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Set theory of derived functors

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O insensata cura de' mortali,
quanto son difettivi silogismi
quei che ti fanno in basso batter l'ali!

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Résumé

Cette thèse explore ce que la théorie des ensembles peut apporter à l'étude des foncteurs dérivés. Ces objets algébriques servent à mesurer dans quelle mesure un foncteur donné s'écarte de l'exactitude. La thèse se concentre en particulier sur les foncteurs dérivés de la limite projective et du foncteur Ext. Ces foncteurs peuvent être décrits comme des quotients issus de certains complexes cohomologiques, et la théorie des ensembles peut leur être appliquée de deux manières principales.

La première application concerne la combinatoire infinie. Pour les limites dérivées de certains systèmes projectifs de groupes abéliens, les co-cycles correspondent à des familles de fonctions dites « cohérentes », c'est-à-dire présentant un comportement « local », tandis que les co-bords traduisent l'extension de ce comportement au niveau global. Ainsi, les foncteurs dérivés mesurent une forme d'incompacité, c'est-à-dire un écart entre le local et le global. Dans cette thèse, nous démontrons plusieurs résultats de cohérence liés à cette incompacité pour des systèmes projectifs indexés par divers idéaux d'ensembles. Pour cela, nous utilisons la technique du forcing, certains principes combinatoires valables dans l'univers constructible, tels que Diamond et Square, ainsi que des faits concernant les invariants cardinaux du continu. Parmi d'autres résultats, nous montrons que les limites dérivées en dimension deux ou plus d'un système projectif largement étudié peuvent simultanément ne pas s'annuler. Nous prouvons également que, pour une certaine classe de systèmes indexés par l'ensemble des fonctions d'un cardinal infini vers le plus petit cardinal infini, toutes les limites dérivées dont l'annulation n'est pas garantie par un théorème de Goblot ne s'annulent pas dans l'univers constructible.

La seconde application de la théorie des ensembles à ces quotients relève de la théorie descriptive des ensembles et de l'étude des modules polonais ou munis d'un revêtement polonais. En particulier, certains résultats de cette thèse concernent les objets injectifs et projectifs dans la catégorie des groupes abéliens polonais pro-Lie. Les modules à revêtement polonais sont les quotients d'un module polonais par un sous-module pouvant être muni d'une topologie polonaise compatible avec la structure borélienne ambiante. Dans la dernière partie de cette thèse, nous étudions la complexité — au sens de la théorie descriptive des ensembles — du sous-module trivial de Ext(C,A), lorsque ce dernier est vu comme un module à revêtement polonais, avec A et C des modules plats dénombrables sur un anneau de Dedekind dénombrable. Cela correspond à la complexité borélienne potentielle de la relation d'isomorphisme entre suites exactes courtes, les premier et troisième modules étant fixés. Cette étude repose sur une suite décroissante de sous-foncteurs indexée par les ordinaux dénombrables, donnant lieu à des analogues supérieurs des modules injectifs et projectifs. Nous caractérisons ainsi ces modules, et montrons notamment que si un module plat dénombrable sur un anneau de Dedekind dénombrable n'est pas injectif, alors il ne l'est pour aucun des analogues supérieurs.

Mots-clés: Combinatoire infinie, forcing, invariants cardinaux du continu, modules à revêtement polonais, complexité borélienne potentielle.

Abstract

This thesis is about what set theory has to say about derived functors. These are algebraic objects that are used to "measure" how far a certain functor is from being exact. The thesis focuses in particular on the derived functors of the inverse limit and the functor Ext. These can be described as quotients coming from certain cohomological complexes, and set theory can be applied to them in two main ways.

The first area of application is infinite combinatorics. For the derived limits of some inverse systems of abelian groups, the co-cycles correspond to some "coherent" families of functions, namely families that exhibit some "local" behavior, while the co-boundaries correspond to that same behavior holding globally. Hence, the derived functors measure some kind of *incompactness*, that is, a discrepancy between local and global behavior. In this thesis, we prove several consistency results that correspond to incompactness for inverse systems indexed by various ideals of sets. For this, we use the technique of forcing, some combinatorial principles holding in the Constructible Universe, such as Diamond and Square, and some facts about cardinal invariants of the continuum. Among other results, we prove that the derived limits in dimension two or higher of a widely studied inverse system may simultaneously fail to vanish. We also prove that, for a certain class of systems indexed by the set of functions from an infinite cardinal to the least infinite cardinal, all the derived limits that need not vanish according to a theorem of Goblot do not vanish in the Constructible Universe.

The other application of set theory to these quotients consists of descriptive set theory and the theory of Polish modules and modules with a Polish cover. In particular, some of the results of this thesis are about injective and projective objects in the category of pro-Lie Polish abelian groups. Modules with a Polish cover are quotients of a Polish module by a submodule that can be given a Polish topology compatible with the Borel structure of the environment. In the last part of this thesis, we study the complexity, in the sense of descriptive set theory, of the trivial submodule of Ext(C,A), when the latter is viewed as a module with a Polish cover and A and C are countable flat modules over a countable Dedekind domain. This corresponds to the potential Borel complexity of the isomorphism relation between short exact sequences, with the first and third modules fixed. This study is based on a decreasing sequence of subfunctors indexed over the countable ordinals, which yield corresponding higher analogues of injective and projective modules. We hence characterize these modules and in particular, show that if a countable flat module over a countable Dedekind domain is not injective, then it is not injective for any of the higher analogues.

Keywords: Infinite combinatorics, forcing, cardinal invariants of the continuum, modules with a Polish cover, potential Borel complexity.

Introduction

Cette thèse explore certaines applications de la combinatoire infinie et de la théorie descriptive des ensembles à la théorie des foncteurs dérivés. Il s'agit donc d'un travail de théorie des ensembles appliquée. Les foncteurs dérivés apparaissent en algèbre homologique comme un moyen de mesurer à quel point un foncteur donné est éloigné de l'exactitude. Pour une discussion de ces notions, nous renvoyons le lecteur à [168]. En particulier, dans cette thèse, nous nous intéresserons aux foncteurs dérivés $\lim^n$ de la limite projective $\lim$ ainsi qu'au premier foncteur dérivé Ext du bifoncteur Hom. Ceux-ci prennent respectivement en entrée des systèmes projectifs (voir [123] pour une définition) et des paires d'objets. En particulier, $\mathrm{Ext}(C, A)$ peut être vu comme le groupe des suites exactes courtes de la forme

$$0 \to A \to X \to C \to 0.$$

L'une des raisons pour lesquelles les foncteurs dérivés sont largement étudiés est le rôle qu'ils jouent en topologie algébrique. En particulier, les systèmes projectifs discutés dans la partie combinatoire de cette thèse sont apparus comme outils dans l'étude de l'*homologie forte* (voir [121]). Mardešić et Prasolov ont identifié un système projectif de groupes abéliens (indexé par l'ensemble ${}^\omega\omega$ des fonctions de l'ensemble ω des ordinaux finis vers lui-même), appelé $\mathbf{A}$, provenant d'une somme disjointe de boules d'oreille hawaïennes, tel que, si l'homologie forte doit être additive, on devrait avoir $\lim^n \mathbf{A} = 0$ pour tout $n \in \mathbb{N}$. Le comportement de ces limites dérivées (par exemple, leur annulation ou non) peut à son tour être caractérisé en termes combinatoires. Plus précisément, les limites dérivées non nulles correspondent à une forme d'*incompacité*, c'est-à-dire à un écart entre le comportement local et global. Cette incompacité s'est avérée en grande partie indépendante des axiomes de la théorie ZFC. Cela signifie que, comme c'est souvent le cas en combinatoire infinie, une réponse affirmative ou négative à une certaine question est cohérente avec ces axiomes. L'outil principal permettant de démontrer des résultats de cohérence en théorie des ensembles est la méthode du forcing développée par Cohen (voir [44] et [45]), qui, à partir d'un modèle V de la théorie des ensembles et d'un filtre générique G pour un poset $\mathbb{P} \in V$, produit une extension $V[G]$ satisfaisant la proposition désirée. Pour une introduction à la théorie du forcing, voir [108]. En particulier, le premier poset de forcing conçu par Cohen avait pour but d'ajouter une certaine quantité de nombres réels de sorte que l'Hypothèse du Continuum soit fausse dans l'extension. Cette même notion de forcing est utilisée dans [20] et [8] pour annuler tous les foncteurs dérivés $\lim^n$ de $\mathbf{A}$ et de certaines généralisations de celui-ci, discutées au Chapitre 3.

Une autre notion de forcing pertinente pour ce sujet est le forcing de Hechler, introduit dans [76]. Celui-ci est utilisé pour réaliser l'existence d'un ensemble cofinal d'une certaine forme dans le quasi-ordre $({}^\omega\omega, \leq^*)$, où $\leq^*$ désigne la domination modulo fini. Par conséquent, le forcing de Hechler peut être utilisé pour contrôler la valeur des invariants

cardinaux du continuum $\mathfrak{b}$ et $\mathfrak{d}$, qui sont respectivement l'additivité et la cofinalité de ce quasi-ordre ; pour la théorie des invariants cardinaux, voir [25]. Le forcing de Hechler apparaît dans [18] et [10], où il est utilisé pour donner une preuve de la cohérence, sous réserve de l'existence d'un cardinal faiblement compact (voir [95]), de l'annulation simultanée des limites dérivées de $\mathbf{A}$ et de l'additivité de l'homologie forte pour les espaces métriques localement compacts et séparables. Il apparaît également aux Chapitres 1 et 2 de cette thèse pour démontrer, au contraire, des résultats de non-annulation.

Afin de prouver la cohérence de $\lim^n \mathbf{A} \neq 0$ pour $n > 1$, à l'état actuel des connaissances, il est également nécessaire d'utiliser certains principes combinatoires tels que $\diamondsuit$, $\square$ et leurs variantes. Jensen a d'abord identifié ces principes dans [91] comme étant valides dans l'Univers Constructible $\mathbb{L}$, qui est apparu dans le travail de Gödel ([74]) comme un moyen de démontrer (relativement à ZF) la cohérence de l'Axiome du Choix et de l'Hypothèse Généralisée du Continuum. La première application de ces principes combinatoires aux limites dérivées supérieures a concerné le cas des systèmes projectifs indexés par les ordinaux, et est consignée dans [172]. Dans [15], ce thème a été repris, et il a été démontré en particulier que le principe $\diamondsuit(S^2_1)$ permet de construire un témoin de $\lim^2 \mathbf{A} \neq 0$ lorsque $\mathfrak{b} = \mathfrak{d} = \aleph_2$. Cela correspond à une construction réalisée le long d'une chaîne $\leq^*$ de longueur ω_2. Dans [164], des versions affaiblies des principes Diamond, notées $\mathrm{w}\diamondsuit(S)$, où le paramètre S est un ensemble stationnaire (voir [50]), ont été utilisées pour obtenir un résultat similaire de manière compatible avec des valeurs plus élevées de $\mathfrak{b}$ et $\mathfrak{d}$, de sorte que la construction d'un témoin pour $\lim^n \mathbf{A} \neq 0$ puisse avoir lieu le long d'une chaîne cofinale $\leq^*$ de longueur ω_n. Les Chapitres 1 et 2 généralisent davantage ces résultats de cohérence. Plus précisément, le Chapitre 1 se dispense de l'hypothèse de l'existence d'une chaîne cofinale dans $({}^\omega\omega, \leq^*)$ grâce à son Théorème principal, et le Chapitre 2 affaiblit encore les hypothèses avec le Corollaire 2.36. Le Chapitre 2 contient également des résultats de *non-annulation simultanée*, où des principes $\square$ sont employés. En particulier, il y est démontré (voir Théorème 2.48) que $\bigwedge_{2\leq n<\omega} \lim^k \mathbf{A} \neq 0$ est cohérent relativement à ZFC. La question de savoir si l'on peut ajouter à cette affirmation une non-annulation de $\lim^1 \mathbf{A}$ reste ouverte.

Au Chapitre 3, nous traitons de $\mathbf{A}_\lambda$, une généralisation de $\mathbf{A}$ qui est indexée par des fonctions dans ${}^\lambda\omega$. Un théorème de [15] affirme que, pour tout cardinal infini λ, on a $\lim^1 \mathbf{A} = 0$ si et seulement si $\lim^1 \mathbf{A}_\lambda = 0$. La question de savoir si la même équivalence vaut pour des limites supérieures a été posée dans le même travail. Le Théorème I apporte une réponse négative à cette question. Dans le même chapitre, une réponse négative est également donnée, via une construction de forcing, dans le cas particulier de $\lim^2$. La question de savoir si l'annulation pour toutes les paires (n, λ) simultanément est cohérente reste ouverte. Cette question est particulièrement importante puisqu'une réponse affirmative impliquerait la cohérence de l'additivité de l'homologie forte sur la classe des espaces métriques localement compacts (voir [9]).

La deuxième partie de la thèse concerne la théorie descriptive des ensembles. La théorie descriptive des ensembles traite principalement des espaces *polonais* (c'est-à-dire séparables et complètement métrisables) et de la complexité de leurs sous-ensembles définissables. Pour une introduction au domaine, voir [103]. Une sous-branche de cette théorie étudie le comportement des *groupes* polonais, c'est-à-dire des groupes topologiques dont la topologie est polonaise. Dans cette thèse, nous discutons en particulier des groupes abéliens polonais pro-Lie. Un groupe abélien polonais est pro-Lie s'il peut être écrit comme une limite pro-

jective de groupes de Lie. Dans ce cas, il est plus précisément la limite projective d'une *tour*, c'est-à-dire d'une ω-suite projective de groupes de Lie. La catégorie des groupes abéliens polonais pro-Lie contient tous les groupes abéliens polonais localement compacts, ainsi que les groupes abéliens polonais non archimédiens (qui sont des limites projectives de groupes discrets dénombrables). En fait, la classe des groupes abéliens polonais pro-Lie est la plus petite classe de groupes abéliens polonais qui contient tous les groupes abéliens polonais localement compacts et les groupes abéliens polonais non archimédiens, et qui est fermée par extensions.

Au Chapitre 4, nous généralisons à la catégorie des groupes abéliens polonais pro-Lie un théorème de structure qui avait été démontré dans [80] pour les groupes abéliens polonais localement compacts. À l'aide de ce théorème de structure, nous prouvons les deux résultats principaux de ce chapitre, à savoir les caractérisations des objets projectifs (4.65) et des objets injectifs (4.71) dans la catégorie des groupes abéliens polonais pro-Lie.

Le Chapitre 5 utilise la théorie des groupes abéliens avec un revêtement polonais et, plus généralement, la théorie des *modules avec un revêtement polonais* (rappelons qu'un groupe abélien est un $\mathbb{Z}$-module). Soit R un anneau commutatif. Un R-module polonais est un module qui est aussi un groupe abélien polonais, tel que pour tout $r \in R$ l'application $M \to M, x \mapsto rx$ est continue. Un module avec un revêtement polonais est un module G présenté explicitement comme un quotient $\hat{G}/N$, où $\hat{G}$ est un module polonais et N est un sous-module *polonisable* de $\hat{G}$. Cela signifie que N est un sous-module borélien de $\hat{G}$ tel qu'il existe une topologie de module polonais (nécessairement unique) sur N dont les boréliens sont précisément les sous-ensembles boréliens de $\hat{G}$ contenus dans N ou, de façon équivalente, telle que l'inclusion $N \to \hat{G}$ soit continue. Les groupes avec un revêtement polonais sont utilisés dans [22] pour donner aux invariants homologiques une structure plus fine que la topologie quotient. Dans le même esprit, nous considérons le module avec un revêtement polonais $\mathrm{Ext}(C, A)$ et la complexité du sous-module $\{0\}$ par rapport à cette structure, laquelle est liée à la complexité borélienne *potentielle* de la relation d'équivalence d'isomorphisme entre extensions de C par A. C'est un thème de la *invariant descriptive set theory*, c'est-à-dire l'étude de la complexité des relations d'équivalence avec les outils de la théorie descriptive des ensembles.

Plus précisément, notre analyse s'appuie sur une généralisation des sous-groupes de Solecki définis dans [159] et sur l'étude de la complexité des sous-groupes polonisables menée dans [113]. Dans le contexte des catégories triangulées, nous définissons des sous-foncteurs $\mathrm{Ph}^\alpha\mathrm{Ext}$ de Ext pour tout $\alpha < \omega_1$. Lorsque, par exemple, A et C sont des modules plats dénombrables sur un anneau de Dedekind dénombrable, $\mathrm{Ph}^\alpha\mathrm{Ext}(C, A)$ coïncide avec le α-ième sous-module de Solecki $s_\alpha(\mathrm{Ext}(C, A))$ du module avec un revêtement polonais $\mathrm{Ext}(C, A)$. Si G est un module avec un revêtement polonais, $s_\alpha(G)$ est le plus petit sous-module de G de complexité $\Pi^0_{1+\alpha+1}$. Pour tout tel G, il existe $\sigma < \omega_1$ tel que $s_\sigma(G) = \{0\}$. Nous appelons le plus petit tel σ la *longueur de Solecki* de G. Pour des modules plats dénombrables A et C, nous appelons *longueur A-projective de C* la longueur de Solecki de $\mathrm{Ext}(C, A)$. Nous disons que C a une longueur projective au plus α s'il a une longueur A-projective au plus α pour tout module plat dénombrable A. Il s'agit, pour chaque $\alpha < \omega_1$, d'une notion plus faible de projectivité, caractérisée dans le Théorème 5.1. De même, nous pouvons définir des notions supérieures d'injectivité ; celles-ci donnent lieu à une dichotomie exprimée par le Théorème 5.162. De plus, la longueur de Solecki de $\mathrm{Ext}(C, A)$ a des implications pour la complexité borélienne potentielle de la relation d'isomorphisme

des extensions de C par A.

Cette thèse est une compilation de plusieurs articles de recherche rédigés par le candidat. En conséquence, chaque chapitre conserve sa structure originale et inclut sa propre introduction, ses définitions et sa notation si nécessaire. Certains recoupements et répétitions — en particulier dans la terminologie et les éléments préliminaires — sont donc délibérés, afin de préserver la clarté et le caractère autonome de chaque contribution. En particulier, le Chapitre 1 correspond à [33], le Chapitre 2 à [34], le Chapitre 3 à [17], le Chapitre 4 à [36], et le Chapitre 5 à [35].

INTRODUCTION

This thesis explores some applications of infinite combinatorics and descriptive set theory to the theory of derived functors. It is therefore a work in applied set theory. Derived functors appear in homological algebra as a way to measure how far a given functor is from being exact. For a discussion of these, we refer the reader to [168]. In particular, in this thesis we will concern ourselves with the derived functors $\lim^n$ of the inverse limit $\lim$ and the first derived functor Ext of the bifunctor Hom. These take as input inverse systems (see [123] for a definition) and pairs of objects, respectively. In particular, $\mathrm{Ext}(C, A)$ can be see as the group of short exact sequences of the form

$$0 \to A \to X \to C \to 0.$$

One of the reasons why derived functors are widely studied is the role they play in algebraic topology. In particular, the inverse systems discussed in the combinatorial part of this thesis arose as tools in the study of *strong homology* (see [121]). Mardešić and Prasolov identified an inverse system of abelian groups (indexed over the set ${}^\omega\omega$ of functions from the set ω of finite ordinals to itself) called $\mathbf{A}$, arising from a disjoint sum of Hawaian earrings, such that if strong homology is to be additive, then one should have $\lim^n \mathbf{A} = 0$ for all $n \in \mathbb{N}$. The behavior of these derived limits (e.g., whether they vanish or not) can in turn be characterized in combinatorial terms. More precisely, nonvanishing derived limits correspond to some form of *incompactness*, that is, a discrepancy between local and global behavior. This incompactness has been proved to be largely independent of the ZFC axioms. This means that, as it often happens in infinite combinatorics, both an affirmative and a negative answer to a certain question are consistent with the axioms. The main tool to prove consistency results in set theory is the method of forcing developed by Cohen (see[44] and [45]), which, starting from a model of set theory V and a generic filter G for some poset $\mathbb{P} \in V$ produces an extension $V[G]$ satisfying the desired proposition. For an introduction to the theory of forcing, we refer the reader to [108]. In particular, the first forcing poset devised by Cohen is meant to add a certain quantity of real numbers so that the Continuum Hypothesis is false in the extension. This same forcing notion is used in [20] and [8] to make all the derived functors $\lim^n$ vanish for $\mathbf{A}$, and for some generalizations thereof that are discussed in Chapter 3.

Another forcing notion that is relevant to this topic is Hechler forcing, introduced in [76]. This is used to realize the existence of a cofinal set of a certain shape in the quasiorder $({}^\omega\omega, \leq^*)$, where $\leq^*$ denotes domination modulo finite. Consequently, Hechler forcing can be used to control the value of the cardinal invariants of the continuum $\mathfrak{b}$ and $\mathfrak{d}$ that are respectively the additivity and the cofinality of said quasiorder; for the theory of cardinal invariants, see [25]. Hechler forcing appears in [18] and [10], where it is used to give a proof of the consistency, modulo the existence of a weakly compact cardinal (see [95]), of

simultaneously vanishing derived limits of $\mathbf{A}$ and the additivity of strong homology for locally compact separable metric spaces, respectively. It also appears in Chapters 1 and 2 of this thesis to prove nonvanishing results instead.

In order to prove the consistency of $\lim^n \mathbf{A} \neq 0$ for $n > 1$, at the present state of knowledge, one also needs some combinatorial principles such as $\diamondsuit$, $\square$, and their relatives. Jensen first identified these principles in [91] as holding in the Constructible Universe $\mathbb{L}$, which appeared in the work of Gödel ([74]) as a way to prove (relative to ZF) the consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis. The first application of these combinatorial principles to higher derived limits was in the case of inverse systems indexed by the ordinals, and is recorded in [172]. In [15], this theme was revived, and it was proved in particular that the principle $\diamondsuit(S^2_1)$ allows one to build a witness to $\lim^2 \mathbf{A} \neq 0$ when $\mathfrak{b} = \mathfrak{d} = \aleph_2$. This corresponds to a construction made along a $\leq^*$-chain of length ω_2. In [164], weaker versions of the Diamond principles denoted by $\mathrm{w}\diamondsuit(S)$, where the parameter S is a stationary set (see [50]) were used to obtain a similar result in a way compatible with higher values of $\mathfrak{b}$ and $\mathfrak{d}$, so that the construction of a witness for $\lim^n \mathbf{A} \neq 0$ can take place along a cofinal $\leq^*$-chain of length ω_n. Chapters 1 and 2 further generalize these consistency results. More precisely, Chapter 1 dispenses with the assumption of the existence of a cofinal chain in $({}^\omega\omega, \leq^*)$ with its Main Theorem, and Chapter 2 further weakens the assumptions with Corollary 2.36. Chapter 2 also contains *simultaneous nonvanishing* results, where $\square$-principles are employed. In particular, it is proved (see Theorem 2.48) that $\bigwedge_{2\leq n<\omega} \lim^k \mathbf{A} \neq 0$ is consistent, relative to ZFC. The question of whether a nonvanishing $\lim^1 \mathbf{A}$ can be added to this statement remains open.

In Chapter 3, we deal with $\mathbf{A}_\lambda$, a generalization of $\mathbf{A}$ which is indexed over functions in ${}^\lambda\omega$ instead. A theorem from [15] states that, for any infinite cardinal λ, $\lim^1 \mathbf{A} = 0$ holds if and only if $\lim^1 \mathbf{A}_\lambda = 0$ does. The question of whether the same equivalence holds for higher limits was asked in the same work. Theorem I gives a negative answer to that question. In the same chapter, a negative answer is also given through a forcing construction for the special case of $\lim^2$. The question of whether vanishing for all pairs (n, λ) simultaneously is consistent remains open. This question is particularly important since an affirmative answer would imply the consistency of the additivity of strong homology on the class of locally compact metric spaces (see [9]).

The second part of the thesis is about descriptive set theory. Descriptive set theory deals mostly with *Polish* spaces (that is, separable completely metrizable) and the complexity of their definable subsets. For an introduction to the field, see [103]. A sub-branch of this theory studies the behavior of Polish *groups*, that is, topological groups whose topology is Polish. In this thesis, we discuss in particular pro-Lie Polish abelian groups. A Polish abelian group is pro-Lie if it can be written as an inverse limit of Lie groups. In that case, it is more specifically the inverse limit of a *tower*, that is, an inverse ω-sequence, of Lie groups. The category of Polish abelian pro-Lie groups contains all the locally compact Polish abelian groups, as well as the non-Archimedean Polish abelian groups (which are inverse limits of countable discrete groups). In fact, the class of pro-Lie Polish abelian groups is the smallest class of Polish abelian groups that contains all locally compact Polish abelian groups and non-Archimedean Polish abelian groups and it is closed under extensions.

In Chapter 4, we generalize to pro-Lie Polish abelian group a structure theorem that was proved in [80] for locally compact Polish abelian groups. Using the structure theorem, we prove the two main results of that chapter, that is, the characterizations of projective

(4.65) and injective (4.71) objects in the category of pro-Lie Polish abelian groups.

Chapter 5 makes use of the theory of abelian groups with a Polish cover and, more generally, of the theory of *modules with a Polish cover* (recall that an abelian group is a $\mathbb{Z}$-module). Let R be a commutative ring. A Polish R-module is a module that is also a Polish abelian group, such that for every $r \in R$ the function $M \to M, x \mapsto rx$ is continuous. A module with a Polish cover is a module G explicitly presented as a quotient $\hat{G}/N$ where $\hat{G}$ is a Polish module and N is a *Polishable* submodule of $\hat{G}$. This means that N is a Borel submodule of $\hat{G}$ such that there exists a (necessarily unique) Polish module topology on N whose Borel sets are precisely the Borel subsets of $\hat{G}$ contained in N or, equivalently, such that the inclusion $N \to \hat{G}$ is continuous. Groups with a Polish cover are used in [22] to give homological invariants a structure finer than the quotient topology. In the same vein, we consider the module with a Polish cover $\mathrm{Ext}(C, A)$ and the complexity of the submodule $\{0\}$ with respect to this structure, which is related to the *potential* Borel complexity of the equivalence relation of isomorphism between extensions of C by A. This is a theme of *invariant* descriptive set theory, that is, the study of the complexity of equivalence relations with the tools of descriptive set theory.

More precisely, our analysis is based on a generalization of the Solecki subgroups defined in [159] and the study of complexity of Polishable subgroups conducted in [113]. In the context of triangulated categories, we define subfunctors $\mathrm{Ph}^\alpha\mathrm{Ext}$ of Ext for all $\alpha < \omega_1$. When, for instance, A and C are countable flat modules over a countable Dedekind domain, $\mathrm{Ph}^\alpha\mathrm{Ext}(C, A)$ equals the α-th Solecki submodule $s_\alpha(\mathrm{Ext(C, A)})$ of the module with a Polish cover $\mathrm{Ext}(C, A)$. If G is a module with a Polish cover, $s_\alpha(G)$ is the smallest submodule of G of complexity $\Pi^0_{1+\alpha+1}$. For every such G, there exists $\sigma < \omega_1$ such that $s_\sigma(G) = \{0\}$. We call the least such σ the Solecki *length* of G. For countable flat modules A and C, we call *A-projective length of C* the Solecki length of $\mathrm{Ext}(C, A)$. We say that C has projective length at most α if it has A-projective length at most α for every countable flat module A. This is, for each $\alpha < \omega_1$, a higher (weaker) notion of projectivity, which is characterized in Theorem 5.1. Similarly, we can define higher notions of injectivity; these yield a dichotomy expressed by Theorem 5.162. Moreover, the Solecki length of $\mathrm{Ext(C, A)}$ bears implications for the potential Borel complexity of the isomorphism relation of extensions of C by A.

This dissertation is a compilation of several research papers authored by the candidate. As a result, each chapter retains its original structure and includes its own introduction, definitions, and notation as needed. Some overlaps and repetitions—particularly in terminology and preliminary material—are therefore deliberate, in order to preserve the clarity and self-contained nature of each contribution. In particular, Chapter 1 corresponds to [33], Chapter 2 to [34], Chapter 3 to [17], Chapter 4 to [36], and Chapter 5 to [35].

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Part I

Infinite Combinatorics

Introduction to Part I

The past few years have seen considerable progress in the application of set-theoretic combinatorial tools to the study of homological algebra, and in particular to the study of the derived functors of the inverse limit functor. This part of the thesis is a contribution to this line of research. In particular, we isolate a number of situations in which certain derived limits are provably nonzero.

The inverse systems that we are primarily concerned with here are of the form $\mathbf{A}_{\mathcal{I}}[H]$, where $\mathcal{I}$ is a $\subseteq$-directed collection of sets and H is an arbitrary abelian group. These systems are indexed along the directed partial order $(\mathcal{I}, \subseteq)$.[1] We are especially interested in the case in which $\mathcal{I} = \{I_f \mid f \in {}^{\omega}\omega\}$, where

$$I_f = \{(k, m) \in \omega \times \omega \mid m < f(k)\}.$$

We will omit $\mathcal{I}$ from the notation $\mathbf{A}_{\mathcal{I}}[H]$ when $\mathcal{I}$ has this value; similarly, we will omit H from the notation when $H = \mathbb{Z}$.

The derived limits of these systems, and questions about their vanishing, show up in a variety of contexts in various fields of mathematical research. To give two prominent examples:

- In [121], Mardešić and Prasolov prove that, if strong homology is additive on the class of all closed subsets of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $1 \leq n < \omega$.
- Clausen and Scholze showed that the assertion that $\lim^n \mathbf{A}[H] = 0$ for all $1 \leq n < \omega$ and all abelian groups H is equivalent to a useful statement about the calculation of derived functors in the category of condensed abelian groups (for a precise statement, see [43, Lecture 4] or [19]).

In the late 1980s and early 1990s, a number of works were published applying set-theoretic tools to the study of the *first* derived limit of the system $\mathbf{A}$ (cf. [52, 93, 121, 162], and see the introduction of [18] for a summary of the contents of these works). It was not until roughly ten years ago that the *higher* derived limits of $\mathbf{A}$ and its relatives began to be explored. This was begun in [15] and has continued in a number of works published since then. We now survey some of the relevant recent highlights of this research program, beginning with results about the consistent *vanishing* of higher derived limits.

[1] We refer the reader to Subsection 2.2.2 for precise definitions.

- In [18], Bergfalk and Lambie-Hanson prove that, relative to the consistency of the existence of certain large cardinals, the statement "for all $1 \leq n < \omega$, $\lim^n \mathbf{A} = 0$" is consistent with ZFC. In particular, they prove that, in any forcing extension obtained by adding weakly compact-many Hechler reals, $\lim^n \mathbf{A} = 0$ for all $1 \leq n < \omega$.
- In [20], Bergfalk, Hrušák, and Lambie-Hanson remove the large cardinal assumptions from the main result of [18]. In particular, they prove that, in any forcing extension obtained by adding $\beth_\omega$-many Cohen reals, $\lim^n \mathbf{A} = 0$ for all $1 \leq n < \omega$.
- In [8], Bannister sharpens the arguments from [18] and [20] to build models where derived limits vanish simultaneously for a broader class of systems. In particular, it follows from his work that, in the model of [20], we in fact have $\lim^n \mathbf{A}[H] = 0$ for all $1 \leq n < \omega$ and all abelian groups H.

In the other direction, there have been some recent results about the consistent *nonvanishing* of derived limits.

- $\mathfrak{d} = \aleph_1$ implies that $\lim^1 \mathbf{A} \neq 0$ ([52]).
- $\lim^1 \mathbf{A}^{\mathcal{F}} \neq 0$ is a ZFC fact, whenever $\mathcal{F}$ is an ω_1-chain in $({}^\omega\omega, \leq^*)$, and $\mathbf{A}^{\mathcal{F}}$ is the restriction of $\mathbf{A}$ to the indexing subset $\mathcal{F}$ ([13]).
- It is consistent that $\lim^1 \mathbf{A} \neq 0$, $\mathfrak{c} = \aleph_2$ and $\mathrm{MA}_{\aleph_1}$ holds (and hence $\mathfrak{b} = \mathfrak{d} = \aleph_2$ ([162])).
- It is consistent that $\lim^n \mathbf{Z}_n \neq 0$ for all n, where $\mathbf{Z}_n$ is a certain pro-group indexed on the ordinal ω_n discussed in ([172]), where this result was proved. In particular, the case $n = 1$ is a ZFC theorem that can be proved for instance using the ρ_1 function ([163]) and one can then use the guessing principles $\diamondsuit(S_i^{i+1})$ for $i < n$ to run an induction through higher n. This is an early instance of the application of guessing principles to this kind of problem.
- It is consistent that $\lim^2 \mathbf{A} \neq 0$. In particular, it is implied by $\mathfrak{b} = \mathfrak{d} = \aleph_2 + \diamondsuit(S_1^2)$([15]). Here once again one starts with a ZFC base case and then moves to the higher case using the guessing principle. Note that the condition $\mathfrak{b} = \mathfrak{d} = \kappa$ is equivalent to the existence of a cofinal κ-chain in $({}^\omega\omega, \leq^*)$, that is, a κ-scale. This allows us to work in a linear situation pretty much like in the case of $\mathbf{Z}_n$. Note also that $\diamondsuit(S_1^2)$ implies $\mathfrak{c} \leq \aleph_2$, and this is the obstruction to using the classical diamonds to get beyond $n = 2$.
- For all $n > 0$ it is consistent that $\lim^n \mathbf{A} \neq 0$. In particular, this is implied by $\mathfrak{b} = \mathfrak{d} = \aleph_n + \bigwedge_{i<n} \mathrm{w}\diamondsuit(S_i^{i+1})$ ([165]). The weak diamonds $\mathrm{w}\diamondsuit(S)$ are some guessing principles that were originally introduced by Devlin and Shelah in [50]. These are sufficient to run an argument similar to that for $\mathbf{Z}_n$ and are compatible with a longer scale.

A number of questions remained open in the wake of the work described above. One obvious question, given that all previous nonvanishing results assumed a cofinal linearly

ordered subset of the indexing set, is whether $\lim^n \mathbf{A} \neq 0$ for some n is consistent with $\mathfrak{b} < \mathfrak{d}$. We answer this question in the affirmative in the main theorem of Chapter 1:

Theorem 0.1. *For all $1 \leq k \leq n < \omega$, it is consistent that $\mathfrak{b} = \aleph_k$, $\mathfrak{d} = \aleph_n$ and $\lim^n \mathbf{A} \neq 0$.*

This in turn implies (see [121]) the following.

Corollary 0.2. *For all $1 \leq k \leq n < \omega$, it is consistent that $\mathfrak{b} = \aleph_k$, $\mathfrak{d} = \aleph_n$, and strong homology is not additive on separable locally compact metric spaces.*

This is complementary to the work of Bannister in [8].

Let us now recall two other prominent questions, which together comprise the primary motivation behind Chapter 2.

Question 0.3. *What is the minimum value of $2^{\aleph_0}$ compatible with the statement "for all $1 \leq n < \omega$, $\lim^n \mathbf{A} = 0$"? What is the minimum value of $2^{\aleph_0}$ compatible with the statement "for all $1 \leq n < \omega$ and all abelian groups H, $\lim^n \mathbf{A}[H] = 0$"?*

The first half of Question 0.3 was asked in [18], and also in [20] and [8]. The second half of Question 0.3 is closely related to Bannister's [8, Question 7.7], which asks an analogous question about a slighly broader class of inverse systems. The first half of the question remains open; we provide here an answer to the second half and, in the process, to [8, Question 7.7]. In particular, we prove the following theorem.

Theorem A.2. *Suppose that $1 \leq n < \omega$ and $\mathfrak{d} = \omega_n$. Then*

1. *$\lim^n \mathbf{A}[\mathbb{Z}^{(\omega_n)}] \neq 0$, where $\mathbb{Z}^{(\omega_n)}$ denotes the direct sum of ω_n-many copies of $\mathbb{Z}$;*
2. *if, in addition, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds for all $k < n$, then $\lim^n \mathbf{A} \neq 0$.*

In particular, if $\lim^n \mathbf{A}[H] = 0$ for all $1 \leq n < \omega$ and all abelian groups H, then $2^{\aleph_0} \geq \aleph_{\omega+1}$.

[8, Question 7.7] asks about the minimum value of the continuum compatible with additivity of derived limits for Ω_ω-systems. We refer the reader to [8] for the relevant definitions; we just note here that a statement of the form $\lim^n \mathbf{A}[H] \neq 0$ provides a counterexample to the additivity of derived limits for Ω_ω-systems, and Bannister showed in [8] that additivity of derived limits for Ω_ω-systems is compatible with $2^{\aleph_0} = \aleph_{\omega+1}$. Therefore, Theorem A.2 provides an answer to [8, Question 7.7].

We now turn to the second question motivating Chapter 2.

Question 0.4. *Let X be an arbitrary set of positive integers. Is there a model of ZFC in which $\lim^n \mathbf{A} = 0$ if and only if $n \in X$? In particular, is the statement "for all $1 \leq n < \omega$, $\lim^n \mathbf{A} \neq 0$" consistent with ZFC?*

Question 0.4 was asked in [20], and we make some partial progress towards it here. Notably, prior to the paper [34] from which Chapter 2 is taken, there was only one instance in which the simultaneous *nonvanishing* of $\lim^n \mathbf{A}$ for multiple values of $1 \leq n < \omega$ was known to be consistent. Namely, it follows from results in [162] and [15] that it is consistent with ZFC that $\lim^1 \mathbf{A} \neq 0$ and $\lim^2 \mathbf{A} \neq 0$ simultaneously (in the model witnessing this, $\lim^n \mathbf{A} = 0$ for all $3 \leq n < \omega$). Here, we construct models witnessing the consistency of the simultaneous nonvanishing of $\lim^n \mathbf{A}$. In particular, we prove the following theorem.

Theorem B.2. *1. Fix $2 \leq n < \omega$. There exists a model of* ZFC *in which $\mathfrak{b} = \mathfrak{d} = \aleph_n$ and $\bigwedge_{2 \leq k \leq n} \lim^k \mathbf{A} \neq 0$.*

2. There exists a model of ZFC *in which $\bigwedge_{2 \leq k < \omega} \lim^k \mathbf{A} \neq 0$.*

Note that, by a theorem of Goblot ([71], although our formulation incorporates the variant [165, Theorem 2.4]) the following holds:

Goblot's vanishing theorem. *If the indexing order of a directed inverse system $\mathbf{X}$ of abelian groups is of cofinality $\aleph_n$ for some $n < \omega$ then $\lim^k \mathbf{X} = 0$ for all $k > n+1$; if, in addition, the transition maps of $\mathbf{X}$ are all surjective, then $\lim^k \mathbf{X} = 0$ for $k = n+1$ as well.*

Hence, if $1 \leq n < \omega$ and $\mathfrak{d} = \aleph_n$, then $\lim^k \mathbf{A} = 0$ for all $k > n$. Thus, clause (1) of Theorem B.2 is in one sense optimal. In our model for clause (2) of Theorem B.2, we have $\mathfrak{b} = \mathfrak{d} = \aleph_{\omega+2}$. By the aforementioned result of Goblot, $\bigwedge_{2 \leq k < \omega} \lim^k \mathbf{A} \neq 0$ implies that $\mathfrak{d}$ is at least $\aleph_{\omega+1}$. Our result is thus almost optimal; it remains open whether one can obtain the same conclusion with $\mathfrak{d} = \aleph_{\omega+1}$. Also, the models that we construct in proving Theorem B.2 will satisfy $\lim^1 \mathbf{A} = 0$. We therefore fall just short of answering the "in particular" clause of Question 0.4, which remains open.

We now briefly discuss some of the methods underlying our results and the structure of the chapter. As has become standard in this area of research, given $1 \leq n < \omega$, our verifications that a derived limit $\lim^n \mathbf{A}_{\mathcal{I}}[H]$ is nonzero in a particular model will reduce to the construction of a combinatorial object known as a *nontrivial coherent n-family*. In Section 2.2, we recall some of the basic definitions and facts surrounding nontrivial coherent n-families and their connections with derived limits.

Our proof of Theorem A.2 builds upon the work of Veličković and Vignati in [165]. In that paper, the assumption that $\mathfrak{b} = \mathfrak{d}$ yields the existence of a $\subseteq^*$-increasing cofinal sequence from $\{I_f \mid f \in {}^{\omega}\omega\}$ along which to perform recursive constructions of objects witnessing instances of $\lim^n \mathbf{A} \neq 0$. In the absence of the assumption that $\mathfrak{b} = \mathfrak{d}$, such sequences need not exist. As shown in [33], the weaker assumption of the existence of an unbounded chain is sufficient. To generalize these results even further, in Section 2.3 we introduce the notion of an *ascending sequence* of sets. "Ascending" is a weakening of "$\subseteq^*$-increasing", but we prove that it is strong enough to allow one to carry out modified versions of the constructions from [165].

In Section 2.4 we apply the technical results of Section 2.3 to obtain nonvanishing results for derived limits. In particular, we isolate natural conditions on an ideal $\mathcal{I}$ that imply the existence of an ascending $\subseteq^*$-cofinal sequence of elements of $\mathcal{I}$ to which one can apply the

methods developed in Section 2.3. We then apply these ideas to the specific ideal $\emptyset \times \mathrm{Fin}$, which is the ideal generated by the sets $\{I_f \mid f \in {}^{\omega}\omega\}$, to obtain a proof of Theorem A.2.

Finally, Section 2.5 contains our proof of Theorem B.2. This proof again builds on the techniques of [165], combining them with the essential use of certain square sequences to enable the recursive construction of nontrivial coherent n-families of length greater than ω_n. The heart of the section is a technical stepping-up lemma allowing us to use an appropriate combination of square principles and weak diamonds to construct nontrivial coherent $(n+1)$-families out of shorter nontrivial coherent n-families. This lemma is then applied in various forcing extensions to yield a proof of Theorem B.2.

Now we focus on the contents of Chapter 3, which deals with generalizations of the system $\mathbf{A}$. Let $\mathbf{A}_1$ denote the inverse sequence

$$\cdots \to \mathbb{Z}^{n+1} \to \mathbb{Z}^n \to \cdots \to \mathbb{Z}^2 \to \mathbb{Z}$$

with bonding maps the natural projections. More generally, for any cardinal λ, let $\mathbf{A}_\lambda$ denote the sum, in the pro-category of abelian groups, of λ many copies of $\mathbf{A}_1$; its terms are then $\bigoplus_{i<\lambda} \mathbb{Z}^{f(i)}$, as f ranges within the pointwise partial ordering of functions from λ to $\mathbb{N}$, with bonding maps again the natural projections.

The higher derived limits of $\mathbf{A}_1$ are all easily seen to vanish, hence for any $n > 0$, any nonvanishing $\lim^n \mathbf{A}_\lambda$ witnesses the failure of the functor $\lim^n : \mathsf{Pro}(\mathsf{Ab}) \to \mathsf{Ab}$ to be additive. This, in turn, carries consequences in more general mathematics, such as those in homology theory first underscored in [121] and already discussed above, or those in condensed mathematics more recently surveyed in [19].

In this context, three main sorts of theorems shape the terrain. The first of these concern generalizations of $\mathbf{A}$ at the level of its terms and transition maps. The inverse systems comprising the natural generalizations are termed *Ω-systems* and, more broadly, *Ω_λ-systems* in [10] and [8, 9], respectively, wherein the fundamental results on $\mathbf{A}$ are shown to extend to these much more general classes. The second of these concerns generalizations of the "vertical" or codomain parameter in $\mathbf{A}$'s index-set to cardinals $\kappa > \aleph_0$. Here the representative theorem is [19, Theorem 2.10], showing that the associated derived limits are nonzero for any $\kappa \geq \aleph_1$. Thus it's only for sums of inverse systems of countable height (i.e., of *towers*) that we may hope for positive additivity results, and it is this which has determined our focus on $\mathbf{A}$'s "horizontal" generalizations $\mathbf{A}_\lambda$. This brings us to the area's third main type of theorem [15, Theorem 5.1]:

Proposition 0.5. $\lim^1 \mathbf{A}_{\aleph_0} = 0$ *implies that* $\lim^1 \mathbf{A}_\lambda = 0$ *for every cardinal* λ.

More colloquially, the functor $\lim^1 : \mathsf{Pro}(\mathsf{Ab}) \to \mathsf{Ab}$ is additive over arbitrary sums of towers if and only if it is additive over countable sums of towers. The status of this implication in higher degrees has long been the obvious next question; it figures as both [15, Question 1] and [8, Question 7.2], and has thus far formed the main avenue of approach to the more general additivity question recorded in our conclusion.

Question 0.6. *Does the above proposition hold for* $\lim^n$ *in degrees* $n > 1$*?*

In the present work, we show, in two distinct ways, that the answer is no, in the process answering the question's Ω_λ-system variants [8, Question 7.4] and [8, Question 7.5] as well.

Theorem A.3. *Assuming the continuum hypothesis, there exists an $\aleph_2$-closed poset $\mathbb{P}$ forcing that $\lim^2 \mathbf{A}_{\aleph_0} = 0$ and $\lim^2 \mathbf{A}_{\aleph_1} \neq 0$.*

This theorem derives additional interest from both the relative simpicity and the combinatorial mainspring of its argument; the latter, in particular, consists in a construction of coherent families satisfying a strengthened version of the classical nontriviality condition of finite-to-one-ness.

We then turn to a stronger theorem, one which forms Chapter 3's main labor. As background, recall how many of the main results on the derived limits of $\mathbf{A}_\lambda$ and related systems have amounted to essentially linear constructions [10, 15, 18, 121, 165]. Such approaches, however, are inherently limited in their reach (cf. Lemma 3.13 below); accordingly, the development of nonlinear construction techniques has been a particular focus of this thesis' author [33, 34]. The following theorem represents a further advance in these techniques; its key device is the construction of nontrivial n-coherent families not along *chains* but along *special λ-Aronszajn trees* in the partial order $({}^\lambda\omega, \leq)$.

Theorem B.3. *Assume the generalized continuum hypothesis and that $\diamondsuit(S^{i+1}_i)$ holds for all positive $i < \omega$. Then for all $n < \omega$ and cardinals λ, $\lim^{n+1} \mathbf{A}_\lambda = 0$ if and only if $\lambda < \aleph_n$.*

In plainer English, it is consistent with the ZFC axioms that the groups $\lim^n \mathbf{A}_\lambda$ are simultaneously nonzero everywhere that they can be — or, more precisely, wherever they're not required to be zero by Goblot's vanishing theorem. The transition maps of any $\mathbf{A}_\lambda$ are all surjective and, as is well-known, the premises of our theorem all hold in, for example, the constructible universe $\mathbb{L}$. Hence:

Corollary C.3. *In Gödel's constructible universe, $\lim^n \mathbf{A}_\lambda \neq 0$ in every instance not proscribed by Goblot's vanishing theorem.*

Chapter 3's organization is as follows. We record our most fundamental notations in the course of Section 3.1; further notations are recorded in Section 3.3 as well. We prove Theorem A.3 in Section 3.2 and Theorem B.3 in Sections 3.3 through 3.6. Our conclusion then records what figure as the main remaining open questions.

Chapter 1

Nonvanishing derived limits without scales

1.1 Preliminaries and notation

Let $\mathbf{G} = (G_\lambda, p^\mu_\lambda, \Lambda)$ be an inverse system of abelian groups, with groups G_λ indexed over the directed quasi-order $(\Lambda, \leq)$ and bonding morphisms $p^\mu_\lambda : G_\mu \to G_\lambda$, with $\lambda \leq \mu \in \Lambda$, between them. Let moreover $\Lambda^{(n)}$ be the set of $\leq$-ordered $n+1$-tuples in Λ. If $\bar{\lambda} = (\lambda_0, ..., \lambda_n) \in \Lambda^{(n)}$ we write $\bar{\lambda^i} = (\lambda_0, ..., \lambda_{i-1}, \widehat{\lambda_i}, \lambda_{i+1}, ..., \lambda_n)$, meaning that we have removed the element λ_i.

The derived limit $\lim^n$, for $n > 0$, is the n-th derived functor of the inverse limit $\lim$. We can also write $\lim^0$ for $\lim$. We refer the reader interested in the definition of derived functors through resolutions or the total derived functor to [168] and [151] respectively. Throughout the proof of the main theorem of this chapter, we will use the following characterization due to Roos ([144]).

Definition 1.1 (Roos complex). *We define the cohomological Roos complex as*

$$K^\bullet(\mathbf{G}) = (\, 0 \to K^0(\mathbf{G}) \xrightarrow{\delta^1} K^1(\mathbf{G}) \xrightarrow{\delta^2} K^2(\mathbf{G}) \xrightarrow{\delta^3} ...\,),$$

where

$$K^n(\mathbf{G}) = \prod_{\bar{\lambda} \in \Lambda^{(n)}} X_{\lambda_0} = \prod_{\lambda_0 \leq ... \leq \lambda_n} G_{\lambda_0}.$$

In other words, K^n consists of all functions $\bar{x} : \Lambda^{(n)} \to \bigcup_\lambda G_\lambda$ (we think of the G_λ as disjoint) such that $\bar{x}(\bar{\lambda}) \in G_{\lambda_0}$. We will also write $x_{\bar{\lambda}}$ for $\bar{x}(\bar{\lambda})$ and $\mathbf{G}^{(n)}$ for $K^n(\mathbf{G})$ and $\mathbf{G}^{(n)} \restriction C$ for $K^n(\mathbf{G} \restriction C)$. The coboundary maps are

$$\delta^n(\bar{x})_{\bar{\lambda}} = p^{\lambda_1}_{\lambda_0}(x_{\bar{\lambda^0}}) + \sum_{1 \leq i \leq n} (-1)^i x_{\bar{\lambda^i}}.$$

Note that we stipulate that $K^{-1}(\mathbf{G}) = 0$ and $\delta^0 = 0$.

We also define a couple of terms related to this construction.

Definition 1.2 (Coherent and trivial elements). *We call an element n-coherent if it is in the kernel of δ^{n+1}, and we call it n-trivial if it is in the image of δ^n. If $\delta^n(\bar{y}) = \bar{x}$ we say that $\bar{y}$ trivializes $\bar{x}$.*

The derived limits can be computed as the cohomology of the Roos complex.

Theorem 1.3. *Let $\mathbf{G}$ be an inverse system of groups, $K^\bullet$ be the associated Roos complex, and $n \geq 0$. Then*

$$\lim{}^n \mathbf{G} \cong H^n(K^\bullet).$$

This is part of a theorem for which the literature on applications of set theory to this topic usually references the rather long proof of [122] in the case of modules. In the appendix, we offer a shorter categorical proof that may be more amenable to logicians. There we expand upon the argument of [138] by clarifying some constructions and proving an intermediate claim that may be obscure to some readers.

We now record two basic results we will need in the next section.

Definition 1.4. *Let*

$$\boldsymbol{G} = \{G_\lambda; p^\mu_\lambda : \lambda \leq \mu \in \Lambda\}.$$

be an inverse system of abelian groups. We say that the system is surjective if each p^μ_λ is surjective.

Theorem 1.5 (Surjective Goblot's Theorem [165])**.** *Let $n \geq 0$. If $\mathbf{G}$ is a surjective inverse system of abelian groups with $cf(\mathbf{G}) = \aleph_n$, then $\lim^k \mathbf{G} = 0$, for all $k \geq n+1$.*

If M is a directed subset of the indexing set Λ we will write $\mathbf{G} \restriction M$ for the system $\{G_\lambda; p^\mu_\lambda : \lambda \leq \mu \in M\}$.

Theorem 1.6 ([130])**.** *Let $\mathbf{G}$ be an inverse system indexed by Λ, let M be another directed set and let $\phi : M \to \Lambda$ be an order preserving cofinal map. Let moreover $\phi^*(\mathbf{G})$ be the system indexed on M defined by pre-composing the functor ϕ between the two orders thought of as categories to the functor $\mathbf{G}$. Then, for all $n \geq 0$,*

$$\lim^n \mathbf{G} \cong \lim^n \phi^*(\mathbf{G}).$$

In particular, the case where ϕ is an inclusion gives us that for a cofinal subset $C \subseteq \Lambda$, we have $\lim^n \mathbf{G} \cong \lim^n \mathbf{G} \restriction C$.

To more easily apply the last two theorems we will introduce the setting of quotient systems described in the following section and will apply it to the systems in which we are interested.

1.2 The inverse system A and its relatives

The main object of our interest is the inverse system $\mathbf{A}$. It is indexed by the set ${}^\omega\omega$ of all functions $f : \omega \to \omega$ and such that the object A_f corresponding to f is defined as

$$A_f = \bigoplus_{i \in \omega} \mathbb{Z}^{f(i)}.$$

The order relation on this set is everywhere domination (i.e., $f \leq g$ if and only if $f(i) \leq g(i)$ for all $i \in \omega$). The bonding morphisms are the obvious projections. We write $I_f = \{(i,j) \mid j \leq f(i)\}$ for the region on which the elements of the groups seen as functions to $\mathbb{Z}$ are supported. Analogously, we define the system $\mathbf{B}$ where we replace infinite sums with infinite products:

$$B_f = \prod_{i \in \omega} \mathbb{Z}^{f(i)}.$$

This yields a short exact sequence of inverse systems

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{B}/\mathbf{A} \to 0,$$

which induces a long exact sequence

$$0 \to ... \to \lim^n \mathbf{A} \to \lim^n \mathbf{B} \to \lim^n \mathbf{B}/\mathbf{A} \to \lim^{n+1} \mathbf{A} \to ... \quad .$$

Now, [121, Lemma 4] says that $\lim^n \mathbf{B} = 0$ for $n \geq 1$, so that $\lim^{n+1} \mathbf{A} \cong \lim^n \mathbf{B}/\mathbf{A}$ and $\lim^1 \mathbf{A} \cong \frac{\lim^0 \mathbf{B}/\mathbf{A}}{\lim^0 \mathbf{B}}$. The proof straightforwardly generalizes to restrictions to directed subsets and groups different than $\mathbb{Z}$. In other words, for any directed $\mathcal{F} \subseteq {}^\omega\omega$, and any nontrivial abelian group G, we can define $\mathbf{A}_G^\mathcal{F}$ and $\mathbf{B}_G^\mathcal{F}$ whose objects are $A_f = \bigoplus_{i \in \omega} G^{f(i)}$ and $B_f = \prod_{i \in \omega} G^{f(i)}$ respectively, for each $f \in \mathcal{F}$. Then the reasoning above still yields isomorphisms $\lim^1 \mathbf{A}_G^\mathcal{F} \cong \frac{\lim^0 \mathbf{B}_G^\mathcal{F}/\mathbf{A}_G^\mathcal{F}}{\lim^0 \mathbf{B}_G^\mathcal{F}}$ and $\lim^{n+1} \mathbf{A}_G^\mathcal{F} \cong \lim^n \mathbf{B}_G^\mathcal{F}/\mathbf{A}_G^\mathcal{F}$.

For the case $n > 1$, we further observe that the inclusion map $\iota : (\mathcal{F}, \leq) \hookrightarrow (\mathcal{F}, \leq^*)$ is order-preserving and cofinal, hence by Theorem 1.6 we can further substitute for $\lim^n \mathbf{B}_G^\mathcal{F}/\mathbf{A}_G^\mathcal{F}$ the limit $\lim^n(\mathbf{B}_G^\mathcal{F}/\mathbf{A}_G^\mathcal{F})_{\leq^*}$ of a system with the same quotient groups as objects and with bonding morphisms analogous to the previous ones for each instance of $\leq^*$. One can justify the same conclusions by the approach of [165]. This last system is surjective and has the property of being indexed on the very order whose cofinality is by definition $\mathfrak{d}$. This will be relevant for the use we are going to make of Theorem 1.5.

Let $=^*$ stand for equality except possibly on finitely many points. The isomorphism for $\lim^1$ yields the following characterization, where one can indifferently use $\omega \times \omega$ or $\bigcup_{f \in \mathcal{F}} I_f$ as the domain of ψ.

Proposition 1.7. *Let G be a nontrivial group, and let $\mathcal{F} \subseteq {}^\omega\omega$ be directed. Then $\lim^1 \mathbf{A}_G^\mathcal{F} \neq 0$ if and only if there exists a family $\langle \phi_f : I_f \to G \mid f \in \mathcal{F} \rangle$ of functions such that:*

1. *For every two $f, g \in \mathcal{F}$, $\phi_f \restriction I_f \cap I_g =^* \phi_g \restriction I_f \cap I_g$.*
2. *There exists no function $\psi : \omega \times \omega \to G$ such that $\psi \restriction I_f =^* \phi_f$ for all $f \in \mathcal{F}$.*

We call a family that satisfies the first condition *coherent*. If it satisfies the second condition we say moreover that it is *nontrivial*.

Remark 1.8. *Note that* $\lim^1 \mathbf{A}^{\mathcal{F}}_G \neq 0$ *implies* $\lim^1 \mathbf{A}^{\mathcal{F}}_{G'} \neq 0$ *whenever* $|G| \leq |G'|$*, so that the vanishing of* $\lim^1 \mathbf{A}^{\mathcal{F}}_G$ *is equivalent for groups of the same cardinality, while the group structure is relevant for higher limits. This is because the equation* $a - b = 0$ *reduces to the logical notion of equality* $a = b$*, so that a witness for* $\lim^1 \mathbf{A}^{\mathcal{F}}_G \neq 0$ *can be seen modulo injection as a witness for* $\lim^1 \mathbf{A}^{\mathcal{F}}_{G'} \neq 0$*, for any* G' *of greater or equal size, in contrast, there is an inescapable algebraic aspect to longer alternating sums.*

The characterization for higher derived limits can be expressed in terms of n-coherent and n-nontrivial families for the quotient system.

Proposition 1.9. *For* $n \geq 1$*,* $\lim^{n+1} \mathbf{A}^{\mathcal{F}}_G \neq 0$ *if and only if there exists a family* $\bar{x} \in ((\mathbf{B}^{\mathcal{F}}_G/\mathbf{A}^{\mathcal{F}}_G)_{\leq^*})^{(n)}$ *such that* $\delta^{n+1}(\bar{x}) = 0$ *and such that for no* $\bar{y} \in ((\mathbf{B}^{\mathcal{F}}_G/\mathbf{A}^{\mathcal{F}}_G)_{\leq^*})^{(n-1)}$ *we have* $\delta^n(\bar{y}) = \bar{x}$*.*

We note however that the established usage is to refer to these witnesses (or, more precisely, their representatives in $\mathbf{B}^{\mathcal{F}}_G$ as nontrivial $(n+1)$-coherent families, since they witness a nontrivial $\lim^{n+1} \mathbf{A}^{\mathcal{F}}_G$.

1.3 Unbounded nontriviality

Before we start the proof by induction of the main result of this section, we need a lemma.

Definition 1.10. *Let* $(P, \leq)$ *be a quasi-order. We say that* $C \subseteq P$ *is a* κ*-chain if there is a bijection* $\sigma : \kappa \to C$ *such that* $\alpha \leq \beta$ *implies* $\sigma(\alpha) \leq \sigma(\beta)$*.*

Remark 1.11. *If* κ *is regular and* C *is unbounded in* P*, then we can find* $C' \subseteq C$ *ordered by the strict order* $<$ *such that* $a < b \leftrightarrow (a \leq b \wedge \neg(a \geq b))$*.*

Lemma 1.12. *Let* $n \geq 0$ *and* $(P, \leq)$ *be a quasi-order such that* $\mathrm{cf}(P) = \aleph_{n+1}$ *and there exists an* ω_{n+1}*-chain* C *that is unbounded in* P*. Let moreover* P *be equipped with a function* $d : P^2 \to P$ *such that:*

1. $d(x, y) \geq x, y$ *(witnessing the directedness of* P*);*
2. $d(x', y') \leq d(x, y)$ *whenever* $x' \leq x$ *and* $y' \leq y$*.*

Then we can find a continuous increasing union

$$P = \bigcup_{\alpha < \omega_{n+1}} P_\alpha,$$

and a $C' \subseteq C$ *cofinal in* C *such that for each* $\alpha < \omega_{n+1}$*,* P_α *is* d*-closed, downward closed, with* $\mathrm{cf}(P_\alpha) \leq \aleph_n$*,* $\mathrm{otp}(P_\alpha \cap C') = \alpha$ *and* $P_\alpha \cap C'$ *unbounded in* P_α *for every limit* α*.*

Proof. Let $c : \omega_{n+1} \to P$ be an enumeration of a cofinal subset of P, and assume without loss of generality, thanks to Remark 1.11, that C is a chain of order-type ω_{n+1} for the strict order relation.

We construct by mutual transfinite recursion the increasing union and a function $b : \omega_{n+1} \to C$ such that $b(\alpha)$ is the least element of $C \setminus P_\alpha$, where P_α is the downward closure of the d-closure of $c[\alpha] \cup b[\alpha]$ (note that it stays d-closed by condition 2). This is increasing and continuous by construction. The d-closure of $c[\alpha] \cup b[\alpha]$ is cofinal in P_α and of cardinality $\leq \aleph_n$. For each element in it, we can find one in the chain that is not dominated by it; their supremum (which exists because the chain has cofinality $> \aleph_n$) will not be in the d-closure P_α. This shows that b is well-defined, strictly increasing, and hence cofinal. Finally, we put $C' = b[\omega_{n+1}]$. The conditions on the order-type and unboundedness are immediate by construction. □

Remark 1.13. *Note that, for every limit λ, $cf(P_\lambda) \geq cf(\lambda)$. This follows from the fact that for every limit $\lambda < \omega_{n+1}$, $C' \cap P_\lambda$ is unbounded in P_λ. Otherwise, for each element in a cofinal set of lesser cardinality we pick one of the chain that is not dominated by it, and taking the supremum we get a contradiction, as in the previous proof.*

Remark 1.14. *In case $P = {}^\omega\omega$, one natural choice for the function d is the lowest upper bound $\vee : ({}^\omega\omega)^2 \to {}^\omega\omega$ defined as $(f \vee g)(i) = \max\{f(i), g(i)\}$. The same is true for every subset $P \subseteq {}^\omega\omega$ that is $\vee$-closed.*

The proof in [165] served as a basis for the following argument. The central idea allowing the present advances is the construction of some nontriviality for the base case having the additional property of being "localized", meaning that the restriction of a larger coherent family to a certain chain is nontrivial, and the observation that this additional property is preserved in the induction.

1.3.1 Base case

To get a result that holds for all nontrivial groups G, we need an extra cardinal arithmetic assumption for the base case.

Lemma 1.15. *Assume that $2^{\aleph_0} < 2^{\aleph_1}$. Let $P \subseteq ({}^\omega\omega, \leq^*)$ be $\vee$-closed and such that $cf(P) = \aleph_1$ and there exists an ω_1-chain C which is unbounded in P, and let $G \neq 0$ be an abelian group. Then there is a coherent 1-family $\Phi = \langle \phi_f : I_f \to G \mid f \in P \rangle$ such that $\Phi \restriction C$ is (coherent) nontrivial.*

Proof. By the reasoning that shows Remark 1.8, we can restrict to the case $G = \mathbb{Z}/2\mathbb{Z}$.

Apply Lemma 3.11 with respect to $d = \vee$ to get a continuous increasing union $\bigcup_{\alpha<\omega_1} P_\alpha = P$ and C' satisfying the lemma. Recall in particular that the α-th element of C' (henceforth g_α) is not in the downward-closed set P_α, and $\mathrm{cf}(P_\alpha) \leq \omega$. So let $\{\hat{f}_{\alpha,n} \mid n < \omega\}$ be a cofinal

subset of P_α. Since P_α is $\vee$-directed we can let $f_{\alpha,n} = \bigvee_{i\leq n} \hat{f}_{\alpha,i}$ so that $\langle f_{\alpha,n} \mid n < \omega\rangle$ is cofinal for $\leq^*$ and a chain with respect to $\leq$.

Since $g_\alpha \not\leq^* f_{\alpha,n}$, the set $I_{g_\alpha} \setminus I_{f_{\alpha,n}}$ is infinite (in particular non-empty) for all n. Pick $x_{\alpha,n} \in I_{g_\alpha} \setminus I_{f_{\alpha,n}}$. Then $X_\alpha = \{x_{\alpha,n}\}_{n<\omega} \subseteq I_{g_\alpha}$ is infinite and almost disjoint from all of the $I_{f_{\alpha,n}}$ as $X_\alpha \cap I_{f_{\alpha,n}} \subseteq \{x_{\alpha,0}, ..., x_{\alpha,n-1}\}$.

We are now going to define a tree of coherent families ordered by end-extension, such that among the unions of the branches one yields a nontrivial family by counting arguments.

For all $s \in 2^{<\omega_1}$ we iteratively construct a coherent family Φ^s on P_α, where $\mathrm{dom}(s) = \alpha$ as follows. If $\alpha \in \lim(\omega_1)$, then $\Phi^s = \bigcup_{\alpha<\mathrm{dom}(s)} \Phi^{s\restriction\alpha}$. Suppose now that $\alpha = \beta + 1$, we have already defined Φ^s with $\mathrm{dom}(s) = \beta$ and we want to define $\Phi^{s,0}$ and $\Phi^{s,1}$. So we find a trivialization ψ^s of Φ^s that exists since P_β is of countable cofinality, so we can consider its cofinal chain $\langle f_{\beta,n} \mid n < \omega\rangle$ defined above; it is then easy to see that we can inductively find finite sets F_n such that for all $m \leq n$ we have that $\phi^s_{f_{\beta,m}} \restriction (I_{f_{\beta,m}} \setminus F_m)$ and $\phi^s_{f_{\beta,n}} \restriction (I_{f_{\beta,n}} \setminus F_n)$ coincide on their common domain. We can then define

$$\psi^s(x) = \begin{cases} \phi^s_{f_{\beta,n}} & \text{if } x \in (I_{f_{\beta,n}} \setminus F_n) \text{ for some } n \\ 0 & \text{otherwise.} \end{cases}$$

Then we let $\psi^{s,0} = \psi^s$ and

$$\begin{cases} \psi^{s,1}(x) = \psi^s(x) + 1 & \text{if } x \in X_\beta \\ \psi^s(x) & \text{otherwise.} \end{cases}$$

Finally, we let $\Phi^{s,0} = \delta^1(\psi^{s,0})$ and $\Phi^{s,1} = \delta^1(\psi^{s,1})$.

Now for all $h : \omega_1 \to 2$ we put $\Phi^h = \bigcup_{\alpha<\omega_1} \Phi^{h\restriction\alpha}$. Note that if $h \neq h'$, then $\phi^h_{g_\gamma} \neq^* \phi^{h'}_{g_\gamma}$ where γ is such that $h(\gamma) \neq h'(\gamma)$. But then the same trivialization cannot trivialize both $\Phi^h \restriction C'$ and $\Phi^{h'} \restriction C'$. Since there are only $2^{\aleph_0}$ many putative trivializations and we are assuming that $2^{\aleph_0} < 2^{\aleph_1}$, there must be $\bar{h} : \omega_1 \to 2$ such that $\Phi^{\bar{h}} \restriction C'$ is nontrivial. Hence, by Theorem 1.6, $\Phi^{\bar{h}} \restriction C$ is also coherent nontrivial. □

Remark 1.16. *In case G is infinite, the cardinal arithmetic assumption $2^{\aleph_0} < 2^{\aleph_1}$ is not needed, as one can simply apply the reasoning of [165, Proposition 3.2] with the sets X_α above instead of the sets b_ξ of the cited paper, after fixing a bijection between $\omega \times \omega$ and ω, and an injection of the latter into G.*

1.3.2 Induction step

Definition 1.17. *We say that $R(n, G)$ holds for some group G if for every $\vee$-closed $P_n \subseteq ({}^\omega\omega, \leq^*)$ with $\mathrm{cf}(P_n) = \aleph_n$ that contains an ω_n-chain C_n that is unbounded in it, there exists an n-coherent family on P_n for G such that its restriction to C_n is nontrivial.*

Remark 1.18. *By Proposition 1.7 and Proposition 1.9,* $R(n,G)$ *implies in particular* $\lim^n \mathbf{A}_G^{P_n} \neq 0$ *for every* P_n *as in the definition above.*

Whenever S is a stationary set, $\mathrm{w}\diamondsuit(S)$ denotes a guessing principle called *weak diamond* of S, introduced in [50]. Recall that $S_n^{n+1} = \{\alpha < \omega_{n+1} \mid \mathrm{cf}(\alpha) = \omega_n\}$.

Lemma 1.19. *Suppose* $n \geq 1$, $|G| \leq 2^{\aleph_1}$, *and both* $R(n,G)$ *and* $\mathrm{w}\diamondsuit(S_n^{n+1})$ *hold. Then* $R(n+1,G)$ *holds.*

Proof. Let P be as in the statement of $R(n+1,G)$. Using Lemma 3.11 we can write $P = \bigcup_{\alpha<\omega_{n+1}} P_\alpha$ with further assumptions on the cofinality of the latter sets and their relation to the unbounded chain. Throughout the proof, we are going to use the quotient system discussed in section 3, so for any $f \in P$ we define $Q_f = \prod_{i\in\omega} G^{f(i)} / \bigoplus_{i\in\omega} G^{f(i)}$ and let $p_f^{f'} : Q_{f'} \to Q_f$ for the natural projection whenever $f \leq^* f'$. Finally, let

$$\mathbf{Q} = \{Q_f; p_f^{f'}; P\}.$$

Let $T = \{s \in 2^{<\omega_{n+1}} \mid \mathrm{supp}(s) \subseteq S_n^{n+1}\}$ with the ordering being end-extension. For $s \in T$ we let $r(s) = P_{\mathrm{dom}(s)}$.

We now define, for all $s \in T$, a sequence $\bar{x}^s$ such that:

1. $\bar{x}^s \in \mathbf{Q} \restriction r(s)^{(n)}$ and is coherent.
2. If $t \geq s$ then $\bar{x}^t$ extends $\bar{x}^s$.
3. If $\mathrm{cf}(\mathrm{dom}(s)) = \aleph_n$ and $\bar{u} \in \mathbf{Q} \restriction (C \cap r(s))^{(n-1)}$ trivializes $\bar{x}^s \restriction (C \cap r(s))^{(n)}$ then there exists $\epsilon \in \{0,1\}$ such that there is no extension of $\bar{u}$ to a trivialization of $\bar{x}^{s,\epsilon} \restriction (C \cap P_{\mathrm{dom}(s)+1})^{(n)}$.

If $\mathrm{dom}(s)$ is limit, and $\bar{x}^t$ has already been constructed for all t with $\mathrm{dom}(t) < \mathrm{dom}(s)$, we let $\bar{x}^s = \bigcup_{\alpha<\mathrm{dom}(s)} \bar{x}^{s\restriction\alpha}$.

Now for the successor step, since $\mathrm{cf}(r(s)) \leq \aleph_n$ then $\bar{x}^s \in \mathbf{Q} \restriction r(s)^{(n)}$ is trivial by Goblot (Theorem 1.5). Then for $\epsilon \in \{0,1\}$, let $s,\epsilon = s^\frown\{\epsilon\}$ and $\bar{w}^s \restriction r(s)^{(n-1)}$ be a trivialization of $\bar{x}^s$ as in the induction, and $\bar{v}^s \in \mathbf{Q} \restriction (r(s,0))^{(n-1)}$ be

$$\bar{v}^s_{\bar{\gamma}} = \begin{cases} \bar{w}^s_{\bar{\gamma}} & \text{if } \bar{\gamma} \in r(s)^{(n-1)} \\ 0 & \text{if } \bar{\gamma} \in (r(s,0))^{(n-1)} \setminus r(s)^{(n-1)}. \end{cases}$$

Finally, if $\mathrm{cf}(\mathrm{dom}(s)) < \aleph_n$ we let $\bar{x}^{s,0} = \delta^n(\bar{v}^s)$.

Otherwise, $\aleph_n = \mathrm{cf}(\mathrm{dom}(s)) \leq \mathrm{cf}(r(s)) \leq \aleph_n$. In that case, we pick a family $\bar{z}^s \in \mathbf{Q} \restriction r(s)^{(n-1)}$ that witnesses the induction hypothesis, that is, it is coherent on $P_{\mathrm{dom}(s)}$

and nontrivial even when restricted to the ω_n-chain unbounded in it that corresponds to a witness of the cofinality of $\mathrm{dom}(s)$ (or equivalently to its $\leq^*$-closure) and let $\bar{y}^s \in \mathbf{Q} \restriction (r(s,0))^{(n-1)}$

$$\bar{y}^s_{\bar{\gamma}} = \begin{cases} \bar{z}^s_{\bar{\gamma}} & \text{if } \bar{\gamma} \in r(s)^{(n-1)} \\ 0 & \text{if } \bar{\gamma} \in (r(s,0))^{(n-1)} \setminus r(s)^{(n-1)}. \end{cases}$$

Finally, we let: $\bar{x}^{s,0} = \delta^n(\bar{v}^s)$ and $\bar{x}^{s,1} = \delta^n(\bar{v}^s + \bar{y}^s)$.

Let us now check that conditions (1) to (3) are satisfied. An increasing union of coherent sequences is coherent, and it extends the sequences of which it is a union, so conditions 1 and 2 are preserved at limit steps. They are also preserved at successor steps because trivial sequences are coherent and

$$\delta^n(\bar{v}^s + \bar{y}^s) \restriction r(s)^{(n)} = \delta^n(\bar{v}^s) \restriction r(s)^{(n)} + \delta^n(\bar{y}^s) \restriction r(s)^{(n)} = \delta^n(\bar{v}^s) \restriction r(s)^{(n)} =$$

$$= \delta^n(\bar{v}^s \restriction r(s)^{(n)}) = \delta^n(\bar{w}^s) = \bar{x}^s.$$

The only thing left is to prove condition 3. Suppose by contradiction that $\bar{u}^0 \restriction (C \cap r(s))^{(n-1)} = \bar{u} = \bar{u}^1 \restriction (C \cap r(s))^{(n-1)}$, $\delta^n(\bar{u}^0) = x^{s,0} \restriction (C \cap r(s,0)))^{(n)}$ and $\delta^n(\bar{u}^1) = x^{s,1} \restriction (C \cap r(s,0)))^{(n)}$.

Let $f = \max\{C \cap r(s,0)\}$ and $\bar{u} \in \mathbf{Q} \restriction P^{(n-1)}_{\mathrm{dom}(s)+1}$. We define $d_f(\bar{u}) \in \mathbf{Q} \restriction (C \cap r(s))^{(n-2)}$ by $(d_f(\bar{u}))_{\bar{h}} = \bar{u}_{\bar{h},f}$, Then, if $\bar{g}$ ranges over $(C \cap r(s))^{(n-1)}$, we have (omitting restrictions for readability):

$$\delta^{n-1}(d_f(\bar{u}^0))_{\bar{g}} = \sum_{0 \leq i \leq n-1} (-1)^i (d_f \bar{u}^0)_{\bar{g}^i} = \sum_{0 \leq i \leq n-1} \bar{u}^0_{\bar{g}^i,f} = (-1)^{n-1} \bar{u}^0_{\bar{g}} + \bar{x}^{s,0}_{\bar{g},f} = (-1)^{n-1} \bar{u}^0_{\bar{g}} + (-1)^n \bar{w}_{\bar{g}}.$$

By the same reasoning

$$\delta^{n-1}(d_f(\bar{u}^1))_{\bar{g}} = (-1)^{n-1} \bar{u}^0_{\bar{g}} + (-1)^n \cdot (\bar{w}_{\bar{g}} + \bar{z}^s_{\bar{g}}),$$

so that, since $\bar{u}^0$ and $\bar{u}^1$ agree on $(C \cap r(s))^{(n-1)}$

$$\delta^{n-1}(d_f(\bar{u}^1 - \bar{u}^0)) = (-1)^n \bar{z}^s \restriction (C \cap r(s))^{(n-1)},$$

contradicting the fact that the restriction of $\bar{z}^s$ to the unbounded chain is nontrivial.

It should be noticed that the case $n = 1$ is computationally identical but conceptually different insofar as we need to stipulate that a family indexed by 0-tuples is a singleton and that a function defined on I_f can be regarded as a trivialization of a 1-family below f as it can be extended to the entire grid $\omega \times \omega$.

This concludes the construction of the $\bar{x}^s$ for $s \in T$. For $g \in 2^{\omega_{n+1}}$ with $\mathrm{supp}(g) \subseteq S^{n+1}_n$ let

$$\bar{x}^g = \bigcup_{\lambda<\omega_{n+1}} \bar{x}^{g\restriction\lambda}.$$

Note that $\bar{x}^g \in \mathbf{Q}^{(n)}$ is coherent (increasing union of coherent sequences).

Now we need a club $D \subseteq \omega_{n+1}$ such that for all $\lambda \in D$ and $s \in T$ with $\mathrm{dom}(s) = \lambda$, the latter codes a sequence $\bar{u}^s \in \mathbf{Q} \restriction (C \cap r(s))^{(n-1)}$ and moreover:

- If $\lambda \in D$ then every $\bar{u}^s \in \mathbf{Q} \restriction (C \cap r(s))^{(n-1)}$ is coded by some $s : \lambda \to 2$.
- If $s \leq t$ then $\bar{u}^t$ extends $\bar{u}^s$.

For this, we will use

$$D = \{\lambda < \omega_{n+1} \mid \omega_n \cdot \lambda = \lambda\},$$

since for any two elements $\lambda, \lambda' \in D$ there is a subset of order-type at least ω_n between them in ω_{n+1}. Now let λ_+ be the successor of λ in the increasing enumeration of the elements of D. Then there are enough end-extensions of $s : \lambda \to 2$ to an element in 2^{λ_+} to code all the different extensions of $\bar{u}^s \in \mathbf{Q} \restriction (C \cap P_\lambda)^{(n-1)}$ to a sequence $\bar{u}^t \in \mathbf{Q} \restriction (C \cap P_{\lambda_+})^{(n-1)}$. Indeed, these are at most $(|G|^{\aleph_0})^{\aleph_n} \leq 2^{\aleph_n}$. The base case of the transfinite induction follows similarly from the fact that $\omega_n \leq \lambda_0 = \min D$ Since we are dealing with a club, the limits of the enumeration are limits in ω_{n+1} of the previous elements, and we can continue the process at limit steps just by taking unions. Finally, for $b : \omega_{n+1} \to 2$, we let

$$\bar{u}^b = \bigcup_{\lambda\in D} \bar{u}^{b\restriction\lambda}.$$

Notice that this implies that for all $\bar{u} \in \mathbf{Q} \restriction C^{(n-1)}$ there is a b that codes it and for every such b we have

$$\bar{u}^{b\restriction\lambda} = \bar{u}^b \restriction (C \cap P_\lambda)^{(n-1)}$$

on a club.

Before defining our guessing function, we need the following claim. Such an observation and the consequent definition of F seem needed in the argument of [165] as well, yet they are not present.

Claim 1.20. *Let $\bar{u} \in \mathbf{Q} \restriction (C \cap P_\beta)^{(n-1)}$ for some $\beta < \omega_{n+1}$. Then there is at most one $s \in 2^\beta$ such that $\bar{u}$ trivializes $\bar{x}^s \restriction (C \cap P_\beta)^{(n)}$.*

Proof of Claim. Assume, towards a contradiction, that there exist distinct t and t' such that $\bar{u}$ trivializes both $\bar{x}^t \restriction (C \cap P_\beta)^{(n)}$ and $\bar{x}^{t'} \restriction (C \cap P_\beta)^{(n)}$. Let $\alpha \in S^{n+1}_n \cap \beta$ be least such that $t(\alpha) \neq t'(\alpha)$. Then $\bar{u} \restriction (C \cap P_\alpha)^{(n-1)}$ trivializes $x^t \restriction (C \cap P_\alpha)^{(n)} = x^{t\restriction\alpha} \restriction (C \cap P_\alpha)^{(n)} = x^{t'\restriction\alpha} \restriction (C \cap P_\alpha)^{(n)} = x^{t'} \restriction (C \cap P_\alpha)^{(n)}$.

For the same reasons, $\bar{u} \restriction (C \cap P_{\alpha+1})^{(n-1)}$ trivializes both $x^t \restriction (C \cap P_{\alpha+1})^{(n)} = x^{t\restriction\alpha+1} \restriction (C \cap P_{\alpha+1})^{(n)}$ and $x^{t'} \restriction (C \cap P_{\alpha+1})^{(n)} = x^{t'\restriction\alpha+1} \restriction (C \cap P_{\alpha+1})^{(n)}$. This contradicts property 3 of our construction. □

Then let $T \restriction D$ be the set of all nodes whose level belongs to D. We define $F : T \restriction D \to 2$ as follows. If $\bar{u}^s$ trivializes $\bar{x}^t \restriction (C \cap P_{\mathrm{dom}(t)})^{(n)}$ for some t such that $\mathrm{dom}(s) = \mathrm{dom}(t)$, which is unique by the previous claim, then $F(s) = \epsilon \in \{0,1\}$ such that $\bar{u}^s$ does not extend to a trivialization of $x^{t,\epsilon}$. Otherwise, let $F(s)$ take an arbitrary value in $\{0,1\}$.

Now $\mathrm{w}\diamondsuit(S^{n+1}_n)$ means precisely that for all $F : 2^{<\omega_{n+1}} \to 2$ there exists $g : \omega_{n+1} \to 2$ such that for all $b : \omega_{n+1} \to 2$.

$$S = \{\lambda \in S^{n+1}_n \mid g(\lambda) = F(b \restriction \lambda)\}$$

is stationary. Then $\bar{x}^g \restriction C^{(n)}$ is nontrivial because for any putative trivialization $\bar{u}^b$ we get $\lambda \in S \cap D$ such that $g(\lambda) = F(b \restriction \lambda)$. Now, given the assumption we made towards a contradiction,

$$\bar{u}^{b\restriction\lambda} = \bar{u}^b \restriction (C \cap P_\lambda)^{(n-1)}$$

trivializes

$$\bar{x}^g \restriction (C \cap P_\lambda)^{(n)} = \bar{x}^{g\restriction\lambda} \restriction (C \cap P_\lambda)^{(n)}.$$

So, if we use the definition of F above for $s = b \restriction \lambda$ and $t = g \restriction \lambda$ we find that cannot be extended to a sequence in $\mathbf{Q} \restriction (C \cap P_{\lambda+1})^{(n-1)}$ that trivializes $\bar{x}^{g\restriction(\lambda+1)} \restriction (C \cap P_{\lambda+1})^{(n)} = \bar{x}^g \restriction (C \cap P_{\lambda+1})^{(n)}$, while $\bar{u}^b \restriction (C \cap P_{\lambda+1})^{(n-1)}$ is an extension that should do precisely that if $\bar{u}^b$ is to be a trivialization. □

Finally, we can prove the following.

Theorem 1.21. *Let $n \geq 1$ and assume that $\mathrm{w}\diamondsuit(S^{k+1}_k)$ for $k < n$ and $2^{\aleph_0} < 2^{\aleph_1}$ hold. Then for every directed family $\mathcal{F} \subseteq ({}^\omega\omega, \leq^*)$ containing an ω_n-chain C that is unbounded in it and such that $\mathrm{cf}(\mathcal{F}) = \aleph_n$, and for every nontrivial group G with $|G| \leq 2^{\aleph_1}$, there exists a coherent family $\bar{x} \in \mathbf{A}_G^{\mathcal{F}^{(n)}}$ for G such that $\bar{x} \restriction C^{(n)}$ is nontrivial. In particular, $\lim^n \mathbf{A}_G^{\mathcal{F}} \neq 0$. So, if $\mathfrak{d} = \aleph_n$ and $({}^\omega\omega, \leq^*)$ contains an unbounded chain of length ω_n, then $\lim^n \mathbf{A}_G \neq 0$.*

Proof. The cardinal arithmetic assumption yields $R(1, G)$ by Lemma 1.15. Then by the weak diamonds and Lemma 1.19 one can deduce that $R(n, G)$ holds. Then, by Remark 1.18,

we deduce that $\lim^n \mathbf{A}_G^{\mathcal{F}} \neq 0$. for every $\mathcal{F}$ as in the statement of $R(n,G)$. Finally, if $\mathfrak{d} = \aleph_n$ and there exists an unbounded ω_n-chain, then $({}^{\omega}\omega, \leq^*)$ is among the posets $R(n,G)$ is about. □

Remark 1.22. *The cardinal arithmetic assumption $2^{\aleph_0} < 2^{\aleph_1}$ is not required for $G = \mathbb{Z}$, i.e. for the pro-group $\mathbf{A}$, as it is only needed in the base case for finite groups, as Remark 1.16 shows.*

1.4 Cofinal rectangles and weak diamonds

We will now show that all the set-theoretic assumptions that we needed for the proof of the main theorem of this chapter are relatively consistent with ZFC.

First, we need to introduce a "nonlinear" finite support iteration of Hechler's forcings. The following forcing notion was first introduced in [76].

Let $\mathbb{Q}$ be any well-founded, countably directed poset. Let moreover $\mathbb{Q}^+ = \mathbb{Q} \cup \{*\}$ be the poset that consists of $\mathbb{Q}$ with one more element $*$ on top of it (i.e. greater than all other elements).

For each $a \in \mathbb{Q}^+$, we define a forcing notion $\mathbb{P}_a$. First, let $\mathbb{Q}/a = \{c \in \mathbb{Q} \mid c <_{\mathbb{Q}} a\}$. Suppose now that we have already defined $\mathbb{P}_b$ for all $b <_{\mathbb{Q}} a$.

1. $p \in \mathbb{P}_a$ if and only if
   (a) p is a finite function with $\mathrm{dom}(p) \subseteq \mathbb{Q}/a$.
   (b) For all $b \in \mathrm{dom}(p)$, $p(b) = (t, \dot{g})$ where $t \in {}^{<\omega}\omega$, and $\dot{g}$ is a $\mathbb{P}_b$-name for an element of ${}^{\omega}\omega$.
2. For $p, q \in \mathbb{P}_a$, we write $p \leq q$ if and only if
   (a) $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$.
   (b) For all $b \in \mathrm{dom}(q)$, if $p(b) = (s, \dot{f})$ and $q(b) = (t, \dot{g})$ then
      i. $t = s \restriction \mathrm{dom}(t)$
      ii. $p \restriction (\mathbb{Q}/b) \Vdash_{\mathbb{P}_b} \mathrm{dom}(t) \leq n < \mathrm{dom}(s) \Rightarrow s(n) > \dot{g}(n)$.
      iii. $p \restriction (\mathbb{Q}/b) \Vdash_{\mathbb{P}_b} \forall n\ \dot{f}(n) \geq \dot{g}(n)$.

The following is the case $\lambda = \omega$ of [48, Theorem 1].

Theorem 1.23. *Suppose $\mathbb{Q}$ is any countably directed well-founded poset. Let $\mathbb{H}_{\mathbb{Q}} = \mathbb{P}_*$ be the poset corresponding to the maximum $*$ of $\mathbb{Q}^+$ as in the definition above.*

1. *$\mathbb{H}_{\mathbb{Q}}$ is ccc.*

2. $V^{\mathbb{H}_{\mathbb{Q}}} \models \mathbb{Q}$ *embeds cofinally into* $({}^{\omega}\omega, <^*)$.

3. *If* $V \models$ "*the bounding number of* $\mathbb{Q}$ *is* β" *then* $V^{\mathbb{H}_{\mathbb{Q}}} \models \mathfrak{b} = \beta$.

4. *If* $V \models$ "*the dominating number of* $\mathbb{Q}$ *is* δ" *then* $V^{\mathbb{H}_{\mathbb{Q}}} \models \mathfrak{d} = \delta$.

We now focus on the specific instance of Hechler's forcing that we are going to need for our consistency result.

Proposition 1.24. $\mathbb{H}_{\omega_k \times \omega_n}$ *is* κ*-Knaster for every uncountable regular cardinal* κ.

Proof. Let A be a set of conditions of size κ. By the Δ-system lemma, we can thin out to an $A' \subseteq A$ with $|A'| = \kappa$ such that the supports of the conditions in A' form a Δ-system. Then we thin out to $A'' \subseteq A'$, again with $|A''| = \kappa$, and such that the stems of all conditions in the root of the Δ-system are equal. Now for any two $q_0, q_1 \in A''$ we find $p \leq q_0, q_1$ as follows. Let $\mathrm{dom}(p) = \mathrm{dom}(q_0) \cup \mathrm{dom}(q_1)$ and $p(b) = (s_b, \dot{f}_b)$ when $q_i(b) = (s_b, \dot{g}^i_b)$ for $i \in \{0, 1\}$. Finally, by the existential completeness lemma we can choose names such that $\mathbb{1}_{\mathbb{P}_b} \Vdash \dot{f}_b \geq \dot{g}^i_b$ for $i \in \{0, 1\}$. □

Proposition 1.25. *Let* $k \leq n < \omega$. *If* $V \models \mathrm{CH}$ *then* $\mathbb{H}_{\omega_k \times \omega_n}$ *has size* $\aleph_n$ *and* $V^{\mathbb{H}_{\omega_k \times \omega_n}} \models \mathfrak{c} = \aleph_n$.

Proof. Let $(\omega_k, \omega_n) = *$ be the maximum element of the associated poset. One proves by mutual transfinite induction on the well-founded order relation that for all $(\omega_1, \omega_1) \leq (\alpha, \beta) \leq (\omega_k \times \omega_n)$ we have $|\mathbb{P}_{(\alpha,\beta)}| = \max\{|\alpha|, |\beta|\}$ and $V^{\mathbb{P}_{(\alpha,\beta)}} \models \mathfrak{c} = \max\{|\alpha|, |\beta|\}$. One implication follows from the fact that the nonlinear iteration is finite support and the other follows by the technique of nice names [108, Lemma IV 3.11]. □

This allows us to apply the following lemma.

Lemma 1.26 (see [165])**.** *Let* κ *be an uncountable regular cardinal, and let* λ *and* μ *be cardinals such that* $\lambda^{<\kappa} < \mu$. *Suppose* $\mathbb{P}$ *is a* κ*-cc poset of size* λ. *Let* $\mathbb{Q} = \mathrm{Fn}(\mu \times \kappa, 2, \kappa)$ *that is, the poset of partial functions from* $\mu \times \kappa$ *to* 2 *of size strictly less than* κ. *Then, for every stationary subset* S *of* κ *in* V, $\mathrm{w}\diamondsuit(S)$ *holds in the generic extension by* $\mathbb{P} \times \mathbb{Q}$.

In fact, Lemma 1.26 allows us to prove that a certain forcing extension satisfies all of our requirements.

Theorem 1.27. *Let* $k, n \leq \omega$. *Then it is relatively consistent with* ZFC *that there is a cofinal set in* ${}^{\omega}\omega$ *isomorphic to* $\omega_k \times \omega_n$ *with respect to* $<^*$, *that* $2^{\aleph_0} < 2^{\aleph_1}$, *and* $\mathrm{w}\diamondsuit(S^{i+1}_i)$ *for all* $1 \leq i < n$.

Proof. We start with a ground model $V \models \mathrm{GCH}$. For $1 \leq i \leq n$, let $\mathbb{C}_i = Fn(\omega_{n+i} \times \omega_i, 2, \omega_i)$ and $\mathbb{C} = \mathbb{H}_{\omega_k \times \omega_n} \times \mathbb{C}_1 \times \ldots \times \mathbb{C}_n$ be the product forcing notion. Let $G = G_0 \times$

$... \times G_n$ be a generic filter over $\mathbb{C}$. It is well known [108, Lemma V.2.6] that $\mathbb{C}_1 \times ... \times \mathbb{C}_n$ preserves cofinalities; moreover, it is σ-closed and so does not add any reals. Hence, by the product lemma [108, Theorem V.1.2] $V[G] = V[G_1 \times ... \times G_n][G_0]$ and so we have indeed a cofinal subset of shape $\omega_k \times \omega_n$ in the extension, as well as preservation of cofinalities and $2^{\aleph_i} = \aleph_{n+i}$ for $0 \leq i \leq n$ (again, by nice names) as $\mathbb{H}_{\omega_k \times \omega_n}$ is absolute between V and $V[G_1 \times ... \times G_n]$.

As for $\mathrm{w}\diamondsuit(S_i^{i+1})$, note that for all $i < n$ the notion $\mathbb{C}_{i+2} \times ... \times \mathbb{C}_n$ is ω_{i+2}-closed. Hence if we let $W = V[G_{i+2} \times ... \times G_n]$ we have $2^{\aleph_i} = \aleph_{i+1}$ in W. Now we apply Lemma 1.26 over W with $\mathbb{P} = \mathbb{H}_{\omega_k \times \omega_n} \times \mathbb{C}_1 \times ... \times \mathbb{C}_i$, $\mathbb{Q} = \mathbb{C}_{i+1}$, $\kappa = \omega_{i+1}$, $\lambda = \omega_{n+i}$, $\mu = \omega_{n+i+1}$, and $S = S_i^{i+1}$. Here $\mathbb{P}$ is $\kappa - cc$ because the factors are κ-Knaster; this is well-known for the generalized Cohen forcings and has been proved for $\mathbb{H}_{\omega_k \times \omega_n}$ in Proposition 1.24. We get the conclusion again by the product lemma. □

Proof of Main Theorem of Chapter 1. Theorem 1.27 together with 1.21 easily implies the consistency of the statement $\lim^n \mathbf{A}_G \neq 0$ with the relevant values of $\mathfrak{b}$ and $\mathfrak{d}$. The case $G = \mathbb{Z}$ is precisely the main theorem of this chapter.

□

1.5 Appendix: the Roos complex

As mentioned in the introduction, in this appendix we give a categorical proof of Roos' method of computation for derived limits. This consists of the argument presented in [138, pp.345-348], with the addition of a proof for Claim 1.38 and an explanation of how some suitable functors F_a^i act on morphisms. These are contravariant, while Neeman claims that the corresponding functors in his proof are covariant.

We start by defining one of the so-called Grothendieck's AB conditions.

Definition 1.28. *Let λ be an infinite cardinal. An abelian category $\mathcal{A}$ is said to be $AB4^*(\lambda)$ if it has all products of fewer than λ-many object and the products of less than λ-many exact sequences in $\mathcal{A}$ is exact.*

To make sense of the definition, one can define a category ExSeq($\mathcal{A}$) of exact sequences in $\mathcal{A}$ and compute the limit of a functor from a discrete category to it.

Definition 1.29 (Simplicial and cosimplicial objects)**.** *Let Δ be the category where objects are finite ordinals and morphisms are weakly increasing maps between them, and put $[n] := \{0, ..., n\}$. Then a simplicial object in the category $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$.*

The i-th face is $d_i = X(\delta^i) : X([n]) \to X([n-1])$ where $\delta^i : [n-1] \to [n]$ is the unique injection in the category Δ whose range omits $i \in [n]$.

The i-th degeneracy is $s_i = X(\sigma^i) : X([n]) \to X([n-1])$ where $\sigma^i : [n+1] \to [n]$ is the unique surjection in the category Δ such that $i \in [n]$ has two elements in its preimage.

Similarly, a cosimplicial object in the category $\mathcal{C}$ is a functor $Y : \Delta \to \mathcal{C}$.

Cofaces and codegeneracies are the morphisms $Y(\delta^i)$ and $Y(\sigma^i)$ respectively, for $i \in [n]$.

Definition 1.30. *The nerve of the category $\mathcal{J}$ is defined as the simplicial object $\mathscr{N}_\bullet(\mathcal{J})$ where $\mathscr{N}_k(\mathcal{J})$ (the k-simplices) is the set of composable sequences of morphisms of length k*

$$i_0 \to i_1 \to i_2 \to ... \to i_k.$$

The j-th face map is that which composes morphisms at i_j and the j-th degeneracy map is the one that adds the identity at i_j.

Definition 1.31. *An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called effaceable if for any object A in $\mathcal{A}$ we can find a monomorphism $u : A \to M$ such that $F(u) = 0$.*

Definition 1.32 (δ-functor)**.** *Given two abelian categories $\mathcal{A}$ and $\mathcal{B}$, a (cohomological covariant) δ-functor is a sequence of additive functors $S^n : \mathcal{A} \to \mathcal{B}$ for $n \geq 0$ and of connecting morphisms $\delta_E^n : S^n(X'') \to S^{n+1}(X')$ for each short exact sequence in $\mathcal{A}$*

$$E := (\ 0 \to X' \xrightarrow{f'} X \xrightarrow{f} X'' \to 0\ )$$

making the following sequence exact

$$0 \to S^0(X') \xrightarrow{S^0(f')} S^0(X) \xrightarrow{S^0(f)} S^0(X'') \xrightarrow{\delta_E^0} S^1(X') \to ...$$

$$... \to S^n(X') \xrightarrow{S^n(f')} S^n(X) \xrightarrow{S^n(f)} S^n(X'') \xrightarrow{\delta_E^n} S^{n+1}(X')$$

and such that if $\boldsymbol{h} := (h', h, h'') : E \to F$ is a morphism of exact sequences as below

$$\begin{array}{rcccccccccc} E &=& (\ 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0\ ) \\ & & & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h''} & & \\ F &=& (\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0\ ) \end{array}$$

then, for every $n \geq 0$, the following diagram commutes:

$$\begin{array}{ccc} S^n(X'') & \xrightarrow{\delta_E^n} & S^{n+1}(X') \\ \downarrow{\scriptstyle S^n(h'')} & & \downarrow{\scriptstyle S^{n+1}(h')} \\ S^n(Y'') & \xrightarrow{\delta_F^n} & S^{n+1}(Y'). \end{array}$$

If $\boldsymbol{S} = (S^n, \theta_E^n)$ and $\boldsymbol{T} = (T^n, \omega_E^n)$ are two δ-functors with homomorphisms $\theta_E^n : S^n(X'') \to S^{n+1}(X')$ and $\omega_E^n : T^n(X'') \to T^{n+1}(X')$, a morphism of δ-functors is a sequence of natural transformations $\phi^n : S^n \to T^n$ such that, for every exact sequence E as above, the following diagram commutes:

$$\begin{array}{ccc} S^n(X'') & \xrightarrow{\theta_E^n} & S^{n+1}(X') \\ \downarrow{\scriptstyle \phi^n_{X''}} & & \downarrow{\scriptstyle \phi^{n+1}_{X'}} \\ T^n(X'') & \xrightarrow{\omega_E^n} & T^{n+1}(X') \end{array}$$

An isomorphism of δ-functors is a morphism such that ϕ^n is a natural isomorphism for all n.

Definition 1.33 (Universal δ-functor)**.** *A δ-functor $\boldsymbol{S} = (S^n, \theta_E^n)$ is said to be universal if for any other connected sequence of functors $\boldsymbol{T} = (T^n, \omega_E^n)$ and any natural transformation $\phi : S^0 \to T^0$ there exists a unique morphism $(\phi^n)_{n\geq 0}$ from $\boldsymbol{S}$ to $\boldsymbol{T}$ such that $\phi^0 = \phi$.*

Remark 1.34. *If two δ-functors $\boldsymbol{S}$ and $\boldsymbol{T}$ are both universal and $\phi : S^0 \to T^0$ is a natural isomorphism, then there exists a unique (iso)morphism (ϕ^n) from $\boldsymbol{S}$ to $\boldsymbol{T}$ such that $\phi^0 = \phi$. This is because the compositions of the unique morphisms of δ-functors arising from ϕ and ϕ^{-1} must be the identity on $\boldsymbol{S}$ and $\boldsymbol{T}$ as they are the identity on level zero.*

Fact 1.35. *$(\lim^n, \theta_E^n)$ form a universal δ-functor, for some connecting morphisms θ_E^n.*

The following lemma was first proved in Grothendieck's celebrated *Tohoku paper* [72].

Lemma 1.36. *Every δ-functor $T = (T^i)_{i\geq 0}$ with T^i effaceable for all $i \geq 1$ is universal.*

Theorem 1.37. *Let λ be an infinite cardinal. Let $\mathcal{A}$ be an abelian category satisfying $AB4^*(\lambda)$. Suppose moreover that $\mathcal{J}$ is a small category with set of all morphisms of cardinality $< \lambda$. Then the functor*

$$\lim : \mathcal{A}^{\mathcal{J}^{op}} \to \mathcal{A}$$

has right derived functors $\lim^n$ and these can be computed as the cohomology of a complex.

Proof. Here we will construct the general version of the Roos complex and prove that its cohomology is the same as the derived limits by showing that it gives rise to a universal δ-functor.

Now, if the cardinality of the total set of morphisms of $\mathcal{J}$ is strictly less than λ, then the same holds for $\mathscr{N}_k(\mathcal{J})$, and since $\mathcal{A}$ is $AB4^*(\lambda)$, given a functor $F : \mathcal{J}^{op} \to \mathcal{A}$ we can form a complex

$$N_0(F) \xrightarrow{\partial_0} N_1(F) \xrightarrow{\partial_1} N_2(F) \xrightarrow{\partial_2} \dots$$

where

$$N_k(F) = \prod_{\{i_0 \to i_1 \to \dots \to i_k\} \in \mathcal{N}_k(\mathcal{J})} F(i_0).$$

The differential $\partial_k : N_k(F) \to N_{k+1}(F)$ is given by the alternating sum

$$\partial_k = \sum_{j=0}^{k+1} (-1)^j \partial_k^j,$$

where ∂_k^j is the morphism induced by the j-th face map of the nerve meaning that for $j > 0$ we define the $i_0 \to \dots \to i_{j-1} \to i_j \to i_{j+1} \to \dots \to i_k \to i_{k+1}$-th component as the projection of $N_k(F)$ on the factor of index $i_0 \to \dots i_{j-1} \to i_{j+1} \to i_k \to i_{k+1}$ and for $j = 0$ we post-compose $F(i_o \to i_1)$ to the obvious projection. One could similarly define maps induced by the degeneracies, and these maps constitute respectively the cofaces and codegeneracies of a cosimplicial object. The differential is therefore the alternating sum of the cofaces.

We write $T^n(F)$ for the n-th cohomology of the Roos complex

$$N_0(F) \xrightarrow{\partial_0} N_1(F) \xrightarrow{\partial_1} N_2(F) \xrightarrow{\partial_2} \dots$$

Suppose now that we have a short exact sequence in $\mathcal{A}^{\mathcal{J}^{op}}$, namely, a sequence of functors $\mathcal{J}^{op} \to \mathcal{A}$

$$0 \to F' \to F \to F'' \to 0.$$

By $AB4^*(\lambda)$, the sequence

$$0 \to N_k(F') \to N_k(F) \to N_k(F'') \to 0$$

is also exact for all k. So we have a short exact sequence of complexes and by Zig-Zag Lemma a long exact sequence in cohomology. It follows that $(T^n)_{n \geq 0}$ is a δ-functor. It is easy to see that T^n commutes with products. The fact that $\lim$ can be expressed as the equalizer of two morphisms between products and the definition of the differentials gives us a natural isomorphism $T^0 \cong \lim^0$.

We now show that every object F in $\mathcal{A}^{\mathcal{J}^{op}}$ can be embedded in an object F' such that $T^n(F') = 0$, for all $n > 0$. This will imply the Theorem by Lemma 1.36, Remark 1.34, and Fact 1.35.

For every object i of $\mathcal{J}$ and every object a of $\mathcal{A}$ we define a functor $F_a^i : \mathcal{J}^{op} \to \mathcal{A}$ by the rule:

$$F_a^i(j) = \prod_{\mathcal{J}(i,j)} a.$$

That is, $F_a^i(j)$ is the product of copies of a indexed over all morphisms from i to j. The functor is defined on morphisms as follows. A morphism $\beta : j_1 \to j_2$ gets mapped to the morphism $\prod_{\mathcal{J}(i,j_2)} a \to \prod_{\mathcal{J}(i,j_1)} a$ such that the component corresponding to the factor $\alpha : i \to j_1$ is $id_a \circ \pi_{\beta\circ\alpha}$, where id_a is the identity on a and π_γ, for $\gamma : i \to j_2$, is the projection of $\prod_{\mathcal{J}(i,j_2)}$ on the corresponding factor.

Claim 1.38. *The complex*

$$N_0(F_a^i) \xrightarrow{\partial_0} N_1(F_a^i) \xrightarrow{\partial_1} N_2(F_a^i) \xrightarrow{\partial_2} ...$$

is homotopy equivalent to

$$a \to 0 \to 0 \to ...$$

Proof of Claim. First, one defines a category $\mathcal{C}$ whose objects are copies of a, one for each morphism $i \to \cdot$ and such that

$$\mathrm{Hom}(\underset{\alpha:i\to j_1}{a}, \underset{\gamma:i\to j_2}{a}) = \{\beta : j_1 \to j_2 \mid \beta \circ \alpha = \gamma\}.$$

Then note that

$$N_k(F_a^i) = \prod_{i_0\to...\to i_k} F_a^i(i_0) = \prod_{i_0\to...\to i_k} \prod_{i\to i_0} a = \prod_{i\to i_0\to...\to i_k} a.$$

So the cohomological complex above is simply what we get by applying an operation reminiscent of the $\mathrm{Hom}(_, G)$ functor to get cohomology with coefficients in a group G (here a takes the role analogous to that of the target in the contravariant Hom) to the chain complex associated to the nerve of $\mathcal{C}$. Now call $[a]$ the trivial category on the initial object $a_{id:i\to i}$, $\eta : \mathcal{C} \to [a]$ the natural projection and $\iota : [a] \to \mathcal{C}$ the inclusion. Now, $\eta \circ \iota = Id_{[a]}$ and $\iota \circ \eta$ is connected to $Id_{\mathcal{C}}$ by the obvious natural transformation represented in the commutative square below:

$$\begin{array}{ccc} a_{i\xrightarrow{id}i} & \xrightarrow{i\xrightarrow{id}i} & a_{i\xrightarrow{id}i} \\ {\scriptstyle i\xrightarrow{\rho}j_1}\downarrow & & \downarrow{\scriptstyle i\xrightarrow{\tau}j_2} \\ a_{i\xrightarrow{\rho}j_1} & \xrightarrow{j_1\xrightarrow{\sigma}j_2} & a_{i\xrightarrow{\tau}j_2} \end{array}$$

Now, by [111, Proposition 7], the existence of such a natural transformation implies that the induced simplicial maps $M(Id_{\mathcal{C}})$ and $M(\iota \circ \eta) = M(\iota) \circ M(\eta)$ are homotopic. Since the same holds trivially between $M(Id_{[a]})$ and $M(\eta) \circ M(\iota)$, we can deduce the desired homotopy equivalence between the associated homological complexes, and then between the cohomological complexes "with coefficients in a". □

So $T^n(F^i_a) = 0$ for all $n > 0$, and since T^n commutes with products, the same holds for any product of F^i_a's. Now, any functor G can be embedded in the product over all i of $F^i_{G(i)}$, the embedding of a factor into the product been given by the composition of the isomorphism $A \xrightarrow{\sim} A \times 0$ and the monomorphism $A \times 0 \hookrightarrow A \times B$, both defined for general abelian categories. This concludes the proof. □

1.6 Open Questions

Question 1.39. *Can the condition $|G| \leq 2^{\aleph_1}$ be removed from Lemma 1.19 and similar results?*

Question 1.40. *What assumptions on the value of $\mathfrak{b}$ and $\mathfrak{d}$, together with the weak diamonds, are sufficient for these nonvanishing results?*

Question 1.41. *Are the weak diamond assumptions necessary for $\lim^n \mathbf{A} \neq 0$ to hold for $n > 1$?*

Chapter 2

Simultaneously nonvanishing derived limits

2.1 Notation and conventions

Our notation is for the most part standard. For undefined notions in set theory, we refer the reader to [90], and in homological algebra, to [168].

We identify an ordinal with the set of all ordinals strictly less than it. In particular, we identify the natural number n with the set $\{0, 1, \dots, n-1\}$. If C is a set of ordinals, then we let $\mathrm{Lim}(C)$ denote the set $\{\alpha \in C \setminus \{0\} \mid \sup(C \cap \alpha) = \alpha\}$. If a and b are sets of ordinals, then we let $a < b$ denote the assertion $(\forall \alpha \in A)(\forall \beta \in B)(\alpha < \beta)$. In particular, for any set a of ordinals, we have $\emptyset < a$ and $a < \emptyset$, so this is only a partial order when restricted to nonempty sets.

If α is an ordinal, then $\mathrm{cf}(\alpha)$ denotes its cofinality. If λ is a regular infinite cardinal and δ is an ordinal, then $S^\delta_\lambda = \{\alpha < \delta \mid \mathrm{cf}(\alpha) = \lambda\}$. If $m < n < \omega$, then we let S^n_m denote $S^{\omega_n}_{\omega_m}$.

Given a positive integer n, we will identify functions with domain n and sequences of length n, i.e., we will make no distinction between a function u with domain n and the sequence $\langle u(0), u(1), \dots, u(n-1)\rangle$. Given such a u and a natural number $i < n$, we let u^i denote the sequence of length $n-1$ formed by removing $u(i)$ from u. This is often denoted

$$\langle u(0), u(1), \dots, \widehat{u(i)}, \dots, u(n-1)\rangle;$$

formally, it is the function u^i with domain $n-1$ defined by

$$u^i(j) = \begin{cases} u(j) & \text{if } j < i \\ u(j+1) & \text{if } j \geq i \end{cases}$$

for all $j < n-1$. If $n < \omega$ and $\sigma : n \to n$ is a permutation, then $\mathrm{sgn}(\sigma) \in \{-, +\}$ denotes the parity of σ.

Given a class X and a cardinal κ, $[X]^\kappa$ denotes the class of all subsets of X of cardinality κ. Given a set of ordinals a, we let $\mathrm{otp}(a)$ denote its order-type. We will customarily identify sets of ordinals with the functions enumerating them in increasing order, i.e., if $a \subseteq \mathrm{Ord}$ has order-type δ then, for all $i < \delta$, we let $a(i)$ denotes the unique $\beta \in a$ such that $\mathrm{otp}(a \cap \beta) = i$. We will also regularly identify a sequence of length 1 with its unique value, e.g., we will write φ_α instead of $\varphi_{\langle\alpha\rangle}$. Similarly, we may write, e.g., $\varphi_{\alpha\beta}$ instead of $\varphi_{\langle\alpha,\beta\rangle}$. If σ is a sequence indexed by an ordinal, then $\mathrm{lh}(\sigma)$ denotes its domain. If $\alpha < \beta$ are ordinals and σ and τ are sequences of length α and β, respectively, then $\sigma \sqsubseteq \tau$ denotes the assertion that $\tau \restriction \alpha = \sigma$.

Given two functions φ_0 and φ_1, we write $\varphi_0 =^* \varphi_1$ to denote the assertion that, on their common domains, φ_0 and φ_1 agree at all but finitely many places. More formally, this is the assertion that the set

$$\{x \in \mathrm{dom}(\varphi_0) \cap \mathrm{dom}(\varphi_1) \mid \varphi_0(x) \neq \varphi_1(x)\}$$

is finite. As a special case, if j is an element of the codomain of φ, then $\varphi =^* j$ is the assertion that $\{x \in \mathrm{dom}(\varphi_0) \mid \varphi(x) \neq j\}$ is finite. Similarly, if u and v are two sets, then we let $u \subseteq^* v$ denote the assertion that $u \setminus v$ is finite. If φ is a function from a set x into

an abelian group A, then the *support* of φ, denoted $\operatorname{supp}(\varphi)$, is $\{i \in x \mid \varphi(i) \neq 0\}$. If f is a function and $X \subseteq \operatorname{dom}(f)$, then $f[X]$ denotes the pointwise image of X under f. Similarly, if X is a subset of the codomain of f, then $f^{-1}[X]$ denotes the preimage of X under f.

We let Ab denote the category of abelian groups. We will be interested in functions mapping into abelian groups. For a set u and an abelian group H, the set of functions from u to H itself has a natural abelian group structure defined by pointwise addition. For improved readability, we will slightly abuse notation in the following way: given sets u_0 and u_1, an abelian group H, and functions $\varphi_i : u_i \to H$ for $i < 2$, we will write $\varphi_0 + \varphi_1$ instead of the more precise $\varphi_0 \restriction (u_0 \cap u_1) + \varphi_1 \restriction (u_0 \cap u_1)$. This generalizes straightforwardly to longer finite sums: if n is a positive integer and $\langle \varphi_i \mid i < n \rangle$ is a family of functions mapping into H, then by convention the domain of the function $\sum_{i<n} \varphi_i$ is $\bigcap_{i<n} \operatorname{dom}(\varphi_i)$.

2.2 Coherence, triviality, and derived limits

In this section, we introduce the primary objects of study of this chapter and review some of their basic properties.

2.2.1 Nontrivial coherent families

Definition 2.1. *Suppose that $\mathcal{I}$ is a collection of sets, n is a positive integer, and H is an abelian group. An H-*valued *n-*family indexed along *$\mathcal{I}$ is a family of functions of the form*

$$\Phi = \left\langle \varphi_u : \bigcap_{i<n} u(i) \to H \;\middle|\; u \in \mathcal{I}^n \right\rangle.$$

*If H and $\mathcal{I}$ are either clear from context or irrelevant, then we may refer to such an object simply as an n-*family*. An n-family is said to be*

- alternating *if, for every $u \in \mathcal{I}^n$ and every permutation $\sigma : n \to n$, we have*

$$\varphi_{u \circ \sigma} = \operatorname{sgn}(\sigma) \varphi_u;$$

- coherent *if it is alternating and, for all $v \in \mathcal{I}^{n+1}$, we have*

$$\sum_{i<n+1} (-1)^i \varphi_{v^i} =^* 0;$$

- trivial *if*
  - *$n = 1$ and there is a function $\psi : \bigcup \mathcal{I} \to H$ such that, for all $u \in \mathcal{I}$, we have $\psi \restriction u =^* \varphi_u$; or*

– $n > 1$ *and there is an alternating* $(n-1)$*-family*

$$\Psi = \left\langle \psi_t : \bigcap_{i<n-1} t(i) \to H \;\middle|\; t \in \mathcal{I}^{n-1} \right\rangle$$

such that, for all $u \in \mathcal{I}^n$*, we have*

$$\varphi_u =^* \sum_{i<n} (-1)^i \psi_{u^i}.$$

In this case, we say that ψ *or* Ψ trivializes Φ.

Remark 2.2. *It is easily verified that a trivial* n*-family is always coherent. The question of when the converse holds is the primary motivating question of this work.*

Moreover, if an n*-family* $\langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ *is alternating, then, for every non-injective* $u \in \mathcal{I}^n$ *we must have* $\varphi_u = 0$*. Therefore, when constructing alternating* n*-families indexed along* $\mathcal{I}$*, it suffices to specify* φ_u *for injective* $u \in \mathcal{I}^n$*. In practice, we will often be working with sets* $\mathcal{I}$ *equipped with a natural linear order* $<_{\mathcal{I}}$*. In this setting, when constructing an alternating* n*-family indexed along* $\mathcal{I}$*, it suffices to specify* φ_u *for functions* $u \in \mathcal{I}^n$ *that are strictly increasing with respect to* $<_{\mathcal{I}}$*, since the requirement that the family be alternating will then determine all other values. Similarly, when verifying that such a family is either coherent or trivial, it suffices to consider functions* $v \in \mathcal{I}^{n+1}$ *or* $u \in \mathcal{I}^n$*, respectively, that are strictly increasing with respect to* $<_{\mathcal{I}}$*. In particular, given a sequence of sets* $\vec{x} = \langle x_\alpha \mid \alpha < \delta \rangle$*, by an* n-family indexed along $\vec{x}$ *we mean a family of the form*

$$\Phi = \langle \varphi_a : \bigcup_{i<n} x_{a(i)} \mid a \in \delta^n \rangle \qquad \text{or} \qquad \Phi = \langle \varphi_a : \bigcup_{i<n} x_{a(i)} \mid a \in [\delta]^n \rangle$$

Definition 2.3. *Suppose that* $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ *is an* n*-family. If* $\mathcal{J} \subseteq \mathcal{I}$*, then we let* $\Phi \restriction \mathcal{J}$ *denote* $\langle \varphi_u \mid u \in \mathcal{J}^n \rangle$*.*

If $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ *and* $\Psi = \langle \psi_u \mid u \in \mathcal{J}^n \rangle$ *are* n*-families, then* $\Psi \sqsubseteq \Phi$ *denotes the assertion that* $\Psi = \Phi \restriction \mathcal{J}$ *(we will sometimes say in this situation that* Φ extends Ψ*).*

Note that, if $\mathcal{J} \subseteq \mathcal{I}$ and Φ as above is coherent (*resp.* trivial), then $\Phi \restriction \mathcal{J}$ is also coherent (*resp.* trivial). The following partial converse to this follows from the arguments of [18, Lemma 2.7]; we provide a proof for completeness.

Proposition 2.4. *Suppose that* $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ *is a coherent* H*-valued* n*-family,* $\mathcal{J}$ *is a* $\subseteq^*$*-cofinal subset of* $\mathcal{I}$*, and* $\Phi \restriction \mathcal{J}$ *is trivial. Then* Φ *is trivial.*

Proof. We assume that $n > 1$. The case in which $n = 1$ is easier and left to the reader. Let $\Psi = \langle \psi_t \mid t \in \mathcal{J}^{n-1} \rangle$ trivialize $\Phi \restriction \mathcal{J}$. For each $x \in \mathcal{I}$, choose an $x^+ \in \mathcal{J}$ such that $x \subseteq^* x^+$. Given $m < \omega$ and $v \in \mathcal{I}^m$, define

- $v^+ \in \mathcal{J}^m$ by letting $v^+(i) = v(i)^+$ for all $i < m$; and

- for $i < m$, define $\pi_i(v) \in \mathcal{I}^{m+1}$ as

$$\pi_i(v) = \langle v(0), \dots, v(i), v(i)^+, \dots, v(m-1)^+ \rangle.$$

Now define an alternating H-valued $(n-1)$-family $\Psi' = \langle \psi'_t \mid t \in \mathcal{I}^{n-1} \rangle$ by letting

$$\psi'_t = \psi_{t^+} - \sum_{i=0}^{n-2} (-1)^i \varphi_{\pi_i(t)}$$

for all $t \in \mathcal{I}^{n-1}$ (the sum on the right is defined on all but finitely many elements of $\bigcap_{i<n-1} t(i)$; ψ'_t is defined more precisely by extending that sum to the domain $\bigcap_{i<n-1} t(i)$ by setting it equal to 0 on all otherwise undefined arguments).

We claim that Ψ' trivializes Φ. To verify this, fix $u \in \mathcal{I}^n$, and consider

$$\sum_{i=0}^{n-1} (-1)^i \psi'_{u^i} = \sum_{i=0}^{n-1} (-1)^i \left(\psi_{(u^i)^+} - \sum_{j=0}^{n-2} (-1)^j \varphi_{\pi_j(u^i)} \right). \tag{2.1}$$

Note first that, since Ψ trivializes $\Phi \restriction \mathcal{J}$, we have

$$\sum_{i=0}^{n-1} (-1)^i \psi_{(u^i)^+} =^* \varphi_{u^+},$$

and therefore the sum in (2.1) reduces to

$$\varphi_{u^+} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-2} (-1)^{i+j+1} \varphi_{\pi_j(u^i)}. \tag{2.2}$$

For each $\ell < n$, let

$$A_\ell = \{(i,j) \in (n-1) \times (n-2) \mid \ell = j < i \text{ or } i \leq j = \ell - 1\}.$$

Note that A_ℓ consists precisely of those pairs (i,j) for which the formal definition of the sequence $\pi_j(u^i)$ contains u_ℓ and $(u_\ell)^+$ as consecutive elements. Now the sum in (2.2) becomes

$$\varphi_{u^+} + \sum_{\ell<n} \sum_{(i,j)\in A_\ell} (-1)^{i+j+1} \varphi_{\pi_j(u^i)}.$$

By the coherence of Φ, we have

$$\varphi_{u^+} + \sum_{(i,j)\in A_0} (-1)^{i+j+1} \varphi_{\pi_j(u^i)} =^* \varphi_{\langle u(0), u(1)^+, \dots, u(n-1)^+ \rangle}.$$

More generally, for $\ell < n$, we have

$$\varphi_{\langle u(0),\dots,u(\ell-1),u(\ell)^+,\dots,u(n-1)^+\rangle} + \sum_{(i,j)\in A_\ell} (-1)^{i+j+1} \varphi_{\pi_j(u^i)} =^* \varphi_{\langle u(0),\dots,u(\ell),u(\ell+1)^+,\dots,u(n-1)^+\rangle}.$$

By repeatedly applying these equalities, we obtain:

$$\begin{aligned}
\sum_{i=0}^{n-1}(-1)^i\psi'_{\varphi_{u^i}} &= u^+ + \sum_{\ell<n}\sum_{(i,j)\in A_\ell}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\
&=^* \varphi_{\langle u(0),u(1)^+,\dots,u(n-1)^+\rangle} + \sum_{\ell=1}^{n-1}\sum_{(i,j)\in A_\ell}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\
&=^* \varphi_{\langle u(1),u(1),u(2)^+,\dots,u(n-1)^+\rangle} + \sum_{\ell=2}^{n-1}\sum_{(i,j)\in A_\ell}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\
&=^* \cdots \\
&=^* \varphi_{\langle u(1),\dots,u(n-2),u(n-1)^+\rangle} + \sum_{(i,j)\in A_{n-1}}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\
&=^* \varphi_u.
\end{aligned}$$

Thus, Ψ' does trivialize Φ, as desired. □

We can also lift coherence from a $\subseteq^*$-cofinal subset of $\mathcal{I}$ to all of $\mathcal{I}$, in the following sense.

Proposition 2.5. *Suppose that $\mathcal{J}$ is a $\subseteq^*$-cofinal subset of $\mathcal{I}$ and $\Phi = \langle \varphi_u \mid u \in \mathcal{J}^n\rangle$ is a coherent H-valued n-family. Then there is a coherent H-valued n-family $\Phi' = \langle \varphi'_u \mid u \in \mathcal{I}^n\rangle$ such that $\Phi' \restriction \mathcal{J} = \Phi$.*

Proof. For each $x \in \mathcal{I}$, choose an $x^+ \in \mathcal{J}$ such that $x \subseteq^* x^+$. If $x \in \mathcal{J}$, then choose $x^+ = x$. Given $m < \omega$ and $v \in \mathcal{I}^m$, define $v^+ \in \mathcal{J}^m$ by letting $v^+(i) = v(i)^+$ for all $i < m$. Now, for all $u \in \mathcal{I}^n$, define $\varphi'_u : \bigcap_{i<n} u(i) \to H$ by setting $\varphi'_u = \varphi_{u^+}$. Note that φ_{u^+} is a function defined on all but finitely many elements of $\bigcap_{i<n} u(i)$; φ'_u is defined more precisely by extending that sum to the domain $\bigcap_{i<n} u(i)$ by setting it equal to 0 on all otherwise undefined arguments.

It is now immediate from the construction that Φ' is a coherent n-family with $\Phi' \restriction \mathcal{J} = \Phi$. □

Remark 2.6. *In light of Proposition 2.5, if we are given $0 < n < \omega$, an abelian group H, a collection $\mathcal{I}$ of sets, and a $\subseteq^*$-cofinal subset $\mathcal{J} \subseteq \mathcal{I}$ and we seek to construct a nontrivial coherent H-valued n-family indexed along $\mathcal{I}$, then it suffices to construct such a family indexed along $\mathcal{J}$. Indeed, by Proposition 2.5, this family extends to one indexed along $\mathcal{I}$, and the nontriviality of the original family clearly implies the nontriviality of the extension.*

The following is a variation on Goblot's vanishing theorem [71] indicating that if the cofinality of $(\mathcal{I}, \subseteq^*)$ is less than ω_n, then every coherent n-family indexed along $\mathcal{I}$ is trivial.

Proposition 2.7. *Suppose that H is an abelian group, $0 < n < \omega$, $\mathcal{I}$ is a collection of sets with $\mathrm{cf}(\mathcal{I}, \subseteq^*) < \omega_n$, and $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n\rangle$ is an H-valued coherent n-family. Then Φ is trivial.*

Proof. For concreteness, assume that $\mathrm{cf}(\mathcal{I}, \subseteq^*)$ is infinite; the case in which it is finite is much easier. Let $\kappa = \mathrm{cf}(\mathcal{I}, \subseteq^*)$. By replacing $\mathcal{I}$ with some $\subseteq^*$-cofinal subset and invoking Proposition 2.4, we can assume that $|\mathcal{I}| = \kappa$. Enumerate $\mathcal{I}$ as $\langle x_\eta \mid \eta < \kappa \rangle$. For ease of notation, and recalling Remark 2.2, we will think of Φ as $\langle \varphi_b \mid b \in [\kappa]^n \rangle$, where elements of $[\kappa]^n$ are thought of as strictly increasing n-tuples from κ and, formally, given $b \in [\kappa]^n$, $\varphi_b = \varphi_{\langle x_{b(i)} | i<n \rangle}$. Note that this does not involve any loss of information due to the fact that Φ is assumed to be alternating.

The proof is by induction on n. Suppose first that $n = 1$, and hence $\kappa = \omega$. We will define a function $\psi : \bigcup \mathcal{I} \to H$ witnessing that Φ is trivial. For each $k < \omega$, let $y_k = x_k \setminus (\bigcup \{x_j \mid j < k\})$; note that $\{y_k \mid k < \omega\}$ is a partition of $\bigcup \mathcal{I}$. Now, for each $k < \omega$, set $\psi \restriction y_k = \varphi_k \restriction y_k$. Using the coherence of Φ, it is straightforward to verify that ψ thus defined witnesses that Φ is trivial.

Now suppose that $n > 1$. We must construct an alternating $(n-1)$-family

$$\Psi = \langle \psi_t \mid t \in \mathcal{I}^{n-1} \rangle$$

witnessing that Φ is trivial. Precisely as above, using the assumption that our families are alternating we will in fact construct a family of the form

$$\Psi = \langle \psi_a \mid a \in [\kappa]^{n-1} \rangle$$

such that, for all $b \in [\kappa]^n$, we have

$$\varphi_b =^* \sum_{i<n} (-1)^i \psi_{b^i}.$$

The construction is slightly different depending on whether $n = 2$ or $n > 2$. Suppose first that $n = 2$, in which case $\kappa \leq \omega_1$. We will construct $\psi_\eta : x_\eta \to H$ for $\eta < \kappa$ by recursion on η.

Suppose that $\xi < \kappa$ and we have constructed $\langle \psi_\eta \mid \eta < \xi \rangle$ such that, for all $\eta_0 < \eta_1 < \xi$, we have $\varphi_{\eta_0 \eta_1} =^* \psi_{\eta_1} - \psi_{\eta_0}$. Define a 1-family of functions

$$T_\xi = \langle \tau_\eta : x_\eta \cap x_\xi \to H \mid \eta < \xi \rangle$$

by letting $\tau_\eta = \psi_\eta + \varphi_{\eta\xi}$.

Claim 2.8. *T_ξ is coherent.*

Proof. Fix $\eta_0 < \eta_1 < \xi$. Then

$$\begin{aligned} \tau_{\eta_1} - \tau_{\eta_0} &= (\psi_{\eta_1} + \varphi_{\eta_1 \xi}) - (\psi_{\eta_0} + \varphi_{\eta_0 \xi}) \\ &=^* \varphi_{\eta_1 \xi} - \varphi_{\eta_0 \xi} + \varphi_{\eta_0 \eta_1} \\ &=^* 0, \end{aligned}$$

where all functions are restricted to the domain $x_{\eta_0} \cap x_{\eta_1} \cap x_\xi$, the passage from the first line to the second follows from our assumptions about $\langle \psi_\eta \mid \eta < \xi \rangle$, and the passage from the second line to the third follows from the coherence of Φ. □

Since $|\xi| < \omega_1$, the inductive hypothesis implies that T_ξ is trivial. We can therefore fix a function $\psi_\xi : x_\xi \to H$ such that, for all $\eta < \xi$, we have $\psi_\xi \restriction (x_\eta \cap x_\xi) =^* \tau_\eta$. Note that, for all $\eta < \xi$, we have

$$\psi_\xi - \psi_\eta =^* \tau_\eta - \psi_\eta = (\psi_\eta + \varphi_{\eta\xi}) - \psi_\eta = \varphi_{\eta\xi},$$

so this choice satisfies the requirements of the construction. This completes the case $n = 2$.

Suppose now that $n > 2$. We will construct $\psi_a : \bigcap_{i<n-1} x_{a(i)} \to H$ for $a \in [\kappa]^{n-1}$ by recursion on $\max(a)$.

Suppose that $\xi < \kappa$ and we have constructed $\langle \psi_a \mid a \in [\xi]^{n-1} \rangle$ such that, for all $b \in [\xi]^n$, we have

$$\varphi_b =^* \sum_{i<n} (-1)^i \psi_{b^i}.$$

We now describe how to define $\varphi_{d\cup\{\xi\}}$ for $d \in [\xi]^{n-2}$. First, define an $(n-1)$-family of functions

$$T_\xi = \langle \tau_a : x_\xi \cap \bigcap \{x_{a(i)} \mid i < n-1\} \to H \mid a \in [\xi]^{n-1} \rangle$$

by letting $\tau_a = (-1)^n \psi_a + \varphi_{a\cup\{\xi\}}$.

Claim 2.9. *T_ξ is coherent.*

Proof. Fix $b \in [\xi]^n$, and let $c = b \cup \{\xi\}$. Note that, for $i < n-1$, we have $c^i = b^i \cup \{\xi\}$, and $c^{n-1} = b$. Then

$$\begin{aligned}
\sum_{i<n} (-1)^i \tau_{b^i} &= (-1)^n \sum_{i<n} (-1)^i \psi_{b^i} + \sum_{i<n} (-1)^i \varphi_{b^i\cup\{\xi\}} \\
&=^* (-1)^n \varphi_b + \sum_{i<n} (-1)^i \varphi_{b^i\cup\{\xi\}} \\
&= (-1)^n \varphi_{c^{n-1}} + \sum_{i<n} (-1)^i \varphi_{c^i} \\
&= \sum_{i<n+1} (-1)^i \varphi_{c^i} =^* 0,
\end{aligned}$$

where all functions are restricted to the domain $x_\xi \cap \bigcap \{x_{b(i)} \mid i < n\}$, the passage from the first line to the second follows from our assumptions about $\langle \psi_a \mid a \in [\xi]^{n-1} \rangle$, the passage from the second line to the third follows from the observations at the beginning of this proof, the passage from the third line to the fourth is a simple rearranging of terms, and the final equality (mod finite) follows from the coherence of Φ. □

Since $|\xi| < \kappa \leq \omega_{n-1}$, the inductive hypothesis implies that T_ξ is trivial. We can thus fix an $(n-2)$-family of functions

$$\langle \psi_{d\cup\{\xi\}} : x_\xi \cap \bigcap \{x_{d(i)} \mid i < n-2\} \to H \mid d \in [\xi]^{n-2} \rangle$$

such that, for all $a \in [\xi]^{n-1}$, we have

$$\tau_a =^* \sum_{i<n-1} (-1)^i \psi_{a^i \cup \{\xi\}}.$$

We claim that this assignment of $\psi_{d\cup\{\xi\}}$ works. To check this, fix an arbitrary $a \in [\xi]^{n-1}$, and let $b = a \cup \{\xi\}$. Then

$$\begin{aligned}\sum_{i<n}(-1)^i \psi_{b^i} &= (-1)^{n-1}\psi_a + \sum_{i<n-1}(-1)^i \psi_{a^i\cup\{\xi\}} \\ &=^* (-1)^{n-1}\psi_a + \tau_a \\ &= (-1)^{n-1}\psi_a + (-1)^n \psi_a + \varphi_{a\cup\{\xi\}} \\ &= \varphi_b.\end{aligned}$$

Thus, we can carry out the construction of a family $\langle \psi_a \mid a \in [\kappa]^{n-1}\rangle$ witnessing that Φ is trivial, thus completing the proof. □

Definition 2.10. *Suppose that $\vec{x} = \langle x_\alpha \mid \alpha < \delta\rangle$ is a sequence of sets, $1 \leq n < \omega$, H is an abelian group, and $\Phi = \langle \varphi_a : \bigcap_{i<n} x_{a(i)} \to H \mid u \in \delta^n\rangle$ is an n-family of functions indexed along $\vec{x}$. If E is a set, then $\Phi \restriction\restriction E$ denotes the n-family*

$$\left\langle \varphi_a \restriction E \cap \bigcap_{i<n} x_{a(i)} \;\middle|\; a \in \delta^n \right\rangle$$

indexed along $\vec{x} \restriction\restriction E := \langle x_\alpha \cap E \mid \alpha < \delta\rangle$.

Note that, if Φ is an alternating, coherent, or trivial n-family, then these properties are inherited by $\Phi \restriction\restriction E$ for every set E. In particular, if Φ is a coherent n-family and $\Phi \restriction\restriction E$ is nontrivial for some set E, then also Φ is nontrivial.

Definition 2.11. *Suppose that $1 < n < \omega$, δ is an ordinal, H is an abelian group, $\langle x_\alpha \mid \alpha < \delta\rangle$ is a sequence of sets, and $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n\rangle$ is a coherent n-family. Let $\mathrm{Triv}_H(\Phi)$ be the set of all $(n-1)$-families $\Psi = \langle \psi_a : \bigcap_{i<n-1} x_{a(i)} \to H \mid a \in \delta^{n-1}\rangle$ that trivialize Φ (so $\mathrm{Triv}_H(\Phi)$ is empty if Φ is nontrivial).*

If the group H is clear from context, we may omit it from the notation $\mathrm{Triv}_H(\Phi)$.

Proposition 2.12. *Suppose that $1 < n < \omega$, δ is an ordinal, H is an abelian group, $\langle x_\alpha \mid \alpha < \delta\rangle$ is a sequence of sets, and $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n\rangle$ is a coherent n-family. For all $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$, the $(n-1)$-family*

$$\Psi^1 - \Psi^0 = \langle \psi^1_a - \psi^0_a \mid a \in \delta^{n-1}\rangle$$

is coherent.

Proof. Fix $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$. The fact that $\Psi^1 - \Psi^0$ is alternating follows immediately from the fact that both Ψ^0 and Ψ^1 are alternating. To check coherence, fix $b \in \delta^n$. For each $\ell < 2$, the fact that $\Psi^\ell \in \mathrm{Triv}_H(\Phi)$ implies that $\sum_{i<n} \psi^\ell_{b^i} =^* \varphi_b$. Therefore, we have

$$\sum_{i<n}(\psi^1_{b^i} - \psi^0_{b^i}) = \sum_{i<n} \psi^1_{b^i} - \sum_{i<n} \psi^0_{b^i} =^* \varphi_b - \varphi_b = 0,$$

as desired. □

Definition 2.13. *Given $1 < n < \omega$, an abelian group H, and an H-valued coherent n-family Φ, define an equivalence relation $\cong_{\Phi,H}$ on $\mathrm{Triv}_H(\Phi)$ by setting $\Psi^0 \cong_{\Phi,H} \Psi^1$ if and only if the coherent $(n-1)$-family $\Psi^1 - \Psi^0$ is trivial (via an H-valued trivialization).*

The following basic facts are easily established; their proofs are left to the reader.

Proposition 2.14. *Suppose that $1 < n < \omega$, H is an abelian group, Φ is an H-valued coherent n-family, and $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$.*

1. *For every set E, we have $\Psi^0 \restriction\restriction E, \Psi^1 \restriction\restriction E \in \mathrm{Triv}_H(\Phi \restriction\restriction E)$.*
2. *If $E \subseteq F$ are sets and $\Psi^0 \restriction\restriction F \cong_{\Phi\restriction\restriction F,H} \Psi^1 \restriction\restriction F$, then $\Psi^0 \restriction\restriction E \cong_{\Phi\restriction\restriction E,H} \Psi^1 \restriction\restriction E$.* □

Proposition 2.15. *Suppose that $1 < n < \omega$, δ is an ordinal, H is an abelian group, $\langle x_\alpha \mid \alpha < \delta\rangle$ is a sequence of sets, and $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n\rangle$ is a coherent n-family. Suppose also that $\Psi \in \mathrm{Triv}_H(\Phi)$ and $T = \langle \tau_a : \bigcap_{i<n-1} x_{a(i)} \to H \mid a \in \delta^{n-1}\rangle$ is a coherent $(n-1)$-family. Then the family*

$$\Psi + T = \langle \psi_v + \tau_v \mid v \in \delta^{n-1}\rangle$$

is in $\mathrm{Triv}_H(\Phi)$. Moreover, we have $\Psi + T \cong_{\Phi,H} \Psi$ if and only if T is trivial.

Proof. We first verify that $\Psi + T$ is in $\mathrm{Triv}_H(\Phi)$. To this end, fix $b \in \delta^n$. Then we have

$$\begin{aligned}\sum_{i<n}(\psi_{b^i} + \tau_{b^i}) &= \sum_{i<n} \psi_{b^i} + \sum_{i<n} \tau_{b^i} \\ &=^* \varphi_b + 0 = \varphi_b,\end{aligned}$$

as desired. The "moreover" clause follows immediately from the definition of the equivalence relation $\cong_{\Phi,H}$. □

Definition 2.16. *Suppose that*

- *$1 < n < \omega$;*
- *δ is an ordinal;*
- *$\langle x_\alpha \mid \alpha < \delta + 1\rangle$ is a sequence of sets;*

- *H is an abelian group;*
- *$\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n \rangle$ is a coherent n-family;*
- *$\Psi = \langle \psi_a : x_\delta \cap \bigcap_{i<n-1} a_{a(i)} \to H \mid a \in \delta^{n-1} \rangle \in \mathrm{Triv}_H(\Phi \restriction\restriction x_\delta)$.*

Then let $\Phi^\frown\langle\Psi\rangle$ denote the coherent n-family $\langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in (\delta+1)^n \rangle$ extending Φ defined by letting $\varphi_{a^\frown\langle\delta\rangle} = (-1)^{n+1}\psi_a$ for all $a \in \delta^{n-1}$.

Remark 2.17. *Implicit in Definition 2.16 is the assertion that the n-family $\Phi^\frown\langle\Psi\rangle$ is indeed coherent. The verification of this fact is routine and left to the reader.*

Proposition 2.18. *Suppose that*

- *$1 < n < \omega$;*
- *δ is an ordinal;*
- *$\langle x_\alpha \mid \alpha < \delta + 1 \rangle$ is a sequence of sets;*
- *H is an abelian group;*
- *$\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n \rangle$ is a coherent n-family;*
- *$\Psi \in \mathrm{Triv}_H(\Phi \restriction\restriction x_\delta)$, $\Psi' \in \mathrm{Triv}_H(\Phi)$, and $\Psi \not\approx_{\Phi\restriction\restriction x_\delta, H} \Psi' \restriction\restriction x_\delta$.*

Then Ψ' does not extend to a trivialization of $\Phi^\frown\langle\Psi\rangle$.

Proof. Assume that $n > 2$. The case $n = 2$ is similar but easier, and left to the reader. Let $\Phi^\frown\langle\Psi\rangle = \langle \varphi_b \mid b \in (\delta+1)^n \rangle$. Suppose for the sake of contradiction that $\langle \psi'_a \mid a \in (\delta+1)^{n-1} \rangle$ extends Ψ' and trivializes $\Phi^\frown\langle\Psi\rangle$. Then, for all $a \in \delta^{n-1}$, we have

$$\begin{aligned}(-1)^{n+1}\psi_a &= \varphi_{a^\frown\langle\delta\rangle} \\ &=^* \sum_{i<n} (-1)^i \psi'_{(a^\frown\langle\delta\rangle)^i} \\ &= (-1)^{n-1}\psi'_a + \sum_{i<n-1} \psi'_{a^i\frown\langle\delta\rangle}.\end{aligned}$$

Rearranging the above equation, we see that, for all $a \in \delta^{n-1}$, we have

$$\psi_a - \psi'_a =^* (-1)^{n+1} \sum_{i<n-1} \psi'_{a^i\frown\langle\delta\rangle}.$$

In particular, the family $\langle (-1)^{n+1}\psi'_{e^\frown\langle\delta\rangle} \mid e \in \delta^{n-2} \rangle$ trivializes $\Psi - \Psi' \restriction\restriction x_\delta$, contradicting the fact that $\Psi \not\approx_{\Phi\restriction\restriction x_\delta, H} \Psi' \restriction\restriction x_\delta$. □

2.2.2 Derived limits

One of the reasons for our interest in coherent n-families is their connection with the calculation of the derived limits of certain natural inverse systems of abelian groups.

Definition 2.19. *If $(\Lambda, \leq_\Lambda)$ is a directed partial order, then an* inverse system of abelian groups indexed by Λ *is a structure of the form*

$$\mathbf{B} = \langle B_u, \pi_{uv} \mid u, v \in \Lambda,\ u \leq_\Lambda v \rangle$$

such that

1. *for all $u \in \Lambda$, B_u is an abelian group;*
2. *for all $u \leq_\Lambda v$, $\pi_{uv} : B_v \to B_u$ is a group homomorphism;*
3. *for all $u \leq_\Lambda v \leq_\Lambda w$, $\pi_{uw} = \pi_{uv} \circ \pi_{vw}$.*

Given an inverse system $\mathbf{B}$, one can form the inverse limit $\lim \mathbf{B} \in \mathsf{Ab}$. *Concretely,* $\lim \mathbf{B}$ *can be represented as*

$$\{x \in \prod_{u \in \Lambda} B_u \mid \forall u \leq_\Lambda v,\ x_u = \pi_{uv} x(v)\}.$$

Given a directed set Λ, one can form the category $\mathsf{Ab}^{\Lambda^{\mathrm{op}}}$ of all inverse systems of abelian groups indexed by Λ. (Since the details of this and what follows will not be necessary for us in this chapter, we refer the reader to [122, Section III] for precise definitions and proofs of facts presented in the remainder of this section.) The inverse limit operation then induces a functor $\lim : \mathsf{Ab}^{\Lambda^{\mathrm{op}}} \to \mathsf{Ab}$. This functor is left exact but in general not exact; it therefore has (right) derived functors $\lim^n : \mathsf{Ab}^{\Lambda^{\mathrm{op}}} \to \mathsf{Ab}$ for $1 \leq n < \omega$. The nontrivial coherent families of functions introduced earlier in this section play a key role in the computation of the derived limits of a particular class of inverse systems, which we now introduce.

Definition 2.20. *Suppose that H is an abelian group.*

1. *Given a set x, let $A_x[H]$ denote the group $\bigoplus_x H$. Concretely, this is the group of all finitely-supported functions from x to H.*
2. *Given sets $x \subseteq y$, let $\pi^H_{xy} : A_y[H] \to A_x[H]$ be the natural restriction map. More precisely, if $f : y \to H$ is in $A_y[H]$, then $\pi^H_{xy}(f) = f \restriction x$. Note that this is an abelian group homomorphism. If the group H is clear from context, then we will omit it from the notation.*
3. *Suppose that $\mathcal{I}$ is a collection of sets that is $\subseteq$-directed, i.e., for all $x, y \in \mathcal{I}$, there is $z \in \mathcal{I}$ such that $x \cup y \subseteq z$. Then $\mathbf{A}_\mathcal{I}[H]$ denotes the inverse system of abelian groups*

$$\langle A_x[H], \pi^H_{xy} \mid x, y \in \mathcal{I},\ x \subseteq y \rangle.$$

There is a particular $\subseteq$-directed collection of sets in which we will be especially interested in this chapter. Given a function $f : \omega \to \omega$, let $I_f = \{(k, m) \in \omega \times \omega \mid m < f(k)\}$; i.e., I_f is the region "under the graph of f" in the plane $\omega \times \omega$. If we omit $\mathcal{I}$ from the notation $\mathbf{A}_{\mathcal{I}}[H]$, then $\mathcal{I}$ should be understood to be the set $\{I_f \mid f \in {}^{\omega}\omega\}$. Moreover, given $f \leq g \in {}^{\omega}\omega$, we will typically write $A_f[H]$ in place of $A_{I_f}[H]$ and π_{fg} in place of $\pi_{I_f I_g}$, i.e., the inverse system $\mathbf{A}[H]$ is the system

$$\langle A_f[H], \pi_{fg} \mid f \leq g \in {}^{\omega}\omega \rangle.$$

We will also omit mention of the group H in case $H = \mathbb{Z}$, e.g., we will write $\mathbf{A}$ for $\mathbf{A}[\mathbb{Z}]$.

We can now state the connection between nontrivial coherent n-families and the derived limits of these inverse systems. For a proof of the fact, see [122, Section III] or [18, Section 2.2].[1]

Fact 2.21. *Suppose that $1 \leq n < \omega$, $\mathcal{I}$ is a $\subseteq$-directed collection of sets, and H is an abelian group. Then the following are equivalent:*

1. $\lim^n \mathbf{A}_{\mathcal{I}}[H] = 0$;
2. *every coherent H-valued n-family indexed along $\mathcal{I}$ is trivial.*

Fact 2.21, together with Propositions 2.4 and 2.5 yield the following observation.

Fact 2.22. *Suppose that $1 \leq n < \omega$, $\mathcal{I}$ is a $\subseteq$-directed collection of sets, $\mathcal{J}$ is a $\subseteq^*$-cofinal subset of $\mathcal{J}$, and H is an abelian group. Then the following are equivalent:*

1. $\lim^n \mathbf{A}_{\mathcal{I}}[H] = 0$;
2. $\lim^n \mathbf{A}_{\mathcal{J}}[H] = 0$.

2.3 Ascending sequences and nontrivial coherence

In this section, we isolate the notion of an *ascending sequence* of sets and prove that under certain assumptions one can recursively construct nontrivial coherent n-families along them.

For a nonempty set Y, we let $\mathrm{Fn}^+(Y)$ denote the collection of all finite partial functions w from Y to 2 such that $w^{-1}\{1\} \neq \emptyset$.

Definition 2.23. *Suppose that $\vec{x} = \langle x_\alpha \mid \alpha < \delta \rangle$ is a sequence of sets, $1 \leq n < \omega$, H is an abelian group, and $\Phi = \langle \varphi_a : \bigcap_{i<n} x_{a(i)} \to H \mid a \in \delta^n \rangle$ is an alternating n-family.*

1. *If E is a set, then we say that Φ is* supported on E *if* $\mathrm{supp}(\varphi_a) \subseteq E$ *for all $a \in \delta^n$.*

[1] The contents of [18, Section 2.2] are specifically about the system $\mathbf{A}$, but the arguments therein straightforwardly generalize to $\mathbf{A}_{\mathcal{I}}[H]$ for any $\subseteq$-directed collection $\mathcal{I}$ and any abelian group H.

2. *Suppose that* $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\delta)\rangle$ *is a sequence of sets. For all* $a \in \delta^n$*, let* $\beta_a := \max\{a(i) \mid i < n\}$*. We say that* Φ *is* supported on $\vec{e}$ *if for all* $a \in \delta^n$*, we have* $\mathrm{supp}(\varphi_a) \subseteq \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\beta_a)\}$.

The following proposition is immediate.

Proposition 2.24. *Suppose that* $1 < n < \omega$*,* H *is an abelian group,* Φ *is an* H*-valued coherent* n*-family, and* $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$*. If* E *is a set,* Ψ^0 *and* Ψ^1 *are both supported on* E*, and* $\Psi^0 \restriction\restriction E \cong_{\Phi\restriction\restriction E, H} \Psi^1 \restriction\restriction E$*, then* $\Psi^0 \cong_{\Phi,H} \Psi^1$*.* □

Definition 2.25. *Suppose that* δ *is a limit ordinal of uncountable cofinality and* $\vec{x} = \langle x_\alpha \mid \alpha < \delta\rangle$ *is a sequence of sets.*

1. *For* $w \in \mathrm{Fn}^+(\delta)$*, we say that* $\vec{x}$ respects w *if the set*

$$d^{\vec{x}}_w := \bigcap_{\beta \in w^{-1}\{1\}} x_\beta \setminus \bigcup_{\alpha \in w^{-1}\{0\}} x_\alpha$$

is infinite.

2. *We say that* $\vec{x}$ *is* ascending *if there is a sequence* $\langle e_w \mid w \in \mathrm{Fn}^+(\delta)\rangle$ *and a club* $C \subseteq \delta$ *satisfying the following requirements:*

   (a) *for all* $w \in \mathrm{Fn}^+(\delta)$*, if* $\vec{x}$ *respects* w*, then* $e_w \in [d^{\vec{x}}_w]^{\aleph_0}$ *(otherwise* $e_w = \emptyset$*);*

   (b) *for all* $\beta \in C$ *and all* $a \in [\beta]^{<\omega}$*, we have* $x_\beta \not\subseteq^* \bigcup_{\alpha\in a} x_\alpha$*;*

   (c) *for all* $a \in [C]^{<\omega}$ *and all* $w \in \mathrm{Fn}^+(\min(a))$*, if* $\vec{x}$ *respects* w*, then the set*

$$e_w \cap \bigcap_{\beta\in a} x_\beta$$

   is infinite. In particular, for all $\beta \in C$ *and all* $w \in \mathrm{Fn}^+(\beta)$*, if* $\vec{x}$ *respects* w*, then* $\vec{x}$ *respects* $w \cup \{(\beta, 1)\}$.

We now prove that, for $1 \leq n < \omega$, ascending sequences of length ω_n carry nontrivial coherent n-families. We first prove the existence of n-families mapping into arbitrary sufficiently small nonzero abelian groups under an additional weak diamond assumption. We will later eliminate this weak diamond assumption, at the cost of increasing the size of the groups that we are mapping into. Before beginning, let us recall the definition of weak diamond.

Definition 2.26. *Let* λ *be a regular uncountable cardinal and let* $S \subseteq \lambda$ *be stationary in* λ*. The principle* $\mathrm{w}\diamondsuit(S)$ *asserts that for every* $F : 2^{<\lambda} \to 2$ *there exists* $g : \lambda \to 2$ *such that for every* $b : \lambda \to 2$ *the set*

$$\{\alpha \in S \mid g(\alpha) = F(b \restriction \alpha)\}$$

is stationary in λ*.*

We begin with the special case of $n = 1$.

Theorem 2.27. *Suppose that $\vec{x} = \langle x_\alpha \mid \alpha < \omega_1 \rangle$ is an ascending sequence of sets, H is a nonzero abelian group, and $\mathrm{w}\diamondsuit(\omega_1)$ holds. Then there is a nontrivial coherent 1-family $\Phi = \langle \varphi_\alpha : x_\alpha \to H \mid \alpha < \omega_1 \rangle$.*

Moreover, if $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_1) \rangle$ and C witness that $\vec{x}$ is ascending, then we can arrange so that Φ is supported on $\vec{e}$.

Proof. Fix for the duration of the proof an arbitrary nonzero element H, and denote it by 1_H. We will denote the zero element of H by 0_H. For each $\sigma \in {}^{<\omega_1}2$, we will construct a coherent H-valued n-family $\Phi^\sigma = \langle \varphi^\sigma_\alpha \mid \alpha < \mathrm{lh}(\sigma) \rangle$ in such a way that if $\sigma \sqsubseteq \tau \in {}^{<\omega_1}2$, then $\Phi^\sigma \sqsubseteq \Phi^\tau$. We will then find some $g \in {}^{\omega_1}2$ for which the family $\Phi^g := \bigcup_{\beta<\omega_1} \Phi^{g\restriction\beta}$ is nontrivial.

Let $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_1) \rangle$ and $C \subseteq \omega_1$ witness that $\vec{a}$ is ascending. Before we begin the construction of $\langle \Phi^\sigma \mid \sigma \in {}^{<\omega_1}2 \rangle$, we isolate, for each $\gamma \in \mathrm{Lim}(C)$, a relevant set $y_\gamma \in [x_\gamma]^{\aleph_0}$.

Fix $\gamma \in \mathrm{Lim}(C)$, as well as a bijection $\pi_\gamma : \gamma \to \omega$. Let $\langle \gamma_n \mid n < \omega \rangle$ be an increasing cofinal sequence from $C \cap \gamma$ such that $\langle \pi_\gamma(\gamma_n) \mid n < \omega \rangle$ is also increasing. For each $n < \omega$, define a function $w_{\gamma,n} \in \mathrm{Fn}^+(\gamma)$ by letting

$$\mathrm{dom}(w_{\gamma,n}) := \{\gamma_n\} \cup \{\alpha < \gamma_n \mid \pi_\gamma(\alpha) < \pi_\gamma(\gamma_n)\}$$

and setting $w_{\gamma,n}(\gamma_n) := 1$ and $w_{\gamma,n}(\alpha) := 0$ for all $\alpha \in \mathrm{dom}(w_n) \cap \gamma_n$. Then let $w^+_{\gamma,n} := w_{\gamma,n} \cup \{(\gamma, 1)\}$. The fact that $\gamma_n < \gamma$ are both in C implies that $\vec{x}$ respects both $w_{\gamma,n}$ and $w^+_{\gamma,n}$ and that $x_\gamma \cap e_{w_{\gamma,n}}$ is infinite. Note also that, for all $m < n < \omega$, we have $e_{w_{\gamma,m}} \cap e_{w_{\gamma,n}} = \emptyset$, since $w_{\gamma,n}(\gamma_m) = 0$. Now, for each $n < \omega$, choose a single element $i_{\gamma,n} \in x_\gamma \cap e_{w_{\gamma,n}}$, and let $y_\gamma := \{i_{\gamma,n} \mid n < \omega\}$. Note that, by construction, the set $y_\gamma \cap x_\alpha$ is finite for all $\alpha < \gamma$, since $w_{\gamma,n}(\alpha) = 0$ for all sufficiently large $n < \omega$.

We now turn to the construction of $\langle \Phi^\sigma \mid \sigma \in {}^{<\omega_1}2 \rangle$. We will ensure throughout the construction that each Φ^σ is supported on $\vec{e}$. If $\sigma \in {}^{<\omega_1}2$ and $\mathrm{lh}(\sigma)$ is a limit ordinal, then we are obliged to set $\Phi^\sigma := \bigcup_{\beta<\mathrm{lh}(\sigma)} \Phi^{\sigma\restriction\beta}$. Thus, to complete the construction, it suffices to specify how to produce $\Phi^{\sigma^\frown\langle 0\rangle}$ and $\Phi^{\sigma^\frown\langle 1\rangle}$ from Φ_σ. To this end, fix $\gamma < \omega_1$ and $\sigma \in {}^\gamma 2$, and suppose that we have constructed Φ_σ. For readability, we will write $\Phi^{\sigma,\ell}$ in place of $\Phi^{\sigma^\frown\langle\ell\rangle}$ for $\ell < 2$.

Fix $\ell < 2$. To construct $\Phi^{\sigma,\ell}$, it is enough to define $\varphi^{\sigma,\ell}_\gamma$. To begin, since $\gamma < \omega_1$, we can use Proposition 2.7 to find a function $\psi : x_\gamma \to H$ such that $\psi =^* \varphi^\sigma_\alpha$ for all $\alpha < \gamma$. Moreover, since Φ^σ is supported on $\vec{e}$, we can assume that the support of ψ is a subset of $\bigcup\{e_w \mid w \in \mathrm{Fn}^+(\gamma)\}$. If $\gamma \notin \mathrm{Lim}(C)$, then simply let $\varphi^{\sigma,\ell}_\gamma = \psi$. If $\gamma \in \mathrm{Lim}(C)$, then define $\varphi^{\sigma,\ell}_\gamma$ by setting

$$\varphi^{\sigma,\ell}_\gamma(x) = \begin{cases} \ell_H & \text{if } x \in y_\gamma; \\ \psi(x) & \text{if } x \in x_\gamma \setminus y_\gamma. \end{cases}$$

Note that we still have $\varphi^{\sigma,\ell}_\gamma =^* \varphi^\sigma_\alpha$ for all $\alpha < \gamma$, due to the fact that $y_\gamma \cap x_\alpha$ is finite for all such α. We have also satisfied the requirement that each $\Phi^{\sigma,\ell}$ is supported on $\vec{e}$. This completes the construction.

We now prepare for an application of $\mathrm{w}\diamondsuit(\omega_1)$. Let $E := \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\omega_1)\}$, and note that $|E| \leq \aleph_1$ and $y_\gamma \subseteq E$ for all $\gamma \in \mathrm{Lim}(C)$. Let $\langle j_\alpha \mid \alpha < |E|\rangle$ be an injective enumeration of E. For $\gamma < \omega_1$, let $E_\gamma := \{j_\alpha \mid \alpha < \gamma\}$. Let D be the set of $\gamma \in \lim(C)$ such that, for all $w \in \mathrm{Fn}^+(\gamma)$, we have $e_w \subseteq E_\gamma$, and note that D is club in ω_1. Additionally note that, by construction, for each $\gamma \in D$ we have $y_\gamma \subseteq E_\gamma$, and hence every $\tau \in {}^\gamma 2$ induces a function $h_\tau : y_\gamma \to 2$ by setting $h_\tau(j_\alpha) = \tau(\alpha)$ for all $\alpha < \gamma$ such that $j_\alpha \in y_\gamma$.

Now define a function $F : {}^{<\omega_1}2 \to 2$ by setting $F(\tau) := 1$ for all τ such that $\gamma := \mathrm{lh}(\tau) \in D$ and $h_\tau =^* 0$, and setting $F(\tau) := 0$ for all other $\tau \in {}^{<\omega_1}2$. Let $g \in {}^{\omega_1}2$ witness the instance of $\mathrm{w}\diamondsuit(\omega_1)$ associated with F. We claim that $\Phi^g = \bigcup_{\beta<\omega_1} \Phi^{g\restriction\beta} = \langle \varphi_\alpha \mid \alpha < \omega_1\rangle$ is nontrivial. Suppose for the sake of contradiction that ψ trivializes Φ^g. Define a function $f \in {}^{\omega_1}2$ by setting, for all $\alpha < \omega_1$,

$$f(\alpha) = \begin{cases} 0 & \text{if } \psi(j_\alpha) = 0_H \\ 1 & \text{otherwise.} \end{cases}$$

By our choice of g, we can find $\gamma \in D$ such that $g(\gamma) = F(f \restriction \gamma)$.

Let $\ell := g(\gamma)$ and $\tau := f \restriction \gamma$. By construction of Φ^g, we know that $\varphi_\gamma \restriction y_\gamma =^* \ell_H$. By the assumption that ψ trivializes Φ^g, it follows that $\psi \restriction y_\gamma =^* \ell_H$. Our definition of h_τ then implies that $h_\tau =^* \ell$, but then our definition of F implies that $F(\tau) = 1 - \ell$, contradicting the fact that $g(\gamma) = F(f \restriction \gamma)$ and completing the proof. □

We now turn to higher dimensions.

Theorem 2.28. *Suppose that $1 \leq n < \omega$, $\vec{x} = \langle x_\alpha \mid \alpha < \omega_n\rangle$ is an ascending sequence of sets, H is a nonzero abelian group with $|H| \leq 2^{\omega_1}$, and $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds for all $k < n$. Then there is a nontrivial coherent n-family*

$$\Phi = \left\langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \;\middle|\; b \in (\omega_n)^n \right\rangle.$$

Moreover, if $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_n)\rangle$ and $C \subseteq \omega_n$ witness that $\vec{x}$ is ascending, then we can arrange so that Φ is supported on $\vec{e}$.

Proof. The proof is by induction on n. The case of $n = 1$ is precisely Theorem 2.27, so assume that $n > 1$ and that we have established the theorem for all $1 \leq m < n$.

Let $\langle e_w \mid w \in \mathrm{Fn}^+(\omega_n)\rangle$ and $C \subseteq \omega_n$ witness that $\vec{x}$ is ascending. Let θ be a sufficiently large regular cardinal, and let $\lhd$ be a fixed well-ordering of $H(\theta)$. For $\gamma < \omega_n$, let $E_\gamma := \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\gamma)\}$, and let $\bar{E}_\gamma := E_\gamma \cap x_\gamma$.

Let $\mathcal{S} := \{\sigma \in {}^{<\omega_n}2 \mid \mathrm{supp}(\sigma) \subseteq \mathrm{Lim}(C) \cap S^n_{n-1}\}$, and let $\mathcal{S}^+ := \{g \in {}^{\omega_n}2 \mid \mathrm{supp}(g) \subseteq \mathrm{Lim}(C) \cap S^n_{n-1}\}$. Similarly to the proof of Theorem 2.27, for each $\sigma \in \mathcal{S}$, we will construct a coherent H-valued n-family $\Phi^\sigma = \langle \varphi^\sigma_b \mid b \in \mathrm{lh}(\sigma)^n\rangle$ in such a way that if $\sigma \sqsubseteq \tau \in \mathcal{S}$, then $\Phi^\sigma \sqsubseteq \Phi^\tau$. We will then find some $g \in \mathcal{S}^+$ such that $\Phi^g := \bigcup_{\alpha<\omega_n} \Phi^{g\restriction\alpha}$ is nontrivial.

We now describe the construction of $\langle \Phi^\sigma \mid \sigma \in \mathcal{S}\rangle$, which will be done by recursion on $\mathrm{lh}(\sigma)$. We will ensure along the way that, for all $\gamma < \omega_n$ and all $\sigma \in \mathcal{S} \cap {}^\gamma 2$, the family Φ^σ is supported on $\langle e_w \mid w \in \mathrm{Fn}^+(\gamma)\rangle$. As in the proof of Theorem 2.27, the case in which $\mathrm{lh}(\sigma)$ is a limit ordinal is trivial, so fix a $\sigma \in \mathcal{S}$ and assume that Φ^σ has been constructed. We describe how to construct $\Phi^{\sigma,\ell}$ for $\ell < 2$ if $\mathrm{lh}(\sigma) \in \mathrm{Lim}(C) \cap S^n_{n-1}$ and for $\ell = 0$ otherwise.

Let $\gamma := \mathrm{lh}(\sigma)$. Since $\gamma < \omega_n$, Proposition 2.7 implies that $\Phi^\sigma \restriction\restriction x_\gamma$ is trivial. Let $\Psi^\sigma = \langle \psi^\sigma_a \mid a \in \gamma^{n-1}\rangle$ be an element of $\mathrm{Triv}_H(\Phi^\sigma \restriction\restriction x_\gamma)$. Since $\Phi^\sigma \restriction\restriction x_\gamma$ is supported on $\bar{E}_\gamma$, we can assume that Ψ^σ is also supported on $\bar{E}_\gamma$. If $\gamma \notin \mathrm{Lim}(C) \cap S^n_{n-1}$, then simply let $\Phi^{\sigma,0} = \Phi^\sigma {}^\frown \langle \Psi^\sigma\rangle$. It is easily verified that this maintains the requirements of the construction; to see that it maintains coherence, recall Remark 2.17.

If $\gamma \in \mathrm{Lim}(C) \cap S^n_{n-1}$, then we work a little bit harder. Let $D_\gamma \subseteq C \cap \gamma$ be a club in γ with $\mathrm{otp}(D_\gamma) = \omega_{n-1}$, and let $\pi_\gamma : \gamma \to \omega_{n-1}$ be a bijection such that $\pi \restriction D_\gamma$ is order-preserving and continuous. In particular, $\bar{D}_\gamma := \pi[D_\gamma]$ is a club in ω_{n-1}. Note that $\{\eta \in \bar{D}_\gamma \mid \pi_\gamma^{-1}[\eta] \subseteq \pi^{-1}(\eta)\}$ is a club in ω_{n-1}. By thinning out D_γ (and hence also $\bar{D}_\gamma$), we can assume that $\pi_\gamma^{-1}[\eta] \subseteq \pi_\gamma^{-1}(\eta)$ for all $\eta \in \bar{D}_\gamma$.

For all $\eta < \omega_{n-1}$, let $y^\gamma_\eta := x_\gamma \cap x_{\pi_\gamma^{-1}(\eta)}$. Given $w \in \mathrm{Fn}^+(\omega_{n-1})$, let $\hat{w}^\gamma \in \mathrm{Fn}^+(\gamma)$ be defined by letting $\mathrm{dom}(\hat{w}^\gamma) := \pi_\gamma^{-1}[\mathrm{dom}(w)]$ and, for all $\alpha \in \mathrm{dom}(\hat{w}^\gamma)$, setting $\hat{w}^\gamma(\alpha) = w(\pi_\gamma(\alpha))$. For such w, let $\bar{e}^\gamma_w := x_\gamma \cap e_{\hat{w}^\gamma}$.

Claim 2.29. *$\vec{y} = \langle y^\gamma_\eta \mid \eta < \omega_{n-1}\rangle$ is an ascending sequence of sets, as witnessed by $\langle \bar{e}^\gamma_w \mid w \in \mathrm{Fn}^+(\omega_{n-1})\rangle$ and $\bar{D}_\gamma$.*

Proof. We must verify conditions (2)(a–c) of Definition 2.25. To see (a), fix $w \in \mathrm{Fn}^+(\omega_{n-1})$, and note that

$$d^{\vec{y}}_w = d^{\vec{x}}_{\hat{w}^\gamma} \cap x_\gamma = d^{\vec{x}}_{\hat{w}^\gamma \cup \{(\gamma,1)\}}.$$

By the fact that $\gamma \in C$, we know that

$$(\vec{y} \text{ respects } w) \Leftrightarrow (\vec{x} \text{ respects } \hat{w}^\gamma \cup \{(\gamma,1\}) \Leftrightarrow (\vec{x} \text{ respects } \hat{w}^\gamma).$$

Thus, if $\vec{y}$ respects w, the fact that $\vec{x}$ is ascending implies that

$$\bar{e}^\gamma_w = x_\gamma \cap e_{\hat{w}^\gamma} \in [d^{\vec{y}}_w]^{\aleph_0},$$

so (a) is satisfied.

For item (b), fix $\eta \in \bar{D}_\gamma$ and $a \in [\eta]^{<\omega}$. Let $\hat{a}^\gamma := \pi_\gamma^{-1}[a]$ and $\beta = \pi_\gamma^{-1}(\eta)$. Since $\eta \in \bar{D}_\gamma$, we have $\hat{a}^\gamma \subseteq \beta$ and $\beta \in C$, and hence $x_\beta \not\subseteq^* \bigcup_{\alpha\in\hat{a}^\gamma} x_\alpha$. In particular, if we define

$\hat{w} \in \mathrm{Fn}^+(\gamma)$ by letting $\mathrm{dom}(\hat{w}) = \hat{a}^\gamma \cup \{\beta\}$, $\hat{w}(\beta) = 1$, and $\hat{w} \restriction \hat{a}^\gamma = 0$, then $\vec{x}$ respects $\hat{w}$. Since $\gamma \in C$, $\vec{x}$ respects $\hat{w} \cup \{(\gamma, 1)\}$. But

$$d^{\vec{x}}_{\hat{w}\cup\{(\gamma,1)\}} = (x_\gamma \cap x_\beta) \setminus \bigcup_{\alpha\in\hat{a}^\gamma} x_\alpha = y_\eta \setminus \bigcup_{\xi\in a} y_\xi.$$

Since this set is infinite, we have $y_\eta \not\subseteq^* \bigcup_{\xi\in a} y_\xi$, as desired.

For (c), fix $a \in [\bar{D}_\gamma]^{<\omega}$ and $w \in \mathrm{Fn}^+(\min(a))$ such that $\vec{y}$ respects w. Let $\hat{a}^\gamma := \pi_\gamma^{-1}[a]$. Note that $\hat{w}^\gamma \in \mathrm{Fn}^+(\min(\hat{a}^\gamma))$, $\hat{a}^\gamma \in [C]^{<\omega}$, and $\vec{x}$ respects $\hat{w}^\gamma$. Therefore, since $\gamma \in C$, we know that $e_{\hat{w}^\gamma} \cap x_\gamma \cap \bigcap_{\alpha\in\hat{a}^\gamma} x_\alpha$ is infinite. But $e_{\hat{w}^\gamma} \cap x_\gamma \cap \bigcap_{\alpha\in\hat{a}^\gamma} x_\alpha = \bar{e}^\gamma_w \cap \bigcap_{\eta\in a} y_\eta$, so this instance of (c) is satisfied. □

By the inductive hypothesis, we can find a nontrivial coherent $(n-1)$-family

$$\bar{T}^\gamma = \left\langle \bar{\tau}^\gamma_a : \bigcap_{i<n-1} y_{a(i)} \to H \;\middle|\; a \in (\omega_{n-1})^{n-1} \right\rangle$$

that is supported on $\bar{E}_\gamma$. Through re-indexing via π_γ, this yields a nontrivial coherent $(n-1)$-family

$$T^\gamma = \left\langle \tau^\gamma_a : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H \;\middle|\; a \in \gamma^{n-1} \right\rangle$$

that is also supported on $\bar{E}_\gamma$. Then Proposition 2.15 implies that $(\Psi^\sigma + T^\gamma) \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction x_\gamma)$ and $\Psi^\sigma \not\cong_{\Phi^\sigma\restriction\restriction x_\gamma, H} (\Psi^\sigma + T^\gamma)$. Finally, set $\Phi^{\sigma,0} := \Phi^\sigma{}^\frown\langle\Psi^\sigma\rangle$, and set $\Phi^{\sigma,1} := \Phi^\sigma{}^\frown\langle\Psi^\sigma + T^\gamma\rangle$. It is routine to verify that these definitions maintain coherence and that the resulting families $\Phi^{\sigma,\ell}$ are supported on $\langle e_w \mid w \in \mathrm{Fn}^+(\gamma+1)\rangle$. This completes the construction of $\langle \Phi^\sigma \mid \sigma \in \mathcal{S}\rangle$.

Let $E = E_{\omega_n} := \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\omega_n)\}$. Note that the sequence $\langle E_\gamma \mid \gamma \leq \omega_n\rangle$ is $\subseteq$-increasing and continuous, and that $|E_\gamma| < \omega_n$ for all $\gamma < \omega_n$.

For all $\gamma \leq \omega_n$, let $\mathcal{P}^\gamma$ denote the set of all alternating $(n-1)$-families of the form

$$\Psi = \left\langle \psi_a : E_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H \;\middle|\; a \in \gamma^{n-1} \right\rangle.$$

If $\gamma < \delta \leq \omega_n$, $\Psi \in \mathcal{P}^\gamma$, and $\Psi' \in \mathcal{P}^\delta$, then we say that Ψ' *extends* Ψ, written $\Psi' \trianglerighteq \Psi$, if, for all $a \in \gamma^{n-1}$, we have $\psi_a = \psi'_a \restriction (E_\gamma \cap \bigcap_{i<n-1} x_{a(i)})$.

Claim 2.30. *For all $\gamma < \omega_n$ and all $\Psi \in \mathcal{P}^\gamma$, there is at most one $\sigma \in \mathcal{S} \cap {}^\gamma 2$ such that $\Psi \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction E_\gamma)$.*

Proof. Suppose for the sake of contradiction that there are $\gamma < \omega_n$, $\Psi \in \mathcal{P}_\gamma$, and distinct $\sigma_0, \sigma_1 \in \mathcal{S} \cap {}^\gamma 2$ such that $\Psi \in \mathrm{Triv}_H(\Phi^{\sigma_\ell} \restriction\restriction E_\gamma)$ for $\ell < 2$. Let $\beta < \gamma$ be the least ordinal at which σ_0 and σ_1 disagree; without loss of generality, assume that $\sigma_\ell(\beta) = \ell$ for $\ell < 2$.

By the definition of $\mathcal{S}$, we must have $\beta \in \mathrm{Lim}(C) \cap S^n_{n-1}$. Let $\tau = \sigma_0 \restriction \beta = \sigma_1 \restriction \beta$. In the construction, we fixed a $\Psi^\tau \in \mathrm{Triv}_H(\Phi^\tau \restriction\restriction x_\beta)$ and a nontrivial coherent $(n-1)$-family T^β, both supported on $\bar{E}_\beta$, such that $\Phi^{\tau,0} = \Phi^\tau{}^\frown\langle \Psi^\tau\rangle$ and $\Phi^{\tau,1} = \Phi^\tau{}^\frown\langle \Psi^\tau + T^\beta\rangle$. Then our assumptions imply that, for all $a \in \beta^{n-1}$, we have

$$\begin{aligned}(-1)^{n+1}(\psi^\tau_a) \restriction E_\gamma &= \varphi^{\sigma_0}_{a^\frown\langle\beta\rangle} \restriction E_\gamma \\ &=^* (-1)^{n-1}\psi_a \restriction E_\gamma + \sum_{i<n-1}(-1)^i \psi_{a^i{}^\frown\langle\beta\rangle} \restriction E_\gamma \quad =^* \varphi^{\sigma_1}_{a^\frown\langle\beta\rangle} \restriction E_\gamma \\ &= (-1)^{n+1}(\psi^\tau_a + \tau^\beta_a) \restriction E_\gamma.\end{aligned}$$

In particular, from the first and last terms of the above equation we see that $\tau^\beta_a \restriction E_\gamma =^* 0$, contradicting the fact that T^β is a nontrivial coherent $(n-1)$-family supported on $\bar{E}_\beta \subseteq E_\gamma$. □

Fix a coding function G with domain ${}^{\leq\omega_n}2$ and a club $D \subseteq \mathrm{Lim}(C)$ with the following properties:

- for all $\gamma \in D \cup \{\omega_n\}$, $G \restriction {}^\gamma 2$ is a surjection from ${}^\gamma 2$ to $\mathcal{P}^\gamma$ (this is where we use the assumption that $|H| \leq 2^{\omega_1}$);
- for all $\gamma < \delta$, both in $D \cup \{\omega_n\}$, and all $\sigma \in {}^\gamma 2$ and $\sigma' \in {}^\delta 2$, if $\sigma' \sqsupseteq \sigma$, then $G(\sigma') \trianglerighteq G(\sigma)$.

As long as all intervals between successive elements of D have size ω_{n-1}, it is routine to construct such a function G, using the fact that $|\mathcal{P}_\gamma| \leq 2^{\omega_{n-1}}$ for each $\gamma < \omega_n$.

Now define a function $F : {}^{<\omega_n}2 \to 2$ as follows. If $\gamma \notin D \cap S^n_{n-1}$, simply let $F \restriction {}^\gamma 2 = 0$. Suppose now that $\gamma \in D \cap S^n_{n-1}$ and $\rho \in {}^\gamma 2$. If there does not exist $\sigma \in \mathcal{S} \cap {}^\gamma 2$ such that $G(\rho) \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction E_\gamma)$, then let $F(\rho) = 0$. If there is $\sigma \in \mathcal{S} \cap {}^\gamma 2$ such that $G(\rho) \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction E_\gamma)$, then, by Claim 2.30, there is a unique such σ; denote this by $\sigma(\rho)$. By Proposition 2.18 and the construction of $\Phi^{\sigma(\rho),\ell}$, there is at most one $\ell < 2$ such that $G(\rho)$ extends to an element of $\mathrm{Triv}_H(\Phi^{\sigma(\rho),\ell})$. More precisely, if $G(\rho) = \Psi$, then there is at most one $\ell < 2$ for which there exists a $\Psi' \in \mathrm{Triv}_H(\Phi^{\sigma(\rho),\ell})$ such that, for all $a \in \gamma^{n-1}$, we have $\psi'_a \restriction E_\gamma = \psi_a$. Then choose $F(\rho) \in 2$ so that $G(\rho)$ does *not* extend to an element of $\mathrm{Triv}_H(\Phi^{\sigma(\rho),\ell})$.

Let $g \in {}^{\omega_n}2$ witness the instance of $\mathrm{w}\diamondsuit(S^n_{n-1})$ associated with F. Since this only depends on the values g takes on S^n_{n-1}, or any relatively club subset thereof, we can assume that $g \in \mathcal{S}$. We claim that $\Phi^g = \langle \varphi_u \mid u \in (\omega_n)^n\rangle$ is nontrivial. Suppose for the sake of contradiction that $\Psi = \langle \psi_a \mid a \in (\omega_n)^{n-1}\rangle$ trivializes Φ^g. By the construction of G, we can find a function $f \in {}^{\omega_n}2$ such that, for all $\gamma \in D \cup \{\omega_n\}$, we have

$$G(f \restriction \gamma) = \left\langle \psi_a \restriction E_\gamma \cap \bigcap_{i<n-1} a_{v(i)} \;\middle|\; a \in \gamma^{n-1} \right\rangle.$$

Moreover, for all $\gamma \in D$, it follows that $G(f \restriction \gamma)$ trivializes $\Phi^{g\restriction\gamma} \restriction\restriction E_\gamma$.

By our choice of g, we can find $\gamma \in D \cap S^n_{n-1}$ such that $g(\gamma) = F(f \restriction \gamma)$. By our construction of F, we then know that $\sigma(f \restriction \gamma) = g \restriction \gamma$ and that $G(f \restriction \gamma)$ does not extend to an element of $\mathrm{Triv}_H(\Phi^{g\restriction(\gamma+1)})$. Therefore, *a fortiori*, $G(f \restriction \gamma)$ does not extend to an element of $\mathrm{Triv}_H(\Phi^g)$. But this is contradicted by the fact that Ψ extends $G(f \restriction \gamma)$ and is an element of $\mathrm{Triv}_H(\Phi^g)$. This contradiction shows that Φ^g is indeed nontrivial, thus completing the proof of the theorem. □

We can get rid of the weak diamond assumptions in the previous theorem at the cost of increasing the size of the group into which we are mapping. For a group H and an ordinal β, we write $H^{(\beta)}$ to denote the direct sum of β-many copies of H; concretely, this is the group consisting of all finitely-supported functions $f : \beta \to H$. If $\alpha < \beta$, then we consider $H^{(\alpha)}$ as a subgroup of $H^{(\beta)}$ in the obvious way. If we have fixed a particular nonzero element $1_H \in H$, then, for $\eta < \beta$, we let 1_η denote the element of $H^{(\beta)}$ whose support is precisely $\{\eta\}$ and takes value 1_H at η.

Theorem 2.31. *Suppose that $1 \leq n < \omega$, H is a nonzero abelian group, and $\vec{x} = \langle x_\alpha \mid \alpha < \omega_n\rangle$ is an ascending sequence of sets. Then there is a nontrivial coherent n-family*

$$\Phi = \left\langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H^{(\omega_n)} \;\middle|\; b \in (\omega_n)^n \right\rangle.$$

Moreover, if $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_n)\rangle$ and $C \subseteq \omega_n$ witness that $\vec{x}$ is ascending, then we can arrange so that Φ is supported on $\vec{e}$.

Proof. Fix for the remainder of the proof an arbitrary nonzero element of H, and denote it by 1_H. The proof is by induction on n. Suppose first that $n = 1$, and fix $\vec{e}$ and C witnessing that $\vec{x}$ is ascending. We will construct a nontrivial coherent 1-family $\Phi = \langle \varphi_\beta : x_\beta \to H^{(\omega_1)} \mid \beta < \omega_1\rangle$ that is supported on $\vec{e}$ by recursion on β, maintaining the recursive hypothesis that, for all $\beta < \omega_1$, φ_β maps into $H^{(\beta+1)}$ (i.e., for all $z \in x_\beta$, the support of $\varphi_\beta(z)$ is a subset of $\beta + 1$).

Suppose that $\beta < \omega_1$ and we have constructed $\Phi^\beta = \langle \varphi_\alpha \mid \alpha < \beta\rangle$. As before, let $E_\beta = \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\beta)\}$ and $\bar{E}_\beta = E_\beta \cap x_\beta$. Since $\beta < \omega_1$, Proposition 2.7 implies that $\Phi_\beta \restriction\restriction x_\beta$ is trivial. Fix a trivialization $\psi^\beta : x_\beta \to H^{(\omega_1)}$. By our recursive assumption, $\Phi_\beta \restriction\restriction x_\beta$ is supported on $\bar{E}_\beta$ and, for each $\alpha < \beta$, φ_α maps into $H^{(\beta)}$. Therefore, we can assume that ψ^β is also supported on $\bar{E}_\beta$ and maps into $2^{(\beta)}$. If $\beta \notin \mathrm{Lim}(C)$, then simply let $\varphi_\beta = \psi^\beta$; this readily maintains the recursive hypotheses.

Suppose now that $\beta \in \mathrm{Lim}(C)$. Precisely as in the proof of Theorem 2.27, fix a set $y_\beta \in [\bar{E}_\beta]^\omega$ such that, for all $\alpha < \beta$, $y_\beta \cap x_\alpha$ is finite. Now define φ_β by setting, for all $z \in x_\beta$,

$$\varphi_\beta(z) = \begin{cases} \psi^\beta(z) & \text{if } z \notin y_\beta \\ 1_\beta & \text{if } z \in y_\beta. \end{cases}$$

Since $y_\beta \cap x_\alpha$ is finite for all $\alpha < \beta$, this maintains the coherence of Φ. It is easily verified that all other recursive requirements are satisfied by this definition. This completes the construction of Φ.

We claim that Φ is nontrivial. Suppose for the sake of contradiction that $\psi : \bigcup\{x_\beta \mid \beta < \omega_1\} \to H^{(\omega_1)}$ trivializes Φ. Note that $\langle E_\beta \mid \beta < \omega_1\rangle$ is a continuous $\subseteq$-increasing sequence of countable sets. Therefore, since, for each $z \in \mathrm{dom}(\psi)$, $\mathrm{supp}(\psi(z))$ is a finite subset of ω_1, we can find $\beta \in \mathrm{Lim}(C)$ such that $\psi \restriction E_\beta$ maps into $H^{(\beta)}$. However, when we constructed φ_β, we found an infinite subset $y_\beta \subseteq \bar{E}_\beta$ and ensured that $\beta \in \mathrm{supp}(\varphi_\beta(z))$ for all $z \in y_\beta$. This implies that $\psi \restriction x_\beta \neq^* \varphi_\beta$, contradicting the assumption that ψ trivializes Φ.

Suppose now that $1 < n < \omega$ and we have established the theorem for all $1 \leq m < n$. Again fix $\vec{e}$ and C witnessing that $\vec{x}$ is ascending. For notational convenience, we will index our construction of a nontrivial coherent n-family by $[\omega_n]^n$, whose elements we may think of as strictly increasing sequences of length n, rather than by the full $(\omega_n)^n$. Recall that this involves no loss of generality by Remark 2.2. We will construct a nontrivial coherent n-family $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H^{(\omega_n)} \mid b \in [\omega_n]^n\rangle$ by recursion on $\max(b)$. We will ensure that Φ is supported on $\vec{e}$ and will maintain the recursive hypothesis that, for all $b \in [\omega_n]^n$, φ_b maps into $H^{(\max(b)+\omega_{n-1})}$.

Fix $\gamma < \omega_n$ and suppose that we have specified $\Phi^\gamma = \langle \varphi_b \mid b \in [\gamma]^n\rangle$. We describe how to specify $\varphi_{a\cup\{\gamma\}}$ for $a \in [\gamma]^{n-1}$. As before, let $E_\gamma := \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\gamma)\}$, and let $\bar{E}_\gamma := E_\gamma \cap x_\gamma$. Since $\mathrm{cf}(\gamma) < \omega_n$, Proposition 2.7 implies that $\Phi^\gamma \restriction\restriction x_\gamma$ is trivial. Fix $\Psi^\gamma = \langle \psi^\gamma_a : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H^{(\omega_{n+1})} \mid a \in [\gamma]^{n-1}\rangle$ in $\mathrm{Triv}_{H^{(\omega_n)}}(\Phi^\gamma \restriction\restriction x_\gamma)$. Since $\Phi^\gamma \restriction\restriction x_\gamma$ is supported on $\bar{E}_\gamma$ and, for each $b \in [\gamma]^n$, φ_b maps into $H^{(\gamma+\omega_{n-1})}$, we can assume that Ψ^γ is also supported on $\bar{E}_\gamma$ and, for each $a \in [\gamma]^{n-1}$, ψ^γ_a maps into $H^{(\gamma+\omega_{n-1})}$. Moreover, if $\mathrm{cf}(\gamma) = \omega_{n-1}$, then we in fact know that, for each $b \in [\gamma]^n$, φ_b maps into $H^{(\gamma)}$, so we can require that ψ^γ_a does for each $a \in [\gamma]^{n-1}$ as well.

If $\gamma \notin \mathrm{Lim}(C) \cap S^n_{n-1}$, then simply let $\varphi_{a\cup\{\gamma\}} = (-1)^{n+1}\psi^\gamma_a$ for all $a \in [\gamma]^{n-1}$. Suppose now that $\gamma \in \mathrm{Lim}(C) \cap S^n_{n-1}$. Recall in this case that ψ^γ_a maps into $2^{(\gamma)}$ for each $a \in [\gamma]^{n-1}$. First, exactly as in the proof of Theorem 2.28, the inductive hypothesis implies that we can find a nontrivial coherent $(n-1)$-family

$$T^\gamma = \left\langle \tau^\gamma_a : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H^{(\omega_{n-1})} \;\middle|\; a \in [\gamma]^{n-1} \right\rangle$$

that is supported on $\bar{E}_\gamma$. The idea now is to add a "shifted" version of T^γ to Ψ^γ. More precisely, for each $a \in [\gamma]^{n-1}$, define a function

$$\mathrm{sh}_\gamma(\tau^\gamma_a) : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H^{(\gamma+\omega_{n-1}))}$$

by setting, for all $z \in x_\gamma \cap \bigcap_{i<n-1} x_{a(i)}$ and all $\eta < \gamma + \omega_{n-1}$,

$$\mathrm{sh}_\gamma(\tau^\gamma_a)(z)(\eta) = \begin{cases} 0 & \text{if } \eta < \gamma \\ \tau^\gamma_a(z)(\xi) & \text{if } \eta = \gamma + \xi. \end{cases}$$

Then let $\varphi_{a\cup\{\gamma\}} = (-1)^{n+1}(\psi^\gamma_a + \mathrm{sh}_\gamma(\tau^\gamma_a)))$ for all $a \in [\gamma]^{n-1}$. It is easily verified, using the coherence of T^γ and the fact that Ψ^γ trivializes $\Phi^\gamma \restriction\restriction x_\gamma$, that this maintains coherence and all of the other requirements of the recursive construction.

We claim that the family Φ thus constructed is nontrivial. For the sake of contradiction, suppose that it is trivial, and fix a $\Psi = \langle \psi_a \mid a \in [\omega_n]^{n-1}\rangle$ in $\mathrm{Triv}_{H^{(\omega_n)}}(\Phi)$. Find $\gamma \in \mathrm{Lim}(C) \cap S^n_{n-1}$ such that, for all $a \in [\gamma]^n$, ψ_a maps into $H^{(\gamma)}$.

The argument now differs slightly depending on whether $n = 2$ or $n > 2$. Suppose first that $n = 2$. Define a function $\rho : x_\gamma \to H^{(\omega_{n-1})}$ by setting, for all $z \in x_\gamma$ and all $\xi < \omega_{n-1}$, $\rho(z)(\xi) = -\psi_\gamma(z)(\gamma + \xi)$. By our choice of γ and the construction of Φ, we know that, for all $\alpha < \gamma$, we have

- $\psi_\gamma - \psi_\alpha =^* \varphi_{\alpha\gamma}$;
- ψ_α maps into $H^{(\gamma)}$;
- for all $z \in x_\alpha \cap x_\gamma$ and all $\xi < \omega_{n-1}$, $\varphi_{\alpha\gamma}(z)(\gamma + \xi) = -\tau^\gamma_\alpha(z)(\xi)$.

Putting this all together, we see that $\rho \restriction (x_\alpha \cap x_\gamma) =^* \tau^\gamma_\alpha$ for all $\alpha < \gamma$, and hence ρ witnesses that T^γ is trivial, contradicting our choice of T^γ.

Suppose next that $n > 2$. For each $d \in [\gamma]^{n-2}$, define a function $\rho_d : x_\gamma \cap \bigcap_{i<n-2} x_{d(i)} \to H^{(\omega_{n-1})}$ by setting, for all $z \in x_\gamma \cap \bigcap_{i<n-2} x_{d(i)}$ and all $\xi < \omega_{n-1}$, $\rho_d(z)(\xi) = (-1)^{n+1}\psi_{d\cup\{\gamma\}}(z)(\gamma + \xi)$. By our choice of Φ, we know that, for all $a \in [\gamma]^{n-1}$, we have

- $(-1)^{n-1}\psi_a + \sum_{i<n-1}(-1)^i \psi_{a^i\cup\{\gamma\}} =^* \varphi_{a\cup\{\gamma\}}$;
- ψ_a maps into $H^{(\gamma)}$;
- for all $z \in x_\gamma \cap \bigcap_{i<n-1} x_{a(i)}$ and all $\xi < \omega_{n-1}$,

$$\varphi_{a\cup\{\gamma\}}(z)(\gamma + \xi) = (-1)^{n+1}\tau^\gamma_a(z)(\xi).$$

Putting this together, we see that, for all $a \in [\gamma]^{n-1}$, we have

$$\sum_{i<n-1} (-1)^i \rho_{a^i} =^* \tau^\gamma_a,$$

and hence $\langle \rho_d \mid d \in [\gamma]^{n-2}\rangle$ witnesses that T^γ is trivial, contradicting our choice of T^γ and completing the proof of the theorem. □

2.4 Ideals and their characteristics

In this section, we isolate some properties of ideals $\mathcal{I}$ that ensure that they carry $\subseteq^*$-cofinal ascending sequences; this will then yield a proof of Theorem A.2.

The following cardinal characteristics associated with ideals were introduced in [78] in the context of tall ideals on ω. We reproduce the definition here in a more general context.

Definition 2.32. *Suppose that $\mathcal{I}$ is an ideal on a set X properly extending the ideal of all finite subsets of X with no $\subseteq^*$-maximal set.*

- $\mathrm{cof}^*(\mathcal{I}) = \min\{|\mathcal{Y}| \mid \mathcal{Y} \subseteq \mathcal{I} \wedge (\forall x \in \mathcal{I})(\exists y \in \mathcal{Y})(x \subseteq^* y)\}$.
- $\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{E}| \mid \mathcal{E} \subseteq \mathcal{I} \cap [X]^\omega \wedge (\forall x \in \mathcal{I})(\exists e \in \mathcal{E})(|e \cap x| < \aleph_0)\}$.[2]

The next lemma demonstrates the relevance of these characteristics to our results by showing that, if $\mathrm{cof}^*(\mathcal{I}) = \mathrm{non}^*(\mathcal{I})$, then $\mathcal{I}$ carries an ascending $\subseteq^*$-cofinal sequence.

Lemma 2.33. *Suppose that $\mathcal{I}$ is an ideal on a set X properly extending the ideal of all finite subsets of X with no $\subseteq^*$-maximal set, and suppose that $\mathrm{cof}^*(\mathcal{I}) = \mathrm{non}^*(\mathcal{I}) = \kappa$. Then there is an ascending sequence $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ of elements of $\mathcal{I}$ that is cofinal in $(\mathcal{I}, \subseteq^*)$.*

Proof. Let $\langle z_\alpha \mid \alpha < \kappa\rangle$ enumerate a $\subseteq^*$-cofinal subset of $\mathcal{I}$. We will simultaneously construct $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ and $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\kappa)\rangle$ by recursion on α and $\max(\mathrm{dom}(w))$ such that each x_α is in $\mathcal{I}$ and such that $\langle e_w \mid w \in \mathrm{Fn}^+(\kappa)\rangle$ and κ witness that $\vec{x}$ is ascending. We will ensure that, for all $\alpha < \kappa$, we have $z_\alpha \subseteq x_\alpha$, which will in turn ensure that $\vec{x}$ is $\subseteq^*$-cofinal in $\mathcal{I}$. To this end, suppose that $\beta < \kappa$ and we have constructed $\vec{x} \restriction \beta$ and $\vec{e} \restriction \mathrm{Fn}^+(\beta)$ in such a way that $\vec{e} \restriction \mathrm{Fn}^+(\beta)$ and β witness that $\vec{x} \restriction \beta$ is ascending. We describe how to choose x_β and $\langle e_w \mid w \in \mathrm{Fn}^+(\beta+1) \wedge \beta \in \mathrm{dom}(w)\rangle$.

First, let $W_\beta = \{w \in \mathrm{Fn}^+(\beta) \mid \vec{x} \restriction \beta \text{ respects } w\}$, and let

$$\mathcal{E}_\beta = \{e_w \cap \bigcap_{\alpha\in a} x_\alpha \mid w \in W_\beta,\ a \in [\beta]^{<\omega},\ \text{and } \mathrm{dom}(w) < a\}.$$

By the inductive hypothesis, we know that $\mathcal{E}_\beta \subseteq [X]^\omega$. Moreover, since $\beta < \kappa = \mathrm{non}^*(\mathcal{I})$, we can find $x_{\beta,0} \in \mathcal{I}$ such that $e \cap x_{\beta,0}$ is infinite for all $e \in \mathcal{E}_\beta$. Since $\beta < \kappa = \mathrm{cof}^*(\mathcal{I})$, we can find $x_{\beta,1} \in \mathcal{I}$ such that, for all $a \in [\beta]^{<\omega}$, we have $x_{\beta,1} \not\subseteq^* \bigcup_{\alpha\in a} x_\alpha$.

Let $x_\beta = x_{\beta,0} \cup x_{\beta,1} \cup z_\beta$. Our choice of $x_{\beta,0}$ ensures that this maintains clause (2)(c) of Definition 2.25, and our choice of $x_{\beta,1}$ ensures that this maintains clause (2)(b) of the same definition. Now, for every $w \in \mathrm{Fn}^+(\beta+1)$ such that $\beta \in \mathrm{dom}(w)$ and $\vec{x} \restriction (\beta+1)$ respects w, let e_w be an arbitrary element of $[d_w^{\vec{x}\restriction(\beta+1)}]^{\aleph_0}$. This completes the inductive step of the construction and hence the proof of the lemma. □

[2] In [78] the set $\mathcal{E}$ is only assumed to be a subset of $[X]^\omega$, not $\mathcal{I} \cap [X]^\omega$. In the context of tall ideals on ω this does not change the definition; in the general context, the definition we give here seems the more useful one, at least for our purposes.

An ideal of particular interest to us is the ideal $\emptyset \times \mathrm{Fin}$, which can be concretely defined as an ideal on $\omega \times \omega$ in the following way. First, for all functions $f : \omega \to \omega$, recall that $I_f = \{(k, n) \in \omega \times \omega \mid n < f(k)\}$. Then $\emptyset \times \mathrm{Fin}$ is the set

$$\{x \subseteq \omega \times \omega \mid (\exists f \in {}^{\omega}\omega)(x \subseteq I_f)\}.$$

Recall that $\mathfrak{d}$ denotes the *dominating number*, i.e., the least size of a cofinal subset of $({}^{\omega}\omega, \leq^*)$.

Proposition 2.34. $\mathfrak{d} = \mathrm{cof}^*(\emptyset \times \mathrm{Fin}) = \mathrm{non}^*(\emptyset \times \mathrm{Fin})$.

Proof. The fact that $\mathfrak{d} = \mathrm{cof}^*(\emptyset \times \mathrm{Fin})$ follows immediately from the definitions. Let us show that $\mathfrak{d} = \mathrm{non}^*(\emptyset \times \mathrm{Fin})$. First, suppose that $\mathcal{F} \subseteq {}^{\omega}\omega$ has size $\mathfrak{d}$ and is cofinal in $({}^{\omega}\omega, \leq^*)$. For each $f \in {}^{\omega}\omega$, let I_f^+ denote the graph of f, i.e., the set $\{(k, f(k)) \mid k < \omega\}$. Let $\mathcal{E} = \{I_f^+ \mid f \in \mathcal{F}\}$. We claim that $\mathcal{E}$ witnesses that $\mathrm{non}^*(\emptyset \times \mathrm{Fin}) \leq \mathfrak{d})$. To see this, fix an arbitrary $x \in \emptyset \times \mathrm{Fin}$. Then there is $g \in {}^{\omega}\omega$ such that $x \subseteq I_g$. Then $I_g^+ \in \mathcal{E}$ and $I_g \cap I_g^+ = \emptyset$. Thus, $\mathrm{non}^*(\emptyset \times \mathrm{Fin}) \leq \mathfrak{d}$.

For the other inequality, suppose that $\mathcal{E}$ is a family of infinite elements of $\emptyset \times \mathrm{Fin}$ with $|\mathcal{E}| < \mathfrak{d}$. We must find $x \in \emptyset \times \mathrm{Fin}$ such that $|e \cap x| = \aleph_0$ for all $e \in \mathcal{E}$. For each $e \in \mathcal{E}$, fix $f_e \in {}^{\omega}\omega$ such that $e \subseteq I_{f_e}$. Let $a_e = \{k < \omega \mid (\exists m < \omega)((k, m) \in e)\}$. Since e is an infinite element of $\emptyset \times \mathrm{Fin}$, a_e must be infinite. Moreover, a_e partitions ω into pairwise disjoint intervals $\langle J_n^e \mid n < \omega \rangle$, where $J_0^e = [0, a_e(0)]$ and, for all $n < \omega$, $J_{n+1}^e = (a_e(n), a_e(n+1)]$. Define a function $g_e \in {}^{\omega}\omega$ as follows. For all $k < \omega$, let $n < \omega$ be such that $k \in J_n^e$, and then let $g_e(k) = f_e(a_e(n))$.

Using the fact that $|\mathcal{E}| < \mathfrak{d}$, find $h \in {}^{\omega}\omega$ such that, for all $e \in \mathcal{E}$, we have $h \not\leq^* g_e$. By increasing h if necessary, we may assume that it is weakly increasing, i.e., $h(k_0) \leq h(k_1)$ for all $k_0 < k_1 < \omega$. For each $e \in \mathcal{E}$, let $b_e := \{k < \omega \mid g_e(k) < h(k)\}$, and let $c_e := \{n < \omega \mid b_e \cap J_n^e\} \neq \emptyset$. Since each J_n^e is finite, we have $c_e \in [\omega]^{\omega}$. Moreover, for all $n \in c_e$, we can fix $k \in b_e \cap J_n^e$ and conclude that

$$f_e(a_e(n)) = g_e(k) < h(k) \leq h(a_e(n)),$$

and hence there is $m \leq h(a_e(n))$ such that $(a_e(n), m) \in e$. In particular, letting $x = I_h$, it follows that $x \in \emptyset \times \mathrm{Fin}$ and $|e \cap x| = \aleph_0$ for all $e \in \mathcal{E}$. □

Putting the results of this section together with Theorems 2.28 and 2.31 immediately yields the following corollary.

Corollary 2.35. *Suppose that $1 \leq n < \omega$, $\mathcal{I}$ is an ideal on a set X properly extending the ideal of all finite subsets of X with no $\subseteq^*$-maximal set. Suppose also that $\mathrm{cof}^*(\mathcal{I}) = \mathrm{non}^*(\mathcal{I}) = \omega_n$. Then*

1. $\lim^n \mathbf{A}_{\mathcal{I}}[\mathbb{Z}^{(\omega_n)}] \neq 0$;

2. *if* $\mathrm{w}\diamondsuit(S^{k+1}_k)$ *holds for all* $k < n$*, then* $\lim^n \mathbf{A}_{\mathcal{I}} \neq 0$.

Setting $\mathcal{I} = \emptyset \times \mathrm{Fin}$ and invoking Proposition 2.34 and Fact 2.22 then yields the following important special case, which is Theorem A.2 from the introduction.

Corollary 2.36. *Suppose that* $1 \leq n < \omega$ *and* $\mathfrak{d} = \omega_n$*. Then*

1. $\lim^n \mathbf{A}[\mathbb{Z}^{(\omega_n)}] \neq 0$*;*
2. *if* $\mathrm{w}\diamondsuit(S^{k+1}_k)$ *holds for all* $k < n$*, then* $\lim^n \mathbf{A} \neq 0$.

In particular, if $\lim^n \mathbf{A}[H] = 0$ *for all* $1 \leq n < \omega$ *and all abelian groups* H*, then* $2^{\aleph_0} > \aleph_\omega$.

We end this section by noting one other important special case. Given a regular uncountable cardinal κ, let $\mathcal{I}_\kappa$ denote the ideal $[\kappa]^{<\kappa}$. It is easily verified that $\mathrm{cof}^*(\mathcal{I}_\kappa) = \mathrm{non}^*(\mathcal{I}_\kappa) = \kappa$. Our framework therefore immediately yields the following corollary.

Corollary 2.37. *Suppose that* $1 \leq n < \omega$*. Then*

1. $\lim^n \mathbf{A}_{\mathcal{I}_\kappa}[\mathbb{Z}^{(\omega_n)}] \neq 0$*;*
2. *if* $\mathrm{w}\diamondsuit(S^{k+1}_k)$ *holds for all* $k < n$*, then* $\lim^n \mathbf{A}_{\mathcal{I}_\kappa} \neq 0$.

This corollary is not new, as the inverse systems $\mathbf{A}_{\mathcal{I}_\kappa}[H]$ have been somewhat extensively studied, although under different names (for instance, in [20] they are denoted $\mathbf{C}(\kappa, H)$). The first derived limits of these systems are also familiar to set theorists, although in a different guise: the assertion "$\lim^1 \mathbf{A}_{\mathcal{I}_\kappa}[\mathbb{Z}] \neq 0$" is equivalent to the existence of a coherent κ-Aronszajn subtree of ${}^{<\kappa}\omega$.

Clause (1) of Corollary 2.37 is implicit in [129]; for a more set-theoretic proof, see [16]. Clause (2) has, to the best of our knowledge, not explicitly appeared anywhere, but it follows from the techniques in [165]. We include the corollary here to indicate that these systems can be incorporated into the general framework we develop in this chapter.

2.5 Simultaneous nonvanishing from squares and weak diamonds

In this final section, we prove Theorem B.2 from the introduction. We first recall the following definition.

Definition 2.38. *Suppose that* λ *is a regular uncountable cardinal and* $S \subseteq \lambda$ *is stationary. The principle* $\square(\lambda, S)$ *asserts the existence of a sequence* $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ *such that:*

1. *for every limit ordinal* $\alpha < \lambda$*,* C_α *is a club in* α*;*

2. *for every limit ordinal* $\alpha < \beta$*, if* $\alpha \in \mathrm{Lim}(C_\beta)$*, then* $C_\beta \cap \alpha = C_\alpha$*;*

3. *for every limit ordinal* $\alpha < \lambda$*,* $C_\alpha \cap S = \emptyset$*.*

Such a sequence is called a $\square(\lambda, S)$*-sequence.*

Remark 2.39. *Condition* (3) *implies that there exists no club* $C \subseteq \lambda$ *such that for all* $\alpha \in \mathrm{Lim}(\lambda)$*,* $C \cap \alpha = C_\alpha$*, i.e., there is no* thread *through* $\vec{C}$*. For a proof, if it were not so, let* $\gamma \in C \cap S$ *and* $\alpha \in \mathrm{Lim}(\lambda) \setminus (\gamma + 1)$*. Then* $\gamma \in C \cap \alpha = C_\alpha$*, a contradiction. In particular, if* $S \subseteq \lambda$ *is stationary, then a* $\square(\lambda, S)$*-sequence is* a fortiori *a* $\square(\lambda)$*-sequence.*

If δ is an ordinal, then a *δ-chain* in $({}^\omega\omega, \leq^*)$ is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \delta \rangle$ such that, for all $\alpha < \beta < \delta$, we have $f_\alpha <^* f_\beta$. If H is an abelian group and $\vec{f}$ is a δ-chain in $({}^\omega\omega, \leq^*)$, then when we speak of an *H-valued n-family indexed over $\vec{f}$*, we mean an n-family of the form

$$\left\langle \varphi_a : \bigcap_{\alpha \in b} I_{f_\alpha} \to H \;\middle|\; a \in [\delta]^n \right\rangle.$$

Lemma 2.40. *Suppose that* $\kappa < \lambda$ *are uncountable regular cardinals,* n *is a positive integer, and* H *is an abelian group with* $|H| \leq 2^\kappa$*. Suppose moreover that*

1. *for every* κ*-chain in* $({}^\omega\omega, \leq^*)$ *there exists a nontrivial coherent* H*-valued* n*-family indexed over that chain;*

2. *there exists a stationary* $S \subseteq S^\lambda_\kappa$ *such that both* $\mathrm{w}\diamondsuit(S)$ *and* $\square(\lambda, S)$ *hold.*

Then for every λ*-chain in* $({}^\omega\omega, \leq^*)$ *there exists a nontrivial coherent* H*-valued* $(n+1)$*-family indexed over that chain.*

Proof. Fix a λ-chain $\vec{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ in $({}^\omega\omega, \leq^*)$ and a $\square(\lambda, S)$-sequence $\langle C_\alpha \mid \alpha < \lambda \rangle$. Let $\mathcal{S}$ be the set of $\sigma \in {}^{<\lambda}2$ such that $\mathrm{supp}(\sigma) \subseteq S$, and let $\mathcal{S}^+$ be the set of $g \in {}^\lambda 2$ such that $\mathrm{supp}(g) \subseteq S$. For all $\sigma \in \mathcal{S}$, we will construct a coherent (and trivial) $(n+1)$-family $\Phi^\sigma = \langle \varphi^\sigma_b : \bigcap_{\alpha \in b} I_{f_\alpha} \to H \mid b \in [\mathrm{lh}(\sigma)]^{n+1} \rangle$. We will arrange so that, if $\sigma, \tau \in \mathcal{S}$ and $\sigma \sqsubseteq \tau$, then $\Phi^\sigma \sqsubseteq \Phi^\tau$. At the end of the proof, we will find $g \in \mathcal{S}$ such that $\Phi^g = \bigcup_{\alpha < \lambda} \Phi^{g \restriction \alpha}$ is nontrivial. The construction is by recursion on $\mathrm{lh}(\sigma)$, and we will use $\vec{C}$ to ensure coherence by maintaining the following recursion hypothesis for each $\sigma \in \mathcal{S}$:

$$(\dagger) \qquad \forall \gamma \in \mathrm{Lim}(\mathrm{lh}(\sigma)) \setminus S \; \forall \beta \in \mathrm{Lim}(C_\gamma) \; \forall a \in [\beta]^n \left[\varphi^\sigma_{a \cup \{\beta\}} =^* \varphi^\sigma_{a \cup \{\gamma\}} \right]$$

We now turn to the construction. First, if $\sigma \in \mathcal{S}$, $\mathrm{lh}(s) = \delta \in \mathrm{Lim}(\lambda)$, and $\Phi^{\sigma \restriction \alpha}$ has been constructed for all $\alpha < \delta$, then let $\Phi^\sigma = \bigcup_{\alpha < \delta} \Phi^{\sigma \restriction \alpha}$.

It remains to describe the successor step of the construction. Thus, suppose that $\sigma \in \mathcal{S}$, $\mathrm{lh}(\sigma) = \delta$, and we have constructed Φ^σ. We describe how to construct $\Phi^{\sigma,\ell}$ for $\ell < 2$ if $\delta \in S$ and $\ell = 0$ otherwise. We distinguish between a number of cases.

Case 1: δ **is a successor ordinal**. In this case there is nothing new to check with respect to (†), so any coherent extension $\Phi^{\sigma,0}$ of Φ^σ will do. Such an extension exists by Theorem 2.7 and Remark 2.17.

Case 2: $\delta \in \mathrm{Lim}(\lambda) \setminus S$ **and** $\delta > \sup(\mathrm{Lim}(C_\delta))$. Let $\gamma = \sup(\mathrm{Lim}(C_\delta))$. In the present case, $\gamma = \max(\mathrm{Lim}(C_\delta))$ and $\mathrm{otp}(C_\delta \setminus \gamma) = \omega$, thus witnessing that $\mathrm{cf}(\delta) = \omega$. Hence again by Theorem 2.7 there exists $\Psi^\sigma = \langle \psi^\sigma_a \mid a \in [\delta]^n \rangle$ that trivializes Φ^σ.

In order to construct $\Phi^{\sigma,0}$, we need to define $\varphi^{\sigma,0}_{a\cup\{\delta\}}$ for all $a \in [\delta]^n$. Fix such an a if $\gamma \in a$, then set $\varphi^{\sigma,0}_{a\cup\{\delta\}} = 0$. If $\gamma \notin a$, then let $h = |a \setminus \gamma|$, and define $\varphi^{s,0}_{a\cup\{\delta\}} : \bigcap_{i<n} I_{f_{\alpha_i}} \cap I_{f_\delta} \to H$ by setting

$$\varphi^{s,0}_{a\cup\{\delta\}}(x) = \begin{cases} (-1)^h \varphi^\sigma_{a\cup\{\gamma\}}(x) & \text{if } x \in I_{f_\gamma}; \\ (-1)^n \psi^\sigma_a(x) & \text{if } x \notin I_{f_\gamma}. \end{cases}$$

We now verify that this choice of $\Phi^{\sigma,0}$ satisfies the requirements of the construction. For readability, we will omit the superscripts σ and 0 during this verification. To verify condition (†), we must show that, for all $\beta \in \mathrm{Lim}(C_\delta)$ and all $a \in [\beta]^n$, we have $\varphi_{a\cup\{\beta\}} =^* \varphi_{a\cup\{\delta\}}$. Fix such β and a. By the construction of $\Phi^{s,0}$, we know that $\varphi_{a\cup\{\gamma\}} =^* \varphi_{a\cup\{\delta\}}$. By the properties of $\vec{C}$, we know that either $\beta = \gamma$ or $\beta \in \mathrm{Lim}(C_\gamma)$. In the latter case, the fact that Φ^σ satisfies (†) implies that $\varphi_{a\cup\{\beta\}} =^* \varphi_{a\cup\{\gamma\}}$. Since $I_{f_\beta} \subseteq^* I_{f_\gamma} \subseteq^* I_{f_\delta}$, this in turn implies that $\varphi_{a\cup\{\beta\}} =^* \varphi_{a\cup\{\delta\}}$, as desired.

We next verify coherence. It suffices to show that, for all $b \in [\delta]^{n+1}$, we have $(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i \varphi_{b^i\cup\{\delta\}} =^* 0$. Fix such a b. We distinguish between two cases. First suppose that $\gamma \in b$ (say $\gamma = b(k)$, so $|b \setminus \gamma| = n - k$). Then

$$\begin{aligned} (-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i \varphi_{b^i\cup\{\delta\}} &= (-1)^{n+1}\varphi_b + (-1)^k \varphi_{b^k\cup\{\delta\}} \\ &= (-1)^{n+1}\varphi_b + (-1)^k(-1)^{n-k}\varphi_{b^k\cup\{\gamma\}} \\ &= (-1)^{n+1}\varphi_b + (-1)^n \varphi_b = 0. \end{aligned}$$

Suppose on the other hand that $\gamma \notin b$, and let $k = |b \cap \gamma|$ (so $|b \setminus \gamma| = n + 1 - k$). Then,

when restricting to I_{f_γ}, we have

$$\begin{aligned}
&(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\delta\}}\\
=&(-1)^{n+1}\varphi_b + \sum_{i<k}(-1)^i(-1)^{n+1-k}\varphi_{b^i\cup\{\gamma\}} + \sum_{i=k}^{n}(-1)^i(-1)^{n-k}\varphi_{b^i\cup\{\gamma\}}\\
=&(-1)^{n+1-k}\Big((-1)^k\varphi_b + \sum_{i<k}(-1)^i\varphi_{b^i\cup\{\gamma\}} + \sum_{i=k}^{n}(-1)^{i+1}\varphi_{b^i\cup\{\gamma\}}\Big)\\
=&(-1)^{n+1-k}\sum_{i<n+2}(-1)^i\varphi_{(b\cup\{\gamma\})^i} =^* 0.
\end{aligned}$$

Outside of I_{f_γ}, we have

$$\begin{aligned}
(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\delta\}} &= (-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i(-1)^n\psi_{b^i}\\
&=^* (-1)^{n+1}\varphi_b + (-1)^n\varphi_b = 0.
\end{aligned}$$

Thus, we have verified coherence.

Case 3: $\delta \in \mathrm{Lim}(\lambda)\setminus S$ **and** $\sup(\mathrm{Lim}(C_\delta)) = \delta$**.** For $a \in [\delta]^n$, let $\gamma(a) = \min(\mathrm{Lim}(C_\delta)\setminus(\max(a)+1))$. Then, for all $x \in I_{f_\delta}\cap\bigcap_{\alpha\in a} I_{f_\alpha}$, set

$$\varphi^{\sigma,0}_{a\cup\{\delta\}} = \begin{cases}\varphi^\sigma_{a\cup\{\gamma(a)\}}(x) & \text{if } x\in I_{f_{\gamma(a)}}\\ 0 & \text{otherwise.}\end{cases}$$

Note that the "otherwise" case of the above definition only occurs for finitely many $x \in \mathrm{dom}(\varphi^{\sigma,0}_{a\cup\{\delta\}})$.

We verify condition (†), again omitting superscripts for readability. For all $\beta \in \mathrm{Lim}(C_\delta)$ and $a \in [\beta]^n$ we have $\varphi_{a\cup\{\delta\}} =^* \varphi_{a\cup\{\gamma(a)\}} =^* \varphi_{a\cup\{\beta\}}$, where the second equality holds by the induction assumption since either $\gamma(a) = \beta$ or $\gamma(a) \in \mathrm{Lim}(C_\beta) = \mathrm{Lim}(C_\delta)\cap\beta$. Since moreover $I_{f_{\gamma(a)}} \subseteq^* I_{f_\beta} \subseteq^* I_{f_\delta}$, we have $\varphi_{a\cup\{\delta\}} =^* \varphi_{a\cup\{\beta\}}$, as desired.

We now verify coherence. Let $b \in [\delta]^{n+1}$, and let $\gamma = \gamma(b^0)$. Then

$$\begin{aligned}
(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\delta\}} &=^* (-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\gamma\}}\\
&=^* \sum_{i<n+2}(-1)^i\varphi_{(b\cup\{\gamma\})^i} =^* 0,
\end{aligned}$$

where the first equality (mod finite) uses the fact that $\gamma(b^i) = \gamma$ for all $i < n$ and $\varphi_{b^n\cup\{\gamma\}} =^* \varphi_{b^n\cup\{\gamma(b^n)\}}$ by (†), and the last equality holds because Φ^σ satisfies coherence.

Case 4: $\delta \in S$. In this case there is nothing to check with respect to (†), so we only need to produce two coherent families $\Phi^{\sigma,0}$ and $\Phi^{\sigma,1}$. First observe that $\sup(\mathrm{Lim}(C_\delta)) = \delta$, since $\delta \in S \subseteq S^\lambda_\kappa$. We can thus define $\Phi^{\sigma,0}$ exactly as in Case 3.

By the hypothesis of the theorem, every κ-chain in $({}^{\omega}\omega, \leq^*)$ carries a nontrivial coherent H-valued n-family. By Proposition 2.5 and Remark 2.6, the same is true for every chain whose length has cofinality κ. In particular, we can fix a nontrivial coherent H-valued n-family

$$T^\delta = \langle \tau^\delta_a : \bigcap_{\alpha\in a} I_{f_\alpha} \to H \mid a \in [\delta]^n \rangle.$$

Note that, since $I_{f_\alpha} \subseteq^* I_{f_\delta}$ for all $\alpha < \delta$, the restriction $T^\delta \restriction\restriction I_{f_\delta}$ remains nontrivial. Then define $\Phi^{\sigma,1}$ by setting $\varphi^{\sigma,1}_{a\cup\{\delta\}} = \varphi^{\sigma,0}_{a\cup\{\delta\}} + \tau^\delta_a$ for all $a \in [\delta]^n$. The coherence of $\Phi^{\sigma,1}$ then follows from the coherence of $\Phi^{\sigma,0}$ and T^δ.

This completes the construction. We now prove that there is $g \in \mathcal{S}^+$ such that $\Phi^g = \bigcup_{\beta<\lambda} \Phi^{g\restriction\beta}$ is nontrivial. We first record a couple of simple claims.

Claim 2.41. *Suppose that $\delta \in S$ and $\sigma \in {}^{\delta}2$. Then no trivialization of Φ^σ extends both to a trivialization of $\Phi^{\sigma,0}$ and a trivialization of $\Phi^{\sigma,1}$.*

Proof. This follows from the construction of $\Phi^{\sigma,0}$ and $\Phi^{\sigma,1}$ and from Propositions 2.15 and 2.18. □

Claim 2.42. *Let Υ be a n-family over $\vec{f} \restriction \beta$ for some $\beta < \lambda$. Then there exists at most one $\sigma \in {}^{\beta}2$ such that Υ trivializes Φ^σ.*

Proof. This is proven in exactly the same way as Claim 2.30. □

Now we use a coding similar to that used in the proof of Theorem 2.28. More precisely, let $\mathcal{P}^\gamma$ be the set of alternating H-valued n-families indexed over $\vec{f} \restriction \gamma$. Then let G be a function with domain ${}^{\leq\lambda}2$ such that for every $\gamma \in S^\lambda_\kappa \cup \{\lambda\}$ the map $G \restriction {}^{\gamma}2$ is a surjection onto $\mathcal{P}^\gamma$, and such that if $\rho \sqsubseteq \rho' \in \mathrm{dom}(G)$, then $G(\rho) \sqsubseteq G(\rho')$. Such a G exists because the intervals between successive elements of S^λ_κ have size κ, and $|H| \leq 2^\kappa$. We also write Υ^ρ for $G(\rho)$.

Finally, we define $F : {}^{<\lambda}2 \to 2$ as follows. If $\mathrm{lh}(\rho) \in S$ and Υ^ρ trivializes Φ^σ for some σ with $\mathrm{lh}(\sigma) = \mathrm{lh}(\rho)$ (such a σ is unique by Claim 2.42) then we let $F(\rho) = \ell \in \{0,1\}$ be such that Υ^ρ does not extend to a trivialization of $\Phi^{\sigma,\ell}$ (such an ℓ exists by Claim 2.41). Otherwise, let $F(\rho) \in \{0,1\}$ be arbitrary.

By $\mathrm{w}\diamondsuit(S)$ there exists $g : \lambda \to 2$ such that for all $b : \lambda \to 2$ there exists $\alpha \in S$ such that $g(\alpha) = F(b \restriction \alpha)$. Since this only depends on the values that g takes on S, we can assume that $g \in \mathcal{S}^+$. Now suppose for the sake of contradiction that some Υ trivializes $\Phi^g = \bigcup_{\alpha<\lambda} \Phi^{g\restriction\alpha}$. Find b such that $\Upsilon = \Upsilon^b$, and find $\alpha \in S$ such that $g(\alpha) = F(b \restriction \alpha)$. Then $\Upsilon^b \restriction (\vec{f} \restriction \alpha) = \Upsilon^{b\restriction\alpha}$ trivializes $\Phi^g \restriction (\vec{f} \restriction \alpha) = \Phi^{g\restriction\alpha}$. By our choice of F, g, and α, then, no extension of $\Upsilon^{b\restriction\alpha}$ can trivialize $\Phi^{g\restriction(\alpha+1)}$ and hence, *a fortiori*, no extension of $\Upsilon^{b\restriction\alpha}$ can trivialize Φ^g, contradicting the fact that Υ extends $\Upsilon^{b\restriction\alpha}$ and trivializes Φ^g and completing the proof of the theorem. □

We can now show the consistency of the simultaneous nonvanishing of $\lim^k \mathbf{A}$ for many values of k. We first show that for all n it is consistent that $\bigwedge_{2\leq k\leq n} \lim^k \mathbf{A} \neq 0$. This can be realized in the kind of model considered in [165]. Recall that, for a regular cardinal κ and a set I, the poset $\mathrm{Fn}(I, 2, \kappa)$ consists of all partial functions $p : I \to 2$ such that $|p| < \kappa$, ordered by reverse inclusion. Recall also that Hechler forcing $\mathbb{H}$ consists of conditions of the form $p = (s_p, f_p)$ such that s_p is a finite partial function from ω to ω and $f_p \in {}^\omega\omega$. If $p, q \in \mathbb{H}$, then $q \leq p$ if and only if $s_q \supseteq s_p$, $f_q \geq f_p$ and, for all $n \in \mathrm{dom}(s_q) \setminus \mathrm{dom}(s_p)$, we have $s_q(n) \geq f_p(n)$.

Lemma 2.43. *Suppose that $1 \leq n < \omega$. Then there exists a model of ZFC in which*

1. *$\mathfrak{b} = \mathfrak{d} = \omega_n$;*
2. *for all $k < n$, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds;*
3. *for all $k < n$, there exists a stationary set $S \subseteq S^n_k$ such that $\square(\omega_n, S) + \mathrm{w}\diamondsuit(S)$ holds.*

Proof. Our model will be the following forcing extension of L:

$$V' = L^{\mathbb{H}_{\omega_n} \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_n},$$

where $\mathbb{H}_{\omega_n}$ is the finite support iteration of Hechler forcings of length ω_n and $\mathbb{C}_k = \mathrm{Fn}(\omega_{n+k}, 2, \omega_k)$ for all $1 \leq k \leq n$, i.e., $\mathbb{C}_k$ is the forcing to add ω_{n+k}-many Cohen subsets to ω_k.

It follows from [165, Theorem 4.5] that V' has all of the same cardinals and cofinalities as L and satisfies $\mathfrak{b} = \mathfrak{d} = \omega_n$ as well as $\mathrm{w}\diamondsuit(S^{k+1}_k)$ for all $k < n$. To verify item (3) in the statement of the lemma, fix $k < n$, and temporarily work in L. By [91], we can find a stationary subset $S \subseteq S^n_k$ and a $\square(\omega_n, S)$-sequence $\vec{C}$. Let $\mathbb{P} = \mathbb{H}_{\omega_n} \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_{n-1}$. Then $\mathbb{P}$ is an ω_n-cc poset of size ω_{2n-1}, $\mathbb{C}_n$ is ω_n-closed, and $V' = L^{\mathbb{P}\times\mathbb{C}_n}$. In particular, S remains stationary in V'. Since all of the defining properties of $\square(\omega_n, S)$ are upwards absolute from L to V', $\vec{C}$ remains a $\square(\omega_n, S)$-sequence in V'. Moreover, [165, Lemma 4.3] implies that $\mathrm{w}\diamondsuit(S)$ holds in V', so $S \subseteq S^n_k$ is as desired. □

Lemma 2.44. *Assume that*

1. *for all $k < n$, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds;*
2. *for all $1 \leq k < n$, there exists a stationary set $S \subseteq S^n_k$ such that $\square(\omega_n, S) + \mathrm{w}\diamondsuit(S)$ holds.*

Then every ω_n-chain in $({}^\omega\omega, \leq^*)$ carries a nontrivial coherent H-valued k-family, for all $2 \leq k \leq n$ and all countable abelian groups H.

Proof. Fix a countable abelian group H. In ZFC, every ω_1-chain in $({}^\omega\omega, \leq^*)$ carries a nontrivial coherent H-valued 1-family (cf. [165, Proposition 3.4]). Using [165, Lemma 3.6], one

can prove by induction that, for every $1 \leq k \leq n$, every ω_k-chain carries a nontrivial coherent H-valued k-family. Finally, suppose that $2 \leq k \leq n$, and pick $S \subseteq S^{\omega_n}_{\omega_{k-1}}$ such that $\square(\omega_n, S) + \mathrm{w}\diamondsuit(S)$ holds. Then Lemma 2.40 implies that every ω_n-chain carries a nontrivial coherent H-valued k-family, as desired. □

The following theorem is now immediate, yielding clause (1) of Theorem B.2 from the introduction.

Theorem 2.45. *Fix $2 \leq n < \omega$. Relative to the consistency of* ZFC*, it is consistent that $\mathfrak{b} = \mathfrak{d} = \omega_n$ and $\bigwedge_{2\leq k\leq n} \lim^k \boldsymbol{A} \neq 0$ holds.*

Proof. In any model witnessing the conclusion of Lemma 2.43, there exists an ω_n-chain that is cofinal in $({}^\omega\omega, \leq^*)$. Lemma 2.44 then implies that, for all $2 \leq k \leq n$, this ω_n-chain carries a nontrivial coherent $\mathbb{Z}$-valued k-family. Then Proposition 2.4, Remark 2.6, and Fact 2.21 imply that $\lim^k \mathbf{A} \neq 0$ for all $2 \leq k \leq n$. □

We now show that it is consistent that $\lim^n \mathbf{A}$ does not vanish for any $n \geq 2$. By Goblot's Theorem (cf. Proposition 2.7), this requires the dominating number to be at least $\aleph_{\omega+1}$. For technical reasons, we were unable to obtain this simultaneous nonvanishing with $\mathfrak{d} = \aleph_{\omega+1}$, but we were able to achieve it with $\mathfrak{d} = \aleph_{\omega+2}$.

Lemma 2.46. *There exists a model of* ZFC *in which*

1. *$\mathfrak{b} = \mathfrak{d} = \omega_{\omega+2}$;*
2. *for all $k < \omega$, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds;*
3. *for all $k < \omega$, there exists a stationary set $S \subseteq S^{\omega_{\omega+2}}_{\omega_k}$ such that $\square(\omega_{\omega+2}, S) + \mathrm{w}\diamondsuit(S)$ holds.*

Proof. Our model will be the following forcing extension of L:

$$V' = L^{\mathbb{H}\times\prod_{1\leq n<\omega}\mathbb{C}_n\times\mathbb{C}_\omega}$$

satisfies the properties in question, where $\mathbb{H}$ is the finite-support iteration of Hechler forcings of length $\omega_{\omega+2}$, $\mathbb{C}_n = \mathrm{Fn}(\omega_{\omega+n+2}, 2, \omega_n)$ for all $1 \leq n < \omega$, and $\mathbb{C}_\omega = \mathrm{Fn}(\omega_{\omega\cdot 2+2}, 2, \omega_{\omega+2})$.

We first verify clause (1) in the statement of the theorem. Let G be a generic filter for $\mathbb{H} \times \prod_{1\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$. By standard arguments, forcing over L with this product preserves all cardinalities and cofinalities. We can write G as the product $G = G_0 \times G'$ of generic filters on $\mathbb{H}$ and $\prod_{1\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$, respectively. Note that $\prod_{1\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$ is σ-closed, so that in particular it does not add any reals, preserves CH and $\mathbb{H}$ is absolute with respect to it. Therefore, forcing with $\mathbb{H}$ over $L[G']$ forces that $\mathfrak{b} = \mathfrak{d} = \omega_{\omega+2}$, as desired.

We now verify clause (2). Let us further decompose G' as a product $\prod_{1\leq n<\omega} G_n \times G_\omega$, where G_n is generic for $\mathbb{C}_n$ for all $1 \leq n \leq \omega$. Fix some $k < \omega$. Then $\prod_{k+2\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$

is ω_{k+2}-closed, so that $W = L[\prod_{k+2\leq n<\omega} G_n \times G_\omega]$ satisfies $2^{\omega_k} = \omega_{k+1}$. Now apply [165, Lemma 4.3] in W with $\mathbb{P} = \mathbb{H} \times \prod_{j\leq k} \mathbb{C}_k$ and $\mathbb{Q} = \mathbb{C}_{k+1}$ to conclude that $\mathrm{w}\diamondsuit(S)$ holds for every stationary in $S \subseteq \omega_{k+1}$ with $S \in W$; in particular, for $S = S^{k+1}_k$. More precisely, in W, $\mathbb{P}$ is ω_{k+1}-cc and has size $\omega_{\omega+k+2}$, $\mathbb{Q} = \mathrm{Fn}(\omega_{\omega+k+3}, 2, \omega_{k+1})$ and, in W, we have $(\omega_{\omega+k+2})^{\omega_k} = \omega_{\omega+k+2}$, so [165, Lemma 4.3] implies that $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds in the extension of W by $\mathbb{P} \times \mathbb{Q}$, i.e., in V'.

We finally verify clause (3). Fix $k < \omega$, and temporarily work in L. By [91], we can find a stationary subset $S \subseteq S^{\omega_{\omega+2}}_{\omega_k}$ and a $\square(\omega_{\omega+2}, S)$-sequence $\vec{C}$. Since $\mathbb{H} \times \prod_{1\leq n<\omega} \mathbb{C}_n$ is $\omega_{\omega+2}$-cc and $\mathbb{C}_\omega$ is $\omega_{\omega+2}$-closed, S remains stationary in V', so $\vec{C}$ remains a $\square(\omega_{\omega+2}, S)$-sequence in V'. Another application of [165, Lemma 4.3], this time with $\mathbb{P} = \mathbb{H} \times \prod_{1\leq n<\omega} \mathbb{C}_n$ and $\mathbb{Q} = \mathbb{C}_\omega$, implies that $\mathrm{w}\diamondsuit(S)$ holds in V'. □

Lemma 2.47. *Assume that $\lambda \geq \omega_{\omega+1}$ is a regular cardinal and*

1. *for all $1 \leq k < \omega$, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds;*
2. *for all $1 \leq k < \omega$, there is a stationary set $S \subseteq S^\lambda_{\omega_k}$ such that $\square(\lambda, S) + \mathrm{w}\diamondsuit(S)$ holds.*

Then every λ-chain in $({}^\omega\omega, \leq^*)$ carries a nontrivial coherent H-valued k-family for all $2 \leq k < \omega$ and all countable abelian groups H.

Proof. Fix a countable abelian group H. As above, we know that every ω_1-chain in $({}^\omega\omega, leq^*)$ carries a nontrivial coherent H-valued 1-family. Using [165, Lemma 3.6], one can prove by induction that, for all $1 \leq k < \omega$, every ω_k-chain carries a nontrivial coherent H-valued k-family. Finally, fix $2 \leq k < \omega$ and fix a stationary $S \subseteq S^\lambda_{\omega_{k-1}}$ such that $\square(\lambda, S) + \mathrm{w}\diamondsuit(S)$ holds. Then Lemma 2.40 implies that every λ-chain carries a nontrivial coherent H-valued k-family. □

These yield the following result, which is clause (2) of Theorem B.2 from the introduction.

Theorem 2.48. *Relative to the consistency of* ZFC*, it is consistent that $\mathfrak{b} = \mathfrak{d} = \omega_{\omega+2}$ and $\bigwedge_{2\leq k<\omega} \lim^k \mathbf{A} \neq 0$.*

Proof. This follows immediately from Lemmas 2.46 and 2.47, together with the arguments from the proof of Theorem 2.45. □

It is noted in [18] that if one adds $\aleph_2$-many Hechler reals over a model of CH, then $\lim^1 \mathbf{A} = 0$ in the resulting forcing extension. The same argument easily adapts to forcing extensions resulting from adding λ-many Hechler reals to a model of GCH, where $\lambda \geq \aleph_2$ is a regular cardinal. Therefore, the models we construct to witness Theorems 2.45 and 2.48 will satisfy $\lim^1 \mathbf{A} = 0$. The following question therefore remains open.

Question 2.49. *Is $\bigwedge_{1\leq k<\omega} \lim^k \mathbf{A} \neq 0$ consistent with* ZFC*?*

Chapter 3

Higher limits of wider systems

3.1 Nontrivial coherence and the systems $\mathbf{A}_\lambda$

Broadly speaking, the study of the derived limits of $\mathbf{A}_\lambda$ and of related systems is a study of *nontrivial coherence*; let us therefore begin by recalling a classical example of the latter.

Example 3.1. *There exists a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to \omega \mid \alpha < \omega_1 \rangle$ with the following two properties:*

1. *φ_α and $\varphi_\beta \restriction \alpha$ differ only finitely often for every $\alpha < \beta < \omega_1$;*
2. *$\varphi_\alpha^{-1}(i)$ is finite for every $\alpha < \omega_1$ and $i < \omega$.*

Functions φ_α satisfying the weak injectivity condition of item (2) are often called finite-to-one, *for concision. Since no function $\varphi : \omega_1 \to \omega$ is finite-to-one, condition (2) limits the extent of the phenomena of item (1); in particular, any Φ satisfying (1) and (2) satisfies the following as well:*

(2') for every $\varphi : \omega_1 \to \omega$, φ_α and $\varphi \restriction \alpha$ differ infinitely often for some $\alpha < \omega_1$.

Families Φ satisfying conditions (1) and (2') are called coherent *and* nontrivial, *respectively; more precisely, they are nontrivially coherent modulo the ideal of finite subsets of ω_1. In what follows, we denote such* mod finite *relations with an asterisk, recording those as in item (1) as $\varphi_\alpha =^* \varphi_\beta \restriction \alpha$, for instance.*

See [163] for canonical examples of such Φ, as well as for their close connection to a range of infinitary combinatorics and to walks on the ordinals in particular. Such Φ also witness the nonvanishing both of the first sheaf cohomology group of ω_1 and of $\lim^1$ of the inverse system whose terms are the free abelian groups $\mathbb{Z}^{(\alpha)}$ indexed by $\alpha < \omega_1$ and whose transition maps $\pi_{\alpha\beta} : \mathbb{Z}^{(\beta)} \to \mathbb{Z}^{(\alpha)}$ $(\alpha < \beta < \omega_1)$ are the natural projections.

The nonvanishing of a significant range of derived limit expressions traces to basic variations on the above parameters. For example, for any cardinal λ and function $f : \lambda \to \omega$, write $I(f)$ for the set $\{(i, j) \in \lambda \times \omega \mid j \leq f(i)\}$ of coordinate pairs falling below the graph of f (i.e., for the *sottografico* of f; pl. *sottografici*); let also $\leq$ denote the product ordering of such functions, so that $f \leq g$ if and only if $I(f) \subseteq I(g)$.

Definition 3.2. *A family*

$$\langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in {}^\lambda\omega \rangle$$

is coherent *if*

$$\varphi_f =^* \varphi_g \restriction I(f)$$

for all $f \leq g$ in ${}^\lambda\omega$, and is trivial *if there exists a $\varphi : \lambda \times \omega \to \mathbb{Z}$ such that*

$$\varphi_f =^* \varphi \restriction I(f)$$

for all f in ${}^\lambda\omega$.

Henceforth, we will sometimes omit restriction-notations; sums or comparisons of functions should in all cases be interpreted on the intersections of those functions' domains. These notions more concretely relate to the systems $\mathbf{A}_\lambda$ as follows.

Proposition 3.3 ([15]). $\lim^1 \mathbf{A}_\lambda = 0$ *if and only if every coherent family* $\langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in {}^\lambda\omega \rangle$ *is trivial. More precisely, observe that both the collection of coherent such families and its subcollection of trivial such families inherit a group structure from* $\mathbb{Z}$*;* $\lim^1 \mathbf{A}_\lambda$ *is exactly the quotient of the former by the latter.*

Put differently, the instances of *coherence* and *triviality* appearing in Definition 3.2 are, respectively, reframings of cocycle and coboundary conditions on the first term of a cochain complex whose n^{th} cohomology group is the nth derived limit of $\mathbf{A}_\lambda$, and our analyses of higher $\lim^n \mathbf{A}_\lambda$ will center on parallel reframings of higher-dimensional cocycle and coboundary conditions. Let us turn now to more formal definitions, beginning with that of the system $\mathbf{A}_\lambda$ itself.

Definition 3.4. *For any cardinal* λ*,* $\mathbf{A}_\lambda$ *denotes the inverse system of abelian groups indexed by* $({}^\lambda\omega, \leq)$*, with*

- *terms* $A_f := \bigoplus_{I(f)} \mathbb{Z}$*, and*
- *transition maps* $\pi_{f,g} : A_g \to A_f$ *the natural projections*

for every $f \leq g$ *in* ${}^\lambda\omega$*.*

We adopt the convention that $0 \notin \mathbb{N}$. For any quasi-order $(X, \trianglelefteq)$ and $n \in \mathbb{N}$, we write $X^{(n)}$ for the collection of n-tuples $\vec{x} = (x_0, \ldots, x_{n-1})$ in X for which $x_i \trianglelefteq x_j$ for all $i \leq j < n$; for any $j < n$ we then write $\vec{x}^j$ for the $(n-1)$-tuple obtained from $\vec{x}$ by removing x_i.

Definition 3.5. *Fix* $n \in \mathbb{N}$*. If* $n = 1$ *then the family*

$$\Phi = \left\langle \varphi_{\vec{f}} : I(f_0) \to \mathbb{Z} \mid \vec{f} \in ({}^\lambda\omega)^{(n)} \right\rangle$$

is n-coherent *or* n-trivial*, respectively, if it is coherent or trivial in the sense of Definition 3.2.*

When $n > 1$*, the family* Φ *is* n-coherent *if*

$$\sum_{i=0}^{n} (-1)^i \varphi_{\vec{f}^i} =^* 0$$

for all $\vec{f} \in ({}^\lambda\omega)^{(n+1)}$*, and is* n-trivial *if there exists a family*

$$\Psi = \left\langle \psi_{\vec{f}} : I(f_0) \to \mathbb{Z} \mid \vec{f} \in ({}^\lambda\omega)^{(n-1)} \right\rangle$$

such that

$$\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{f}^i} =^* \varphi_{\vec{f}}$$

for all $\vec{f} \in ({}^{\lambda}\omega)^{(n)}$. *We write* $\partial(\Psi)$ *for the* n*-family whose functions are obtained from those of* Ψ *through the alternating sum on the left-hand side. If these equal modulo finite* $\phi_{\vec{f}}$ *for every multi-index* $\vec{f}$ *we write* $\partial(\Psi) = \Phi$ *and say that* Ψ trivializes Φ. *Sometimes we will apply the multi-index as a subscript to the uppercase Greek letter that stands for a certain family to denote the corresponding function.*

Proposition 3.6. *For any cardinal* λ *and* $n \in \mathbb{N}$, $\lim^n \mathbf{A}_\lambda$ *is exactly the quotient of the group of* ${}^{\lambda}\omega$*-indexed* n*-coherent families by the group of* ${}^{\lambda}\omega$*-indexed* n*-trivial families. In particular,* $\lim^n \mathbf{A}_\lambda = 0$ *if and only if every such* n*-coherent family is* n*-trivial.*

See Section 2 of [18], [20], or of [34] for fuller discussions of these characterizations. We conclude this section by observing that analyses of the above relations tend to involve analyses of their restrictions to subsystems; a few last points will therefore be important for us below:

- The relations together described by Definitions 3.4 and 3.5 and Proposition 3.6 all relativize in the obvious manner to any directed suborder of $({}^{\lambda}\omega, \leq)$.
- More particularly, to define an n-coherent family, it will frequently suffice to define it over only a cofinal subset of indices: the essential point here is that such definitions then admit natural n-coherent extensions to the full index-set, and that the latter are n-trivial if and only if the cofinal subfamily is. For further discussion, see [34, Props. 2.4 and 2.5].
- It is frequently convenient to consider, in place of those of Definition 3.5, families indexed by $({}^{\lambda}\omega, \leq^*)^{(n)}$, where the * again signals the *with only finitely many exceptions* modification of a coordinatewise relation. The n-coherence and nontriviality of such families are defined exactly as before, and bear, broadly, the same relation to our other objects of interest (and in particular to the vanishing of $\lim^n \mathbf{A}$) as those of Definition 3.5; see [33, §3], for example, for further discussion.
- If $\mathcal{F}$ is a family of functions and X is a set, then $\mathcal{F} \restriction\restriction X$ denotes the family determined by restricting each function in $\mathcal{F}$ to the intersection of its domain with X.

3.2 Theorem A.3

In this section, we will show that the following partial order satisfies the conclusions of our introduction's Theorem A.3.

Definition 3.7. *Let* $\mathbb{P}$ *denote the forcing poset with*

- ***conditions*** 2*-coherent families*

$$p = \left\langle \varphi^p_{fg} : I(f) \to \mathbb{Z} \mid f \leq g \text{ in } E_p \right\rangle$$

with E_p *a* $\leq$*-directed subset of* ${}^{\omega_1}\omega$ *of cardinality at most* $\aleph_1$,

- ***ordered by*** $p \leq q$ *if and only if* $p \supseteq q$.

Proof of Theorem A.3. By assumption, the continuum hypothesis (CH) holds in our ground model V; hence by Goblot's vanishing theorem, $\lim^2 \mathbf{A}_{\aleph_0} = 0$ does as well. The poset $\mathbb{P}$ is $\aleph_2$-closed in the strong sense that, for any $\alpha < \omega_2$, the union of any descending chain $\langle p_i \mid i < \alpha \rangle$ of conditions forms a lower bound for all of them. $\mathbb{P}$ therefore adds no functions from ω_1 to ω and, in particular, preserves both CH and its consequence $\lim^2 \mathbf{A}_{\aleph_0} = 0$. It is also straightforward to see, again via Goblot's vanishing theorem, that for any $f : \omega_1 \to \omega$ the set $\{p \in \mathbb{P} \mid f \in E_p\}$ is dense in $\mathbb{P}$. Thus the union of a $\mathbb{P}$-generic filter G is, in $V[G]$, a 2-coherent family of the form $\Phi = \langle \varphi_{fg} : I(f) \to \mathbb{Z} \mid f \leq g \text{ in } {}^{\omega_1}\omega \rangle$, and only the following remains to be seen:

Claim 3.8. $\Phi = \cup G$ *is nontrivial in* $V[G]$*; in particular,* $V[G] \vDash \lim^2 \mathbf{A}_{\aleph_0} \neq 0$.

We will argue our claim by way of the lemma just below, which we prove in this section's conclusion. First, fix the following notations: for any $f : \omega_1 \to \omega$ and $\alpha < \omega_1$, define $f^\alpha : \omega_1 \to \omega$ by

$$f^\alpha(\xi) = \begin{cases} f(\xi) & \xi < \alpha \\ 0 & \xi \geq \alpha \end{cases}$$

and let $B = \{h \in {}^{\omega_1}\omega \mid h = f^\alpha \text{ for some } f \in {}^{\omega_1}\omega \text{ and } \alpha < \omega_1\}$. Put differently, B is the set of functions $h : \omega_1 \to \omega$ of bounded support; under the continuum hypothesis, it is of size $\aleph_1$.

Lemma 3.9. *For any* $E \subseteq {}^{\omega_1}\omega$ *of cardinality at most* $\aleph_1$ *there exists a* $g : \omega_1 \to \omega$ *and nontrivial coherent family* $\Theta = \langle \theta_\alpha : I(g^\alpha) \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$ *such that* $\Theta \restriction\restriction I(f)$ *is trivial for every* $f \in E$.

Let us see how the lemma implies Claim 3.8, thereby completing the proof of Theorem A.3. Suppose for contradiction that the claim is false, and hence that

$$p \Vdash \text{“}\dot\Psi = \langle \dot\psi_f : I(f) \to \mathbb{Z} \mid f \in {}^{\omega_1}\omega \rangle \text{ trivializes } \dot\Phi\text{”}$$

for some $p \in G$. As $\mathbb{P}$ is (strongly) $\aleph_2$-closed, we may extend p to a q with $B \subseteq E_q$ and E_q $\wedge$-closed which decides the functions $\dot\psi_f$ for all $f \in E_q$. The contradiction then takes the form of an $r \leq q$ forcing that $\dot\Psi$ does not trivialize $\dot\Phi$; more precisely, r will force that (dropping the dot notation and working in $V[G]$) for some $g \in E_r$, the family

$$\langle \varphi_{g^\alpha g} + \psi_{g^\alpha} \mid \alpha < \omega_1 \rangle$$

is nontrivially coherent. This, though, implies that no value of ψ_g can satisfy $\psi_g \restriction I(g^\alpha) =^* \varphi_{g^\alpha g} + \psi_{g^\alpha}$ (or, equivalently, $\psi_g \restriction I(g^\alpha) - \psi_{g^\alpha} =^* \varphi_{g^\alpha g}$) for all $\alpha < \omega_1$, and thus that Ψ does not trivialize Φ. It is Lemma 3.9 that ensures the existence of such an r: to apply it, take E_q for the E of the lemma's premise; then, taking g and Θ as in its conclusion, let $\varphi^r_{g^\alpha g} = \theta_\alpha - \psi_{g^\alpha}$ for all $\alpha < \omega_1$. It remains then only to extend these assignments, and

those of q, to a 2-coherent family r indexed by a $\leq$-directed $E_r \supseteq E_q \cup \{g\}$ of cardinality $\aleph_1$. In fact E_r will be the $\vee$-closure of $E_q \cup \{g\}$, and we will argue below that it essentially suffices, by our assumptions on E_q, to extend our assignments in a 2-coherent manner to $\varphi^r_{f,f\vee g}$ and $\varphi^r_{g,f\vee g}$ for all $f \in E_q$.

To this end, well-order E_q and consider at stage $f \in E_q$ the family of equations

$$\varphi_{g^\alpha,(f\vee g)^\alpha} + \varphi_{(f\vee g)^\alpha,f\vee g} - \varphi_{g^\alpha,g} - \varphi_{g,f\vee g} =^* 0 \tag{3.1}$$

as α ranges below ω_1. For readability, we've suppressed notation of the restriction of these functions to the intersection of their domains; note that these are equations which any 2-coherent family of functions over such indices must satisfy. Note also that we've already assigned values and relations to these equations' first and third terms, so that the equations (3.1) unpack as

$$\begin{aligned}\varphi_{g,f\vee g} &=^* \varphi_{g^\alpha,(f\vee g)^\alpha} + \varphi_{(f\vee g)^\alpha,f\vee g} - \varphi_{g^\alpha,g}\\ &=^* \psi_{(f\vee g)^\alpha} - \psi_{g^\alpha} + \varphi_{(f\vee g)^\alpha,f\vee g} - (\theta_\alpha - \psi_{g^\alpha})\\ &=^* \psi_{(f\vee g)^\alpha} - \theta_\alpha + \varphi_{(f\vee g)^\alpha,f\vee g}\end{aligned}$$

for every $\alpha < \omega_1$. Assume that $f\vee g \neq f'\vee g$ for any f' appearing earlier in the well-ordering of E_q (if this is false then the above requirements have already been addressed). The last line then consists for each α of two pregiven terms together with a term, $\varphi_{(f\vee g)^\alpha,f\vee g}$, which we are free to define. We must do so so that as α ranges, the resulting expressions form, over the domain $I(g)$, a coherent family trivialized by $\varphi_{g,f\vee g}$. More particularly, our choice must offset both the nontriviality of Θ and the potential noncoherence of $\langle \psi_{(f\vee g)^\alpha} \mid \alpha < \omega_1\rangle$; the natural choices are thus given by

$$\varphi^r_{(f\vee g)^\alpha,f\vee g}(\xi,i) = \begin{cases}\theta_\alpha(\xi,i) - \psi_{(f\vee g)^\alpha}(\xi,i) & \text{if } (\xi,i) \in I(g^\alpha)\\ -\psi_{(f\vee g)^\alpha}(\xi,i) & \text{if } (\xi,i) \in I((f\vee g)^\alpha)\backslash I(g^\alpha)\end{cases} \tag{3.2}$$

These together determine $\varphi_{g,f\vee g}$ up to finite error: the constantly zero function is now the natural choice for $\varphi^r_{g,f\vee g}$.

Next, consider for the same f the family of equations

$$\varphi_{f^\alpha,(f\vee g)^\alpha} + \varphi_{(f\vee g)^\alpha,f\vee g} - \varphi_{f^\alpha,f} - \varphi_{f,f\vee g} =^* 0 \tag{3.3}$$

as α ranges below ω_1. These unpack much as above, but to a system of equations in some tension with the procedure just described. For we now have that

$$\begin{aligned}\varphi_{f,f\vee g} &=^* \varphi_{f^\alpha,(f\vee g)^\alpha} + \varphi_{(f\vee g)^\alpha,f\vee g} - \varphi_{f^\alpha,f}\\ &=^* \psi_{(f\vee g)^\alpha} - \psi_{f^\alpha} + \varphi_{(f\vee g)^\alpha,f\vee g} - (\psi_f - \psi_{f^\alpha})\\ &=^* \psi_{(f\vee g)^\alpha} + \varphi_{(f\vee g)^\alpha,f\vee g} - \psi_f\end{aligned}$$

for every $\alpha < \omega_1$, and under the definitions in (3.2), the α-dependent terms in the last line sum, over the domain $I(g)$, to the nontrivial coherent family Θ. Crucially, however, only

the portion $I(f \wedge g)$ of that domain is in play in the equations (3.3): Lemma 3.9 ensures us a trivialization $\upsilon : I(f \wedge g) \to \mathbb{Z}$ of $\Theta \restriction\restriction I(f)$, and thus

$$\varphi^r_{f,f\vee g}(\xi,i) = \begin{cases} \upsilon(\xi,i) - \psi_f(\xi,i) & \text{if } (\xi,i) \in I(f \wedge g) \\ -\psi_f(\xi,i) & \text{if } (\xi,i) \in I(f) \backslash I(f \wedge g) \end{cases}$$

combines with the other elements of r so far defined to satisfy the equations of (3.3).

Proceeding in this fashion, one defines the functions $\varphi^r_{f,f\vee g}$ and $\varphi^r_{g,f\vee g}$ for all $f \in E_q$. These determine, in turn, definitions (mod finite) of all yet-undefined elements of r: for $h, f \in E_q$, for example, $h \leq f \vee g$ implies that $f \vee g = (f \vee h) \vee g$, and hence that the associated $\varphi^r_{h,f\vee g}$ must (mod finite) equal the sum of the defined terms $\varphi^r_{h,f\vee h}$ and $\varphi^r_{f\vee h,(f\vee h)\vee g}$. The somewhat tedious but elementary remaining details, and in particular the verification that these assignments indeed 2-cohere, are left to the reader. Modulo the proof of Lemma 3.9, this completes the proof of Theorem A.3. □

Before proving Lemma 3.9, let us briefly expand on how it's interesting. For each $n \in \omega$, write $\bar{n}$ for the function $\omega_1 \to \omega$ constantly taking the value n and E for the collection $\{\bar{n} \mid n \in \omega\}$. Next, recalling the Φ of Example 3.1, define a family $\Psi = \langle \psi_\beta : \beta \times \omega \to \mathbb{Z} \mid \beta < \omega_1 \rangle$ by

$$\psi_\beta(\alpha, j) = \begin{cases} 1 & \text{if } \varphi_\beta(\alpha) = j \\ 0 & \text{otherwise} \end{cases}$$

Each ψ_β is, in other words, the characteristic function of the graph of φ_β, and the finite-to-oneness of Φ translates as the fact that $\Psi \restriction\restriction I(f)$ is trivial for every $f \in E$ (more precisely, it translates as the fact that each such $\Psi \restriction\restriction I(f)$ is, mod finite, 0, in the natural sense). The connection to Lemma 3.9 should be clear; for a better fit, for any $\alpha < \omega_1$ and $n < \omega$ let $g(\omega \cdot \alpha + n) = n$, and for any $\alpha < \beta < \omega_1$ and $j \leq n < \omega$ let

$$\theta_{\omega\cdot\beta}(\omega \cdot \alpha + n, j) = \begin{cases} 1 & \text{if } j = n = \varphi_\beta(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

This g and the obvious extension of our definition to a

$$\Theta = \langle \theta_\gamma : I(g \restriction \gamma) \to \mathbb{Z} \mid \gamma < \omega_1 \rangle$$

are then as the lemma envisions for the collection $E = \{\bar{n} \mid n \in \omega\}$.[1] From this perspective, the lemma may be viewed as generalizing the classical anti-triviality condition of finite-to-oneness to larger and more arbitrary collections of functions E.

Proof of Lemma 3.9. Fix $E = \langle f_\alpha \mid \alpha < \omega_1 \rangle$ as in the statement of the lemma; by expanding E if necessary, we may assume that $f_n = \bar{n}$ for each $n \in \omega$. Fix also a bijection $b[\alpha] : \omega \to \alpha$

[1] Note a slight imprecision here: the lemma posits, in the language most natural to the proof of Theorem A.3, families of functions θ_γ of domain $I(g^\gamma)$. Domains of $I(g \restriction \gamma)$ are somewhat more natural to lemma's proof, though, and the former so readily derive from the latter (via extension by zeros) that we ignore the difference for the remainder of this section.

for each infinite $\alpha < \omega_1$, and define $g : \omega_1 \to \omega$ by

$$g(\omega \cdot \alpha + n) = \begin{cases} n+1 & \text{if } \alpha < \omega \\ \sup\{f_{b[\alpha](k)}(n) \mid k \leq n\} + 1 & \text{otherwise} \end{cases}$$

We define a coherent $\Theta = \langle \theta_\gamma : I(g \restriction \gamma) \to \mathbb{Z} \mid \gamma < \omega_1 \rangle$ as desired by recursion on $\beta < \omega_1$ — or more precisely on $\gamma = \omega \cdot \beta$ for such β — as follows.

At successor steps $\beta = \alpha + 1$, define $\theta_\gamma : I(g \restriction \gamma) \to \mathbb{Z}$ for any $\gamma \in (\omega \cdot \alpha, \omega \cdot \beta]$ by

$$\theta_\gamma(i,j) = \begin{cases} \theta_{\omega \cdot \alpha}(i,j) & \text{if } i < \omega \cdot \alpha \\ 1 & \text{if } i = \omega \cdot \alpha \text{ and } j = g(i) \\ 0 & \text{otherwise} \end{cases}$$

Note that this step conserves the condition that, on any full subdomain of the form $I(g \restriction [\omega \cdot \eta, \omega \cdot \eta + \omega))$, the support of any θ_γ constructed so far is of size exactly 1. We will denote the unique element (i,j) of this support by $s_\gamma(\eta)$.

At limit steps β, let $\xi_\ell = \max\{b[\beta](k) \mid k \leq \ell\}$ and let $C_\beta = \{\xi_\ell \mid \ell < \omega\}$. Fix next an injection $h : \beta \to C_\beta$ so that $h(\alpha) > \alpha$ for all $\alpha \in \beta$. Define $\hat{\theta}_{\omega \cdot \beta} : I(g \restriction \omega \cdot \beta) \to \mathbb{Z}$ by

$$\hat{\theta}_{\omega \cdot \beta}(\omega \cdot \alpha + n, j) = \theta_{\omega \cdot \min(C_\beta \setminus (\alpha+1))}(\omega \cdot \alpha + n, j)$$

for any $\alpha < \beta$ and $n < \omega$. This definition conserves both coherence and the support condition noted above. We will modify it, though, to conserve something deeper, namely the condition that

$$\text{supp}(\theta_\gamma \restriction I(f_\alpha \restriction [\omega \cdot \alpha, \gamma))) \text{ is finite for any } \alpha < \beta \text{ and } \gamma \in (\omega \cdot \alpha, \omega \cdot \beta]. \qquad (\text{RH}(\beta))$$

We do so via finite modifications (thereby conserving coherence) to each of the constituent "blocks"

$$\hat{\theta}_{\omega \cdot \beta} \restriction I(g \restriction [\omega \cdot \xi_\ell, \omega \cdot \xi_{\ell+1})) = \theta_{\omega \cdot \xi_{\ell+1}} \restriction I(g \restriction [\omega \cdot \xi_\ell, \omega \cdot \xi_{\ell+1}))$$

of $\hat{\theta}_{\omega \cdot \beta}$. The idea of these modifications is simply, for $\eta \in [\xi_\ell, \xi_{\ell+1})$ with $h(\alpha) \leq \xi_\ell$, to "shift" any $s_{\omega \cdot \xi_{\ell+1}}(\eta)$ falling within $I(f_\alpha)$ out of it, along the graph of g; by our inductive hypotheses $\text{RH}(\xi_\ell)$, for each ℓ this is only necessary for a finite collection F of such η. More precisely, for each such $\eta \in F$, let

$$n = \max b[\eta]^{-1}\{\alpha \mid h(\alpha) \leq \xi_\ell \text{ and } s_{\omega \cdot \xi_{\ell+1}}(\eta) \in I(f_\alpha)\}$$

and define $\theta_{\omega \cdot \beta}$ so that

$$s_{\omega \cdot \beta}(\eta) = \begin{cases} (\omega \cdot \eta + n, g(\omega \cdot \eta + n)) & \text{if } \eta \in F \\ s_{\omega \cdot \xi_{\ell+1}}(\eta) & \text{otherwise} \end{cases}$$

for any $\eta \in [\xi_\ell, \xi_{\ell+1})$. This completes our description of the construction of Θ.

To conclude, let us check that Θ is as desired. Its coherence was noted above. Its non-triviality follows from our choice of $\{f_n \mid n \in \omega\}$, which ensure that Θ exhibits the finite-to-one-ness condition of Example 3.1 in the manner discussed just before this proof. And the triviality of $\Theta \restriction\restriction I(f_\alpha)$ for any $f_\alpha \in E$ follows from the fact that for any $\gamma \in (\omega \cdot \alpha, \omega_1)$, the support of $\theta_\gamma \restriction I(f_\alpha \restriction (\omega \cdot \alpha, \gamma))$ is finite. □

3.3 Spine and tree filtrations

We turn now to the proof of Theorem B.3 — or, more precisely, to its essential preliminaries; as indicated, the theorem's argument involves organizations of the family $({}^\lambda\omega, \leq^*)$ which are, in some sense, unavoidably nonlinear, and the focus of the present section is their specification.

Let us begin by fixing a few further notations. Both the quasi-orders $\leq$ and $\leq^*$ will be in play below; we reiterate that a * affixed to $=$, $\leq$, or $<$ signals that the relation in question holds on all but finitely many coordinates. The letters κ and λ will always denote cardinals; cf denotes the cofinality function. S^λ_κ denotes the stationary set of cofinality-κ ordinals below λ; we adopt the more readable variant S^n_m when $\kappa = \aleph_m$ and $\lambda = \aleph_n$. Much as in Section 3.2, we will have occasion to consider the Euclidean division of an ordinal α by ω_n. We denote the quotient and the remainder by $q_n(\alpha)$ and $r_n(\alpha)$, so that $r_n(\alpha) < \omega_n$ and $\alpha = \omega_n \cdot q_n(\alpha) + r_n(\alpha)$. The following definition's a bit more general than it needs to be: in all our applications of it below, Q will itself already be a chain.

Definition 3.10. *Let $\trianglelefteq$ be a quasi-order on sets $Q \subseteq P$. We say that P* possesses a κ-spine *R in Q for some cardinal κ if there exists a $\triangleleft$-increasing enumeration $\langle r_\alpha \mid \alpha < \kappa\rangle$ of a $\trianglelefteq$-cofinal subset R of Q. We say that (P, Q) is a* κ-unbounded pair *if*

- *P is $\trianglelefteq$-directed,*
- *$\mathrm{cf}(P) = \mathrm{cf}(Q) = \mathrm{cf}(R) = \kappa$,*
- *Q is $\trianglelefteq$-unbounded in P, and*
- *P possesses a κ-spine in Q.*

A κ-spine filtration *for a κ-unbounded pair (P, Q) is a continuous filtration of*

$$P = \bigcup_{\alpha<\kappa} P_\alpha$$

by $\trianglelefteq$-order-ideals P_α together with a κ-spine R in Q such that $\mathrm{cf}(P_\alpha) < \kappa$ and $\mathrm{otp}(R \cap P_\alpha) = \alpha$ for all $\alpha < \kappa$.

In our contexts, unbounded pairs admit nice spine filtrations, for the join operation $(f, g) \mapsto f \vee g$ (computed in $({}^\lambda\omega, \leq)$) is a function j as below for both the quasi-orders $\leq$ and $\leq^*$.

Lemma 3.11. *Let κ be an uncountable cardinal and let (P, Q) be a κ-unbounded pair equipped with a function $j : P^2 \to P$ such that:*

1. $x, y \trianglelefteq j(x, y)$ *and*
2. $j(x, y) \trianglelefteq j(x', y')$

for every $x \trianglelefteq x'$ and $y \trianglelefteq y'$ in P. Then there exists a κ-spine filtration for (P, Q) such that each P_α is closed with respect to j (and thus, in particular, is directed).

Proof. Observe that κ is regular by definition. Fix a spine $R' \subseteq Q$ witnessing that (P, Q) is κ-unbounded and let $e : \kappa \to P$ enumerate a cofinal subset of P.

We construct an $R = \langle r_\alpha \mid \alpha < \kappa \rangle$ and the family $\langle P_\alpha \mid \alpha < \kappa \rangle$ by joint recursion on $\beta < \kappa$: at each stage β, the set $\langle r_\alpha \mid \alpha < \beta \rangle$ is already defined and we let P_β denote the $\trianglelefteq$-downwards closure of the j-closure of $\{r_\alpha \mid \alpha < \beta\} \cup \{e(\alpha) \mid \alpha < \beta\}$. Because this j-closure is cofinal in P_β (and of size less than the regular cardinal κ) and for each of its elements x there exists some $r' \in R'$ unbounded by x, the set $R' \setminus P_\beta$ is nonempty; we let r_β equal its $\trianglelefteq$-minimal element. This describes the construction, from which the asserted ordertype, continuity, j-closure, and cofinality properties of $\langle P_\alpha \mid \alpha < \kappa \rangle$ and R all immediately follow. □

Below, the quasi-orderings $\leq$ and $\leq^*$ of subcollections P of the families ${}^\lambda\omega$ will often be simultaneously significant.

Definition 3.12. *Let (P, Q) be a κ-unbounded pair within the ordering $({}^\lambda\omega, \leq^*)$. If Q is well-ordered and its identity map induces an order-isomorphism of $(Q, \leq)$ with $(Q, \leq^*)$ then we call Q a* strong chain *and call (P, Q) a* strong κ-unbounded pair.

For better or worse, however, such substructures are, in our main settings of interest, less extensive than one might wish.

Lemma 3.13. *If λ is an uncountable cardinal and $\mathcal{F}$ is a $\leq^*$-increasing chain in ${}^\lambda\omega$ then the cofinality of $\mathcal{F}$ is at most $2^{<\lambda}$. In particular, under the generalized continuum hypothesis, for every $n \in \mathbb{N}$ there exists no $\leq^*$-chain in ${}^{\omega_n}\omega$ of ordertype ω_{n+1}; more particularly, in the language of Definition 3.10, $({}^{\omega_n}\omega, \leq^*)$ possesses no ω_{n+1}-spine in itself.*

Proof. Fix, for contradiction, an $\mathcal{F}'$ witnessing the failure of the lemma and an $<^*$-increasing enumeration of a cofinal subset $\mathcal{F} = \langle f_\xi : \lambda \to \omega \mid \xi < \lambda \rangle$ of $\mathcal{F}'$ which is of regular cardinal ordertype $\mu > 2^{<\lambda}$. For any $\alpha < \lambda$, let $\nu(\alpha)$ denote the $\leq^*$-cofinality of $\mathcal{F} \restriction\restriction \alpha = \langle f_\xi \restriction \alpha : \alpha \to \omega \mid \xi < \lambda \rangle$; in particular, for each $\alpha < \lambda$ there exists a $c_\alpha : \nu(\alpha) \to \mu$ with $\langle f_{c_\alpha(i)} \restriction \alpha \mid i < \nu(\alpha) \rangle$ an increasing cofinal subset of $\mathcal{F} \restriction\restriction \alpha$. Since $\nu(\alpha) \leq 2^{<\lambda}$, the image of c_α is bounded by some $f \in \mathcal{F}$ and since, therefore, $f_{c_\alpha(i)} \restriction \alpha \leq^* f \restriction \alpha$ for all $i < \nu(\alpha)$, it follows that $\nu(\alpha) = 1$.

Thus we have for each $\alpha < \lambda$ an $f(\alpha) \in \mathcal{F}$ whose α-truncation bounds $\mathcal{F} \restriction\restriction \alpha$. Choose $f \leq^* g$ in $\mathcal{F}$ with both $\leq^*$-above all $f(\alpha)$ $(\alpha < \lambda)$ and a $\beta < \lambda$ such that $f \restriction \beta \not\geq^* g \restriction \beta$. By definition, though, $g \restriction \beta \leq^* f(\beta) \leq^* f \restriction \beta$, a contradiction. □

By the above combined with the easy fact below (see [131, Proposition 2.12]) the generalized continuum hypothesis implies that the following holds for every $n \in \mathbb{N}$: there exist no $(n+1)$-unbounded pairs $((^{\omega_n}\omega, \leq^*), Q)$.

Fact 3.14. *If λ is an infinite cardinal then $2^\lambda = \lambda^+$ implies that $\mathrm{cf}(^\lambda\omega, \leq^*) = \lambda^+$.*

The broad implication is that analyses of the derived limits of uncountably wide systems $\mathbf{A}_\lambda$ will tend to involve more general order relations than linear ones. Underlying our Theorem B.3, however, is the observation that, even as it constrains the appearance of linear suborders of $^{\omega_n}\omega$, the generalized continuum hypothesis furnishes the next best thing, namely ω_{n+1}-Aronszajn tree structures organizing length-ω_{n+1} filtrations of $^{\omega_n}\omega$ by families $\{(P_\alpha, Q_\alpha) \mid \alpha < \omega_{n+1}\}$ of unbounded pairs. To fix notation, let us briefly recall these structures' basic features.

Definition 3.15. *A partial order $(T, \prec)$ is a* tree *if the set $t\downarrow := \{s \in T \mid s \prec t\}$ is well-ordered for any $t \in T$. The* height $\mathrm{ht}_T(t)$ *of such a t is the ordertype of $t\downarrow$. We write* $\mathrm{lev}_\alpha(T)$ *for the αth level $\{t \in T \mid \mathrm{ht}_T(t) = \alpha\}$ of T; the height of T is the least α for which $\mathrm{lev}_\alpha(T) = \varnothing$. For any $n \geq 0$ we write Λ_n for the class of multiples of ω_n; for any class X of ordinals, we write $T \restriction X$ for the restriction of $\prec$ to $\{t \in T \mid \mathrm{ht}_T(t) \in X\}$. For any regular infinite cardinal λ, a λ^+-*Aronszajn tree *is a tree T of height λ^+, with levels all of cardinality at most λ; such a T is* special *if it admits an $F : T \to \lambda$ for which $F(s) = F(t)$ implies that s is incomparable with t.*

The derivation of special Aronszajn trees from cardinal arithmetic hypotheses traces essentially to Specker in 1949.

Theorem 3.16 ([160])**.** *Let λ be a regular infinite cardinal. If $2^{<\lambda} = \lambda$ then there exists a special λ^+-Aronszajn tree.*

The following lemma is this section's main result. For simplicity, when n is finite, we will tend below to write *n-unbounded pair* for what is more properly denoted an *$\aleph_n$-unbounded pair*, and we will similarly write *n-spine filtration* for *$\aleph_n$-spine filtration* in the sections which follow.

Lemma 3.17. *Suppose that $n > 0$ and $2^{\aleph_n} = \aleph_{n+1}$ and, hence, that there exists a special ω_{n+1}-Aronszajn tree T. There then exists*

1. *a length-ω_{n+1} continuous filtration of $^{\omega_n}\omega$ by $\leq^*$-order-ideals P_α each of cofinality at most $\aleph_n$, and*
2. *an order-preserving function $H : T \restriction \Lambda_n \to (^{\omega_n}\omega, \leq)$*

such that for all $\alpha \in S^{n+1}_n$ there exists a $Q_\alpha \subseteq H[T \restriction (\Lambda_n \cap \omega_n \cdot \alpha)]$ for which:

3. *(P_α, Q_α) is a strong n-unbounded pair, and*
4. *there exists an $f_\alpha \in P_{\alpha+1}$ such that $g \leq f_\alpha$ for every $g \in Q_\alpha$, and, moreover, every such g falls strictly below f_α on infinitely many coordinates.*

Proof. Fix a bijection $e : \omega_{n+1} \to {}^{\omega_n}\omega$ and a specializing function $F : T \to \omega_n$. Write T' for $T \restriction \Lambda_n$ (so that $\mathrm{lev}_\alpha(T') = \mathrm{lev}_{\omega_n \cdot \alpha}(T)$) and for $\beta < \omega_n$ let $B_\beta = [\omega \cdot \beta, \omega \cdot \beta + \omega)$. We will define an order-preserving function $H : T' \to ({}^{\omega_n}\omega, \leq)$ and filtration

$$ {}^{\omega_n}\omega = \bigcup_{\alpha < \omega_{n+1}} P_\alpha $$

such as the lemma posits by recursion on the ordinals $\alpha < \omega_{n+1}$, conserving the condition that at each stage α, each of the following objects is already defined:

- $H \restriction (T' \restriction \alpha)$,
- the $\leq^*$-order-ideal P_α generated by $H[T' \restriction \alpha] \cup e[\alpha]$, and
- functions $G_\xi : \omega_n \to P_{\xi+1}$ with $\leq^*$-cofinal image in $P_{\xi+1}$ for each $\xi < \alpha$.

At the stage $\alpha = 1$, for example, we may assume that H of the root of T is the constant function $\omega_n \to \{0\}$, so that P_1 is the $\leq^*$-order-ideal generated by $e(0)$, and we may let G_0 constantly take the value $e(0)$.

At any given stage α, we extend the definition of H to $\mathrm{lev}_\alpha(T')$ as follows: for $x \in \mathrm{lev}_\alpha(T')$ and $\gamma \in \omega_n$, let

$$ H(x)(\gamma) = \begin{cases} G_{q_n(\mathrm{ht}_T(z))}(r_n(\mathrm{ht}_T(z)))(\gamma) + 1 & \text{if } \gamma \in B_{F(z)} \text{ for some } z <_T x \\ 0 & \text{otherwise.} \end{cases} $$

As indicated above, $P_{\alpha+1}$ is then the $\leq^*$-order-ideal generated by $H[T' \restriction \alpha + 1] \cup c[\alpha + 1]$. That T' is Aronszajn thus ensures that $P_{\alpha+1}$ is of $\leq^*$-cofinality at most ω_n, and hence that selecting a map $G_\alpha : \omega_n \to P_{\alpha+1}$ with $\leq^*$-cofinal image presents no difficulties. These definitions furnish the material for subsequent stages; this describes the construction.

To see that H is order-preserving, fix $x \leq_{T'} y$ and observe that $\gamma \in \mathrm{supp}(x)$ implies that $\gamma \in B_{F(z)}$ for some $z <_T x \leq_T y$ and thus that

$$ H(y)(\gamma) = G_{q_n(\mathrm{ht}_T(z))}(r_n(\mathrm{ht}_T(z)))(\gamma) + 1 = H(x)(\gamma). $$

This verifies the lemma's item (2); item (1) is immediate from our definition of the ideals P_α. Furthermore, by the continuity of this filtration, if $\alpha < \omega_{n+1}$ is a limit then $\bigcup_{\xi<\alpha} \mathrm{im}(G_\xi)$ is $\leq^*$-cofinal in P_α; put differently, any f in this cofinal set is of the form $f = G_\eta(\varepsilon)$ for some

$\eta < \alpha$ and $\varepsilon < \omega_n$. For any $x \in \mathrm{lev}_\alpha(T')$, then, letting z be the unique node below x in T such that $\mathrm{ht}_T(z) = \omega_n \cdot \eta + \varepsilon$, we have that

$$H(x)(\gamma) = G_\eta(\varepsilon)(\gamma) + 1 > G_\eta(\varepsilon)(\gamma) = f(\gamma)$$

for each of the infinitely many γ in the interval $B_{F(z)}$. This shows that $H(x) \notin P_\alpha$, or, more generally, that $H[\mathrm{lev}_\alpha(T')] \cap P_\alpha = \emptyset$ whenever $\alpha < \omega_{n+1}$ is a limit. It also implies that if $x <_{T'} y$ are both of limit heights then $H(x)$ falls strictly below $H(y)$ on infinitely many coordinates, and a similar argument shows that if $C \subseteq T'$ is a chain of limit-of-limits ordertype then $H[C]$ is $\leq^*$-unbounded in P_α for $\alpha = \sup\{\mathrm{ht}_{T'}(x) \mid x \in C\}$. In particular, choosing for each $\alpha \in S_n^{n+1}$ a $y_\alpha \in \mathrm{lev}_\alpha(T')$ and letting

$$Q_\alpha = H[\{x \in T' \mid \mathrm{ht}_{T'}(x) \text{ is a limit and } x <_T y_\alpha\}]$$

and letting $f_\alpha = H(y_\alpha)$ defines families of functions as posited in the lemma's items (3) and (4). □

3.4 The continuity and existence of twistable sets

Assume now the hypotheses and notations of Lemma 3.17. To prove Theorem B.3, in Sections 3.5 and 3.6 we will construct witnesses to $\lim^{n+1} \mathbf{A}_{\aleph_n} \neq 0$ which restrict to an $(n+1)$-coherent nontrivial family on $\mathrm{im}(H)$. This witness is in turn built from n-coherent but non-n-trivial families on P_α whose restriction to Q_α is nontrivial for every α in S_n^{n+1}. The construction of these n-coherent families from guessing principles is much as in [33]; the setting of *wide* inverse systems, however, introduces an additional difficulty. This is the following: the inductive arguments of Section 3.5 require a small list (e.g., a $\diamondsuit(S_k^{k+1})$-sequence) of codes for initial fragments of putative trivializations of $(k+1)$-coherent families, as k ranges below n. However, since $|I(f)| = \aleph_n$ for any $f \in {}^{\omega_n}\omega$, we cannot even hope to code all functions $I(f) \to \mathbb{Z}$ in a sequence $\langle S_\alpha \subseteq \alpha \mid \alpha < \omega_{k+1}\rangle$. To address this issue, we introduce the notion of a *k-twistable set*, which is subset of the grid $\omega_n \times \omega$ just large enough to support nontrivial k-coherence for (P_α, Q_α); within our argument, it will then be sufficient to code putative trivializations on twistable sets.

Definition 3.18. *Let S be a set of ordinals. We say that α is a limit point of S if $\alpha \in S$ and $\alpha = \sup(\alpha \cap S)$.*

Definition 3.19. *Let (P, Q) be such that $Q \subseteq P \subseteq {}^{\omega_n}\omega$ and $\mathrm{cf}(P) \leq \aleph_0$ with respect to $\leq^*$. We say that $E \subseteq \omega_n \times \omega$ is* 0-twistable for (P, Q) *if $|E| = \aleph_0$ and $|E \cap I(f)| < \aleph_0$ for all $f \in P$.*

Next, fix $k \geq 0$ and let $(\langle P_\alpha \mid \alpha < \omega_{k+1}\rangle, Q')$ be a $(k+1)$-spine filtration for some strong $(k+1)$-unbounded pair (P, Q). We say that E is $(k+1)$-twistable for $(\langle P_\alpha \mid \alpha < \omega_{k+1}\rangle, Q')$ *if E is the union of a sequence $\mathcal{E} = \langle E_\alpha \mid \alpha \in \{0\} \cup S_k^{k+1}\rangle$ of sets such that*

(0) $|E_0| \leq \aleph_k$ and $E_0 \subseteq I(Q'(0))$,

1. $E_\alpha \subseteq I(Q'(\alpha))$ *and*
2. E_α *is* k*-twistable for* $(P_\alpha, Q' \restriction \alpha)$

for all $\alpha \in S^{k+1}_k$*, and*

(3) $E_\alpha \subseteq \bigcup(\mathcal{E} \restriction \alpha)$ *for any limit point* α *of* S^{k+1}_k*.*

Similarly, we will say that E *is* $(k+1)$*-twistable for a strong unbounded pair* (P, Q) *if and only if there exists a spine filtration of* (P, Q) *for which* E *is* $(k+1)$*-twistable.*

Twistable sets are small by definition.

Lemma 3.20. *If* E *is* k*-twistable, then* $|E| \leq \aleph_k$*.*

Proof. This is definitional for $k = 0$ and immediate for all higher k by induction. □

It's equally clear that any order-ideal of countable cofinality admits a 0-twistable set; more particularly:

Lemma 3.21. *Let* $P \subseteq {}^{\omega_n}\omega$ *be an* $\leq^*$*-ideal of countable cofinality; suppose also that* $g : \omega_n \to \omega$ *is not in* P*. Then there exists a* 0*-twistable* $E \subseteq I(g)$ *for* P*.*

Proof. Fix a $\leq^*$-cofinal subset $\{f_n \mid n \in \omega\}$ of P and choose an $x_n \in I(g) \setminus \bigcup_{k \leq n} I(f_k)$ for each n; clearly, $E = \{x_n \mid n \in \omega\}$ is as desired. □

The following lemma expresses the fact that the notion of twistable set only depends on the tail of a spine filtration.

Lemma 3.22. *Let* $k \geq 0$ *and suppose that* (P, Q_1) *and* (P, Q_2) *are two strong* $(k+1)$*-unbounded pairs such that there exists* $f \in Q_1 \cap Q_2$ *such that* $\{g \in Q_1 \mid g \geq f\} = \{g \in Q_2 \mid g \geq f\}$*. Then a set* E *is* $(k+1)$*-twistable for* (P, Q_1) *if and only if it is* $(k+1)$*-twistable for* (P, Q_2)*.*

Proof. As the hypotheses are symmetrical we only show one implication. Also, the proof will be by induction, even though inductive reasoning is only required for one of the conditions to be checked. Call $P(k)$ the statement above relative to a fixed $k \geq 0$.

Let $(\langle P_\alpha \mid \alpha < \omega_{k+1}\rangle, Q')$ be a spine filtration of (P, Q_1) for which E is $(k+1)$-twistable, that is, it admits a decomposition $E = \bigcup\{E_\alpha \mid \alpha \in \{0\} \cup S^{k+1}_k\}$ as in the definition. There exists a successor ordinal α_0 such that $Q'(\alpha_0) \geq f$ and hence thanks to our assumptions $Q' \restriction (\omega_{k+1} \setminus \alpha_0) \subseteq Q_2$. So, once we reset the indexing of this final segment by letting $Q^*(\beta) = Q'(\alpha_0 + \beta)$ for all $\beta < \omega_{k+1}$ we obtain that $(\langle P_{\alpha_0+\beta}\rangle_{\beta<\omega_{k+1}}, Q^*)$ is a spine filtration for (P, Q_2). As α_0 is a successor, $(\alpha_0 + \beta) \in S^{k+1}_k$ if and only if $\beta \in S^{k+1}_k$.

Hence we let $E'_0 = \bigcup\{E_\alpha \mid \alpha \in \{0\} \cup (\alpha_0 \cap S_k^{k+1})\}$ and $E'_\beta = E_{\alpha_0+\beta}$. Clearly, $|E'_0| \leq \aleph_k$ by an easy application of Lemma 3.20, and $E'_0 \subseteq I(Q'(\alpha_0)) = I(Q^*(0))$. Now we check conditions $(1) - (3)$.

(1) is satisfied since $E'_\beta = E_{\alpha_0+\beta} \subseteq I(Q'(\alpha_0 + \beta)) = I(Q^*(\beta))$.

(3) is satisfied since if $\beta = \sup(\beta \cap S_k^{k+1})$ then $\alpha_0 + \beta = \sup((\alpha_0 + \beta) \cap S_k^{k+1})$, hence

$$E'_\beta = E_{\alpha_0+\beta} \subseteq \bigcup\{E_\alpha \mid \alpha \in \{0\} \cup ((\alpha_0 + \beta) \cap S_k^{k+1})\}$$

$$= (E_0 \cup \bigcup_{\alpha \in \alpha_0 \cap S_k^{k+1}} E_\alpha) \cup \bigcup_{\beta \in S_k^{k+1}} E_{\alpha_0+\beta} = E'_0 \cup \bigcup_{\beta \in S_k^{k+1}} E'_\beta$$

Finally, (2) is immediate if $k = 0$ and follows easily from $P(k-1)$ otherwise. In fact, what we need to prove is that $E'_\beta = E_{\alpha_0+\beta}$ is k-twistable for $(P_{\alpha_0+\beta}, Q^* \restriction \beta) = (P_{\alpha+\beta}, Q' \restriction (\alpha_0 + \beta \setminus \alpha_0))$, and we know that it is for the k-unbounded pair $(P_{\alpha_0+\beta}, Q' \restriction \alpha_0 + \beta)$ of which that other chain is a final segment. So we conclude by $P(k-1)$.

Finally, notice that $E = \bigcup\{E'_\beta \mid \{0\} \cup \beta \in S_k^{k+1}\}$. □

With the next few lemmas, we show by induction on k that the notion of k-twistable set is *continuous*, in the sense that when taking the increasing union of certain sequences of unbounded pairs, the union of the corresponding twistable sets contains a twistable set for that increasing union. The proofs of both the base case and the induction step are based on some form of diagonalization. In the remainder of this section, we work in ${}^{\omega_n}\omega$ for some fixed n.

Lemma 3.23. *Let $P = \bigcup_{i<\omega} P_i$ be an increasing union of ideals of countable cofinality, each with its 0-twistable set X_i. Then $X = \bigcup_{i<\omega} X_i$ contains a 0-twistable set for P.*

Proof. Note that since each P_i is $\leq$-directed, for each we have a $\leq^*$-cofinal, $\leq$-ordered chain $\langle f_{i,j} \mid j < \omega\rangle$. Using the fact that $P_i \subseteq P_{i'}$ for $i < i'$, we can iteratively find $j(i)$ such that:

1. $f_{i,j(i)} \leq f_{i+1,j(i+1)}$
2. $f_{l,m} \leq f_{i+1,j(i+1)}$ for all $l, m \leq i$.

Then we consider the diagonal chain $\langle f_{i,j(i)} \mid i < \omega\rangle$ and observe that it is cofinal in P. For each $i < \omega$ pick $x_i \in X_i \setminus I(f_{i,j(i)})$. Then $\{x_i\}_{i<\omega}$ is almost disjoint from all the $I(f_{i,j})$ and hence from all the *sottografici* of functions in P. □

For $k \geq 0$ we define the statement $R(k)$, which we also call k-continuity, as follows. $R(k)$ holds if and only if whenever

1. (P, Q) is a strong $(k+1)$-unbounded pair.

2. $(\langle P_\xi \mid \xi < \omega_{k+1}\rangle, Q')$ is a spine filtration of it,
3. ζ is a limit point of S^{k+1}_k,
4. $(\xi_\alpha)_{\alpha<\omega_k} \subseteq \zeta \cap S^{k+1}_k$ is increasing and cofinal in ζ,
5. For all $\alpha < \omega_k$, there exist a k-twistable $E_{\xi_\alpha} \subseteq I(Q'(\xi_\alpha))$ for $(P_{\xi_\alpha}, Q' \restriction \xi_\alpha)$,

there exists $E_\zeta \subseteq \bigcup_{\alpha<\omega_k} E_{\xi_\alpha} \subseteq I(Q'(\zeta))$ that is k-twistable for the unbounded pair $(P_\zeta, Q' \restriction \zeta)$.

Lemma 3.24. *$R(k)$ implies $R(k+1)$.*

Proof. Throughout the proof, let $q_k(\alpha)$ and $r_k(\alpha)$ denote the quotient and the remainder of α in the Euclidean division by ω_k, respectively.

We now assume $R(k)$ and the hypotheses of $R(k+1)$ to prove the conclusion of $R(k+1)$.

Hypothesis $R(k+1).(5)$ gives us, for each index $\alpha < \omega_{k+1}$, a spine filtration $(\langle P_{\xi_\alpha,\beta}\rangle_{\beta<\omega_{k+1}}, Q''_{\xi_\alpha})$ and k-twistable sets $E_{\xi_\alpha,\beta}$ for all $\beta \in S^{k+1}_k$, such that $E_{\xi_\alpha} = \bigcup\langle E_{\xi_\alpha,\beta} \mid \beta \in \{0\} \cup S^{k+1}_k\rangle$.

We recursively define a function $F : \omega_{k+1} \to \omega_{k+1}$ such that

$$P_{\xi_\alpha,F(\alpha)} \supseteq \bigcup_{\gamma<\alpha} P_{\xi_\gamma,F(\gamma)+\omega_k+1} \cup \bigcup_{\epsilon,\eta<\alpha} P_{\xi_\epsilon,\eta}.$$

This is possible because the right-hand side is a union of at most $\aleph_k$-many ideals of cofinality at most $\aleph_k$, and is included into P_{ξ_α}.

Then we let

$$P'_\gamma := P_{\xi_{q_k(\gamma)},F(q_k(\gamma))+r_k(\gamma)+1},$$

and finally

$$P''_\alpha := \bigcup_{\gamma<\alpha} P'_\gamma.$$

As for the chain, we let

$$Q''_\zeta(\alpha) = Q''_{\xi_{q_k(\alpha)}}(F(q_k(\alpha)) + r_k(\alpha)).$$

Claim 3.25. *$(\langle P''_\alpha \mid \alpha < \omega_{k+1}\rangle, Q''_\zeta)$ is a spine filtration for $(P_\zeta, Q' \restriction \zeta)$.*

Proof of Claim 3.25. The condition satisfied by F implies that P'_γ is increasing and hence P''_α is increasing and continuous by construction. Again by the property that defines F, we have $\bigcup_{\alpha<\omega_{k+1}} P''_\alpha = P_\zeta$. Finally, for each $\alpha < \omega_{k+1}$, $\mathrm{cf}(P'_\alpha) \leq \aleph_k$, and hence $\mathrm{cf}(P''_\alpha)$ as the latter is the union of at most $\aleph_k$-many ideals of cofinality less or equal than $\aleph_k$.

As for the chain, we have the inclusions $Q''_\zeta \subseteq \bigcup_{\alpha<\omega_{k+1}} Q''_{\xi_\alpha} \subseteq Q' \restriction \zeta$, following from $Q''_{\xi_\alpha} \subseteq Q' \restriction \xi_\alpha \subseteq Q' \restriction \zeta$. It remains to prove that, for every $\alpha < \omega_{k+1}$, $Q''_\zeta \restriction \alpha = P''_\alpha \cap Q''_\zeta$, or equivalently that $P''_\alpha \not\ni Q''_\zeta(\alpha) \in P''_{\alpha+1}$. This holds because for every $\gamma < \alpha$, we have

$$Q''_\zeta(\alpha) = Q''_{\xi_{q_k(\alpha)}}(F(q_k(\alpha)) + r_k(\alpha)) \notin P'_\gamma = P_{\xi_{q_k(\gamma)},F(q_k(\gamma))+r_k(\gamma)+1}.$$

In fact, if $q_k(\gamma) < q_k(\alpha)$,then, by construction of F,

$$\begin{aligned} Q''_\zeta(\alpha) &= Q''_{\xi_{q_k(\alpha)}}(F(q_k(\alpha)) + r_k(\alpha)) \\ &\notin P_{\xi_{q_k(\alpha)},F(q_k(\alpha))+r_k(\alpha)} \\ &\supseteq P_{\xi_{q_k(\alpha)},F(q_k(\alpha))} \\ &\supseteq P_{\xi_{q_k(\gamma)},F(q_k(\gamma))+\omega_k+1} \\ &\supseteq P_{\xi_{q_k(\gamma)},F(q_k(\gamma))+r_k(\gamma)+1}. \end{aligned}$$

If $q_k(\gamma) = q_k(\alpha)$ and $r_k(\gamma) < r_k(\alpha)$ then $r_k(\gamma) + 1 \leq r_k(\alpha)$. Hence,

$$P_{\xi_{q_k(\alpha)},F(q_k(\alpha))+r_k(\alpha)} \supseteq P_{\xi_{q_k(\gamma)},F(q_k(\gamma))+r_k(\gamma)+1},$$

and we can apply the same reasoning as above.

Moreover,

$$Q''_\zeta(\alpha) = Q''_{\xi_{q_k(\alpha)}}(F(q_k(\alpha)) + r_k(\alpha)) \in P_{\xi_{q_k(\alpha)},F(q_k(\alpha))+r_k(\alpha)+1} = P'_\alpha = P''_{\alpha+1}.$$

This ends the proof of the claim. □

We are finally ready to define the E_ζ witnessing $R(k+1)$. This is

$$E_\zeta = \bigcup_{\alpha<\omega_{k+1}} E_{\xi_{q_k(\alpha)},F(q_k(\alpha))+\omega_k}.$$

We now define a decomposition $E_\zeta = \bigcup\langle E''_\alpha \mid \alpha \in \{0\} \cup S_k^{k+1}\rangle$ as in the definition. We let $E''_0 = \emptyset$, and for every $\alpha \in S_k^{k+1}$ we find a set E''_α such that

1. $E''_\alpha \subseteq I(Q''_\zeta(\alpha))$,
2. E''_α is k-twistable for $(P''_\alpha, Q''_\zeta \restriction \alpha)$,
3. if α is a limit point of S_k^{k+1}, then $E''_\alpha \subseteq \bigcup_{\gamma\in\alpha\cap S_k^{k+1}} E''_\gamma$,

and finally

$$\bigcup_{\alpha\in S_k^{k+1}} E''_\alpha = E_\zeta.$$

If $q_k(\alpha)$ is a successor we let

$$E''_\alpha = E_{\xi_{q_k(\alpha)-1}, F(q_k(\alpha)-1)+\omega_k} \subseteq I(Q''_{\xi_{q_k(\alpha)-1}}(F(q_k(\alpha)-1)+\omega_k)) \subseteq I(Q''_\zeta(\alpha)),$$

where the last inclusion follows from the fact that

$$\begin{aligned} Q''_\zeta(\alpha) &= Q''_{\xi_{q_k(\alpha)}}(F(q_k(\alpha)) + r_k(\alpha)) \\ &\notin P_{\xi_{q_k(\alpha)}, F(q_k(\alpha))+r_k(\alpha)} \\ &\supseteq P_{\xi_{q_k(\alpha)}, F(q_k(\alpha))} \\ &\supseteq P_{\xi_{q_k(\alpha)-1}, F(q_k(\alpha)-1)+\omega_k+1} \\ &\ni Q''_{\xi_{q_k(\alpha)-1}}(F(q_k(\alpha)-1)+\omega_k), \end{aligned}$$

which implies $Q''_\zeta(\alpha) = Q''_{\xi_{q_k(\alpha)}}(F(q_k(\alpha)) + r_k(\alpha)) \not\leq Q''_{\xi_{q_k(\alpha)-1}}(F(q_k(\alpha)-1)+\omega_k)$ and hence that $Q''_{\xi_{q_k(\alpha)-1}}(F(q_k(\alpha)-1)+\omega_k) \leq Q''_{\xi_{q_k(\alpha)}}(F(q_k(\alpha)) + r_k(\alpha))$ since we are dealing with a strong chain.

That $E''_\alpha = E_{\xi_{q_k(\alpha)-1}, F(q_k(\alpha)-1)+\omega_k}$ is k-twistable for $(P''_\alpha, Q''_\zeta \restriction \alpha)$ follows from the fact that

$$\begin{aligned} P''_\alpha &= \bigcup_{\gamma<\alpha} P_{\xi_{q_k(\gamma)}, F(q_k(\gamma))+r_k(\gamma)+1} \\ &= \bigcup_{\beta<\omega_k} P_{\xi_{q_k(\alpha)-1}, F(q_k(\alpha)-1)+\beta+1} \\ &= P_{\xi_{q_k(\alpha)-1}, F(q_k(\alpha)-1)+\omega_k} \end{aligned}$$

(which is sufficient for the $k = 0$ case) and the fact that

$$Q''_\zeta \restriction (\alpha \setminus q_k(\alpha) - 1) = Q''_{\xi_{q_k(\alpha)-1}} \restriction [F(q_k(\alpha)-1), F(q_k(\alpha)-1)+\omega_k),$$

hence $Q''_\zeta \restriction \alpha$ and $Q''_{\xi_{q_k(\alpha)-1}} \restriction (F(q_k(\alpha)-1)+\omega_k)$ share a final segment and since $E''_\alpha = E_{\xi_{q_k(\alpha)-1}, F(q_k(\alpha)-1)+\omega_k}$ is k-twistable for the unbounded pair

$$(P_{\xi_{q_k(\alpha)-1}, F(q_k(\alpha)-1)+\omega_k}, Q''_{\xi_{q_k(\alpha)-1}} \restriction (F(q_k(\alpha)-1)+\omega_k))$$

by assumption, then it is for $(P''_\alpha, Q''_\zeta \restriction \alpha)$, thanks to Lemma 3.22 in case $k > 0$.

If $\mathrm{cf}(q_k(\alpha)) = \omega_k$ instead, that is, $\alpha = \sup(\alpha \cap S^{k+1}_k)$, then we can find an increasing sequence $(\rho_\delta)_{\delta<\omega_k} \subseteq \alpha \cap S^{k+1}_k$ cofinal in α such that $q_k(\rho_\delta)$ is a successor for all $\delta < \omega_k$. Then we can apply $R(k)$ and find a k-twistable

$$E''_\alpha \subseteq \bigcup_{\delta<\omega_k} E''_{\rho_\delta} \subseteq \bigcup_{\gamma\in\alpha\cap S^{k+1}_k} E''_\gamma.$$

Notice that

$$E''_\alpha \subseteq \bigcup_{\delta<\omega_k} E''_{\rho_\delta} \subseteq I(Q''_\zeta(\alpha)).$$

In fact, for all $\delta < \omega_k$,

$$E''_{\rho_\delta} \subseteq I(Q''_{\xi_{q_k(\rho_\delta)-1}}(F(q_k(\rho_\delta)-1)+\omega_k)) \in P_{\xi_{q_k(\rho_\delta)-1}, F(q_k(\rho_\delta)-1)+\omega_k+1}$$

and

$$\begin{aligned} P_{\xi_{q_k(\rho_\delta)-1}, F(q_k(\rho_\delta)-1)+\omega_k+1} &\subseteq P_{q_k(\alpha), F(q_k(\alpha))} \\ &\not\ni Q''_{q_k(\alpha)}(F(q_k(\alpha))) \\ &= Q''_{q_k(\alpha)}(F(q_k(\alpha))+r_k(\alpha)) \\ &= Q''_\zeta(\alpha), \end{aligned}$$

hence $I(Q''_{\xi_{q_k(\rho_\delta)-1}}(F(q_k(\rho_\delta)-1))+\omega_k) \subseteq I(Q''_\zeta(\alpha))$, since we are dealing with a strong chain.

Finally, $E_\zeta \subseteq I(Q'(\zeta))$ because the chain is strong and $E''_\alpha \subseteq I(Q''_\zeta(\alpha)) \subseteq I(Q'(\zeta))$ for all $\alpha \in S_k^{k+1}$. □

Lemma 3.26. *For all $k \geq 0$, $R(k)$ holds.*

Proof. Lemma 3.23 easily implies the base case $R(0)$ since we are dealing with a strong chain and hence we find a 0-twistable $E_\zeta \subseteq \bigcup_{i<\omega} E_{\xi_i} \subseteq \bigcup_{i<\omega} I(Q'(\xi_i)) \subseteq I(Q'(\zeta))$. Lemma 3.24 gives us the induction step. We conclude by induction. □

Finally, by continuity, we conclude the existence of a twistable set for any unbounded pair.

Lemma 3.27. *For every $k \geq 0$ and for every $(k+1)$-spine filtration*

$$(\langle P_\alpha \mid \alpha < \omega_{k+1}\rangle, Q')$$

of a strong $(k+1)$-unbounded pair (P, Q), there exists a $(k+1)$-twistable set

$$E \subseteq \bigcup_{f\in Q} I(f).$$

Proof. We prove the result by induction. Let $(\langle P_\alpha \mid \alpha < \omega_{k+1}\rangle, Q')$ be our spine filtration. If $\alpha \in S_k^{k+1}$ and $\alpha > \sup(\alpha \cap S_k^{k+1})$ then $(P_\alpha, Q' \restriction \alpha)$ is a strong k-unbounded pair. If $k = 0$ then use Lemma 3.21 to find a 0-twistable set $E_\alpha \subseteq I(Q'(\alpha))$. If $k > 1$ then, by the induction assumption, there exists a k-twistable $E \subseteq \bigcup_{f\in Q'\restriction\alpha} I(f) \subseteq I(Q'(\alpha))$, where the last inclusion follows from the fact that Q' is a strong chain. In both cases we find $E_\alpha \subseteq \bigcup_{\gamma\in\alpha\cap S_k^{k+1}} E_\gamma$ for the case of $\alpha \in S_k^{k+1}$ with $\alpha = \sup(\alpha \cap S_k^{k+1})$ as follows.

First, we find $(\rho_\delta)_{\delta<\omega_k} \subseteq \alpha \cap S_k^{k+1}$ such that $\rho_\delta > \sup(\rho_\delta \cap S_k^{k+1})$ for all $\delta < \omega_k$. Then we apply continuity, that is, $R(k)$ of Lemma 3.26 to find E_α.

Finally, we let $E_0 = \emptyset$ and obtain that $E = \bigcup\langle E_\alpha \mid \alpha \in \{0\} \cup S_k^{k+1}\rangle$ is $(k+1)$-twistable and $E \subseteq \bigcup_{\alpha\in S_k^{k+1}} I(Q'(\alpha)) \subseteq \bigcup_{f\in Q} I(f)$. □

3.5 Nontriviality on unbounded chains

Let $\langle (P_\alpha, Q_\alpha) \mid \alpha \in S^{n+1}_n \rangle$ be a sequence of n-unbounded pairs coming from a tree filtration of ${}^{\omega_n}\omega$. In this section, we prove by induction the existence of a nontrivial coherent n-family on P_α, whose nontriviality is witnessed by the values taken on an n-twistable set by the functions whose multi-index is in the chain Q_α. The base case of the induction is a ZFC fact.

Lemma 3.28. *For any* 1*-twistable set* $E = \bigcup_{\gamma \in S^1_0 \cup \{0\}} E_\gamma$ *for a spine filtration* $(\langle P_\alpha \rangle_{\alpha<\omega_1}, Q')$ *of a strong* $\aleph_1$*-unbounded pair* (P, Q) *in* $({}^\lambda\omega, \leq)$*, there exists a coherent family*

$$\Phi = \langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in P \rangle$$

with the property that

$$(\Phi \restriction Q') \restriction\restriction E = \langle \varphi_f \restriction I(f) \cap E \mid f \in Q' \rangle$$

is nontrivial.

Proof. Here the broad idea is that the decomposition $E = \bigcup_{\gamma \in S^1_0} E_\gamma$ sufficiently resembles the decomposition $\omega_1 \times \omega = \bigcup_{\gamma \in \omega_1} \{\gamma\} \times \omega$ to support constructions along the lines of Lemma 3.9 (which trace, in turn, to those of Example 3.1). Before more precisely describing such a construction, let us record a few observations. First, it is among our assumptions that P possesses a cofinal directed set of the form $P' = \langle p_\alpha \mid \alpha < \omega_1 \rangle$; thus to prove the lemma, it will suffice by the remarks concluding Section 3.1 to construct a coherent family $\Phi \restriction P' \cup Q' = \langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in P' \cup Q' \rangle$ with the property that $(\Phi \restriction Q') \restriction\restriction E$ is nontrivial. Second, note that $E_\alpha \subseteq I(Q'(\beta))$ for all $\alpha \leq \beta \in S^1_0$, thus if γ is a successor of some β in the natural enumeration of S^1_0 then $\bigcup_{\alpha \in S^1_0 \cap \gamma} E_\alpha$ has finite intersection with E_γ; write $S \subset S^1_0$ for the collection of such γ. Relatedly and third, note that for every $f \in P$ there exists a least $\beta \in S^1_0$ such that $\gamma \in S^1_0 \backslash \beta$ implies that $E_\gamma \cap I(f)$ is finite. Write A for the set of accumulation points of S^1_0 below ω_1 and let C denote the closed unbounded subset

$$\{\beta \in A \mid E_\gamma \cap I(p_\alpha) \text{ and } E_\gamma \cap I(Q'(\alpha)) \text{ are finite for all } \alpha < \beta \text{ and } \gamma \in S^1_0 \backslash \beta\}$$

of ω_1; note that we may without loss of generality assume the collection of functions of the form p_α or $Q(\alpha)$ for some $\alpha < \beta$ to be $\leq^*$-directed for each $\beta \in C$ as well. Fix also a filtration of $E = \bigcup_{i<\omega} E^i$ by sets E^i whose intersection with E_γ is finite for every $(i, \gamma) \in \omega \times S^1_0$, and choose $e^i_\gamma \in E_\gamma \backslash E^i$ so that $i \neq j$ implies $e^i_\gamma \neq e^j_\gamma$ for all $i, j \in \omega$ and $\gamma \in S$. We will construct $\Phi \restriction P' \cup Q'$ by recursion on the elements ξ of C, maintaining as we go the hypotheses that for all $\eta \in C \cap \xi$ and $\alpha < \eta \leq \delta < \xi$,

$$\{\gamma \in S^1_0 \backslash \eta \mid I(p_\alpha) \cap \mathrm{supp}(\varphi_{Q'(\delta)}) \cap E_\gamma \neq \varnothing\}$$

and

$$\{\gamma \in S^1_0 \backslash \eta \mid I(Q'(\alpha)) \cap \mathrm{supp}(\varphi_{Q'(\delta)}) \cap E_\gamma \neq \varnothing\}$$

are finite, as is

$$\{\gamma \in S^1_0 \mid E^i \cap \mathrm{supp}(\varphi_{Q'(\delta)}) \cap E_\gamma \neq \varnothing\} \tag{3.4}$$

for every $i \in \omega$ and $\delta < \xi$. The third of these hypotheses is, in essence, a finite-to-one-ness condition which, alongside the condition that

$$\mathrm{supp}(\varphi_{Q'(\delta)}) \cap E_\gamma \neq \varnothing \text{ for all } \gamma \in S \cap \delta \text{ and } \delta < \xi \tag{3.5}$$

will ensure, at the construction's conclusion, the nontriviality of $\Phi \restriction Q'$, while the first two hypotheses will facilitate our construction's continuation. Write $\mathrm{RH}(\xi)$ for the conjunction of these conditions, and note that it's trivially conserved at limit steps; thus it will suffice to establish $\mathrm{RH}(\xi)$ at base and successor steps $\xi \in C$.

For the base case $\xi = \min C$, write Φ_ξ for the family

$$\langle \varphi_f : I(f) \to \mathbb{Z} \mid f = p_\alpha \text{ or } f = Q'(\alpha) \text{ for some } \alpha < \xi \rangle$$

and define the functions of $\Phi_\xi \restriction\restriction (\lambda \backslash \xi \times \omega)$ all to constantly equal 0, and, using an ordertype-ω enumeration of $S \cap \xi$, define $\Phi_\xi \restriction\restriction \xi \times \omega$ to satisfy the only non-vacuous conditions of $\mathrm{RH}(\xi)$, namely items (3.4) and (3.5) above; note that it's straightforward to additionally arrange that the supports of all functions in Φ_ξ fall within E.

Suppose next that ξ is the successor in C of some η and that $\mathrm{RH}(\eta)$ holds. By Goblot's theorem, there exists a family $\varphi : \eta \times \omega \to \mathbb{Z}$ trivializing $\Phi_\eta \restriction\restriction \eta \times \omega$; fix also a sequence $C_\eta = \langle \eta_i \mid \omega \rangle$ which is increasing and cofinal in η. To define a $\varphi_{Q'(\eta)}$ satisfying the $\eta = \delta$ instance of (3.4) and (3.5) above, we revise φ along C_η as follows: for $x \in I(Q'(\eta))$, we let $\varphi_{Q'(\eta)}(x) = \varphi(x)$ *unless* $x \in E_\gamma$ *for some* $\gamma \in S \cap [\eta_j, \eta_{j+1})$ *with* $\mathrm{supp}(\varphi) \cap E_\gamma \cap E_j \neq \varnothing$. For such x and E_γ, let

$$\varphi_{Q'(\eta)}(x) = \begin{cases} 1 & \text{if } x = e^j_\gamma \\ 0 & \text{otherwise} \end{cases}$$

Next, fix bijections $b : \omega \to \eta$ and $a : \omega \to S^1_0 \cap [\eta, \xi)$ and for each $\beta \in S^1_0 \cap [\eta, \xi)$ define $\varphi_{Q'(\beta)}(x)$ by

$$\varphi_{Q'(\beta)}(x) = \begin{cases} \varphi_{Q'(\eta)}(x) & \text{if } x \in I(Q'(\eta)) \\ 1 & \text{if } x = e^i_\delta \text{ for some } \delta \in S \cap [\eta, \beta) \text{ with } i \text{ the minimum of} \\ & \quad \{j \in \omega \mid e^j_\delta \notin I(p_{b(\alpha)}) \cup I(Q'(b(\alpha))) \text{ for any } \alpha \in a^{-1}(\delta)\} \\ 0 & \text{otherwise} \end{cases}$$

Again by Goblot's theorem, there exists a trivialization $\psi : \xi \times \omega \to \mathbb{Z}$ of the (suitable restrictions of the) functions so far defined; one then defines the restriction to $\xi \times \omega$ of all yet-undefined φ_{p_α} and $\varphi_{Q'(\alpha)}$ with $\alpha < \xi$ as $\psi \restriction I(p_\alpha)$ and $\psi \restriction I(Q'(\alpha))$, respectively, and lets the remainder of these functions constantly equal 0. This completes the construction. □

The easy verification of the following fact, which will be invoked several times below, is left to the reader.

Lemma 3.29. *For every pair of directed sets $P \subseteq P' \subseteq {}^{\omega_n}\omega$ and domains $E \subseteq E' \subseteq \omega_n \times \omega$, if $\Psi = \{\psi_{\vec{f}} : E' \to \mathbb{Z} \mid \vec{f} \in P'^{(k)}\}$ trivializes Φ, then $(\Psi \restriction P^{(k)}) \restriction\restriction E$ trivializes $(\Phi \restriction P^{(k)}) \restriction\restriction E$.*

We turn now to the induction step of our argument; this is based on some guessing principles. These allow us to construct nontrivial coherent families by trasfinite recursion. More precisely, they allow us to negate all putative trivializations by choosing between a pair of extensions one that is not trivialized by any extension of the family coded by the set A_α in the $\diamondsuit(S^{k+1}_k)$-sequence $\langle A_\alpha \mid \alpha \in S^{k+1}_k\rangle$.

For $k \geq 1$, let $T(k)$ denote the claim that for every strong k-unbounded pair (P, Q), every one of its k-spine filtrations $(\langle P_\alpha \mid \alpha < \omega_k \rangle, Q')$ and every k-twistable set E for $(\langle P_\alpha \mid \alpha < \omega_k\rangle, Q')$ there exists a k-coherent family Φ such that $(\Phi \restriction Q'^{(k)}) \restriction\restriction E$ is nontrivial. Note that Lemma 3.28 above is $T(1)$.

Lemma 3.30. *For every $k \geq 1$, if $T(k)$ and $\diamondsuit(S^{k+1}_k)$ hold, then $T(k+1)$ holds.*

Proof. Let $E = \bigcup \mathcal{E}$ with $\mathcal{E} = \langle E_\alpha \mid \alpha \in \{0\} \cup S^{k+1}_k\rangle$ be our $(k+1)$-twistable set. For $\alpha \in S^{k+1}_k$, let $\mathcal{P}^\alpha$ be the set of all families of the form $\{\psi_{\vec{f}} : \bigcup \mathcal{E} \restriction (\alpha+1) \cap I(f_0) \to \mathbb{Z} \mid \vec{f} \in (Q' \restriction \alpha)^{(k)}\}$. Similarly, let $\mathcal{P}^{\omega_{k+1}}$ be the set of all families of the form $\{\psi_{\vec{f}} : E \cap I(f_0) \to \mathbb{Z} \mid \vec{f} \in Q'^{(k)}\}$

We fix a coding function G with domain ${}^{\leq\omega_{k+1}}2$ such that

- for all $\alpha \in S^{k+1}_k \cup \{\omega_{k+1}\}$, $G \restriction {}^\alpha 2$ is a surjection from ${}^\alpha 2$ to $\mathcal{P}^\alpha$.
- for all $\alpha < \beta$, both in $S^{k+1}_k \cup \{\omega_{k+1}\}$ and all $\sigma \in {}^\alpha 2$ and $\sigma' \in {}^\beta 2$, if σ' end-extends σ, then the family $G(\sigma')$ extends the family $G(\sigma)$.

We show that such a G exists by recursion on the increasing enumeration of $S^{k+1}_k \cup \{\omega_{k+1}\}$.

Let $\alpha \in S^{k+1}_k$ and assume that the conditions are satisfied for all lower members of the stationary set. If $\alpha > \sup(\alpha \cap S^{k+1}_k)$, then $\alpha = \sup(\alpha \cap S^{k+1}_k) + \omega_k$. Since $|\bigcup \mathcal{E} \restriction (\alpha+1)| \leq \aleph_k$ by Lemma 3.20, we can use the last ω_k values of $\sigma \in {}^\alpha 2$ to code the values of all possible extensions of $\bigcup_{\gamma \in \alpha \cap S^{k+1}_k} G(\sigma \restriction \gamma)$ on $\bigcup \mathcal{E} \restriction (\alpha+1)$ and for multi-indices in $(Q' \restriction \alpha)^{(k)}$, thus preserving the two conditions above.

If $\alpha = \sup(\alpha \cap S^{k+1}_k)$, then we use the fact that $E_\alpha \subseteq \bigcup \mathcal{E} \restriction \alpha$ and let $G(\sigma) = \bigcup_{\gamma \in \alpha \cap S^{k+1}_k} G(\sigma \restriction \gamma)$.

Finally, for $\sigma \in {}^{\omega_{k+1}}2$ we let $G(\sigma) = \bigcup_{\alpha \in S^{k+1}_k} G(\sigma \restriction \alpha)$.

Now we define a sequence of $(k+1)$-coherent (and trivial) families $\langle \Phi^\alpha \mid \alpha < \omega_{k+1}\rangle$ such that $\Phi = \bigcup_{\alpha < \omega_{k+1}} \Phi^\alpha$ is nontrivial.

At limit λ, let $\Phi^\lambda = \bigcup_{\alpha<\lambda} \Phi^\alpha$. At successor step, we know by Goblot's theorem that Φ^α is trivial, so we let $\partial(\Theta^\alpha) = \Phi^\alpha$, then we just consider the extension $\widetilde{\Theta}^\alpha$ defined as

$$\widetilde{\Theta}^\alpha_{\vec{f}} = \begin{cases} \Theta^\alpha_{\vec{f}} & \text{if } \vec{f} \in P^{(k)}_\alpha \\ 0 & \text{if } \vec{f} \in P^{(k)}_{\alpha+1} \setminus P^{(k)}_\alpha \end{cases}$$

Now, if $\mathrm{cf}(\alpha) < \omega_k$, then we simply let $\Phi^{\alpha+1} = \partial(\widetilde{\Theta}^\alpha)$.

If $\mathrm{cf}(\alpha) = \omega_k$ instead, then $(P_\alpha, Q' \restriction \alpha)$ is a strong k-unbounded pair, so by the induction assumption $T(k)$, there exists a k-coherent family Ξ^α on P_α such that $\Xi^\alpha \restriction (Q' \restriction \alpha)^{(k)} \restriction\restriction E_\alpha$ is nontrivial. Similarly, we let

$$\widetilde{\Xi}^\alpha_{\vec{f}} = \begin{cases} \Xi^\alpha_{\vec{f}} & \text{if } \vec{f} \in P^{(k)}_\alpha \\ 0 & \text{if } \vec{f} \in P^{(k)}_{\alpha+1} \setminus P^{(k)}_\alpha \end{cases}$$

Then we let $\Phi^{\alpha,0} = \partial(\widetilde{\Theta}^\alpha)$, and $\Phi^{\alpha,1} = \partial(\widetilde{\Theta}^\alpha + \widetilde{\Xi}^\alpha)$. We need to decide which to use as our $\Phi^{\alpha+1}$. Let $\langle A_\alpha \mid \alpha \in S^{k+1}_k \rangle$ be a $\diamondsuit(S^{k+1}_k)$-sequence. Consider $A_\alpha \in {}^\alpha 2$. If $\Psi_{A_\alpha} = G(A_\alpha)$ does not trivialize the family $(\Phi^\alpha \restriction (Q' \restriction \alpha)^{(k+1)}) \restriction\restriction E_\alpha$, then put $\Phi^{\alpha+1} = \Phi^{\alpha,\epsilon}$ for any $\epsilon \in \{0,1\}$. If it does, consider the following claim:

Claim 3.31. *For every trivialization Υ of $(\Phi^\alpha \restriction (Q' \restriction \alpha)^{(k+1)}) \restriction\restriction E_\alpha$ there exists $\epsilon \in \{0,1\}$ such that Υ does not extend to a trivialization of the family $(\Phi^{\alpha,\epsilon} \restriction (Q' \restriction (\alpha+1))^{(k+1)}) \restriction\restriction E_\alpha$*

Proof of Claim. Suppose toward a contradiction that there exist Υ^0 and Υ^1 that trivialize $(\Phi^{\alpha,0} \restriction (Q' \restriction (\alpha+1))^{(k+1)}) \restriction\restriction E_\alpha$ and $(\Phi^{\alpha,1} \restriction (Q' \restriction (\alpha+1))^{(k+1)}) \restriction\restriction E_\alpha$ respectively, and $\Upsilon^0 \restriction (Q' \restriction \alpha)^{(k)} = \Upsilon = \Upsilon^1 \restriction (Q' \restriction \alpha)^{(k)}$.

We define $d_f(\Upsilon)$ by $(d_f(\Upsilon))_{\vec{h}} = \Upsilon_{\vec{h},f}$ for $\vec{h} = (h_0, ..., h_{k-2})$. If $f = Q'(\alpha)$ and $\vec{g} = (g_0, ..., g_{k-1})$ ranges over $(Q' \restriction \alpha)^{(k)}$ we have (omitting restrictions to $I_{g_0} \cap E_\alpha$ for readability):

$$\begin{aligned} \partial(d_f(\Upsilon^0))_{\vec{g}} &= \sum_{0\le i\le k-1} (-1)^i (d_f(\Upsilon^0))_{\vec{g}^i} \\ &= \sum_{0\le i\le k-1} (-1)^i \Upsilon^0_{\vec{g}^i,f} \\ &=^* (-1)^{k-1}\Upsilon^0_{\vec{g}} + \Phi^{\alpha,0}_{\vec{g},f} \\ &= (-1)^{k-1}\Upsilon^0_{\vec{g}} + (-1)^k \Theta^\alpha_{\vec{g}}. \end{aligned}$$

By the same reasoning

$$\partial(d_f(\Upsilon^1))_{\vec{g}} =^* (-1)^{k-1}\Upsilon^1_{\vec{g}} + (-1)^k \cdot (\Theta^\alpha_{\vec{g}} + \Xi^\alpha_{\vec{g}}),$$

so that, since Υ^0 and Υ^1 agree on $(Q' \restriction \alpha)^{(k)}$,

$$\partial(d_f(\Upsilon^1 - \Upsilon^0)) = (-1)^k (\Xi^\alpha \restriction (Q' \restriction \alpha)^{(k)}) \restriction\restriction E_\alpha,$$

thus contradicting the fact that $(\Xi^\alpha \restriction (Q' \restriction \alpha)^{(k)}) \restriction\restriction E_\alpha$ is nontrivial.

It should be noticed that the case $k = 1$ is computationally identical but conceptually different insofar as we need to stipulate that a family indexed by 0-tuples is a singleton and that a function defined on $I(f) \cap E_\alpha$ can be regarded as a trivialization of a 1-family below f as it can be extended to the entire grid $\omega_n \times \omega$. □

Then we let $\Phi^{\alpha+1} = \Phi^{\alpha,\epsilon}$ for the $\epsilon \in \{0,1\}$ such that Ψ_{A_α} does not extend to a trivialization of $(\Phi^{\alpha,\epsilon} \restriction (Q' \restriction (\alpha+1))^{(k+1)}) \restriction\restriction E_\alpha$.

Finally, we let $\Phi = \bigcup_{\alpha<\omega_{k+1}} \Phi^\alpha$.

We check that $(\Phi \restriction Q'^{(k+1)}) \restriction\restriction E$ is nontrivial. Let Ψ be any putative trivialization, and let $A \in {}^{\omega_{k+1}}2$ be such that $G(A) = \Psi$. Let $\alpha \in S^{k+1}_k$ be such that $A_\alpha = A \cap \alpha$. So, $G(A_\alpha) = \Psi_{A_\alpha} = (\Psi \restriction (Q' \restriction \alpha)^{(k)}) \restriction\restriction \bigcup \mathcal{E} \restriction (\alpha+1)$. By Lemma 3.29, Ψ_{A_α} should trivialize $(\Phi^\alpha \restriction (Q' \restriction \alpha)^{(k+1)}) \restriction\restriction E_\alpha$. Therefore, $(\Phi^{\alpha+1} \restriction (Q' \restriction (\alpha+1))^{(k+1)}) \restriction\restriction E_\alpha$ is not trivialized by any extension of Ψ_{A_α} such as $(\Psi \restriction (Q' \restriction (\alpha+1)^{(k)})) \restriction\restriction \bigcup \mathcal{E} \restriction (\alpha+1)$. Since $Q' \restriction (\alpha+1) \subseteq Q'$ and $\bigcup \mathcal{E} \restriction (\alpha+1) \subseteq E$, again by Lemma 3.29, Ψ is not a trivialization of $(\Phi \restriction Q'^{(k+1)}) \restriction\restriction E$. □

Theorem 3.32. *Let $k \geq 1$. If $\bigwedge_{i<k} \diamondsuit(S^{i+1}_i)$ holds, then the statement $T(k)$ holds.*

Proof. The result easily follows from Lemma 3.28 and Lemma 3.30. □

3.6 Theorem B.3

The following implies Theorem B.3 and will therefore conclude our argument. To simplify its statement, we invoke stronger hypotheses here than are really necessary. This is true in two senses: first, by Lemma 3.28, $\diamondsuit(S^1_0)$ is unneeded, and the statement of Theorem B.3 reflects this fact. Second, even for $i > 0$, the assumptions $\diamondsuit(S^{i+1}_i)$ are a little stronger than we need. In fact, the weak diamond principles $\mathrm{w}\diamondsuit(S^{i+1}_i)$ suffice. To see this, one simply needs to run the argument of [33, Claim 4.11] with domains $\bigcup \mathcal{E} \restriction (\alpha+1)$, and use the fact that these are monotone increasing in α.

Theorem 3.33. *For any $n > 0$, if $2^{\aleph_i} = \aleph_{i+1}$ and $\diamondsuit(S^{i+1}_i)$ hold for all $i \leq n$ then $\lim^{n+1} \mathbf{A}_{\aleph_n} \neq 0$.*

Proof. First note our premises subsume those of Lemma 3.17, thus below, we can and will employ the conclusions and notations of the latter. We have then a continuous filtration ${}^{\omega_n}\omega = \bigcup_{\alpha<\omega_{n+1}} P_\alpha$ and strong n-unbounded pairs (P_α, Q_α) for all $\alpha \in S^{n+1}_n$. From Lemma 3.27 we deduce the existence of an n-twistable set $E^\alpha \subseteq \bigcup_{f\in Q_\alpha} I(f) \subseteq I(f_\alpha)$ for (P_α, Q_α). From Theorem 3.32 and $\bigwedge_{i<n} \diamondsuit(S^{i+1}_i)$ we derive the existence of an n-coherent nontrivial family Ξ^α on P_α such that $(\Xi^\alpha \restriction Q'^{(n)}_\alpha) \restriction\restriction E^\alpha$ is nontrivial. By Lemma 3.29, we conclude that $\Xi^\alpha \restriction Q^{(n)}_\alpha$ itself is nontrivial.

Now we want to code the restriction of any putative trivialization Ψ to $\mathrm{im}(H)$ in an efficient way.

For $\alpha \in S_n^{n+1}$, let $\mathcal{R}^\alpha$ be the set of all n-families indexed over $(H[T' \restriction \alpha])^{(n)}$. Similarly, let $\mathcal{R}^{\omega_{n+1}}$ be the set of all families indexed over $(\mathrm{im}(H))^{(n)}$.

We fix a coding function G with domain ${}^{\leq\omega_{n+1}}2$ such that

- for all $\alpha \in S_n^{n+1} \cup \{\omega_{n+1}\}$, $G \restriction {}^{\alpha}2$ is a surjection from ${}^{\alpha}2$ to $\mathcal{R}^\alpha$.
- for all $\alpha < \beta$, both in $S_n^{n+1} \cup \{\omega_{n+1}\}$ and all $\sigma \in {}^{\alpha}2$ and $\sigma' \in {}^{\beta}2$, if σ' end-extends σ, then the family $G(\sigma')$ extends the family $G(\sigma)$.

We show that such a G exists by recursion on the increasing enumeration of $S_n^{n+1} \cup \{\omega_{n+1}\}$. Let $\alpha \in S_n^{n+1}$ and assume the coding has already been defined for all ordinals in $\alpha \cap S_n^{n+1}$.

If $\alpha > \sup(\alpha \cap S_n^{n+1})$ then $\alpha = \sup(\alpha \cap S_n^{n+1}) + \omega_n$, and we can use the last ω_n values of $\sigma \in {}^{\alpha}2$ to code the values of all possible extensions of $\bigcup_{\gamma\in\alpha\cap S_n^{n+1}} G(\sigma \restriction \gamma)$ (here we are using that $|\mathrm{lev}_\alpha(T')| \leq \aleph_n$ for all α.

If $\alpha = \sup(\alpha \cap S_n^{n+1})$, we simply let $G(\sigma) = \bigcup_{\gamma\in\alpha\cap S_n^{n+1}} G(\sigma \restriction \gamma)$.

Finally, if $\sigma \in {}^{\omega_{n+1}}2$, we let $G(\sigma) = \bigcup_{\alpha\in S_n^{n+1}} G(\sigma \restriction \alpha)$.

Now we construct a sequence $\langle \Phi^\alpha \mid \alpha < \omega_{n+1}\rangle$ of $(n+1)$-coherent families extending one another, where Φ^α is indexed over P_α, and $\Phi = \bigcup_{\alpha<\omega_{n+1}} \Phi^\alpha$ is nontrivial.

At limits let $\Phi^\lambda = \bigcup_{\alpha<\lambda} \Phi^\alpha$. At successors use Goblot's theorem and find $\partial(\Theta^\alpha) = \Phi^\alpha$. Again we consider the extension $\widetilde{\Theta}^\alpha$ defined as

$$\widetilde{\Theta}^\alpha_{\vec{f}} = \begin{cases} \Theta^\alpha_{\vec{f}} & \text{if } \vec{f} \in P_\alpha^{(n)} \\ 0 & \text{if } \vec{f} \in P_{\alpha+1}^{(n)} \setminus P_\alpha^{(n)} \end{cases}$$

Now, if $\mathrm{cf}(\alpha) < \omega_n$, then we simply let $\Phi^{\alpha+1} = \partial(\widetilde{\Theta}^\alpha)$.

If $\mathrm{cf}(\alpha) = \omega_n$ instead, take the coherent nontrivial family Ξ^α P_α defined above such that $\Xi^\alpha \restriction Q_\alpha^{(n)}$ is nontrivial. Then let

$$\widetilde{\Xi}^\alpha_{\vec{f}} = \begin{cases} \Xi^\alpha_{\vec{f}} & \text{if } \vec{f} \in P_\alpha^{(n)} \\ 0 & \text{if } \vec{f} \in P_{\alpha+1}^{(n)} \setminus P_\alpha^{(n)} \end{cases}$$

Finally, let $\Phi^{\alpha,0} = \partial(\widetilde{\Theta}^\alpha)$, and $\Phi^{\alpha,1} = \partial(\widetilde{\Theta}^\alpha + \widetilde{\Xi}^\alpha)$. We need to decide which to use as our $\Phi^{\alpha+1}$. Let $\langle A_\alpha \mid \alpha \in S_n^{n+1}\rangle$ be a $\diamondsuit(S_n^{n+1})$-sequence. Consider $A_\alpha \in {}^{\alpha}2$. Recall that it

codes a family $\Psi_{A_\alpha} = G(A_\alpha)$ indexed over $(H[T' \upharpoonright \alpha])^{(n)}$, so in particular it codes some $\Psi_{A_\alpha} \upharpoonright Q_\alpha^{(n)}$. If $\Psi_{A_\alpha} \upharpoonright Q_\alpha^{(n)}$ does *not* trivialize the family $\Phi^\alpha \upharpoonright Q_\alpha^{(n)}$, then let $\Phi^{\alpha+1} = \Phi^{\alpha,\epsilon}$ for any $\epsilon \in \{0,1\}$. If it does, consider the following claim:

Claim 3.34. *For every trivialization Υ of $\Phi^\alpha \upharpoonright Q_\alpha^{(n+1)}$ there exists $\epsilon \in \{0,1\}$ such that Υ does not extend to a trivialization of the family $\Phi^{\alpha,\epsilon} \upharpoonright (Q_\alpha \cup \{f_\alpha\})^{(n+1)}$.*

Proof of Claim. Suppose toward a contradiction that there exist Υ^0 and Υ^1 that trivialize $\Phi^{\alpha,0} \upharpoonright (Q_\alpha \cup \{f_\alpha\})^{(n+1)}$ and $\Phi^{\alpha,1} \upharpoonright (Q_\alpha \cup \{f_\alpha\})^{(n+1)}$ respectively, and $\Upsilon^0 \upharpoonright Q_\alpha^{(n)} = \Upsilon = \Upsilon^1 \upharpoonright Q_\alpha^{(n)}$.

We define $d_f(\Upsilon)$ by $(d_f(\Upsilon))_{\vec{h}} = \Upsilon_{\vec{h},f}$ where $\vec{h} = (h_0, ..., h_{n-2})$. If $f = f_\alpha$ and $\vec{g} = (g_0, ..., g_{n-1})$ ranges over $Q_\alpha^{(n)}$ we have (omitting restrictions to I_{g_0} for readability):

$$\begin{aligned}
\partial(d_f(\Upsilon^0))_{\vec{g}} &= \sum_{0\le i\le n-1} (-1)^i (d_f(\Upsilon^0))_{\vec{g}^i} \\
&= \sum_{0\le i\le n-1} (-1)^i \Upsilon^0_{\vec{g}^i,f} \\
&=^* (-1)^{n-1}\Upsilon^0_{\vec{g}} + \Phi^{\alpha,0}_{\vec{g},f} \\
&= (-1)^{n-1}\Upsilon^0_{\vec{g}} + (-1)^n \Theta^\alpha_{\vec{g}}.
\end{aligned}$$

By the same reasoning

$$\partial(d_f(\Upsilon^1))_{\vec{g}} = (-1)^{n-1}\Upsilon^1_{\vec{g}} + (-1)^n \cdot (\Theta^\alpha_{\vec{g}} + \Xi^\alpha_{\vec{g}}),$$

so that, since Υ^0 and Υ^1 agree on $Q_\alpha^{(k)}$,

$$\partial(d_f(\Upsilon^1 - \Upsilon^0)) = (-1)^n \Xi^\alpha \upharpoonright Q_\alpha^{(n)},$$

contradicting the fact that $\Xi^\alpha \upharpoonright Q_\alpha^{(n)}$ is nontrivial.

Once again, the case $n = 1$ is computationally identical but conceptually different insofar as we need to stipulate that a family indexed by 0-tuples is a singleton and that a function defined on $I(f)$ can be regarded as a trivialization of a 1-family below f as it can be extended to the entire grid $\omega_n \times \omega$. □

Then we let $\Phi^{\alpha+1} = \Phi^{\alpha,\epsilon}$ for the $\epsilon \in \{0,1\}$ such that $\Psi_{A_\alpha} \upharpoonright Q_\alpha^{(n)}$ does not extend to a trivialization of $\Phi^{\alpha,\epsilon} \upharpoonright (Q_\alpha \cup \{f_\alpha\})^{(n+1)}$. Finally, let $\Phi = \bigcup_{\alpha<\omega_{n+1}} \Phi^\alpha$.

We check that Φ is nontrivial. Let Ψ be any putative trivialization, and let $A \in {}^{\omega_{n+1}}2$ be such that $G(A) = \Psi \upharpoonright (\mathrm{im}(H))^{(n)}$. Let $\alpha \in S_n^{n+1}$ be such that $A_\alpha = A \cap \alpha$. Then $\Psi_{A_\alpha} \upharpoonright Q_\alpha^{(n)} = \Psi \upharpoonright Q_\alpha^{(n)}$. By Lemma 3.29, $\Psi \upharpoonright Q_\alpha^{(n)}$ should trivialize $\Phi^\alpha \upharpoonright Q_\alpha^{(n+1)}$. Therefore, $\Phi^{\alpha+1} \upharpoonright (Q_\alpha \cup \{f_\alpha\})^{(n+1)}$ is not trivialized by any extension of $\Psi_{A_\alpha} \upharpoonright Q_\alpha^{(n)}$ such as $\Psi \upharpoonright (Q_\alpha \cup \{f_\alpha\})^{(n)}$. Again by Lemma 3.29, Ψ is not a trivialization of Φ. □

3.7 Conclusion

Let us return, now, to item (ii) of our introduction's second paragraph, or in other words to the problem of understanding more general variants of the classical system $\mathbf{A}$. For reasons outlined thereafter, the central remaining question in this direction is the following:

Question 3.35. *Is it consistent with the* ZFC *axioms that* $\lim^n \mathbf{A}_\lambda = 0$ *for all cardinals* λ *and integers* $n > 0$*?*

The question is, at this point, a well-documented one: minor variations on it appear as [15, Question 3], [20, Question 7.10], and [19, Question 5.7]; see also [8, Questions 7.3 and 7.1], respectively, for a subinstance and more general formulation. Moreover, as Bannister has very recently shown [9], an affirmative answer is in fact equivalent to the additivity of strong homology on all locally compact metric spaces. Any of these variants of Question 3.35 should ultimately be regarded as asking just how additive a derived limit functor is compatible with the ZFC axioms. This paper's results lend this question additional interest, both by ruling out the most optimistic scenario for its solution (namely, the extension of the result [15, Theorem 5.1] recorded in our introduction as *Theorem* to all degrees $n > 0$), and, more broadly, by identifying fundamental combinatorial possibilities which any positive solution to the question must sidestep. Note also the relation to our introduction's item (i): if a main theme in the study of $\lim^n \mathbf{A}$ has been *vertical* interactions (via the cardinal invariant $\mathfrak{d}$, for example) of ordinal combinatorics with product orderings like $({}^\omega\omega, \leq)$, it's their persistent *horizontal* interactions in our analyses that seem to account for some of the difficulty of Question 3.35, and of research line (ii) more generally. Note lastly two variations on this work's initiating question left open:

Question 3.36. *Does* $\lim^n \mathbf{A} = 0$ *for all integers* $n > 0$ *imply that* $\lim^n \mathbf{A}_\lambda = 0$ *for all cardinals* λ *and integers* $n > 0$*?*

Question 3.37. *Does there exist for each integer* $n > 0$ *an accessible cardinal* κ *such that* $\lim^n \mathbf{A}_\kappa = 0$ *implies that* $\lim^n \mathbf{A}_\lambda = 0$ *for all cardinals* $\lambda \geq \kappa$*?*

Affirmative answers to either continue to figure as central scenarios for a positive solution to Question 3.35.

Part II

Descriptive set theory

Chapter 4

Homological algebra of pro-Lie Polish abelian groups

4.1 Introduction

In this chapter, we initiate the study of pro-Lie Polish abelian groups from the viewpoint of homological algebra. *Pro-Lie groups* [81] are the topological groups that can be written as limits of an inverse system of Lie groups. We will focus on pro-Lie groups that are furthermore *Polish*, in which case they can be written as limits of an inverse sequence of Lie groups. The class of pro-Lie Polish groups is a natural extension of the class of Lie groups, and it satisfies desirable closure properties, as it is closed under countable products, closed subgroups, and quotients by closed subgroups within the class of Polish groups.

We will focus on the case of *abelian* pro-Lie Polish groups, as they form a category enriched over abelian groups, where the machinery from homological algebra can be applied. The category of abelian Polish pro-Lie groups contains all the *locally compact* Polish abelian groups, as well as the *non-Archimedean* Polish abelian groups (which are inverse limits of countable discrete groups). In fact, the class of pro-Lie Polish abelian groups is the smallest class of Polish abelian groups that contains all locally compact Polish abelian groups and non-Archimedean Polish abelian groups and it is closed under extensions.

The study of homological invariants in the context of topological abelian groups followed rapidly their introduction in the purely algebraic setting, going as far back as the 1940s [41, 136]; see also [29, 32, 65–67, 118, 134]. This study has been most commonly undertaken for *locally compact* topological Hausdorff (Polish) abelian groups. Injective and projective objects in this category were characterized by Moskowitz in [134], showing in particular that this category does *not* have enough injective and projective objects. The Yoneda Ext groups in terms of extensions were introduced and studied by Fulp and Griffith in [67], where it is proved that Ext^2 vanishes. The functor Ext for locally compact Hausdorff topological abelian groups was later studied from the viewpoint of derived categories by Hoffmann and Spitzweck [80], who recognized it as the cohomological right derived functor of Hom in the sense of Verdier [166], despite the category of locally compact Hausdorff topological groups not having enough injectives or projectives.

Derived categories and functors had been traditionally studied in the context of *abelian categories*. However, this framework does not include the category of locally compact topological Hausdorff (Polish) abelian groups, as the inclusion map $\mathbb{Q} \to \mathbb{R}$ of the rationals with the discrete topology into the reals with the Euclidean topology is an arrow that is both monic and epic but not an isomorphism. This and other categories of algebraic structures endowed with a topology can be seen as *quasi-abelian categories*, where the axioms of abelian categories (such as the requirement that every epimorphism be the cokernel of its kernel) are replaced with suitable relaxations. Most of the notions and constructions usually considered in the context of abelian categories can be generalized to the quasi-abelian case. Furthermore, Schneiders describes in [151], building on work of Beilinson–Bernstein–Deligne [12], a canonical way to enlarge a given quasi-abelian category $\mathcal{A}$ to an abelian category $\mathrm{LH}(\mathcal{A})$ called its *left heart*.

The quasi-abelian viewpoint affords Hoffmann and Spitzwek to *refine* the functor Ext for locally compact topological (Polish) abelian groups, by seeing it as a functor to the *left heart* of the category of topological (Polish) abelian groups. The category $\mathbf{PAb}$ of Polish

abelian groups, or any of its full subcategories closed by taking closed subgroups, quotients by closed subgroups, and extensions (*thick subcategories*), is easily seen to be quasi-abelian. Using tools from descriptive set theory, an explicit description of its left heart as concrete category was recently provided in [114] in terms of *groups with a Polish cover* and *Borel-definable group homomorphisms*; see also [22].

Building on the work of Hoffmann and Spitzweck, Bergfalk, Sarti, and the third-named author of the paper from which this chapter is taken [21] gave a short purely homological proof of the fact that the functor Hom on the category $\mathbf{LCPAb}$ of locally compact Polish abelian groups has a *total* right derived functor, from which Ext^n for $n \geq 0$ is recovered by taking cohomology, and the homological dimension of $\mathbf{LCPAb}$ is 1. The latter result recovers the theorem that $\mathrm{Ext}^n = 0$ for $n \geq 2$ that had previously been obtained by Fulp and Griffith via a much more laborious argument. The enrichment of Ext as group with a Polish cover is used in [21] to completely characterize injective and projective objects in the left heart of $\mathbf{LCPAb}$ and many of its salient thick subcategories, showing in particular that $\mathrm{LH}(\mathbf{LCPAb})$ has no nonzero injectives.

In this chapter, we extend the applications of Borel-definable methods to the category of pro-Lie Polish abelian groups. We begin with showing that the category $\mathbf{proLiePAb}$ of pro-Lie Polish abelian groups is quasi-abelian, and in fact a thick subcategory of $\mathbf{PAb}$. In order to prove that an extension of pro-Lie Polish abelian groups is pro-Lie, we use recently developed tools from the theory of groups with a Polish cover and Borel and continuous cocycles, including a continuous selection theorem from [22]. We then completely characterize projective and injective objects in $\mathbf{proLiePAb}$, showing in particular that $\mathbf{proLiePAb}$ has enough projectives (but not enough injectives) and homological dimension 1. Thus $\mathbf{proLiePAb}$ is in some ways better-behaved than $\mathbf{LCPAb}$ from the viewpoint of homological algebra, and the total derived functor of Hom on $\mathbf{proLiePAb}$ can be defined in terms of projective resolutions, as in the case of discrete groups. In the case when the first argument is locally compact, and the second one merely pro-Lie, we prove that Ext can be regarded as a functor to the left heart of $\mathbf{PAb}$, and hence enriched with the structure of group with a Polish cover, extending from the locally compact case the analysis in [21].

We perform a similar analysis on the category $\mathbf{PAb}_{\mathrm{nA}}$ of *non-Archimedean* Polish abelian groups, which is also a thick subcategory of $\mathbf{PAb}$. In this case, we completely characterize injective and projective objects, and show that $\mathbf{PAb}_{\mathrm{nA}}$ has enough injectives and projectives and homological dimension 1. We also completely characterize injective and projective objects in the categories of pro-p pro-Lie Polish abelian groups and topological torsion pro-Lie Polish abelian groups, and show that these categories have enough injectives and homological dimension 1.

We also extend from the locally compact case—considered by Hoffmann and Spitzweck in [80]—to pro-Lie Polish abelian groups the notions of type $\mathbb{Z}$, type $\mathbb{A}$, and type $\mathbb{S}^1$ groups, as well as the decomposition of an arbitrary pro-Lie Polish abelian groups in terms of groups of these types.

Finally, we show that the functor $\mathrm{Hom} : \mathbf{LCPAb} \times \mathbf{PAb} \to \mathbf{PAb}$ admits a total right derived functor, extending a result from [21] in the locally compact case.

The rest of this chapter is divided into three sections. In Section 5.2 we recall some preliminary notions from category theory, including the notions of exact and quasi-abelian category and the corresponding notions of derived category and functor. In Section 4.3 we obtain our main results outlined above concerning the category of pro-Lie Polish abelian groups. Finally, in Section 4.4 we consider several natural thick (or just fully exact) subcategories of $\mathbf{proLiePAb}$, characterizing injective and projective objects in each of them, and determining whether they have enough injectives or projectives.

4.2 Category theory background

4.2.1 Exact categories

An *additive category* [117, Section VIII.2] is a category enriched over the category $\mathbf{Ab}$ of abelian groups that also has a terminal object (which is necessarily also initial, and called the *zero object*) and binary products (which are necessarily also coproducts, and called *biproducts*). A kernel-cokernel pair in an additive category $\mathcal{A}$ is a pair (f, g) of arrows in $\mathcal{A}$ such that f is the kernel of g and g is the cokernel of f. This can be seen as an object in the double arrow category $\mathcal{A}^{\cdot\to\cdot\to\cdot}$ of $\mathcal{A}$.

An *exact structure* $\mathcal{E}$ on $\mathcal{A}$ [30, Definition 2.1] is a collection of short exact sequences closed under isomorphism in $\mathcal{A}^{\cdot\to\cdot\to\cdot}$ that satisfies the following axioms as well as their duals obtained by reversing all the arrows. In order to formulate the axioms, we say that a morphism f is an admissible monic (or inflation) or an admissible epic (or deflation) if it appears in an exact sequence in $\mathcal{E}$:

1. for every object A of $\mathcal{A}$, the corresponding identity arrow 1_A is an admissible monic;
2. the collection of admissible monics is closed under composition;
3. the push-out of an admissible monic along an arbitrary morphism exists and it is an admissible monic.

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is an additive category and $\mathcal{E}$ is an exact structure on $\mathcal{A}$. A short exact sequence in $(\mathcal{A}, \mathcal{E})$ is by definition an element of $\mathcal{E}$. In an exact category, an isomorphism is an admissible monic and an admissible epic; see [30, Remark 2.3]. Furthermore, by [30, Remark 2.3, Proposition 2.15] we have the following—see [7, Sections 5.2, 5.3] for the notion of pull-back along a morphism:

Lemma 4.1. *In an exact category, the pull-back of an admissible monic along an admissible epic is an admissible monic.*

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are exact categories, and $F : \mathcal{A} \to \mathcal{B}$ is a functor. One says that F is *exact* if it maps short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathcal{B}$ [30, Remark 2.3, Proposition 2.15]. By [30, Proposition 5.2], this implies that F preserves push-outs along admissible monics, and pull-backs along admissible epics.

Remark 4.2. *In the context of finitely complete and finitely cocomplete categories, a stronger notion of exact functor is considered. If $\mathcal{C}$ and $\mathcal{D}$ are categories that are finitely complete and finitely cocomplete (i.e., have finite limits and finite colimits), then a functor $F : \mathcal{C} \to \mathcal{D}$ is called* exact *if it is finitely continuous and finitely cocontinuous (i.e., commutes with finite limits and finite colimits); see [100, Section 3.3]. An exact functor in the sense of [30, Proposition 5.2] between finitely complete and finitely cocomplete exact categories is not necessarily exact in the sense of [100, Section 3.3].*

Suppose that $\mathcal{A}$ is an exact category. A *fully exact* subcategory [30, Definition 10.21] of $\mathcal{A}$ is a full subcategory $\mathcal{B}$ of $\mathcal{A}$ such that, whenever $0 \to B' \to A \to B'' \to 0$ is a short exact sequence in $\mathcal{A}$ with B' and B'' in $\mathcal{B}$, one also has that A is isomorphic to an object of $\mathcal{B}$. In this case, we have that $\mathcal{B}$ is an exact category, where a short exact sequence in $\mathcal{B}$ is a short exact sequence in $\mathcal{A}$ whose morphisms are in $\mathcal{B}$ [30, Definition 10.20].

Let $\mathcal{A}$ be an exact category and f be a morphism in $\mathcal{A}$. Then we say that f is *admissible* [30, Definition 8.1] if $f = m \circ e$ for some admissible epic e and admissible monic m. The pair (e, m) is called an admissible factorization of f and its essentially unique [30, Lemma 8.4]. The notion of admissible morphism recovers the notion of admissible monic (respectively, epic) in the particular case when the given morphism is monic (respectively, epic) [30, Remark 8.3]. The class of admissible morphisms is closed under push-out along admissible monics and pull-back along admissible epics [30, Lemma 8.7].

Definition 4.3. *A sequence of admissible morphisms*

$$A \xrightarrow{f} B \xrightarrow{g} A'$$

in an exact category $\mathcal{A}$ is exact if

$$\bullet \xrightarrow{m} A \xrightarrow{e'} \bullet$$

is short exact, where $f = m \circ e$ and $g = m' \circ e'$ are admissible factorizations.

4.2.2 Derived categories

Let $\mathcal{A}$ be an additive category. One can then define the category $\mathrm{Ch}(\mathcal{A})$ of complexes, and the homotopy category $\mathrm{K}(\mathcal{A})$; see [30, Section 9]. We have that $\mathrm{K}(\mathcal{A})$ has a canonical structure of *triangulated category* [30, Remark 9.8], graded by the translation functor T defined by $TA = A[1]$ where $A[k]^n = A^{n+k}$ for $n, k \in \mathbb{Z}$. The distinguished triangles in $\mathrm{K}(A)$ are those isomorphic to a strict triangle, which is one of the form

$$(A, B, \mathrm{cone}(f), f, i, j)$$

for some morphism of complexes $f : A \to B$, where $i : B \to \mathrm{cone}(f)$ and $j : \mathrm{cone}(f) \to TA$ are the canonical morphisms of complexes.

A morphism $f : A \to A$ in $\mathcal{A}$ is idempotent if $f = f \circ f$. The category $\mathcal{A}$ is idempotent-complete if every idempotent morphism has a kernel. Suppose now that $\mathcal{A}$ is an *idempotent-complete exact category*. A complex A over $\mathcal{A}$ is *acyclic* [30, Section 10] if it has admissible

differentials and it is exact at each degree as in Definition 4.3. We let $\mathrm{N}(\mathcal{A})$ be the full subcategory of $\mathrm{K}(\mathcal{A})$ spanned by acyclic complexes. Then we have that $\mathrm{N}(\mathcal{A})$ is a *thick subcategory* [137, Definition 2.1.6] of the triangulated category $\mathrm{K}(\mathcal{A})$ [30, Corollary 10.11]. This means that $\mathrm{N}(\mathcal{A})$ is a *triangulated subcategory* of $\mathrm{K}(\mathcal{A})$ [137, Definition 1.5.1] and contains all the direct summands of its objects.

One can thus define the corresponding Verdier quotient $\mathrm{D}(\mathcal{A}) := \mathrm{K}(\mathcal{A})/\mathrm{N}(\mathcal{A})$ [137, Theorem 2.1.8], which is called the *derived category* of $\mathcal{A}$ [30, Section 10]. By definition, this is the localization $\mathrm{K}(\mathcal{A})[\Sigma^{-1}]$ where Σ is the multiplicative system in $\mathrm{K}(\mathcal{A})$ consisting of homotopy classes of chain maps whose mapping cone is acyclic, called *quasi-isomorphisms* [30, Definition 10.16]. If $Q_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is the quotient functor, then a morphism in $\mathrm{K}(\mathcal{A})$ is a quasi-isomorphism if and only if its image in $\mathrm{D}(\mathcal{A})$ is invertible, i.e. the multiplicative system Σ is *saturated* [30, Remark 10.19].

The full subcategories $\mathrm{K}^+(\mathcal{A})$, $\mathrm{K}^-(\mathcal{A})$, and $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ of $\mathrm{K}(\mathcal{A})$ and the corresponding full subcategories $\mathrm{D}^+(\mathcal{A})$, $\mathrm{D}^-(\mathcal{A})$, and $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ of $\mathrm{D}(\mathcal{A})$ are defined in a similar fashion, by considering only left-bounded, right-bounded, and bounded complexes respectively.

4.2.3 Abelian and quasi-abelian categories

A *quasi-abelian category* [30, Definition 4.1] is a finitely complete and finitely cocomplete additive category such that the push-out of a kernel along an arbitrary morphism is a kernel, and dually the pull-back of a cokernel along an arbitrary morphism is a cokernel. If $\mathcal{A}$ is a quasi-abelian category, then $(\mathcal{A}, \mathcal{E})$ is an exact category, where $\mathcal{E}$ is the collection of all the kernel-cokernel pairs in $\mathcal{A}$ [30, Proposition 4.4]. In what follows, we will regard every quasi-abelian category as an exact category with respect to such a canonical (maximal) exact structure. An *abelian category* is a quasi-abelian category where every morphism is admissible or, equivalently, every monomorphism is a kernel and every epimorphism is a cokernel [30, Remark 4.7].

Let $\mathcal{A}$ be a quasi-abelian category, and $\mathcal{B}$ be a (not necessarily full) subcategory of $\mathcal{A}$. Then we say that $\mathcal{B}$ is a quasi-abelian subcategory of $\mathcal{A}$ if $\mathcal{B}$ is a quasi-abelian category and the inclusion functor $\mathcal{B} \to \mathcal{A}$ is finitely continuous and finitely cocontinuous. We say that $\mathcal{B}$ is a *fully exact quasi-abelian subcategory* if it is a quasi-abelian subcategory that is also a fully exact subcategory (where $\mathcal{A}$ is endowed with its canonical exact structure). In this case, we also say that $\mathcal{B}$ is a *thick* subcategory of $\mathcal{A}$; see [100, Definition 8.3.21]. (Notice that this is different from the notion of thick subcategory of a triangulated category from [137, Definition 2.1.6].)

Let $\mathcal{A}$ be a quasi-abelian category and $\mathrm{D}(\mathcal{A})$ be its derived category. The *left heart* of $\mathcal{A}$ is by definition the heart of $\mathrm{D}(\mathcal{A})$ with respect to its canonical left truncation structure [151, Definition 1.2.18]. Concretely, $\mathrm{LH}(\mathcal{A})$ is the full subcategory spanned by complexes A with $A^n = 0$ for $n \in \mathbb{Z} \setminus \{-1, 0\}$ and such that the differential $A^{-1} \to A^0$ is monic. Then $\mathrm{LH}(\mathcal{A})$ is an abelian category that contains $\mathcal{A}$ as a thick subcategory [151, Proposition 1.2.29]. Furthermore, the inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$ satisfies the following universal property: for every abelian category $\mathcal{M}$ and finitely continuous and exact functor $F : \mathcal{A} \to \mathcal{M}$ there exists an essentially unique functor $\mathrm{LH}(\mathcal{A}) \to \mathcal{M}$ whose restriction to $\mathcal{A}$ is isomorphic

to F [151, Proposition 1.2.34]. The inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$ extends to an equivalence of categories $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathrm{LH}(\mathcal{A}))$ [151, Proposition 1.2.32], and it has a left adjoint $\kappa : \mathrm{LH}(\mathcal{A}) \to \mathcal{A}$ [151, Definition 1.2.26 and Proposition 1.2.27].

The canonical left truncation structure on $\mathrm{D}(\mathcal{A})$ yields the *cohomology functors* $\mathrm{H}^n : \mathrm{D}(\mathcal{A}) \to \mathrm{LH}(\mathcal{A})$ for $n \in \mathbb{Z}$ [151, Definition 1.2.18]. We have that a morphism f in $\mathrm{D}(\mathcal{A})$ is an isomorphism if and only if $\mathrm{H}^n(f)$ is an isomorphism for every $n \in \mathbb{Z}$.

Recall that a *torsion pair* [161, Definition 2.4] in a quasi-abelian category $\mathcal{M}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories, where $\mathcal{T}$ is called a torsion class and $\mathcal{F}$ called a torsion-free class, such that:

1. for all objects T of $\mathcal{T}$ and F of $\mathcal{F}$, $\mathrm{Hom}(T, F) = 0$;
2. for all objects M of $\mathcal{M}$ there exists a short exact sequence

$$0 \to {}_{\mathcal{T}}M \to M \to M_{\mathcal{F}} \to 0$$

where ${}_{\mathcal{T}}M$ is in $\mathcal{T}$ and $M_{\mathcal{F}}$ is in $\mathcal{F}$.

In this case, we have that an object X of $\mathcal{M}$ is in $\mathcal{T}$ if and only if $\mathrm{Hom}(X, C) = 0$ for all C in $\mathcal{F}$, and it is in $\mathcal{F}$ if and only if $\mathrm{Hom}(T, X) = 0$ for all T in $\mathcal{T}$. If $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{M}$, then $\mathcal{T}$ and $\mathcal{F}$ are full subcategories of $\mathcal{M}$ (essentially) closed under extensions [147, Theorem 2]. Conversely, if $\mathcal{A}$ is a quasi-abelian category, let $\kappa : \mathrm{LH}(\mathcal{A}) \to \mathcal{A}$ be the left adjoint of the inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$. Thus, we have that

$$\mathrm{Hom}_{\mathcal{A}}(\kappa(X), B) \cong \mathrm{Hom}_{\mathrm{LH}(\mathcal{A})}(X, B)$$

for all objects B of $\mathcal{A}$. Define $\mathcal{T}$ to be the full subcategory of $\mathrm{LH}(\mathcal{A})$ spanned by the objects X such that $\kappa(X) = 0$, and define $\mathcal{F}$ to be equal to $\mathcal{A}$. Then we have that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathrm{LH}(\mathcal{A})$ [147, Theorem 2].

Recall that if $\mathcal{C}$ is a category with finite products and initial object 1, then an (abelian) group in a category is a tuple (G, m, i, u) where G is an object of $\mathcal{C}$, $m : G \times G \to G$, $i : G \to G$, and $u : 1 \to G$ are morphisms in $\mathcal{C}$ that satisfy the "group axioms" (and the commutativity of the operation), expressed in terms of morphisms of $\mathcal{C}$ [7, Definition 4.1]. A homomorphism between group objects in $\mathcal{C}$ is a morphism in $\mathcal{C}$ between group objects that commutes with the "group operations" [7, Definition 4.2]. In the category of (abelian) groups, every object has a unique group structure, which is automatically abelian [7, Proposition 4.5].

Definition 4.4. *A category of abelian groups is a category $\mathcal{C}$ with finite products such that every object of $\mathcal{C}$ has a unique abelian group structure.*

We regard an object of a category of abelian groups as an abelian group object in $\mathcal{C}$ with respect to its unique abelian group structure. Notice that a morphism in $\mathcal{C}$ is automatically a homomorphism between group objects, with respect to such a unique abelian group structure.

For example, the category of abelian Polish groups is a category of abelian groups. Likewise, any quasi-abelian subcategory of the category of abelian Polish groups is a category of abelian groups. If R is a ring, then the category of (left) R-modules is a category of abelian groups.

Recall that an $\mathbf{Ab}$-category is a category enriched over the (monoidal) category $\mathbf{Ab}$ of abelian groups. An additive category is an $\mathbf{Ab}$-category with that has a *zero object* and *biproducts* [117, Chapter VIII].

Suppose that $\mathcal{M}$ is a category of abelian groups. For objects A, B of $\mathcal{M}$ and morphisms $\varphi, \psi : A \to B$ define $\varphi + \psi : A \to B$ to be the morphism $m_B \circ (\varphi \times \psi) \circ \Delta_A$ where $\Delta_A : A \to A \times A$ is the canonical diagonal morphism, m_B is the group operation in B.

Proposition 4.5. *Let $\mathcal{M}$ be a locally small category of abelian groups. Then $\mathcal{M}$ is an additive category with respect to the operation on* Hom*-sets defined above.*

Proof. Associativity of the operation follows from associativity of m_B. It is easily seen that the unique morphism $0 : A \to B$ that factors through the initial object of $\mathcal{M}$ is the neutral element with respect to the operation, while $i_B \circ \varphi$ is the opposite of a morphism φ. Since by assumption $\mathcal{M}$ has all finite products, it is an additive category [117, Chapter VIII, Section 2, Proposition 1 and Theorem 2]. □

We define a (quasi-)abelian category of abelian groups to be a locally small category of abelian groups that is (quasi-)abelian with respect to the additive category structure as in the previous proposition.

We would like to consider the natural generalization of $\mathbf{Ab}$-category where $\mathbf{Ab}$ is replaced with a category of abelian groups that is not necessarily monoidal, such as the category of pro-Lie Polish abelian groups.

Definition 4.6. *Let $\mathcal{M}$ be a quasi-abelian category of abelian groups. An $\mathcal{M}$-category is a category $\mathcal{A}$ such that for every object A and B of $\mathcal{A}$,* $\mathrm{Hom}_{\mathcal{A}}(A, B)$ *is an object of $\mathcal{M}$, and* $\mathrm{Hom}_{\mathcal{A}}(-, B)$ *an* $\mathrm{Hom}_{\mathcal{A}}(A, -)$ *are functors $\mathcal{A} \to \mathcal{M}$. Given $\mathcal{M}$-categories $\mathcal{A}$ and $\mathcal{B}$, an additive $\mathcal{M}$-functor $F : \mathcal{A} \to \mathcal{B}$ is a functor such that, for every object A, B in $\mathcal{A}$,* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(FA, FB)$ *is a morphism in $\mathcal{M}$.*

Example 4.7. *If R is a commutative ring, then R-modules form an abelian category of abelian groups* $\mathrm{Mod}(R)$*. A* $\mathrm{Mod}(R)$*-category in the sense of Definition 4.6 is the same as an R-linear category.*

Notice that, if $\mathcal{A}$ is an $\mathcal{M}$-category, then $\mathcal{A} \times \mathcal{A}$ has a natural $\mathcal{M}$-category structure obtained by the identification

$$\mathrm{Hom}((A_0, A_1), (B_0, B_1)) = \mathrm{Hom}(A_0, B_0) \times \mathrm{Hom}(A_1, B_1).$$

Likewise, the category of arrows $\mathcal{A}^{\to}$ in $\mathcal{A}$ has a natural $\mathcal{M}$-category structure. More generally, for every finite category J, the category $\mathcal{A}^J$ of diagrams of shape J in $\mathcal{A}$ is an $\mathcal{M}$-category.

Definition 4.8. *Let $\mathcal{M}$ be a quasi-abelian category of abelian groups.*

- *An* additive *$\mathcal{M}$-category is an $\mathcal{M}$-category $\mathcal{A}$ that has a zero object and biproducts, and such that the biproduct functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is an $\mathcal{M}$-functor;*
- *A* quasi-abelian *$\mathcal{M}$-category is an additive $\mathcal{M}$-category that is also quasi-abelian, and such that the kernel and cokernel functors are $\mathcal{M}$-functors, which implies that all the finite limit and colimit functors are $\mathcal{M}$-functors;*
- *A* triangulated *$\mathcal{M}$-category is an additive $\mathcal{M}$-category $\mathcal{T}$ that is also triangulated, and such that the shift functor Σ on $\mathcal{T}$ is an $\mathcal{M}$-functor, and for al objects A, B of $\mathcal{T}$, $\mathrm{Hom}(A, -)$ is a homological functor in the sense of [137, Definition 1.1.7] when regarded as a functor $\mathcal{T} \to \mathrm{LH}(\mathcal{M})$, and $\mathrm{Hom}(-, B)$ is a cohomological functor when regarded as a functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{LH}(\mathcal{M})$.*

4.2.4 Derived functors

Suppose that $\mathcal{A}$ and $\mathcal{R}$ are exact categories, and $F : \mathcal{A} \to \mathcal{R}$ is a functor. Then F induces a *triangulated* functor $\mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{R})$ [137, Definition 2.1.1], which we still denote by F. A *total right derived functor* for $F : \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{R})$ [30, Section 10.6]—see also [151, Definition 1.3.1]—is a triangulated functor $\mathrm{R}F : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{R})$ together with a morphism $\mu : Q_{\mathcal{R}} \circ F \Rightarrow \mathrm{R}F \circ Q_{\mathcal{A}}$ of triangulated functors that satisfies the following universal property: for every triangulated functor $G : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{R})$ and morphism $\nu : Q_{\mathcal{R}} \circ F \to G \circ Q_{\mathcal{A}}$ of triangulated functors, there exists a unique morphism of triangulated functors $\sigma : \mathrm{R}F \Rightarrow G$ such that $\nu = (\sigma Q_{\mathcal{A}}) \circ \mu$.

If $\mathcal{R}$ is quasi-abelian, then $\mathrm{H}^0 \circ F$ is a *cohomological functor* [166, Section 1.1, Definition 3.1]; see also [137, Definition 1.1.7]. A *right cohomological derived functor* for $\mathrm{H}^0 \circ F$ is defined in a similar fashion, via a suitable universal property [166, Section 2.2, Definition 1.4].

Definition 4.9. *Suppose that $\mathcal{A}$ is an exact category, and $\mathcal{C}$ is a full subcategory of $\mathcal{A}$. We say that $\mathcal{C}$ is:*

1. generating *[100, Definition 8.3.21(v)] if for each object A of $\mathcal{A}$ there exists an admissible epimorphism $C \to A$ with C in $\mathcal{C}$;*
2. cogenerating *if $\mathcal{C}^{\mathrm{op}}$ is generating in $\mathcal{A}^{\mathrm{op}}$;*
3. closed under quotients *if for all short exact sequences*

$$0 \to A' \to C \to A'' \to 0$$

in $\mathcal{A}$ with C in $\mathcal{C}$, A'' is isomorphic to an object of $\mathcal{C}$;

4. closed under subobjects *if $\mathcal{C}^{\mathrm{op}}$ is closed under quotients in $\mathcal{A}^{\mathrm{op}}$.*

We have the following result [100, Proposition 13.2.2(b)]; see also [30, Theorem 10.22 and Remark 10.23], [100, Proposition 13.2.2], [151, Lemma 1.3.3 and Lemma 1.3.4], and [80, Corollary 3.10].

Lemma 4.10. *Suppose that $\mathcal{A}$ is an exact category and $\mathcal{C}$ is a cogenerating fully exact subcategory of $\mathcal{A}$ closed under quotients. Then for any bounded complex A in $\mathcal{A}$ there exists a bounded complex C in $\mathcal{C}$ and a quasi-isomorphism $\eta : A \to C$ with $\eta^k : A^k \to C^k$ an admissible monic for every $k \in \mathbb{Z}$.*

Furthermore, the inclusion $\mathcal{C} \to \mathcal{A}$ induces an equivalence of categories

$$\mathrm{D}^b(\mathcal{C}) \to \mathrm{D}^b(\mathcal{A}).$$

As a consequence of Lemma 4.10 we have the following; see [30, Section 10.6], [100, Definition 10.3.2, Proposition 10.3.3, Corollary 13.3.8], [151, Proposition 1.3.5], and [80, Remark 4.10].

Proposition 4.11. *Suppose that $F : \mathcal{A} \to \mathcal{R}$ is a functor between exact categories, and $\mathcal{C}$ is an cogenerating full subcategory of $\mathcal{A}$ closed under quotients such that $F|_{\mathcal{C}}$ is exact. For a bounded complex A over $\mathcal{A}$, pick a bounded complex C_A over $\mathcal{C}$ together with a quasi-isomorphism $\eta_A : A \to C_A$. For bounded complexes A, B over $\mathcal{A}$ and morphisms $f : A \to B$ in $\mathrm{D}^b(\mathcal{A})$, define*

$$(\mathrm{R}F)(A) := F(C_A)$$

and

$$(\mathrm{R}F)(f) := Q_{\mathcal{R}}(F(g)) \circ Q_{\mathcal{R}}(F(\sigma))^{-1},$$

where $\sigma : C \to C_A$ and $g : C \to C_B$ are morphisms in $\mathrm{K}(\mathcal{C})$ such that

$$Q_{\mathcal{A}}(g)\, Q_{\mathcal{A}}(\sigma)^{-1} = Q_{\mathcal{A}}(\eta_B) \circ f \circ Q_{\mathcal{A}}(\eta_A)^{-1}.$$

This yields a triangulated functor $\mathrm{R}F : \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{R})$. Defining $\mu_A := F(\eta_A) : F(A) \to (\mathrm{R}F)(A)$ for each bounded complex A over $\mathcal{A}$ yields a morphism $\mu : Q_{\mathcal{R}}F \Rightarrow (\mathrm{R}F)\, Q_{\mathcal{A}}$ of triangulated functors. We have that $(\mathrm{R}F, \mu)$ is the total right derived functor of F. Furthermore, if $\mathcal{R}$ is a quasi-abelian category, then $\mathrm{H}^0 \circ \mathrm{R}F$ is a cohomological right derived functor of $\mathrm{H}^0 \circ F$.

Suppose that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ are exact categories, and $F : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{R}$ is a functor. Given bounded complexes A over $\mathcal{A}$ and B over $\mathcal{B}$, we have a corresponding double complex $F(A, B)$ over $\mathcal{R}$. We let $F^\bullet(A, B)$ be the total complex associated with $F(A, B)$, which is well-defined and bounded since A and B are bounded and hence $F(A, B)$ has only finitely many nonzero entries in each diagonal. This defines a triangulated functor $F^\bullet : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{B}) \to \mathrm{K}^b(\mathcal{R})$; see [100, Proposition 11.6.4]. The same proof as Proposition 4.11 gives the following.

Proposition 4.12. *Suppose that $F : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{R}$ is a functor between exact categories, $\mathcal{C}$ is an generating full subcategory of $\mathcal{A}$ closed under subobjects, and $\mathcal{D}$ is a cogenerating full subcategory of $\mathcal{B}$ closed under quotients. Assume that for every object C of $\mathcal{C}$ and D of $\mathcal{D}$, $F(C, -)$ is exact on $\mathcal{D}$ and $F(-, D)$ is exact on $\mathcal{C}$. For bounded complexes A and B over $\mathcal{A}$ and $\mathcal{B}$, respectively, pick bounded complexes C_A and D_B over $\mathcal{C}$ and $\mathcal{D}$, respectively, together with quasi-isomorphisms $\eta_A : C_A \to A$ and $\eta_B : B \to D_B$, and define*

$$(\mathrm{R}F)(A, B) := F(C_A, D_B).$$

This yields a triangulated functor $\mathrm{R}F^{\bullet} : \mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{B}) \to \mathrm{D}^b(\mathcal{R})$. Defining $\mu_A := F(\eta_A, \eta_B) : F^{\bullet}(A, B) \to (\mathrm{R}F)(A)$ for each bounded complex A over $\mathcal{A}$ yields a morphism $\mu : Q_{\mathcal{R}} F^{\bullet} \Rightarrow (\mathrm{R}F)(Q_{\mathcal{A}} \times Q_{\mathcal{B}})$ of triangulated functors. We have that $(\mathrm{R}F, \mu)$ is the total right derived functor of $F^{\bullet}$. Furthermore, if $\mathcal{R}$ is a quasi-abelian category, then $\mathrm{H}^0 \circ \mathrm{R}F^{\bullet}$ is a cohomological right derived functor of $\mathrm{H}^0 \circ F^{\bullet}$.

4.2.5 Injective and projective objects

Suppose that $\mathcal{A}$ is an exact category. An object I of $\mathcal{A}$ is called *injective* [30, Definition 11.1] if the functor $\mathrm{Ext}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact; see also [30, Proposition 11.3]. The category $\mathcal{A}$ *has enough injectives* [30, Definition 11.9] if for every object A of $\mathcal{A}$ there exists an admissible monic $m : A \to I$ where I is injective. Let $\mathcal{A}$ be an exact category with enough injectives, and let $\mathcal{I}$ be the class of injective objects in $\mathcal{A}$. Then for every left-bounded complex A in $\mathcal{A}$ there exists a left-bounded complex I_A over $\mathcal{I}$ and a quasi-isomorphism $\mu_A : A \to I_A$ [30, Theorem 12.7]. In particular, when A is an object of $\mathcal{A}$, regarded as a complex concentrated in degree zero, then I_A is called an *injective resolution* of A [30, Definition 12.1].

Furthermore, the inclusion $\mathcal{I} \to A$ induces an equivalence of triangulated categories $\mathrm{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{A})$. If $\mathcal{B}$ is an exact category, and $F : \mathcal{A} \to \mathcal{B}$ is a functor, then F has a total right derived functor $\mathrm{R}F : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$. This is defined by setting $(\mathrm{R}F)(A) := F(I_A)$ and $(\mathrm{R}F)(f) = Q_{\mathcal{B}}(F(g))$ for left-bounded complexes A and B over $\mathcal{A}$ and morphism $f : A \to B$ in $\mathrm{D}^+(\mathcal{A})$, where $g : I_{\mathcal{A}} \to I_{\mathcal{B}}$ is a morphism in $\mathrm{K}(\mathcal{A})$ satisfying $Q_{\mathcal{A}}(g) = Q_{\mathcal{A}}(\mu_B) \circ f \circ Q_{\mathcal{A}}(\mu_A)^{-1}$.

For objects A and B of $\mathcal{A}$ and $n \in \mathbb{Z}$, one defines $\mathrm{Ext}^n(A, B) := \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[n])$. One says that $\mathcal{A}$ has *finite homological dimension* if there exists $d \in \mathbb{Z}$ such that $\mathrm{Ext}^n(A, B) = 0$ for all $n > d$ and objects A and B of $\mathcal{A}$. In this case, the least such d is called the *homological dimension* $\mathrm{hd}(\mathcal{A})$ of $\mathcal{A}$ [100, Exercise 13.8]. If $\mathcal{A}$ has enough injectives and $d \geq 0$, then $\mathrm{hd}(\mathcal{A}) \leq d$ if and only if for every exact sequence $X^0 \to X^1 \to \cdots \to X^d \to 0$ in $\mathcal{A}$ with $X^0, X^1, \ldots, X^{d-1}$ injective, we have that X^d is also injective [100, Exercise 13.8]. In this case, every object of $\mathcal{A}$ has an injective resolution I_A with $I_A^k = 0$ for $k \in \mathbb{Z} \setminus \{0, 1, \ldots, d\}$.

If $\mathcal{A}$ has enough injectives and finite homological dimension, then for every (bounded) complex A over $\mathcal{A}$ there exists a quasi-isomorphism $\mu_A : A \to I_A$ where I_A is a (bounded) complex over $\mathcal{I}$ [86, Corollary I.7.7]. Furthermore, the inclusion $\mathcal{I} \to \mathcal{A}$ induces an equivalence of categories $\mathrm{K}(\mathcal{I}) \to \mathrm{D}(\mathcal{A})$, which restricts to an equivalence of categories $\mathrm{K}^b(\mathcal{I}) \to$

$\mathrm{D}^b(\mathcal{A})$ [86, Proposition IX.2.12]; see also [100, Proposition 13.2.2]. As a consequence of Proposition 4.12 one obtains the following:

Proposition 4.13. *Let $\mathcal{A}$ and $\mathcal{R}$ be exact categories. Let $F : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{R}$ be a functor. Suppose that either:*

- *$\mathcal{A}$ has enough injectives, and $F(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathcal{R}$ is exact for every injective object I of $\mathcal{A}$, or*
- *$\mathcal{A}$ has enough projectives, and $F(P, -) : \mathcal{A}^{\mathrm{op}} \to \mathcal{R}$ is exact for every projective object P of $\mathcal{A}$.*

Suppose furthermore that $\mathcal{A}$ has homological dimension at most 1. Then $F^\bullet : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathcal{R})$ has a total right derived functor $\mathrm{R}F : \mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{R})$. If $\mathcal{R}$ is quasi-abelian, then $\mathrm{H}^0 \circ \mathrm{R}F$ is a cohomological derived functor of $\mathrm{H}^0 \circ F^\bullet$.

As a particular instance of Proposition 4.12, one has the following.

Corollary 4.14. *Suppose that $\mathcal{M}$ is a quasi-abelian category of abelian groups, and $\mathcal{A}$ is a quasi-abelian $\mathcal{M}$-category. Assume that $\mathcal{A}$ has enough injectives or enough projectives, and homological dimension at most 1. Then $\mathrm{Hom}^\bullet : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathcal{M})$ has a total right derived functor $\mathrm{RHom} : \mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{M})$, and $\mathrm{Ext}^0 := \mathrm{H}^0 \circ \mathrm{RHom}$ is a cohomological derived functor of $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})} : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{M})$. Furthermore, for bounded complexes A and B over $\mathcal{A}$, $\mathrm{Ext}^0(A, B)$ is naturally isomorphic as an abelian group to $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B)$. This isomorphism turns $\mathrm{D}^b(\mathcal{A})$ into a triangulated category enriched over $\mathrm{LH}(\mathcal{M})$.*

Proof. The first assertion is a particular instance of Proposition 4.12.

For the second assertion, observe that by [100, Proposition 11.7.3], we have that $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}$ is naturally isomorphic to $\mathrm{H}^0 \circ \mathrm{Hom}^\bullet$. Therefore, by the first assertion we have that RHom is a total right derived functor of $\mathrm{Hom}^\bullet$, and $\mathrm{Ext}^0 := \mathrm{H}^0 \circ \mathrm{RHom}$ is a cohomological right derived functor of $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})} \cong \mathrm{H}^0 \circ \mathrm{Hom}^\bullet$. Furthermore, we have that Ext^0 is naturally isomorphic to $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}$ by [100, Proposition 11.7.3]. This allows one to regard $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}$ as a functor to $\mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{M})$, which turns $\mathrm{D}^b(\mathcal{A})$ into a triangulated category enriched over $\mathrm{LH}(\mathcal{M})$. □

Under the hypotheses of Corollary 4.14, one sets $\mathrm{Ext}^n(A, B) := \mathrm{Ext}^0(A, B[n])$ for complexes A and B over $\mathcal{A}$ and $n \in \mathbb{Z}$. Then we have that $\mathrm{Ext}^n = 0$ for $n \geq 2$. Furthermore, for objects A, B of $\mathcal{A}$, $\mathrm{Ext}(A, B) := \mathrm{Ext}^1(A, B)$ is isomorphic to the group $\mathrm{Ext}_{\mathrm{Yon}}(A, B)$ of *isomorphism classes of extensions* of A by B; see [86, Section XI.4]. Notice that if $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$, then for objects A and B of $\mathcal{B}$, the group $\mathrm{Ext}_{\mathrm{Yon}}(A, B)$ is unchanged whether it is computed in $\mathcal{B}$ or in $\mathcal{A}$; see [100, Exercise 13.17].

4.2.6 The left heart of thick categories of Polish abelian groups

Let $\mathbf{PAb}$ be the category of Polish abelian groups and continuous group homomorphisms. This is a quasi-abelian category [114, Lemma 6.1]. An explicit description of $\mathrm{LH}(\mathbf{PAb})$ was given in [114] in terms of *groups with a Polish cover*. An abelian group with a Polish cover is a group G explicitly presented as a quotient $\hat{G}/N$, where $\hat{G}$ is a Polish abelian group and $N \subseteq \hat{G}$ is a Polishable subgroup of $\hat{G}$. This means that N is a Polish group with respect to some Polish topology such that the inclusion $N \to \hat{G}$ is continuous. The Polish topology on N is in general not the subspace topology inherited from $\hat{G}$ (unless N is closed in $\hat{G}$). Nonetheless, the topology on N is in some sense *induced* by the topology on $\hat{G}$, as it is the unique Polish topology on N whose open sets are Borel in $\hat{G}$. Every Polish abelian group G can be identified with the group with a Polish cover $\hat{G}/N$ where $G = \hat{G}$ and N is the trivial subgroup of G. A group homomorphism $\varphi : G \to H$ between groups with a Polish cover $G = \hat{G}/N$ and $H = \hat{H}/M$ is *Borel-definable* if has a *lift* to a Borel function $f : \hat{G} \to \hat{H}$, such that $\varphi(x + N) = f(x) + M$ for every $x \in \hat{G}$.

The category $\mathrm{LH}(\mathbf{PAb})$ is (equivalent to) the category of groups with a Polish cover and Borel-definable group homomorphisms [114, Theorem 6.2]. More generally, if $\mathcal{B}$ is a thick subcategory of $\mathbf{PAb}$, then $\mathrm{LH}(\mathcal{B})$ is (equivalent to) the full subcategory of the category of abelian groups with a Polish cover spanned by *groups with a $\mathcal{B}$-cover*, which are the groups with a Polish cover of the form $\hat{G}/N$ where both $\hat{G}$ and N belong to $\mathcal{B}$ [114, Theorem 6.13 and Proposition 6.15]. As it is remarked therein, a Borel-definable homomorphism between groups with a $\mathcal{B}$-cover is an isomorphism in $\mathrm{LH}(\mathcal{B})$ if and only if it is a bijection.

An abelian Polish group is *non-Archimedean* if it has a basis of zero neighborhoods consisting of subgroups. This is equivalent to the assertion that A is isomorphic to an inverse limit of countable groups. Non-Archimedean Polish abelian groups form a thick subcategory of the category of Polish abelian groups; see [114, Theorem 6.17].

4.3 Homological algebra for pro-Lie Polish abelian groups

4.3.1 Borel cocycles for Polish groups

Let G, H be abelian Polish groups. A (symmetric) 2-*cocycle* on G with values in H is a function $c : G \times G \to H$ satisfying the following identities for all $x, y, z \in G$:

- $c(x + y, z) + c(x, y) = c(x, y + z) + c(y, z)$;
- $c(x, y) = c(y, x)$;
- $c(0, x) = 0$.

We say that a 2-cocycle is Borel (respectively, continuous) if it is Borel (respectively, continuous) as a function $G \times G \to H$. Given a function $t : G \to H$, we define $\delta t : G \times G \to H$ to be the function $(x, y) \mapsto t(x) + t(y) - t(x + y)$. A Borel 2-cocycle

c on G with values in H is a coboundary if there exists a Borel function $t : G \to H$ such that $c = \delta t$. Borel 2-cocycles on G with values in H form a group $\mathrm{Z}^1(G, H)$ with respect to pointwise addition. Coboundaries form a subgroup $\mathrm{B}^1(G, H)$ of $\mathrm{Z}^1(G, H)$. We define $\mathrm{Ext_c}(G, H)$ to be the quotient group $\mathrm{Z}^1(G, H)/\mathrm{B}^1(G, H)$. This defines a functor $\mathrm{Ext_c} : \mathbf{PAb}^{\mathrm{op}} \times \mathbf{PAb} \to \mathbf{Ab}$.

We let $\mathrm{Ext_{Yon}}(C, A)$ be the group whose elements are the isomorphism classes of abelian Polish group extensions $A \to X \to C$, where the group operation is induced by Baer sum of extensions and the trivial element is the class of split extensions. A short exact sequence $A \to X \to C$ in $\mathbf{PAb}$ yields an element of $\mathrm{Ext_c}(C, A)$ as follows. Pick a Borel right inverse $t : C \to X$ for the map $X \to C$, which exists by [103, Theorem 12.17]. Identifying A with a closed subgroup of X, we have that for every $x, y \in C$, $\kappa(x, y) := t(x + y) - t(x) - t(y)$ belongs to A. This defines a Borel 2-cocycle on C with values in A, whose corresponding element of $\mathrm{Ext_c}(C, A)$ is independent on the choice of t. This assignment defines an injective group homomorphism $\mathrm{Ext_{Yon}}(C, A) \to \mathrm{Ext_c}(C, A)$.

Lemma 4.15. *Suppose that C and A are locally compact Polish abelian groups. Then the group homomorphism $\mathrm{Ext_{Yon}}(C, A) \to \mathrm{Ext_c}(C, A)$ is an isomorphism.*

Proof. It suffices to show that it is surjective. Given a Borel cocycle $\kappa : C \times C \to A$ one can define a corresponding extension $A \to X \to C$ as follows. One let X be the group that has $A \times C$ as set of objects, and group operation defined by

$$(a, c) + (a', c') = (a + a' + \kappa(c, c'), c + c').$$

We also endow $A \times C$ with the product Borel structure and the Borel measure defined as the product of Haar measures on A and G. Then we have that this is a σ-finite invariant Borel measure, and so by [119, Theorem 1] there exists a unique locally compact Polish group topology on X that is compatible with its Borel structure and has the given measure as a Haar measure. When endowed with this topology, the canonical maps $A \to X \to C$ give a short exact sequence in $\mathbf{PAb}$. The image of the corresponding element in $\mathrm{Ext_{Yon}}(C, A)$ under the homomorphism $\mathrm{Ext_{Yon}}(C, A) \to \mathrm{Ext_c}(C, A)$ is equal to the element of $\mathrm{Ext_c}(C, A)$ represented by κ. □

Lemma 4.16. *Suppose that C and A are Polish abelian groups, where C is non-Archimedean. Then the image of the group homomorphism $\mathrm{Ext_{Yon}}(C, A) \to \mathrm{Ext_c}(C, A)$ is equal to the subgroup of elements represented by* continuous *cocycles.*

Proof. Suppose that $A \to X \to C$ is a short exact sequence in $\mathbf{PAb}$, where C is non-Archimedean. Then by [22, Proposition 4.6] we have that the map $X \to C$ has a *continuous* right inverse $t : C \to X$. This yields a cocycle δt that represents an element of $\mathrm{Ext_c}(C, A)$ that is in the image of the element of $\mathrm{Ext_{Yon}}(C, A)$ represented by the given extension.

Conversely if $\kappa : C \times C \to A$ is a continuous cocycle, then one can define the extension $A \to X \to C$ letting X be the group that has $A \times C$ as set of objects, endowed with the

product topology, and group operation defined by

$$(a, c) + (a', c') = (a + a' + \kappa(c, c'), c + c').$$

□

The following lemma is [21, Lemma 4.7].

Lemma 4.17. *Suppose that C is an abelian Polish group and A is an abelian Polish group. Suppose that $c : C \times C \to A$ is a continuous cocycle, and $t : C \to A$ is a function such that $c = \delta t$. If t is Borel, then t is continuous, and c is a coboundary.*

Proof. Define the Polish group $X := A \rtimes_c C$ obtained by endowing the product $A \times C$ with the product topology and the group operation defined by $(a, x)+(a', x') := (a + a' + c(x, x'), x + x')$ for $a, a' \in A$ and $x, x' \in C$. Observe that the map $a \mapsto (a, 0)$ is an inclusion of A into X as a closed subgroup, giving an extension $A \to X \to C$.

Let also $A \times C$ be the product Polish group. We can define a Borel group homomorphism $\varphi : X \to A \times C$ by setting

$$\varphi(a, x) := (a + t(x), x).$$

Being a Borel group homomorphism, we must have that φ is continuous. Hence, $t : C \to A$ is continuous. □

Given a short exact sequence $A \to B \to C$ with the corresponding cocycle κ as above, and abelian Polish groups X and Y, we define the corresponding *boundary homomorphisms* $\mathrm{Hom}(X, C) \to \mathrm{Ext}_{\mathrm{c}}(X, A)$, $\varphi \mapsto \kappa \circ (\varphi \times \varphi) + \mathrm{B}^1(X, A)$ and $\mathrm{Hom}(A, Y) \to \mathrm{Ext}_{\mathrm{c}}(C, Y)$, $\psi \mapsto \psi \circ \kappa + \mathrm{B}^1(C, Y)$. The same proof as in the case of discrete groups gives the following lemma; see [63, Chapter IX]. When X, C are non-Archimedean, one can choose κ to be continuous, whence the boundary homomorphism takes values in $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$.

Lemma 4.18. *Let X, Y be abelian Polish groups, and let $A \to B \xrightarrow{\pi} C$ be a short exact sequence of abelian Polish groups.*

1. *We have exact sequences of abelian groups:*

$$\begin{aligned} 0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \\ \to \mathrm{Ext}_{\mathrm{c}}(X, A) \to \mathrm{Ext}_{\mathrm{c}}(X, B) \to \mathrm{Ext}_{\mathrm{c}}(X, C) \end{aligned}$$

2. *If X, C are non-Archimedean, then we have an exact sequence of abelian groups:*

$$\begin{aligned} 0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \\ \to \mathrm{Ext}_{\mathrm{Yon}}(X, A) \to \mathrm{Ext}_{\mathrm{Yon}}(X, B) \to \mathrm{Ext}_{\mathrm{Yon}}(X, C) \end{aligned}$$

3. If A, B, C are non-Archimedean, then we have an exact sequence of abelian groups:

$$\begin{aligned} 0 \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,Y) \\ \to \mathrm{Ext}_{\mathrm{Yon}}(C,Y) \to \mathrm{Ext}_{\mathrm{Yon}}(B,Y) \to \mathrm{Ext}_{\mathrm{Yon}}(A,Y) \end{aligned}$$

Proof. Let us fix a Borel right inverse $\sigma : C \to B$ for π, and let $\kappa = \delta\sigma : C \times C \to A$. We identify A with a closed subgroup of B. By [22, Proposition 4.6], if C is non-Archimedean, then one can choose σ to be continuous.

(1) and (2): It is clear that the image of $\mathrm{Hom}\,(X,C) \to \mathrm{Ext_c}\,(X,A)$ is contained in the kernel of $\mathrm{Ext_c}\,(X,A) \to \mathrm{Ext_c}\,(X,B)$. We prove the converse implication. Suppose that $c : X \times X \to A$ is a Borel cocycle such that there exists a function $t : X \to B$ such that $c = \delta t$. Then we have that $\pi t : X \to C$ is a group homomorphism. Then $\kappa \circ (\pi t \times \pi t)$ is a cocycle cohomologous to c, as we are about to show. Define $g : X \to A, x \mapsto \sigma\pi t\,(x) - t\,(x)$. For $x, y \in X$,

$$\begin{aligned} (\kappa \circ (\pi t \times \pi t))\,(x,y) &= \kappa\,(\pi t\,(x)\,, \pi t\,(y)) \\ &= \sigma\,(\pi t\,(x)) + \sigma\,(\pi t\,(y)) - \sigma\,(\pi t\,(x+y)) \\ &= t\,(x) + t\,(y) - t\,(x+y) + \delta g\,(x,y) \\ &= c\,(x,y) + \delta g\,(x,y)\,. \end{aligned}$$

This concludes the proof that the image of $\mathrm{Hom}\,(X,C) \to \mathrm{Ext_c}\,(X,A)$ is equal to the kernel of $\mathrm{Ext_c}\,(X,A) \to \mathrm{Ext_c}\,(X,B)$.

We now prove that the kernel of $\mathrm{Ext_c}\,(X,B) \to \mathrm{Ext_c}\,(X,C)$ is contained in the image of $\mathrm{Ext_c}\,(X,A) \to \mathrm{Ext_c}\,(X,B)$. Suppose that $c : X \times X \to B$ is a Borel cocycle such that there is a Borel function $f : X \to C$ such that $\pi c = \delta f$. Then we have that $c_0 := c - \delta\,(\sigma f)$ is a Borel cocycle $X \times X \to A$ such that $c = c_0 + \delta\,(\sigma f)$ and hence c, c_0 are cohomologous as Borel cocycles $X \times X \to B$. This shows that the kernel of $\mathrm{Ext_c}\,(X,B) \to \mathrm{Ext_c}\,(X,C)$ is contained in the image of $\mathrm{Ext_c}\,(X,A) \to \mathrm{Ext_c}\,(X,B)$.

When X, C are non-Archimedean, σ is continuous. If c is continuous, then f is continuous, and hence c_0 is continuous as well. This shows that the kernel of $\mathrm{Ext}_{\mathrm{Yon}}\,(X,B) \to \mathrm{Ext}_{\mathrm{Yon}}\,(X,C)$ is contained in the image of $\mathrm{Ext}_{\mathrm{Yon}}\,(X,A) \to \mathrm{Ext}_{\mathrm{Yon}}\,(X,B)$ when X, C are non-Archimedean.

(3) Suppose now that $c : C \times C \to Y$ is a Borel cocycle that represents an element of the kernel of $\mathrm{Ext_c}\,(C,Y) \to \mathrm{Ext_c}\,(B,Y)$. Thus, we have that $c \circ (\pi \times \pi)$ is a coboundary, and there exists a Borel function $f : B \to Y$ such that $\delta f = c \circ (\pi \times \pi)$. Hence,

$$f\,(b + b') = f\,(b) + f\,(b') + c\,(\pi\,(b)\,, \pi\,(b'))$$

for $b, b' \in B$. This implies that

$$f\,(0) = 0$$

and

$$f(b) + f(-b) = -c(\pi(b), -\pi(b)).$$

Then we have that $\varphi := f|_A : A \to Y$ is a continuous group homomorphism. We have that $\varphi \circ \kappa$ is a Borel cocycle cohomologous to c. Indeed, define $g := f\sigma : C \to Y$. For $x, y \in C$ we have that

$$\begin{aligned}
(\varphi \circ \kappa)(x,y) &= \varphi(\sigma(x) + \sigma(y) - \sigma(x+y)) \\
&= f(\sigma(x) + \sigma(y) - \sigma(x+y)) \\
&= f(\sigma(x) + \sigma(y)) + f(-\sigma(x+y)) + c(x+y, -(x+y)) \\
&= f(\sigma(x)) + f(\sigma(y)) + c(x,y) - f(\sigma(x+y)) \\
&\quad - c(x+y, -(x+y)) + c(x+y, -(x+y)) \\
&= (c + \delta g)(x,y).
\end{aligned}$$

This shows that the kernel of $\mathrm{Ext_c}(C,Y) \to \mathrm{Ext_c}(B,Y)$ is contained in the image of $\mathrm{Hom}(A,Y) \to \mathrm{Ext_c}(C,Y)$. In particular, we have that the kernel of $\mathrm{Ext_{Yon}}(C,Y) \to \mathrm{Ext_{Yon}}(B,Y)$ is contained in the image of $\mathrm{Hom}(A,Y) \to \mathrm{Ext_{Yon}}(C,Y)$.

We now prove that the kernel of $\mathrm{Ext_{Yon}}(B,Y) \to \mathrm{Ext_{Yon}}(A,Y)$ is contained in the image of $\mathrm{Ext_{Yon}}(C,Y) \to \mathrm{Ext_{Yon}}(B,Y)$. Suppose that $Y \to H \xrightarrow{p} B$ is an extension that represents an element of the kernel of $\mathrm{Ext_{Yon}}(B,Y) \to \mathrm{Ext_{Yon}}(A,Y)$. This means that the induced extension $Y \to p^{-1}(A) \to A$ splits. Thus, there exists a continuous group homomorphism $\xi : A \to H$ with closed image such that $p\xi$ is the inclusion of A in B. In particular, we have that $\pi p \xi = 0$. Thus, πp induces a continuous group homomorphism $\lambda : H/\xi(A) \to C$ such that $\lambda(h + \xi(A)) = \pi(p(h))$ for $h \in H$. This yields a commuting diagram

$$\begin{array}{ccccccccc}
 & & & & A & = & A & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & H & \to & B & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & H/\xi(A) & \to & C & \to & 0
\end{array}$$

where the diagram

$$\begin{array}{ccc}
H & \to & B \\
\downarrow & & \downarrow \\
H/\xi(A) & \to & C
\end{array}$$

is a push-out. This yields a short exact sequence $Y \to H/\xi(A) \to C$ that represents an element of $\mathrm{Ext_{Yon}}(C,Y)$ whose image in $\mathrm{Ext_{Yon}}(B,Y)$ is the element represented by $Y \to H \to B$. □

Suppose that G, H are abelian Polish groups, where $H = \prod_{n \in \alpha} H_n$ for some $\alpha \leq \omega$. It is clear from the definition that

$$\mathrm{Ext_c}(G,H) \cong \prod_{n \in \alpha} \mathrm{Ext_c}(G, H_n).$$

Furthermore, if $\alpha < \omega$ then, since $\mathrm{Ext_c}(-, G)$ is an additive functor, we also have

$$\mathrm{Ext_c}(H, G) \cong \prod_{n \in \alpha} \mathrm{Ext_c}(H_n, G)$$

4.3.2 Preliminaries on pro-Lie groups

In this section, we recall fundamental facts about the theory of pro-Lie groups as can be found in [81]. We will only consider abelian Polish groups. The category **PAb** of abelian Polish groups and continuous group homomorphisms is a quasi-abelian category [114, Theorem 6.2]. Locally compact Polish abelian groups form a thick subcategory **LCPAb** of **PAb** [114, Theorem 6.17].

An abelian Polish group is a *Lie group* if and only if it is of the form $V \oplus T \oplus D$ where D is countable discrete, V is a finite-dimensional vector group (isomorphic to $\mathbb{R}^n$ for some $n \in \omega$), and T is a finite-dimensional torus (isomorphic to $\mathbb{T}^d$ for some $d \in \omega$); see [82, Exercise E5.18]. Lie groups form a thick subcategory **LiePAb** of the category of locally compact abelian Polish groups; see [114, Theorem 6.17].

A Polish abelian group G has *no small subgroups* if and only if it has a zero neighborhood U such that for every subgroup N of G contained in U, one has that $N = \{0\}$ [134]. When G is locally compact, this is equivalent to the assertion that the Pontryagin dual $G^\vee$ is compactly generated, as well as to the assertion that G is a Lie group [134, Theorem 2.4 and Corollary 1].

Let G be an abelian Polish group. A closed subgroup N of G is co-Lie if G/N is a Lie group. Following [81], we let $\mathcal{N}(G)$ be the collection of co-Lie closed subgroups of G.

Definition 4.19. *An abelian Polish group is* pro-Lie *if every zero neighborhood in G contains an element of $\mathcal{N}(G)$.*

An abelian Polish group G is a *pro-Lie group* if every zero neighborhood of G contains a closed subgroup N such that G/N is a Lie group. This implies that $\mathcal{N}(G)$ is closed under intersections, and hence a filter basis [81, page 148]. It also implies that $G \cong \lim_{N \in \mathcal{N}(G)} G/N$. For a decreasing chain $(N_k)_{k \in \omega}$ in $\mathcal{N}(G)$, we say that $N_k \to 0$ if for every zero neighborhood U in G, N_k is contained in U eventually. Notice that such a sequence exists for a pro-Lie Polish abelian group (as a Polish abelian group has a countable basis of zero neighborhoods).

Lemma 4.20. *Suppose that G is a pro-Lie abelian Polish group. Let $(N_k)_{k \in \omega}$ be a decreasing chain in $\mathcal{N}(G)$. Then we have that $N_k \to 0$ if and only if $(N_k)_{k \in \omega}$ is cofinal in $\mathcal{N}(G)$. If this holds, then $G \cong \lim_k G/N_k$.*

Proof. Suppose that $N_k \to 0$. For $M \in \mathcal{N}(G)$, we have that G/M is a Lie group. Thus, it has a zero neighborhood U that does not contain any subgroup. Since $N_k \to 0$, we have that there exists $k \in \omega$ such that $\pi(N_k) \subseteq U$, where $\pi : G \to G/M$ is the quotient map. This implies that $\pi(N_k) = \{0\}$ and $N_k \subseteq M$. The converse implication is obvious. The second assertion follows from the fact that $G \cong \lim_{N \in \mathcal{N}(G)} G/N$. □

It is proved in [81, Theorem 3.39] that a Polish abelian group is pro-Lie if and only if it is the inverse limit of an inverse system of Polish abelian Lie groups.

A Polish group G is *almost connected* if $G/c\,(G)$ is compact, where $c\,(G)$ is the connected component of the trivial element of G (which is a closed subgroup of G) [81, Definition 5.6]. If G is an abelian pro-Lie group, then $G/c\,(G)$ is non-Archimedean [81, Theorem 5.20(iii)].

The class of abelian Polish pro-Lie groups contains all locally compact abelian Polish groups [81, Example 5.1], all non-Archimedean abelian Polish groups, and all almost connected abelian Polish groups [170, 171], and is closed within the category of Polish groups under the following operations [81, Chapter 3]:

- taking closed subgroups;
- taking quotients by closed subgroups;
- taking countable limits.

An abelian pro-Lie group is called:

- a *vector group* if it is isomorphic to $\mathbb{R}^\alpha$ for some $\alpha \leq \omega$;
- a *torus group* if it is isomorphic to $\mathbb{T}^\beta$ for some $\beta \leq \omega$.

We also say that an abelian Polish pro-Lie group is *vector-free* if it has no nonzero closed subgroups that are vector groups. By [81, Theorem 5.19] we have the following:

Lemma 4.21. *Let G be an abelian Polish pro-Lie group and H a closed subgroup of G. If H is isomorphic to an abelian Polish pro-Lie group of the form $T \oplus V$ where T is a torus group and V is a vector group, then the short exact sequence $H \to G \to G/H$ splits.*

An abelian Polish pro-Lie group G admits a closed subgroup V, called *maximal vector subgroup* (or vector group complement), that is a vector group, and such that $H := G/V$ is vector-free; see [81, Theorem 5.20]. In this case, by Lemma 4.21 we have that $G \cong V \oplus H$. Weil's Lemma for pro-Lie groups asserts the following; see [81, Theorem 5.3].

Lemma 4.22. *Let G be an abelian Polish pro-Lie group and let E be either $\mathbb{Z}$ or $\mathbb{R}$. If $f : E \to G$ is a continuous homomorphism, then exactly one of the following alternatives holds: either the image of f has compact closure, or f is injective with closed image.*

Let G be an abelian Polish group. Define $\mathrm{comp}\,(G)$ to be the subgroup of $g \in G$ such that the subgroup generated by g has compact closure. If G is an abelian Polish pro-Lie group, then $\mathrm{comp}\,(G)$ is a closed subgroup of G [81, Theorem 5.5]. One says that G is *elementwise compact* if $G = \mathrm{comp}\,(G)$ and *compact-free* if $\mathrm{comp}\,(G) = \{0\}$; see [81, Definition 5.4]. We have that $G/\mathrm{comp}\,(G) \cong V \oplus S$ where V is a maximal vector subgroup and S is non-Archimedean and compact-free [81, Theorem 5.20(iv)]. By [81, Proposition 5.43], we have the following:

Lemma 4.23. *If G is an abelian Polish group, then there exists a non-Archimedean closed subgroup D of G such that G/D is a torus group.*

The following result is established in [81, Theorem 4.1].

Lemma 4.24. *Suppose that G is a pro-Lie group, and H is a closed subgroup of G. Suppose that (N_k) is a cofinal sequence in $\mathcal{N}(G)$. Then G/H is pro-Lie, and $((N_k \dot{+} H)/H)_{k\in\omega}$ is a cofinal sequence in $\mathcal{N}(G/H)$.*

We can describe the groups of morphisms between abelian pro-Lie groups as follows.

Lemma 4.25. *Suppose that G is a pro-Lie Polish abelian group and H is a Lie abelian group. Then* $\mathrm{Hom}(G,H) \cong \mathrm{colim}_{N\in\mathcal{N}(G)}\mathrm{Hom}(G/N,H)$.

Proof. Since H has no small subgroups, for every continuous group homomorphism $\varphi : G \to H$ there exists $N \in \mathcal{N}(G)$ such that $N \subseteq \mathrm{Ker}(\varphi)$ and hence φ factors through a homomorphism $G/N \to H$. □

Proposition 4.26. *Suppose that G and H are pro-Lie Polish abelian groups. Then*

$$\mathrm{Hom}(G,H) \cong \lim_{M\in\mathcal{N}(H)}\mathrm{colim}_{N\in\mathcal{N}(G)}\mathrm{Hom}(G/N,H/M).$$

Proof. Since $H \cong \lim_{M\in\mathcal{N}(H)} H/M$, by the universal property of the limit we have

$$\mathrm{Hom}(G,H) \cong \lim_{M\in\mathcal{N}(H)}\mathrm{Hom}(G,H/M).$$

By definition, for $M \in \mathcal{N}(H)$, H/M is a Polish Lie abelian group. Thus, the conclusion follows from Lemma 4.25. □

We regard pro-Lie Polish abelian groups as a full subcategory $\mathbf{proLiePAb}$ of the quasi-abelian category $\mathbf{PAb}$ of Polish abelian groups. We will soon prove that $\mathbf{proLiePAb}$ is in fact a thick subcategory of $\mathbf{PAb}$.

When G is a locally compact Polish abelian group and H is a pro-Lie Polish abelian group, we have that $\mathrm{Hom}(G,H)$ is a *Polish* abelian group endowed with the compact-open topology [68, Exercise 1.1.6], and the isomorphism

$$\mathrm{Hom}(G,H) \cong \lim_{M\in\mathcal{N}(H)}\mathrm{Hom}(G,H/M)$$

is as topological groups.

Lemma 4.27. *If V is a Polish $\mathbb{R}$-vector space that is a pro-Lie abelian Polish group, then $V \cong \mathbb{R}^\alpha$ for some $\alpha \leq \omega$.*

Proof. Let (N_k) be a cofinal sequence in $\mathcal{N}(V)$. Since V is a Polish $\mathbb{R}$-vector space, the function $\mathrm{Hom}(\mathbb{R}, V) \to V$, $\varphi \mapsto \varphi(1)$ is a topological isomorphism. We have that

$$\mathrm{Hom}(\mathbb{R}, V) \cong \lim_k \mathrm{Hom}(\mathbb{R}, V/N_k)$$

Since V/N_k is a connected Polish Lie abelian group, we have that $V/N_k \cong \mathbb{R}^{d_k} \oplus \mathbb{T}^{m_k}$, and $W_k := \mathrm{Hom}(\mathbb{R}, V/N_k)$ is a finite-dimensional $\mathbb{R}$-vector space. Since $\mathrm{Ker}(W_{k+1} \to W_k)$ is a closed $\mathbb{R}$-subspace of W_{k+1}, it is a finite-dimensional $\mathbb{R}$-vector space and a direct summand of W_{k+1}. Thus, we have that

$$V \cong \mathrm{Hom}(\mathbb{R}, V) \cong \lim_k W_k$$

is isomorphic to a product of finite-dimensional $\mathbb{R}$-vector spaces, and hence to $\mathbb{R}^\alpha$ for some $\alpha \leq \omega$. □

By Lemma 4.27, the vector groups are precisely the pro-Lie Polish abelian groups that are Polish $\mathbb{R}$-vector spaces. These are also called *weakly complete topological Polish $\mathbb{R}$-vector spaces in* [81, Proposition 5.43]. We have that if V, W are vector groups, and $\varphi : V \to W$ is a continuous homomorphism, then it is $\mathbb{R}$-linear and its image is a closed subspace of W which is a direct summand [81, Theorem A2.12].

4.3.3 Extensions of abelian Polish pro-Lie groups

The goal of this section is to prove the following result.

Theorem 4.28. *Pro-Lie Polish abelian groups form a thick subcategory of the category of Polish abelian groups.*

In view of the results of [81, Chapter 3], in order to establish Theorem 4.28 it suffices to prove that the class of pro-Lie Polish abelian groups is closed under extensions.

Lemma 4.29. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups, where $A = \prod_{n\in\omega} A_n$ for Polish groups A_n for $n \in \omega$. Then we have an injective continuous homomorphism with closed image $\eta : B \to \prod_{n\in\omega} B_n$ where, for every $n \in \omega$, there is a short exact sequence of abelian Polish groups $A_n \to B_n \to C$. Furthermore, we have a commuting diagram*

$$\begin{array}{ccc} A & = & \prod_n A_n \\ \downarrow & & \downarrow \\ B & \xrightarrow{\eta} & \prod_n B_n \end{array}$$

where the map

$$\prod_{n\in\omega} A_n \to \prod_{n\in\omega} B_n$$

is induced by the maps $A_n \to B_n$ for $n \in \omega$.

Proof. We identify A with a closed subgroup of B. Let $\pi : B \to C$ be the quotient map. For $n \in \omega$, consider the pushout

$$\begin{array}{ccc} A & \to & B \\ p_n \downarrow & & \downarrow \eta_n \\ A_n & \stackrel{\sigma_n}{\to} & B_n \end{array}$$

where $p_n : A \to A_n$ is the canonical projection. Explicitly, we have that

$$B_n := A_n \oplus_A B$$

is the quotient of $A_n \oplus B$ by the closed subgroup

$$\Xi_n := \{(-p_n(a), a) : a \in A\}.$$

The map $\eta_n : B \to B_n$ is given by $b \mapsto (0, b) + \Xi_n$. Then we have a short exact sequence

$$A_n \to B_n \stackrel{\pi_\eta}{\to} C$$

where

$$\pi_n((t, b) + \Xi_n) = \pi(b).$$

Define the continuous group homomorphism $\eta : B \to \prod_n B_n$, $b \mapsto (\eta_n(b))_{n\in\omega}$. We claim that η is injective and has closed image. Indeed, suppose that $b \in B$ is such that $\eta(b) = 0$. This gives that $b \in A$ and $p_n(b) = 0$ for every $n \in \omega$, and hence $b = 0$. This shows that η is injective.

Consider the continuous homomorphism $\tau : \prod_{n\in\omega} B_n \to C^\omega$, $(x_n) \mapsto (\pi_n(x_n))$, and let

$$\Delta_C = \left\{(c_n)_{n\in\omega} \in C^\omega : \forall n \in \omega, c_n = c_0\right\} \subseteq C^\omega.$$

Then we have that the image of η is equal to the preimage of Δ_C under τ. Indeed, it is clear that $\tau \circ \eta$ has image contained in Δ_C. Conversely, suppose that

$$((t_n, b_n) + \Xi_n)_{n\in\omega} \in \prod_{n\in\omega} B_n$$

is mapped to Δ_C by τ. This implies that $\pi(b_n) = \pi(b_0)$ for every $n \in \omega$. Thus, for every $n \in \omega$ there exists $a_n \in A$ such that

$$b_n = b_0 + a_n.$$

Define now $s_n = p_n(a_n)$ for $n \in \omega$, $s := (s_n) \in A$, and $t = (t_n) \in A$. We have

$$\begin{aligned} &((t_n, b_n) + \Xi_n)_{n\in\omega} = ((t_n, b_0 + a_n) + \Xi_n)_{n\in\omega} \\ &= ((t_n + s_n, b_0) + \Xi_n)_{n\in\omega} = ((0, b_0 + t + s) + \Xi_n)_{n\in\omega} \\ &= \eta(b_0 + t + s) \end{aligned}$$

This concludes the proof. □

Lemma 4.30. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is isomorphic to a closed subgroup of an abelian Polish group A', then there exists a short exact sequence of abelian Polish groups $A' \to B' \to C$ such that B is isomorphic to a closed subgroup of B'.*

Proof. It suffices to consider the pushout diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A' & \to & B' \end{array}$$

where the vertical map $A \to A'$ is the inclusion of A into A' as a closed subgroup. □

Lemma 4.31. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is non-Archimedean and C is locally compact, then B is pro-Lie.*

Proof. If A is countable, then B is locally compact, and hence pro-Lie. If $A \cong \prod_{n\in\omega} A_n$ where, for every $n \in \omega$, A_n is countable, then the conclusion follows from Lemma 4.29. Finally, if A is isomorphic to a closed subgroup of $\prod_{n\in\omega} A_n$ where, for every $n \in \omega$, A_n is countable, then the conclusion follows from Lemma 4.30. □

Lemma 4.32. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is pro-Lie and C is locally compact, then B is pro-Lie.*

Proof. By Lemma 4.23, we have a short exact sequence $D \to A \to T$ where D is non-Archimedean and T is a torus group. Considering the pushout diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ T & \to & B_T \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} D & \to & D & \to & 0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & B_T & \to & C \end{array}$$

whose rows and columns are short exact sequences. Since C and T are locally compact, we have that B_T is locally compact. Since D is non-Archimedean, we have that B is pro-Lie by Lemma 4.31. □

If X is a countable set, then for a bounded function $f : X \to \mathbb{R}$ we set

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

Lemma 4.33. *For every $\varepsilon > 0$, countable discrete group A, and bounded 2-cocycle $c : A \times A \to \mathbb{R}$ with $\|c\|_\infty \leq \varepsilon$, there exists a bounded $t : A \to \mathbb{R}$ with $\|t\|_\infty \leq \varepsilon$ and $\delta t = c$.*

Proof. Since A is abelian and, in particular, amenable, we have that the bounded cohomology group $\mathrm{H}^2_\mathrm{b}(A, \mathbb{R})$ is trivial by [62, Theorem 3.6]. Furthermore, A has n-th vanishing modulus of 1; see [59, Definition 4.10 and Example 4.11]. The statement of the lemma is just a reformulation of this fact. □

Lemma 4.34. *Suppose that $V \to X \xrightarrow{\pi} C$ is a short exact sequence of abelian Polish groups. If $V \cong \mathbb{R}$ and $C \cong \prod_{n\in\omega} A_n$ where A_n is a countable abelian group for every $n \in \omega$, then the sequence $V \to X \to C$ splits.*

Proof. We identify C with $\prod_{n\in\omega} A_n$, and we identify V with $\mathbb{R}$ and with a closed subgroup of X. For $n \in \omega$, we write $C_{>k} = \{x \in C : \forall i \leq k, x_i = 0\}$. Fix a compatible complete invariant metric d on X [68, Exercise 2.2.9]. For $x \in V$ we let $|x|$ be its absolute value in $\mathbb{R}$.

By [22, Proposition 4.6], there exists a continuous right inverse $\varphi : C \to X$ for π. Define by recursion a strictly increasing sequence $(k_n)_{n\in\omega}$ in ω such that, for all $n \in \omega$ and $x, y \in C_{>k_n}$, $d(\varphi(x), 0) \leq 2^{-n}$ and $|\delta\varphi(x, y)| \leq 2^{-n}$.

After replacing A_n with $\prod_{i=k_n+1}^{k_{n+1}} A_i$ for $n \in \omega$, we can assume without loss of generality that $k_n = n$. We have that $\|\delta\varphi|_{A_n\times A_n}\|_\infty \leq 2^{-n}$. By Lemma 4.33, we have that exists a bounded function $t_n : A_n \to V \cong \mathbb{R}$ such that $\delta t_n = \delta\varphi|_{A_n\times A_n}$ and $\|t_n\|_\infty \leq 2^{-n}$.

We define now $t : C \to V$ by setting

$$t(x) = \sum_{n\in\omega} t_n(x_n),$$

where we identify x_n as an element of A_n. Define also the continuous function $\psi : C \to X$ by setting

$$\psi(x) = \sum_{n\in\omega} \varphi(x_n)$$

We notice that ψ is a right inverse for π. Furthermore, we have that, for $x, y \in C$,

$$\delta\psi(x, y) = \sum_{n\in\omega} \delta\varphi(x_n, y_n) = \sum_{n\in\omega} \delta t_n(x_n, y_n) = \delta t(x, y).$$

Therefore, we have that $\eta := \psi - t : C \to X$ is a continuous homomorphism that is a right inverse for π. This shows that the short exact sequence $V \to X \to C$ splits. □

Lemma 4.35. *Suppose that $\mathbb{T} \to X \xrightarrow{\pi} C$ is a short exact sequence of abelian Polish groups. If $C \cong \prod_{n\in\omega} A_n$ where A_n is a countable abelian group for every $n \in \omega$, then the sequence $\mathbb{T} \to X \to C$ splits.*

Proof. We identify $\mathbb{T}$ with a closed subgroup of X and with $\mathbb{R}/\mathbb{Z}$. We let $d_\mathbb{T}$ be a compatible complete invariant metric on $\mathbb{T}$. For $\varepsilon > 0$ define $\mathbb{T}_\varepsilon = \{x \in \mathbb{T} : d_\mathbb{T}(x, 0) < \varepsilon\}$. Fix $\varepsilon > 0$

such that there exists a continuous function $\rho : \mathbb{T}_{3\varepsilon} \to \mathbb{R}$ such that $\rho(z) + \mathbb{Z} = z$ for every $z \in \mathbb{T}_{3\varepsilon}$, and $\rho(x+y) = \rho(x) + \rho(y)$ for all $x, y \in \mathbb{T}_{\varepsilon}$.

By [22, Proposition 4.6], we have a continuous right inverse $\varphi : C \to X$ for π. Define the continuous $\kappa : C \times C \to \mathbb{T}$ by $\kappa(x,y) = \varphi(x+y) - \varphi(x) - \varphi(y)$. Since κ is continuous, there exists $n_0 \in \omega$ such that $d_{\mathbb{T}}(\kappa(x,y),0) < \varepsilon$ for every $x \in C_{\geq n_0} := \{x \in C : \forall i < n_0, x_i = 0\}$.

Since $\mathrm{Ext}_{\mathrm{Yon}}(C_{<n_0}, \mathbb{T}) = 0$ by injectivity of $\mathbb{T}$ in the category of locally compact abelian Polish groups, where $C_{<n_0} = \{x \in C : \forall i \geq n_0, x_i = 0\}$, it suffices to show that the function $\pi|_{\pi^{-1}(C_{\geq n})} : \pi^{-1}(C_{\geq n}) \to C_{\geq n}$ has a right inverse that is a continuous group homomorphism. Thus, we can assume without loss of generality that $n = 0$ and $d_{\mathbb{T}}(\kappa(x,y),0) < \varepsilon$ for every $x \in C$.

This implies that, setting $c := \rho \circ \kappa$, one obtains a continuous 2-cocycle $c : C \times C \to \mathbb{R}$ such that $c(x,y) + \mathbb{Z} = \kappa(x,y)$ for every $x, y \in C$. By Lemma 4.34 and Lemma 4.16, we have that $\mathrm{Ext}_{\mathrm{Yon}}(C, \mathbb{R}) = 0$. Therefore, c is a coboundary, and hence κ is a coboundary as well. This implies that φ has a right inverse that is a continuous homomorphism. □

Lemma 4.36. *Suppose that C is a non-Archimedean abelian Polish group. Then there exists a continuous surjective homomorphism $\left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to C$.*

Proof. We have that C is isomorphic to a closed subgroup of $\prod_{n\in\omega} A_n$ where, for every $n \in \omega$, A_n is countable. For every $n \in \omega$, there exists a surjective homomorphism $\mathbb{Z}^{(\omega)} \to A_n$. Hence, C is isomorphic to a quotient of a closed subgroup $\hat{C}$ of $\left(\mathbb{Z}^{(\omega)}\right)^{\omega}$. Let S be a pruned tree on $\mathbb{Z}^{(\omega)}$ such that $\hat{C}$ is equal to closed subgroup $[S] \subseteq \left(\mathbb{Z}^{(\omega)}\right)^{\omega}$ consisting of the branches of S. For every $n \in \omega$, let $B_n := S \cap \left(\mathbb{Z}^{(\omega)}\right)^n$, which is a subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^n$. Since S is pruned, we have that the projection map $B_{n+1} \to B_n$ is onto. Furthermore, we have that $\hat{C} \cong [S] \cong \lim_n B_n$. Since $B_n \cong \mathbb{Z}^{(\omega)}$ and $\mathbb{Z}^{(\omega)}$ is projective for countable abelian groups, we have that $B_{n+1} \cong B_n \oplus \mathrm{Ker}(\pi_{n+1})$. Thus, we have that $\hat{C} \cong B_0 \oplus \prod_{n\geq 1} \mathrm{Ker}(\pi_n)$. Since for every countable group B there exists a surjective homomorphism $\mathbb{Z}^{(\omega)} \to B$, the conclusion follows. □

Lemma 4.37. *Suppose that $L \to X \xrightarrow{\pi} C$ is a short exact sequence of abelian Polish groups. If $L \cong \mathbb{R}^{\alpha} \oplus \mathbb{T}^{\beta}$ for $\alpha, \beta \leq \omega$ and C is non-Archimedean, then the sequence $L \to X \to C$ splits.*

Proof. By Lemma 4.29 it suffices to consider the case when $L = \mathbb{R}$ or $L = \mathbb{T}$. In this case, Lemma 4.34 and Lemma 4.35 prove the result when C is product of countable groups. If C is an arbitrary non-Archimedean Polish abelian group, then by Lemma 4.36 there exists a surjective continuous homomorphism $g : \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to C$. Considering the pullback diagram

$$\begin{array}{ccc} Y & \to & \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \\ \downarrow & & \downarrow g \\ X & \to & C \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} 0 & \to & \ker(g) & \to & \ker(g) \\ \downarrow & & \downarrow & & \downarrow \\ L & \to & Y & \to & \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \\ \downarrow & & \downarrow & & \downarrow \\ L & \to & X & \to & C \end{array}$$

whose rows and columns are short exact sequences. Then the exact sequence $L \to Y \to \left(\mathbb{Z}^{(\omega)}\right)^{\omega}$ splits as explained at the beginning of the proof of this lemma, hence $Y \cong L \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\omega}$ is pro-Lie. Therefore, X is pro-Lie, being quotient of a pro-Lie group by a closed subgroup. (Notice that a surjective continuous homomorphism between Polish groups is a quotient mapping by the Open Mapping Theorem for Polish groups; see [68, Corollary 2.3.4].) By Lemma 4.21, we conclude that the short exact sequence $L \to X \to C$ splits. □

Lemma 4.38. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is pro-Lie and C is non-Archimedean, then B is pro-Lie.*

Proof. As in the proof of Lemma 4.32, by Lemma 4.23 we have a short exact sequence $D \to A \to T$ where D is non-Archimedean and T is a torus group. Considering the pushout diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ T & \to & B_T \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} D & \to & D & \to & 0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & B_T & \to & C \end{array}$$

whose rows and columns are short exact sequences. Since C is non-Archimedean and T is a torus group, we have that the short exact sequence $T \to B_T \to C$ splits by Lemma 4.37. Consider the commuting diagram

$$\begin{array}{ccccc} D & \to & B_0 & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \to & B & \to & B_T \cong T \oplus C \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & T & \to & T \end{array}$$

with short exact rows and columns. We have that B_0 is non-Archimedean since D and C are non-Archimedean. Hence, B is pro-Lie by Lemma 4.31. □

Theorem 4.39. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. Then we have that B is pro-Lie if and only if A and C are pro-Lie.*

Proof. We just need to prove that if A and C are pro-Lie, then B is pro-Lie, as the other implication is established in [81, Chapter 3]. By Lemma 4.23, we have a short exact sequence $D \to C \to T$ where D is non-Archimedean and T is a torus group. Considering the pullback diagram

$$\begin{array}{ccc} B_D & \to & D \\ \downarrow & & \downarrow \\ B & \to & C \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} A & \to & B_D & \to & D \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & T & \to & T \end{array}$$

with exact rows and columns. Since D is non-Archimedean, we have that B_D is pro-Lie by Lemma 4.38. Since T is locally compact, we have that B is pro-Lie by Lemma 4.32. □

Theorem 4.28 is an immediate consequence of Theorem 4.39.

4.3.4 Type decomposition for pro-Lie Polish abelian groups

We recall the type decomposition for locally compact Polish abelian groups as described by Hoffmann and Spitzweck in [80, Section 2].

Definition 4.40. *Let A be a locally compact Polish abelian group. Then we say that:*

- *A is a topological p-group if, for every $x \in A$, the sequence $(p^n x)_{n\in\omega}$ is vanishing or, equivalently, A has a basis of zero neighborhoods consisting of open subgroups U such that A/U is a p-group [3, Chapter 2];*
- *A is a topological torsion group if, for every $x \in A$, the sequence $(n!x)_{n\in\omega}$ is vanishing or, equivalently, A has a basis of zero neighborhoods consisting of open subgroups U such that A/U is a torsion group [3, Chapter 3];*
- *A is type $\mathbb{Z}$ if it is discrete and torsion-free;*
- *A is type $\mathbb{S}^1$ if it is compact and connected or, equivalently, its Pontryagin dual $A^\vee$ is type $\mathbb{Z}$;*
- *A is type $\mathbb{R}$ if it is a vector group;*

- *A is type $\mathbb{A}$ if $A \cong V \oplus B$ where V is a vector group and B is a topological torsion group.*

For a locally compact Polish abelian group A, it is proved in [80, Proposition 2.2] that there exist canonical short exact sequences $F_{\mathbb{Z}}A \to A \to A_{\mathbb{Z}}$ and $A_{\mathbb{S}^1} \to F_{\mathbb{Z}}A \to A_{\mathbb{A}}$ where $A_{\mathbb{Z}}$, $A_{\mathbb{S}^1}$, and $A_{\mathbb{A}}$ have type $\mathbb{Z}$, $\mathbb{S}^1$, and $\mathbb{A}$, respectively. Furthermore, we have $A_{\mathbb{A}} = A_{\mathbb{R}} \oplus A_{\mathrm{t}}$ where $A_{\mathbb{R}}$ is a finite-dimensional vector group and A_{t} is a topological torsion group.

If P is any of the properties from Definition 4.40, we say a pro-Lie Polish abelian group A satisfies P if and only if $\{N \in \mathcal{N}(A) : A/N \text{ satisfies } P\}$ is cofinal in $\mathcal{N}(A)$ (ordered by reverse inclusion). We say that A has sub-type $\mathbb{Z}$ if it is a Polishable subgroup of a type $\mathbb{Z}$ pro-Lie Polish abelian group.

Lemma 4.41. *Suppose that A is an abelian Polish pro-Lie group.*

1. *A is a topological p-group if and only if A is isomorphic to a closed subgroup of $\mathbb{Z}(p^\infty)^\omega$ if and only if the sequence $(p^n x)_{n\in\omega}$ is vanishing for every $x \in A$;*
2. *A is a topological torsion group if and only if A is isomorphic to a closed subgroup of $\prod_p \mathbb{Z}(p^\infty)^\omega$ if and only if the sequence $(n!x)_{n\in\omega}$ is vanishing for every $x \in A$;*
3. *A is type $\mathbb{Z}$ if and only if A is isomorphic to a closed subgroup of $(\mathbb{Q}^{(\omega)})^\omega$;*
4. *A is type $\mathbb{S}^1$ if and only if A is compact connected;*
5. *A is type $\mathbb{R}$ if and only if it is a vector group;*
6. *A is type $\mathbb{A}$ if and only if $A \cong V \oplus B$ where B is a vector group and B is a topological torsion group.*

Proof. We give details for (6), the other assertions being easy to see.

Suppose that A is type $\mathbb{A}$. Then we have that $A \cong V \oplus B$ where V is a vector group and B is vector-free. We claim that B is a topological torsion group. If $N \in \mathcal{N}(B)$ then there exist $L \in \mathcal{N}(V)$ and $M \in \mathcal{N}(B)$ contained in N such that

$$\frac{V \oplus B}{L \oplus M} \cong (V/L) \oplus (B/M)$$

is type $\mathbb{A}$, and hence B/M is type $\mathbb{A}$. We can write $B/M = W \oplus S$ where W is a vector group and S is a topological torsion group. Since B is vector-free, we must have that the composition $B \to B/M \to W$ where $B \to B/M$ is the quotient map and $B/M \to W$ is the coordinate projection, is the zero homomorphism. This shows that $W = 0$ and B/M is a topological torsion group.

The converse implication follows from (1) and (5). □

For locally compact abelian Polish groups A and B, we have that $\mathrm{Hom}(A, B) = 0$ whenever:

- A has type $\mathbb{S}^1$ and B has type $\mathbb{A}$ or $\mathbb{Z}$;
- A has type $\mathbb{A}$ and B has type $\mathbb{Z}$.

It follows from this and Proposition 4.26 that the same conclusions hold when A and B are pro-Lie Polish abelian groups.

If A is a pro-Lie Polish abelian group, then by [81, Theorem 5.20] we have that A has a largest closed vector subgroup, which we denote by $A_{\mathbb{R}}$. Then we have that $A = A_{\mathbb{R}} \oplus B$ where B is vector-free. We also let $A_{\mathbb{S}^1}$ be the connected component of the trivial element in B, which is *compact* [81, Theorem 24(ii)]. We let $c(A)$ be the connected component of the trivial element in A.

Lemma 4.42. *Suppose that $\left(A^{(k)}\right)$ is a tower of compact connected Lie groups. Then $\lim_k A^{(k)}$ is connected.*

Proof. Define $B^{(k)} = \bigcap_{n>k} \mathrm{Ran}\left(A^{(n)} \to A^{(k)}\right) \subseteq A^{(k)}$ for $k \in \omega$. Then, by compactness, $\left(B^{(k)}\right)$ is an epimorphic tower of compact connected Lie groups with $\lim_k B^{(k)} = \lim_k A^{(k)}$. Thus, without loss of generality we can assume that $\left(A^{(k)}\right)$ is an epimorphic tower. If $U \subseteq \lim_k A^{(k)}$ is an open subgroup, then for every $k \in \omega$, $\pi^{(k)}(U) \subseteq A^{(k)}$ is an open subgroup. Thus, $\pi^{(k)}(U) = A^{(k)}$ and hence U is dense in $\lim_k A^{(k)}$. This implies that $U = \lim_k A^{(k)}$, showing that $\lim_k A^{(k)}$, being a compact group, is connected. □

Lemma 4.43. *If V is a* locally compact *vector group, then V is projective in the category of pro-Lie Polish abelian groups.x*

Proof. Suppose that Y is a pro-Lie Polish abelian group and $\varphi : Y \to V$ is a surjective continuous group homomorphism. Since V is a Lie group, we can find a tower $(Y_n, \eta_n : Y_{n+1} \to Y_n)$ of abelian Lie groups and a continuous group homomorphism $\varphi_0 : Y_0 \to V$ such that $Y = \lim_n Y_n$ and $\varphi = \varphi_0 \circ \pi_0$ where $\pi_0 : Y \to Y_0$ is the canonical map. By projectivity of V in the category of locally compact Polish abelian groups, we can find a continuous homomorphism $\psi_0 : V \to Y_0$ that is a right inverse for φ_0. Since the image of ψ_0 is a closed subgroup of Y_0 isomorphic to V, we can find a continuous homomorphism $\psi_1 : \psi_0(V) \to Y_1$ that is a right inverse for $\eta_0|_{\eta_0^{-1}(\psi_0(V))}$, whence $\psi_1 \circ \psi_0$ is a continuous homomorphism that is a right inverse for $\varphi_0 \circ \eta_0$. Proceeding in this fashion, we can define recursively a continuous homomorphisms $\psi_n : (\psi_{n-1} \circ \cdots \circ \psi_0)(V) \to Y_n$ that is a right inverse for $\eta_n|_{\eta_n^{-1}((\psi_n \circ \cdots \circ \psi_0)(V))}$, whence $\psi_n \circ \cdots \circ \psi_0$ is a continuous homomorphism that is a right inverse for $\varphi_0 \circ \eta_0 \circ \cdots \circ \eta_{n-1}$.

One can then define the continuous group homomorphism $\psi : V \to Y, x \mapsto ((\psi_n \circ \cdots \circ \psi_0)(x))_{n\in\omega}$, which is a right inverse for φ. □

Corollary 4.44. *If Y is a vector-free pro-Lie Polish abelian group, then for every $N \in \mathcal{N}(Y)$, Y/N is a vector-free abelian Lie group.*

Lemma 4.45. *Suppose that $\pi : B \to A$ is a continuous surjective homomorphism between Lie Polish abelian groups. Then its restriction $B_0 \to A_0$ is surjective.*

Proof. We can write $B = B_0 \oplus C$ where C is countable. Thus, we have that $\pi(B_0)$ is an analytic subgroup of A_0 of countable index, whence it is open. Since A_0 is connected, this implies that $\pi(B_0) = A_0$. □

Lemma 4.46. *Suppose that A is a pro-Lie Polish abelian group. Let $\left(A^{(k)}\right)$ be an epimorphic tower with $A = \lim_k A^{(k)}$. Then $(c(A^{(k)}))$ is an epimorphic tower.*

Proof. This follows from Lemma 4.45. □

Lemma 4.47. *Let A be a vector-free pro-Lie Polish abelian group. Suppose that $\left(A^{(k)}\right)$ is an epimorphic tower of Lie groups with $A = \lim_k A^{(k)}$. Then $c(A) = \lim_k c\left(A^{(k)}\right)$ is a compact connected subgroups of A.*

Proof. Notice that, for every $k \in \omega$, $A^{(k)}$ is vector-free. We have that $\lim_k c(A^{(k)})$ is compact, and connected by Lemma 4.42. This shows that $\lim_k c(A^{(k)}) \subseteq c(A)$. The converse inclusion follow from functoriality of $c(A)$. □

Lemma 4.48. *Suppose that A is a vector-free pro-Lie Polish abelian group. Then $A/c(A)$ is a non-Archimedean Polish abelian group.*

Proof. Let $\left(A^{(k)}\right)$ be an epimorphic tower of (necessarily vector-free) Lie Polish abelian groups with $A = \lim_k A^{(k)}$. For every $k \in \omega$, we have a short exact sequence

$$c(A^{(k)}) \to A^{(k)} \to C^{(k)}$$

where $C^{(k)}$ is countable. Since $(c(A)^{(k)})$ is an epimorphic tower by Lemma 4.46, this implies that we have a short exact sequence

$$\lim_k c(A^{(k)}) \to \lim_k A^{(k)} \to \lim_k C^{(k)}$$

Since $c(A) = \lim_k c(A^{(k)})$ by Lemma 4.46, $A = \lim_k A^{(k)}$, and $\lim_k C^{(k)}$ is non-Archimedean, the conclusion follows. □

Lemma 4.49. *Suppose that A is a non-Archimedean Polish group. Then A has a largest topological torsion closed subgroup A_{t}. Furthermore, A/A_{t} has sub-type $\mathbb{Z}$.*

Proof. Let $\left(A^{(k)}\right)$ be an epimorphic tower of countable groups with $A = \lim_k A^{(k)}$. For $k \in \omega$, let $T\left(A^{(k)}\right)$ be the torsion subgroup of $A^{(k)}$, and set $C^{(k)} := A^{(k)}/T\left(A^{(k)}\right)$. Then we have an exact sequence

$$0 \to \lim_k T(A^{(k)}) \to \lim_k A^{(k)} \to \lim_k C^{(k)}.$$

Setting $A_{\mathrm{t}} := \lim_k T\left(A^{(k)}\right)$ and $C := \lim_k C^{(k)}$, we have that A_{t} is a topological torsion group and C has type $\mathbb{Z}$. The above argument shows that A/A_{t} is a Polishable subgroup of C, and hence of sub-type $\mathbb{Z}$.

If B is a topological torsion closed subgroup of A, then $\mathrm{Hom}(B, C) = 0$ and hence $B \subseteq A_{\mathrm{t}}$. □

Lemma 4.50. *Suppose that $\left(A^{(k)}\right)$ is a tower of connected Lie groups. Then $\lim_k A^{(k)}$ is connected.*

Proof. As in the proof of Lemma 4.42, we can assume without loss of generality that $\left(A^{(k)}\right)$ is an epimorphic tower. We can write $A^{(k)} = V^{(k)} \oplus T^{(k)}$ where $V^{(k)}$ is a finite-dimensional vector group and $T^{(k)}$ is a finite-dimensional torus group. Furthermore, since $\mathrm{Hom}\left(T^{(k+1)}, V^{(k)}\right) = 0$ for every $k \in \omega$, $\left(V^{(k)}\right)$ is an epimorphic tower. Therefore, we have an exact sequence

$$0 \to \lim_k T^{(k)} \to \lim_k A^{(k)} \to \lim_k V^{(k)} \to \lim_k{}^1 T^{(k)}.$$

Notice that, for every $k \in \omega$, if d_k is the dimension of $T^{(k)}$, then $d_k + 1$ is the maximum length of a chain of compact connected subgroups of $T^{(k)}$. It follows that the chain of subgroups $\left(\mathrm{Ran}\left(T^{(n)} \to T^{(k)}\right)\right)_{n>k}$ stabilizes. As this holds for every $k \in \omega$, $\lim_k{}^1 T^{(k)} = 0$. By projectivity of vector groups, and since $\mathrm{Ran}\left(V^{(k+1)} \to V^{(k)}\right)$ is a closed vector subgroup of $V^{(k)}$ for every $k \in \omega$ by [81, Theorem A2.12], we have that $V := \lim_k V^{(k)}$ is a vector group. Hence, by projectivity of vector groups, we have that $\lim_k A^{(k)} \cong \lim_k T^{(k)} \oplus V$, where $\lim_k T^{(k)}$ is connected by the previous lemma. □

Corollary 4.51. *Let A be a pro-Lie Polish abelian group.*

1. *Suppose that $\left(A^{(k)}\right)$ is an epimorphic tower of Lie groups with $A = \lim_k A^{(k)}$. Then $c(A) = \lim_k c\left(A^{(k)}\right)$.*
2. *$A/c(A)$ is a non-Archimedean Polish abelian group.*

Proof. The proofs of Lemma 4.47 and Lemma 4.48 apply verbatim with Lemma 4.42 replaced with Lemma 4.50. □

Lemma 4.52. *Suppose that A is a pro-Lie Polish abelian group. Then $A/c(A)$ is a non-Archimedean Polish abelian group.*

Proof. Let $\left(A^{(k)}\right)$ be an epimorphic tower of Lie Polish abelian groups with $A = \lim_k A^{(k)}$. For every $k \in \omega$, we have a short exact sequence

$$c\left(A^{(k)}\right) \to A^{(k)} \to C^{(k)}$$

where $C^{(k)}$ is countable. Since $\left(c(A)^{(k)}\right)$ is an epimorphic tower by Lemma 4.46, we have a short exact sequence

$$\lim_k c(A^{(k)}) \to \lim_k A^{(k)} \to \lim_k C^{(k)}$$

By Corollary 4.51, $c(A) = \lim_k c(A^{(k)})$. Hence, $A/c(A) \cong \lim_k C^{(k)}$ is non-Archimedean. □

Lemma 4.53. *Suppose that A is a non-Archimedean Polish group. Then A has a largest topological torsion closed subgroup A_{t}. Furthermore, A/A_{t} has sub-type $\mathbb{Z}$.*

Proof. Let $\left(A^{(k)}\right)$ be an epimorphic tower of countable groups with $A = \lim_k A^{(k)}$. For $k \in \omega$, let $T\left(A^{(k)}\right)$ be the torsion subgroup of $A^{(k)}$, and set $C^{(k)} := A^{(k)}/T\left(A^{(k)}\right)$. Then we have an exact sequence

$$0 \to \lim_k T(A^{(k)}) \to \lim_k A^{(k)} \to \lim_k C^{(k)}.$$

Setting $A_{\mathrm{t}} := \lim_k T\left(A^{(k)}\right)$ and $C := \lim_k C^{(k)}$, we have that A_{t} is a topological torsion group and C has type $\mathbb{Z}$. The above argument shows that A/A_{t} is a Polishable subgroup of C, and hence of sub-type $\mathbb{Z}$.

If B is a topological torsion closed subgroup of A, then $\mathrm{Hom}(B, C) = 0$ and hence $B \subseteq A_{\mathrm{t}}$. □

Lemma 4.54. *We have that $\mathbb{R}^\omega$ is projective in* **proLiePAb**.

Proof. Suppose that X is a pro-Lie Polish abelian group and $\varphi : X \to \mathbb{R}^\omega$ is a continuous surjective homomorphism. We prove that it has a right inverse that is a continuous group homomorphism. We can write $X = V \oplus Y$ where V is a vector group and Y has no nonzero closed vector subgroups. We have short exact sequences $Y_{\mathbb{S}^1} \to F_{\mathbb{Z}}Y \to Y_{\mathrm{t}}$ and $F_{\mathbb{Z}}Y \to Y \to Y_{\mathbb{Z}}$, where $\varphi|_{Y_{\mathbb{S}^1}} = 0$, and the induced function $Y_{\mathrm{t}} \to \mathbb{R}^\omega$ is also trivial. Thus, if $\psi : Y_{\mathbb{Z}} \to \mathbb{R}^\omega$ is the continuous homomorphism induced by $\varphi|_Y$, then $\mathrm{Ran}(\psi) = \mathrm{Ran}(\varphi|_Y)$.

By [81, Theorem A2.12], $\mathrm{Ran}(\varphi|_V)$ is a closed subgroup and a topological direct summand of $\mathbb{R}^\omega$.

Let $\left(N_k^V\right)$ be a cofinal sequence in $\mathcal{N}(V)$, $\left(N_k^Y\right)$ be a cofinal sequence in $\mathcal{N}(Y)$, and set $N_k := N_k^V \oplus N_k^Y$ for $k \in \omega$. Thus, (N_k) is a cofinal sequence in $\mathcal{N}(X)$. For $n \in \omega$, set

$$M_n = \{x \in \mathbb{R}^\omega : \forall i \leq n, x_i = 0\} \subseteq \mathbb{R}^\omega.$$

We claim that $\mathrm{Ran}(\varphi|_V) = \mathbb{R}^\omega$. It suffices to prove that, for every $n \in \omega$, $\pi_{M_n} \circ \varphi|_V : V \to \mathbb{R}^n$ is surjective. We have that $\pi_{M_n} \circ \varphi : X \to \mathbb{R}^n$ factors through X/N_k for some $k \in \omega$. Set $Z_k := X/N_k$ and let $\pi_{N_k} : X \to Z_k$ be the quotient map. Then we have that there exists a continuous homomorphism $\psi : Z_k \to \mathbb{R}^n$ such that $\psi\pi_{N_k} = \pi_{M_n}\varphi$. We have $Z_k \cong V/N_k^V \oplus Y/N_k^Y$. Since Y has no nonzero vector subgroups, it is easily seen considering the type decompositions of Y and Y/N_k that the image of $\psi|_{Y/N_k^Y} : Y/N_k^Y \to \mathbb{R}^n$ is countable. Since $\mathbb{R}^n$ has no nontrivial analytic subgroups of countable index (as any such a subgroup must be open), it follows that $\psi|_{V/N_k^V}$ is surjective, and hence $\pi_{M_n}\varphi|_V$ is surjective.

We have therefore shown that $\varphi|_V : V \to \mathbb{R}^\omega$ is surjective. Since $\mathrm{Ker}\,(\varphi|_V)$ is a closed $\mathbb{R}$-subspace of V, it is a vector group. Therefore, the short exact sequence $\mathrm{Ker}\,(\varphi|_V) \to V \to \mathbb{R}^\omega$ splits, and there exists a continuous homomorphism $\psi : \mathbb{R}^\omega \to V \subseteq X$ that is a right inverse for $\varphi|_V$. Thus, ψ is also a right inverse for φ, if regarded as a continuous group homomorphism with codomain X. □

If A is a pro-Lie Polish abelian group, we define A_{t} to be the largest topological torsion closed subgroup of $A/c(A)$, and $F_{\mathbb{Z}}A$ to be the preimage of A_{t} under the quotient map $A \to A/c(A)$. Define also $A_{\mathbb{Z}} := A/F_{\mathbb{Z}}A$ and $A_{\mathbb{S}^1} = \mathrm{comp}\,(c(A))$, which is the set of elements of $c(A)$ that generate a subgroup with compact closure.

We have that $c(A) = A_{\mathbb{R}} \oplus A_{\mathbb{S}^1}$ where $A_{\mathbb{R}}$ is the largest closed vector subgroup of A. We also set $A_{\mathbb{A}} = A_{\mathbb{R}} \oplus A_{\mathrm{t}}$. Then we have canonical exact sequences

$$A_{\mathbb{S}^1} \to F_{\mathbb{Z}}A \to A_{\mathbb{A}}$$

and

$$F_{\mathbb{Z}}A \to A \to A_{\mathbb{Z}}$$

where $A_{\mathbb{S}^1}$, $A_{\mathbb{A}}$ have type $\mathbb{S}^1$ and $\mathbb{A}$, respectively, and $A_{\mathbb{Z}}$ has sub-type $\mathbb{Z}$.

Remark 4.55. *In general it is not the case that $A_{\mathbb{Z}}$ has type $\mathbb{Z}$. For example, consider the reduced product*

$$G := \prod_n (\mathbb{Z} : 2\mathbb{Z})$$

consisting of sequences of integers that are eventually even. Then G is a Polishable subgroup of $\mathbb{Z}^\omega$ such that every open subgroup U of G is such that G/U has nontrivial torsion subgroup. Indeed, there exists $n \in \omega$ such that the set

$$U_n := \{x \in G : \forall i < \omega,\, x_i \in 2\mathbb{Z} \text{ and } \forall i < n,\, x_i = 0\}$$

is contained U, and hence G/U is a quotient of $G/U_n \cong \mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{(\omega)}$.

It easily follows that, if A is a pro-Lie Polish abelian groups, then setting $A_{\mathbb{S}^1} := \lim_{N \in \mathcal{N}(G)} (A/N)_{\mathbb{S}^1}$ etcetera, we obtain short exact sequences $A_{\mathbb{S}^1} \to F_{\mathbb{Z}}A \to A_{\mathbb{A}}$ and $F_{\mathbb{Z}}A \to A \to A_{\mathbb{Z}}$. Furthermore, we have that $A_{\mathbb{S}^1}$ is the largest closed subgroup of A of type $\mathbb{S}^1$ and the smallest closed subgroup of $F_{\mathbb{Z}}A$ such that $F_{\mathbb{Z}}A/A_{\mathbb{S}^1}$ is of type $\mathbb{A}$, while $F_{\mathbb{Z}}A$ is the smallest closed subgroup of A such that $A/F_{\mathbb{Z}}A$ is of sub-type $\mathbb{Z}$.

This decomposition for pro-Lie Polish abelian groups can be reformulated in terms of *torsion pars*, showing that pro-Lie Polish abelian groups of type $\mathbb{A}$, type $\mathbb{S}^1$, and subtype $\mathbb{Z}$ form fully exact subcategories of the category **proLiePAb**. This follows from the fact that locally compact pro-Lie Polish abelian groups of type $\mathbb{A}$, $\mathbb{S}^1$, and type $\mathbb{Z}$, respectively, form a fully exact subcategories of **LCPAb**.

4.3.5 Injective and projective pro-Lie Polish abelian groups

In this section, we completely characterize the projective objects in **proLiePAb**, and obtain the following.

Theorem 4.56. *The quasi-abelian category* **proLiePAb** *has enough projectives but not enough injectives, and homological dimension* 1.

Recall that an abelian Lie group A is of the form $V \oplus T \oplus D$ where V is a finite-dimensional vector group, T is a finite-dimensional torus, and D is discrete. (This is equivalent to the assertion that A is a locally compact Polish abelian group *with no small subgroups*; see [134, Theorem 2.4].) It easily follows that there exists a surjective homomorphism $\mathbb{R}^n \oplus \mathbb{Z}^{(\omega)} \to A$ for some $n \in \omega$. Recall that the projective objects in **LiePAb** are precisely those of the form $V \oplus F$ where V is a finite-dimensional vector group and F is a countable free abelian group [134, Theorem 3.3]. The injective objects in **LiePAb** are precisely those of the form $V \oplus T$ where V is a finite-dimensional vector group and T is a finite-dimensional torus [134, Theorem 3.2].

Lemma 4.57. *Suppose that $\mathcal{A}$ be a quasi-abelian subcategory of* **PAb** *closed under countable products, and let $\mathcal{D}$ be a class of projective objects in $\mathcal{A}$ closed under direct sums. Suppose that (G_n) is an inverse sequence of objects of $\mathcal{A}$ with surjective continuous homomorphisms $\pi_{n+1} : G_{n+1} \to G_n$ as bonding maps. We also set $G_{-1} = 0$ and $\pi_0 = 0$. Suppose that for every $n \in \omega$ there exists a surjective continuous homomorphism $P \to \mathrm{Ker}\,(\pi_n : G_n \to G_{n-1})$ for some $P \in \mathcal{D}$. Then there exists a surjective continuous homomorphism $\lim_n D_n \to \lim_n G_n$ where $D = (D_n)$ is an inverse sequence with $D_n \in \mathcal{D}$ and surjective continuous homomorphisms $p_{n+1} : D_{n+1} \to D_n$ as bonding maps.*

Proof. We define by recursion objects D_n of $\mathcal{D}$, continuous homomorphisms $p_{n+1} : D_{n+1} \to D_n$, and $\varphi_n : D_n \to G_n$ such that $\pi_{n+1}\varphi_{n+1} = \varphi_n p_{n+1}$. We have that φ_0 exists by hypothesis. Suppose that φ_i and p_i have been defined for $i \leq n$. Then we consider a pushout diagram

$$\begin{array}{ccc} Y_{n+1} & \to & G_{n+1} \\ \downarrow & & \downarrow \\ D_n & \to & G_n \end{array}$$

Since $D_n \to G_n$ and $G_{n+1} \to G_n$ are continuous surjective homomorphisms, the same holds for $Y_{n+1} \to D_n$ and $Y_{n+1} \to G_n$. Since D_n is projective, we have that $Y_{n+1} \cong D_n \oplus \mathrm{Ker}\,(\pi_{n+1})$. By the inductive hypothesis, and since $\mathcal{D}$ is closed under direct sums, there exists a surjective continuous homomorphism $D_{n+1} \to Y_{n+1}$. This concludes the recursive construction.

One then has that $\lim_n \varphi_n : \lim_n D_n \to \lim_n G_n$ is a continuous surjective homomorphism. □

Lemma 4.58. *Let G be a pro-Lie Polish abelian group. Then there exists a surjective continuous homomorphism $\mathbb{R}^\omega \oplus \left(\mathbb{Z}^{(\omega)}\right)^\omega \to G$.*

Proof. By Lemma 4.57 it suffices to prove that the conclusion holds when G is an abelian Lie group, in which case the conclusion follows from the above remarks. □

Lemma 4.59. *If G is a closed subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^{\omega}$, then $G \cong \mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \in \omega + 1$.*

Proof. Define $V_n = \left\{x \in \left(\mathbb{Z}^{(\omega)}\right)^{\omega} : \forall i < n, x_i = 0\right\}$. Then (V_n) is a basis of zero neighborhoods for $\left(\mathbb{Z}^{(\omega)}\right)^{\omega}$, and $(G \cap V_n)$ is a basis of zero neighborhoods of G. For $n \in \omega$, $G/(G \cap V_n)$ is a subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^{n}$, and hence free abelian. Since countable free abelian groups are projective, we have that $G \cong \lim_n G/(G \cap V_n)$ is isomorphic to a product of free abelian groups. The conclusion easily follows. □

Lemma 4.60. *If G is a projective object in* **proLiePAb**, *then $G \cong \mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta} \oplus \mathbb{R}^{\gamma}$ for some $\alpha, \beta, \gamma \in \omega + 1$.*

Proof. By Lemma 4.58, we have a surjective epimorphism $\left(\mathbb{Z}^{(\omega)}\right)^{\omega} \oplus \mathbb{R}^{\omega} \to G$. If G is projective, then we have that G is a topological direct summand of $\left(\mathbb{Z}^{(\omega)}\right)^{\omega} \oplus \mathbb{R}^{\omega}$. This implies that $G_{\mathbb{S}^1} = 0$, $G_{\mathrm{t}} = 0$, and hence $G \cong V \oplus G_0$ where V is a vector group and G_0 is subtype $\mathbb{Z}$. Since V is injective in **proLiePAb** and $\mathrm{Hom}\left(V, \left(\mathbb{Z}^{(\omega)}\right)^{\omega}\right) = 0$, we can assume without loss of generality that $V = 0$ and $G = G_0$ is subtype $\mathbb{Z}$. In this case, we have that $\mathrm{Hom}\left(\mathbb{R}^{\omega}, G\right) = 0$ and $G \subseteq \left(\mathbb{Z}^{(\omega)}\right)^{\omega}$. The conclusion thus follows from Lemma 4.59. □

Lemma 4.61. *Suppose that $(C_i)_{i \in \omega}$ is a sequence of countable abelian groups, $C = \prod_{i \in \omega} C_i$, and A is a countable abelian group. If $\mathrm{Ext}_{\mathrm{Yon}}(C_i, A) = 0$ for every $i \in \omega$, then $\mathrm{Ext}_{\mathrm{Yon}}(C, A) = 0$.*

Proof. By Lemma 4.16 we can identify $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ with the subgroup of $\mathrm{Ext}_{\mathrm{c}}(C, A)$ corresponding to continuous cocycles. For $n \in \omega$, define

$$C^{>n} = \{x \in C : \forall i \leq n, x_i = 0\}$$

and

$$C^{\leq n} = \{x \in C : \forall i > n, x_i = 0\}.$$

Suppose that $c : C \times C \to A$ is a continuous cocycle. Since c is continuous, there exists $n_0 \in \omega$ such that $c(x, y) = 0$ whenever $x, y \in C^{>n}$. Since

$$\mathrm{Ext}_{\mathrm{Yon}}\left(C^{\leq n}, A\right) \cong \mathrm{Ext}(C_0, A) \oplus \cdots \oplus \mathrm{Ext}(C_n, A) \cong 0,$$

it follows that the inclusion $C^{>n} \to C$ induces an isomorphism

$$\mathrm{Ext}_{\mathrm{Yon}}(C, A) \to \mathrm{Ext}_{\mathrm{Yon}}(C^{>n}, A).$$

Thus, we have that c is a coboundary. □

Lemma 4.62. *Suppose that C and A are non-Archimedean Polish abelian groups. Suppose that (V_k) is a basis of zero neighborhoods of A with $V_0 = A$. If $\mathrm{Ext}_{\mathrm{Yon}}(C, V_k/V_{k+1}) = 0$ for every $k \in \omega$, then $\mathrm{Ext}_{\mathrm{Yon}}(C, A) = 0$.*

Proof. Suppose that $c : C \times C \to A$ is a continuous cocycle. By hypothesis, there exists a continuous function $\varphi_0 : C \to A$ such that $c_0 := \delta\varphi_0 + c$ defines an element of $\mathrm{Ext}_{\mathrm{c}}(C, V_1)$. Proceeding in this fashion, we can define a sequence of continuous 2-cocycles $c_n : C \times C \to V_n$ and continuous functions $\varphi_n : C^\omega \to V_n$ with $c_0 = c$ such that $c_{n+1} = \delta\varphi_n + c_n$ for every $n \in \omega$.

Setting $\varphi := \sum_{n\in\omega} \varphi_n$ we obtain a continuous function $C \to A$ such that $\delta\varphi + c = 0$, concluding the proof. □

Lemma 4.63. *Suppose that $(C_i)_{i\in\omega}$ is a sequence of countable abelian groups, and A is a non-Archimedean abelian Polish group. Suppose that (V_k) is a basis of zero neighborhoods of A with $V_0 = A$. Set also $C := \prod_{i\in\omega} C_i$. If $\mathrm{Ext}_{\mathrm{Yon}}(C_i, V_k/V_{k+1}) = 0$ for every $k, i \in \omega$, then $\mathrm{Ext}_{\mathrm{Yon}}(C, A) = 0$.*

Proof. This is obtained combining Lemma 4.61 and Lemma 4.62. □

Lemma 4.64. *Suppose that $(C_i)_{i\in\omega}$ is a sequence of countable free abelian groups. Then, setting $C := \prod_{k\in\omega} C_k$, we have that C is projective for pro-Lie Polish abelian groups.*

Proof. By Lemma 4.16, if A is a pro-Lie Polish abelian group, then we can identify $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ with the subgroup of $\mathrm{Ext}_{\mathrm{c}}(C, A)$ corresponding to continuous cocycles. By Lemma 4.21, since pro-Lie Polish abelian groups form a thick subcategory of $\mathbf{PAb}$, we have that $\mathrm{Ext}_{\mathrm{Yon}}(C, T) = 0$ for every torus T. Furthermore, by Lemma 4.63 we have that $\mathrm{Ext}_{\mathrm{Yon}}(C, A) = 0$ for every non-Archimedean Polish abelian group A. If B is an arbitrary pro-Lie abelian Polish group, then by Lemma 4.23 we have a short exact sequence $A \to B \to T$ where A is non-Archimedean and T is a torus. By Lemma 4.18(2), this induces an exact sequence $0 = \mathrm{Ext}_{\mathrm{Yon}}(C, A) \to \mathrm{Ext}_{\mathrm{Yon}}(C, B) \to \mathrm{Ext}_{\mathrm{Yon}}(C, T) = 0$, showing that $\mathrm{Ext}_{\mathrm{Yon}}(C, B) = 0$. □

Theorem 4.65. *Let G be an abelian pro-Lie Polish group. The following assertions are equivalent:*

1. *G is projective in $\mathbf{proLiePAb}$;*
2. *G is isomorphic to $\mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta \oplus \mathbb{R}^\gamma$ for some $\alpha, \beta, \gamma \in \omega + 1$;*
3. *$\{N \in \mathcal{N}(G) : G/N$ is projective in $\mathbf{LieAb}\}$ is cofinal in $\mathcal{N}(G)$.*

Proof. (2)⇒(1) We have that $\mathbb{R}^\omega$ is projective. We have that $\left(\mathbb{Z}^{(\omega)}\right)^\omega$ is projective by Lemma 4.64. Any group of the form $\mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta \oplus \mathbb{R}^\gamma$ for $\alpha, \beta, \gamma \in \omega + 1$ is a direct summand of $\mathbb{R}^\omega \oplus \left(\mathbb{Z}^{(\omega)}\right)^\omega$, and hence projective.

(1)⇒(2) Any projective object is isomorphic to one of this form by Lemma 4.60.

(3)⇒(1) Let (N_k) be a cofinal sequence in $\mathcal{N}(G)$ such that $G^{(k)} := G/N_k$ is projective in $\mathbf{LieAb}$ for every $k \in \omega$. Then we have that $c(G) = G_{\mathbb{R}}$ and $G/c(G) \cong \lim_k G_{\mathbb{Z}}^{(k)}$ where $(G_{\mathbb{Z}}^{(k)})_{k\in\omega}$ is an epimorphic tower of free abelian groups. Whence, $G_{\mathbb{Z}}$ is isomorphic to a product of free abelian groups and hence projective in $\mathbf{proLiePAb}$. Hence, $G \cong G_{\mathbb{R}} \oplus G_{\mathbb{Z}}$ is projective in $\mathbf{proLiePAb}$.

(2)⇒(3) This follows from the fact that the projective objects in $\mathbf{LieAb}$ are of the form $\mathbb{R}^n \oplus \mathbb{Z}^{(\alpha)}$ for $n < \omega$ and $\alpha \leq \omega$. □

Theorem 4.66. *The category* $\mathbf{proLiePAb}$ *has enough projectives and homological dimension* 1.

Proof. We have that $\mathbf{proLiePAb}$ has enough projectives by Theorem 4.65 and Lemma 4.58. We now prove that closed subgroups of projectives are projective. Suppose that G is a closed subgroup of $(F \oplus \mathbb{R})^{\omega}$ where F is a countable free abelian group of infinite rank. Then we have that $G_{\mathbb{S}^1} = 0$ and $G_{\mathrm{t}} = 0$. Hence, we have that $G \cong G_{\mathbb{R}} \oplus G_{\mathbb{Z}}$. Since $G_{\mathbb{R}}$ is injective, we can assume without loss of generality that $G = G_{\mathbb{Z}}$.

Set

$$N_k := \{x \in (F \oplus \mathbb{R})^{\omega} : \forall i < k, x_i = 0\}.$$

Then (N_k) is a vanishing sequence in $\mathcal{N}((F \oplus \mathbb{R})^{\omega})$, and $(M_k)_{k\in\omega}$ is a vanishing sequence in $\mathcal{N}(G)$, where $M_k := N_k \cap G$. We have that G/M_k is isomorphic to a closed subgroup L of

$$(F \oplus \mathbb{R})^{\omega}/N_k \cong (F \oplus \mathbb{R})^k \cong F^k \oplus \mathbb{R}^k.$$

We consider the cohomological derived functor of $\mathrm{H}^0 \circ \mathrm{Hom}^{\bullet}$ for locally compact Polish abelian groups introduced in [21]; see also [80, Section 4]. For every locally compact Polish abelian group A, we have an exact sequence

$$0 = \mathrm{Ext}\left(F^k \oplus \mathbb{R}^k, A\right) \to \mathrm{Ext}(L, A) \to \mathrm{Ext}^2\left(\left(F^k \oplus \mathbb{R}^k\right)/L, A\right) = 0.$$

Therefore, we have that L is a projective locally compact Polish abelian group. Since L is also countable, we have that is a free abelian group.

Thus, we have that $G \cong \lim_k G/M_k$ where, for every $k \in \omega$, G/M_k is a countable free abelian group. This shows that G is isomorphic to a product of countable free abelian groups, whence projective in $\mathbf{proLiePAb}$. □

Corollary 4.67. *The functor* $\mathrm{Hom}^{\bullet} : \mathrm{K}^b(\mathbf{proLiePAb})^{\mathrm{op}} \times \mathrm{K}^b(\mathbf{proLiePAb}) \to \mathrm{K}^b(\mathbf{Ab})$ *admits a total right derived functor* $\mathrm{RHom} : \mathrm{D}^b(\mathbf{proLiePAb})^{\mathrm{op}} \times \mathrm{D}^b(\mathbf{proLiePAb}) \to \mathrm{D}^b(\mathbf{Ab})$.

Corollary 4.68. *A pro-Lie Polish abelian group* G *is projective if and only if* $\mathrm{Ext}(G, \mathbb{Z}) = 0$ *and* $\mathrm{Ext}\left(G, \mathbb{Z}^{(\omega)}\right) = 0$.

Proof. We prove sufficiency, as necessity is obvious. By Theorem 4.66, we have a short exact sequence $P^0 \to P^1 \to G$ where P^0 and P^1 are projectives. By Theorem 4.65, we have that $P^0 \cong \mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta \oplus \mathbb{R}^\gamma$ for some $\alpha, \beta, \gamma \leq \omega$. We have that $\mathrm{Ext}\left(G, \mathbb{R}^\gamma\right) = \mathrm{Ext}\left(G, \mathbb{R}\right)^\gamma = 0$ since $\mathbb{R}$ is injective. By assumption, we have $\mathrm{Ext}\left(G, \mathbb{Z}^\alpha\right) \cong \mathrm{Ext}\left(G, \mathbb{Z}\right)^\alpha = 0$ and $\mathrm{Ext}(G, \left(\mathbb{Z}^{(\omega)}\right)^\beta) \cong \mathrm{Ext}\left(G, \mathbb{Z}^{(\omega)}\right)^\beta = 0$ by hypothesis. This implies that $\mathrm{Ext}\left(G, P^0\right) = 0$. Thus, G is a topological direct summand of P^1, and hence projective. □

Lemma 4.69. *Every countable divisible group is injective in the category of non-Archimedean Polish abelian groups.*

Proof. Let D be a countable divisible group, and A be a non-Archimedean Polish abelian group. Then there exists a sequence (C_i) of countable abelian groups such that, setting

$$C := \prod_n C_n,$$

there exists an injective continuous homomorphism $A \to C$. This induces a surjective homomorphism

$$\mathrm{Ext}\left(C, D\right) \to \mathrm{Ext}\left(A, D\right).$$

As D is injective in the category of countable abelian groups, for every $i \in \omega$ we have that $\mathrm{Ext}\left(C_i, D\right) = 0$. By Lemma 4.61 this implies that $\mathrm{Ext}\left(C, D\right) = 0$ and hence $\mathrm{Ext}\left(A, D\right) = 0$. □

For pro-Lie Polish abelian groups A and B, we set $\mathrm{Ext}^n\left(A, B\right) = \mathrm{H}^n\left(\mathrm{RHom}\left(A, B\right)\right)$. We thus have that $\mathrm{Ext}^n\left(A, B\right) = 0$ for $n \geq 2$, while $\mathrm{Ext}^1\left(A, B\right)$ (which we simply denote by $\mathrm{Ext}\left(A, B\right)$) is isomorphic to the group $\mathrm{Ext}^1_{\mathrm{Yon}}\left(A, B\right)$ (which we have been denoting by $\mathrm{Ext}_{\mathrm{Yon}}\left(A, B\right)$) of isomorphism classes of (1-fold) extensions of A by B.

Notice that, if C is a countable discrete group and A is a pro-Lie Polish abelian group, then any extension $A \to X \to C$ in $\mathbf{Ab}$ can be turned uniquely into an extension in $\mathbf{proLiePAb}$. Whence, we have that $\mathrm{Ext}\left(C, A\right) \cong \mathrm{Ext}\left(C, A_{\mathrm{disc}}\right)$, where A_{disc} is the group A endowed with the discrete topology.

We now characterize the injective objects in the category of pro-Lie Polish abelian groups.

Lemma 4.70. *Every abelian Polish pro-Lie group is isomorphic to a closed subgroup of* $\mathbb{R}^\omega \oplus \mathbb{T}^\omega \oplus U^\omega$ *where* $U = \mathbb{Q}^{(\omega)} \oplus \bigoplus_p \mathbb{Z}\left(p^\infty\right)^{(\omega)}$.

Proof. Let A be a pro-Lie abelian Polish group. Then A is isomorphic to a closed subgroup of $\prod_{k\in\omega} A_k$ where, for every $k \in \omega$, A_k is an abelian Polish Lie group. Thus, A_k is isomorphic to $\mathbb{R}^{n_k} \oplus \mathbb{T}^{m_k} \oplus D_k$ for some $n_k, m_k \in \omega$ and countable discrete group D_k. Thus, A_k is isomorphic to a closed subgroup of $\mathbb{R}^\omega \oplus \mathbb{T}^\omega \oplus U$. The conclusion about A follows. □

Theorem 4.71. *Let G be an abelian Polish pro-Lie group. The following assertions are equivalent:*

1. *G is an* injective object *in* **proLiePAb**;
2. *G is isomorphic to $\mathbb{R}^\alpha \oplus \mathbb{T}^\beta$ for some $\alpha, \beta \in \omega + 1$;*
3. *G is path-connected;*
4. $\mathrm{Ext}(\mathbb{T}, G) = 0$.

Proof. (2)⇒(1) This follows from Lemma 4.21 considering that **proLiePAb** is a thick subcategory of **PAb** by Theorem 4.28.

(4)⇒(3) Suppose that $\mathrm{Ext}(\mathbb{T}, G) = 0$. Then we have an exact sequence

$$\mathrm{Hom}(\mathbb{R}, G) \to \mathrm{Hom}(\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{T}, G) = 0$$

This shows that G is path-connected.

(3)⇒(2) Since G is connected, we have that $G \cong G_{\mathbb{R}} \oplus G_{\mathbb{S}^1}$. Furthermore, $G_{\mathbb{S}^1}$ is path-connected, and hence a torus by [3, Theorem 8.27]. □

Corollary 4.72. *The class of injective objects in* **proLiePAb** *is closed under quotients.*

Corollary 4.73. *If D is a nonzero countable discrete divisible group, then there exists no injective object I in* **proLiePAb** *such that D is isomorphic to a closed subgroup of I. Hence,* **proLiePAb** *does not have enough injectives.*

Proof. Suppose that D is a countable discrete divisible group that is a closed subgroup of $\mathbb{R}^\omega \oplus \mathbb{T}^\omega$. Then we have that $\mathbb{T}^\omega \cap D$ is a divisible subgroup of D. Hence, we have that $D \cong H \oplus (\mathbb{T}^\omega \cap D)$ for some (necessarily divisible) subgroup H of D. The composition $H \to \mathbb{R}^\omega \oplus \mathbb{T}^\omega \to \mathbb{R}^\omega$ is injective. Since $\mathbb{T}^\omega$ is compact and H is closed in $\mathbb{T}^\omega \oplus \mathbb{R}^\omega$, we have that $\mathbb{T}^\omega + H$ is also closed in $\mathbb{T}^\omega \oplus \mathbb{R}^\omega$. This implies that the image of the function $H \to \mathbb{R}^\omega \oplus \mathbb{T}^\omega \to \mathbb{R}^\omega$ is a closed subgroup of $\mathbb{R}^\omega$. Every closed divisible subgroup of $\mathbb{R}^\omega$ is a closed $\mathbb{R}$-subspace. Since H is countable, this implies that $H = 0$.

Thus, we have that $D = D \cap \mathbb{T}^\omega \subseteq \mathbb{T}^\omega$. Thus, we have that the inclusion $D \to \mathbb{T}^\omega$ induces a surjective homomorphism $\mathbb{Z}^{(\omega)} \to D^\vee$. Since $D^\vee$ is compact and connected, it is uncountable whenever it is nonzero. Hence, $D^\vee = 0$ and $D = 0$. □

4.4 Thick categories of pro-Lie Polish abelian groups

4.4.1 Pro-p Polish abelian groups

Let us say that a *pro-p* Polish group is a pro-Lie Polish abelian topological p-group. We let $\mathbf{PAb}(p)$ be the category of pro-p Polish abelian groups, which is easily seen to be a thick subcategory of **proLiePAb**. A locally compact Polish abelian group G is pro-p if

and only if it has a basis of zero neighborhoods consisting of subgroups U such that G/U is a p-group. A pro-Lie Polish abelian group is a pro-p group if and only if G/N is pro-p for every $N \in \mathcal{N}(G)$.

For an abelian group G, we define $pG = \{px : x \in G\}$. An abelian group G is p-divisible if $pG = G$. We also define recursively for every ordinal α:

$$p^0 G = G$$

and, for $\alpha > 0$,

$$p^\alpha G = \bigcap_{\beta<\alpha} p\left(p^\beta G\right).$$

If σ is the least ordinal such that $p^\sigma G = p^{\sigma+1}G$, then $p^\sigma G$ is the largest p-divisible subgroup of G. Recall that a group G is p-local if it is q-divisible for every prime number q other than p. When G is p-local, σ is the Ulm rank of G and $p^\sigma G$ is the largest divisible subgroup $d(G)$ of G. When G is a Polish group, the least ordinal σ such that $p^\sigma G = p^{\sigma+1}G$ is countable. We define, for an ordinal α,

$$L^p_\alpha(G) = \lim_{\beta<\alpha} G/p^\beta G$$

and $E^p_\alpha(G)$ to be the quotient of $L^p_\alpha(G)$ by the image of G under the canonical homomorphism $G \to L^p_\alpha(G)$.

For an ordinal α, as a particular instance of [139, Theorem 1.5] in the case of the cotorsion functor with enough projectives $G \mapsto p^\alpha G$, we have the following:

Lemma 4.74. *Let A, G be groups. The short exact sequences*

$$p^\alpha A \to A \to A/p^\alpha A$$

and

$$p^\alpha G \to G \to G/p^\alpha G$$

induce homomorphisms

$$\mathrm{Ext}(A, p^\alpha G) \to \mathrm{Ext}(p^\alpha A, p^\alpha G)$$

and

$$\mathrm{Hom}(A, G/p^\alpha G) \to \mathrm{Ext}(A, p^\alpha G).$$

These induce an isomorphism

$$\eta : \frac{\mathrm{Ext}(A, p^\alpha G)}{\mathrm{Ran}\left(\mathrm{Hom}(A, G/p^\alpha G) \to \mathrm{Ext}(A, p^\alpha G)\right)} \to \mathrm{Ext}(p^\alpha A, p^\alpha G)$$

Furthermore, the exact sequence $p^\alpha G \to G \to G/p^\alpha G$ induces an exact sequence

$$\mathrm{Hom}(A, G/p^\alpha G) \to \mathrm{Ext}(A, p^\alpha G) \to p^\alpha \mathrm{Ext}(A, G) \to p^\alpha \mathrm{Ext}(A, G/p^\alpha G) \to 0$$

which induces an exact sequence

$$0 \to \frac{\mathrm{Ext}(A, p^\alpha G)}{\mathrm{Ran}\left(\mathrm{Hom}(A, G/p^\alpha G) \to \mathrm{Ext}(A, p^\alpha G)\right)} \xrightarrow{\rho} p^\alpha \mathrm{Ext}(A, G) \to p^\alpha \mathrm{Ext}(A, G/p^\alpha G) \to 0.$$

Consider the homomorphism

$$r := \rho \circ \eta^{-1} : \mathrm{Ext}\left(p^\alpha A, p^\alpha G\right) \to p^\alpha \mathrm{Ext}\left(A, G\right).$$

We have an exact sequence

$$0 \to \mathrm{Ext}\left(p^\alpha A, p^\alpha G\right) \xrightarrow{r} p^\alpha \mathrm{Ext}\left(A, G\right) \to p^\alpha \mathrm{Ext}\left(A, G/p^\alpha G\right) \to 0$$

and r restricts to an isomorphism

$$\gamma : \mathrm{PExt}\left(p^\alpha A, p^\alpha G\right) \to u_1\left(p^\alpha \mathrm{Ext}\left(A, G\right)\right)$$

where u_1 is the first Ulm subgroup and $\mathrm{PExt}\left(p^\alpha A, p^\alpha G\right) = u_1\left(\mathrm{Ext}\left(p^\alpha A, p^\alpha G\right)\right)$.

Theorem 4.75. *Suppose that G is a pro-p Polish abelian group. The following assertions are equivalent:*

1. *G is injective in $\mathbf{PAb}\left(p\right)$;*
2. *G is divisible and it has a basis of zero neighborhoods consisting of divisible subgroups;*
3. *$G \cong \mathbb{Z}\left(p^\infty\right)^\alpha \oplus \left(\mathbb{Z}\left(p^\infty\right)^{(\omega)}\right)^\beta$ for $\alpha, \beta \leq \omega$;*
4. *$\mathrm{Ext}(\mathbb{Z}\left(p^\infty\right)^{(\omega)}, G) = 0$.*

Proof. (2)⇒(3) Let $\left(V_k\right)_{k\in\omega}$ be a decreasing basis of zero neighborhoods of G consisting of divisible subgroups such that $V_0 = G$. For $n \in \omega$ define $H_n := V_n/V_{n+1}$. Since V_{n+1} is divisible, the short exact sequence $V_{n+1} \to V_n \to V_n/V_{n+1}$ splits. Let $\pi_n : V_n \to V_n/V_{n+1}$ be the quotient map and $\rho_n : V_n/V_{n+1} \to V_n$ be a group homomorphism that is a right inverse for π_n. Given $g \in G$ one defines $g_n \in V_n$ for $n \in \omega$ recursively by setting $g_0 = g$ and $g_{n+1} = g_n - \rho_n\pi_n\left(g_n\right)$. The continuous group homomorphism $G \to \prod_{n\in\omega} H_n$, $g \mapsto \left(g_n + V_{n+1}\right)_{n\in\omega}$ is a group isomorphism. The conclusion now follows from the structure theorem for countable divisible p-groups.

(3)⇒(1) Suppose that, $G = \prod_{n\in\omega} D_n$ where, for every $n \in \omega$, D_n is a countable divisible group. If H is a pro-p Polish abelian group, we have that

$$\mathrm{Ext}\left(H, G\right) \cong \prod_{n\in\omega} \mathrm{Ext}\left(H, D_n\right).$$

For $n \in \omega$, we have that $\mathrm{Ext}\left(H, D_n\right) = 0$ by Lemma 4.69.

(1)⇒(2) If G is not divisible, then $\mathrm{Ext}\left(\mathbb{Z}/p\mathbb{Z}, G\right) \neq 0$. Since $\mathbb{Z}/p\mathbb{Z}$ is a closed subgroup of $\mathbb{Z}\left(p^\infty\right)^{(\omega)}$, it follows that $\mathrm{Ext}(\mathbb{Z}\left(p^\infty\right)^{(\omega)}, G) \neq 0$.

Suppose that G does not have a basis of zero neighborhoods consisting of divisible subgroups. Then it has a basis $\left(V_k\right)_{k\in\omega}$ of open subgroups that are not divisible and such

that $d(V_k)$ is not open. For $k \in \omega$, let $\rho_k \geq 1$ be the least countable ordinal such that $p^{\rho_k} V_k$ is *not* open. Set $H := V_0$ and $\alpha := \rho_0$.

Suppose initially that $\alpha = \beta + 1$ is a successor ordinal. Then after replacing H with $p^\beta H$ we can assume that $\alpha = 1$. Thus, we have that H is an open subgroup such that pH is not open. Then we have that, for every $k \geq 1$, V_k is not contained in pH. Thus, for every $k \geq 1$,

$$\mathrm{Ran}\left(V_k/pV_k \to H/pH\right) \cong \mathrm{Ran}\left(\mathrm{Ext}\left(\mathbb{Z}/p\mathbb{Z}, V_k\right) \to \mathrm{Ext}\left(\mathbb{Z}/p\mathbb{Z}, H\right)\right) \neq 0.$$

Thus, for every $k \geq 1$ there exists a cocycle $c_k : \mathbb{Z}/p\mathbb{Z} \to V_k$ that is not a coboundary even when regarded as a cocycle with values in H. Define the continuous cocycle $c : \left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}} \to H$ by setting

$$c\left((x_i), (y_i)\right) = \sum_i c_i\left(x_i, y_i\right).$$

We claim that c is not a coboundary even when regarded as a cocycle with values in G. We have an exact sequence

$$\mathrm{Hom}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, G/H) \to \mathrm{Ext}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, H) \to \mathrm{Ext}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, G)$$

Thus, it suffices to prove that c defines an element $[c]$ of $\mathrm{Ext}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, H)$ that does not belong to

$$\mathrm{Ran}\left(\mathrm{Hom}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, G/H) \to \mathrm{Ext}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, H)\right).$$

Considering the isomorphism

$$\mathrm{Hom}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, G/H) \cong \mathrm{Hom}\left(\mathbb{Z}/p\mathbb{Z}, G/H\right)^{(\mathbb{N})}$$

due to the fact that G/H is countable, it suffices to prove that, for every $k \in \omega$, the image of $[c]$ in $\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, H)$ induced by the inclusion $\mathbb{Z}/p\mathbb{Z} \to \left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}$ in the k-th coordinate, is nonzero. As this image is the element $[c_k]$ represented by the cocycle c_k, we have that it is nontrivial by the choice of c_k. This shows that $\mathrm{Ext}(\left(\mathbb{Z}/p\mathbb{Z}\right)^{\mathbb{N}}, G) \neq 0$.

Suppose now that α is a limit ordinal. For $k \geq 1$ let α_k be the least countable ordinal such that V_k is *not* contained in $p^{\alpha_k} H$. Notice that, since α is a limit ordinal, $\alpha_k < \alpha$. After passing to a subsequence of $(V_k)_{k \geq 1}$, we can assume that $(\alpha_k)_{k \geq 1}$ is nondecreasing. We also have that $\alpha = \sup_k \alpha_k$. We notice that, for $k \geq 1$, α_k must be a successor ordinal, and we let β_k be its predecessor.

Fix $k \geq 1$ and let T_k be a countable p-group with $p^{\beta_k} T_k = \mathbb{Z}/p\mathbb{Z}$. (Notice that this exists by Ulm's classification of countable p-groups; see [64, Chapter XII].) Consider the exact sequence

$$0 \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H\right) \xrightarrow{r} p^{\beta_k} \mathrm{Ext}\left(T_k, H\right) \to p^{\beta_k} \mathrm{Ext}\left(T_k, H/p^{\beta_k} H\right) \to 0.$$

The inclusion $V_k \subseteq p^{\beta_k} H$ induces an exact sequence

$$\mathrm{Ext}\left(p^{\beta_k} T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H/V_k\right) \to 0$$

that is isomorphic to

$$V_k/pV_k \to p^{\beta_k}H/p^{\alpha_k}H \to p^{\beta_k}H/(p^{\alpha_k}H + V_k) \to 0$$

Since V_k is not contained in $p^{\alpha_k}H$, we have that the homomorphism

$$p^{\beta_k}H/p^{\alpha_k}H \to p^{\beta_k}H/(p^{\alpha_k}H + V_k)$$

is not injective, and hence the homomorphism

$$V_k/pV_k \to p^{\beta_k}H/p^{\alpha_k}H$$

is nonzero. Thus, the homomorphism

$$\mathrm{Ext}\left(p^{\beta_k}T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k}T_k, p^{\beta_k}H\right)$$

is nonzero. The inclusion $p^{\beta_k}T_k \to T_k$ induces a surjective homomorphism

$$\mathrm{Ext}\left(T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k}T_k, V_k\right).$$

In view of the above remarks, there exists a cocycle $c_k : T_k \times T_k \to V_k$ that represents an element of $\mathrm{Ext}\left(T_k, V_k\right)$ whose image under the composition

$$\mathrm{Ext}\left(T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k}T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k}T_k, p^{\beta_k}H\right) \xrightarrow{r} p^{\beta_k}\mathrm{Ext}\left(T_k, H\right) \subseteq \mathrm{Ext}\left(T_k, H\right)$$

is nonzero. Thus, c_k is not a coboundary when regarded as a cocycle with values in H.

Proceeding as above, we can conclude that, setting $T := \prod_{k\geq 1} T_k$, $\mathrm{Ext}\left(T, H\right)$ is nonzero.

(4)⇔(1) This follows from the fact that every object of $\mathbf{PAb}\left(p\right)$ is isomorphic to a closed subgroup of $\left(\mathbb{Z}\left(p^\infty\right)^{(\omega)}\right)^\omega$. □

Remark 4.76. *The proof of Theorem 4.75 shows that* $\mathrm{Ext}((\mathbb{Z}/p\mathbb{Z})^\omega, \mathbb{Q}_p)$ *is nonzero.*

Let us denote by $\mathbf{LCPAb}\left(p\right)$ the category of *locally compact* pro-p Polish abelian groups.

Theorem 4.77. *Suppose that G is a pro-p locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is injective in $\mathbf{LCPAb}\left(p\right)$;*
2. *G is injective in $\mathbf{PAb}\left(p\right)$;*
3. *G is a countable divisible p-group.*

Proof. (1)⇒(2) By Theorem 4.75, it suffices to prove that G is divisible and has a basis of zero neighborhoods consisting of divisible subgroups. As in the proof of (1)⇒(2) of Theorem 4.75, we have that G is divisible. Suppose that G does not have a basis of zero neighborhoods

consisting of divisible subgroups. Then it has a basis $(V_k)_{k\in\omega}$ of *compact* open subgroups that are not divisible and such that $d(V_k)$ is not open. For $k \in \omega$, let $\rho_k \geq 1$ be the least countable ordinal such that $p^{\rho_k}V_k$ is *not* open. By compactness of V_k, we have that $\rho_k \leq \omega$. Indeed, if $\rho_k > \omega$, then we have that $\{p^nV \setminus p^{n+1}V : n \in \omega\}$ is an infinite clopen partition of the compact set $V \setminus \rho^\omega V$, which is impossible. Thus, proceeding as in the proof of (1)⇒(2) of Theorem 4.75, one obtains that there exists a sequence (T_k) of *finite* p-groups such that, setting $T := \prod_{k\in\omega} T_k$, one has that $\mathrm{Ext}(T, G) \neq 0$, which is a contradiction.

(2)⇒(3) This follows from (1)⇒(3) in Theorem 4.75, since G is by hypothesis locally compact. □

Corollary 4.78. *Suppose that G is a pro-p locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is projective in $\mathbf{LCPAb}(p)$;*
2. *G is a compact torsion-free p-group.*

Proof. We have that Pontryagin duality establishes an equivalence between $\mathbf{LCPAb}(p)$ and its opposite. Thus, it maps injective objects to projective objects and vice versa. Furthermore, by [3, Theorem 4.15], a locally compact Polish abelian group G is compact and torsion-free if and only if its dual is countable and divisible. □

Corollary 4.79. *The quasi-abelian category $\mathbf{PAb}(p)$ has enough injectives and homological dimension 1.*

Proof. Let G be a pro-p Polish abelian group. Then G is isomorphic to a closed subgroup of $\prod_{n\in\omega} G_n$ for some sequence (G_n) of countable p-groups. In turn, for every $n \in \omega$, G_n is a subgroup of the countable divisible p-group $D := \mathbb{Z}(p^\infty)^{(\omega)}$. Thus, G is isomorphic to a closed subgroup of D^ω, which is injective by Theorem 4.75. This shows that $\mathbf{PAb}(p)$ has enough injectives.

Suppose that A is an injective pro-p Polish abelian group, B is a Polish abelian group, and $\pi : A \to B$ is a surjective continuous homomorphism. Since A is divisible, B is also divisible. If (V_k) is a basis of zero neighborhoods of A consisting of divisible subgroups, then $(\pi(V_k))$ is a basis of zero neighborhoods of B consisting of divisible subgroups by the Open Mapping Theorem for Polish groups. Thus, B is an injective pro-p Polish abelian group. This shows that quotients of injectives are injective in $\mathbf{PAb}(p)$, and hence $\mathbf{PAb}(p)$ has homological dimension at most 1. □

Problem 4.80. *Characterize the projective objects in $\mathbf{PAb}(p)$. Does $\mathbf{PAb}(p)$ have enough projectives?*

4.4.2 Topological torsion groups

Recall that a pro-Lie Polish abelian group G is topological torsion if and only if it is an inverse limit of countable torsion groups. We thus say that a Polish abelian group is topological torsion if it is an inverse limit of countable torsion groups. Topological torsion Polish abelian groups form a thick subcategory $\mathbf{proTorPAb}$ of $\mathbf{proLiePAb}$. We denote by $\mathbf{proTorLCPAb}$ the category of locally compact topological torsion Polish abelian groups. These are precisely the locally compact Polish abelian groups that are totally disconnected and whose Pontryagin dual is also totally disconnected [3, Theorem 3.5].

Given a topological torsion Polish abelian group, we let G_p be the subgroup of G consisting of elements x such that $\lim_{n\to\infty} p^n x = 0$ [3, Definition 2.1], called the *p-component* of G. Then we have that G_p is a closed subgroup of G [3, Lemma 3.8]. We can write $G \cong G_p \oplus G_p^{\#}$ where $G_p^{\#}$ has trivial p-component, and

$$\mathrm{Hom}\left(G_p^{\#}, T\right) = \mathrm{Hom}\left(T, G_p^{\#}\right) = 0$$

for any topological p-group T. It follows that, when G and H are topological torsion groups, that

$$\mathrm{Hom}\left(G, H\right) \cong \mathrm{Hom}\left(G_p, H_p\right) \oplus \mathrm{Hom}\left(G_p^{\#}, H_p^{\#}\right)$$

and

$$\mathrm{Ext}\left(G, H\right) \cong \mathrm{Ext}\left(G_p, H_p\right) \oplus \mathrm{Ext}\left(G_p^{\#}, H_p^{\#}\right).$$

The same proofs as the ones in Section 4.4.1 give the following results.

Theorem 4.81. *Suppose that G is a topological torsion Polish abelian group. The following assertions are equivalent:*

1. *G is injective in $\mathbf{proTorPAb}$;*
2. *G is divisible and it has a basis of zero neighborhoods consisting of divisible subgroups;*
3. *G is a product of countable divisible abelian groups;*
4. *$\mathrm{Ext}((\mathbb{Z}\left(p^\infty\right)^{(\omega)})^\omega, G) = 0$ for every prime p;*
5. *G_p is injective in $\mathbf{PAb}\left(p\right)$ for every prime p.*

Theorem 4.82. *Suppose that G is a topological torsion locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is injective in $\mathbf{proTorLCPAb}$;*
2. *G is injective in $\mathbf{proTorPAb}$;*
3. *G is a countable and divisible.*

Corollary 4.83. *Suppose that G is a topological torsion locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is projective in* $\mathbf{proTorLCPAb}$*;*
2. *G is compact and torsion-free.*

Corollary 4.84. *The category* $\mathbf{proTorPAb}$ *has enough injectives and homological dimension* 1*.*

Problem 4.85. *Characterize the projective objects in* $\mathbf{proTorPAb}$*. Does* $\mathbf{proTorPAb}$ *have enough projectives?*

4.4.3 Injective non-Archimedean Polish abelian groups

In this section, we characterize injective and projective objects in the category of non-Archimedean Polish abelian groups. Recall that a Polish abelian group G is non-Archimedean if it has a basis of zero neighborhoods consisting of subgroups. This is equivalent to the assertion that G is isomorphic to a closed subgroup of U^{ω}, where U is the universal countable discrete group $\mathbb{Q}^{(\omega)} \oplus \bigoplus_p \mathbb{Z}\left(p^{\infty}\right)^{(\omega)}$. Non-Archimedean Polish abelian groups form a thick subcategory $\mathbf{PAb}_{\mathrm{nA}}$ of the quasi-abelian category of Polish abelian groups.

Recall that a locally compact Polish abelian group has type $\mathbb{Z}$ if and only if it is discrete and torsion-free. A pro-Lie Polish abelian group G has type $\mathbb{Z}$ if $\{N \in \mathcal{N}(G) : G/N \text{ has type } \mathbb{Z}\}$ is cofinal in $\mathcal{N}(G)$. Notice that every type $\mathbb{Z}$ Polish abelian group is isomorphic to a subgroup of $(\mathbb{Q}^{(\omega)})^{\omega}$.We let $\mathbf{proLiePAb}_{\mathbb{Z}}$ be the fully exact subcategory of $\mathbf{proLiePAb}$ consisting of Polish abelian groups of type $\mathbb{Z}$.

Theorem 4.86. *Let A be a non-Archimedean Polish abelian group. The following assertions are equivalent:*

1. *A is of type $\mathbb{Z}$, and projective in* $\mathbf{proLiePAb}$*;*
2. *A is projective in* $\mathbf{proLiePAb}_{\mathbb{Z}}$*;*
3. *A is isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$;*
4. *$\mathrm{Ext}(A, \mathbb{Z}) = 0$ and $\mathrm{Ext}\left(A, \mathbb{Z}^{(\omega)}\right) = 0$.*

Proof. The equivalence (3)⇔(1) follows from Theorem 4.65, while the implications (1)⇒(4) and (1)⇒(2) is obvious.

(2)⇒(1) Suppose that A is projective in $\mathbf{proLiePAb}_{\mathbb{Z}}$. Then by Theorem 4.66 there exists a surjective homomorphism $P \oplus V \to A$ where P is isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$ and V is a vector group. Since A is type $\mathbb{Z}$, $\mathrm{Hom}(V, A) = 0$. Thus, there

exists a surjective homomorphism $P \to A$. Since A is projective in $\mathbf{proLiePAb}_{\mathbb{Z}}$, we have that A is a direct summand of P. Since P is projective in $\mathbf{proLiePAb}$ by Theorem 4.65, also A is projective in $\mathbf{proLiePAb}$, concluding the proof.

(4)⇒(1) For $\alpha \leq \omega$ we have that $\mathrm{Ext}\left(A, \mathbb{Z}^{\alpha}\right) \cong \mathrm{Ext}\left(A, \mathbb{Z}\right)^{\alpha} = 0$ and $\mathrm{Ext}\left(A, \left(\mathbb{Z}^{(\omega)}\right)^{\alpha}\right) \cong \mathrm{Ext}\left(A, \left(\mathbb{Z}^{(\omega)}\right)\right)^{\alpha} = 0$. By Lemma 4.58 there exists a surjective epimorphism $\mathbb{R}^{\omega} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to A$. Since $\mathrm{Hom}\left(\mathbb{R}^{\omega}, A\right) = 0$, there exists a surjective epimorphism $\pi : \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to A$. Let $C := \mathrm{Ker}\left(\pi\right)$ and observe that C is a closed subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^{\omega}$, and hence non-Archimedean. Furthermore, C is projective in $\mathbf{proLiePAb}$, and hence isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$. Thus, by the above remarks we have that $\mathrm{Ext}\left(A, C\right) = 0$ and hence A is isomorphic to a topological direct summand of $\left(\mathbb{Z}^{(\omega)}\right)^{\omega}$. □

Theorem 4.87. *Let A be a non-Archimedean pro-Lie Polish abelian group. The following assertions are equivalent:*

1. *A is projective in $\mathbf{PAb}_{\mathrm{nA}}$;*
2. *A is isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$;*
3. *$\mathrm{Ext}\left(A, \mathbb{Z}\right) = 0$ and $\mathrm{Ext}\left(A, \mathbb{Z}^{(\omega)}\right) = 0$.*

Proof. This is an immediate consequence of Theorem 4.86. □

The same proof as Theorem 4.75 gives the following.

Theorem 4.88. *Suppose that G is a non-Archimedean Polish abelian group. The following assertions are equivalent:*

1. *G is injective in $\mathbf{PAb}_{\mathrm{nA}}$;*
2. *G is divisible and it has a basis of zero neighborhoods consisting of divisible subgroups;*
3. *G is a product of countable divisible abelian groups;*
4. *$\mathrm{Ext}((\mathbb{Z}\left(p^{\infty}\right)^{(\omega)})^{\omega}, G) = 0$ for every prime p.*

Theorem 4.89. *Let A be a type $\mathbb{Z}$ pro-Lie Polish abelian group. The following assertions are equivalent:*

1. *A is injective in $\mathbf{proLiePAb}_{\mathbb{Z}}$;*
2. *A is divisible;*
3. *$A \cong \mathbb{Q}^{\alpha} \oplus (\mathbb{Q}^{(\omega)})^{\beta}$ for some $\alpha, \beta \leq \omega$.*

Proof. (1)⇒(2) Suppose that A is an injective type $\mathbb{Z}$ pro-Lie Polish abelian group. Then A is isomorphic to a closed subgroup of $(\mathbb{Q}^{(\omega)})^{\omega}$. Since A is injective, it is a direct summand of $(\mathbb{Q}^{(\omega)})^{\omega}$, and hence it is divisible.

(2)⇒(3) If A is divisible, then $A \cong \lim_k A_k$ where, for every $k \in \omega$, A_k is countable, divisible, and torsion-free and $A_{k+1} \to A_k$ is surjective with divisible kernel (notice that the kernel of $A_{k+1} \to A_k$ is a pure subgroup of A_{k+1} since A_k is torsion-free, and since A_{k+1} is torsion-free divisible, the same holds for the kernel of $A_{k+1} \to A_k$). Since countable divisible groups are injective for countable abelian groups, it follows that A is isomorphic to the product of countable divisible groups.

(3)⇒(1) We show that if D is a countable divisible group and $\alpha \leq \omega$, then D^{α} is injective in $\mathbf{proLiePAb}_{\mathbb{Z}}$. For a type $\mathbb{Z}$ pro-Lie Polish abelian group A, we have that

$$\mathrm{Ext}\left(A, D^{\alpha}\right) \cong \mathrm{Ext}\left(A, D\right)^{\alpha}.$$

Thus, it suffices to consider the case when $\alpha = 1$. In this case, we have a short exact sequence $A \to B \to C$ where $B = \prod_k B_k$ is a product of countable abelian groups. This induces a surjective homomorphism

$$\mathrm{Ext}\left(B, D\right) \to \mathrm{Ext}\left(A, D\right).$$

Since D is injective for countable abelian groups, we have that $\mathrm{Ext}\left(B, D\right) = 0$ by Lemma 4.63. □

Corollary 4.90. *We have that:*

1. *The category* $\mathbf{PAb}_{\mathrm{nA}}$ *has enough injective objects and enough projective objects, and homological dimension* 1*;*
2. *The category* $\mathbf{proLiePAb}_{\mathbb{Z}}$ *has enough projective objects but not enough injective objects, and homological dimension* 1*. An object in* $\mathbf{proLiePAb}_{\mathbb{Z}}$ *has an injective resolution if and only if it is injective.*

Proof. (1) The same proof as Lemma 4.58 shows that, for every non-Archimedean Polish abelian group A there exists a surjective continuous homomorphism $\left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to A$. By Theorem 4.66 this implies that $\mathbf{PAb}_{\mathrm{nA}}$ has enough projective objects and homological dimension 1. The same proof as Corollary 4.79 shows that $\mathbf{PAb}_{\mathrm{nA}}$ has enough injective objects.

(2) We have that $\mathbf{proLiePAb}_{\mathbb{Z}}$ has enough projectives by Lemma 4.57 and Theorem 4.86. Furthermore, it has homological dimension 1 by Lemma 4.59. Suppose that A is a type $\mathbb{Z}$ Polish abelian group. Suppose that $A \to D \to B$ is a short exact sequence where D and B are type $\mathbb{Z}$ Polish abelian groups, and D is injective in $\mathbf{proLiePAb}_{\mathbb{Z}}$. By Theorem 4.89, we have that D is divisible. Thus, for $p \in \mathbb{Z}$ we have that $\mathrm{Hom}\left(\mathbb{Z}/p\mathbb{Z}, B\right) = 0$ and $\mathrm{Ext}\left(\mathbb{Z}/p\mathbb{Z}, D\right) = 0$. This implies that $\mathrm{Ext}\left(\mathbb{Z}/p\mathbb{Z}, A\right) = 0$ and A is divisible. □

4.4.4 Ext as a group with a Polish cover

Suppose that A, B are countable abelian groups. Then we have that $\mathrm{Hom}(A, B)$ is a Polish abelian group when endowed with the compact-open topology. We let $\mathbf{PAb}_{\aleph_0}$ be the thick subcategory of $\mathbf{proLiePAb}$ consisting of countable abelian groups. Since $\mathbf{PAb}_{\aleph_0}$ has enough injective and projective objects and homological dimension 1, we have that the functor

$$\mathrm{Hom} : \mathbf{PAb}_{\aleph_0}^{\mathrm{op}} \times \mathbf{PAb}_{\aleph_0} \to \mathbf{PAb}$$

admits a total right derived functor.

Suppose now more generally that A is a *locally compact* Polish abelian group, and B is a pro-Lie Polish abelian group. Then we still have that $\mathrm{Hom}(A, B)$ is a Polish abelian group with respect to the compact-open topology. We let $\mathbf{LCPAb}$ be the thick subcategory of $\mathbf{proLiePAb}$ consisting of locally compact Polish abelian groups, and consider the functor

$$\mathrm{Hom} : \mathbf{LCPAb}^{\mathrm{op}} \times \mathbf{proLiePAb} \to \mathbf{PAb}.$$

We will show that this functor admits a total right derived functor.

Definition 4.91. *A pro-Lie Polish abelian group G is* essentially injective *if it has a closed subgroup U injective in $\mathbf{PAb}$ such that G/U is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$.*

By Theorem 4.71, a pro-Lie Polish abelian group is essentially injective if and only if G is isomorphic to $T \oplus V \oplus A$ where T is a torus, V is a vector group, and A is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. We let $\mathcal{D}$ be the collection of essentially injective pro-Lie Polish abelian groups. By Lemma 4.70 and Theorem 4.88, every pro-Lie Polish abelian group is isomorphic to a closed subgroup of an element of $\mathcal{D}$. We will now show that the class $\mathcal{D}$ is closed under taking quotients by closed subgroups. Towards this goal, we isolate a few lemmas concerning Ext of countable abelian groups, regarded as a group with a Polish cover.

Lemma 4.92. *Suppose that T is a countable torsion group and K is a countable torsion-free group. Then $\mathrm{Ext}(T, K)$ is a Polish group isomorphic to $\mathrm{Hom}(T, D/K)$ where D is the divisible hull of K.*

Proof. The short exact sequence $K \to D \to D/K$ induces an exact sequence

$$\mathrm{Hom}(T, D) \to \mathrm{Hom}(T, D/K) \to \mathrm{Ext}(T, K) \to \mathrm{Ext}(T, D).$$

Since T is torsion and D is torsion-free, $\mathrm{Hom}(T, D) = 0$. Since D is divisible, $\mathrm{Ext}(T, D) = 0$. The conclusion follows. □

Lemma 4.93. *Suppose that A is a countable abelian group. If $\mathrm{Ext}(A, \mathbb{Z})$ is a Polish group, then $A_{\mathbb{Z}}$ is free abelian.*

Proof. Since A is countable, $F_{\mathbb{Z}}A = A_{\mathrm{t}}$ is the torsion subgroup of A. Consider the exact sequence

$$0 = \mathrm{Hom}\left(A_{\mathrm{t}}, \mathbb{Z}\right) \to \mathrm{Ext}\left(A_{\mathbb{Z}}, \mathbb{Z}\right) \to \mathrm{Ext}\left(A, \mathbb{Z}\right) \to \mathrm{Ext}\left(A_{\mathrm{t}}, \mathbb{Z}\right) \to 0.$$

Since A_{t} is torsion, we have that $\mathrm{Ext}\left(A_{\mathrm{t}}, \mathbb{Z}\right)$ is Polish by Lemma 4.92. It follows that $\mathrm{Ext}\left(A_{\mathbb{Z}}, \mathbb{Z}\right)$ is Polish as well.

Since $A_{\mathbb{Z}}$ is torsion-free, it has no finite subgroups. Thus, by [56, Corollary 11.6], $\{0\}$ is dense in $\mathrm{Ext}\left(A_{\mathbb{Z}}, \mathbb{Z}\right)$. This implies that $\mathrm{Ext}\left(A_{\mathbb{Z}}, \mathbb{Z}\right) = 0$. Hence, $A_{\mathbb{Z}}$ is a free abelian group by [60, Theorem 3.2]. □

Lemma 4.94. *The class $\mathcal{D}$ of essentially injective pro-Lie Polish abelian groups is closed under taking quotients by closed subgroups.*

Proof. Let H be a pro-Lie Polish group. Then we have that the connected component $c\left(H\right)$ of zero in H is $H_{\mathbb{S}^1} \oplus H_{\mathbb{R}}$, and $H/c\left(H\right)$ is non-Archimedean. Thus, we have that $H = V \oplus A$ for some vector group A and non-Archimedean abelian Polish group A if and only if $H_{\mathbb{S}^1} = 0$.

Suppose that G is an essentially injective pro-Lie Polish abelian group and N is a closed subgroup of G. Set $H := G/N$. Since G is essentially injective, we have that $G = T \oplus V \oplus A$ where V is a vector group, T is a torus, and A is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. We need to prove that G/N is essentially injective.

We have that $T/\left(T \cap N\right)$ is injective, being the quotient of a torus group by a closed subgroup. Furthermore, letting $\pi : G \to V \oplus A$ be the canonical quotient map, we have that $\pi^{-1}\left(\pi\left(N\right)\right) = N + T$ is closed in G, and hence $\pi\left(N\right)$ is closed is $V \oplus A$. Thus, we have a short exact sequence

$$\frac{T}{T \cap N} \to G/N \to \frac{V \oplus A}{\pi\left(N\right)}$$

that splits by injectivity of $T/\left(T \cap N\right)$. It follows that, after replacing G with G/T, we can assume without loss of generality that $T = 0$.

Since N is a closed subgroup of $G = V \oplus A$, we have that $c\left(N\right) \subseteq V = c(G)$ and hence $c\left(N\right) = N_{\mathbb{R}}$ is a vector group and $N_{\mathbb{S}^1} = 0$. Thus, we can write $N = N_{\mathbb{R}} \oplus \Xi$ where Ξ is non-Archimedean. Since $N_{\mathbb{R}} \subseteq V$ is a vector group, we can assume without loss of generality, possibly after replacing G with $G/N_{\mathbb{R}}$, that $N_{\mathbb{R}} = 0$ and $N = \Xi$ is non-Archimedean.

Then we have that $N \cap V$ is a closed subgroup of V and $V/\left(N \cap V\right)$ is injective in $\mathbf{PAb}$. Thus, $V/\left(N \cap V\right) = W \oplus T$ where T is a torus group and W is a vector group. Since W is projective in $\mathbf{PAb}$, we can assume without loss of generality that $W = 0$ and $V/\left(N \cap V\right)$ is a torus group.

Thus, we have that $N/\left(N \cap V\right)$ is a closed subgroup of G/N. Being also injective in $\mathbf{PAb}$, it is a direct summand. Thus, after replacing G with G/V and N with $N/\left(N \cap V\right)$,

we can assume that $G = A$ is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. In this case, we have that also G/N is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$, concluding the proof. □

Recall that a locally compact Polish group G is *codivisible* if its Pontryagin dual $G^\vee$ is divisible. We let $\mathcal{C}$ be the collection of locally compact groups G that are codivisible and satisfy $G_{\mathbb{S}^1} = 0$. Such a group G can be written as $V \oplus A$ where V is a vector group and A is totally disconnected. Clearly, $\mathcal{C}$ is closed under taking closed subgroups.

Lemma 4.95. *If C is a locally compact Polish group with $C_{\mathbb{Z}} = 0$, then there exists a short exact sequence $C \to D \to D/C$ where D is divisible with $D_{\mathbb{Z}} = 0$ and D/C is a countable torsion group.*

Proof. Without loss of generality, we can assume that C has no nonzero closed vector subgroups. By [80, Proposition 3.8], there exists a locally compact Polish group D containing C as an open subgroup such that D/C is a countable torsion group. Consider the pushout diagram

$$\begin{array}{ccc} C & \to & C_{\mathrm{t}} \\ \downarrow & & \downarrow \\ D & \to & H \end{array}$$

This gives rise to a commutative diagram

$$\begin{array}{ccccc} C_{\mathbb{S}^1} & \to & C & \to & C_{\mathrm{t}} \\ \downarrow & & \downarrow & & \downarrow \\ C_{\mathbb{S}^1} & \to & D & \to & H \\ & & \downarrow & & \downarrow \\ & & D/C & \to & H/C_{\mathrm{t}} \end{array}$$

where $D/C \to H/C_{\mathrm{t}}$ is an isomorphism. Since C_{t} is a topological torsion group and $H/C_{\mathrm{t}} \cong D/C$ is a countable torsion group (and in particular a topological torsion group), we conclude that H is a topological torsion group, whence $D_{\mathbb{Z}} = 0$. □

Corollary 4.96. *If C is a locally compact Polish group with $C_{\mathbb{S}^1} = 0$, then there exists a short exact sequence $A \to D \to C$ where D is a codivisible locally compact Polish abelian group with $D_{\mathbb{S}^1} = 0$ and A is a profinite Polish group.*

Lemma 4.97. *For every locally compact Polish group C there exists a continuous surjective homomorphism $G \to C$ for some element G of $\mathcal{C}$.*

Proof. Without loss of generality, we can assume that C has no nonzero closed vector groups as a direct summand, whence $C/C_{\mathbb{S}^1}$ is totally disconnected. We have that $C^\vee_{\mathbb{S}^1}$ is a countable torsion-free group. Thus there exists a short exact sequence $E \to C^\vee_{\mathbb{S}^1} \to S$ where E is a countable free abelian group and S is a countable torsion group. By duality, this gives a short exact sequence $A \to C_{\mathbb{S}^1} \to T$, where $A = S^\vee$ is a profinite abelian Polish

group and $T = E^\vee$ is a torus group. Considering the inclusion $C_{\mathbb{S}^1} \to C$ one obtains by pushout a diagram

$$\begin{array}{ccc} C_{\mathbb{S}^1} & \to & T \\ \downarrow & & \downarrow \\ C & \to & H \end{array}$$

which gives a commutative diagram

$$\begin{array}{ccccc} A & \to & C_{\mathbb{S}^1} & \to & T \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & C & \to & H \\ & & \downarrow & & \downarrow \\ & & C/C_{\mathbb{S}^1} & \to & T/H \end{array}$$

where $A \to C \to H$ is a short exact sequence and the continuous group homomorphism $T \to H$ is injective with closed image, and the continuous group homomorphism $C/C_{\mathbb{S}^1} \to T/H$ is an isomorphism. By injectivity of T, we have that $H \cong T \oplus C/C_{\mathbb{S}^1}$. Since T is a locally compact torus group, there exists a continuous surjective homomorphism $V \to T$ for some locally compact vector group V, which induces a continuous surjective homomorphism $V \oplus C/C_{\mathbb{S}^1} \to H$. We consider the pullback diagram

$$\begin{array}{ccc} C & \to & H \\ \uparrow & & \uparrow \\ C' & \to & V \oplus C/C_{\mathbb{S}^1} \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccccc} A & \to & C & \to & H \\ \uparrow & & \uparrow & & \uparrow \\ A & \to & C' & \to & V \oplus C/C_{\mathbb{S}^1} \end{array}$$

By projectivity of V, we can write $C' = V \oplus C''$, where we have an extension $A \to C'' \to C/C_{\mathbb{S}^1}$. Hence, C'' is totally disconnected. The conclusion thus follows from Corollary 4.96. □

Proposition 4.98. *We have that, for a locally compact Polish abelian group $C \in \mathcal{C}$ and a pro-Lie Polish abelian group $D \in \mathcal{D}$,* $\mathrm{Ext}(C, D) = 0$.

Proof. We can write $D = T \oplus V \oplus A$ where T is a torus, V is a vector group, and A is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. We can also write $C = W \oplus B$ where W is a vector group and B is totally disconnected. By injectivity of T and V, and projectivity of W, in the category of pro-Lie Polish abelian groups, we have that $\mathrm{Ext}(C, D) \cong \mathrm{Ext}(B, A)$. By Theorem 4.88, we have that $A \cong \prod_{n \in \omega} A_n$ where A_n is a countable divisible group. Thus,

$$\mathrm{Ext}(B, A) \cong \prod_{n \in \omega} \mathrm{Ext}(B, A_n) = 0.$$

This concludes the proof. □

It follows from Proposition 4.12 in the case when $\mathcal{A} = \mathbf{LCPAb}$, $\mathcal{B} = \mathbf{proLiePAb}$, and $\mathcal{C}$ and $\mathcal{D}$ are the classes defined above, Proposition 4.98, Lemma 4.97, and Lemma 4.94, that $\mathrm{Hom}^\bullet : \mathrm{K}^b(\mathbf{LCPAb}) \times \mathrm{K}^b(\mathbf{proLiePAb}) \to \mathrm{K}^b(\mathbf{PAb})$ has a total right derived functor $\mathrm{RHom} : \mathrm{D}^b(\mathbf{LCPAb}) \times \mathrm{D}^b(\mathbf{proLiePAb}) \to \mathrm{D}^b(\mathbf{PAb})$, and $\mathrm{H}^0 \circ \mathrm{RHom} : \mathrm{D}^b(\mathbf{LCPAb}) \times \mathrm{D}^b(\mathbf{proLiePAb}) \to \mathrm{LH}(\mathbf{PAb})$ is a cohomological derived functor of $\mathrm{H}^0 \circ \mathrm{Hom}^\bullet : \mathrm{K}^b(\mathbf{LCPAb}) \times \mathrm{K}^b(\mathbf{proLiePAb}) \to \mathrm{LH}(\mathbf{PAb})$.

The same argument as in the proof of [21, Proposition 4.13] yields the following description of $\mathrm{Ext}^1(G, A)$ for locally compact Polish abelian group G and pro-Lie Polish abelian group A. Recall that if (X, μ) is a standard probability space and A is a Polish space, then we let $L^0(X, A)$ be the Polish space of μ-a.e. classes of functions $X \to A$ endowed with the topology of convergence in measure.

Proposition 4.99. *Suppose that G is a locally compact abelian Polish group and A is a pro-Lie abelian Polish group. Define $\mathrm{Z}(\mathrm{Ext}_{\mathrm{Yon}}(G, A))$ to be the group of Borel 2-cocycles on G with coefficients in A.*

Endow $\mathrm{Z}(\mathrm{Ext}_{\mathrm{Yon}}(G, A))$ with the topology given by letting a net (c_i) converge to c if and only if $(c_i(x, y, z))$ converges to $c(x, y, z)$ in A for every $x, y, z \in G$. Define $\mathrm{B}(\mathrm{Ext}_{\mathrm{Yon}}(G, A))$ to be the subgroup of $\mathrm{Z}(\mathrm{Ext}_{\mathrm{Yon}}(G, A))$ consisting of Borel cocycles of the form $\delta f(x, y) = f(y) - f(x + y) + f(x)$ for some Borel function $f : G \to A$. The function

$$\Psi : \mathrm{Z}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G, A)\right) \to L^0\left(G^2, A\right)$$

defined by mapping c to its a.e.-class establishes a continuous isomorphism with a closed subgroup of $L^0(G^2, A)$. Furthermore, $\mathrm{B}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G, A)\right)$ is a Polishable subgroup of $\mathrm{Z}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G, A)\right)$. Consider the group with a Polish cover

$$\mathrm{Ext}^1_{\mathrm{Yon}}(G, A) := \mathrm{Z}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G, A)\right) / \mathrm{B}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G, A)\right).$$

Then $\mathrm{Ext}^1_{\mathrm{Yon}}(G, A)$ is naturally Borel-definably isomorphic to the group with a Polish cover $\mathrm{Ext}^1(G, A)$.

Chapter 5

Projective length and phantom extensions of flat modules

5.1 Introduction

The subject of homological algebra can be traced back to the seminal paper "Group Extensions and Homology" [56] where Eilenberg and MacLane introduced the group $\mathrm{Ext}\,(C, A)$ parametrizing abelian group extensions of an abelian group C by A. In this work, they were interested in the natural *topology* this group is endowed with, coming from the canonical Polish topology on the cochain complex that has Ext as cohomology group. In terms of this topology, they characterized the submodule $\mathrm{PExt}\,(C, A)$ parametrizing pure extensions of C by A as the closure of the trivial submodule, a manifestation of the fact that the topology on Ext fails in general to be Hausdorff. This failure is drastic when C is torsion-free, in which case all extensions are pure, and the topology is trivial.

In this work, we refine the original analysis of Eilenberg and MacLane by considering additional structure on Ext obtained by regarding it as an object of the *left heart* of the category of abelian Polish groups. The objects of this category can be regarded as abelian *groups with a Polish cover*, which are groups explicitly presented as a quotient G/N where G is an abelian Polish group and N is a *Polishable subgroup* of G, i.e., a (not necessarily closed) subgroup that is also Polish, such that the inclusion map $N \to G$ is continuous. This structure allows one to define the *Borel complexity* of a subgroup $H/N \subseteq G/N$ as the Borel complexity of H within the Polish group G. It is proved by Solecki in [159]—see also [114, 115]—that a group with a Polish cover G/N has a canonical chain of subgroups $s_\alpha\,(G/N)$ indexed by countable ordinals, called *Solecki subgroups*, where $s_\alpha\,(G/N)$ is characterized as the smallest subgroup of G/N of Borel complexity $\mathbf{\Pi}^0_{1+\alpha+1}$. The least α for which $s_\alpha\,(G/N)$ is trivial (the *Solecki length* of G) corresponds to the Borel rank of N in G. As it happens, for $\alpha = 0$ one recovers the closure of the trivial subgroup with respect to the quotient topology. We say that G/N is a *phantom Polish group* when N is dense in G, as in this case G/N has "no classical trace".

We generalize the Eilenberg–MacLane Theorem, working in the more general context of modules over a Prüfer domain. Prüfer domains are a very well-studied class of rings, which admits several equivalent characterizations [11]. In particular, a domain is Prüfer if and only if each of its finitely-generated ideals is projective (as a module). This entails that a module is flat if and only if it is torsion-free, and that every finitely-generated flat module is projective. Prüfer domains generalize Dedekind domains, which are precisely the Noetherian Prüfer domains. In particular, every PID is a Prüfer domain.

We consider the modules $\mathrm{Ext}\,(C, A)$ for countable flat modules C and A over a Prüfer domain as phantom Polish modules, characterizing the Solecki submodule $s_\alpha\mathrm{Ext}\,(C, A)$ as the submodule $\mathrm{Ph}^\alpha\mathrm{Ext}\,(C, A)$ parametrizing the *phantom* (or pure) *extensions of order* α. (Pure extensions correspond to the case $\alpha = 0$.) This concept originates from topology, as it was initially defined for phantom maps between CW complexes [85]. The notion of phantom map has been generalized to an arbitrary triangulated category in [42]. The *order* of a phantom map also admits a natural analogue in a triangulated context. In particular, by regarding $\mathrm{Ext}\,(C, A)$ as the module of morphisms $C \to A[1]$ in the derived category of countable flat modules, one obtains the notion of phantom extension of order α. More generally, we define the α-th *phantom subfunctor* of a cohomological functor on a triangulated category, which yields the submodule of phantom morphisms of order α when applied to

Hom with a fixed target. The characterization of pure extensions in terms of existence of lifts of elements of the same order admits a natural generalization in terms of "higher order lifts", characterizing phantom (or pure) extensions of order α.

We identify $\mathrm{Ph}^{\alpha}\mathrm{Ext}$ for countable flat modules as the derived functor $\mathrm{Ext}_{\mathcal{E}_\alpha}$ of Hom with respect to a canonical exact structure $\mathcal{E}_\alpha$ on the category $\mathbf{Flat}\,(R)$ of flat modules. This exact structure $\mathcal{E}_\alpha$ can be identified as the exact structure *projectively generated* by a class $\mathcal{S}_\alpha$ of countable flat modules. Recall that for a category $\mathcal{C}$, a presheaf of finite flat modules over $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to the category $\mathbf{FinFlat}\,(R)$ of finite flat modules. Particularly, this applies when $\mathcal{C}$ is an ordered set, such as a rooted tree with the order defined for distinct nodes x, y, $x \prec y$ if and only if y belongs to the unique path from the root to x. By a *forest*, we mean a disjoint union of rooted trees, with the induced order.

The classes $\mathcal{S}_\alpha$ for $\alpha < \omega_1$ are explicitly described as the colimits of presheaves of finite flat modules over *well-founded tree of rank* α, when α is a successor, or less than α, when α is a limit. We prove that the exact category $(\mathbf{Flat}\,(R)\,, \mathcal{E}_\alpha)$ is hereditary with enough projectives. Furthermore, when R is a Dedekind domain, $(\mathcal{E}_\alpha)_{\alpha<\omega_1}$ is a strictly decreasing chain of nontrivial exact structures on $\mathbf{Flat}\,(R)$.

We say that a countable flat module C has *projective length* at most α if it is $\mathcal{E}_\alpha$-projective, i.e., $\mathrm{Ph}^{\alpha}\mathrm{Ext}\,(C, A) = 0$ for every countable flat module A, and we let $\mathcal{P}_\alpha$ be the class of $\mathcal{E}_\alpha$-projective objects. We prove the following *Structure Theorem* for $\mathcal{P}_\alpha$:

Theorem A.5. *Let R be a countable Dedekind domain. Suppose that C is a countable flat module. Then C has projective length at most α if and only if it is a direct summand of a colimit of a presheaf of finite flat modules over a countable well-founded forest of rank α.*

In the companion paper [116], an analogous result is obtained in the torsion case, showing that a countable *torsion* module has projective length at most α if and only it is a colimit presheaf of finite *torsion* modules over a well-founded forest of rank α. In this case, this is also equivalent to being a reduced countable torsion module of Ulm length at most α. Thus, the notion of projective length can be seen as an extension to the flat case of the Ulm length of torsion modules. (The usual definition of the Ulm length in terms of the Ulm submodules is not applicable in the flat case since the Ulm submodules of a reduced flat module are all trivial.) Furthermore, Theorem 5.1 can be seen as the analogue in the flat case of the classical Ulm Classification Theorem of reduced torsion modules of Ulm length at most α.

We also obtain another Structure Theorem applicable to arbitrary countable flat modules. Let us say that a countable flat module C has A-projective length at most α for some other countable flat module A if and only if $\mathrm{Ph}^{\alpha}\mathrm{Ext}\,(C, A) = 0$. For each $\alpha < \omega_1$, a countable flat module C admits a canonical largest submodule $\sigma_\alpha C$ of R-projective length at most α. Such submodules can be explicitly defined by recursion on α, and are characterized by the isomorphism

$$\mathrm{Ext}\,(\partial_\alpha C, R) \cong \mathrm{Ph}^{\alpha}\mathrm{Ext}\,(C, R)$$

induced by the quotient map $C \to \partial_\alpha C = C/\sigma_\alpha C$. The R-projective length of C is then the least α for which $\partial_\alpha C$ is trivial. This provides a description of C as colimit of the (eventually

constant, continuous at limits) chain of pure submodules $(\sigma_\alpha C)_{\alpha<\omega_1}$ such that $\sigma_{\alpha+1}C/\sigma_\alpha C$ has R-projective length at most 1 for every $\alpha < \omega_1$.

We also isolate a natural class of countable flat modules—here named *extractable*—which are recursively constructed starting from the trivial one by taking countable direct sums, direct summands, and extensions of finite-rank modules. Letting the *extractable length* of such a countable flat module C to be the index of the stage at which it is obtained, one has that the extractable length coincides with the projective length and the R-projective length of C, and is simply called the *length* of C.

For a countable flat module A, we consider those extractable modules that are obtained as above using building blocks F that are not only finite-rank, but also satisfy $\mathrm{Hom}\,(F, A) = 0$. We call these extractable modules A-*independent*. For such modules, the length is also equal to the A-projective length. Using modules of this form, we prove the following Dichotomy Theorem concerning extensions of a given countable flat module.

Theorem B.5. *Let R be a countable Dedekind domain. Suppose that A is a countable flat module. Then either A is injective, and hence $\mathrm{Ext}\,(-, A) = 0$, which happens precisely when A is divisible, or for every $\alpha < \omega_1$ there exists an A-independent extractable countable flat module of length α. In particular, in this second case A admits phantom extensions of arbitrarily high order.*

Considering the complexity-theoretic characterization of phantom extensions provides a reformulation of the Dichotomy Theorem in terms of the complexity of classifying extensions of countable flat modules over a Dedekind domain in the sense of Borel complexity theory [68]. Thus, if A is a countable flat module, then either for all countable flat modules C the relation of isomorphism of extensions of C by A is trivial (which happens precisely when A is divisible), or there exist countable flat modules C such that the relation of isomorphism of extensions of C by A has arbitrarily high (potential) Borel complexity. In fact, in the latter case, for each possible potential complexity class Γ of a Borel orbit equivalence relation that is classifiable by countable structures, there exists a countable flat module C_Γ such that the relation of isomorphism of extensions of C_Γ by A has potential complexity class Γ.

We also prove that for every $\alpha < \omega_1$, the class $\mathcal{P}_\alpha$ of countable flat modules of projective length at most α is a Borel subset of the Polish space of all countable flat modules; see Theorem 5.191(2). However, the union $\mathcal{P}_\infty$ of the classes $\mathcal{P}_\alpha$ for $\alpha < \omega_1$ is a co-analytic set that is not Borel; Theorem 5.191(3).

The proofs of our main results hinge on the canonical contravariant equivalence

$$\mathrm{colim}_{n\in\omega} C_n \mapsto (\mathrm{Hom}\,(C_n, R))_{n\in\omega}$$

between the category of countable flat modules and the category of *towers of finite* (i.e., finitely-generated) *flat modules*. Via this equivalence, the functor $\mathrm{Ext}(-, R)$ corresponds to the functor $\lim^1$. We analyze the latter regarded as a functor to the category of phantom Polish modules, characterizing the α-th Solecki submodule of $\lim^1 \boldsymbol{A}$ as $\lim^1 \boldsymbol{A}_\alpha$. Here, $\boldsymbol{A}_\alpha$ is the α-th *derived tower* of $\boldsymbol{A}$, obtained via the operation of derivative applied to a combinatorial tree canonically associated with $\boldsymbol{A}$. We also observe that the functor $\mathrm{Ext}\,(-, R)$

provides an equivalence between the category of countable flat modules that are *coreduced* (have no nonzero homomorphisms to R) and the category of phantom pro-finiteflat Polish modules—which are phantom Polish modules of the form G/N where both G and N are inverse limit of towers of finite flat modules. Whence, $\lim^1$ provides an equivalence between the category of towers of countable finite flat modules that are *reduced* (have trivial inverse limit) and the category of phantom pro-finiteflat modules.

The main results of this work have a host of implications for homological invariants from algebraic, complex, and coarse geometry, topology, and operator algebras, such as Čech cohomology [23, 56], coarse cohomology [143], Lie group and Lie algebra cohomology [89, 109, 167], discrete, measurable, and continuous cohomology of topological groups and groupoids [5, 6, 37, 38, 132, 133], cohomology of Banach algebras [92, 94], Ext of Banach spaces [31], cohomology of group actions on standard probability spaces [104, 148–150] von Neumann algebras [101, 140] C*-algebras [77, 87, 88] and Banach spaces [125, 126, 156], Dolbeault cohomology [39, 46, 73, 102, 157], and KK-theory [106, 127, 152, 153, 155]. All these invariants can be described as groups with a Polish cover. The study of the corresponding quotient topology, as well as the problem of determining when it is Hausdorff, or more generally to compute the closure of the trivial subgroup, has attracted in each of these cases a lot of attention, as the references above show. A group with a Polish cover is Hausdorff precisely when it is in fact a Polish group or, equivalently, it has Solecki length 0. More generally, the closure of the trivial subgroup is precisely the Solecki subgroup of index 0. Thus, the problem of determining the Solecki length, which can be an arbitrary countable ordinal, can be seen as a wide-reaching generalization of the Hausdorffness problem. In turn, the problem of describing the Solecki subgroup of index α for an arbitrary countable ordinal α subsumes the problem of describing the closure of the trivial subgroup as the particular case when $\alpha = 0$.

In many cases the invariants mentioned above can be expressed in terms of Ext via so-called Universal Coefficient Theorems, thus allowing one to characterize their *phantom* cohomological subfunctors as those corresponding to $\mathrm{Ph}^\alpha\mathrm{Ext}$. Alternatively, one can consider Milnor exact sequences expressing the closure of $\{0\}$ in these invariants as a $\lim^1$ of a tower of countable modules, whose derived towers correspond via $\lim^1$ to their phantom subfunctors. This perspective allows one to refine the classical approach by providing for these invariants a canonical family of subfunctors indexed by countable ordinals.

This article is divided into 13 sections, including this introduction. In Section 5.2 we recall some necessary background from category theory, including the notion of *(quasi-)abelian category* and the construction of *derived categories* and *derived functors*. In Section 5.3 we present some results from commutative algebra concerning *Prüfer and Dedekind domains* and their *modules*. Fundamental properties of the class of inverse limits of towers countable modules (*pro-countable Polish modules*) and its notable subcategories are established in Section 5.4. Building on this analysis, *modules with a Polish cover*, phantom Polish modules, and their *Solecki submodules*, as previously defined in [22, 114], are introduced in Section 5.5.2.

This work considers various notions of "length" and "rank". Ordinal-valued ranks are a well-studied topic in descriptive set theory, particularly the notion of *co-analytic rank* on a co-analytic set. We recall in Section 5.6 these notions and fundamental results concerning

them. The definition and fundamental properties of the first derived functor $\lim^1$ of the limit functor for towers of countable modules are recalled in Section 5.7. The definition of *derived tower* and the characterization of the Solecki subfunctors of $\lim^1$ in terms of derived towers are also included in this section.

Phantom morphisms as previously defined by Christensen [42] are introduced in Section 5.8. Here, we define the analogue for morphisms in a triangulated category of the *order* of a phantom map in the topological setting. More generally, we consider the *phantom subfunctors* of a given cohomological functor on a triangulated category—phantom morphisms being recovered in the particular case of corepresentable functors.

The derived category of the category of countable (flat) modules, and the derived functor Ext of Hom are introduced in Section 5.9. Here the *order of a phantom extension* is introduced as a particular instance of the order of a phantom morphism in the triangulated category. The *projective length* and A-projective length of a countable module are defined as relaxations of projectivity in terms of the phantom subfunctors $\mathrm{Ph}^\alpha\mathrm{Ext}$ of Ext. A characterization of modules of projective length at most one and R-projective length at most one is also obtained.

In Section 5.10 we produce phantom projective resolutions of order α. We realize $\mathrm{Ph}^\alpha\mathrm{Ext}$ as a derived functor of Hom with respect to the exact structure $\mathcal{E}_\alpha$ projectively generated by the class $\mathcal{S}_\alpha$ of colimits of presheaves of finite flat modules over well-founded rooted trees of rank α, when α is a successor, or less than α, when α is limit. We also completely characterize the countable flat modules that have projective length at most α, i.e., are $\mathcal{E}_\alpha$-projective, as the direct summands of colimits of presheaves of finite flat modules over well-founded forests of rank α.

In Section 5.11, higher order phantom extensions are further analyzed. In particular, we construct for every countable flat module a canonical chain of submodules characterized by their projective length, providing the first *Structure Theorem* for arbitrary countable flat modules over a Prüfer domain. *Extractable modules* and the corresponding length are also introduced here, and their main homological properties are established.

Section 5.12 provides explicit constructions of phantom extensions in the case of modules over countable Dedekind domains. For every countable flat module A that admits nontrivial extensions (i.e., it is not divisible) we produce extensions of arbitrarily high order. We conclude in Section 5.13 by isolating the complexity-theoretic content of phantom extensions, in terms of the *Borel complexity* of sets and equivalence relations as studied in descriptive set theory and invariant complexity theory. In particular, we notice how the main results of the chapter can be seen as offering a *Dichotomy Theorem* for the complexity of isomorphism of extensions of a given countable flat module A, a classification problem that is either trivial or has arbitrarily high complexity.

Notationally, we denote by $\mathbb{N}$ the set of positive integers *excluding zero*. The set of nonnegative integers is denoted by ω and identified with the first infinite ordinal. A nonnegative integer $n \in \omega$ is identified with the set $\{0, 1, \ldots, n-1\}$ of its predecessors. We let ω_1 be the set of countable ordinals, which is the first uncountable ordinal. Thus, we write $\alpha < \omega_1$ to mean that α is a countable ordinal.

5.2 Category theory background

In this section we recall some notions from category theory that will be used in the rest of the chapter.

5.2.1 Quasi-abelian categories

A (locally small) *preadditive category* or $\mathbf{Ab}$-category is a category $\mathcal{A}$ enriched over the category $\mathbf{Ab}$ of abelian groups. Thus, for all objects x, y of $\mathcal{A}$, the corresponding hom-set $\mathrm{Hom}_{\mathcal{A}}(x, y)$ is an abelian group, such that composition of arrows is *bilinear*. An *additive category* [117, Section VIII.2] is a preadditive category that has a terminal object, which is automatically also an initial object and called a *zero object*, and all binary products, which are also binary coproducts and called *biproducts*. A functor between preadditive categories is additive if it preserves biproducts or, equivalently, it induces group homomorphisms at the level of hom-sets.

An additive category $\mathcal{A}$ is *quasi-abelian* [151, Section 1.1] if it satisfies the following axioms, together with their duals obtained by reversing all the arrows:

- $\mathcal{A}$ has all kernels (and hence all finite limits);
- the class of kernels is closed under pushout along arbitrary arrows.

In a quasi-abelian category, an arrow $f : A \to B$ is called *strict* or *admissible* if the canonical arrow $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ that it induces is an isomorphism, where $\mathrm{Coim}(f) = \mathrm{Coker}(\mathrm{Ker}(f))$ and $\mathrm{Im}(f) = \mathrm{Ker}(\mathrm{Coker}(f))$. This is equivalent to the assertion that one can write $f = m \circ e$ where e is a cokernel and m is a kernel [151, Section 1.1]. A monic arrow is strict if and only if it is a kernel, while an epic arrow is strict if and only if it is a cokernel.

An *abelian category* is a quasi-abelian category where every arrow is strict [151, Section 1.1]. Equivalently, an additive category is abelian if and only if it satisfies the following axioms, together with their duals:

- $\mathcal{A}$ has all kernels (and hence all finite limits);
- every monic arrow is a kernel.

An additive category is called *idempotent-complete* if every idempotent arrow p in $\mathcal{A}$ (satisfying $p \circ p = p$) has a kernel [30, Definition 6.1]. This implies that every idempotent arrow in $\mathcal{A}$ also has an image [30, Remark 6.2].

5.2.2 Exact categories

Exact categories provide the natural context to develop *relative homological algebra* and other generalizations of notions concerning homological algebra and derived functors.

Let $\mathcal{A}$ be an additive category. A kernel-cokernel pair in $\mathcal{A}$ is a pair (f, g) of arrows of $\mathcal{A}$ such that f is a kernel of g and g is a cokernel of f. An *exact structure* on $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs (f, g), where f is called an *admissible monic* of $\mathcal{E}$ or $\mathcal{E}$-inflation and g is called an *admissible epic* of $\mathcal{E}$ or $\mathcal{E}$-deflation, satisfying the following axioms, as well as their duals:

1. For every object of $\mathcal{A}$, the corresponding identity morphism is an admissible monic;
2. The class of admissible monics is closed under composition;
3. The pushout of an admissible monic along an arbitrary morphism exists and yields an admissible monic.

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is an additive category and $\mathcal{E}$ is an exact structure on $\mathcal{A}$ [30, Section 2]. The elements of $\mathcal{E}$ are called the *short exact sequences* for the given exact category. A morphism f of $\mathcal{A}$ is called *admissible* if it admits a factorization $f = m \circ e$ where m is an admissible monic and e is an admissible epic [30, Section 2]. In this case, such a factorization is essentially unique [30, Lemma 8.4].

A sequence

$$A' \xrightarrow{f} A \xrightarrow{g} A''$$

of admissible arrows with admissible factorizations $f = i \circ p$ and $g = j \circ q$ is *exact* or *acyclic* if the sequence

$$\bullet \xrightarrow{i} A \xrightarrow{q} \bullet$$

is short-exact. A sequence

$$A^0 \to A^1 \to \cdots \to A^n$$

for $n \geq 2$ is exact if each of the sequences

$$A^i \to A^{i+1} \to A^{i+2}$$

for $0 \leq i \leq n-2$ are exact.

If $\mathcal{A}$ is a quasi-abelian category, then the collection $\mathcal{E}$ of all kernel-cokernel pairs in $\mathcal{A}$ is an exact structure on $\mathcal{A}$ [30, Proposition 4.4]. Unless specified otherwise, we will regard a quasi-abelian category as an exact category with respect to this (maximal) exact structure.

The natural notion of morphism between exact categories is provided by exact functors; see [30, Definition 5.1].

Definition 5.1. *An additive functor $F : \mathcal{A} \to \mathcal{B}$ between exact categories is* exact *if it maps short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathcal{B}$.*

An additive functor is exact if and only if it preserves kernels and cokernels of admissible morphisms. An exact functor between exact categories preserves pushouts along admissible monics, and pullbacks along admissible epics [30, Definition 5.1].

Definition 5.2. *An additive functor $F : \mathcal{A} \to \mathcal{B}$ between exact categories is* left exact *if for every exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, the sequence

$$0 \to FA \to FB \to FC$$

in $\mathcal{B}$ is exact.

An additive functor is left exact if and only if it preserves kernels of admissible arrows. The notion of *right exact* functor is obtained by duality.

A *fully exact subcategory* $\mathcal{B}$ of an exact category $\mathcal{A}$ is a full subcategory whose collection of objects is closed under extensions, in the sense that if

$$0 \to B' \to A \to B'' \to 0$$

is an exact sequence with B' and B'' in $\mathcal{B}$, then A is isomorphic to an object of $\mathcal{B}$ [30, Definition 10.21]. We then have that $\mathcal{B}$ is itself an exact category, where a short exact sequence in $\mathcal{B}$ is a short exact sequence in $\mathcal{A}$ whose objects are in $\mathcal{B}$.

A *thick* (*quasi-*)*abelian* subcategory $\mathcal{B}$ of a (quasi-)abelian category $\mathcal{A}$ is a fully exact subcategory of $\mathcal{A}$ that is also (quasi-)abelian and such that the inclusion functor $\mathcal{B} \to \mathcal{A}$ is exact, finitely continuous, and finitely cocontinuous.

5.2.3 Triangulated categories

A *triangulated category* graded by the translation functor T is an additive category $\mathcal{A}$ endowed with an additive automorphism T together with a collection of *distinguished triangles* (X, Y, Z, u, v, w) where $u \in \mathrm{Hom}^0(X, Y)$, $v \in \mathrm{Hom}^0(Y, Z)$, and $w \in \mathrm{Hom}^1(Z, X)$, satisfying the axioms of triangulated categories [166, Section 1.1]; see also [100, Definition 10.1.6]. In this definition, we let $\mathrm{Hom}^k(X, Y)$ be the group $\mathrm{Hom}_{\mathcal{A}}\left(X, T^k Y\right)$. The composition of $f \in \mathrm{Hom}^n(X, Y)$ and $g \in \mathrm{Hom}^m(Y, Z)$ is defined to be $T^n g \circ f \in \mathrm{Hom}^{n+m}(X, Z)$. This defines a new category with graded groups as hom-sets.

A functor $F : \mathcal{A} \to \mathcal{B}$ between triangulated categories is *triangulated* if it is additive, graded (which means that TF and FT are naturally isomorphic), and it maps distinguished triangles to distinguished triangles. A *morphism* $\mu : F \Rightarrow G$ between triangulated functors $\mathcal{A} \to \mathcal{B}$ is a natural transformation μ such that $\mu T \circ \alpha_F \cong \alpha_G \circ Tu$ where α_F is the isomorphism $TF \cong FT$ and α_G is the isomorphism $TG \cong GT$ as in the definition of triangulated functor.

An additive functor $F : \mathcal{A} \to \mathcal{E}$ from a triangulated category $\mathcal{A}$ to an abelian category $\mathcal{E}$ is *cohomological* if for every distinguished triangle (X, Y, Z, u, v, w) of $\mathcal{A}$, the sequence

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ$$

is exact. This yields long exact sequence

$$\cdots \to F^d X \overset{F^d u}{\to} F^d Y \overset{F^d v}{\to} F^d Z \overset{F^{d+1} w}{\to} F^{d+1} X \to \cdots$$

where we set $F^d := F \circ T^d$ for $d \in \mathbb{Z}$. Dually, a functor is *homological* if it is a cohomological functor on the opposite category.

For example, for objects X, Y of $\mathcal{A}$, the functors $\mathrm{Hom}\,(X, -)$ and, respectively, $\mathrm{Hom}\,(-, Y)$ from $\mathcal{A}$ to the category of abelian groups are homological and cohomological, respectively.

Suppose now that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are triangulated categories, and $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor. One says that F is *graded* if there exist natural isomorphisms

$$F\,(TX, Y) \cong_{\ell_{X,Y}} TF\,(X, Y)$$

and

$$F\,(X, TY) \cong_{r_{X,Y}} TF\,(X, Y)$$

for $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ such that

$$T\ell_{X,Y} \circ r_{TX,Y} = Tr_{X,Y} \circ \ell_{X,TY};$$

see [100, Definition 10.1.1]. The functor F is *triangulated* if it is graded, as witnessed by natural transformations ℓ and r as above, and for every distinguished triangle

$$X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} TX$$

in $\mathcal{A}$ and object B of $\mathcal{B}$,

$$F\,(X, B) \to F\,(Y, B) \to F\,(Z, B) \to TF\,(X, B)$$

is a distinguished triangle in $\mathcal{C}$, where the rightmost arrow is the composition

$$F\,(Z, B) \overset{F(w, 1_B)}{\to} F\,(TX, B) \overset{\ell_{X,B}}{\to} TF\,(X, B)\,,$$

and for every distinguished triangle as above in $\mathcal{B}$ and object A of $\mathcal{A}$,

$$F\,(A, X) \to F\,(A, Y) \to F\,(A, z) \to TF\,(A, X)$$

is a distinguished triangle in $\mathcal{C}$, where the rightmost arrow is the composition

$$F\,(A, Z) \overset{F(1_A, w)}{\to} F\,(A, TX) \overset{r_{A,X}}{\to} TF\,(A, X)\,;$$

see [100, Definition 10.3.6].

5.2.4 Derived categories

Suppose that $(\mathcal{A}, \mathcal{E})$ is an idempotent-complete exact category. We let $\mathrm{Ch}^b\,(\mathcal{A})$ be the category of bounded complexes over $\mathcal{A}$, and $\mathrm{K}^b\,(\mathcal{A})$ be the category of bounded complexes of $\mathcal{A}$ whose morphisms are the *homotopy classes* of morphisms of complexes [30, Section 10].

Then we have that $\mathrm{K}^b(\mathcal{A})$ is a triangulated category with translation functor T defined by $(TA)^n = A^{n+1}$ for $n \in \mathbb{Z}$. A *strict triangle* in $\mathrm{K}^b(\mathcal{A})$ over a morphism $f : A \to B$ in $\mathrm{Ch}^b(\mathcal{A})$ is $(A, B, \mathrm{cone}(f), f, i, j)$ where $i : B \to \mathrm{cone}(f)$ and $j : \mathrm{cone}(f) \to TA$ are the canonical morphisms of complexes [30, Definition 9.2]. A distinguished triangle in $\mathrm{K}^b(\mathcal{A})$ is one that is isomorphic to a strict triangle. Similarly one defines the categories $\mathrm{K}^+(\mathcal{A})$ and $\mathrm{K}^-(\mathcal{A})$ of left-bounded complexes and right-bounded complexes, respectively.

A complex A over $\mathcal{A}$ is *acyclic* if for every $n \in \mathbb{Z}$,

$$A^n \to A^{n+1} \to A^{n+2}$$

is an exact sequence of admissible morphisms. The subcategory $\mathrm{N}^b(\mathcal{A})$ of $\mathrm{K}^b(\mathcal{A})$ spanned by bounded acyclic complexes is *thick* [30, Corollary 10.11], namely it is strictly full and it is closed under taking direct summands of its objects [30, Remark 10.10]. One can thus define the Verdier quotient

$$\mathrm{D}^b(\mathcal{A}) := \frac{\mathrm{K}^b(\mathcal{A})}{\mathrm{N}^b(\mathcal{A})}.$$

By definition, this is obtained as the *category of fractions* $\mathrm{K}^b(\mathcal{A})[\Sigma^{-1}]$. Here, Σ is the *multiplicative system* of *quasi-isomorphisms* in $\mathrm{K}^b(\mathcal{A})$. By definition, a morphism $u = [f] : X \to Y$ is a quasi-isomorphism if and only if it fits into a distinguished triangle (X, Y, Z, u, v, w). This is equivalent to the assertion that $\mathrm{cone}(f)$ is acyclic; see [30, Definition 10.16 and Remark 10.17]. We let $Q_{\mathcal{A}} : \mathrm{K}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A})$ be the canonical quotient functor.

One similarly define the categories $\mathrm{D}^+(\mathcal{A})$ and $\mathrm{D}^-(\mathcal{A})$ as quotients of $\mathrm{K}^+(\mathcal{A})$ and $\mathrm{K}^-(\mathcal{A})$, respectively.

5.2.5 Left hearts of quasi-abelian categories

Let $\mathcal{A}$ be a quasi-abelian category. A canonical "completion" of $\mathcal{A}$ to an abelian category is described in [12, 151]. Let us consider $\mathcal{A}$ as an exact category, endowed with its maximal exact structure. Let $\mathrm{LH}(\mathcal{A})$ be the heart of $\mathrm{D}^b(\mathcal{A})$ with respect to its canonical left truncation structure. Explicitly, $\mathrm{LH}(\mathcal{A})$ is the full subcategory of $\mathrm{D}^b(\mathcal{A})$ spanned by complexes A with $A^n = 0$ for $n \in \mathbb{Z} \setminus \{-1, 0\}$ and such that the coboundary morphism $A^{-1} \to A^0$ is a monomorphism. For brevity, we call $\mathrm{LH}(\mathcal{A})$ the *left heart* of $\mathcal{A}$. The definition of $\mathrm{LH}(\mathcal{A})$ as the heart of $\mathrm{D}^b(\mathcal{A})$ with respect to its canonical left truncation structure yields a canonical *cohomological* functor $\mathrm{H}^0 : \mathrm{D}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{A})$.

It is proved in [151] that the left heart of $\mathcal{A}$ is an abelian category satisfying the following universal property, which characterizes it up to equivalence: the inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$ is exact and finitely continuous, and for any abelian category $\mathcal{M}$ and exact and finitely continuous functor $F : \mathcal{A} \to \mathcal{M}$ there exists an essentially unique exact (and finitely continuous) functor $\mathrm{LH}(\mathcal{A}) \to \mathcal{M}$ whose restriction to $\mathcal{A}$ is isomorphic to F.

Recall that a *torsion pair* [161, Definition 2.4]—see also [58, 75]—in a quasi-abelian category $\mathcal{M}$ with torsion class $\mathcal{T}$ and torsion-free class $\mathcal{F}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{M}$ such that:

1. for all objects T of $\mathcal{T}$ and F of $\mathcal{F}$, $\mathrm{Hom}(T, F) = 0$;
2. for all objects M of $\mathcal{M}$ there exists a short exact sequence

$$0 \to {}_{\mathcal{T}}M \to M \to M_{\mathcal{F}} \to 0$$

where ${}_{\mathcal{T}}M$ is in $\mathcal{T}$ and $M_{\mathcal{F}}$ is in $\mathcal{F}$.

In this case, we say that $\mathcal{A}$ is an extension of $\mathcal{F}$ by $\mathcal{T}$ and write

$$0 \to \mathcal{T} \to \mathcal{A} \to \mathcal{F} \to 0.$$

We have that an object X of $\mathcal{M}$ is in $\mathcal{T}$ if and only if $\mathrm{Hom}(X, F) = 0$ for all F in $\mathcal{F}$, and it is in $\mathcal{F}$ if and only if $\mathrm{Hom}(T, X) = 0$ for all T in $\mathcal{T}$. Furthermore, $\mathcal{T}$ is closed under quotients and $\mathcal{F}$ under subobjects. The torsion pair $(\mathcal{T}, \mathcal{F})$ is *hereditary* when $\mathcal{T}$ is also closed under subobjects, in which case $\mathcal{T}$ is a thick quasi-abelian subcategory of $\mathcal{A}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{M}$, then $\mathcal{T}$ and $\mathcal{F}$ are fully exact quasi-abelian subcategories of $\mathcal{M}$ [147, Theorem 2]. Conversely, if $\mathcal{A}$ is a quasi-abelian category, let $\kappa : \mathrm{LH}(\mathcal{A}) \to \mathcal{A}$ be the left adjoint of the inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$; see [151, Defintion 1.2.26 and Proposition 1.2.27]. Thus, we have that

$$\mathrm{Hom}_{\mathcal{A}}(\kappa(X), B) \cong \mathrm{Hom}_{\mathrm{LH}(\mathcal{A})}(X, B)$$

for all objects B of $\mathcal{A}$. Define $\mathrm{Ph}(\mathcal{A})$ to be the full subcategory of $\mathrm{LH}(\mathcal{A})$ spanned by the objects X such that $\kappa(X) = 0$. Then we have that $(\mathrm{Ph}(\mathcal{A}), \mathcal{A})$ is a torsion pair in $\mathrm{LH}(\mathcal{A})$ [147, Theorem 2].

Definition 5.3. *Let $\mathcal{A}$ be a quasi-abelian category. The full subcategory $\mathrm{Ph}(\mathcal{A})$ of $\mathrm{LH}(\mathcal{A})$ defined as above is the* phantom category *of $\mathcal{A}$, and its object are called phantom objects of $\mathcal{A}$.*

Lemma 5.4. *If $\mathcal{B}$ is a fully exact quasi-abelian subcategory of $\mathcal{A}$, then the inclusion $\mathcal{B} \to \mathcal{A}$ induces a fully faithful exact functor $\mathrm{LH}(\mathcal{B}) \to \mathrm{LH}(\mathcal{A})$.*

Proof. Since the inclusion $\mathcal{B} \to \mathcal{A}$ is exact and finitely continuous, the functor $\mathrm{LH}(\mathcal{B}) \to \mathrm{LH}(\mathcal{A})$ that it induces is exact. It remains to prove that the functor $\mathrm{LH}(\mathcal{B}) \to \mathrm{LH}(\mathcal{A})$ is fully faithful. We will use the description of $\mathrm{LH}(\mathcal{A})$ as a localization as in [151, Defintion 1.2.26 and Proposition 1.2.27]. Let $\mathrm{LK}(\mathcal{A})$ be the full subcategory of $\mathrm{K}(\mathcal{A})$ spanned by complexes A such that $A^n = 0$ for $n \in \mathbb{Z} \setminus \{-1, 0\}$ and such that the coboundary morphism $A^{-1} \to A^0$ is a monomorphism. Then quasi-isomorphisms between such complexes are precisely the chain maps that induce a square commutative diagram that is both a pullback and a pushout. The corresponding localization of $\mathrm{LK}(\mathcal{A})$ is equivalent to $\mathrm{LH}(\mathcal{A})$.

Suppose that X, Y are objects of $\mathrm{LH}(\mathcal{B})$. Let $f : X \to Y$ be morphism in $\mathrm{LH}(\mathcal{A})$. Let $\mathcal{R}(f)$ be the collection of triples (Z, σ, g) that *represent* f, where Z is an object of $\mathrm{LK}(\mathcal{A})$, $\sigma : Z \to X$ is a morphism in $\mathrm{N}(\mathcal{A})$, and $g : Z \to Y$ is a morphism in $\mathrm{K}(\mathcal{A})$. We endow $\mathcal{R}(f)$ with the preorder relation defined by setting $(Z, \sigma, g) \leq (Z', \sigma', g')$ if and only if

there exists a morphism $\eta : Z \to Z'$ in $\mathrm{K}(\mathcal{A})$ such that $\sigma'\eta = \sigma$ and $g'\eta = g$. It suffices to prove that the collection $\mathcal{R}_{\mathcal{B}}(f)$ triples (Z, σ, g) in $\mathcal{R}(f)$ with Z in $\mathrm{LK}(\mathcal{B})$ is cofinal.

Suppose that (Z, σ, g) is in $\mathcal{R}(f)$. After replacing $\mathcal{A}$ with a small quasi-abelian subcategory containing $\mathcal{A}$, we can assume without loss of generality that $\mathcal{A}$ is small. In this case, we have that $\mathrm{LH}(\mathcal{A})$ is also small. By the Freyd–Mitchell Embedding Theorem [99, Definition 8.3.21], we can assume that $\mathrm{LH}(\mathcal{A})$ is a fully exact abelian subcategory of the category Mod_R of R-modules for some ring R. Whence, $\mathcal{A}$ is a fully exact quasi-abelian subcategory of Mod_R.

Consider the morphism

$$\tau = \sigma \oplus g : Z \to X \oplus Y.$$

After replacing Z_1 with $Z_1 \oplus X_1 \oplus Y_1 \oplus X_0$ and Z_0 with $Z_0 \oplus X_1 \oplus Y_1 \oplus X_0$, we can assume that τ_1 and σ_0 are epimorphisms in $\mathrm{LH}(\mathcal{A})$ (and hence admissible epimorphisms in $\mathcal{A}$).

After replacing Z_1 with $\mathrm{Coker}(\mathrm{Ker}(\tau_1))$ and Z_0 with $\mathrm{Coker}(\ker(\tau_0))$, we can assume that $\tau_1 : Z_1 \to X_1 \oplus Y_1$ is an isomorphism. Identifying Z_1 with $X_1 \oplus Y_1$ via this isomorphism, we have that

$$\mathrm{Ker}(\sigma_1) = 0 \oplus Y_1.$$

Thus, we have a short exact sequence

$$0 \to 0 \oplus Y_1 \to Z_0 \xrightarrow{\sigma_0} X_0 \to 0$$

in $\mathcal{A}$. Since $\mathcal{B}$ is a fully exact subcategory of $\mathcal{A}$, this implies that Z_0 is in $\mathcal{B}$. Thus, we have that Z is in $\mathrm{K}(\mathcal{B})$, concluding the proof. □

5.2.6 Relative homological algebra

Suppose that $\mathcal{A}$ is a category. Recall that an arrow $r : A \to B$ is a *retraction* if it has a right inverse, which is an arrow $g : B \to A$ satisfying $r \circ g = 1_B$. A class $\mathbb{E}$ of cokernels in $\mathcal{A}$ is called a proper class of cokernels [70, Section 1] if it satisfies the following:

1. Every retraction is in $\mathbb{E}$;
2. The class $\mathbb{E}$ is closed under composition;
3. The class $\mathbb{E}$ is closed under pullback along arbitrary arrows;
4. If σ, τ are arrows such that the composition $\sigma\tau$ exists, and $\sigma\tau$ and τ belong to $\mathbb{E}$, then σ belongs to $\mathbb{E}$.

A class $\mathbb{M}$ of kernels is called a proper class of kernels if it is a proper class of cokernels in the opposite category $\mathcal{A}^{\mathrm{op}}$.

Lemma 5.5. *Suppose that $\mathcal{E}$ is a class of short exact sequences in a finitely complete and finitely cocomplete additive category $\mathcal{A}$. The following assertions are equivalent:*

1. *$(\mathcal{A}, \mathcal{E})$ is an exact category;*
2. *The class $\mathbb{E}$ of cokernels that appear in $\mathcal{E}$ is a proper class of cokernels, and the class $\mathbb{M}$ of kernels that appear in $\mathcal{E}$ is a proper class of kernels.*

Proof. (1)⇒(2) It suffices to prove that $\mathbb{M}$ is a proper class of kernels, as the other assertion follows by duality. Since $\mathcal{A}$ is in particular idempotent complete, every coretraction is an admissible monic by [30, Corollary 7.5]. By definition, we also have that the class $\mathbb{M}$ is closed under composition and pushout by arbitrary arrows. Suppose now that σ, τ are arrows such that the composition $\sigma\tau$ exists, and $\sigma\tau$ and σ are admissible monics. We need to prove that τ is an admissible monic. This follows from the "Obscure Axiom" of exact categories [30, Proposition 7.6], asserting that if f, g are morphisms in an idempotent-complete exact category such that gf is defined and is an admissible monic, then f is an admissible monic.

(2)⇒(1) This is obvious. □

Suppose that $\mathfrak{I}$ is a class of objects in a quasi-abelian category $\mathcal{A}$. Let $\mathcal{E}_\mathfrak{I}$ be the collection of kernel-cokernel pairs $A \to B \to C$ in $\mathcal{A}$ such that for every object X of $\mathfrak{I}$ the induced morphism

$$\mathrm{Hom}\,(X, B) \to \mathrm{Hom}\,(X, C)$$

is surjective. We say that $\mathcal{E}_\mathfrak{I}$ is the exact structure *projectively generated* by $\mathfrak{I}$.

Proposition 5.6. *The exact structure $\mathcal{E}_\mathfrak{I}$ projectively generated by a class of objects $\mathfrak{I}$ in a quasi-abelian category $\mathcal{A}$ is indeed an exact structure on $\mathcal{A}$.*

Proof. We verify the axioms of exact structures. It is clear that the identity morphisms are $\mathcal{E}_\mathfrak{I}$-inflations and $\mathcal{E}_\mathfrak{I}$-deflations, and that composition of $\mathcal{E}_\mathfrak{I}$-deflations is an $\mathcal{E}_\mathfrak{I}$-deflation. Let us prove that composition of $\mathcal{E}_\mathfrak{I}$-inflations is an $\mathcal{E}_\mathfrak{I}$-inflation. Suppose that $A \to B \to B/A$ and $B \to C \to C/B$ are kernel-cokernel pairs that belong to $\mathcal{E}_\mathfrak{I}$. We let $\pi^B_{B/A} : B \to B/A$ and $\pi^C_{C/B} : C \to C/B$ be the corresponding $\mathcal{E}_\mathfrak{I}$-deflations. We claim that $A \to C \to C/A$ belongs to $\mathcal{E}_\mathfrak{I}$, where $C \to C/A$ is the cokernel of $A \to C$. Let X be an object of $\mathfrak{I}$ and let $f : X \to C/A$ be a morphism. Let $B/A \to C/A \to C/B$ be the kernel-cokernel pair induced by $B \to C \to C/B$ and the strict monic $A \to B$. Let $\pi^{C/A}_{C/B} : C/A \to C/B$ be the corresponding strict epic and $\iota^{B/A}_{C/A} : B/A \to C/A$ be the corresponding strict monic. Consider $\pi^{C/A}_{C/B} f : X \to C/B$. Since $B \to C \to C/B$ is $\mathcal{E}_\mathfrak{I}$-exact, there exists a morphism $\varphi : X \to C$ such that $\pi^C_{C/B}\varphi = \pi^{C/A}_{C/B} f$. Consider $\eta := f - \pi^C_{C/A}\varphi : X \to C/A$ and observe that $\pi^{C/A}_{C/B}\eta = \pi^{C/A}_{C/B} f - \pi^C_{C/B}\varphi = 0$. Therefore we have that $\eta = \iota^{B/A}_{C/A}\eta_0$ for some $\eta_0 : X \to B/A$. Since $A \to B \to B/A$ is $\mathcal{E}_\mathfrak{I}$-exact, there exists a morphism $\tau : X \to B$ such that $\eta_0 = \pi^B_{B/A}\tau$.

Define now $g := \iota_C^B \tau + \varphi : X \to C$. Then we have that

$$\begin{aligned} \pi_{C/A}^C g &= \pi_{C/A}^C \iota_C^B \tau + \pi_{C/A}^C \varphi = \iota_{C/A}^{B/A} \pi_{B/A}^B \tau + \pi_{C/A}^C \varphi \\ &= \iota_{C/A}^{B/A} \eta_0 + \pi_{C/A}^C \varphi = \eta + \pi_{C/A}^C \varphi = f. \end{aligned}$$

This concludes the proof that $A \to C \to C/A$ is $\mathcal{E}_{\mathfrak{I}}$-exact.

We now show that the pullback of an $\mathcal{E}_{\mathfrak{I}}$-deflation along an arbitrary morphism is an $\mathcal{E}_{\mathfrak{I}}$-deflation. Suppose that $A \to B \to B/A$ is $\mathcal{E}_{\mathfrak{I}}$-exact and that $t : L \to B/A$ is a morphism. Let P be the pullback of $\pi_{B/A}^B : B \to B/A$ and t with canonical arrows $a : P \to B$ and $b : P \to L$. We claim that b is an $\mathcal{E}_{\mathfrak{I}}$-deflation. We have that b is a cokernel by the axioms of quasi-abelian categories. Suppose that $f : X \to L$ is a morphism, and let $f_0 := tf : X \to B/A$. Since $A \to B \to B/A$ is $\mathcal{E}_{\mathfrak{I}}$-exact, we have that there exists an arrow $g_0 : X \to B$ such that $\pi_{B/A}^B g_0 = f_0 = tf$. By the universal property of the pullback, there exists a unique arrow $\xi : X \to P$ such that $b\xi = f$ and $a\xi = g_0$. This shows that b is an $\mathcal{E}_{\mathfrak{I}}$-deflation.

We now prove that the pushout of an $\mathcal{E}_{\mathfrak{I}}$-inflation along an arbitrary morphism is an $\mathcal{E}_{\mathfrak{I}}$-inflation. Suppose that $A \to B \to B/A$ is $\mathcal{E}_{\mathfrak{I}}$-exact. Let $t : A \to L$ be a morphism, and P be the pushout of t and $\iota_B^A : A \to B$, with canonical arrows $a : L \to P$ and $b : B \to P$. By the axioms of quasi-abelian categories, a is a kernel. Let $c = \mathrm{Coker}\,(a) : P \to Q$. By the universal property of the pushout, there exists a unique arrow $\sigma_0 : P \to B/A$ such that $\sigma_0 a = 0$ and $\sigma_0 b = \pi_{B/A}^B$. Since $\sigma_0 a = 0$, by the universal property of the cokernel there exists a unique arrow $\sigma : Q \to B/A$ such that $\sigma c = \sigma_0$. Suppose that X is an object of $\mathfrak{I}$ and $f : X \to Q$ is a morphism. We need to show that there exists a morphism $g : X \to P$ such that $cg = f$. Consider the morphism $\sigma f : X \to B/A$. Since $A \to B \to B/A$ is $\mathcal{E}_{\mathfrak{I}}$-exact, there exists a morphism $g : X \to B$ such that $\pi_{B/A}^B g = \sigma f$. Thus, we have that $bg : X \to P$ is a morphism such that $\sigma cbg = \sigma_0 bg = \pi_{B/A}^B g = \sigma f$. It remains to prove that $\sigma : Q \to B/A$ is an isomorphism. To this purpose, it suffices to prove that the arrow $cb : B \to Q$ is a cokernel of $\iota_B^A : A \to B$. Suppose that Z is an object and $\xi : Q \to Z$ is a morphism such that $\xi cb = 0$. Since c is the cokernel of a, we have $\xi ca = 0$. Hence, by the universal property of the pushout P, $\xi c = 0$. Hence we have that $\xi = 0$ since c is an epimorphism. Suppose now that $\eta : B \to Z$ is an arrow such that $\eta \iota_B^A = 0$. Then considering the arrow $0 : L \to Z$ and the universal property of the pushout P, there exists an arrow $\tau : P \to Z$ such that $\tau b = \eta$ and $\tau a = 0$. By the universal property of $c = \mathrm{Coker}\,(a)$ there exists an arrow $\xi : Q \to Z$ such that $\xi c = \tau$ and hence $\xi cb = \tau b = \eta$. This concludes the proof. □

5.2.7 Injective and projective objects

Let $\mathcal{A}$ be an exact category. An object I of $\mathcal{A}$ is *injective* if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact. The category $\mathcal{A}$ has *enough injectives* if for every object A of $\mathcal{A}$ there exists an admissible monic $A \to I$, where I is injective. Dually, one defines the notion of *projective* object, and

the notion of category with *enough projectives*. The following lemma is the content of [151, Proposition 1.3.24].

Lemma 5.7 (Schneiders)**.** *Suppose that $\mathcal{A}$ is a quasi-abelian category. Then $\mathcal{A}$ has enough projectives if and only if $\mathrm{LH}(\mathcal{A})$ has enough projectives. Furthermore, in this case every projective object of $\mathcal{A}$ is projective in $\mathrm{LH}(\mathcal{A})$, and every projective object of $\mathrm{LH}(\mathcal{A})$ is isomorphic to one of this form.*

Suppose that $\mathcal{A}$ is an exact category. Then $\mathrm{Hom}_{\mathcal{A}}$ is a functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$. The corresponding functor

$$\mathrm{Hom}^{\bullet} : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathbf{Ab})$$

is such that $(\mathrm{H}^0 \circ \mathrm{Hom}^{\bullet})(A, B)$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{K}^b(\mathcal{A})}(A, B)$; see [100, Proposition 11.7.3]. For objects X, Y of $\mathcal{A}$ and $k \in \mathbb{Z}$, define $\mathrm{Ext}^k(X, Y) := \mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}\left(X, T^k Y\right)$; see [100, Notation 13.1.9]. In particular, $\mathrm{Ext}^0(X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mathrm{Ext}^k(X, Y) = 0$ for $k < 0$ [100, Proposition 13.1.10]. One says that $\mathcal{A}$ is *hereditary* or has homological dimension at most 1 if $\mathrm{Ext}^n(X, Y) = 0$ for $n > 1$ [100, Definition 13.1.18]. In this case, we set $\mathrm{Ext}(X, Y) := \mathrm{Ext}^1(X, Y)$ for objects X, Y of $\mathcal{A}$. If $\mathcal{A}$ has enough injectives, this is equivalent to the assertion that for some (equivalently, every) short exact sequence $A \to I \to C$ with I injective, C is also injective [100, Exercise 13.8]. When $\mathcal{A}$ has enough projectives, this is equivalent to the assertion that for some (equivalently, every) short exact sequence $A \to P \to C$ with P projective, A is also projective. The group $\mathrm{Ext}(X, Y)$ parameterizes the *extensions* of X by Y [69, Section II.2].

Lemma 5.8. *Suppose that $\mathcal{A}$ is a quasi-abelian category with enough projectives. The left heart $\mathrm{LH}(\mathcal{A})$ is equivalent to its full subcategory consisting of objects of the form $P/S := \mathrm{Coker}_{\mathrm{LH}(\mathcal{A})}(S \to P)$ for some monic arrow $S \to P$ in $\mathcal{A}$ with P projective. Furthermore, the morphisms $P/S \to P'/S'$ are precisely those induced by pairs of morphisms $S \to S'$ and $P \to P'$ that make the diagram*

$$\begin{array}{ccc} P & \to & P' \\ \uparrow & & \uparrow \\ S & \to & S' \end{array}$$

commute.

Proof. The first assertion follows from the fact that $\mathcal{A}$ has enough projectives, and the pullback of a monic arrow is monic. Suppose now that $\varphi : P/S \to P'/S'$ is a morphism in $\mathrm{LH}(\mathcal{A})$. Then by [151, Corollary 1.2.20] there exist an object Q/R and morphisms $\sigma : Q/R \to P/S$ and $\psi : Q/R \to P'/S'$ that are induced by commuting diagrams

$$\begin{array}{ccc} Q & \overset{\psi_Q}{\to} & P' \\ \uparrow & & \uparrow \\ R & \overset{\psi_R}{\to} & S' \end{array}$$

and

$$\begin{array}{ccc} Q & \stackrel{\sigma_Q}{\to} & P \\ i_R \uparrow & & \uparrow i_S \\ R & \stackrel{\sigma_R}{\to} & S \end{array}$$

where the latter is both a pullback and a pushout, σ is invertible, and $\varphi = \psi \circ \sigma^{-1}$. After replacing Q with $Q \oplus P$ and R with $R \oplus S$ we can assume that the morphism $\sigma_Q : Q \to P$ is a strict epimorphism. Thus, since P is projective, σ_Q admits a right inverse $\eta_P : P \to Q$. We also have by the universal property of the pullback a morphism $\eta_S : S \to R$ such that $i_R \eta_S = \eta_P i_S$ and $\sigma_R \eta_S = 1_S$. This shows that the morphism $\sigma^{-1} : P/S \to Q/R$ is induced by the commuting diagram

$$\begin{array}{ccc} P & \stackrel{\eta_P}{\to} & Q \\ i_S \uparrow & & \uparrow i_R \\ S & \stackrel{\eta_S}{\to} & R \end{array}$$

and hence φ is induced by the commuting diagram

$$\begin{array}{ccc} P & \stackrel{\psi_Q \circ \eta_P}{\to} & P' \\ i_S \uparrow & & \uparrow i_{S'} \\ S & \stackrel{\psi_R \circ \eta_S}{\to} & S'. \end{array}$$

This concludes the proof. □

The same proof as [30, Theorem 10.22 and Remark 10.23] gives the following:

Lemma 5.9. *Suppose that $\mathcal{A}$ is an idempotent-complete exact category and $\mathcal{B}$ is a fully exact subcategory of $\mathcal{A}$ closed under direct summands. Suppose that:*

1. *for each object A of $\mathcal{A}$ there exists an admissible monic $A \to B$ with B in $\mathcal{B}$;*
2. *for all short exact sequences $B' \to B \to A''$ in $\mathcal{A}$ with B' and B in $\mathcal{B}$, one has that A'' is isomorphic to an object of $\mathcal{B}$.*

Then the inclusion $\mathcal{B} \to \mathcal{A}$ induces an equivalence of categories $\mathrm{D}^b(\mathcal{B}) \to \mathrm{D}^b(\mathcal{A})$.

Corollary 5.10. *Suppose that $\mathcal{A}$ is a hereditary idempotent-complete exact category with enough projectives, and $\mathcal{P}$ is the fully exact subcategory of $\mathcal{A}$ consisting of projective objects. The canonical quotient map $\mathrm{D}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{A})$ is an equivalence of categories, while the inclusion $\mathcal{P} \to \mathcal{A}$ induces an equivalence of categories $\mathrm{D}^b(\mathcal{P}) \to \mathrm{D}^b(\mathcal{A})$.*

The following more general version of Lemma 5.9 is (the dual of) [30, Theorem 10.22 and Remark 10.23]:

Lemma 5.11. *Suppose that $\mathcal{A}$ is an idempotent-complete exact category and $\mathcal{B}$ is a fully exact subcategory of $\mathcal{A}$ closed under direct summands. Suppose that:*

1. *for each object A of $\mathcal{A}$ there exists an admissible epic $B \to A$ with B in $\mathcal{B}$;*

2. *for all short exact sequences $A' \to A \to B''$ in $\mathcal{A}$ with B'' in $\mathcal{B}$, there exists a commutative diagram with exact rows*

$$\begin{array}{ccccc} B' & \to & B & \to & B'' \\ \downarrow & & \downarrow & & \downarrow \\ A' & \to & A & \to & B'' \end{array}$$

with B' and B in $\mathcal{B}$.

Then the inclusion $\mathcal{B} \to \mathcal{A}$ induces an equivalence $\mathrm{D}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{A})$.

Corollary 5.12. *Suppose that $\mathcal{A}$ is an idempotent-complete exact category with enough projectives, and $\mathcal{P}$ is the fully exact subcategory of $\mathcal{A}$ consisting of projective objects. The canonical quotient map $\mathrm{K}^-(\mathcal{P}) \to \mathrm{D}^-(\mathcal{P})$ is an equivalence of categories, while the inclusion $\mathcal{P} \to \mathcal{A}$ induces an equivalence of categories $\mathrm{D}^-(\mathcal{P}) \to \mathrm{D}^-(\mathcal{A})$.*

5.2.8 Derived functors

Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a functor between exact categories. Then F induces a triangulated functor $\mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathcal{B})$, which we still denote by F. If F is exact, then its extension $\mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathcal{B})$ maps acyclic complexes to acyclic complexes, and hence it induces a canonical triangulated functor $G : \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{B})$ such that $GQ_{\mathcal{A}}$ is isomorphic to $Q_{\mathcal{B}}F$.

In general, we say that F has a *total right derived functor* if there exists an (essentially unique) triangulated functor $\mathrm{R}F : \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{B})$ with an isomorphism $\mu : Q_{\mathcal{B}}F \Rightarrow (\mathrm{R}F)\, Q_{\mathcal{A}}$ of triangulated functors, and such that for any other triangulated functor $G : \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{B})$ and morphism $\nu : Q_{\mathcal{B}}F \Rightarrow GQ_{\mathcal{A}}$ of triangulated functors there exists a unique morphism $\sigma : \mathrm{R}F \Rightarrow G$ of triangulated functors such that $\mu \circ \sigma Q_{\mathcal{A}} = \nu$.

Suppose that $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is an additive functor, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are exact categories. Let $\mathrm{Ch}^b(\mathcal{A})$ be the category of bounded complexes over $\mathcal{A}$ and $\mathrm{Ch}^b(\mathcal{A})^2$ be the category of bounded *double* complexes over $\mathcal{A}$. Then we have that F induces a functor $F : \mathrm{Ch}^b(\mathcal{A}) \times \mathrm{Ch}^b(\mathcal{B}) \to \mathrm{Ch}^b(\mathcal{C})^2$ defined by

$$F(A,B)^{i,j} := F\left(A^i, B^j\right)$$

see [100, Section 11.6]. The vertical and horizontal maps in $F(A,B)$ are given by

$$\delta_{\mathrm{v}}^{ij} := F\left(\delta_A^i, \mathrm{id}_{B^j}\right) : F\left(A^i, B^j\right) \to F\left(A^{i+1}, B^j\right)$$

$$\delta_{\mathrm{h}}^{ij} := F(\mathrm{id}_{A_i}, \delta_B^j) : F\left(A^i, B^j\right) \to F\left(A^i, B^{j+1}\right).$$

The functor $F^\bullet : \mathrm{Ch}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{Ch}^b(\mathcal{B}) \to \mathrm{Ch}^b(\mathcal{C})$ is defined by

$$F^\bullet(A,B) = \mathrm{Tot}\left(F(A,B)\right)$$

for bounded complexes A and B, where $\mathrm{Tot}\left(F\left(A,B\right)\right)$ denotes the *total complex* of $F\left(A,B\right)$ [100, Section 11.5]. This induces a *triangulated* bifunctor $F^{\bullet}:\mathrm{K}^{b}\left(\mathcal{A}\right)\times\mathrm{K}^{b}\left(\mathcal{B}\right)\to\mathrm{K}^{b}\left(\mathcal{C}\right)$ in the sense of [100, Definition 10.3.6]; see [100, Proposition 11.6.4].

If $\Phi:\mathrm{K}^{b}\left(\mathcal{A}\right)\times\mathrm{K}^{b}\left(\mathcal{B}\right)\to\mathrm{K}^{b}\left(\mathcal{C}\right)$ is a triangulated bifunctor, then a *total right derived functor* $\mathrm{R}\Phi$ of Φ is a triangulated bifunctor $\mathrm{D}^{b}\left(\mathcal{A}\right)\times\mathrm{D}^{b}\left(\mathcal{B}\right)\to\mathrm{D}^{b}\left(\mathcal{C}\right)$ with a natural isomorphism $g:Q_{\mathcal{C}}\Phi\Rightarrow\mathrm{R}\Phi\left(Q_{\mathcal{A}}\times Q_{\mathcal{B}}\right)$ such that for any other triangulated bifunctor $\Psi:\mathrm{D}^{b}\left(\mathcal{A}\right)\times\mathrm{D}^{b}\left(\mathcal{B}\right)\to\mathrm{D}^{b}\left(\mathcal{C}\right)$ with natural transformation $h:Q_{\mathcal{C}}\Phi\Rightarrow\Psi\left(Q_{\mathcal{A}}\times Q_{\mathcal{B}}\right)$ there exists a unique morphism $\gamma:\mathrm{R}\Phi\Rightarrow\Psi$ of triangulated functors such that $\gamma\left(Q_{\mathcal{A}}\times Q_{\mathcal{B}}\right)\circ g=h$.

Similar consideration apply if one replaces the bounded categories of complexes with the left-bounded or right-bounded categories of complexes; see [100, Proposition 11.6.4]. The notion of *left derived functor* is obtained by duality.

5.2.9 Construction of derived functors

The most common way to produce derived functors is by using *projective resolutions*. The following can be sen as a particular instance of the version for bounded complexes of [100, Proposition 13.4.3].

Proposition 5.13. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be idempotent-complete exact categories. Suppose that $\mathcal{A}$ is hereditary with enough projectives. Let also $F:\mathcal{A}\times\mathcal{B}\to\mathcal{C}$ be an additive functor such that $F\left(P,-\right):\mathcal{B}\to\mathcal{C}$ is exact for every projective object P of $\mathcal{A}$. Then the triangulated bifunctor $F^{\bullet}:\mathrm{K}^{b}(\mathcal{A})\times\mathrm{K}^{b}(\mathcal{B})\to\mathrm{K}^{b}\left(\mathcal{C}\right)$ has a total left derived functor $\mathrm{L}F^{\bullet}:\mathrm{D}^{b}(\mathcal{A})\times\mathrm{D}^{b}(\mathcal{B})\to\mathrm{D}^{b}(\mathcal{C})$. Furthermore, this can be defined as follows: For a bounded complex A over $\mathcal{A}$, pick a bounded complex P_{A} over $\mathcal{P}$ together with quasi-isomorphisms $\eta_{A}:P_{A}\to A$. Then one can define*

$$\mathrm{L}F^{\bullet}\left(A,B\right):=F^{\bullet}\left(P_{A},B\right)$$

for $A\in\mathrm{D}^{b}\left(\mathcal{A}\right)$ and $B\in\mathrm{D}^{b}\left(\mathcal{A}\right)$.

When $\mathcal{C}$ is quasi-abelian, one sets

$$\mathrm{L}^{k}F:=\left(\mathrm{H}^{k}\circ\mathrm{L}F\right)|_{\mathcal{A}\times\mathcal{B}}:\mathcal{A}\times\mathcal{B}\to\mathrm{LH}\left(\mathcal{C}\right)$$

If F is right exact in each variable, then $\mathrm{L}^{0}F\cong F$ and $\mathrm{L}^{k}F=0$ for $k>0$.

Proof. Let $\mathcal{P}$ be the class of projective objects of $\mathcal{A}$. Consider the full subcategory $\mathcal{P}\times\mathcal{B}$ of $\mathcal{A}\times\mathcal{B}$. By Lemma 5.9, the inclusion $\mathcal{P}\times\mathcal{B}\to\mathcal{A}\times\mathcal{B}$ induces an equivalence between their bounded derived categories. Suppose that P is a bounded complex in $\mathcal{P}$ and A is a bounded complex in $\mathcal{A}$. Considering that $\mathcal{P}$ is the class of projective objects of $\mathcal{A}$, and that $F\left(P,-\right):\mathcal{B}\to\mathcal{C}$ is exact for every P in $\mathcal{P}$, one can see that $F^{\bullet}\left(P,B\right)$ is acyclic whenever P is an acyclic complex in $\mathcal{P}$ or B is an acyclic in $\mathcal{B}$.

The conclusion that $\mathrm{L}F^{\bullet}$ as defined in the statement is a total left derived functor for $F^{\bullet}$ is a consequence of the above remarks; see also [100, Section 13.4]. When F is right-exact in

each variable, it follows as in (the dual of) [151, Proposition 1.3.11] that $\mathrm{L}^0 F|_{\mathcal{A}\times\mathcal{B}} \cong F$ and $\mathrm{L}^k F = 0$ for $k > 0$. (Notice that in the terminology of [151], "RL left exact" is equivalent to left exact; see [151, Proposition 1.1.15].) □

A similar proof as Proposition 5.13 gives the following result (a particular instance of [100, Proposition 13.4.3]), which applies to exact categories that are not necessarily hereditary.

Proposition 5.14. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be idempotent-complete exact categories. Suppose that $\mathcal{A}$ has enough projectives. Let also $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be an additive functor such that $F(P, -) : \mathcal{B} \to \mathcal{C}$ is exact for every projective object P of $\mathcal{A}$. Then the triangulated bifunctor $F^\bullet : \mathrm{K}^+(\mathcal{A}) \times \mathrm{K}^+(\mathcal{B}) \to \mathrm{K}^+(\mathcal{C})$ has a total left derived functor $\mathrm{L}F^\bullet : \mathrm{D}^+(\mathcal{A}) \times \mathrm{D}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{C})$. Furthermore, this can be defined as follows: For a left-bounded complex A over $\mathcal{A}$, pick a left-bounded complex P_A over $\mathcal{P}$ together with a quasi-isomorphism $\eta_A : P_A \to A$. Then one can define*

$$\mathrm{L}F^\bullet(A, B) := F^\bullet(P_A, B)$$

for $A \in \mathrm{D}^+(\mathcal{A})$ and $B \in \mathrm{D}^+(\mathcal{A})$.

When $\mathcal{C}$ is quasi-abelian, one sets

$$\mathrm{L}^k F := \left(\mathrm{H}^k \circ \mathrm{L}F\right)|_{\mathcal{A}\times\mathcal{B}} : \mathcal{A} \times \mathcal{B} \to \mathrm{LH}(\mathcal{C})$$

If F is right exact in each variable, then $\mathrm{L}^0 F \cong F$.

5.2.10 Monoidal categories

Let $\mathcal{C}$ be a category. A *monoidal structure* on $\mathcal{C}$ consists of a quintuple $(\otimes, I, \alpha, \lambda, \rho)$ where $\otimes$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called *tensor product*, I is an object of $\mathcal{C}$ called *tensor identity*, α is a natural isomorphism

$$\otimes \circ (\otimes \times 1_{\mathcal{C}}) \cong \otimes \circ (1_{\mathcal{C}} \times \otimes)$$

called *associator*, ρ is a natural isomorphism

$$\otimes \circ (1_{\mathcal{C}} \times I) \cong 1_{\mathcal{C}}$$

called *right unitor*, and λ is a natural isomorphism

$$\otimes \circ (I \times 1_{\mathcal{C}}) \cong 1_{\mathcal{C}}$$

called *left unitor*, such that α, λ, ρ satisfy the associativity and unit coherence axioms; see [117, Section VII.1] or [28, Definition 6.1.1]. Thus, for objects x, y, z of $\mathcal{C}$, $\alpha_{x,y,z}$ is an isomorphism

$$(x \otimes y) \otimes z \cong x \otimes (y \otimes z),$$

ρ_x is an isomorphism

$$x \otimes I \cong x,$$

and λ_x is an isomorphism

$$I \otimes x \cong x.$$

A *monoidal category* is a category endowed with a monoidal structure.

A *tensor category* or *symmetric monoidal category* is a monoidal category together with a natural isomorphism β, called *braiding*, between $\otimes$ and $\otimes \circ \sigma$, where $\sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the flip, satisfying $\beta \circ (\beta\sigma) = 1_{\otimes}$ (*symmetry*), and the associativity and unit cohere axioms [28, Definition 6.1.2]. Thus, for objects x, y of $\mathcal{C}$, $\beta_{x,y}$ is an isomorphism

$$x \otimes y \cong y \otimes x.$$

The symmetry requirement asserts that $\beta_{y,x} \circ \beta_{x,y}$ is the identity of $x \otimes y$.

A functor $F : \mathcal{A} \to \mathcal{B}$ between monoidal categories is strict monoidal if $F(I) = I$ and $F(A \otimes B) = FA \otimes FB$ for objects A, B of $\mathcal{A}$. The notion of (lax) monoidal functor is obtained in a similar fashion by replacing equalities with natural transformations that satisfy coherence conditions that involve the associator and unitors of $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{V}$ be a monoidal category. A $\mathcal{V}$-category is given by a collection of objects and, for objects x, y, z, an object $\mathrm{Hom}(x, y)$ of $\mathcal{V}$, a morphism $1_x : I \to \mathrm{Hom}(x, x)$ (the *identity*), and a morphism $\mathrm{Hom}(y, z) \otimes \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z)$ (*composition*), that satisfy the natural version of the associativity and identity axioms of a category, phrased in terms of commuting diagrams involving the associator and the left and right unitor of $\mathcal{V}$. A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories is given by a function $x \mapsto F(x)$ mapping objects of $\mathcal{C}$ to objects of $\mathcal{D}$ together with morphisms $\mathrm{Hom}(x, y) \to \mathrm{Hom}(Fx, Fy)$ in $\mathcal{V}$ for objects x, y of $\mathcal{C}$ that preserve identities and composition in the obvious sense.

5.2.11 Tensor quasi-abelian categories

A tensor additive category is a tensor category that is also additive, and such that the tensor product functor is additive in each variable. The notion of tensor exact category is defined in the same fashion.

A tensor quasi-abelian category is a tensor additive category that is also quasi-abelian, where the quasi-abelian and monoidal structures are compatible in a suitable sense. For our purposes, it will be convenient to consider the following:

Definition 5.15. *A* tensor quasi-abelian category *is a tensor category which is also quasi-abelian, such that the tensor identity I is projective and the tensor product functor is right exact in each variable.*

An object X of a tensor quasi-abelian category is flat *if the functor $X \otimes -$ is exact.*

The reason for this definition is that it includes the categories of modules over a commutative ring, and other categories that we will consider. If $\mathcal{A}$ is a monoidal quasi-abelian category, then by the universal property of $\mathrm{LH}(\mathcal{A})$, the tensor product functor $\otimes_{\mathcal{A}}$ extend to a functor $\otimes_{\mathrm{LH}(\mathcal{A})} : \mathrm{LH}(\mathcal{A}) \times \mathrm{LH}(\mathcal{A}) \to \mathrm{LH}(\mathcal{A})$ which is still right exact in each variable.

The tensor identity for $\mathcal{A}$ is also a tensor identity for $\mathrm{LH}(\mathcal{A})$. Furthermore, the associator, unitors, and braiding for $\otimes_{\mathcal{A}}$ induce associator, unitors, and braiding for $\otimes_{\mathrm{LH}(\mathcal{A})}$. This endows $\mathrm{LH}(\mathcal{A})$ with a canonical structure of tensor abelian category.

5.2.12 Tensor triangulated categories

A variety of notions of tensor triangulated category have been considered in the literature. Informally, a tensor triangulated category is a triangulated category that is also a symmetric monoidal category, with compatibility requirements between the triangulated and monoidal structures. Depending on the source and the desired applications, the compatibility requirements between the triangulated and monoidal structures may vary. In this chapter, we consider the ones from [49]; see also [169], [53, Definition 3.1], and [83, Appendix A]. Unlike these sources, we consider monoidal categories that are not necessarily closed, i.e., we do not require the tensor product functor to have a right adjoint.

Let thus $\mathcal{T}$ be a triangulated category with translation functor T (called *suspension* in [83, Appendix A]) endowed with a tensor structure which we denote as above (the tensor functor is denoted by $\wedge$ and called smash product in [83, Appendix A]). We require that the tensor product functor be triangulated in the sense of [100, Definition 10.3.6], as witnessed by natural isomorphisms

$$(Tx) \otimes y \cong_{\ell_{x,y}} T(x \otimes y)$$

and

$$x \otimes (Ty) \cong_{r_{x,y}} T(x \otimes y)$$

for objects x, y of $\mathcal{T}$. The triangulated and monoidal structures together with the natural transformations ℓ and r, called *left and right translator* respectively, define a *tensor triangulated category* provided they satisfy the following compatibility requirements as in [49] for all objects x, y, z:

1. the right translator and the left unitor satisfy
$$T\lambda_x \circ r_x = \lambda_{Tx};$$
2. the left translator and the right unitor satisfy
$$T\rho_x \circ \ell_x = \rho_{Tx}$$
3. the left and right translator satisfy
$$T\ell_{x,y} \circ r_{Tx,y} = Tr_{x,y} \circ \ell_{x,Ty}$$
4. the associator and the left translator satisfy
$$\ell_{x,y\otimes z} \circ \alpha_{Tx,y,z} = T\alpha_{x,y,z} \circ \ell_{x\otimes y,z} \circ (\ell \otimes 1_z)$$
5. the left and right translator determine each other via the braiding by the identity
$$r_{x,y} = (T\beta_{y,x}) \circ \ell_{y,x} \circ \beta_{x,Ty}.$$

Suppose that $\mathcal{A}$ is a tensor additive category. Then the tensor product functor extends as in Section 5.2.8 to a triangulated bifunctor $\otimes^{\bullet}$ on $\mathrm{K}^{b}(\mathcal{A})$ which turns it into a tensor triangulated category with the same tensor identity as $\mathcal{A}$, and associators, unitors, and braiding induced by those of $\mathcal{A}$. The same applies to $\mathrm{K}^{+}(\mathcal{A})$.

Proposition 5.16. *Let $\mathcal{A}$ be a tensor exact category. Suppose that:*

- *the tensor product functor on $\mathcal{A}$ is right exact in each variable;*
- *$\mathcal{A}$ has enough* flat *projectives.*

Then the triangulated bifunctor $\otimes : \mathrm{K}^{+}(\mathcal{A}) \times \mathrm{K}^{+}(\mathcal{A}) \to \mathrm{K}^{+}(\mathcal{A})$ has a total left derived functor $\otimes : \mathrm{D}^{+}(\mathcal{A}) \times \mathrm{D}^{+}(\mathcal{A}) \to \mathrm{D}^{+}(\mathcal{M})$. This turns $\mathrm{D}^{+}(\mathcal{A})$ into a tensor triangulated category, such that the inclusion $\mathcal{A} \to \mathrm{D}^{+}(\mathcal{A})$ is monoidal and triangulated.

If $\mathcal{A}$ is hereditary, the same conclusions hold replacing the category of left-bounded complexes with the category of bounded complexes.

Proof. By Proposition 5.14, the tensor product functor $\otimes^{\bullet}$ on $\mathrm{K}^{+}(\mathcal{A})$ admits a total left derived functor $\otimes$ on $\mathrm{D}^{+}(\mathcal{A})$. By definition, for each complex A one picks a quasi-isomorphism $A \to P_A$, where P_A is a complex of *flat* projectives. On then has

$$A \otimes B := P_A \otimes^{\bullet} P_B.$$

The associators, unitors, and braiding for $\otimes^{\bullet}$ on $\mathrm{K}^{+}(\mathcal{A})$ induce those for $\otimes$ on $\mathrm{D}^{+}(\mathcal{A})$. □

5.2.13 Tilting

Suppose that $(\mathcal{E}, \mathcal{F})$ is a torsion pair in a quasi-abelian category $\mathcal{A}$. The pair is called *tilting*, and $\mathcal{E}$ a tilting torsion class, if $\mathcal{E}$ cogenerates $\mathcal{A}$, i.e., for every object A of $\mathcal{A}$ there exists a strict monic $A \to E$ with E in $\mathcal{A}$ [75]; see also [27, Appendix B]. Dually, the pair $(\mathcal{E}, \mathcal{F})$ is called cotilting, and $\mathcal{F}$ a cotilting torsion-free class, if $\mathcal{F}$ generates $\mathcal{A}$, i.e., for every object A of $\mathcal{A}$ there exists a strict epic $F \to \mathcal{A}$ with F in $\mathcal{F}$. In this case, $\mathcal{A}$ is equivalent to $\mathrm{LH}(\mathcal{E})$ via an equivalence that is the identity of $\mathcal{E}$.

Suppose that $(\mathcal{E}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$. Then the inclusion $\mathcal{E} \to \mathcal{A}$ is exact and cocontinuous. For an arrow f in $\mathcal{E}$ one has $\mathrm{Ker}_{\mathcal{E}}(f) = \pi_{\mathcal{E}}\mathrm{Ker}_{\mathcal{A}}(f)$. For an object A of $\mathcal{A}$, by definition we have a short exact sequence

$$0 \to E_A \to A \to F_A \to 0$$

with E_A in $\mathcal{E}$ and F_A in $\mathcal{F}$. The assignment $A \to E_A$ defines a functor $\pi_{\mathcal{E}} : \mathcal{A} \to \mathcal{E}$ that is right adjoint of the inclusion functor $\mathcal{E} \to \mathcal{A}$ and satisfies $\pi_{\mathcal{E}}|_{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$ and $\pi_{\mathcal{E}}|_{\mathcal{F}} = 0$. It is transparent that $\mathcal{E}$ is a $\pi_{\mathcal{E}}$-injective category in the sense of [151, Proposition 1.3.5]. Therefore by [151, Proposition 1.3.5], $\pi_{\mathcal{E}}$ is explicitly right derivable in the sense of [151,

Definition 1.3.6]. The derived functor $\mathrm{R}^1\pi_{\mathcal{E}} : \mathcal{A} \to \mathrm{LH}(\mathcal{E})$ is defined as follows. For an object A of $\mathcal{A}$, since $\mathcal{E}$ is a tilting torsion class, there exists a strict monic $A \to I$. Then the complex

$$0 \to I \to \mathrm{Coker}_{\mathcal{A}}(I \to A) \to 0$$

with I in degree 0, is quasi-isomorphic to A. Its cohomology in degree 1 is $\mathrm{R}^1\pi_{\mathcal{E}}(A)$. Thus,

$$\mathrm{R}^1\pi_{\mathcal{E}}(A) = \mathrm{Coker}_{\mathrm{LH}(\mathcal{E})}(I \to \mathrm{Coker}_{\mathcal{A}}(I \to A)).$$

Recall the definition of the phantom category of a quasi-abelian category from Definition 5.3.

Proposition 5.17. *Let $\mathcal{A}$ be a quasi-abelian category, and $(\mathcal{E}, \mathcal{F})$ be a tilting torsion pair in $\mathcal{A}$:*

1. *the restriction of $\mathrm{R}^1\pi_{\mathcal{E}}$ to $\mathcal{F}$ is faithful, and its essential image is the phantom category $\mathrm{Ph}(\mathcal{E})$ of $\mathcal{E}$;*
2. *if for every object F of $\mathcal{F}$ there exists a strict monic $F \to P$ where P is projective in $\mathcal{E}$, $\mathrm{R}^1\pi_{\mathcal{E}}$ establishes an equivalence of categories between $\mathcal{F}$ and $\mathrm{Ph}(\mathcal{E})$.*

Proof. (1) Suppose that $F \in \mathcal{F}$. We need to prove that $\mathrm{R}^1\pi_{\mathcal{E}}(F) \in \mathrm{Ph}(\mathcal{E})$. Let $\kappa : \mathrm{LH}(\mathcal{E}) \to \mathcal{E}$ be the left adjoint of the inclusion $\mathcal{E} \to \mathrm{LH}(\mathcal{E})$. Then we have that

$$\begin{aligned}
\kappa\left(\mathrm{R}^1\pi_{\mathcal{E}}(A)\right) &= \kappa\left(\mathrm{Coker}_{\mathrm{LH}(\mathcal{E})}(I \to \mathrm{Coker}_{\mathcal{A}}(I \to A))\right) \\
&= \mathrm{Coker}_{\mathcal{E}}(I \to \mathrm{Coker}_{\mathcal{A}}(I \to A)) \\
&= \mathrm{Coker}_{\mathcal{A}}(I \to \mathrm{Coker}_{\mathcal{A}}(I \to A)) \\
&= 0.
\end{aligned}$$

Conversely, suppose that X is a phantom object of $\mathcal{E}$, i.e., an object of $\mathrm{Ph}(\mathcal{E})$. Then

$$X \cong \mathrm{Coker}_{\mathrm{LH}(\mathcal{E})}\left(X^0 \to X^1\right)$$

for some monic arrow $X^0 \to X^1$ in $\mathcal{E}$ satisfying

$$\mathrm{Coker}_{\mathcal{E}}\left(X^0 \to X^1\right) = 0.$$

Define now

$$F := \mathrm{Ker}_{\mathcal{A}}\left(X^0 \to X^1\right).$$

By definition F is an object of $\mathcal{A}$, but since $X^0 \to X^1$ is a monic arrow of $\mathcal{E}$ and $(\mathcal{E}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$, we must have that F is an object of $\mathcal{A}$. Since

$$\mathrm{Coker}_{\mathcal{E}}\left(X^0 \to X^1\right) = \mathrm{Coker}_{\mathcal{A}}\left(X^0 \to X^1\right) = 0$$

we have that the arrow $X^0 \to X^1$ is the cokernel in $\mathcal{A}$ of the canonical arrow $F \to X^0$. Thus, we have that by definition

$$\mathrm{R}^1\pi_{\mathcal{E}}(F) = \mathrm{Coker}_{\mathcal{A}}\left(F \to X^0\right) = \mathrm{Coker}_{\mathrm{LH}(\mathcal{E})}\left(X^0 \to X^1\right) \cong X.$$

This shows that the essential image of $\mathcal{F}$ under $\mathrm{R}^1\pi_{\mathcal{E}}$ is $\mathrm{Ph}\,(\mathcal{E})$.

We now prove that $\mathrm{R}^1\pi_{\mathcal{E}}$ is faitfhul on $\mathcal{F}$. Suppose that $f : F \to F'$ is an arrow in $\mathcal{F}$ with $\mathrm{R}^1\pi_{\mathcal{E}}\,(f) = 0$. By definition, we have that $\mathrm{R}^1\pi_{\mathcal{E}}\,(f)$ is obtained by choosing strict monic arrows

$$i : F \to E$$

and

$$i' : F' \to E'$$

and an arrow

$$\varphi : E \to E'$$

such that

$$\varphi i = f i'.$$

Then the pair (f, φ) induces an arrow

$$\rho : \mathrm{Coker}_{\mathcal{A}}\,(i) \to \mathrm{Coker}\,(i')$$

which in turn induces together with f an arrow

$$\mathrm{R}^1\pi_{\mathcal{E}}\,(f) : \mathrm{Coker}_{\mathrm{LH}(\mathcal{E})}\,(E \to \mathrm{Coker}_{\mathcal{A}}\,(i)) \to \mathrm{Coker}_{\mathrm{LH}(\mathcal{E})}\,(E' \to \mathrm{Coker}_{\mathcal{A}}\,(i'))\,.$$

This implies that ρ factors through i'. In turn, this implies that the arrow φ factors through F'. Since $\mathrm{Hom}\,(\mathcal{E}, \mathcal{F}) = 0$ this implies $\varphi = 0$ and hence $f = 0$.

(2) We now assume that for every object F of $\mathcal{F}$ there exists a strict monic $i : F \to P$ where P is projective in $\mathcal{E}$. In this case, in the definition of $\mathrm{R}^1\pi_{\mathcal{E}}\,(F)$ we can use such a strict monic i. Suppose that F, F' are objects of $\mathcal{F}$ and $i : F \to P$ and $i' : F' \to P'$ are the corresponding strict monics in $\mathcal{A}$ with P and P' projective in $\mathcal{E}$. Consider a morphism $g : \mathrm{R}^1\pi_{\mathcal{E}}\,(F) \to \mathrm{R}^1\pi_{\mathcal{E}}\,(F')$. Since P is projective in $\mathcal{E}$, by Lemma 5.8 g is induced by a pair of morphisms $\gamma : P \to P'$, $\psi : \mathrm{Coker}_{\mathcal{A}}\,(i) \to \mathrm{Coker}_{\mathcal{A}}\,(i')$. By the universal property of $\mathrm{Coker}_{\mathcal{A}}\,(i')$, we must have that $\psi \circ i$ factors through i', whence there exists a morphism $f : F \to F'$ such that $i'f = \gamma i$. It follows from the construction that $\mathrm{R}^1\pi_{\mathcal{E}}\,(f) = \gamma$. □

5.2.14 Categories of towers

Let $\mathcal{C}$ be a category. We let $\mathrm{pro}\,(\mathcal{C})$ be the full subcategory of the pro-category $\mathrm{Pro}\,(\mathcal{C})$ spanned by the pro-objects with countable index set. Up to equivalence, one can identify $\mathrm{pro}\,(\mathcal{C})$ with the category whose objects are towers (i.e., ω^{op}-diagrams) $\left(A^{(n)}\right)$ of objects of $\mathcal{C}$, and the hom-set $\left(A^{(n)}\right) \to \left(B^{(n)}\right)$ is by definition

$$\mathrm{lim}_i \mathrm{colim}_j \mathrm{Hom}(A^{(j)}, B^{(i)});$$

see [142, Definition 6.1.1] and [100, Chapter 6]. When $\mathcal{A}$ is a quasi-abelian category, $\mathrm{pro}\,(\mathcal{A})$ is a countably complete quasi-abelian category, and the canonical inclusion $\mathcal{A} \to \mathrm{pro}\,(\mathcal{A})$

is exact and finitely continuous [142, Proposition 7.1.5, Proposition 7.1.7]; see also [100, Section 8.6] and [55, Section 2.1].

Let us say that a tower $\left(A^{(n)}\right)$ is *reduced* if $\lim_n A^{(n)} = 0$, the limit being computed in $\mathrm{pro}\left(\mathcal{A}\right)$. A tower $\left(A^{(n)}\right)$ is *epimorphic* if the bonding maps $A^{(n+1)} \to A^{(n)}$ are admissible epimorphisms, and *essentially epimorphic* if it is isomorphic to an epimorphic tower. We define $\mathbf{\Pi}\left(\mathcal{A}\right)$ to be the full subcategory of $\mathrm{pro}\left(\mathcal{A}\right)$ spanned by the essentially epimorphic towers, and $\mathrm{P}\left(\mathcal{A}\right)$ to be the full subcategory of $\mathrm{pro}\left(\mathcal{A}\right)$ spanned by the reduced towers. Then the pair $\left(\mathbf{\Pi}\left(\mathcal{A}\right), \mathrm{P}\left(\mathcal{A}\right)\right)$ defines a torsion pair on $\mathrm{pro}\left(\mathcal{A}\right)$, i.e., we have an extension

$$0 \to \mathbf{\Pi}\left(\mathcal{A}\right) \to \mathrm{pro}\left(\mathcal{A}\right) \to \mathrm{P}\left(\mathcal{A}\right) \to 0.$$

Furthermore, $\mathbf{\Pi}\left(\mathcal{A}\right)$ is a tilting torsion class, and the right adjoint of the inclusion $\mathbf{\Pi}\left(\mathcal{A}\right) \to \mathrm{pro}\left(\mathcal{A}\right)$ is the functor $\lim : \mathrm{pro}\left(\mathcal{A}\right) \to \mathbf{\Pi}\left(\mathcal{A}\right)$ that maps a tower to its limit. In this case, the derived functor $\mathrm{R}^1\lim$ is denoted by $\lim^1$. Explicitly, given a tower $\left(A^{(n)}\right)$ of objects of $\mathcal{A}$ one can realize $\lim_n^1 A^{(n)}$ as the cokernel of the morphism

$$\prod_n A^{(n)} \to \prod_n A^{(n)}$$

defined in terms of generalized elements by

$$\left(x_n\right) \mapsto \left(x_n - p^{(n,n+1)}\left(x_{n+1}\right)\right)$$

where $p^{(n,n+1)} : A^{(n+1)} \to A^{(n)}$ is the bonding map. (Here, the product is computed in $\mathbf{\Pi}\left(\mathcal{A}\right)$.) As a particular instance of Proposition 5.17 we have the following:

Proposition 5.18. *Let $\mathcal{A}$ be a quasi-abelian category:*

1. *the restriction of the functor $\lim^1 : \mathrm{pro}\left(\mathcal{A}\right) \to \mathbf{\Pi}\left(\mathcal{A}\right)$ to the full subcategory $\mathrm{P}\left(\mathcal{A}\right)$ comprising the reduced towers is faithful, and its essential image is the phantom category $\mathrm{Ph}\left(\mathbf{\Pi}\left(\mathcal{A}\right)\right)$ of $\mathbf{\Pi}\left(\mathcal{A}\right)$;*

2. *if every object of $\mathcal{A}$ is projective, then $\lim^1$ establishes an equivalence of categories between the category of reduced towers over $\mathcal{A}$ and the phantom category of $\mathbf{\Pi}\left(\mathcal{A}\right)$.*

It is easy to see that if $\mathcal{A}$ is hereditary with enough projectives, then $\mathbf{\Pi}(\mathcal{A})$ is also hereditary with enough projectives. Furthermore, products of projectives of $\mathcal{A}$ are projective in $\mathbf{\Pi}\left(\mathcal{A}\right)$.

Suppose that $\mathcal{A}$ is a tensor quasi-abelian category. As by assumption the tensor product functor is right exact in each variable, it preserves strict epimorphisms. Therefore, it induces a tensor product functor on $\mathbf{\Pi}\left(\mathcal{A}\right)$. The associator, unitors, and braiding for the tensor product of $\mathcal{A}$ induce associator, unitors, and braiding for the tensor product on $\mathbf{\Pi}\left(\mathcal{A}\right)$, which turn it into a tensor quasi-abelian category.

5.2.15 Categories of modules

We would like to consider generalizations of additive, quasi-abelian, and triangulated categories, where the hom-sets are not just abelian groups, but objects in a more general category, such as the category of pro-countable Polish modules, or its left heart. To this purpose, we consider the abstract notion of *category of modules*. We think of a category of modules as a category whose objects are modules "with additional structure" and the morphisms are the module homomorphisms that "preserve" this additional structure. For example, the additional structure can be a topology, in which case the module homomorphisms that preserve the additional structure are the continuous ones. Consistently with this interpretation, we will notice that a category of modules in the sense we are about to define is always a (not necessarily full) subcategory of the category of R-modules for some ring R.

Recall that one can define the notion of *(abelian) group object* in an abstract category with finite products [7, Definition 4.1] (here and in the following, we only consider group objects with respect to a cartesian monoidal structure). The group objects in the category of groups are precisely the abelian groups [7, Corollary 4.6]. Furthermore, such a group object structure is unique, and it is induced by the given operation on the abelian group. When R is a ring, R itself is not only a tensor identity, but also a projective generator for the category of R-modules. (This means that R is projective, and two parallel arrows f, g are equal if and only if $fx = gx$ for every arrow x with source R.)

Suppose now that $\mathcal{M}$ is a category with finite products such that every object has a unique group object structure. By the *Eckmann–Hilton argument* [7, Example 4.4], the unique group object structure on a given object is necessarily abelian. Furthermore, every morphism in $\mathcal{M}$ is necessarily a homomorphism [7, Definition 4.2] with respect to the unique group object structures on the source and target. The group structure on the objects induce a group operation on the hom-sets, that turns $\mathcal{M}$ into an $\mathbf{Ab}$-category. We say that $\mathcal{M}$ is (quasi-)abelian if it is a (quasi-)abelian category with respect to this canonical $\mathbf{Ab}$-category structure.

Definition 5.19. *A* category of modules *is a* locally small tensor category $\mathcal{M}$ *such that:*

1. *$\mathcal{M}$ has finite products, and every object has a unique group object structure;*
2. *with respect to the induced $\mathbf{Ab}$-category structure and the given tensor structure, $\mathcal{M}$ is a quasi-abelian tensor category with enough projectives;*
3. *the tensor identity R is projective and a generator;*
4. *for projective objects P and Q, $P \otimes Q$ is projective;*
5. *every projective object P is flat, i.e., the functor $P \otimes -$ is exact.*

If $\mathcal{M}$ is a category of modules, then it is easily seen considering the remarks above that $\mathbf{\Pi}(\mathcal{M})$ is also a category of modules. The fact that each object of $\mathbf{\Pi}(\mathcal{M})$ admits a

unique group object structure follows from instance from [55, Proposition 2.1.5]. Likewise, the left heart $\mathrm{LH}(\mathcal{M})$ is also a category of modules, where the unique group structure on an object P/R is the one induced by P. The uniqueness follows for example from the Eckmann–Hilton argument [54] together with Lemma 5.8.

If R is the tensor identity of $\mathcal{M}$, one can identify $\mathcal{M}$ with the concrete category whose objects are the abelian groups $\mathrm{Hom}(R, A)$ for $A \in \mathcal{M}$, and whose morphisms $\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, B)$ are the group homomorphisms induced by arrows $A \to B$ in $\mathcal{M}$. Notice that the group $\mathrm{Hom}(R, R)$ is in fact a ring, and $\mathrm{Hom}(R, A)$ a right $\mathrm{Hom}(R, R)$-module for every $A \in \mathcal{M}$.

As $\mathcal{M}$ is a monoidal category, one can consider the notion of $\mathcal{M}$-category. For example, if R is a commutative ring, then the category $\mathrm{Mod}(R)$ of R-modules is an abelian category of modules in the sense above. A $\mathrm{Mod}(R)$-category is then precisely an R-linear category in the usual sense.

Given an $\mathcal{M}$-category $\mathcal{A}$, for every finite category J, one can endow the category of diagrams $\mathcal{A}^J$ of shape J with a natural structure of $\mathcal{M}$-category. This allows one to define a *(quasi-)abelian $\mathcal{M}$-category* to be an $\mathcal{M}$-category that is also (quasi-)abelian, and such that for every finite category J, finite limits and finite colimits of shape J are given by $\mathcal{M}$-functors $\mathcal{A}^J \to \mathcal{A}$. Likewise, one defines the notion of *exact $\mathcal{M}$-category*.

One can similarly generalize the notion of triangulated category to the context of $\mathcal{M}$-categories. A *triangulated $\mathcal{M}$-category* will be an $\mathcal{M}$-category $\mathcal{T}$ that is also triangulated, such that the shift functor is an $\mathcal{M}$-functor, and the Hom-functor is a cohomological functor in the first variable and a homological functor in the second variable [137, Definition 1.1.7], when regarded as a functor with values in the left heart of $\mathcal{M}$.

As a particular instances of Corollary 5.10 and Proposition 5.13 we obtain the following; see also [100, Section 13.4]. Notice that we can identify $\mathrm{Ch}^b(\mathcal{A})^{\mathrm{op}}$ with $\mathrm{Ch}^b(\mathcal{A}^{\mathrm{op}})$ using the convention on signs from [100, Proposition 11.3.12].

Proposition 5.20. *Suppose that $\mathcal{M}$ is a category of modules. Let $\mathcal{P}$ be its full subcategory spanned by its projective objects. Then the inclusion $\mathcal{P} \to \mathcal{M}$ induces an equivalence $\mathrm{K}^b(\mathcal{P}) \to \mathrm{D}^b(\mathcal{M})$. In particular, $\mathrm{D}^b(\mathcal{M})$ is a triangulated $\mathrm{LH}(\mathcal{P})$-category.*

Proposition 5.21. *Suppose that $\mathcal{M}$ is a category of modules. Let $\mathcal{A}$ be a hereditary idempotent-complete exact $\mathcal{M}$-category with enough projectives. Then the triangulated bifunctor $\mathrm{Hom}^\bullet_{\mathcal{A}} : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathcal{M})$ has a total right derived functor $\mathrm{RHom}^\bullet_{\mathcal{A}} : \mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{M})$. Furthermore, this can be defined as follows: For a bounded complex A over $\mathcal{A}$, pick a bounded complex P_A over $\mathcal{P}$ together with quasi-isomorphisms $\eta_A : P_A \to A$. Then one can define $(\mathrm{RHom}^\bullet_{\mathcal{A}})(A, B) := \mathrm{Hom}^\bullet_{\mathcal{A}}(P_A, B)$ for $A \in \mathrm{D}^b(\mathcal{A})^{\mathrm{op}}$ and $B \in \mathrm{D}^b(\mathcal{A})$. We have that*

$$\mathrm{H}^0 \circ \mathrm{RHom}^\bullet_{\mathcal{A}} : \mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{M})$$

is a cohomological right derived functor of $\mathrm{Hom}^\bullet_{\mathcal{A}}$ such that

$$\mathrm{H}^k \circ \mathrm{RHom}^\bullet(A, B) \cong \mathrm{Ext}^k(A, B) \in \mathrm{LH}(\mathcal{M})$$

for objects A, B of $\mathcal{A}$.

5.3 Countable modules

By a *domain* we will mean a *unital commutative* ring R with no zero divisors. We will denote by K its field of fractions. A module will always mean an R-module. We say that a module M is *finite* or finitely-generated if there exist $n \in \omega$ and $a_0, \ldots, a_n \in M$ such that M is equal to the submodule of M generated by $a_0, \ldots, a_n$. An ideal is finite or finitely-generated if it is so as a module.

5.3.1 Some ring terminology

A ring is called:

- *semihereditary* if every finite ideal is projective [145, Section 4.4];
- *hereditary* if every ideal is projective [145, Section 4.3].

A *Prüfer domain* is a semihereditary domain, while a *Dedekind domain* is a hereditary domain. Prüfer and Dedekind domains are well-studied classes of rings, which admit several equivalent characterizations and extend many other important classes of rings.

A *Bezout domain* is a domain in which every finite ideal is principal, while a *principal ideal domain* (PID) is a domain where every ideal is principal. A *valuation ring* is a domain R that satisfies $a|b$ or $b|a$ for all $a, b \in R$. A *local ring* is a ring with a unique maximal ideal. A domain is *Noetherian* if every ideal is finitely generated. Every PID is a Dedekind domain, every valuation ring is a Bezout domain, and every Bezout domain is a Prüfer domain. Conversely, a domain R is Prüfer if and only if its localization R_P with respect to any prime (or, equivalently, to any maximal) ideal P is a valuation ring [145, Example 4.28]. A ring is a Dedekind domain if and only if it is a Noetherian Prüfer domain, and a valuation domain if and only if it is a local Prüfer domain.

A module M is *finitely-presented* if there exists an exact sequence $R^d \to R^n \to M \to 0$. When R is a Prüfer domain, this is equivalent to the assertion that there exists a short exact sequence $0 \to Q \to P \to M \to 0$ where P and Q are finite projective modules. When R is a Dedekind domain, every finite module is finitely-presented. A module M is *flat* if the functor $M \otimes -$ is exact [145, Section 3.3]. We say that M is *finiteflat* if it is both finite and flat, and *countableflat* if it is both countable and flat. A short exact sequence

$$0 \to A \to B \to C \to 0$$

of modules is *pure-exact* if, for every module N,

$$0 \to N \otimes A \to N \otimes B \to N \otimes C \to 0$$

is exact [145, Section 3.3.1]. In this case, one says that A is a pure submodule of B. The pure-exact sequences are precisely those that belong to the exact structure projectively generated by the finitely-presented modules [14, Theorem 2.1.2].

Lemma 5.22. *Suppose that $M' \to M \to M''$ is a short exact sequence of modules.*

- *If M'' is flat, then M is flat if and only if M' is flat;*
- *If M is flat, then $M' \to M \to M''$ is pure if and only if M'' is flat;*
- *If M'' is flat, then $M' \to M \to M''$ is pure.*

Proof. This combines [145, Corollary 7.4], [110, Corollary 4.86], and [14, Section 2.7]. □

A nonzero element a of a module M is *torsion* if $ra = 0$ for some $r \in R$. A module is called *torsion* if every nonzero element is torsion, and *torsion-free* if no nonzero element is torsion. If M is a module, then the set M_{t} of torsion elements is a submodule such that M/M_{t} is torsion-free. Every flat module is torsion-free, while the converse does not holds in general.

A submodule A of a torsion-free module B is *relatively divisible* if $rA = rB \cap A$ for every $r \in R$. Every pure submodule is relatively divisible, but the converse does not necessarily hold.

5.3.2 Characterization of Prüfer and Dedekind domains

One can characterize the classes of Prüfer and Dedekind domains in terms of properties of modules.

Proposition 5.23. *Let R be a domain. The following assertions are equivalent:*

1. *R is a Prüfer domain;*
2. *every torsion-free R-module is flat;*
3. *every finite torsion-free R-module is projective;*
4. *Every finite R-submodule of a projective R-module is projective;*
5. *If M is a flat R-module and N is a relatively divisible R-submodule of M, then N is pure in M;*
6. *an R-module is projective if and only if it is a sum of finite ideals of R;*
7. *if M is a finite R-module, then M_{t} is a direct summand of M.*

Proof. See [145, Section 4.4] for the equivalence of (1)–(4).

(2)⇒(5) Suppose that N is relatively divisible in M. Then M/N is torsion-free, whence flat. Thus, N is pure in M by Lemma 5.22.

(5)⇒(2) Suppose that M is a torsion-free module. Then there exists a short-exact sequence $A \to P \to M$ where P is projective. Since M is torsion-free, A is relatively divisible in P, whence pure in P. Thus, by Lemma 5.22, M is flat.

(1)⇔(6) This is [97, Theorem 3].

(1)⇔(7) This is proved in [96, Theorem 1] and [98]. □

Proposition 5.24. *Let R be a domain. Then the following assertions are equivalent:*

1. *R is a Dedekind domain;*
2. *every R-submodule of a projective R-module is projective;*
3. *an R-module is projective if and only if it is a sum of ideals of R.*

Likewise, we have that R is a PID if and only if every R-submodule of a free R-module is free.

Proof. See [145, Section 4.3]. □

Let M be a flat module over a Prüfer domain. We define its *rank* to be the dimension of $K \otimes M$ as a K-vector space. Clearly, if M is finite, then it has finite rank. The converse holds if M is projective, as in this case it is a sum of finite ideals of R.

By Proposition 5.23, if X is a subset of M, then there exists a smallest pure submodule $\langle X \rangle_*$ of M containing X, called the *pure hull* of X in M. If X has n elements, then $\langle X \rangle_*$ has rank at most n.

5.3.3 Finite flat modules

In this section, we assume that R is a Prüfer domain. Let M be a finite flat (whence, projective) module. We set $M^* := \mathrm{Hom}(M, R)$, which is also a finite flat module. We have a canonical homomorphism $M \to M^{**}$.

Lemma 5.25. *Suppose that R is a Prüfer domain. Let M be a finite flat module. Then the canonical homomorphism $\eta_M : M \to M^{**}$ is an isomorphism.*

Proof. As M is a finite direct sum of finite ideals of R, it suffices to consider the case when M is a finite ideal I of R. We claim that the image of the canonical homomorphism $\eta_I : I \to I^{**}$ is a pure submodule of I^{**}. Suppose that $\Phi \in I^{**}$ and $r \in R$ are such that $r\Phi \in \mathrm{Ran}(\eta_I)$. Thus, there exists a such that $r\Phi = \eta_I(a)$. For every $\psi \in I^*$ we have that

$$r\Phi(\psi) = (r\Phi)(\psi) = (\eta_I(a))(\psi) = \psi(a)$$

In particular when ψ is the inclusion $\iota : I \to R$,

$$r\Phi(\iota) = \iota(a) = a.$$

Thus, setting

$$b := \Phi(\iota) = \frac{1}{r}a$$

we have that, for every $\psi \in I^*$,

$$\Phi(\psi) = \frac{1}{r}\psi(a) = \psi(\frac{1}{r}a) = \psi(b).$$

This shows that $\mathrm{Ran}(\iota)$ is pure in I^{**}. Thus, we can write $I^{**} = I \oplus I^{\perp}$ for some pure submodule $I^{\perp}$ of I^{**}.

Since I is projective and finite, it is a direct summand of a free finite module. Thus, there exist $n \in \omega$ such that $R^n = I \oplus M$ for some pure submodule M of R^n. Thus,

$$R^n \cong (R^n)^{**} \cong I^{**} \oplus M^{**} \cong I \oplus I^{\perp} \oplus M^{**}.$$

This implies

$$\begin{aligned} n &= \mathrm{rk}(R^n) = \mathrm{rk}(I) + \mathrm{rk}\left(I^{\perp}\right) + \mathrm{rk}(M^{**}) \\ &\geq \mathrm{rk}\left(I^{\perp}\right) + \mathrm{rk}(I) + \mathrm{rk}(M) \\ &= n + \mathrm{rk}\left(I^{\perp}\right). \end{aligned}$$

This forces $I^{\perp} = 0$, concluding the proof. □

5.3.4 Finitely-presented modules

In this section, we continue to assume that R is a Prüfer domain with field of fractions K. A *fractional ideal* of R is a submodule $I \subseteq K$ such that there exists $a \in R$ with $aI \subseteq R$. Given two fractional ideals I and J, one defines IJ to be the submodule of K generated by xy for $x \in I$ and $y \in J$. A fractional ideal I is *invertible* if there exists a fractional ideal J such that $IJ = R$, in which case J is the ideal $I^{-1} = \{a \in R : aI \subseteq R\}$.

Lemma 5.26. *Suppose that I is a fractional ideal of R. Then I is invertible if and only if I is projective, and in this case I is finite.*

Proof. Notice that a projective ideal is necessarily finite. Indeed, if F is a free module, $F \to I$ is a surjective homomorphism, and $I \to F$ is a right inverse, then $I \to F$ factors through a finite direct summand of F, since I is an ideal of R and R is a domain. The conclusion thus follows from [11, Theorem 2.5]. □

Notice that a module is finite flat of rank 1 if and only if it is isomorphic to a finite (fractional) ideal of R. Let us say that a finitely-presented torsion module is *cyclic* if it is of the form L/I for some rank 1 finite flat module L and finite submodule $I \subseteq L$.

Lemma 5.27. *Every finitely-presented torsion module is a finite direct sum of finitely-presented cyclic torsion modules.*

Proof. Let M be a finitely-presented torsion module. Since M is finitely-presented, $M \cong F/S$ for some finite free module F and finite submodule $S \subseteq F$. Since M is torsion, $\langle S \rangle_* = F$. As S is finite and flat, there exist $n \in \omega$ and finite ideals $I_0, \dots, I_n$ in R such that $S = I_0 \oplus \cdots \oplus I_n$. If $F_k := \langle I_k \rangle_* \subseteq F$ for $0 \leq k \leq n$, then we have that $F = F_0 \oplus \cdots \oplus F_n$, and

$$F/S \cong F_0/I_0 \oplus \cdots \oplus F_n/I_n.$$

As F_k is a pure submodule of F, it is a finite projective module, which has rank 1 since so does I_k. □

Lemma 5.28. *Every finitely-presented module is a finite direct sum of finitely-presented cyclic torsion modules and finite ideals.*

Proof. Let M be a finitely-presented module. We have a short exact sequence

$$0 \to M_{\mathrm{t}} \to M \to A \to 0$$

where A is finite and torsion-free, whence projective. Therefore $M \cong M_{\mathrm{t}} \oplus A$. Being direct summands of M, M_{t} and A are finitely-presented. The conclusion thus follows from Lemma 5.27 and Proposition 5.23. □

Every Prüfer domain is, in particular, a coherent ring [112, Section II.3]. It follows that finitely-presented modules form a thick subcategory of the category of modules [112, Section IV.3].

5.3.5 Countably presented modules

A module M is *countably presented* if there exists an exact sequence

$$R^{(\omega)} \to R^{(\omega)} \to M \to 0.$$

This is equivalent to the assertion that $M \cong \operatorname{colim}_n M_n$ for some inductive sequence (M_n) of *finitely presented* modules, as well as to the assertion that $M \cong P/Q$ where $Q \subseteq P$ are countable projective modules. Similar arguments as for finitely-presented modules show that countably presented modules form an abelian category $\mathbf{Mod}_{\aleph_0}(R)$. When R is countable, every countably-presented module is countable, and $\mathbf{Mod}_{\aleph_0}(R)$ is a thick abelian subcategory of the abelian category of countable modules.

5.4 Pro-countable Polish modules

Throughout this section, we assume that R is a *countable* domain, and assume that all the modules are R-modules.

5.4.1 Polish modules

A *Polish* module is a module that is also a Polish abelian group, such that for every $r \in R$ the function $M \to M$, $x \mapsto rx$ is continuous. We have that the category $\mathbf{PolMod}(R)$ of *Polish* R-modules and *continuous* homomorphisms is a quasi-abelian category [114, Section 6]. The admissible arrows in this category are the continuous homomorphisms with closed image. We let $\mathbf{Mod}(R)$ be the category of countable modules, $\mathbf{Flat}(R)$ be its full subcategory of countable flat modules, and $\mathbf{FinFlat}(R)$ be its full subcategory of finite flat modules.

We say that a Polish module M is *pro-countable* if it is isomorphic to the inverse limit of a tower of countably presented modules, and *pro-finite* if it is isomorphic to the inverse limit of a tower of finite modules. More generally, if $\mathcal{A}$ is a fully exact subcategory of the category $\mathbf{Mod}_{\aleph_0}(R)$ of countably presented modules, we say that a Polish module is pro-$\mathcal{A}$ if it is isomorphic to the inverse limit of a tower of modules in $\mathcal{A}$. We let $\mathbf{\Pi}(\mathcal{A})$ be the full subcategory of $\mathbf{PolMod}(R)$ spanned by the pro-$\mathcal{A}$ modules.

The same proof as [51, Proposition 2.1] shows that, when $\mathcal{A}$ is closed under submodules, M is pro-$\mathcal{A}$ if and only if M is isomorphic to a closed submodule of a countable product of modules in $\mathcal{A}$. In particular, in this case $\mathbf{\Pi}(\mathcal{A})$ is closed under taking closed submodules. Note that, for pro-countable Polish modules $N \cong \lim_k N_k$ and $M \cong \lim_k M_k$ with N_k, M_k in $\mathcal{A}$, we have that the group of continuous homomorphisms $\mathrm{Hom}(N, M)$ is isomorphic to

$$\lim_k \mathrm{colim}_j \mathrm{Hom}(N_j, M_k).$$

This establishes an equivalence between $\mathbf{\Pi}(\mathcal{A})$ and the full subcategory of the category $\mathrm{pro}(\mathcal{A})$ of towers of modules in $\mathcal{A}$ spanned by essentially epimorphic towers. Thus, the notation adopted here is consistent with the notation from Section 5.2.14.

Lemma 5.29. *Let $\mathcal{A}$ be a thick abelian subcategory of $\mathbf{Mod}(R)$. Then $\mathbf{\Pi}(\mathcal{A})$ is closed under quotients by closed submodules.*

Proof. Suppose that $(A_i)_{i\in\omega}$ is a sequence of modules in $\mathcal{A}$ and N is a closed submodule of $A := \prod_{i\in\omega} A_i$. Define $A_{\leq k} := \prod_{i\leq k} A_i$, which we can identify with a closed submodule of A, and $N_{\leq k} := A_{\leq k} \cap N$. Then we have that A/N is isomorphic to the inverse limit

$$\lim_k (A_{\leq k}/N_{\leq k}).$$

Since $A_{\leq k}/N_{\leq k}$ is isomorphic to an object of $\mathcal{A}$, this shows that A/N is pro-countable. □

Lemma 5.30. *Suppose $\mathcal{A}$ is a fully exact subcategory of $\mathbf{Mod}(R)$ closed under submodules. Suppose that $A \to B \to C$ is a short exact sequence in $\mathbf{PolMod}(R)$. If A is pro-$\mathcal{A}$ and C is in $\mathcal{A}$, then B is pro-$\mathcal{A}$.*

Proof. Suppose initially that $A = \prod_{i\in\omega} A_i$ for a sequence (A_i) of modules in $\mathcal{A}$. For $i \in \omega$, consider the commutative diagram

$$\begin{array}{ccccc} A & \to & B & \to & C \\ \pi_i \downarrow & & \downarrow \xi_i & & \downarrow \\ A_i & \xrightarrow{\tau_i} & B_i & \xrightarrow{p_i} & C \end{array}$$

obtained via pushout, where $\pi_i : A \to A_i$ is the canonical projection. Thus, we have that B_i is the quotient of $A_i \oplus B$ by the submodule

$$\Xi_i := \{(-\pi_i(x), x) : x \in A\}.$$

Since B_i is an extension of C by A_i, it is isomorphic to an object of $\mathcal{A}$. Define

$$B_\omega := \prod_{i\in\omega} B_i$$

and the continuous homomorphism

$$\xi : B \to B_\omega, b \mapsto (\xi_i(b))_{i\in\omega}.$$

Then ξ is injective, and its image is equal to the preimage of the diagonal $\Delta_C \subseteq C^\omega$ under the continuous homomorphism $B_\omega \to C^\omega$, $(b_i) \mapsto (p_i(b_i))$. Thus, B is isomorphic to a closed submodule of B_ω, and therefore pro-$\mathcal{A}$.

If A is an arbitrary pro-$\mathcal{A}$ module, then we have an admissible monic $A \to \prod_{k\in\omega} A_k$ for some sequence (A_k) of modules in $\mathcal{A}$. By taking pushouts, we obtain a commutative diagram

$$\begin{array}{ccccc} A & \to & X & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{k\in\omega} A_k & \to & Y & \to & C \end{array}$$

where the arrow $X \to Y$ is an admissible monic. By the above discussion, Y is pro-$\mathcal{A}$, and hence also X is pro-$\mathcal{A}$. □

Lemma 5.31. *Suppose that $\mathcal{A}$ is a fully exact subcategory of $\mathbf{Mod}(R)$. Suppose that $\mathcal{P} \subseteq \mathcal{A}$ is a class of projective objects of $\mathcal{A}$ closed under direct summands such that for every $A \in \mathcal{A}$ there exists $Q \in \mathcal{P}$ and a strict epimorphism $Q \to A$. For every pro-$\mathcal{A}$ module C there exists a surjective continuous module homomorphism $P \to C$ where P is a countable product of objects of $\mathcal{P}$.*

Proof. Let $\left(A^{(n)}\right)$ be a tower of modules in $\mathcal{A}$ with epimorphic bonding maps $A^{(n+1)} \to A^{(n)}$ such that $A = \lim_n A^{(n)}$. We define by recursion on $k \in \omega$ modules $P^{(k)}$ in $\mathcal{P}$, surjective homomorphisms $P^{(k)} \to A^{(k)}$ and $P^{(k+1)} \to P^{(k)}$ for $k \geq 1$ such that the diagram

$$\begin{array}{ccc} P^{(k+1)} & \to & A^{(k+1)} \\ \downarrow & & \downarrow \\ P^{(k)} & \to & A^{(k)} \end{array}$$

commutes. Define $P^{(0)}$ to be a module in $\mathcal{P}$ together with a surjective homomorphism $P^{(0)} \to A^{(0)}$. Supposing that $P^{(i)}$ has been defined for $i < k$, let $B^{(k)}$ be the pullback

$$\begin{array}{ccc} B^{(k)} & \to & A^{(k)} \\ \downarrow & & \downarrow \\ P^{(k-1)} & \to & A^{(k-1)} \end{array}$$

Define $P^{(k)}$ to be a module in $\mathcal{P}$ together with a surjective homomorphism $P^{(k)} \to B^{(k)}$. Consider then the homomorphisms $P^{(k)} \to B^{(k)} \to P^{(k-1)}$ and $P^{(k)} \to B^{(k)} \to A^{(k)}$. This concludes the recursive definition. One then sets $P := \lim_n P^{(k)}$, and $P \to A$ be the surjective continuous homomorphism induced by the morphisms $P^{(k)} \to A^{(k)}$ for $k \in \omega$. Since $\left(P^{(k)}\right)$ is an epimorphic tower of projective modules of $\mathcal{A}$ and $\mathcal{P}$ is closed under direct summands, P is isomorphic to a product of objects of $\mathcal{P}$. □

5.4.2 Extensions of modules and cocycles

Suppose that C, A are Polish modules. We can define the group $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ of isomorphism classes of extensions $A \to X \to C$, where the group operation is induced by Baer sum of extensions [145, Section 7.2]. The trivial element of $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ corresponds to *split* extensions of C by A. One can obtain an equivalent description of such a group in terms of cocycles.

A (2-)cocycle on C with values in A is a pair (c, ρ) where $c : C \times C \to A$ and $\rho : R \times C \to A$ satisfy the following identities, for all $r, s \in R$ and $x, y, z \in C$:

1. $c(x, 0) = 0$;
2. $c(x, y) = c(y, x)$;
3. $c(x + y, z) + c(x, y) = c(x, y + z) + c(y, z)$;
4. $\rho(r + s, x) = c(rx, ry) + \rho(r, x) + \rho(s, x)$;
5. $c(rx, ry) + \rho(r, x) + \rho(r, y) = \rho(r, x + y) + rc(x, y)$.

We say that a cocycle is Borel (respectively, continuous) if each of the functions c and ρ is Borel (respectively, continuous). A Borel cocycle is a coboundary if there exists a Borel function $\xi : C \to A$ such that

$$(c, \rho) := \delta\xi$$

where $\delta\xi$ is the pair of functions (c_ξ, ρ_ξ) defined by

$$c_\xi(x, y) := \xi(x + y) - \xi(x) - \xi(y)$$

and

$$\rho_\xi(r, x) := \xi(rx) - r\xi(x).$$

Given a continuous cocycle (c, ρ) on C with values in A, one can define an extension

$$A \to X_{(c,\rho)} \to C$$

where the Polish module $X_{(c,\rho)}$ is defined as follows. As a topological space, $X_{(c,\rho)}$ is equal to $C \times A$, with module operation defined by setting

$$(x, a) + (y, b) := (x + y, c(x, y) + a + b)$$

$$r(x, a) := (rx, \rho(r, x) + a)$$

for $r \in R$, $x, y \in C$, and $a, b \in A$.

Lemma 5.32. *Suppose that C, A are Polish modules, and (c, ρ) is a cocycle on C with values in A. Let ξ be a Borel function such that $c(x, y) = \xi(x + y) - \xi(x) - \xi(y)$ for all $x, y \in C$. Then ξ is continuous.*

Proof. Define the Polish module $X_{(c,\rho)}$ as above. Consider also the direct sum $C \oplus A$. We can define a Borel group isomorphism $\varphi : X_{(c,\rho)} \to C \oplus A$, $(x, a) \mapsto (x, \xi(x) + a)$. Since $X_{(c,\rho)}$ and $C \oplus A$ are Polish groups, we must have that φ is continuous [103, Theorem 9.9], and ξ is continuous. □

Borel cocycles on C with values in A form a module $\mathrm{Z}(C, A)$ that has coboundaries as a submodule $\mathrm{B}(C, A)$. We let $\mathrm{Ext_c}(C, A)$ be the module obtained as the quotient $\mathrm{Z}(C, A)/\mathrm{B}(C, A)$.

When C, A are pro-countable, we define a homomorphism $\mathrm{Ext_{Yon}}(C, A) \to \mathrm{Ext_c}(C, A)$ as follows. Given an extension $A \to X \to C$, then by [22, Theorem 4.5], the map $X \to C$ has a continuous right inverse $f : C \to X$. Identifying A with a closed submodule of X, we can define $c(x, y) = f(x + y) - f(x) - f(y)$ and $\rho(r, x) = \rho(rx) - r\rho(x)$. This defines a continuous cocycle (c, ξ) on C with values in A, whose corresponding element of $\mathrm{Ext_c}(C, A)$ only depends on the isomorphism class of the given extension.

Lemma 5.33. *Suppose that C and A are pro-countable Polish modules. The assignment described above defines an injective homomorphism $\mathrm{Ext_{Yon}}(C, A) \to \mathrm{Ext_c}(C, A)$, whose image is the submodule of $\mathrm{Ext_c}(C, A)$ consisting of elements that are represented by a continuous cocycle.*

Proof. Suppose that $A \to B \to C$ is a short exact sequence of pro-countable Polish modules. Then by [22, Proposition 4.6] the homomorphism $B \to C$ has a continuous right inverse $t : C \to B$. This yields a continuous cocycle representing an element of $\mathrm{Ext_c}(C, A)$ that is the image of the element of $\mathrm{Ext_{Yon}}(C, A)$ represented by the given extension. Conversely, if (c, ρ) is a continuous cocycle on C with values in A, then one can define an extension $A \to B \to C$ associated with (c, ρ) as above. This represents an element of $\mathrm{Ext_{Yon}}(C, A)$ whose image in $\mathrm{Ext_c}(C, A)$ is represented by the cocycle (c, ρ). □

One can also easily prove directly the following:

Lemma 5.34. *Suppose that X, Y are Polish modules, and $A \to B \to C$ is a short exact sequence of Polish modules.*

1. *If X and C are pro-countable, then we have an exact sequence of abelian groups:*

$$\begin{aligned} 0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \\ \to \mathrm{Ext_{Yon}}(X, A) \to \mathrm{Ext_{Yon}}(X, B) \to \mathrm{Ext_{Yon}}(X, C) \end{aligned}$$

2. *If A, B, C are pro-countable, then we have an exact sequence of abelian groups:*

$$\begin{aligned} 0 \to \mathrm{Hom}(C, Y) \to \mathrm{Hom}(B, Y) \to \mathrm{Hom}(A, Y) \\ \to \mathrm{Ext_{Yon}}(C, Y) \to \mathrm{Ext_{Yon}}(B, Y) \to \mathrm{Ext_{Yon}}(A, Y) \end{aligned}$$

5.4.3 The thick subcategory of pro-countable Polish modules

Using the representation of extensions in terms of cocycles we can show that pro-countable modules form a thick subcategory of the category of Polish modules. The same proof as [36, Lemma 3.37] gives the following result.

Lemma 5.35. *Suppose that $\mathcal{A}$ is a fully exact subcategory of $\mathbf{Mod}\,(R)$ closed under submodules, and $(C_i)_{i\in\omega}$ is a sequence of projective objects in $\mathcal{A}$. Then, setting $C := \prod_{i\in\omega} C_i$, we have that $\mathrm{Ext}_{\mathrm{Yon}}\,(C, A) = 0$ for every pro-$\mathcal{A}$ module A.*

Proof. Assume initially that A is isomorphic to an object of $\mathcal{A}$. Let (c, ρ) be a continuous cocycle representing an element of $\mathrm{Ext}_{\mathrm{Yon}}\,(C, A)$. For $n_0 < n_1 \leq \omega$ define

$$C_{[n_0,n_1)} = \prod_{n_0\leq i<n_1} C_i,$$

which we identify with a closed submodule of C. Since (c, ρ) is continuous, there exists $n_0 \in \omega$ such that (c, ρ) is trivial on $C_{[n_0,\omega)}$. Since $C_{[0,n_0)}$ is isomorphic to a projective object of $\mathcal{A}$, and A is isomorphic to an object of $\mathcal{A}$, we also have that (c, ρ) is trivial on $C_{[0,n_0)}$. This easily implies that (c, ρ) is trivial.

Consider now the general case. Let (V_n) be a basis of open submodules of A such that $V_0 = A$ and A/V_n is isomorphic to an object of $\mathcal{A}$ for every $n \in \omega$. Thus, there exists a continuous function $\xi_0 : C \to V_0$ such that

$$(c, \rho) \equiv \delta\xi_0 \,\mathrm{mod} V_1,$$

which means that the cocycle

$$(c_1, \rho_1) = (c, \rho) - \delta\xi_0$$

takes values in V_1. Again, there exists a continuous function $\xi_1 : C \to V_1$ such that

$$(c_1, \rho_1) \equiv \delta\xi_1 \,\mathrm{mod} V_2.$$

Proceeding in this fashion one defines a sequence of continuous cocycles (c_n, ρ_n) on C with values in V_n with $(c_0, \rho_0) = (c, \rho)$ and continuous functions $\xi_n : C \to V_n$ such that

$$(c_{n+1}, \rho_{n+1}) = (c_n, \rho_n) - \delta\xi_n.$$

Setting

$$\xi := \sum_{n\in\omega} \xi_n : C \to A$$

we obtain a continuous function such that $(c, \rho) = \delta\xi$. □

Proposition 5.36. *Let R be a countable domain.*

1. *If $\mathcal{A}$ is a fully exact subcategory of $\mathbf{Mod}(R)$ with enough projectives and closed under submodules, then $\mathbf{\Pi}(\mathcal{A})$ is a fully exact subcategory of $\mathbf{PolMod}(R)$ with enough projectives and closed under closed submodules;*

2. *If $\mathcal{A}$ is a thick subcategory of $\mathbf{Mod}(R)$ with enough projectives, then $\mathbf{\Pi}(\mathcal{A})$ is a thick subcategory of $\mathbf{PolMod}(R)$ with enough projectives.*

Proof. In view of Lemma 5.29, it remains to prove that if $A \to X \to C$ is a short exact sequence of Polish modules where A and C are pro-$\mathcal{A}$, then X is pro-$\mathcal{A}$. As $\mathcal{A}$ has enough projectives, it is easy to see that there exists a continuous surjective homomorphism $P \to C$ where P is a product of projective objects of $\mathcal{A}$. By Lemma 5.31, P is projective in $\mathbf{\Pi}(\mathcal{A})$. Via pullback we obtain a commuting diagram

$$\begin{array}{ccccc} A & \to & Y & \to & P \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & X & \to & C \end{array}$$

By Lemma 5.35, Y is isomorphic to $A \oplus P$. Thus, Y is pro-$\mathcal{A}$, and hence X is pro-$\mathcal{A}$ by Lemma 5.29, being isomorphic to a quotient of Y by a closed submodule. □

5.4.4 Projective pro-countable Polish modules

We now characterize the projective objects in $\mathbf{\Pi}(\mathcal{A})$.

Lemma 5.37. *Let $\mathcal{A}$ be a* hereditary *fully exact subcategory of $\mathbf{Mod}(R)$ with enough projectives. Then $\mathbf{\Pi}(\mathcal{A})$ is hereditary, and its projective objects are precisely the products of projective objects of $\mathcal{A}$.*

Proof. Let P be a projective pro-$\mathcal{A}$ module. Then we have that there exists a surjective continuous homomorphism $Q \to P$ where

$$Q = \prod_{k \in \omega} Q_k$$

where Q_k is a projective object of $\mathcal{A}$. Thus, P is a direct summand of Q. In order to prove that $\mathcal{A}$ is hereditary, it suffices to prove that if L is a closed submodule of Q, then L is a product of projective objects of $\mathcal{A}$. For $k \in \omega$ define

$$N_k = \{(x_i) \in Q : \forall i \leq k, x_i = 0\}.$$

Then we have that N_k is an open submodule of Q, and (N_k) is a basis of zero neighborhoods of Q. Thus, we have that $(N_k \cap L)$ is a basis of zero neighborhoods of L. Hence, we have $L \cong \lim_k (L/(N_k \cap L))$. For $k \in \omega$, $L/(N_k \cap L)$ is isomorphic to a closed submodule of $Q/N_k \cong Q_0 \oplus \cdots \oplus Q_k$, and hence a projective object of $\mathcal{A}$. Thus, L is isomorphic to a product of projective objects of $\mathcal{A}$. □

5.4.5 Pro-flat modules

We let $\mathbf{Flat}\,(R)$ be the category of countably-presented flat modules, and $\mathbf{FinFlat}\,(R)$ the category of finite flat modules.

Lemma 5.38. *Suppose that R is a Prüfer domain. Let M be a pro-countable module. The following assertions are equivalent:*

1. *M is torsion-free and has a basis of zero neighborhoods consisting of pure open submodules;*
2. *M is pro-countableflat;*

Proof. (1)⇒(2) Since R is a Prüfer domain, M is flat. Suppose that $U \subseteq M$ is a pure open submodule. Then M/U is a countable flat module by Lemma 5.22.

(2)⇒(1) If $M = \lim_k M_k$ where for every $k \in \omega$, M_k is flat, then M is torsion-free, whence flat. Furthermore, for every $k \in \omega$, $\mathrm{Ker}\,(M \to M_k)$ is a pure open submodule by Lemma 5.22, since M_k and M are flat. □

Lemma 5.39. *Suppose that R is a Prüfer domain. Let M be a pro-countableflat module and $N \subseteq M$ be a closed submodule. The following assertions are equivalent:*

1. *N is pure in M;*
2. *$U \cap N$ is pure in N for every pure open submodule U of M.*

Furthermore, in this case N is pro-countableflat.

Proof. We have that M is flat, whence N is also flat. If (2) holds, then N has a basis of zero neighborhoods consisting of pure open submodules.

(1)⇒(2) Let U be a pure open submodule of M. Then $U \cap N$ is pure in N. Indeed, if $x \in rU \cap N$, then $x = rz$ for some $z \in U$. Since $x \in N$ and N is pure in M, necessarily $z \in U \cap N$.

(2)⇒(1) Obvious. □

Theorem 5.40. *Suppose that R is a countable domain.*

1. *The category $\mathbf{\Pi}\,(\mathbf{Mod}\,(R))$ of pro-countable modules is a thick subcategory of $\mathbf{PolMod}\,(R)$ with enough projectives.*
2. *If R is Prüfer, then the category $\mathbf{\Pi}\,(\mathbf{Mod}_{\aleph_0}\,(R))$ is a thick subcategory of $\mathbf{\Pi}\,(\mathbf{Mod}\,(R))$ with enough projectives;*

3. *If R is Dedekind, then $\mathbf{\Pi}(\mathbf{Mod}(R))$ is hereditary, and its projective objects are precisely the countable products of countable direct sums of ideals of R;*

4. *If R is Prüfer, then the category $\mathbf{\Pi}(\mathbf{Flat}(R))$ is a hereditary quasi-abelian category with enough projectives, which is a fully exact subcategory of $\mathbf{\Pi}(\mathbf{Mod}(R))$ closed under taking closed submodules, and its projective objects are precisely the countable products of countable direct sums of finite ideals of R;*

5. *If R is Prüfer, then the category $\mathbf{\Pi}(\mathbf{FinFlat}(R))$ is a thick subcategory of $\mathbf{\Pi}(\mathbf{Flat}(R))$ that is hereditary with enough projectives, whose projective objects are precisely the countable products of finite sums of finite ideals of R.*

Proof. (1) This follows from Proposition 5.36 applied when $\mathcal{A} = \mathbf{Mod}(R)$.

(2) This follows from Proposition 5.36 applied when $\mathcal{A} = \mathbf{Mod}_{\aleph_0}(R)$.

(3) This follows from Proposition 5.24 and Lemma 5.37 applied when $\mathcal{A} = \mathbf{Mod}(R)$.

(4) This follows from 5.23, Lemma 5.37 applied when $\mathcal{A} = \mathbf{Flat}(R)$, Lemma 5.38, and Lemma 5.39, together with the following observations.

Recall that Polish modules form a quasi-abelian category, whose kernels are the injective homomorphisms with closed image and whose cokernels are the surjective continuous homomorphisms. If $\varphi : M \to N$ is a continuous homomorphism between Polish modules in $\mathbf{\Pi}(\mathbf{Flat}(R))$, then it is easily seen that its kernel in $\mathbf{\Pi}(\mathbf{Flat}(R))$ is $\{x \in M : \varphi(x) = 0\}$ while its cokernel is the quotient of N by the closed pure hull $\langle \varphi(X) \rangle_*$ of $\varphi(X)$. Suppose that

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ g \uparrow & & \uparrow h \\ A & \xrightarrow{f} & B \end{array}$$

is a pushout diagram, where f is a kernel in $\mathbf{\Pi}(\mathbf{Flat})$. We have that

$$N := \{(g(a), -f(a)) : a \in A\}$$

is a pure submodule of $A' \oplus B$. Indeed, if $(a', b) \in A' \oplus B$ and $r \in R \setminus \{0\}$ are such that $r(a', b) = (g(a), -f(a))$ for some $a \in A$, then, since f is a kernel, there exists $a_0 \in A$ such that $-f(a_0) = b$ and hence $g(a_0) = a'$. Since N is also closed, we have that B' is the quotient of $A' \oplus B$ by N. We claim that f' is kernel. If $b' \in B'$ and $a' \in A'$ and $r \in R \setminus \{0\}$ are such that $rb' = f'(a')$, then we have that $b' = (x, y) + N$ for some $x \in A'$ and $y \in B$ and $(rx, ry) \equiv (a', 0) \bmod N$. Thus, there exists $a \in A$ such that $rx + g(a) = a'$ and $ry = f(a)$. Since f is a kernel, there exists $a_0 \in A$ such that $y = f(a_0)$ and hence $r(x + g(a_0)) = a'$. Thus, we have that $(x, y) \equiv (x + g(a_0), 0) \bmod N$. Thus, if $a'' := x + g(a_0) \in A'$ then we have $f'(a'') = b'$.

Suppose now that

$$\begin{array}{ccc} A' & \stackrel{f'}{\to} & B' \\ g \uparrow & & \uparrow h \\ A & \stackrel{f}{\to} & B \end{array}$$

is a pullback diagram. Observe that $\{(a', b) \in A' \oplus B : f'(a') = h(b)\}$ is pure in $A' \oplus B$. Since it is also closed, is equal to A. It follows now that f is a cokernel.

(5) This follows from Lemma 5.37 applied when $\mathcal{A} = \mathbf{FinFlat}(R)$. □

It follows from Proposition 5.14 that the functor Hom on $\mathbf{\Pi}(\mathbf{Mod}(R))$ admits a total right derived functor RHom. We let $\mathrm{Ext}^n = \mathrm{H}^n \circ \mathrm{RHom}$ for $n \in \mathbb{Z}$. For pro-countable Polish modules C, A we denote $\mathrm{Ext}^1(C, A)$ by $\mathrm{Ext}(C, A)$. By [69, Section III.2], $\mathrm{Ext}(C, A)$ is isomorphic to $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$. If R is Prüfer and C, A are pro-flat, then $\mathrm{Ext}^n(C, A) = 0$ for $n \geq 2$. If R is Dedekind, then $\mathrm{Ext}^n = 0$ for $n \geq 2$.

5.5 Modules with a Polish cover

In this section we recall the definition and fundamental properties of *modules with a Polish cover*, and how they provide an explicit description of the left heart of the quasi-abelian category of Polish modules. We denote by R a countable Prüfer domain and assume that all the modules are R-modules.

5.5.1 Modules with a Polish cover

The following notion was introduced in [23]. A *module with a Polish cover* is a module G explicitly presented as a quotient $\hat{G}/N$ where $\hat{G}$ is a Polish module and N is a Polishable submodule of $\hat{G}$. This means that N is a Borel submodule of $\hat{G}$ such that there exists a (necessarily unique) Polish module topology on N whose Borel sets are precisely the Borel subsets of $\hat{G}$ contained in N or, equivalently, such that the inclusion $N \to \hat{G}$ is continuous. Equivalently, there exist a Polish module L and a continuous (injective) homomorphism $L \to \hat{G}$ with image equal to N. In what follows, unless otherwise specified, all the topological notions pertaining to subsets of N will be considered in reference its unique Polish module topology that makes the inclusion $N \to \hat{G}$ is continuous.

Suppose that $G = \hat{G}/N$ and $H = \hat{H}/M$ are modules with a Polish cover. A homomorphism $\varphi : G \to H$ is *Borel* if there exists a *Borel lift* for φ, namely a Borel function $\hat{\varphi} : \hat{G} \to \hat{H}$ such that $\varphi(x + N) = \hat{\varphi}(x) + M$ for $x \in G$. We regard modules with a Polish cover as objects of a category with Borel group homomorphisms as morphisms. It is proved in [114, Section 6] that this is an abelian category, where the sum of two Borel homomorphisms $\varphi, \psi : G \to H$ is defined by setting $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ for $x \in G$. By [114, Remark 3.10], a Borel homomorphism between modules with a Polish cover is an isomorphism in the category of modules with a Polish cover if and only if it is a bijection.

We identify a Polish module G with the module with a Polish cover $\hat{G}/N$ where $\hat{G} = G$ and $N = \{0\}$. By [103, Theorem 9.10 and Theorem 12.17], this realizes the category of Polish modules and a continuous module homomorphisms as a full subcategory of the category of modules with a Polish cover. A module with a Polish cover $\hat{G}/N$ is Borel isomorphic to a Polish module if and only if N is a closed submodule of $\hat{G}$.

The category $\mathbf{PolMod}(R)$ of Polish modules is a quasi-abelian category, whose left heart is equivalent to the category of modules with a Polish cover [114, Theorem 6.3]. More generally, if $\mathcal{B}$ is a thick subcategory of $\mathbf{PolMod}(R)$, then $\mathrm{LH}(\mathcal{B})$ can be identified with the full subcategory of the category of groups with a Polish cover spanned by modules with a Polish cover of the form $\hat{G}/N$ where both $\hat{G}$ and N are in $\mathcal{B}$; see [114, Theorem 6.14 and Proposition 6.16]. For example, this applies to the category $\mathbf{\Pi}(\mathbf{Mod}(R))$ of pro-countable Polish modules. We say that $\hat{G}/N$ is a pro-countable module with a Polish cover if both $\hat{G}$ and N are pro-countable. A submodule with a pro-countable Polish cover of $\hat{G}/N$ is a submodule of the form $\hat{H}/N$ where $\hat{H}$ is a pro-countable Polishable submodule of $\hat{G}$.

Lemma 5.41. *Suppose that $\mathcal{A}$ is a strictly full thick subcategory of $\mathbf{PolMod}(R)$ and $\mathcal{B}$ is a strictly full thick subcategory of $\mathcal{A}$. Suppose that for every object B of $\mathcal{B}$ there exists an object P of $\mathcal{B}$ that is projective in $\mathcal{A}$ and a strict epimorphism $P \to B$. Then $\mathrm{LH}(\mathcal{B})$ is a thick subcategory of $\mathrm{LH}(\mathcal{A})$.*

Proof. Suppose that

$$0 \to L/K \to G/N \to H/M \to 0$$

is a short exact sequence in $\mathrm{LH}(\mathcal{A})$ such that L, K, H, M are objects of $\mathcal{B}$. We need to prove that G/N is isomorphic in $\mathrm{LH}(\mathcal{A})$ to an object of $\mathrm{LH}(\mathcal{B})$. By hypothesis, we can assume that H and L are projective in $\mathcal{B}$. Without loss of generality, after replacing G/N with an isomorphic object in $\mathrm{LH}(\mathcal{A})$, we can assume that the morphism $G/N \to H/M$ is induced by a continuous group homomorphism $G \to H$. Furthermore, after replacing G with $G \oplus M$ and N with $N \oplus M$, we can assume without loss of generality that the continuous group homomorphism $G \to H$ is surjective. Thus, we have that $G = H \oplus H^{\perp}$ and the homomorphism $G \to H$ is the first-coordinate projection. Thus, we have that $H^{\perp} \subseteq N$. As $H^{\perp}$ is a closed submodule of G, and hence of N, after replacing G with $G/H^{\perp}$ and N with $N/H^{\perp}$, we can assume without loss of generality that $H^{\perp} = 0$ and hence $G = H$.

Arguing as before, we can assume that the morphism $L/K \to G/N$ is induced by an injective continuous homomorphism $L \to G$. By identifying L with its (not necessarily closed) image inside of G we have that $K = L \cap N$ and $L + N = M$. Thus, the inclusion $N \to L + N$ induces a Borel isomorphism

$$N/(L \cap N) \cong (L + N)/L.$$

Consider

$$\Gamma = \{(x, y) \in N \oplus (L + N) : x + (L \cap N) = y + L\}.$$

Then we have a short exact sequence

$$0 \to L \to \Gamma \to N \to 0.$$

Since $\mathcal{B}$ is a strictly thick subcategory of $\mathcal{A}$, this implies that Γ is in $\mathcal{B}$. We also have an exact sequence

$$0 \to L \cap N \to \Gamma \to L + N \to 0.$$

This implies that $K = L \cap N$ is in $\mathcal{B}$, concluding the proof. □

If G is a module with a Polish cover, and N is the closure of $\{0\}$ in G, then G/N is a Polish module. Furthermore, we have an extension

$$0 \to N \to G \to G/N \to 0.$$

As a particular instance of Definition 5.3 we have the following:

Definition 5.42. *A* phantom Polish module *is a module with a Polish cover whose trivial submodule is dense. Likewise, a phantom pro-countable Polish module is a module with a pro-countable Polish cover whose trivial submodule is dense.*

We let $\mathrm{Ph}\left(\mathbf{\Pi}(\mathbf{Mod}\left(R\right))\right)$ be the category of phantom pro-countable Polish modules. Then $\mathrm{Ph}\left(\mathbf{\Pi}(\mathbf{Mod}\left(R\right))\right)$ is a thick abelian subcategory of $\mathbf{\Pi}(\mathbf{Mod}\left(R\right))$, and we have an extension

$$0 \to \mathrm{Ph}(\mathbf{\Pi}(\mathbf{Mod}\left(R\right))) \to \mathrm{LH}(\mathbf{\Pi}(\mathbf{Mod}\left(R\right))) \to \mathbf{\Pi}(\mathbf{Mod}\left(R\right)) \to 0$$

Lemma 5.43. *Suppose that* $\boldsymbol{G} = \left(G^{(n)}\right)$ *is an inverse sequence of phantom Polish modules. Then its limit* $\lim \boldsymbol{G}$ *in* $\mathbf{\Pi}(\mathbf{Mod}\left(R\right))$ *is a phantom Polish module.*

Proof. Write $G^{(n)} = \hat{G}^{(n)}/N^{(n)}$ for $n \in \omega$, where $\hat{G}^{(n)}$ and $N^{(n)}$ are Polish modules. Thus, we have that $\lim \boldsymbol{G} = \hat{G}/N$ where

$$\hat{G} = \left\{ \left(x_n, z_n\right)_{n\in\omega} \in \prod_{n\in\omega} \hat{G}^{(n)} \oplus N^{(n)} : \forall n \in \omega, x_n - p^{(n,n+1)}\left(x_{n+1}\right) = z_n \right\}$$

and

$$N = \left\{ \left(x_n, z_n\right)_{n\in\omega} \in \hat{G} : \forall n \in \omega, x_n \in N^{(n)} \right\}.$$

Suppose that $\left(a_n, c_n\right)_{n\in\omega} \in \hat{G}$ and W is an open neighborhood of $\left(a_n, c_n\right)_{n\in\omega}$ in $\hat{G}$. Without loss of generality, we can assume that there exist $n_0 \geq 1$ and, for $n < n_0$, open neighborhoods U_n of 0 in $\hat{G}^{(n)}$ and V_n of 0 in $N^{(n)}$ such that $V_n \subseteq U_n$ and

$$W = \left\{ \left(x_n, z_n\right)_{n\in\omega} : \forall n < n_0, \left(x_n - a_n, z_n - c_n\right) \in U_n \times V_n \right\}.$$

For $n \leq n_0$, we have that

$$p^{(n,n_0)} a_{n_0} - a_n = p^{(n,n_0-1)} c_{n_0-1} + p^{(n,n_0-2)} c_{n_0-2} + \cdots + p^{(n,n+1)} c_{n+1} + c_n.$$

By assumption, $N^{(n_0)}$ is dense in $\hat{G}^{(n_0)}$. Thus, there exists a sequence $(a_{n_0}^{(k)})$ in $N^{(n_0)}$ that converges to a_{n_0} in $\hat{G}^{(n_0)}$. Thus,

$$(p^{(n,n_0)}a_{n_0}^{(k)} - (p^{(n,n_0-1)}c_{n_0-1} + p^{(n,n_0-2)}c_{n_0-2} + \cdots + p^{(n,n+1)}c_{n+1} + c_n))$$

converges in $\hat{G}^{(n)}$ to a_n for $n < n_0$. Thus, there exists $k \in \omega$ such that, for $n < n_0$,

$$b_n := p^{(n,n_0)}a_{n_0}^{(k)} - (p^{(n,n_0-1)}c_{n_0-1} + p^{(n,n_0-2)}c_{n_0-2} + \cdots + p^{(n,n+1)}c_{n+1} + c_n) \in U_n + a_n.$$

Set $b_{n_0} := a_{n_0}^{(k)}$, $d_{n_0} := a_{n_0}^{(k)}$, and $d_n := c_n$ for $n < n_0$. Define for $n > n_0$, $b_n = d_n = 0$. Then we have that $(b_n, d_n)_{n\in\omega}$ defines an element of $\hat{G}$, since for $n < n_0$ we have that

$$\begin{aligned}
& p^{(n,n+1)}b_{n+1} - b_n \\
= \; & p^{(n,n+1)}\left(p^{(n+1,n_0)}a_{n_0}^{(k)} - (p^{(n+1,n_0-1)}c_{n_0-1} + p^{(n+1,n_0-2)}c_{n_0-2} + \cdots + c_{n+1})\right) \\
& - \left(p^{(n,n_0)}a_{n_0}^{(k)} - (p^{(n,n_0-1)}c_{n_0-1} + p^{(n,n_0-2)}c_{n_0-2} + \cdots + p^{(n,n+1)}c_{n+1} + c_n)\right) \\
= \; & c_n = d_n.
\end{aligned}$$

Furthermore, we have that $(b_n, d_n)_{n\in\omega}$ belongs to W, since for $n < n_0$ we have that

$$d_n - c_n = 0$$

and

$$b_n - a_n \in U_n.$$

This concludes the proof. □

5.5.2 Submodules with a Polish cover

Suppose that $G = \hat{G}/N$ is a module with a Polish cover, and let $H = \hat{H}/N$ be a submodule of G. Then we say that H is a *submodule with a Polish cover* of G if $\hat{H}$ is a Polishable submodule of $\hat{G}$. We denote by $\bar{H}^G$ the closure of H in G with respect to the quotient topology. It is proved in [114, Section 4] that images and preimages of submodules with a Polish cover of modules with a Polish cover under Borel homomorphisms are submodules with a Polish cover.

Recall that a *complexity class* Γ is an assignment $X \mapsto \Gamma(X)$ where X is a Polish space and $\Gamma(X)$ is a collection of subsets of X, such that $f^{-1}(A) \in \Gamma(X)$ for every continuous function $f : X \to Y$ between Polish spaces and for every $A \in \Gamma(Y)$. We say that a submodule with a Polish cover $H = \hat{H}/N$ of a module with a Polish cover $G = \hat{G}/N$ is Γ in G or belongs to $\Gamma(G)$ if $\hat{H} \in \Gamma(\hat{G})$. Furthermore, H has complexity class Γ in G if $\hat{H} \in \Gamma(\hat{G})$ and $\hat{G} \setminus \hat{H} \notin \Gamma(\hat{G})$. Let Γ be the complexity class $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, or $D(\mathbf{\Pi}^0_\alpha)$ for $\alpha < \omega_1$; see [68, Section 1.3]. (Here, $D(\mathbf{\Pi}^0_\alpha)$ is the class of sets that can be written as intersection of a $\mathbf{\Sigma}^0_\alpha$ set and a $\mathbf{\Pi}^0_\alpha$ set.) It is proved in [115, Section 3] that if $\varphi : G \to H = \hat{H}/M$ is a Borel homomorphism between modules with a pro-countable Polish cover and $H_0 = \hat{H}_0/M$ is a submodule with a pro-countable Polish cover of H, then:

- $H_0 \in \Gamma(H)$ implies $\varphi^{-1}(H_0) \in \Gamma(G)$;
- $H_0 \in \Gamma(H)$ if and only if the coset equivalence relation of $\hat{H}_0$ in $\hat{H}$ is *potentially* Γ, namely Borel reducible to an equivalence relation E on a Polish space X such that $E \in \Gamma(X \times X)$.

5.5.3 Solecki submodules

Let $G = \hat{G}/N$ be a module with a Polish cover. The first Solecki submodule $s_1(G)$ of G is the smallest $\mathbf{\Pi}^0_3$ submodule with a Polish cover of G. It was shown by Solecki in [159, Theorem 2.1] that such a submodule exists, and $s_1(G) \subseteq \{\bar{0}\}^G$. In fact, one can explicitly define $s_1(G)$ as $\hat{G}_1/N$ where $\hat{G}_1$ is the submodule of $\hat{G}$ comprising the $x \in \hat{G}$ such that for every zero neighborhood V of N there exists $z \in N$ such that $x + z$ belongs to the closure $\bar{V}^{\hat{G}}$ of V in $\hat{G}$ [159, Lemma 2.3].

One then defines the sequence $(s_\alpha(G))_{\alpha<\omega_1}$ of Solecki submodules of G by recursion on α by setting:

- $s_0(G) = \{\bar{0}\}^G$;
- $s_{\alpha+1}(G) = s_1(s_\alpha(G))$;
- $s_\lambda(G) = \bigcap_{\alpha<\lambda} s_\alpha(G)$ for λ limit.

It is shown in [159, Theorem 2.1] that there exists $\sigma < \omega_1$ such that $s_\sigma(G) = \{0\}$. The least such a countable ordinal is called the *Solecki length* of G in [114]. When α is a successor ordinal, we say that G has plain Solecki length α if it has Solecki length α and $\{0\} \in \mathbf{\Sigma}^0_2(s_{\alpha-1}(G))$. In this case, we also say that G is plain.

For every $\alpha < \omega_1$, $s_\alpha(G)$ is the smallest $\mathbf{\Pi}^0_{1+\alpha+1}$ submodule of G [114, Theorem 5.4]. If G is a pro-countable module with a Polish cover, then $s_\alpha(G)$ is a pro-countable submodule of G for every $\alpha < \omega_1$. The following characterization of the first Solecki submodule can be easily obtained with the methods of [159]; see [115, Lemma 4.2].

Lemma 5.44. *Suppose that $G = \hat{G}/N$ is a module with a Polish cover, and let $H = \hat{H}/N$ be a $\mathbf{\Pi}^0_3$ submodule with a Polish cover of G. Suppose that:*

1. *N is dense in $\hat{H}$;*
2. *for every open neighborhood V of the identity in N, $\bar{V}^{\hat{G}}$ contains an open neighborhood of the identity in $\hat{H}$.*

Then $H = s_1(G)$.

To simplify the statement of the following result, we introduce some notation. For $\lambda < \omega_1$ either zero or limit and $n < \omega$ we define the complexity class:

$$\Gamma_{\lambda+n} := \begin{cases} \mathbf{\Pi}^0_1 & \text{for } \lambda = n = 0; \\ \mathbf{\Pi}_\lambda & \text{for } n = 0 \text{ and } \lambda > 0; \\ \mathbf{\Pi}^0_{1+\lambda+n+1} & \text{for } \lambda > 0 \text{ and } n > 0; \end{cases}$$

and

$$\Gamma^{\text{semiplain}}_{\lambda+n} := \begin{cases} \Gamma_{\lambda+n} & \text{for } n = 0; \\ D(\mathbf{\Pi}^0_{1+\lambda+n}) & \text{for } n \geq 1; \end{cases}$$

and

$$\Gamma^{\text{plain}}_{\lambda+n} := \begin{cases} \Gamma^{\text{semiplain}}_{\lambda+n} & \text{for } n \neq 1; \\ \mathbf{\Sigma}^0_{1+\lambda+1} & \text{for } n = 1. \end{cases}$$

By [115, Theorem 6.1] we have the following description of the complexity of $\{0\}$ in a module with a pro-countable Polish cover.

Theorem 5.45. *Suppose that G is a module with a pro-countable Polish cover. Let $\alpha < \omega_1$ be the Solecki length of G.*

1. *If G is plain, then the complexity class of $\{0\}$ in G is $\Gamma^{\text{plain}}_\alpha$;*
2. *If G is not plain, then the complexity class of $\{0\}$ in G is Γ_α.*

Corollary 5.46. *Let G be a module with a pro-countable Polish cover. The the complexity class of $\{0\}$ in G is either Γ_α or $\Gamma^{\text{plain}}_\alpha$ for some $\alpha < \omega_1$.*

In fact, it is also proved in [115, Theorem 1.2] that the ones in Corollary 5.46 are the only possible complexity classes of a non-Archimedean Polishable subgroup of a Polish group. Recall that Γ^{plain}_1 is by definition the complexity class $\mathbf{\Sigma}^0_2$.

Lemma 5.47. *Suppose that (G_i) is an inverse sequence of phantom pro-countable modules of plain Solecki length at most 1, and H is a phantom pro-countable module. Suppose that $G = \lim_i G_i$ and $\Phi : G \to H$ is a Borel homomorphism. If H has plain Solecki length at most 1, then there exists $n \in \omega$ such that Φ factors through G_n.*

Proof. As $\{0\} \in \Sigma^0_2(G_i)$, we can write $G_i = \hat{G}_i / L_i$ for some pro-countable Polish module $\hat{G}_i$ and some countable submodule L_i. Set

$$\hat{G} = \left\{ (x_i, y_i)_{i\in\omega} \in \prod_{i\in\omega} \hat{G}_i \oplus L_i : \forall i, p^{(i,i+1)}(x_{i+1}) - x_i = y_i \right\}$$

and

$$L := \left\{ (x_i, y_i)_{i\in\omega} \in \hat{G} : \forall i, x_i \in L_i \right\}.$$

Then we have that $G = \hat{G}/L$ and $\operatorname{Ker}(\Phi) = K/L$ for some $\mathbf{\Sigma}^0_2$ pro-countable Polishable submodule K of $\hat{G}$ containing L.

Since K is a $\mathbf{\Sigma}^0_2$ pro-countable Polishable submodule of $\hat{G}$, by the Baire Category Theorem it has an open submodule U that is closed in $\hat{G}$. As U is open in K, $U \cap L$ is open in L. Thus, there exists $n \in \omega$ such that $U \cap L$ contains

$$\left\{(x_i, y_i)_{i\in\omega} \in L : \forall i \leq n, x_i = 0\right\}.$$

As U is closed in $\hat{G}$, this implies that U contains

$$\left\{(x_i, y_i)_{i\in\omega} \in \hat{G} : \forall i \leq n, x_i = 0\right\}.$$

Thus, the same holds for K, and the conclusion follows. □

Corollary 5.48. *Let $\left(G^{(n)}, p^{(n,n+1)} : G^{(n+1)} \to G^{(n)}\right)$ be an inverse sequence of phantom pro-countable modules such that $p^{(n,n+1)}$ is surjective for every $n \in \omega$. Assume that $G^{(n)}$ has plain Solecki length at most 1 for every $n \in \omega$. Set $G := \lim_n G^{(n)}$. The following assertions are equivalent:*

1. *G has plain Solecki length at most 1;*
2. *there exists $k \in \omega$ such that the canonical map $G \to G^{(k)}$ is an isomorphism;*
3. *there exists $k \in \omega$ such that for all $n \geq k$, $p^{(n,n+1)}$ is an isomorphism.*

Lemma 5.49. *Suppose that G, H are modules with a pro-countable cover. Let $\varphi : G \to H$ be a Borel group homomorphism. If G has plain Solecki length at most 1, then so does $\varphi(G)$.*

Proof. Since G is a module with a pro-countable cover, we can write $G = \hat{G}/N$ where $\hat{G}$ and N are pro-countable modules. Since $\{0\} \in \mathbf{\Sigma}^0_2(G)$, we have that N has a zero neighborhood V that is a closed submodule of $\hat{G}$. After replacing $\hat{G}$ with $\hat{G}/V$ and N with N/V, we can assume that N is countable. We can also assume without loss of generality that φ is surjective. If $\mathrm{Ker}(\varphi)$ is the submodule K/N, then we have that $H \cong \hat{G}/(N+K)$. Since K is closed in $\hat{G}$, we have that $N + K \in \mathbf{\Sigma}^0_2(\hat{G})$, and hence $\{0\} \in \mathbf{\Sigma}^0_2(H)$. □

5.6 Ranks and games

In this section, we recall some notions from descriptive set theory concerning *ranks* and their description in terms of games. More information can be found in [103].

5.6.1 Rank of well-founded binary relation

Suppose that Y is a set, and $\prec$ is an binary relation on Y. For a subset A of Y, define its *$\prec$-derivative* to be the set

$$D_\prec(A) := \{y \in Y : \exists a \in A, a \prec y\}.$$

Define the recursively, for every ordinal α,

$$D^0_{\prec}(A) := A,$$

$$D^{\alpha+1}_{\prec}(A) := D_{\prec}(D^{\alpha}_{\prec}(A)),$$

and for λ limit

$$D^{\lambda}_{\prec}(A) := \bigcap_{\alpha<\lambda} D^{\alpha}_{\prec}(A).$$

The binary relation $\prec$ is *ill-founded* if there exists a sequence (x_n) in Y with $x_{n+1} \prec x_n$ for every $n \in \omega$, and *well-founded* otherwise. If $\prec$ is well-founded, one defines the rank $\rho_{\prec} : Y \to \mathbf{ORD}$, where $\mathbf{ORD}$ is the class of ordinals, by well-founded recursion on $\prec$ by setting

$$\rho_{\prec}(x) = \sup\{\rho_{\prec}(y) + 1 : y \in Y \text{ and } y \prec x\}.$$

One defines then the rank $\rho(\prec)$ of $\prec$ to be the least ordinal not in the range of $\rho_{\prec}$, thus

$$\rho(\prec) = \sup\{\rho_{\prec}(y) + 1 : y \in Y\};$$

see [103, Appendix B]. When $\prec$ is ill-founded, one sets $\rho(\prec) = \infty$. It is clear that we have the equality

$$D^{\alpha}_{\prec}(Y) = \{y \in Y : \alpha \leq \rho_{\prec}(y)\}.$$

Therefore, the following result is immediate.

Lemma 5.50. *Suppose that Y is a set, and $\prec$ is a binary relation on Y. Then $\prec$ is well-founded if and only if there exists an ordinal α such that $D^{\alpha}_{\prec}(Y) = \varnothing$. In this case, we have that $\rho(\prec)$ is the least such an ordinal.*

Suppose now that Y is a *standard Borel space*, and $\prec$ is *analytic* binary relation on Y. (Recall that a subset A of a standard Borel space Y is analytic if it is the image of a continuous function $f : Z \to Y$ for some standard Borel space Z, and co-analytic if its complement in Y is analytic.) In this case, if $\prec$ is well-founded, then its rank is a *countable* ordinal [103, Theorem 31.1]. Furthermore, for an analytic subset $A \subseteq Y$, its derivative $D^{\alpha}_{\prec}(A)$ is still analytic.

5.6.2 Coanalytic ranks

Recall that if X is a standard Borel space, then a co-analytic rank on X is a function $\varphi : X \to \omega_1 \cup \{\infty\}$ such that there exist analytic binary relations $\leq_{\varphi}$ and $<_{\varphi}$ on X satisfying, for $x, y \in X$ with $\varphi(y) < \infty$:

$$(\varphi(x) < \infty \wedge \varphi(x) \leq \varphi(y)) \Leftrightarrow x \leq_{\varphi} y$$

and

$$(\varphi(x) < \infty \wedge \varphi(x) < \varphi(y)) \Leftrightarrow x <_{\varphi} y;$$

see [103, Section 34.B].

Suppose now that X, Y are standard Borel spaces, and $\prec \subseteq X \times Y \times Y$ is an analytic set. For $x \in X$ we define the analytic binary relation $\prec_x$ on Y by setting, for $y, y' \in Y$,

$$y \prec_x y' \Leftrightarrow (x, y, y') \in \prec .$$

We think of $\prec$ as an analytic assignment of binary relations on Y to elements of X. Define now

$$\varphi_\prec : X \to \omega_1, x \mapsto \rho(\prec_x).$$

Define LO to be the standard Borel space of $t \in 2^{\omega \times \omega}$ such that $D(t) := \{i \in \omega : t(i,i) = 1\}$ is a nonempty subset of ω, and setting

$$i <_t j \Leftrightarrow t(i,j) = 1$$

defines a linear order on $D(t)$ with least element 0. Define also $\mathrm{WO} \subseteq \mathrm{LO}$ to be the co-analytic set of $t \in \mathrm{LO}$ such that $<_t$ is a well-order, in which case we define $\alpha_t < \omega_1$ to be its order type.

Proposition 5.51. *Suppose that X, Y are standard Borel spaces, and $x \mapsto \prec_x$ is an analytic assignment of binary relations on Y to elements of X. Then the relation*

$$\{(t, x) \in \mathrm{LO} \times X : \alpha_t \leq \varphi_\prec(x)\}$$

is analytic, and $\varphi_\prec$ is a co-analytic rank of X.

Proof. For $x \in X$ with $\varphi_\prec(x) < \infty$ and $t \in \mathrm{WO}$ we have that $\alpha_t \leq \varphi_\prec(x)$ if and only if for every $n \in \omega$ we have

$$\forall a_0 \exists b_0 \forall a_1 \exists b_1 \cdots \forall a_n \exists b_n, \ (\text{if } a_n <_t a_{n-1} <_t \cdots <_t a_0 \text{ then } b_n \prec_x \cdots \prec_x b_0)$$

where $a_0, \ldots, a_n \in D(t)$ and $b_0, \ldots, b_n \in X$. More generally, for $a \in D(t)$ and $b \in X$ we have that $\rho_{<_t}(a) \leq \rho_{\prec_x}(b)$ if and only if

$$\forall a_0 \exists b_0 \forall a_1 \exists b_1 \cdots \forall a_n \exists b_n, \ (\text{if } a_n <_t a_{n-1} <_t \cdots <_t a_0 <_t a \text{ then } b_n \prec_y \cdots \prec_y b_0 \prec_y b).$$

The second assertion follows from the first one as in the proof of [103, Lemma 34.11]. □

A fundamental property of co-analytic ranks is that their restriction to Borel subsets is necessarily bounded; see [103, Theorem 35.23].

Proposition 5.52. *Suppose that X is a standard Borel space and $\varphi : X \to \omega_1 \cup \{\infty\}$ is a co-analytic rank on X. If A is a Borel subset of X such that $\varphi|_A$ takes values in ω_1, then there exists $\alpha < \omega_1$ such that $\varphi|_A : A \to \alpha$.*

5.6.3 Trees

A canonical example of co-analytic rank is the rank of combinatorial trees. Let X be a countable set. A *tree* on X is a set T of finite tuples of elements of X (called the *nodes* of T) that is closed under taking initial segments. The *root* of the tree is the empty tuple $\varnothing$. The *body* $[T]$ of T is the closed subsets of X^ω comprising the infinite sequences whose initial segments all belong to T. The binary relation $\prec_T$ on the set $X^{<\omega}$ of finite tuples of elements of X is defined by setting $a \prec_T b$ if and only if $a, b \in T$ and b is an initial segment of a. Then we have that $\prec_T$ is *well-founded* if and only if the body of T is empty. Furthermore, the α-th derivative $D^\alpha_{\prec_T}(T)$ is a subtree T^α of T. The rank $\rho(T)$ of T is defined to be the rank of $\prec_T$, which is thus the least α such that T^α is empty. As the root is easily seen to have the largest rank among the nodes of T, we have by definition

$$\rho(T) = \rho_{\prec_T}(\varnothing) + 1.$$

In particular, the rank of T is always a successor ordinal.

Trees form a closed subspace Tree of $2^{X^{<\omega}}$, of which the well-founded trees are a co-analytic subspace. If Z is a standard Borel space, and $z \mapsto T_z$ is an analytic assignment of trees to elements of Z, then $z \mapsto \prec_{T_z}$ is an analytic assignments of binary relations on $X^{<\omega}$ to elements of Z. It follows from this and Proposition 5.51 that the map $z \mapsto \rho(T_z)$ is a co-analytic rank on Z.

Graph-theoretically, a rooted tree is a connected acyclic graph with a distinguished vertex (the root). When X is the set ω of natural numbers, one can think of a tree T on ω as a graph-theoretic countable rooted tree such that each level-set is endowed with a well-order of order type at most ω. We regard a rooted tree T as an ordered set with respect to the relation between nodes of T

$$v \prec w \Leftrightarrow w \text{ belongs to the unique path from the root to } v.$$

We define a forest to be a disjoint union of rooted trees, with the induced order.

5.6.4 Games

Let X be a countable discrete set, and denote by X^ω the Polish space of infinite sequences of elements of X. A subset A of X^ω defines a game G_A between two players, Alice and Bob. The game starts with Alice picking an element x_0 of X, to which Bob must reply with an element x_1 of X. The games continues in this fashion, with Alice and Bob alternating to pick elements x_{2n} and, respectively, x_{2n+1} at Alice's and, respectively, Bob's turn of index n. A run of the game produces an infinite sequence $(x_n)_{n\in\omega}$. Alice wins the game if such a sequence belongs to A, otherwise Bob wins the game. One can similarly define a run of the game with Bob as starting player. The payoff set for Bob is clearly $B := X^\omega \setminus A$.

One says that Alice has a winning strategy for such a game if there exists a way for Alice to choose her n-th move (as given by a function from finite tuples of elements of X to X) in a way that lets her win any run of the game. A winning strategy for Bob is defined

in the same fashion. The game is called *determined* if either Alice or Bob has a winning strategy. If A is a *Borel* subset of X^ω, then the game is determined.

Following [1], we define the notion of *game rank* for a game G_A with *open* payoff set A for Alice (and, hence, closed payoff set for Bob), and more generally the notion of *game rank* for a finite tuple $x \in X^{<\omega}$ representing a position of the game after finitely many steps. In fact, we define two ranks, the σ-game rank σ_A and π-game rank π_A, depending on whether one considers Alice and Bob as starting player. We say that a finite tuple x is a *winning position* for Alice if the open set N_x of sequences that have x as initial segment is a subset of A. Define:

- $\pi_A(x) = 0$ if and only if $\sigma_A(x) = 0$ if and only if x is a winning position for Alice;
- $\sigma_A(x) \leq \alpha$ if and only if $\exists z \in X, \pi_A(x, z) < \alpha$;
- $\pi_A(x) \leq \alpha$ if and only if $\forall z \in X, \sigma_A(x, z) < \alpha$.

We then define the rank $\rho(G_A)$ of the game G_A to be the least α such that $\pi_A(\varnothing) \leq \alpha$.

This description of ranks is a reformulation of the one in terms of trees, considering the following: If the payoff set A for Alice is an open subset of X^ω, then the payoff set $B := X^\omega \setminus A$ for Bob is a *closed* subset of X^ω. Thus, there exists a tree T such that B is the body of T. Then we have that the rank of G_A is at most α if and only if the rank of T is at most α.

5.6.5 Indices

We define by recursion on countable ordinals sets of indices which can be seen as a version of *walks on ordinals* [163, Chapter 2]. Recall that for each limit ordinal λ we have fixed an increasing sequence of successor ordinals (λ_i) cofinal in λ. Likewise, if α is a successor or zero, we set $\alpha_i = \alpha$ for every $i < \omega$. We now define by recursion on $\alpha < \omega_1$ the sets of indices I_α and, when α is a successor, I_α^{plain}. Define $I_1^{\text{plain}} = \{0\}$. If I_β and I_β^{plain} has been defined for $1 \leq \beta < \alpha$ define I_α to be the set of pairs $(i; n)$ with $i \in I_{\alpha_n}^{\text{plain}}$. Define then I_α^{plain} to be $I_{\alpha-1} \cup \{\alpha - 1\}$.

We also define, for $\beta < \gamma$:

- $\sigma_\gamma I_\beta = I_\beta$;
- $\sigma_\gamma I_\beta^{\text{plain}} = I_\beta^{\text{plain}}$;
- $\partial_\gamma I_\beta = \varnothing$;
- $\partial_\gamma I_\beta^{\text{plain}} = \varnothing$;
- $\sigma_\gamma I_{\gamma+1}^{\text{plain}} = I_\gamma$;

- $\partial_\gamma I^{\mathrm{plain}}_{\gamma+1} = \{\gamma\}$.

If $\sigma_\gamma I^{\mathrm{plain}}_\beta$ and $\partial_\gamma I^{\mathrm{plain}}_\beta$ have been defined for $\gamma+1 \leq \beta \leq \alpha$ define $\sigma_\gamma I_\alpha$ to be the set of pairs $(i;n)$ with $i \in \sigma_\gamma I^{\mathrm{plain}}_{\alpha_n}$, and $\partial_\gamma I_\alpha$ to be the set of pairs $(i;n)$ with $i \in \partial_\beta I^{\mathrm{plain}}_{\alpha_n}$. We also set, for $\gamma < \alpha$,

$$\sigma_\gamma I^{\mathrm{plain}}_\alpha = \sigma_\gamma I_{\alpha-1} \text{ and } \partial_\gamma I^{\mathrm{plain}}_\alpha = \partial_\gamma I_{\alpha-1} \cup \{\alpha-1\}\,.$$

Thus, by definition, σ_γ and ∂_γ describe complementary subsets of I_α and $I^{\mathrm{plain}}_\alpha$.

We consider I_α and $I^{\mathrm{plain}}_\alpha$ as ordered set, with respect to the relation $\prec$ defined recursively as follows. Say that $\alpha-1$ is the largest element of $I^{\mathrm{plain}}_\alpha$, and the map $I^{\mathrm{plain}}_{(\alpha-1)_n} \to I^{\mathrm{plain}}_\alpha$, $i \mapsto (i;n)$ is order-preserving. This can be seen as the relation of belonging to the same branch on a canonical tree structure on $I^{\mathrm{plain}}_\alpha$ with root α. In fact, this can be regarded as a tree on ω, where the order on each level set in $I^{\mathrm{plain}}_\alpha$ is recursively defined by setting $((\alpha-1)_n\,;n) \prec ((\alpha-1)_k\,;k) \Leftrightarrow n<k$. It is easily seen by induction that $I^{\mathrm{plain}}_\alpha$ is a countable well-founded tree of rank α. Furthermore, every countable well-founded rooted tree of rank at most α is isomorphic to a subtree of $I^{\mathrm{plain}}_\alpha$, i.e., a nonempty subset $S \subseteq I^{\mathrm{plain}}_\alpha$ that contains, for each of its nodes, the path from it to the root.

Clearly, if $(x_i)_{i\in S}$ is a diagram of shape S in a category $\mathcal{C}$ that has a limit x, then one can extend it to a diagram of shape over $I^{\mathrm{plain}}_\alpha$ in $\mathcal{C}$ with the same limit. This shows that every limit of a diagram whose shape is a countable well-founded rooted tree of rank at most α can be realized as a limit of a diagram of shape $I^{\mathrm{plain}}_\alpha$. Likewise, we have that I_α is a countable forest of rank α, and every limit of a diagram whose shape is a countable forest of rank at most α is also a limit of a diagram of shape I_α. By duality, the same applies to colimits of diagrams of shape S^{op}, i.e., presheaves on S.

There is a canonical linear order $<$ on $I^{\mathrm{plain}}_\alpha$ defined recursively by letting $\alpha-1$ be the largest element, every descendent of $((\alpha-1)_n\,;n)$ less than every descendent of $((\alpha-1)_k\,;k)$ for $n<k$, and the map $I^{\mathrm{plain}}_{(\alpha-1)_n} \to I^{\mathrm{plain}}_\alpha$, $i \mapsto (i;n)$ is order-preserving. In terms of the structure as tree on ω on $I^{\mathrm{plain}}_\alpha$, this is the order on nodes defined by setting $x<y$ if and only if x is less than y in the lexicographic order, or $x \prec y$. Such a linear order on $I^{\mathrm{plain}}_\alpha$ is easily seen by induction to be isomorphic to $\omega^{\alpha-1}+1$. The subset of elements of $I^{\mathrm{plain}}_\alpha$ whose rank with respect to the tree structure is at most $\beta < \alpha-1$ corresponds via this isomorphism to the set of ordinals whose Cantor normal form is $\omega^{\beta_0}k_0 + \cdots + \omega^{\beta_\ell}k_\ell$ for $k_0,\ldots,k_\ell \in \omega$ and $\alpha-1 > \beta_0 \geq \beta_1 \geq \cdots \geq \beta_\ell \geq \beta$. In particular, for $\beta=1$ we obtain that the *terminal nodes* of $I^{\mathrm{plain}}_\alpha$ correspond to the *successor ordinals* in $\omega^{\alpha-1}$.

5.7 The functor $\lim^1$

In this section we regard the $\lim^1$ of a tower of countable modules as a phantom procountable module. We explicitly describe its Solecki submodules in terms of an ordinal sequence of towers obtained from the given one by iterating the operation which we call *derivation*. As an application, we give a purely algebraic characterization of the towers $\boldsymbol{A}$

of countable modules such that $\{0\}$ has a given complexity within $\lim^1 \boldsymbol{A}$. We continue to let R be a countable domain and to consider all modules to be R-modules.

5.7.1 $\lim$ and $\lim^1$ for a tower of modules

We adopt the notations from Section 5.2.14 concerning the category of towers of Polish modules. Thus, a *tower* of Polish modules is a sequence $\boldsymbol{G} = \left(G^{(n)}, p^{(n,n+1)}\right)_{n\in\omega}$ of Polish modules such that, for every $n \in \omega$, $p^{(n,n+1)} : G^{(n+1)} \to G^{(n)}$ is a continuous homomorphism. For $n \leq m$, we define recursively $p^{(n,n)} := \mathrm{id}_{G^{(n)}}$ and $p^{(n,m+1)} := p^{(n,m)} \circ p^{(m,m+1)}$.

Suppose that $\boldsymbol{G} = (G^{(n)})$ and $\boldsymbol{H} = (H^{(n)})$ are towers of Polish modules. A morphism from $\boldsymbol{G}$ to $\boldsymbol{H}$ is represented by a sequence $\left(m_k, f^{(k)}\right)$ such that (m_k) is an increasing sequence in ω, and $f^{(k)} : G^{(m_k)} \to H^{(k)}$ is a continuous homomorphism such that $p^{(k,k+1)} f^{(m_{k+1})} = f^{(k)} p^{(m_k,m_{k+1})}$ for every $k \in \omega$. Two such sequences $\left(m_k, f^{(k)}\right)$ and $\left(m'_k, f'^{(k)}\right)$ define the same morphism if and only if there exists an increasing sequence $(\tilde{m}_k)_{k\in\omega}$ in ω such that $\tilde{m}_k \geq \max\{m_k, m'_k\}$ for every $k \in \omega$, and

$$f^{(k)} p^{(m_k,\tilde{m}_k)} = f'^{(k)} p^{(m'_k,\tilde{m}_k)}$$

for every $k \in \omega$. This defines an additive category $\mathrm{pro}\,(\mathbf{PolMod}\,(R))$ of towers of Polish modules.

We let:

- $\mathrm{pro}\,(\mathbf{Mod}\,(R))$ be the thick subcategory of $\mathrm{pro}\,(\mathbf{PolMod}\,(R))$ spanned by towers of countable modules;
- $\mathrm{pro}\,(\mathbf{Fin}\,(R))$ be the thick subcategory of $\mathrm{pro}\,(\mathbf{Mod}\,(R))$ spanned by towers of finite modules.

Suppose that $\boldsymbol{G} = (G^{(n)})_{n\in\omega}$ is a tower of Polish modules. For $n \in \omega$, set

$$\Lambda_n := \{(\lambda_0, \ldots, \lambda_n) \in \omega \times \cdots \times \omega : \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n\}.$$

Define $K(\boldsymbol{G})$ to be the cochain complex of Polish modules obtained by setting

$$K^n(\boldsymbol{G}) = \{0\}$$

for $n < 0$ and

$$K^n(\boldsymbol{G}) = \prod_{(\lambda_0,\ldots,\lambda_n)\in\Lambda_n} G^{(\lambda_0)}$$

for $n \geq 0$. The continuous homomorphism $\delta^n : K^{n-1}(G) \to K^n(G)$ is given by

$$\delta^n(c)_{(\lambda_0,\ldots,\lambda_n)} = p^{(\lambda_0,\lambda_1)}\left(c_{(\lambda_1,\ldots,\lambda_n)}\right) + \sum_{i=1}^{n} (-1)^i c_{(\lambda_0,\ldots,\widehat{\lambda_i},\ldots,\lambda_n)}.$$

One then defines, for $n \in \omega$,

$$\mathrm{Z}^n\left(K(\boldsymbol{G})\right) = \mathrm{Ker}\left(\delta^n : K^n\left(\boldsymbol{G}\right) \to K^{n+1}\left(\boldsymbol{G}\right)\right)$$

$$\mathrm{B}^n\left(K(\boldsymbol{G})\right) = \mathrm{Ran}\left(\delta^{n-1} : K^{n-1}\left(\boldsymbol{G}\right) \to K^n\left(\boldsymbol{G}\right)\right) \subseteq \mathrm{Z}^n\left(K(\boldsymbol{G})\right)$$

and

$$H^n\left(K(\boldsymbol{G})\right) = \frac{\mathrm{Z}^n\left(K(\boldsymbol{G})\right)}{\mathrm{B}^n\left(K(\boldsymbol{G})\right)}.$$

By definition one has that

$$\lim\boldsymbol{G} = \mathrm{H}^0\left(K\left(\boldsymbol{G}\right)\right)$$

and

$$\lim{}^1\boldsymbol{G} = \mathrm{H}^1\left(K\left(\boldsymbol{G}\right)\right);$$

see [123, Section 11.5]. Furthermore, $\mathrm{H}^n\left(K\left(\boldsymbol{G}\right)\right) = 0$ for $n \geq 2$.

We have that $\lim$ defines a countably continuous functor $\mathrm{pro}\left(\boldsymbol{\Pi}(\mathbf{Mod}\left(R\right))\right) \to \boldsymbol{\Pi}(\mathbf{Mod}\left(R\right))$, while $\lim^1$ defines a countably continuous functor $\mathrm{pro}\left(\boldsymbol{\Pi}(\mathbf{Mod}\left(R\right))\right) \to \mathrm{Ph}\left(\boldsymbol{\Pi}(\mathbf{Mod}\left(R\right))\right)$. A short exact sequence

$$0 \to \boldsymbol{A} \to \boldsymbol{B} \to \boldsymbol{C} \to 0$$

in $\mathrm{pro}\left(\boldsymbol{\Pi}(\mathbf{Mod}\left(R\right))\right)$ gives rise to a six-term exact sequence

$$0 \to \lim\boldsymbol{A} \to \lim\boldsymbol{B} \to \lim\boldsymbol{C} \to \lim{}^1\boldsymbol{A} \to \lim{}^1\boldsymbol{B} \to \lim{}^1\boldsymbol{C} \to 0$$

in $\mathrm{LH}\left(\boldsymbol{\Pi}\left(\mathbf{Mod}\left(R\right)\right)\right)$.

As in Section 5.2.14, we say that $\boldsymbol{G}$ is an epimorphic (respectively, monomorphic) tower if $p^{(n,n+1)} : G^{(n+1)} \to G^{(n)}$ is a strict epimorphism (respectively, a monomorphism) for every $n \in \omega$. We say that $\boldsymbol{G}$ is essentially epimorphic (respectively, essentially monomorphic) if it is isomorphic to an epimorphic (respectively, monomorphic) tower.

A tower $\boldsymbol{G}$ is essentially epimorphic if and only if it satisfies the so-called *Mittag-Leffler condition*: for every $n \in \omega$ there exists $k_0 \in \omega$ such that, for every $k \geq k_0$, $p^{(n,n+k)}\left(G^{(n+k)}\right) = p^{(n,n+k_0)}\left(G^{(n+k_0)}\right)$. This implies that $\lim^1\boldsymbol{G}$ is trivial, where the converse holds if $\boldsymbol{G}$ is a tower of countable modules; see [124, Ch. II, Section 6.2]. We let $\boldsymbol{\Pi}\left(\mathbf{Mod}\left(R\right)\right)$ be the full subcategory of $\mathrm{pro}\left(\mathbf{Mod}\left(R\right)\right)$ spanned by the (essentially) epimorphic towers.

Definition 5.53. *Let $\boldsymbol{A} = \left(A^{(n)}\right)$ be a tower of pro-countable modules. Define $A^{(n)}_\infty$ to be the submodule of $A^{(n)}$ consisting of $a \in A^{(n)}$ such that there exist $b_i \in A^{(i)}$ for $i \geq n$ such that $b_n = a$ and $p^{(i,i+1)}\left(b_{i+1}\right) = b_i$ for $i \geq n$. Then $\boldsymbol{A}_\infty = \left(A^{(n)}_\infty\right)$ is a subtower of $\boldsymbol{A}$.*

Since $\boldsymbol{A}_\infty$ satisfies the Mittag-Leffler condition, $\lim^1\boldsymbol{A}_\infty = 0$, and the quotient $\boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{A}_\infty$ induces an isomorphism at the level of $\lim^1$ modules. We say that a tower $\boldsymbol{A}$ is *reduced* if $\boldsymbol{A}_\infty = 0$ or, equivalently, $\lim\boldsymbol{A} = 0$.

Notice that the full subcategory $\mathrm{P}\left(\mathbf{Mod}\left(R\right)\right)$ of $\mathrm{pro}(\mathbf{Mod}\left(R\right))$ spanned by reduced towers is a fully exact quasi-abelian category that is closed under subobjects but not under

quotients. Let $\boldsymbol{B}$ be a tower of countable modules, and $\boldsymbol{A}$ be a subtower of $\boldsymbol{B}$. Consider the quotient $\boldsymbol{B}/\boldsymbol{A}$ of $\boldsymbol{B}$ by $\boldsymbol{A}$ in $\mathbf{\Pi}(\mathbf{Mod}(R))$. Then we have an exact sequence

$$\lim \boldsymbol{B} \to \lim \boldsymbol{B}/\boldsymbol{A} \to \lim{}^1 \boldsymbol{A} \to \lim{}^1 \boldsymbol{B}.$$

Thus, if $\boldsymbol{B}$ is reduced, then $\boldsymbol{B}/\boldsymbol{A}$ is reduced if and only if the inclusion $\lim^1 \boldsymbol{A} \to \lim^1 \boldsymbol{B}$ is injective, in which case $\boldsymbol{B}/\boldsymbol{A}$ is the quotient of $\boldsymbol{B}$ by $\boldsymbol{A}$ in the category of reduced towers of countable modules.

5.7.2 Derived towers

Let $\boldsymbol{A}$ be a reduced tower of countable modules. We define a sequence $(\boldsymbol{A}_\alpha)$ of reduced subtowers of $\boldsymbol{A}$, indexed by countable ordinals, which we called the *derived towers* of $\boldsymbol{A}$. For every limit ordinal λ we fix a strictly increasing sequence (λ_n) of successor ordinals with $\sup_n \lambda_n = \lambda$. We also set $\alpha_k = \alpha$ if α is either zero or a successor ordinal.

Given a tower $\boldsymbol{A} = \left(A^{(n)}\right)$ we define $A_0^{(n)} = A^{(n)}$ and for each successor ordinal α,

$$A_\alpha^{(n)} := \bigcap_{k\in\omega} p^{(n,n+k)}(A_{(\alpha-1)_k}^{(n+k)}).$$

We then define for each ordinal α, $\boldsymbol{A}_\alpha$ to be the tower $(A_{\alpha_n}^{(n)})_{n\in\omega}$, which we call the α-th derived tower of $\boldsymbol{A}$.

Notice that for limit λ we have

$$A_{\lambda+1}^{(n)} = \bigcap_{\alpha<\lambda} A_\alpha^{(n)}.$$

Definition 5.54. *Let $\boldsymbol{A}$ be a reduced tower of countable modules, and $\alpha < \omega_1$. We say that $\boldsymbol{A}$ has (Mittag-Leffler)* length *at most α if $\boldsymbol{A}_\alpha \cong 0$. When α is a successor ordinal, we say that $\boldsymbol{A}$ has plain length at most α if $\boldsymbol{A}_{\alpha-1}$ is essentially monomorphic.*

A similar definition is also considered in [26, Definition 3.2]; see also [85].

Let X be the disjoint union of $A^{(n)}$ for $n \in \omega$. Then we have that the set $T_{\boldsymbol{A}}$ of $x = (a_0, \ldots, a_n)$ such that, for all $i \in \{0, 1, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$, $a_i \in A^{(i)}$ and $p^{(j-1,j)}(a_j) = a_{j-1}$, is a tree on X. Clearly, we have that $\boldsymbol{A}$ is reduced if and only if $T_{\boldsymbol{A}}$ is well-founded, and $\boldsymbol{A} \cong 0$ if and only if $T_{\boldsymbol{A}}$ is finite. By recursion on α one can see that

$$a_n \in A_\alpha^{(n)} \Leftrightarrow \left(p^{(0,n)}(a_n), \ldots, p^{(n-1,n)}(a_n), a_n\right) \in T_{\boldsymbol{A}}^{(\omega\alpha)}.$$

Thus, the length of $\boldsymbol{A}$ is at most α if and only if $T_{\boldsymbol{A}}^{(\omega\alpha)}$ is finite. This also shows the length defines a co-analytic rank on towers of countable modules.

Definition 5.55. *Let $\boldsymbol{A}$ be a reduced tower of countable modules, and $\alpha < \omega_1$. For $n \in \omega$, define the chain of submodules*

$$S^{(n)}(\boldsymbol{A}) = (p^{(n,k)}(A^{(k)}))_{k\geq n}$$

of $A^{(n)}$. We regard $S^{(n)}(\boldsymbol{A})$ as a monomorphic tower, where the bonding maps are the inclusion maps.

We let $L^{(n)}(\boldsymbol{A})$ be the completion of $A^{(n)}$ with respect to the chain of submodules $(S^{(n)}(\boldsymbol{A}))$ and $\kappa_{\boldsymbol{A}}^{(n)} : A^{(n)} \to L^{(n)}(\boldsymbol{A})$ be the canonical homomorphism. The cokernel $E^{(n)}(\boldsymbol{A}) = \mathrm{Coker}(\kappa_{\boldsymbol{A}}^{(n)})$ is then isomorphic to $\lim^1 S^{(n)}(\boldsymbol{A})$; see [22, Theorem 5.13]. This gives an epimorphic tower $(E^{(n)}(\boldsymbol{A}))$ of phantom pro-countable modules. By definition, $A_1^{(n)}$ is equal to $\mathrm{Ker}(\kappa_{\boldsymbol{A}}^{(n)})$. We have an exact sequence

$$0 \to \boldsymbol{A}_1 \to \boldsymbol{A} \to \lim_n S^{(n)}(\boldsymbol{A}) \to 0,$$

and hence an exact sequence

$$0 \to \lim^1 \boldsymbol{A}_1 \to \lim^1 \boldsymbol{A} \to \lim_n E^{(n)}(\boldsymbol{A}) \to 0.$$

If λ is a limit ordinal, then

$$\lim^1 \boldsymbol{A}_\lambda = \bigcap_{\beta<\lambda} \lim^1 \boldsymbol{A}_\beta \subseteq \lim^1 \boldsymbol{A}$$

and we have an exact sequence

$$0 \to \boldsymbol{A}_\lambda \to \boldsymbol{A} \to \lim_k \boldsymbol{A}/\boldsymbol{A}_{\lambda_k} \to 0.$$

The sequence of submodules $(\mathrm{Ker}(\lim^1 \boldsymbol{A} \to E^{(n)}(\boldsymbol{A})))_{n\in\omega}$ is sometimes called the *Gray filtration*; see [84].

5.7.3 The Solecki submodules of $\lim^1$

In this section we describe the Solecki submodules of the $\lim^1$ of a tower of countable modules in terms of its derived towers.

Proposition 5.56. *Suppose that $\boldsymbol{A} = \left(A^{(n)}\right)$ is a tower of countable modules. Then*

$$\lim^1 \boldsymbol{A}_1 = s_1\left(\lim^1 \boldsymbol{A}\right).$$

Proof. By the remarks in the previous section, we have that

$$\lim_n{}^1 A_1^{(n)} = \mathrm{Ker}\left(\lim_n{}^1 A^{(n)} \to \lim_n E_1(A^{(n)})\right).$$

For $n \in \omega$, since $A^{(n)}$ is countable, $\{0\}$ is $\mathbf{\Sigma}_2^0$ in $E_1(A^{(n)})$. As this holds for every $n \in \omega$, we have that $\{0\}$ is $\mathbf{\Pi}_3^0$ in $\lim_n E_1\left(A^{(n)}\right)$. Hence, $\lim_n{}^1 A_1^{(n)}$ is $\mathbf{\Pi}_3^0$ in $\lim_n{}^1 A^{(n)}$. We have that $\{0\}$ is dense in $\lim_n{}^1 A_1^{(n)}$ by Proposition 5.17(1). By Lemma 5.44, it remains to prove that if V is an open neighborhood of 0 in $\mathrm{B}^1(K(\boldsymbol{A}))$, then $\bar{V}^{\mathrm{Z}^1(K(\boldsymbol{A}))}$ contains an open neighborhood of 0 in $\mathrm{Z}^1(K(\boldsymbol{A}_1)) \subseteq \mathrm{Z}^1(K(\boldsymbol{A}))$. Without loss of generality, we can assume that there exists $N \in \omega$ such that

$$V = \left\{\left(p^{(n_0,n_1)}(b_{n_1}) - b_{n_0}\right) \in \mathrm{B}^1(K(\boldsymbol{A})) : \forall n \leq N, b_n = 0\right\}.$$

Consider now

$$U = \left\{ (a_{n_0,n_1}) \in \mathrm{Z}^1\left(K\left(\boldsymbol{A}_1\right)\right) : a_{n_0,n_1} \in A_1^{(n_0)}, \forall n_0, n_1 \leq N, a_{n_0,n_1+1} = 0 \right\}.$$

Then we have that U is an open neighborhood of 0 in $\mathrm{Z}^1\left(K\left(\boldsymbol{A}_1\right)\right)$.

We claim that $U \subseteq \bar{V}^{\mathrm{Z}^1(K(\boldsymbol{A}))}$. Suppose that

$$\tilde{x} = (\tilde{a}_{n_0,n_1}) \in U.$$

Thus, we have that $\tilde{a}_{n_0,n_1} = 0$ for $n_0, n_1 \leq N$. Fix an open neighborhood W of $\tilde{x}$ in $\mathrm{Z}^1\left(K\left(\boldsymbol{A}\right)\right)$. Then there exists $M > N$ such that W contains

$$\left\{ (a_{n_0,n_1}) \in \mathrm{Z}^1\left(K\left(\boldsymbol{A}\right)\right) : a_{n_0,n_1} \in A^{(n_0)}, \forall n_0, n_1 \leq M, a_{n_0,n_1} = \tilde{a}_{n_0,n_1} \right\}.$$

We need to find $b_n \in A^{(n)}$ for $n \in \omega$ such that:

1. $b_n = 0$ for $n \leq N$;
2. $p^{(n,n+1)}\left(b_{n+1}\right) - b_n = \tilde{a}_{n,n+1}$ for $n < M$.

This will give an element $\left(p^{(n,n+1)}\left(b_{n+1}\right) - b_n\right)_{n\in\omega}$ in $V \cap W$. Set $b_n = 0$ for $n \leq N+1$ and for $n > M$. We now define recursively $b_{N+2}, b_{N+2}, \ldots, b_M$ such that

$$b_n \in p^{(n,M)}(A^{(M)}) \subseteq A^{(n)}$$

and

$$p^{(n,n+1)}\left(b_{n+1}\right) = b_n + \tilde{a}_{n,n+1}$$

for $n < M$. Since $\tilde{a}_{N+1,N+2} \in A_1^{(N+1)}$ there exists $b_{N+2} \in p^{(N+1,M)}(A^{(M)})$ such that

$$p^{(N+1,N+2)}\left(b_{N+2}\right) = \tilde{a}_{N+1,N+2} = b_{N+1} + \tilde{a}_{N+1,N+2}.$$

Suppose that $b_{N+2}, \ldots, b_k$ have been defined for some $k < M$. Since $\tilde{a}_{k,k+1} \in A_1^{(N+1)}$, we have that

$$b_k + \tilde{a}_{k,k+1} \in p^{(k,M)}(A^{(M)}) + A_1^{(k)} \subseteq p^{(k,M)}(A^{(M)}).$$

Thus, there exists $b_{k+1} \in p^{(k+1,M)}\left(A^{(M)}\right)$ such that

$$p^{(k,k+1)}\left(b_{k+1}\right) = b_k + \tilde{a}_{k,k+1}.$$

This concludes the recursive definition, and the proof. □

Lemma 5.57. *Let $\boldsymbol{A} = \left(A^{(n)}\right)$ be a reduced tower of countable modules. The following assertions are equivalent:*

1. *$\boldsymbol{A}$ has plain length at most 1;*

2. $\lim^1 \boldsymbol{A}$ *has plain Solecki length at most* 1*;*

3. $\boldsymbol{A} \cong S_1^{(n)}(\boldsymbol{A})$ *for some* $n \in \omega$*.*

Proof. (1)⇒(2) If $\boldsymbol{A}$ is a monomorphic tower, then the $E^{(n)}(\boldsymbol{A}) \to E^{(0)}(\boldsymbol{A})$ is an isomorphism for every $n \in \omega$. Furthermore, by Proposition 5.56 the map $\lim^1 \boldsymbol{A} \to E^{(0)}(\boldsymbol{A})$ is an isomorphism. As $\{0\}$ is $\mathbf{\Sigma}^0_2$ in $E^{(0)}(\boldsymbol{A})$, the same holds for $\lim^1 \boldsymbol{A}$.

(2)⇒(1) If $\{0\}$ is $\mathbf{\Sigma}^0_2$ in $\lim^1 \boldsymbol{A}$, then $\boldsymbol{A}$ has length 1. As $\boldsymbol{A}$ is reduced, after passing to an isomorphic tower we can assume that for every $n \in \omega$, $\kappa^{(n)}_{\boldsymbol{A}} : A^{(n)} \to L^{(n)}(\boldsymbol{A})$ is injective. We can thus identify $A^{(n)}$ with its image inside $L^{(n)}(\boldsymbol{A})$. By Proposition 5.56, we have that $\{0\}$ is $\mathbf{\Sigma}^0_2$ in $\lim_n E^{(n)}(\boldsymbol{A})$. Hence, by Corollary 5.48, without loss of generality we can assume that for every $n \in \omega$, $E^{(n+1)}(\boldsymbol{A}) \to E^{(n)}(\boldsymbol{A})$ is an isomorphism. The fact that $E^{(1)}(\boldsymbol{A}) \to E^{(0)}(\boldsymbol{A})$ is injective implies that the closure of $N^{(1)} := \operatorname{Ker}\left(p^{(0,1)} : A^{(1)} \to A^{(0)}\right)$ within $L^{(1)}(\boldsymbol{A})$ is contained in $A^{(1)}$. Therefore, we have that the chain of submodules $\left(N^{(1)} \cap p^{(1,n)}\left(A^{(n)}\right)\right)_{n\in\omega}$ of $N^{(1)}$ is eventually constant. Hence, there exists $n_2 \geq 1$ such that $p^{(1,n_2)}\left(A^{(n_2)}\right) \cap N^{(1)} = \{0\}$. Proceeding in this fashion, we can define an increasing sequence (n_k) in ω with $n_0 = 0$ and $n_1 = 1$ such that $p^{(n_i,n_{i+1})}\left(A^{(n_{i+1})}\right) \cap \operatorname{Ker}\left(p^{(n_{i-1},n_i)}\right) = \{0\}$ for every $i \geq 1$. The tower $\left(p^{(n_i,n_{i+1})}\left(A^{(n_{i+1})}\right)\right)_{i\in\omega}$ is the desired monomorphic tower isomorphic to $\boldsymbol{A}$.

(1)∧(2)⇒(3) If $\boldsymbol{A}$ is a monomorphic tower, then $E^{(n)}(\boldsymbol{A}) \to E^{(0)}(\boldsymbol{A})$ is an isomorphism. Furthermore, if $\{0\}$ is $\mathbf{\Sigma}^0_2$ in $\lim^1 \boldsymbol{A}$ then by Proposition 5.56,

$$\operatorname{Ker}\left(\lim^1 \boldsymbol{A} \to E^{(0)}(\boldsymbol{A})\right) = \operatorname{Ker}\left(\lim^1 \boldsymbol{A} \to \lim_n E^{(n)}(\boldsymbol{A})\right) = 0$$

for every $n \in \omega$.

(3)⇒(2) This follows from the fact that $\{0\}$ is $\mathbf{\Sigma}^0_2$ in $E^{(n)}(\boldsymbol{A})$ for every $n \in \omega$. □

Theorem 5.58. *Suppose that* $\left(A^{(n)}\right)$ *is a tower of countable modules and* $\alpha < \omega_1$*. Then we have that*

$$s_\alpha\left(\lim^1 \boldsymbol{A}\right) = \lim^1 \boldsymbol{A}_\alpha.$$

Proof. We prove that the conclusion holds by induction on α. The limit case follows from the observation that, when λ is limit,

$$\lim^1 \boldsymbol{A}_\lambda = \bigcap_{\beta<\lambda} \lim^1 \boldsymbol{A}_\beta.$$

The successor stage follows from Proposition 5.56 applied to $\boldsymbol{A}_\alpha$. □

Corollary 5.59. *A reduced tower* $\boldsymbol{A}$ *has (plain) length at most* α *if and only if* $\lim^1 \boldsymbol{A}$ *has (plain) Solecki length at most* α*.*

For a tower $\boldsymbol{A} = \left(A^{(n)}\right)$, we let $\boldsymbol{A}^{\oplus\omega}$ be the tower whose n-th term is the countable direct sum of infinitely many copies of $A^{(n)}$, with bonding maps induced by those of $\boldsymbol{A}$. It

is easy to prove by induction on $\alpha < \omega$ that $(\boldsymbol{A}^{\oplus\omega})_\alpha = (\boldsymbol{A}_\alpha)^{\oplus\omega}$. Furthermore, $\boldsymbol{A}$ is plain if and only if $\boldsymbol{A}^{\oplus\omega}$ is plain. As a consequence of these observations and Corollary 5.59 we obtain the following.

Corollary 5.60. *If $\boldsymbol{A}$ is a reduced tower of countable modules, then $\{0\}$ has the same complexity class in $\lim^1 \boldsymbol{A}$ and in $\lim^1 \boldsymbol{A}^{\oplus\omega}$.*

Lemma 5.61. *Suppose that $\left(A^{(k)}\right)$ is a tower of countable modules, and $0 \to B \to A^{(k)} \to A^{(k)}/B \to 0$ is an exact sequence for every $k \in \omega$. Then the map $\lim^1_k A^{(k)} \to \lim^1_k (A^{(k)}/B)$ is an isomorphism, which restricts to an isomorphism $\lim^1_k A^{(k)}_\alpha \to \lim^1_k (A^{(k)}/B)_\alpha$ for every $\alpha < \omega_1$.*

Proof. It follows from the six-term exact sequence relating $\lim$ and $\lim^1$, considering that $\lim^1_k B^{(k)} = 0$ if $B^{(k)} = B$ for every $k \in \omega$. □

Lemma 5.62. *Suppose that $\boldsymbol{A}$ is a reduced tower of countable modules, and let $\boldsymbol{A}_\alpha$ be its derived tower for $\alpha < \omega_1$. For $\ell \leq i < \omega$ and $\alpha < \omega_1$, define*

$$A^{(i)}_\alpha[\ell] = \operatorname{Ker}(A^{(i)}_\alpha \to A^{(\ell)}_\alpha),$$

For fixed $\ell < \omega$ and $\alpha < \omega_1$ we let $\boldsymbol{A}_\alpha[\ell]$ be the tower $(A^{(i)}_\alpha[\ell])_{i \geq \ell}$. The canonical homomorphism $\lim^1 \boldsymbol{A}_\alpha[\ell] \to \lim^1 \boldsymbol{A}[\ell]$ is injective. For $\ell \leq i < \omega$ and $\alpha < \omega_1$, we have a short exact sequence

$$0 \to A^{(i)}_\alpha[\ell] \to A^{(i)}_\alpha \to \operatorname{Ran}(A^{(i)}_\alpha \to A^{(\ell)}_\alpha) \to 0.$$

Considering the six-term exact sequence relating $\lim$ and $\lim^1$, this gives rise to an exact sequence

$$\lim^1 A^{(i)}_\alpha \to A^{(\ell)}_{\alpha+1} \to \lim^1 \boldsymbol{A}_\alpha[\ell] \to \lim^1 \boldsymbol{A}_\alpha \to \lim^1_i \operatorname{Ran}(A^{(i)}_\alpha \to A^{(\ell)}_\alpha) \to 0.$$

We have that

$$\lim^1 \boldsymbol{A}_{\alpha+1} = \bigcap_{\ell \in \omega} \operatorname{Ker}(\lim^1 \boldsymbol{A}_\alpha \to \lim^1_i \operatorname{Ran}(A^{(i)}_\alpha \to A^{(\ell)}_\alpha)) = \bigcap_{\ell \in \omega} \operatorname{Ran}(\lim^1 \boldsymbol{A}_\alpha[\ell] \to \lim^1 \boldsymbol{A}_\alpha).$$

Proof. After replacing $\boldsymbol{A}$ with $\boldsymbol{A}_\alpha$, it suffices to consider the case $\alpha = 1$.

Suppose that a is an element of $\lim^1 \boldsymbol{A}_1$, represented by $(a_{ij}) \in \prod_{i \leq j} A^{(i)}_1$. We fix $\ell < \omega$ and show that a belongs to

$$\operatorname{Ker}(\lim^1 \boldsymbol{A} \to \lim^1_i \operatorname{Ran}(A^{(i)} \to A^{(\ell)})).$$

Consider the element $\left(p^{(\ell,i)}\,(a_{ij})\right)_{i \leq j} \in \prod_{\ell \leq i \leq j} p^{(\ell,i)}\left(A^{(i)}\right)$. Using the fact that

$$a_{ij} \in A^{(i)}_1 = \bigcap_{t > i} p^{(i,t)}(A^{(t)})$$

we can define recursively $b_i \in A^{(i)}$ for $i \geq \ell$ such that $b_\ell = 0$ and

$$p^{(\ell,i)}\left(a_{i,i+1}\right) = p^{(\ell,i)}\left(b_i\right) - p^{(\ell,i+1)}\left(b_{i+1}\right)$$

for $i \geq \ell$. This shows that $\left(p^{(\ell,i)}\left(a_{ij}\right)\right)_{i\leq j}$ represents the trivial element of $\lim_i^1 \mathrm{Ran}\left(A^{(i)} \to A^{(\ell)}\right)$. Therefore, a belongs to

$$\mathrm{Ker}(\lim{}^1 \boldsymbol{A} \to \lim_i^1 \mathrm{Ran}(A^{(i)} \to A^{(\ell)})).$$

Conversely, suppose that a is an element of

$$\bigcap_{\ell\in\omega} \mathrm{Ker}(\lim{}^1 \boldsymbol{A} \to \lim_i^1 p^{(\ell,i)}(A^{(i)})),$$

represented by $(a_{ij}) \in \prod_{i\leq j} A^{(i)}$. Fix $\ell \in \omega$ and define $\hat{A}^{(\ell)} = \lim_i(A^{(\ell)}/p^{(\ell,i)}\left(A^{(i)}\right))$. We also let $\kappa^{(\ell)} : A^{(\ell)} \to \hat{A}^{(\ell)}$ be the canonical homomorphism obtained by mapping a to $\left(a + p^{(\ell,i)}\left(A^{(i)}\right)\right)_{i\geq\ell}$. Consider the isomorphism

$$\lim_i^1 p^{(\ell,i)}(A^{(i)}) \cong \hat{A}^{(\ell)}/\kappa(A^{(\ell)})$$

obtained by mapping an element b of $\lim_i^1 p^{(\ell,i)}\left(A^{(i)}\right)$ represented by (b_{ij}) to $(\lim_j \kappa\left(b_{\ell j}\right)) + \kappa\left(A^{(\ell)}\right)$; see [22, Theorem 5.13]. Since (a_{ij}) represents an element of $\mathrm{Ker}(\lim{}^1 \boldsymbol{A} \to \lim_i^1 p^{(\ell,i)}\left(A^{(i)}\right))$, we have that $\left(p^{(\ell,i)}\left(a_{ij}\right)\right)$ represents the trivial element in $\lim_i^1 p^{(\ell,i)}\left(A^{(i)}\right)$. Thus, there exists $c_\ell \in A^{(\ell)}$ such that $\lim_j \kappa^{(\ell)}(a_{\ell j}) = \kappa^{(\ell)}(c_\ell)$. This means that

$$\kappa^{(\ell)}\left(a_{\ell,\ell+1}\right) = \lim_i \kappa^{(\ell)}(p^{(\ell,i)}\left(a_{\ell i}\right)) - \lim_i \kappa^{(\ell)}(p^{(\ell,\ell+1)}\left(a_{\ell+1,i}\right)) = \kappa^{(\ell)}\left(c_\ell - p^{(\ell,\ell+1)}\left(c_{\ell+1}\right)\right).$$

Thus,

$$b_{\ell,\ell+1} := a_{\ell,\ell+1} - (c_\ell - p^{(\ell,\ell+1)}\left(c_{\ell+1}\right)) \in \mathrm{Ker}(\kappa^{(\ell)}) = A_1^{(\ell)}.$$

As $(b_{\ell,\ell+1})$ and (a_{ij}) represent the same element $\lim{}^1 \boldsymbol{A}$, this shows that a belongs to $\lim{}^1 \boldsymbol{A}_1$. □

5.7.4 Fishbone towers

We now present a way to construct new towers from old. In order to present this construction more succinctly, we introduce the notion of *fishbone* of towers.

Definition 5.63. *Consider a nontrivial reduced essentially monomorphic tower* $\boldsymbol{A} = \left(A^{(n)}\right)_{n\in\omega}$ *and reduced towers* $\boldsymbol{B}[k] = \left(B^{(n)}[k]\right)_{n\in\omega}$ *for* $k \in \omega$.

- *We say that* $(\boldsymbol{A}, \boldsymbol{B}[k])_{k\in\omega}$ *is a* fishbone *with spine* $\boldsymbol{A}$ *and ribs* $\boldsymbol{B}[k]$ *for* $k \in \omega$ *if* $A^{(n)} = B^{(0)}[n]$ *for every* $n \in \omega$.

- *If α is a successor ordinal, then we say that such a fishbone is* straight *of length α if for every $k \in \omega$, $\boldsymbol{B}[k]$ has plain length $\beta[k] < \alpha$, $\sup_k \beta[k] = \alpha - 1$, and*

$$p^{(\ell,0)}(B^{(\ell)}_{\beta[k]-1}[k]) + p^{(k,\ell)}(A^{(\ell)}) = A^{(k)}$$

for every $\ell \geq k$.

- *Suppose that $(\boldsymbol{A}, \boldsymbol{B}[k])_{k\in\omega}$ is a* fishbone *with spine $\boldsymbol{A}$ and ribs $\boldsymbol{B}[k]$ for $k \in \omega$. We define the corresponding* fishbone tower *to be the tower $\left(C^{(n)}\right)_{n\in\omega}$ where*

$$C^{(n)} := B^{(n)}[0] \oplus B^{(n-1)}[1] \oplus \cdots \oplus B^{(1)}[n-1] \oplus A^{(n)}$$

and bonding maps $C^{(n+1)} \to C^{(n)}$,

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto \left(p^{B[0]}(x_0), p^{B[1]}(x_1), \ldots, p^{B[n-1]}(x_{n-1}), p^{B[n]}(x_n) + p^{A}(x_{n+1})\right).$$

The notion of straight fishbone allows one to easily compute the derived towers.

Proposition 5.64. *Suppose that $(\boldsymbol{A}, \boldsymbol{B}[k])_{k\in\omega}$ is a fishbone with spine $\boldsymbol{A}$ and ribs $\boldsymbol{B}[k]$ for $k \in \omega$. Let $\boldsymbol{C}$ be the corresponding fishbone tower. Suppose that $(\boldsymbol{A}, \boldsymbol{B}[k])_{k\in\omega}$ is* straight *of length α. Let $\beta[k]$ be the plain length of $\boldsymbol{B}[k]$.*

1. *If $k \in \omega$, $\beta[k] \leq \beta < \beta[k+1]$, then for $n \leq k$,*

$$C^{(n)}_\beta = 0 \oplus \cdots \oplus 0 \oplus p^{(n,k)}(A^{(k)})$$

and for $n > k$,

$$C^{(n)}_\beta = B^{(n)}_\beta[0] \oplus \cdots \oplus B^{(1)}_\beta[n-1] \oplus A^{(n)};$$

2. *$\boldsymbol{C}$ has plain length α.*

Proof. Recall that given a tower $\boldsymbol{D} = \left(D^{(n)}\right)$ we have defined $D^{(n)}_0 = D^{(n)}$ and for each successor ordinal β

$$D^{(n)}_\beta := \bigcap_{i\in\omega} p^{(n,n+i)}(A^{(n+i)}_{(\beta-1)_i}).$$

(1) Suppose that the conclusion holds for ordinals less then β. Suppose that $\beta[k] \leq \beta < \beta[k+1]$. By definition we have

$$C^{(n)}_\beta = \bigcap_{i\in\omega} p^{(n,n+i)}(C^{(n+i)}_{(\beta-1)_{n+i}}).$$

Suppose initially that $\beta - 1 > \beta[k]$. In this case for $n > k$ we have by the inductive hypothesis

$$C^{(n)}_\beta = B^{(n)}_\beta[0] \oplus B^{(n-1)}_\beta[1] \oplus \cdots \oplus B^{(1)}_\beta[n-1] \oplus A^{(n)}$$

as above, and for $n \leq k$ by the inductive hypothesis

$$C_\beta^{(n)} = \bigcap_{i\in\omega} p^{(n,n+i)}(C_{(\beta-1)_i}^{(n+i)}) = 0 \oplus \cdots \oplus 0 \oplus p^{(n,k)}(A^{(k)}).$$

Suppose now that $\beta - 1 = \beta[k]$. This means that $\beta - 1$ is a successor ordinal. Therefore we have $(\beta-1)_i = \beta[k]$ for every $i \in \omega$. Therefore as before we conclude from the inductive hypothesis that, for $n > k$,

$$C_\beta^{(n)} = B_\beta^{(n)}[0] \oplus B_\beta^{(n-1)}[1] \oplus \cdots \oplus B_\beta^{(1)}[n-1] \oplus A^{(n)}$$

and for $n \leq k$,

$$C_\beta^{(n)} = 0 \oplus \cdots \oplus 0 \oplus p^{(n,k)}(A^{(k)}).$$

(2) It follows from (1) that $\boldsymbol{C}_{\alpha-1} \cong \boldsymbol{A}$, whence $\boldsymbol{C}$ has plain length α. □

5.8 Phantom morphisms

In this section, we recall the notion of *phantom morphism* in a triangulated category. We also consider the natural analogue in the context of triangulated category of the notion of *higher order phantom map* from topology.

5.8.1 Phantom morphisms

Suppose that $\mathcal{T}$ is a triangulated category with translation functor Σ. An object C of $\mathcal{T}$ is *compact* [24, Section 3] if, for every sequence $(X_i)_{i\in\omega}$ of objects of $\mathcal{T}$, if X is its coproduct, then

$$\mathrm{Hom}(C, X) \cong \bigoplus_{i\in\omega} \mathrm{Hom}\,(C, X_i)\,.$$

Let $\mathcal{C}$ be a class of compact objects in $\mathcal{T}$. As the functor $\mathrm{Hom}\,(-, Y)$ is triangulated, a distinguished triangle

$$C \to X \to X/C$$

in $\mathcal{T}$ induces an exact sequence

$$\mathrm{Hom}\,(X/C, Y) \to \mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(C, Y)\,.$$

Given objects X, Y in $\mathcal{T}$, the collection $\mathrm{PhHom}\,(X, Y)$ of *phantom morphisms* is the intersection

$$\begin{aligned}\mathrm{PhHom}\,(X, Y) &= \bigcap \mathrm{Ker}(\mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(C, Y)) \\ &= \bigcap \mathrm{Ran}\,(\mathrm{Hom}\,(X/C, Y) \to \mathrm{Hom}\,(X, Y))\end{aligned}$$

ranging over the distinguished triangles $C \to X \to X/C$ with C in $\mathcal{C}$; see [42, Section 5]. This is a notion inspired by the notion of *phantom map* from topology; see [128]. One can likewise generalize the notion of *higher order* phantom map introduced in the topological context in [85].

Definition 5.65. *We define by recursion on an ordinal α the set $\mathrm{Ph}^{\alpha}\mathrm{Hom}(X,Y)$ of phantom morphisms of order α from X to Y by setting*

$$\mathrm{Ph}^{0}\mathrm{Hom}(X,Y) = \mathrm{PhHom}(X,Y);$$

$$\mathrm{Ph}^{\alpha+1}\mathrm{Hom}(X,Y) = \bigcap \mathrm{Ran}\left(\mathrm{Ph}^{\alpha}\mathrm{Hom}(X/C,Y) \to \mathrm{Hom}(X,Y)\right)$$

where the intersection is ranging over the distinguished triangles $C \to X \to X/C$ with C in $\mathcal{C}$;

$$\mathrm{Ph}^{\lambda}\mathrm{Hom}(X,Y) = \bigcap_{\beta<\lambda} \mathrm{Ph}^{\beta}\mathrm{Hom}(X,Y)$$

for λ limit.

Suppose that $\boldsymbol{X} = (X_i, \phi_i : X_i \to X_{i+1})_{i\in\omega}$ is an inductive sequence of compact objects in $\mathcal{T}$. An object $\mathrm{hocolim}\boldsymbol{X}$ of $\mathcal{T}$ is called a *homotopy colimit* of $\boldsymbol{X}$ if there exists a distinguished triangle

$$\bigoplus \boldsymbol{X} \overset{\mathrm{id}-\phi}{\to} \bigoplus \boldsymbol{X} \to \mathrm{hocolim}\boldsymbol{X}$$

where $\bigoplus \boldsymbol{X}$ is the coproduct of $(X_i)_{i\in\omega}$ with canonical maps $\eta_i : X_i \to \bigoplus \boldsymbol{X}$ for $i \in \omega$, and $\phi : \bigoplus \boldsymbol{X} \to \bigoplus \boldsymbol{X}$ is such that

$$\phi \circ \eta_i = \eta_{i+1} \circ \phi_i$$

for $i \in \omega$.

If C is a compact object in $\mathcal{T}$, and X is the homotopy colimit of $\boldsymbol{X}$, then by [107, Lemma 3.4.3] we have that $\mathrm{Hom}(C,X)$ is the colimit of $(\mathrm{Hom}(C,X_i))_{i\in\omega}$. Thus, we have that

$$\begin{aligned}\mathrm{PhHom}(X,Y) &= \bigcap_{\ell\in\omega} \mathrm{Ker}(\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X_\ell,Y)) \\ &= \bigcap_{\ell\in\omega} \mathrm{Ran}\left(\mathrm{Hom}(X/X_\ell,Y) \to \mathrm{Hom}(X,Y)\right).\end{aligned}$$

Furthermore, by [107, Lemma 5.2.6], the exact triangle defining the homotopy colimit induces a natural exact sequence

$$0 \to \mathrm{PhHom}(X,Y) \to \mathrm{Hom}(X,Y) \to \lim_n \mathrm{Hom}(X_n,Y) \to 0$$

and a natural isomorphism

$$\mathrm{PhHom}(X,Y) \cong \lim\nolimits^1_\ell \mathrm{Hom}\left(X_\ell, \Sigma^{-1}Y\right) \cong \lim\nolimits^1_\ell \mathrm{Hom}(\Sigma X_\ell, Y).$$

5.8.2 Phantom cohomological subfunctors

Suppose that $\mathcal{T}$ is a triangulated category, $\mathcal{C}$ is the class of compact objects of $\mathcal{T}$, $\mathcal{M}$ is a *countably complete* abelian category, and $F : \mathcal{T} \to \mathcal{M}$ is a contravariant cohomological functor. We generalize the notion of (higher order) phantom morphisms to obtain a chain of *phantom subfunctors* of F.

Define $\mathrm{Pro}(\mathcal{M})$ to be the pro-category of $\mathcal{M}$, which is the opposite category of $\mathrm{Ind}(\mathcal{M}^{\mathrm{op}})$ [100, Section 6.1]. Then we have that $\mathrm{Pro}(\mathcal{M})$ is still abelian and contains $\mathcal{M}$ as a thick subcategory [100, Proposition 8.6.11, Theorem 8.6.5]. Given an object F of X, the phantom subobject $\mathrm{Ph}F(X)$ of $F(X)$ (which is computed in $\mathrm{Pro}(\mathcal{M})$) is defined to be

$$\bigcap \mathrm{Ker}(F(X) \to F(C)) = \bigcap \mathrm{Ran}(F(X/C) \to F(X))$$

ranging over the distinguished triangles $C \to X \to X/C$ with C in $\mathcal{C}$. Recursively, one defines the phantom subobject of order α for every ordinal α by setting

$$\mathrm{Ph}^{\alpha+1}F(X) := \bigcap \mathrm{Ran}(\mathrm{Ph}^{\alpha}F(X/C) \to F(X))$$

ranging over the distinguished triangles $C \to X \to X/C$, and for a limit ordinal λ

$$\mathrm{Ph}^{\lambda}F(X) := \bigcap_{\beta<\lambda} \mathrm{Ph}^{\beta}F(X).$$

This defines for each ordinal α a subfunctor $\mathrm{Ph}^{\alpha}F$ of F.

Suppose that $\boldsymbol{X} = (X_i, \phi_i : X_i \to X_{i+1})$ is an inductive sequence of compact objects in $\mathcal{T}$, with homotopy colimit $X := \mathrm{hocolim}\boldsymbol{X}$. The same argument as in the case of the functor $\mathrm{Hom}(-, Y)$ shows that

$$\mathrm{Ph}F(X) = \bigcap_{\ell\in\omega} \mathrm{Ker}(F(X) \to F(X_i)) = \bigcap_{\ell\in\omega} \mathrm{Ran}(F(X/X_\ell) \to F(X))$$

Likewise, we have that for every ordinal α:

$$\mathrm{Ph}^{\alpha+1}F(X) := \bigcap_{\ell\in\omega} \mathrm{Ran}(\mathrm{Ph}^{\alpha}F(X/X_\ell) \to F(X))$$

As $\mathcal{M}$ is countably complete, this guarantees that $\mathrm{Ph}^{\alpha}F(X)$ is (naturally isomorphic to) an object of $\mathcal{M}$ for every countable ordinal α.

Furthermore, the exact triangle defining X as the homotopy colimit of $\boldsymbol{X}$ induces a natural exact sequence

$$0 \to \mathrm{Ph}F(X) \to F(X) \to \lim\nolimits_n F(X_n) \to 0$$

and a natural isomorphism

$$\mathrm{Ph}F(X) \cong \lim\nolimits^1_\ell F(\Sigma X_\ell).$$

Theorem 5.66. *Let R be a countable domain. Suppose that $\mathcal{T}$ is a triangulated category, and*

$$F : \mathcal{T} \to \mathrm{LH}(\mathbf{\Pi}(\mathbf{Mod}(R)))$$

is a contravariant cohomological functor. Suppose that $\boldsymbol{X} = (X_i, \phi_i : X_i \to X_{i+1})$ is an inductive sequence of compact objects in $\mathcal{T}$, with homotopy colimit $X := \mathrm{hocolim}\boldsymbol{X}$. If $F(X_i)$ is countable for every $i \in \omega$, then the isomorphism

$$\mathrm{Ph}F(X) \cong \lim\nolimits^1_\ell F(\Sigma X_\ell)$$

maps $\mathrm{Ph}^{\alpha}F(X)$ to $s_\alpha\left(\lim^1_\ell F(\Sigma X_\ell)\right)$ for $\alpha < \omega_1$. As a consequence, $\mathrm{Ph}^{\alpha}F(X) = s_\alpha(F(X))$ for every $\alpha < \omega_1$.

Proof. Define $F^n := F \circ \Sigma^n$ for $n \in \mathbb{Z}$. Since $\{0\}$ is dense in $\lim^1_\ell F^1(X_\ell)$ and $\operatorname{Ker}(F(X) \to \lim_n F(X_n))$ is closed, we have that the latter is equal to $s_0(F(X))$. Without loss of generality, we can assume $X_0 = 0$. Fix $\ell \in \omega$ and for $i \geq \ell$ consider distinguished triangles

$$X_\ell \to X_i \to X_i/X_\ell.$$

Then we have an exact sequence

$$F^2(X_\ell) \to F^1(X_i/X_\ell) \to F^1(X_i) \to F^1(X_\ell).$$

It follows from the Octahedral Axiom of triangulated categories [100, Definition 10.1.6 (TR5)] that X/X_ℓ is the homotopy colimit of the inductive sequence $(X_i/X_\ell)_{i\geq\ell}$. Thus, we have a natural isomorphism

$$F(X/X_\ell) \cong \lim^1_{i\geq\ell} F^1(X_i/X_\ell).$$

We prove by induction on $\alpha < \omega_1$ that under such an isomorphism,

$$\mathrm{Ph}^\alpha F(X/X_\ell)$$

corresponds to

$$s_\alpha\left(\lim^1_i F^1(X_i/X_\ell)\right).$$

Suppose that the conclusion holds for α. Define $\boldsymbol{A}$ to be the tower $(F^1(X_i))_{i\in\omega}$. Adopting the notation from Lemma 5.62, we set

$$\begin{aligned}\boldsymbol{A}[\ell] &= \left(\operatorname{Ker}\left(F^1(X_i) \to F^1(X_\ell)\right)\right)_{i\geq\ell} \\ &= \left(\operatorname{Ran}\left(F^1(X_i/X_\ell) \to F^1(X_i)\right)\right)_{i\geq\ell}.\end{aligned}$$

Then we have by Lemma 5.62,

$$\begin{aligned}s_{\alpha+1}\left(\lim^1_i \boldsymbol{A}\right) &= \bigcap_{\ell\in\omega} \operatorname{Ran}\left(s_\alpha(\lim{}^1 \boldsymbol{A}[\ell]) \to \lim{}^1 \boldsymbol{A}\right) \\ &\cong \bigcap_{\ell\in\omega} \operatorname{Ran}\left(s_\alpha(\lim{}^1 \frac{F^1(X_i/X_\ell)}{F^2(X_\ell)}) \to \lim{}^1 \boldsymbol{A}\right) \\ &\cong \bigcap_{\ell\in\omega} \operatorname{Ran}\left(s_\alpha(\lim{}^1 F^1(X_i/X_\ell)) \to \lim{}^1 \boldsymbol{A}\right) \\ &\cong \bigcap_{\ell\in\omega} \operatorname{Ran}\left(\mathrm{Ph}^\alpha F(X/X_\ell) \to F(X)\right) \\ &= \mathrm{Ph}^{\alpha+1} F(X).\end{aligned}$$

The case of limit stages is trivial. □

5.8.3 Phantom extensions in derived categories

Suppose that $\mathcal{M}$ is a category of modules, and $\mathcal{A}$ is an idempotent-complete hereditary exact $\mathcal{M}$-category with enough projectives; see Section 5.2.15. Then we have that $\mathrm{D}^b(\mathcal{A})$ is a triangulated $\mathrm{LH}(\mathcal{M})$-category. Fix a bounded complex X over $\mathcal{A}$. As $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}(-, X) : \mathrm{D}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{M})$ is a cohomological functor, we can consider for every ordinal α its α-th phantom subfunctor

$$\mathrm{Ph}^{\alpha}\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}(-, X) : \mathrm{D}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{M}).$$

For objects A, B of $\mathcal{A}$, recall that one sets

$$\mathrm{Ext}(A, B) := \mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}(A, B[1]).$$

We define its α-th phantom subobject

$$\mathrm{Ph}^{\alpha}\mathrm{Ext}(A, B) := \mathrm{Ph}^{\alpha}\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}(A, B[1]).$$

As a particular instance of [107, Lemma 5.2.6] and Theorem 5.66 we have the following:

Theorem 5.67. *Suppose that $\mathcal{A}$ is an idempotent-complete hereditary exact $\mathbf{\Pi}(\mathbf{Mod}(R))$-category with enough projectives. Suppose that $\boldsymbol{X} = (X_i, \phi_i : X_i \to X_{i+1})$ is an inductive sequence of compact objects in $\mathcal{A}$ with colimit $X := \mathrm{colim}\boldsymbol{X}$, and Y is an object of A. If $\mathrm{Hom}(X_i, Y)$ is countable for every $i \in \omega$, then we have an exact sequence*

$$0 \to \mathrm{PhExt}(X, Y) \to \mathrm{Ext}(X, Y) \to \lim_n \mathrm{Ext}(X_n, Y) \to 0$$

and an isomorphism

$$\mathrm{PhExt}(X, Y) \cong \lim\nolimits^1_{\ell} \mathrm{Hom}(X_{\ell}, Y)$$

in $\mathrm{LH}(\mathbf{\Pi}(\mathbf{Mod}(R)))$. *Under such an isomorphism,* $\mathrm{Ph}^{\alpha}\mathrm{Ext}(X, Y)$ *corresponds to* $s_{\alpha}(\mathrm{Ext}(X, Y))$ *for every* $\alpha < \omega_1$.

5.9 The functors Ext and PExt

In this section we assume that R is a countable Prüfer domain, and consider all modules to be R-modules.

5.9.1 Extensions of countable modules

We let $\mathbf{Mod}_{\aleph_0}(R)$ be the category of countably-presented modules. For countably-presented modules C, A, we regard $\mathrm{Hom}(C, A)$ as a pro-countably-presented Polish module endowed with the topology of pointwise convergence, where A and C have the discrete topology. This turns $\mathbf{Mod}_{\aleph_0}(R)$ into an abelian $\mathbf{\Pi}(\mathbf{Mod}_{\aleph_0}(R))$-category with enough projectives.

(See Section 5.2.15 for the notion of abelian $\mathcal{M}$-category, where $\mathcal{M}$ is a category of modules.) By Proposition 5.14 we have that the triangulated functor

$$\mathrm{Hom}^{\bullet} : \mathrm{K}^{-}(\mathbf{Mod}_{\aleph_0}(R))^{\mathrm{op}} \times \mathrm{K}^{+}(\mathbf{Mod}_{\aleph_0}(R)) \to \mathbf{\Pi}(\mathbf{Mod}_{\aleph_0}(R))$$

has a total right derived functor

$$\mathrm{RHom}^{\bullet} : \mathrm{D}^{-}(\mathbf{Mod}_{\aleph_0}(R))^{\mathrm{op}} \times \mathrm{D}^{+}(\mathbf{Mod}_{\aleph_0}(R)) \to \mathrm{D}^{+}(\mathbf{\Pi}(\mathbf{Mod}_{\aleph_0}(R))).$$

We let $\mathrm{Ext}^n = \mathrm{H}^n \circ \mathrm{RHom}^{\bullet}$ for $n \geq 0$.

Suppose that $\mathcal{C}$ is the class of *finitely-presented* modules. We let $\mathcal{E}_0$ be the exact structure *projectively generated* by $\mathcal{C}$ on the abelian category $\mathbf{Mod}_{\aleph_0}(R)$. The sequences in $\mathcal{E}_0$ are precisely the *pure short-exact* sequences of countable modules; see [145, Theorem 3.69]. An admissible arrow in this exact category is called *pure-admissible*. A countable module is projective in $(\mathbf{Mod}_{\aleph_0}(R), \mathcal{E}_0)$, in which case it is called *pure-projective*, if and only if it is a direct summand of a countable direct sum of finitely-presented modules [14, Theorem 2.2.3]. The exact category $(\mathbf{Mod}_{\aleph_0}(R), \mathcal{E}_0)$ is hereditary with enough projectives [14, Section 2.6]. We let PExt^n for $n \geq 0$ be the corresponding derived functor of Hom. We have that $\mathrm{PExt}^n = 0$ for $n \geq 2$. We write PExt for PExt^1. Furthermore, if C is a countable flat module and A is any countable module, then $\mathrm{Ext}^n(C, A) = \mathrm{PExt}^n(C, A)$ for every $n \geq 0$ [14, Proposition 2.7.1]. In this case, we write $\mathrm{Ext}(C, -)$ for $\mathrm{Ext}^1(C, -)$.

Lemma 5.68. *Let C be a finitely-presented module and A be a countable flat module. Then $\mathrm{Ext}^1(C, A)$ is countably presented.*

Proof. By Lemma 5.28, without loss of generality we can assume that $C = L/I$ is a cyclic torsion module, for some finite ideals $I \subseteq L \subseteq R$. Considering the short exact sequence

$$0 \to I \to L \to C \to 0$$

we have an induced exact sequence

$$\mathrm{Hom}(L, A) \to \mathrm{Hom}(I, A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(L, A) = 0.$$

Thus, $\mathrm{Ext}(C, A)$ is countably presented. □

We let $\mathbf{Flat}(R)$ be the full subcategory of $\mathbf{Mod}(R)$ spanned by the countable *flat* modules. A short exact sequence in $\mathbf{Flat}(R)$ is necessarily pure. By Lemma 5.69, the projective objects in $\mathbf{Flat}(R)$ are precisely the projective countable modules. As any countable flat module is countably presented, $\mathbf{Flat}(R)$ is a *hereditary* quasi-abelian category with enough projectives; see [14, Corollary 2.7.3].

Lemma 5.69. *Let M be a countable torsion-free module. Then M is pure-projective if and only if it is projective.*

Proof. Since R is a Prüfer domain, M is the colimit of a sequence of finite projective modules. Whence there is a pure short-exact sequence

$$0 \to P \to Q \to M \to 0$$

where P and Q are projective. If M is pure-projective, then M is isomorphic to a direct summand of Q, whence it is projective. □

By definition, for countably-presented modules C, A, $\mathrm{Ph}^0\mathrm{Ext}^1(C,A)$ is the submodule $\mathrm{PExt}(C,A)$ of $\mathrm{Ext}^1(C,A)$ parametrizing *pure* extensions, which can be identified with the derived functor $\mathrm{Ext}_{\mathcal{E}_0}$ of Hom on $(\mathbf{Mod}(R), \mathcal{E}_0)$. When both C and A are *flat*, one has that $\mathrm{PExt}^1(C,A) = \mathrm{Ext}^1(C,A)$; see [14, Proposition 2.7.1]. If A is flat, then it follows from [107, Lemma 5.2.6] and Lemma 5.68 that $\mathrm{PExt}(C,A)$ is the closure of the trivial submodule in $\mathrm{Ext}^1(C,A)$. Furthermore, if $C \cong \mathrm{colim}_n C_n$, where (C_n) is an inductive sequence of finitely-presented modules, then

$$\mathrm{PExt}(C,A) \cong \lim\nolimits^1_n \mathrm{Hom}(C_n, A);$$

see also [14, Theorem 2.6.1]. If C is a countably-presented torsion module, and A is a countable flat module, then $\mathrm{Hom}(C_n, A) = 0$ for every $n \in \omega$, and hence $\mathrm{Ext}^1(C,A) \cong \lim_n \mathrm{Ext}^1(C_n, A)$ is a pro-countably-presented Polish module.

Lemma 5.70. *Suppose that C, A are countably-presented modules, with A flat. Then the quotient map $C \to C/C_{\mathrm{t}}$ induces an injective homomorphism $\mathrm{Ext}(C/C_{\mathrm{t}}, A) \to \mathrm{Ext}(C,A)$ whose image is equal to $\mathrm{PExt}(C,A)$.*

Proof. Notice that $\mathrm{Hom}(C_{\mathrm{t}}, A) = 0$. Furthermore, $\mathrm{Ext}(C_{\mathrm{t}}, A)$ is a pro-countably-presented Polish module. Thus, the homomorphism $\mathrm{Ext}(C/C_{\mathrm{t}}, A) \to \mathrm{Ext}(C,A)$ is injective with closed image. Since $\{0\}$ is dense in $\mathrm{Ext}(C/C_{\mathrm{t}}, A)$, being C/C_{t} flat, the conclusion follows. □

Corollary 5.71. *Suppose that C, A are countably-presented modules, with A flat. Then the quotient map $C \to C/C_{\mathrm{t}}$ induces an isomorphism $\mathrm{Ph}^\alpha\mathrm{Ext}(C/C_{\mathrm{t}}, A) \to \mathrm{Ph}^\alpha\mathrm{Ext}(C,A)$ for every $\alpha \in \omega_1$. Furthermore, $\{0\}$ has the same complexity class in $\mathrm{Ext}^1(C,A)$ and $\mathrm{Ext}^1\left(C, A^{(\omega)}\right)$.*

Proof. The first assertion is an immediate consequence of Lemma 5.70.

Let (C_n) be an increasing sequence of finite submodules of C with union equal to C. Then we have that $\mathrm{PExt}(C,A) \cong \lim^1_n \mathrm{Hom}(C_n, A)$ is the closure of $\{0\}$ in $\mathrm{Ext}(C,A)$. Thus, the complexity class of $\{0\}$ in $\mathrm{Ext}^1(C,A)$ is the same as the complexity class of $\{0\}$ in $\lim^1_n \mathrm{Hom}(C_n, A)$. The same holds replacing A with $A^{(\omega)}$. Thus, the conclusion follows from Corollary 5.60. □

Corollary 5.72. *Suppose that C is a countably-presented module. If $\mathrm{Ext}(C,R)$ has (plain) Solecki length at most α, then the same holds for $\mathrm{Ext}(C,P)$ for any countable projective nontrivial module.*

Proof. By Corollary 5.71, if $\mathrm{Ext}\,(C,R)$ has (plain) Solecki length at most α, then the same holds for $\mathrm{Ext}\left(C,R^{(\omega)}\right)$. As P is a direct summand of $R^{(\omega)}$, the same holds for $\mathrm{Ext}\,(C,P)$. □

The module with a Polish cover $\mathrm{Ext}^1\,(C,A)$ is naturally isomorphic to the module with a Polish cover $\mathrm{Ext}_{\mathrm{Yon}}\,(C,A)$ of continuous cocycles modulo coboundaries as defined in Section 5.4.2. The isomorphism is defined by mapping a given extension to the corresponding cocycle; see [69, Section II.2].

Suppose that C,A are countable flat modules. Let $(C_n)_{n\in\omega}$ be an increasing sequence of submodules of C. For every $n\in\omega$, the inclusion map $\iota_n : C_n \to C$ induces a surjective homomorphism $\mathrm{Ext}\,(C,A)\to\mathrm{Ext}\,(C_n,A)$. Together, these induce a homomorphism $\zeta:\mathrm{Ext}\,(C,A)\to\lim_{n\in\omega}\mathrm{Ext}\,(C_n,A)$. Roos' Theorem expresses the kernel of such a homomorphism as the $\lim^1$ of the tower of pro-countable modules $(\mathrm{Hom}\,(C_n,A))_{n\in\omega}$; see [154, Theorem 6.2].

Theorem 5.73 (Roos). *Adopt the notation above. Then we have an exact sequence*

$$0\to\lim\nolimits^1_n\mathrm{Hom}\,(C_n,A)\to\mathrm{Ext}\,(C,A)\to\lim\nolimits_n\mathrm{Ext}\,(C_n,A)\to 0$$

in $\mathrm{LH}\,(\mathbf{\Pi}\,(\mathbf{Mod}\,(R)))$.

5.9.2 Pure extensions

In this section we assume that R is a PID. For an element x of a module M we let its order $\mathfrak{o}\,(x)$ to be the kernel of $R\to M$, $\lambda\mapsto\lambda x$. If N is a submodule of M then we define $\mathfrak{o}\,(x)\ \mathrm{mod}N$ to be the order of $x+N$ in M/N. The notion of pure extension can be phrased in terms of liftings of elements of given order. Thus, an extension

$$A\to X\xrightarrow{\pi}C$$

is *pure* if for every $c\in C$ there exist a *lift* $\hat{c}\in X$ of c (so that $\pi(\hat{c})=c$) such that $\mathfrak{o}(\hat{c})=\mathfrak{o}(c)$. The natural "order one" version of this consists in requiring that for $c_0\in C$, there exists a lift $\hat{c}_0\in X$ of c_0 such that $\mathfrak{o}(\hat{c}_0)=\mathfrak{o}(c_0)$ and for every $c_1\in C$ there exists a lift $\hat{c}_1\in X$ of c_1 such that $\mathfrak{o}\,(\hat{c}_1)\ \mathrm{mod}\langle\hat{c}_0\rangle=\mathfrak{o}\,(c_1)\ \mathrm{mod}\langle c_0\rangle$. This motivates the following:

Definition 5.74. *Let R be a countable PID. Consider an extension*

$$\mathfrak{S}:A\to X\to C.$$

Say that a triple

$$(c,C_0,\hat{C}_0)$$

with $c\in C$, C_0 submodule of C, and $\hat{C}_0$ submodule of X with $\pi\hat{C}_0=C_0$ has:

- *a sharp lift of order 0 if there exists $\hat{c}\in X$ with $\pi\hat{c}=c$ and $\mathfrak{o}\,(\hat{c})\ \mathrm{mod}\hat{C}_0=\mathfrak{o}\,(c)\ \mathrm{mod}C_0$, and*

- *a sharp lift of order α if for every $\beta < \alpha$ there exists $\hat{c} \in X$ with $\pi\hat{c} = c$ and $\mathfrak{o}(\hat{c}) \bmod \hat{C}_0 = \mathfrak{o}(x) \bmod C_0$ and for every $d \in C$,*

$$(d, \langle C_0, c\rangle, \langle \hat{C}_0, \hat{c}\rangle)$$

has a sharp lift of order β.

The extension $\mathfrak{S}$ is pure *of order α if and only if for every $c \in C$ and $r \in R$, the triple $(c, 0, 0)$ has a sharp lift of order α.*

Proposition 5.75. *Let R be a countable PID. Consider an extension*

$$\mathfrak{S} : A \to X \to C$$

of C by A as above. Then the following assertions are equivalent:

1. *$\mathfrak{S}$ is phantom of order α;*
2. *$\mathfrak{S}$ is pure of order α.*

Proof. By recursion of α. For $\alpha = 0$ this is remarked above. Suppose that the conclusion holds for $\beta < \alpha$.

(1)⇒(2) Assume that $\mathfrak{S}$ is pure of order α. We claim that it is phantom of order α. Fix $\beta < \alpha$. We need to prove that $\mathfrak{S}$ represents an element of

$$\mathrm{Ran}\left(\mathrm{Ph}^{\beta}\mathrm{Ext}\left(C/\langle c\rangle, A\right) \to \mathrm{Ext}\left(C, A\right)\right).$$

By hypothesis, the triple $(c, 0, 0)$ has a sharp lift of order α. Thus, there exists $\hat{c} \in X$ with $\pi\hat{c} = c$ and $\mathfrak{o}(\hat{c}) = \mathfrak{o}(c)$ such that for every $d \in C$, $(d, \langle c\rangle, \langle \hat{c}\rangle)$ has a sharp lift of order β. Consider the homomorphism $\pi_1 : X/\langle \hat{c}\rangle \to C/\langle c\rangle$ induced by $\pi : X \to C$. As $A \cap \langle \hat{c}\rangle = 0$ we have a short exact sequence

$$A \to X/\langle \hat{c}\rangle \xrightarrow{\pi_1} C/\langle c\rangle.$$

Such an extension is pure of order β. Therefore, by the inductive hypothesis it is phantom of order at β. As the extension of C by A that it induces via the quotient map $C \to C/\langle c\rangle$ is isomorphic to $\mathfrak{S}$, this concludes the proof.

(2)⇒(1) Assume that $\mathfrak{S}$ is phantom of order α. We claim that it is pure of order α. Fix $\beta < \alpha$. Then $\mathfrak{S}$ represents an element of

$$\mathrm{Ran}\left(\mathrm{Ph}^{\beta}\mathrm{Ext}\left(C/\langle c\rangle, A\right) \to \mathrm{Ext}\left(C, A\right)\right).$$

Thus, there exists an extension

$$A \to Y \to C/\langle c\rangle$$

that is phantom of order β that yields via pullback along the quotient map $C \to C/\langle c \rangle$ an extension isomorphic to $\mathfrak{S}$. Consider the pullback diagram

$$\begin{array}{ccc} X & \stackrel{\pi}{\to} & C \\ \rho \downarrow & & \downarrow \\ Y & \to & C/\langle c \rangle \end{array}$$

Let $\hat{c}$ be the unique element of X such that $\rho\hat{c} = 0$ and $\pi\hat{c} = c$. Then we have that $\mathfrak{o}(\hat{c}) = \mathfrak{o}(c)$. By the inductive hypothesis, the extension

$$A \to Y \to C/\langle c \rangle$$

is pure of order β. Thus, for every $d \in C$, the triple $(d + \langle c \rangle, 0, 0)$ has a sharp lift of order β. This implies that the triple $(d, \langle c \rangle, \langle \hat{c} \rangle)$ has a sharp lift of order β. This shows that $(c, 0, 0)$ has a sharp lift of order α, concluding the proof. □

5.9.3 Thin modules

Let us say that a countable module M is *thin* if its torsion submodule M_{t} is finitely-presented. We denote by $\mathbf{Thin}(R)$ the full subcategory of $\mathbf{Mod}(R)$ consisting of thin modules. Every countable flat module is, in particular, thin.

Proposition 5.76. *Suppose that X is a countable thin module. Then there exists a short exact sequence*

$$0 \to A \to B \to X \to 0$$

where A and B are countable projective modules.

Proof. Consider an extension

$$0 \to T \to X \to M \to 0$$

where M is flat and T is finitely-presented and torsion. Then since M is flat it has a projective resolution

$$0 \to Q \to P \to M \to 0.$$

Considering a pullback we obtain a commuting diagram

$$\begin{array}{ccccc} T & \to & Y & \to & P \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & X & \to & M \end{array}$$

Since P is projective, $Y \cong T \oplus P$. Since T is finitely-presented, it has a projective resolution

$$0 \to S \to F \to T \to 0.$$

Thus,

$$0 \to S \oplus Q \to F \oplus P \to X \to 0$$

is as desired. □

Lemma 5.77. *Let M be a countable flat module and T be a finitely-presented torsion module. Then* $\operatorname{Ext}(M,T)=0$.

Proof. As in Section 5.3.3, for a finite flat module M we let M^* be the finite flat module $\operatorname{Hom}(M,R)$. For a finite flat module N and a countable flat module X, we have natural isomorphisms

$$\operatorname{Hom}(X,N)\cong\operatorname{Hom}(X,\operatorname{Hom}(N^*,R))\cong\operatorname{Hom}(X\otimes N^*,R)$$

and hence

$$\operatorname{Ext}(X,N)\cong\operatorname{Ext}(X\otimes N^*,R).$$

Since T is a finite direct sum of cyclic finitely-presented torsion modules, we can assume without loss of generality that T is cyclic. Thus, we have that $T=L/I$ for some finite ideals $I\subseteq L\subseteq R$. Notice that the inclusion $I\to L$ induces a morphism $L^*\to I^*$ that is injective. Indeed, if $\varphi:L\to R$ is a homomorphism such that $\varphi|_I=0$, if $a\in I$ then for every $x\in L$ we have $a\varphi(x)=\varphi(ax)=0$ and hence $\varphi(x)=0$. Notice also that $\operatorname{Hom}(I^*/L^*,R)=0$. Indeed, if $\varphi:I^*\to R$ is a homomorphism such that $\varphi|_{L^*}=0$ then considering $a\in I$ such that $\varphi(\xi)=\xi(a)$ for $\xi\in I^*$, and considering $\xi\in L^*$ to be the inclusion $L\to R$, we have that $\xi|_I$ is the inclusion $I\to R$, and $0=\varphi(\xi|_I)=\xi(a)=a$. This shows that $\varphi=0$.

We have an exact sequence

$$\operatorname{Ext}(M,I)\to\operatorname{Ext}(M,L)\to\operatorname{Ext}(M,T)\to 0$$

which is naturally isomorphic to

$$\operatorname{Ext}(M\otimes I^*,R)\to\operatorname{Ext}(M\otimes L^*,R)\to\operatorname{Ext}(M,T)\to 0.$$

Since the morphism $L^*\to I^*$ is injective and M is flat, the morphism $M\otimes L^*\to M\otimes I^*$ is injective. Hence, it induces a surjective homomorphism

$$\operatorname{Ext}(M\otimes I^*,R)\to\operatorname{Ext}(M\otimes L^*,R).$$

This shows that $\operatorname{Ext}(M,T)=0$. □

Corollary 5.78. *If X is a countable thin module, then $X=T\oplus M$ where T is a finitely-presented torsion module and M is a countable flat module.*

Lemma 5.79. *Suppose that*

$$0\to A\to B\to C\to 0$$

is a short exact sequence, where A and C are countable thin modules. Then B is a countable thin module.

Proof. We need to prove that B_{t} is finitely-presented. Since B_{t} maps to C_{t}, after replacing C with C_{t} and B with the preimage of C_{t}, we can assume that C is a finitely-presented torsion module. By the previous corollary, we can write $A = N \oplus S$. Considering the canonical isomorphism

$$\mathrm{Ext}^1(C, A) \cong \mathrm{Ext}^1(C, N) \oplus \mathrm{Ext}^1(C, S) \cong \mathrm{Ext}^1(C, S)$$

we have that

$$B \cong N \oplus T$$

where T is an extension $S \to T \to C$. Since C and S are finitely-presented torsion modules, the same holds for T. □

Corollary 5.80. *The category* $\mathbf{Thin}(R)$ *is a fully exact subcategory of* $\mathbf{Mod}(R)$.

It follows from the above that $\mathbf{Thin}(R)$ is a *hereditary* quasi-abelian $\mathbf{\Pi}(\mathbf{Mod}(R))$-category with enough projectives. Thus, the categories $\mathrm{K}^b(\mathbf{Thin}(R))$ and $\mathrm{D}^b(\mathbf{Thin}(R))$ are triangulated $\mathrm{LH}(\mathbf{\Pi}(\mathbf{Mod}(R)))$-categories. The *compact objects* in $\mathrm{D}^b(\mathbf{Thin}(R))$ are the bounded complexes of finitely-presented modules. We let Ext^n for $n \geq 0$ be the derived functor of Hom on $\mathbf{Thin}(R)$ as defined in Section 5.2.8. Since $\mathbf{Thin}(R)$ is hereditary, $\mathrm{Ext}^n = 0$ for $n \geq 2$. We let Ext be Ext^1. Since $\mathbf{Thin}(R)$ is a fully exact subcategory of $\mathbf{Mod}(R)$, this definition of Ext is consistent with the one previously considered in reference to the derived functor of Hom on $\mathbf{Mod}(R)$.

Let now $\mathcal{E}_0$ be the exact structure on $\mathbf{Thin}(R)$ *projectively generated* by the class of finitely-presented (torsion) modules. Then the exact sequences in $\mathcal{E}_0$ are precisely the pure short-exact sequences. A countable module is projective in $(\mathbf{Thin}(R), \mathcal{E}_0)$ if and only if it is of the form $T \oplus P$ where T is a finitely-presented torsion module and P is projective in $\mathbf{Mod}(R)$.

5.9.4 Largest free summand

For a module C, define ΣC to be the intersection of the kernels of the idempotent homomorphisms $C \to C$ whose image is a (finite) projective module, and ΦC to be the union of the images of the idempotent homomorphisms $C \to C$ whose image is a direct summand of C that is a (finite) projective module. We say that C is *coreduced* if $\Phi C = 0$.

Lemma 5.81. *Suppose that* C *is a countable flat module. Then:*

1. ΦC *is a countable projective module;*
2. $C = \Sigma C \oplus \Phi C$*;*
3. $\Sigma(\Sigma C) = \Sigma C$*, and* ΣC *has no projective direct summands.*

Proof. Suppose that $f, g : C \to C$ are idempotent homomorphisms with images P, Q, respectively. Suppose that P, Q are finite projective modules, and direct summands of C. Then we have that $P \cap Q$ is a direct summand of Q. Write $Q = Q' \oplus (P \cap Q)$. Thus, there exists an idempotent homomorphism $g' : C \to C$ whose image is Q'. Thus, $f + g' : C \to C$ is an idempotent homomorphism whose image is $P \oplus Q' = P \oplus Q$.

Let (ϕ_n) be a sequence of idempotent homomorphisms $C \to C$ whose image is a finite projective module, such that

$$\Phi C = \bigcup_n \mathrm{Ran}\,(\phi_n)\,.$$

By the above remarks, one can obtain another sequence (ψ_n) such that, for every n,

$$\mathrm{Ran}\,(\psi_0 + \cdots + \psi_n) = \bigoplus_i \mathrm{Ran}\,(\psi_i) = \sum_i \mathrm{Ran}\,(\varphi_i)\,.$$

Thus,

$$\Phi_{\mathcal{C}} C = \bigoplus_{i \in \omega} \mathrm{Ran}\,(\psi_i)\,.$$

The rest of the assertions easily follow from this. □

It follows from Lemma 5.81 that $C \mapsto \Sigma C$ is a radical in the sense of [40, 57].

5.9.5 Injective and projective objects

Suppose that M is a module. Let $\mathcal{F}$ be a countable downward-directed collection of submodules of M, and let $\mathcal{G}$ be the collection of cosets of elements of $\mathcal{F}$. A family $\mathcal{G}_0 \subseteq \mathcal{G}$ satisfies the *finite intersection property* (FIP) if every finite subset of $\mathcal{G}_0$ has nonempty intersection. By definition, the module M is $\mathcal{G}$-compact if every family $\mathcal{G}_0 \subseteq \mathcal{G}$ with the FIP has nonempty intersection. The family $\mathcal{F}$ satisfies the descending chain condition (DCC) if every decreasing sequence of elements of $\mathcal{F}$ is eventually constant.

Lemma 5.82. *Adopt the notations above. Then M is $\mathcal{G}$-compact if and only if for every decreasing chain (S_n) of elements of $\mathcal{F}$, the canonical homomorphism*

$$\varphi_{(S_n)} : M \to \lim_n M/S_n$$

is surjective.

Proof. We have that M is $\mathcal{G}$-compact if and only if $\varphi_{(S_n)}$ has closed image for each decreasing chain (S_n). Since $\varphi_{(S_n)}$ has dense image, the conclusion follows. □

Corollary 5.83. *Adopt the notations above. If M is countable and $\mathcal{G}$-compact, then $\mathcal{F}$ satisfies the* DCC.

Proof. If (S_n) is a strictly decreasing sequence of elements of $\mathcal{F}$, then $\lim_n M/S_n$ is an uncountable Polish module. □

A submodule N of M is pp-definable if it is of the form

$$\{x \in M : \exists y_0, \dots, y_\ell \in M, rx = r_0 y_0 + \cdots + r_\ell y_\ell\}$$

for some $\ell \in \omega$ and $r, r_0, \dots, r_\ell \in R$. The module M is *algebraically compact* if it is $\mathcal{G}$-compact with respect to the family $\mathcal{G}$ of cosets of pp-definable submodules [141, Section 4].

The module M is r-divisible for some $r \in R$ if $rM = M$, and it is *divisible* if it is r-divisible for every nonzero $r \in R$. For $r \in R$ define $R[r^{-1}] \subseteq K$ to be the submodule generated by r^{-n} for $n \in \mathbb{N}$. More generally, for a finite ideal I, we say that M is I-divisible if $IM = M$, where IM is the submodule of M generated by rx for $r \in I$ and $x \in M$. We also let $R[I^{-1}] \subseteq K$ be the union of I^{-n} for $n \in \mathbb{N}$.

Lemma 5.84. *Let A be a countable flat module, and $I \subseteq M$ be a finite ideal. Then A is I-divisible if and only if* $\mathrm{Ext}\left(R[I^{-1}], A\right) = 0$.

Proof. We have that $R[I^{-1}]$ is isomorphic to the colimit of the inductive sequence $\left(I^{-n}\right)_{n\in\omega}$. We have

$$\mathrm{Ext}\left(R[I^{-1}], A\right) \cong \lim\nolimits^1_n \mathrm{Hom}\left(I^{-n}, A\right).$$

The tower $\left(\mathrm{Hom}\left(I^{-n}, A\right)\right)$ is isomorphic to the tower $\left(I^n A\right)_{n\in\omega}$ with inclusions as bonding maps. Thus, we have that $\mathrm{Ext}\left(R[I^{-1}], A\right) = 0$ if and only if such a tower satisfies the Mittag-Leffler condition, if and only if the sequence $\left(I^n A\right)$ of submodules of A is eventually constant. Since R is a Prüfer domain, I is invertible. Thus, there exists $n \in \omega$ such that $I^{n+1}A = I^n A$ if and only if A is I-divisible, concluding the proof. □

Lemma 5.85. *Let A be a countable flat module. Then A is algebraically compact if and only if it is divisible.*

Proof. If A is algebraically compact and countable, then by Corollary 5.83 the family of pp-definable submodules satisfies the DCC. In particular, for every nonzero $r \in A$ the sequence $\left(r^n A\right)$ is eventually constant, whence A is r-divisible.

Conversely, if A is divisible every pp-definable submodule is either $\{0\}$ or A and the conclusion follows. □

Proposition 5.86. *Suppose that A is a countable flat module. The following assertions are equivalent:*

1. *A is injective in* $\mathbf{Flat}\left(R\right)$*;*
2. *A is pure-injective in* $\mathbf{Thin}\left(R\right)$*;*
3. *A is pure-injective in* $\mathbf{Mod}\left(R\right)$*;*
4. *A is algebraically compact;*

5. *A is divisible;*

6. $\mathrm{Ext}\,(R[1/r], A) = 0$ *for every* $r \in R \setminus \{0\}$*;*

7. $\mathrm{Ext}\,(R[I^{-1}], A) = 0$ *for every finite ideal* $I \subseteq R$*.*

Proof. Considering that every short-exact sequence of flat modules is pure, the implications (3)⇒(2)⇒(1)⇒(7)⇒(6) are obvious. The equivalence of (3) and (4) is part of [141, Theorem 4.1]. The equivalence of (4) and (5) is Lemma 5.85, and the equivalence of (5) and (6) is Lemma 5.84. □

Suppose that A is a countable flat module. Then A has a largest divisible submodule, denoted by ΔA, which is pure in A. Therefore, ΔA is a direct summand of A. We say that A is *reduced* if $\Delta A = 0$.

Proposition 5.87. *Suppose that C is a countable flat module. The following assertions are equivalent:*

1. *C is projective in* $\mathbf{Flat}\,(R)$*;*

2. *C is projective in* $\mathbf{Mod}\,(R)$*;*

3. *C is a direct sum of finite ideals of R;*

4. $\mathrm{Ext}\,(C, R) = 0$*.*

Proof. (4)⇒(2) Suppose that $\mathrm{Ext}\,(C, R) = 0$. Write $C = \mathrm{colim}_n C_n$ for some increasing sequence (C_n) of finite submodules of C. Then we have that

$$0 = \mathrm{Ext}\,(C, R) = \lim\nolimits^1_n \mathrm{Hom}\,(C_n, R) = \lim\nolimits^1_n C_n^*$$

where $C_n^* = \mathrm{Hom}\,(C_n, R)$. Notice that $X \mapsto X^*$ is an equivalence between the category of finite projective modules and its opposite. As $\lim^1_n C_n^* = 0$, the tower (C_n^*) is isomorphic to an epimorphic tower. Thus, after replacing (C_n) with an isomorphic inductive sequence, we can assume that (C_n^*) is an epimorphic tower. Since C_n^* is projective, the map $C_{n+1}^* \to C_n^*$ is a split epimorphism, whence the map $C_n \to C_{n+1}$ is a split monomorphism. This shows that C is isomorphic to a direct sum of finite projective modules, whence projective.

The implication (2)⇒(3) is part of Proposition 5.23, and the other implications are trivial. □

The following result is a particular case of [120, Theorem 13].

Lemma 5.88. *Let R be a countable Prüfer domain. A module M is projective if and only if there exists an increasing sequence of projective pure submodules of M with union equal to M.*

5.9.6 Duality

Let M be a module. If M is either countable or pro-finite, we set $M^* := \mathrm{Hom}\,(M, R)$. Notice that if M is countable and flat, then M^* is pro-finiteflat, while if M is pro-finiteflat, then M^* is countable and flat. Furthermore, there is a canonical morphism $M \to M^{**}$.

Lemma 5.89. *Let R be a countable Prüfer domain.*

1. *If M is either countable projective or pro-finiteflat, then the canonical morphism $M \to M^{**}$ is an isomorphism.*
2. *The functor $M \mapsto M^*$ establishes a duality between countable projective modules and pro-finiteflat modules.*

Proof. (1) A countable projective module is a direct sum of finite flat modules, while a pro-finiteflat module is a direct product of finite flat modules. The conclusion easily follows from this and Lemma 5.25.

(2) This is a consequence of (1). □

Let M be a module. If M is either countable coreduced and flat, or phantom pro-finiteflat, define $M^\vee := \mathrm{Ext}\,(M, R)$. Notice that if M is countable coreduced and flat, then $M^\vee$ is phantom pro-finiteflat, while if M is phantom pro-finiteflat, then $M^\vee$ is countable and flat. Furthermore, there is a canonical morphism $M \mapsto M^{\vee\vee}$, obtained by considering a projective resolution $S \to P \to M$ of M and, adopting the notation above, considering the identifications

$$M^\vee \cong \frac{S^*}{P^*}$$

and

$$M^{\vee\vee} \cong \frac{S^{**}}{P^{**}}.$$

Proposition 5.90. *Let R be a countable Prüfer domain.*

1. *If M is either countable coreduced and flat or phantom pro-finiteflat, then the canonical morphism $M \to M^{\vee\vee}$ is an isomorphism;*
2. *The functor $M \mapsto M^\vee$ establishes a duality between countable coreduced flat modules and phantom pro-finiteflat modules.*

Proof. The first assertions easily follows from Lemma 5.89. The second assertion is a consequence of the first one. □

The following corollary is in fact a particular instance of Proposition 5.17(2).

Corollary 5.91. *Let R be a countable Prüfer domain. The functor $\lim^1$ establishes an equivalence from the category of reduced towers of finite flat modules to the category of phantom pro-finiteflat modules.*

Proof. Choosing for each countable flat module M a cofinal increasing sequence (M_n) of finite submodules defines a functor $M \mapsto (\mathrm{Hom}\,(M_n, R))$ that establishes an equivalence between countable coreduced flat modules and reduced towers of finite flat modules. By Theorem 5.67, via this equivalence, the functor $\lim^1$ corresponds to $\mathrm{Ext}\,(-, R)$. The conclusion thus follow from Proposition 5.90. □

As a particular instance of Proposition 5.17(1) we also have the following:

Proposition 5.92. *Let R be a countable Prüfer domain. The functor $\lim^1$ from the category $\mathrm{Ph}\,(\mathbf{Mod}_{\aleph_0}(R))$ of reduced towers of countably-presented modules to the category of phantom pro-countably-presented modules is faithful and essentially surjective. Furthermore:*

- *the essential image of the category of reduced towers of flat modules is the category of phantom pro-countableflat modules;*
- *the essential image of the category of reduced towers of finiteflat modules is the category of phantom pro-finiteflat modules.*

For a phantom pro-countably-presented module G/N, let (U_n) be a basis open submodules of G, and (W_n) be a basis of open submodules of N with $W_n \subseteq U_n$ for every $n \in \omega$. Let $A^{(n)}$ be the countably-presented module

$$\frac{U_n \cap N}{W_n}.$$

This defines a reduced tower $\boldsymbol{A}$ of countably-presented modules with $\lim^1 \boldsymbol{A} \cong G/N$.

Proof. We just prove the last assertion. For every $k \in \omega$, the inclusion $N \to G$ induces a continuous homomorphism $N/W_k \to G/U_k$, which is surjective since $\{0\}$ is dense in G/N. For every $k \in \omega$ we have a short exact sequence

$$Q^{(k)} \to N/W_k \to G/U_k$$

where

$$Q^{(k)} = \frac{(N \cap U_k)}{W_k}$$

Taking the limit we obtain an exact sequence

$$N = \lim_k (N/W_k) \to \lim_k (G/U_k) \to \lim{}^1 Q^{(k)} \to \lim{}^1 N/W_k = 0.$$

This yields an isomorphism

$$\lim{}^1 Q^{(k)} \cong G/N.$$

□

5.9.7 Duality over a PID

In this subsection we assume that R is a countable PID with field of fractions K, to extend the duality from Section 5.9.6.

Lemma 5.93. *Let R be a countable PID. The functor $\mathrm{Hom}(-, K/R)$ from countable modules to pro-finite modules is faithful*

Proof. Suppose that $\varphi : C \to C'$ is a homomorphism between countable modules that induces the trivial homomorphism $\mathrm{Hom}(C', K/R) \to \mathrm{Hom}(C, K/R)$. If $x \in C'$ is nonzero and $\langle x \rangle$ is the submodule of C' generated by x, then $\langle x \rangle \cong R$ or $\langle x \rangle \cong R/rR$ for some $r \in R$. Either way, there exists a homomorphism $\langle x \rangle \to K/R$ that maps x to a nonzero element of K/R. As K/R is injective, this extends to a nonzero homomorphism $\alpha : C' \to K/R$ such that $\alpha(x) \neq 0$. By hypothesis, we have that $\alpha \circ \varphi = 0$. This implies that x does not belong to the image of φ. As this holds for every nonzero element of C', $\varphi = 0$. □

Let us say that a Polish module C is pro-finite-torsion if it is the limit of a tower of finite torsion modules. When C is either a countable torsion module or a pro-finite-torsion module, define as before $C^\vee := \mathrm{Ext}(C, R)$. Considering the exact sequence

$$\mathrm{Hom}(C, K) \to \mathrm{Hom}(C, K/R) \to \mathrm{Ext}(C, R) \to 0$$

shows that in this case $C^\vee \cong \mathrm{Hom}(C, K/R)$.

Proposition 5.94. *Let R be a countable PID. The assignment $C \mapsto C^\vee$ establishes a duality:*

1. *between countable torsion modules and pro-finite-torsion modules;*
2. *between countable modules and modules with a Polish cover G such that $G/s_0(G)$ is pro-finite-torsion;*

Proof. (1) is easily established using the Ulm classification of countable torsion modules by induction on the Ulm length. (2) follows from (1) and Proposition 5.90, considering the short exact sequence

$$C_{\mathrm{t}} \to C \to C/C_{\mathrm{t}}$$

associated with a countable torsion module C, and the short exact sequence

$$s_0(G) \to G \to G/s_0(G)$$

associated with a module with a Polish cover G. □

5.9.8 Module trees for phantom modules

Pro-countable modules can be described in terms of "module trees" on a sequence of countable modules. Let $\boldsymbol{M} := (M_i)_{i\in\omega}$ be a sequence of countable modules. A *module tree* on $\boldsymbol{M}$ is a rooted tree T on

$$M^{(\omega)} := \bigoplus_{i\in\omega} M_i$$

such that, for every $i \in \omega$, the set $T^{(n)}$ of nodes of height n in T is a submodule of

$$M^{(n)} := \bigoplus_{i\in n} M_i.$$

Then the body M_T of T is a closed submodule of

$$\prod_{n\in\omega} M_n.$$

Every pro-countable module can be represented in this fashion; see [51, 158] for more on "module trees" in the case of $\mathbb{Z}$-modules.

Likewise, one can represent a *phantom* pro-countable module by a module tree T on a sequence $\boldsymbol{M}$ of countable modules. Indeed, such a tree defines a tower of countable modules $\left(T^{(n)}\right)$ with bonding maps given by truncation, whence a phantom module $\mathfrak{M}_T := \lim_n{}^1 T^{(n)}$. Proposition 5.92 shows that every phantom pro-countable module arises in this fashion for some well-founded module tree. Furthermore, the Solecki submodules of $\mathfrak{M}_T$ correspond to the derivatives of T, as $s_\alpha\mathfrak{M}_T$ is the phantom module associated with the $\omega\alpha$-th derivative $T^{(\omega\alpha)}$ of T; see Theorem 5.58.

5.9.9 Projective length

We define the notion of *projective length* of a countable flat module, as a relaxation of the notion of projective module.

Definition 5.95. *Suppose that C is a coreduced countable flat module.*

- *If A is a countable flat module, then the (plain) A-projective length of C is the (plain) Solecki length of* $\operatorname{Ext}(C, A)$*;*
- *C has (plain) projective length at most α if and only if it has (plain) A-projective length at most α for every countable flat module A;*
- *C is a* phantom projective *if it has projective length at most α for some $\alpha < \omega_1$.*

Consider the quasi-abelian category $\mathbf{Flat}(R)$ of countable *flat* modules. Suppose that $\mathcal{C}_1$ is the class of finite-rank flat modules. We let $\mathcal{E}_1$ be the exact structure *projectively generated* by $\mathcal{C}_1$ on the quasi-abelian category $\mathbf{Flat}(R)$. A similar proof as the one of [64, Theorem 87.1] yields the following.

Lemma 5.96. *A countable flat module A is a direct sum of finite-rank modules if and only if every finite subset of A is contained in a finite-rank direct summand of A.*

Proof. Suppose that A is a countable flat module such that every finite subset F of A is contained in a finite-rank direct summand of A. Let (a_n) be an enumeration of A. By assumption, there exists a finite-rank direct summand A_1 of A containing a_1. Suppose that we have defined an ascending chain $A_1 \subseteq \cdots \subseteq A_{n-1}$ of finite-rank direct summands of A such that $\{a_1, \ldots, a_i\} \subseteq A_i$ for $i < n$. Then by assumption there exists a finite-rank direct summand A_n of A that contains a_n and a finite subset G_{n-1} of A_{n-1} such that $A_{n-1} = \langle G_{n-1} \rangle_*$. Since A_n is a direct summand of A, this implies that $A_{n-1} \subseteq A_n$. This procedure defines an increasing sequence (A_n) of finite-rank direct summands of A with union equal to A. For every $n \in \omega$, one has that $A_{n+1} = A_n \oplus B_{n+1}$, where B_{n+1} has finite rank. Setting $B_0 = A_0$, one has that $A = \bigoplus_{n\in\omega} B_n$. As the converse implication is obvious, this concludes the proof. □

Proposition 5.97. *Suppose that C is a countable flat module. Then there exist a short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{E}_1$ where A and B are countable direct sums of finite-rank flat modules.

Proof. Let (C_n) be an increasing sequence of finite-rank pure submodules of C with union equal to C. Define

$$A = B = \bigoplus_{n\in\omega} C_n.$$

Set $\phi : B \to C$ be the homomorphism induced by the inclusion map $C_n \to C$ for $n \in \omega$, and $\psi : A \to B$ be the homomorphism induced by the morphisms $C_i \to C_j$, for $i, j \in \omega$, defined by

$$x \mapsto \begin{cases} x & \text{if } i = j; \\ -x & \text{if } i + 1 = j; \\ 0 & \text{otherwise.} \end{cases}$$

This defines a short exact sequence $A \to B \to C$ in $\mathcal{E}_1$. □

Recall that a *retraction* in a category is a morphism that has a right inverse, and a *section* is a morphism that has a left inverse.

Lemma 5.98. *Suppose that $r : \boldsymbol{B} \to \boldsymbol{A}$ is a retraction in the category of towers of countable modules. Then the induced map $\lim^1 \boldsymbol{B} \to \lim^1 \boldsymbol{A}$ maps $s_\alpha\left(\lim^1 \boldsymbol{B}\right)$ onto $s_\alpha\left(\lim^1 \boldsymbol{A}\right)$ for every $\alpha < \omega_1$.*

Proof. As r is a retraction, the induced map $\lim^1 \boldsymbol{B} \to \lim^1 \boldsymbol{A}$ is also a retraction. Since $G \mapsto s_\alpha G$ is a subfunctor of the identity, the conclusion follows. □

It is clear that a countable direct sum of finite-rank flat modules is projective in $(\mathcal{A}, \mathcal{E}_1)$. As a consequence of this observation and Proposition 5.97 we obtain the following.

Proposition 5.99. *The idempotent-complete exact category* $(\mathbf{Flat}\,(R)\,, \mathcal{E}_1)$ *is hereditary and has enough projectives. Its projective objects are precisely the direct summands of the countable direct sums of finite-rank flat modules.*

It follows that the functor $\mathrm{Hom} : (\mathbf{Flat}\,(R)\,, \mathcal{E}_1) \times (\mathbf{Flat}\,(R)\,, \mathcal{E}_1) \to \mathbf{\Pi}(\mathbf{Flat}\,(R))$ has a total right derived functor $\mathrm{RHom}^{\bullet}_{\mathcal{E}_1}$. We can then define $\mathrm{Ext}^n_{\mathcal{E}_1} := \mathrm{H}^n \circ \mathrm{RHom}^{\bullet}_{\mathcal{E}_1}$ for $n \in \mathbb{Z}$. For $n \geq 2$ we have that $\mathrm{Ext}^n_{\mathcal{E}_1} = 0$.

Corollary 5.100. *Suppose that* C *and* A *are countable flat module. If* (F_n) *is an increasing sequence of finite-rank pure submodules of* C *such that* $\bigcup_{n\in\omega} F_n = C$*, then*

$$\mathrm{Ext}_{\mathcal{E}_1}(C, A) \cong \lim\nolimits^1_n \mathrm{Hom}(F_n, A)$$

Proof. This follows from the definition of $\mathrm{Ext}_{\mathcal{E}_1}$ by applying Lemma 5.73. □

Suppose that C and A are countable flat modules. Notice that, by definition, $\mathrm{Ext}_{\mathcal{E}^1}(C, A)$ is the submodule of $\mathrm{Ext}(C, A)$ obtained as the intersection of the kernels of the homomorphisms $\mathrm{Ext}(C, A) \to \mathrm{Ext}(B, A)$ induced by the inclusion $B \to C$, where B varies among the finite-rank pure submodules of C.

Lemma 5.101. *Suppose that* A, C *are countable flat modules such that* C *is finite-rank. Then* $\mathrm{Ext}(C, A)$ *has plain Solecki length at most* 1.

Proof. Let $D := K \otimes A$ be the divisible hull of A. Thus, D is a divisible flat module. Consider then the exact sequence

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, D) \to \mathrm{Hom}(C, D/A) \to \mathrm{Ext}(C, A) \to 0.$$

Thus, we have that $\mathrm{Ext}(C, A)$ is isomorphic to

$$\frac{\mathrm{Hom}(C, D/A)}{\mathrm{Ran}(\mathrm{Hom}(C, D) \to \mathrm{Hom}(C, D/A))}.$$

Notice that $\mathrm{Hom}(C, D)$ is countable since C has finite-rank. □

Lemma 5.102. *Suppose that* C, A *are countable flat modules. Then* $\{0\}$ *is dense in* $\mathrm{Ext}_{\mathcal{E}^1}(C, A)$.

Proof. It suffices to prove that $\{0\}$ is dense in $\mathrm{Ker}(\mathrm{Ext}(C, A) \to \mathrm{Ext}(B, A))$ for every finite-rank pure submodule B of C. Let B be a finite-rank submodule of C. Then we have an exact sequence

$$\mathrm{Hom}(B, A) \to \mathrm{Ext}(C/B, A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(B, A) \to 0$$

Thus, we have an isomorphism

$$\mathrm{Ker}(\mathrm{Ext}\,(C,A)\to\mathrm{Ext}\,(B,A))\cong\frac{\mathrm{Ext}\,(C/B,A)}{\mathrm{Ran}(\mathrm{Hom}\,(B,A)\to\mathrm{Ext}\,(C/B,A))}$$

Thus, it suffices to prove that $\mathrm{Ran}(\mathrm{Hom}\,(B,A)\to\mathrm{Ext}\,(C/B,A))$ is dense in $\mathrm{Ext}\,(C/B,A)$. Since B is pure in C, C/B is flat and hence $\mathrm{Ext}\,(C/B,A)=\mathrm{PExt}\,(C/B,A)$. Thus, $\{0\}$ is dense in $\mathrm{Ext}\,(C/B,A)$, and *a fortiori* $\mathrm{Ran}(\mathrm{Hom}\,(B,A)\to\mathrm{Ext}\,(C/B,A))$ is dense in $\mathrm{Ext}\,(C/B,A)$. □

Theorem 5.103. *Suppose that A,C are countable flat modules. Then $\mathrm{Ext}_{\mathcal{E}_1}(C,A)$ is equal to the first Solecki submodule of $\mathrm{Ext}\,(C,A)$.*

Proof. By Lemma 5.101, $\{0\}$ is a $\mathbf{\Sigma}^0_2$ submodule of $\mathrm{Ext}\,(B,A)$ for every finite-rank submodule B of C. Since $\mathrm{Ext}_{\mathcal{E}_1}(C,A)$ is the intersection of the kernels of the Borel homomorphisms $\mathrm{Ext}\,(C,A)\to\mathrm{Ext}\,(B,A)$ where B varies among the finite-rank pure submodules of C, it follows that $\mathrm{Ext}_{\mathcal{E}_1}(C,A)$ is a $\mathbf{\Pi}^0_3$ submodule of $\mathrm{Ext}\,(C,A)$ as there are countably many such submodules B.

Consider a projective resolution $S\to F\to C$ of C, where F is a free module. Thus, we have a natural isomorphism

$$\mathrm{Ext}\,(C,A)\cong\frac{\mathrm{Hom}\,(S,A)}{\mathrm{Hom}\,(F|S,A)},$$

where

$$\mathrm{Hom}\,(F|S,A)=\mathrm{Ran}\left(\mathrm{Hom}\,(F,A)\to\mathrm{Hom}\,(S,A)\right).$$

For a submodule B of C, define $F_B=\{x\in F:x+S\in B\}$. Under such an isomorphism $\mathrm{Ext}_{\mathcal{E}_1}(C,A)$ corresponds to

$$\frac{\mathrm{Hom}^1\,(S,A)}{\mathrm{Hom}\,(F|S,A)}$$

where $\mathrm{Hom}^1\,(S,A)$ is the submodule of homomorphisms $S\to A$ that have an extension $F_B\to A$ for every finite-rank (pure) submodule B of C.

By Lemma 5.44 and Lemma 5.102, it suffices to prove that if $V\subseteq\mathrm{Hom}\,(F|S,A)$ is an open neighborhood of 0 in $\mathrm{Hom}\,(F|S,A)$, then the closure $\bar{V}^{\mathrm{Hom}(S,A)}$ of V inside $\mathrm{Hom}\,(S,A)$ contains an open neighborhood of 0 in $\mathrm{Hom}^1\,(S,A)$. Suppose that $V\subseteq\mathrm{Hom}\,(F|S,A)$ is an open neighborhood of 0. Thus, there exist $x_0,\ldots,x_n\in F$ such that

$$\{\varphi|_S:\varphi\in\mathrm{Hom}\,(F,A)\,,\forall i\leq n,\varphi\,(x_i)=0\}\subseteq V.$$

Let B be a finite-rank pure submodule of C such that $\{x_0,\ldots,x_n\}\subseteq F_B$. Thus, we have that

$$U=\left\{\varphi\in\mathrm{Hom}^1\,(S,A):\exists\hat{\varphi}\in\mathrm{Hom}(F_B,A),\forall i\leq n,\hat{\varphi}\,(x_i)=0,\hat{\varphi}|_S=\varphi\right\}$$

is an open neighborhood of 0 in $\mathrm{Hom}^1(S,A)$. We claim that $U \subseteq \bar{V}^{\mathrm{Hom}(S,A)}$. Indeed, suppose that $\varphi_0 \in U$ and let W be an open neighborhood of φ_0 in $\mathrm{Hom}(S,A)$. Thus, there exist $y_0,\ldots,y_\ell \in S$ such that

$$\{\varphi \in \mathrm{Hom}(S,A) : \forall i \leq \ell, \varphi(y_i) = \varphi_0(y_i)\} \subseteq W.$$

We need to prove that $V \cap W \neq \varnothing$. To this purpose, it suffices to show that there exists $\varphi \in \mathrm{Hom}(F,A)$ such that $\varphi(y_i) = \varphi_0(y_i)$ for $i \leq \ell$ and $\varphi(x_i) = 0$ for $i \leq n$. Since $\varphi_0 \in U$, there exists $\hat{\varphi}_0 \in \mathrm{Hom}(F_B,A)$ such that $\hat{\varphi}_0(x_i) = 0$ for $i \leq n$ and $\hat{\varphi}_0|_S = \varphi_0$.

Let $(z_k)_{k\in I}$ be a free R-basis of F. The elements $x_0,\ldots,x_n,y_0,\ldots,y_\ell$ of F are contained in the submodule F_0 of F generated by $\{z_i : i \in I_0\}$ for some *finite* $I_0 \subseteq I$. Letting $\langle F_0, F_B\rangle$ be the submodule of F generated by F_0 and F_B, we have that, since B is pure in C,

$$\langle F_0, F_B\rangle/F_B \subseteq F/F_B \cong C/B$$

is finitely-generated and flat, and hence projective. Therefore, $\mathrm{Ext}(\langle F_0,F_B\rangle/F_B, A) = 0$ and there exists $\psi \in \mathrm{Hom}(\langle F_0,F_B\rangle, A)$ that extends $\hat{\varphi}_0$. One can then define $\varphi \in \mathrm{Hom}(F,A)$ by setting

$$\varphi(z_i) = \begin{cases} \psi(z_i) & \text{if } i \in I_0, \\ 0 & \text{if } i \in I \setminus I_0. \end{cases}$$

This concludes the proof. □

Corollary 5.104. *An extension of countable flat modules is in $\mathcal{E}_1$ if and only if it is phantom of order 1.*

We now completely characterize the countable flat modules that have (plain) projective rank at most one.

Theorem 5.105. *Given a countable coreduced flat module C, the following assertions are equivalent:*

1. *C has plain R-projective length at most one;*
2. *C has plain projective length at most one;*
3. *C has finite rank.*

Proof. Let (C_n) be an increasing of finite-rank pure submodules of C with union equal to C.

(1)⇒(2) By Corollary 5.100 and Theorem 5.103, we have that

$$\mathrm{Ext}(C,R) \cong \lim_n \mathrm{Ext}(C_n,R).$$

By Lemma 5.57, after passing to a subsequence of (C_n) we can assume that $\mathrm{Ext}\,(C_n, R) \to \mathrm{Ext}\,(C_{n-1}, R)$ is an isomorphism for every $n \geq 1$. For $n \geq 0$ we have

$$0 = \mathrm{Hom}\,(C_n, R) \to \mathrm{Ext}\,(C_{n+1}/C_n, R) \to \mathrm{Ext}\,(C_{n+1}, R) \to \mathrm{Ext}\,(C_n, R) \to 0.$$

This shows that $\mathrm{Ext}\,(C_{n+1}/C_n, R) = 0$ and hence C_{n+1}/C_n is a projective module by Proposition 5.87. As this holds for every $n \geq 1$, and since C is coreduced, $C = C_0$.

(3)⇒(2) This follows from Lemma 5.101. □

Theorem 5.106. *Given a countable coreduced flat module C, the following assertions are equivalent:*

1. *C has R-projective length at most one;*
2. *$C \cong \mathrm{colim}_k C_k$ for some inductive sequence (C_k) of coreduced flat modules of finite rank.*

Proof. Let (C_n) be an increasing of finite-rank pure submodules of C with union equal to C. By Theorem 5.103, we have

$$s_1\mathrm{Ext}\,(C, A) \cong \lim{}^1\mathrm{Hom}\,(C_n, R)\,.$$

(2)⇒(1) If C_n is coreduced for every $n \in \omega$, then $\mathrm{Hom}\,(C_n, R) = 0$, whence $s_1\mathrm{Ext}\,(C, A) = 0$.

(1)⇒(2) If $s_1\mathrm{Ext}\,(C, A) = 0$, then since C is coreduced we have that the tower $(\mathrm{Hom}\,(C_n, R))$ of finite flat modules is isomorphic to a trivial tower. Thus, after passing to a subsequence, we can assume that $\mathrm{Hom}\,(C_n, R) = 0$ for every $n \in \omega$, i.e., that C_n is coreduced for every $n \in \omega$. □

Theorem 5.107. *Given a countable coreduced flat module C, the following assertions are equivalent:*

1. *C has projective length at most one;*
2. *C has A-projective length at most one for every finite-rank countable flat module A;*
3. *C is a direct summand of a countable direct sum of finite-rank flat modules.*

Proof. (2)⇒(3) The hypothesis implies that $\mathrm{Ext}\,(C, A)$ has Solecki length at most 1 for every finite-rank decomposable countable flat module. By Proposition 5.97 we have a 1-pure short exact sequence

$$0 \to A \to B \to C \to 0$$

where A and B are finite-rank decomposable. By Theorem 5.103 we have $\mathrm{Ext}_{\mathcal{E}_1}\,(C, A) = 0$ and hence such a sequence splits. This shows that C is a direct summand of B.

(3)⇒(1) This follows from Lemma 5.101. □

Note that a direct summand of a countable direct sum of finite-rank flat modules is not necessarily a countable direct sum of finite-rank flat modules; see [47] for an example in the case of $\mathbb{Z}$-modules.

5.10 Phantom projective resolutions

For each limit ordinal α, we let (α_n) be a sequence of successor ordinals converging to α. When α is a successor we set $\alpha_n = \alpha$ for every $n \in \omega$. In this section we provide a description of the functor $\mathrm{Ph}^{\alpha}\mathrm{Ext}$ on the category $\mathbf{Flat}\,(R)$ of countable flat modules as a derived functor with respect to a canonical exact structure $\mathcal{E}_\alpha$ on $\mathbf{Flat}\,(R)$.

5.10.1 Wedge sums

We begin with introducing a natural operation between modules which we term *wedge sum*. Recall the sets of indices I_α and $I_\alpha^{\mathrm{plain}}$ from Section 5.6.5. In particular, I_2^{plain} is the complete countable rooted tree of height 1. A presheaf of flat modules over I_2^{plain} is given by a flat module M_1 together with flat modules $M_{(i;0)}$ for $i \in \omega$ and module homomorphisms $\psi_i : M_1 \to M_{(i;0)}$ for $i \in \omega$. We define the *wedge sum*

$$M := \bigoplus_{(M_1, \psi_i)} M_{(i;0)}$$

to be its colimit in $\mathbf{Flat}\,(R)$. We can identify M with the quotient of the direct sum M_∞ of M_1 and $M_{(i;0)}$ for $i \in \omega$ by the pure submodule generated by the submodule N consisting of elements of the form

$$\left(\sum_{i\in\omega} y_i, -\psi_0(y_0), -\psi_1\left(y_1\right), \ldots\right)$$

for some $(y_i) \in M_1^{(\omega)}$. We can also identify M_1 with a submodule of M, via the map that assigns $a \in M_1$ to the element of M represented by the sequence $(a, 0, 0, \ldots)$. Notice that the sequence

$$M_1 \to M \to \bigoplus_i \mathrm{Coker}\left(\psi_i : M_1 \to M_{(i;0)}\right)$$

is exact at M.

It is easily verified that, adopting the notation above, if ψ_i is injective with pure image for every $i \in \omega$, then N is pure in M_∞ and M_1 is pure in M.

5.10.2 Tree length

Recall that a countable flat module C is *finite* if it is finitely-generated. We say that a countable flat module has plain tree length 0 if and only if it is finite. We now define by recursion on α the notion of (plain) tree length at most α.

Definition 5.108. *Let C be a countable flat module, and $\alpha < \omega_1$:*

- *if α is a successor, then C has plain tree length at most α if and only if there exist countable flat modules $M_{(i;0)}$ of plain tree length at most $(\alpha - 1)_i$, a finiteflat module M_1, and injective module homomorphisms $\psi_i : M_1 \to M_{(i;0)}$ such that C is a direct summand of the wedge sum*
$$\bigoplus_{(M_1,\psi_n)} M_{(i;0)};$$
- *C has tree length at most α if and only if C is a direct summand of a direct sum of modules C_n of plain tree length at most α_n for $n < \omega$.*

Definition 5.109. *For $\alpha < \omega_1$, we let:*

- *$\mathcal{S}_\alpha$ be the class of countable flat modules of plain tree length at most α;*
- *$\mathcal{E}_\alpha$ be the exact structure on $\mathbf{Flat}\,(R)$ projectively generated by the union of $\mathcal{S}_{\alpha_n}$ for $n \in \omega$;*
- *$\mathcal{P}_\alpha$ be the class of $\mathcal{E}_\alpha$-projective countable flat modules.*

Considering that colimits commute with colimits, a straightforward induction on α proves the following characterization of modules of (plain) tree length at most α.

Proposition 5.110. *Suppose that C is a countable flat module. Then the following assertions are equivalent:*

1. *C has plain tree length at most α;*
2. *C is isomorphic to a direct summand of a colimit of a presheaf of finiteflat modules over $I^{\mathrm{plain}}_{1+\alpha}$;*
3. *C is isomorphic to a direct summand of a colimit of a presheaf of finiteflat modules over a countable well-founded tree of rank at most $1 + \alpha$.*

Furthermore, the following assertions are equivalent:

1. *C has tree length at most α;*
2. *C is isomorphic to a direct summand of a colimit of a presheaf of finiteflat modules over $I_{1+\alpha}$;*
3. *C is isomorphic to a direct summand of a colimit of a presheaf of finiteflat modules over a countable well-founded forest of rank at most $1 + \alpha$.*

We now relate the (plain) tree length with the (plain) projective length of a module.

Lemma 5.111. *Suppose that a countable flat module C is a wedge sum*

$$\bigoplus_{(M_1,\psi_i)} M_{(i;0)}$$

for some finiteflat module M_1, flat modules $M_{(i;0)}$, and injective module homomorphisms $\psi_i : M_1 \to M_{(i;0)}$. For $n \in \omega$, let C_n be the submodule of C corresponding to $M_1 \oplus M_{(i;0)} \oplus \cdots \oplus M_{(i;n-1)}$. Let A be a countable flat module, and $\alpha, \beta < \omega_1$. For $n < \omega$, set

$$\mathrm{Hom}\left(C|C_n, A\right) := \mathrm{Ran}\left(\mathrm{Hom}\left(C, A\right) \to \mathrm{Hom}\left(C_n, A\right)\right).$$

1. *The tower*
$$\left(\frac{\mathrm{Hom}\left(C_n, A\right)}{\mathrm{Hom}\left(C|C_n, A\right)}\right)_{n\in\omega}$$
is monomorphic;

2. *If $M_{(i;0)}, \ldots, M_{(i;n)}$ have plain A-projective length at most β, then C_n has plain A-projective length at most β;*

3. *If $M_{(i;0)}$ has plain A-projective length at most $(\alpha - 1)_i$ for every $i \in \omega$, then C has plain A-projective length at most α.*

Proof. (1) Suppose that $\varphi \in \mathrm{Hom}\left(C_{n+1}, A\right)$ is such that $\varphi|_{C_n} \in \mathrm{Hom}\left(C|C_n, A\right)$. Thus, there exists a homomorphism $\Psi \in \mathrm{Hom}\left(C, A\right)$ such that $\Psi|_{C_n} = \varphi|_{C_n}$. One can then define a homomorphism $\Phi \in \mathrm{Hom}\left(C, A\right)$ by setting

$$\Phi\left(x\right) := \begin{cases} \varphi\left(x\right) & x \in C_{n+1}; \\ \Psi\left(x\right) & x \notin C_{n+1}. \end{cases}$$

Then we have that $\Phi|_{C_{n+1}} = \varphi$.

(2) Fix $n \in \omega$. Notice that we have a short exact sequence

$$M_1^n \to M_1 \oplus M_{(0;0)} \oplus \cdots \oplus M_{(0;n-1)} \to C_n.$$

Since $M_{(0;0)}, \ldots, M_{(0;n-1)}$ have plain A-projective length at most β, the same holds for $M_1 \oplus M_{(0;0)} \oplus \cdots \oplus M_{(0;n-1)}$. Set

$$G_n := \mathrm{Ext}\left(C_n, A\right)$$

and

$$\begin{aligned} H_n &:= \mathrm{Ran}\left(\mathrm{Hom}(M_1^n, A) \to \mathrm{Ext}(C_n, A)\right) \\ &= \mathrm{Ker}\left(\mathrm{Ext}(C_n, A) \to \mathrm{Ext}\left(M_1 \oplus M_{(0;0)} \oplus \cdots \oplus M_{(0;n-1)}, A\right)\right) \end{aligned}$$

Then G_n/H_n is isomorphic to a submodule of $\mathrm{Ext}\left(M_1 \oplus M_{(0;0)} \oplus \cdots \oplus M_{(0;n-1)}, A\right)$, whence it has plain Solecki length at most β. Since M_1^n is finite, H_n is countable, and hence G_n has plain Solecki length at most β by Lemma 5.179(3).

(3) By (2), we have

$$\mathrm{Ph}^{\alpha-1}\mathrm{Ext}\left(C,A\right) \subseteq \bigcap_{n\in\omega} \mathrm{Ker}\left(\mathrm{Ext}\left(C,A\right) \to \mathrm{Ext}\left(C_n,A\right)\right) \cong \lim\nolimits_n^1 \mathrm{Hom}\left(C_n,A\right) \cong \lim\nolimits_n^1 \frac{\mathrm{Hom}\left(C_n,A\right)}{\mathrm{Hom}\left(C|C_n,A\right)}$$

By (1), the tower

$$\left(\frac{\mathrm{Hom}\left(C_n,A\right)}{\mathrm{Hom}\left(C|C_n,A\right)}\right)_{n\in\omega}$$

is monomorphic. Furthermore,

$$\lim\nolimits_n^1\left(\frac{\mathrm{Hom}\left(C_n,A\right)}{\mathrm{Hom}\left(C|C_n,A\right)}\right) \cong \lim\nolimits_n^1 \mathrm{Hom}\left(C_n,A\right).$$

It follows that $\lim_n^1 \mathrm{Hom}\left(C_n,A\right)$ has plain Solecki length at most 1. As $\mathrm{Ph}^{\alpha-1}\mathrm{Ext}\left(C,A\right)$ is isomorphic to a submodule of $\lim_n^1 \mathrm{Hom}\left(C_n,A\right)$, the same holds for $\mathrm{Ph}^{\alpha-1}\mathrm{Ext}\left(C,A\right)$. Thus, we have that $\mathrm{Ext}\left(C,A\right)$ has plain Solecki length at most α. □

Lemma 5.112. *Suppose that C is a countable flat module, and $\alpha < \omega_1$:*

1. *if C has plain tree length at most α, then C has plain projective length at most α;*
2. *if C has tree length at most α, then C has projective length at most α.*

Proof. We prove this by induction on α. It suffices to notice that (2) for α is a consequence of (1) for α_n for $n < \omega$, while (1) for α follows immediately from (1) for α_n for $n < \omega$ by Lemma 5.111(3). □

5.10.3 Resolutions

Let C be a countable flat module. Write $C = \mathrm{colim}_n C_n$ for some finite submodules C_n of C. We define by recursion, for every countable successor ordinal α, countable flat modules $P_n^\alpha C$ of plain length at most α containing C_n as a pure submodule, together with module homomorphisms $P_n^{(\alpha-1)_n} C \to P_{n+1}^{(\alpha-1)_{n+1}} C$ and $P_n^\alpha C \to C$ that restrict to the inclusion $C_n \to C$ and yield an isomorphism

$$C \cong \mathrm{colim}_n P_n^{\alpha_n} C.$$

Define $P_n^0 C = C_n$ for $n \in \omega$. Suppose that $P_n^\beta\left(C\right)$ has been defined for every successor ordinal $\beta < \alpha$ and $n \in \omega$. Define then $P_n^\alpha\left(C\right)$ to be the wedge sum of $P_k^{(\alpha-1)_k} C$ for $k \geq n$ over C_n. The inclusion $C_n \to P_n^\alpha C$ is the canonical one given by the definition of wedge sum. The homomorphism $P_n^\alpha C \to C$ is induced by the homomorphisms $P_k^{(\alpha-1)_k} C \to C$

for $k \geq n$. By definition, $P_n^\alpha C$ has plain tree length at most α for every $n \in \omega$. Let $\mathfrak{R}^\alpha C$ be the short exact sequence

$$0 \to P^\alpha C \to P^\alpha C \to C \to 0$$

realizing C as $\mathrm{colim}_n P_n^{\alpha_n} C$.

Theorem 5.113. *Let C and A be countable flat modules. Let (C_n) be an increasing sequence of finite submodules of C with union equal to C. Define*

$$\mathrm{Hom}\left(C_n, A\right) := \mathrm{Hom}_0\left(C_n, A\right)$$

and recursively

$$\mathrm{Hom}_\alpha\left(C_n, A\right) := \bigcap_{k \geq n} \mathrm{Ran}(\mathrm{Hom}_{\alpha_k}\left(C_k, A\right) \to \mathrm{Hom}\left(C, A\right)).$$

Then we have that the tower $\left(\mathrm{Hom}\left(C_n, A\right)\right)_{n\in\omega}$ has $\left(\mathrm{Hom}_{\alpha_n}\left(C_n, A\right)\right)_{n\in\omega}$ as its α-th derived tower.

1. *for $\alpha < \omega_1$ and $n < \omega$*

$$\mathrm{Hom}_\alpha\left(C_n, A\right) = \mathrm{Hom}\left(P_n^\alpha C | C_n, A\right) := \mathrm{Ran}\left(\mathrm{Hom}\left(P_n^\alpha C, A\right) \to \mathrm{Hom}\left(C_n, A\right)\right);$$

2. *$P_n^\alpha\left(C\right)$ has plain projective length at most α;*

3. $$\mathrm{Ph}^\alpha \mathrm{Ext}\left(C, A\right) = \bigcap_{n\in\omega} \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}(P_n^{\alpha_n} C, A)\right) \cong \lim\nolimits_n^1 \mathrm{Hom}_{\alpha_n}\left(C, A\right)$$

4. *a countable flat module has tree length at most α if and only if it has projective length at most α;*

5. *for every countable flat module B of plain projective length α, finite submodule $A \subseteq B$, $d_0 \in \omega$, and homomorphism $\varphi : B \to C$ that maps A to C_{d_0}, there exist $d \geq d_0$ and a homomorphism $\bar{\varphi} : B \to P_d^\alpha C$ that maps A to $C_d \subseteq P_d^\alpha C$ and such φ is equal to the composition of $\bar{\varphi}$ and the canonical homomorphism $P_n^\alpha C \to C$;*

6. *$\mathfrak{R}^\alpha C$ represents an element of $\mathrm{Ph}^\alpha \mathrm{Ext}\left(C, P^\alpha C\right)$ and it is a projective resolution of C in the exact category $\left(\mathbf{Flat}\left(R\right), \mathcal{E}_\alpha\right)$;*

7. *the exact category $\left(\mathbf{Flat}\left(R\right), \mathcal{E}_\alpha\right)$ is hereditary with enough projectives;*

8. *if $\mathbf{Ext}_{\mathcal{E}_\alpha}$ is the total right derived functor of Hom on the exact category $\left(\mathbf{Flat}\left(R\right), \mathcal{E}_\alpha\right)$, then*

$$\mathrm{Ph}^\alpha \mathrm{Ext} = \mathrm{Ext}^1_{\mathcal{E}_\alpha} = \mathrm{H}^1 \circ \mathbf{Ext}_{\mathcal{E}_\alpha}.$$

Proof. By induction on α. As the case $\alpha = 0$ is obvious, we can assume $\alpha > 0$.

(1) By definition, we have

$$\mathrm{Hom}_\alpha\left(C_n, A\right) = \bigcap_{k \geq n} \mathrm{Hom}_{(\alpha-1)_k}\left(C_k | C_n, A\right).$$

By the inductive hypothesis, we have that

$$\mathrm{Hom}_{(\alpha-1)_k}\left(C_k, A\right) = \mathrm{Hom}(P_k^{(\alpha-1)_k} C | C_k, A)$$

and hence

$$\mathrm{Hom}_\alpha\left(C_n, A\right) = \bigcap_{k \geq n} \mathrm{Hom}(P_k^{(\alpha-1)_k} C | C_n, A).$$

Considering that by definition $P_n^\alpha C$ is the wedge sum of $(P_k^{(\alpha-1)_k} C)$ over C_n, we obtain

$$\mathrm{Hom}_\alpha\left(C_n, A\right) = \mathrm{Hom}\left(P_n^\alpha C | C_n, A\right).$$

(2) Since $P_n^\alpha C$ has plain tree length at most α, it also has plain projective length at most α by Lemma 5.112.

(3) When α is limit, we have

$$\mathrm{Ph}^\alpha \mathrm{Ext}\left(C, A\right) = \bigcap_{n \in \omega} \mathrm{Ph}^{\alpha_n} \mathrm{Ext}\left(C, A\right).$$

By the inductive hypothesis, we have

$$\mathrm{Ph}^{\alpha_n} \mathrm{Ext}\left(C, A\right) = \bigcap_{k \geq n} \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}(P_k^{\alpha_n} C, A)\right).$$

Therefore,

$$\begin{aligned} \mathrm{Ph}^\alpha \mathrm{Ext}\left(C, A\right) &= \bigcap_{n \in \omega} \bigcap_{k \geq n} \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}(P_k^{\alpha_n} C, A)\right) \\ &= \bigcap_{n \in \omega} \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}(P_n^{\alpha_n} C, A)\right) \\ &\cong \lim\nolimits_n^1 \mathrm{Hom}_{\alpha_n}\left(C, A\right). \end{aligned}$$

Suppose now that α is a successor. By the inductive hypothesis, we have

$$\begin{aligned} \mathrm{Ph}^{\alpha-1} \mathrm{Ext}\left(C, A\right) &= \bigcap_{n \in \omega} \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}\left(P_n^{\alpha-1} C, A\right)\right) \\ &\cong \lim\nolimits_n^1 \mathrm{Hom}_{\alpha-1}\left(C_n, A\right). \end{aligned}$$

Under such an isomorphism we have that

$$\mathrm{Ph}^\alpha \mathrm{Ext}\left(C, A\right) = s_1 \mathrm{Ph}^{\alpha-1} \mathrm{Ext}\left(C, A\right)$$

corresponds to

$$\lim_n^1 \mathrm{Hom}_\alpha\left(C_n, A\right) = \lim_n^1 \mathrm{Ran}(\mathrm{Hom}\left(P_n^\alpha C, A\right) \to \mathrm{Hom}\left(C, A\right))$$

which in turn corresponds to

$$\bigcap_{n\in\omega} \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}\left(P_n^\alpha C, A\right)\right).$$

(4) If C has tree length at most α, then it has projective length at most α by Lemma 5.112. Conversely, suppose that C has projective length at most α. Then we have that $\mathrm{Ph}^\alpha\mathrm{Ext}\left(C, P^\alpha C\right) = 0$ and hence the short exact sequence $\mathfrak{R}^\alpha C$ splits, and C is a direct summands of $P^\alpha C$. This shows that C has tree length at most α.

(5) We have that B is the wedge sum

$$\bigoplus_{(B_1,\psi_i)} B_{(i;0)}$$

of a sequence $\left(B_{(i;0)}\right)_{i\in\omega}$ of modules of plain tree length $(\alpha-1)_i$ over a finite module B_1 with respect to homomorphisms $\psi_i : B_1 \to B_{(i;0)}$. After replacing A with the submodule generated by A and B_1, and d_0 with a larger integer, we can assume that $B_1 \subseteq A$. For every $n \in \omega$ let $\varphi_n : B_{(n;0)} \to C$ be the homomorphism induced by $\varphi : B \to C$. Then we have that there exists a homomorphism $\bar{\varphi}_n : B_{(n;0)} \to P_{d_n}^{(\alpha-1)_n} C$ for some $d_n \geq d_0$ that maps $A_n := A \cap B_{(n;0)}$ to C_{d_n} and such that the φ_n is equal to the composition of $\bar{\varphi}_n$ with the canonical homomorphism $P_{d_n}^{(\alpha-1)_n} C$. If $n_0 \in \omega$ is such that $A_n = B_n$ for $n \geq n_0$, then $(\bar{\varphi}_n)$ induce a homomorphism $\bar{\varphi} : B \to P_{d_{n_0}}^\alpha C$ that maps A to $C_{d_{n_0}}$.

(6) The fact that $\mathfrak{R}_\alpha C$ represents an element of $\mathrm{Ph}^\alpha\mathrm{Ext}\left(C, P^\alpha C\right)$ follows from (3). The fact that $\mathfrak{R}_\alpha C$ is $\mathcal{E}_\alpha$-exact follows from (5) and the fact that $P_n^\alpha C$ has plain tree length at most α for every $n \in \omega$. By definition, we have that $P^\alpha C$ is $\mathcal{E}_\alpha$-projective.

(7) This is an immediate consequence of (6).

(8) Let $\mathfrak{S}$ be the exact sequence $A \to X \to C$. If $\mathfrak{S}$ represents an element of $\mathrm{Ph}^\alpha\mathrm{Ext}\left(C, A\right)$, then by (3) we have that $\mathfrak{S}$ represents an element of

$$\mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}(P_n^{\alpha_n} C, A)\right).$$

If B has plain tree length at most α and $\varphi : B \to C$ is a homomorphism, then by (5) φ factors through $P_n^{\alpha_n} C \to C$ for some $n \in \omega$. Thus, $\mathfrak{S}$ represents an element of

$$\mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}\left(B, A\right)\right).$$

As this holds for every countable flat module B of plain tree length at most α, $\mathfrak{S}$ is $\mathcal{E}_\alpha$-exact. Conversely, if $\mathfrak{S}$ is $\mathcal{E}_\alpha$-exact, then since $P_n^{\alpha_n} C$ has plain tree length at most α_n, $\mathfrak{S}$ represents an element of

$$\mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}(P_n^{\alpha_n} C, A)\right).$$

As this holds for every $n \in \omega$, we have that $\mathfrak{S}$ represents an element of $\mathrm{Ph}^{\alpha}\mathrm{Ext}\left(C, A\right)$ by (3). □

Corollary 5.114. *Fix $\alpha < \omega_1$. The functor*

$$\mathrm{Hom} : \left(\mathbf{Flat}\left(R\right), \mathcal{E}_{\alpha}\right) \times \left(\mathbf{Flat}\left(R\right), \mathcal{E}_{\alpha}\right) \to \mathbf{\Pi}(\mathbf{Flat}\left(R\right))$$

has a total right derived functor $\mathrm{RHom}^{\bullet}_{\mathcal{E}_{\alpha}}$. We can then define $\mathrm{Ext}^{n}_{\mathcal{E}_{\alpha}} := \mathrm{H}^{n} \circ \mathrm{RHom}^{\bullet}_{\mathcal{E}_{\alpha}}$ for $n \in \mathbb{Z}$. Then we have that $\mathrm{Ext}^{0}_{\mathcal{E}_{\alpha}} \cong \mathrm{Hom}$, $\mathrm{Ext}^{1}_{\mathcal{E}_{\alpha}} \cong \mathrm{Ph}^{\alpha}\mathrm{Ext}$, and, for $n \in \mathbb{Z} \setminus \{0, 1\}$, $\mathrm{Ext}^{n}_{\mathcal{E}_{\alpha}} = 0$.

Proof. This follows from Theorem 5.113 by Proposition 5.13. □

Corollary 5.115. *Let C be a countable flat module. The following assertions are equivalent:*

1. *C projective in $\left(\mathbf{Flat}\left(R\right), \mathcal{E}_{\alpha}\right)$;*
2. *C has projective length at most α;*
3. *C has tree length at most α;*
4. *C has A-projective length at most α for every $n \in \omega$ and countable flat module A of plain tree length at most α_n.*

Proof. This follows from Theorem 5.113. □

5.10.4 Tree* length

We now define a modified version of the notion of tree length, which we call tree* length, obtained by replacing finite modules with finite-rank modules. Thus, we say that a countable flat module has plain tree* length 1 if and only if it has finite rank. We now define by recursion on α the notion of (plain) tree* length at most α.

Definition 5.116. *Let C be a countable flat module, and $2 \leq \alpha < \omega_1$:*

- *if α is a successor, then C has plain tree* length at most α if and only if there exist countable flat modules $M_{(i;0)}$ of plain tree* length at most $(\alpha - 1)_i$, a finite-rank module M_1, and injective module homomorphisms $\psi_i : M_1 \to M_{(i;0)}$ with pure image such that C is isomorphic to the wedge sum*

$$\bigoplus_{(M_1, \psi_i)} M_{(i;0)};$$

- *C has tree* length at most α if and only if C is a direct summand of a direct sum of modules C_n of plain tree* length at most α_n for $n < \omega$.*

Definition 5.117. *For $\alpha < \omega_1$, we let:*

- *$\mathcal{S}^*_\alpha$ be the class of countable flat modules of plain tree* length at most α;*
- *$\mathcal{E}^*_\alpha$ be the exact structure on $\mathbf{Flat}\,(R)$ projectively generated $\mathcal{S}_{\alpha_n}$ for $n \in \omega$.*

The same proofs as Proposition 5.110 and Lemma 5.112 yield the following:

Proposition 5.118. *Suppose that C is a countable flat module. Then the following assertions are equivalent:*

1. *C has plain tree* length at most α;*
2. *C is isomorphic to a direct summand of a colimit of a presheaf of finite-rank flat modules over $I^{\mathrm{plain}}_\alpha$;*
3. *C is isomorphic to a direct summand of a colimit of a presheaf of finite-rank flat modules over a countable well-founded tree of rank at most α.*

Furthermore, the following assertions are equivalent:

1. *C has tree* length at most α;*
2. *C is isomorphic to a direct summand of a colimit of a presheaf of finite-rank flat modules over I_α;*
3. *C is isomorphic to a direct summand of a colimit of a presheaf of finite-rank flat modules over a countable well-founded forest of rank at most α.*

Lemma 5.119. *Suppose that C is a countable flat module, and $\alpha < \omega_1$:*

1. *if C has plain tree* length at most α, then C has plain projective length at most α;*
2. *if C has tree* length at most α, then C has projective length at most α.*

Let C be a countable flat module. Write $C = \mathrm{colim}_n C_n$ for some *finite-rank* pure submodules C_n of C. We define as before by recursion for every countable successor ordinal α countable flat modules $Q^\alpha_n C$ of plain tree* length at most α containing C_n as a pure submodule. We then let $\mathfrak{R}^\alpha_* C$ be the short exact sequence

$$0 \to Q^\alpha C \to Q^\alpha C \to C \to 0$$

realizing C as $\mathrm{colim}_n Q^{\alpha_n}_n C$. The same proof as Theorem 5.113 yields.

Theorem 5.120. *Let C and A be countable flat modules. Let (C_n) be an increasing sequence of finite-rank pure submodules of C with union equal to C. Define*

$$\mathrm{Hom}\,(C_n, A) := \mathrm{Hom}_1\,(C_n, A)$$

and recursively

$$\mathrm{Hom}_\alpha\,(C_n, A) := \bigcap_{k\geq n} \mathrm{Ran}(\mathrm{Hom}_{\alpha_k}\,(C_k, A) \to \mathrm{Hom}\,(C, A)).$$

Then we have that $(\mathrm{Hom}\,(C_n, A))$ has $(\mathrm{Hom}_{\alpha_n}\,(C_n, A))$ as its α-th derived tower.

1. *for $\alpha < \omega_1$ and $n < \omega$*

$$\mathrm{Hom}_\alpha\,(C_n, A) = \mathrm{Hom}\,(Q^\alpha_n C|C, A) := \mathrm{Ran}\,(\mathrm{Hom}\,(Q^\alpha_n C, A) \to \mathrm{Hom}\,(C, A))\,;$$

2. *$Q^\alpha_n\,(C)$ has plain projective length at most α;*

3. $$\mathrm{Ph}^\alpha\mathrm{Ext}\,(C, A) = \bigcap_{n\in\omega} \mathrm{Ker}\,(\mathrm{Ext}\,(C, A) \to \mathrm{Ext}(Q^{\alpha_n}_n C, A)) \cong \lim\nolimits^1_n \mathrm{Hom}_{\alpha_n}\,(C, A)$$

4. *a countable flat module has tree* length at most α if and only if it has projective length at most α;*

5. *for every countable flat module B of plain projective length α and finite-rank submodule $A \subseteq B$, for every homomorphism $\varphi : B \to C$ that maps A to C_{d_0} there exists $d \geq d_0$ and a homomorphism $\bar\varphi : B \to Q^\alpha_d C$ that maps A to $C_d \subseteq Q^\alpha_d C$ and such φ is equal to the composition of $\bar\varphi$ and the canonical homomorphism $Q^\alpha_n C \to C$;*

6. *$\mathfrak{R}^\alpha_* C$ represents an element of $\mathrm{Ph}^\alpha\mathrm{Ext}\,(C, Q^\alpha C)$ and it is a projective resolution of C in the exact category $(\mathbf{Flat}\,(R)\,, \mathcal{E}^*_\alpha)$;*

7. *the exact structures $\mathcal{E}_\alpha$ and $\mathcal{E}^*_\alpha$ coincide.*

Proof. By induction on α. The case $\alpha = 1$ is established in Section 5.9.9. The inductive step is the same as in the proof of Theorem 5.113. □

Corollary 5.121. *For every $\alpha < \omega_1$, the tree length and the tree* length of a countable flat module coincide, as they are both equal to the projective length.*

5.11 Extractable modules

In this section we continue to assume that R is a countable Prüfer domain, and all modules are R-modules. For each countable limit ordinal α we let (α_k) be an increasing sequence of *successor* ordinals cofinal in α. If α is a successor ordinal, we let α_k be equal to α.

5.11.1 Derivation

Suppose that C is a coreduced flat module. We define, by recursion on $\alpha < \omega_1$, functorial *pure coreduced* submodules $\sigma_\alpha C \subseteq C$ and flat quotients $\partial_\alpha C = C/\sigma_\alpha C$, as well as a countable upward-directed family $\mathcal{B}_\alpha(C)$ of pure submodules of C, which we call the α-constituents of C.

Let $\mathcal{B}_1(C)$ be the family of finite-rank pure submodules of C. For $B \in \mathcal{B}_1(C)$, we can write

$$B = \Sigma B \oplus \Phi B$$

by Lemma 5.81. We let $\left(B_1^k C\right)$ be a cofinal increasing sequence in $\mathcal{B}_1(C)$. Define

$$\sigma_1 C := \mathrm{colim}_{B\in\mathcal{B}_1(C)} \Sigma B = \mathrm{colim}_k \Sigma B_1^k C$$

and

$$\partial_1 C = C/\sigma_1 C = \mathrm{colim}_{B\in\mathcal{B}_1(C)} \Phi B = \mathrm{colim}_k \Phi B_1^k C.$$

Define $\mathcal{B}_{\alpha+1}(C)$ to be the collection of preimages of finite-rank pure submodules of $\partial_\alpha C$ under the quotient map $C \to \partial_\alpha C$. Define then

$$\sigma_{\alpha+1} C := \mathrm{colim}_{B\in\mathcal{B}_{\alpha+1}(C)} \Sigma B = \mathrm{colim}_k \Sigma B_{\alpha+1}^k C$$

and

$$\partial_{\alpha+1} C := C/\sigma_{\alpha+1} C \cong \mathrm{colim}_{B\in\mathcal{B}_{\alpha+1}(C)} \Phi B \cong \mathrm{colim}_k \Phi B_{\alpha+1}^k C.$$

Notice that $\sigma_{\alpha+1} C$ is the preimage of $\sigma_1(\partial_\alpha C)$ under the quotient map $C \to \partial_\alpha C$. If α is limit, define $\mathcal{B}_\alpha(C)$ to be the union of $\mathcal{B}_\beta(C)$ for $\beta < \alpha$, and $\sigma_\alpha C$ to be the union of $\sigma_\beta C$ for $\beta < \alpha$. We have that $\left(B_{\alpha_k}^k C\right)$ is a cofinal increasing sequence in $\mathcal{B}_\alpha(C)$. Notice that

$$\sigma_\alpha C = \mathrm{colim}_{B\in\mathcal{B}_\alpha(C)} \Sigma B = \mathrm{colim}_k \Sigma B_{\alpha_k}^k C.$$

We also set

$$\partial_\alpha C := C/\sigma_\alpha C \cong \mathrm{colim}_{B\in\mathcal{B}_\alpha(C)} \Phi B \cong \mathrm{colim}_k \Phi B_{\alpha_k}^k C.$$

For an arbitrary α, define now $F_\alpha C$ to be the direct sum of $B_{\alpha_k}^k C$ for $k \in \omega$. Consider the short exact sequence

$$0 \to L_\alpha C \xrightarrow{\eta_\alpha^C} F_\alpha C \xrightarrow{\pi_\alpha^C} C \to 0$$

that realizes C as the colimit $\mathrm{colim}_{k\in\omega} B_{\alpha_k}^k C$, where the homomorphism $\pi_\alpha^C : F_\alpha C \to C$ is induced by the inclusion maps $B_{\alpha_k}^k C \to C$ for each $k \in \omega$. Notice that $L_\alpha C$ is also isomorphic to a direct sum of copies $B_{\alpha_k}^k C$ for $k < \omega$.

5.11.2 Extractable length

We define by induction on $\alpha < \omega_1$ what it means for a coreduced countable flat module to have (plain) *extractable length* at most α.

Definition 5.122. *A coreduced countably-presented module C:*

- *has extractable length* 0 *if and only if it is trivial;*
- *has plain extractable length at most* $\alpha+1$ *if and only if there exists a short exact sequence*

$$0 \to B \to C \to A$$

where B *is a countably-presented module of extractable length at most* α *and* A *is a countably-presented module with* A/A_{t} *finite-rank;*

- *has extractable length at most* α *if and only if it is a direct summand of a countable direct sum of modules of plain extractable length at most* α*.*

We say that a countably-presented flat module is extractable *if it has extractable length* α *for some* $\alpha < \omega_1$*.*

Proposition 5.123. *Suppose that* C *is a extractable flat module. Then for every* $\alpha < \omega_1$*,* $\sigma_\alpha C$ *and* $\partial_\alpha C$ *are extractable.*

Proof. We prove that the conclusion holds by induction on the extractable length of C. If

$$C = \bigoplus_n C_n$$

and the conclusion holds for every $n \in \omega$, then it clearly holds for C. The same applies if C is a direct summand of a module D for which the conclusion applies. Suppose now that there is an extension

$$0 \to D \to C \to E \to 0$$

where E/E_{t} has finite-rank and for every $\alpha < \omega_1$ we have that $\sigma_\alpha D$ and $\partial_\alpha D$ are extractable. Then for every $\alpha < \omega_1$ we have short exact sequences

$$0 \to \sigma_\alpha D \to \sigma_\alpha C \to E_\alpha \to 0$$

and

$$0 \to \partial_\alpha D \to \partial_\alpha D \to E'_\alpha \to 0$$

where $E_\alpha/E_{\alpha,\mathrm{t}}$ and $E'_\alpha/E'_{\alpha,\mathrm{t}}$ have finite rank. Thus, $\sigma_\alpha C$ and $\partial_\alpha D$ are extractable. □

Proposition 5.124. *Suppose that* C *is a countably-presented module of (plain) extractable length at most* α*, and* $E \subseteq C$ *has finite rank. Then* C/E *has (plain) extractable length at most* α*.*

Proof. By induction on α. This is obvious for length 0. Suppose that this holds for length α. If C has plain extractable length at most $\alpha + 1$, then we have an extension

$$0 \to B \to C \to A \to 0$$

where B has extractable length at most α and A/A_{t} has finite rank. If $E \subseteq B$ then we have a short exact sequence

$$0 \to B/E \to C/E \to A \to 0$$

and the conclusion follows from the inductive hypothesis. After replacing E with $E/(B \cap E)$ and C with $C/(B \cap E)$ we can assume that $E \cap B = 0$. In this case, we have a short exact sequence

$$0 \to B \to C/E \to A' \to 0$$

for some module A' such that A'/A'_{t} has finite rank. The conclusion again follows.

Suppose now that the conclusion holds for plain extractable length less than α. If C has extractable length α, then C is isomorphic to the direct sum of a sequence (C_n) where for every $n \in \omega$, C_n has plain extractable length less than α. Without loss of generality we can assume that $E \subseteq C_0$, whence the conclusion follows from the inductive hypothesis. □

Proposition 5.125. *Suppose that C is a extractable countable flat module. Then the (plain) projective length of C is less than or equal to the (plain) extractable length of C.*

Proof. Fix a countable flat module A. We prove by induction on $\alpha < \omega_1$ that if C has (plain) extractable length at most α, then it has (plain) A-projective length at most α. Suppose that C has plain extractable length at most α. Then we have an exact sequence

$$0 \to D \to C \to E \to 0$$

where E/E_{t} has finite rank and D has extractable length at most $\alpha - 1$. Then for every countable flat module A, we have an exact sequence

$$\mathrm{Ext}(E, A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(D, A) \to 0.$$

By the inductive hypothesis, we have that $s_{\alpha-1}\mathrm{Ext}(D, A) = 0$, whence

$$s_{\alpha-1}\mathrm{Ext}(C, A) \subseteq \mathrm{Ran}(\mathrm{Ext}(E, A) \to \mathrm{Ext}(C, A)).$$

Since E/E_{t} has finite rank, $\{0\} \in \mathbf{\Sigma}^0_2(\mathrm{Ext}(E, A))$ by Theorem 5.105, and hence

$$\{0\} \in \mathbf{\Sigma}^0_2(\mathrm{Ran}(\mathrm{Ext}(E, A) \to \mathrm{Ext}(C, A)))$$

by Lemma 5.49. Thus,

$$\{0\} \in \mathbf{\Sigma}^0_2(s_{\alpha-1}\mathrm{Ext}(C, A)).$$

This shows that C is has plain projective length at most α.

Suppose now that C has extractable length at most α. Without loss of generality, we can assume that C is a direct sum of countable coreduced modules C_i for $i \in \omega$ of plain extractable length at most α. Then for every $i \in \omega$ we have that C_i has projective length at most α. Therefore the same conclusion holds for C. □

5.11.3 Extensions by projective modules

We now characterize the Solecki submodules of $\mathrm{Ext}\,(C, A)$ in terms of the derivatives of C when A is a projective module.

Theorem 5.126. *Let C be a countable flat coreduced module, and A be a countable* projective *module. Fix $\alpha < \omega_1$.*

1. *the homomorphism*
$$\mathrm{Ext}\,(\partial_\alpha C, A) \to \mathrm{Ext}\,(C, A)$$
is injective;

2. *if α is limit,*
$$\mathrm{Ext}\,(\partial_\alpha C, A) = \bigcap_{\beta<\alpha} \mathrm{Ext}\,(\partial_\beta C, A)\,;$$

3. *for α arbitrary,*
$$s_\alpha \mathrm{Ext}\,(C, A) = \mathrm{Ext}\,(\partial_\alpha C, A)\,;$$

4. $\mathrm{Ext}\,(C, A)$ *has (plain) Solecki length α if and only if $\partial_\alpha C = 0$ (and $\partial_{\alpha-1} C$ has finite rank).*

Proof. (1) Since C is coreduced, we have that $\mathrm{Hom}\,(\sigma_\alpha C, A) = 0$. Thus, the conclusion follows from the long exact sequence relating Hom and Ext;

(2) If α is limit, we have a short exact sequence

$$0 \to \lim_k{}^1 \mathrm{Hom}\,(\sigma_{\alpha_k} C, A) \to \mathrm{Ext}\,(\sigma_\alpha C, A) \to \lim_k \mathrm{Ext}\,(\sigma_{\alpha_k} C, A) \to 0$$

As $\lim_k{}^1 \mathrm{Hom}\,(\sigma_{\alpha_k} C, A) = 0$, this yields an isomorphism

$$\mathrm{Ext}\,(\sigma_\alpha C, A) \cong \lim_k \mathrm{Ext}\,(\sigma_{\alpha_k} C, A)\,.$$

Therefore, we have that

$$\begin{aligned}
\bigcap_{k\in\omega} \mathrm{Ext}\,(\partial_{\alpha_k} C, A) &= \mathrm{Ker}\,(\mathrm{Ext}\,(C, A) \to \lim_k \mathrm{Ext}\,(\sigma_{\alpha_k} C, A)) \\
&= \mathrm{Ker}\,(\mathrm{Ext}\,(C, A) \to \mathrm{Ext}\,(\sigma_\alpha C, A)) \\
&= \mathrm{Ran}\,(\mathrm{Ext}\,(\partial_\alpha C, A) \to \mathrm{Ext}\,(C, A))\,.
\end{aligned}$$

(3) We prove that the conclusion holds by induction on α. The case of limits follows from (2). Suppose that α is a successor ordinal. Then by the inductive hypothesis, we have that

$$s_{\alpha-1} \mathrm{Ext}\,(C, A) = \mathrm{Ext}\,(\partial_{\alpha-1} C, A)\,.$$

Thus, we have that

$$\begin{aligned} s_1\mathrm{Ext}\left(\partial_{\alpha-1}C, A\right) &= \bigcap_{k\in\omega} \mathrm{Ker}\left(\mathrm{Ext}\left(\partial_{\alpha-1}C, A\right) \to \mathrm{Ext}\left(B_1^k\left(\partial_{\alpha-1}C\right), A\right)\right) \\ &\cong \lim_k{}^1\mathrm{Hom}\left(B_1^k\left(\partial_{\alpha-1}C\right), A\right) \\ &\cong \lim_k{}^1\mathrm{Hom}\left(\Phi\left(B_{\alpha-1}^k C\right), A\right) \\ &\cong \mathrm{Ext}\left(\mathrm{colim}_k\Phi\left(B_{\alpha-1}^k C\right), A\right) \\ &\cong \mathrm{Ext}\left(\partial_\alpha C, A\right). \end{aligned}$$

(4) This is a consequence of (3) and Theorem 5.107. □

Corollary 5.127. *Suppose that C is a countable flat module. If $\alpha < \omega_1$, then $\sigma_\alpha C$ is the largest coreduced pure submodule of C of R-projective length at most α.*

The same proof as Theorem 5.126 gives the following more general result.

Theorem 5.128. *Let C, A are countable flat modules, with C coreduced. Suppose that*

$$\mathrm{Hom}\left(\sigma_\beta C, A\right) = 0 \text{ and } \mathrm{Hom}\left(\Sigma\left(B_1^k\left(\partial_\beta C\right)\right), A\right) = 0$$

for every $\beta < \omega_1$ and for all but finitely many $k \in \omega$. Then for every $\alpha < \omega_1$, we have:

1. *the homomorphism*
$$\mathrm{Ext}\left(\partial_\alpha C, A\right) \to \mathrm{Ext}\left(C, A\right)$$
is injective;

2. *if α is limit*
$$\mathrm{Ext}\left(\partial_\alpha C, A\right) = \bigcap_{\beta<\alpha} \mathrm{Ext}\left(\partial_\beta C, A\right);$$

3. *for arbitrary α,*
$$s_\alpha\mathrm{Ext}\left(C, A\right) = \mathrm{Ext}\left(\partial_\alpha C, A\right);$$

4. *C has (plain) A-projective length at most α if and only if $\partial_\alpha C = 0$ (and $\mathrm{Ext}\left(\partial_{\alpha-1}C, A\right) \to \mathrm{Ext}\left(B_1^k\left(\partial_{\alpha-1}C\right), A\right)$ is an isomorphism for some $k \in \omega$).*

Proof. (1) This follows from the hypothesis that $\mathrm{Hom}\left(\sigma_\alpha C, A\right) = 0$ and the long exact sequence relating Hom and Ext;

(2) If α is limit, we have a short exact sequence

$$0 \to \lim_k{}^1\mathrm{Hom}\left(\sigma_{\alpha_k} C, A\right) \to \mathrm{Ext}\left(\sigma_\alpha C, A\right) \to \lim_k\mathrm{Ext}\left(\sigma_{\alpha_k} C, A\right) \to 0.$$

As $\lim_k{}^1\mathrm{Hom}\left(\sigma_{\alpha_k} C, A\right) = 0$, this yields an isomorphism

$$\mathrm{Ext}\left(\sigma_\alpha C, A\right) \cong \lim_k\mathrm{Ext}\left(\sigma_{\alpha_k} C, A\right).$$

Therefore, we have that

$$\begin{aligned}\bigcap_{k\in\omega}\mathrm{Ext}\left(\partial_{\alpha_k}C,A\right) &= \mathrm{Ker}\left(\mathrm{Ext}\left(C,A\right)\to \lim_k\mathrm{Ext}\left(\sigma_{\alpha_k}C,A\right)\right)\\ &= \mathrm{Ker}\left(\mathrm{Ext}\left(C,A\right)\to \mathrm{Ext}\left(\sigma_{\alpha}C,A\right)\right)\\ &= \mathrm{Ran}\left(\mathrm{Ext}\left(\partial_{\alpha}C,A\right)\to \mathrm{Ext}\left(C,A\right)\right).\end{aligned}$$

(3) We prove that the conclusion holds by induction on α. The case of limits follows from (2). Suppose that α is a successor ordinal. Then by the inductive hypothesis, we have that

$$s_{\alpha-1}\mathrm{Ext}\left(C,A\right)=\mathrm{Ext}\left(\partial_{\alpha-1}C,A\right).$$

Thus, using that

$$\mathrm{Hom}\left(\Sigma\left(B_1^k\left(\partial_{\alpha-1}C\right)\right),A\right)=0$$

for all but finitely many $k\in\omega$, we have

$$\begin{aligned}s_1\mathrm{Ext}\left(\partial_{\alpha-1}C,A\right) &= \bigcap_{k\in\omega}\mathrm{Ker}\left(\mathrm{Ext}\left(\partial_{\alpha-1}C,A\right)\to\mathrm{Ext}\left(B_1^k\left(\partial_{\alpha-1}C\right),A\right)\right)\\ &\cong \lim\nolimits_k^1\mathrm{Hom}\left(B_1^k\left(\partial_{\alpha-1}C\right),A\right)\\ &\cong \lim\nolimits_k^1\mathrm{Hom}\left(\Phi\left(B_{\alpha-1}^kC\right),A\right)\\ &\cong \mathrm{Ext}\left(\mathrm{colim}_k\Phi\left(B_{\alpha-1}^kC\right),A\right)\\ &\cong \mathrm{Ext}\left(\partial_{\alpha}C,A\right).\end{aligned}$$

(4) This is a consequence of (3) and Lemma 5.57. □

5.11.4 Structure

We collect in the following theorem a number of results we will establish by induction on countable ordinals. As in the inductive step we will need the inductive hypotheses to include all these statements, we need to run a single inductive proof of all these results simultaneously.

Theorem 5.129. *For every countable ordinal α, and coreduced countable extractable flat module C:*

1. *if C has (plain) extractable length at most α, then C has (plain) projective length at most α;*
2. *if α is a successor, then C has plain extractable length at most α if and only if C has plain R-projective length at most α;*
3. *if α is a successor, then each α-constituent of C has plain extractable length at most α;*

4. *if E is a pure coreduced submodule of C of plain extractable length at most α, then there exists an α-constituent B of C such that $E \subseteq \Sigma B$;*

5. *if (E_k) is an increasing sequence of coreduced submodules of C of plain extractable length at most α containing $\sigma_\alpha C$ with union equal to C, then for every α-constituent B of C there exists $\ell \in \omega$ such that $\Sigma B \subseteq E_\ell$;*

6. *for every countable flat module A,*

$$s_\alpha \mathrm{Ext}\left(C, A\right) = \bigcap_B \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}\left(B, A\right)\right) = \bigcap_E \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}\left(E, A\right)\right)$$

   where B ranges among the α-constituents of C and E ranges among the pure coreduced submodules of C of plain extractable length at most α;

7. *we have that*

$$0 \to L_\alpha C \to F_\alpha C \to C \to 0$$

   is a phantom exact sequence of order α, where $L_\alpha C$ and $F_\alpha C$ have extractable length at most α;

8. *C has (plain) extractable length at most α if and only if it has (plain) projective length at most α;*

9. *C has (plain) extractable length at most α if and only if it has (plain) R-projective length at most α;*

10. *for every countable flat module A, the short exact sequence*

$$0 \to L_\alpha C \to F_\alpha C \to C \to 0$$

    induces an isomorphism

$$s_\alpha \mathrm{Ext}\left(C, A\right) \cong \frac{\mathrm{Hom}\left(L_\alpha C, A\right)}{\mathrm{Hom}\left(F_\alpha C | L_\alpha C, A\right)}$$

    where $\mathrm{Hom}\left(F_\alpha C | L_\alpha C, A\right)$ is the Polishable submodule of $\mathrm{Hom}\left(L_\alpha C, A\right)$ comprising the homomorphisms $L_\alpha C \to A$ that extend to a homomorphism $F_\alpha C \to A$.

Proof. We can assume without loss of generality that C is coreduced. We prove that the conclusion holds by induction on $\alpha < \omega_1$. Suppose that the conclusion holds for α. We set $S^k_\alpha C = B^k_\alpha C / \sigma_{\alpha-1} C = B^k_1 \partial_{\alpha-1} C$ for a successor ordinal $\alpha < \omega_1$ and $k < \omega$.

(1) This is the content of Proposition 5.125.

(2) Suppose that α is a successor and $\{0\} \in \mathbf{\Sigma}^0_2\left(s_{\alpha-1}\mathrm{Ext}\left(C, R\right)\right)$. By Theorem 5.126, we have that

$$s_{\alpha-1}\mathrm{Ext}\left(C, R\right) = \mathrm{Ext}\left(\partial_{\alpha-1} C, R\right).$$

Therefore, by Theorem 5.105 we have that $\partial_{\alpha-1}C$ is finite-rank. Furthermore, we have a short exact sequence

$$0 \to \sigma_{\alpha-1}C \to C \to \partial_{\alpha-1}C \to 0$$

where $\sigma_{\alpha-1}C$ has extractable length at most $\alpha - 1$. This shows that C has plain extractable length at most α. The converse implication follows from (1).

(3) If B is an α-constituent of C, then we have an extension

$$0 \to \sigma_{\alpha-1}C \to B \to B/\sigma_{\alpha-1}C \to 0$$

where $B/\sigma_{\alpha-1}C$ is a finite-rank submodule of $\partial_{\alpha-1}C$ by definition, and $\sigma_{\alpha-1}C$ has extractable length at most $\alpha - 1$ by the inductive hypothesis;

(4) Suppose that $E \subseteq C$ is a coreduced submodule of plain extractable length at most α. Then we have that $\sigma_{\alpha-1}E \subseteq \sigma_{\alpha-1}C$ and $\partial_{\alpha-1}E$ is finite-rank. Thus the inclusion $E \to C$ induces a map $\partial_\alpha E \to \partial_\alpha C$ whose image is contained in $S^k_\alpha C$ for some $k \in \omega$. This shows that E is contained in $B^k_\alpha C$ and hence in $\Sigma\left(B^k_\alpha C\right)$ (being coreduced).

(5) Suppose that (E_k) is an increasing sequence of submodules of C of plain projective length at most α with union equal to C such that $\sigma_{\alpha-1}C \subseteq E_\ell$ for every $\ell \in \omega$. We have that $\partial_{\alpha-1}C$ is the union of the images $\partial_{\alpha-1}E_\ell$ of the maps $E_\ell \to \partial_{\alpha-1}C$ for $\ell \in \omega$. As E_ℓ is coreduced of plain projective length at most α, we have that $\partial_{\alpha-1}E_\ell$ has finite rank for every $\ell \in \omega$. Therefore, for every $k \in \omega$ there exists $\ell \in \omega$ such that $S^k_\alpha C \subseteq \partial_{\alpha-1}E_\ell$ and hence $B^k_\alpha C \subseteq E_\ell$.

(6) By (3) and (4), the second and third term are equal. Furthermore, by (1) and (3), we have that $\mathrm{Ext}\left(B^k_{\alpha_k}C, A\right)$ has Solecki length at most α_k for every $k \in \omega$. Whence we have

$$G := \bigcap_{k\in\omega} \mathrm{Ker}\left(\mathrm{Ext}\left(C, A\right) \to \mathrm{Ext}\left(B^k_{\alpha_k}C, A\right)\right)$$

has Borel rank at most α in $\mathrm{Ext}\left(C, A\right)$, and contains $s_\alpha\mathrm{Ext}\left(C, A\right) = s_1 s_{\alpha-1}\mathrm{Ext}\left(C, A\right)$. We now prove that G is contained in $s_1 s_{\alpha-1}\mathrm{Ext}\left(C, A\right)$. If α is limit, this follows from the inductive hypothesis, considering that in this case

$$s_\alpha\mathrm{Ext}\left(C, A\right) = \bigcap_k s_{\alpha_k}\mathrm{Ext}\left(C, A\right).$$

Suppose now that α is a successor. By the inductive hypothesis (10) applied to $\alpha - 1$, the short exact sequence

$$L_{\alpha-1}C \to F_{\alpha-1}C \to C$$

induces an isomorphism

$$s_{\alpha-1}\mathrm{Ext}\left(C, A\right) \cong \frac{\mathrm{Hom}\left(L_{\alpha-1}C, A\right)}{\mathrm{Hom}\left(F_{\alpha-1}C | L_{\alpha-1}C, A\right)}$$

For $k \in \omega$, define

$$\hat{B}^k_{\alpha-1}C := \left\{x \in F_{\alpha-1}C : x + L_{\alpha-1}C \in B^k_{\alpha-1}C\right\}.$$

Under such an isomorphism G corresponds to

$$\frac{\operatorname{Hom}^1\left(L_{\alpha-1}C, A\right)}{\operatorname{Hom}\left(F_{\alpha-1}C|L_{\alpha-1}C, A\right)}$$

where $\operatorname{Hom}^1\left(L_{\alpha-1}C, A\right)$ is the submodule comprising the homomorphisms $L_{\alpha-1}C \to A$ that have an extension $\hat{B}^k_{\alpha-1}C \to A$ for every $k \in \omega$. Therefore, it remains to prove that

$$\frac{\operatorname{Hom}^1\left(L_{\alpha-1}C, A\right)}{\operatorname{Hom}\left(F_{\alpha-1}C|L_{\alpha-1}C, A\right)} \subseteq s_1\left(\frac{\operatorname{Hom}\left(L_{\alpha-1}C, A\right)}{\operatorname{Hom}\left(F_{\alpha-1}C|L_{\alpha-1}C, A\right)}\right).$$

By Lemma 5.44 and Lemma 5.102, it suffices to prove that if $V \subseteq \operatorname{Hom}\left(F_{\alpha-1}C|L_{\alpha-1}C, A\right)$ is an open neighborhood of 0 in $\operatorname{Hom}\left(F_{\alpha-1}C|L_{\alpha-1}C, A\right)$, then the closure $\bar{V}^{\operatorname{Hom}(L_{\alpha-1}C,A)}$ of V inside $\operatorname{Hom}\left(L_{\alpha-1}C, A\right)$ contains an open neighborhood of 0 in $\operatorname{Hom}^1\left(L_{\alpha-1}C, A\right)$. Suppose that $V \subseteq \operatorname{Hom}\left(F_{\alpha-1}C|L_{\alpha-1}C, A\right)$ is an open neighborhood of 0. Thus, there exist $x_0, \ldots, x_n \in F_{\alpha-1}C$ such that

$$\left\{\varphi|_{L_{\alpha-1}C} : \varphi \in \operatorname{Hom}\left(F_{\alpha-1}C, A\right) \text{ and } \forall i \leq n, \varphi\left(x_i\right) = 0\right\} \subseteq V.$$

Let $k \in \omega$ be such that $\{x_0, \ldots, x_n\} \subseteq \hat{B}^k_{\alpha-1}C$. Thus, we have that

$$U = \left\{\varphi \in \operatorname{Hom}^1\left(L_{\alpha-1}, A\right) : \exists \hat{\varphi} \in \operatorname{Hom}(\hat{B}^k_{\alpha-1}C, A), \forall i \leq n, \hat{\varphi}\left(x_i\right) = 0, \hat{\varphi}|_{L_{\alpha-1}C} = \varphi\right\}$$

is an open neighborhood of 0 in $\operatorname{Hom}^1\left(L_{\alpha-1}C, A\right)$. We claim that $U \subseteq \bar{V}^{\operatorname{Hom}(L_{\alpha-1}C,A)}$. Indeed, suppose that $\varphi_0 \in U$ and let W be an open neighborhood of φ_0 in $\operatorname{Hom}\left(L_{\alpha-1}C, A\right)$. There exist $y_0, \ldots, y_\ell \in L_{\alpha-1}C$ such that

$$\left\{\varphi \in \operatorname{Hom}\left(L_{\alpha-1}C, A\right) : \forall i \leq \ell, \varphi\left(y_i\right) = \varphi_0\left(y_i\right)\right\} \subseteq W.$$

Define

$$F_{\alpha-1}C_\Sigma := \bigoplus_k \Sigma B^k_{(\alpha-1)_k}C$$

and

$$F_{\alpha-1}C_\Phi := \bigoplus_k \Phi B^k_{(\alpha-1)_k}C$$

Then we have that, by definition,

$$F_{\alpha-1}C = F_{\alpha-1}C_\Sigma \oplus F_{\alpha-1}C_\Phi.$$

We need to prove that $V \cap W \neq \varnothing$. To this purpose, it suffices to show that there exists $\varphi \in \operatorname{Hom}(F_{\alpha-1}C, A)$ such that $\varphi\left(y_i\right) = \varphi_0\left(y_i\right)$ for $i \leq \ell$ and $\varphi\left(x_i\right) = 0$ for $i \leq n$. Since $\varphi_0 \in U$, there exists $\hat{\varphi}_0 \in \operatorname{Hom}(\hat{B}^k_{\alpha-1}C, A)$ such that $\hat{\varphi}_0\left(x_i\right) = 0$ for $i \leq n$ and $\hat{\varphi}_0|_{L_{\alpha-1}C} = \varphi_0$. For $i \leq \ell$ and $j \leq n$, we can write

$$y_i = y_i^\Phi + y_i^\Sigma$$

$$x_j = x_j^\Phi + x_j^\Sigma$$

where $x_j^\Phi, y_i^\Phi \in F_{\alpha-1}C_\Phi$ and $x_j^\Sigma, y_i^\Sigma \in F_{\alpha-1}C_\Sigma$. Let L be a finite direct summand of $F_{\alpha-1}C_\Phi$ that contains $y_0^\Phi, \dots, y_\ell^\Phi, x_0^\Phi, \dots, x_n^\Phi$ (which exists since $F_{\alpha-1}C_\Phi$ is a sum of finite projective modules), and write $F_{\alpha-1}C_\Phi = L \oplus L'$. Let T be the submodule of $F_{\alpha-1}C_\Phi$ generated by L and by $\hat{B}^k_{\alpha-1}C \cap F_{\alpha-1}C_\Phi$. Then we have that

$$\frac{T}{\hat{B}^k_{\alpha-1}C \cap F_{\alpha-1}C_\Phi} \cong \frac{T + \hat{B}^k_{\alpha-1}}{\hat{B}^k_{\alpha-1}} \subseteq \frac{F_{\alpha-1}C}{\hat{B}^k_{\alpha-1}C} \cong C/B^k_{\alpha-1}C$$

is finite (since L is finite) and flat (since $C/B^k_{\alpha-1}C$ is flat), and hence projective. Thus there exists $\psi \in \mathrm{Hom}\,(T, A)$ that extends $\hat{\varphi}_0|_{\hat{B}^k_{\alpha-1}C \cap F_{\alpha-1}C_\Phi}$. One can define $\varphi \in \mathrm{Hom}\,(F_{\alpha-1}C, A)$ by setting

$$\varphi\,(z + w + w') = \hat{\varphi}_0\,(z) + \psi\,(w)$$

for $z \in F_{\alpha-1}C_\Sigma$ and $w \in L$ and $w' \in L'$. For $i \leq \ell$ and $j \leq n$ we have

$$\varphi\,(y_i) = \hat{\varphi}_0\left(y_i^\Sigma\right) + \psi\left(y_i^\Phi\right) = \hat{\varphi}_0\,(y_i) = \varphi\,(y_i)$$

and

$$\varphi\,(x_j) = \hat{\varphi}_0(x_j^\Sigma) + \psi(x_j^\Phi) = \hat{\varphi}_0\,(x_j) = 0.$$

This concludes the proof.

(7) It follows from (4) that

$$0 \to L_\alpha C \to F_\alpha C \to C \to 0$$

is a phantom short exact sequence of order α, where $L_\alpha C$ and $F_\alpha C$ have extractable length at most α by definition and the inductive hypothesis.

(8) If C has (plain) extractable length at most α, then C has (plain) projective length at most α by (1). Conversely, if C has L_α-projective length at most α, then the short exact sequence in (7) splits. Thus, C is a direct summand of $F_\alpha C$, whence it has extractable length at most α.

If C has plain projective length at most α, then $\partial_{\alpha-1}C$ has finite rank by Theorem 5.126. By the inductive hypothesis, $\sigma_{\alpha-1}C$ has extractable length at most $\alpha - 1$. Considering the short exact sequence $\sigma_{\alpha-1}C \to C \to \partial_{\alpha-1}C$ we conclude that C has plain extractable length at most α.

(9) If C has plain R-projective length at most α, then the proof of (8) above shows that C has plain extractable length at most α. Suppose now that C has R-projective length at most α. We claim that C has extractable length at most α. Without loss of generality, we can assume that C is isomorphic to the direct sum of a sequence (C_n) of countable flat modules such that C_n has plain extractable length β_n for every $n \in \omega$. Then we have that

$$\mathrm{Ext}\,(C, R) \cong \prod_n \mathrm{Ext}\,(C_n, R)\,.$$

Therefore, the projective length of C is equal by the inductive hypothesis to $\sup_n \beta_n$. This shows that $\sup_n \beta_n \leq \alpha$ and hence C has extractable length at most α.

(10) Let A be a countable flat module. By (7) we have that the extension $\mathfrak{R}_C$

$$0 \to L_\alpha C \to F_\alpha C \to C \to 0$$

is phantom of order α. If $\varphi : L_\alpha C \to A$ is a homomorphism, then the image of φ under the homomorphism

$$\mathrm{Hom}\,(L_\alpha C, A) \to \mathrm{Ext}\,(C, A)$$

is the extension obtained via the pushout diagram

$$\begin{array}{ccccc} L_\alpha C & \to & F_\alpha C & \to & C \\ \varphi\downarrow & & \downarrow & & \downarrow \\ A & \to & X & \to & C \end{array}$$

Since $\mathfrak{R}_C$ is phantom of order α, by (6) the exact sequence induced by

$$0 \to L_\alpha C \to F_\alpha C \to C \to 0$$

and the inclusion $B^k_{\alpha_k} C \to C$ splits. Thus,t he same holds for the exact sequence induced by

$$0 \to A \to X \to C \to 0$$

and the inclusion $B^k_{\alpha_k} C \to C$. As this holds for every $k \in \omega$, by (6) again the short exact sequence

$$A \to X \to C$$

is phantom of order α. This shows that the image of the homomorphism

$$\mathrm{Hom}\,(L_\alpha C, A) \to \mathrm{Ext}\,(C, A)$$

is contained in $s_\alpha \mathrm{Ext}\,(C, A)$. Conversely, suppose that

$$A \to X \to C$$

is a phantom short exact sequence of order α. Then by (6) we have that for every $k \in \omega$ the induced short exact sequence

$$\begin{array}{ccccc} A & \to & X & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & Y_k & \to & B^k_{\alpha_k} C \end{array}$$

splits. Thus, there exists a homomorphism $\psi_k : B^k_{\alpha_k} C \to Y_k$ that is a right inverse for $Y_k \to B^k_{\alpha_k} C$. For every $k \in \omega$, we have that

$$\varphi_k := \psi_{k+1}|_{B^k_{\alpha_k} C} - \psi_k$$

is a homomorphism $B^k_{\alpha_k} C \to A$. For $k \in \omega$, these define a homomorphism $\varphi : L_\alpha C \to C$. The image of φ under the homomorphism

$$\mathrm{Hom}\,(L_\alpha C, A) \to \mathrm{Ext}\,(C, A)$$

is the extension of C by A represented by

$$A \to X \to C.$$

This concludes the proof. □

Corollary 5.130. *Let C be an extractable countable flat module. Then the extractable length, the R-projective length, and the projective length of C coincide, as well as the extractable rank, the R-projective rank, and the projective rank.*

In consideration of Corollary 5.130, we can formulate the following:

Definition 5.131. *We define the (plain) length of a countable* extractable *flat module to be the common value of its (plain) extractable, projective, and R-projective lengths.*

5.11.5 Independence

Suppose that A is a countable flat module. If C is an extractable coreduced flat module of (plain) length α, then in general the (plain) A-projective length of A is *at most* α. We now consider a natural condition on C, which we term A-independence, which guarantees that the (plain) A-projective length of A is indeed equal to α.

Definition 5.132. *Suppose that C is a extractable coreduced countable flat module, and A is a countable flat module. We say that C is* A-independent *if*

$$\mathrm{Hom}\,(\sigma_\beta C, A) = \mathrm{Hom}\,(\partial_\beta C, A) = 0$$

for every $\beta < \omega_1$.

Notice that every extractable coreduced countable flat module is R-independent. The following lemma is easily established by induction on the length.

Lemma 5.133. *Suppose that C is a extractable coreduced countable flat module. The following assertions are equivalent:*

1. *C is A-independent;*
2. $\mathrm{Hom}\left(\Sigma(B^k_1(\partial_\beta C)), A\right) = 0$ *for every $\beta < \omega_1$ and $k \in \omega$.*

Theorem 5.134. *Suppose that C is a extractable coreduced countable flat module, and A is a countable flat module. Assume that C is A-independent. Then:*

1. *the (plain) length of C is equal to the (plain) A-projective length of C;*

2. *for every $\beta < \omega_1$, the quotient map $M \to \partial_\beta M$ induces an isomorphism*

$$\mathrm{Ext}\left(\partial_\beta M, A\right) \cong \mathrm{Ph}^\beta \mathrm{Ext}\left(M, A\right).$$

Proof. This follows from Theorem 5.128 considering that its hypotheses are verified by Lemma 5.133. □

5.11.6 Rigidity

Suppose that M is a countable flat module. Let $\mathrm{Aut}\left(M\right)$ be the group of automorphisms of M endowed with the topology of pointwise convergence. Then we have that $\mathrm{Aut}\left(M\right)$ is a non-Archimedean Polish group. This means that $\mathrm{Aut}\left(M\right)$ is a topological group whose topology is Polish, and admits a basis of neighborhoods of the identity consisting of open subgroups. The automorphisms $\mu_r : x \mapsto rx$ for $r \in R^\times$, which we call *inner*, form a central closed subgroup isomorphic to $R^\times$.

Likewise, we have that *endomorphisms* of M form a Polish ring $\mathrm{End}\left(M\right)$. An endomorphism of M is called *inner* if it is of the form $x \mapsto rx$ for some $x \in X$. Inner endomorphisms form a central closed subring isomorphic to R. We say that M is *rigid* if every endomorphism of M is inner.

We say that two countable flat modules N and M are *orthogonal* if $\mathrm{Hom}\left(M, N\right) = 0$ and $\mathrm{Hom}\left(N, M\right) = 0$. We say that $\left(N_i\right)_{i \in I}$ is an *orthonormal family* if for every $i \in I$, N_i is rigid, and for every distinct $i, j \in I$, N_i and N_j are orthogonal.

5.11.7 Minimal elements

We say that an element of a module M is *minimal* if it generates a pure submodule. A *pointed module* is a module with a distinguished element, and a *well-pointed module* is a module with a distinguished minimal element. Recall that a sequence $\left(\xi_n\right)$ of elements of a module M is *independent* if the submodule N generated by $\left(\xi_n\right)$ is the direct sum of the submodules N_n generated by ξ_n for $n \in \omega$.

Definition 5.135. *A family $\left(\xi_i\right)_{i \in I}$ of elements of a module M is* minimal *if it is independent and the submodule N generated by $\left(\xi_i\right)$ is* pure.

Notice that a family $\left(\xi_i\right)_{i \in I}$ in M is minimal if and only if, letting N be the submodule generated by $\left(\xi_i\right)$, an element of $rM \cap N$ can be written uniquely as

$$\sum_{i \in I} \mu_i \xi_i$$

for $\mu_i \in R$, and in this case one has $\mu_i \in rR$ for every $i \in I$.

Lemma 5.136. *Suppose that B is a countable flat module, $A \subseteq B$ is a pure submodule, and $C := B/A$. Let $(\xi_i)_{i\in I}$ be family of elements of B. Suppose $I_A \subseteq I$ is such that $(\xi_i)_{\in I_A}$ is a minimal family of elements of A. Set $I_C := I \setminus I_A$, and assume that $(\xi_i + A)_{i\in I_C}$ is a minimal family of elements of C. Then $(\xi_i)_{i\in I}$ is a minimal family of elements of B.*

Proof. Suppose that $(\lambda_i) \in R^{(I)}$ is such that

$$\sum_{i\in I} \lambda_i \xi_i = 0.$$

Then we have that

$$\sum_{i\in I_C} \lambda_i \xi_i + A = A$$

and hence $\lambda_i = 0$ for $i \in I_C$. Hence

$$\sum_{i\in I_A} \lambda_i \xi_i = 0$$

and $\lambda_i = 0$ for $i \in I_A$. This shows that (ξ_i) is independent.

Suppose that $b \in B$ and $r \in R \setminus \{0\}$ are such that rb belongs to the submodule N generated by (ξ_i). We claim that $b \in N$. We can write

$$rb = \sum_{i\in I} \lambda_i \xi_i$$

for $(\xi_i) \in R^{(I)}$. Then we have that $rb + A$ belongs to the submodule of C generated by $(\xi_i + A)_{i\in I_C}$. As such a submodule is pure, the same holds for $b + A$. Thus, there exist $(\mu_i) \in R^{(I_C)}$ and $a \in A$ such that

$$b - a = \sum_{i\in I_C} \mu_i \xi_i$$

and hence

$$\sum_{i\in I} \lambda_i \xi_i - ra = rb - ra = \sum_{i\in I_C} r\mu_i \xi_i$$

Thus, we have $r\mu_i = \lambda_i$ for $i \in I_C$. Hence,

$$\sum_{i\in I_C} (\lambda_i - r\mu_i)\, \xi_i + \sum_{i\in I_A} \lambda_i \xi_i = ra \in A.$$

Thus, we have

$$\sum_{i\in I_C} (\lambda_i - r\mu_i)\, \xi_i \equiv 0 \bmod A.$$

This implies that $\lambda_i = r\mu_i$ for $i \in I_C$. Thus, we have that

$$r(b - \sum_{i\in I_C} \mu_i \xi_i) = \sum_{i\in I} \lambda_i \xi_i - \sum_{i\in I_C} r\mu_i \xi_i = \sum_{i\in I_A} \lambda_i \xi_i$$

belongs to the submodule N_A of A generated by $(\xi_i)_{i\in A}$. As such a submodule is pure, we also have

$$b - \sum_{i\in I_C} \mu_i \xi_i \in N_A \subseteq N.$$

Thus, we have

$$b = (b - \sum_{i\in I_C} \mu_i \xi_i) + \sum_{i\in I_C} \mu_i \xi_i \in N.$$

This concludes the proof. □

5.11.8 Types

In this section we consider the natural generalization of the notion of type of an element from torsion-free abelian groups to flat modules [64, Chapter XIII].

Definition 5.137. *Define $\mathfrak{I}(R)$ to be the collection of* finite *ideals of R ordered by inclusion. Let P and P' be countable upward-directed ordered sets. Suppose that $\mathfrak{t} : P \to \mathfrak{I}(R)$ and $\mathfrak{t}' : P' \to \mathfrak{I}(R)$ are order-reversing functions. Say that $\mathfrak{t}$ and $\mathfrak{t}'$ are* cofinally equivalent *if for all $p \in P$ there exists $p' \in P'$ such that $\mathfrak{t}'(p') \subseteq \mathfrak{t}(p)$ and vice versa. A* type *(of finite ideals) is a cofinal equivalence class of order-reversing functions $\mathfrak{t} : P \to \mathfrak{I}(R)$, where P is an upward directed ordered set.*

Clearly, any type admits a representative $\mathfrak{t}$ that is *tower of ideals*, i.e., a decreasing sequence of ideals. In what follows by abuse of notation we identify a type with any of its representatives. We define an order relation among types by setting

$$(I_n) \leq (J_n) \Leftrightarrow \forall n \exists k,\, I_n \subseteq J_k.$$

The smallest type (I_n) is such that $I_n = 0$ for every $n \in \omega$, and the largest type (I_n) is such that $I_n = R$ for every $n \in \omega$. Types form a lattice with

$$(I_n) \vee (J_n) := (I_n + J_n)$$

and meet

$$(I_n) \wedge (J_n) := (I_n \cap J_n).$$

Definition 5.138. *Two types $\mathfrak{t}$ and $\mathfrak{t}'$ are* orthogonal, *in which case we write $\mathfrak{t} \perp \mathfrak{t}'$, if their join is the largest type.*

We can define the type of an element, generalizing the usual notion in the case of $\mathbb{Z}$-modules; see [64, Chapter XIII].

Definition 5.139. *Suppose that M is a coreduced countable flat module, and $a \in M$ is a nonzero element. Fix an increasing sequence (M_n) of finite submodules of M with $a \in M_0$ and union of $\{M_n : n \in \omega\}$ equal to M. Let*

$$\boldsymbol{A} = (\mathrm{Hom}(M_n, R))_{n\in\omega}$$

be the corresponding tower of finite flat modules. Given a subtower $\boldsymbol{B}$ *of* $\boldsymbol{A}$*, one defines the* $\boldsymbol{B}$*-type of* a *to be*

$$\boldsymbol{B}[a] = (\{\varphi(a) : \varphi \in B_n\})_{n\in\omega}.$$

In particular, we define the α*-type* $\mathfrak{t}_\alpha(a)$ *of* a *as the* $\boldsymbol{A}_\alpha$*-type, where* $\boldsymbol{A}_\alpha$ *is the* α*-th derived tower of* $\boldsymbol{A}$*. The* 0*-type is also called the type and denoted by* $\mathfrak{t}(a)$*.*

Notice that $(\mathfrak{t}_\alpha(a))_{\alpha<\omega_1}$ forms a decreasing sequence of types.

5.11.9 Straight length

We now introduce a more stringent notion than the one of extractable module, which we term straight extractable module.

Definition 5.140. *Suppose that* M *is a countable coreduced flat module. We say that* M *has plain straight length* 1 *if and only if* M *is nontrivial and has finite rank. For a countable successor ordinal* $\alpha > 1$*,* M *has plain straight length* α *if and only if there exist:*

1. *a submodule* $L \subseteq M$*;*
2. *an inductive sequence* (J_n, φ_n) *of finite free modules;*
3. *bases* $(e_{n,i})_{i<d_n}$ *for* J_n *for* $n < \omega$ *with* $\sup_n d_n < \infty$*;*
4. *elements* $f_{n,i} \in M$ *for* $n < \omega$ *and* $i < d_n$*;*
5. *pointed submodules* $(L_{n,i}, \xi_{n,i}) \subseteq L$ *for* $n < \omega$ *and* $i < d_n$*;*

such that

- $L_{n,i}$ *has plain straight length* $\beta[n]$ *and* $\sup_n \beta[n] = \alpha - 1$*;*
- L *is the direct sum of* $L_{n,i}$ *for* $n < \omega$ *and* $i < d_n$*;*
- *there exists an isomorphism between the quotient* J *of* M *by* L *and* $\operatorname{colim}_n(J_n, \varphi_n)$*, which maps* $f_{n,i} + L$ *to the element of* $\operatorname{colim}_n(J_n, \varphi_n)$ *corresponding to* $e_{n,i} \in J_n$*;*
- *for* $n < \omega$ *and* $i < d_n$*, if*

$$\varphi_n(e_{n,i}) = \sum_{j<d_{n+1}} a_j e_{n+1,j}$$

for $a_j \in R$*, then*

$$f_{n,i} = \sum_{j<d_{n+1}} a_j f_{n+1,j} + \xi_{n,i}$$

- *for $n < \omega$ and $i < d_n$,*
$$\mathfrak{t}^{L_{n,i}}_{\alpha_n - 1}(\xi_{n,i}) \perp \mathfrak{t}^{J}(e_{n,i}).$$

We say that M has straight length α if and only if M is isomorphic to the direct sum of a sequence of modules M_n of plain straight length $\beta[n]$ with $\sup_n \beta[n] = \alpha$.

The module M is straight extractable *if and only if it has (plain) straight length α for some $\alpha < \omega_1$.*

Any straight extractable module is, in particular, extractable. In particular, its (plain) R-projective and extractable lengths coincide.

Proposition 5.141. *Suppose that $\alpha > 1$ is a successor ordinal, A is a countable flat module, and M is an countable coreduced flat module of plain straight length α. Let $L_{n,i} \subseteq L \subseteq M$ for $n < \omega$ and $i < d_n$ and J be as in the definition of plain straight length α.*

1. *$L = \sigma_{\alpha-1} M$ and $J \cong \partial_{\alpha-1} M$;*
2. *M is A-independent if and only if $L_{n,i}$ is A-independent for $n < \omega$ and $i < d_n$ and $\operatorname{Hom}(J, A) = 0$;*
3. *the plain length of M is α.*

Proof. We prove that the conclusion holds by induction on α. Observe that (2) and (3) easily follow from (1), so it remains to prove (1).

For $i < d_n$ let $(L_{n,i,k})_{k\in\omega}$ be an increasing sequence of finite submodules with union equal to $L_{n,i}$ and $\xi_{n,i} \in L_{n,i,0}$. Set
$$L[n] := L_{n,0} \oplus \cdots \oplus L_{n,d_n-1}$$
and
$$L_k[n] := L_{n,0,k} \oplus \cdots \oplus L_{n,d_n-1,k.}$$
Define
$$M_k := L_k[0] \oplus \cdots \oplus L_k[k] \oplus L_k[-1]$$
where $L_k[-1]$ is the submodule of M generated by $f_{k,i}$ for $i < d_k$. Then we have that (M_k) is an increasing sequence of finite submodules of M with union equal to M. We have that
$$\boldsymbol{C} := (\operatorname{Hom}(M_n, A))_{n\in\omega}$$
is the tower associated with the fishbone of towers $(\boldsymbol{A}, \boldsymbol{B}[n])_{n\in\omega}$, which is straight of length α; see Definition 5.63. Here $\boldsymbol{A}$ is the tower
$$(\operatorname{Hom}(J_n, A))_{n\in\omega}$$

while $\boldsymbol{B}[n]$ is the tower

$$(\mathrm{Hom}\,(L_k[n], A))_{k\in\omega}\,.$$

The isomorphism

$$\mathrm{Ext}\,(M, A) \cong \lim{}^1\boldsymbol{C},$$

induces an isomorphism

$$\mathrm{Ph}^{\beta}\mathrm{Ext}\,(M, A) \cong \lim{}^1\boldsymbol{C}_{\beta}$$

for $\beta < \omega_1$, considering the description of the Solecki subgroups of Ext and $\lim^1$; see Theorem 5.67 and Theorem 5.58.

The description of the derived towers of a fishbone tower given by Proposition 5.64 yields

$$\mathrm{Ph}^{\alpha-1}\mathrm{Ext}\,(M, A) \cong \lim{}^1\boldsymbol{A}.$$

This shows that the canonical projection $M \to J$ corresponds to $M \to \partial_{\alpha-1}M$ and hence $\sigma_{\alpha-1}M = L$. □

Corollary 5.142. *If C is a countable coreduced straight extractable flat module, then its (plain) straight length equals its (plain) length.*

5.11.10 Fishbones of modules

We now define the analogue for modules of the notion of *fishbone* of towers.

Definition 5.143. *A* fishbone *of modules $((L_{n,i}, \xi_{n,i}, J_n, e_{n,i}, \varphi_n)_{i<d_n})_{n\in\omega}$ is given by:*

1. *an inductive sequence (J_n, φ_n) of nonzero finite free modules with $\mathrm{rank}(J_n) = d_n$ and $d_0 = 1$;*
2. *free bases $(e_{n,i})_{i<d_n}$ of J_n for $n < \omega$;*
3. *pointed countable coreduced flat modules $(L_{n,i}, \xi_{n,i})$ for $n \in \omega$ and $i < d_n$.*

The module

$$L_n := L_{n,0} \oplus \cdots \oplus L_{n,d_n-1}$$

is called the n-th rib of the fishbone.

We can define the homomorphism $\psi_n : J_n \to L_n$ by way of the universal property of the free module by $e_{n,i} \mapsto \xi_{n,i}$. The fishbone colimit *$M$ is the colimit of the sequence*

$$(L_0 \oplus \cdots \oplus L_n \oplus J_n)$$

with respect to the morphisms

$$\begin{array}{rcl} L_0 \oplus \cdots \oplus L_n \oplus J_n & \to & L_0 \oplus \cdots \oplus L_{n+1} \oplus J_{n+1} \\ (x_0, \ldots, x_n, a_n) & \mapsto & (x_0, \ldots, x_n, \psi_n(a_n), \varphi_n\,(a_n))\,. \end{array}$$

Notice that by definition M contains the direct sum L of $(L_n)_{n\in\omega}$. This defines a short exact sequence

$$0 \to L \to M \to J \to 0.$$

Furthermore, M is isomorphic to the quotient of $L \oplus J$ by the (pure) submodule generated by

$$\xi_{n,i} - e_{n,i}$$

for $n < \omega$ and $i < d_n$.

We define the notion of *fishbone length* of a module, capturing the complexity of describing it in terms of fishbones.

Definition 5.144. *Let M be a countable coreduced flat module, and A be a countable flat module.*

- *We say that M has plain fishbone length 1 if and only if it is nontrivial and finite-rank.*
- *The module M has plain fishbone length a successor ordinal $\alpha > 1$ if and only if there exists a fishbone of modules $((L_{n,i}, \xi_{n,i}, J_n, e_{n,i}, \varphi_n)_{i<d_n})_{n\in\omega}$ with fishbone colimit M such that for every $n \in \omega$ and $i < d_n$:*
  1. *$L_{n,i}$ has plain fishbone length $\beta[n]$ where $\sup_n \beta[n] = \alpha - 1$;*
  2. *$\mathfrak{t}^{L_{n,i}}_{\beta[n]-1}(\xi_{n,i}) \perp \mathfrak{t}^J(e_{n,i})$.*
- *The module M has fishbone length α if and only if it is the direct sum of a sequence of modules M_n of plain fishbone length $\beta[n]$ with $\sup_n \beta[n] = \alpha$.*

A module is fishbone extractable *if and only if it has (plain) fishbone length α for some countable ordinal α.*

It is easily shown by induction that any fishbone extractable module is straight extractable.

Proposition 5.145. *Suppose that M is a countable coreduced flat module of plain fishbone length $\alpha > 1$, and let $((L_{n,i}, \xi_{n,i}, J_n, e_{n,i}, \varphi_n)_{i<d_n})_{n\in\omega}$ be a fishbone of modules as in Definition 5.144 with fishbone colimit L witnessing that M has plain fishbone length α. Let L be the module*

$$\operatorname{colim}_n(L_{n,0} \oplus \cdots \oplus L_{n,d_{n-1}})$$

identified with a submodule of M. Suppose that A is a countable flat module. Then

1. *$L = \sigma_{\alpha-1}M$ and $\partial_{\alpha-1}M \cong J$;*
2. *M is A-independent if and only if $L_{n,i}$ is A-independent for $n < \omega$ and $i < \omega$ and $\operatorname{Hom}(J, A) = 0$;*

3. *M has plain length α.*

Proof. This is a consequence of Proposition 5.141. □

Corollary 5.146. *If C is a countable coreduced fishbone extractable flat module, then its (plain) fishbone length equals its (plain) length.*

5.12 Constructions for modules over a Dedekind domain

In this section, we assume that R is not only a Prüfer domain but a *Dedekind domain*. We let $\mathbb{P}$ be the collection of nonzero *prime ideals* of R. For $\mathfrak{p} \in \mathbb{P}$, we let $R_{\mathfrak{p}}$ be the localization of R at $\mathfrak{p}$, which is a discrete valuation domain (DVR), namely a PID that is also a local ring. We furthermore assume that R is countable and not a field, which implies that R is countably infinite (as a finite division ring is necessarily a field).

5.12.1 Factorization of ideals

Recall that in a Dedekind domain, every ideal can be written uniquely (up to the order of the factors) as a product of prime ideals [4, Corollary 9.4]. Furthermore, it has *Krull dimension* 1, whence the nonzero prime ideals are precisely the maximal ideals. Every ideal is generated by two elements [146, Proposition C-5.99]. Notice that if $\mathfrak{p}, \mathfrak{q}$ are distinct elements of $\mathbb{P}$, then $\mathfrak{p}$ and $\mathfrak{q}$ are *coprime*, namely $\mathfrak{p} + \mathfrak{q} = R$. More generally, if $\mathfrak{p}_0, \ldots, \mathfrak{p}_n, \mathfrak{q}_0, \ldots, \mathfrak{q}_n \in \mathbb{P}$ satisfy

$$\{\mathfrak{p}_0, \ldots, \mathfrak{p}_n\} \cap \{\mathfrak{q}_0, \ldots, \mathfrak{q}_n\} = \varnothing$$

then $\mathfrak{p}_0 \cdots \mathfrak{p}_n$ and $\mathfrak{q}_0 \cdots \mathfrak{q}_n$ are coprime. If I is an ideal of R, then by the Krull Intersection Theorem [2], the sequence $(I^n)_{n\in\omega}$ has trivial intersection.

Lemma 5.147. *Suppose that B is a countable flat module. Let $(\xi_i)_{i\in\omega}$ be a family of minimal elements of B. Let N be the submodule generated by $(\xi_i)_{i\in\omega}$. Then $rB \cap N$ is the submodule generated by $(r\xi_i)_{i\in\omega}$. Suppose that $a_n \in R^{(\omega)}$ for $n \in \omega$ are such that, for every $\mathfrak{p} \in \mathbb{P}$, the elements $a_n + \mathfrak{p} \in (R/\mathfrak{p})^{(\omega)}$ for $n \in \omega$ are linearly independent. Define $\eta_n := \sum_{i\in\omega} a_n(i)\,\xi_i$ for $n \in \omega$. Then $(\eta_n)_{n\in\omega}$ is a family of minimal elements of B.*

Proof. Since every finite ideal J of R is a product of prime ideals, we also have that $(a_n + J)_{n\in\omega} \in (R/J)^{(\omega)}$ are linearly independent.

The assertion about $rB \cap N$ follows from minimality of (ξ_i). Furthermore, if (λ_n) are such that

$$\sum_n \lambda_n \eta_n = 0$$

then we have

$$\sum_i \left(\sum_n \lambda_n a_n(i) \right) \xi_i = 0$$

Hence, for every i

$$\sum_n \lambda_n a_n(i) = 0$$

This shows

$$\sum_n \lambda_n a_n = 0 \in R^{(\omega)}.$$

By hypothesis, (a_n) are an independent family, whence $\lambda_n = 0$ for every $n \in \omega$. This shows that (η_n) is an independent family.

Let M be the submodule generated by (η_n). Consider $x \in rB \cap M \subseteq rB \cap N$. Then we have

$$x = \sum_i x_i \xi_i$$

for some $x_i = ry_i \in rR$. We also have

$$x = \sum_n \mu_n \eta_n$$

for some $\mu_n \in R$. Thus, we have

$$\sum_i x_i \xi_i = \sum_j \left(\sum_n \mu_n a_n(j) \right) \xi_j.$$

This implies

$$ry_i = x_i = \sum_n \mu_n a_n(i)$$

for $i \in \omega$. Hence, we have that $\mu_n \in rR$ for every $n \in \omega$, and $x \in rM$. This concludes the proof. □

5.12.2 Completions

We define the $\mathfrak{p}$-*completion* of R to be $\hat{R}_\mathfrak{p} := \lim_n R/\mathfrak{p}^n R$, with the canonical embedding $\iota_\mathfrak{p} : R \to \hat{R}_\mathfrak{p}$. As this can be seen as the completion of the DVR $R_\mathfrak{p}$, we have that $\hat{R}_\mathfrak{p}$ is a DVR. The $\mathfrak{p}$-completion of a countable flat module M is the pro-countable Polish flat $\hat{R}_\mathfrak{p}$-module

$$\hat{M}_\mathfrak{p} := \lim_n M/\mathfrak{p}^n M.$$

We let the completion of R to be $\hat{R} := \lim_I R/I$ where I ranges among the ideals of R. We identify R with a subring of $\hat{R}$. By the Chinese Remainder Theorem, we have an isomorphism

$$\hat{R} \cong \prod_{\mathfrak{p} \in \mathbb{P}} \hat{R}_\mathfrak{p},$$

and under this isomorphism the inclusion $R \to \hat{R}$ corresponds to the diagonal embedding

$$a \mapsto (\iota_\mathfrak{p}(a))_{\mathfrak{p} \in \mathbb{P}}.$$

We define $\hat{K}_{\mathfrak{p}}$ to be the field of fractions of the DVR $\hat{R}_{\mathfrak{p}}$, and set $\hat{K}$ to be the divisible hull $K \otimes \hat{R}$ of the R-module $\hat{R}$. This can be identified with the *restricted product*

$$\prod_{\mathfrak{p}\in\mathbb{P}}(\hat{K}_{\mathfrak{p}} : \hat{R}_{\mathfrak{p}}) = \left\{ (a_{\mathfrak{p}}) \in \prod_{\mathfrak{p}\in\mathbb{P}} \hat{K}_{\mathfrak{p}} : \left\{ \mathfrak{p} \in \mathbb{P} : a_{\mathfrak{p}} \notin \hat{R}_{\mathfrak{p}} \right\} \text{ is finite} \right\}.$$

If M is a flat module, we let

$$\hat{M} := \lim_I M/IM \cong \prod_{\mathfrak{p}\in\mathbb{P}} \hat{M}_{\mathfrak{p}}$$

be its completion, which we can regard as an $\hat{R}$-module. Its divisible hull $K \otimes \hat{M}$ as an $\hat{R}$-module can be seen as a $\hat{K}$-module.

The kernel of the canonical homomorphism $M \to \hat{M}$ is the largest divisible submodule ΔM of M, which is a direct summand of M. We say that M is *reduced* if ΔM is trivial. In this case, we can and will identify M with a submodule of its completion. Notice that if M is a reduced nonzero flat module, then it must be infinite, whence $\hat{M}$ is uncountable.

For $\mathfrak{p} \in \mathbb{P}$, the kernel of the canonical homomorphism $M \to \hat{M}_{\mathfrak{p}}$ is the intersection $\mathfrak{p}^\infty M$ of $\mathfrak{p}^n M$ for $n \in \mathbb{N}$. Notice that

$$\mathfrak{p}^\infty M \cap \mathfrak{q}^\infty M = (\mathfrak{p}\mathfrak{q})^\infty M$$

for $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}$. Furthermore, by the Ideal Factorization Theorem,

$$\bigcap_{\mathfrak{p}\in\mathbb{P}} \mathfrak{p}^\infty M = \Delta M.$$

We say that M is *completely reduced* if $\mathfrak{p}^\infty M = 0$ for every $\mathfrak{p} \in \mathbb{P}$. In this case, we can identify M with a submodule of $\hat{M}_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathbb{P}$.

5.12.3 Divisibility

For $\mathfrak{p} \in \mathbb{P}$ and $\lambda \in \hat{K}_{\mathfrak{p}}$, we define $K_{\mathfrak{p}}(\lambda)$ to be the (countable) subfield of $\hat{K}_{\mathfrak{p}}$ generated by K and λ.

Definition 5.148. *Given elements λ and μ of $\hat{R}_{\mathfrak{p}}$, we say that λ and μ are* essentially equivalent *if $K_{\mathfrak{p}}(\lambda) = K_{\mathfrak{p}}(\mu)$.*

Thus, we have that λ and μ are essentially equivalent if there exist nonzero $s, t \in R$ such that $r\lambda = s\mu$. Notice that the relation of essential equivalence has countable equivalence classes, since R is countable.

Lemma 5.149. *Fix $\mathfrak{p} \in \mathbb{P}$. The set of $\lambda \in \hat{R}_{\mathfrak{p}}$ such that the elements $(\lambda^n)_{n\in\omega}$ of $\hat{K}_{\mathfrak{p}}$ are linearly dependent over K is* countable.

Proof. Suppose by contradiction that there exists an uncountable set $X \subseteq \hat{K}_{\mathfrak{p}}$ such that for every $\lambda \in X$, $(\lambda^n)_{n\in\omega}$ are linearly dependent over K. Then for every $\lambda \in X$ there exists $a \in K^{(\omega)}$ such that $\sum_n a_n\lambda^n = 0$. Since X is uncountable while $K^{(\omega)}$ is countable, after passing to an uncountable subset of X we can assume that there exists $a \in K^{(\omega)}$ such that $\sum_n a_n\lambda^n = 0$ for all $\lambda \in X$. Since $\hat{K}_p$ is infinite, a nonzero polynomial function has only finitely many roots. This implies that $a = 0$. □

We let:

- for $\mathfrak{p} \in \mathbb{P}$, $J_{\mathfrak{p}}(R)$ be the (uncountable) set of $\lambda \in \hat{R}_{\mathfrak{p}} \setminus K$ such that $\lambda \equiv 1 \bmod \mathfrak{p}$ and $(\lambda^n)_{n\in\omega}$ are linearly independent over K in $\hat{K}_{\mathfrak{p}}$;
- for $\mathfrak{p} \in \mathbb{P}$, $\boldsymbol{J}_{\mathfrak{p}}(R)$ be the set of essential equivalence classes of elements of $J_{\mathfrak{p}}(R)$;
- $J(R)$ to be the product of $J_{\mathfrak{p}}(R)$ for $\mathfrak{p} \in \mathbb{P}$;
- $\boldsymbol{J}(R)$ to be the product of $\boldsymbol{J}_{\mathfrak{p}}(R)$ for $\mathfrak{p} \in \mathbb{P}$.

If $\lambda \in J_{\mathfrak{p}}(R)$, we let $[\lambda] \in \boldsymbol{J}_{\mathfrak{p}}(R)$ be its essential equivalence class. Likewise, if $\lambda = (\lambda_{\mathfrak{p}}) \in J(R)$, we let $[\lambda] = ([\lambda_{\mathfrak{p}}])_{p\in\mathbb{P}} \in \boldsymbol{J}(R)$.

Definition 5.150. *Let M be a countable flat module. For $\mathfrak{p} \in \mathbb{P}$, identify $M/\mathfrak{p}^\infty M$ with a submodule of $\hat{M}_{\mathfrak{p}}$. If $\tau \in J(R)$, and $\boldsymbol{\tau} = [\tau] \in \boldsymbol{J}(R)$, we let $E_\tau(M) = E_{\boldsymbol{\tau}}(M)$ be the pure submodule of M generated by the set of $x \in M$ such that for every $\mathfrak{p} \in \mathbb{P}$ there exists μ essentially equivalent to $\tau_{\mathfrak{p}}$ and $z \in M$ such that either*

$$\mu(x + \mathfrak{p}^\infty M) = z + \mathfrak{p}^\infty M \in M/\mathfrak{p}^\infty M \subseteq \hat{M}_{\mathfrak{p}}$$

or

$$\mu(z + \mathfrak{p}^\infty M) = x + \mathfrak{p}^\infty M \in M/\mathfrak{p}^\infty M \subseteq \hat{M}_{\mathfrak{p}}.$$

It is clear that $M \mapsto E_\tau(M)$ is a subfunctor of the identity.

Definition 5.151. *Suppose that A is countable flat module. For $\mathfrak{p} \in \mathbb{P}$, define $\Omega_{\mathfrak{p}}(A)$ to be the set of $\tau \in J_{\mathfrak{p}}(R)$ such that for every μ essentially equivalent to τ and for every nonzero $x \in A/\mathfrak{p}^\infty A$, $\mu x \notin A/\mathfrak{p}^\infty A$. Let also $\Omega(A) = \prod_{\mathfrak{p}\in\mathbb{P}} \Omega_{\mathfrak{p}}(A)$.*

Lemma 5.152. *Suppose that A is a countable flat module. For every $\mathfrak{p} \in \mathbb{P}$, $J_{\mathfrak{p}}(R) \setminus \Omega_{\mathfrak{p}}(A)$ is countable.*

Proof. If A is $\mathfrak{p}$-divisible there is nothing to prove. If A is not $\mathfrak{p}$-divisible, then after replacing A with $A/\mathfrak{p}^\infty A$ we can assume without loss of generality that $\mathfrak{p}^\infty A = 0$ and hence A can be identified with an R-submodule of its $\mathfrak{p}$-completion $\hat{A}_{\mathfrak{p}}$. As $\hat{A}_{\mathfrak{p}}$ is a flat $\hat{R}_{\mathfrak{p}}$-module, and A is countable, the set of $\tau \in \hat{R}_{\mathfrak{p}}$ such that there exists a nonzero $x \in A$ such that $\tau x \in A$ is also countable. This concludes the proof. □

Lemma 5.153. *Suppose that A is a countable reduced flat module, and $\tau \in \Omega(A)$. Then $E_\tau(A) = 0$. If M is a countable coreduced flat module, then* $\mathrm{Hom}(E_\tau(M), A) = 0$.

Proof. The second assertion follows from the first one since E_τ is a subfunctor of the identity.

If $x \in E_\tau(A)$, then we have that for every $\mathfrak{p} \in \mathbb{P}$ there exists μ essentially equivalent to $\tau_\mathfrak{p}$ and $z \in A$ such that either

$$\mu(x + \mathfrak{p}^\infty A) = z + \mathfrak{p}^\infty A \in A/\mathfrak{p}^\infty A \subseteq \hat{A}_\mathfrak{p}$$

or

$$\mu(z + \mathfrak{p}^\infty A) = x + \mathfrak{p}^\infty A \in A/\mathfrak{p}^\infty A \subseteq \hat{A}_\mathfrak{p}.$$

Since $\tau_\mathfrak{p} \in \Omega_\mathfrak{p}(A)$, this implies that $x \in \mathfrak{p}^\infty A$. As this holds for every $\mathfrak{p} \in \mathbb{P}$, $x \in \Delta A$. Hence, $x = 0$ since A is reduced. □

5.12.4 The modules $\Xi(\tau)$

Fix an enumeration $(\mathfrak{p}_n)$ of $\mathbb{P}$. Consider $\tau = (\tau_\mathfrak{p})_{\mathfrak{p}\in\mathbb{P}} \in J(R)$. For $\mathfrak{p} \in \mathbb{P}$ and $n \in \omega$ pick $\tau_{\mathfrak{p},n} \in R$ such that

$$\tau_\mathfrak{p} \equiv \tau_{\mathfrak{p},n} \bmod \mathfrak{p}^{n+1}.$$

For $n \geq 1$, let $\Xi_n(\tau)$ be submodule of $K \oplus K$ generated by $e_{\mathfrak{p},0} := (1,0) = e$, $f_{\mathfrak{p},n} := (0,1) = f$, and

$$I_{\mathfrak{p},n} := \mathfrak{p}^{-n}(e - \tau_{\mathfrak{p},n} f)$$

for $\mathfrak{p} \in \{\mathfrak{p}_0, \ldots, \mathfrak{p}_n\}$. Clearly,

$$\mathfrak{p}^n I_{\mathfrak{p},n} + R\tau_{\mathfrak{p},n} f = Re$$

and in particular

$$I_{\mathfrak{p},0} + R\tau_{\mathfrak{p},0} = Re.$$

Notice that $\Xi_n(\tau) \subseteq \Xi_{n+1}(\tau)$ for $n \in \omega$. Indeed, we have for every $\mathfrak{p} \in \mathbb{P}$,

$$I_{\mathfrak{p},n} = \mathfrak{p}^{-n}(e - \tau_{\mathfrak{p},n} f) \subseteq \mathfrak{p}^{-n}(e - \tau_{\mathfrak{p},n+1} f + \mathfrak{p}^n f) \subseteq I_{\mathfrak{p},n+1} + Rf.$$

Define $\Xi(\tau) = \mathrm{colim}_n \Xi_n(\tau) \subseteq K \oplus K$.

Lemma 5.154. *Suppose that M is a countable flat module. If M has rank one, then $E_\tau(M) = \{0\}$ for every $\tau \in J(R)$.*

Proof. If M is not reduced, then $M = K$. In this case, if $\mathfrak{p} \in \mathbb{P}$, $x, z \in M$, and $r \in R$ are nonzero such that $\tau_\mathfrak{p} z = rx$ then $\tau_\mathfrak{p} \in K$, which is not possible since $\tau_\mathfrak{p} \notin K$.

Assume that M is reduced. Suppose that $\mathfrak{p} \in \mathbb{P}$, $x, z \in M$, and $r \in R \setminus \{0\}$ are such that $\tau_\mathfrak{p} z = rx$. If z is nonzero, then there exists $s \in K$ such that $x = sz$ and hence $\tau_\mathfrak{p} z = rsz$ and $\tau_\mathfrak{p} = rs \in K$, which is again a contradiction. □

Lemma 5.155. *Suppose that R is a DVR with maximal ideal generated by p and M is a reduced R-module. Then M is a pure submodule of its completion $\hat{M}$.*

Proof. Suppose that $x \in \hat{M}$ is such that $p^k x = a \in M$ for some $k \in \mathbb{N}$. We can write $a = p^d b$ where $b \in M$, $d \geq 0$, and p does not divide b. Then for every $n \geq d$ there exists $x_n \in M$ such that $p^k x_n \equiv p^d b \bmod p^n M$. This implies that $k = d$, and $x_n \equiv b \bmod p^n M$ for every $n \geq d$. Thus, $x = b \in M$. □

Recall that, for a countable flat R-module M, we let $M_{\mathfrak{p}}$ be $R_{\mathfrak{p}}$-module obtained as the localization of M at $\mathfrak{p}$. Explicitly, $M_{\mathfrak{p}}$ is the $R_{\mathfrak{p}}$-submodule of $K \otimes_R M$ generated by M, thus its elements are of the form ax for $x \in M$ and $a \in R_{\mathfrak{p}}$.

Lemma 5.156. *Suppose that M is a countable flat R-module, and E is a submodule of M. Then:*

1. *M is completely reduced if and only if $M_{\mathfrak{p}}$ is reduced for every $\mathfrak{p} \in \mathbb{P}$;*
2. *E is pure in M if and only if $E_{\mathfrak{p}}$ is pure in $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathbb{P}$.*

Proof. (1) Suppose that $\mathfrak{p} \in \mathbb{P}$. If $M_{\mathfrak{p}}$ is reduced then $\mathfrak{p}^\infty M \subseteq \mathfrak{p}^\infty M_{\mathfrak{p}} = \{0\}$. If M is completely reduced, and $y \in \mathfrak{p}^\infty M_{\mathfrak{p}}$, then one can write $y = ax$ with $a \in R_{\mathfrak{p}}$ and $x \in M$. Since $y \in \mathfrak{p}^\infty M_{\mathfrak{p}}$, for every $n \in \omega$, $\mathfrak{p}^{-n} y \subseteq M_{\mathfrak{p}}$. Thus, there is an ideal J_n coprime with $\mathfrak{p}$ such that $\mathfrak{p}^{-n} y \subseteq J_n^{-1} M$. Thus, $J_n y \subseteq \mathfrak{p}^n M$. Since J_n is coprime with $\mathfrak{p}$ we have $J_n + \mathfrak{p}^n = R$. Thus, we have

$$Ry = (J_n + \mathfrak{p}^n)\, y \subseteq J_n y + \mathfrak{p}^n y \subseteq \mathfrak{p}^n M.$$

As this holds for every $n \in \omega$, we conclude $y \in \mathfrak{p}^\infty M = 0$. This concludes the proof.

(2) Suppose that E is pure in M. Then if $y \in M_{\mathfrak{p}}$ and $r \in R_{\mathfrak{p}}$ are such that $ry \in E_{\mathfrak{p}}$, there exists an ideal J coprime with $\mathfrak{p}$ such that $ry \in J^{-1}E$ and $r \in J^{-1}R$. Thus, we have that $Jry \subseteq E$ and $Jr \subseteq R$. If $s \in Jr$ then we have $sy \in E$ and hence $y \in E$ since E is pure in M.

Conversely, suppose that $E_{\mathfrak{p}}$ is pure in $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathbb{P}$. Suppose that $y \in M$ and $r \in R$ are such that $ry \in E$. Then there exist prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $Rr = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. We prove that $y \in E$ by induction on n. If $n = 1$ then we have $Rr = \mathfrak{p}$. Thus, we have that $\mathfrak{p}y \subseteq E \subseteq E_{\mathfrak{p}}$. Since $E_{\mathfrak{p}}$ is pure in $M_{\mathfrak{p}}$, this implies that $y \in E_{\mathfrak{p}}$. Thus, there exists an ideal J coprime with $\mathfrak{p}$ such that $y \in J^{-1}E$. Thus, we have $Jy \subseteq E$. Hence $Ry = (J + \mathfrak{p})\, y \subseteq E$ and $y \in E$. This shows that $y \in E$ when $n = 1$. If the conclusion holds for $n - 1$, then we have $\mathfrak{p}_1 \cdots \mathfrak{p}_{n-1} y \subseteq E$ by the case $n = 1$, and then $y \in E$ by the case $n - 1$. □

We now establish some properties of the modules $\Xi(\tau)$. Recall the definition of $\Omega(A)$ for some countable flat module A from 5.151.

Lemma 5.157. *For $\tau = (\tau_{\mathfrak{p}})_{\mathfrak{p}\in\mathbb{P}} \in J(R)$, consider the pointed module $(\Xi(\tau), e)$ defined above. Denote by E the submodule of $\Xi(\tau)$ generated by e. Then:*

1. *$\Xi(\tau)$ is completely reduced;*
2. *$(\Xi(\tau), e)$ is a well-pointed module;*
3. *E is the submodule of $\Xi(\tau)$ comprising the τ-divisible elements;*
4. *E is a characteristic submodule of $\Xi(\tau)$;*
5. *$E_\tau(\Xi(\tau)) = \Xi(\tau)$;*
6. *$\Xi(\tau)$ is rigid;*
7. *for $\lambda \in J(R)$ not essentially equivalent to τ, $E_\lambda(\Xi(\tau)) = 0$;*
8. *if A is a countable reduced flat module and $\tau \in \Omega(A)$, then $\mathrm{Hom}(\Xi(\tau), A) = 0$;*

Proof. We have that (8) is a consequence of (5) in view of Lemma 5.153.

By Lemma 5.156, after localizing at a prime, we can assume that R is a DVR. In this case, $\mathbb{P}$ contains a single maximal ideal $\mathfrak{p}$, which is furthermore principal [145, Example 4.82]. We can thus fix a generator p of $\mathfrak{p}$. In the notation below, we will use p for the subscript instead of $\mathfrak{p}$, to simplify the notation. In this case, the ideal $I_{p,n}$ is principal, and generated by $g_{p,n} := p^{-n}(e - \tau_{p,n}f)$. Therefore, we have that $e \equiv \tau_p f \bmod p^n$ for every $n \in \omega$, and $e \in E_\tau(\Xi(\tau))$.

(1) We need to prove that $\Xi(\tau)$ is reduced. If $\Xi(\tau)$ is not reduced, as it has rank 2, it must be isomorphic to $K \oplus L$ where L has rank one. This implies that $E_\tau(\Xi(\tau)) = \{0\}$ by Lemma 5.154, contradicting the fact that $e \in E_\tau(\Xi(\tau))$.

(2) Suppose that $x \in \Xi(\tau)$ and $d \geq 0$ are such that $p^d x = e$. We can write $x = ag_{p,n} + bf$ for some $a, b \in R$ and $n \geq d$. Thus, we have that

$$\begin{aligned} e &= p^d x = p^d a g_{p,n} + p^d b f \\ &= p^d a\left(p^{-n}(e - \tau_{p,n}f)\right) + p^d b f \\ &= p^{d-n} a e + \left(p^d b - p^{d-n} a \tau_{p,n}\right) f \end{aligned}$$

This implies that

$$p^{d-n} a = 1$$

and

$$p^d b = p^{d-n} a \tau_{p,n} = \tau_{p,n}.$$

This shows that p^d divides τ_p. Since $\tau_p \equiv 1 \bmod p$ this implies that $d = 0$;

(3) Suppose that $x \in \Xi(\tau)$ is τ_p-divisible, whence $x = \tau_p y$ for some $y \in \Xi(\tau_p)$. We work in $K_p(\tau_p) \otimes \Xi(\tau)$, which is a 1-dimensional $K_p(\tau_p)$-vector space. We can write $x = se + tf$ and $y = s'e + t'f$ for some $s, s', t, t' \in K$. Thus, we have

$$s\tau_p f + tf = x = \tau_p y = s'\tau_p^2 f + t'\tau_p f$$

This implies that

$$s'\tau_p^2 + (t' - s)\tau_p - t = 0.$$

Since 1, τ_p, τ_p^2 are linearly independent over K by the choice of τ_p, this implies $t = s' = 0$ and $t' = s$. Thus, we have $x \in E$.

(4) This is a consequence of (3).

(5) This follows from the fact that $e, f \in E_\tau(\Xi(\tau))$.

(6) If φ is an endomorphism of $\Xi(\tau)$, we have $\varphi(e) = re$ for some $r \in R$ by (4). Thus, since $e = \tau f$ we also have

$$\tau r f = re = \varphi(\tau f) = \tau\varphi(f)$$

and hence $\varphi(f) = rf$. Since the pure submodule of $\Xi(\tau)$ generated by e and f is $\Xi(\tau)$, this shows that φ is the endomorphism $x \mapsto rx$.

(7) Suppose that $\lambda \in J(R)$ and $x, y \in \Xi(\tau)$ are such that $\lambda_p y = x$. Consider the subfield $K(\tau_p, \lambda_p)$ of $\hat{K}_p$ generated by τ_p, λ_p. We work in $K(\tau_p, \lambda_p) \otimes \Xi(\tau)$. We can write $x = se + tf$ and $y = s'e + t'f$ for some $s, s', t, t' \in K$. Then we have that

$$(s\tau_p + t) f = se + tf = \lambda_p y = \lambda_p (s'\tau_p + t') f$$

and hence

$$(s\tau_p + t) = \lambda_p (s'\tau_p + t').$$

This shows that $\lambda_p \in K(\tau_p)$ and hence λ_p, τ_p are essentially equivalent. □

5.12.5 Ξ-extractable modules

We define what it means for a countable flat module to have (plain) Ξ-*extractable* length α. We also define the Ξ-invariant τ_M and the minimal family ξ_M of M. Recall the definition of the sets of indices I_α and $I_\alpha^{\mathrm{plain}}$ for $\alpha < \omega_1$.

Definition 5.158. *Let M be a coreduced countable flat module.*

- *M has plain Ξ-length 1, Ξ-invariant $\boldsymbol{\tau} \in M^{I_1^{\mathrm{plain}}}$ and minimal family $\xi \in M^{I_1^{\mathrm{plain}}}$ if, for some $\tau \in J(R)$,*

$$M = \Xi(\tau),$$
$$\xi(0) = e_{\Xi(\tau)},$$
$$\boldsymbol{\tau}(0) = [\tau];$$

- *M has Ξ-length α, Ξ-invariant $\boldsymbol{\tau} \in M^{I_\alpha}$ and minimal family $\xi \in M^{I_\alpha}$ if*

$$M = \bigoplus_n M_n$$

$$\boldsymbol{\tau}(-; n) = \boldsymbol{\tau}_n$$

$$\xi(-; n) = \xi_n$$

where M_n has plain Ξ-length α_n, Ξ-invariant $\boldsymbol{\tau}_n \in M_n^{I_{\alpha_n}^{\text{plain}}}$, and minimal family $\xi \in M^{I_{\alpha_n}^{\text{plain}}}$, and the function $\boldsymbol{\tau}$ is injective;

- *M has plain Ξ-length α, Ξ-invariant $\boldsymbol{\tau} \in M^{I_\alpha^{\text{plain}}}$, and minimal family $\xi \in M^{I_\alpha^{\text{plain}}}$ if there exists a fishbone of modules*

$$\left(\left(L_{n,i}, \xi_{n,i}\left(\left(\alpha - 1\right)_n - 1\right), J_n, e_{n,i}, \varphi_n\right)_{i<d_n}\right)_{n\in\omega}$$

with fishbone colimit M and $\lambda \in J(R)$ such that:

1. *for every $n \in \omega$ and $i < d_n$, $L_{n,i}$ has plain Ξ-length $(\alpha - 1)_n$, Ξ-invariant $\boldsymbol{\tau}_{n,i}$, and minimal family $\xi_{n,i}$;*
2. *if L is the submodule of the fishbone colimit M corresponding to*

$$\operatorname{colim}_n\left(L_{n,0} \oplus \cdots \oplus L_{n,d_n-1}\right),$$

   then there is an isomorphism between the quotient of M by L and $\Xi(\lambda)$ that maps the element corresponding to $\xi(\alpha - 1)$ to $e_{\Xi(\lambda)}$;
3. *$\boldsymbol{\tau}(\alpha - 1) = [\lambda]$;*
4. *$\boldsymbol{\tau}(-; d_0 + \cdots + d_{n-1} + i) = \boldsymbol{\tau}_{n,i}$ for $n < \omega$ and $i < d_n$;*
5. *$\xi(-; d_0 + \cdots + d_{n-1} + i) = \xi_{n,i}$ for $n < \omega$ and $i < d_n$;*
6. *$\boldsymbol{\tau}$ is injective.*

The module M is Ξ-extractable if it has (plain) Ξ-length α for some $\alpha < \omega_1$.

It follows easily by induction that the (plain) fishbone length of a module is less than or equal to the (plain) Ξ-length. Thus, any Ξ-extractable module is fishbone extractable. Furthermore, the minimal family of a Ξ-extractable module is indeed a minimal family in the sense of Definition 5.135.

Proposition 5.159. *For every $\alpha < \omega_1$ and infinite $S \subseteq \boldsymbol{J}(R)$ there exists a well-pointed countable flat module M that has plain Ξ-length α and Ξ-invariant $\boldsymbol{\lambda}$ with image contained in S.*

Proof. By induction on α. For $\alpha = 1$ define M to be $\Xi(\lambda)$ for some $\lambda \in S$.

Suppose that this holds for $\beta < \alpha$. Pick $\lambda \in S$ and define $J := \Xi(\lambda)$ with $e = e_{\Xi(\lambda)}$ as a distinguished element. Write $J = \operatorname{colim}_n J_n$ for finite submodules $J_n \subseteq J$ with $\dim(J_n) = d_n$ and $d_0 = 1$. Let $(e_{n,i})_{i<d_n}$ be a free basis of J_n. By the inductive hypothesis, for $n < \omega$ and $i < d_n$ there exists a countable flat module $M_{n,i}$ of plain Ξ-length $(\alpha-1)_n$, Ξ-invariant $\boldsymbol{\lambda}_n$, and minimal family ξ_n such that $(\boldsymbol{\lambda}_n)_{n\in\omega}$ have images that are pairwise disjoint and contained in $S \setminus \{\lambda\}$. Consider the fishbone of modules

$$(M_{n,i}, \xi_{n,i}((\alpha-1)_n - 1), J_n, \varphi_n, e_{n,i})$$

Then its corresponding fishbone colimit has plain Ξ-length α, and Ξ-invariant $\boldsymbol{\lambda}$ with image contained in S. □

The following proposition is easily established by induction on α. Recall the definition of $\Omega(A)$ for some countable flat module A from 5.151.

Proposition 5.160. *Let M be a coreduced countable flat module of (plain) Ξ-length α and Ξ-invariant $\boldsymbol{\lambda}$. Let also A be a countable flat module with $\Omega(A)$ disjoint from the image of $\boldsymbol{\lambda}$. Then M is extractable of (plain) length α and A-independent.*

5.12.6 Zipping

Let α be a countable ordinal. We define a *successor* function $s_\alpha : I_\alpha \to I_\alpha$ by induction on α. For $\alpha = 1$ define

$$s_\alpha(0; n) := (0; n+1).$$

If it has been defined for ordinals less than α, define

$$s_\alpha((\alpha-1)_n) := (\alpha-1)_{n+1}$$

and

$$s_\alpha(i; n) = (s_{\alpha_n}(i); n)$$

for $i \in I_{(\alpha-1)_n}$ and $n \in \omega$. This is indeed the successor function when I_α is regarded as a linear order isomorphic to ω^α as described in Section 5.6.5.

Given a module and a function $\xi \in M^{I_\alpha}$, we define $\delta\xi \in M^{I_\alpha}$ to be the function $\xi - \xi \circ s_\alpha$. Suppose now that M is a coreduced and completely reduced Ξ-extractable module of length $\alpha \geq 1$ with Ξ-invariant $\boldsymbol{\tau}$ and minimal family ξ. Let also $\boldsymbol{\lambda} \in \boldsymbol{J}(R)^{I_\alpha}$ be an injective function whose image is disjoint from the image of τ. For each $i \in I_\alpha$ we let $\lambda(i) \in J(R)$ be such that $[\lambda(i)] = \boldsymbol{\lambda}(i)$. Let S be the direct sum of $\Xi(\lambda(i))$ for $i \in I_\alpha$. Define then the *zipping* $Z(M, \boldsymbol{\lambda})$ of M along $\boldsymbol{\lambda}$ to be the quotient of $M \oplus S$ by the submodule N generated by $(\delta\xi)(i) - e_{\lambda(i)}$ for $i \in I_\alpha$. We can identify M and S as submodules of $Z(M, \boldsymbol{\lambda})$.

Lemma 5.161. *Adopt the notation above. Then we have that:*

1. $\left((\delta\xi)(i) - e_{\lambda(i)}\right)_{i\in I_\alpha}$ *is a minimal family in $M \oplus S$;*

2. *the submodule* N *of* $M \oplus S$ *generated by* $\left((\delta\xi)(i) - e_{\lambda(i)}\right)_{i \in I_\alpha}$ *is pure;*

3. $Z(M, \boldsymbol{\lambda})$ *is a countable flat module;*

4. $\sigma_\alpha Z(M, \boldsymbol{\lambda}) = Z(\sigma_\alpha M, \boldsymbol{\lambda})$ *for* $0 \leq \alpha < \omega_1$*;*

5. $\partial_\alpha Z(M, \boldsymbol{\lambda}) \cong \partial_\alpha M$ *for* $0 \leq \alpha < \omega_1$*;*

6. *if* A *is a countable flat module,* $E_{\lambda(i)}(A) = 0$ *for every* $i \in I_\alpha$*, and* M *is* A*-independent, then* $Z(M, \boldsymbol{\lambda})$ *has* A*-projective length* α*, and*

$$s_\beta \mathrm{Ext}(Z(M, \boldsymbol{\lambda}), A) \cong \mathrm{Ext}(\partial_\beta M, A)$$

   for $\beta \geq 1$*;*

7. M *is a characteristic submodule of* $Z(M, \boldsymbol{\lambda})$*;*

8. $Z(M, \boldsymbol{\lambda})$ *is rigid;*

9. $Z(M, \boldsymbol{\lambda})$ *is completely coreduced.*

Proof. We prove that the conclusion holds by induction on α. Define $\eta_i := (\delta\xi)(i) - e_{\lambda(i)}$ for $i \in I_\alpha$.

(1) For $\alpha = 1$, this follows from the fact that $(\delta\xi(i))_{i \in I_1}$ is the image of $(\xi(i))_{i \in I_1}$ under the isomorphism $M \to M$ that maps (x_i) to $(x_i - x_{i+1})$. Suppose that the conclusion holds for $\beta < \alpha$. Then we can write

$$M \cong \bigoplus_n M_n$$

where for $n \in \omega$, M_n is Ξ-extractable of plain length α_n. Define

$$L := \bigoplus_n \sigma_{\alpha_n - 1} M_n \subseteq M$$

and

$$J := \bigoplus_n \partial_{\alpha_n - 1} M_n.$$

We have an exact sequence

$$L \to M \to J.$$

For $n \in \omega$, $\sigma_{\alpha_n - 1} M_n$ is extractable of length $\alpha_n - 1$. Thus, by the inductive hypothesis, $(\eta_i)_{i \in I_{\alpha_n - 1}}$ is minimal. By the case $\alpha = 1$, the family

$$\left(\eta_{(\alpha_n - 1, n)} + L\right)_{n \in \omega}$$

is minimal in J. It follows that $(\eta_i)_{i \in I}$ is minimal in M by Lemma 5.136.

(2) and (3) are consequences of (1).

(4) and (5) Write $M = \operatorname{colim}_n M_n$ for some finite-rank pure submodules $M_n \subseteq M$. Define $Z(M_n, \boldsymbol{\lambda})$ to be the submodule of $Z(M, \boldsymbol{\lambda})$ generated by M_n and $\Xi(\lambda(i))$ for $i \leq n$. Then $Z(M, \boldsymbol{\lambda}) = \operatorname{colim}_n Z(M_n, \boldsymbol{\lambda})$. Since $\Xi(\lambda(i))$ is coreduced for every $i \in \omega$, the inclusion $M_n \to Z(M_n, \boldsymbol{\lambda})$ induces an isomorphism

$$\operatorname{Hom}(M_n, R) \cong \operatorname{Hom}(Z(M_n, \boldsymbol{\lambda}), R).$$

Thus, the inclusion $M \to Z(M, \boldsymbol{\lambda})$ induces an isomorphism

$$\begin{aligned} \operatorname{Ext}(\partial_1 M, R) &\cong \operatorname{Ph}^1\operatorname{Ext}(M, R) \\ &\cong \lim{}^1\operatorname{Hom}(M_n, R) \\ &\cong \lim{}^1\operatorname{Hom}(Z(M_n, \boldsymbol{\lambda}), R) \\ &\cong \operatorname{Ph}^1\operatorname{Ext}(Z(M, \boldsymbol{\lambda}), R) \\ &\cong \operatorname{Ext}(\partial_1 Z(M, \boldsymbol{\lambda}), R). \end{aligned}$$

The conclusion thus follows by induction.

(6) Since M is A-independent, it has projective A-rank α by Theorem 5.134. Since $E_{\lambda(i)}(A) = 0$ for every $i \in I_\alpha$, we have $\operatorname{Hom}(\Xi(\lambda(i)), A) = 0$ for every $i \in I_\alpha$ by Lemma 5.153. The conclusion thus can be obtained as in (4) and (5) above.

(7) We have that $\sigma_1 Z(M, \boldsymbol{\lambda}) = Z(\sigma_1 M, \boldsymbol{\lambda})$ is a characteristic submodule of $Z(M, \boldsymbol{\lambda})$. Thus, it suffices to consider the case when $\alpha = 1$. In this case, we have that M is the submodule $E_{\boldsymbol{\tau}}(Z(M, \boldsymbol{\lambda}))$ and hence it is characteristic.

(8) By induction on α. For an element a of $Z(M, \boldsymbol{\lambda})$, we denote by $\langle a \rangle$ the submodule of $Z(M, \boldsymbol{\lambda})$ generated by a. For $\alpha = 1$, we have

$$E_{\lambda(k)}(Z(M, \boldsymbol{\lambda})) = \langle \xi_{(0;k)} - \xi_{(0;k+1)} \rangle$$

and

$$E_{\tau(0;k)}(Z(M, \boldsymbol{\lambda})) = \langle \xi_{(0;k)} \rangle$$

for every $k \in \omega$. This implies that for every $k \in \omega$ there exists $r_k, s_k \in R$ such that

$$\varphi\left(\xi_{(0;k)}\right) = r_k \xi_{(0;k)}$$

and

$$\varphi\left(\xi_{(0;k)} - \xi_{(0;k+1)}\right) = s_k \xi_{(0;k)} - s_k \xi_{(0;k+1)}.$$

In turn, this implies that $r_k = s_k = r_0$ for every $k \in \omega$. Thus, φ is the endomorphism $x \mapsto r_0 x$.

Assume the conclusion holds for $\beta < \alpha$. As before, we can write

$$M \cong \bigoplus_n M_n$$

where for $n \in \omega$, M_n is Ξ-extractable of plain rank α_n. Again, define

$$L := \bigoplus_n \sigma_{\alpha_n} M_n,$$

$$J := \bigoplus_n \partial_{\alpha_n} M_n,$$

and consider the short exact sequence

$$L \to M \to J.$$

By the inductive hypothesis, for every $n \in \omega$ there exists $r_n \in R$ such that $\varphi|_{M_n} : x \mapsto r_n x$. Furthermore, we have that

$$E_{\tau(\alpha_n - 1;n)} \left(Z \left(M, \boldsymbol{\lambda} \right) \right) = \langle \xi_{(\alpha_n - 1;n)} \rangle$$

and

$$E_{\lambda(\alpha_n - 1;n)} \left(Z \left(M, \boldsymbol{\lambda} \right) \right) = \langle \xi_{(\alpha_n - 1;n)} - \xi_{(\alpha_{n+1} - 1;n+1)} \rangle.$$

As above, this implies that $r_n = r_0$ for every $n \in \omega$ and hence φ is the endomorphism $x \mapsto r_0 x$.

(9) After localization one can assume that R is a DVR. The conclusion is then obtained by induction on α. For $\alpha = 1$ it is an easy a calculation. The case of α follows from the cases of 1 and $\alpha_n - 1$ for $n \in \omega$ and considering the extension

$$L \to M \to J$$

as above. □

5.12.7 Dichotomy

Summarizing the results established above, we obtained the following Dichotomy Theorem for modules over a countable Dedekind domain.

Theorem 5.162. *Suppose that R is a countable Dedekind domain, and A is a countable flat module. If A is divisible, then* $\operatorname{Ext}(-, A) = 0$. *If A is not divisible, then for every $\alpha < \omega_1$:*

1. *there exists an extractable A-independent countable coreduced and completely reduced module of (plain) length α and (plain) A-projective length α;*

2. *there exists a rigid countable coreduced and completely reduced module of R-projective and A-projective length α.*

Proof. If A is divisible, then $\operatorname{Ext}(-, A) = 0$ by Proposition 5.86. If A is not divisible, then by Proposition 5.160 for every $\alpha < \omega_1$ there exists a extractable A-independent countable coreduced flat module M of (plain) length α. By Theorem 5.134, M has (plain) A-projective length α.

Considering such an M of length α, we can obtain by Lemma 5.161 a new *rigid* module which is still countable, coreduced, completely reduced, and of R-projective and A-projective length α. □

Corollary 5.163. *Suppose that R is a countable Dedekind domain. Let $\mathcal{E}_\alpha$ be the exact structure projectively generated by the countable flat module of plain tree length at most α_n for $n \in \omega$. Then $(\mathcal{E}_\alpha)_{\alpha<\omega_1}$ is a strictly decreasing chain of nontrivial exact structures on the category $\mathbf{Flat}(R)$ of countable flat modules over R.*

Proof. This follows from Theorem 5.162 and Theorem 5.113. □

5.12.8 Modules that are not phantom projective

While the modules we have considered so far are extractable, there exist modules that fail to be so. Indeed, we now provide examples of modules of that are not phantom projective, which necessarily are not extractable.

Theorem 5.164. *Suppose that R is a countable Dedekind domain. Then there exists a countable module C of R-projective length 1 such that:*

1. *for every phantom pro-finiteflat module $\boldsymbol{M}$ there exists a countable flat module A such that $\operatorname{Ph}^1\operatorname{Ext}(C, A) \cong \boldsymbol{M}$;*
2. *C is not a phantom projective.*

Proof. After replacing R with the localization of R at a maximal ideal, we can assume that R is a DVR with maximal ideal generated by $p \in R$. Let $\omega^{<\omega}$ be the set of finite tuples of elements of ω. The idea is to construct a module with generators and relations that code any possible tower of finite flat modules. Let $\mu(\varphi, i)$ and $\lambda(\varphi, i, \psi)$ be pairwise not essentially equivalent elements of $J(R)$ for $n \in \omega$, $\varphi \in \operatorname{Hom}(R^n, R^{n-1})$, and $\psi \in \operatorname{Hom}(R^{n-1}, R^{n-2})$, where by convention $\operatorname{Hom}(R^n, R^{n-1}) = \varnothing$ for $n \in \{0, 1\}$. Set

$$B(\varphi, i) := \Xi(\mu(\varphi, i))$$

and

$$e(\varphi, i) := e_{\Xi(\mu(\varphi,i))} \in \Xi(\mu(\varphi, i)) = B(\varphi, i).$$

Likewise we define $B(\varphi, i, \psi)$ and $e(\varphi, i, \psi)$. Let B be the direct sum of $B(\varphi, i)$ and $B(\varphi, i, \psi)$ as above. Thus, B is the pure submodule of itself generated by $e(\varphi, i)$ and

$e(\varphi, i, \psi)$. Fix $\varphi \in \mathrm{Hom}(R^n, R^{n-1})$, and let $a_\varphi(j,i) \in R$ such that

$$\varphi(\delta_i) = \sum_{j<n-1} a_\varphi(j,i)\,\delta_j$$

for $i < n$, where $(\delta_i)_{i<n}$ is the canonical basis of R^n. We want to add the relation

$$e(\varphi, i) = \sum_{j<n-1} a_\varphi(j,i)\,e(\psi, j) + e(\varphi, i, \psi)$$

for any $\varphi \in \mathrm{Hom}(R^n, R^{n-1})$, $\psi \in \mathrm{Hom}(R^{n-1}, R^{n-2})$, and $i < n$. We therefore consider the submodule N of B generated by the elements

$$\rho(\varphi, i, \psi) := e(\varphi, i) - \left(\sum_{j<n-1} a_\varphi(j,i)\,e(\psi, j) + e(\varphi, i, \psi) \right).$$

It follows easily from Lemma 5.147 that $(\rho(\varphi, i, \psi))$ is a minimal family in B. Thus, N is a pure submodule of B. We let $c(\varphi, i)$ be the element $e(\varphi, i) + N \in C$, and likewise $c(\varphi, i, \psi)$ be the element $\rho(\varphi, i, \psi) + N \in C$.

(1) Given a phantom pro-finiteflat module $\boldsymbol{M}$, consider a tower (M_n) of finiteflat modules such that $\lim^1 M_n \cong \boldsymbol{M}$. We can assume that $M_n \cong R^n$ and the bonding map $M_{n+1} \to M_n$ corresponds to a homomorphism $\varphi_{n+1} : R^{n+1} \to R^n$. Now consider the countable flat module A obtained as follows. Set $\varphi_0 = \varnothing$. Consider

$$A = \bigoplus_{n\in\omega} \bigoplus_{i<n} B(\varphi_n, i).$$

Then it is easily seen that one can write an increasing sequence (C_k) of finite-rank pure submodules of C with $C = \mathrm{colim}_k C_k$ such that $(\mathrm{Hom}(C_k, A)) \cong (M_k)$. Indeed, let for every $n \in \omega$, $(\psi_{n,i})$ be an enumeration of $\mathrm{Hom}(R^n, R^{n-1})$ with $\psi_{n,0} = \varphi_n$. Define C_k to be the pure submodule of C generated by $c(\psi_{n,i}, j)$ for $n, i \leq k$ and $j \leq n$. Then we have

$$\mathrm{Hom}(C_k, A) \cong R^k$$

and under such an isomorphism the inclusion $C_{k-1} \to C_k$ induces the map $\varphi_k : R^k \to R^{k-1}$. Thus, $s_1\mathrm{Ext}(C, A) \cong \lim^1_k \mathrm{Hom}(C_k, A) \cong \boldsymbol{M}$.

(2) is a consequence of (1). □

5.12.9 Phantom projective modules that are not extractable

We now prove that when R is a Dedekind domain, there exist phantom projective modules that are not extractable. Again, after replacing R with its localization at a prime, we can assume that R is a DVR with maximal ideal generated by p. We recall the definition of the module $\Xi(\tau)$ for $\tau \in J(R)$ in this case. For $n \in \omega$ pick $\tau_n \in R$ such that $\tau \equiv \tau_n \bmod p^n$. For

$n \geq 1$, let $\Xi_n(\tau)$ be submodule of $K \oplus K$ generated by $e_0 := (1,0) = e$, $f_n := (0,1) = f$, and

$$g_n := p^{-n}(e - \tau_n f).$$

Then

$$p^n g_n + \tau_n f = e$$

for every $n \in \omega$. Notice that $\Xi_n(\tau) \subseteq \Xi_{n+1}(\tau)$ for $n \in \omega$. Define $\Xi(\tau) = \mathrm{colim}_n \Xi_n(\tau) \subseteq K \oplus K$. Then we have that $\Xi(\tau)$ is a rigid module.

Notice that if M is a countable coreduced flat module of plain tree length 1, then it has plain projective length 1. This implies that M has finite rank, and M has also plain R-projective length 1. We now show that this fails for 2.

Proposition 5.165. *Let R be a countable Dedekind domain. Then there exists a countable flat module that has plain tree length 2 and R-projective length 1. In particular, M is a phantom projective module that is not extractable.*

Proof. As noticed above, we can assume without loss of generality that R is a DVR. Fix an element τ of $J(R)$. We define M as a wedge sum

$$\bigoplus_{(C_1,\varphi_i)} C_{(i;0)}$$

where $C_1 = R$, $C_{(i;0)} = \Xi(\tau)$, and $\psi_i : R \to C_{(i;0)}$, $1 \mapsto p^i e$. Let M_n be the submodule of M corresponding to $C_1 \oplus C_{(0;0)} \cdots \oplus C_{(0;n-1)}$. Then we have that

$$\mathrm{Ph}^1\mathrm{Ext}(M,R) \cong \lim{}^1\mathrm{Hom}(M_n|C_1, R) = 0.$$

On the other hand,

$$\mathrm{Ph}^1\mathrm{Ext}(M,\Xi(\tau)) \cong \lim{}^1\mathrm{Hom}(M_n|C_1, \Xi(\tau)) \cong \lim{}^1(p^n R) \cong \hat{R}/R.$$

This shows that M has plain tree length 2 and plain projective length 2, but R-projective length 1. In particular, M is not extractable by Corollary 5.130. □

5.13 Complexity of extensions

In this last section we recall some fundamental notions from descriptive set theory, and isolate some complexity-theoretic consequence of the main results of the previous sections.

5.13.1 Complexity of classification problems

Invariant complexity theory provides a framework to compare the complexity of classification problems in mathematics. Every (reasonable) classification problem in mathematics

can be regarded (perhaps after a suitable parametrization) as an analytic equivalence relation E on a Polish space $\boldsymbol{X}$. Recall that a subset A of a Polish space $\boldsymbol{X}$ is *analytic* or $\boldsymbol{\Sigma}^1_1$ if it is the image of a Borel function $\boldsymbol{Z} \to \boldsymbol{X}$ for some standard Borel space $\boldsymbol{Z}$. Every Borel set is analytic, but the converse does not hold. An equivalence relation E on $\boldsymbol{X}$ is analytic if it is analytic as a subset of $\boldsymbol{X} \times \boldsymbol{X}$.

The notion of *Borel reducibility* allows one to compare the complexity of different classification problems.

Definition 5.166. *Let E, F be analytic equivalence relations on Polish spaces $\boldsymbol{X}, \boldsymbol{Y}$, respectively. The equivalence relation E is* Borel reducible *to F if there exists a Borel function $f : \boldsymbol{X} \to \boldsymbol{Y}$ such that, for $x, x' \in \boldsymbol{X}$,*

$$xEx' \Leftrightarrow f(x)\, F\, f(x').$$

The relations E, F are Borel bireducible if both E is Borel reducible to F and F is Borel reducible to E.

Special equivalence relations are used as *benchmarks* to assess the complexity of classification problems.

If G is a Polish group and X is a Polish G-space, the one can consider the corresponding *orbit equivalence relation* E^X_G on X. An equivalence relation is *classifiable by orbits* if it is Borel reducible to E^X_G for some Polish group G and Polish G-space X. By restricting G to smaller classes of Polish groups one obtains stronger notions of classifiability. Recall that a Polish group G is *non-Archimedean* if it has a basis of neighborhoods of the identity consisting of open subgroups. This is equivalent to the assertion that G is isomorphic to a closed subgroup of the Polish group S_∞ of permutations of $\mathbb{N}$.

Definition 5.167. *An equivalence relation E is* classifiable by countable structures *if it is Borel reducible to the orbit equivalence relation of a Polish G-space for some non-Archimedean Polish group G.*

In fact, one can always take G to be the Polish group S_∞ by [68, Theorem 3.5.2]. If $\mathcal{L}$ is a countable first-order language, then the space of countable $\mathcal{L}$-structures can be regarded as a Polish space $\mathrm{Mod}(\mathcal{L})$. An equivalence relation E is classifiable by countable structures if and only if it is Borel reducible to the relation of *isomorphism* of countable $\mathcal{L}$-structures for some countable language $\mathcal{L}$, thus justifying the name.

One obtains other benchmarks of complexity by considering distinguished equivalence relations. We let $\wp(\mathbb{N})$ be the collection of subsets of $\mathbb{N}$. This is a Polish space when identified with the product $\{0,1\}^{\mathbb{N}}$. As any other uncountable Polish space, it admits a Borel bijection with $\mathbb{R}$, whence its elements are also called reals. The relations of equality in $\wp(\mathbb{N})$, $\mathbb{R}$, or any other uncountable Polish space are all Borel bireducible.

Definition 5.168. *An equivalence relation E is smooth or concretely classifiable if it is Borel reducible to the relation of equality of real numbers.*

A more generous notion of classifiability is obtained by considering on the space $\{0,1\}^{\mathbb{N}}$ of binary sequences a coarser relation than equality.

Definition 5.169. *An equivalence relation E is* essentially hyperfinite *if it is Borel reducible to the relation of* tail equivalence *of binary sequences.*

The relation of tail equivalence of binary sequences is the orbit equivalence relation associated with a Borel action of a *countable* group on $\{0,1\}^{\mathbb{N}}$. More generally, a Borel equivalence relation on a Polish space is called *countable* if it is the orbit equivalence relation associated with a Borel action of a countable group or, equivalently, if its equivalence classes are countable.

Definition 5.170. *An equivalence relation E is* essentially countable *if it is Borel reducible to a countable Borel equivalence relations.*

Every essentially hyperfinite equivalence relation is essentially countable. The converse holds for orbit equivalence relations associated with actions of *abelian* non-Archimedean Polish groups on Polish spaces; see [51, Corollary 6.3]. Among the (essentially) countable Borel equivalence relations there exists one of maximum complexity, usually denoted by E_∞.

We will consider an equivalent presentation of the study of Borel complexity of classification problems, in terms of Borel-definable and $\mathbf{\Sigma}^1_1$-definable sets.

5.13.2 Definable and semidefinable set

By a $\mathbf{\Sigma}^1_1$-definable set we mean a pair $(\boldsymbol{X}, E)$ where $\boldsymbol{X}$ is a Polish space and E is an analytic equivalence relation on $\boldsymbol{X}$. We think of the pair $(\boldsymbol{X}, E)$ as an explicit presentation of the space $X = \boldsymbol{X}/E$ of E-equivalence classes. A subset Z of X is Borel (respectively, analytic) if its *lift* $\boldsymbol{Z} := \{z \in \boldsymbol{X} : [z]_E \in Z\}$ is a Borel (respectively, analytic) subset of $\boldsymbol{X}$. A function $f : X \to Y$ between $\mathbf{\Sigma}^1_1$-definable sets is Borel if it is induced by a Borel function $\boldsymbol{X} \to \boldsymbol{Y}$. We consider $\mathbf{\Sigma}^1_1$-definable sets as objects of a category with Borel functions as morphisms. The more restrictive notion of Borel-definable set $X := \boldsymbol{X}/E$ is obtained by requiring that E be an equivalence relation on $\boldsymbol{X}$ that satisfies stronger definability requirements.

We recall the notion of idealistic equivalence relation, which we present in a slightly stronger formulation than in [22, Definition 3.1]. Given an equivalence relation E on $\boldsymbol{X}$ we define the equivalence relation $E^{\mathbb{N}}$ on $\boldsymbol{X}^{\mathbb{N}}$ by setting

$$(x_n)\, E^{\mathbb{N}}\, (y_n) \Leftrightarrow \forall n \in \mathbb{N},\, x_n E y_n.$$

Definition 5.171. *An equivalence relation E on a Polish space X is* idealistic *if there is a Borel function $s : X \to X$ such that $xEs(x)$ for every $x \in X$, and an assignment $[x]_E \mapsto \mathcal{F}_{[x]_E}$ of a nontrivial σ-filter on $[x]_E$ to each E-class $[x]_E$, that is Borel in the sense that for every Polish space Z and every Borel set $A \subseteq X \times Z \times X$ the set*

$$\left\{(x,z) \in X \times Z : \mathcal{F}_{[x]_E} y,\ (s(x), z, y) \in A\right\}$$

is Borel, where

$$\mathcal{F}_{[x]_E} y,\ (s(x), z, y) \in A \Leftrightarrow \{y \in [x]_E : (s(x), z, y) \in E\} \in \mathcal{F}_{[x]_E}.$$

We say that E is stably idealistic *if the equivalence relation $E^{\mathbb{N}}$ on $X^{\mathbb{N}}$ defined by is idealistic.*

Definition 5.172. *A* Borel-definable set *is a $\mathbf{\Sigma}^1_1$-definable set $X := \boldsymbol{X}/E$ where E is Borel and stably idealistic.*

It is proved in [22, Section 3] that Borel-definable sets form an isomorphism-invariant full subcategory of $\mathbf{\Sigma}^1_1$-definable sets, which satisfies natural properties that can be seen as analogues of corresponding elementary facts concerning the category of sets. Furthermore, a function $f : X \to Y$ between Borel-definable sets is Borel if and only if its graph is a Borel subset of $X \times Y$.

Every group with a Polish cover yields a Borel-definable set, as the orbit-equivalence relation associated with a continuous action of a Polish group on a Polish space is stably idealistic [68, Proposition 5.4.19]. The notion of *complexity class* of the trivial subgroup of a group with a Polish cover can be extended naturally to Borel-definable sets.

Definition 5.173. *Let Γ be a complexity class of Borel subsets of Polish spaces. The Borel-definable set X is Γ-definable or potentially Γ if it is isomorphic in the category of Borel-definable sets to $Y = \boldsymbol{Y}/E$ where E is an equivalence relation on $\boldsymbol{Y}$ such that $E \in \Gamma(\boldsymbol{Y} \times \boldsymbol{Y})$.*

If Γ is a complexity class of Borel subsets of a Polish space, then its *dual class* $\check{\Gamma}$ is defined by letting, for every Polish space X, $\check{\Gamma}(X)$ be the collection of *complements* of elements of $\Gamma(X)$. The class Γ is self-dual if $\Gamma = \check{\Gamma}$. If Γ is *not* self-dual, then we say that Γ is the complexity class of a Borel-definable set X if X is Γ-definable and not $\check{\Gamma}$-definable.

5.13.3 Orbit spaces

Suppose that G is a Polish group. A Polish G-space is a Polish space $\boldsymbol{X}$ endowed with a continuous action $G \curvearrowright X$, $(g, x) \mapsto g \cdot x$. The corresponding orbit equivalence relation E^X_G is always stably idealistic, although it might fail to be Borel. Therefore, the space of G-orbits $\boldsymbol{X}/G$ is a $\mathbf{\Sigma}^1_1$-definable set, which is Borel-definable if and only if E^X_G is Borel. This holds if and only if there exists $\alpha < \omega_1$ such that every G-orbit Gx is $\mathbf{\Pi}^0_\alpha$; see [68, Therem 8.2.2].

More generally, suppose that $X = \boldsymbol{X}/E$ is a Borel-definable set, and G is a Polish group. A Borel action of G on X is an action that is defined by a Borel function $G \times X \to X$. The corresponding orbit equivalence relation E^X_G is the equivalence relation on $\boldsymbol{X}$ defined by $xE^X_G y \Leftrightarrow g \cdot [x]_E = [y]_E$. In this case, we say that X is a Borel-definable G-space. The orbit space is the $\mathbf{\Sigma}^1_1$-definable set $X/G := \boldsymbol{X}/E^X_G$. A similar proof as [68, Therem 8.2.2] gives the following.

Proposition 5.174. *Suppose that G is a Polish group, and X is a Borel-definable G-space. Then the orbit equivalence relation E^X_G is stably idealistic. Thus, if E^X_G is also Borel, then X/G is a Borel-definable set.*

For example, given a Borel-definable set $X = \boldsymbol{X}/E$, one can consider the Borel-definable set $\wp_{\aleph_0}(X)$ of countable subsets of X. This can be seen as the orbit space Y/S_∞, where $Y \subseteq X^{\mathbb{N}}$ is the S_∞-invariant Borel set of sequences that are injective modulo E and the action $S_\infty \curvearrowright X^{\mathbb{N}}$ is the shift. The corresponding equivalence relation F is the *Friedman–Stanley jump* of the equivalence relation E [68, Definition 8.3.1]. In particular, when $X = \mathbb{R}$ (or any other uncountable Polish space), then $\wp_{\aleph_0}(\mathbb{R})$ can be seen as the definable set of *countable sets of reals*.

Definition 5.175. *A definable set X can be* parametrized by countable sets of reals *if it admits a Borel bijection onto a Borel subset of $\wp_{\aleph_0}(\mathbb{R})$.*

5.13.4 Homogenous spaces with a Polish cover

In Section 5.4 we have introduced the notion of a module with a Polish cover. This is a module M explicitly presented as a quotient $M = \hat{M}/N$ where $\hat{M}$ is a Polish module and N is a Polishable submodule of $\hat{M}$ (not necessarily closed in $\hat{M}$). More generally, one can consider a Borel-definable set X presented as the quotient G/N of a (not necessarily commutative) Polish group G by a (not necessarily closed nor normal) Polishable subgroup N of G. The fact that N is a Polishable subgroup of G means that N is itself a Polish group with respect to a topology that makes the inclusion $N \to G$ continuous. Such a Polish group topology on N is unique, and it is characterized by the fact that is Borel sets are precisely the Borel subsets of G contained in N. (In this sense, the Polish topology on N is "induced" by the one of G, albeit not in the sense of the subspace topology.) In particular, N is a Borel subset of G, whence the orbit equivalence relation associated with the left translation action $N \curvearrowright G$ is Borel and stably idealistic.

Classically, a homogeneous space is a space X endowed with a transitive a action of a group G. Letting N be the stabilizer of a fixed point of X, one obtains an identification of X with the space G/N of left N-cosets of G. This motivates the following definition:

Definition 5.176. *A* homogeneous space with a Polish cover *is a Borel-definable set X of the form $\boldsymbol{X}/E$ where $\boldsymbol{X}$ is a Polish group G and E is the left coset relation of a Polishable subgroup N of G. In this case, we denote X by G/N.*

A phantom homogeneous Polish space *is a homogeneous space with a Polish cover G/N where N is dense in G.*

If G/N is a homogeneous space with a Polish cover, then a homogeneous subspace with a Polish cover of G/N is a subspace of the form H/N where H is a Polishable subgroup of G containing N. Clearly, H/N is itself a homogeneous space with a Polish cover, such that the inclusion $H/N \to G/N$ is Borel. Likewise, the quotient $G/H = (G/N)/(H/N)$ is a homogeneous space with a Polish cover. When H is *normal* in G, G/H is actually a group, and in this case we call it a group with a Polish cover. Plainly, every module with a Polish cover is, in particular, a group with a Polish cover.

The results about complexity of modules with a Polish cover from Section 5.5 holds more generally in the context of homogeneous spaces with a Polish cover; see [115]. The same

holds for the notion and properties of Solecki submodules, which in the context of homogeneous spaces with a Polish cover will be called Solecki subspaces; see [115]. The notion of (plain) Solecki length of a module with a Polish cover can be defined for homogeneous spaces with a Polish cover in the same manner. Notice in particular that a homogeneous space with a Polish cover G/N has Solecki length 0 if and only if N is closed in G, which is equivalent to the assertion that G/N is a Polish space (with the quotient topology). Thus, the homogeneous *Polish* spaces can be thought of as the homogeneous spaces with a Polish cover of Solecki length zero. If G/N is a homogeneous space with a Polish cover, and H is the closure of N in G, then H/N is a phantom homogeneous Polish space, and G/H is a homogeneous Polish space. We let $\rho\left(G/N\right)$ be the Solecki length of G/N, which is equal to the Solecki length of H/N. If $\alpha := \rho\left(G/N\right)$, then we say that G/N is:

- *semiplain* if α is a successor and $s_{\alpha-1}\left(G/N\right)$ is $D(\mathbf{\Sigma}^0_2)$-definable, which is equivalent to the assertion that it is $\mathbf{\Sigma}^0_3$-definable as shown in [115];
- *plain* if α is a successor and $s_{\alpha-1}\left(G/N\right)$ is $\mathbf{\Sigma}^0_2$-definable.

Clearly, if G/N is semiplain, then it is plain. The converse holds when N is non-Archimedean a shown in [115]. Let us recall the definition of the complexity classes Γ_α, $\Gamma^{\text{semiplain}}_\alpha$, and $\Gamma^{\text{plain}}_\alpha$ for $\alpha < \omega_1$ from Section 5.5.3:

$$\Gamma_{\lambda+n} := \begin{cases} \mathbf{\Pi}^0_1 & \text{for } \lambda = n = 0; \\ \mathbf{\Pi}_\lambda & \text{for } n = 0 \text{ and } \lambda > 0; \\ \mathbf{\Pi}^0_{1+\lambda+n+1} & \text{for } \lambda > 0 \text{ and } n > 0; \end{cases}$$

and

$$\Gamma^{\text{semiplain}}_{\lambda+n} := \begin{cases} \Gamma_{\lambda+n} & \text{for } n = 0; \\ D(\mathbf{\Pi}^0_{1+\lambda+n}) & \text{for } n \geq 1; \end{cases}$$

and

$$\Gamma^{\text{plain}}_{\lambda+n} := \begin{cases} \Gamma^{\text{semiplain}}_{\lambda+n} & \text{for } n \neq 1; \\ \mathbf{\Sigma}^0_{1+\lambda+1} & \text{for } n = 1. \end{cases}$$

The following is a reformulation of [115, Theorem 6.1].

Theorem 5.177. *Suppose that X is phantom homogeneous Polish space. Let α be the Solecki length of X.*

1. *If X is plain, then $\Gamma^{\text{plain}}_\alpha$ is the complexity class of X;*
2. *If X is semiplain and not plain, then $\Gamma^{\text{semiplain}}_\alpha$ is the complexity class of X;*
3. *If X is not semiplain, then Γ_α is the complexity class of X.*

Lemma 5.178. *Suppose that Suppose that $G = \hat{G}/N$ is a homogeneous space with a Polish cover and $H = \hat{H}/N$ a homogeneous subspace with a Polish cover of G.*

1. *If H has length at most 1 and G/H has plain length at most 1, then G has length at most length 1;*

2. *If H has plain length at most 1 and G/H has plain length at most 1, then G has plain length at most 1;*

3. *If H has plain length at most 1 and G/H has semiplain length at most 1, then G has semiplain length at most 1.*

Proof. (1) By hypothesis, we have that $\hat{H}$ contains a closed neighborhood V of the identity that is closed in $\hat{G}$. Since H has length at most 1, we have that $\hat{H}$ has a basis of neighborhoods of the identity U contained in V such that $\bar{U}^{\hat{H}} \cap N \subseteq U$. For such a set U, since $U \subseteq V$ and V is closed in $\hat{H}$, we have that $\bar{U}^{\hat{H}} = \bar{U}^{\hat{G}}$, and hence $\bar{U}^{\hat{G}} \cap N \subseteq U$. This shows that $\hat{G}$ has length at most 1.

(2) By hypothesis, we have that $\hat{H}$ contains a closed neighborhood V of the identity that is closed in $\hat{G}$. Likewise, N contains a closed neighborhood W of the identity contained in V that is closed in $\hat{H}$. This implies that W is closed in $\hat{G}$, whence G has plain length 1.

(3) As in (2), N contains a neighborhood W of the identity that is closed in $\hat{H}$. By [115, Lemma 5.1], since $\hat{H}$ is $D(\mathbf{\Pi}^0_2)$ in G, we conclude that W is $D(\mathbf{\Pi}^0_2)$ in $\hat{G}$. Thus, N is $\mathbf{\Sigma}^0_3$ in $\hat{G}$, and hence also $D(\mathbf{\Pi}^0_2)$ by Theorem 5.177. □

Lemma 5.179. *Suppose that $G = \hat{G}/N$ is a homogeneous space with a Polish cover and $H = \hat{H}/N$ a homogeneous subspace with a Polish cover of G.*

1. *We have that*
$$\rho(G) \leq \rho(G/H) + \rho(H).$$

2. *If $\rho(G/H)$ is successor, and G/H is plain, then*
$$\rho(G) \leq (\rho(G/H) - 1) + \rho(H).$$

3. *If H has plain length 1 and G/H has (semi)plain length at most γ, then G has (semi)plain length at most γ;*

4. *If H has (semi)plain length 1 and G/H has length at most γ, then G has (semi)plain length at most $\gamma + 1$;*

5. *If H is countable, then $\{0\}$ is $\mathbf{\Sigma}^0_{1+\gamma}$ in G if and only if H is $\mathbf{\Sigma}^0_{1+\gamma}$ in G.*

Proof. Set $\alpha := \rho(G/H)$ and $\beta := \rho(H)$.

(1) We have that $H \in \Gamma_\alpha(G)$. Thus, $s_\alpha(G) \subseteq H$. We also have $\{0\} \in \Gamma_\beta(H)$. Thus, $\{0\} \in \Gamma_\beta(s_\alpha(G))$. Hence, $\{0\} = s_\beta(s_\alpha(G)) = s_{\alpha+\beta}(G)$. Thus shows that $\rho(G) \leq \alpha + \beta$.

(2) For $\alpha = 1$ and $\beta = 1$ the conclusion follows from Lemma 5.178(1) By induction on β one can then show that the conclusion holds for $\alpha = 1$ and every $\beta < \omega_1$. Let L be the

preimage of $s_{\alpha-1}(G/H)$ under the quotient map $G \to G/H$. The conclusion for arbitrary α follows from the case $\alpha = 1$ and Part (1) applied to the exact sequences

$$H \to L \to s_{\alpha-1}(G/H)$$

and

$$L \to G \to G/L = \frac{G/H}{s_{\alpha-1}(G/H)}.$$

(3) Since G/H has (semi)plain length at most γ, there is $H \subseteq K \subseteq G$ such that G/K has length at most $\gamma - 1$ and K/H has (semi)plain length at most 1. Thus, by Lemma 5.178(2) and Lemma 5.178(3), since H has plain length 1, K has (semi)plain length at most 1. Thus shows that G has (semi)plain length at most γ.

(4) Suppose that H has plain length 1, whence $\{0\}$ is $\mathbf{\Sigma}^0_2$ in H. Since G/H has length at most γ, we have $s_\gamma(G) \subseteq H$. Thus, $\{0\}$ is $\mathbf{\Sigma}^0_2$ in $s_\gamma(G)$, and G has plain length at most 1. The same proof shows that if H has semiplain length 1, then G has semiplain length at most 1.

(5) If $\{0\}$ is $\mathbf{\Sigma}^0_{1+\gamma}$ in G, then H is $\mathbf{\Sigma}^0_{1+\gamma}$ in G as a countable union of translates of $\{0\}$. Conversely, suppose that H is $\mathbf{\Sigma}^0_{1+\gamma}$ in G. We can write $\gamma = \lambda + n$ for a limit ordinal λ and $n < \omega$. If $n = 0$ then we must have that G/H has length less than λ, whence the same holds for G. This implies that $\{0\}$ is $\mathbf{\Pi}^0_{1+\beta}$ for some $\beta < \lambda$ and hence $\mathbf{\Sigma}^0_{1+\lambda}$. If $n = 1$, then we have that G/H has plain length at most $\lambda + 1$, whence the same holds for G by (3). Thus, we have that $\{0\}$ is $\mathbf{\Sigma}^0_{1+\lambda+1}$. If $n \geq 2$ then we have that H is $\mathbf{\Pi}^0_{1+\lambda+n-1}$ in G, and G/H has length at most $1 + \lambda + n - 2$. Therefore, we have that G has plain length at most $1 + \lambda + n - 1$. This shows that $\{0\}$ is $D(\mathbf{\Pi}^0_{1+\lambda+n-1})$ and hence also $\mathbf{\Sigma}^0_{1+\lambda+n}$ in G. This concludes the proof. □

5.13.5 Strict actions

Given a homogeneous space with a Polish cover G/H and a Polish group Λ, we define a notion of left *strict action* of Λ on G/H.

Definition 5.180. *A left strict action α of a Polish group Λ on a homogeneous space with a Polish cover $X = G/H$ is a given by a continuous action α of Λ on G that leaves H setwise invariant. We define the space $X/\Lambda = \mathrm{Out}(\Lambda \curvearrowright G/H)$ of orbits of the action to be the $\mathbf{\Sigma}^1_1$-definable set $X = \boldsymbol{X}/E$ where $\boldsymbol{X} = G$ and*

$$E = \{(g, g') \in G \times G : \exists \lambda \in \Lambda,\, \lambda H g = H g'\}.$$

Suppose that M is a countable R-module. Let $\mathrm{Aut}(M)$ be the group of automorphisms of M endowed with the topology of pointwise convergence. Then we have that $\mathrm{Aut}(M)$ is a non-Archimedean Polish group. The automorphisms $\mu_r : x \mapsto rx$ for $r \in R^\times$, which we call *inner*, form a central closed subgroup denoted by $\mathrm{Inn}(M)$.

Suppose now that C and A are countable thin modules. Let $\mathrm{Aut}(C)$ be the automorphism group of C, which is a Polish group with respect to the topology of pointwise convergence. Then one can define the canonical right action $\mathrm{Ext}(C,A) \curvearrowleft \mathrm{Aut}(C)$ as follows. Up to isomorphism of modules with a Polish cover, we can write $\mathrm{Ext}(C,A)$ as $\mathrm{Z}(C,A)/\mathrm{B}(C,A)$. Here, as in Section 5.4, $\mathrm{Z}(C,A)$ is the Polish group of cocycles on C with values in A, while $\mathrm{B}(C,A)$ is the Polishable subgroup of cocycles that are trivial. We have a canonical action $\mathrm{Z}(C,A) \curvearrowleft \mathrm{Aut}(C)$ by automorphisms that leave $\mathrm{B}(C,A)$ (setwise) invariant. This defines a strict right action $\mathrm{Ext}(C,A) \curvearrowleft \mathrm{Aut}(C)$. In the same fashion, one can define a strict left action of $\mathrm{Aut}(A)$ on $\mathrm{Ext}(C,A)$.

Notice that if M is a phantom pro-finiteflat module, then by Proposition 5.90 we have that $M \cong \mathrm{Ext}(M^\vee, R)$ where $M^\vee := \mathrm{Ext}(M,R)$ is a countable flat module. Furthermore, $\mathrm{Aut}(M)$ is a Polish group canonically isomorphic to $\mathrm{Aut}(M^\vee)^{\mathrm{op}}$, in such a way that the canonical left action $\mathrm{Aut}(M) \curvearrowright M$ corresponds to the canonical right action $\mathrm{Ext}(M^\vee, R) \curvearrowleft \mathrm{Aut}(M^\vee)$. Thus, the action $\mathrm{Aut}(M) \curvearrowright M$ can be seen as a strict action of the Polish group $\mathrm{Aut}(M)$ on the homogeneous space with a Polish cover M. Incidentally, this also shows that every phantom pro-finiteflat module is *rigid* in the sense of [22, Definition 4.2].

Lemma 5.181. *Suppose that $M = G/N$ is a module with a Polish cover. For $\alpha < \omega_1$ and $g \in G$, the following assertions are equivalent:*

1. *N is $\mathbf{\Sigma}^0_{1+\alpha}$ in G;*
2. *the relation*
$$\{(x,y) \in G : \exists r \in R^\times, rx \equiv y \bmod N\}$$
is potentially $\mathbf{\Sigma}^0_{1+\alpha}$;
3. *$N + R^\times g$ is $\mathbf{\Sigma}^0_{1+\alpha}$ in G for every $g \in G \setminus N$;*
4. *$N + Rg$ is $\mathbf{\Sigma}^0_{1+\alpha}$ in G for some $g \in G$;*

Proof. (1)⇒(2) and (3)⇒(4) are obvious. (2)⇒(3) follows from [115, Proposition 3.1]. The equivalence of (1) and (4) follows from Lemma 5.179(5). □

5.13.6 Parametrization by hereditarily countable sets

The following lemmas are essentially [105, Lemma 3.7] and [135, Proposition 2.3].

Lemma 5.182 (Kechris–Macdonald)**.** *Suppose that E, F are equivalence relations on Polish spaces $\boldsymbol{X}, \boldsymbol{Y}$, respectively. If E is idealistic, F is Borel, and E is Borel reducible to F, then there exists a Borel F-invariant subset $\boldsymbol{A}$ of $\boldsymbol{Y}$ such that E is Borel bireducible to $F|_{\boldsymbol{A}}$.*

In particular, it follows that a Borel injection $X \to Y$ between Borel-definable sets has Borel image.

Lemma 5.183 (Motto Ros). *Suppose that X and Y are Borel-definable sets. If there exist a Borel-injection $X \to Y$ and a Borel-injection $Y \to X$, then there exists a Borel bijection between X and Y.*

Combining these with [61, Theorem 1.5] we obtain the following.

Lemma 5.184 (Friedman). *Suppose that $X = \boldsymbol{X}/E$ is a Borel-definable set. Then E is classifiable by countable structures if and only if X is Borel-isomorphic to an orbit space $\boldsymbol{Y}/G$ for some non-Archimedean Polish group G and Polish G-space $\boldsymbol{Y}$. In fact, one can take $G = S_\infty$.*

A canonical hierarchy of Borel S_∞-orbit spaces is considered in [79]. One defines recursively $\wp^0(\mathbb{N}) := \mathbb{N}$, $\wp^{<\alpha}(\mathbb{N})$ to be the union of $\wp^\beta(\mathbb{N})$ for $\beta < \alpha$, and $\wp^\alpha(\mathbb{N})$ to be the collection of countable subsets of $\wp^{<\alpha}(\mathbb{N})$. Thus for example $\wp^1(\mathbb{N})$ comprises all the subsets of $\mathbb{N}$, whence $\wp^1(\mathbb{N}) \cong \mathbb{R}$. In turn, $\wp^2(\mathbb{N}) \cong \wp_{\aleph_0}(\mathbb{R})$ is the collection of all countable sets of reals. More generally, one can think of $\wp^\alpha(\mathbb{N})$ as the collection of *hereditarily countable sets of rank* α. These can be parametrized by the isomorphism classes of the countable models of some infinitary sentence σ_α in a suitable countable language $\mathcal{L}_\alpha$. One lets $\cong_\alpha$ be the relation of isomorphism of such countable $\mathcal{L}_\alpha$-structures, which is a Borel equivalence relation [79, Section 1]. The relation $\cong_1$ recovers the relation of equality of reals, while $\cong_2$ coincides with the Friedman–Stanly jump of $\cong_1$. Thus,

Lemma 5.185. *Let X be a $\mathbf{\Sigma}^1_1$-definable set. We say that X is parametrized by* hereditarily countable sets of rank α *if it Borel-isomorphic to a Borel subset of $\wp^\alpha(\mathbb{N})$.*

Note that this implies that X is in fact a Borel-definable set.

In [79] it is also defined for a countable successor ordinal $\alpha \geq 3$ a Borel subset $\wp^\alpha_*(\mathbb{N})$ of $\wp^\alpha(\mathbb{N})$. Again, $\wp^a_*(\mathbb{N})$ can be parametrized as the isomorphism classes of countable modules of some infinitary $\mathcal{L}_\alpha$-sentence σ^*_α. One then lets $\cong^*_\alpha$ be the corresponding isomorphism relation, which can be regarded as a subequivalence relation of $\cong_\alpha$. We say that $\wp^\alpha_*(\mathbb{N})$ comprises the hereditary countable sets of *plain* rank α. For $\alpha = 2$ we let $\wp^2_*(\mathbb{N})$ be the Borel-definable set associated with the countable Borel equivalence relation of maximum complexity, while for $\alpha = 1$ we let $\wp^1_*(\mathbb{N}) = \wp^1(\mathbb{N})$. For a limit ordinal λ we define $\wp^\lambda_*(\mathbb{N}) = \wp^\lambda(\mathbb{N})$. Combining Lemma 5.184 with [79, Theorem 2 and Theorem 3] we obtain the following.

Theorem 5.186 (Hjorth–Kechris–Louveau). *Let $X := \boldsymbol{X}/E$ be a $\mathbf{\Sigma}^1_1$-definable set, where $\boldsymbol{X}$ is a Polish space E be an idealistic equivalence relation classifiable by countable structures. Then X is a Borel-definable set if and only if there exists $\alpha < \omega_1$ such that X is parametrized by countable sets of rank α for some $\alpha < \omega_1$.*

Furthermore, for every $\alpha < \omega_1$ the following assertions are equivalent:

1. *X is Γ_α-definable if and only if it is parametrized by hereditarily countable sets of rank $1 + \alpha$;*

2. *X is $\Gamma_\alpha^{\mathrm{plain}}$-definable if and only if it is parametrized by hereditarily countable sets of plain rank $1+\alpha$.*

5.13.7 The classification problem for extensions

We conclude by isolating the complexity-theoretic content of our main results, in terms of the classification problem for extensions of countable flat modules.

Theorem 5.187. *Let R be a countable Dedekind domain and A be a countable flat module. Then for every $\alpha < \omega_1$ there exist countable flat modules C_α and $C_\alpha^{\mathrm{plain}}$ such that the potential Borel complexity class of the isomorphism relation of extension of C_α (respectively, $C_\alpha^{\mathrm{plain}}$) by A is Γ_α (respectively, $\Gamma_\alpha^{\mathrm{plain}}$). Furthermore, for every $\lambda < \omega_1$ either zero or limit and for every $n < \omega$ we have that:*

- *for $n = 0$, isomorphism classes of extensions of C_λ by A can be parametrized by hereditarily countable sets of rank $1+\lambda$ but not by hereditarily countable sets of rank $1+\beta$ for any $\beta < \lambda$;*
- *for $n \geq 1$, isomorphism classes of extensions of $C_{\lambda+n}$ by A for $n \geq 1$ can be parametrized by hereditarily countable sets of rank $1+\lambda+n$ but not by hereditarily countable sets of* plain *rank $1+\lambda+n$;*
- *for $n \geq 1$, isomorphism classes of extensions of $C_{\lambda+n}^{\mathrm{plain}}$ by A for $n \geq 1$ can be parametrized by hereditarily countable sets of* plain *rank $1+\lambda+n$ but not by hereditarily countable sets of rank $1+\lambda+n-1$.*

Proof. This follows from Theorem 5.177, Theorem 5.162, and Theorem 5.186. □

Corollary 5.188. *Let R be a countable Dedekind domain. Suppose that A is a countable flat module. Then either:*

1. *for all countable flat module C, the relation of isomorphisms of extensions of C by A is trivial (which happens precisely when A is divisible), or*
2. *there exist countable flat modules C such that the relation of isomorphisms of extensions of C by A has arbitrarily high potential Borel complexity.*

Proof. This is a consequence of Theorem 5.177 and Theorem 5.162. □

Theorem 5.189. *Suppose that R is a countable Dedekind domain and A is a countable flat module that is not divisible. Then for every $\lambda + n < \omega_1$ with λ limit and $n < \omega$ there exists a rigid countable flat module C of plain extractable length and plain A-projective length $\lambda + n$. For such a module C, we have that*

$$\mathrm{Out}\left(\mathrm{Ext}\left(C, A\right) \curvearrowleft \mathrm{Aut}\left(C\right)\right)$$

is $\mathbf{\Sigma}^0_{\lambda+1}$-definable and not $\mathbf{\Sigma}^0_\lambda$-definable for $n = 0$, and $\mathbf{\Sigma}^0_{1+\lambda+n+2}$-definable and not $\mathbf{\Sigma}^0_{1+\lambda+n+1}$-definable for $n > 0$.

Proof. This follows from Theorem 5.177, Theorem 5.162, and Lemma 5.181. □

Theorem 5.190. *Suppose that R is a countable Dedekind domain. Then there exists a countable flat module C such that the isomorphism classes of extensions of C by R can be parametrized by countable sets of reals, and for every $\alpha < \omega_1$ there exist countable flat modules A_α and $A_\alpha^{\mathrm{plain}}$ such that the potential Borel complexity class of the isomorphism relation of extension of C by A_α (respectively, by $A_\alpha^{\mathrm{plain}}$) is Γ_α (respectively, $\Gamma_\alpha^{\mathrm{plain}}$).*

Proof. This follows from Theorem 5.164 and Theorem 5.177. □

5.13.8 Complexity of classes of modules

We have defined $\mathbf{Flat}\,(R)$ to be the category of countable flat modules. The category $\mathbf{Flat}\,(R)$ is (equivalent to) a category whose objects can be seen as points of a Polish space $\mathrm{Flat}\,(R)$. Indeed, a countable flat module is isomorphic to one whose set of elements is the set $\mathbb{N}$ of natural numbers. Its module operation is then a subset of $\mathbb{N} \times \mathbb{N}$, whence the space of such modules can be seen as a subspace of $2^{\mathbb{N}\times\mathbb{N}}$. This subspace is easily seen to be G_δ, whence Polish with the subspace topology [103, Theorem 3.11].

It is therefore meaningful to talk about the Borel complexity of isomorphism-invariant classes of countable flat modules, identified as subspaces of $\mathrm{Flat}\,(R)$.

Theorem 5.191. *Let R be a countable Dededind domain.*

1. *The class of coreduced countable flat modules is co-analytic and not Borel;*
2. *For every $\alpha < \omega_1$, the class $\mathcal{P}_\alpha$ of countable flat modules of projective length at most α is Borel;*
3. *The class of $\mathcal{P}_\infty$ of countable flat phantom projective modules, which is the union of $\mathcal{P}_\alpha$ for $\alpha < \omega_1$, is co-analytic not Borel.*

Proof. (1) Consider the co-analytic rank φ on $\mathrm{Flat}\,(R)$ defined by letting $\varphi\,(C)$ be the least α such that $\partial_\alpha C = 0$ if it exists, and ∞ otherwise. Then we have that φ is a co-analytic rank that is unbounded on the class of coreduced countable flat modules by Theorem 5.162. The conclusion follows from the Boundedness Theorem for Coanalytic Ranks on Borel sets; see Proposition 5.52.

(2) A countable flat module C has projective length at most α if and only if $\mathrm{Ph}^\alpha \mathrm{Ext}\,(C, A) = 0$ for every countable flat module A. This shows that $\mathcal{P}_\alpha$ is co-analytic. We also have that C has projective length at most α if and only if it has tree length at most α; see Corollary

5.115. The class of such modules is easily seen to be analytic by induction on α. Being both co-analytic and analytic, $\mathcal{P}_\alpha$ is Borel.

(3) A countable flat module C has projective length at most α if and only if $\mathrm{Ph}^\alpha\mathrm{Ext}\,(C,A)=0$ for every countable flat module A. Thus defining $\varphi\,(C)$ to be the least α such that C is in $\mathcal{P}_\alpha$ and ∞ otherwise, defines a co-analytic rank on $\mathbf{Flat}\,(R)$. As such a rank is unbounded by Theorem 5.162, the class $\mathcal{P}_\infty$ is not Borel.

In order to see that $\mathcal{P}_\infty$ is co-analytic, it suffices to observe that its complement $\mathcal{C}$ within the class of countable flat modules is analytic. Indeed, a countable flat module C is $\mathcal{C}$ if and only if for every countable ordinal α there exist a countable flat module A and a nontrivial extension of C by A that is phantom of order α. As in the proof of Proposition 5.51 one can write this as an analytic condition on C. □

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