optimal value for the objective function, which is $3.33333e-01$. This consistency in objective value further confirms the non-uniqueness of the Wasserstein projection when $\rho = 1$.

5.4.1.2 WP with $\rho = 2$: reduction to a quadratic problem

When $\rho = 2$, we build upon methodologies from the linear case. Specifically, we extend the ideas by incorporating a quadratic term on the objective function and the constraints. This type of problem is well-suited to be solved using optimizers that support quadratic objectives and constraints, such as Gurobi [128], HiGHS [141], and Ipopt [225], etc. We numerically solved a quadratic optimization problem involving the Wasserstein projection with index $\rho = 2$ with the following setting:

$$\min_{p_{j|i}} \sum_{i=1}^{I} \mu_i \left(x_i - \sum_{j=1}^{J} p_{j|i}\, \delta_{y_j} \right)^2$$

$$\text{subject to} \quad \begin{cases} p_{j|i} \geq 0, & \forall i = 1, \dots, I, \quad \forall j = 1, \dots J, \\ \sum_{j=1}^{J} p_{j|i} = 1, & \forall i = 1, \dots, I, \\ \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j & \forall j = 1, \dots J. \end{cases}$$

The objective was to minimize a cost function that captures the squared Euclidean distance between points x_i and their projections based on the conditinal probabilities $p_{j|i}$. Additionally, the problem included quadratic constraints on auxiliary variables z_i, which enforce bounds on the squared distances between the projections and the original points. The optimization was subject to non-negativity, normalization, and marginal consistency constraints on the probabilities.

Experimental Results. We construct the probability measures μ, ν based on Example 5.4.1. Several problem sizes are tested:

$$(I, J) \in \{(20, 20), (100, 100), (200, 200), (500, 500), (1000, 1000)\},$$

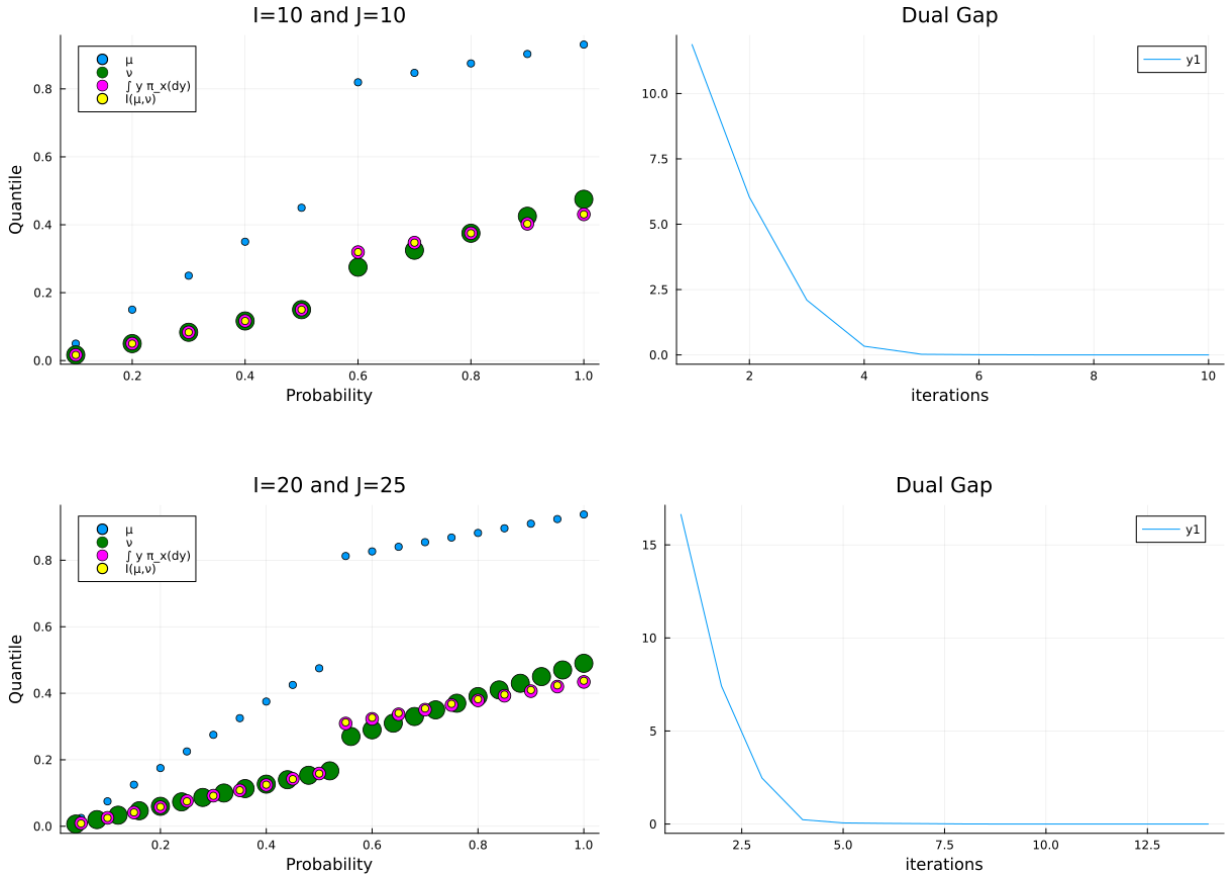
as well as a case where $I = 20$ and $J = 25$ have different values. All the optimizers discussed in this section successfully solved these cases, providing efficient and accurate optimal solutions. Since the objective values obtained by the different solvers are nearly identical, we demonstrated

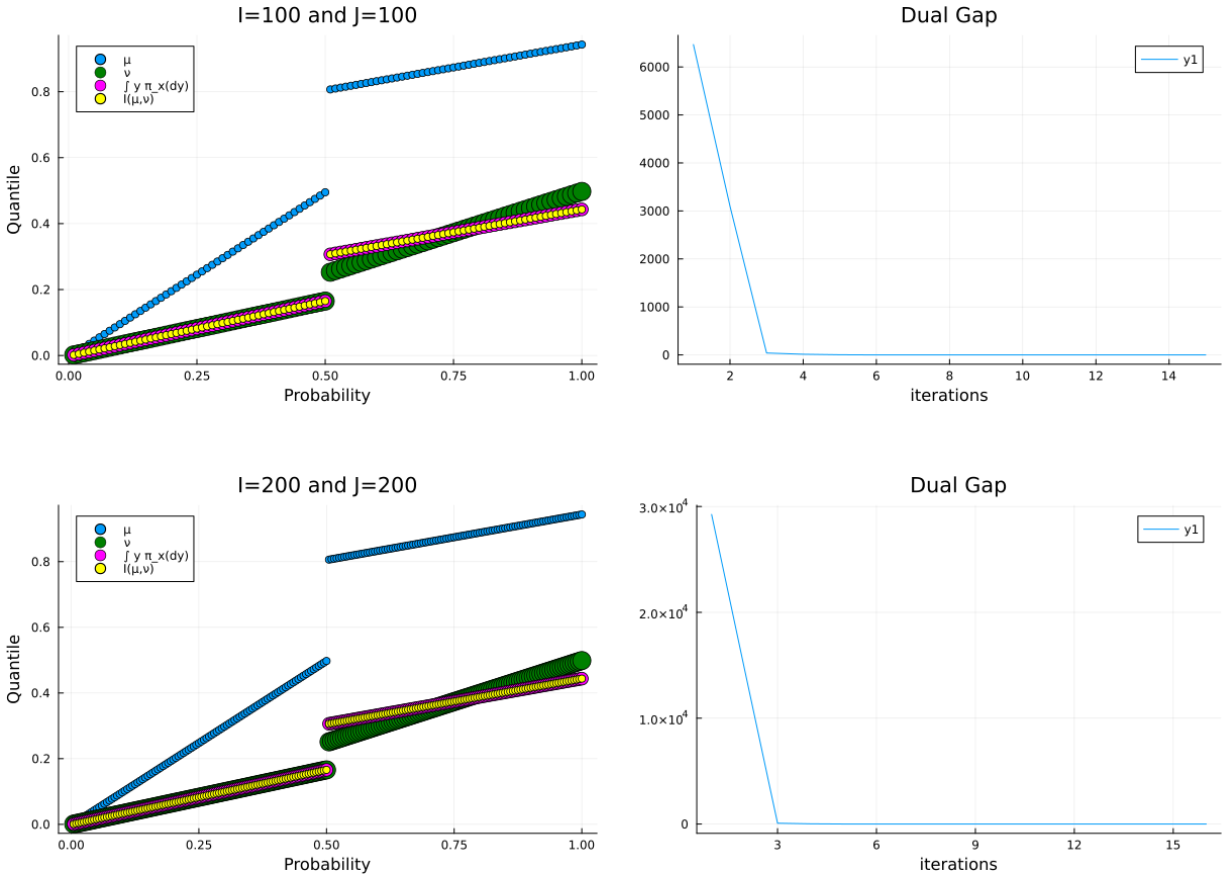
below the results produced by Gurobi [128].

I	J	Objective Value	Dual Gap	Time (s)
10	10	1.43333e-01	2.621401e-7	0.01
20	25	1.45229e-01	6.298232e-7	0.01
100	100	1.43517e-01	2.840158e-7	0.14
200	200	1.43518e-01	2.981635e-8	0.92
500	500	1.43519e-01	2.243310e-7	13.96
1000	1000	1.43518e-01	2.459260e-7	233.99

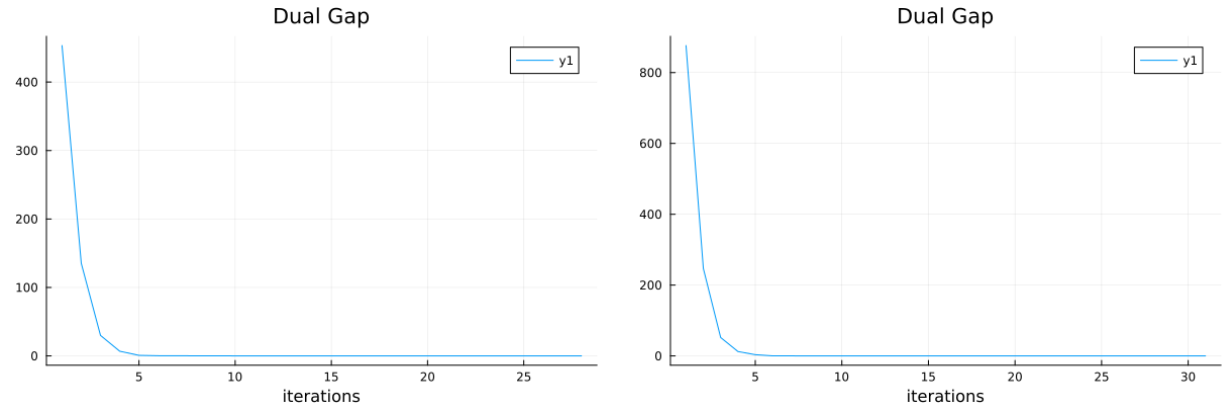
Table 5.1: Results for Wasserstein Projection (WP) with $\rho = 2$.

Our numerical example verified [12, Theorem 2.1] that the value $x \mapsto \int_{\mathbb{R}^d} \pi_x^*(y)\, dy$ is coincides with $\mathcal{I}_\rho(\mu, \nu)$, where π^* is an optimizer of (5.3). We also plot the quantile functions of the distributions $\hat{\mu}$ and $\hat{\nu}$ along each combination of I and J with $\mathcal{I}_\rho(\mu, \nu)$. These plots provided valuable insights into the alignment of the optimized solutions with our theoretical expectations. As the problem size increased, the results remained consistent.





Since we are using scatter plots to present the quantile functions, the plots became somewhat crowded for $I = J = 500$ and $I = J = 1000$. Therefore, for these larger cases, we only plot the minimization of the dual gap over iterations, which provide a clearer representation of the optimization progress. The value of $x \mapsto \int_{\mathbb{R}^d} \pi_x^*(y)\, dy$ and $\mathcal{I}_\rho(\mu, \nu)$ are always coincide in our experiments when $\rho = 2$.



The Wasserstein projection problem for $\rho = 2$ can be solved either using software packages that handle quadratic constraints or via the Frank-Wolfe algorithm, both approaches produce virtually identical objective values. For $(I, J) = (1000, 1000)$, both methods produced accurate solutions, as shown in the results above. However, when the problem size increased to $(I, J) =$

$(1200, 1200)$, we tested the solvers Gurobi [128], HiGHS [141], and Ipopt [225] and the Frank-Wolfe algorithm, all of them encountered issues with the kernel crashing during the computation. This suggests that while these approaches perform well for moderately large problem sizes, further optimization or alternative methods may be needed for significantly larger instances.

5.4.1.3 WP solved by Frank Wolfe algorithm

For $\rho > 2$, the Wasserstein projection problem can be efficiently solved using the Frank-Wolfe algorithm with the following formulation:

$$\min_{p_{j|i}} \sum_{i=1}^{I} \mu_i \left| x_i - \sum_{j=1}^{J} p_{j|i}\, \delta_{y_j} \right|^{\rho}$$

$$\text{subject to} \quad \begin{cases} p_{j|i} \geq 0, & \forall i = 1, \dots, I, \quad \forall j = 1, \dots J, \\ \sum_{j=1}^{J} p_{j|i} = 1, & \forall i = 1, \dots, I, \\ \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j & \forall j = 1, \dots J.. \end{cases}$$

The Frank-Wolfe algorithm is particularly effective for this formulation due to the convexity of the objective when $\rho > 2$, and the fact that the feasible set forms a convex polytope, which aligns well with the Frank-Wolfe method's ability to solve large-scale linear sub-problems efficiently at each iteration.

We applied the Frank-Wolfe algorithm for $\rho = 3, 5, 7, 10, 15$ with $I = J = 30$ and confirmed that the computed solutions match the theoretical expectations. Specifically, the value of the map $x \mapsto \int_{\mathbb{R}^d} \pi_x^*(y)\, dy$ coincides with the metric projection $\mathcal{I}_\rho(\mu, \nu)$, confirming the algorithm's accuracy across different values of ρ.

ρ	Primal Value	Dual Value	Dual Gap	Time (s)	Last Iteration
3	6.71203e-02	6.71189e-02	1.36904e-06	3.12127e+02	7320
5	1.59669e-02	1.59658e-02	1.15611e-06	2.57709e+02	6109
7	3.93540e-03	3.93416e-03	1.24464e-06	1.40058e+02	3347
10	4.89933e-04	4.88619e-04	1.31451e-06	3.95366e+01	952
15	1.60198e-05	1.50128e-05	1.00708e-06	6.61879e+00	165

Table 5.2: Results for Wasserstein Projection solved by Frank Wolfe algorithm.

The resulting plots of the Wasserstein projection for these values of ρ are identical to those obtained in the previous section, illustrating consistent results across different exponents. Additionally, we observed that as ρ increases, the number of iterations required by the algorithm decreases. This reduction is likely due to the objective function becoming smoother and more convex for larger ρ, which facilitates faster convergence.

5.4.2 Stretched Brownian motion

In [23], a Benamou-Brenier type formulation for (MOT) is suggested. In dimension one, this problem consists in maximising for two probability measures $\mu \leq_{cx} \nu$,

$$MT(\mu,\nu) := \sup \mathbb{E}\left[\int_0^1 \sigma_t dt\right] \tag{MBB}$$

over all filtered probability spaces $(\Omega, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$, real-valued $(\mathcal{F}_t)_{t\in[0,1]}$-progressive process $(\sigma_t)_{t\in[0,1]}$ and real-valued $(\mathcal{F}_t)_{t\in[0,1]}$-Brownian motions $(B_t)_{t\in[0,1]}$ such that the process

$$(M_t)_{t\in[0,1]} = \left(M_0 + \int_0^t \sigma_s dB_s\right)_{t\in[0,1]}$$

is a continuous martingale which satisfies $M_0 \sim \mu$ and $M_1 \sim \nu$. When the second moment of ν is finite, then (MBB) has a unique maximiser $(M_t^*)_{t\in[0,1]}$, [23, Theorem 1.5] called the stretched Brownian motion from μ to ν, since it is the martingale subject to the constraints $M_0^* \sim \mu$ and $M_1^* \sim \nu$ which correlates the most with the Brownian motion.

Let $C_2 : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be defined by $C_2(x,p) = W_2^2(p, \mathcal{N}(0,1))$ for all $(x,p) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, where $\mathcal{N}(0,1)$ denotes the uni-dimensional standard normal distribution. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$ and $V_{C_2}^M(\mu,\nu)$ be the value function given by (WMOT) with the cost function C_2. Let $\pi^* \in \Pi_M(\mu,\nu)$ be optimal for $V_{C_2}^M(\mu,\nu)$ and M^* be the stretched Brownian motion from μ to ν. Then, by Remark 2.1, Theorem 2.2 and Remark 2.3 from [23], there exists a direct link between the martingale Benamou-Brenier formulation (MBB) and the weak formulation $V_{C_2}^M(\mu,\nu)$:

(a) $MT(\mu,\nu) = \frac{1}{2}\left(1 + \int_{\mathbb{R}} |y|^2 \nu(dy) - V_{C_2}^M(\mu,\nu)\right)$

(b) π^* is the joint probability distribution of (M_0^*, M_1^*), and conversely

$$M_t^* = \mathbb{E}\left[F^{-1}_{\pi_X^*}\left(F_{\mathcal{N}(0,1)}(B_1)\right) \,\middle|\, X, (B_s)_{0\leq s\leq t}\right],$$

where $X \sim \mu$ is a random variable independent of the Brownian motion $(B_t)_{t\in[0,1]}$.

The discretized $V_{C_2}^M$ can be formulated as:

$$\min_{p_{j|i}} \sum_{i=1}^{I} \mu_i \, C_2\Big(x_i, \sum_{j=1}^{J} p_{j|i}\, \delta_{y_j}\Big)$$

$$\text{subject to} \quad \begin{cases} p_{j|i} \geq 0, & \forall i = 1,\dots,I, \quad \forall j = 1,\dots J, \\ \sum_{j=1}^{J} p_{j|i} = 1, & \forall i = 1,\dots,I, \\ \sum_{j=1}^{J} p_{j|i}\, y_j = x_i, & \forall i = 1,\dots,I, \\ \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j & \forall j = 1,\dots J. \end{cases}$$

According to [3, Theorem 3.9], given that μ is continuous, for a non-convex cost function, one can always replace the original cost by its convex envelope without changing the infimum value of the problem. In this context, the Frank-Wolfe algorithm is an efficient method for solving this problem.

It is well-known that the comonotonic coupling minimizes the Wasserstein distance $\mathcal{W}_\rho(\mu,\nu)$. For any probability measures $\mu,\nu\in\mathcal{P}_\rho(\mathbb{R})$, the Wasserstein distance of order ρ can be expressed in terms of their quantile functions F_μ^{-1} and F_ν^{-1} as:

$$\mathcal{W}_\rho(\mu,\nu)=\left(\int_0^1\left|F_\mu^{-1}(u)-F_\nu^{-1}(u)\right|^\rho du\right)^{1/\rho}=\left\|F_\mu^{-1}-F_\nu^{-1}\right\|_\rho.$$

Let $\gamma\sim\mathcal{N}(0,1)$ denote the standard normal distribution. We can then express the cost function $C_2(x,p)=\mathcal{W}_2^2(p,\gamma)$ explicitly in terms of the quantile functions $F_p^{-1}(u)$ and $F_\gamma^{-1}(u)$. The analytical expression for $C_2(x,p)$ is given by:

$$C_2(x,p)=\mathcal{W}_2^2(p,\gamma)=\int_0^1\left(F_p^{-1}(u)-F_\gamma^{-1}(u)\right)^2 du.$$

This formulation allows us to compute $C_2(x,p)$ directly using the quantile functions, facilitating the analytical approximation of the cost function in our optimization problem.

Denote by $p=\sum_{j=1}^{J} q_j\,\delta_{y_j}$, we have

$$\begin{aligned}
&C_2(x,p)=\mathcal{W}_2^2(p,\gamma)\\
&=\int_0^1\left(F_p^{-1}(u)-F_\gamma^{-1}(u)\right)^2 du\\
&=\sum_{j=1}^{J}\int_{p_{1|i}+\ldots+p_{j-1|i}}^{p_{1|i}+\ldots+p_{j|i}}\left(y_j-F_\gamma^{-1}(u)\right)^2 du\\
&=\sum_{j=1}^{J}p_{j|i}\,y_j^2+1-2\sum_{j=1}^{J}y_j\int_{p_{1|i}+\ldots+p_{j-1|i}}^{p_{1|i}+\ldots+p_{j|i}}F_\gamma^{-1}(u)du\\
&=\sum_{j=1}^{J}p_{j|i}\,y_j^2+1\\
&\quad-\sqrt{\frac{2}{\pi}}\sum_{j=1}^{J}y_j\left(\exp\left\{\frac{\left(F_\gamma^{-1}(p_{1|i}+\ldots+p_{j-1|i})\right)^2}{2}\right\}-\exp\left\{-\frac{\left(F_\gamma^{-1}(p_{1|i}+\ldots+p_{j|i})\right)^2}{2}\right\}\right)\\
&=\sum_{j=1}^{J}p_{j|i}\,y_j^2+1+\sqrt{\frac{2}{\pi}}\,y_1\exp\left\{-\frac{\left(F_\gamma^{-1}(p_{1|i})\right)^2}{2}\right\}-\sqrt{\frac{2}{\pi}}\,y_J\exp\left\{-\frac{\left(F_\gamma^{-1}(p_{1|i}+\ldots+p_{J-1|i})\right)^2}{2}\right\}\\
&\quad-\sqrt{\frac{2}{\pi}}\sum_{j=2}^{J-1}y_j\left(\exp\left\{-\frac{\left(F_\gamma^{-1}(p_{1|i}+\ldots+p_{j-1|i})\right)^2}{2}\right\}-\exp\left\{-\frac{\left(F_\gamma^{-1}(p_{1|i}+\ldots+p_{j|i})\right)^2}{2}\right\}\right).
\end{aligned}$$

Since the Frank-Wolfe algorithm relies on calculating gradients to iteratively optimize the objective function, having an analytical expression for the gradient of C_2 is essential for efficient implementation. For each $k = 1, \dots, J$, the gradient of $C_2(x, p)$ with respect to $p_{k|i}$ is given by:

$$\frac{\partial}{\partial p_{k|i}} C_2(x,p) = y_{k|i}^2 - 2\sum_{j=1}^{J} y_{j|i} \cdot \Delta_{j,k},$$

where $\Delta_{j,k}$ is defined based on the indices j and k as follows:

$$\Delta_{j,k} = \begin{cases} F_\gamma^{-1}(p_{1|i}), & \text{if } j = k = 1, \\ F_\gamma^{-1}(p_{1|i}, \dots, p_{j|i}), & \text{if } 2 \le j \le J-1 \text{ and } k = j, \\ F_\gamma^{-1}(p_{1|i}, \dots, p_{j|i}) - F_\gamma^{-1}(p_{1|i}, \dots, p_{j-1|i}), & \text{if } 2 \le j \le J-1 \text{ and } k < j, \\ -F_\gamma^{-1}(p_{1|i}, \dots, p_{J-1|i}), & \text{if } j = J \text{ and } k < J. \end{cases}$$

The analytical expression involves quantile functions, it is important to account for its behavior at the boundaries where $F_\gamma^{-1}(0) = -\infty$ and $F_\gamma^{-1}(1) = +\infty$. This introduces the challenge of ensuring that all $p_{j|i} > 0$ for each $1 \le i \le I$ and $1 \le j \le J$, as any zero value could destabilize the optimization process. To address this, we must carefully select the initial point. Specifically, for a given $z \in \mathbb{R}$, we consider the following optimization to ensure positivity:

$$\max_{i,j} \; z$$

$$\text{subject to} \quad \begin{cases} p_{j|i} \ge z, & \forall i = 1, \dots, I, \quad \forall j = 1, \dots J, \\ p_{j|i} \ge 0, & \forall i = 1, \dots, I, \quad \forall j = 1, \dots J, \\ \sum_{j=1}^{J} p_{j|i} = 1, & \forall i = 1, \dots, I, \\ \sum_{j=1}^{J} p_{j|i}\, y_j = x_i, & \forall i = 1, \dots, I, \\ \sum_{i=1}^{I} \mu_i\, p_{j|i} = \nu_j & \forall j = 1, \dots J. \end{cases}$$

By solving this auxiliary problem, we obtain a matrix

$$P := (p_{j|i})_{\substack{1 \le i \le I \\ 1 \le j \le J}},$$

whose entries are all strictly positive, and which achieves the minimal value of z subject to the required conditions. with positive entries with the minimal value of z that satisfies the necessary conditions. This matrix P serves as our carefully chosen initial point for the Frank-Wolfe algorithm.

For couples (μ, ν) with log-normal distributions up to re-centering and $\mu \le_{cx} \nu$, we first construct finitely supported approximations $(\hat{\mu}, \hat{\nu})$ such that the convex order $\hat{\mu} \le_{cx} \hat{\nu}$ is pre-

served. We combine random drawings with Baker's construction [92]. For different combination of values of $I = J$ and $n \in \mathbb{N}^*$, we denote by

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(nI)} \quad \text{and} \quad Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(nI)}$$

the order statistics of nI random variables i.i.d. according to μ and ν respectively. We compute

$$\hat{\mu} = \frac{1}{I}\sum_{i=1}^{I} \delta_{\hat{X}_{(i)}} \quad \text{and} \quad \hat{\nu} = \frac{1}{I}\sum_{i=1}^{I} \delta_{\hat{Y}_{(i)}}$$

where,

$$\hat{X}_{(i)} = \frac{1}{n}\sum_{j=1}^{n} X_{(n(i-1)+j)} - \frac{1}{nI}\sum_{j=1}^{nI} X_{(j)}, \qquad \text{for } 1 \leq i \leq I,$$
$$\hat{Y}_{(i)} = \frac{1}{n}\sum_{j=1}^{n} Y_{(n(i-1)+j)} - \frac{1}{nI}\sum_{j=1}^{nI} Y_{(j)}, \qquad \text{for } 1 \leq i \leq I.$$

For this numerical experiment, we consider two log-normal distributions $\mu = \exp\sharp\mathcal{N}(0, 0.24)$ and $\nu = \exp\sharp\mathcal{N}(-0.0104, 0.28)$ after re-centering, we apply Baker's method as described earlier to generate the empirical measures $\hat{\mu}$ and $\hat{\nu}$. We conducted experiments with varying problem sizes $(I, J) \in \{(20, 20), (30, 30), (50, 50)(100, 100)\}$.

I, J	Primal Value	Dual Value	Dual Gap	Time (s)	Last Iteration
20	1.897068	1.897057	1.01915e-05	1.41408e+02	18857
30	1.733103	1.733093	1.02563e-05	3.74879e+02	15385
50	1.819929	1.819914	1.45341e-05	2.05579e+03	20001
100	1.837175	1.837160	1.51842e-05	2.68000e+04	20001

Table 5.3: Results for Stretched Brownian motion solved by Frank Wolfe algorithm.

The results are summarized in Table. 5.3, demonstrate that the Frank-Wolfe algorithm effectively converges to optimal solutions across all tested sizes. The primal and dual values remain nearly identical as the dual gap remains consistently small. As the problem size increases, both the computational time and the number of iterations required grow, reflecting the algorithm's scalability.

We visualize the optimal martingale coupling using 3D plots with Fig. 5.7. Let π^* denote the minimizer obtained for the cost function C_2. The axes of each plot are defined as follows:

- The x-axis represents the values of $\hat{X}_i$ for $i = 1, \ldots, I$.
- The y-axis represents the values of $\hat{Y}_j$ for $j = 1, \ldots, J$.
- The z-axis displays the values of the coupling matrix $\mu(dx)\pi^*_x(dy)$.

Fig. 5.7 presents the 3D visualizations of the optimal martingale couplings for different problem sizes.

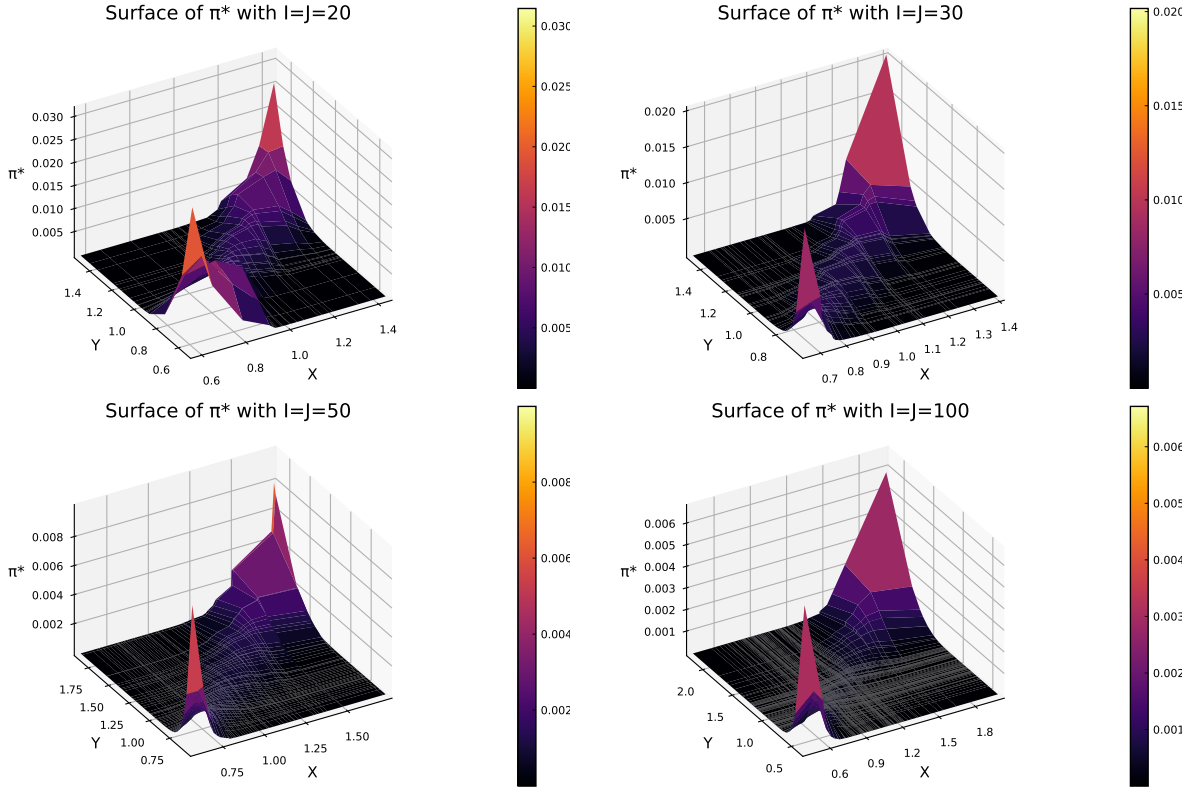


Figure 5.7: Optimal martingale coupling $\mu(dx)\pi_x^*(dy)$ for different problem sizes.

Chapter 6

Extended graphon mean field games in discrete time

Abstract

We study games involving a continuum of heterogeneous players in the discrete-time setting with finite state spaces and continuous action spaces. We introduce a new model that incorporates joint state-action interactions within the graphon aggregate, described by a coupled forward-backward system. We rigorously establish the existence of solutions to this system and demonstrate a one-to-one correspondence between *Graphon Nash equilibrium* (GNE) and the solutions of the forward-backward system. Additionally, we prove the uniqueness of the GNE under various structural assumptions. To illustrate the practical relevance of our framework, we provide an example of optimal investment with a relative performance criterion and solve it numerically using fixed-point iterations.

Contents

6.1 Introduction . . . **147**
6.2 Graphon mean field game model . . . **149**
6.2.1 Setup and preliminaries . . . 149
6.2.2 Graphons . . . 149
6.2.3 Model . . . 150
6.2.4 Assumptions . . . 151
6.3 Link between Graphon Nash Equilibrium (GNE) and forward-backward system . . . **152**
6.4 Existence of equilibrium . . . **153**
6.4.1 Equilibrium condition and existence . . . 153
6.4.2 Proofs . . . 154
6.5 Uniqueness Results . . . **168**
6.5.1 Uniqueness by monotonicity . . . 168
6.5.2 Uniqueness by contraction . . . 170
6.6 Approximate Equilibrium in Finite Player Games . . . **172**
6.6.1 From GNE to N-player Nash equilibrium . . . 173
6.6.2 Useful lemmas . . . 175
6.6.3 Proofs of the lemmas in Section 6.6 . . . 176
6.7 Application in finance - optimal investment with relative performance criterion . . . **185**
6.7.1 Model . . . 185
6.7.2 Numerical results . . . 191

6.1 Introduction

Mean Field Games Many real-world applications involve multi-agent systems such as flocking [189], epidemiology [84], autonomous driving [138], and financial systems [71]. In these systems, each agent aims to optimize her own cumulative rewards by selecting sequential actions over finite or infinite time horizons, while simultaneously interacting and adapting to the strategic behavior of others. From the aspect of learning, one can refer to [168, 68, 232, 127, 102, 190, 189, 188, 169]. In general, studying such systems is extremely challenging when the number of agents is very large. *Mean Field Games* (MFGs) were introduced independently by Lasry and Lions [165] and Caines, Huang, and Malhamé [139] to analyze games involving an infinite number of players under assumptions of symmetry and indistinguishability. An MFG describes the limiting behavior of finite-player games as the number of players $N \to +\infty$. The central objective is to determine a *Nash equilibrium* (NE) – a situation where no agent can improve their individual outcome by unilaterally changing their action. For more details, we refer to Lions' lectures at Collège de France [175] and the notes by Cardaliaguet [69].

Graphon Mean Field Games In the classical MFG framework, each player interacts with the rest of the population through the probability distribution of the population. This means that each player perceives the influence of every other player in the same way, independently of the identity of that player. However, in more realistic settings, player interactions are often heterogeneous, there is a necessary use of graphs to represent these complex relationships. In this context, interactions are captured by weighted graphs, where vertices correspond to players, and edge weights quantify the intensity of interactions between them. Each player responds to an interaction-weighted average of others' empirical state measures. As the number of players goes to infinity, this framework generalizes to incorporate graphons with MFG, the concept of *Graphon Mean Field Games* (GMFG) was introduced, and it has attracted a growing attention from different groups of researchers [63, 112, 72, 19, 83, 219, 170].

GMFGs offer significantly greater flexibility compared to classical MFGs, which enable applications across diverse fields such as epidemic control [20, 171, 17], cyber-security [159, 16], and finance [219, 170, 113, 75]. Unlike MFGs, where interactions are homogeneous and uniform, GMFGs account for heterogeneous interactions on large, complex networks. Given that equilibrium solutions for GMFGs are often intractable, recent research efforts have increasingly focused on learning-based methods to approximate equilibria. This active area of study includes contributions on scalable algorithms and convergence guarantees [78, 79, 226, 83, 234].

Static Graphon Games GMFGS can be classified based on their time scale and state space. According to Lovász et al. [177, 55, 56], graphons serve as continuum limits for large dense graphs. Incorporating the convergence results of graphons in large population analysis, Parise and Ozdaglar [185] introduced a new class of static infinite-player games, known as *graphon games*. They established that the NE of static graphon games serves as a good approximation for the NE of finite network games when the underlying graphs are sampled from the graphon. For additional contributions to static graphon games, readers can refer to Carmona et al. [72]

and subsequent work by Parise and Ozdaglar [186].

Continuous-Time Graphon Games Dynamic stochastic graphon games can be further divided into continuous-time and discrete-time frameworks. In the continuous-time setting, much of the literature focuses on continuous state spaces. Foundational work in this area includes contributions by Bayraktar et al. [31, 33, 32] and Caines et al. [65, 64, 63]. These studies analyze classical results such as the existence and uniqueness of *Graphon Nash Equilibrium* (GNE) and the convergence of ε-NE to GNE. Among continuous-time graphon games, the *linear-quadratic (LQ) setting* has gained particular attention. Gao et al. [112] derived approximate NE for finite-player games on large graphs under the LQ setting. This work was later extended and refined by Aurell et al. [19], Lacker and Soret [164], and Hu et al. [137].

In the continuous-time and discrete-state space framework, Aurell et al. [20] conducted a theoretical analysis of finite-state graphon games with applications to epidemic dynamics. Their work also propose a numerical approach based on machine learning methods.

Discrete-Time Graphon Games and Related Work In the discrete-time setting, Cui et al. [83] analyzed GMFGs with continuum players, where both the state and action spaces are finite. They demonstrated the existence of Nash equilibria and approximate equilibria under the assumption of Lipschitz continuity in transition kernels and graphons. Fabian et al. [103] extended this analysis to sparse GMFGs within the same framework. Zhang et al. [234] studied GMFGs with entropic regularization for continuum players in finite state and action spaces. Under the assumption of oracle access for population dynamics and optimal policies, Zhang et al. [234] derived convergence rates for their proposed algorithms, whereas Cui et al. [83] and Fabian et al. [103] established only asymptotic convergence. On the other hand, Zhou et al. [235] shifted focus to a representative-player framework on $\mathbb{R}^d$ with a compact action space. They provided a comprehensive theoretical analysis, proving the existence, uniqueness, and approximate equilibrium of solutions in this setting.

Our Contributions In this paper, we perform a theoretical analysis of GMFGs in the discrete-time setting, where the state space is finite and the action space is continuous. Additionally, we incorporate joint state-action interactions within the graphon aggregate. The model is formulated as a coupled forward-backward system, introduced in Section 6.2. In Section 6.4, we establish the existence of solutions to this system. We further demonstrate in Section 6.3 a one-to-one correspondence between GNE and the solutions of the forward-backward system. In Section 6.5, we prove that the uniqueness of the GNE holds under various assumptions. Finally, we present an example of optimal investment with a relative performance criterion and solve it numerically using fixed-point iterations in Section 6.7.

6.2 Graphon mean field game model

6.2.1 Setup and preliminaries

The following notation will be adopted throughout this paper. We focus on discrete time setting with finite horizon $T > 0$ and denote by $\mathcal{T} := \{0, 1, \dots, T-1\}$. Let $\overline{\mathcal{T}} = \mathcal{T} \cup \{T\} = \{0, 1, \dots, T\}$. For a random variable X taking values in a measurable space, we write $\mathcal{L}(X)$ for its law.

We denote by $\|\cdot\|$ the Euclidean norm.

Let $I := [0, 1]$ be the unit interval equipped with the Euclidean distance. We denote the Lebesgue measure and Borel σ-field on I by $\mathrm{Unif}(I)$ and $\mathcal{B}(I)$ respectively. The set I is an index set labeling continuum of players in the graphon game.

Given two sets A and B, we denote by A^B the set of functions from B to A.

Given an N-tuple $(x^1, \dots, x^N)$ and an index $i \in [N]$, we denote by x^{-i} the $(N-1)$-tuple of the x^j with $j \neq i$ and $j \in [N]$. Furthermore, with a slight abuse of notation, we identify (x^i, x^{-i}) with the N-tuple $(x^1, \dots, x^{i-1}, x^i, x^{i+1}, \dots, x^N)$.

6.2.2 Graphons

A graphon is defined as a symmetric measurable function $G : I \times I \to I$, which is regarded as the limits of edge matrices of weighted graphs when the size of graphs (number of vertices) tends to infinity. We denote $\mathcal{W}$ the space of graphons.

We can also view a graphon as an operator from $L^\infty(I)$ to $L^1(I)$, associating any $\phi \in L^1(I)$ with:

$$G\phi(u) := \int_I G(u, v)\phi(v)dv.$$

If $G \in L^p(I \times I)$, we denote by $\|G\|_p = \left(\int_{I\times I} G(u, v)^p du dv\right)^{1/p}$ its $L^p(I \times I)$ norm.

The cut norm of a graphon is defined by

$$\|G\|_\square := \sup_{A,B\in\mathcal{B}(I)} \left|\int_{A\times B} G(u, v)\, du\, dv\right|.$$

For any simple graph $\mathbb{G} = (V, E)$ with sets of vertices and edges $V = \{1, 2, \dots, N\}$ and $E \subseteq V \times V$ respectively, we define its corresponding step-graphon as:

$$G_N(u, v) = \sum_{i,j\in V} \mathbb{1}_{(i,j)\in E}\, \mathbb{1}_{u\in\left(\frac{i-1}{N}, \frac{i}{N}\right]}\, \mathbb{1}_{v\in\left(\frac{j-1}{N}, \frac{j}{N}\right]}. \tag{6.1}$$

By Lovász [177, Lemma 8.11], the resulting operator norm turns out to be equivalent to the cut norm

$$\|G\|_\square \leq \|G\|_{\infty\to 1} \leq 4\|G\|_\square,$$

with

$$\|G\|_{\infty\to 1} := \sup_{|\phi|\leq 1} \|G\phi\|_{L^1}.$$

When studying the connection with N-player games, we will consider sequences of graphs and their associated step graphons. We use the following assumption.

Assumption 6.2.1. *The sequence of step-graphons $(G_N)_{N\in\mathbb{N}}$ converges in cut norm $\|\cdot\|_\square$ or equivalently in operator norm $\|\cdot\|_{\infty\to 1}$ as $N\to\infty$ to some graphon $G\in\mathcal{W}$ as $N\to+\infty$, i.e.*

$$\|G_N - G\|_\square \to 0 \qquad \text{and} \qquad \|G_N - G\|_{\infty\to 1} \to 0.$$

By [177, Theorem 11.59], a sufficient condition for the above assumption to be satisfied is that the sequence of step graphons stems from a converging sequence of graphs: if $(\mathbb{G}_N)_{N\in\mathbb{N}}$ is a sequence of graphs such that $\mathbb{G}_N \to G$, then $\mathbb{G}_N$ can be labeled as step graphon G_N so that $\|G_N - G\|_\square \to 0$.

6.2.3 Model

We first introduce the model.

Definition 6.2.2. *A (finite-horizon, discrete time) Graphon Mean Field Games (GMFG) is defined by a tuple $(\overline{\mathcal{T}}, G, \mathcal{X}, \mathcal{A}, p, r, \varphi, \mu_0)$ where*

- *$\overline{\mathcal{T}} = \{0,\dots,T\}$ is a set of discrete time steps,*
- *G is a graphon,*
- *$\mathcal{X}$ is a finite state space and $\mathcal{A}$ is an action set, which is a convex and compact subset of $\mathbb{R}^k$ for some integer $k>0$; we let $C_{\mathcal{A}}>0$ be a bound on the norm of the elements of $\mathcal{A}$,*
- *$P:\mathcal{X}\times\mathcal{A}\times\mathbb{R}\to\mathcal{P}(\mathcal{X})$ is a one-step transition probability kernel,*
- *$r:\mathcal{X}\times\mathcal{A}\times\mathbb{R}\to\mathbb{R}$ is a reward function,*
- *$\varphi:\mathcal{X}\times\mathcal{A}\to\mathbb{R}$ is a measurable function for the interactions,*
- *$\mu_0^u := \mu_0 \in \mathcal{P}(\mathcal{X})$ is an initial distribution for all $u\in I$.*

We use the following notation:

- $\mathcal{M} := L^2(I;\mathcal{P}(\mathcal{X}))^{\overline{\mathcal{T}}}$ denotes the set of graphon mean fields.
- $\Pi := L^2(I;\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}}$ denotes the set of graphon policies.
- $\mathcal{Z} := L^2(I)^{\overline{\mathcal{T}}}$ denotes the set of graphon aggregates.

A mean field $\mu\in\mathcal{P}(\mathcal{X})$, $(u,t,x)\mapsto\mu_t^u(x) := \mu(u,t,x)$ gives a probability distribution over states such that

$$\begin{aligned}
&\forall\,(t,x)\in\overline{\mathcal{T}}\times\mathcal{X}, \quad \left(\int_I |\mu_t^u(x)|^2\,du\right) < +\infty,\\
&\forall\,(t,u)\in\overline{\mathcal{T}}\times I, \quad \sum_{x\in\mathcal{X}}\mu_t^u(x) = 1.
\end{aligned}$$

A policy $\pi \in \Pi$, $(u,t,x) \mapsto \pi_t^u(x) := \pi(u,t,x) \in \mathcal{A}$ provides an action given each state x at time t and $u \in I$.

Definition 6.2.3. *Given $\pi \in \Pi$, the mean field $\mu^\pi \in \mathcal{M}$ and the aggregate $Z^\pi \in \mathcal{Z}$ are defined as follows: For all $u \in I$,*

$$\begin{cases} \mu_0^{\pi,u}(x) = \mu_0(x), & x \in \mathcal{X} \\ Z_0^{\pi,u} = \displaystyle\int_I G(u,v) \sum_{x \in \mathcal{X}} \varphi\left(x, \pi_0^v(x)\right) \mu_0^{\pi,v}(x)\, dv, \end{cases} \tag{6.2}$$

and for every $t \in \mathcal{T}$, for all $u \in I$,

$$\begin{cases} \mu_{t+1}^{\pi,u}(x) = \displaystyle\sum_{x' \in \mathcal{X}} \mu_t^{\pi,u}(x') P(x \mid x', \pi_t^u(x'), Z_t^{\pi,u}), & x \in \mathcal{X}, \\ Z_{t+1}^{\pi,u} = \displaystyle\int_I G(u,v) \sum_{x \in \mathcal{X}} \varphi\left(x, \pi_{t+1}^v(x)\right) \mu_{t+1}^{\pi,v}(x)\, dv. \end{cases} \tag{6.3}$$

The total reward of player $u \in I$ in the GMFG setting, given an aggregate $Z^u \in \mathcal{Z}$, is:

$$J(\pi^u; Z^u) = \mathbb{E}\left[\sum_{t \in \mathcal{T}} r(X_t^u, a_t^u, Z_t^u)\right], \tag{6.4}$$

subject to

$$X_0^u \sim \mu_0, \qquad a_t^u = \pi_t^u(X_t^u), \qquad X_{t+1}^u \sim P(\cdot \mid X_t^u, a_t^u, Z_t^u), \qquad \forall\, t \in \mathcal{T}. \tag{6.5}$$

Definition 6.2.4 (Graphon MFG Nash Equilibrium)**.** *A policy profile $\hat{\pi} \in \Pi$ is a Graphon Nash equilibrium (GNE) if, for every $u \in I$, $\hat{\pi}^u$ maximizes $J(\,\cdot\,; Z^{\hat{\pi},u})$. Alternatively, a pair $(\hat{\pi}, \hat{Z}) \in \Pi \times \mathcal{Z}$ of policy and aggregate is a (graphon MFG) Nash equilibrium if the following two conditions are satisfied:*

$$J(\hat{\pi}^u; \hat{Z}^u) \geq J(\pi^u; \hat{Z}^u), \qquad \forall\, u \in I, \pi^u \in \Pi,$$

and

$$\hat{Z}^u = Z^{\hat{\pi},u}.$$

6.2.4 Assumptions

We use the following assumptions.

Assumption 6.2.5. *We assume that the graphon $G : I \times I \to I$ is continuous.*

Let us denote

$$f(Z,V,x,a) := r(x,a,Z) + \sum_{x' \in \mathcal{X}} P(x' \mid x,a,Z)\, V(x'), \qquad (Z,V,x,a) \in \mathbb{R} \times \mathbb{R}^{\mathcal{X}} \times \mathcal{X} \times \mathcal{A}. \tag{6.6}$$

Assumption 6.2.6. *Let us assume the following:*

(1) *r and P are twice differentiable with respect to a, and there exists $C_r, C_P, \epsilon_r > 0$ such that for all $(x, a, Z) \in \mathcal{X} \times \mathcal{A} \times \mathbb{R}$,*

- $|\, r(x,a,Z)\,| \le C_r$.
- $\left| \dfrac{\partial^2}{\partial a^2} P(\cdot \mid x, a, Z) \right| \le C_P$.
- $\dfrac{\partial^2}{\partial a^2} r(x,a,Z) < -C_P\, C_r |\mathcal{X}|\, T - \epsilon_r$.

(2) *$(Z, V) \mapsto \partial_a f(Z, V, x, a)$ is Lipschitz continuous, uniformly in $a \in \mathcal{A}$ and $x \in \mathcal{X}$, i.e. for $(Z, V), (\tilde{Z}, \tilde{V}) \in \mathbb{R} \times \mathbb{R}^{\mathcal{X}}$: there exists $L_{f,Z}, L_{f,V}$ such that:*

$$\max_{a \in \mathcal{A},\, x \in \mathcal{X}} \left| \partial_a f(Z, V, x, a) - \partial_a f(\tilde{Z}, \tilde{V}, x, a) \right| \le L_{f,Z} \left| Z - \tilde{Z} \right| + L_{f,V} \max_x \left| V(x) - \tilde{V}(x) \right|.$$

Assumption 6.2.7. *For every $x \in \mathcal{X}, a \in \mathcal{A}, Z \in \mathbb{R}$, we assume*

(1) *the function $\mathbb{R} \ni Z \mapsto P(\cdot \,|\, x, a, Z)$ is Lipschitz continuous with Lipschitz constant $L_{P,Z}$, uniformly in x, a.*

(2) *the function $\mathcal{A} \ni a \mapsto P(\cdot \,|\, x, a, Z)$ is Lipschitz continuous with Lipschitz constant $L_{P,a}$, uniformly in x, Z.*

(3) *the function $\mathbb{R} \ni Z \mapsto r(x, a, Z)$ is Lipschitz continuous with Lipschitz constant $L_{r,Z}$, uniformly in x, a.*

Assumption 6.2.8. *There exist constants $L_\varphi, C_\varphi > 0$ such that for all $a, a' \in \mathcal{A}$ and $x \in \mathcal{X}$, we have*

$$|\, \varphi(x,a) - \varphi(x,a')| \le L_\varphi \|a - a'\|_{\mathcal{A}} \quad \text{and} \quad |\varphi(x,a)| \le C_\varphi.$$

Assumption 6.2.9. *We assume that $C_\Phi := L_\varphi \, \|G\|_2 \, L_{a,Z} < 1$.*

For the approximate result, we shall also need the following assumption.

Assumption 6.2.10. *For every $x \in \mathcal{X}$, $t \in \overline{\mathcal{T}}$ and $\pi \in \Pi$, $I \ni u \mapsto \pi_t^u(x)$ is Lipschitz continuous with Lipschitz constant L_π.*

6.3 Link between Graphon Nash Equilibrium (GNE) and forward-backward system

Theorem 6.3.1. *$(\hat{\pi}, \hat{Z}) \in \Pi \times \mathcal{Z}_{C_\varphi}$ is a graphon MFG Nash equilibrium if and only if it satisfies the forward-backward system (6.7).*

Proof of Theorem 6.3.1. **From GNE to FB system.**

Assume $(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{Z}})$ is a (graphon MFG) Nash equilibrium. By Definition 6.2.4, we have the following two properties:

1. For every u, $\hat{\pi}^u$ is an optimal policy for player of index u against aggregate $\hat{Z}^u$. So, by the theory of MDPs (see e.g. [50, Lemma 8.7, p 206]), $\hat{\pi}^u$ satisfies:

$$\hat{\pi}_t^u(x) = \underset{a\in\mathcal{A}}{\operatorname{argmax}} \left\{ r(x,a,\hat{Z}_t^u) + \sum_{x'\in\mathcal{X}} P(x' \mid x,a,\hat{Z}_t^u)\, \hat{V}_{t+1}^u(x') \right\}, \qquad \forall x\in\mathcal{X}, t\in\mathcal{T},$$

where $\hat{V}^u$ satisfies: $\hat{V}_T^u(x) = 0$ for all $x\in\mathcal{X}$ and

$$\hat{V}_t^u(x) = \max_{a\in\mathcal{A}} \left\{ r(x,a,\hat{Z}_t^u) + \sum_{x'\in\mathcal{X}} P(x' \mid x,a,\hat{Z}_t^u)\, \hat{V}_{t+1}^u(x') \right\}, \qquad \forall x\in\mathcal{X}, t\in\mathcal{T}.$$

2. Moreover, $\hat{Z}^u = Z^{\hat{\pi},u}$, using the notation introduced in Definition 6.2.3. By (6.3), it means that $\hat{Z}^u$ satisfies:

$$\begin{cases} \mu_{t+1}^{\hat{\pi},u}(x) = \sum_{x'\in\mathcal{X}} \mu_t^{\hat{\pi},u}(x') P(x \mid x', \hat{\pi}_t^u(x'), \hat{Z}^u), & x\in\mathcal{X}, \\ \hat{Z}^u = \int_I G(u,v) \sum_{x\in\mathcal{X}} \varphi\left(x, \hat{\pi}_{t+1}^v(x)\right)\, \mu_{t+1}^{\hat{\pi},v}(x)\, dv. \end{cases}$$

Combining the above equations, we obtain the FB system (6.7).

From FB system to GNE. Let (μ, V) of the FB system (6.7) be the solution to the FB system, with associated (π, Z). By the equations satisfied by V and π, we deduce that π is a best response against Z, see e.g. [50, Lemma 8.7, p206]). By the equations satisfied by μ and Z, we see that $Z = Z^\pi$, i.e., it is the aggregate generated by π. We thus find that the conditions in Definition 6.2.4 are satisfied. So (π, Z) is a GNE.

□

6.4 Existence of equilibrium

6.4.1 Equilibrium condition and existence

The following result shows that the forward-backward system admits a solution.

Theorem 6.4.1. *Let μ_0^u be a probability distribution on $\mathcal{X}$. Let G be a graphon satisfying Assumption 6.2.5. Under Assumption 6.2.6 and 6.2.7 on r and P, Assumption 6.2.8 on φ, and*

Assumption 6.2.9, the system

$$\begin{cases} \mu^u_{t+1}(x) = \sum_{x' \in \mathcal{X}} \mu^u_t(x') P(x \mid x', \pi^u_t(x'), Z^u_t), & \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T} \\ V^u_t(x) = \max_{a \in \mathcal{A}} \left\{ r(x, a, Z^u_t) + \sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z^u_t)\, V^u_{t+1}(x') \right\}, & \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T} \\ Z^u_t = \int_I G(u, v) \sum_{x \in \mathcal{X}} \varphi\left(x, \pi^v_t(x)\right)\, \mu^v_t(x)\, dv, & \forall u \in I, t \in \mathcal{T} \\ \pi^u_t(x) = \operatorname{argmax}_{a \in \mathcal{A}} \left\{ r(x, a, Z^u_t) + \sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z^u_t)\, V^u_{t+1}(x') \right\}, & \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T} \\ \mu^u_0(x) = \mu_0(x),\ V^u_T(x) = 0 \qquad \forall x \in \mathcal{X}, u \in I. \end{cases} \tag{6.7}$$

admits a bounded solution (V, μ) in $L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ such that for each $u \in I$, μ_0 is a probability distribution function on $\mathcal{X}$.

The proof of Theorem 6.4.1 is provided in Section 6.4.2.

6.4.2 Proofs

The proof of Theorem 6.4.1, given in Section 6.4.2.2, relies on definitions and lemmas given in Section 6.4.2.1.

6.4.2.1 Useful lemmas

Definition 6.4.2. *We endow $L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ with the following norm:*

$$\|\psi\|_{\infty,\infty,2} = \max_{t \in \overline{\mathcal{T}}, x \in \mathcal{X}} \sqrt{\int_I \psi^u_t(x)^2 du}, \qquad \psi \in L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}. \tag{6.8}$$

We endow $\mathcal{Z}$ with the following norm:

$$\|\psi\|_{\infty,2} = \max_{t \in \overline{\mathcal{T}}} \sqrt{\int_I |\psi^u_t|^2\, du}, \qquad \psi \in \mathcal{Z}. \tag{6.9}$$

Definition 6.4.3. *Let us define*

$$\mathcal{Z}_{C_\varphi} := \left\{ \psi \in \mathcal{Z} : \|\psi\|_{\infty,2} \le C_\varphi \right\}, \tag{6.10}$$

where C_φ is the same constant as in Assumption 6.2.8.

Lemma 6.4.4. *The function $\mathcal{A} \ni a \mapsto f(Z, V, x, a)$ is continuously differentiable and strongly concave in a, uniformly in $\theta := (Z, V, x) \in \mathbb{R} \times [-TC_r, TC_r]^{\mathcal{X}} \times \mathcal{X}$, i.e. there exists a constant $\lambda > 0$ such that for every $a, \tilde{a} \in \mathcal{A}$ and $\epsilon \in [0, 1]$:*

$$f(\theta, \epsilon a + (1-\epsilon)\tilde{a}) \ge \epsilon f(\theta, a) + (1-\epsilon) f(\theta, \tilde{a}) + \frac{\lambda}{2} \epsilon(1-\epsilon)|a - \tilde{a}|^2, \quad \forall\, \theta \in \mathbb{R} \times [-TC_r, TC_r]^{\mathcal{X}} \times \mathcal{X}.$$

Proof of Lemma 6.4.4. We want to show that $\partial^2_{aa} f(\theta, a) \le -\lambda < 0$ for some $\lambda > 0$, uniformly in $\theta = (Z, V, x)$. Taking the second derivative of f with respect to a,

$$\frac{\partial^2}{\partial a^2} f(\theta, a) = \frac{\partial^2}{\partial a^2} r(x, a, Z) + \frac{\partial^2}{\partial a^2} \Big[\sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z)\, V(x') \Big].$$

By Assumption 6.2.6 (1), there is a constant $C_P > 0$ such that for all x, a, Z,

$$\left| \frac{\partial^2}{\partial a^2} P(\cdot \mid x, a, Z) \right| \le C_P,$$

then, by the boundness of V and P and triangle inequality,

$$\begin{aligned} \left| \frac{\partial^2}{\partial a^2} \sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z)\, V(x') \right| &= \left| \sum_{x' \in \mathcal{X}} \frac{\partial^2}{\partial a^2} P(x' \mid x, a, Z)\, V(x') \right| \\ &\le \sum_{x' \in \mathcal{X}} \left| \frac{\partial^2}{\partial a^2} P(x') \right| \cdot |V(x')| \\ &\le T\, C_r\, C_P\, |\mathcal{X}|. \end{aligned}$$

Combining the bound of second derivative of the reward function in Assumption 6.2.6 (1),

$$\begin{aligned} \frac{\partial^2}{\partial a^2} f(\theta, a) &= \frac{\partial^2}{\partial a^2} r(x, a, Z) + \frac{\partial^2}{\partial a^2} \Big[\sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z)\, V(x') \Big] \\ &\le -T\, C_r\, C_P\, |\mathcal{X}| - \epsilon_r + T\, C_r\, C_P\, |\mathcal{X}| = -\epsilon_r, \end{aligned}$$

the statement is satisfied with $\lambda = \epsilon_r$. □

Lemma 6.4.5. *For all $(Z, V, x) \in \mathbb{R} \times [-TC_r, TC_r]^{\mathcal{X}} \times \mathcal{X}$, $\mathcal{A} \ni a \mapsto f(Z, V, x, a)$ admits a unique maximizer, which we denote by $\hat{a}(Z, V, x)$. Moreover, $\mathbb{R} \times [-TC_r, TC_r]^{\mathcal{X}} \ni (Z, V) \mapsto \hat{a}(Z, V, x) \in \mathcal{A}$ is Lipschitz continuous:*

$$\max_{x \in \mathcal{X}} \left| \hat{a}(Z, V, x) - \hat{a}(\tilde{Z}, \tilde{V}, x) \right| \le L_{a,Z} \left| Z - \tilde{Z} \right| + L_{a,V} \max_{x \in \mathcal{X}} \left| V(x) - \tilde{V}(x) \right|.$$

Proof of Lemma 6.4.5. Due to the strong concavity of f shown in Lemma 6.4.4, there exists a unique solution $\hat{a}(Z, V, x)$ of the following variational inequality (where a is the unknown variable):

$$\forall\, b \in \mathcal{A}, \qquad (b - a)\, \partial_a f(Z, V, x, a) \le 0.$$

Let $\theta := (Z, V, x)$ and $\tilde{\theta} := (\tilde{Z}, \tilde{V}, x)$, we denote $a = \hat{a}(\theta)$ and $\tilde{a} = \hat{a}(\tilde{\theta})$. By the inequality above, we have

$$(\tilde{a} - a)\, \partial_a f(\theta, a) \le 0 \quad \text{and} \quad (a - \tilde{a})\, \partial_a f(\tilde{\theta}, \tilde{a}) \le 0,$$

so that

$$(\tilde{a}-a)\left[\partial_a f(\theta,a)-\partial_a f(\tilde{\theta},\tilde{a})\right]\leq 0$$

hence:

$$(\tilde{a}-a)\left[\partial_a f(\theta,a)-\partial_a f(\theta,\tilde{a})+\partial_a f(\theta,\tilde{a})-\partial_a f(\tilde{\theta},\tilde{a})\right]\leq 0$$

and we deduce:

$$(\tilde{a}-a)\left[\partial_a f(\theta,a)-\partial_a f(\theta,\tilde{a})\right]\leq(\tilde{a}-a)\left[\partial_a f(\tilde{\theta},\tilde{a})-\partial_a f(\theta,\tilde{a})\right].$$

By the strong concavity of f [179, Lemma A.1.], we have $\lambda|\tilde{a}-a|^2\leq(\tilde{a}-a)[\partial_a f(\theta,a)-\partial_a f(\theta,\tilde{a})]$, combine the inequality above, we deduce

$$\lambda|\tilde{a}-a|^2\leq(\tilde{a}-a)[\partial_a f(\theta,a)-\partial_a f(\theta,\tilde{a})]\leq(\tilde{a}-a)[\partial_a f(\tilde{\theta},\tilde{a})-\partial_a f(\theta,\tilde{a})].$$

By the Assumption 6.2.6 (2), we have

$$\max_{x\in\mathcal{X}}|\hat{a}(Z,V,x)-\hat{a}(\tilde{Z},\tilde{V},x)|\leq\frac{L_{f,Z}}{\lambda}\left|Z-\tilde{Z}\right|+\frac{L_{f,V}}{\lambda}\max_{x\in\mathcal{X}}\left|V-\tilde{V}\right|.$$

We take $L_{a,Z}=\frac{L_{f,Z}}{\lambda}$ and $L_{a,V}=\frac{L_{f,V}}{\lambda}$, then we conclude the proof. □

Corollary 6.4.6. *According to Lemma 6.4.5, for all $(Z,V,x)\in\mathbb{R}\times[-TC_r,TC_r]^{\mathcal{X}}\times\mathcal{X}$, $\mathcal{A}\ni a\mapsto f(Z,V,x,a)$ admits a unique maximizer $\hat{a}(Z,V,x)$, hence*

$$\pi_t^{*,Z,V,u}(x)=\operatorname*{argmax}_{a\in\mathcal{A}}f(Z_t^u,V_{t+1}^u,x,a)=\hat{a}(Z_t^u,V_{t+1}^u,x),$$

where $\pi_t^{*,Z,V,u}(x)$ is defined in (6.15).

Remark 6.4.7. *If the action space $\mathcal{A}$ is discrete and $\mathcal{A}\ni a\mapsto f(Z,V,x,a)$ is assumed to admit a unique maximizer, then $(Z,V,x)\mapsto\hat{a}(Z,V,x)$ is constant for all $(Z,V,x)\in\mathbb{R}\times[-TC_r,TC_r]^{\mathcal{X}}\times\mathcal{X}$. According to Lemma 6.4.5, the continuity of $(Z,V,u)\mapsto\pi_t^{*,Z,V,u}(x)$ defined in (6.15) will stem from the continuity of Z^u and V^u with respect to u and of $\pi_t^{*,Z,V,u}(x)$ with respect to Z and V.*

Lemma 6.4.8. *For $Z\in\mathbb{R}^{\overline{\mathcal{T}}}$, let us define:*

$$\begin{cases}V_T^Z(x)=0,\quad\forall x\in\mathcal{X}\\ V_t^Z(x)=\max_{a\in\mathcal{A}}\left\{r(x,a,Z_t)+\sum_{x'\in\mathcal{X}}P(x'\mid x,a,Z_t)\,V_{t+1}^Z(x')\right\},\quad\forall x\in\mathcal{X},t=T-1,\dots,0.\end{cases}\tag{6.11}$$

Then,

(i) $\mathbb{R}^{\overline{\mathcal{T}}} \ni Z \mapsto V^Z \in \mathbb{R}^{\overline{\mathcal{T}} \times \mathcal{X}}$ *is Lipschitz continuous, i.e. there exists* $\tilde{K}_{V,Z} \geq 0$ *such that*

$$\left\| V^Z - V^{\tilde{Z}} \right\|_{\infty,\infty} \leq \tilde{K}_{V,Z} \left\| Z - \tilde{Z} \right\|_{\infty}, \tag{6.12}$$

where $\left\| V^Z \right\|_{\infty,\infty} = \max_x \max_t \left| V_t^Z(x) \right|$ *and* $\|Z\|_{\infty} = \max_t |Z_t|$.

(ii) $\mathcal{Z}_{C_\varphi} \ni Z = (Z^u)_{u \in I} \mapsto V^Z = (V^{Z^u})_{u \in I} \in L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ *is Lipschitz continuous with* $K_{V,Z} \geq 0$ *such that:*

$$\left\| V^Z - V^{\tilde{Z}} \right\|_{\infty,\infty,2} \leq K_{V,Z} \left\| Z - \tilde{Z} \right\|_{\infty,2}. \tag{6.13}$$

where the norms $\|\cdot\|_{\infty,\infty,2}$ *and* $\|\cdot\|_{\infty,2}$ *are introduced in Definition 6.4.2 (6.8) and (6.9) respectively.*

(iii) *For all* $x \in \mathcal{X}$ *and* $t \in \overline{\mathcal{T}}$, $\left| V_t^Z(x) \right| \leq T\, C_r$ *under Assumption 6.2.6.*

Proof of Lemma 6.4.8.

Step 1: Proof of statement (i) by induction:

Initial step: $t = T$

At the terminal time, the value function is 0, so it is Lipschitz continuous with respect to Z with constant $L_{V,Z,T} = 0$.

Inductive steps: $t \in \mathcal{T}$

Let $t \in \mathcal{T}$. As induction hypothesis, we assume that $Z \mapsto V_{t+1}^Z$ is Lipschitz continuous with respect to Z with constant denoted by $L_{V,Z,t+1}$. Let us prove that $Z \mapsto V_t^Z$ is Lipschitz continuous as well.

Let $Z, \tilde{Z} \in \mathbb{R}^{\mathcal{T}}$. We have:

$$\begin{aligned}
\max_x \left| V_t^Z(x) - V_t^{\tilde{Z}}(x) \right| &= \max_x \left| \max_a \left\{ r(x,a,Z_t) + \sum_{x' \in \mathcal{X}} P(x' \mid x,a,Z_t)\, V_{t+1}^Z(x') \right\} \right. \\
&\qquad \left. - \max_{a'} \left\{ r(x,a',\tilde{Z}_t) + \sum_{x' \in \mathcal{X}} P(x' \mid x,a',\tilde{Z}_t)\, V_{t+1}^{\tilde{Z}}(x') \right\} \right| \\
&\leq \max_{x,a} \left| r(x,a,Z_t) - r(x,a,\tilde{Z}_t) \right. \\
&\qquad \left. + \sum_{x' \in \mathcal{X}} P(x' \mid x,a,Z_t)\, V_{t+1}^Z(x') - \sum_{x' \in \mathcal{X}} P(x' \mid x,a,\tilde{Z}_t)\, V_{t+1}^{\tilde{Z}}(x') \right| \\
&\leq \max_{x,a} \left(\left| r(x,a,Z_t) - r(x,a,\tilde{Z}_t) \right| \right. \\
&\qquad + \left| \sum_{x' \in \mathcal{X}} \Big(P(x' \mid x,a,Z_t) - P(x' \mid x,a,\tilde{Z}_t) \Big)\, V_{t+1}^Z(x') \right| \\
&\qquad \left. + \left| \sum_{x' \in \mathcal{X}} P(x' \mid x,a,\tilde{Z}_t) \left(V_{t+1}^Z(x') - V_{t+1}^{\tilde{Z}}(x') \right) \right| \right).
\end{aligned}$$

For the first term, by Assumption 6.2.7 (3),

$$\left|r(x,a,Z_t) - r(x,a,\tilde{Z}_t)\right| \leq L_{r,Z}\|Z-\tilde{Z}\|_\infty.$$

For the second term, by Assumption 6.2.7 (1),

$$\begin{aligned}
&\left|\sum_{x'\in\mathcal{X}}\Big(P(x'\mid x,a,Z_t) - P(x'\mid x,a,\tilde{Z}_t)\Big)V^Z_{t+1}(x')\right| \\
&\quad\leq \max_x |V^Z_{t+1}(x)| \sum_{x'\in\mathcal{X}}\Big|P(x'\mid x,a,Z_t) - P(x'\mid x,a,\tilde{Z}_t)\Big| \\
&\quad\leq L_{P,Z}\,|\mathcal{X}|\,\max_x |V^Z_{t+1}(x)|\|Z-\tilde{Z}\|_\infty.
\end{aligned}$$

For the third term, we use the fact that $P(x'\mid x,a,\tilde{Z}_t)\in[0,1]$, and by induction hypothesis, there exists some $L'_{V,Z,t+1}\geq 0$ such that

$$\begin{aligned}
\left|\sum_{x'\in\mathcal{X}} P(x'\mid x,a,\tilde{Z}_t)\left(V^Z_{t+1}(x') - V^{\tilde{Z}}_{t+1}(x')\right)\right| &\leq \sum_{x'\in\mathcal{X}}\left|V^Z_{t+1}(x') - V^{\tilde{Z}}_{t+1}(x')\right| \\
&\leq L'_{V,Z,t+1}\,|\mathcal{X}|\|Z-\tilde{Z}\|_\infty.
\end{aligned}$$

Collecting all the terms and letting $L_{V,Z,t} = \Big(L_{r,Z} + L_{P,Z}\,|\mathcal{X}|\max_x |V^Z_{t+1}(x)| + L'_{Z,V,t+1}\,|\mathcal{X}|\Big)$, we have

$$\max_x \left|V^Z_t(x) - V^{\tilde{Z}}_t(x)\right| \leq L_{V,Z,t}\,\|Z-\tilde{Z}\|_\infty.$$

Hence, we deduce that $Z\mapsto V^Z_t$ is Lipschitz continuous for all $t\in\overline{\mathcal{T}}$.

Taking the max over $x\in\mathcal{X}$ and $t\in\overline{\mathcal{T}}$, and let $\tilde{K}_{V,Z} = \max_t L_{V,Z,t}$, we obtain the Lipschitz continuity of $Z\mapsto V^Z$:

$$\left\|V^Z - V^{\tilde{Z}}\right\|_{\infty,\infty} \leq \tilde{K}_{V,Z}\left\|Z-\tilde{Z}\right\|_\infty.$$

Step 2: Proof of statement (ii):
For every $Z\in\mathcal{Z}_{C_\varphi}$ and $V^Z\in L^2(I)^{\overline{\mathcal{T}}\times\mathcal{X}}$,

$$\begin{aligned}
\left\|V^Z - V^{\tilde{Z}}\right\|_{\infty,\infty,2} &\leq \left(\int_I \left\|V^{Z^u} - V^{\tilde{Z}^u}\right\|^2_{\infty,\infty} du\right)^{1/2} \\
&\leq \left(\int_I \tilde{K}_{V,Z}\,\|Z^u - \tilde{Z}^u\|^2_\infty du\right)^{1/2} \\
&\leq K_{V,Z}\,\|Z-\tilde{Z}\|_{\infty,2},
\end{aligned}$$

where $K_{V,Z} = \sqrt{\tilde{K}_{V,Z}}$.

Step 3: Proof of statement (iii) by induction:
Initial step: $t=T$

At the terminal time, $V^Z_T(x) = 0 \leq T\,C_r$.

Inductive steps: $t \in \mathcal{T}$

Let $t \in \mathcal{T}$. As induction hypothesis, we assume that $|V_{t+1}^Z(x)| \leq (T-t-1)\, C_r$. According to Assumption 6.2.6, $|r(x,a,Z)| \leq C_r$

$$\begin{aligned}\left|V_t^Z(x)\right| &= \left|\max_a \left\{ r(x,a,Z_t) + \sum_{x' \in \mathcal{X}} P(x' \mid x,a,Z_t)\, V_{t+1}^Z(x') \right\}\right| \\ &\leq \max_a \left\{ |\, r(x,a,Z_t)| + \sum_{x' \in \mathcal{X}} |\, P(x' \mid x,a,Z_t)| \cdot |\, V_{t+1}^Z(x')| \right\} \\ &\leq C_r + (T-t-1)C_r = (T-t)\, C_r.\end{aligned}$$

As a consequence of $(T-t)\, C_r \leq T\, C_r$ for all $t \in \mathcal{T}$, we conclude the proof.

□

Lemma 6.4.9. *For $Z \in \mathcal{Z}_{C_\varphi}$, let us define:*

$$\begin{cases} \mu_{t+1}^{Z,u}(x) = \sum_{x' \in \mathcal{X}} \mu_t^{Z,u}(x') P(x \mid x', \pi_t^{*,Z,V^Z,u}(x'), Z_t^u), & \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T} \\ \mu_0^{Z,u}(x) = \mu_0(x),\ \forall x \in \mathcal{X}, u \in I, \end{cases} \tag{6.14}$$

where $\pi^{*,Z,V^Z,u} \in L^2(I;\mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}}$ define by:

$$\pi_t^{*,Z,V^Z,u}(x) = \operatorname*{argmax}_a \left\{ r(x,a,Z_t^u) + \sum_{x' \in \mathcal{X}} P(x' \mid x,a,Z_t^u)\, V_{t+1}^u(x') \right\}, \qquad \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T}. \tag{6.15}$$

Then,

(i) $\mathcal{Z}_{C_\varphi} \ni Z \mapsto \pi^{*,Z,V^Z,u} \in L^2(I;\mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}}$ *is Lipschitz continuous with $K_{\pi,Z} \geq 0$ such that:*

$$\left\| \pi^{*,Z,V^Z,u} - \pi^{*,\tilde{Z},V^{\tilde{Z}},u} \right\|_{\infty,\infty,2} \leq K_{\pi,Z}\, \|Z - \tilde{Z}\|_{\infty,2}. \tag{6.16}$$

(ii) $\mathcal{Z}_{C_\varphi} \ni Z \mapsto \mu^Z = (\mu^{Z,u})_u \in L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ *is Lipschitz continuous with $K_{\mu,Z} \geq 0$ such that:*

$$\|\mu^Z - \mu^{\tilde{Z}}\|_{\infty,\infty,2} \leq K_{\mu,Z}\, \|Z - \tilde{Z}\|_{\infty,2}, \tag{6.17}$$

Proof of Lemma 6.4.9. The structure of the proof consists of the following three parts:

(1) First, we show the Lipschitz continuity of $\mathcal{A}^{\overline{\mathcal{T}} \times \mathcal{X}} \times \mathbb{R}^{\overline{\mathcal{T}}} \ni (\pi, Z) \mapsto \mu^{(\pi,Z)}$ with Lipschitz constant $K_m \geq 0$ such that

$$\left\| \mu^{\pi,Z} - \mu^{\tilde{\pi},\tilde{Z}} \right\|_{\infty,\infty} \leq K_m \left\| (\pi, Z) - (\tilde{\pi}, \tilde{Z}) \right\|_\infty .$$

where the norms $\|\cdot\|_{\infty,\infty}$ and $\|\cdot\|_\infty$ are introduced in (6.19) and (6.20) respectively.

(2) Proof of statement (i).

(3) Proof of statement (ii).

Step 1: Lipschitz continuity of $(\pi, Z) \mapsto \mu^{(\pi,Z)}$.

For $(\pi, Z) \in \mathcal{A}^{\overline{\mathcal{T}}\times\mathcal{X}} \times \mathbb{R}^{\overline{\mathcal{T}}}$, let us introduce $\mu^{\pi,Z}$ defined as:

$$\begin{cases} \mu_{t+1}^{\pi,Z}(x) = \sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') P(x \mid x', \pi_t(x'), Z_t), & \forall x \in \mathcal{X}, t \in \mathcal{T} \\ \mu_0^{\pi,Z}(x) = \mu_0(x), & \forall x \in \mathcal{X}. \end{cases} \tag{6.18}$$

As a first step, we are going to prove that $\mu^{\pi,Z}$ is Lipschitz continuous with respect to $(\pi, Z) \in \mathcal{A}^{\overline{\mathcal{T}}\times\mathcal{X}} \times \mathbb{R}^{\overline{\mathcal{T}}}$ for the norms:

$$\|\mu\|_{\infty,\infty} = \max_t \max_x |\mu_t(x)|, \tag{6.19}$$

and

$$\|(\pi, Z)\|_\infty = \max\{\|\pi\|_{\infty,\infty}, \|Z\|_\infty\}. \tag{6.20}$$

We will prove this step by induction.

Initial step: $t = 0$
Since the initial state distribution is given independently of (π, Z), it is Lipschitz continuous with respect to (π, Z), with Lipschitz constant $L_{m,0} = 0$.

Inductive steps: $t \in \mathcal{T}$

Assume our induction property is true at time $t \in \mathcal{T}$, i.e., $(\pi, Z) \mapsto \mu_t^{\pi,Z}$ is Lipschitz continuous. Let $L_{m,t}$ be its Lipschitz constant. Let us fix (π, Z) and $(\tilde{\pi}, \tilde{Z}) \in \mathcal{A}^{\overline{\mathcal{T}}\times\mathcal{X}} \times \mathbb{R}^{\overline{\mathcal{T}}}$.

We have: for any $x \in \mathcal{X}$,

$$\begin{aligned}
&\left|\mu_{t+1}^{\pi,Z}(x) - \mu_{t+1}^{\tilde{\pi},\tilde{Z}}(x)\right| \\
&= \left|\sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') P(x \mid x', \pi_t(x'), Z_t) - \sum_{x'\in\mathcal{X}} \mu_t^{\tilde{\pi},\tilde{Z}}(x') P(x \mid x', \tilde{\pi}_t(x'), \tilde{Z}_t)\right| \\
&\leq \left|\sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') P(x \mid x', \pi_t(x'), Z_t) - \sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') P(x \mid x', \pi_t(x'), \tilde{Z}_t)\right| \\
&\quad + \left|\sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') P(x \mid x', \pi_t(x'), \tilde{Z}_t) - \sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') P(x \mid x', \tilde{\pi}_t(x'), \tilde{Z}_t)\right| \\
&\quad + \left|\sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') P(x \mid x', \tilde{\pi}_t(x'), \tilde{Z}_t) - \sum_{x'\in\mathcal{X}} \mu_t^{\tilde{\pi},\tilde{Z}}(x') P(x \mid x', \tilde{\pi}_t(x'), \tilde{Z}_t)\right| \\
&\leq \sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') \left|P(x \mid x', \pi_t(x'), Z_t) - P(x \mid x', \pi_t(x'), \tilde{Z}_t)\right| \\
&\quad + \sum_{x'\in\mathcal{X}} \mu_t^{\pi,Z}(x') \left|P(x \mid x', \pi_t(x'), \tilde{Z}_t) - P(x \mid x', \tilde{\pi}_t(x'), \tilde{Z}_t)\right|
\end{aligned}$$

$$+\left|\sum_{x'\in\mathcal{X}}(\mu_t^{\pi,Z}(x')-\mu_t^{\tilde{\pi},\tilde{Z}}(x'))P(x\mid x',\tilde{\pi}_t(x'),\tilde{Z}_t)\right|. \tag{6.21}$$

For the first term in (6.21), by Assumption 6.2.7 (1), for all $x\in\mathcal{X}$,

$$\sum_{x'\in\mathcal{X}}\mu_t^{\pi,Z}(x')\left|P(x\mid x',\pi_t(x'),Z_t)-P(x\mid x',\pi_t(x'),\tilde{Z}_t)\right|\leq L_{P,Z}\,\|Z-\tilde{Z}\|_\infty.$$

For the second term in (6.21), by Assumption 6.2.7 (2), for all $x\in\mathcal{X}$,

$$\sum_{x'\in\mathcal{X}}\mu_t^{\pi,Z}(x')\left|P(x\mid x',\pi_t(x'),\tilde{Z})-P(x\mid x',\tilde{\pi}_t(x'),\tilde{Z})\right|\leq L_{P,a}\,\|\pi-\tilde{\pi}\|_{\infty,\infty}.$$

For the third term in (6.21), by our induction hypothesis, we have

$$\|\mu_t^{\pi,Z}-\mu_t^{\tilde{\pi},\tilde{Z}}\|_\infty=\max_x|\mu_t^{\pi,Z}(x)-\mu_t^{\tilde{\pi},\tilde{Z}}(x)|\leq L_{m,t}\,\max\{\|\pi-\tilde{\pi}\|_{\infty,\infty},\|Z-\tilde{Z}\|_\infty\}.$$

We deduce that:

$$\begin{aligned}\left|\sum_{x'\in\mathcal{X}}(\mu_t^{\pi,Z}(x')-\mu_t^{\tilde{\pi},\tilde{Z}}(x'))P(x\mid x',\tilde{\pi}_t(x'),\tilde{Z})\right|&\leq\sum_{x'\in\mathcal{X}}\left|\mu_t^{\pi,Z}(x')-\mu_t^{\tilde{\pi},\tilde{Z}}(x')\right|\\&\leq|\mathcal{X}|\max_x|\mu_t^{\pi,Z}(x)-\mu_t^{\tilde{\pi},\tilde{Z}}(x)|\\&\leq L_{m,t}\,|\mathcal{X}|\,\max\{\|\pi-\tilde{\pi}\|_{\infty,\infty},\|Z-\tilde{Z}\|_\infty\}.\end{aligned}$$

Combining the above bounds and going back to (6.21), we have:

$$\begin{aligned}\left|\mu_{t+1}^{\pi,Z}(x)-\mu_{t+1}^{\tilde{\pi},\tilde{Z}}(x)\right|&\leq L_{m,t}\,|\mathcal{X}|\max\{\|\pi-\tilde{\pi}\|_{\infty,\infty},\|Z-\tilde{Z}\|_\infty\}+L_{P,Z}\|Z-\tilde{Z}\|_\infty+L_{P,a}\|\pi-\tilde{\pi}\|_{\infty,\infty}\\&\leq 2\,\max\left\{(L_{m,t}\,|\mathcal{X}|+L_{P,Z})\|Z-\tilde{Z}\|_\infty,\,(L_{m,t}\,|\mathcal{X}|+L_{P,a})\|\pi-\tilde{\pi}\|_{\infty,\infty}\right\}\\&\leq L_{m,t+1}\,\max\left\{\|Z-\tilde{Z}\|_\infty,\|\pi-\tilde{\pi}\|_{\infty,\infty}\right\},\end{aligned}$$

where $L_{m,t+1}=2\max\{(L_{m,t}\,|\mathcal{X}|+L_{P,Z},L_{m,t}\,|\mathcal{X}|+L_{P,a}\}$. Hence, we deduce that $(\pi,Z)\mapsto\mu_{t+1}^{\pi,Z}$ is Lipschitz continuous.

By induction, we obtain that:

$$\left\|\mu^{\pi,Z}-\mu^{\tilde{\pi},\tilde{Z}}\right\|_{\infty,\infty}\leq K_m\,\left\|(\pi,Z)-(\tilde{\pi},\tilde{Z})\right\|_\infty, \tag{6.22}$$

where $K_m=\max_{t\in\overline{T}}L_{m,t}$ and the norms are defined in (6.19) and (6.20).

Step 2: Lipschitz continuity of $Z\mapsto\pi^Z$ in statement (i).

For $Z, \tilde{Z} \in \mathcal{Z}_{C_\varphi}$ and $\pi^{*,Z,V,u} \in L^2(I;\mathcal{A})^{\overline{T}\times\mathcal{X}}$ defined in (6.15),

$$\begin{aligned}
\left\|\pi^{*,Z,V^Z,u} - \pi^{*,\tilde{Z},V^{\tilde{Z}},u}\right\|_{\infty,\infty,2} &\leq \sqrt{\int_I \max_t \max_x |\pi_t^{*,Z,V^Z,u}(x) - \pi_t^{*,\tilde{Z},V^{\tilde{Z}},u}(x)|^2 du} \\
&= \sqrt{\int_I \max_t \max_x |\hat{a}(Z_t^u, V_{t+1}^{Z^u}, x) - \hat{a}(\tilde{Z}_t^u, V_{t+1}^{\tilde{Z}^u}, x)|^2 du} \\
&\leq \sqrt{\int_I \max_t (L_{a,Z}|Z_t^u - \tilde{Z}_t^u| + L_{a,V} \max_x |V_{t+1}^{Z^u}(x) - V_{t+1}^{\tilde{Z}^u}(x)|)^2 du} \\
&\leq \sqrt{\int_I \max_t ((L_{a,Z} + L_{a,V}\,\tilde{K}_{V,Z})|Z_t^u - \tilde{Z}_t^u|)^2 du}
\end{aligned}$$

where the second inequality is deduced by Lemma 6.4.5 and the third inequality is deduced by Lipschitz continuity of $Z \mapsto V^Z$ proved in Lemma 6.4.8 (i).

Take $K_{\pi,Z} = \sqrt{L_{a,Z} + L_{a,V}\,\tilde{K}_{V,Z}}$, we deduce:

$$\left\|\pi^{*,Z,V^Z,u} - \pi^{*,\tilde{Z},V^{\tilde{Z}},u}\right\|_{\infty,\infty,2} \leq K_{\pi,Z}\,\|Z - \tilde{Z}\|_{\infty,2}.$$

Step 3: Lipschitz continuity of $Z \mapsto \mu^Z$ in statement (ii).

For $Z, \tilde{Z} \in \mathcal{Z}_{C_\varphi}$, we have

$$\begin{aligned}
\|\mu^Z - \mu^{\tilde{Z}}\|_{\infty,\infty,2} &\leq \sqrt{\int_I \|\mu^{Z,u} - \mu^{\tilde{Z},u}\|_{\infty,\infty}^2 du} \\
&= \sqrt{\int_I \left\|\mu^{\pi^{*,Z,V^Z,u},Z^u} - \mu^{\pi^{*,\tilde{Z},V^{\tilde{Z}},u},\tilde{Z}^u}\right\|_{\infty,\infty}^2 du} \\
&\leq \sqrt{\int_I K_m^2 \max\left\{\|Z^u - \tilde{Z}^u\|_\infty,\, \|\pi^{*,Z,V^Z,u} - \pi^{*,\tilde{Z},V^{\tilde{Z}},u}\|_{\infty,\infty}\right\}^2 du} \\
&\leq \sqrt{\int_I K_m^2 \left(\|Z^u - \tilde{Z}^u\|_\infty^2 + \|\pi^{*,Z,V^Z,u} - \pi^{*,\tilde{Z},V^{\tilde{Z}},u}\|_{\infty,\infty}^2\right) du} \\
&= K_m \left(\|Z - \tilde{Z}\|_{\infty,2} + \|\pi^{*,Z,V^Z,u} - \pi^{*,\tilde{Z},V^{\tilde{Z}},u}\|_{\infty,\infty,2}\right) \\
&\leq K_m\,(1 + K_{\pi,Z})\,\|Z - \tilde{Z}\|_{\infty,2},
\end{aligned}$$

where the second inequality comes from the Lipschitz continuity of $(\pi, Z) \mapsto \mu^{\pi,Z}$ that we have shown in Step 1, see (6.22); the last inequality is deduced by the Lipschitz continuity of $Z \mapsto \pi^Z$ in Step 2, see (6.16).

Collecting all the terms, we have

$$\|\mu^Z - \mu^{\tilde{Z}}\|_{\infty,\infty,2} \leq K_{\mu,Z}\,\|Z - \tilde{Z}\|_{\infty,2},$$

where $K_{\mu,Z} = K_m\,(1 + K_{\pi,Z})$.

□

6.4.2.2 Proof of Theorem 6.4.1

Proof. This proof is divided into three steps:

1. We define the solution space and restrict (V, μ) into a feasible set $\mathcal{K}_{C_1}$ as in (6.24).
2. We apply Banach fixed point theorem to the map:

$$\Phi^{(V,\mu)}(Z) = \left(\int_{v \in I} G(u,v) \sum_{x \in \mathcal{X}} \varphi\left(x, \pi_t^{*,Z,V,v}(x)\right) \mu_t^v(x)\, dv \right)_{t \in \overline{\mathcal{T}}, u \in I}, \tag{6.23}$$

   where $\pi^{*,Z,V,v}$ is defined as in (6.15). We show $\Phi^{(V,\mu)}(\hat{Z}) = \hat{Z}$ has a unique fixed point.
3. We apply Schauder's fixed point theorem to prove that the map $\Psi : \mathcal{K}_{C_1} \ni (V, \mu) \mapsto (\hat{V}^{\hat{Z}^{(V,\mu)}}, \hat{\mu}^{\hat{Z}^{(V,\mu)}})$ which is defined through the inner fixed point $\hat{Z}$, admits a fixed point $(\hat{V}, \hat{\mu})$.

These three steps enable us to conclude the existence of solution of the system (6.7).

Step 1. Definition of the solution space.

We endow $L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ with the norm $\|\cdot\|_{\infty,\infty,2}$ as in Definition 6.4.2 (6.8). With this norm, it is a Banach space.

For every $C_1 > 0$, let $\mathcal{K}_{C_1}$ be the closed set of functions (V, μ) in $L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$ such that (V, μ) is bounded by C_1, i.e.

$$\mathcal{K}_{C_1} := \left\{ (V,\mu) \in L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}} : \|V\|_{\infty,\infty,2} \le C_1, \|\mu\|_{\infty,\infty,2} \le C_1 \right\}. \tag{6.24}$$

$\mathcal{K}_{C_1}$ is a convex subset of the Banach space $L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}} \times \mathcal{X}}$.

To alleviate notations, we denote:

$$\|(V,\mu)\|_{\mathcal{K}} = \max\{\|V\|_{\infty,\infty,2}, \|\mu\|_{\infty,\infty,2}\}. \tag{6.25}$$

So for every $(V, \mu) \in \mathcal{K}_{C_1}$, $\|(V,\mu)\|_{\mathcal{K}} \le C_1$ by definition.

Step 2. Definition of the aggregate mapping

We endow $\mathcal{Z}$ with the norm $\|\cdot\|_{\infty,2}$ as in Definition 6.4.2 (6.9). Recall Definition 6.4.3,

$$\mathcal{Z}_{C_\varphi} := \{ Z \in \mathcal{Z} : \|Z\|_{\infty,2} \le C_\varphi \}.$$

For each $(V, \mu) \in \mathcal{K}_{C_1}$ and $Z \in \mathcal{Z}_{C_\varphi}$, let us introduce $\pi^{*,Z,V,u} \in L^2(I; \mathcal{A})^{\overline{\mathcal{T}} \times \mathcal{X}}$

$$\pi_t^{*,Z,V,u}(x) = \underset{a \in \mathcal{A}}{\operatorname{argmax}} \left\{ r(x,a,Z_t^u) + \sum_{x' \in \mathcal{X}} P(x' \mid x, a, Z_t^u)\, V_{t+1}^u(x') \right\}, \qquad \forall x \in \mathcal{X}, u \in I, t \in \mathcal{T},$$

as (6.15) in Lemma 6.4.9.

Let us define the map $\Phi^{(V,\mu)} : \mathcal{Z}_{C_\varphi} \to \mathcal{Z}$ by:

$$\Phi^{(V,\mu)}(Z) = \left(\int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \varphi\left(x, \pi_t^{*,Z,V,v}(x)\right) \mu_t^v(x)\, dv \right)_{t\in\overline{\mathcal{T}},u\in I}.$$

For $Z, \tilde{Z} \in \mathcal{Z}_{C_\varphi}$,

$$\begin{aligned}
&\left\|\Phi^{(V,\mu)}(Z) - \Phi^{(V,\mu)}(\tilde{Z})\right\|_{\infty,2} \\
&= \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left| (\Phi^{(V,\mu)}(Z)_t)^u - (\Phi^{(V,\mu)}(\tilde{Z})_t)^u \right|^2 du \right)^{\frac{1}{2}} \\
&= \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left| \int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \left(\varphi(x, \pi_t^{*,Z,V,v}(x)) - \varphi(x, \pi_t^{*,\tilde{Z},V,v}(x)) \right) \mu_t^v(x) dv \right|^2 du \right)^{\frac{1}{2}}
\end{aligned}$$

Recall f is defined by (6.6) and by Corollary 6.4.6, we have

$$\pi_t^{*,Z,V,v}(x) = \hat{a}(Z_t^v, V_{t+1}^v, x).$$

By Assumption 6.2.8 and Cauchy-Schwarz, we deduce

$$\begin{aligned}
\left| \sum_{x\in\mathcal{X}} \left(\varphi(x, \pi_t^{*,Z,V,v}(x)) - \varphi(x, \pi_t^{*,\tilde{Z},V,v}(x)) \right) \mu_t^v(x) \right|^2 &\le \sum_{x\in\mathcal{X}} \left| \varphi(x, \pi_t^{*,Z,V,v}(x)) - \varphi(x, \pi_t^{*,\tilde{Z},V,v}(x)) \right|^2 \\
&\le \sum_{x\in\mathcal{X}} L_\varphi^2 \left| \pi_t^{*,Z,V,v}(x) - \pi_t^{*,\tilde{Z},V,v}(x) \right|^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left\|\pi^{*,Z,V,v} - \pi^{*,\tilde{Z},V,v}\right\|_{\infty,\infty,2} &\le \sqrt{\int_I \max_t \max_x |\pi_t^{*,Z,V,v}(x) - \pi_t^{*,\tilde{Z},V,v}(x)|^2 dv} \\
&= \sqrt{\int_I \max_t \max_x |\hat{a}(Z_t^v, V_{t+1}^v, x) - \hat{a}(\tilde{Z}_t^v, V_{t+1}^v, x)|^2 dv} \\
&\le \sqrt{\int_I \max_t (L_{a,Z} |Z_t^v - \tilde{Z}_t^v|)^2 dv} \\
&= L_{a,Z} \left\|Z - \tilde{Z}\right\|_{\infty,2},
\end{aligned}$$

where the last inequality is deduced by Lemma 6.4.5. Then,

$$\begin{aligned}
\left\|\Phi^{(V,\mu)}(Z) - \Phi^{(V,\mu)}(\tilde{Z})\right\|_{\infty,2} &\le L_\varphi \|G\|_2 \max_{t\in\overline{\mathcal{T}}} \sqrt{\int_{v\in I} \sum_{x\in\mathcal{X}} \left| \pi_t^{*,Z,V,v}(x) - \pi_t^{*,\tilde{Z},V,v}(x) \right|^2 dv} \\
&\le L_\varphi \|G\|_2 \left\|\pi_t^{*,Z,V,v} - \pi_t^{*,\tilde{Z},V,v}\right\|_{\infty,\infty,2} \\
&\le L_\varphi \|G\|_2 L_{a,Z} \|Z - \tilde{Z}\|_{\infty,2}.
\end{aligned}$$

Under Assumption 6.2.8, we have $\Phi^{(V,\mu)}(\mathcal{Z}_{C_\varphi}) \subset \mathcal{Z}_{C_\varphi}$ and $|\varphi(x,a)| \le C_\varphi$ for all $x \in \mathcal{X}$ and

$a \in \mathcal{A}$. Furthermore, $\mathcal{Z}_{C_\varphi}$ is a closed subset of $\mathcal{Z}$, which is also a Banach space. Hence $\mathcal{Z}_{C_\varphi}$ is complete.

Since $L_\varphi \|G\|_2 L_{a,Z} =: C_\Phi < 1$ by Assumption 6.2.9, with Banach fixed point theorem, we conclude that $\Phi^{(V,\mu)}$ has a unique fixed point in $\mathcal{Z}_{C_\varphi}$, which we denote by $\hat{Z}^{(V,\mu)}$.

Step 3. Application of Schauder's theorem

For $(V,\mu) \in \mathcal{K}_{C_1}$, Step 2 has shown that there exists a unique fixed point of $\Phi^{(V,\mu)}$, that we denote by $\hat{Z}^{(V,\mu)}$. Conversely, given $Z \in \mathcal{Z}_{C_\varphi}$, we define $V^Z = \left(V_t^{Z^u}\right)_{t,u}$ and $\mu^Z = (\mu_t^{Z,u})_{t,u}$ as in (6.11) and (6.14) respectively.

Let us define $\Psi : \mathcal{K}_{C_1} \ni (V,\mu) \mapsto \hat{Z}^{(V,\mu)} \mapsto (\hat{V}^{\hat{Z}^{(V,\mu)}}, \hat{\mu}^{\hat{Z}^{(V,\mu)}})$. We apply Schauder's fixed point theorem (see for instance [61, Theorem (Schauder), p179]) to prove the existence of a fixed point of Ψ after verifying the following conditions:

(i) $\Psi : \mathcal{K}_{C_1} \to \mathcal{K}_{C_1}$ is continuous.

(ii) $\Psi(\mathcal{K}_{C_1}) \subset \mathcal{F}$, where $\mathcal{F}$ is some relatively compact subset of $\mathcal{K}_{C_1}$.

Step 3.1: Continuity of Ψ. We prove that (i) holds. First, let us consider $(V,\mu) \mapsto \hat{Z}^{(V,\mu)}$.

$$
\begin{aligned}
&\left\|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\right\|_{\infty,2} \\
&= \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left|\hat{Z}^{(V,\mu),u} - \hat{Z}^{(\tilde{V},\tilde{\mu}),u}\right|^2 du\right)^{\frac{1}{2}} \\
&= \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left|\int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \left(\varphi(x,\pi_t^{*,Z,V,v}(x))\,\mu_t^v(x) - \varphi(x,\pi_t^{*,Z,\tilde{V},v}(x)\,\tilde{\mu}_t^v(x))\right) dv\right|^2 du\right)^{\frac{1}{2}} \\
&\le \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left|\int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \left(\varphi(x,\pi_t^{*,Z,V,v}(x)) - \varphi(x,\pi_t^{*,Z,\tilde{V},v}(x)\right) \mu_t^v(x)\, dv\right|^2 du\right)^{\frac{1}{2}} \\
&\quad + \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left|\int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \varphi(x,\pi_t^{*,Z,\tilde{V},v}(x))\left(\mu_t^v(x) - \tilde{\mu}_t^v(x)\right) dv\right|^2 du\right)^{\frac{1}{2}} \\
&\le L_\varphi \|G\|_2 \|\pi_t^{*,Z,V,v} - \pi_t^{*,Z,\tilde{V},v}\|_{\infty,\infty,2} + C_\varphi \|G\|_2 \|\mu_t^v - \tilde{\mu}_t^v\|_{\infty,\infty,2},
\end{aligned}
$$

where the last inequality is deduced by Assumption 6.2.8. Since $\pi_t^{*,Z,V,v}(x) = \hat{a}(Z_t^v, V_{t+1}^v, x)$ under Assumption 6.2.6, by Lemma 6.4.5, we have

$$
\begin{aligned}
\left\|\pi^{*,Z,V,v} - \pi^{*,Z,\tilde{V},v}\right\|_{\infty,\infty,2} &\le \sqrt{\int_I \max_t \max_x |\pi_t^{*,Z,V,v}(x) - \pi_t^{*,Z,\tilde{V},v}(x)|^2 dv} \\
&= \sqrt{\int_I \max_t \max_x |\hat{a}(Z_t^v, V_{t+1}^v, x) - \hat{a}(Z_t^v, \tilde{V}_{t+1}^v, x)|^2 dv} \\
&\le \sqrt{\int_I \max_t \max_x (L_{a,V}|V_t^v - \tilde{V}_t^v|)^2 dv} \\
&= L_{a,V} \|V - \tilde{V}\|_{\infty,\infty,2},
\end{aligned}
$$

Therefore,

$$\left\|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\right\|_{\infty,2} \leq L_{a,V}\, L_\varphi\, \|G\|_2\, \|V - \tilde{V}\|_{\infty,\infty,2} + C_\varphi\, \|G\|_2\, \|\mu_t^v - \tilde{\mu}_t^v\|_{\infty,\infty,2}.$$

Then, we prove the continuity of $\hat{Z}^{(V,\mu)} \mapsto (\hat{V}^{\hat{Z}^{(V,\mu)}}, \hat{\mu}^{\hat{Z}^{(V,\mu)}})$. Let $(V,\mu),(\tilde{V},\tilde{\mu}) \in \mathcal{K}_{C_1}$, recall that $\|\cdot\|_{\mathcal{K}}$ is defined in (6.25), we have

$$\begin{aligned}
\left\|\tilde{\Psi}(V,\mu) - \tilde{\Psi}(\tilde{V},\tilde{\mu})\right\|_{\mathcal{K}} &= \left\|(\hat{V}^{\hat{Z}^{(V,\mu)}}, \hat{\mu}^{\hat{Z}^{(V,\mu)}}) - (\hat{V}^{\hat{Z}^{(\tilde{V},\tilde{\mu})}}, \hat{\mu}^{\hat{Z}^{(\tilde{V},\tilde{\mu})}})\right\|_{\mathcal{K}} \\
&= \max\left\{\|\hat{V}^{\hat{Z}^{(V,\mu)}} - \hat{V}^{\hat{Z}^{(\tilde{V},\tilde{\mu})}}\|_{\infty,\infty,2}, \|\hat{\mu}^{\hat{Z}^{(V,\mu)}} - \hat{\mu}^{\hat{Z}^{(\tilde{V},\tilde{\mu})}}\|_{\infty,\infty,2}\right\} \\
&\leq \max\{K_{V,Z}, K_{\mu,Z}\}\, \|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\|_{\infty,2}, \qquad (6.26)
\end{aligned}$$

where the Lipschitz continuity results of $Z \mapsto V^Z$ and $Z \mapsto \mu^Z$ has been addressed in Lemma 6.4.8 (ii) and Lemma 6.4.9 (ii) respectively, then it enable us to deduce that $\Psi : \mathcal{K}_{C_1} \to \mathcal{K}_{C_1}$ is continuous.

Step 3.2: Definition of $\mathcal{F}$ and proof of relative compactness. Now, we prove that (ii) holds. We apply Fréchet-Kolmogorov theorem (see for instance [61, Theorem 4.26]) on functions from $\mathbb{R}$ to $\mathbb{R}$. To this end, we consider, for each (t,x), the functions $u \mapsto V_t^{Z^u}(x)$ and $u \mapsto \mu_t^{u,Z}(x)$, but we need to define them over the whole real line instead of just I.

Let $\tilde{u} : \mathbb{R} \to I$ be an extended function of index such that

$$\tilde{u}(i) = \begin{cases} i, & \text{if } i \in [0,1], \\ 0, & \text{if } i < 0, \\ 1, & \text{if } i > 1. \end{cases} \qquad (6.27)$$

We denote $\tau_h f^{\tilde{u}(i)} = f^{\tilde{u}(i+h)}, i \in \mathbb{R}, h \in \mathbb{R}$. Let

$$\mathcal{F} := \left\{ f = (f^{\tilde{u}(i)})_{i\in\mathbb{R}} = \left(\hat{V}^{\hat{Z}^{(V,\mu),\tilde{u}(i)}}, \hat{\mu}^{\hat{Z}^{(V,\mu),\tilde{u}(i)}}\right)_{i\in\mathbb{R}} : (V,\mu) \in \mathcal{K}_{C_1} \right\},$$

We view $\mathcal{F}$ as a subset of $L^2(\mathbb{R};\mathbb{R}^n)$ with $n = 2 \times |\mathcal{X}| \times |\overline{\mathcal{T}}|$, and for $f \in \mathcal{F}$, we can write $f = (f_1, \ldots, f_n)$ with an arbitrary ordering of the coordinates. We denote

$$\mathcal{G}_k := \{f_k : f \in \mathcal{F}\} \in L^2(\mathbb{R}), \qquad \forall\, k = 1, \ldots, n.$$

Note that for all $g \in \mathcal{G}_k$, $\|g\|_2 \leq C_1$.

Moreover, we denote $\mathcal{G} := \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. We have that $\mathcal{F}$ is a subset of $\mathcal{G}$, hence, in order to prove that $\mathcal{F}$ is relatively compact, it is sufficient to prove that $\mathcal{G}$ is relatively compact. By comparing the norms, we realize that if

$$\lim_{h\to 0} \sup_{f\in\mathcal{F}} \|\tau_h f - f\|_{\infty,\infty,2} = 0, \qquad (6.28)$$

then,

$$\lim_{h\to 0} \sup_{f\in\mathcal{F}} \|\tau_h f_k - f_k\|_2 = 0, \qquad \forall\, k = 1, \ldots, n,$$

from which we deduce that the closure of $\mathcal{G}_{k|I}$ in $L^2(\mathbb{R})$ is compact, where $\mathcal{G}_{k|I}$ denotes the set of the restrictions to $I \subset \mathbb{R}$ of the functions in $\mathcal{G}_k$. This yields that $\mathcal{F} \subseteq \mathcal{G}$ is relatively compact.

It remains to verify the statement (6.28), which can be rephrased as follows: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \left(\hat{V}^{\hat{Z}^{(V,\mu)}},\ \hat{\mu}^{\hat{Z}^{(V,\mu)}}\right) \in \mathcal{F},\ \forall h \in \mathbb{R} \ \text{ with } \ |h| < \delta, \qquad \begin{aligned} \left\|\tau_h \hat{V}^{\hat{Z}^{(V,\mu)}} - \hat{V}^{\hat{Z}^{(V,\mu)}}\right\|_{\infty,\infty,2} &\leq \varepsilon, \\ \left\|\tau_h \hat{\mu}^{\hat{Z}^{(V,\mu)}} - \hat{\mu}^{\hat{Z}^{(V,\mu)}}\right\|_{\infty,\infty,2} &\leq \varepsilon. \end{aligned}$$

Since both $\hat{V}^{\hat{Z}^{(V,\mu)}}$ and $\hat{\mu}^{\hat{Z}^{(V,\mu)}}$ are computed through the aggregate $\hat{Z}^{(V,\mu)}$, let us start by checking

$$\lim_{h\to 0} \sup_{(V,\mu)\in\mathcal{K}_{C_1}} \left\|\tau_h \hat{Z}^{(V,\mu)} - \hat{Z}^{(V,\mu)}\right\|_{\infty,2} = 0.$$

Since $|\varphi(x,a)| \leq C_\varphi$ by Assumption 6.2.8 for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$, μ_t^w below is a probability distribution for all $t \in \mathcal{T}$, and we recall that $Z^{(V,\mu)}$ is defined as the fixed point of $\Phi^{(V,\mu)}$ in (6.23), we have:

$$\begin{aligned}
&\left\|\tau_h \hat{Z}^{(V,\mu)} - \hat{Z}^{(V,\mu)}\right\|_{\infty,2}^2 \\
&= \max_{t\in\mathcal{T}} \int_{v\in I} \left|\hat{Z}_t^{(V,\mu),\tilde{u}(v+h)} - \hat{Z}_t^{(V,\mu),\tilde{u}(v)}\right|^2 dv \\
&= \max_{t\in\mathcal{T}} \int_{v\in I} \left| \int_{w\in I} G(\tilde{u}(v+h),w) \sum_{x\in\mathcal{X}} \varphi(x, \pi_t^{*,\hat{Z}^{(V,\mu)},V,w}(x))\, \mu_t^w(x)\, dw \right. \\
&\qquad\qquad \left. - \int_{w\in I} G(\tilde{u}(v),w) \sum_{x\in\mathcal{X}} \varphi(x, \pi_t^{*,\hat{Z}^{(V,\mu)},V,w}(x))\, \mu_t^w(x)\, dw \right|^2 dv \\
&= \max_{t\in\mathcal{T}} \int_{v\in I} \left| \int_{w\in I} [G(\tilde{u}(v+h),w) - G(\tilde{u}(v),w)] \sum_{x\in\mathcal{X}} \varphi(x, \pi_t^{*,\hat{Z}^{(V,\mu)},V,w}(x))\, \mu_t^w(x)\, dw \right|^2 dv \\
&\leq C_\varphi^2 \max_{t\in\mathcal{T}} \int_{v\in I} \left| \int_{w\in I} [G(\tilde{u}(v+h),w) - G(\tilde{u}(v),w)]\, dw \right|^2 dv \\
&\leq C_\varphi^2 \max_{t\in\mathcal{T}} \int_{v\in I} \int_{w\in I} |G(\tilde{u}(v+h),w) - G(\tilde{u}(v),w)|^2\, dw\, dv
\end{aligned}$$

Notice that the last right-hand side is independent of (V,μ). Furthermore, we recall that, under Assumption 6.2.5, the graphon $G : I \times I \to I$ is continuous, and $\tilde{u}$ is also continuous by definition, see (6.27). So for all $w \in I$, $G(\tilde{u}(\cdot),w)$ is continuous, i.e., for all $v \in I$,

$$\lim_{h\to 0} G(\tilde{u}(v+h),w) = G(\tilde{u}(v),w).$$

With Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{h\to 0} \sup_{(V,\mu)\in\mathcal{K}_{C_1}} \left\|\tau_h \hat{Z}_t^{(V,\mu)} - \hat{Z}_t^{(V,\mu)}\right\|_{\infty,2} = 0. \tag{6.29}$$

We note that $V^{\hat{Z}^{(V,\mu),u}}$ and $\mu^{\hat{Z}^{(V,\mu),u}}$ depend on u only through $\hat{Z}^{(V,\mu),u}$. Furthermore, V^Z

and μ^Z are Lipschitz continuous with respect to Z by Lemma 6.4.8 (ii) and Lemma 6.4.9 (ii) respectively. Then, for all $v \in I$,

$$(\tau_h \hat{V}^{\hat{Z}^{(V,\mu)}})^v = \hat{V}^{(\tau_h \hat{Z}^{(V,\mu)})^v} \qquad \text{and} \qquad (\tau_h \hat{\mu}^{\hat{Z}^{(V,\mu)}})^v = \hat{\mu}^{(\tau_h \hat{Z}^{(V,\mu)})^v}.$$

Consequently,

$$\begin{aligned}\left\|\tau_h \hat{V}^{\hat{Z}^{(V,\mu)}} - \hat{V}^{\hat{Z}^{(V,\mu)}}\right\|_{\infty,\infty,2} &= \left\|\hat{V}^{\tau_h \hat{Z}^{(V,\mu)}} - \hat{V}^{\hat{Z}^{(V,\mu)}}\right\|_{\infty,\infty,2} \\ &\leq C \left\|\tau_h \hat{Z}^{(V,\mu)} - \hat{Z}^{(V,\mu)}\right\|_{\infty,2},\end{aligned}$$

where C depends on the Lipschitz constant of $\hat{V}$. Likewise,

$$\left\|\tau_h \hat{\mu}^{\hat{Z}^{(V,\mu)}} - \hat{\mu}^{\hat{Z}^{(V,\mu)}}\right\|_{\infty,\infty,2} \leq C' \left\|\tau_h \hat{Z}^{(V,\mu)} - \hat{Z}^{(V,\mu)}\right\|_{\infty,2},$$

where C' depends on the Lipschitz constant of $\hat{\mu}$. Combining this with the uniform continuity of $\hat{Z}^{(V,\mu)}$, see (6.29), we deduce that:

$$\lim_{h\to 0} \sup_{(V,\mu)\in\mathcal{K}_{C_1}} \left\|\tau_h \hat{V}^{\hat{Z}^{(V,\mu)}} - \hat{V}^{\hat{Z}^{(V,\mu)}}\right\|_{\infty,\infty,2} = 0,$$

and

$$\lim_{h\to 0} \sup_{(V,\mu)\in\mathcal{K}_{C_1}} \left\|\tau_h \hat{\mu}^{\hat{Z}^{(V,\mu)}} - \hat{\mu}^{\hat{Z}^{(V,\mu)}}\right\|_{\infty,\infty,2} = 0.$$

This shows that (6.28) holds.

We conclude that, for every $k = 1, \ldots, n$, $\mathcal{G}_k$ is a family of equicontinuous functions. By Fréchet–Kolmogorov theorem, the closure of $\mathcal{G}_{k|I}$ is compact for all $k = 1, \ldots, n$. Then, $\mathcal{F}$ is relatively compact, which means that (ii) holds.

Step 3.3: Conclusion of the proof. By Schauder's fixed point theorem, we conclude that the system (6.7) admits a bounded solution (V, μ) in $L^2(I)^{\overline{\mathcal{T}}\times\mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}}\times\mathcal{X}}$.

□

6.5 Uniqueness Results

We establish the uniqueness results through two different approaches: monotonicity and contraction, presented in the next two sections.

6.5.1 Uniqueness by monotonicity

Definition 6.5.1. *For every state-action distribution sequence $\nu \in \mathcal{P}(\mathcal{X}\times\mathcal{A})^{\overline{\mathcal{T}}}$, every $u \in I$ and every given time $t \in \overline{\mathcal{T}}$, the aggregate $Z_t^{\nu,u} \in \mathbb{R}$ is defined as:*

$$Z_t^{\nu,u} = \int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \int_{\mathcal{A}} \varphi(x,a)\, \nu_t^v(x,da)\, dv.$$

Remark 6.5.2. *Given $\mu \in \mathcal{P}(\mathcal{X})$ and $\pi \in \mathcal{A}^{\mathcal{X}}$, the joint state-action distribution $\nu^{\pi,\mu} \in \mathcal{P}(\mathcal{X} \times \mathcal{A})$ generated by π consists in:*

$$\nu^{\pi,\mu}(x, da) = \mu(x)\, \delta_{\pi(x)}(da). \tag{6.30}$$

We extend the notation to time and index-dependent objects. Furthermore, for $(Z_t^{\pi,u})_{u\in I, t\in\overline{\mathcal{T}}} \in \mathcal{Z}$ in Definition 6.2.3, $Z_t^{\pi,u} = Z_t^{\nu^\pi,u}$ for all $t \in \overline{\mathcal{T}}, u \in I$.

Assumption 6.5.3 (monotonicity condition)**.** *The function r is said to be (strictly) monotone if, for every state-action distribution $\nu, \tilde{\nu} \in \mathcal{P}(\mathcal{X} \times \mathcal{A})$,*

$$\int_{u\in I} \sum_{x\in\mathcal{X}} \int_{\mathcal{A}} (r(x, a, Z^{\nu,u}) - r(x, a, Z^{\tilde{\nu},u}))(\nu(x, da) - \tilde{\nu}(x, da))\, du < 0,$$

where $Z^{\nu,u}$ is from Definition 6.5.1.

Theorem 6.5.4. *Suppose Assumption 6.5.3 holds. Then, there exists at most one pair of (graphon MFG) Nash equilibrium $(\hat{\pi}, \hat{Z})$.*

Proof of Theorem 6.5.4. Suppose there exist two different graphon MFG Nash equilibria, denoted by (π, Z) and $(\tilde{\pi}, \tilde{Z})$. By definition we have:

$$\int_{u\in I} \Big(J(\pi^u; Z^u) - J(\tilde{\pi}^u; Z^u)\Big) du \geq 0,$$
$$\int_{u\in I} \Big(J(\tilde{\pi}^u; \tilde{Z}^u) - J(\pi^u; \tilde{Z}^u)\Big)\, du \geq 0,$$

and we must have $Z = Z^\pi$ (resp. $\tilde{Z} = Z^{\tilde{\pi}}$) by Definition 6.2.4.

Notice that $Z^\pi = Z^{\nu^\pi}$ where ν^π is the state-action distribution generated by π. Likewise for $Z^{\tilde{\pi}} = Z^{\nu^{\tilde{\pi}}}$.

Furthermore, according to Theorem 6.3.1, (π, Z) have the form as in system (6.7) and the existence of solution of the system (6.7) is shown in Theorem 6.4.1.

By Remark 6.5.2, the joint state-action distribution $\left(\nu^{\pi,\mu,u}\right)_{u\in I}$ generated by GNE π is the form

$$\nu^{\pi,\mu,u}(x, da) = \mu^u(x)\, \delta_{\pi^u(x)}(da), \qquad \forall\, u \in I,$$

and $Z^{\nu^{\pi,\mu},u} = Z^{\pi,u}$. Similarly, for another GNE $(\tilde{\pi}, \tilde{Z})$, we have $\nu^{\tilde{\pi},\tilde{\mu},u}(x, da) = \tilde{\mu}^u(x)\, \delta_{\tilde{\pi}^u(x)}(da)$ and $Z^{\nu^{\tilde{\pi},\tilde{\mu}},u} = Z^{\tilde{\pi},u}$ for all $u \in I$.

For all $u \in I$, let us denote by $\nu^u := \nu^{\pi,\mu,u}$ and $\tilde{\nu}^u := \nu^{\tilde{\pi},\tilde{\mu},u}$ for simplification below. Combining the two inequalities above, we have:

$$\begin{aligned}
0 \leq & \int_{u\in I} J(\pi^u; Z^{\pi,u}) - J(\pi^u; Z^{\tilde{\pi},u}) + J(\tilde{\pi}^u; Z^{\tilde{\pi},u}) - J(\tilde{\pi}^u; Z^{\pi,u})\, du \\
&= \int_{u\in I} \sum_{t\in\mathcal{T}} \sum_{x\in\mathcal{X}} \Big[\Big(r(x, \pi_t^u(x), Z_t^{\pi,u}) - r(x, \pi_t^u(x), Z_t^{\tilde{\pi},u})\Big)\, \mu_t^u(x) \\
&\qquad\qquad + \Big(r(x, \tilde{\pi}_t^u(x), Z_t^{\tilde{\pi},u}) - r(x, \tilde{\pi}_t^u(x), Z_t^{\pi,u})\Big)\, \tilde{\mu}_t^u(x) \Big] du
\end{aligned}$$

$$
\begin{aligned}
&= \int_{u\in I} \sum_{t\in\mathcal{T}} \sum_{x\in\mathcal{X}} \Big[\Big(r(x,a,Z_t^{\pi,u}) - r(x,a,Z_t^{\tilde{\pi},u}) \Big) \, \mu_t^u(x) \, \delta_{\pi_t^u(x)}(da) \\
&\qquad\qquad + \Big(r(x,a,Z_t^{\tilde{\pi},u}) - r(x,a,Z_t^{\pi,u}) \Big) \, \tilde{\mu}_t^u(x) \, \delta_{\tilde{\pi}_t^u(x)}(da) \Big] du \\
&= \int_{u\in I} \sum_{t\in\mathcal{T}} \sum_{x\in\mathcal{X}} \int_{\mathcal{A}} \Big[\Big(r(x,a,Z_t^{\nu^{\pi},u}) - r(x,a,Z_t^{\nu^{\tilde{\pi}},u}) \Big) \, \nu_t^u(x,da) \\
&\qquad\qquad + \Big(r(x,a,Z_t^{\nu^{\tilde{\pi}},u}) - r(x,a,Z_t^{\nu^{\pi},u}) \Big) \, \tilde{\nu}_t^u(x,da) \Big] du \\
&= \int_{u\in I} \sum_{t\in\mathcal{T}} \sum_{x\in\mathcal{X}} \int_{\mathcal{A}} \Big(r(x,a,Z_t^{\nu^{\pi},u}) - r(x,a,Z_t^{\nu^{\tilde{\pi}},u}) \Big) \Big[\nu_t^u(x,da) - \tilde{\nu}_t^u(x,da) \Big] du.
\end{aligned}
$$

After integrating over u, the right hand side must be strictly negative by the monotonicity Assumption 6.5.3, which leads to a contradiction with the above lower bound.

□

6.5.2 Uniqueness by contraction

Theorem 6.5.5. *Suppose Assumptions 6.2.5, 6.2.6, 6.2.7, 6.2.8, 6.2.9 hold. Moreover, suppose that*

$$
C_{\tilde{\Psi}} := \max\{K_{V,Z}, K_{\mu,Z}\} \frac{L_\varphi \|G\|_2 (L_{a,V} + C_\varphi)}{1 - L_\varphi L_{a,Z} \|G\|_2} < 1, \qquad \text{and} \qquad L_\varphi L_{a,Z} \|G\|_2 < 1.
$$

Then, there exists a unique pair of (graphon MFG) Nash equilibrium $(\hat{\pi}, \hat{Z})$.

Proof of Theorem 6.5.5. We prove the existence and uniqueness of a bounded solution (V,μ) in $L^2(I)^{\overline{\mathcal{T}}\times\mathcal{X}} \times L^2(I)^{\overline{\mathcal{T}}\times\mathcal{X}}$ to the system (6.7) in three steps.

The first two steps of the proof are adapted by the proof Theorem 6.4.1, in the third step, we use Banach fixed point theorem instead of Schauder's fixed point theorem:

1. We define the solution space and restrict (V,μ) into a feasible set $\mathcal{K}_{C_1}$ as in (6.24).

2. We apply Banach fixed point theorem to the map as in (6.23):

$$
\Phi^{(V,\mu)}(Z) = \left(\int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \varphi\left(x, \pi_t^{*,Z,V,v}(x)\right) \mu_t^v(x) \, dv \right)_{t\in\overline{\mathcal{T}}, u\in I},
$$

   where $\pi^{*,Z,V,v}$ is defined as in (6.15). We show $\Phi^{(V,\mu)}(\hat{Z}) = \hat{Z}$ has a unique fixed point.

3. We apply Banach fixed point theorem theorem to prove the map

$$
\tilde{\Psi} : \mathcal{K}_{C_1} \to \mathcal{K}_{C_1}, \qquad (V,\mu) \mapsto (\hat{V}^{\hat{Z}^{(V,\mu)}}, \hat{\mu}^{\hat{Z}^{(V,\mu)}}),
$$

   which is defined through the inner fixed point $\hat{Z}$, admits a fixed point $(\hat{V},\hat{\mu})$.

Note that the first two steps above are identical within the proof of Theorem 6.4.1, so we start with step 3 directly. Let $(V,\mu),(\tilde{V},\tilde{\mu}) \in \mathcal{K}_{C_1}$, recall (6.26), we have

$$
\left\| \tilde{\Psi}(V,\mu) - \tilde{\Psi}(\tilde{V},\tilde{\mu}) \right\|_{\mathcal{K}} \le \max\{K_{V,Z}, K_{\mu,Z}\} \, \|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\|_{\infty,2}.
$$

With triangle inequality, we have:

$$
\begin{aligned}
&\|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\|_{\infty,2} \\
&= \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left| \hat{Z}_t^{(V,\mu),u} - \hat{Z}_t^{(\tilde{V},\tilde{\mu}),u} \right|^2 du \right)^{\frac{1}{2}} \\
&\leq \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left| \int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \left(\varphi(x, \pi_t^{*,\hat{Z}^{(V,\mu),u},V,v}(x)) - \varphi(x, \pi_t^{*,\hat{Z}^{(\tilde{V},\tilde{\mu})},\tilde{V},v}(x)) \right) \mu_t^v(x) dv \right|^2 du \right)^{\frac{1}{2}} \\
&\quad + \max_{t\in\overline{\mathcal{T}}} \left(\int_{u\in I} \left| \int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \varphi(x, \pi_t^{*,\hat{Z}^{(\tilde{V},\tilde{\mu})},\tilde{V},v}(x)) \left(\mu_t^v(x) - \tilde{\mu}_t^v(x)\right) dv \right|^2 du \right)^{\frac{1}{2}}.
\end{aligned}
$$

For the first term, we apply Cauchy-Schwarz inequality and Assumption 6.2.8, we deduce

$$
\begin{aligned}
&\left| \sum_{x\in\mathcal{X}} \left(\varphi(x, \pi_t^{*,\hat{Z}^{(V,\mu)},V,v}(x)) - \varphi(x, \pi_t^{*,\hat{Z}^{(\tilde{V},\tilde{\mu})},\tilde{V},v}(x)) \right) \mu_t^v(x) \right|^2 \\
&\leq \sum_{x\in\mathcal{X}} \left| \varphi(x, \pi_t^{*,\hat{Z}^{(V,\mu)},V,v}(x)) - \varphi(x, \pi_t^{*,\hat{Z}^{(\tilde{V},\tilde{\mu})},\tilde{V},v}(x)) \right|^2 \\
&\leq \sum_{x\in\mathcal{X}} L_\varphi^2 \left| \pi_t^{*,\hat{Z}^{(V,\mu)},V,v}(x) - \pi_t^{*,\hat{Z}^{(\tilde{V},\tilde{\mu})},\tilde{V},v}(x) \right|^2.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| \pi^{*,\hat{Z}^{(V,\mu)},V} - \pi^{*,\hat{Z}^{(\tilde{V},\tilde{\mu})},\tilde{V}} \right\|_{\infty,\infty,2} \\
&\leq \sqrt{\int_I \max_t \max_x \left| \pi_t^{*,\hat{Z}^{(V,\mu)},V,v}(x) - \pi_t^{*,\hat{Z}^{(\tilde{V},\tilde{\mu})},\tilde{V},v}(x) \right|^2 dv} \\
&= \sqrt{\int_I \max_t \max_x \left| \hat{a}(\hat{Z}_t^{(V,\mu),v}, V_{t+1}^v, x) - \hat{a}(\hat{Z}_t^{(\tilde{V},\tilde{\mu}),v}, \tilde{V}_{t+1}^v, x) \right|^2 dv} \\
&\leq \sqrt{\int_I \max_t \left((L_{a,Z} |\hat{Z}_t^{(V,\mu),v} - \hat{Z}_t^{(\tilde{V},\tilde{\mu}),v}|^2 + \max_x L_{a,v} |V_{t+1}^v(x) - \tilde{V}_{t+1}^v(x)|^2 \right) dv} \\
&= L_{a,Z} \|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\|_{\infty,2} + L_{a,V} \|V - \tilde{V}\|_{\infty,\infty,2},
\end{aligned}
$$

where the last inequality is deduced by Lemma 6.4.5. We obtain,

$$
\begin{aligned}
\|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\|_{\infty,2} &\leq L_\varphi \|G\|_2 \left(L_{a,Z} \|\hat{Z}^{(V,\mu)} - \hat{Z}^{(\tilde{V},\tilde{\mu})}\|_{\infty,2} + L_{a,V} \|V - \tilde{V}\|_{\infty,\infty,2} + C_\varphi \|\mu - \tilde{\mu}\|_{\infty,\infty,2} \right) \\
&\leq \frac{L_\varphi \|G\|_2}{1 - L_\varphi L_{a,Z} \|G\|_2} \left(L_{a,V} \|V - \tilde{V}\|_{\infty,\infty,2} + C_\varphi \|\mu - \tilde{\mu}\|_{\infty,\infty,2} \right) \\
&\leq \frac{L_\varphi \|G\|_2}{1 - L_\varphi L_{a,Z} \|G\|_2} (L_{a,V} + C_\varphi) \max \left\{ \|V - \tilde{V}\|_{\infty,\infty,2}, \|\mu - \tilde{\mu}\|_{\infty,\infty,2} \right\} \\
&\leq \frac{L_\varphi \|G\|_2 (L_{a,V} + C_\varphi)}{1 - L_\varphi L_{a,Z} \|G\|_2} \left\| (V,\mu) - (\tilde{V},\tilde{\mu}) \right\|_{\mathcal{K}}.
\end{aligned}
$$

Collecting all the terms, we have

$$\begin{aligned}\left\|\tilde{\Psi}(V,\mu)-\tilde{\Psi}(\tilde{V},\tilde{\mu})\right\|_{\mathcal{K}} &\leq \max\{K_{V,Z},K_{\mu,Z}\}\,\|\hat{Z}^{(V,\mu)}-\hat{Z}^{(\tilde{V},\tilde{\mu})}\|_{\infty,2}\\ &\leq \max\{K_{V,Z},K_{\mu,Z}\}\,\frac{L_\varphi\,\|G\|_2(L_{a,V}+C_\varphi)}{1-L_\varphi L_{a,Z}\,\|G\|_2}\left\|(V,\mu)-(\tilde{V},\tilde{\mu})\right\|_{\mathcal{K}}\\ &= C_{\tilde{\Psi}}\left\|(V,\mu)-(\tilde{V},\tilde{\mu})\right\|_{\mathcal{K}}.\end{aligned}$$

Since $C_{\tilde{\Psi}}<1$, and $\mathcal{K}_{C_1}$ is complete, then, with Banach fixed point theorem, we conclude that $\tilde{\Psi}(V,\mu)$ has a unique fixed point.

According to Theorem 6.3.1, $(\hat{\pi},\hat{Z})$ is a (graphon MFG) Nash equilibrium if and only if $(\hat{\pi},\hat{Z})$ has the form of the system (6.7), hence we conclude the existence and the uniqueness of the graphon MFG Nash equilibrium.

□

6.6 Approximate Equilibrium in Finite Player Games

Let G_N be a step graphon associated with some simple graph $\mathbb{G}_N=(V_N,E_N)$ as defined in (6.1). Let $\zeta_{ij}^N:=\mathbb{1}_{(i,j)\in E_N}$ be a $N\times N$ interaction matrix generated by G_N which describes the interaction between players i and j. In particular, $\zeta_{ij}^N=G_N(\frac{i}{N},\frac{j}{N})$.

N-player graph game model We first introduce the model.

Definition 6.6.1. *A N-player graph model is defined by a tuple $(N,\overline{\mathcal{T}},\mathcal{X},\mathcal{A},p,r,\varphi,\mu_0)$ where*

- *$N\in\mathbb{N}$ is number of players,*
- *$\overline{\mathcal{T}}=\{0,\dots,T\}$ is a set of discrete time steps,*
- *$\mathcal{X}$ and $\mathcal{A}$ are state space and action space respectively, same as in the GMFG model,*
- *$P:\mathcal{X}\times\mathcal{A}\times\mathbb{R}\to\mathcal{P}(\mathcal{X})$ is a one-step transition probability kernel,*
- *$r:\mathcal{X}\times\mathcal{A}\times\mathbb{R}\to\mathbb{R}$ is a reward function,*
- *$\varphi:\mathcal{X}\times\mathcal{A}\to\mathbb{R}$ is a measurable function for the interactions,*
- *$\mu_0^i:=\mu_0\in\mathcal{P}(\mathcal{X})$ is an initial distribution for all $i\in[N]$.*

Let us define the set of policies for one player as $\mathbf{\Pi}:=\mathcal{A}^{\mathcal{T}\times\mathcal{X}}$. A policy $\pi\in\mathbf{\Pi}$, $(t,x)\mapsto\pi_t(x):=\pi(t,x)\in\mathcal{A}$ provides an action given each state x at time t.

Recall the notation $\Pi:=L^2(I;\mathcal{A})^{\overline{\mathcal{T}}\times\mathcal{X}}$ we have defined earlier is the set of graphon policies, and $\mathbf{\Pi}$ is set of policies for a single player where it is assumed to be homogeneous for each player.

In the finite graph game setting, given a tuple of N-player policies $\boldsymbol{\pi}^N=(\pi^1,\dots,\pi^N)\in\mathbf{\Pi}^N$, we define the finite graph aggregate for player $i\in[N]$ as:

$$Z_t^{N,\boldsymbol{\pi}^N,i}:=\sum_{j\in V_N}\zeta_{ij}^N\,\varphi(X_t^j,\pi_t^j(X_t^j)),\qquad\forall\, i\in V_N,t\in\overline{\mathcal{T}},$$

where the player i evolves with the following dynamic:

$$X_0^i \sim \mu_0, \qquad a_t^i = \pi_t^i(X_t^i), \qquad X_{t+1}^i \sim P(\cdot \,|\, X_t^i, a_t^i, Z_t^{N,\boldsymbol{\pi}^N,i}), \qquad \forall\, i \in V_N, t \in \mathcal{T}. \tag{6.31}$$

Note that the evolution of the players' states is coupled through the aggregate. The total reward of player i in the finite graph game setting is defined as: for $\pi^i \in \mathbf{\Pi}$ and $\pi^{-i} \in \mathbf{\Pi}^{N-1}$,

$$\mathcal{J}^{N,i}(\pi^i; \pi^{-i}) := \mathbb{E}\left[\sum_{t\in\mathcal{T}} r(X_t^i, a_t^i, Z_t^{N,\boldsymbol{\pi}^N,i})\right]. \tag{6.32}$$

Definition 6.6.2 (ϵ-Nash)**.** *For any $\epsilon > 0$, a tuple of policies $\hat{\boldsymbol{\pi}}^N = (\hat{\pi}^1, \dots, \hat{\pi}^N) \in \mathbf{\Pi}^N$ is an ϵ-Nash equilibrium if there exists $N' \in \mathbb{N}$ such that for all $N > N'$,*

$$\sup_{\tilde{\pi}\in\mathbf{\Pi}} \mathcal{J}^{N,i}(\tilde{\pi}; \hat{\pi}^{-i}) - \mathcal{J}^{N,i}(\hat{\boldsymbol{\pi}}^N) \le \epsilon, \qquad \forall\, i \in [N].$$

A Nash equilibrium is an ϵ-Nash equilibrium with $\epsilon = 0$.

6.6.1 From GNE to N-player Nash equilibrium

In this section, we show that any GNE provides an ϵ-NE for the N-player graph game, and $\epsilon \to 0$ as $N \to +\infty$. To build a connection between N-player graph game and GMFG settings, for N-player tuple $\boldsymbol{\pi}^N = (\pi^1, \dots, \pi^N) \in \mathbf{\Pi}^N$, we define the step policy as:

$$\pi_t^{N,\boldsymbol{\pi}^N,u}(x) := \sum_{i\in V_N} \mathbb{1}_{u\in\left(\frac{i-1}{N}, \frac{i}{N}\right]} \pi_t^i(x)\,, \qquad \forall\, x \in \mathcal{X}, u \in I, t \in \overline{\mathcal{T}}. \tag{6.33}$$

Given $\boldsymbol{\pi}^N$, let us define the distribution:

$$\mu_t^{N,\boldsymbol{\pi}^N,u} := \sum_{i\in V_N} \mathbb{1}_{u\in\left(\frac{i-1}{N}, \frac{i}{N}\right]} \delta_{X_t^i}, \qquad \forall\, i \in V_N, t \in \overline{\mathcal{T}}, \tag{6.34}$$

where X^i follows the evolution in (6.31).

Let us introduce an equivalent notation of the GMFG total reward function with respect to a given state-action distribution profile $\nu = (\nu_t^u)_{u\in I, t\in\overline{\mathcal{T}}}$, $\nu_t^u \in \mathcal{P}(\mathcal{X}\times\mathcal{A})$ and graphon aggregate $Z^{\nu,u}$ as in Definition 6.5.1,

$$\mathcal{J}^{\nu,u}(\pi^u) := \mathbb{E}\left[\sum_{t=0}^{T-1} r(X_t^u, a_t^u, Z_t^{\nu,u})\right], \qquad \forall\, u \in I, \tag{6.35}$$

subject to

$$X_0^u \sim \mu_0, \qquad a_t^u = \pi_t^u(X_t^u), \qquad X_{t+1}^u \sim P(\cdot \,|\, X_t^u, a_t^u, Z_t^{\nu,u}), \qquad \forall\, t \in \mathcal{T},$$

where $Z_t^{\nu,u}$ is defined in Definition 6.5.1.

Let us define the map $\Lambda : \Pi \to \mathbf{\Pi}^N$ such that for any $\pi \in \Pi$,

$$\Lambda(\pi) = (\pi^{\frac{1}{N}}, \pi^{\frac{2}{N}}, \dots, \pi^{\frac{N}{N}}).$$

In the following, we consider deviations of some i-th player for $i \in [N]$. Let $\pi \in \Pi$ and $\tilde{\pi} \in \mathbf{\Pi}$, let X^i be the state of player i in the N-player game such that player i uses $\tilde{\pi}$ and the other players use $\Lambda(\pi)^{-i}$, i.e.

$$X_0^i \sim \mu_0, \qquad a_t^i = \tilde{\pi}_t(X_t^i), \qquad X_{t+1}^i \sim P(\cdot \,|\, X_t^i, a_t^i, Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}), \qquad \forall\, t \in \mathcal{T}. \tag{6.36}$$

As an auxiliary, we introduce $X^{\frac{i}{N}}$ as the state of player $\frac{i}{N}$ in the graphon game, where this player uses $\tilde{\pi}$ and the rest of the population uses π, i.e.

$$X_0^{\frac{i}{N}} \sim \mu_0^{\frac{i}{N}}, \qquad a_t^{\frac{i}{N}} = \tilde{\pi}_t(X_t^{\frac{i}{N}}), \qquad X_{t+1}^{\frac{i}{N}} \sim P(\cdot \,|\, X_t^{\frac{i}{N}}, a_t^{\frac{i}{N}}, Z_t^{\nu^\pi,\frac{i}{N}}), \qquad \forall\, t \in \mathcal{T}. \tag{6.37}$$

Since this player has a no impact on the rest of the population, $Z_t^{\nu^\pi,\frac{i}{N}}$ depends only on π through ν^π and not on $\tilde{\pi}$.

Theorem 6.6.3. *Let $\hat{\pi} \in \Pi$ be a GNE. Under Assumption 6.2.10, for any $\epsilon > 0$, there exists $N' \in \mathbb{N}$ such that for all $N > N'$, $\Lambda(\hat{\pi})$ is an ϵ-Nash equilibrium to the N-player game, i.e. for every $i \in V_N$,*

$$\sup_{\tilde{\pi} \in \mathbf{\Pi}} \mathcal{J}^{N,i}(\tilde{\pi}; \Lambda(\hat{\pi})^{-i}) - \mathcal{J}^{N,i}(\Lambda(\hat{\pi})) \leq \epsilon.$$

The proof of Theorem 6.6.3 relies on Lemma 6.6.4, which requires several intermediate results. The useful lemmas and the proofs of these lemmas are postponed to Section 6.6.2 and 6.6.3 respectively. For now, we provide the proof of Theorem 6.6.3 using Lemma 6.6.4.

Proof of Theorem 6.6.3. Let $\nu = (\nu_t^{\hat{\pi},\mu,u})_{u \in I, t \in \mathcal{T}}$, $\nu_t^{\hat{\pi},\mu,u} \in \mathcal{P}(\mathcal{X} \times \mathcal{A})$, be the corresponding state-action distribution profile defined in (6.30) generated by GNE $\hat{\pi}$ and μ. Then, there exists $N' \in \mathbb{N}$ such that for all $N > N'$,

$$\begin{aligned}
&\sup_{\tilde{\pi} \in \mathbf{\Pi}} \mathcal{J}^{N,i}(\tilde{\pi}; \Lambda(\hat{\pi})^{-i}) - \mathcal{J}^{N,i}(\Lambda(\hat{\pi})) \\
&\leq \sup_{\tilde{\pi} \in \mathbf{\Pi}} \left(\mathcal{J}^{N,i}(\tilde{\pi}; \Lambda(\hat{\pi})^{-i}) - \mathcal{J}^{\nu,\frac{i}{N}}(\tilde{\pi}) \right) + \sup_{\tilde{\pi} \in \mathbf{\Pi}} \left(\mathcal{J}^{\nu,\frac{i}{N}}(\tilde{\pi}) - \mathcal{J}^{\nu,\frac{i}{N}}(\hat{\pi}^{\frac{i}{N}}) \right) + \mathcal{J}^{\nu,\frac{i}{N}}(\hat{\pi}^{\frac{i}{N}}) - \mathcal{J}^{N,i}(\Lambda(\hat{\pi})) \\
&< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where in the first step, we apply the triangle inequality, and in the second step, we use Lemma 6.6.4 for the first and last terms, while the middle term is derived from the optimality of the GNE policy.

□

6.6.2 Useful lemmas

Lemma 6.6.4. *Let $\pi \in \Pi$ satisfying Assumption 6.2.10 and let $\nu^\pi = (\nu_t^{\pi,u})_{u\in I, t\in\mathcal{T}}$, $\nu_t^{\pi,u} \in \mathcal{P}(\mathcal{X}\times\mathcal{A})$ be the corresponding state-action distribution profile generated by π. Let $\tilde{\pi} \in \mathbf{\Pi}$. Then, for any $\epsilon > 0$, there exists $N' \in \mathbb{N}$ such that for all $N > N'$,*

$$\left| \mathcal{J}^{N,i}(\tilde{\pi}\,; \Lambda(\pi)^{-i}) - \mathcal{J}^{\nu^\pi, \frac{i}{N}}(\tilde{\pi}) \right| < \epsilon, \qquad \forall\, i \in [N].$$

Lemma 6.6.5. *Let $\pi \in \Pi$ and $\tilde{\pi} \in \mathbf{\Pi}$. For $i \in [N]$, let X^i and $X^{\frac{i}{N}}$ be defined in (6.36) and (6.37) respectively. Let $\mathcal{H}$ be the family of measurable functions from $\mathcal{X}$ to $\mathbb{R}$ bounded by some constant $C_h > 0$. For any $\epsilon > 0$, there exists $N' \in \mathbb{N}$ such that for all $N > N'$, we have*

$$\sup_{h\in\mathcal{H}} \left| \mathbb{E}\Big[h(X_t^i) \Big] - \mathbb{E}\Big[h(X_t^{\frac{i}{N}}) \Big] \right| < \epsilon \qquad \forall\, t \in \overline{\mathcal{T}}, i \in V_N. \tag{6.38}$$

Similarly, let $\tilde{\mathcal{H}}$ be the family of measurable functions $h : \mathcal{X}\times\mathcal{Z} \ni (x,z) \mapsto h(x,z) \in \mathbb{R}$ bounded by C_h, Lipschitz continuous with respect to $z \in \mathcal{Z}$ with Lipschitz constant $L_h > 0$ uniform in $x \in \mathcal{X}$. For any $\epsilon > 0$, there exists $N' \in \mathbb{N}$ such that for all $N > N'$, we have

$$\sup_{h\in\tilde{\mathcal{H}}} \left| \mathbb{E}\Big[h(X_t^i, Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}) \Big] - \mathbb{E}\Big[h(X_t^{\frac{i}{N}}, Z_t^{\nu^\pi,\frac{i}{N}}) \Big] \right| < \epsilon \qquad \forall\, t \in \overline{\mathcal{T}}, i \in V_N. \tag{6.39}$$

Lemma 6.6.6. *Let $\pi \in \Pi$, $\tilde{\pi} \in \mathbf{\Pi}$, and $\pi_t^N = \pi_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i})} \in L^2(I;\mathcal{A})^{\mathcal{X}}$ be defined as in (6.41). Let $\mu \in \mathcal{M}$ and μ^N be as defined in Lemma 6.6.7. Given $G \in \mathcal{W}$ and its approximated step graphon of G_N, then*

$$\lim_{N\to\infty} \left| \mathbf{Z}^u\left(G, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t^N, \pi_t^N\right) \right| = 0, \qquad \forall\, u \in I, t \in \overline{\mathcal{T}},$$

where $\mathbf{Z}$ is defined in (6.40).

Lemma 6.6.7. *Given $\pi \in \Pi$ and $\tilde{\pi} \in \mathbf{\Pi}$. For $i \in [N]$, let us consider the state distribution sequences*

$$\mu^N = (\mu_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),u})_{u\in I, t\in\overline{\mathcal{T}}} \qquad \text{and} \qquad \mu = (\mu_t^{\pi,u})_{u\in I, t\in\overline{\mathcal{T}}},$$

where we recall (6.34) and Definition 6.2.3 respectively. For any measurable and bounded function $\mathcal{X}\times I \ni (x,u) \mapsto f^u(x) := f(x,u) \in \mathbb{R}$, we have

$$\mathbb{E}[|\mu_t^N(f) - \mu_t(f)|] \to 0 \text{ as } N \to +\infty, \qquad \forall\, t \in \overline{\mathcal{T}}.$$

Lemma 6.6.8. *Let us define the function $\mathbf{Z} : I \times \mathcal{W} \times L^2(I;\mathcal{P}(\mathcal{X})) \times L^2(I;\mathcal{A})^{\mathcal{X}} \to \mathbb{R}$,*

$$\mathbf{Z}^u(G,\mu,\pi) := \int_{v\in I} G(u,v) \sum_{x\in\mathcal{X}} \varphi(x,\pi^v(x))\, \mu^v(x)\, dv, \qquad \forall\, u \in I. \tag{6.40}$$

Let $\pi \in \Pi$, $\tilde{\pi} \in \mathbf{\Pi}$, and $\mu \in \mathcal{M}$. Let us introduce the step policy $\pi^N_t = \pi^{N,(\tilde{\pi},\Lambda(\pi)^{-i})}_t \in L^2(I;\mathcal{A})^{\mathcal{X}}$, see (6.33). Let $(G_N)_N$ be a sequence of step-graphons and G be a graphon such that Assumption 6.2.1 holds. Then,

$$\lim_{N\to\infty} \int_{u\in I} \left| \mathbf{Z}^u\left(G, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t, \pi^N_t\right) \right| du = 0, \qquad \forall\, t \in \overline{\mathcal{T}}.$$

We note that with $\pi^N_t = \pi^{N,(\tilde{\pi},\Lambda(\pi)^{-i})}_t$, player i uses $\tilde{\pi}$ and the other players use $\Lambda(\pi)^{-i}$, i.e.

$$\pi^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),u}_t(x) := \sum_{j\in V_N\setminus\{i\}} \mathbb{1}_{u\in\left(\frac{j-1}{N},\frac{j}{N}\right]} \Lambda(\pi)^j_t(x) + \mathbb{1}_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} \tilde{\pi}_t(x), \qquad \forall\, x\in\mathcal{X}, u\in I, t\in\mathcal{T}. \tag{6.41}$$

6.6.3 Proofs of the lemmas in Section 6.6

Proof of Lemma 6.6.4. Let us introduce the notation $r_\pi : \mathcal{X}\times\mathbb{R}\to\mathbb{R}$ with respect to some given $\pi\in\mathcal{A}^{\mathcal{T}\times\mathcal{X}}$:

$$r_{\pi_t}(x, Z_t) := r(x, \pi_t(x), Z_t), \qquad \forall\, t\in\mathcal{T},$$

Recall that X^i is the state of player i in the N-player game, when player i uses $\tilde{\pi}$ and the other players use $\Lambda(\pi)^{-i}$; $X^{\frac{i}{N}}$ is the state of player $\frac{i}{N}$ in the graphon game, when this player uses $\tilde{\pi}$ and the rest of the population uses π. The evolution of X^i and $X^{\frac{i}{N}}$ are given by (6.36) and (6.37) respectively.

Then, by triangle inequality and (6.39) in Lemma 6.6.5, we deduce that

$$\begin{aligned}
&\left| \mathcal{J}^{N,i}(\tilde{\pi}\,;\Lambda(\pi)^{-i}) - \mathcal{J}^{\nu^\pi,\frac{i}{N}}(\tilde{\pi}) \right| \\
&= \left| \mathbb{E}\left[\sum_{t\in\mathcal{T}} r(X^i_t, \tilde{\pi}_t(X^i_t), Z^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}_t) \right] - \mathbb{E}\left[\sum_{t\in\mathcal{T}} r(X^{\frac{i}{N}}_t, \tilde{\pi}_t(X^{\frac{i}{N}}_t), Z^{\nu^\pi,\frac{i}{N}}_t) \right] \right| \\
&= \left| \mathbb{E}\left[\sum_{t\in\mathcal{T}} r_{\tilde{\pi}_t}(X^i_t, Z^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}_t) \right] - \mathbb{E}\left[\sum_{t\in\mathcal{T}} r_{\tilde{\pi}_t}(X^{\frac{i}{N}}_t, Z^{\nu^\pi,\frac{i}{N}}_t) \right] \right| \\
&\le \sum_{t\in\mathcal{T}} \left| \mathbb{E}\left[r_{\tilde{\pi}_t}(X^i_t, Z^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}_t) \right] - \mathbb{E}\left[r_{\tilde{\pi}_t}(X^{\frac{i}{N}}_t, Z^{\nu^\pi,\frac{i}{N}}_t) \right] \right| \\
&< \epsilon.
\end{aligned}$$

□

Proof of Lemma 6.6.5. We prove (6.38) first, then (6.39).

Proof of (6.38): Let $t\in\mathcal{T}$. Since $|h|<C_h$, by triangle inequality and Assumption 6.2.7 (1), we have:

$$\begin{aligned}
&\left| \mathbb{E}\Big[h(X^i_{t+1}) \Big] - \mathbb{E}\Big[h(X^{\frac{i}{N}}_{t+1}) \Big] \right| \\
&= \left| \sum_{x\in\mathcal{X}} h(x) \sum_{x'\in\mathcal{X}} \Big(P(x\,|\,x', \tilde{\pi}_t(x'), Z^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}_t)\, \mu^i_t(x') - P(x\,|\,x', \tilde{\pi}_t(x'), Z^{\nu^\pi,\frac{i}{N}}_t)\, \mu^{\frac{i}{N}}_t(x') \Big) \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big| \sum_{x\in\mathcal{X}} h(x) \sum_{x'\in\mathcal{X}} \Big(P(x\,|\,x', \tilde{\pi}_t(x'), Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),\,i}) - P(x\,|\,x', \tilde{\pi}_t(x'), Z_t^{\nu^\pi,\frac{i}{N}}) \Big)\, \mu_t^i(x') \Big| \\
&\quad + \Big| \sum_{x\in\mathcal{X}} h(x) \sum_{x'\in\mathcal{X}} P(x\,|\,x', \tilde{\pi}_t(x'), Z_t^{\nu^\pi,\frac{i}{N}}) \Big(\mu_t^i(x') - \mu_t^{\frac{i}{N}}(x') \Big) \Big| \\
&\leq |\mathcal{X}|\, C_h\, L_{P,Z} \Big| Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),\,i} - Z_t^{\nu^\pi,\frac{i}{N}} \Big| + |\mathcal{X}|\, C_h.
\end{aligned}
$$

Recall the function $\mathbf{Z}$ defined in (6.40), and the state distribution $\mu^N = (\mu_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),u})_{u\in I, t\in\overline{\mathcal{T}}}$ induced by the step policy $\pi^N = (\pi^{N,(\tilde{\pi},\,\Lambda(\pi)^{-i}),u})_{u\in I, t\in\overline{\mathcal{T}}}$ defined in (6.34) and (6.41) respectively. We have

$$
\begin{aligned}
&\mathbf{Z}^{\frac{i}{N}}\Big(G_N, \mu_t^N, \pi_t^N\Big) \\
&= \int_I G_N(\frac{i}{N}, v) \sum_{x\in\mathcal{X}} \varphi\Big(x, \sum_{j\in V_N\setminus\{i\}} \mathbb{1}_{u\in\left(\frac{j-1}{N},\frac{j}{N}\right]}\, \Lambda(\pi)_t^j(x) + \mathbb{1}_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]}\, \tilde{\pi}_t(x)\Big) \sum_{j\in V_N} \mathbb{1}_{v\in\left(\frac{j-1}{N},\frac{j}{N}\right]}\, \delta_{X_t^j}(x)\, dv \\
&= \sum_{x\in\mathcal{X}} \Bigg(\sum_{j\in V_N\setminus\{i\}} \int_{v\in\left(\frac{j-1}{N},\frac{j}{N}\right]} G_N(\frac{j}{N}, v) \sum_{x\in\mathcal{X}} \varphi(x, \Lambda(\pi)_t^j(x))\, \delta_{X_t^j}(x)\, dv \\
&\qquad\qquad + \int_{v\in\left(\frac{i-1}{N},\frac{i}{N}\right]} G_N(\frac{i}{N}, v) \sum_{x\in\mathcal{X}} \varphi(x, \tilde{\pi}_t(x))\, \delta_{X_t^i}(x)\, dv \Bigg) \\
&= \sum_{j\in V_N\setminus\{i\}} \zeta_{ij}^N\, \varphi(X_t^j, \Lambda(\pi)_t^j(X_t^j)) + \zeta_{ii}^N\, \varphi(X_t^i, \tilde{\pi}_t(X_t^i)) \\
&= Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}.
\end{aligned}
$$

On the other hand, with graphon π and $\mu_t = (\mu_t^{\pi,u})_{u\in I}$ as in Definition 6.2.3, we see that $Z_t^{\pi,u} = \mathbf{Z}^u(G, \mu_t, \pi_t)$. By Remark 6.5.2, $Z_t^{\nu^\pi,\frac{i}{N}} = Z_t^{\pi,\frac{i}{N}} = \mathbf{Z}^{\frac{i}{N}}(G, \mu_t, \pi_t)$.

Therefore, with Lemma 6.6.6, we deduce that for all $t \in \overline{\mathcal{T}}$,

$$
\Big| Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),\,i} - Z_t^{\nu^\pi,\frac{i}{N}} \Big| = \Big| \mathbf{Z}^{\frac{i}{N}}\Big(G_N, \mu_t^N, \pi_t^N\Big) - \mathbf{Z}^{\frac{i}{N}}(G, \mu_t, \pi_t) \Big| \to 0 \quad \text{as} \quad N \to +\infty.
$$

Consequently,

$$
\sup_{h\in\mathcal{H}} \Big| \mathbb{E}\Big[h(X_t^i) \Big] - \mathbb{E}\Big[h(X_t^{\frac{i}{N}}) \Big] \Big| < \epsilon \qquad \forall\, t \in \overline{\mathcal{T}}, i \in V_N,
$$

where $\epsilon = |\mathcal{X}|\, C_h$.

Proof of (6.39): For all $t \in \mathcal{T}$,

$$
\begin{aligned}
&\Big| \mathbb{E}\Big[h(X_t^i, Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}) \Big] - \mathbb{E}\Big[h(X_t^{\frac{i}{N}}, Z_t^{\nu^\pi,\frac{i}{N}}) \Big] \Big| \\
&\leq \Big| \mathbb{E}\Big[h(X_t^i, Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i}) \Big] - \mathbb{E}\Big[h(X_t^i, Z_t^{\nu^\pi,\frac{i}{N}}) \Big] \Big| \\
&\quad + \Big| \mathbb{E}\Big[h(X_t^i, Z_t^{\nu^\pi,\frac{i}{N}}) \Big] - \mathbb{E}\Big[h(X_t^{\frac{i}{N}}, Z_t^{\nu^\pi,\frac{i}{N}}) \Big] \Big|
\end{aligned}
$$

$$\leq L_h \, \mathbb{E}\Big[\Big| Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i} - Z_t^{\nu^\pi,\frac{i}{N}}\Big|\Big] + \Big|\mathbb{E}\Big[\tilde{h}_{Z_t^{\nu^\pi,\frac{i}{N}}}(X_t^i)\Big] - \mathbb{E}\Big[\tilde{h}_{Z_t^{\nu^\pi,\frac{i}{N}}}(X_t^{\frac{i}{N}})\Big]\Big|,$$

where $\tilde{h}_Z(X) = h(X,Z)$ for all $Z \in \mathbb{R}$. Then, the first term goes to 0 as $N \to +\infty$ by Lemma 6.6.6; the second term is bounded by ϵ from (6.38). Hence, we concluded that for all $t \in \overline{\mathcal{T}}$ and $i \in V_N$,

$$\sup_{h\in\tilde{\mathcal{H}}} \Big|\mathbb{E}\Big[h(X_t^i, Z_t^{N,(\tilde{\pi},\Lambda(\pi)^{-i}),i})\Big] - \mathbb{E}\Big[h(X_t^{\frac{i}{N}}, Z_t^{\nu^\pi,\frac{i}{N}})\Big]\Big| < \epsilon.$$

□

Proof of Lemma 6.6.6. By triangle inequality:

$$\begin{aligned}
&|\mathbf{Z}^u\left(G_N,\mu_t^N,\pi_t^N\right) - \mathbf{Z}^u\left(G,\mu_t,\pi_t\right)| \\
&\quad\leq |\mathbf{Z}^u\left(G_N,\mu_t^N,\pi_t^N\right) - \mathbf{Z}^u\left(G_N,\mu_t,\pi_t^N\right)| + |\mathbf{Z}^u\left(G_N,\mu_t,\pi_t^N\right) - \mathbf{Z}^u\left(G_N,\mu_t,\pi_t\right)| \\
&\qquad + |\mathbf{Z}^u\left(G_N,\mu_t,\pi_t\right) - \mathbf{Z}^u\left(G,\mu_t,\pi_t\right)|. \qquad (6.42)
\end{aligned}$$

With the same proof of the $\boldsymbol{I_4}$ term in Lemma 6.6.7, the first term of (6.42)

$$|\mathbf{Z}^u\left(G_N,\mu_t^N,\pi_t^N\right) - \mathbf{Z}^u\left(G_N,\mu_t,\pi_t^N\right)| \to 0 \quad \text{as} \quad N \to +\infty.$$

Then, we apply the same reasoning as in Lemma 6.6.8 for the last two terms of (6.42). Since π^N and π are independent with $u \in I$, the second term of (6.42) is still bounded by $\frac{L_\varphi(L_\pi+2C_{\mathcal{A}})}{N}$.
The third term of (6.42) becomes:

$$\begin{aligned}
|\mathbf{Z}^u\left(G,\mu_t,\pi_t\right) - \mathbf{Z}^u\left(G_N,\mu_t,\pi_t\right)| &\leq C_\varphi \sup_{\|\Phi\|_\infty\leq 1} \left|\int_{v\in I}[(G(u,v) - G_N(u,v))\Phi(v)]dv\right| du \\
&= C_\varphi \, \|G - G_N\|_\square \to 0.
\end{aligned}$$

To conclude with all three terms, for each $u \in I$,

$$\lim_{N\to\infty} \left|\mathbf{Z}^u\left(G,\mu_t,\pi_t\right) - \mathbf{Z}^u\left(G_N,\mu_t,\pi_t^N\right)\right| = 0, \qquad \forall\, t \in \overline{\mathcal{T}}.$$

□

Proof of Lemma 6.6.7. Consider f as in the statement and let $C_{\varphi,f} = \max\{C_\varphi, \sup_{x,u}|f(x,u)|\}$, which is finite by assumption. We proceed by induction.
Initial step: $t = 0$. For all $i \in V_N$, we have $\mathcal{L}(X_0^i) = \mu_0 = \mu_0^u$ for all $u \in I$. By Cauchy-Schwartz inequality and the bound $|f| < C_{\varphi,f}$, we have:

$$\begin{aligned}
&\mathbb{E}[|\mu_0^N(f) - \mu_0(f)|] \\
&= \mathbb{E}\left[\left|\int_{u\in I}\left(\sum_{x\in\mathcal{X}} f^u(x)\,\mu_0^{N,u}(x) - \sum_{x\in\mathcal{X}} f^u(x)\,\mu_0^u(x)\right) du\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left|\sum_{i\in V_N}\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]}\left(\sum_{x\in\mathcal{X}} f^u(x)\,\delta_{X_0^i}(x) - \sum_{x\in\mathcal{X}} f^u(x)\,\mu_0(x)\right)du\right|\right]\\
&= \mathbb{E}\left[\left|\sum_{i\in V_N}\left(\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_0^i)\,du - \mathbb{E}\left[\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_0^i)\,du\right]\right)\right|\right]\\
&\leq \left(\mathbb{E}\left[\left|\sum_{i\in V_N}\left(\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_0^i)\,du - \mathbb{E}\left[\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_0^i)\,du\right]\right)\right|^2\right]\right)^{\frac{1}{2}}\\
&= \left(\sum_{i\in V_N}\mathbb{E}\left[\left|\left(\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_0^i)\,du - \mathbb{E}\left[\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_0^i)\,du\right]\right)\right|^2\right]\right)^{\frac{1}{2}}\\
&\leq \sqrt{N\cdot\left(\frac{2C_{\varphi,f}}{N}\right)^2} = \frac{2C_{\varphi,f}}{\sqrt{N}},
\end{aligned}
$$

where we note that the last equality holds by independence of $(X_0^i)_{i\in V_N}$. Hence, the initial step goes to 0 as $N\to+\infty$.

Inductive steps: Let $t\in\mathcal{T}$. For the sake of induction, we assume that:

$$\mathbb{E}[|\mu_t^N(f)-\mu_t(f)|]\to 0 \text{ as } N\to+\infty.$$

Given $\pi\in\mathcal{A}^{\mathcal{X}}$, $G\in\mathcal{W}, \tilde{\eta}\in L^2(I,\mathcal{P}(\mathcal{X}))$, and $\tilde{\pi}\in L^2(I,\mathcal{A})^{\mathcal{X}}$, we define the operator $\mathbf{T}^{\pi}_{G,\tilde{\eta},\tilde{\pi}}: L^2(I;\mathcal{P}(\mathcal{X}))\to L^2(I;\mathcal{P}(\mathcal{X}))$ such that:

$$\left(\eta\,\mathbf{T}^{\pi}_{G,\tilde{\eta},\tilde{\pi}}\right)^u = \sum_{x\in\mathcal{X}}\eta^u(x)\,P\left(\cdot\mid x,\pi^u(x),\mathbf{Z}^u\left(G,\tilde{\eta},\tilde{\pi}\right)\right),\qquad \forall\, u\in I.$$

With this notation, $\mu_{t+1}=\mu_t\,\mathbf{T}^{\pi_t}_{G,\,\mu_t,\pi_t}$.

We denote $\pi_t^N=\pi_t^{N,(\tilde{\pi},\,\Lambda(\pi)^{-i})}\in L^2(I;\mathcal{A})^{\mathcal{X}}$ defined as in (6.41). By triangle inequality, we have

$$
\begin{aligned}
\mathbb{E}[|\mu_{t+1}^N(f)-\mu_{t+1}(f)|] \leq\ & \mathbb{E}\left[\left|\mu_{t+1}^N(f)-\mu_t^N\,\mathbf{T}^{\pi_t^N}_{G_N,\,\mu_t^N,\,\pi_t^N}(f)\right|\right]\\
&+\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t^N}_{G_N,\,\mu_t^N,\,\pi_t^N}(f)-\mu_t^N\,\mathbf{T}^{\pi_t}_{G_N,\,\mu_t^N,\,\pi_t}(f)\right|\right]\\
&+\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t}_{G_N,\,\mu_t^N,\,\pi_t}(f)-\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t^N,\,\pi_t}(f)\right|\right]\\
&+\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t^N,\,\pi_t}(f)-\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t,\,\pi_t}(f)\right|\right]\\
&+\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t,\,\pi_t}(f)-\mu_{t+1}(f)\right|\right]\\
=\ & I_1+I_2+I_3+I_4+I_5,
\end{aligned}
$$

where the terms $I_1,\ldots,I_5$ are defined in an obvious manner.

Term I_1:
We denote $\boldsymbol{X}_t:=(X_t^i)_{i\in V_N}$, and we note that $(X_{t+1}^i)_{i\in V_N}$ are conditionally independent given

$\boldsymbol{X}_t$. Similarly with the initial case, we apply the Cauchy-Schwartz inequality and the bound of $|f| < C_{\varphi,f}$, and we obtain:

$$
\begin{aligned}
&\mathbb{E}\left[\left|\mu_{t+1}^N(f) - \mu_t^N \mathbf{T}^{\pi_t^N}_{G_N,\mu_t^N,\pi_t^N}(f)\right|\right] \\
&= \mathbb{E}\left[\left|\int_{u\in I}\sum_{x\in\mathcal{X}} f^u(x)\,\mu_{t+1}^{N,u}(x)\,du\right.\right. \\
&\qquad \left.\left. - \int_{u\in I}\sum_{x\in\mathcal{X}} f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x \mid x', \pi_t^{N,u}(x'), \mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t^N\right)\right)du\right|\right] \\
&= \mathbb{E}\left[\left|\sum_{i\in V_N}\left(\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_{t+1}^i)\,du - \mathbb{E}\left[\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_{t+1}^i)\,du \,\middle|\, \boldsymbol{X}_t\right]\right)\right|\right] \\
&\le \left(\mathbb{E}\left[\left|\sum_{i\in V_N}\left(\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_{t+1}^i)\,du - \mathbb{E}\left[\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_{t+1}^i)\,du \,\middle|\, \boldsymbol{X}_t\right]\right)\right|^2\right]\right)^{\frac{1}{2}} \\
&= \left(\sum_{i\in V_N}\mathbb{E}\left[\left|\left(\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_{t+1}^i)\,du - \mathbb{E}\left[\int_{u\in\left(\frac{i-1}{N},\frac{i}{N}\right]} f^u(X_{t+1}^i)\,du \,\middle|\, \boldsymbol{X}_t\right]\right)\right|^2\right]\right)^{\frac{1}{2}} \\
&\le \sqrt{N\cdot\left(\frac{2C_{\varphi,f}}{N}\right)^2} = \frac{2C_{\varphi,f}}{\sqrt{N}},
\end{aligned}
$$

which goes to 0 as $N \to +\infty$.

Term I_2:

By triangle inequality, Assumption 6.2.7 (1) (2), and the bound of $|f| < C_{\varphi,f}$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left|\mu_t^N \mathbf{T}^{\pi_t^N}_{G_N,\mu_t^N,\pi_t^N}(f) - \mu_t^N \mathbf{T}^{\pi_t}_{G_N,\mu_t^N,\pi_t}(f)\right|\right] \\
&= \mathbb{E}\left[\left|\mu_t^N \mathbf{T}^{\pi_t^N}_{G_N,\mu_t^N,\pi_t^N}(f) \mp \mu_t^N \mathbf{T}^{\pi_t^N}_{G_N,\mu_t^N,\pi_t}(f) - \mu_t^N \mathbf{T}^{\pi_t}_{G_N,\mu_t^N,\pi_t}(f)\right|\right] \\
&= \mathbb{E}\left[\left|\int_{u\in I}\sum_{x\in\mathcal{X}} f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x \mid x', \pi_t^{N,u}(x'), \mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t^N\right)\right)du\right.\right. \\
&\qquad \mp \int_{u\in I}\sum_{x\in\mathcal{X}} f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x \mid x', \pi_t^{N,u}(x'), \mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)\right)du \\
&\qquad \left.\left. - \int_{u\in I}\sum_{x\in\mathcal{X}} f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x \mid x', \pi_t^{u}(x'), \mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)\right)du\right|\right] \\
&\le |\mathcal{X}|\,C_{\varphi,f}\,L_{P,Z}\,\mathbb{E}\left[\int_{u\in I}\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\left|\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t^N\right) - \mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)\right|du\right] \\
&\qquad + |\mathcal{X}|\,C_{\varphi,f}\,L_{P,a}\,\mathbb{E}\left[\int_{u\in I}\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\left|\pi_t^{N,u}(x') - \pi_t^u(x')\right|du\right] \\
&= J_1 + J_2.
\end{aligned}
$$

Since $\sum_{x'} \mu_t^{N,u}(x') = 1$, by Lemma 6.6.8, we deduce that J_1 goes to 0 as $N \to +\infty$.

For J_2, with Assumption 6.2.10 and the bound of $C_{\mathcal{A}}$ over the action space, we have:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{u\in I}\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\left|\pi_t^{N,u}(x')-\pi_t^u(x')\right|du\right]\\
&=\mathbb{E}\left[\int_{u\in I}\sum_{x'\in\mathcal{X}}\sum_{i\in V_N}\mathbb{1}_{u\in(\frac{i-1}{N},\frac{i}{N}]}\,\delta_{X_t^i}\left|\pi_t^{N,u}(x')-\pi_t^u(x')\right|du\right]\\
&=\mathbb{E}\left[\sum_{i\in V_N}\int_{u\in(\frac{i-1}{N},\frac{i}{N}]}\left|\pi_t^{N,u}(X_t^i)-\pi_t^u(X_t^i)\right|du\right]\\
&=\mathbb{E}\left[\sum_{j\in V_N\setminus\{i\}}\int_{u\in(\frac{j-1}{N},\frac{j}{N}]}\left|\pi_t^{\frac{\lceil Nu\rceil}{N}}(X_t^j)-\pi_t^u(X_t^j)\right|du\right]+\mathbb{E}\left[\int_{u\in(\frac{i-1}{N},\frac{i}{N}]}\left|\tilde{\pi}_t(X_t^i)-\pi_t^u(X_t^i)\right|du\right]\\
&\leq L_\pi\left|\frac{\lceil Nu\rceil}{N}-u\right|+\frac{2C_{\mathcal{A}}}{N}\\
&\leq\frac{L_\pi+2C_{\mathcal{A}}}{N},
\end{aligned}
$$

where we apply the same reasoning as for (6.43) in Lemma 6.6.8 for the inequality. This term goes to 0 as $N \to +\infty$.

Term I_3:

$$
\begin{aligned}
&\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t}_{G_N,\mu_t^N,\pi_t}(f)-\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\mu_t^N,\pi_t}(f)\right|\right]\\
&=\mathbb{E}\left[\left|\int_{u\in I}\sum_{x\in\mathcal{X}}f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)\right)du\right.\right.\\
&\qquad\left.\left.-\int_{u\in I}\sum_{x\in\mathcal{X}}f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right)du\right|\right].
\end{aligned}
$$

By Assumption 6.2.7 (1) and triangle inequality, we have:

$$
\begin{aligned}
&\left|\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\left(P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)\right)-P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right)\right)\right|\\
&\leq\sum_{x'\in\mathcal{X}}\left|P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)\right)-P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right)\right|\mu_t^{N,u}(x')\\
&\leq\sum_{x'\in\mathcal{X}}L_{P,Z}\left|\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)-\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right|\mu_t^{N,u}(x')\\
&=L_{P,Z}\left|\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)-\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right|.
\end{aligned}
$$

Then, the term I_3 becomes:

$$
\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t}_{G_N,\mu_t^N,\pi_t}(f)-\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\mu_t^N,\pi_t}(f)\right|\right]
$$

$$\leq L_{P,Z}\,\mathbb{E}\left[\int_{u\in I}\sum_{x\in\mathcal{X}}|f^u(x)|\left|\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)-\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right|du\right]$$
$$\leq L_{P,Z}\,|\mathcal{X}|\,C_{\varphi,f}\,\mathbb{E}\left[\int_{u\in I}\left|\mathbf{Z}^u\left(G_N,\,\mu_t^N,\,\pi_t\right)-\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right|du\right].$$

Applying Lebesgue's dominated convergence theorem, with the same reasoning in the proof of Lemma 6.6.8, the term I_3 goes to 0 as $N\to+\infty$.

Term I_4:
Similarly with the reasoning in I_3, we use Assumption 6.2.7 (1), $|f|\leq C_{\varphi,f}$, and triangle inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t^N,\,\pi_t}(f)-\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t,\,\pi_t}(f)\right|\right]\\
&=\mathbb{E}\left[\left|\int_{u\in I}\sum_{x\in\mathcal{X}}f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)\right)du\right.\right.\\
&\qquad\left.\left.-\int_{u\in I}\sum_{x\in\mathcal{X}}f^u(x)\sum_{x'\in\mathcal{X}}\mu_t^{N,u}(x')\,P\left(x\mid x',\pi_t^u(x'),\mathbf{Z}^u\left(G,\,\mu_t,\,\pi_t\right)\right)du\right|\right]\\
&\leq|\mathcal{X}|\,C_{\varphi,f}\,L_{P,Z}\,\mathbb{E}\left[\int_{u\in I}\left|\mathbf{Z}^u\left(G,\,\mu_t^N,\,\pi_t\right)-\mathbf{Z}^u\left(G,\,\mu_t,\,\pi_t\right)\right|du\right]\\
&=|\mathcal{X}|\,C_{\varphi,f}\,L_{P,Z}\,\mathbb{E}\left[\int_{u\in I}\left|\int_{v\in I}G(u,v)\sum_{y\in\mathcal{X}}\varphi(y,\pi_t^v(y))\left(\mu_t^{N,v}(y)-\mu_t^v(y)\right)dv\right|du\right],\\
&=|\mathcal{X}|\,C_{\varphi,f}\,L_{P,Z}\int_{u\in I}\mathbb{E}\left[\left|\int_{v\in I}G(u,v)\sum_{y\in\mathcal{X}}\varphi(y,\pi_t^v(y))\left(\mu_t^{N,v}(y)-\mu_t^v(y)\right)dv\right|\right]du,
\end{aligned}$$

where the last equality is derived from Fubini's theorem. Let $I\times I\times\mathcal{X}\ni(u,v,y)\mapsto\hat{f}^{u,v}(y):=G(u,v)\,\varphi(y,\pi_t^v(y))$ be such that

$$I\times\mathcal{X}\ni(u,y)\mapsto\hat{f}^u(y)=\int_{v\in I}G(u,v)\,\varphi(y,\pi_t^v(y))\,dv.$$

Note that $|\hat{f}^u|\leq C_\varphi\leq C_{\varphi,f}$ by Assumption 6.2.8. Then, we have

$$\begin{aligned}
&\int_{u\in I}\mathbb{E}\left[\left|\int_{v\in I}G(u,v)\sum_{y\in\mathcal{X}}\varphi(y,\pi_t^v(y))\left(\mu_t^{N,v}(y)-\mu_t^v(y)\right)dv\right|\right]du\\
&=\int_{u\in I}\mathbb{E}\left[\left|\int_{v\in I}\sum_{y\in\mathcal{X}}\hat{f}^{u,v}(y)\left(\mu_t^{N,v}(y)-\mu_t^v(y)\right)dv\right|\right]du\\
&=\int_{u\in I}\mathbb{E}\left[\left|\,\mu_t^N(\hat{f}^u)-\mu_t(\hat{f}^u)\right|\right]du,
\end{aligned}$$

and the fourth term becomes:

$$\mathbb{E}\left[\left|\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t^N,\,\pi_t}(f)-\mu_t^N\,\mathbf{T}^{\pi_t}_{G,\,\mu_t,\,\pi_t}(f)\right|\right]\leq|\mathcal{X}|\,L_{P,Z}\,C_{\varphi,f}\int_{u\in I}\mathbb{E}\left[\left|\,\mu_t^N(\hat{f}^u)-\mu_t(\hat{f}^u)\right|\right]du.$$

By the dominated convergence theorem and our induction hypothesis, the fourth term goes to 0 as $N \to +\infty$.

Term I_5:

$$
\begin{aligned}
&\mathbb{E}\left[\left|\mu_t^N \mathbf{T}^{\pi_t}_{G,\,\mu_t,\,\pi_t}(f) - \mu_{t+1}(f)\right|\right] \\
&= \mathbb{E}\left[\left|\int_{u\in I} \sum_{x\in\mathcal{X}} f^u(x) \sum_{x'\in\mathcal{X}} \mu_t^{N,u}(x')\, P\left(x \mid x', \pi_t^u(x'), \mathbf{Z}^u\left(G,\, \mu_t,\, \pi_t\right)\right) du\right.\right. \\
&\qquad \left.\left. - \int_{u\in I} \sum_{x\in\mathcal{X}} f^u(x) \sum_{x'\in\mathcal{X}} \mu_t^{u}(x')\, P\left(x \mid x', \pi_t^u(x'), \mathbf{Z}^u\left(G,\, \mu_t,\, \pi_t\right)\right) du\right|\right] \\
&= \mathbb{E}\left[\left|\int_{u\in I} \sum_{x\in\mathcal{X}} f^u(x) \sum_{x'\in\mathcal{X}} \left(\mu_t^{N,u}(x') - \mu_t^{u}(x')\right) P\left(x \mid x', \pi_t^u(x'), \mathbf{Z}^u\left(G,\, \mu_t\, \pi_t\right)\right) du\right|\right] \\
&= \mathbb{E}\left[\left|\int_{u\in I} \sum_{x'\in\mathcal{X}} \tilde{f}^u(x') \left(\mu_t^{N,u}(x') - \mu_t^{u}(x')\right) du\right|\right] \\
&= \mathbb{E}\left[\left|\mu_t^N(\tilde{f}) - \mu_t(\tilde{f})\right|\right],
\end{aligned}
$$

where

$$
\tilde{f}^u(x') = \sum_{x\in\mathcal{X}} f^u(x)\, P\left(x \mid x', \pi_t^u(x'), \mathbf{Z}^u\left(G,\, \mu_t,\, \pi_t\right)\right).
$$

We apply the induction hypothesis, the fifth term goes to 0 as $N \to +\infty$.

□

Proof of Lemma 6.6.8. For all $t \in \overline{\mathcal{T}}$ and $u \in I$, by triangle inequality, we have

$$
\begin{aligned}
&\left|\mathbf{Z}^u\left(G, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t, \pi_t^N\right)\right| \\
&\quad \le \left|\mathbf{Z}^u\left(G, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t, \pi_t\right)\right| + \left|\mathbf{Z}^u\left(G_N, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t, \pi_t^N\right)\right|.
\end{aligned}
$$

The first term: By triangle inequality and Assumption 6.2.8, we have

$$
\begin{aligned}
&\int_{u\in I} \left|\mathbf{Z}^u\left(G, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t, \pi_t\right)\right| du \\
&= \int_{u\in I} \left| \int_{v\in I} \left(G(u,v) - G_N(u,v)\right) \sum_{x\in\mathcal{X}} \varphi(x, \pi_t^v(x))\, \mu_t^v(x)\, dv \right| du \\
&= C_\varphi \int_{u\in I} \left| \int_{v\in I} \left(G(u,v) - G_N(u,v)\right) \frac{\sum_{x\in\mathcal{X}} \varphi(x, \pi_t^v(x))\, \mu_t^v(x)}{C_\varphi}\, dv \right| du \\
&\le C_\varphi \underbrace{\sup_{\|\Phi\|_\infty \le 1} \int_{u\in I} \left| \int_{v\in I} \left(G(u,v) - G_N(u,v)\right) \Phi(v) dv \right| du}_{= \|G - G_N\|_{\infty\to 1}},
\end{aligned}
$$

where the last inequality holds due to the fact that the function

$$
\Phi(v) = \frac{\sum_{x\in\mathcal{X}} \varphi(x, \pi_t^v(x))\, \mu_t^v(x)}{C_\varphi}
$$

is in the set of measurable functions Φ such that $\|\Phi\|_\infty \leq 1$. We conclude by Assumption 6.2.1 that the first term goes to 0 as $N \to +\infty$.

The second term:
Using the fact that $|G_N(u,v)| \leq 1$ for all $u, v \in I$, Assumption 6.2.8 and triangle inequality, we have:

$$\begin{aligned}
&\int_{u\in I} |\, \mathbf{Z}^u\left(G_N, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t, \pi_t^N\right)|du \\
&= \int_{u\in I} \left| \int_{v\in I} G_N(u,v) \sum_{x\in\mathcal{X}} \left(\varphi(x, \pi_t^{N,v}(x)) - \varphi(x, \pi_t^v(x))\right) \mu_t^v(x)\, dv \right| du \\
&\leq \int_{u\in I}\int_{v\in I} |\, G_N(u,v)| \sum_{x\in\mathcal{X}} \left|\varphi(x, \pi_t^{N,v}(x)) - \varphi(x, \pi_t^v(x))\right| \mu_t^v(x)\, dv\, du \\
&\leq L_\varphi \int_{v\in I} \sum_{x\in\mathcal{X}} \left|\, \pi_t^{N,v}(x) - \pi_t^v(x)\right| \mu_t^v(x)\, dv\,.
\end{aligned}$$

Furthermore, by Assumption 6.2.10,

$$\begin{aligned}
&\int_{v\in I} \sum_{x\in\mathcal{X}} \left|\, \pi_t^{N,v}(x) - \pi_t^v(x)\right| \mu_t^v(x)\, dv \\
&= \sum_{j\in V_N\setminus\{i\}} \int_{v\in\left(\frac{j-1}{N}, \frac{j}{N}\right]} \sum_{x\in\mathcal{X}} \left|\, \pi_t^{\frac{\lceil Nv\rceil}{N}}(x) - \pi_t^v(x)\right| \mu_t^v(x)\, dv + \int_{v\in\left(\frac{i-1}{N}, \frac{i}{N}\right]} \sum_{x\in\mathcal{X}} \left|\, \tilde{\pi}_t(x) - \pi_t^v(x)\right| \mu_t^v(x)\, dv \\
&\leq \sum_{j\in V_N\setminus\{i\}} \int_{v\in\left(\frac{j-1}{N}, \frac{j}{N}\right]} L_\pi \left|\frac{\lceil Nv\rceil}{N} - v\right| \sum_{x\in\mathcal{X}} \mu_t^v(x)\, dv + 2\, C_{\mathcal{A}} \int_{v\in\left(\frac{i-1}{N}, \frac{i}{N}\right]} \sum_{x\in\mathcal{X}} \mu_t^v(x)\, dv \\
&\leq \frac{L_\pi + 2C_{\mathcal{A}}}{N},
\end{aligned}$$

where the last inequality stems from the fact that $|\,\lceil Nv\rceil - Nv\,| \leq 1$ so:

$$\frac{|\,\lceil Nv\rceil - Nv\,|}{N} = \left|\frac{\lceil Nv\rceil}{N} - v\right| \leq \frac{1}{N}. \tag{6.43}$$

Hence,

$$\int_{u\in I} |\mathbf{Z}^u\left(G_N, \mu_t, \pi_t\right) - \mathbf{Z}^u\left(G_N, \mu_t, \pi_t^N\right)|\, du \leq \frac{L_\varphi(L_\pi + 2C_{\mathcal{A}})}{N},$$

which goes to 0 as $N \to +\infty$.

□

6.7 Application in finance - optimal investment with relative performance criterion

6.7.1 Model

We consider a one dimensional investment problem. Let us assume that the players invest in a stock $S \in \mathbb{R}$ with *Geometric Brownian motion* (GBM) dynamics

$$dS_t = S_t\,(\kappa\,dt + \sigma\,dW_t), \qquad \forall\, u \in I, S_0 > 0,$$

where $\kappa, \sigma \in \mathbb{R}$ and W_t is a real-valued Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Given a real-valued $(\mathcal{F}_t)_{t\in[0,T]}$-adapted trading strategy π_t^u which represents the amount of wealth invested in the stock at time t employed by agent u, the wealth follows the dynamics

$$dX_t^u = \pi_t^u\,(\kappa\,dt + \sigma dW_t)\,, \qquad \forall\, u \in I.$$

Since we focus on a finite-state, discrete-time problem, we approximate the controlled diffusion process by a controlled Markov chain according to [162, page 100].

- State space $\mathcal{X}$: we fix a bound $B > 0$, and restrict the value of X to stay in the interval $[-B, B]$. We choose a space step h such that

$$B \in h\mathbb{Z} \quad \text{and we define} \quad \mathcal{X} = h\mathbb{Z} \cap [-B, B].$$

- Action space $\mathcal{A} \subseteq \mathbb{R}$ such that for all $a \in \mathcal{A}$, $a \in [-C_{\mathcal{A}}, C_{\mathcal{A}}]$.

- $P : \mathcal{X} \times \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is a one-step transition probability kernel. Denote

$$D^h := \max_{a\in\mathcal{A}} \left\{(\sigma\,a)^2 + h|a\,\kappa|\right\}, \qquad \Delta t^h := \frac{h^2}{D^h}$$

  Then, the transition probabilities from x to $x \pm h$ for all $x \in h\mathbb{Z} \cap (-B, B)$ are

$$\begin{aligned} P(x + h\,|\,x, a) &= \frac{(\sigma\,a)^2/2 + h(a\,\kappa)^+}{D^h}, \\ P(x - h\,|\,x, a) &= \frac{(\sigma\,a)^2/2 + h(a\,\kappa)^-}{D^h}, \\ P(x\,|\,x, a) = 1 - \sum_{y\neq x} P(y\,|\,x, a) &= \frac{D^h - \left[(\sigma\,a)^2 + h|a\,\kappa^u|\right]}{D^h}. \end{aligned}$$

  We consider the "absorbing" boundary conditions:

$$P(B|B, a) = 1, \quad P(-B| - B, a) = 1.$$

- $r : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ is a reward function

$$r(x, Z_t^u) = -\exp\left\{\frac{1}{\eta}|\, x - Z_t^u\,|\right\}\Delta t^h.$$

  This can be interpreted as an exponential utility function. Since the reward is negative, maximizing it effectively amounts to minimizing the exponential penalty $\exp\{\frac{1}{\eta}\,|x - Z_t^u|\}$. In other words, as the absolute distance $|x - Z_t^u|$ increases, the exponential penalty grows, thereby making the reward more negative. Consequently, to maximize the reward, players of index u are encouraged to choose states x close to the aggregate Z_t^u. The parameter η serves as a "risk-aversion" coefficient: smaller values of η make the penalty more sensitive to deviations $|x - Z_t^u|$.

- $\varphi(x, a) = \varphi(x) = \rho\, x$, where ρ is a parameter that models how much market participants prioritize competition.

- Given an initial distribution $\mu_0^u := \mu_0 \in \mathcal{P}(\mathcal{X})$ for the continuous-time process for all $u \in I$, we define its discretization on $\mathcal{X}$ by:

$$\begin{cases} \mu_0^h\left(\{lh\}\right) = \mu_0\left(\left[lh - \dfrac{h}{2}, lh + \dfrac{h}{2}\right)\right), \quad \forall l \in \left\{-\dfrac{B}{h}, \ldots, \dfrac{B}{h}\right\}, \\ \mu_0^h\left(\{-B\}\right) = \mu_0\left(\left(-\infty, -B + \dfrac{h}{2}\right)\right), \\ \mu_0^h\left(\{B\}\right) = \mu_0\left(\left[B - \dfrac{h}{2}, +\infty\right)\right). \end{cases}$$

Consider a discretization $0 = u_0, \ldots, u_M = 1$, $u_m = m\Delta u$, of the interval I with $\Delta u = 1/M$. For $m = 0, \ldots, M$ and $t \in \mathcal{T}$, the discretized version of the system (6.7) can be written as:

$$\begin{cases} \mu_{t+1}^{u_m}(x) = \displaystyle\sum_{x' \in \mathcal{X}} \mu_t^{u_m}(x') P(x \mid x', \pi_t^{u_m}(x')), & \forall x \in \mathcal{X}, \\ V_t^{u_m}(x) = -\exp\left\{\dfrac{1}{\eta}|\, x - Z_t^{u_m}\,|\right\}\Delta t^h + \displaystyle\sum_{x' \in \mathcal{X}} P(x' \mid x, \pi_t^{u_m}(x))\, V_{t+1}^{u_m}(x'), & \forall x \in \mathcal{X}, \\ Z_t^{u_m} = \displaystyle\sum_{\ell=0}^{M} G(u_m, u_\ell) \sum_{x \in \mathcal{X}} \varphi(x)\, \mu_t^{u_\ell}(x)\, \Delta u, & \\ \pi_t^{u_m}(x) = \displaystyle\operatorname*{argmax}_a \left\{-\exp\left\{\dfrac{1}{\eta}|\, x - Z_t^{u_m}\,|\right\}\Delta t^h + \sum_{x' \in \mathcal{X}} P(x' \mid x, a)\, V_{t+1}^{u_m}(x')\right\}, & \forall x \in \mathcal{X}, \\ \mu_0^{u_m}(x) = \mu_0(x),\ V_T^{u_m}(x) = 0, & \forall x \in \mathcal{X}. \end{cases} \tag{6.44}$$

We apply the fixed-point iterations described in Algorithm 4 to solve the discretized system (6.44). If we choose the geometric Brownian motion parameters κ and σ for the stocks, along with the spatial step size h, such that

$$C_{\mathcal{A}} > \frac{h\,|\kappa|}{\sigma^2},$$

then the policy admits a closed-form solution, as detailed in Proposition 6.7.1 below.

Algorithm 4 Fixed point for graphon game

Input: Initial policy π^0, bound of the state space B, space-step h, bound of the action space $C_{\mathcal{A}}$, risk aversion parameter η, market prioritization parameter ρ, number of indices M, dimension of the stock d, final time T, GBM parameter for stocks κ and σ, graphon matrix G, number of iterations K;

Output: Approximation of μ^*, V^*

Initialize μ_0 at time 0 with the bound of the state space B space-step h

Initialize the sequence of $\mu^0 := (\mu_t^{\pi^0,u_m})_{t\in\overline{\mathcal{T}},m\in[M]}$ with given initial policy π^0

Initialize the sequence of $Z^0 := (Z_t^{u_m})_{t\in\overline{\mathcal{T}},m\in[M]}$ with μ^0

Initialize the sequence of $V^0 := (V_t^{u_m})_{t\in\overline{\mathcal{T}},m\in[M]}$ with μ^0 and Z^0

for $k = 1, \dots, K$: **do**

Update π^k: for all $x \in \mathcal{X}$, $m = 0, \dots, M$ and $t \in \mathcal{T}$:

$$\pi_t^{k,u_m} := \operatorname*{argmax}_a \left\{ -e^{\frac{1}{\eta}\left| x - Z_t^{k-1,u_m} \right|} \Delta t^h + \sum_{x'\in\mathcal{X}} P(x' \mid x, a)\, V_{t+1}^{k-1,u_m}(x') \right\}$$

Update $\mu^k := \mu^{\pi^k}$: for all $x \in \mathcal{X}$ and $m = 0, \dots, M$:

$$\begin{cases} \mu_0^{k,u_m}(x) = \mu_0 \\ \mu_{t+1}^{k,u_m}(x) = \sum_{x'\in\mathcal{X}} \mu_t^{k,u_m}(x') P(x \mid x', \pi_t^{k,u_m}(x')), \quad t \in \mathcal{T} \end{cases}$$

Update $Z^k := Z^{\mu^k}$: for all $m = 0, \dots, M$ and $t \in \overline{\mathcal{T}}$:

$$Z_t^{k,u_m} = \frac{1}{M} \sum_{\ell=0}^{M} G(u_m, u_\ell) \sum_{x\in\mathcal{X}} \varphi(x)\, \mu_t^{k,u_\ell}(x)$$

Update $V^k := V^{\pi^k, Z^k}$: for all $x \in \mathcal{X}$, $m = 0, \dots, M$ and $t \in \mathcal{T}$:

$$\begin{cases} V_T^{k,u_m}(x) = 0 \\ V_t^{k,u_m}(x) = -e^{\frac{1}{\eta}\left| x - Z_t^{k,u_m} \right|} \Delta t^h + \sum_{x'\in\mathcal{X}} P(x' \mid x, \pi_t^{k,u_m}(x))\, V_{t+1}^{k,u_m}(x'), \quad t \in \mathcal{T} \end{cases}$$

end for

return π^K, μ^K

Proposition 6.7.1. *Let $\kappa > 0$ and we choose h such that $C_{\mathcal{A}} > \frac{h|\kappa|}{\sigma^2}$ is satisfied. For $t \in \mathcal{T}$, $m = 0, \dots, M$, and $x \in \mathcal{X}$, consider $V_t^{u_m}(x)$ arising in the discretized system (6.44). Fix $x \in \mathcal{X}$,*

$m = 0, \dots, M$ and $t \in \mathcal{T}$ and define

$$A = \sigma^2 \left(\frac{V_{t+1}(x+h) + V_{t+1}(x-h)}{2} - V_{t+1}(x) \right), \qquad C = D^h V_{t+1}(x),$$
$$B^+ = h\kappa[V_{t+1}(x+h) - V_{t+1}(x)], \qquad B^- = h\kappa[V_{t+1}(x) - V_{t+1}(x-h)].$$

Then, the optimal action $\pi_t^{u_m}(x)$ is given by:

- $\pi_t^{u_m}(x) = C_{\mathcal{A}}$, *if one of the following cases holds:*

$$\begin{cases} A > 0 \ \text{ and } \ C_{\mathcal{A}} \geq \dfrac{-B^+}{A} \ \text{ and } \ (B^+ + B^-) \geq 0, \\ A = 0 \ \text{ and } \ B^+ > 0 \ \text{ and } \ (B^+ + B^-) \geq 0, \\ A < 0 \ \text{ and } \ B^+ > 0 \ \text{ and } \ C_{\mathcal{A}} < -\dfrac{B^+}{2A}. \end{cases}$$

- $\pi_t^{u_m}(x) = -C_{\mathcal{A}}$, *if one of the following cases holds:*

$$\begin{cases} A > 0 \ \text{ and } \ C_{\mathcal{A}} \geq \dfrac{B^-}{A} \ \text{ and } \ (B^+ + B^-) < 0, \\ A = 0 \ \text{ and } \ B^- < 0 \ \text{ and } \ (B^+ + B^-) < 0, \\ A < 0 \ \text{ and } \ B^- < 0 \ \text{ and } \ C_{\mathcal{A}} < \dfrac{B^-}{2A}. \end{cases}$$

- $\pi_t^{u_m}(x) = -\frac{B^+}{2A}$, *if $A < 0$ and $B^+ > 0$ and $C_{\mathcal{A}} \geq -\frac{B^+}{2A}$.*
- $\pi_t^{u_m}(x) = -\frac{B^-}{2A}$, *if $A < 0$ and $B^- < 0$ and $C_{\mathcal{A}} \geq \frac{B^-}{2A}$.*
- $\pi_t^{u_m}(x) = 0$ *otherwise.*

Proof of Proposition 6.7.1. For each $x \in \mathcal{X}$, $m = 0, \dots, M$, and $t \in \mathcal{T}$, observe that $Z_t^{u_m}$ is independent of a. Consequently, the policy $\pi_t^{u_m}(x)$ depends on a only through the sum

$$\sum_{x' \in \mathcal{X}} P(x' \mid x, a)\, V_{t+1}^{u_m}(x').$$

For simplicity, denote $V(\cdot) := V_{t+1}^{u_m}(\cdot)$ for all $t \in \mathcal{T}$ and $m = 0, \dots, M$. Define

$$\mathrm{PV}(x, a) = \sum_{x' \in \mathcal{X}} P(x' \mid x, a)\, V(x').$$

Recall that

$$D^h \ := \ \max_{a \in \mathcal{A}} \Big\{ (\sigma a)^2 \ + \ h\, |a\, \kappa| \Big\}.$$

Since $C_{\mathcal{A}} \ > \ \dfrac{h\,|\kappa|}{\sigma^2}$, the maximum is attained at $a = \pm\, C_{\mathcal{A}}$. Hence,

$$D^h = (\sigma\, C_{\mathcal{A}})^2 \ + \ h\, |C_{\mathcal{A}}\, \kappa|.$$

We split the analysis into two cases: $a \geq 0$ and $a \leq 0$. The point $a = 0$ is included in both cases, as it might be the maximizer of $\mathrm{PV}(x, a)$.

Case 1: $a \geq 0$. In this region, $(a\kappa)^+ = a\kappa$, $(a\kappa)^- = 0$, and $|a\kappa| = a\kappa$. Hence,

$$\begin{aligned} P(x+h \mid x, a) &= \frac{(\sigma\, a)^2/2 + h\, a\, \kappa}{D^h}, \\ P(x-h \mid x, a) &= \frac{\sigma\, a)^2/2}{D^h}, \\ P(x \mid x, a) &= \frac{D^h - \left[(\sigma\, a)^2 + h\, a\, \kappa\right]}{D^h}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{PV}(x, a) &= P(x+h \mid x, a)\, V(x+h) + P(x-h \mid x, a)\, V(x-h) + P(x \mid x, a)\, V(x) \\ &= \frac{\left(\frac{(\sigma\, a)^2}{2} + ha\kappa\right) V(x+h) + \left(\frac{(\sigma\, a)^2}{2}\right) V(x-h) + \left[D^h - (\sigma\, a)^2 - ha\kappa\right] V(x)}{D^h} \\ &= \frac{\sigma^2 \left(\frac{V(x+h)+V(x-h)}{2} - V(x)\right) a^2 + h\kappa[V(x+h) - V(x)]a + D^h V(x)}{D^h} \\ &= \frac{A\, a^2 + B^+ a + C}{D^h}. \end{aligned}$$

Subcases for $a \geq 0$:

- $\boldsymbol{A > 0}$: The function $\mathrm{PV}(x, a)$ is strictly convex in a.
  - If $C_{\mathcal{A}} \geq -\frac{B^+}{A}$, the maximum is attained at $a = C_{\mathcal{A}}$. Hence,
  $$\mathrm{PV}(x, a) = \frac{A\left(C_{\mathcal{A}}\right)^2 + B^+ C_{\mathcal{A}}}{D^h} + V(x).$$
  - If $C_{\mathcal{A}} < -\frac{B^+}{A}$, the maximum is attained at $a = 0$.
- $\boldsymbol{A = 0}$: The function $\mathrm{PV}(x, a)$ is linear in a.
  - If $V(x+h) > V(x)$ $\left(B^+ > 0\right)$, the maximum is attained at $a = C_{\mathcal{A}}$.
  $$\mathrm{PV}(x, a) = \frac{B^+ C_{\mathcal{A}}}{D^h} + V(x).$$
  - If $V(x+h) \leq V(x)$ $\left(B^+ \leq 0\right)$, the maximum is attained at $a = 0$.
  $$\mathrm{PV}(x, a) = V(x).$$
- $\boldsymbol{A < 0}$: The function $\mathrm{PV}(x, a)$ is concave in a. Let $a^* = -\frac{B^+}{2A}$.

– If $B^+ > 0$ and $C_{\mathcal{A}} \geq a^*$, then $a^* > 0$ is the maximizer:

$$\mathrm{PV}(x, a^*) = \frac{4AC - (B^+)^2}{4AD^h}.$$

If instead $C_{\mathcal{A}} < a^*$, the maximum is attained at $a = C_{\mathcal{A}}$.

– If $B^+ \leq 0$, then $a^* \leq 0$ and does not lie in $[0, C_{\mathcal{A}}]$, so the maximum is attained at $a = 0$.

$$\mathrm{PV}(x, 0) = V(x).$$

Case 2: $a \leq 0$. For $a \leq 0$, $(a\kappa)^+ = 0$, $(a\kappa)^- = -\,a\,\kappa$, and $|a\kappa| = -\,a\,\kappa$. Thus,

$$\begin{aligned} P(x+h \mid x, a) &= \frac{(\sigma\, a)^2/2}{D^h}, \\ P(x-h \mid x, a) &= \frac{(\sigma\, a)^2/2 - ha\kappa}{D^h}, \\ P(x \mid x, a) &= \frac{D^h - ((\sigma\, a)^2 - ha\kappa)}{D^h}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{PV}(x, a) &= P(x+h \mid x, a)\, V(x+h) + P(x-h \mid x, a)\, V(x-h) + P(x \mid x, a)\, V(x) \\ &= \frac{(\frac{(\sigma\, a)^2}{2})V(x+h) + [(\frac{(\sigma\, a)^2}{2}) - ha\kappa]V(x-h) + [D^h - (\sigma\, a)^2 + ha\kappa]V(x)}{D^h} \\ &= \frac{\sigma^2 \left(\frac{V(x+h)+V(x-h)}{2} - V(x)\right) a^2 + h\kappa[V(x) - V(x-h)]a + D^h V(x)}{D^h} \\ &= \frac{A\, a^2 + B^- a + C}{D^h}. \end{aligned}$$

Notably, the same A and C appear in both cases, while the linear term is B^- here:

$$B^- = h\,\kappa\,[V(x) - V(x-h)].$$

Subcases for $a \leq 0$:

- $\boldsymbol{A > 0}$: The function $\mathrm{PV}(x, a)$ is strictly convex in a.

  – If $C_{\mathcal{A}} \geq \frac{B^-}{A}$, then the boundary $a = -C_{\mathcal{A}}$ is the maximizer:

  $$\mathrm{PV}(x, a) = \frac{A\,(C_{\mathcal{A}})^2 - B^-\, C_{\mathcal{A}}}{D^h} \;+\; V(x).$$

  – If $C_{\mathcal{A}} < \frac{B^-}{A}$, the maximum is reached at $a = 0$.

- $\boldsymbol{A = 0}$: The function $\mathrm{PV}(x, a)$ is linear in a.

  – If $V(x) > V(x-h)$ $(B^- \geq 0)$, the maximum occurs at $a = 0$.

  $$\mathrm{PV}(x, 0) = V(x).$$

– If $V(x) < V(x-h)$ $(B^- < 0)$, the maximum occurs at $a = -C_{\mathcal{A}}$:

$$\mathrm{PV}(x,a) = \frac{B^-\,(-C_{\mathcal{A}})}{D^h} + V(x).$$

- $\boldsymbol{A < 0}$: The function $\mathrm{PV}(x,a)$ is concave in a. Let $\underline{a}^* = -\dfrac{B^-}{2\,A}$.

  – If $V(x) > V(x-h)$ $(B^- \geq 0)$, then $\underline{a}^* \geq 0$. Since here $a \leq 0$, the maximum is attained at $a = 0$:

  $$\mathrm{PV}(x,0) = V(x).$$

  – If $V(x) < V(x-h)$ $(B^- < 0)$ and $C_{\mathcal{A}} \geq |\underline{a}^*|$, then $\underline{a}^* < 0$ is in $[-C_{\mathcal{A}}, 0]$. The maximum is attained at $a = \underline{a}^*$:

  $$\mathrm{PV}(x,a) = \frac{4AC - (B^-)^2}{4AD^h}.$$

  Otherwise, if $C_{\mathcal{A}} < |\underline{a}^*|$, the maximum is at $a = -C_{\mathcal{A}}$.

□

6.7.2 Numerical results

In this section, we present various numerical experiments highlighting different types of graphon interactions. Specifically, we focus on the following graphons:

- *Constant Graphon.* The function G_1 is given by

$$G_1(u,v) \;=\; 1, \qquad \forall\,(u,v) \in I \times I.$$

  In this setting, all players are symmetric and each player interacts equally with every other player, thus recovering a standard *Mean Field Games* framework.

- *Two-block graphon with no interaction.* The function G_2 is defined by

$$G_2(u,v) \;=\; \begin{cases} 1, & \text{if } (u,v) \in [0,\frac{1}{2}] \times [0,\frac{1}{2}], \\ 1, & \text{if } (u,v) \in (\frac{1}{2},1] \times (\frac{1}{2},1], \\ 0, & \text{otherwise.} \end{cases}$$

  Here, the population splits naturally into two independent groups. Players with index $u \leq \frac{1}{2}$ only interact with each other, while those with index $u > \frac{1}{2}$ also only interact within their own group.

- *Two-block graphon with interaction.* The function G_3 is given by

$$G_3(u,v) \;=\; \begin{cases} 0.6, & \text{if } (u,v) \in [0,\frac{1}{2}] \times [0,\frac{1}{2}], \\ 0.6, & \text{if } (u,v) \in (\frac{1}{2},1] \times (\frac{1}{2},1], \\ 0.4, & \text{otherwise.} \end{cases}$$

  In this case, two population groups interact more frequently within their respective groups (at intensity 0.6) and less frequently with the other group (at intensity 0.4).

Parameters We set the initial policy to $\pi^0 = 1$ for all players, choose a state-space bound $B = 5$, and take a space-step $h = 0.5$. The action-space bound is $C_{\mathcal{A}} = 10$, the risk-aversion parameter is $\eta = 3$, and the market prioritization parameter is $\rho = 1$. We consider $M = 30$ agents, and a final time $T = 5$. We consider a uniform discretization of the interval $[0, T]$ with step size Δt^h. Concretely, we define the time steps

$$t_{\tilde{\ell}} = \tilde{\ell}\,\Delta t^h, \qquad \tilde{\ell} \;=\; 0, 1, 2, \ldots, \tfrac{T}{\Delta t^h}.$$

The stock price follows a discrete time approximation of geometric Brownian motion with parameters $\kappa = 0.1$ and $\sigma = 0.2$. We run $K = 20$ iterations of Algorithm 4, which converges within these 20 iterations in our numerical experiments.

Convergence analysis We adapt the definition of exploitability from the introduction (1.10) to our setting. Specifically, we define the function $\mathcal{E}\colon \Pi \to \mathbb{R}$ by

$$\mathcal{E}(\pi^u) \;:=\; \max_{\hat{\pi}} J(\hat{\pi}; \hat{Z}^u) \;-\; J(\pi^u; \hat{Z}^u), \qquad \forall\, u \in I,\ \pi^u \in \Pi.$$

To measure convergence, we compute the element-wise absolute differences of the state distribution and value function, namely $|\mu^k - \mu^{k-1}|$ and $|V^k - V^{k-1}|$ for $k = 1, \ldots, K$, along with the exploitability $\mathcal{E}(\pi^k)$. Our stopping criterion is

$$\min\Big\{ |\mu^k - \mu^{k-1}|,\ |V^k - V^{k-1}|,\ \mathcal{E}(\pi^k) \Big\} \;<\; 10^{-5}.$$

In the figures below, Fig. 6.1 shows the convergence results for the constant graphon G_1, corresponding to the standard mean-field game setup. Fig. 6.2 shows the convergence for the two-block graphon G_2 with no interaction, effectively modeling two isolated mean-field game populations. Finally, Fig. 6.3 demonstrates the convergence for the two-block graphon G_3, where partial interaction occurs between the two groups.

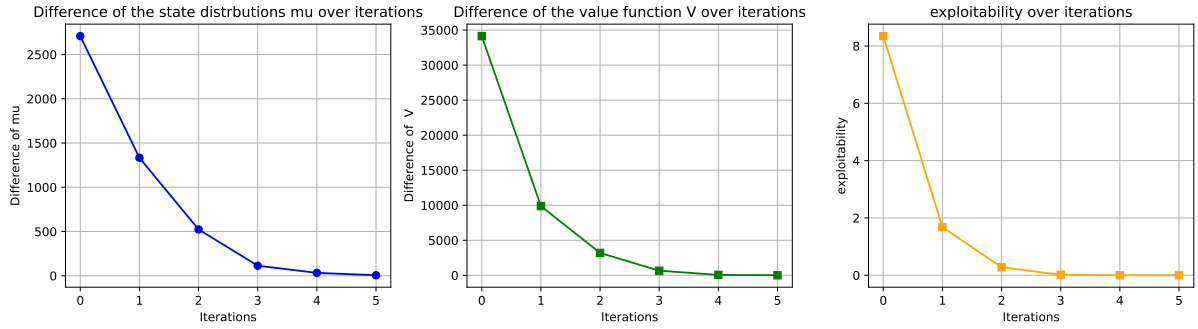


Figure 6.1: Convergence results with constant graphon G_1

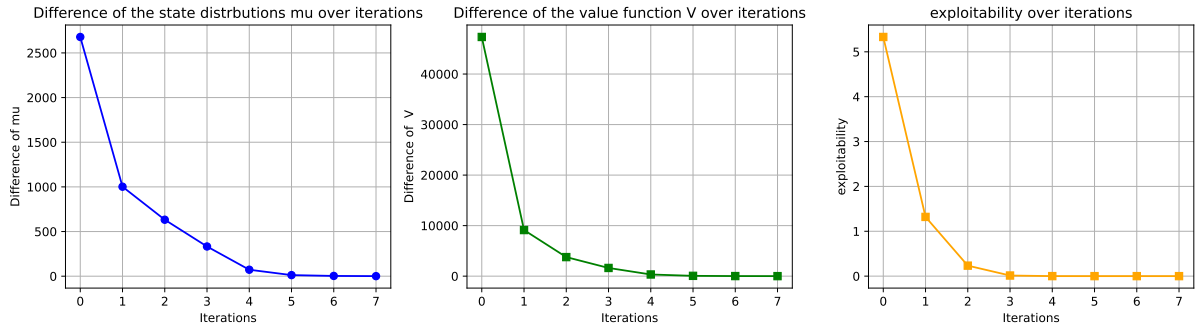


Figure 6.2: Convergence results with two-block graphon with zero interaction G_2

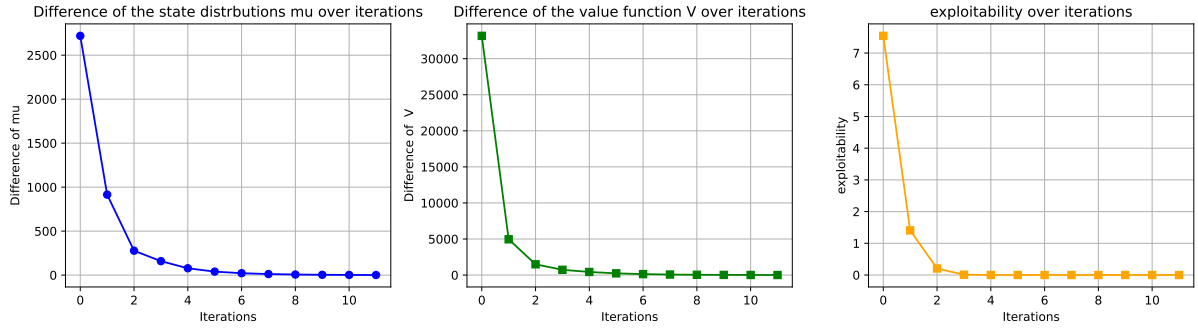


Figure 6.3: Convergence results with two-block graphon with interaction G_3

State distributions. We generate the initial distribution by assigning the state of players with index $u \leq \frac{1}{2}$ to be uniformly distributed over $[0, \frac{1}{2}]$, while players with index $u > \frac{1}{2}$ are uniformly distributed over $(\frac{1}{2}, 1]$.

Recall that our reward function involves the absolute difference between the player's state and the graphon aggregate:

$$r(x, Z_t^u) \;=\; -\exp\Big\{\tfrac{1}{\eta}\,|\,x - Z_t^u\,|\Big\}\,\Delta t^{\,h}.$$

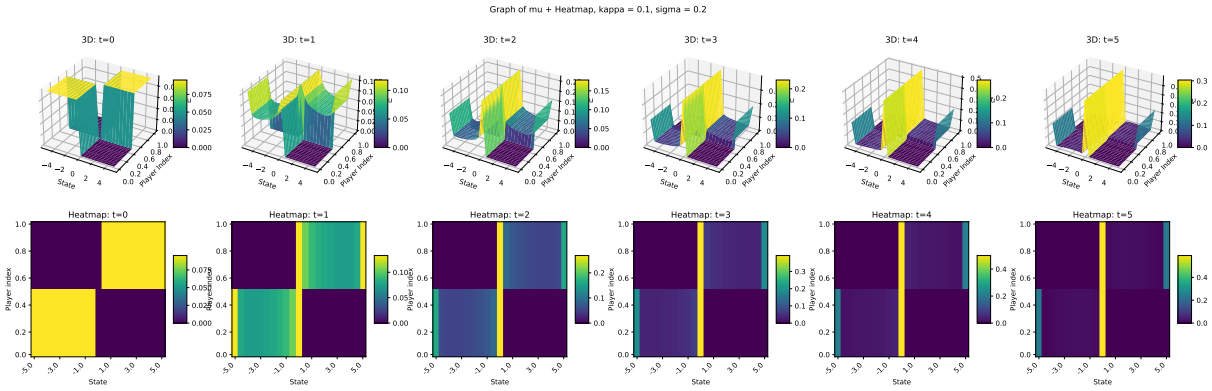


Figure 6.4: The state distribution μ over constant graphon G_1

In the classical MFG setting above, corresponding to the constant graphon G_1, all players interact symmetrically. As illustrated in Fig. 6.4, since the reward encourages players to minimize the distance to the aggregate, the final distribution concentrates around the center of the state space, even though the initial distribution is uniform over two separate groups.

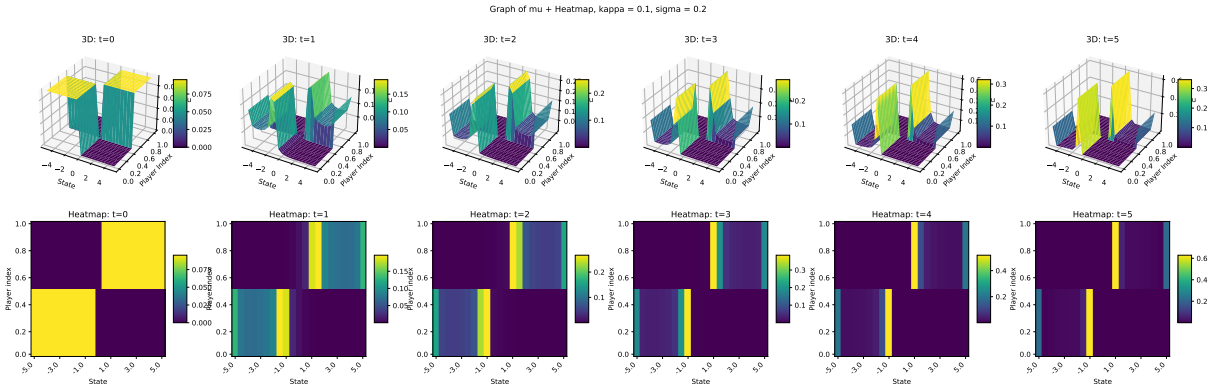


Figure 6.5: The state distribution μ over two-block graphon with zero interaction G_2

Fig. 6.5 above displays the dynamics under the two-block graphon without between the blocks G_2. Here, players with $u \leq \frac{1}{2}$ interact only among themselves, and players with $u > \frac{1}{2}$ also form an isolated group. Consequently, the final distribution exhibits two distinct peaks, each associated with its respective sub-population. Moreover, because of absorbing boundaries in the model, a portion of the mass remains at the state-space extremes.

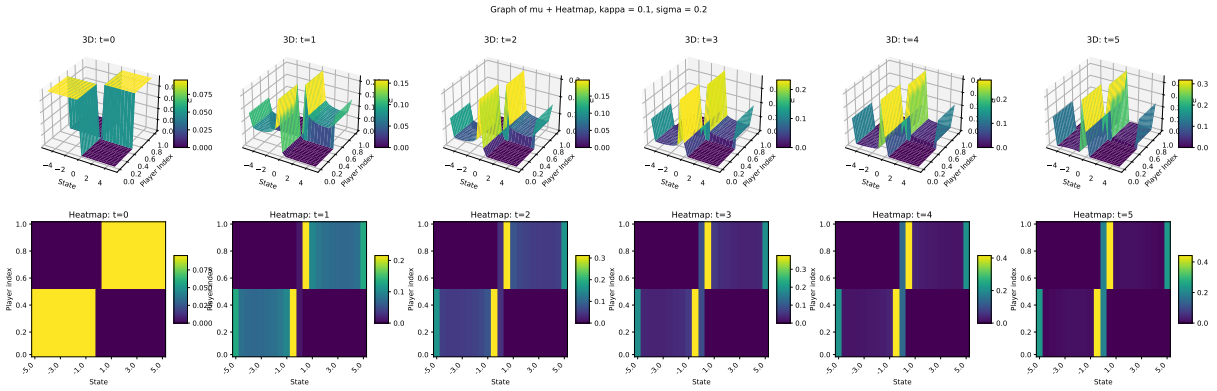


Figure 6.6: The state distribution μ over two-block graphon with interaction G_3

Finally, Fig. 6.6 shows the case of the two-block graphon with interaction, G_3. In this scenario, players still form two groups, but they interact more frequently within their own group (with intensity 0.6) and less frequently with the other group (with intensity 0.4). As a result, two peaks remain in the final distribution, but they shift closer together compared to the non-interacting case, as seen by comparing Fig. 6.5 and Fig. 6.6.

Value functions. In the same spirit as the state distributions in the previous figures, the classical MFG setting (see Fig. 6.7) exhibits a single peak, whereas the two-block graphon with no interaction (Fig. 6.8) shows two peaks that are farther apart. When partial interaction is introduced (Fig. 6.9), these two peaks move closer together.

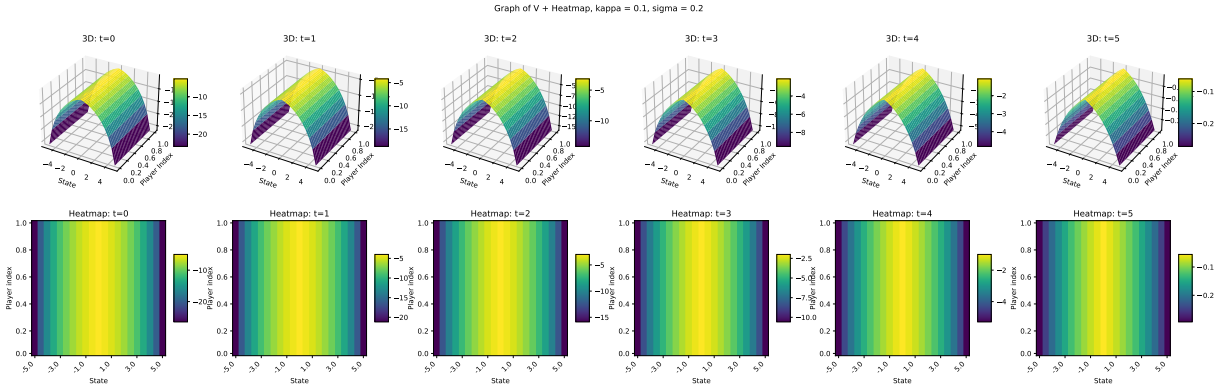


Figure 6.7: Value functions over constant graphon G_1

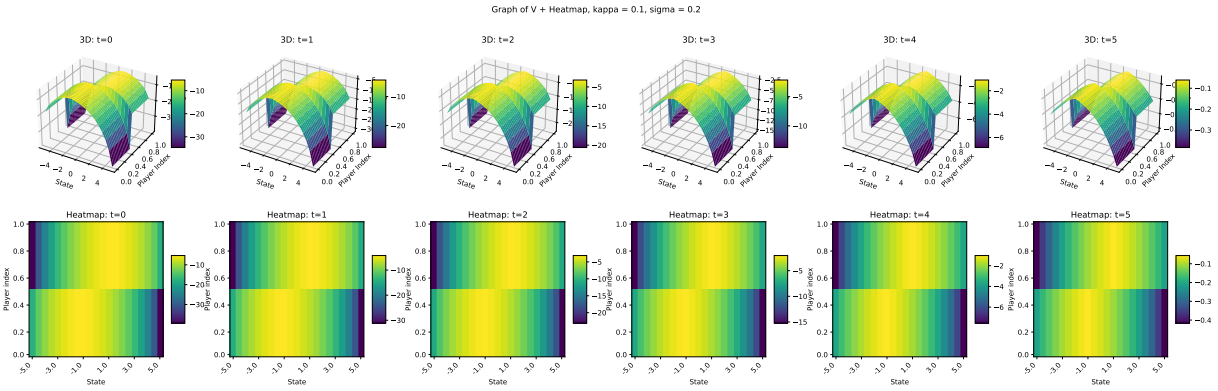


Figure 6.8: Value functions over two-block graphon with zero interaction G_2

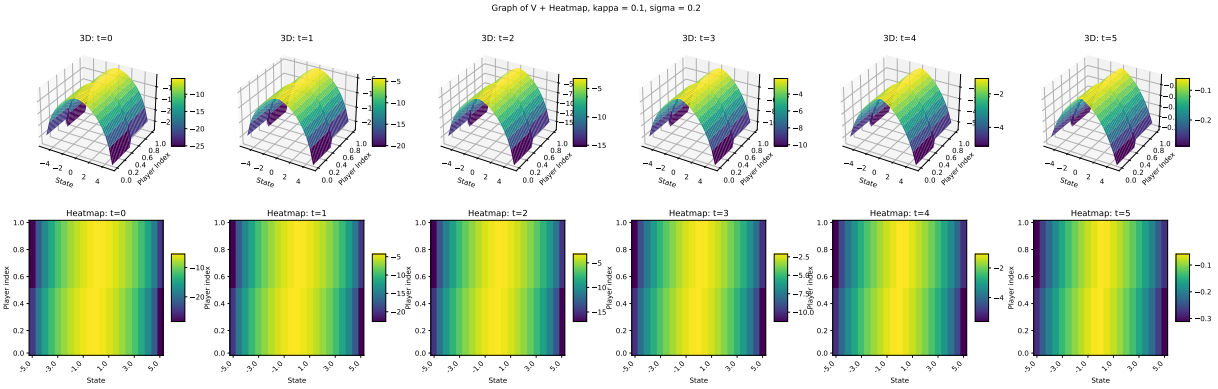


Figure 6.9: Value functions over two-block graphon with interaction G_3

Since the reward function is a negative exponential of the distance to the aggregate, the resulting value functions are themselves negative. Nonetheless, the objective is to maximize (i.e., make as close to zero as possible) these negative values. Consequently, points near the peak of each distribution represent the highest achievable payoff, as those states minimize the negative contribution of the reward.

Policies. In line with the state distributions and value functions from the previous figures, we observe pronounced peaks in the optimal policies as well. Under fully symmetric interactions (Fig. 6.10), the policies remain largely symmetric. In contrast, in the two-block graphon with zero interaction (Fig. 6.11), agents initially select distinctly different actions that ultimately split into two peaks. When partial interaction is introduced (Fig. 6.12), these peaks move closer together.

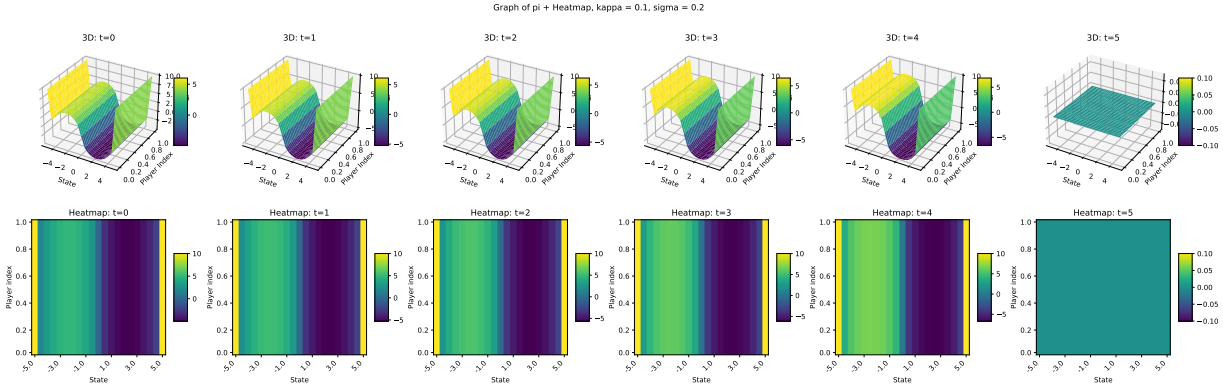


Figure 6.10: Policies over constant graphon G_1

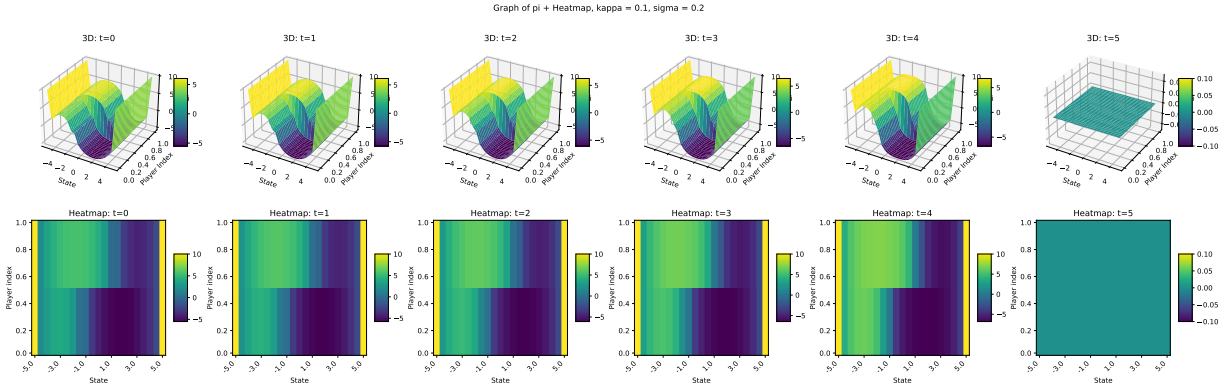


Figure 6.11: Policies functions over two-block graphon with zero interaction G_2

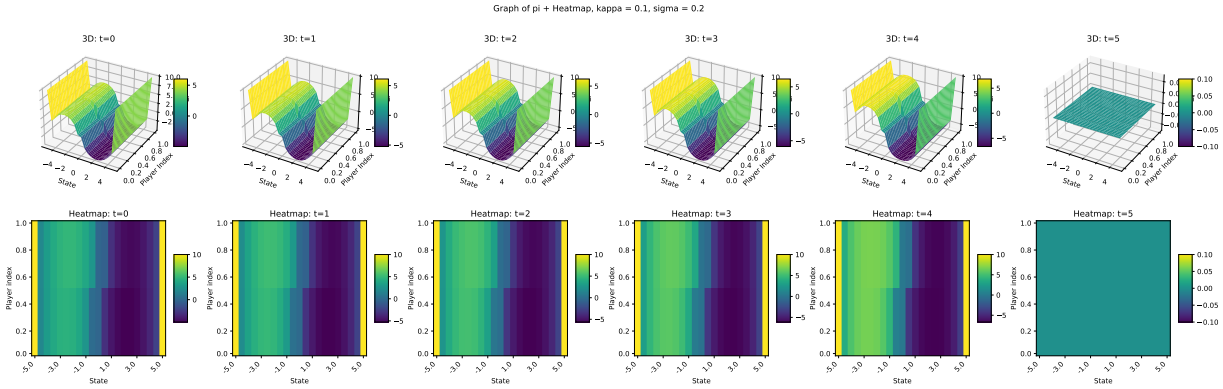


Figure 6.12: Policies functions over two-block graphon with interaction G_3

List of publications

Benjamin JOURDAIN, Kexin SHAO. "*Maximal Martingale Wasserstein inequality*". Electronic Communications in Probability 29 (2024), pp. 1–8.
DOI: 10.1214/24-ECP593

Benjamin JOURDAIN, Kexin SHAO. "*Non-decreasing martingale couplings*". ESAIM: PS, 29 (2025) 1-44.
DOI: 10.1051/ps/2024012

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RÉSUMÉ

Cette thèse s'intéresse à deux sujets distincts: le *Transport Optimal Martingale* (MOT) et la théorie des *Jeux à champ moyen* (MFG). Dans le cadre du MOT, trois projets indépendants sont développés : l'inégalité de Wasserstein martingale maximale dans le Chapitre 3, les couplages martingales croissants dans Chapitre 4, et les méthodes numériques pour le transport optimal faible au Chapitre 5. Dans le cadre des MFG, la thèse comprend un projet unique : les jeux à champ moyen étendus sur un graphon en temps discret, au Chapitre 6. Au Chapitre 3, nous achevons l'analyse de l'inégalité de Wasserstein martingale et démontrons qu'une inégalité maximale de Wasserstein martingale est valable quelle que soit la dimension d avec un paramètre d'intégrabilité $\rho \geq 2$. Au Chapitre 4, nous observons numériquement que le couplage martingale de Hobson et Neuberger, connu pour satisfaire une certaine propriété de croissance, permet d'approcher le maximum de l'intégrale de $|y-x|^\rho$ pour $\rho \in (0,2)$, et le minimum pour $\rho > 2$. Nous étudions l'optimalité de ce couplage d'un point de vue théorique. Nous identifions une décomposition de la seconde loi marginale qui est en bijection avec des couplages martingales croissants au sens généralisé. Au Chapitre 5, nous appliquons l'algorithme de Frank-Wolfe pour approcher les problèmes de *Weak Optimal Transport* et de *Weak Martingale Optimal Transport* dans une perspective d'optimisation. Au Chapitre 6, nous étudions des jeux de champ moyen étendus sur un graphon en temps discret, en intégrant des interactions conjointes entre les états et les actions dans l'agrégat du graphon. Nous établissons l'existence et l'unicité de l'équilibre de Nash sur le graphon sous diverses hypothèses et proposons un exemple numérique illustrant un investissement optimal avec un critère de performance relative.

MOTS CLÉS

Transport optimal martingale, couplages martingales, jeux à champ moyen sur graphons, finance robuste

ABSTRACT

This thesis investigates two distinct directions: *Martingale Optimal Transport* (MOT) and *Mean Field Games* (MFG). Within the MOT framework, three independent projects are developed: maximal martingale Wasserstein inequality in Chapter 3, non-decreasing martingale couplings in Chapter 4, and numerical methods of weak optimal transport in Chapter 5. Under the MFG framework, the thesis includes a single project in Chapter 6: extended graphon mean field games in discrete time. In Chapter 3, we complete the analysis of the martingale Wasserstein inequality and prove that a maximal martingale Wasserstein inequality holds whatever the dimension d when integrability parameter $\rho \geq 2$. In Chapter 4, we observe numerically that the Hobson and Neuberger martingale coupling, which is known to satisfy some non-decreasing property, is still very close to maximize this integral of $|y-x|^\rho$ for $\rho \in (0,2)$ and minimize it for $\rho > 2$. We investigate the optimality of this coupling from a theoretical perspective. We find some decomposition of the second marginal which is in one-to-one correspondence with martingale couplings that are non-decreasing in a generalized sense. In Chapter 5, we apply the Frank-Wolfe algorithm to approximate the *Weak Optimal Transport* and *Weak Martingale Optimal Transport* problems from an optimization perspective. In Chapter 6, we studied extended graphon mean field games in discrete time setting, incorporating joint state-action interactions within the graphon aggregate. We establish the existence and uniqueness of graphon Nash Equilibrium under various assumptions and provide a numerical example of optimal investment with a relative performance criterion.

KEYWORDS

Martingale optimal transport, martingale couplings, graphon mean field games, robust finance