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Commande optimale de l'irrigation : Double modélisation mathématique et agronomique vers une application au modèle Optirrig

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Résumé

Cette thèse étudie, à l'aide de la théorie de la commande optimale, le sujet de l'optimisation des systèmes d'irrigation des cultures en agronomie, à l'échelle de la parcelle, sous serre ou en champ extérieur, dans un contexte de ressources en eau limitées. Ce contexte spécifique fait jouer un rôle crucial au pilotage tactique de l'irrigation et aux prises de décisions de l'opérateur au cours du temps. En utilisant une formalisation mathématique de l'irrigation des cultures en termes de problème de commande optimale, l'objectif est de développer des outils théoriques afin de fournir des stratégies de décision efficaces pouvant se transposer au monde opérationnel.

La première partie de cette thèse se concentre sur l'analyse d'un problème d'irrigation dans un contexte de culture sous serre. Nous utilisons le modèle CCI (Controlled Crop Irrigation) qui est un modèle générique écrit sous forme de système dynamique contrôlé (non lisse), décrivant de façon simple les processus biophysiques d'ordre 1 et permettant une étude des propriétés intrinsèques liés à ces phénomènes physiques principaux. Notre étude nous amène d'abord à revisiter le principe Bang-Bang usuel dans une version portant sur les dynamiques affines par morceaux sous contraintes d'état et une version stochastique, permettant d'en déduire des propriétés structurelles de contrôles optimaux et de leurs trajectoires d'état associées. Ensuite, nous dérivons les conditions nécessaires d'optimalité en utilisant le Principe du Maximum de Pontryagin dans un cadre non-lisse, afin de caractériser plus précisément les solutions optimales de problèmes d'irrigation sujets à différents types de contraintes et de critères. Ces conditions, combinées avec les précédentes propriétés structurelles, aboutissent à une formulation d'une classe de politiques optimales sous forme de retours d'état paramétrés. Ceci signifie que la résolution du problème de commande optimale a priori difficile (en dimension infinie) revient à faire une optimisation en petite dimension. Cette petite dimensionnalité nous permet alors d'étudier à l'aide d'une exploration numérique globale les propriétés des solutions optimales, notamment la sensibilité de l'instant de prise de décision en fonction des paramètres du système (plante, sol, matériel), et les conséquences (bienfaits et inconvénients) de décisions sous-optimales.

La seconde partie se penche sur la question de l'application de ces résultats théoriques dans un cadre opérationnel plus proche du monde réel. Pour ce faire, nous adoptons tout d'abord une approche de double modélisation, consistant à établir des connections entre le modèle CCI (simple, théorique) et le modèle Optirrig (opérationnel, pratique) afin de confronter et d'affiner leurs hypothèses et résultats. Nous décrivons en particulier l'objectif de tester les scénarios théoriques d'irrigation optimale extraits du modèle CCI, en les implémentant dans le modèle Optirrig et en évaluant leurs performances dans un cadre pratique. De plus, nous tirons profit des techniques de "contrôle à coût moyenné" pour formaliser le problème d'irrigation sous certains types d'incertitudes du monde réel. En particulier, nous nous intéressons à la recherche de solutions optimales robustes par rapport à la variabilité de paramètres du modèle les plus sensibles, ainsi qu'à la recherche de solutions qui optimisent "en moyenne" la production de biomasse vis-à-vis d'incertitudes météorologiques, décrites sous forme de sauts des variables d'état.

Abstract

This thesis investigates, with the help of optimal control theory, the topic of optimizing crop irrigation systems in agronomy, at the plot scale, in greenhouse or in open-field conditions, in a context of limited water resources. This specific context gives a crucial role to the tactical irrigation piloting and to the decisions made by the operator over time. Using a mathematical formalism of crop irrigation in terms of an optimal control problem, the goal is to develop theoretical tools in order to provide efficient decision strategies that can be transposed to the operational world.

The first part of this thesis focuses on the analysis of an irrigation problem in a context of greenhouse agriculture. We utilize the CCI (Controlled Crop Irrigation) model, which is a generic model written as a controlled (non-smooth) dynamical system, describing order-1 biophysical processes in a simple way, and enabling a study of intrinsic properties related to these central physical phenomena. Our study first leads us to revisit the usual Bang-Bang principle to a version dealing with piecewise affine dynamics under state constraints and a stochastic version, allowing to deduce

structural properties of optimal controls and their associated state trajectories. Then we derive the optimality necessary conditions through the Pontryagin Maximum Principle in a non-smooth framework, in order to more precisely characterize the optimal solutions to irrigation problems subject to various types of constraints and criteria. These conditions, combined with the aforementioned structural properties, result in a formulation of a class of optimal policies in terms of parameterized feedback. This means that solving the a priori difficult optimal control problem (in infinite dimension) amounts to performing an optimization in small dimension. This small dimension thus allows us to study, through a global numerical exploration, the properties of optimal solutions, especially the sensitivity of the timing of decision-making with respect to the system parameters (plant, soil, material), and the outcomes (benefits and drawbacks) of sub-optimal decisions.

The second part addresses the question of applying these theoretical results in an operational framework closer to real-world conditions. To this end, we employ a double modeling approach, consisting in building connections between the CCI model (simple, theoretical) and the Optirrig model (operational, practical) in order to compare and refine their hypotheses and results. In particular we aim for testing the theoretical scenarios of optimal irrigation taken from the CCI model, by implementing them in the Optirrig model and evaluating their performances in a practical framework. In addition, we take advantage of "averaged cost control" techniques to formalize the irrigation problem under different types of real world uncertainties. In particular, we are interested in finding optimal solutions that are robust regarding the variability of the most sensitive model parameters, as well as finding solutions that optimize the biomass production "on average" with respect to meteorological uncertainties, described as jumps on the state variable.

Contents

1 General introduction **11**
1.1 Application context: expectations in irrigation science . . . 11
1.1.1 Water management in agriculture . . . 11
1.1.2 Main issues in crop irrigation: recent advances and current challenges . . . 13
1.1.3 Motivation of the thesis . . . 15
1.2 Framework for mathematical and methodological answers . . . 16
1.2.1 Double modeling approach and challenges . . . 16
1.2.2 Presentation of the Controlled Crop Irrigation model . . . 17
1.2.3 Issues, difficulties and theoretical background . . . 21
1.3 Contributions to the optimization of crop irrigation . . . 23
1.3.1 Contribution to the CCI model and to the irrigation problem formulation . . . 23
1.3.2 Extension of the bang-bang principle for piecewise affine dynamics . . . 26
1.3.3 Extension of the bang-bang principle in stochastic settings . . . 27
1.3.4 Optimal structures of crop irrigation strategies in greenhouse context . . . 28
1.3.5 Exploring the optimality of Threshold-Based Irrigation Strategies . . . 30
1.4 On-going works and perspectives . . . 31
1.4.1 Correspondence between the CCI and Optirrig models . . . 31
1.4.2 Averaged cost control for crop irrigation management in operational contexts 32
1.5 *Introduction générale* . . . 37
1.5.1 *Cadre de travail pour des réponses mathématiques et méthodologiques* . . . 37
1.5.2 *Contributions à l'optimisation de l'irrigation* . . . 45
1.5.3 *Travaux en cours et perspectives* . . . 54

I Contributions to optimal control theory and analysis of the optimal irrigation policies in a greenhouse context **61**

2 Preliminary results in optimal control theory **63**
2.1 Extension of the bang-bang principle for PieceWise Affine Continuous dynamics . . . 63
2.1.1 Introduction . . . 63
2.1.2 Settings and assumptions . . . 64
2.1.3 Extended bang-bang principle . . . 66
2.1.4 Examples . . . 69
2.1.5 Conclusions . . . 73
2.2 Extension of the bang-bang principle in Stochastic settings . . . 74
2.2.1 Introduction . . . 74
2.2.2 Stochastic setting and preliminaries . . . 74
2.2.3 Convexity and bang-bang properties . . . 76
2.2.4 Conclusion . . . 81

3 Optimal structures of crop irrigation strategies under state constraints **83**
3.1 Introduction . . . 83
3.2 Model description and hypotheses . . . 84
3.2.1 Basis model description . . . 84
3.2.2 Framework and constraints . . . 85
3.2.3 Problems formulation . . . 86

3.2.4 Preliminaries about trajectories of soil moisture . . . 87
3.3 Equivalent formulation and preliminary properties . . . 89
3.3.1 Budget compensation technique . . . 89
3.3.2 Properties of optimal solutions . . . 92
3.4 Optimal synthesis . . . 96
3.4.1 Application of the Pontryagin Maximum Principle . . . 96
3.4.2 Structures of optimal solutions . . . 99
3.4.3 Main result . . . 101
3.5 Conclusion . . . 102

4 Exploring the optimality of Threshold-Based crop Irrigation feedback Strategies 105
4.1 Introduction . . . 105
4.2 Material and methods . . . 107
4.2.1 Threshold-Based Irrigation Strategies . . . 107
4.2.2 The Controlled Crop-Irrigation model . . . 107
4.2.3 Methodological tools for analysis . . . 109
4.3 Results . . . 115
4.3.1 Soil filling from possible irrigation strategies . . . 115
4.3.2 Numerical exploration . . . 118
4.4 Discussion . . . 123
4.5 Conclusion . . . 124

II On-going works towards an application to the Optirrig model and an extension to open-field conditions 127

5 The Optirrig and CCI models 129
5.1 Models comparison guidelines . . . 129
5.2 Presentation of the Optirrig model . . . 131
5.2.1 A wide-purpose irrigation optimization tool . . . 131
5.2.2 Model rationale . . . 132
5.3 Towards a correspondence with the CCI model . . . 139
5.3.1 Comparison of structures . . . 139
5.3.2 Term-to-term correspondence . . . 142
5.4 Discussion and perspectives . . . 143

6 Formulation and study of optimal averaged cost control problems in view of operational irrigation management 147
6.1 Robustness of optimal strategies with respect to crop growth variability . . . 147
6.1.1 Problem formulation . . . 147
6.1.2 Properties of the optimal solutions . . . 149
6.1.3 Towards an optimal synthesis . . . 150
6.1.4 Conclusion . . . 152
6.2 Integrating rainfall uncertainties and soil moisture jumps . . . 153
6.2.1 Framework of rainfed irrigation . . . 153
6.2.2 Problem reformulation with an abstract time . . . 155
6.2.3 Perspectives . . . 157

Conclusions and perspectives 159

Chapter 1

General introduction

1.1 Application context: expectations in irrigation science

1.1.1 Water management in agriculture

1.1.1.1 Current situation of water management and the role of agriculture

Over the past 60 years, the global water demand has increased, and rapid population growth, urban expansion, intensive agricultural practices have exacerbated the tensions on freshwater resources [136, 43, 126, 159, 192]. This continuously growing pressure on water resources causes concerns about long-term sustainability of the competitive use of water for irrigation practices and other sectors in general [80, 119, 193]. Water use conflicts, choice of crops and rotations, varietal selection, soil conservation practices, unconventional waters are all levers that may constrain or influence irrigation management. Furthermore, warming climate and human activities highlight the need to anticipate and make decisions under uncertainties [125, 86].

Freshwater resources are vital for human and economic development of society, in particular through the production and supply of food. In many countries, particularly in arid and semi-arid regions, water scarcity has become a crucial topic, especially since the frequency, intensity and duration of droughts are increasing [188, 70, 185]. In fact, areas affected by droughts went up by 20%, leading to significant economic damages [126]. For example, summer months water scarcity is particularly high in southern Europe, and the impacts (such as water consumption increase or soil water content decline) are already happening in the Mediterranean basin [2, 116, 64, 56, 96, 135], especially when the water resources are not enough to meet all the local demands [16].

In this context, agriculture (in particular irrigation) plays a main role for water management. To ensure food security under rapid population growth, a large amount of water is extracted from rivers (70% of freshwater diverted for agriculture irrigation comes from rivers) [118, 187]. Globally, food production accounts for 90% of the freshwater consumption, making irrigated agriculture the largest user of world's freshwater resources [176, 201]. Only approximately 20% of the world's croplands are irrigated, but this fraction contributes to about 40% of the global food production [147].

Thus global food supply increasingly relies on irrigation, raising questions about the viability of freshwater resources, especially given the supply-demand imbalances and the recent drought events. In drylands, rainwater alone cannot sustain healthy crop production, and irrigation is essential [46]. As a results, in such regions, irrigation management (water abstraction, pumping, irrigation) directly decides crop yields and the consequent incomes [197]. Growing agricultural water demands have led to shortages and decreases in availability for other sectors, in particular in arid and semi-arid regions [183, 182].

1.1.1.2 Expected evolution for the next decades

In the next few decades, global water management is expected to face various challenges induced by combined pressures of demography, climate and agriculture. Climate change is expected

to increase the variability of weather events and rainfall patterns and intensity, leading to more vulnerability to droughts and uncertainties about water availability.

Global population keeps rising rapidly and is projected to reach approximately 9.3 billion by 2050 (up from 7 billion in 2011) [180]. This growth is expected to generate at least a 50% increase in food demand and a 30% increase in water demand by 2030, thus potentially leading to shortages [197]. In order to match global food needs, agricultural production will have to increase by an estimated 33% by 2050 [197]. As a result, global freshwater consumption is also expected to rise significantly due to agriculture practices [61, 180].

The intensification of competitive uses of water resources being exacerbated by growing demands for domestic, industrial and hydroelectric purposes, one can foresee a decrease in overall water availability [163]. In arid and semi-arid regions, the expected rise in frequency, intensity and duration of droughts (due to climate change dynamics) will likely reduce the current stock of freshwater resources, limiting crop development and yields [151, 168, 188]. Then irrigation is supposed to expand in the future in order to meet food security, however this expansion will take place in conditions where long-term agricultural management will be facing various sources of uncertainties and increasing pressure on water systems.

1.1.1.3 Changes in irrigation practices and targets in the near future

The anticipated evolution in climate, population and water demand has significant impacts on both agriculture and other water-dependent sectors [84, 85]. Increasing pressure on water resources may threaten the availability and quality of water for competitive uses, including agriculture [111]. The sustainable management of water resources will thus become a major issue, with global environmental and socio-economic consequences.

In the agricultural context, irrigation plays a crucial role in food security, however it is also a major driver of environmental impacts. The overuse of irrigation might lead to the depletion of water resources, the deterioration of water quality, and the degradation of ecosystems, notably through soil salinization. These impacts are particularly severe in dryland regions, where land and water resources are already under stress [147, 50, 55, 68]. The expansion of cultivated areas to increase production is constrained by urbanization (and unwillingness to disturb environment), making it essential to improve productivity on existing farmland. This includes increasing water productivity per unit of area of available land and water resources (or "crop per drop") to help close the "yield gap" losses, exacerbated by climate vulnerability [20, 132, 128].

Moreover, crop growth and yield are affected non-linearly by water availability, particularly under soil moisture deficit scenarios. Deriving reliable relationships between crop water use and yield thus remains a complex task [51, 169]. As a result, agricultural planning must move beyond the traditional practices. Instead, agriculture must head towards decision-making frameworks that integrate water limitations and able to account for uncertainty, while keeping the objective to improve the system performances [103, 196]. While "precision" or "smart" irrigation practices suit the purpose of deficit irrigation and avoiding water waste, one might however obtain a "rebound effect" when small-scale efficiency induces an adverse effect on water resources at the regional scale.

These challenges highlight the importance of evaluating the joint impacts of multiple drivers on agricultural performance, as well as identifying how the constraints affect water management practices and efficiency. In the context of reliable, economically viable, and sustainable agricultural production increase, optimizing the allocation and use of limited water resources is not only a necessity for food security, but also a key research priority for the years to come. It further underlines the need to inform public policies with scientific results and prospectives, seeking social equity and fair data in relation to water management, connecting scientific approaches and operational declensions.

1.1.2 Main issues in crop irrigation: recent advances and current challenges

1.1.2.1 Questions in irrigation related to the aforementioned context, with a focus on operational decision making

In regions affected by severe water scarcity, optimally managing water resources is critical [140]. The operational decisions taken in these contexts have direct implications for the sustainability and efficiency of agricultural practices. Among the key decision making clues is the timing of water allocation : crop yields are closely related not only to the total availability of water but also to the temporal distribution of soil water content throughout the growing season. In water-limited conditions, the ability to prevent crop water stress (especially at important development stages) depends on thoroughly timed irrigation, making irrigation scheduling a central issue in decision-making.

Achieving a sustainable equilibrium between available water resources and irrigation demands requires following robust strategies able to account for limitations while still being productive. The decisions must then integrate multiple factors, including climate variability, resource availability, and production targets. Poorly planned irrigation might lead to numerous insufficiencies and long-term problems, such as resource depletion, waterlogging or salinization (especially in arid and semi-arid areas having a thin room for maneuver) [164].

On a wider scale than the plot, irrigation scheduling is only one part of water management challenges. For example, these challenges include the conjunctive use planning (with multiple sources of water), groundwater level management, seawater intrusion prevention, cropping patterns, reservoir systems, or even solid waste water management [164]. Each of these issues adds complexity in the global decision-making process and highlights the important of integrated approaches. Ultimately, operational irrigation management would have to aim for improving production efficiency as well as anticipating the environmental impacts associated with unsustainable practices, using for example the "environmental footprint", possibly in a way to discriminate relevant practices from those that are not (or less). Further, enlarging the targets would mean heading towards a different logic, as "robustness" instead of "performance".

1.1.2.2 Advantages and drawbacks of existing approaches

To address the growing pressure on water resources and improve irrigation outcomes, several approaches and modelling tools have been developed, particularly for regions in which "rainfed" crops are not sufficiently productive anymore (as in arid and semi-arid areas). Among these, supplemental irrigation and deficit irrigation have emerged as potentially effective approaches to improve water productivity while dealing with limited water availability [132]. These strategies tend to parsimoniously allocate irrigation water in order to target critical crop growth stages, ensuring convenient yield without requiring full irrigation coverage. It is possible to represent the water use and yield relationship, as resulting outcomes of different irrigation schedules, with crop kites [165]. These graphs can provide the capacity to visualize and assess water efficiency over a wide range of strategies and agroclimatic conditions.

Modeling and simulation tools, including those based on the FAO guidelines (such as Irrigation and Drainage papers 33, 56 and 66 [3, 169, 51, 100, 114, 195]) or dynamic models like AquaCrop [137, 110, 67, 63] and alternative approaches [33, 21, 148], are commonly used in order to develop decision-making support. The purpose of such models is to provide useful estimations of crop development and enable meaningful design of irrigation strategies, yet being conscious that modeling approximations embedded in simulations might carry error margins comparable to (or greater than) climate uncertainties, which might complexify the decision support process [130, 171].

From a modeling point of view, various optimization frameworks have been developed to improve irrigation efficiency. Their formulations vary from complete to constrained, from discrete to continuous, from strategy-based to calendar-based, and may or may not take in account weather uncertainties [164]. It is also possible to consider ex-post analyses, real-time decision support, or ex-ante prospective works. The exhaustive optimization problem remains challenging due to the high

dimension of the problem (a naive formulation including daily decision for the irrigation volume would introduce numerous variables and constraints, and lead to computationally expensive resolution). Besides, the simplifications induced by models might be questionable regarding biophysical processes, especially in situations with deficit irrigation (involving additional processes due to water stress) [25], needing cautious interpretation of results, and a proper scientific assessment of strategies.

The design of such models enables to develop Decision Support Systems (DSS) which help farmers by giving suggestions for irrigation timing, fertilization, crop rotation or other decisions. They can integrate soil moisture variables, plant growth stage, weather patterns, market conditions [200]. Various methodologies have been applied to DSS, such as Dynamic Programming, Stochastic approaches, Artificial Intelligence for example, with their respective strengths and weaknesses (ability to adapt to sequential decision-making, dealing with non-linearity, limited in variables and memory required, dependent on data acquisition) [164]. Nevertheless, every DSS should face the challenges, difficulties and limitations due to the need to adapt in accordance to different farming types, objectives, philosophy and constraints [1]. However, we found relatively few approaches in this literature that propose tactical policies (over the whole agronomic season) in terms of feedback (most of the approaches deal with open-loop controls).

While these approaches contribute meaningfully to the development of practical irrigation strategies, they often require computationally expensive processes. In contrast, when model structure is compatible, analytical frameworks (such as tools from open-loop or closed-loop control theory) offer the possibility to derive irrigation policies with a more "high-level modeling" point of view. Optimizing irrigation can be formulated through deterministic approaches (using Optimal Control for example) or approaches incorporating rainfall variability (as Stochastic Control) as well as adaptive approaches (such as Model Predictive Control) [157, 158, 199, 83, 142, 189]. Analytical frameworks provide a solid mathematical exploration and enable some flexibility in managing the tradeoff between the complexification of models and the search space approximation (i.e. finding a compromise in terms of formulation and resolution accuracies). This kind of approach facilitates a deeper understanding of intrinsic properties of the system dynamics, and might help investigating decision policies in a structural point of view.

1.1.2.3 Why the use of optimal control theory is relevant to be investigated for irrigation problems

Among the many promising directions of improvement for crop irrigation management, the integration of automatic control and real-time feedback (or closed-loop) systems seems particularly interesting. Controlled irrigation potentially offers notable advantages: increased accuracy in water distribution, reduced waste, lower costs, and decrease in human error [120]. In particular, these benefits are relevant in the context of water scarcity where parsimonious use of water is essential to ensure both healthy crop production and sustainable resource management.

Recent advances in sensor-informed agricultural decisions and smart farming technologies involve the development of automated systems capable of monitoring soil moisture and crop growth in real time [23, 45, 123, 184]. These sensor-based monitoring systems (seen as feedback control systems), can dynamically adjust irrigation inputs according to current operating conditions, seeking to optimize the chosen criterion. In particular, closed-loop strategies, where soil moisture measurements directly indicate the recommended irrigation volume, represent a natural application of control theory to agriculture. Such approaches enable the design of irrigation systems able to respond efficiently to varying environmental conditions, and to adapt the decisions in order to better follow the farmer's objectives.

Several recent studies based on classical control methods (such as on-off control [71], PID [146], MPC [117, 121]) or machine learning techniques (such as fuzzy logic, multi-agent systems) [62] have shown potential, however they often assume full water availability or omit legal or technical constraints on the use of water resources, making those considerations improvable regarding real-world conditions, especially for arid and semi-arid regions.

There remains significant room for improvement, aiming for an irrigation system able to be

robust under unexpected environmental change, taking advantage of real-time optimization and control methodologies in a more systematic way [115]. Open key challenges include the resource allocation problem with competitive demands, the integration of long-term objectives or weather uncertainties, and sensor and actuator placement [37]. A generic, adaptable control framework that can account for various farming objectives and operational constraints would be a considerable progress.

In this context, the application of optimal control theory to irrigation management provides a generic methodology for optimizing water use, integrating real-time decision-making (by seeking feedback formulations) and long-term operating constraints (by setting a terminal target for the budget). This suggests a co-existence between different time horizons: the operational horizon (decision-making, few hours or days), the tactical horizon (cropping season or part of it), the strategic horizon (long-term view in which practices should be viable). Such considerations, combined with a probabilistic formalism of weather and climate, certainly represent a research direction to investigate in order to enhance both the efficiency and the robustness of controlled irrigation policies. Achieving the development of such strategies would make their implementation in realistic operational conditions meaningful and impactful.

1.1.3 Motivation of the thesis

1.1.3.1 Research positioning

As mentioned in the above section, among the existing approaches involving the control theory for studying crop irrigation problems, we choose to focus on an optimal control framework, in order to take advantage of the possibility to derive necessary conditions to seek optimal solutions. Moreover, optimal control formulations allow us to carry out an optimization that goes beyond the local (or small window of time) optimization by integrating constraints and conditions that apply to the global time span. Aside from the numerical studies (see [108, 153, 95, 110]) whose optimization study either depends on some model restrictions or rely on open-loop scenarios, there exists several works that aim for a more analytical use of the optimal control theory. They usually consist in defining a dynamical system representing their crop irrigation model, then introducing some hypotheses, and optimizing a cost function (which is linear in [157, 112, 19, 101] and quadratic in [113, 102]). Typically, their study follows the process of writing and deriving necessary conditions, in order to obtain at least properties on optimal control that can be explored through direct or indirect method, or even directly the expression of optimal controls. Moreover those structures of strategies will be generally valid for a wide class of operating conditions and parameter values [177, 75]. Theoretical works rely on simple dynamics (rather abstract as in [112, 101] or more physical as in [157, 19]), might contain an additional state constraint (as in [113]), might contain water use limitations ([19]) or assume full water availability ([112, 101]), and might ([112, 101]) or might not ([157, 19]) consider rainfall. The study [102] takes a different point of view and addresses the question of robustness, evaluating the sensitivity of the optimal solution with respect to poorly known parameters, through a non-linear programming formulation.

The previous work of [19] appears to be a good compromise between model simplicity and flexibility, and explicit physical sense. Its main weaknesses are the lack of consideration for critically low soil moisture levels, and the absence of rainfall and drainage contributions. This work uses the Pontryagin Maximum Principle in a non-differentiable framework to show, under certain conditions based on a simplified irrigation model ([91]), that there exists a non-trivial (yet parameterized in low dimension) structure of irrigation strategies (or controls) which maximize the biomass production in a situation of water scarcity. The originality of such structures relies in the emergence of "singular arcs" specifically at some given soil moisture setpoints. However this study is still incomplete since it does not take into account the water contributions from unknown precipitations. Moreover, the derived strategies present a major issue: under some conditions it is possible that the computed optimal solution involves undergoing critically low soil water contents that lead to the crop's death. A following study ([72]) investigated the problem with rainfall considerations, using the optimization numerical tool Bocop ([173]) based on a direct method (using the Ipopt library), foreseeing new directions for the development of solutions to the problem with rainfall and weather forecast.

However, the solutions found in these studies are unfortunately not directly applicable to operational situations, either because of the lack of a suitable complex operational formulation, or because their optimization relies on an external numerical resolution via a direct method. For these reasons, this thesis will integrate a double modeling approach (see [73]) detailed in Chapter 5, consisting in making a correspondence (in some sense) between a small irrigation model and the Optirrig model ([34, 33]) which has been developed in view of operational purposes. The double modeling approach takes advantage of both small theoretical model and detailed physical model to derive efficient control laws for a complex system. Double modeling has rarely been considered with the primary objective of designing control strategies, but rather as a mean to conduct analyses on a calibrated reduced model to enhance the performances of a complex model. In this thesis, we seek to develop irrigation strategies on a small theoretical model and assess their efficiency on the model Optirrig. Notice, however, that the small model won't be a reduced version of Optirrig, making their correspondence a real challenge.

1.1.3.2 Application framework: the Optirrig model

The application framework relies in the Optirrig model, a crop irrigation numerical model aiming for the generation, analysis and optimization of irrigation scenarios (we use the same name "Optirrig" for the model and its numerical implementation as a software). The specificity of Optirrig compared to other existing model-embedded DSS is that it focuses on the declension of irrigation strategies rather than the detailed description of crop processes. The Optirrig model considers a relatively parsimonious modeling of the plant biophysical processes and the water transfer physical processes. Typically, the issue to be solved consists in "using less water than a given quota to obtain more than a given yield, with maximal economical benefits".

To this end, the development of Optirrig requires a multisciplinary collaboration, from agronomy, hydrology, physics, mathematics and numerical science research fields, and its aim is to contribute to various operational purposes in crop irrigation management. These purposes divide in four main categories: ex-post analysis (assessment of passed practices and alternative choices), ex-ante analysis (guidelines for adapting to climate change and to increasing constraints on water resources), real-time decision support (decision rules, watering thresholds, closed-loop control if measures are available), and tactical or strategic analysis (needs of major crops for support to public policies).

Most of the current developments aim for establishing reasoned irrigation strategies, in classical agricultural contexts (such as great crops in open-field conditions with aspersion irrigation), taking into account a restrictive usage of water and the need to think and promote alternative irrigation strategies (such as deficit irrigation). Those strategies might get economical "rewards" for their efficiency or virtuous features, modifying the nature of the economical problem and potentially shifting the optimal solutions for irrigation.

In this thesis, the objective is to provide innovative strategies embedded from theoretical framework into Optirrig. In particular, this thesis aims for enabling an efficient real-time operational use, by conceiving short-term decisions integrating long-term considerations, thus improving the current way to make such scenarios with uncorrelated considerations. In other words, the goal is to suggest irrigation decisions according to the yield that is still possible to obtain.

1.2 Framework for mathematical and methodological answers

1.2.1 Double modeling approach and challenges

The double modeling approach consists in using two models, that may be coupled or sequential or interactive, in order to capture different aspects of a complex system, by taking advantage of the respective strengths of each model. This method follows the need to overcome single-model studies limitations, by integrating two points of view about the subject of study, that may correspond to different scales, processes or domains of application. For instance, in the crop irrigation field, one may consider models focusing on biophysical, economical or control aspects, for a management at the plant scale, the plot scale or the basin scale, and describing processes in a more detailed or more

abstract way. The usual methodology consists in identifying the compatibility of the two models, calibrating them, then performing integrated simulations. This double modeling method provides interesting features, such as improved realism, modularity and flexibility, and is a promising tool for enhanced decision support systems. However it might face several difficulties, such as model incompatibility, calibration complexity, or might increase the simulation costs.

In this thesis, the double-modeling challenge consists in establishing the connection between formal analytical approaches in the field of optimal control, represented by the home-made Controlled Crop Irrigation model (see Section 1.2.2), and a more physics-based approach in the field of water management, represented by the Optirrig model. The purpose is to take advantage of both types to derive efficient control laws for a quite complex crop irrigation problem, possibly allowing us to highlight and understand the emergence of decision-making phenomena (expected or not). This model connection and the progressive drift of results (see Figure 1.1) across disciplinary fields goes along with the necessity of keeping a single, overall objective: it is the same method or methodological framework that is first established on firm theoretical grounds, then illustrated in sufficiently controlled conditions, with eventually the objective of being applied slightly out of its conditions of validity.

As a result, this thesis project addresses the optimization of irrigation strategies in a context of climate change and water resources limitations, and their application to the Optirrig model. The suggested solution should provide a decision : in real-time, integrating seasonal management constraints like water quotas, integrating weather conditions with mid-term and long-term uncertainties. The double modeling approach allows the identification of intrinsic possibilities of the crop irrigation system to reach some given objectives while fulfilling some given constraints (see [4]). It is reasonable to apply this methodology from a control problem formulation (the CCI model) into Optirrig, since Optirrig was designed by focusing on the irrigation scenarios (or strategies) rather than a detailed crop physiology. Notice though that the compatibility of the CCI model and Optirrig is yet to be proven before considering to apply strategies from CCI into Optirrig. Moreover, underline that strategies that are optimal for the simple model are sub-optimal for the Optirrig model, but they can be compared with existing strategies to evaluate the benefit of the approach. Such association of optimal control theory and a more complex numerical model through the double modeling approach is original, since double modeling is a renown methodology but with very few applications in agronomy and water sciences.

1.2.2 Presentation of the Controlled Crop Irrigation model

1.2.2.1 Basis formalisms and dynamics

Let us introduce the foundations of the model which is the main subject of analysis of this thesis, and that we named: “Controlled Crop Irrigation” model (CCI). In this section, we present its initial formulation and hypotheses, and we will detail the complementary elements specific to this thesis in Section 1.3.1. The CCI model is based on an initial formulation in [133] related to the FAO crop-water simulation model AquaCrop [170]. The formulation in [133] uses a system of Ordinary Differential Equations (ODE) to describe several biophysical processes involved in crop development, such as soil water contents dynamics, nitrogen uptake, canopy cover, biomass and yield. It addresses a focus on water management, then assuming than nitrogen is not a limiting factor and omitting its description. As a result, the formulation used is a simplification of the one in [133], in particular there is only three state variables : soil moisture (S), crop biomass (B) and water budget (V) (see [91, 19, 28]). Basically, the CCI model is a parsimonious dynamical system of ODEs, suitable for applying the optimal control theory while containing the order-1 biophysical processes at play.

As in [133], soil properties are assumed to be spatially homogeneous with an even plant spacing, not needing to consider any spatial pattern or heterogeneity within the modeled area. With characteristics seen in such a spatially-averaged way, the CCI model consists in a small set of ordinary differential equations representing the interplay between soil moisture, biomass production and the remaining water amount from the original quota, during the crop growth period of the cropping season, denoted by a time interval $[0, T]$. The CCI model is written as a continuous-time dynamical

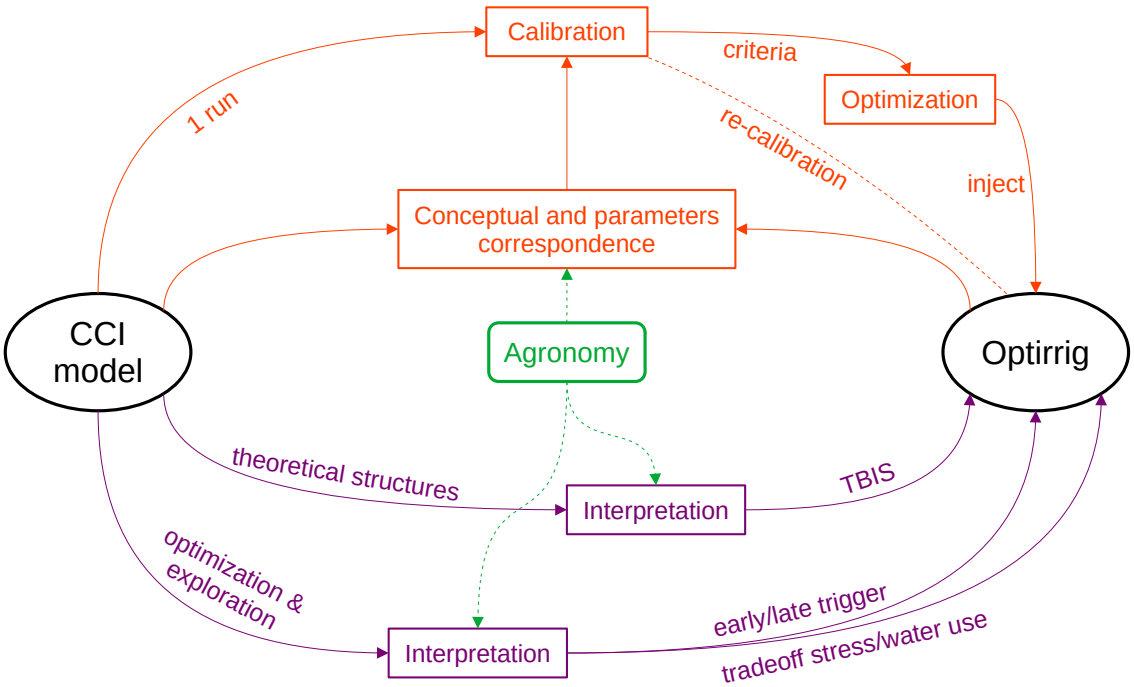


Figure 1.1: Conceptual scheme of the double modeling possibilities between the CCI model and Optirrig, representing the standard process (in orange), the contributions from theory to practice (in violet, see Chapter 4 and Chapter 5), and the influence of physical sens in the loop (in green).

system, that can be understood as a system evolving with a certain variation per unit of time (for example per day), and parameters representing rates can thus be seen as quantities per unit of time (for example per day). The basis equation of study is the following, describing the balance of soil water content:

$$Vol\frac{\Delta S}{\Delta t} = -TR - EV + IR + RA - DR, \tag{1.1}$$

where $Vol\frac{\Delta S}{\Delta t}$ represents the variation of quantity of water ($Vol\Delta S$) over a time interval (Δt) and in a soil reservoir of size Vol (recalling that the spatial homogeneity allows us to consider this quantity normalized per unit square meter). The notations TR, EV, IR, RA and DR respectively denote plant transpiration, soil evaporation, irrigation, rainfall and drainage rates, i.e. variations during the time interval Δt.

Following [91] and [73], we now detail the expression of each term.

Plant transpiration TR is calculated as the reference evapotranspiration ET_0 (or climatic demand) multiplied by the water stress related coefficient K_S (or transpiration efficiency) and by the percentage of canopy cover φ (or radiation interception efficiency, see [11]). Given that the transpiration efficiency depends on the current soil moisture, and considering that quantities are varying over time, the instantaneous plant transpiration rate can be written as

$$TR(t) = ET_0(t)K_S(S(t))\varphi(t). \tag{1.2}$$

Soil evaporation EV is calculated as the reference evapotranspiration ET_0 (or climatic demand) multiplied by the soil water content related coefficient K_R (or evaporation efficiency) and by the percentage of soil not covered by canopy $1-\varphi$ (or radiation not intercepted). Given that the evaporation efficiency depends on the current soil moisture, and considering that quantities are varying over time, the instantaneous soil evaporation rate can be written as

$$EV(t) = ET_0(t)K_R(S(t))(1-\varphi(t)). \tag{1.3}$$

The irrigation rate is taken as the maximal irrigation rate F_{max} (or capacity of the material) multiplied by the percentage of irrigation valve opening u (or normalized irrigation). Considering that quantity u is time-dependent, we can write the instantaneous irrigation rate as

$$IR(t) = F_{max}u(t). \tag{1.4}$$

Soil water variation induced by rainfall RA is calculated as the quantity of precipitation r (per unit square meter) multiplied by the coefficient α (or rainfall infiltration efficiency). Considering that quantity r is time-dependent, we can write the instantaneous variation induced by rainfall as

$$RA(t) = \alpha r(t). \tag{1.5}$$

Drainage DR is calculated as the maximal drainage rate k_{sat} (or saturation water losses) multiplied by the coefficient K_D (or drainage efficiency). Given that the drainage efficiency depends on the current soil moisture, and considering that quantities are varying over time, we can write the instantaneous drainage rate as

$$DR(t) = k_{sat}K_D(S(t)). \tag{1.6}$$

All quantities $K_S(S(t))$, $K_R(S(t))$, $K_D(S(t))$, $\varphi(t)$, $u(t)$, α are defined as efficiencies (or percentages), thus belong to $[0, 1]$. The crop functions K_S, K_R and K_D are piecewise affine and continuous functions of the soil moisture S, and are parameterized by characteristic values of soil water content [3, 73], in ascending order : S_{rw} the residual water content, S_{pw} the permanent wilting point, S^* the point of incipient water stress, S_{fc} the point of field capacity, S_{sat} the soil moisture saturation point. Typical shapes found in the literature are represented in Figure 1.2. In the following, we will always set S_{fc} to be equal to 1, as we interpret the quantity S to be the ratio between the amount of water in the system and the available space in the soil that the water may fill. As a consequence, situations where S would be greater than 1 correspond to overfilling, and that means there is too much water compared to the capacity of the soil to assimilate it, thus generating drainage.

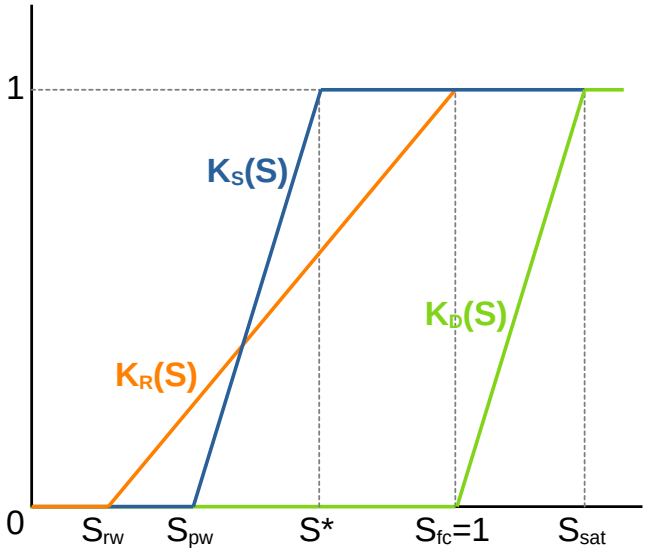


Figure 1.2: Shape of the crop coefficients of plant transpiration K_S (in blue), soil evaporation K_R (in orange) and drainage K_D (in green).

The quantity ET_0 is time-varying, depending on the climatic conditions, and roughly oscillates in a not too wide range regarding the length of the time interval. For the sake of simplification (and again for homogenization purposes) the parameter ET_0 will be considered constant over time, most likely near the average of its real values.

Now Equation (1.1) may be rewritten as an ODE (making the time duration Δt tend to 0), using all the above additional modeling details:

$$\dot{S}(t) = k_1 \left(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t) - \frac{k_{sat}}{ET_0}K_D(S(t)) + \frac{\alpha}{ET_0}r(t) \right), \tag{1.7}$$

where $k_1 = \frac{ET_0}{Vol}$ and $k_2 = \frac{F_{max}}{ET_0}$.

An equation describing the variation of biomass is also given, assuming that the production of biomass is proportional to the transpiration rate, as follows:

$$\dot{B}(t) = k_3 \varphi(t) K_S(S(t)), \tag{1.8}$$

where k_3 is a potential biomass growth rate (or water use productivity), which acts as a normalizing constant (and it might be taken equal to 1 without loss of generality).

Finally, the consumption of irrigation water is tracked by the following:

$$\dot{V}(t) = F_{max} u(t), \tag{1.9}$$

where the coefficient F_{max} might be dropped from Equation (1.9) without loss of generality.

The system of Equations (1.7) to (1.9) is supposed to progress over a fixed time period $[0, T]$ corresponding to the crop growth phase. The decision variable is the flow rate irrigation u, that we will also call the control variable. This control is a measurable function defined on $[0, T]$ and taking values in $[0, 1]$.

1.2.2.2 Model hypotheses

First let us precise the conditions given over the different parameters related to biophysical processes, whose information is gathered in the different constants involved in the formulation of Equations (1.7) to (1.9). In fact, all the quantities ET_0, Vol, F_{max}, k_{sat}, α and k_3 are positive by definition, and in a healthy operational context the capacity of the irrigation material is likely to be greater than the climatic demand. Hence the following assumption.

Assumption 1.2.1. *The constants k_1, k_2, k_{sat}, α and k_3 satisfy*

$$k_1 > 0\,, \quad k_2 > 1\,, \quad k_{sat} > 0\,, \quad \alpha > 0\,, \quad k_3 > 0.$$

The radiation interception efficiency φ is an increasing function of time and taking values in $[0, 1]$, related to the canopy growth throughout the season, hence giving more weight to evaporation early in the season then to transpiration later in the season. Increasing φ values also means there is no human intervention on the canopy, like for example cutting branches and leaves of the plants during the crop growth phase. Reliable hypotheses on the shape of φ will be made later for numerical purposes.

Assumption 1.2.2. *The function $\varphi : [0, T] \to [0, 1]$ is increasing and of class C^1.*

The crop coefficients K_S, K_R and K_D follow empirical laws derived from experimental data (see [3]), and we consider only structural hypotheses in order to preserve some flexibility.

Assumption 1.2.3. *The functions K_S, K_R and K_D are defined for $S \geq 0$, and are non-decreasing, piecewise affine, continuous. Typically, K_S, K_R and K_D are found in the literature ([19, 133, 73]) as*

$$K_S(S) = \begin{cases} 0, & S \in [0, S_{pw}), \\ \frac{S - S_{pw}}{S^* - S_{pw}}, & S \in [S_{pw}, S^*), \\ 1, & S \geq S^*; \end{cases} \tag{1.10a}$$

$$K_R(S) = \begin{cases} 0, & S \in [0, S_{rw}), \\ \frac{S - S_{rw}}{1 - S_{rw}}, & S \geq S_{rw}; \end{cases} \tag{1.10b}$$

$$K_D(S) = \begin{cases} 0, & S \in [0, 1), \\ \frac{S - 1}{S_{sat} - 1}, & S \in [1, S_{sat}), \\ 1, & S \geq S_{sat}. \end{cases} \tag{1.10c}$$

Remark 1.2.1. *It is possible to consider more generic expressions than Equations* (1.10a) *and* (1.10b) *(with a greater number of non-differentiable points for example), as long as they remain non-decreasing, piecewise affine, continuous, and verify K_S concave over $\{S \geq S_{pw}\}$ and K_R convex over $\{S \geq 0\}$. Notice that K_S, K_R and K_D are Lipschitz, and denote $L_S > 0$, $L_R > 0$ and $L_D > 0$ their respective Lipschitz constant.*

At the beginning of the cropping season, it is assumed that soil has been partly refilled during the inter-cropping period, and the initial soil moisture is then supposed to be sufficiently high (i.e. no water stress at initial time). Moreover, the soil water contents above the field capacity $S = 1$ are non-equilibrium points and the system naturally tends to evacuate the excess of water. As a consequence, as in [19], we consider initial conditions as follows.

Assumption 1.2.4. *The system has the following initial conditions:*

$$S(0) = S_0 \in (S^*, 1]\,, \quad B(0) = 0\,, \quad V(0) = 0.$$

Remark 1.2.2. *The condition $k_2 > 1$ ensures local controllability of the S variable (for $S \in [0, 1]$), in the sense that it is possible to select controls in order to chose the sign of $\dot{S}$. Moerover, the irrigation keeping soil moisture constant at a fixed value $\widetilde{S} \in [0, 1]$ is a locally admissible control.*

Definition 1.2.1. Under Assumption 1.2.1, provided that $\widetilde{S} \in [0, 1]$, the control defined by

$$u_{sing}(t, \widetilde{S}) = \frac{\varphi(t)K_S(\widetilde{S}) + (1 - \varphi(t))K_R(\widetilde{S})}{k_2}$$

belongs to $[0, 1]$.

Following the dynamics presented in the previous section, and the hypotheses introduced in this section, we aim at studying some formulations of optimal control problems that will be detailed later. These formulations may contain additional conditions or constraints according to the operational context, and the study will carry out all the resulting difficulties.

1.2.3 Issues, difficulties and theoretical background

As discussed previously, this thesis focuses on the study of several optimization problems formulated as optimal control problems under constraints. More precisely, the problems involve a dynamical system (the CCI model) over a time interval with a fixed horizon T, various criteria formulated as terminal costs, various terminal targets, and some state constraints (detailed later). The dynamics are non-linear, non-autonomous and non-smooth with respect to the state, but so that they are affine with respect to the control u. Such optimal control problems are in general non trivial to tackle and require a rigorous and deep analysis [36, 104, 35, 190, 75]. Our study case however presents a particularity, since the dynamics are regionally affine (and continuous) with respect to the state, and we will consider state constraints. As a consequence, the non-smoothness of the dynamics resides in a discrete set (the junctions of the affine parts), a finite number of points where the dynamics is non-differentiable. Everywhere else, i.e. over a dense set of the state space, the dynamics are locally affine.

For globally linear (or affine) optimal control problems, there exists a literature covering a very wide range of topics and a great variety of results. In particular, for time-optimal problems relying on linear (or affine) dynamics, there exists a classical result about structural properties of the optimal solutions. Indeed, it is well-known (see for example [79]) that there exists an optimal control taking values in the set of extreme controls, for a compact convex set of controls (this result is called the "bang-bang principle"). Since our subject of work involves dynamics that are not linear (nor affine), it is not possible to make such a statement. However, as our system dynamics are regionally affine, it may seem intuitive that we should be able to extract some structural information about optimal solutions with the help of the dynamics structure and the well-known classical bang-bang principle.

Note that our irrigation problem will be formulated with intrinsic state constraints that will be detailed in Section 1.3.1. It is well-known that state constrained optimal control problems are far

from easy to solve, and possess the particularity to generate jumps in the adjoint vector. Various works (see [122, 78, 87, 152, 174]) provide some results and techniques able to deal with such difficulties, under different kinds of hypotheses (such as order of the state constraint, junction conditions, state augmentation, regularization). The idea of adding a state variable (whose dynamics typically accounts for the state constraint violation), and replacing the running state constraint by a terminal target, present the promising feature to make the adjoint vector continuous. However, one might observe (due to the addition of an adjoint component) a situation corresponding to the degeneracy of the previous formulation, which does not give any useful information when deriving the necessary conditions. In this thesis, we combine this idea with a budget penalization that avoids the addition of a state variable, but requires to modify the formulation, and thus prove that it remains an equivalent problem.

One of the main objectives of this thesis is to find irrigation scenarios in terms of control strategies, preferably formulated as feedback controls. Thanks to the optimal control formulation, it is possible to write some necessary optimality conditions (using the non-smooth Pontryagin Maximum Principle (PMP) [134, 98]) and derive them in order to aim for constructive properties of optimal controls. Moreover, the non-differentiable feature of our system dynamics brings additional difficulty into the analysis, since it will require to derive a non-differentiable version of the PMP (see for example [36, 35]) using especially the notion of generalized gradients. This method may or may not result in a completely analytical description of optimal solutions, but it can at least typically provide some structural properties over the nature of candidates for optimal controls. It is then customary to complement these properties by a numerical solving through a direct or an indirect method in order to approximately find the optimal solutions. In our study, we will not only derive the optimality necessary conditions, but we will aim for providing a complete feedback solution without the need of a numerical complementary analysis.

Then, thinking of operating application, it is usually interesting not only to know about the structural properties of optimal irrigation policies, but also to explain how this structure depends on the physical components of our irrigation system. In order to provide such insights about the drivers of some model outputs, it is customary to project a numerical model exploration, provided that we have sufficient information about the ranges and distributions of model parameter values. A simple, inexpensive way to explore the model is to conduct parameter perturbations using the "one factor at a time" method (see for example [127]). However such method is not suitable for non-linear models and unable to detect effects from parameters interactions. Therefore we need to consider more expensive methods, such as global sensitivity analysis [41, 166, 57] through a global numerical exploration and variance-based computational methods [88, 150]. This method is suitable for our investigation, but will require a significant amount of runs over sufficiently wide ranges of parameter in order to extract effective results. Moreover, we may consider taking advantage of such a numerical exploration to conduct a complementary analysis about the effects of decision perturbations on the model outputs, over the global exploration settings. The main challenge to overcome for such numerical analyses is the computational cost.

In the open-field conditions context, the dynamics involve a non-deterministic term to take into account the probabilistic behaviour of rainfall sequences. There exists several ways to consider and model such non-deterministic phenomena, from fully stochastic formulations to parameterized laws of probability or the use of markovian processes. In this thesis, the choice went towards a probabilistic formalism, considering that each sequence of annual rainfall will be represented as a time-varying function defined over the cropping season interval $[0, T]$. At each time t, it should represent the current rainfall rate. Since the sequence of annual rainfall is not known a priori, we consider a finite set of all possible sequences of annual rainfall, and we assume that each sequence has some known probability to actually occur. The optimal control problem formulation will thus shift from a single controlled soil moisture equation to a set (finite in our case) of controlled soil moisture equations, each one of these corresponding to a different sequence of rainfall. However the control has to be common to all these scenarios and represents the generic approach, a decision that is able to take into account any possible future. The criterion to be optimized will become an average of the individual costs associated to each equation, weighted by the corresponding given probability that each sequence may occur. Such problem formulation is called "averaged cost control" problem [12, 13],

and provides a framework to apply optimality necessary conditions that are somewhat similar to the case with only one equation, where the additional difficulty obviously relies in the analysis of the consequences derived from the weighted Hamiltonian maximization.

Finally, it has been mentioned that annual rainfall sequences will be represented by time-varying "inputs". In fact, there is different possibilities to chose the type of inputs we could use. In our work we decided to consider that rainfall events might occur on various time scales, while the physical process of infiltration and assimilation by the soil can be considered as a slow process. As a consequence, according to the nature of rainfall that may occur especially in arid and semi-arid regions (relatively high rate over a limited period of time), it is reasonable to assume that rainfall events could be modeled as instantaneous. This results in representing a possible sequence of annual rainfall with a (finite) set of instants (dates of rainfall event t_i) and their corresponding depth (h_i), thus allowing the soil moisture state variable to present some jumps (or discontinuities) at the times of instantaneous rainfall events. The system dynamics will then be written as a differential equation complemented with state jumps and extended theoretical tools will be needed to carry out the aforementioned analysis. To this end, we refer to [160, 161, 139, 124, 44, 191, 198] describing how dynamical systems with state discontinuities may be dealt with, in particular how to transform it into a standard dynamical system (using a change of time scaling), in order to apply and derive standard optimality necessary conditions.

1.3 Contributions to the optimization of crop irrigation

1.3.1 Contribution to the CCI model and to the irrigation problem formulation

First, in order to prevent critically low levels of soil water content that would lead to severe plant degradation or early senescence (or even crop's death), we added to the initial CCI model formulation a state constraint on the soil moisture representing the risk aversion or inclination of the operator regarding the tolerance to water stress. More precisely, the system is subject to the following state constraint:

$$\forall t \in [0,T],\ S_{ope} \leq S(t), \tag{1.11}$$

where S_{ope} is called the operating constraint (or tolerance threshold), and verifies $S_{pw} < S_{ope} < S^*$.

1.3.1.1 Operational contexts

The dynamics Equation (1.7) possess a non-deterministic term due to the rainfall uncertainties. However, it is possible to consider agricultural practices where rainfall does not occur, and where the variations of weather conditions in general have low impact on the crop development. Indeed, greenhouses can even enable to tend towards a fully controlled crop irrigation system, provided that the operator is able to maintain a stable temperature, and uniform wind and air humidity conditions. In such a framework of controlled greenhouse conditions, it justifies the use of deterministic dynamics, and we can just remove the term of rainfall without any more concern.

An implicit consequence of fully controlled greenhouse irrigation systems is that the operator is able to command irrigation with a rather high precision (thanks to sensor-based monitoring for example). As a result, the operator will likely avoid wasting water, i.e. they should not reach a situation where the soil moisture would exceed $S = 1$ (which would generate water losses by drainage). Therefore the state constraint Equation (1.11) in the context of greenhouse becomes

$$\forall t \in [0,T],\ S_{ope} \leq S(t) \leq 1. \tag{1.12}$$

Moreover, as mentioned above, fulfilling the new state constraint Equation (1.12) is equivalent to consider that drainage is no longer involved in the soil water content dynamics. Consequently the equation ruling soil moisture variations in the controlled greenhouse context is now

$$\dot{S}(t) = k_1\big(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t)\big). \tag{1.13}$$

Finally, this context will be investigated in detail in Chapter 3 and Chapter 4, where the deterministic formulation allows us to derive optimality necessary conditions from the Pontryagin

Maximum Principle, leading to the design of a class of optimal structures of solutions capturing some intrinsic properties of the crop dynamics. This way, we will be able to extract information from the inner "soil-plant" system without environmental disturbances, justifying the interest of such a deterministic formulation.

It is also possible to keep the non-deterministic dynamics of Equation (1.7), which means that we accept to take into account the effects of rainfall over our study system. This implies placing our study in the framework where rainfall events occur, i.e. in open-field conditions. Furthermore, such weather events are by nature unknown in advance and present a wide range of variability, thus might have significant impacts on the instantaneous soil moisture balance. In particular, an intense rainfall event, involving great amounts of water in a short period of time, might overfill the soil reservoir and generate a non-negligible drainage, which increases up until a saturation level above the field capacity $S = 1$.

As a consequence, the use of both terms of rainfall and drainage in Equation (1.7) is well justified in open-field conditions. Moreover, even if the operator has full control over the irrigation amounts and their precision, they cannot prevent the soil moisture to reach levels above the field capacity, thus resulting in some undesired yet unavoidable water losses. The state constraint $S(t) \leq 1$ is thus removed, and only Equation (1.11) is sufficient to reflect the margin of risk aversion (or inclination) of the operator.

This way, considering a study case in open-field conditions has the advantage to get closer to actual operational situations, because assuming fully controlled irrigation might be seen as a slighlty too strong hypothesis, due to existing variability in the biophysical processes and the crop itself. Therefore adding uncertainty to the analysis is a significant argument to claim that we are now looking for irrigation strategies that present some robustness with respect to perturbations. Furthermore, taking into account an unknown future implicitly means that the designed strategies will be generic, able to adapt to any upcoming uncertain event, and that the strategy relies on a systematic methodology. This might be particularly interesting as a support of public policies to operating decisions.

Finally, this context is the main subject of investigation in Chapter 5 and Chapter 6. More precisely, in Chapter 5 we will explore the possibility of evaluating the performance of the candidate strategies "greenhouse optimal solutions" in situations close to real-world conditions, and conclude about their eventual robustness to uncertainty. Then, Chapter 6 introduces the analytical framework that aims for deriving optimality necessary conditions in a similar way to the "greenhouse" deterministic context, but presenting much more complicated challenges mostly due to the fact that the soil moisture trajectories associated to all possible rainfall sequences must be controlled by the same uniform decision.

1.3.1.2 Deficit irrigation framework

Remark 1.3.1. *Notice that the worst case scenario of the open-field context would be that of getting zero precipitation throughout the whole season, which would formally coincide with a greenhouse context (at least in the way we are simulating climatic conditions, via $ET0$ only). It is only a formal comparison and does not mean that greenhouse crops will be worse, since greenhouse in fact protects the crop from climate aspects so that some parameters could be different than in open-field conditions. Still, it allows us to define a uniform framework to represent the situation where water resources are limited, such as the already mentioned water scarcity in arid and semi-arid areas.*

In both greenhouse context and open-field worst case scenario, the soil moisture will decrease if there is no irrigation (i.e. $u = 0$). Moreover, according to the biomass production dynamics, the crop does not suffer any stress (hence production is maximal) when the soil moisture is above the setpoint S^*. As mentioned before, our study will consider situations where full water availability can not be satisfied, in other words the limitations in water resources must generate water stress to some extent. Let us then define the least consuming irrigation chronicle such that the soil moisture trajectory is sufficient to ensure no water stress, and denote this irrigation u^*. Since the initial condition of soil

moisture is $S_0 \in (S^*, 1]$, the chronicle u^* consists in no irrigation until the (decreasing) soil moisture trajectory reaches the level S^*, then the chronicle u^* consists in irrigating in such a way that the soil moisture remains constant at the level S^*. In other words, there exists $t^* \in (0, T]$ such that

$$u^*(t) = \begin{cases} 0, & t \in [0, t^*), \\ u_{sing}(t, S^*), & t \in [t^*, T], \end{cases} \tag{1.14}$$

which is possible since $S^* \in [0, 1]$.

This enables us to introduce the limit value of irrigation water that allows avoiding stress throughout the cropping season.

Definition 1.3.1. Using the control u^* defined above, the associated water consumption is

$$V^* = \int_0^T u^*(t)dt.$$

As a result, the context of deficit irrigation may be represented by a supplementary condition over the amount of available irrigation water, that we can write as

$$\text{“}u \text{ takes place in the context of deficit irrigation}\text{”} \quad \Leftrightarrow \quad \int_0^T u(t)dt < V^*.$$

In addition, as aforementioned, the operator must decide to set a soil moisture threshold (S_{ope}) below which they consider it is too risky for the crop health. Let us then define the least consuming irrigation chronicle such that the soil moisture trajectory is sufficient to ensure that this operating condition is satisfied, and denote it u_{ope}. Similarly as above, there exists $t_{ope} \in (0, T]$ such that the soil moisture trajectory decreases then reaches S_{ope} at time t_{ope}, and then remains constant at S_{ope}, as follows:

$$u_{ope}(t) = \begin{cases} 0, & t \in [0, t_{ope}), \\ u_{sing}(t, S_{ope}), & t \in [t_{ope}, T], \end{cases} \tag{1.15}$$

which is possible since $S_{ope} \in [0, 1]$.

This enables us to introduce the limit value of irrigation water that allows avoiding too risky stress throughout the cropping season.

Definition 1.3.2. Using the control u_{ope} defined above, the associated water consumption is

$$V_{ope} = \int_0^T u_{ope}(t)dt.$$

As a result, the operating context of risk aversion (or inclination) may be represented by a supplementary condition over the amount of available irrigation water, that we can write as

$$\text{“}u \text{ fulfills the operating context of risk aversion}\text{”} \quad \Leftrightarrow \quad \int_0^T u(t)dt > V_{ope}.$$

Finally, we combine the context of deficit irrigation and the consideration of risk aversion, and we define the framework of our study as the irrigation chronicles (or controls) u that fulfill both conditions:

$$V_{ope} < \int_0^T u(t)dt < V^*. \tag{1.16}$$

1.3.1.3 Objectives and targets, formulation of the problems

Let us now introduce the different criteria and targets that will be possible to consider in the following, according to different classical agronomic objectives and desires of the operator.

The problem of optimizing crop irrigation might be understood in various ways. As mentioned before, the standard issue of operating irrigation consists in “using less water than a given quota,

producing more than a given yield, while ensuring maximal economic benefits". This issue can lead to different kinds of optimization direction. In particular, we may choose if there is strict obligations or if there is flexibility on some variables. As we want to apply control theory, the strict obligations will be formulated as mathematical constraints, and the flexible variables will play the role of the criterion to be optimized. Typically, we might have a strict limitation of water access and have freedom over the biomass production (as in [19]), or conversely we might have a strict objective for biomass production and keep freedom over the water use (as in [157, 112]). Alternatively, we might not have any strict limitation or objective, and rather looking for economic optimization without any regards on the values of water use and biomass production. In this thesis, we will tackle all these study cases. We then introduce three optimization problems corresponding to three natural points of view about the operational targets and the criteria to be optimized.

- $(P1)$: maximizing the biomass production at the end of the season, when using a limited quota of water, i.e. maximizing $B(T)$ under the terminal condition $V(T) \leq Q$ where Q is the initial available water.
- $(P2)$: minimizing the water consumption to reach a certain target of biomass, i.e. minimizing $V(T)$ under the terminal condition $B(T) \geq Y$ where Y is the objective of biomass production at the end of the season.
- $(P3)$: maximizing a financial balance between biomass sales and water consumption expenses, i.e. maximizing the value $g(B(T)) - c(V(T))$ where g and c are given functions representing the earnings and spendings, with $g(0) = c(0) = 0$, $g' > 0$ and $c' > 0$.

These problems formulations are in fact subject to the "admissibility" question. More precisely, we must look for control functions u that aim to optimizing the respective criteria, while fulfilling the different constraints (over the values taken by u, the values taken by the consequent trajectory of S, the terminal condition). If a control u is able to satisfy all the mentioned conditions, then it will be called "admissible". For each problem $(P1)$, $(P2)$ and $(P3)$, and for each context declension, there exists some set of admissible controls. As a result, the optimization will only take place amongst the admissible controls corresponding to each study case. The nature of such admissible sets will be detailed later for some more specific cases.

Consequently, we prepare ourselves to formulate the irrigation problems as the following optimal control problems.

Definition 1.3.3. For a given context C, and corresponding admissible sets $\mathcal{A}_1^C$, $\mathcal{A}_2^C$, $\mathcal{A}_3^C$ (that will be detailed in Chapter 3), we formally define

$$
\begin{aligned}
(P1): &\quad \sup_{u(\cdot)\in\mathcal{A}_1^C} B(T),\\
(P2): &\quad \inf_{u(\cdot)\in\mathcal{A}_2^C} V(T),\\
(P3): &\quad \sup_{u(\cdot)\in\mathcal{A}_3^C} g(B(T)) - c(V(T)).
\end{aligned}
$$

1.3.2 Extension of the bang-bang principle for piecewise affine dynamics

As mentioned previously, the system dynamics are piecewise affine (and continuous) with respect to the state, and affine in control. Since we aim for providing structural properties about optimal solutions to the irrigation problem, it is reasonable to investigate the well-known bang-bang principle for globally linear (or affine) problems, and to seek to extend such a result in some sense for piecewise affine systems under state constraints. In Section 2.1, as an extension of the classical bang-bang principle for linear systems, we show that for non-autonomous piecewise affine optimal control problems with state constraints, the bang-bang principle does not hold globally even when the dynamics is continuous with respect to the state variable. However, we show that there exist optimal trajectories that are generated by extreme controls at the loci where the dynamics is differentiable, which is a weaker version of the classical bang-bang statement. Moreover we provide two examples which highlight the potential existence of singular arcs exactly at the loci of non-differentiability of

dynamics, i.e. at the boundaries of the regions. More precisely, consider the following dynamical system:

$$\dot{X} = F(t, X) + B(t)u, \tag{1.17}$$

where $X \in \mathbb{R}^n$ and $u \in \mathcal{U}$, with $\mathcal{U}$ the set of measurable functions taking values in a convex compact set $U \subset \mathbb{R}^r$. Also, the drift function F is defined by

$$F(t, X) = A_i(t)X + C_i(t), \quad (t, X) \in P_i, \tag{1.18}$$

where A_i and C_i are $n \times n-$ and $n \times 1-$ time measurable matrix-valued functions, B is a $n \times r-$ time measurable matrix-valued function, and F is assumed to be continuous. In addition, the sets P_i define a countable partition of the set Ω of the trajectory constraints. In this framework, we show the following result, illustrated by Figure 1.3.

Theorem 1.3.2 (Extended bang-bang principle)**.** *Considering a minimum-time control problem subject to the non-autonomous regional affine settings Equation* (1.18)*, if the initial condition $X_0 \in \mathbb{R}^n$ belongs to the attainability set of the target $\Gamma \subset \mathbb{R}^n$, then there exists an optimal solution composed of a succession of bang-bang arcs over time interval where the trajectory belongs to a region $\mathring{P}_i$, and possible singular arcs along the boundaries ∂P_i.*

Remark 1.3.3. *Notice that the locus ∂P_i might be an interface between two regions but might also belong to the boundary of the domain Ω.*

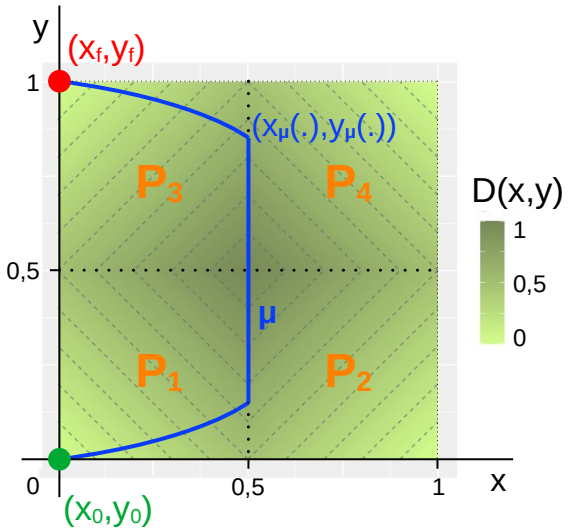


Figure 1.3: Example of a minimum-time trajectory (in blue) of reaching a target (in red) from an initial condition (in green), over a piecewise affine dynamics given by $\dot{x}(t) = u(t)$ and $\dot{y}(t) = D(x(t), y(t))$. We can observe that the trajectory corresponds to a "bang-singular-bang" control, and that the singular part is exactly along boundaries of the space partition.

This result may be then applied to the crop irrigation optimal control problem in the context of greenhouse. Indeed, this context formulation only involves a piecewise affine (and continuous) dynamics, due to the definition of the crop coefficients K_S and K_R. As a consequence, we deduce from the extended bang-bang principle for piecewise affine systems that there exists optimal solutions fulfilling structural properties consisting in bang arcs when the state trajectory travels inside an affine region, and may contain singular arcs only at the non-differentiable points of the dynamics, i.e. at the points S_{rw}, S_{pw}, S^*, or along state constraints. This property will provide a significant help in Chapter 3 where we aim for describing the structures of optimal irrigation policies.

1.3.3 Extension of the bang-bang principle in stochastic settings

In view of similar structural properties of optimal solutions to the irrigation problem taking into account the rainfall uncertainties, we may think once again of a similar bang-bang like property.

However, it may be difficult to directly head towards an extension involving both piecewise affine dynamics and a non-deterministic term. A first step is to investigate the bang-bang principle for globally linear (or affine) dynamics containing a stochastic term. To this end, we wrote the stochastic term as a standard brownian noise, enabling a rather generic analysis for different kinds of applications. In Section 2.2, we revisit the classical bang-bang principle, established for deterministic affine controlled dynamics, in a stochastic framework when the dynamics is subject to a brownian motion. We show that such a principle does not hold in general in this setting, in the sense that the reachable sets (with any control and with only extreme controls) do not coincide. However, with stochastic controls, we show, under some conditions, that Mayer problems with criterion in expectation and a compact control set are equivalent to the same problems with controls taking values in the closed convex hull of the control set. More precisely, consider the following dynamical system:

$$dX(t) = \big[a(t) + A(t)X(t) + B(t)u(t)\big]dt + \big[c(t) + C(t)X(t)\big]dW(t), \quad t \in [0,T], \tag{1.19}$$

where $X \in \mathbb{R}^n$, u taking values in a compact set $U \subset \mathbb{R}^d$, time-dependent matrices $a(\cdot)$, $A(\cdot)$, $B(\cdot)$, $c(\cdot)$ and $C(\cdot)$ are of appropriate dimensions, and W denotes the standard one-dimensional brownian motion. Note that here we assume that the noise does not operate on the controlled term (in our paper [31] we explain the difficulties when relaxing this assumption). Denoting $\mathcal{U}$ the set of adapted U-valued control processes, $\overline{\mathcal{U}}$ the set of adapted $\overline{co}\,U$-valued control processes, and with an initial condition $X(t_0) = x_0$, we define the reachable sets by

$$\mathcal{R}_U^{st}(t_0,x_0;t) = \{X^{t_0,x_0,u}(t); u \in \mathcal{U}\}\,, \quad \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0;t) = \{X^{t_0,x_0,u}(t); u \in \overline{\mathcal{U}}\}, \tag{1.20}$$

where $X^{t_0,x_0,u}$ is the solution to the dynamical system associated to $X(t_0) = x_0$ and the control u. Moreover, given a map $g : \mathbb{R}^n \to \mathbb{R}$, we define the optimization problems

$$(\mathcal{P}_{strict}) : \inf_{\xi \in \mathcal{R}_U^{st}(t_0,x_0,t_1)} \mathbb{E}[g(\xi)]\,, \quad (\mathcal{P}_{convex}) : \inf_{\xi \in \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,t_1)} \mathbb{E}[g(\xi)].$$

In the particular case where U corresponds to the set of extreme points of $\overline{co}\,U$, then $\mathcal{R}_{\overline{co}\,U}^{st}$ may be seen as the reachable set for any admissible control, and $\mathcal{R}_U^{st}$ may be seen as the reachable set associated to bang controls only. Note that controls in $\mathcal{R}_{\overline{co}\,U}^{st}$ may be seen as "relaxed" controls as usually considered in the deterministic framework when the velocity set is not convex. In this framework, we show the following result.

Theorem 1.3.4. *Assume that the function g is lower semi-continuous function, bounded from below and with at most quadratic growth. Then, one has the following properties.*

1. *For every $0 < t_0 \leq t < T$, and every $x_0 \in \mathbb{L}^4(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, the control problems $(\mathcal{P}_{strict})$ and $(\mathcal{P}_{convex})$ are equivalent.*
2. *Furthermore, if g is also convex, then an optimal ξ exists in $\mathcal{R}_U^{st}(t_0,x_0,t) \subset \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,t)$.*

This result states that, even if the reachable sets $\mathcal{R}_U^{st}$ and $\mathcal{R}_{\overline{co}\,U}^{st}$ do not coincide (unlike the case of the classical bang-bang principle where they do coincide), the optimal criteria associated to the optimization problems $(\mathcal{P}_{strict})$ and $(\mathcal{P}_{convex})$ coincide. Furthermore, with reasonable hypotheses on g, there exists an optimal solution whose control is composed of arcs belonging to the set U, i.e. bang arcs in the particular case where $U = \mathrm{Ext}(\overline{co}\,U)$. This property may be seen as a first step towards an application to the crop irrigation problem, in the sense that it would now require to combine this stochastic framework with the aforementioned piecewise affine dynamics framework.

1.3.4 Optimal structures of crop irrigation strategies in greenhouse context

In the context of fully controlled greenhouse crop irrigation (i.e. without environmental variability), we look for structural properties of optimal solutions that are intrinsic to the crop development dynamics. In Chapter 3, we investigate the optimal control problem of crop irrigation with such non-autonomous and non-smooth dynamics corresponding to the aforementioned greenhouse context. Depending on contexts and objectives, several formulations associated to different constraints and criteria can be derived. Non-smoothness makes the problems a priori more difficult to study (from the numerical point of view, a direct method requires a regularization), but our work will

precisely relies on these non-differentiable points to describe the structure of solutions. Our work aims at providing optimal feedback solutions for these problems by deriving and analyzing the optimality necessary conditions. To this end, we assemble the different problems into a common formulation, and we carry out a dedicated way of handling state constraints, before applying the non-differentiable Pontryagin Maximum Principle without state constraint. We show that all optimal irrigation strategies belong to a family of simple parameterized time-varying feedback controls, for a large class of objectives, and suitable for computational purposes. More precisely, consider the soil moisture balance from Equation (1.13), the state contraints Equation (1.12), the hypotheses Assumptions 1.2.1 to 1.2.4, and the optimization problems defined in Definition 1.3.3.

We first show in Proposition 3.3.2 that an optimal solution to a reformulated problem, in the sense that we replace a state constraint (thus allowing more state trajectories and more admissible controls) by a penalization inside the budget consumption dynamics, is in fact an admissible control of the initial constrained problem. We show here that the penalization is exact in the sense that an optimal solution has necessarily a null penalization. As a result, it is equivalent to solve the unconstrained optimal control problem, and it allows us to derive optimality necessary conditions without state constraints. After showing some properties satisfied by optimal solutions (especially that the terminal targets of their respective problem formulation are saturated Proposition 3.3.5), we write the optimality necessary conditions from the non-differentiable Pontryagin Maximum Principle and we derive the consequent properties about the adjoint variables. First we show that there is no abnormal extremal, then we use the extended bang-bang principle for piecewise affine dynamics (see Theorem 2.1.5) to provide structural conditions over the optimal solutions, and a brief analysis of the associated switching function has implications on the possible number of switches (see Proposition 3.4.8). Finally, we gather all the above properties to conclude with Theorem 3.4.9 showing that the optimal irrigation policies can be formulated as time-varying parameterized feedback controls, as follows:

$$U_{t_d,W_c,t_c}(t,S,V) = \begin{cases} 0, & \big(S \geq S_{ope} \wedge t \leq t_d\big) \vee V = W_c \vee t \geq t_c, \\ u_{sing}(t, S_{ope}), & S = S_{ope} \text{ and } t \leq t_d, \\ u_{sing}(t, S^*), & S = S^* \text{ and } \big(V < W_c \wedge t < t_c\big), \\ 1, & \text{otherwise}, \end{cases} \tag{1.21}$$

where t_d, W_c and t_c are parameters whose values and conditions depend on the chosen problem formulation. In particular, we only need one or two parameters according to the chosen problem, thus shifting the initial infinite dimensional optimization problem into a reduced one or two dimensional optimization problem, which is very computationally affordable. We illustrate this result with a soil moisture trajectory and its associated optimal control in Figure 1.4.

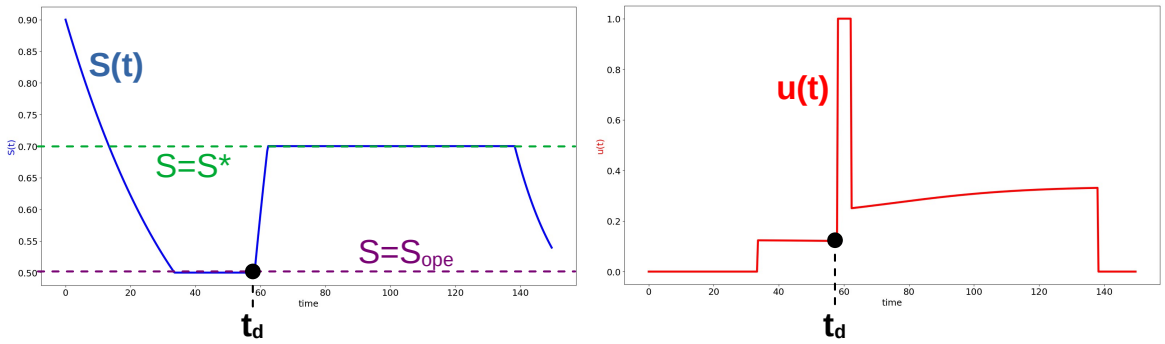


Figure 1.4: Example of optimal soil moisture trajectory (in blue, on the left), associated to the optimal control strategy (in red, on the right), which is "bang-singular-bang-singular-bang" control, and which is parameterized by only the decision time t_d.

1.3.5 Exploring the optimality of Threshold-Based Irrigation Strategies

We provide in Chapter 3 a class of parameterized irrigation strategies in the context of greenhouse water management, but the result is still lacking of some crucial elements. In particular, even if it has been proven that it is computationally affordable to determine the optimal parameter values for the optimal irrigation policy, the theoretical result does not explain the relationship between the context and the computed optimal values, or the relative efficiency of optimal strategies compared to less efficient practices. It turns out that the properties of the Controlled Crop Irrigation (CCI) model led to optimal solutions written as Threshold-Based Irrigation Strategies. In Chapter 4, we aim at studying both the optimal TBIS (the number of thresholds, their values and the most influential parameters which dictate the shape of optimal irrigation scenarios, see their shape declension in Figure 1.5), and their sub-optimal features (the effects of errors made on their determination).

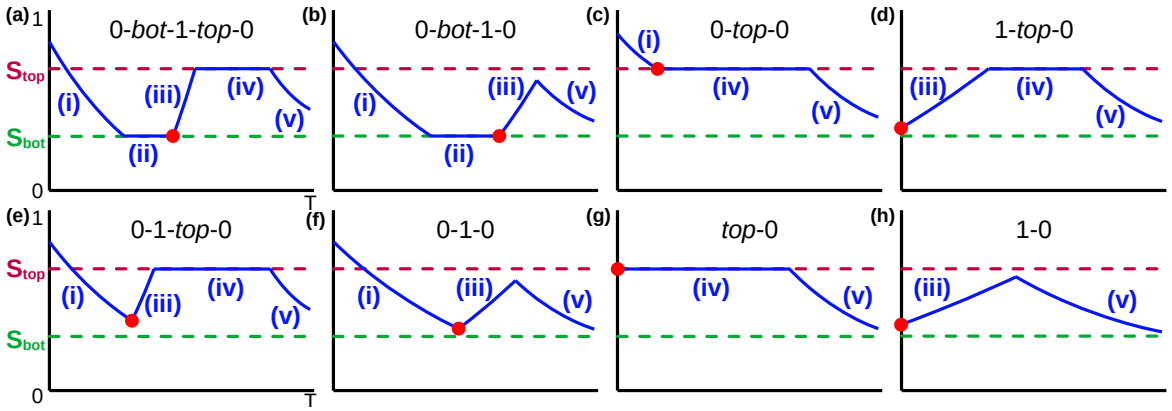


Figure 1.5: Families of possible soil moisture trajectories generated by the 2-TBIS framework (based on thresholds $S_{bot} < S_{top}$). According to the different model parameters values, some of the stages may be skipped. Also indicated is the point corresponding to the decision time t_d (red dot).

Much attention is drawn to the crucial "decision time" at which the irrigator decides to increase soil water content from a "stress tolerance level" to a "maximal biomass production" setpoint. In this work we couple known results from optimal control theory and numerical methods for sensitivity analysis. The main results show that the optimal decision time is mostly driven by parameters related to plant growth dynamics and the evaluation of stress effects on biomass production, and that suboptimal decisions induce a trade-off between biomass production, irrigation water use and plant water stress. The genericity of the CCI model and the chosen methodology (exploiting optimal feedback synthesis for global model exploration) offer promising perspectives for various operational contexts in irrigation planning. More precisely, we consider several operating contexts that cover different situations of water limitation and both high and low risk aversion. In each operating context, we conduct a global sensitivity analysis of the so-called "decision time" with respect to the problem parameters (of soil, plant, material characteristics), thanks to a wide exploration setting, and a multi-local perturbation analysis to assess the sub-optimality of near-optimal irrigation chronicles.

We highlighted that in both alternatives of risk aversion or inclination, the optimal time of triggering maximal irrigation rate is strongly related to plant characteristics, regardless of the initial irrigation water quota. In particular, the values of optimal decision time globally follow an increasingly monotonic behaviour with respect to a characteristic plant growth time, and a decreasingly monotonic behaviour with respect to the soil water content of incipient stress. We observed through our second numerical experiment (see Figure 1.6) that in both alternatives as above and in any case of initial irrigation water quota, mistakes over the top soil moisture threshold may undermine the biomass production compared to the maximal potential biomass. Conversely, sub-optimality (or mistakes) over the decision time tend to have moderate effects over the biomass production. However, early and late decisions lead to noteworthy different outcomes in terms of chances to violate the operating constraint or saving a fraction of irrigation water. The combination of low irrigation quotas, high risk aversion and early decisions, creates strong operational constraints on the use of water resources. It makes more difficult (or unlikely) the identification (or the existence)

of admissible soil moisture trajectories. The opposite extreme combination of high irrigation quotas, high risk inclination and late decisions, provides favorable conditions for unhindered and efficient operational use of water. Our results demonstrate the capabilities of the coupled optimal control and global exploration methodology, to point out some principles of enhancing the management of irrigation scheduling. With our in-depth analysis of TBIS, we opened a perspective to assess their potential in terms of water use efficiency.

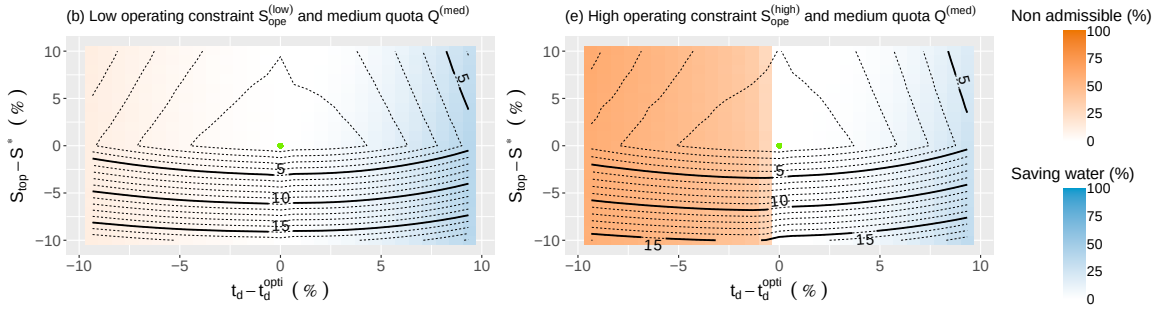


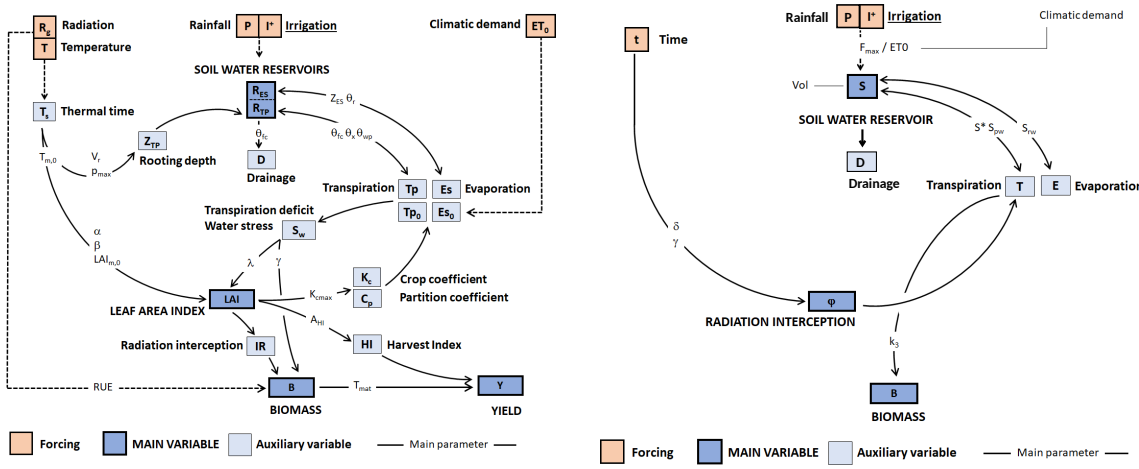
Figure 1.6: Performance of sub-optimal strategies, evaluated as mean percentage losses in final biomass, in the vicinity of the optimal level (S^*) and of the optimal decision time (t_d^{opti}). The isovalues in each sketch correspond to the mean percentage loss of final biomass relative to the "contextual optimum" (i.e. for the indicated parameterization), located in the center of each plot (green dot). In complement, orange shades represent the percentage of exploration parameter sets leading to non-admissible trajectories of soil moisture (too low final S values), and blue shades represent the percentage of exploration parameter sets leading to water saving trajectories (under quota), i.e. for which $V(T) > 0$.

1.4 On-going works and perspectives

1.4.1 Correspondence between the CCI and Optirrig models

As mentioned as the initial motivation and framework of this thesis, we investigate the compatibility between the CCI model and Optirrig, in order to anticipate a double modeling process. We split this investigation in three successive discussions : a conceptual comparison, a term-to-term correspondence attempt, and a discussion about the limitations. Following the formalism of Optirrig, we present the main variables and processes of interest that are described in Optirrig and that can be found in a different formulation in the CCI model. We highlight that, even if the two model formulations were made in independent works and arise from a priori very different purposes, their conceptual schemes are surprisingly similar (see Figure 1.7), which makes the project of double modeling a reasonable and not so far objective.

Once the respective concepts in both sides are identified, we specify the relationship that may be settled between the parameters of Optirrig and the CCI model. Some correspondences are direct transformations (as for example the soil moisture contents or the contributions from rainfall and irrigation), some are more complicated but still achievable (as for example the climatic demand, or the evapotranspiration and drainage efficiencies), and some are at least challenging if not problematic (as for example the crop growth consequences, or the volume of the soil reservoir). We demonstrate our methodology and underlying hypotheses to proceed to the models correspondence, noticing the potential issues emerging from the necessary approximations due to the compatibility difficulties. Moreover, we suggest some model improvements in order to facilitate the transfer or parameters, especially regarding the processes representing the growth of the plant and its roots. That being said, the framework is now set and ready to carry out some inaugural experimental simulations with Optirrig in order to test the strategies extracted from the CCI model (in particular the 2-TBIS that are expected to perform well).



(a) Reference conceptual scheme of Optirrig, showing the causal links between model variables as well as the position and limited number of key model parameters.

(b) Conceptual scheme of the CCI model, showing the causal links between model variables as well as the position and limited number of key model parameters.

Figure 1.7: Conceptual comparison between Optirrig and the CCI model formalisms.

1.4.2 Averaged cost control for crop irrigation management in operational contexts

In order to consolidate the objective of providing irrigation strategies from a theoretical framework that make sense for operational purposes, one needs to be conscious that real-world systems contain some variability or uncertainties of different types. Several ways can deal with such uncertainties, but we chose in this thesis to stick to a "deterministic" way (in some sense), since it will allow us to attempt extending the already investigated arguments in Chapters 2 and 3. To this aim, we introduce a theoretical tool called "averaged cost control" which formulates optimal control problems under some uncertainties as a set of all possible scenarios associated to their probability of occurrence. It then consists in looking for a control which is common to all scenarios, and which is optimal in the sense that it optimizes the averaged cost of all these scenarios. The goal of introducing this approach is to apply it in situations closer to real-world conditions, as frameworks in which such situations may be formalized and are projected to be further investigated.

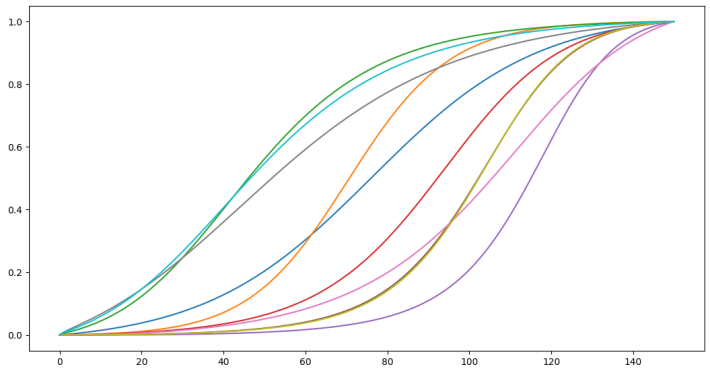


Figure 1.8: Example of a set of ten possible shapes for the parameter function φ. These functions represent the variability of crop growth that may happen in real-world or may be approximately known.

First, we recall that the system parameters are in general approximately known, thus the opti-

mality of a solution might be sensible to the variability of these estimated parameters. In particular, we highlighted in Chapter 4 that the optimal decision time is mainly driven by the shape of the function φ. As a result, we investigate the irrigation problem formulated in the averaged cost control framework, to seek to irrigation strategies that will account for the potential variability of the function φ (see Figure 1.8), which may be seen as a robustness property. We thus look for irrigation strategies that optimize the mean biomass production with respect to the uncertainties over the function φ. To this aim, we consider that φ belongs to a finite set of possible profiles φ_i, associated to their probability of occurrence μ_i. We assume that the function is initially drawn with a certain probability, in a unknown way, but does not change over time. Then we formalize our irrigation problem by duplicating the basis dynamics as many times as the number of φ_i. We obtain a high dimensional "deterministic" optimal control problem for which there exists a dedicated statement of optimality necessary conditions. Deriving these conditions will provide properties for the search and characterization of optimal solutions.

Then, we remind that we aim for applying irrigation strategies in open-field conditions, thus we need to consider the occurrence of rainfall events, which are by definition non-deterministic. As a consequence, we project to investigate the irrigation problem accounting for such rainfall uncertainties, and we want to employ the averaged cost control framework, by considering that an annual rainfall sequence belongs to a finite set of possible sequences associated to their probability of occurrence. However, we mentioned previously that rainfall events could be considered as instantaneous events that generate jumps in the soil moisture state variable. The system dynamics will then be written with differential equations complemented with state jumps, which is not a standard framework suited to a direct application of averaged cost control theory. As a consequence, we seek to employ a time reformulation technique, based on a graph completion, but we need to formalize it in such a way that the new time variable is the same for every considered scenario. This way, we will be able to transform our initial non-standard dynamics into a more suitable one to apply the dedicated optimality necessary conditions, and we then project to search and characterize "average" optimal solutions efficient when facing rainfall uncertainties.

Dissemination and projects related to the thesis

- Journal articles :
  - "Extension of the bang-bang principle for piecewise affine dynamics under state constraints"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    (submitted to Automatica in November 2024, under review)
  - "About the bang-bang principle for controlled affine dynamics with brownian noise"
    Ruben Chenevat, Dan Goreac, Qinlong Li, Alain Rapaport
    Journal of Convex Analysis, 2025
  - "Optimal structures of crop irrigation strategies with state constraints"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    Journal of Optimization Theory and Applications, 2025
    doi: 10.1007/s10957-025-02854-7
  - "Exploring the optimality of threshold-based crop irrigation feedback strategies"
    Ruben Chenevat, Bruno Cheviron, Alain Rapaport, Sébastien Roux
    (to be submitted)

- Conference articles :
  - "About the Bang-Bang Principle for piecewise affine systems"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    63rd IEEE Conference on Decision and Control (CDC), December 2024
    doi: 10.1109/CDC56724.2024.10886617
  - "Common structures of optimal solutions for a crop irrigation problem under various constraints and criteria"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    63rd IEEE Conference on Decision and Control (CDC), December 2024
    doi: 10.1109/CDC56724.2024.10886226

- Talks and presentations :
  - "Optimal structures in irrigation and water use efficiency"
    Annual days of GdR MOA, Perpignan (October 2023)
  - "Optimal structures for a model of crop irrigation with constraints"
    Journées SMAI-MODE (Poster), Lyon (March 2024)
  - "About the Bang-Bang Principle for piecewise affine dynamics under constraints"
    Workshop Optimal control and applications - FRUMAM, Marseille (June 2024)
  - "Optimal structures for a model of crop irrigation with constraints"
    Lab Seminar - MOMAT, Universidad Complutense Madrid (June 2024)
  - "About the Bang-Bang Principle in a non-smooth setting"
    21st French-German-Spanish Conference on Optimization, Gijon (June 2024)
  - "Commande optimale et double modélisation pour l'utilisation de l'eau en irrigation des cultures"
    Annual meeting of INRAE "REUSE" thematic network, Avignon (October 2024)
  - "About the Bang-Bang Principle for piecewise affine systems"
    63rd IEEE Conference on Decision and Control (CDC), Milano (December 2024)
  - "Common structures of optimal solutions for a crop irrigation problem under various constraints and criteria"
    63rd IEEE Conference on Decision and Control (CDC), Milano (December 2024)
  - "Controlled crop irrigation : from roots to leaf"
    Lab Seminar - SYSTEC, FEUP Porto (January 2025)

- "Optimizing crop irrigation under biological and operational constraints with meteorological uncertainty"
  16th Conference on Evolutionary and Deterministic Methods for Design, Optimization, and Control with Applications (EUROGEN), Lahti (September 2025)

- Projects participation :

  - EIR-A (École Internationale de Recherche d'Agreenium) PhD label
    Successful application to the EIR-A PhD program – Session of 2024
    International mobility: 3-month visit to the SYSTEC laboratory (FEUP Porto), from January to April 2025
  - "BOUM" Project grant from the SMAI
    Successful application to the SMAI-BOUM call – Session of Fall 2024
    Organization (with Anas Bouali and Gildas Dadjo) of a Workshop day on Optimization, Modelling and Control (June 12, 2025)
  - Participation to a Hubert Curien partnership France-Portugal (Program PHC-PESSOA 2025-2027)
    Subject : "Optimizing Irrigation efficiency under Technical and Environmental constraints"

- Teaching :

  - Two years of Tutorial sessions (problem-solving classes) : 64 hours per year, at Undergraduate level (1st to 3rd year), for Mathematics students
    Topics covered : Logic & Set theory, Functions of one variable, Sequences, Riemann Integral, Numerical analysis of ODEs and PDEs

Visual summary of the thesis

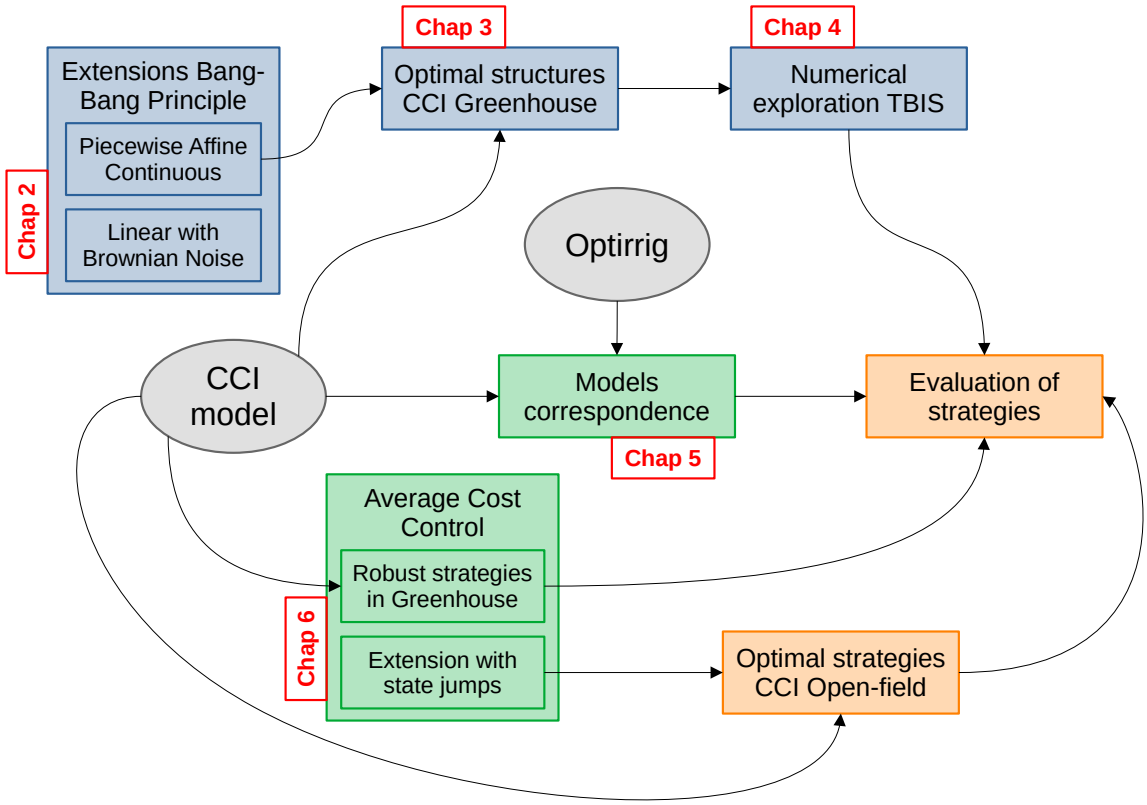


Figure 1.9: Conceptual summary of the thesis. Models are in grey, contributions are in blue, advanced perspectives are in green, further studies are in orange.

1.5 Introduction générale

1.5.1 Cadre de travail pour des réponses mathématiques et méthodologiques

Cette thèse aborde le sujet de l'optimisation des systèmes d'irrigation des cultures en agronomie, à l'échelle de la parcelle, sous serre ou en champ extérieur, dans un contexte de ressources en eau limitées. Ce contexte spécifique fait jouer un rôle crucial au pilotage tactique de l'irrigation et aux prises de décisions de l'opérateur au cours du temps. En utilisant une formalisation de l'irrigation des cultures en terme de problème de commande optimale, l'objectif est de développer des outils théoriques afin de fournir des stratégies de décision efficaces pouvant se transposer au monde opérationnel.

1.5.1.1 Positionnement de la recherche

Parmi les approches existantes impliquant la théorie du contrôle pour l'étude des problèmes d'irrigation des cultures, nous choisissons de nous concentrer sur un cadre de contrôle optimal, afin de tirer parti de la possibilité de dériver des conditions nécessaires pour rechercher des solutions optimales. De plus, les formulations en contrôle optimal permettent de mener une optimisation qui dépasse le cadre local (ou sur une courte fenêtre temporelle) en intégrant des contraintes et des conditions globales qui s'appliquent à l'ensemble de l'horizon temporel. Hormis les travaux numériques (voir [108, 153, 95, 110]) dont l'étude d'optimisation dépend de certaines restrictions du modèle, ou bien repose sur des scénarios en boucle ouverte, il existe plusieurs travaux visant à une utilisation plus analytique de la théorie du contrôle optimal. Ils consistent généralement à définir un système dynamique représentant leur modèle d'irrigation des cultures, puis à introduire certaines hypothèses, et à optimiser une fonction de coût (qui est de type linéaire dans [157, 112, 19, 101] et de type quadratique dans [113, 102]). Typiquement, ces études consistent à d'écrire et dériver les conditions nécessaires, afin d'obtenir au moins des propriétés sur les contrôles optimaux pouvant être explorées via des méthodes directes ou indirectes, au mieux directement l'expression des contrôles optimaux. De plus, ces structures de stratégies sont généralement valables pour une large classe de conditions opératoires et de valeurs de paramètres [177, 75]. Les travaux théoriques reposent sur des dynamiques simples (plutôt abstraites comme dans [112, 101] ou plus physiques comme dans [157, 19]), peuvent contenir une contrainte d'état supplémentaire (comme dans [113]), peuvent contenir des limitations sur l'usage de l'eau ([19]) ou supposer une disponibilité complète de l'eau ([112, 101]), et peuvent ([112, 101]) ou non ([157, 19]) prendre en compte la pluie. L'étude [102] adopte un point de vue différent et aborde la question de la robustesse, en évaluant la sensibilité de la solution optimale par rapport à un paramètre mal connu, à travers une formulation en programmation non linéaire.

Le travail antérieur de [19] semble constituer un bon compromis entre simplicité du modèle, flexibilité, et pertinence physique explicite. Ses principales limitations résident dans l'absence de prise en compte des niveaux critiques d'humidité du sol, et dans l'absence des contributions des précipitations et du drainage. Ce travail utilise le Principe du Maximum de Pontryagin dans un cadre non lisse pour montrer, sous certaines conditions basées sur un modèle d'irrigation simplifié ([91]), qu'il existe une structure non triviale (mais paramétrée en faible dimension) de stratégies d'irrigation (ou de contrôles) qui maximisent la production de biomasse en situation de rareté de l'eau. L'originalité de ces structures réside dans l'apparition d'"arcs singuliers" précisément à certains seuils d'humidité du sol. Cependant, cette étude reste incomplète puisqu'elle ne prend pas en compte les apports en eau issus de précipitations inconnues. De plus, les stratégies dérivées présentent un problème majeur : dans certaines conditions, il est possible que la solution optimale calculée implique de subir des niveaux d'humidité du sol critiques pouvant entraîner la mort de la culture. Une étude suivante ([72]) a traité le problème avec prise en compte des précipitations, en utilisant l'outil numérique d'optimisation Bocop ([173]) basé sur une méthode directe (utilisant la librairie Ipopt), ouvrant de nouvelles perspectives pour le développement de solutions au problème avec précipitations et prévisions météorologiques.

Cependant, les solutions trouvées dans ces études ne sont pas directement applicables à des situations opérationnelles, soit en raison de l'absence d'une formulation opérationnelle suffisamment

complexe, soit parce que leur optimisation repose sur une résolution numérique externe via une méthode directe. Pour ces raisons, cette thèse intègre une approche de double modélisation (voir [73]) détaillée au Chapter 5, consistant à établir une correspondance (en un certain sens) entre un petit modèle d'irrigation et le modèle Optirrig ([34, 33]) qui a été développé en vue d'applications opérationnelles. L'approche de double modélisation tire parti à la fois d'un modèle théorique et d'un modèle physique détaillé pour dériver des lois de commande efficaces pour un système complexe. La double modélisation a rarement été envisagée avec pour objectif principal la conception de stratégies de contrôle, mais plutôt comme un moyen de conduire des analyses sur un modèle réduit calibré afin d'améliorer les performances d'un modèle complexe. Dans cette thèse, nous cherchons à développer des stratégies d'irrigation sur un petit modèle théorique et à évaluer leur efficacité sur le modèle Optirrig. Notons cependant que le modèle théorique ne sera pas une version réduite d'Optirrig, rendant leur correspondance un véritable enjeu.

1.5.1.2 Cadre de la double modélisation et défis

L'approche de double modélisation consiste à utiliser deux modèles, pouvant être couplés, séquentiels ou en interaction, afin de capturer différents aspects d'un système complexe, en tirant parti des forces respectives de chaque modèle. Cette méthode répond à la nécessité de dépasser les limitations des études fondées sur un modèle unique, en intégrant deux points de vue sur le sujet étudié, qui peuvent correspondre à des échelles, des processus ou des domaines d'application différents. Par exemple, dans le domaine de l'irrigation des cultures, on peut envisager des modèles portant sur les aspects biophysiques, économiques ou de contrôle, pour une gestion à l'échelle de la plante, de la parcelle ou du bassin versant, et décrivant les processus de manière plus détaillée ou plus abstraite. La méthodologie habituelle consiste à identifier la compatibilité des deux modèles, à les calibrer, puis à effectuer des simulations intégrées. Cette méthode de double modélisation présente des caractéristiques intéressantes, telles qu'un meilleur réalisme, une modularité et une flexibilité améliorées, et constitue un outil prometteur pour le développement et l'amélioration de systèmes d'aide à la décision. Elle peut toutefois faire face à plusieurs difficultés, comme une incompatibilité entre les modèles, une complexité de calibration, ou encore une augmentation des coûts de simulation.

Dans cette thèse, le défi de la double modélisation consiste à établir la connexion entre les approches analytiques formelles du contrôle optimal, représentées par le modèle développé à INRAE et nommé Controlled Crop Irrigation (voir Section 1.5.1.3), et une approche plus fondée sur la physique dans le domaine de la gestion de l'eau, représentée par le modèle Optirrig. L'objectif est de tirer parti des deux types d'approches pour établir des lois de commande efficaces pour un problème d'irrigation de cultures relativement complexe, permettant potentiellement de mettre en évidence et de comprendre l'émergence de stratégies de décision (anticipées ou non). Cette connexion entre modèles et la déclinaison progressive des résultats (voir Figure 1.10) entre champs disciplinaires s'accompagne de la nécessité de conserver un objectif unique et global : il s'agit d'une même méthode ou d'un même cadre méthodologique, d'abord établi sur des bases théoriques solides, puis illustré dans des conditions suffisamment contrôlées, avec enfin l'objectif d'être appliqué légèrement en dehors de son domaine de validité.

En conséquence, ce projet de thèse aborde le sujet de l'optimisation des stratégies d'irrigation dans un contexte de changement climatique et de limitations des ressources en eau, ainsi qu'à leur application dans le modèle Optirrig. La solution proposée doit aboutir à une décision : en temps réel, intégrant des contraintes de gestion saisonnière comme des quotas d'eau, et tenant compte des conditions météorologiques à travers des incertitudes de moyen et long termes. L'approche de double modélisation permet d'identifier les possibilités intrinsèques du système d'irrigation des cultures à atteindre certains objectifs tout en respectant certaines contraintes données (voir [4]). Il est raisonnable d'appliquer cette méthodologie depuis une formulation en problème de contrôle (le modèle CCI) vers Optirrig, puisque ce dernier a été conçu en mettant l'accent sur les scénarios d'irrigation (ou stratégies) plutôt que sur une description physiologique détaillée. Il convient toutefois de noter que la compatibilité entre le modèle CCI et le modèle Optirrig reste à démontrer avant d'envisager l'application de stratégies issues du CCI dans Optirrig. En outre, il faut souligner que les stratégies optimales pour le modèle simplifié sont sous-optimales pour le modèle Optirrig, mais elles peuvent être comparées aux stratégies existantes afin d'évaluer l'intérêt de l'approche. Une telle association entre théorie du contrôle optimal et modèle numérique complexe via la dou-

ble modélisation est originale, puisque la double modélisation est une méthodologie reconnue mais encore très peu appliquée en agronomie et en sciences de l'eau.

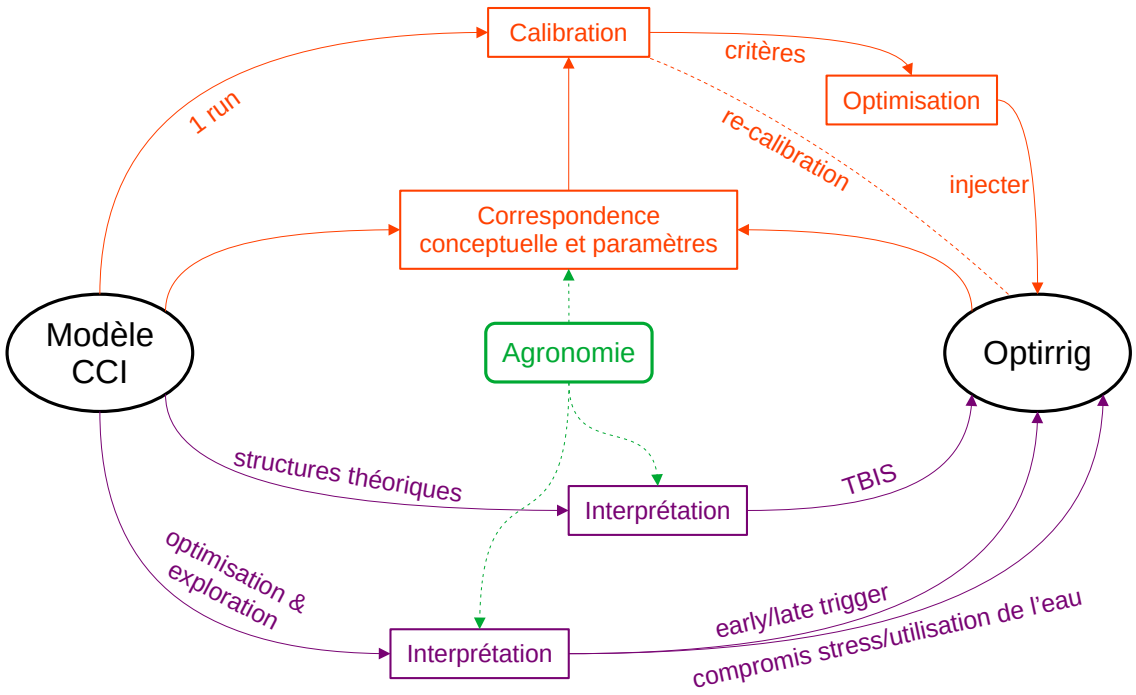


Figure 1.10: Schéma conceptuel de la double modélisation entre le modèle CCI et Optirrig, représentant le processus standard (en orange), les contributions allant de la théorie vers la pratique (en violet, voir Chapitre 4 et Chapitre 5), et l'influence du sens physique dans la boucle (en vert).

1.5.1.3 Présentation du modèle Controlled Crop Irrigation

1.5.1.3.1 Formalismes et dynamique de base

Introduisons les fondements du modèle qui constitue le principal sujet d'analyse de cette thèse, et que nous avons nommé : modèle "Controlled Crop Irrigation" (CCI). Dans cette section, nous présentons ses formalismes et ses hypothèses initiales, et nous détaillerons les éléments complémentaires spécifiques à cette thèse en Section 1.5.2.1. Le modèle CCI est basé sur une formulation initialement présentée dans [133], liée au modèle de simulation agro-hydrologique AquaCrop de la FAO [170]. La formulation de [133] utilise un système d'équations différentielles ordinaires (EDO) pour décrire plusieurs processus biophysiques impliqués dans le développement des cultures, tels que la dynamique de la teneur en eau du sol, l'absorption de l'azote, la couverture végétale, la biomasse et le rendement. Elle se focalise sur la gestion de l'eau, en supposant que l'azote n'est pas un facteur limitant et en omettant sa description. En conséquence, la formulation utilisée est une simplification de celle de [133], en particulier il n'y a que trois variables d'état : l'humidité du sol (S), la production de biomasse (B) et le budget d'eau d'irrigation (V) (voir [91, 19, 28]). Fondamentalement, le modèle CCI est un système dynamique parcimonieux d'EDO, adapté à l'application de la théorie du contrôle optimal tout en contenant les processus biophysiques d'ordre 1 en jeu.

Comme dans [133], les propriétés du sol sont supposées spatialement homogènes avec un espacement uniforme des plantes, ce qui ne nécessite pas de prendre en compte de motif spatial ou d'hétérogénéité au sein de la zone modélisée. Avec des caractéristiques considérées de cette manière spatialement moyennée, le modèle CCI consiste en un petit ensemble d'équations différentielles ordinaires représentant les interactions entre l'humidité du sol, la production de biomasse et la quantité d'eau d'irrigation restante du quota initial, durant la période de croissance des cultures dans la saison culturale, notée par un intervalle de temps $[0, T]$. Le modèle CCI est écrit comme un système dynamique en temps continu, qui peut être compris comme un système évoluant avec une certaine

variation par unité de temps (par exemple par jour), et les paramètres représentant des taux peuvent ainsi être considérés comme des quantités par unité de temps (par exemple par jour). L'équation de base étudiée est la suivante, décrivant le bilan de la teneur en eau du sol :

$$Vol\frac{\Delta S}{\Delta t} = -TR - EV + IR + RA - DR, \tag{1.22}$$

où $Vol\frac{\Delta S}{\Delta t}$ représente la variation de la quantité d'eau ($Vol\Delta S$) sur un intervalle de temps (Δt) et dans un réservoir de sol de taille Vol (rappelant que l'homogénéité spatiale permet de considérer cette quantité normalisée par mètre carré). Les notations TR, EV, IR, RA et DR désignent respectivement les quantités d'eau de transpiration des plantes, d'évaporation du sol, d'irrigation, de précipitation et de drainage, c'est-à-dire les variations au cours de l'intervalle de temps Δt.

En suivant [91] et [73], nous détaillons à présent l'expression de chaque terme.

La transpiration des plantes TR est calculée comme l'évapotranspiration de référence ET_0 (ou demande climatique) multipliée par le coefficient K_S relié au stress hydrique (ou efficacité de transpiration) et par le pourcentage de couverture végétale φ (ou efficacité d'interception du rayonnement, voir [11]). Étant donné que l'efficacité de transpiration dépend de l'humidité du sol à un instant donné, et en considérant que les quantités varient dans le temps, on peut écrire le taux instantané de transpiration des plantes comme :

$$TR(t) = ET_0(t)K_S(S(t))\varphi(t). \tag{1.23}$$

L'évaporation du sol EV est calculée comme l'évapotranspiration de référence ET_0 (ou demande climatique) multipliée par le coefficient K_R relié à la teneur en eau du sol (ou efficacité d'évaporation) et par le pourcentage de sol non couvert par la couverture végétale $1-\varphi$ (ou rayonnement non intercepté). Étant donné que l'efficacité d'évaporation dépend de l'humidité du sol à un instant donné, et en considérant que les quantités varient dans le temps, on peut écrire le taux instantané d'évaporation du sol comme :

$$EV(t) = ET_0(t)K_R(S(t))(1-\varphi(t)). \tag{1.24}$$

Le taux d'irrigation est pris comme le taux d'irrigation maximal F_{max} (ou capacité du matériel) multiplié par le pourcentage d'ouverture de la vanne d'irrigation u (ou irrigation normalisée). En considérant que la quantité u dépend du temps, on peut écrire le taux instantané d'irrigation comme :

$$IR(t) = F_{max}u(t). \tag{1.25}$$

La variation de l'eau du sol induite par les précipitations RA est calculée comme la quantité de précipitations r (par mètre carré) multipliée par le coefficient α (ou efficacité d'infiltration des précipitations). En considérant que la quantité r dépend du temps, on peut écrire la variation instantanée induite par les précipitations comme :

$$RA(t) = \alpha r(t). \tag{1.26}$$

Le drainage DR est calculé comme le taux de drainage maximal k_{sat} (ou pertes en eau par saturation) multiplié par le coefficient K_D (ou efficacité de drainage). Étant donné que l'efficacité de drainage dépend de l'humidité du sol à un instant donné, et en considérant que les quantités varient dans le temps, on peut écrire le taux instantané de drainage comme :

$$DR(t) = k_{sat}K_D(S(t)). \tag{1.27}$$

Toutes les quantités $K_S(S(t))$, $K_R(S(t))$, $K_D(S(t))$, $\varphi(t)$, $u(t)$, α sont définies comme des efficacités (ou pourcentages), et appartiennent donc à l'intervalle $[0,1]$. Les fonctions culturales K_S, K_R et K_D sont des fonctions affines par morceaux et continues de l'humidité du sol S, et sont paramétrées par des valeurs caractéristiques de la teneur en eau du sol [3, 73], dans l'ordre croissant : S_{rw} la teneur en eau résiduelle, S_{pw} le point de flétrissement permanent, S^* le point de début de stress hydrique, S_{fc} la capacité au champ, S_{sat} le point de saturation du sol. Les profils typiques trouvés dans la littérature sont représentés en Figure 1.11. Dans la suite, nous fixerons toujours S_{fc} égal à 1, car nous interprétons la quantité S comme le rapport entre la quantité d'eau dans le

système et l'espace disponible dans le sol que l'eau peut remplir. En conséquence, les situations où S serait supérieur à 1 correspondent à un sur-remplissage, ce qui signifie qu'il y a trop d'eau par rapport à la capacité du sol à l'assimiler, générant ainsi du drainage.

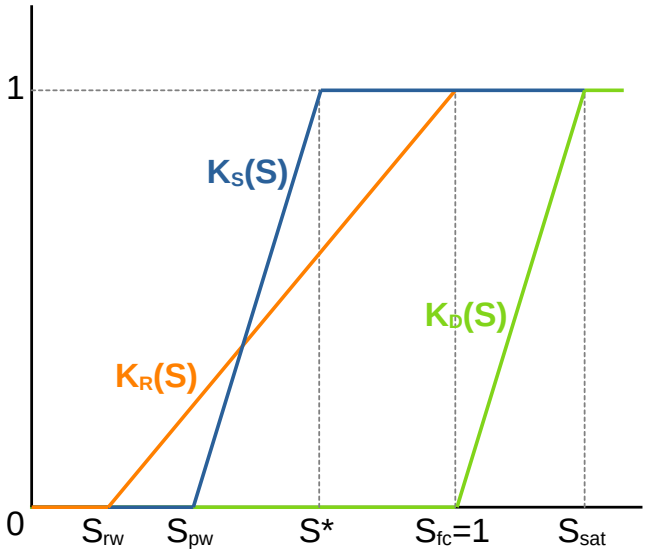


Figure 1.11: Forme des coefficients culturaux de la transpiration des plantes K_S (en bleu), de l'évaporation du sol K_R (en orange) et du drainage K_D (en vert).

La quantité ET_0 varie dans le temps, en fonction des conditions climatiques, et oscille globalement dans une plage pas trop large compte tenu de la durée de l'intervalle de temps. Par souci de simplification (et à nouveau dans un objectif d'homogénéisation), le paramètre ET_0 sera considéré comme une constante dans le temps, vraisemblablement proche de la moyenne de ses valeurs réelles.

L'Equation (1.22) peut maintenant être réécrite sous forme d'une EDO (en faisant tendre la durée de temps Δt vers 0), en utilisant tous les détails de modélisation supplémentaires ci-dessus :

$$\dot{S}(t) = k_1 \left(-\varphi(t) K_S(S(t)) - (1-\varphi(t)) K_R(S(t)) + k_2 u(t) - \frac{k_{sat}}{ET_0} K_D(S(t)) + \frac{\alpha}{ET_0} r(t) \right), \quad (1.28)$$

où $k_1 = \frac{ET_0}{Vol}$ et $k_2 = \frac{F_{max}}{ET_0}$.

La variation de la biomasse est donnée par l'expression suivante, en supposant que la production de biomasse est proportionnelle au taux de transpiration, comme suit :

$$\dot{B}(t) = k_3 \varphi(t) K_S(S(t)), \quad (1.29)$$

où k_3 est un taux potentiel de croissance de la biomasse (ou productivité de l'eau), qui joue le rôle de constante de normalisation (et il peut être pris égal à 1 sans perte de généralité).

Enfin, l'évolution de la consommation de l'eau d'irrigation est décrite par l'équation suivante :

$$\dot{V}(t) = F_{max} u(t), \quad (1.30)$$

où le coefficient F_{max} peut être omis dans Equation (1.30) sans perte de généralité.

Le système de Equations (1.28) to (1.30) est supposé évoluer sur une période de temps fixée $[0, T]$ correspondant à la phase de croissance de la culture. La variable de décision est le débit d'irrigation u, que nous appellerons aussi variable de contrôle. Ce contrôle est une fonction mesurable définie sur $[0, T]$ et prenant ses valeurs dans $[0, 1]$.

1.5.1.3.2 Hypothèses de modèle

Précisons tout d'abord les conditions imposées sur les différents paramètres liés aux processus biophysiques, dont les informations sont regroupées dans les constantes intervenant dans la formulation de Equations (1.28) to (1.30). En effet, toutes les quantités ET_0, Vol, F_{max}, k_{sat}, α et k_3 sont par définition strictement positives, et dans un contexte opérationnel usuel, la capacité du matériel d'irrigation est vraisemblablement supérieure à la demande climatique. D'où l'hypothèse suivante.

Assumption 1.5.1. *Les constantes k_1, k_2, k_{sat}, α et k_3 vérifient*

$$k_1 > 0\,, \quad k_2 > 1\,, \quad k_{sat} > 0\,, \quad \alpha > 0\,, \quad k_3 > 0.$$

L'efficacité d'interception du rayonnement φ est une fonction strictement croissante du temps et prenant ses valeurs dans $[0, 1]$, liée à la croissance du couvert végétal au cours de la saison, donnant ainsi davantage de poids à l'évaporation en début de saison puis à la transpiration plus tard dans la saison. Des valeurs croissantes de φ signifient également qu'il n'y a pas d'intervention humaine sur la canopée, comme par exemple la coupe de branches ou de feuilles des plantes pendant la phase de croissance. Des hypothèses fiables sur la forme de φ seront formulées plus tard lors des applications numériques.

Assumption 1.5.2. *La fonction $\varphi : [0, T] \to [0, 1]$ est strictement croissante et de classe C^1.*

Les coefficients K_S, K_R et K_D suivent des lois empiriques issues de données expérimentales (voir [3]), et nous ne considérons que des hypothèses structurelles peu contraignantes afin de préserver une certaine flexibilité.

Assumption 1.5.3. *Les fonctions K_S, K_R et K_D sont définies pour $S \geq 0$, et sont croissantes, affines par morceaux, continues. Typiquement, K_S, K_R et K_D sont rencontrées dans la littérature ([19, 133, 73]) par*

$$K_S(S) = \begin{cases} 0, & S \in [0, S_{pw}), \\ \frac{S - S_{pw}}{S^* - S_{pw}}, & S \in [S_{pw}, S^*), \\ 1, & S \geq S^*; \end{cases} \tag{1.31a}$$

$$K_R(S) = \begin{cases} 0, & S \in [0, S_{rw}), \\ \frac{S - S_{rw}}{1 - S_{rw}}, & S \geq S_{rw}; \end{cases} \tag{1.31b}$$

$$K_D(S) = \begin{cases} 0, & S \in [0, 1), \\ \frac{S-1}{S_{sat}-1}, & S \in [1, S_{sat}), \\ 1, & S \geq S_{sat}. \end{cases} \tag{1.31c}$$

Remark 1.5.1. *Il est possible de considérer des expressions plus générales que les Equations* (1.31a) *and* (1.31b) *(avec un plus grand nombre de points non différentiables par exemple), tant qu'elles restent croissantes, affines par morceaux, continues, et que K_S est concave sur $\{S \geq S_{pw}\}$ et K_R convexe sur $\{S \geq 0\}$. Remarquons que K_S, K_R et K_D sont lipschitziennes, et notons $L_S > 0$, $L_R > 0$ et $L_D > 0$ leurs constantes de Lipschitz respectives.*

Au début de la saison culturale, on suppose que le sol a été partiellement rechargé durant la période interculturale, et l'humidité initiale du sol est alors supposée suffisamment élevée (c'est-à-dire sans stress hydrique au temps initial). De plus, les teneurs en eau du sol supérieures à la capacité au champ $S = 1$ ne sont pas des points d'équilibre et le système tend naturellement à évacuer l'excès d'eau. En conséquence, comme dans [19], nous considérons les conditions initiales suivantes.

Assumption 1.5.4. *Le système a les conditions initiales suivantes :*

$$S(0) = S_0 \in (S^*, 1]\,, \quad B(0) = 0\,, \quad V(0) = 0.$$

Remark 1.5.2. *La condition $k_2 > 1$ garantit la contrôlabilité locale de la variable S (pour $S \in [0, 1]$), au sens où il est possible de sélectionner des contrôles afin de choisir le signe de $\dot{S}$. De plus, l'irrigation qui maintient l'humidité du sol constante à une valeur fixée $\widetilde{S} \in [0, 1]$ est un contrôle localement admissible.*

Definition 1.5.1. Selon Assumption 1.5.1, et avec $\widetilde{S} \in [0,1]$, le contrôle défini par

$$u_{sing}(t,\widetilde{S}) = \frac{\varphi(t)K_S(\widetilde{S}) + (1-\varphi(t))K_R(\widetilde{S})}{k_2}$$

appartient à $[0,1]$.

En suivant les dynamiques présentées dans la section précédente, ainsi que les hypothèses introduites dans cette section, nous visons à étudier certaines formulations de problèmes de contrôle optimal qui seront détaillées ultérieurement. Ces formulations pourront contenir des conditions ou des contraintes supplémentaires selon le contexte opérationnel, et l'étude devra composer avec toutes les difficultés qui en résultent.

1.5.1.4 Implications, difficultés et cadre théorique

Comme discuté précédemment, cette thèse se concentre sur l'étude de plusieurs problèmes d'optimisa-tion formulés comme des problèmes de contrôle optimal sous contraintes. Plus précisément, les problèmes impliquent un système dynamique (le modèle CCI) sur un intervalle de temps à horizon fixé T, divers critères formulés comme des coûts terminaux, différentes cibles terminales, et certaines contraintes d'état (détaillées ultérieurement). La dynamique est non linéaire, non autonome et non lisse par rapport à l'état, tout en étant affine par rapport au contrôle u. De tels problèmes de contrôle optimal sont en général non triviaux à traiter et nécessitent une analyse rigoureuse et approfondie [36, 104, 35, 190, 75]. Notre cas d'étude présente toutefois une particularité, puisque la dynamique est régionalement affine (et continue) par rapport à l'état, et que nous considérerons des contraintes d'état. En conséquence, le caractère non-lisse de la dynamique réside dans un ensemble discret (les points de raccordement des morceaux affines), c'est-à-dire un nombre fini de points où la dynamique n'est pas différentiable. Partout ailleurs, c'est-à-dire sur un ensemble dense de l'espace d'état, la dynamique est localement affine.

Dans le cas de problèmes de contrôle optimal globalement linéaires (ou affines), il existe une littérature couvrant un très large éventail de thématiques et une grande variété de résultats. En particulier, pour des problèmes de temps minimal reposant sur une dynamique linéaire (ou affine), il existe un résultat classique concernant les propriétés structurelles des solutions optimales. En effet, il est bien connu (voir par exemple [79]) qu'il existe un contrôle optimal prenant ses valeurs dans l'ensemble des contrôles extrêmes, pour un ensemble compact convexe de contrôles (ce résultat est appelé le "principe bang-bang"). Puisque notre sujet d'étude implique une dynamique qui n'est pas linéaire (ni affine), il n'est pas possible d'énoncer un tel résultat. Toutefois, comme notre système dynamique est affine par morceaux, il peut sembler intuitif que l'on soit en mesure d'extraire certaines informations structurelles sur les solutions optimales en s'appuyant sur la structure de la dynamique et sur le principe bang-bang classique.

Notons que notre problème d'irrigation sera formulé avec des contraintes d'état intrinsèques qui seront détaillées en Section 1.5.2.1. Il est bien établi que les problèmes de contrôle optimal avec contrainte d'état sont loin d'être simples à résoudre, et présentent la particularité de générer des discontinuités dans le vecteur adjoint. Divers travaux (voir [122, 78, 87, 152, 174]) proposent des résultats et des techniques pour traiter ces difficultés, sous différents types d'hypothèses (telles que l'ordre de la contrainte d'état, les conditions de raccordement, l'augmentation d'état, la régularisation). L'idée d'ajouter une variable d'état (dont la dynamique rend compte de la violation de la contrainte d'état), et de remplacer la contrainte d'état courante par une cible terminale, présente l'intérêt prometteur de rendre le vecteur adjoint continu. Toutefois, on peut observer (du fait de l'ajout d'une composante adjointe) une situation correspondant à la dégénérescence de la formulation précédente, ce qui ne fournit aucune information utile lors de la dérivation des conditions nécessaires. Dans cette thèse, nous combinons cette idée avec une pénalisation du budget, qui évite l'ajout d'une variable d'état, mais requiert de modifier la formulation, et donc de prouver qu'il s'agit bien d'un problème équivalent.

L'un des objectifs principaux de cette thèse est de trouver des scénarios d'irrigation en termes de stratégies de contrôle, de préférence formulées comme des contrôles en retour d'état. Grâce à la formulation en contrôle optimal, il est possible d'écrire des conditions nécessaires d'optimalité (en

utilisant le Principe du Maximum de Pontryagin (PMP) non lisse [134, 98]) et de les dériver afin de viser des propriétés constructives des contrôles optimaux. De plus, l'aspect non différentiable de notre dynamique de système apporte une difficulté supplémentaire à l'analyse, puisqu'il sera nécessaire de dériver une version non différentiable du PMP (voir par exemple [36, 35]) en utilisant en particulier la notion de gradient généralisé. Cette méthode peut ou non mener à une description entièrement analytique des solutions optimales, mais elle permet au minimum de fournir certaines propriétés structurelles sur la nature des candidats aux contrôles optimaux. Il est alors habituel de compléter ces propriétés par une résolution numérique, via une méthode directe ou indirecte, pour approcher les solutions optimales. Dans notre étude, nous ne nous contenterons pas de dériver les conditions nécessaires d'optimalité, mais nous viserons à fournir une solution en retour d'état complète sans recourir à une analyse numérique complémentaire.

Ensuite, dans une perspective d'application opérationnelle, il est généralement intéressant non seulement de connaître les propriétés structurelles des politiques d'irrigation optimales, mais aussi de comprendre comment cette structure dépend des composantes physiques de notre système d'irrigation. Afin de fournir de telles informations sur les facteurs influençant certaines sorties du modèle, il est courant d'envisager une exploration numérique de modèle, à condition de disposer d'informations suffisantes sur les plages et les distributions des valeurs des paramètres du modèle. Une méthode simple et peu coûteuse pour explorer le modèle consiste à effectuer des perturbations de paramètres selon la méthode du "un facteur à la fois" (voir par exemple [127]). Cependant, cette méthode n'est pas adaptée aux modèles non linéaires et ne permet pas de détecter les effets issus d'interactions entre paramètres. Il est donc nécessaire d'envisager des méthodes plus coûteuses, telles que l'analyse de sensibilité globale [41, 166, 57], à travers une exploration numérique globale et des méthodes de calcul basées sur la variance [88, 150]. Cette méthode est adaptée à notre étude, mais exigera un nombre important de simulations sur des plages suffisamment larges de paramètres afin d'en extraire des résultats significatifs. Par ailleurs, il est envisageable de profiter de cette exploration numérique pour mener une analyse complémentaire sur les effets des perturbations de décision sur les sorties du modèle, dans le cadre de l'exploration globale. Le principal défi à relever pour de telles analyses numériques est le coût de simulations.

Dans le contexte des conditions de plein champ, la dynamique intègre un terme non déterministe pour prendre en compte le comportement probabiliste des séquences de précipitations. Il existe plusieurs façons de considérer et modéliser de tels phénomènes non déterministes, allant de formulations totalement stochastiques à des lois de probabilité paramétrées ou à l'utilisation de processus markoviens. Dans cette thèse, le choix s'est porté sur un formalisme probabiliste, considérant que chaque séquence annuelle de précipitations est représentée comme une fonction dépendant du temps, définie sur l'intervalle de la saison culturale $[0, T]$. À chaque instant t, elle doit représenter le taux de précipitation courant. Puisque la séquence annuelle de précipitations n'est pas connue a priori, nous considérons un ensemble fini de toutes les séquences possibles de précipitations annuelles, et nous supposons que chaque séquence possède une probabilité connue d'apparaître effectivement. La formulation du problème de contrôle optimal passe alors d'une unique équation d'humidité du sol contrôlée, à un ensemble (fini dans notre cas) d'équations d'humidité du sol contrôlées, chacune correspondant à une séquence de précipitations différente. Toutefois, le contrôle doit être commun à tous ces scénarios et représente l'approche générique, c'est-à-dire une décision capable de prendre en compte n'importe quel futur possible. Le critère à optimiser devient alors une moyenne des coûts individuels associés à chaque équation, pondérée par la probabilité correspondante que chaque séquence se produise. Une telle formulation de problème est appelée problème de "contrôle à coût moyenné" ("averaged cost control") [12, 13], et fournit un cadre permettant d'appliquer des conditions nécessaires d'optimalité similaires à celles du cas à une seule équation, la difficulté supplémentaire résidant évidemment dans l'analyse des conséquences issues de la maximisation du Hamiltonien pondéré.

Enfin, il a été mentionné que les séquences de précipitations annuelles seront représentées par des "entrées" variables dans le temps. En réalité, plusieurs possibilités existent pour choisir le type d'entrées que l'on pourrait utiliser. Dans notre travail, nous avons choisi de considérer que les événements de précipitation peuvent se produire à différentes échelles de temps, tandis que le processus physique d'infiltration et d'assimilation par le sol peut être considéré comme un pro-

cessus lent. En conséquence, en tenant compte de la nature des précipitations pouvant survenir notamment dans les régions arides et semi-arides (taux relativement élevé sur une période courte), il est raisonnable de supposer que les événements de précipitations peuvent être modélisés comme instantanés. Cela conduit à représenter une séquence annuelle possible de précipitations par un ensemble (fini) d'instants (dates des événements de pluie t_i) et leur amplitude associée (h_i), ce qui autorise la variable d'état représentant l'humidité du sol de présenter des sauts (ou discontinuités) aux instants des précipitations instantanées. La dynamique du système sera alors exprimée sous forme d'une équation différentielle complétée par des sauts d'état, et des extensions d'outils théoriques seront nécessaires pour mener à bien l'analyse mentionnée. À cette fin, nous renvoyons à [160, 161, 139, 124, 44, 191, 198] qui décrivent comment traiter les systèmes dynamiques à discontinuités d'état, en particulier comment les transformer en un système dynamique standard (par un changement de variable de temps), afin d'appliquer et dériver des conditions nécessaires d'optimalité classiques.

1.5.2 Contributions à l'optimisation de l'irrigation

1.5.2.1 Contribution au modèle CCI et à la formulation du problème d'irrigation

Tout d'abord, afin d'éviter des niveaux critiques de teneur en eau du sol pouvant entraîner une dégradation sévère des plantes ou une sénescence précoce (voire la mort de la culture), nous avons ajouté à la formulation initiale du modèle CCI une contrainte d'état sur l'humidité du sol représentant l'aversion ou l'inclination au risque de l'opérateur vis-à-vis de la tolérance au stress hydrique. Plus précisément, le système est soumis à la contrainte d'état suivante :

$$\forall t \in [0,T],\ S_{ope} \leq S(t), \tag{1.32}$$

où S_{ope} est appelé la contrainte opératoire (ou seuil de tolérance), et vérifie $S_{pw} < S_{ope} < S^*$.

1.5.2.1.1 Contextes opérationels

La dynamique Equation (1.28) possède un terme non déterministe en raison des incertitudes liées aux précipitations. Cependant, il est possible de considérer des pratiques agricoles dans lesquelles les précipitations n'interviennent pas, et où les variations des conditions météorologiques ont un faible impact sur le développement des cultures. En effet, les serres peuvent même permettre de tendre vers un système d'irrigation des cultures totalement contrôlé, à condition que l'opérateur soit capable de maintenir une température stable, ainsi que des conditions homogènes de vent et d'humidité de l'air. Dans un tel cadre de conditions contrôlées sous serre, l'utilisation d'une dynamique déterministe est justifiée, et l'on peut alors simplement retirer le terme des précipitations sans autre considération.

Une conséquence implicite des systèmes d'irrigation sous serre totalement contrôlés est que l'opérateur est capable de commander l'irrigation avec une précision relativement élevée (grâce, par exemple, à un suivi par capteurs d'humidité). Par conséquent, l'opérateur évitera vraisemblablement de gaspiller de l'eau, c'est-à-dire qu'il ne devrait pas atteindre une situation où l'humidité du sol dépasserait $S = 1$ (ce qui générerait des pertes en eau par drainage). Ainsi, la contrainte d'état Equation (1.32) dans le contexte sous serre devient

$$\forall t \in [0,T],\ S_{ope} \leq S(t) \leq 1. \tag{1.33}$$

De plus, comme mentionné ci-dessus, satisfaire la nouvelle contrainte d'état Equation (1.33) revient à considérer que le drainage n'intervient plus dans la dynamique de la teneur en eau du sol. Par conséquent, l'équation régissant les variations de l'humidité du sol dans le contexte sous serre contrôlé devient

$$\dot{S}(t) = k_1\big(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t)\big). \tag{1.34}$$

Enfin, ce contexte sera étudié en détail dans Chapter 3 et Chapter 4, où la formulation déterministe permet d'établir des conditions nécessaires d'optimalité issues du principe du maximum de Pontryagin, menant à la conception d'une classe de structures optimales de solutions capturant certaines propriétés intrinsèques de la dynamique des cultures. De cette façon, nous pourrons extraire

de l'information du système interne "sol-plante" sans perturbation de l'environnement, ce qui justifie l'intérêt d'une telle formulation déterministe.

Il est également possible de conserver la dynamique non déterministe de Equation (1.28), ce qui signifie que l'on accepte de prendre en compte les effets des précipitations sur le système étudié. Cela implique de placer notre étude dans un cadre où des événements pluvieux se produisent, c'est-à-dire dans des conditions de plein champ. En outre, de tels événements météorologiques sont, par nature, inconnus à l'avance et présentent une grande variabilité, pouvant ainsi avoir un impact significatif sur le bilan instantané de l'humidité du sol. En particulier, un épisode de pluie intense, impliquant de grandes quantités d'eau en un court laps de temps, peut saturer le réservoir du sol et générer un drainage non négligeable, qui augmente jusqu'à un niveau de saturation au-delà de la capacité au champ $S = 1$.

En conséquence, l'utilisation conjointe des termes de précipitations et de drainage dans Equation (1.28) est bien justifiée dans des conditions de plein champ. De plus, même si l'opérateur a un contrôle total sur les quantités d'irrigation et leur précision, il ne peut pas empêcher que l'humidité du sol atteigne des niveaux supérieurs à la capacité au champ, ce qui conduit à des pertes en eau indésirables mais inévitables. La contrainte d'état $S(t) \leq 1$ est donc supprimée, et seule Equation (1.32) suffit à refléter la marge d'aversion (ou d'inclination) au risque de l'opérateur.

Ainsi, considérer une étude dans des conditions de plein champ présente l'avantage de se rapprocher de situations opérationnelles réelles, car supposer une irrigation totalement contrôlée peut être vu comme une hypothèse un peu trop forte, en raison de la variabilité existante dans les processus biophysiques et dans la culture elle-même. Par conséquent, l'introduction d'incertitude dans l'analyse constitue un argument important pour affirmer que l'on cherche désormais des stratégies d'irrigation présentant une certaine robustesse vis-à-vis des perturbations. En outre, prendre en compte un futur inconnu signifie implicitement que les stratégies conçues seront génériques, capables de s'adapter à tout événement incertain à venir, et que la stratégie repose sur une méthodologie systématique. Cela peut s'avérer particulièrement intéressant en tant que soutien aux politiques publiques dans les décisions opérationnelles.

Enfin, ce contexte constitue le principal sujet d'étude dans le Chapter 5 et le Chapter 6. Plus précisément, dans Chapter 5, nous explorerons la possibilité d'évaluer la performance des stratégies candidates "solutions optimales sous serre" dans des situations proches des conditions réelles, et de conclure sur leur éventuelle robustesse face à l'incertitude. Ensuite, Chapter 6 introduit le cadre analytique visant à dériver des conditions nécessaires d'optimalité de manière similaire au contexte déterministe "sous serre", mais présentant des défis bien plus complexes, principalement dus au fait que les trajectoires d'humidité du sol associées à toutes les séquences possibles de précipitations doivent être contrôlées par une même décision uniforme.

1.5.2.1.2 Cadre de travail de l'irrigation déficitaire

Remark 1.5.3. *Remarquons que le pire scénario possible dans le contexte de plein champ serait celui où aucune précipitation ne se produirait pendant toute la saison, ce qui coïnciderait formellement avec un contexte sous serre (du moins dans la manière dont nous simulons les conditions climatiques, via ET_0 uniquement). Il ne s'agit que d'une comparaison formelle et cela ne signifie pas que les cultures sous serre seront moins performantes, puisque la serre protège en réalité la culture des effets climatiques, si bien que certains paramètres pourraient différer de ceux des conditions de plein champ. Cela permet néanmoins de définir un cadre uniforme pour représenter la situation où les ressources en eau sont limitées, comme c'est le cas dans les zones arides et semi-arides déjà mentionnées.*

Dans le contexte sous serre comme dans le pire scénario en plein champ, l'humidité du sol décroît en l'absence d'irrigation (c'est-à-dire lorsque $u = 0$). De plus, d'après la dynamique de production de biomasse, la culture ne subit aucun stress (et donc la production est maximale) lorsque l'humidité du sol est supérieure au seuil S^*. Comme mentionné précédemment, notre étude portera sur des

situations où la disponibilité totale en eau ne peut pas être satisfaite ; en d'autres termes, les limitations des ressources en eau doivent engendrer un certain niveau de stress hydrique. Définissons alors la chronique d'irrigation la moins consommatrice telle que la trajectoire d'humidité du sol soit suffisante pour éviter tout stress, et notons cette irrigation u^*. Étant donné que la condition initiale d'humidité du sol est $S_0 \in (S^*, 1]$, la chronique u^* consiste à ne pas irriguer jusqu'à ce que la trajectoire (décroissante) d'humidité atteigne le niveau S^*, puis à irriguer de manière à maintenir l'humidité du sol constante au niveau S^*. Autrement dit, il existe $t^* \in (0, T]$ tel que

$$u^*(t) = \begin{cases} 0, & t \in [0, t^*), \\ u_{sing}(t, S^*), & t \in [t^*, T], \end{cases} \tag{1.35}$$

ce qui est possible puisque $S^* \in [0, 1]$.

Cela nous permet d'introduire la valeur limite d'eau d'irrigation qui permet d'éviter le stress sur toute la durée de la saison culturale.

Definition 1.5.2. En utilisant le contrôle u^* défini ci-dessus, la consommation d'eau associée est

$$V^* = \int_0^T u^*(t)dt.$$

Par conséquent, le cadre de l'irrigation déficitaire peut être représenté par une condition supplémentaire sur la quantité d'eau d'irrigation disponible, que l'on peut écrire

$$\text{“}u \text{ relève du cadre de l'irrigation déficitaire}'' \quad \Leftrightarrow \quad \int_0^T u(t)dt < V^*.$$

De plus, comme mentionné précédemment, l'opérateur doit choisir un seuil d'humidité du sol (S_{ope}) en dessous duquel il considère que la situation devient trop risquée pour la santé de la culture. Définissons alors la chronique d'irrigation la moins consommatrice telle que la trajectoire d'humidité du sol soit suffisante pour que cette condition opératoire soit satisfaite, et notons-la u_{ope}. De manière similaire à ce qui précède, il existe $t_{ope} \in (0, T]$ tel que la trajectoire d'humidité du sol soit décroissante puis atteigne S_{ope} à l'instant t_{ope}, et reste ensuite constante à S_{ope}, comme suit :

$$u_{ope}(t) = \begin{cases} 0, & t \in [0, t_{ope}), \\ u_{sing}(t, S_{ope}), & t \in [t_{ope}, T], \end{cases} \tag{1.36}$$

ce qui est possible puisque $S_{ope} \in [0, 1]$.

Cela nous permet d'introduire la valeur limite d'eau d'irrigation qui permet d'éviter un stress jugé trop risqué pendant toute la saison culturale.

Definition 1.5.3. En utilisant le contrôle u_{ope} défini ci-dessus, la consommation d'eau associée est

$$V_{ope} = \int_0^T u_{ope}(t)dt.$$

Par conséquent, le contexte opérationnel d'aversion (ou d'inclination) au risque peut être représenté par une condition supplémentaire sur la quantité d'eau d'irrigation disponible, que l'on peut écrire

$$\text{“}u \text{ satisfait le contexte opérationnel d'aversion au risque}'' \quad \Leftrightarrow \quad \int_0^T u(t)dt > V_{ope}.$$

Enfin, nous combinons le cadre de l'irrigation déficitaire et la prise en compte de l'aversion au risque, et nous définissons le cadre de notre étude comme l'ensemble des chroniques d'irrigation (ou contrôles) u qui satisfont les deux conditions :

$$V_{ope} < \int_0^T u(t)dt < V^*. \tag{1.37}$$

1.5.2.1.3 Objectifs et cibles, formulation des problèmes

Introduisons à présent les différents critères et cibles qui pourront être envisagés par la suite, selon différents objectifs agronomiques classiques et les souhaits de l'opérateur.

Le problème d'optimisation de l'irrigation des cultures peut être compris de différentes manières. Comme mentionné précédemment, la problématique classique de gestion de l'irrigation consiste à "utiliser moins d'eau qu'un quota donné, produire plus qu'un rendement donné, tout en assurant un bénéfice économique maximal". Cette problématique peut conduire à différents types d'approches en matière d'optimisation. En particulier, on peut choisir s'il y a des obligations strictes ou une certaine flexibilité sur certaines variables. Comme nous souhaitons appliquer la théorie du contrôle, les obligations strictes seront formulées comme des contraintes mathématiques, et les variables flexibles joueront le rôle du critère à optimiser. Typiquement, on peut avoir une limitation stricte d'accès à l'eau et une liberté sur la production de biomasse (comme dans [19]), ou à l'inverse, un objectif strict sur la biomasse produite et une liberté sur l'usage de l'eau (comme dans [157, 112]). Alternativement, il est possible de ne fixer aucune contrainte ni objectif strict, et de rechercher une optimisation économique sans considération particulière sur les valeurs de consommation d'eau et de production de biomasse. Dans cette thèse, nous traiterons l'ensemble de ces cas d'étude. Nous introduisons alors trois problèmes d'optimisation correspondant à trois points de vue naturels sur les cibles opérationnelles et les critères à optimiser.

- $(P1)$: maximiser la production de biomasse en fin de saison, avec un quota d'eau d'irrigation limité, c'est-à-dire maximiser $B(T)$ sous la condition terminale $V(T) \leq Q$ où Q est l'eau disponible initialement.
- $(P2)$: minimiser la consommation d'eau pour atteindre une cible de biomasse donnée, c'est-à-dire minimiser $V(T)$ sous la condition terminale $B(T) \geq Y$ où Y est l'objectif de production de biomasse en fin de saison.
- $(P3)$: maximiser un équilibre financier entre les recettes de vente de la biomasse et les dépenses liées à la consommation d'eau, c'est-à-dire maximiser la valeur $g(B(T)) - c(V(T))$ où g et c sont des fonctions données représentant les gains et les coûts, avec $g(0) = c(0) = 0$, $g' > 0$ et $c' > 0$.

Ces formulations de problèmes sont en réalité soumises à la question d'"admissibilité". Plus précisément, il faut rechercher des fonctions de contrôle u qui visent à optimiser les critères respectifs, tout en satisfaisant les différentes contraintes (sur les valeurs prises par u, sur les valeurs prises par la trajectoire correspondante de S, sur la condition terminale). Si un contrôle u est capable de satisfaire toutes les conditions mentionnées, alors il sera dit "admissible". Pour chaque problème $(P1)$, $(P2)$ et $(P3)$, et pour chaque déclinaison de contexte, il existe un ensemble de contrôles admissibles. L'optimisation s'effectuera donc uniquement parmi les contrôles admissibles correspondant à chaque cas d'étude. La nature de ces ensembles admissibles sera précisée plus tard dans des cas spécifiques.

En conséquence, nous nous préparons à formuler les problèmes d'irrigation comme les problèmes de contrôle optimal suivants.

Definition 1.5.4. Pour un contexte donné C, et les ensembles admissibles correspondants $\mathcal{A}_1^C$, $\mathcal{A}_2^C$, $\mathcal{A}_3^C$ (qui seront détaillés au Chapter 3), on définit formellement

$$
\begin{aligned}
(P1): &\quad \sup_{u(\cdot)\in\mathcal{A}_1^C} B(T),\\
(P2): &\quad \inf_{u(\cdot)\in\mathcal{A}_2^C} V(T),\\
(P3): &\quad \sup_{u(\cdot)\in\mathcal{A}_3^C} g(B(T)) - c(V(T)).
\end{aligned}
$$

1.5.2.2 Extension du principe bang-bang pour les dynamiques affines par morceaux

Comme mentionné précédemment, la dynamique du système est affine par morceaux (et continue) par rapport à l'état, et affine par rapport au contrôle. Puisque nous cherchons à obtenir des

propriétés structurelles des solutions optimales au problème d'irrigation, il est raisonnable d'étudier le principe "bang-bang" pour les systèmes globalement linéaires (ou affines), et de tenter d'étendre ce résultat, dans une certaine mesure, aux systèmes affines par morceaux sujets à des contraintes d'état. En Section 2.1, nous montrons que pour les problèmes de contrôle optimal non autonomes affines par morceaux avec contraintes d'état, le principe bang-bang ne s'applique pas globalement, même lorsque la dynamique est continue par rapport à la variable d'état. Toutefois, nous montrons qu'il existe des trajectoires optimales générées par des contrôles extrêmes aux endroits où la dynamique est différentiable, ce qui constitue une version étendue du principe bang-bang classique. De plus, nous fournissons deux exemples qui mettent en évidence l'existence potentielle d'arcs singuliers précisément aux points de non-différentiabilité de la dynamique, c'est-à-dire aux frontières des régions. Plus précisément, considérons le système dynamique suivant :

$$\dot{X} = F(t, X) + B(t)u, \tag{1.38}$$

où $X \in \mathbb{R}^n$ et $u \in \mathcal{U}$, avec $\mathcal{U}$ l'ensemble des fonctions mesurables à valeurs dans un compact convexe $U \subset \mathbb{R}^r$. De plus, la fonction F est définie par

$$F(t, X) = A_i(t)X + C_i(t), \quad (t, X) \in P_i, \tag{1.39}$$

où A_i et C_i sont des fonctions à valeurs matricielles mesurables en temps, respectivement de tailles $n \times n$ et $n \times 1$, B est une fonction matricielle mesurable dans le temps de taille $n \times r$, et F est supposée continue. De plus, les ensembles P_i définissent une partition dénombrable de l'ensemble Ω des contraintes de trajectoires. Dans ce cadre, nous montrons le résultat suivant, illustré par la Figure 1.12.

Theorem 1.5.4 (Principe bang-bang étendu)**.** *Considérant un problème de contrôle en temps minimal dans le cadre non autonome affine par morceaux Equation* (1.18)*, si la condition initiale $X_0 \in \mathbb{R}^n$ appartient à l'ensemble d'atteignabilité de la cible $\Gamma \subset \mathbb{R}^n$, alors il existe une solution optimale composée d'une succession d'arcs bang-bang sur les intervalles de temps où la trajectoire appartient à une région $\mathring{P}_i$, et éventuellement d'arcs singuliers le long des frontières ∂P_i.*

Remark 1.5.5. *Remarquons que le lieu ∂P_i peut être une interface entre deux régions mais peut aussi appartenir à la frontière du domaine Ω.*

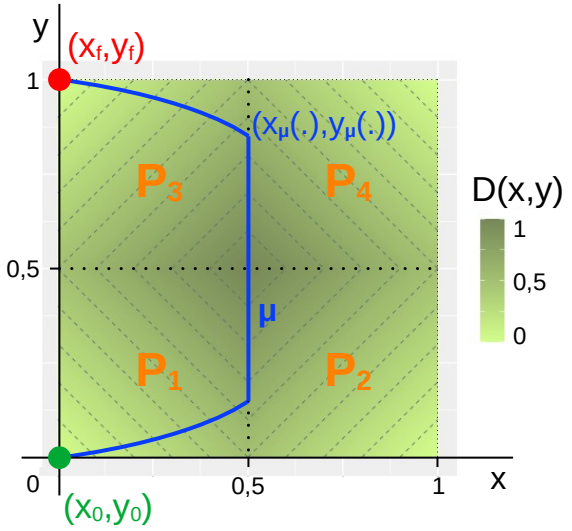


Figure 1.12: Exemple d'une trajectoire de temps minimal (en bleu) pour atteindre une cible (en rouge) depuis une condition initiale (en vert), avec une dynamique affine par morceaux donnée par $\dot{x}(t) = u(t)$ et $\dot{y}(t) = D(x(t), y(t))$. On observe que la trajectoire correspond à un contrôle de type "bang-singulier-bang", et que la partie singulière se trouve exactement le long de frontières de la partition de l'espace.

Ce résultat peut alors être appliqué au problème de contrôle optimal d'irrigation dans le contexte sous serre. En effet, cette formulation contextuelle implique uniquement une dynamique affine

par morceaux (et continue), en raison de la définition des coefficients culturaux K_S et K_R. Par conséquent, on déduit du principe bang-bang étendu pour systèmes affines par morceaux qu'il existe des solutions optimales vérifiant des propriétés structurelles qui consistent en des arcs bang lorsque la trajectoire d'état évolue à l'intérieur d'une région affine, et pouvant contenir des arcs singuliers uniquement aux points de non-différentiabilité de la dynamique, c'est-à-dire aux points S_{rw}, S_{pw}, S^*, ou le long des contraintes d'état. Cette propriété fournira une aide précieuse dans Chapter 3 où nous chercherons à décrire les structures des politiques d'irrigation optimales.

1.5.2.3 Extension du principe bang-bang dans un cadre stochastique

Afin de rechercher des propriétés structurelles similaires des solutions optimales au problème d'irrigation prenant en compte les incertitudes liées aux précipitations, on peut envisager de nouveau une propriété de type bang-bang. Cependant, il peut être difficile d'aller directement vers une extension impliquant à la fois une dynamique affine par morceaux et un terme non déterministe. Une première étape consiste à étudier le principe bang-bang pour des dynamiques globalement linéaires (ou affines) contenant un terme stochastique. À cette fin, nous écrivons le terme stochastique comme un bruit brownien standard, permettant une analyse plutôt générique pour différents types d'applications. Dans Section 2.2, nous revisitons le principe bang-bang classique, établi pour des dynamiques contrôlées affines déterministes, dans un cadre stochastique lorsque la dynamique est soumise à un mouvement brownien. Nous montrons qu'un tel principe n'est pas valide en général dans ce cadre. Cependant, nous montrons qu'il est vrai sous certaines conditions avec des contrôles déterministes. Avec des contrôles stochastiques, nous obtenons, sous certaines conditions, que les problèmes de type Mayer avec critère en espérance et un ensemble de contrôles compact sont équivalents aux mêmes problèmes avec des contrôles prenant leurs valeurs dans l'enveloppe convexe fermée de l'ensemble de contrôles. Plus précisément, considérons le système dynamique suivant :

$$dX(t) = \big[a(t) + A(t)X(t) + B(t)u(t)\big]dt + \big[c(t) + C(t)X(t)\big]dW(t), \quad t \in [0,T], \tag{1.40}$$

où $X \in \mathbb{R}^n$, u prend ses valeurs dans un ensemble compact $U \subset \mathbb{R}^d$, les matrices dépendant du temps $a(\cdot)$, $A(\cdot)$, $B(\cdot)$, $c(\cdot)$ et $C(\cdot)$ sont de dimensions appropriées, et W désigne le mouvement brownien unidimensionnel standard. Remarquons que nous supposons ici que le bruit n'intervient pas dans le terme controlé (dans notre papier [31] nous expliquons les difficultés lorsqu'on relaxe cette hypothèse). En notant $\mathcal{U}$ l'ensemble des processus de contrôle adaptés à valeurs dans U, $\overline{\mathcal{U}}$ l'ensemble des processus de contrôle adaptés à valeurs dans $\overline{co}\,U$, et avec une condition initiale $X(t_0) = x_0$, nous définissons les ensembles atteignables par

$$\mathcal{R}_U^{st}(t_0,x_0;t) = \{X^{t_0,x_0,u}(t); u \in \mathcal{U}\}\,, \quad \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0;t) = \{X^{t_0,x_0,u}(t); u \in \overline{\mathcal{U}}\}, \tag{1.41}$$

où $X^{t_0,x_0,u}$ est la solution du système dynamique associée à $X(t_0) = x_0$ et au contrôle u. De plus, étant donné une application $g : \mathbb{R}^n \to \mathbb{R}$, nous définissons les problèmes d'optimisation

$$(\mathcal{P}_{strict}) : \inf_{\xi \in \mathcal{R}_U^{st}(t_0,x_0,t_1)} \mathbb{E}[g(\xi)]\,, \quad (\mathcal{P}_{convex}) : \inf_{\xi \in \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,t_1)} \mathbb{E}[g(\xi)]$$

Dans le cas particulier où U correspond à l'ensemble des points extrémaux de $\overline{co}\,U$, alors $\mathcal{R}_{\overline{co}\,U}^{st}$ peut être vu comme l'ensemble atteignable pour tout contrôle admissible, et $\mathcal{R}_U^{st}$ comme l'ensemble atteignable associé uniquement aux contrôles de type bang. Remarquons que les contrôles dans $\mathcal{R}_{\overline{co}\,U}^{st}$ peuvent être vus comme des contrôles "relaxés", de la même façon qu'usuellement dans le cadre déterministe lorsque l'ensemble des vitesses n'est pas convexe. Dans ce cadre, nous montrons le résultat suivant.

Theorem 1.5.6. *Supposons que la fonction g soit semi-continue inférieurement, bornée inférieurement et avec une croissance au plus quadratique. Alors, on a les propriétés suivantes.*

1. *Pour tout $0 < t_0 \le t < T$, et tout $x_0 \in \mathbb{L}^4(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, les problèmes de contrôle $(\mathcal{P}_{strict})$ et $(\mathcal{P}_{convex})$ sont équivalents.*
2. *De plus, si g est également convexe, alors il existe ξ optimal dans $\mathcal{R}_U^{st}(t_0,x_0,t) \subset \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,t)$.*

Ce résultat affirme que, même si les ensembles atteignables $\mathcal{R}_U^{st}$ et $\mathcal{R}_{\overline{co}\,U}^{st}$ ne coïncident pas (contrairement au cas du principe bang-bang classique où ils coïncident), les critères optimaux

associés aux problèmes d'optimisation ($\mathcal{P}_{strict}$) et ($\mathcal{P}_{convex}$) coïncident. De plus, avec des hypothèses raisonnables sur g, il existe une solution optimale dont le contrôle est composé d'arcs appartenant à l'ensemble U, c'est-à-dire des arcs de type bang dans le cas particulier où $U = \mathrm{Ext}(\overline{co}\, U)$. Cette propriété peut être vue comme une première étape vers une application au problème d'irrigation des cultures, dans le sens où il s'agirait désormais de combiner ce cadre stochastique avec le cadre précédent des dynamiques affines par morceaux.

1.5.2.4 Structures optimales de stratégies d'irrigation des cultures en contexte sous serre

Dans le contexte de l'irrigation de cultures sous serre entièrement contrôlée (c'est-à-dire sans variabilité environnementale), nous recherchons des propriétés structurelles des solutions optimales qui soient intrinsèques à la dynamique de développement de la culture. Dans le Chapitre 3, nous étudions le problème de contrôle optimal de l'irrigation des cultures avec une dynamique non autonome et non lisse correspondant au contexte de serre précédemment décrit. En fonction des contextes et des objectifs, plusieurs formulations associées à différentes contraintes et différents critères peuvent être envisagées. Le caractère non-lisse rend les problèmes a priori plus difficiles à étudier (du point de vue numérique, une méthode directe nécessite une régularisation), mais notre travail va précisément s'appuyer sur ces points non-différentiables pour décrire la structure des solutions. Notre travail vise à fournir des solutions optimales en retour d'état (feedback) pour ces problèmes en dérivant et en analysant les conditions nécessaires d'optimalité. À cette fin, nous rassemblons les différents problèmes dans une formulation commune, et nous proposons une méthode spécifique pour gérer les contraintes d'état, avant d'appliquer le Principe du Maximum de Pontryagin non lisse sans contrainte d'état. Nous montrons que toutes les stratégies d'irrigation optimales appartiennent à une famille de lois de commande en retour d'état simples, paramétrées et dépendant du temps, adaptées à un large éventail d'objectifs, et convenant aux applications numériques. Plus précisément, on considère le bilan hydrique du sol issu de l'Équation 1.34, les contraintes d'état de l'Équation 1.33, les Hypothèses 1.5.1 à 1.5.4, et les problèmes d'optimisation définis dans la Définition 1.5.4.

Nous montrons d'abord en Proposition 3.3.2 qu'une solution optimale à un problème reformulé, au sens où l'on remplace une contrainte d'état (ce qui permet plus de trajectoires admissibles et plus de contrôles admissibles) par une pénalisation dans la dynamique de la consommation d'eau, constitue en fait un contrôle admissible pour le problème initial avec contrainte. Nous montrons ici que la pénalisation est exacte, au sens où une solution optimale a nécessairement une pénalisation nulle. Par conséquent, il est équivalent de résoudre le problème de contrôle optimal sans contrainte, ce qui nous permet de dériver les conditions nécessaires d'optimalité sans contrainte d'état. Après avoir établi certaines propriétés satisfaites par les solutions optimales (notamment le fait que les cibles terminales de chaque formulation sont saturées, Proposition 3.3.5), nous écrivons les conditions nécessaires d'optimalité issues du Principe du Maximum de Pontryagin non lisse, et nous en déduisons des propriétés sur les variables adjointes. Nous montrons d'abord qu'il n'existe pas d'extrémale anormale, puis nous utilisons le principe du bang-bang étendu pour les dynamiques affines par morceaux (voir Théorème 2.1.5) afin d'obtenir des conditions structurelles sur les solutions optimales, et une brève analyse de la fonction de commutation associée donne des informations sur le nombre possible de commutations (voir Proposition 3.4.8). Enfin, nous rassemblons toutes les propriétés ci-dessus pour conclure avec le Théorème 3.4.9 qui montre que les politiques d'irrigation optimales peuvent être formulées comme des lois de commande en retour d'état paramétrées et dépendant du temps, de la manière suivante :

$$U_{t_d,W_c,t_c}(t,S,V) = \begin{cases} 0, & \big(S \geq S_{ope} \wedge t \leq t_d\big) \vee V = W_c \vee t \geq t_c, \\ u_{sing}(t, S_{ope}), & S = S_{ope} \text{ et } t \leq t_d, \\ u_{sing}(t, S^*), & S = S^* \text{ et } \big(V < W_c \wedge t < t_c\big), \\ 1, & \text{sinon}, \end{cases} \tag{1.42}$$

où t_d, W_c et t_c sont des paramètres dont les valeurs et conditions dépendent de la formulation du problème considéré. En particulier, un ou deux paramètres seulement sont nécessaires selon le problème choisi, ce qui permet de transformer le problème initial d'optimisation en dimension infinie en un problème d'optimisation réduit à une ou deux dimensions, très abordable numériquement. Nous

illustrons ce résultat avec une trajectoire d'humidité du sol et son contrôle optimal correspondant en Figure 1.13.

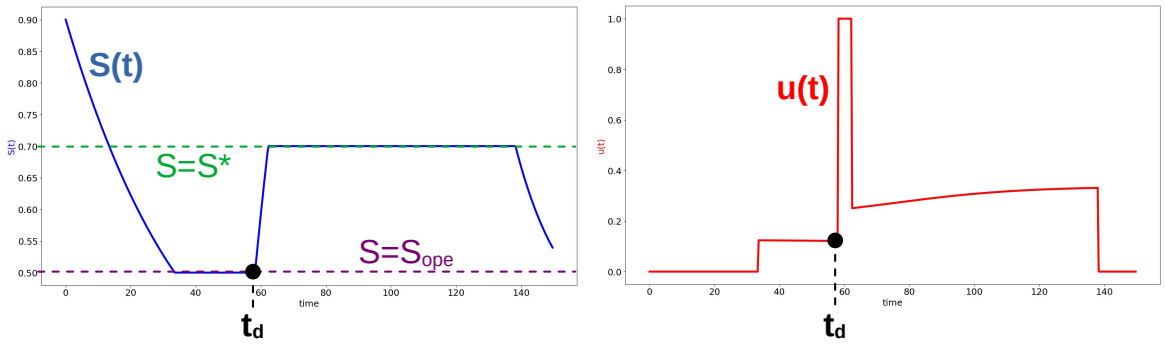


Figure 1.13: Exemple de trajectoire optimale de l'humidité du sol (en bleu, à gauche), associée à une stratégie de contrôle optimale (en rouge, à droite), qui est un contrôle "bang-singulier-bang-singulier-bang", et qui est paramétré uniquement par le temps de décision t_d.

1.5.2.5 Exploration de l'optimalité des stratégies d'irrigation basées sur des seuils

Nous présentons dans le Chapitre 3 une classe de stratégies d'irrigation paramétrées dans le contexte de la gestion de l'eau en serre, mais ce résultat reste insuffisant sur certains points cruciaux. En particulier, bien qu'il ait été démontré qu'il est abordable sur le plan computationnel de déterminer les valeurs optimales des paramètres pour la politique d'irrigation optimale, le résultat théorique n'explique pas la relation entre le contexte et les valeurs optimales calculées, ni l'efficacité relative des stratégies optimales par rapport à des pratiques moins performantes. Il se trouve que les propriétés du modèle Controlled Crop Irrigation (CCI) conduisent à des solutions optimales exprimées sous forme de stratégies d'irrigation à seuils ("Threshold-Based Irrigation Strategies"). Dans le Chapitre 4, nous visons à étudier à la fois les TBIS optimales (le nombre de seuils, leurs valeurs et les paramètres les plus influents qui dictent la forme des scénarios d'irrigation optimaux, voir leurs déclinaisons en Figure 1.14), et leurs caractéristiques sous-optimales (les effets d'erreurs commises sur leur détermination).

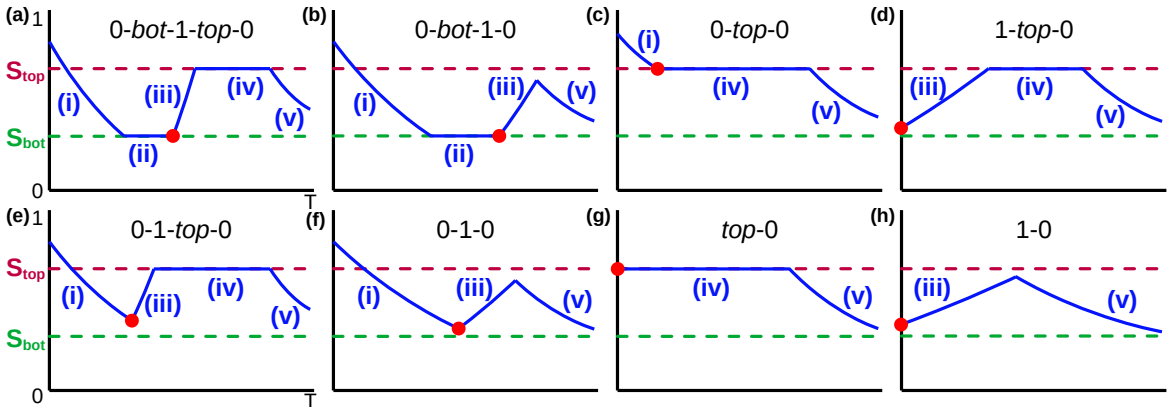


Figure 1.14: Familles des trajectoires d'humidité du sol possibles générées par le cadre des 2-TBIS (basées sur des seuils S_{bot} et S_{top}). Selon les différentes valeurs des paramètres du modèle, certaines étapes peuvent être sautées. Nous indiquons également le point correspondant au temps de décision t_d (point rouge).

Une attention particulière est portée au crucial "temps de décision" auquel l'irrigant choisit d'augmenter la teneur en eau du sol depuis un "niveau de tolérance au stress" jusqu'à un seuil de "production maximale de biomasse". Dans ce travail, nous couplons des résultats connus de la

théorie du contrôle optimal et des méthodes numériques d'analyse de sensibilité. Les principaux résultats montrent que le temps optimal de décision est principalement influencé par des paramètres liés à la dynamique de croissance des plantes et à l'évaluation des effets du stress hydrique sur la production de biomasse, et que des décisions sous-optimales induisent un compromis entre production de biomasse, consommation d'eau d'irrigation et stress hydrique subi par la plante. La généricité du modèle CCI et la méthodologie choisie (exploitation d'une synthèse de retour d'état optimal pour une exploration globale du modèle) offrent des perspectives prometteuses pour divers contextes opérationnels de planification de l'irrigation. Plus précisément, nous considérons plusieurs contextes opérationnels couvrant différentes situations de limitation en eau et des niveaux d'aversion au risque à la fois élevés et faibles. Dans chaque contexte opérationnel, nous menons une analyse de sensibilité globale du "temps de décision" par rapport aux paramètres du problème (relatifs aux caractéristiques du sol, de la plante, et du matériel), grâce à un large plan d'exploration, ainsi qu'une analyse de perturbation multi-locale pour évaluer la sous-optimalité de chroniques d'irrigation proches de l'optimale.

Nous avons mis en évidence que dans les deux alternatives d'aversion ou d'inclination au risque, le moment optimal de déclenchement du débit d'irrigation maximal est fortement lié aux caractéristiques de la plante, indépendamment du quota initial d'eau d'irrigation. En particulier, les valeurs du temps de décision optimal suivent globalement un comportement monotone croissant par rapport à un temps caractéristique de croissance de plante, et un comportement monotone décroissant par rapport à la teneur en eau du sol correspondant au début du stress hydrique. Nous avons observé, à travers notre deuxième expérience numérique (voir Figure 1.15), que dans les deux alternatives mentionnées ci-dessus, et quel que soit le quota initial d'eau d'irrigation, des erreurs sur le seuil supérieur d'humidité du sol peuvent compromettre la production de biomasse par rapport à la biomasse potentielle maximale. En revanche, la sous-optimalité (ou les erreurs) concernant le temps de décision tendent à avoir des effets modérés sur la production de biomasse. Cependant, des décisions précoces ou tardives conduisent à des résultats sensiblement différents en termes de risques de violation de la contrainte opérationnelle, ou de possibilité d'économiser une fraction de l'eau d'irrigation. La combinaison d'un faible quota d'irrigation, d'une forte aversion au risque et de décisions précoces crée des contraintes opérationnelles fortes sur l'utilisation des ressources en eau. Cela rend plus difficile (ou improbable) l'identification (ou l'existence) de trajectoires admissibles de teneur en eau du sol. Á l'extrême opposé, une combinaison d'un quota d'irrigation élevé, d'une forte inclination au risque et de décisions tardives offre des conditions favorables à une utilisation opérationnelle efficace et sans entrave de l'eau. Nos résultats démontrent les capacités de la méthodologie couplée de contrôle optimal et d'exploration globale à faire émerger certains principes d'amélioration de la gestion de la planification de l'irrigation. Par notre analyse détaillée des TBIS, nous avons ouvert une perspective pour évaluer leur potentiel en matière d'efficience de l'usage de l'eau.

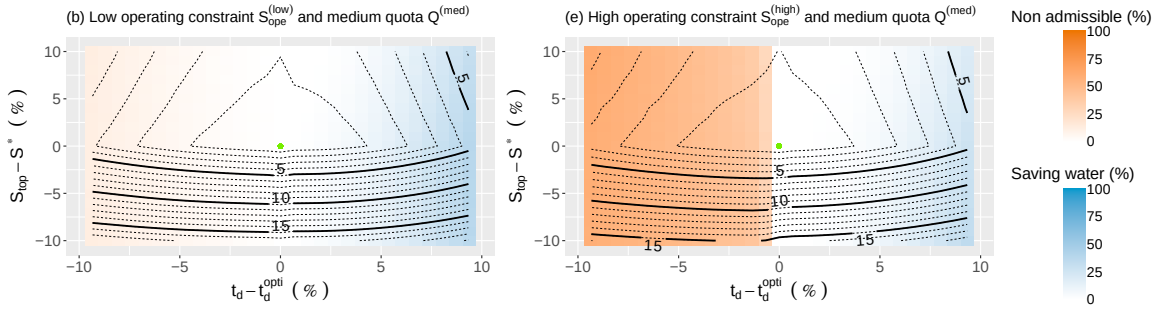
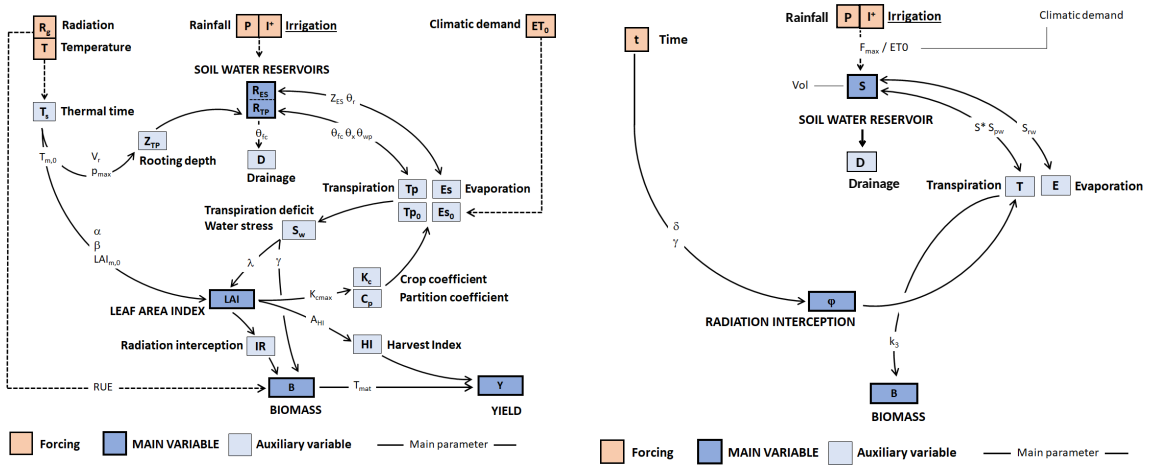


Figure 1.15: Performance des stratégies sous-optimales, évaluée en tant que pertes moyennes en pourcentage de la biomasse finale, au voisinage du seuil optimal (S^*) et du temps de décision optimal (t_d^{opti}). Les isovaleurs correspondent à la perte moyenne en pourcentage de biomasse finale relativement à l'"optimum contextuel" (c'est-à-dire pour la paramétrisation indiquée), situé au centre de chaque graphique (point vert). En complément, les zones orange représentent le pourcentage de jeux de paramètres explorés menant à des trajectoires non admissibles (valeurs finales de S trop faibles), et les zones bleues représentent le pourcentage de jeux de paramètres explorés menant à des trajectoires économes en eau, c'est-à-dire pour lesquelles $V(T) > 0$.

1.5.3 Travaux en cours et perspectives

1.5.3.1 Correspondance entre les modèles CCI et Optirrig

Comme mentionné dans les motivations initiales et le cadre de cette thèse, nous étudions la compatibilité entre le modèle CCI et Optirrig, dans l'objectif d'anticiper un processus de double modélisation. Nous scindons cette étude en trois discussions successives : une comparaison conceptuelle, une tentative de correspondance terme à terme, et une discussion sur les limitations. En suivant le formalisme d'Optirrig, nous présentons les principales variables et processus d'intérêt décrits dans Optirrig et que l'on peut retrouver sous une autre formulation dans le modèle CCI. Nous mettons en évidence que, bien que les deux formulations aient été établies dans des travaux indépendants et émanent a priori d'objectifs très différents, leurs schémas conceptuels sont étonnamment similaires (voir Figure 1.16), ce qui rend le projet de double modélisation raisonnable et pas si lointain.



(a) Schéma conceptuel de référence d'Optirrig, montrant les liens causaux entre les variables du modèle ainsi que la position et le nombre limité des paramètres clés du modèle.

(b) Schéma conceptuel du modèle CCI, montrant les liens causaux entre les variables du modèle ainsi que la position et le nombre limité des paramètres clés du modèle.

Figure 1.16: Comparaison conceptuelle entre les formalismes d'Optirrig et du modèle CCI.

Une fois les concepts respectifs identifiés de part et d'autre, nous précisons les relations qui peuvent être établies entre les paramètres d'Optirrig et ceux du modèle CCI. Certaines correspondances sont des transformations directes (comme par exemple les teneurs en eau du sol ou les apports dus à la pluie et à l'irrigation), d'autres sont plus complexes mais restent réalisables (comme la demande climatique, ou les efficacités d'évapotranspiration et de drainage), et certaines sont au minimum délicates, voire problématiques (comme les conséquences de la croissance de la culture, ou le volume du réservoir du sol). Nous exposons notre méthodologie et les hypothèses sous-jacentes permettant d'établir la correspondance entre les modèles, tout en soulignant les difficultés potentielles liées aux approximations nécessaires pour résoudre les difficultés de compatibilité. Par ailleurs, nous proposons quelques améliorations de modèle afin de faciliter le transfert de paramètres, notamment concernant les processus représentant la croissance de la plante et de son système racinaire. Le cadre étant posé, la prochaine étape sera d'effectuer des simulations expérimentales inaugurales avec Optirrig, dans le but de tester les stratégies extraites du modèle CCI (en particulier les stratégies 2-TBIS qui sont attendues comme étant performantes).

1.5.3.2 Contrôle à coût moyenné pour la gestion de l'irrigation des cultures dans des contextes opérationnels

Afin de consolider l'objectif de proposer des stratégies d'irrigation issues d'un cadre théorique mais pertinentes pour des finalités opérationnelles, il est nécessaire de garder à l'esprit que les systèmes réels comportent des éléments incertains, de natures diverses. Plusieurs méthodes perme-

ttent de traiter ces incertitudes, mais nous avons choisi dans cette thèse d'adopter une approche "déterministe" (en un certain sens), car elle permet d'envisager une extension des arguments déjà explorés dans Chapters 2 and 3. À cette fin, nous introduisons un outil théorique appelé "contrôle à coût moyenné" (ou "averaged cost control"), qui permet de formuler des problèmes de contrôle optimal sous incertitude comme un ensemble de scénarios possibles, chacun associé à une probabilité d'occurrence. Il s'agit alors de rechercher un contrôle commun à tous les scénarios, et qui soit optimal au sens où il optimise le coût moyen sur l'ensemble de ces scénarios. L'objectif de l'introduction de cette approche est de l'appliquer à des situations plus proches des conditions réelles, comme les cadres dans lesquels de telles situations peuvent être formalisées et que nous projetons d'étudier plus en détail.

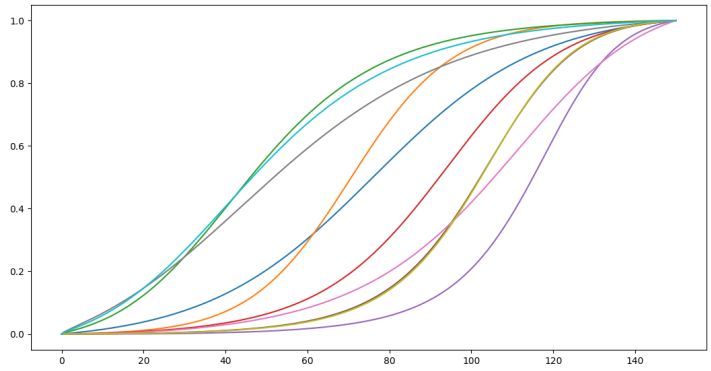


Figure 1.17: Exemple d'un ensemble de dix profils possibles pour la fonction paramètre φ. Ces fonctions représentent la variabilité de la croissance des cultures qui peut se produire en situation réelle, ou qui peut être connue de façon approximative.

Tout d'abord, rappelons que les paramètres du système sont en général connus de manière approximative, ce qui implique que l'optimalité d'une solution peut être sensible à la variabilité de ces paramètres estimés. En particulier, nous avons mis en évidence dans le Chapter 4 que le temps optimal de décision est principalement dicté par la forme de la fonction φ. En conséquence, nous étudions le problème d'irrigation formulé dans le cadre du contrôle à coût moyenné, afin de rechercher des stratégies d'irrigation tenant compte de la variabilité potentielle de la fonction φ (voir Figure 1.17), ce qui peut être vu comme une propriété de robustesse. Nous cherchons donc des stratégies d'irrigation qui optimisent la production moyenne de biomasse en tenant compte des incertitudes sur la fonction φ. Pour cela, nous considérons que φ appartient à un ensemble fini de profils possibles φ_i, chacun associé à une probabilité d'occurrence μ_i. Nous supposons que la fonction est initialement donnée avec une certaine probabilité, de façon inconnue, mais qu'elle n'est pas modifée au cours du temps. Nous formalisons alors notre problème d'irrigation en dupliquant la dynamique de base autant de fois qu'il y a de fonctions φ_i. Nous obtenons ainsi un problème de contrôle optimal "déterministe" de grande dimension, pour lequel il existe une version adaptée des conditions nécessaires d'optimalité. L'établissement de ces conditions permettra d'obtenir des propriétés utiles à la recherche et à la caractérisation des solutions optimales.

Ensuite, rappelons que nous visons l'application des stratégies d'irrigation en conditions de plein champ, et qu'il est donc nécessaire de prendre en compte l'occurrence des précipitations, lesquelles sont par définition non déterministes. Par conséquent, nous projetons d'étudier le problème d'irrigation en tenant compte de ces incertitudes liées aux précipitations, et nous souhaitons utiliser le cadre du contrôle à coût moyenné, en considérant qu'une séquence annuelle de précipitations appartient à un ensemble fini de séquences possibles, chacune associée à une probabilité d'occurrence. Toutefois, nous avons précédemment mentionné que les précipitations pourront être considérées comme des événements instantanés générant des sauts dans la variable d'état d'humidité du sol. Les dynamiques du système seront alors écrites sous forme d'équations différentielles complétées par des sauts d'état, ce qui ne constitue pas un cadre standard pour l'application directe

de la théorie du contrôle à coût moyenné. Par conséquent, nous cherchons à utiliser une technique de reformulation du temps, basé sur une complétion de graphe, mais qu'il convient de formaliser de manière à ce que la nouvelle variable de temps soit la même pour tous les scénarios considérés. Ce procédé permettra de transformer notre dynamique initiale non standard en une dynamique plus adaptée à l'application des conditions nécessaires d'optimalité, et nous projetons ensuite de rechercher et de caractériser des solutions optimales "moyennes" efficaces face aux incertitudes liées aux précipitations.

Dissémination et projets associés à la thèse

- Articles de journal :
  - "Extension of the bang-bang principle for piecewise affine dynamics under state constraints"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    (soumis à Automatica en Novembre 2024, en cours d'évaluation)
  - "About the bang-bang principle for controlled affine dynamics with brownian noise"
    Ruben Chenevat, Dan Goreac, Qinlong Li, Alain Rapaport
    Journal of Convex Analysis, 2025
  - "Optimal structures of crop irrigation strategies with state constraints"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    Journal of Optimization Theory and Applications, 2025
    doi: 10.1007/s10957-025-02854-7
  - "Exploring the optimality of threshold-based crop irrigation feedback strategies"
    Ruben Chenevat, Bruno Cheviron, Alain Rapaport, Sébastien Roux
    (soumission prochaine)

- Articles de conférence :
  - "About the Bang-Bang Principle for piecewise affine systems"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    63rd IEEE Conference on Decision and Control (CDC), Décembre 2024
    doi: 10.1109/CDC56724.2024.10886617
  - "Common structures of optimal solutions for a crop irrigation problem under various constraints and criteria"
    Ruben Chenevat, Bruno Cheviron, Sébastien Roux, Alain Rapaport
    63rd IEEE Conference on Decision and Control (CDC), Décembre 2024
    doi: 10.1109/CDC56724.2024.10886226

- Communications et présentations :
  - "Optimal structures in irrigation and water use efficiency"
    Journées annuelles du GdR MOA, Perpignan (Octobre 2023)
  - "Optimal structures for a model of crop irrigation with constraints"
    Journées SMAI-MODE (Poster), Lyon (Mars 2024)
  - "About the Bang-Bang Principle for piecewise affine dynamics under constraints"
    Journée Contrôle Optimal et applications - FRUMAM, Marseille (Juin 2024)
  - "Optimal structures for a model of crop irrigation with constraints"
    Séminaire de laboratoire - MOMAT, Universidad Complutense Madrid (Juin 2024)
  - "About the Bang-Bang Principle in a non-smooth setting"
    21st French-German-Spanish Conference on Optimization, Gijon (Juin 2024)
  - "Commande optimale et double modélisation pour l'utilisation de l'eau en irrigation des cultures"
    AG du réseau thématique INRAE "REUSE", Avignon (Octobre 2024)
  - "About the Bang-Bang Principle for piecewise affine systems"
    63rd IEEE Conference on Decision and Control (CDC), Milano (Décembre 2024)
  - "Common structures of optimal solutions for a crop irrigation problem under various constraints and criteria"
    63rd IEEE Conference on Decision and Control (CDC), Milano (Décembre 2024)
  - "Controlled crop irrigation : from roots to leaf"
    Séminaire de laboratoire - SYSTEC, FEUP Porto (Janvier 2025)

- "Optimizing crop irrigation under biological and operational constraints with meteorological uncertainty"
  16th Conference on Evolutionary and Deterministic Methods for Design, Optimization, and Control with Applications (EUROGEN), Lahti (Septembre 2025)

- Participation à des projets :

  - Label du parcours doctoral EIR-A (École Internationale de Recherche d'Agreenium)
    Candidature retenue au parcours doctoral EIR-A – Session 2024
    Mobilité internationale : visite de 3 mois au laboratoire SYSTEC (FEUP Porto), de Janvier à Avril 2025
  - Projet "BOUM" de la SMAI
    Candidature retenue à l'appel à projets SMAI-BOUM – Session automne 2024
    Organisation (avec Anas Bouali et Gildas Dadjo) d'une Journée conférence sur l'Optimisation, la Modélisation et le Contrôle (12 Juin 2025)
  - Participation à un Partenariat Hubert Curien France-Portugal (Programme PESSOA 2025-2027)
    Sujet : "Optimizing Irrigation efficiency under Technical and Environmental constraints"

- Enseignements :

  - Deux ans de Travaux Dirigés (TD) : 64 heures par an, au niveau Licence (1ère à 3ème année), pour des étudiants en Mathématiques
    Sujets enseignés : Logique & Théorie des ensembles, Fonctions d'une variable, Suites, Intégrale de Riemann, Analyse numérique des EDO et EDP

Résumé visuel de la thèse

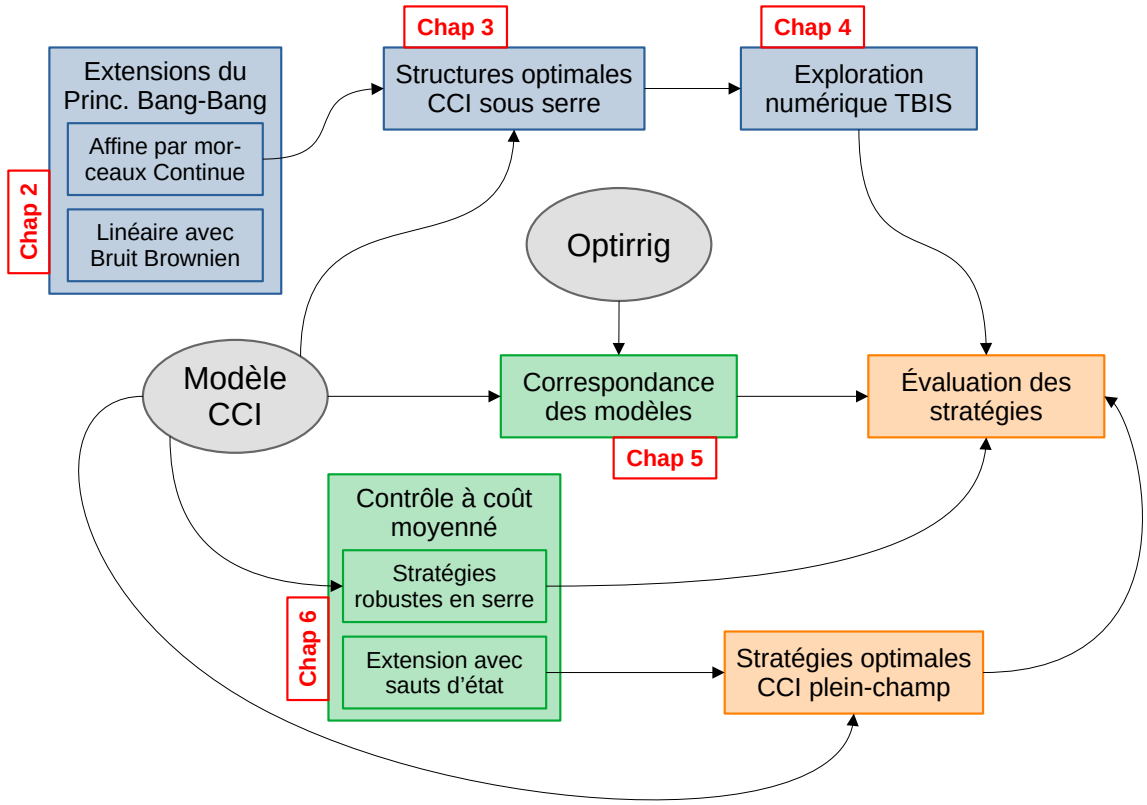


Figure 1.18: Résumé conceptuel de la thèse. Les modèles sont en gris, les contributions sont en bleu, les perspectives avancées sont en vert, les études pour aller plus loin sont en orange.

Part I

Contributions to optimal control theory and analysis of the optimal irrigation policies in a greenhouse context

Chapter 2

Preliminary results in optimal control theory

This chapter is devoted to the introduction of theoretical results in optimal control theory, in particular about the well-known "bang-bang principle" for minimum-time problems ruled by linear dynamics. We revisit this principle in two different frameworks (piecewise affine dynamics under state constraints, and affine dynamics with a stochastic drift term) and we present their respective result formulated as extended bang-bang properties. Such properties provide practical consequences in terms of the nature of the trajectories that reach the target in minimal time. In particular, a numerical resolution will thus be reduced to the class of controls corresponding to the structure induced by its contextual bang-bang principle extension.

2.1 Extension of the bang-bang principle for piecewise affine dynamics under state constraints

This section corresponds to Article [29], which has been submitted to the journal Automatica (in November 2024), and is currently under review. It is an extension of the conference article [27] presented in the IEEE 63rd Conference on Decision and Control, held in December 2024. We investigate the bang-bang principle in a context of non-linear dynamics, more precisely piecewise affine dynamics subject to state constraints. In this framework, even inside an individual region where the dynamics is affine, it is not possible to directly apply the classical bang-bang principle since it cannot guarantee that the trajectory generated by the obtained bang-bang control. A more meticulous study of the local behaviour of the response of trajectories with respect to bang controls is thus needed. We still show that a bang-bang property holds in the sense that there exists an optimal solution composed of extreme controls when the trajectory evolves inside a region of affine dynamics, and might present singular arcs necessarily located at the boundaries, i.e. the non-differentiable parts of the dynamics. We exhibit examples where singular arcs are indeed optimal.

2.1.1 Introduction

The bang-bang principle is a pillar of the theory of linear controlled systems (see [97, 167, 74, 98]). Consider a linear dynamical system

$$\dot{X} = A(t)X + B(t)u, \quad X \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^r, \tag{2.1}$$

where $A(\cdot)$, $B(\cdot)$ are time-measurable matrices of adequate dimensions, and U is a convex compact set. Denote $\mathrm{Ext}(U)$ the set of extreme points of the set U, i.e. points that cannot be written as a convex combination of two distincts points of U (see [98]). For any initial condition (t_0, X_0) in $\mathbb{R}^+ \times \mathbb{R}^n$, the attainability set $\mathcal{A}(t)$, resp. $\mathcal{A}_{BB}(t)$, is defined as the set of $X(t)$ where $X(\cdot)$ is the solution of Equation (2.1) for the initial condition (t_0, X_0) and any time-measurable controls $u(\cdot)$ that takes values in U, resp. in $\mathrm{Ext}(U)$. The bang-bang principle states that $\mathcal{A}(t)$ and $\mathcal{A}_{BB}(t)$ coincide. This result has consequences in terms of controllability and the nature of trajectories that

join two points of the state space in minimal time.

Moreover, it is customary to consider optimal control problems under constraints. These problems usually contain challenging difficulties – especially when it comes to applied areas – and the literature is wide and various, depending on the formulation of the different kinds of constraints and the forms of necessary or sufficient optimality conditions. A certain amount of cases is listed in detail in [78].

We shall consider in this paper non-autonomous regional affine controlled systems, defined by a family of controlled affine systems

$$\dot{X} = f_i(t, X) := A_i(t)X + B(t)u + C_i(t), \quad i \in \mathcal{I}, \tag{2.2}$$

over a partition of $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$ (where Ω may be corresponding to the constraint time-state space), and whose dynamics is given by one of these linear systems on each subset of the partition. Such a dynamics is in general discontinuous and the question of well-posedness of solutions raises (see for example [81], [109], [5]). In particular sliding modes [181] or Filippov solutions [58] may be considered. Controllability of such systems has been well investigated in the literature (see [10, 24, 89, 141, 186]). In the present work, we shall assume that the dynamics are continuous with respect to X (but not necessarily differentiable), which does not bring any issue to define unique solutions as for any Lipschitz continuous differential equations. There is surprisingly much less works dedicated to this precise setting excepted for instance [175] that investigates the controllability of such systems.

For practical interest one may wonder if the bang-bang principle still holds in this simple setting, and if not where the control has to be a non extreme point. The underlying question is to know if it is worth anticipating the change of region with non bang-bang solution in the interior of the regions. To the best of our knowledge, this question has not been addressed specifically in the literature, although the problem of regional optimal control has got recent interests in the literature (see [6, 7, 8]). For instance in [8], a minimal time problem with piecewise affine dynamics is presented, but whose dynamics is discontinuous. Note that considering a family of linear systems with the continuity assumption appears to be very restrictive (as it would impose the common boundary of two domains P_i, P_j to belong to the kernel of $A_i - A_j$). However in many applications one deals with saturation induced by min or max operators. This is why we consider families of affine rather than linear systems. Let us underline that applying directly the bang-bang principle for trajectories joining two points in a same subset P_i does not guarantee that the trajectory with extreme controls remains in the same set P_i. This is precisely the point investigated in the present work that leads to extend the classical bang-bang principle. We will then explore through examples the possible existence of singular arcs, also noticing that a constrained arc in this framework will be seen as a singular arc.

The following is organized as follows. In Section 2.1.2, we make precise our notations and assumptions, making distinctions between linear and affine bang-bang principle for the minimal time control problem. In Section 2.1.3, we give our main result. Section 2.1.4 is devoted to examples.

2.1.2 Settings and assumptions

In this section we present the framework of spatially-temporally regional affine systems, and highlight their similarities and differences with classical linear systems.

We consider an non-autonomous system that is linear with respect to u:

$$\dot{X} = F(t, X) + B(t)u, \tag{2.3}$$

where $X \in \mathbb{R}^n$ and $u \in \mathcal{U}$, with $\mathcal{U}$ the set of measurable functions taking values in $U \subset \mathbb{R}^r$. Consider the following assumptions.

Assumption 2.1.1. *Given the set $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$ such that $\mathring{\Omega}$ is non-empty, connected,* $\mathrm{proj}_t\, \Omega = \mathbb{R}^+$, *and there exists a partition*

$$\Omega = \bigcup_{i \in \mathcal{I}} \overline{P_i}, \tag{2.4}$$

where the $\mathring{P}_i$ are pairwise disjoint, non-empty and connected subsets for any $i \in \mathcal{I}$ (where $\mathcal{I}$ is countable), such that

$$F(t,X) = A_i(t)X + C_i(t), \quad (t,X) \in P_i, \tag{2.5}$$

where A_i and C_i are $n \times n-$ and $n \times 1-$ time measurable matrix-valued functions, B is a $n \times r-$ time measurable matrix-valued function. Moreover F is continuous with respect to X for any t.

Let us stress that we do not impose any regularity of the boundaries of the sets P_i.

Assumption 2.1.2. *The set U is convex compact.*

We consider $(\mathcal{P})$ the minimum time problem to reach a compact target set $\Gamma \subset \mathbb{R}^n$ among controls in $\mathcal{U}$, from an initial state $(t_0, X(t_0)) \in \Omega$ that belongs to the attainability set of Γ, while fulfilling the constraint $(t, X(t)) \in \Omega$ for any time t.

In the case where there is only one area (ie $\mathcal{I} = \{\alpha\}$), the dynamics is linear (ie $C_\alpha = 0$), and there is no constraints (i.e. $\Omega = \mathbb{R}^+ \times \mathbb{R}^n$), we have in fact a classical linear system

$$\dot{X}(t) = A_\alpha(t)X(t) + B(t)u(t). \tag{2.6}$$

For this problem, the classical bang-bang principle is often stated for linear dynamics, as follows.

Theorem 2.1.1 (Classical bang-bang principle [79])**.** *If X_0 belongs to the attainability set of Γ, then there exists a control function $u(\cdot)$ that takes values in $E(U)$ for which the associated trajectory of Equation* (2.6) *reaches Γ in minimal time from the initial condition $X(t_0) = X_0$.*

We recall that this result can be stated as above for a linear system, or similarly for an affine system as in [98]. In fact these two statements are equivalent, as detailed in the following Lemma.

Lemma 2.1.2. *Consider the affine system in $\mathbb{R}^n$*

$$\dot{Y}(t) = L(t)Y(t) + M(t)u(t) + N(t), \tag{2.7}$$

where $u(t) \in U$, a convex compact set of $\mathbb{R}^p$, and L, M and N are $n \times n-$, $n \times r-$ and $n \times 1-$ time mesurable matrix-valued functions. Then, the conclusion of Theorem Theorem 2.1.1 holds for dynamics Equation (2.7)*.*

Proof. The proof only consists in rewriting the affine system into an equivalent linear one in higher dimension, and we explicitly construct it as follows:

$$\dot{\tilde{Y}}(t) = \tilde{L}(t)\tilde{Y}(t) + \tilde{M}(t)u(t), \tag{2.8}$$

where $\tilde{L}$ and $\tilde{M}$ are $(n+1) \times (n+1)-$ and $(n+1) \times r-$ matrix-valued functions. Let us set the additional state variable

$$\dot{y}_{n+1}(t) = 0\,, \quad y_{n+1}(0) = 1.$$

Then we define $\tilde{Y} \in \mathbb{R}^{n+1}$ such that the j-th coefficient of $\tilde{Y}$ is $\tilde{Y}_j = Y_j$ for all $1 \leq j \leq n$, and $\tilde{Y}_{n+1} = y_{n+1}$. We posit $\tilde{L} \in \mathcal{M}_{n+1,n+1}(\mathbb{R})$ such that the coefficients (k,j) are $\tilde{L}_{k,j} = L_{k,j}$ for all $1 \leq k,j \leq n$, the coefficients $(k, n+1)$ are $\tilde{L}_{k,n+1} = N_k$ for all $1 \leq k \leq n$, and the coefficients $(n+1, j)$ are $\tilde{L}_{n+1,j} = 0$ for all $1 \leq j \leq n+1$. We similarly define $\tilde{M} \in \mathcal{M}_{n+1,r}(\mathbb{R})$ such that the coefficients (k,j) are $\tilde{M}_{k,j} = M_{k,j}$ for all $1 \leq k \leq n$ and $1 \leq j \leq r$, and the coefficients $(n+1, j)$ are $\tilde{M}_{n+1,j} = 0$ for all $1 \leq j \leq r$. With such a construction, we easily check that the system Equation (2.8) describes the same dynamics as the system Equation (2.7). □

We may now introduce the main issue of this work. Considering a time-optimal problem subject to the dynamics Equation (2.3), we will discuss about an extension of the bang-bang principle for such settings. It is well known that for hybrid dynamical systems or switched systems the optimal time trajectory may lie on the locus of discontinuity of the dynamics, and it is called a sliding mode. In this work, the systems we consider might be seen as spatially hybrid systems (with a non-autonomous regionalization), but with an additional condition of continuity at the boundaries. One may wonder if such sliding mode behaviour is still possible with continuous piecewise affine dynamics. A straightforward application of the classical bang-bang principle would fail in the framework of Assumption 2.1.1, since we can not ensure that the obtained bang-bang trajectory

doesn't cross any boundary ∂P_i (especially since these boundaries might be moving), which would thus modify the dynamics driving the system. Therefore we have to be a bit more cautious and examine the evolution of the system in order to locally apply the classical bang-bang principle, and then build a candidate bang-bang control step by step.

Remark 2.1.3. *We underline the fact that this work does not focus on the question of the controllability of our systems, and we will always assume that our problems are feasible, i.e. the target is attainable in finite time.*

2.1.3 Extended bang-bang principle

Let us now establish the statement and the proof of the main issue of this work.

Let $u^*(\cdot) \in \mathcal{U}$ be an optimal solution of the time-optimal problem $(\mathcal{P})$ driven by the dynamics Equation (2.3). Denote $X^*(\cdot)$ the associated optimal trajectory and $T^* \in \mathbb{R}_+^*$ the corresponding optimal time. Assume that there exists a time interval $[a, b] \subset [0, T^*]$ such that $(b, X^*(b)) \in \partial P_i$ where P_i is a fixed region of the partition, and $(t, X^*(t)) \in \mathring{P}_i$ for all $t \in [a, b)$.

First we are going to restrict our analysis on a segment strictly inside $\mathring{P}_i$. For any $[t_d, t_f] \subset [a, b)$, we study the evolution of the system over $[t_d, t_f]$, thus following the dynamics of the region P_i:

$$\dot{X}(t) = A_i(t)X(t) + B(t)u(t) + C_i(t). \tag{2.9}$$

We aim to apply the classical bang-bang principle for affine systems, but we need to ensure that all the trajectories undergo the same dynamics. To this aim we will have to study the local behaviour of trajectories and we will define an appropriate neighbourhood of $\zeta_{[t_d,t_f]} = \{(t, X^*(t)),\ t \in [t_d, t_f]\}$ where the classical bang-bang principle may be applied.

Lemma 2.1.4. *With the notations above, there exists a bang-bang control over the time interval $[t_d, t_f] \subset [a, b)$ such that its associated trajectory $Z(\cdot)$ verifies*

$$Z(t_d) = X^*(t_d)\,, \quad Z(t_f) = X^*(t_f),$$
$$(t, Z(t)) \in \mathring{P}_i \quad \textit{for all } t \in [t_d, t_f].$$

Proof. In order to evaluate the distance from $\zeta_{[t_d,t_f]}$ to the boundary ∂P_i, we define the distance d between (t, x) and (t', x') in $\mathbb{R}^+ \times \mathbb{R}^n$ by

$$d((t, x), (t', x')) = \left(|t - t'|^2 + \|x - x'\|^2\right)^{1/2}, \tag{2.10}$$

where $\|\cdot\|$ denotes the euclidian norm. Then we have $r = d(\zeta_{[t_d,t_f]}, \partial P_i) > 0$, and we define $\nu_r = \bigcup_{t\in[t_d,t_f]} \mathcal{B}((t, X^*(t)), r)$.

Step 1 : Assessing the deviation of any trajectory

Let us fix $\hat{t} \in [t_d, t_f)$. Consider the problem defined by the dynamics of Equation (2.9) with the initial condition $X(\hat{t}) = X^*(\hat{t})$, and select an admissible control $u(\cdot)$. We want to evaluate the difference between the associated trajectory $X(\cdot)$ and the reference trajectory $X^*(\cdot)$. In particular, we are looking for a time t^+ such that for all $t \in [\hat{t}, t^+]$ we have $\|X(t) - X^*(t)\| < r$. For $t \geq \hat{t}$ we have

$$\begin{aligned} X(t) - X^*(t) &= X(t) - X(\hat{t}) - (X^*(t) - X^*(\hat{t})) \\ &= \int_{\hat{t}}^{t} A_i(s)(X(s) - X^*(s)) + B(s)(u(s) - u^*(s))ds. \end{aligned}$$

Then

$$\begin{aligned} \|X(t) - X^*(t)\| \leq 2M(t - \hat{t}) \sup_{s\in[a,b]} \|B(s)\| \\ + \int_{\hat{t}}^{t} \|A_i(s)\|\|X(s) - X^*(s)\|ds. \end{aligned}$$

By the Grönwall lemma, we get

$$\begin{aligned}\|X(t) - X^*(t)\| &\leq 2M(t-\hat{t})K^B_{[a,b]} \exp\Big(\int_{\hat{t}}^{t} \|A_i(s)\|ds\Big)\\ &\leq 2M(t-\hat{t})K^B_{[a,b]} \exp\big((t-\hat{t})K^A_{[a,b]}\big),\end{aligned}$$

where $K^A_{[a,b]} = \sup_{s\in[a,b]} \|A_i(s)\| < +\infty$ and $K^B_{[a,b]} = \sup_{s\in[a,b]} \|B(s)\| < +\infty$.

Then there exists $t^+ \in (\hat{t}, t_f]$ such that

$$\|X(t) - X^*(t)\| < r, \quad \forall t \in [\hat{t}, t^+]. \tag{2.11}$$

Moreover, the duration $\delta = |t^+ - \hat{t}|$ only depends on r.

Step 2 : Splitting the trajectory to restrain the deviation

Then there exists $N \in \mathbb{N}^*$ such that $t_d + (N-1)\delta < t_f \leq t_d + N\delta$, and we define a finite sequence $(\theta_k)_{0\leq k\leq N}$ by

$$\begin{aligned}\theta_k &= t_d + k\delta, \quad \text{for } 0 \leq k \leq N-1,\\ \theta_N &= t_f.\end{aligned}$$

Therefore from above we deduce that for any $0 \leq k \leq N-1$, all the trajectories $Y_k(\cdot)$ following the dynamics (6) with initial condition $Y_k(\theta_k) = X^*(\theta_k)$ satisfy

$$\|Y_k(t) - X^*(t)\| < r, \quad \forall t \in [\theta_k, \theta_{k+1}]. \tag{2.12}$$

In particular, we have

$$(t, Y_k(t)) \in \mathcal{B}((t, X^*(t)), r), \quad \forall t \in [\theta_k, \theta_{k+1}], \tag{2.13}$$

and thus the trajectory $\mathcal{S} = \{(t, Y(t)),\ t \in [\theta_k, \theta_{k+1}]\}$ satisfies

$$\mathcal{S} \subset \bigcup_{t\in[\theta_k,\theta_{k+1}]} \mathcal{B}((t, X^*(t)), r) \subset \mathring{P}_i. \tag{2.14}$$

Step 3 : Classical bang-bang principle on each division

As a consequence, we now may apply the bang-bang principle as follows. Consider the dynamics of Equation (2.9) over the whole space $\mathbb{R}^n$, with the initial condition $Y_k(\theta_k) = X^*(\theta_k)$. By assumption, the target $\{X^*(\theta_{k+1})\}$ is attainable at the time θ_{k+1}, thus the bang-bang principle for linear time-optimal problems indicates that there exists a bang-bang control $\bar{u}_k(\cdot)$ such that the corresponding trajectory $\bar{Y}_k(\cdot)$ reaches $X^*(\theta_{k+1})$ at the time θ_{k+1}. Furthermore, we know from the well-chosen construction detailed above that this trajectory verifies

$$\{(t, \bar{Y}(t)),\ t \in [\theta_k, \theta_{k+1}]\} \subset \mathring{P}_i. \tag{2.15}$$

Step 4 : Conclusion for the segment inside $\mathring{P}_i$

Let us now define $v(\cdot)$ as the concatenation of the $\bar{u}_k(\cdot)$ for $0 \leq k \leq N-1$, and denote $Z(\cdot)$ the corresponding trajectory (in $\mathbb{R}^n$) as we can see in Figure 2.1. Finally, we showed that for any $[t_d, t_f] \subset [a, b)$ the obtained control $v(\cdot)$ is a bang-bang control defined over $[t_d, t_f]$, and the associated trajectory $Z(\cdot)$ satisfies $Z(t_d) = X^*(t_d)$, $Z(t_f) = X^*(t_f)$ and the trajectory $\mathcal{T} = \{(t, Z(t)),\ t \in [t_d, t_f]\}$ fulfills

$$\mathcal{T} \subset \bigcup_{0\leq k\leq N-1} \left(\bigcup_{t\in[\theta_k,\theta_{k+1}]} \mathcal{B}((t, X^*(t)), r) \right) \subset \mathring{P}_i. \tag{2.16}$$

□

Let us now deal with the complete trajectory of an optimal solution to a minimum-time problem subject to the settings Equation (2.3).

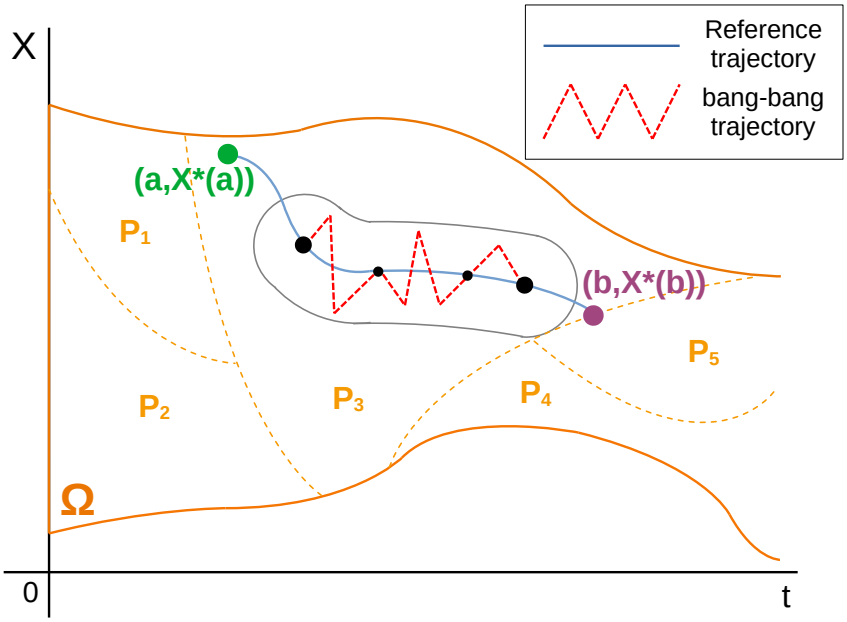


Figure 2.1: Optimal bang-bang trajectory $(\cdot, Z(\cdot))$ (red) built from the reference trajectory $(\cdot, X^*(\cdot))$ (blue) and the intermediate points $(\theta_k, X^*(\theta_k))$ over $[t_d, t_f]$, included in the neighbourhood ν_r

Theorem 2.1.5 (Extended bang-bang principle)**.** *Considering the minimum-time control problem* $(\mathcal{P})$ *subject to the non-autonomous regional affine settings Equation* (2.3)*, if* X_0 *belongs to the attainability set of the target* Γ*, then there exists an optimal solution composed of a succession of bang-bang arcs over time interval where the trajectory belongs to a region* $\mathring{P}_i$*, and possible singular arcs along the boundaries* ∂P_i*.*

Proof. A trajectory going through regions P_i successively, it is sufficient to deal with the case of a part of trajectory in one region P_i. Then we use again the previous notations, and we make our analysis over the time interval $[a, b]$.

To prove the statement, it suffices to apply the Lemma 2.1.4 on well-chosen time sub-intervals. We set a sequence of times $(\sigma_f^j)_{j\geq 0}$ by

$$\sigma_f^j = b - (b-a)/2^j, \quad \text{for } j \in \mathbb{N}. \tag{2.17}$$

We get a corresponding family of radiuses $r_j = d(\zeta_{[\sigma_f^j, \sigma_f^{j+1}]}, \partial P_i) > 0$, and the previous reasoning provides a family of bang-bang controls $(v^j)_{j\geq 0}$, and their associated trajectories $(Z^j)_{j\geq 0}$, defined on each portion of time interval by

$$\begin{aligned}
v^0 : [a, \sigma_f^1] &\longrightarrow U \quad \text{given by Lemma 2.1.4 on } [a, \sigma_f^1], \\
&\text{with } Z^0(a) = X^*(a),\ Z^0(\sigma_f^1) = X^*(\sigma_f^1), \\
&\text{and } \{(t, Z^0(t)),\ t \in [a, \sigma_f^1]\} \subset \mathring{P}_i; \\
v^j : [\sigma_f^j, \sigma_f^{j+1}] &\longrightarrow U \quad \text{given by Lemma 2.1.4 on } [\sigma_f^j, \sigma_f^{j+1}], \\
&\text{with } Z^j(\sigma_f^j) = X^*(\sigma_f^j),\ Z^j(\sigma_f^{j+1}) = X^*(\sigma_f^{j+1}), \\
&\text{and } \{(t, Z^j(t)),\ t \in [\sigma_f^j, \sigma_f^{j+1}]\} \subset \mathring{P}_i.
\end{aligned}$$

Finally, the control defined by a concatenation of the controls v^j above is

$$V(t) = v^j(t) \quad \text{for } t \in [\sigma_f^j, \sigma_f^{j+1}). \tag{2.18}$$

Thus V is well-defined over $[a, b)$, is bang-bang, and its corresponding trajectory $\xi(\cdot)$ satisfies

$$\begin{gathered}
\xi(a) = X^*(a)\,, \quad \xi(t) \underset{t\to b}{\longrightarrow} X^*(b), \\
\{(t, \xi(t)),\ t \in [a, b)\} \subset \mathring{P}_i.
\end{gathered}$$

Hence the existence of an optimal solution composed of bang-bang arcs in the regions where the dynamics is differentiable. □

Remark 2.1.6. *We may have constructed an optimal bang-bang solution that has an infinite (countable) number of commutations. However it would be of interest to look for regularity conditions on the boundaries (such as for example the interior sphere condition in [131]) for which we could be able to show that we have a finite number of commutations.*

Corollary 2.1.7. *Consider an optimal control problem subject to the non-autonomous regional affine settings Equation* (2.3) *with* $\Omega = \mathbb{R}^+ \times \mathbb{R}^n$. *We fix a terminal time* $T > 0$ *and we seek to minimize the integral cost*

$$J(u(\cdot)) = \int_0^T l(t, X(t), u(t))\mathrm{d}t, \tag{2.19}$$

such that the function $l : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ *is piecewise affine, continuous with respect to* X, *and affine with respect to* u. *Then there exists an optimal solution composed of a succession of bang-bang arcs over time interval where the trajectory belongs to a region where the dynamics and the function* l *are differentiable, and possible singular arcs at the loci of non-differentiability of the dynamics or the integrand* l.

Remark 2.1.8. *The results depicted in this section are expected to be still valid (or at least adapted) for a broader class of problems where the dynamics is discontinuous, for example by investigating the augmentation technique (see [49, 9]) in order to reduce to the present case.*

2.1.4 Examples

In this section we are going to present two examples of problems fulfilling the settings described in this work, and we illustrate the result of Theorem 6 by presenting complete optimal trajectories composed of both bang-bang and singular arcs. The first one is a simple autonomous system driving an optimal time problem where the geometry of the (constant) partition of the space gives the intuition of the existence of a singular arc. The second one is from a crop irrigation model [19], and is written as a non-autonomous system driving an optimization problem with a fixed terminal time, an integral cost, and constraints over the input and the state.

2.1.4.1 The ridge path

Let us consider the minimum-time control problem to go from an initial state $(x_0, y_0) = (0, 0)$ to a target $(x_f, y_f) = (0, 1)$, as follows:

$$(R) : \begin{cases} \dot{x}(t) = u(t), \\ \dot{y}(t) = D(x(t), y(t)), \end{cases}$$

where

$$D(x, y) = \min(d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y)),$$

with

$$\begin{aligned} d_1(x, y) &= x + y, & d_2(x, y) &= -x + y + 1, \\ d_3(x, y) &= x - y + 1, & d_4(x, y) &= -x - y + 2. \end{aligned}$$

This systems (R) respects the framework of piecewise affine dynamics and continuous with respect to the state, with the following partition:

$$D(x, y) = \begin{cases} d_1(x, y) & \text{, if } (x, y) \in]-\infty, \frac{1}{2}] \times]-\infty, \frac{1}{2}] = P_1, \\ d_2(x, y) & \text{, if } (x, y) \in [\frac{1}{2}, +\infty[\times]-\infty, \frac{1}{2}] = P_2, \\ d_3(x, y) & \text{, if } (x, y) \in]-\infty, \frac{1}{2}] \times [\frac{1}{2}, +\infty[= P_3, \\ d_4(x, y) & \text{, if } (x, y) \in [\frac{1}{2}, +\infty[\times [\frac{1}{2}, +\infty[= P_4. \end{cases}$$

We set the initial conditions $x(0) = y(0) = 0$, and the target is the singleton $(x_f, y_f) = (0, 1)$. Moreover the control $u(\cdot)$ is a measurable function with values constrained in the compact $U =$

$[-1, 1]$. We seek to minimize the time to reach the target from the initial point.

By definition, the magnitude of $\dot{y}$ is locally higher when y lies on a boundary, then the intuition would say that the trajectory will tend to use the boundaries as much as possible, in order to move as quickly as possible. We are going to show that this trajectory (as illustrated in Figure 2.2) is indeed optimal.

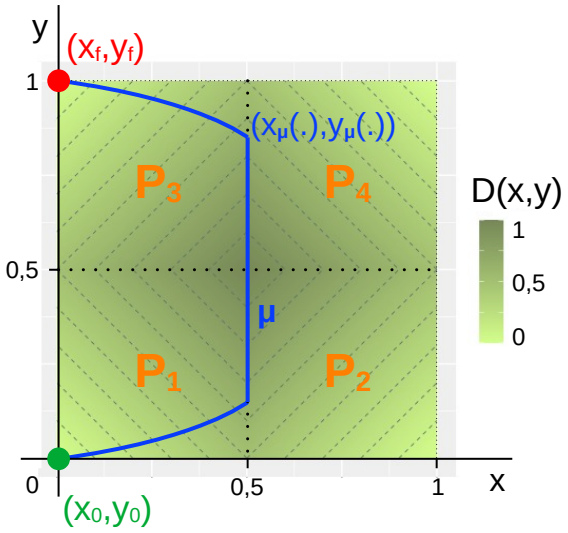


Figure 2.2: Reference trajectory on the heatmap of the dynamics D for the minimum time problem (R)

Consider an optimal control $u^*(\cdot)$, with its associated trajectory $(x^*(\cdot), y^*(\cdot))$ and the corresponding optimal time T^*. By symmetry, there exists an optimal solution following a symmetrical trajectory. Moreover, denoting $t_{1/2}$ the first time when this trajectory crosses the line $y = 1/2$, we deduce that its symmetrization reaches (x_f, y_f) at the time $2t_{1/2}$ and that the trajectory crosses the line $y = 1/2$ only once. Indeed, if there existed another time $t'_{1/2}$ when the trajectory would cross $y = 1/2$, then it would be reaching (x_f, y_f) at a time strictly larger than $2t_{1/2}$, which is therefore non optimal. Furthermore it is then optimal to cross the line $y = 1/2$ at the time $t_{1/2}$.

We would like to show that the strategy of going to the boundary $\{x = 1/2\}$ as fast as possible (with $u = 1$) and then staying on this boundary (with $u = 0$) until reaching $(\frac{1}{2}, \frac{1}{2})$ is time optimal. We define μ the corresponding symmetrical orbit with respect to $\{y = 1/2\}$ (as we can see in Figure 2.2), and we denote (x_μ, y_μ) its associated trajectory, thus obtained when first solving the differential equation with the controls $u = 1$ while $x_\mu < 1/2$ and $u = 0$ until $y_\mu \leq 1/2$, as follows:

$$(x_\mu, y_\mu)(t) = \begin{cases} (t, e^t - 1 - t), & \text{if } t \in [0, \frac{1}{2}), \\ (\frac{1}{2}, \frac{\sqrt{e}-1}{\sqrt{e}} e^t - \frac{1}{2}), & \text{if } t \in \left[\frac{1}{2}, \ln\left(\frac{\sqrt{e}}{\sqrt{e}-1}\right)\right], \end{cases}$$

and then concatenating with its symmetric with respect to $y = \frac{1}{2}$ for $t \in \left[\ln\left(\frac{\sqrt{e}}{\sqrt{e}-1}\right), 2\ln\left(\frac{\sqrt{e}}{\sqrt{e}-1}\right)\right]$.

Suppose that the optimal trajectory $(x^*(\cdot), y^*(\cdot))$ is different than the reference trajectory (x_μ, y_μ) on $[0, t_{1/2}]$. Then denote $J = \{t \in [0, t_{1/2}],\ (x^*(t), y^*(t)) \neq (x_\mu(t), y_\mu(t))\}$, which is a set of non-null measure. For all $t \in J$, we have

$$\begin{cases} x^*(t) < x_\mu(t), & \text{if } (x^*(t), y^*(t)) \in P_1, \\ 1 - x^*(t) < 1 - \frac{1}{2}, & \text{if } (x^*(t), y^*(t)) \in P_2. \end{cases}$$

Therefore we can compare $\dot{y}^*$ and $\dot{y}_\mu$ over J, and by integrating between 0 and $t_{1/2}$ we get

$$y^*(t_{1/2}) < y_\mu(t_{1/2}).$$

Since y_μ is continuous, this would contradict the fact that $t_{1/2}$ is the optimal time for reaching the line $y = 1/2$.

To conclude, we know that $(x^*(\cdot), y^*(\cdot)) = (x_\mu(\cdot), y_\mu(\cdot))$ over the time interval $[0, t_{1/2}]$, and the time for crossing the line $y = 1/2$ is in fact $t_{1/2} = \ln\Big(\frac{\sqrt{e}}{\sqrt{e}-1}\Big)$. By symmetry, the whole trajectory (x_μ, y_μ) is optimal. As a consequence, the minimum time to reach (x_f, y_f) from (x_0, y_0) is $T^* = 2\ln\Big(\frac{\sqrt{e}}{\sqrt{e}-1}\Big)$.

Finally, the corresponding optimal control is "bang-singular-bang" as follows:

$$u^*(t) = \begin{cases} 1, & \text{for } t \in [0, \frac{1}{2}), \\ 0, & \text{for } t \in \Big[\frac{1}{2}, 2\ln\big(\frac{\sqrt{e}}{\sqrt{e}-1}\big) - \frac{1}{2}\Big), \\ -1, & \text{for } t \in \Big[2\ln\big(\frac{\sqrt{e}}{\sqrt{e}-1}\big) - \frac{1}{2}, 2\ln\big(\frac{\sqrt{e}}{\sqrt{e}-1}\big)\Big]. \end{cases}$$

It takes extreme values when the trajectory is inside a region $\mathring{P}_i$, and it is singular when the trajectory follows a boundary.

2.1.4.2 Example from an Irrigation model

Let us consider the optimal control problem, denoted $(Irrig)$, taken from a crop irrigation model (see [19]), and evolving over a fixed time interval $[0, T]$, as follows:

$$\begin{cases} \dot{S}(t) = k_1\Big(- \varphi(t)K_S(S(t)) - (1 - \varphi(t))K_R(S(t)) + k_2 u(t)\Big), \\ \dot{V}(t) = u(t), \end{cases}$$

where S represents the soil moisture, u is the irrigation and V measures the water consumed. Set the initial conditions $S(0) = S_0 \in (S^*, 1]$ and $V(0) = 0$.

The functions K_S and K_R are considered continuous piecewise affine, as follows:

$$K_S(S) = \begin{cases} 0, & S \in [0, S_w], \\ K_M \frac{S - S_w}{S_M - S_w}, & S \in [S_w, S_M], \\ K_M \frac{S - S^*}{S_M - S^*} + \frac{S - S_M}{S^* - S_M}, & S \in [S_M, S^*], \\ 1, & S \in [S^*, 1]; \end{cases}$$

$$K_R(S) = \begin{cases} 0, & S \in [0, S_h], \\ \frac{S - S_h}{1 - S_h}, & S \geq S_h, \end{cases}$$

with $0 < S_h < S_w < S_M < S^* < 1$ and $K_M \in (0, 1)$ such that the function K_S is concave on $[S_w, 1]$, as in Figure 2.3. These functions can typically be written as a combination of min and max of affine functions.

The control $u(\cdot)$ takes values in $[0, 1]$, and the constants verify $k_1 > 0$ and $k_2 > 1$ (local controllability condition). The function φ is supposed to be C^1, increasing, and taking values in $[0, 1]$. The optimization problem consists in fixing a terminal time $T > 0$ and aiming at maximizing the following integral cost:

$$\int_0^T \varphi(t)K_S(S(t))\mathrm{d}t,$$

under some target constraint $V(T) = V_{init}$.

The problem formulation in [19] can be improved by adding a time-varying state constraint that models an important objective for crop irrigation : maintaining high soil moisture level for some specific stage of plant growth. For instance, we consider a tolerance threshold S_{tol} and a humidity level S_{flo} suitable during the flowering period, such that $S_w < S_{tol} < S_{flo} < S^*$. Then we introduce

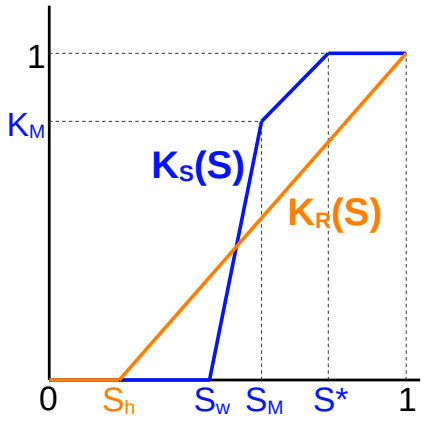


Figure 2.3: Graphs of the example functions K_S (blue) and K_R (orange)

an inequality state constraint as $S_{tol} + F(t) \leq S(t) \leq 1$ for all $t \in [0, T]$, with the function F defined by

$$F(t) = \begin{cases} S_{flo} - S_{tol}, & t \in [f_a, f_b] \text{ (flowering period)}, \\ 0, & \text{otherwise.} \end{cases}$$

This system $(Irrig)$ is a quite complex constrained optimization problem, and does not fulfill the framework of piecewise affine time-optimal problems yet. Let us assume that there exists an optimal solution of the problem $(Irrig)$. Define $B(t) = \int_0^t \varphi(\tau) K_S(S(\tau)) d\tau$. In this case, there is a corresponding optimal terminal cost $B^* > 0$ and a final state $S(T) = \widetilde{S}$. We then consider that it gives in fact an additional terminal condition corresponding to the additional state $B(\cdot)$, i.e. $B(T) = B^*$. Now we may replace all the terminal conditions by a target

$$(S_f, V_f, B_f) = (\widetilde{S}, V_{init}, B^*).$$

With all these transformations, we now have a problem written as a non-autonomous regional affine system with continuous dynamics with respect to the state, and the previous optimization is thus by definition equivalent to a minimum time problem to reach the target (S_f, V_f, B_f). Indeed, if there exists $\hat{t} < T$ such that $B(\hat{t}) = B^*$ and $V(\hat{t}) = V_{init}$, then the control $u = 0$ over $[\hat{t}, T]$ would still get $V(T) = V_{init}$, and the dynamics of B would imply $B(T) > B(\hat{t})$, which would contradict the optimality of B^* in the initial formulation of the problem.

The extended bang-bang principle (Theorem 2.1.5) thus states that there exists an optimal solution consisting in bang arcs over time intervals when the trajectory belongs to a differentiable area, and possible singular arcs on boundaries. We illustrate this result on our irrigation system with a numerical resolution computed via Bocop (see [173]), using $\varphi(t) = (t/T)^\alpha$ and the parametrization in Tables 2.1 and 2.2. We highlight in Figure 2.4 the partition of the time-state set (only showing the state S, the partition being constant with respect to the other states), by $\Omega = P_1 \cup P_2 \cup P_3 \cup P_4$.

S_h	S_w	S_M	K_M	S^*	S_{tol}	S_{flo}
0.2	0.4	0.55	0.8	0.75	0.45	0.6

Table 2.1: Moisture thresholds for $(Irrig)$

T	S_0	k_1	k_2	f_a	f_b	α	V_{init}
150	0.9	0.03	1.9	0.26T	0.37T	2	49.5

Table 2.2: Parametrization of $(Irrig)$

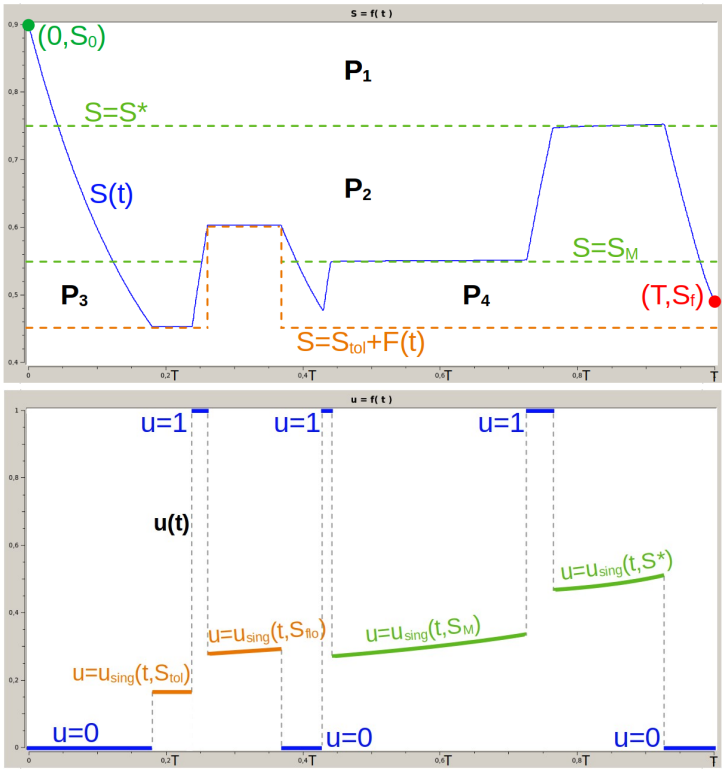


Figure 2.4: Optimal mixed bang, constrained and singular trajectory $S(\cdot)$ (top) and corresponding control $u(\cdot)$ (bottom) for the problem $(Irrig)$

Remark 2.1.9. *The definition of the problem $(Irrig)$ implies that the singular arcs occur only at discrete values of S, then the corresponding singular controls are easy to compute using the equation of $\dot{S}$. For a singular arc at the level $\hat{S}$, we obtain the following singular control:*

$$u_{sing}(t,\hat{S}) = \frac{1}{k_2}\Big(\varphi(t)K_S(\hat{S}) + (1-\varphi(t))K_R(\hat{S})\Big).$$

2.1.5 Conclusions

We focused our study here on extending the classical bang-bang principle for minimum time problems with non-autonomous regional affine continuous dynamics and under constraints. As we showed, an optimal solution to reach a target can always be written as a succession of bang arcs inside the areas of the partition of the time-state set, and eventual singular arcs where the dynamics is non-differentiable, as illustrated by our examples.

We expect that the result of optimal bang arcs in the interior of the regions holds true in a more general framework where the dynamics is discontinuous, which would require redefining the notion of solution suited to such framework. For future perspectives, we may wonder about the conditions of occurrence of singular arcs, and we can discuss about the possible difficulty to compute the value of such singular controls. Moreover, in order to numerically search in an efficient way optimal solutions of an optimal control problem in our framework, we may consider to adapt the multiple shooting method [15].

2.2 Extension of the bang-bang principle for controlled affine dynamics with brownian noise

This section corresponds to Article [31], which has been accepted in the Journal of Convex Analysis (in September 2025). This work is based on a collaborative thinking with Dan Goreac, an associate professor at Université Laval in Québec. The starting point of the thought process is the fact that there doesn't exist any bang-bang like property for dynamics involving a stochastic term, even in the linear case. In fact, we highlight the existence of a pathological behaviour induced by controlled noise where the reachable sets (from any control and from only extreme controls) do not coincide, thus preventing a direct bang-bang property. Therefore we investigate systems with non-controlled noise and problems with terminal cost. We first show that in this context the reachable sets (of "any" and "extreme" controls) are convex, strongly closed and weakly compact. Then we show a bang-bang like property in the sense that the optimal costs of the two problems (with "any" and "extreme" controls) have the same optimal value.

2.2.1 Introduction

The bang-bang principle is a pillar of the control theory, which has important implications in terms of controllability and optimal control of deterministic linear systems in finite dimension. Given a control system in $\mathbb{R}^n$ (for some $n \in \mathbb{N}^*$) of the form

$$\frac{dX(t)}{dt} = a(t) + A(t)X(t) + B(t)u(t), \quad t \geq 0, \tag{2.20}$$

where the control $u(t)$ takes values in a compact set $U \subset \mathbb{R}^d$ (for some $d \in \mathbb{N}^*$), and $a(\cdot)$, $A(\cdot), B(\cdot)$ are time-dependent matrices of adequate dimension, the (linear) bang-bang principle (see e.g. [99]) states that for any initial condition $X(t_0) = X_0$, the reachable set at a given time $T > t_0$ coincides with the reachable set with controls that takes values in the closed convex hull of U, denoted $\overline{co}\, U$. Alternatively, the bang-bang principle is often stated in the literature for convex compact sets U, whose sets of extreme points $\mathrm{Ext}(U)$ is closed, as the equality of reachable sets with control in U or in $\mathrm{Ext}(U)$ (we recall that the extreme points u of a convex set U are such that $U \setminus \{u\}$ is convex). Typically, when the set U takes the form $\prod_{i=1}^d [\underline{u_i}, \overline{u_i}]$, one deduces that a reachable target can be reached in minimal time using controls u_i $(i = 1, \dots, d)$ taking values $\underline{u_i}$ or $\overline{u_i}$ only [97].

The purpose of this work is to investigate the validity of this principle in a stochastic framework, adding a brownian part to the dynamics, considered then as a stochastic differential equation. The paper is organized as follows. In the next Section 2.2.2, we give the stochastic setting we are considering, with some definitions and preliminary comments. Section 2.2.3 is devoted to our main results.

2.2.2 Stochastic setting and preliminaries

A natural way for extending dynamics Equation (2.20) to a stochastic framework, keeping the linearity property with respect to the state and the control, is to consider the following stochastic differential equation:

$$dX(t) = \big(a(t) + A(t)X(t) + B(t)u(t)\big)dt + \big(c(t) + C(t)X(t) + D(t)u(t)\big)dW(t), \quad t \in [0,T], \tag{2.21}$$

where W denotes the standard one-dimensional brownian motion on a complete standard probability space $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, for a fixed finite time horizon $0 < T < +\infty$, and time-dependent matrices $a(\cdot)$, $A(\cdot)$, $B(\cdot)$, $c(\cdot)$, $C(\cdot)$, $D(\cdot)$ are of appropriate dimensions. We are not aiming at complicating the measurability statements by asking specific integrability for the coefficients of these matrices. For this reason, we make the following assumption.

Assumption 2.2.1.

1. *The control space U is a compact subset of the euclidean space $\mathbb{R}^d$ (for some $d \in \mathbb{N}^*$).*
2. *The matrices $a, c \in \mathbb{L}^\infty([0,T];\mathbb{R}^n)$, $A, C \in \mathbb{L}^\infty([0,T];\mathbb{R}^{n\times n})$, and $B, D \in \mathbb{L}^\infty([0,T];\mathbb{R}^{n\times d})$ are deterministic.*

Let us first consider deterministic controls.

Definition 2.2.1. Let $0 \leq t_0 \leq t_1 \leq T$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$. The reachable sets using deterministic controls are denoted by

$$\mathcal{R}_U^{det}(t_0, x_0; t_1) = \{X^{t_0,x_0,u}(t_1); u : \mathbb{R}_+ \longrightarrow U \text{ is Borel measurable}\},$$

respectively

$$\mathcal{R}_{\overline{co}\,U}^{det}(t_0, x_0; t_1) = \{X^{t_0,x_0,u}(t_1); u : \mathbb{R}_+ \longrightarrow \overline{co}\,U \text{ is Borel measurable}\},$$

where $X^{t_0,x_0,u}$ is the associated solution to Equation (2.21) with $X(t_0) = x_0$, governed by the control u (the existence and uniqueness of the solution is standard, since we are dealing with an affine system, and the coefficients are bounded in time).

We shall say that the bang-bang principle holds in this setting if $\mathcal{R}_U^{det}(t_0, x_0; t_1) = \mathcal{R}_{\overline{co}\,U}^{det}(t_0, x_0; t_1)$. Let us begin by a simple case.

Example 2.2.1. Consider a pure controlled diffusion:

$$dX(t) = u(t)dW(t),$$

with $U = \{0, 1\}$ for which $\overline{co}\,U = [0, 1]$. We observe the following features.

1. One has clearly
$$\frac{W(T)}{2} \in \mathcal{R}_{[0,1]}^{det}(0, 0; T),$$
but if u is an adapted process (w.r.t. the brownian filtration) taking its values in $\{0, 1\}$, then one has
$$\mathbb{E}\left[\left|\int_0^T u(s)dW(s) - \frac{W(T)}{2}\right|^2\right] = \mathbb{E}\left[\int_0^T \left|u(s) - \frac{1}{2}\right|^2 ds\right] = \frac{T}{4}.$$
It is, therefore, clear that the bang-bang property cannot hold true for this system. However, we can actually consider notions of reachable sets $\mathcal{R}^{st}(0, 0; T)$ by asking the controls u to be adapted w.r.t. the brownian filtration instead of Borel measurable, as we do in Definition 2.2.3 below.

2. On the other hand, if u is deterministic, then $\mathbb{P}_{X^{0,0,u}(t)}$[1] belongs to the Gaussian family
$$\mathcal{G}\text{auss}_T := \{\mathcal{N}(0, \sigma^2); \sigma^2 \leq T\}.$$
The normality result is rather straightforward, and can be found, for instance, in [22, Section 3.2]. Conversely, if $\sigma^2 \leq T$, consider $u(t) = \mathbf{1}_{[0,\sigma^2]}(t) \in U$ to see that
$$\mathcal{G}\text{auss}_T = \{\mathbb{P}_\xi; \xi \in \mathcal{R}_U^{det}(0, 0; T)\} = \{\mathbb{P}_\xi; \xi \in \mathcal{R}_{[0,1]}^{det}(0, 0; T)\}.$$

3. However, even this bang-bang property in law does not hold true for general linear systems. To see this, we consider the same control system, but $U = \{-1, 1\}$. As hinted before, the Dirac measure δ_0 (seen as a degenerated Gaussian law with $\sigma = 0$) belongs to $\{\mathbb{P}_\xi; \xi \in \mathcal{R}_{[-1,1]}^{det}(0, 0; T)\}$, and it is obtained with the null control $u_0 = 0$. However, if u is adapted and $\{-1, 1\}$-valued, one has
$$\mathbb{E}\left[(X^{0,0,u}(T))^2\right] = T > 0,$$
showing that, in this case, one has
$$\delta_0 \notin \{\mathbb{P}_\xi; \xi \in \mathcal{R}_U^{st}(0, 0; T)\} \quad \text{thus} \quad \{\mathbb{P}_\xi; \xi \in \mathcal{R}_U^{st}(0, 0; T)\} \subsetneq \{\mathbb{P}_\xi; \xi \in \mathcal{R}_{\overline{co}\,U}^{st}(0, 0; T)\}.$$

This example motivated us to consider systems with non-controlled noise:

$$dX(t) = \big[a(t) + A(t)X(t) + B(t)u(t)\big]dt + \big[c(t) + C(t)X(t)\big]dW(t), \quad t \in [0, T], \tag{2.22}$$

along with the following definitions of stochastic controls and reachable sets.

[1] The law of a square-integrable random variable $\xi : \Omega \longrightarrow \mathbb{R}^n$ is denoted by $\mathbb{P}_\xi$.

Definition 2.2.2. Under Assumption 2.2.1, given $t_0 \in [0,T]$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, then

1. $\mathcal{U}$ denotes the set of adapted U-valued control process $u(\cdot)$.
2. When the set U is replaced by its closed convex hull $\overline{co}\, U$, the family of $\overline{co}\, U$-valued adapted controls is denoted by $\overline{\mathcal{U}}$.

Such stochastic controls are referred to as *admissible*.

Definition 2.2.3. Let $0 \leq t_0 \leq t_1 \leq T$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$. The reachable sets using admissible (stochastic) controls are denoted by

$$\mathcal{R}_U^{st}(t_0, x_0; t_1) = \{X^{t_0,x_0,u}(t_1); u \in \mathcal{U}\},$$

respectively

$$\mathcal{R}_{\overline{co}\, U}^{st}(t_0, x_0; t_1) = \{X^{t_0,x_0,u}(t_1); u \in \overline{\mathcal{U}}\},$$

where $X^{t_0,x_0,u}$ is the associated solution to Equation (2.22) with $X(t_0) = x_0$ and control u (existence and uniqueness of the solution is standard, since we are dealing with an affine system, the coefficients are bounded in time and u is adapted).

Given a map $g : \mathbb{R}^n \to \mathbb{R}$, we shall also consider Mayer problems in this stochastic context.

Definition 2.2.4. For $0 \leq t_0 \leq t_1 \leq T$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, let

$$(\mathcal{P}_{strict}) : \inf_{\xi \in \mathcal{R}_U^{st}(t_0,x_0,t_1)} \mathbb{E}[g(\xi)], \tag{2.23a}$$

$$(\mathcal{P}_{convex}) : \inf_{\xi \in \mathcal{R}_{\overline{co}\, U}^{st}(t_0,x_0,t_1)} \mathbb{E}[g(\xi)]. \tag{2.23b}$$

We aim at studying when these problems are equivalent, in the sense that they have the same value. Reachable sets are a natural way of expressing Mayer cost problems. While the bang-bang principle recalled in the introduction for the deterministic setting is a well-established property, it appears to be more difficult to obtain in the stochastic framework (as already suggested by the pathological behavior of Example 2.2.1).

The deterministic setting suggests that, at least for linear, or affine dynamics, optimality for a Mayer cost only requires the convexity of the reachable set (and not of the control family U itself). In the Brownian diffusive case, the standard to obtain strict optimal controls is set by [54], which, even for linear dynamics, amounts to the convexity of U. The alternative, as suggested for instance in [59], [14], [18], is to consider occupation measures and their convex hull in order to guarantee the existence of relaxed optimal policies.

This is why, inspired by the deterministic setting, and with the understanding that convexity is the basis for the spirit of the arguments in [54], we are looking for specialization of bang-bang principles for a class of affine brownian diffusions.

2.2.3 Convexity and bang-bang properties

Let us first give a fundamental representation of solutions of Equation (2.22).

Given $t_0 \in [0,T]$, we consider the following stochastic differential equation with matrix values:

$$d\Phi(t) = -\Phi(t)\big(A(t) - C(t)^2\big)dt - \Phi(t)C(t)dW(t), \quad t \geq 0, \quad \Phi(t_0) = I_n, \tag{2.24}$$

where I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$ (here also, the existence and uniqueness of the solution is standard, since we are dealing with an affine system, and the coefficients are bounded in time). Remind that $C(t)$ is a square matrix and thus $C(t)^2$ is well defined. Because of the boundedness of the coefficients, the solutions of Equation (2.24) are $\mathbb{F}$-adapted and belong to

$\mathbb{L}^\infty\big([0,T];\mathbb{L}^p(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^{n\times n})\big)$, for every $1<p<+\infty$. Itô's formula applied to the product ΦX gives

$$\begin{aligned}d(\Phi(t)X(t)) &= \Big[\Phi(t)\big(a(t)+A(t)X(t)+B(t)u(t)\big)-\Phi(t)\big(A(t)-C(t)^2\big)X(t)\Big]dt\\ &\quad+\Big[\Phi(t)\big(c(t)+C(t)X(t)\big)-\Phi(t)C(t)X(t)\Big]dW(t)\\ &\quad-\Phi(t)C(t)\big[c(t)+C(t)X(t)\big]dt\\ &=\Phi(t)\Big[\big(a(t)-C(t)c(t)+B(t)u(t)\big)dt+c(t)dW(t)\Big],\end{aligned}$$

leading to the following representation of the solutions:

$$X(t)=X^{t_0,x_0,u}(t) \tag{2.25}$$

$$=\Phi^{-1}(t)\left(x_0+\int_{t_0}^t\Phi(r)\Big[\big(a(r)-C(r)c(r)+B(r)u(r)\big)dr+c(r)dW(r)\Big]\right), \tag{2.26}$$

for $t\in[t_0,T]$, where Φ^{-1} is solution of the stochastic differential equation

$$d\Phi^{-1}(t)=A(t)\Phi^{-1}(t)dt+C(t)\Phi^{-1}(t)dW(t),\quad t\geq 0,\quad \Phi^{-1}(t_0)=I_n. \tag{2.27}$$

Indeed, one can check, owing to standard Itô calculus, that one has

$$\begin{aligned}d\big(\Phi^{-1}(t)\Phi(t)\big) &= \Big[-\Phi^{-1}(t)\Phi(t)\big(A(t)-C(t)^2\big)+A(t)\Phi^{-1}(t)\Phi(t)\Big]dt\\ &\quad+\Big[-\Phi^{-1}(t)\Phi(t)C(t)+C(t)\Phi^{-1}(t)\Phi(t)\Big]dW(t)\\ &\quad-C(t)\Phi^{-1}(t)\Phi(t)C(t)dt\\ &=0,\qquad \Phi^{-1}(t_0)\Phi(t_0)=I_n.\end{aligned}$$

2.2.3.1 Convexity of the reachable sets

Similarly to the deterministic systems, we prove the convexity of the stochastic reachable set $\mathcal{R}_U^{st}(t_0,x_0;T)$, under the mere compactness of the set of control U, without assuming convexity.

Theorem 2.2.1. *For every $0\leq t_0\leq t_1\leq T$, and every $x_0\in\mathbb{L}^4(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathbb{R}^n)$, the reachable sets $\mathcal{R}_U^{det}(t_0,x_0,t_1)$ and $\mathcal{R}_U^{st}(t_0,x_0,t_1)$ are convex, strongly closed and (weakly) compact with respect to $\mathbb{L}^2(\Omega,\mathcal{F}_{t_1},\mathbb{P};\mathbb{R}^n)$.*

Remark 2.2.2. *The keen reader will have noticed that we require a fourth-order integrability of the initial condition x_0. This is due to the fact that we are extensively employing the variation-of-constants or mild formulation in equation Equation* (2.25) *and, as already mentioned after equation Equation* (2.24)*, the $\mathbb{P}$-integrability of the fundamental solution Φ is of every finite order $p>1$, but not $\mathbb{L}^\infty$. As a consequence, the result can be adapted, in a straightforward way, to $x_0\in\bigcup_{\delta>0}\mathbb{L}^{2+\delta}(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathbb{R}^n)$.*

Proof. We only prove the assertion for the stochastic reachable set, the one with deterministic controls being quasi-identical. The proof adapts the finite-dimensional counterpart in [99, Theorem 3, Page 76]. The main ingredient is a Hilbert-fitted version of Lyapunov's convexity theorem in [52].

If U is a singleton, there is nothing to prove. We assume that U consists of at least two points.

A glance at the solution expressed in Equation (2.25) shows that one only needs to concentrate on the control-dependent part. As a consequence, we assume, without loss of generality, $c=0$ and $a=0$, such that

$$X^{t_0,x_0,u}(t)=\Phi^{-1}(t)\left(x_0+\int_{t_0}^t\Phi(r)B(r)u(r)dr\right),\quad \forall t_0\leq t\leq T.$$

Let us first prove that the closure of the reachable set is convex. We begin by fixing two admissible U-valued control processes $u\neq v$, and $\lambda\in(0,1)$. The aim is to show that $\lambda X^{t_0,x_0,u}(t_1)+(1-$

$\lambda)X^{t_0,x_0,v}(t_1)$ can be approximated with suitable control processes.

We reason in the setting where $t_1 = T$ and consider $J := [t_0, T]$.

Let $\mathcal{M}(J)$ stand for the family of Lebesgue-measurable subsets of J. For each measurable subset $I \subset J$, we define the map $\eta : \mathcal{M}(J) \to \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^{2n})$, by setting

$$\eta(I) = \begin{pmatrix} \int_I \Phi^{-1}(T)\Phi(r)B(r)u(r)dr \\ \int_I \Phi^{-1}(T)\Phi(r)B(r)v(r)dr \end{pmatrix}, \quad \eta(\emptyset) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order to simplify arguments, let us assume $t_0 = 0$ and $x_0 = 0$. Furthermore, we let $\eta(J) = \begin{pmatrix} X^{0,0,u}(T) \\ X^{0,0,v}(T) \end{pmatrix} =: \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$.

Then, by an extension to Hilbert spaces of Lyapunov's convexity theorem (see [52, Theorem, page 405]) $cl(\{\eta(I) : I \in \mathcal{M}(J)\})$ is a convex subset of the Hilbert space $\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^{2n})$ (the Kuratowski closure operator being intended in the usual topology on this Hilbert space as well).

In particular, for $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists some $I_{\lambda,\varepsilon} \subset J$ (that can even be chosen Borel-measurable) such that

$$\left\| \eta(I_{\lambda,\varepsilon}) - \lambda \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^{2n})} \leq \frac{\varepsilon}{2}.$$

The reader is invited to note that one has

$$\eta(J \setminus I_{\lambda,\varepsilon}) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \eta(I_{\lambda,\varepsilon}).$$

In particular, the following inequalities are satisfied:

$$\left\| \Phi^{-1}(T) \int_{J\setminus I_{\lambda,\varepsilon}} \Phi(r)B(r)v(r)dr - (1-\lambda)\eta_2 \right\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)} \leq \frac{\varepsilon}{2},$$

$$\left\| \Phi^{-1}(T) \int_{I_{\lambda,\varepsilon}} \Phi(r)B(r)u(r)dr - \lambda\eta_1 \right\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)} \leq \frac{\varepsilon}{2}.$$

By defining

$$u^{\lambda,\varepsilon}(t) = u(t)\mathbf{1}_{I_{\lambda,\varepsilon}}(t) + v(t)\mathbf{1}_{J\setminus I_{\lambda,\varepsilon}}(t), \quad t \in J,$$

we get an adapted control, for which one has

$$\left\| X^{0,0,u^{\lambda,\varepsilon}}(T) - \big(\lambda\eta_1 + (1-\lambda)\eta_2\big) \right\| =$$

$$\left\| \Phi^{-1}(T) \int_{I_{\lambda,\varepsilon}} \Phi(r)B(r)u(r)dr - \lambda\eta_1 + \Phi^{-1}(T) \int_{J\setminus I_{\lambda,\varepsilon}} \Phi(r)B(r)v(r)dr - (1-\lambda)\eta_2 \right\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)} \leq \varepsilon,$$

thus concluding that the closure of the reachable set is convex by invoking the arbitrariness of $\varepsilon > 0$ (and $\lambda \in (0, 1)$).

Let us now prove that the reachable set is, in fact, closed.

Let $(u_m)_m$ be a sequence of adapted U-valued control policies, and $x_0 \in \mathbb{L}^4(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, and $\eta_m := X^{t_0,x_0,u_m}(T) \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$. Furthermore, let us assume that η_m converges (as $m \to +\infty$) to some $\eta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$. Due to the boundedness of the (adapted) control sequence with respect to $\mathbb{L}^\infty(\Omega \times [t_0, T], \mathbb{P} \times Leb; U)$, hence in $\mathbb{L}^4(\Omega \times [t_0, T], \mathbb{P} \times Leb; U)$, there exists a subsequence, still denoted by u_m weakly converging to some adapted $u \in \mathbb{L}^4(\Omega \times [t_0, T], \mathbb{P} \times Leb; U)$. Since $\Phi \in \mathbb{L}^\infty\big([0, T]; \mathbb{L}^q(\Omega; \mathbb{R}^{n\times n})\big)$, for every $q > 1$, by picking q large enough, it is then easy to show that $\eta^m := X^{t_0,x_0,u_m}(T)$ weakly converges to some $\tilde{\eta} := X^{t_0,x_0,u}(T)$ in $\mathbb{L}^3(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$. Of course,

the weak convergence is also valid in $\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$, thus implying that $\eta = \tilde{\eta}$, which yields the strong closedness of the reachable set w.r.t. $\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$.

The reader will easily note that the aforementioned procedure actually yields the weak compactness of the reachable sets. □

We then obtain the bang-bang principle with deterministic controls.

Corollary 2.2.3. *For every $0 \leq t_0 \leq t_1 \leq T$, and every $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, one has*

$$\mathcal{R}_U^{det}(t_0, x_0, t_1) = \mathcal{R}_{\overline{co}\, U}^{det}(t_0, x_0, t_1).$$

Proof. In the deterministic framework, the bang-bang principle is inferred from the convexity of these reachable sets. The idea of proof in this setting (i.e. $C = 0, c = 0$) is to use piecewise (time-) constant control policies (since this class is dense in the family of Borel-measurable controls), then approximate elements in $\overline{co}\, U$ with convex combinations of elements in U. The reader may look at the proof of [99, Theorem 1A, last part], for instance. The proof goes through without any change for stochastic dynamics, provided that the considered controls are deterministic. □

Let us make some comments about the bang-bang principle with stochastic controls.

Remark 2.2.4. *A similar argument than the one used in the proof of Corollary 2.2.3 could be tried for stochastic control policies, albeit the fact that constant in time would imply the usage, not of elements of $co\, U$, but $\mathcal{F}_t$-measurable, $\overline{co}\, U$-valued random variables. In this case, piecewise (time-) constant admissible controls are of the form*

$$\sum_i \mathbf{1}_{[t_i, t_{i+1})} u_i(\omega),$$

with u_i being an $\mathcal{F}_{t_i}$-measurable, $\overline{co}\, U$-valued random variable. Although, ω-wise, $u_i(\omega)$ is a convex combination of pure controls in U, the dependence on ω does not allow to conclude the same property for the associated trajectory.

2.2.3.2 A bang-bang property with stochastic controls

Albeit the fact that we do not aim to prove $\mathcal{R}_U^{st}(t_0, x_0, \cdot) = \mathcal{R}_{\overline{co}\, U}^{st}(t_0, x_0, \cdot)$, we are able to provide the following version of the bang-bang property adapted to stochastic problems. This is the main result of the paper.

Theorem 2.2.5. *Assume that the function g is lower semi-continuous function, bounded from below and with at most quadratic growth. Then, one has the following properties.*

1. *For every $0 < t_0 \leq t_1 < T$, and every $x_0 \in \mathbb{L}^4(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, the control problems $(\mathcal{P}_{strict})$ and $(\mathcal{P}_{convex})$ are equivalent.*
2. *Furthermore, if g is also convex, then an optimal ξ exists in $\mathcal{R}_U^{st}(t_0, x_0, t_1) \subset \mathcal{R}_{\overline{co}\, U}^{st}(t_0, x_0, t_1)$.*

Proof. The proof consists in three steps by approximating the dynamics with stochastic systems with bounded coefficients, then applying the duality characterizations in [69]. Finally, the convexity (and weak compactness) of the reachable sets yields the conclusion.

By Theorem 2.2.1, both sets $\mathcal{R}_U^{st}(t_0, x_0, t_1)$ and $\mathcal{R}_{\overline{co}\, U}^{st}(t_0, x_0, t_1)$ are convex. For simplicity, we focus on $t_0 > 0, t_1 = T$, $x_0 \in \mathbb{R}^n$, and drop the dependency on t_0, by simply writing $X^{x_0,u}$ for the solution of Equation (2.22).

Step I. We consider the approximating equation

$$dX^{N,x_0,u}(t) = \big[a(t) + A(t)\Pi_N X^{N,x_0,u}(t) + B(t)u(t)\big]dt + \big[c(t) + C(t)\Pi_N X^{N,x_0,u}(t)\big]dW(t),$$

for $t \geq t_0$. Here, Π_N is the projection onto the closed disc of radius N in $\mathbb{R}^n$. Since all the linear coefficients are bounded and $\overline{co}\, U$ is compact, classical arguments yield the existence of a constant $C_0 > 0$ such that

$$\mathbb{E}\big[|X^{N,x_0,u}(t)|^4\big] \leq C_0\big(1 + |x_0|^4\big), \qquad \mathbb{E}\big[|X^{x_0,u}(t)|^4\big] \leq C_0\big(1 + |x_0|^4\big),$$

for every $x_0 \in \mathbb{R}^n$ and every adapted control with values in $\overline{co}\, U$.

The difference $y^N := X^{x_0,u} - X^{N,x_0,u}$ satisfies

$$\begin{aligned} dy^N(t) = &\big[A(t)y^N(t) + A(t)(I-\Pi_N)X^{N,x_0,u}(t)\big]dt \\ &+ \big[C(t)y^N(t) + C(t)(I-\Pi_N)X^{N,x_0,u}(t)\big]dW(t). \end{aligned}$$

As a consequence, by standard arguments, there exists a (generic) constant $C_1 > 0$ (that may depend on T, but is independent of $N \geq 1$, u and x_0) such that, for every $t_0 \leq t \leq T$, one has

$$\mathbb{E}\big[|y^N(t)|^2\big] \leq C_1 \int_{t_0}^T \mathbb{E}\Big[\big|(I-\Pi_N)X^{N,x_0,u}(s)\big|^2\Big]ds \leq C_1 \int_{t_0}^T \mathbb{E}\Big[\big|X^{N,x_0,u}(s)\big|^2 \mathbf{1}_{|X^{N,x_0,u}(s)|>N}\Big]ds.$$

Using Cauchy-Schwarz inequality followed by Markov's inequality, one gets

$$\begin{aligned} \mathbb{E}\big[|y^N(t)|^2\big] &\leq C_1 \int_{t_0}^T \Big(\mathbb{E}\Big[\big|X^{N,x_0,u}(s)\big|^4\Big]\mathbb{P}\big(|X^{N,x_0,u}(s)| > N\big)\Big)^{\frac{1}{2}} ds \\ &\leq C_1 T C_0\big(1+|x_0|^4\big)\frac{1}{N^2} =: C_2\big(1+|x_0|^4\big)\frac{1}{N^2}. \end{aligned}$$

In particular, if $[g]_1$ denotes the Lipschitz constant for the function g, then it follows that

$$\begin{cases} \Big|\inf_{u\in\mathcal{U}} \mathbb{E}\big[g(X^{x_0,u}(T))\big] - \inf_{u\in\mathcal{U}} \mathbb{E}\big[g(X^{N,x_0,u}(T))\big]\Big| \leq \frac{1}{N}[g]_1\Big(C_2\big(1+|x_0|^4\big)\Big)^{\frac{1}{2}}, \\ \Big|\inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\big[g(X^{x_0,u}(T))\big] - \inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\big[g(X^{N,x_0,u}(T))\big]\Big| \leq \frac{1}{N}[g]_1\Big(C_2\big(1+|x_0|^4\big)\Big)^{\frac{1}{2}}. \end{cases}$$

Step II. We are now going to show that, for every $N \geq 1$, one has

$$\inf_{u\in\mathcal{U}} \mathbb{E}\big[g(X^{N,x_0,u}(T))\big] = \inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\big[g(X^{N,x_0,u}(T))\big].$$

To see this, we fix N, and note that the coefficients of our diffusion $X^{N,x_0,u}$ are bounded. As a consequence, as soon as g is lower semi-continuous, bounded from below and with at most quadratic growth, we can apply the linearization and duality results in [69, Theorem 7] to get

$$\begin{aligned} \inf_{u\in\mathcal{U}} \mathbb{E}\big[g(X^{N,x_0,u}(T))\big] = \sup\Big\{&\eta \in \mathbb{R} : \exists \phi \in C_b^{1,2}, \text{ s.t. } \forall (s,y,z) \in [t_0,T]\times\mathbb{R}^{2n}, \\ &\eta \leq (T-t_0)\inf_{u\in U}\langle B(s)u, D\phi(s,y)\rangle \\ &+ (T-t_0)\mathcal{L}\phi(s,y) + g(z) - \phi(T,z) + \phi(t_0,x_0)\Big\}, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} \inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\big[g(X^{N,x_0,u}(T))\big] = \sup\Big\{&\eta \in \mathbb{R} : \exists \phi \in C_b^{1,2}, \text{ s.t. } \forall (s,y,z) \in [t_0,T]\times\mathbb{R}^{2n}, \\ &\eta \leq (T-t_0)\inf_{u\in\overline{co}\, U}\langle B(s)u, D\phi(s,y)\rangle \\ &+ (T-t_0)\mathcal{L}\phi(s,y) + g(z) - \phi(T,z) + \phi(t_0,x_0)\Big\}, \end{aligned} \tag{2.29}$$

The control-independent part of the generator is

$$\mathcal{L}\phi(s,y) := \frac{1}{2} Tr\Big[C(s)\Pi_N y(\Pi_N y)^\top C^\top(s) D^2\phi(s,y)\Big] + \partial_t\phi(s,y),$$

where $\cdot^\top$ denotes transposition. The set $C_b^{1,2}$ denotes real-valued regular functions on $\mathbb{R}\times\mathbb{R}^n$ that are once differentiable in time and twice in space and are, together with the mentioned partial derivatives, bounded. Looking at the right-hand members of Equation (2.28) and Equation (2.29), it is clear that they provide the same values since, for regular ϕ, one has

$$\inf_{u\in\overline{co}\, U}\langle B(s)u, D\phi(s,y)\rangle = \inf_{u\in U}\langle B(s)u, D\phi(s,y)\rangle.$$

Step III. We are now able to conclude. In view of the convergence in Step I and the equality in Step II, we get

$$\inf_{\xi\in\mathcal{R}_U^{st}(t_0,x_0,T)}\mathbb{E}[g(\xi)]=\inf_{u\in\mathcal{U}}\mathbb{E}\big[g(X^{x_0,u}(T))\big]=\inf_{u\in\overline{\mathcal{U}}}\mathbb{E}\big[g(X^{x_0,u}(T))\big]=\inf_{\xi\in\mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,T)}\mathbb{E}[g(\xi)].$$

If, furthermore, if g is convex, then the functional

$$\xi\in\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)\longmapsto\mathbb{E}[g(\xi)]\in\mathbb{R}$$

is convex, weakly and strongly lower semicontinuous, hence the infimum is attained on both sets $\mathcal{R}_U^{st}(t_0,x_0,T)$ and $\mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,T)$. Our claim follows. □

Remark 2.2.6. *The reader will have noticed that the requirement on g being convex is crucial in the last part of our argument (on the existence of minima). Even when g is Lipschitz-continuous (instead of merely lower semi-continuous), although the functional $\mathbb{E}[g(\cdot)]$ is strongly continuous on $\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$, the reachable subsets are convex, closed subsets, but they are only weakly-compact, and further assumptions guaranteeing the weak (sequential) lower-semicontinuity are required to infer the existence of strict optimal controls. Nonetheless, for the equivalence of the two problems (and, therefore, existence of nearly-optimal controls), convexity plays no role whatsoever.*

2.2.4 Conclusion

In this work, we have proved that the bang-bang principle in a stochastic framework holds if one considers uncontrolled diffusion terms and deterministic controls. With stochastic controls, such a Principle cannot be established, at least in the form of equality of the reachable sets. However, if one considers Mayer problems under some mild hypotheses (but still with uncontrolled diffusion), then we have shown that the value functions coincide, which is probably more useful for concrete applications. Future investigations beyond this work include the consideration of multi-dimensional Brownian motions and piecewise affine dynamics, as in Section 2.1.

Chapter 3

Optimal structures of crop irrigation strategies under state constraints

This chapter corresponds to Article [30], which has been accepted in the Journal of Optimization Theory and Applications (in September 2025). It consists in an improved version, with detailed proofs, of the conference article [28] presented in the IEEE 63rd Conference on Decision and Control, held in December 2024. In this work, we investigate the crop irrigation problem in the context of greenhouse agriculture. Therefore we study the corresponding model formulation, and we aim at showing properties of optimal solutions to several optimization problems representing various operating objectives. Non-smoothness features are a priori difficult to study (a numerical point of view requires a regularisation), but our work precisely relies on these non-differentiable points to describe the structure of solutions. In addition to these various problem formulations, we improve previous modeling works by considering a new state constraint representing the risk aversion of the operator, and preventing low soil moisture that would imply irreversible damage to the crop. Since we aim for deriving the optimality necessary conditions through the Pontryagin Maximum Principle, we first develop an equivalent problem formulation thanks to an original technique (that we call "budget compensation") consisting in penalizing the budget dynamics by a function of constraint violation. This way, we are able to use a standard PMP to derive necessary conditions, which, combined with a structural property coming from the piecewise affine nature of the dynamics, allow us to show that an optimal control belongs to a low-parameterized class of time-varying feedbacks.

3.1 Introduction

The development of agriculture plays a significant role in the worldwide management of natural resources. Among all the consequent challenges, the management of water resources is critical especially in areas where the scarce precipitations need to be more and more supported by irrigation. It is possible to save water through different agronomic and technological approaches, such as selecting stress resilient species, upgrading equipment performances, improving irrigation scheduling. Practices based on enhancing irrigation planning [155], coupled with a framework of deficit irrigation [66, 138] show promising potential to increase efficiency.

Numerous studies tackle the agricultural water management by incorporating modeling tools, such as complex numerical models to describe precisely the physical processes of plant growth [11, 33, 90]. They typically present a detailed structure for predefined crop or regional context [110, 194], and contain significant constraints due to biological requirements. Their complexity implies computational limitations, it is thus reasonable to allow a more parsimonious description in order to improve flexibility and analytical capacity. To this end, irrigation planning is seen as a control problem, and using advanced control theory show good potential to improve efficiency in agriculture [48, 145]. Some studies have explored model predictive control [47, 149, 156] targeting adaptive and robust strategies regarding unknown entries like weather or model uncertainties, through numerical simulations. In a more deterministic approach towards analytical strategies, some studies addressed the irrigation problem using optimal control tools [19, 102, 112, 157]. These

works explore models based on simple dynamical systems and decision making through only a few variables, and require a limited number of hypotheses on the physical processes. According to their various assumptions and objectives some models consider more or less detailed soil water balance and more or less constrained contexts.

The desired benefit of using analytical models is to aim for strategies depicting optimal irrigation thanks to time-varying feedback controls suitable for the low number of model inputs and state variables that are related to quantities accessible in real world. Several major agronomic objectives match with a formulation of a constrained optimization problem : maximizing the biomass production with a water availability constraint, minimizing the consumption of water to reach a yield target, maximizing a financial cost related to production sales and water expenses, avoiding harmful stress due to critically low level of soil moisture. In this work, we investigate these problems with a unified methodological point of view, and we employ a model that we call Controlled Crop Irrigation (CCI) model, inspired from [19, 91, 133]. This model is prone to analytical optimal control techniques (even though it presents non-smooth features) and is thought to possess enough physical sense to describe the order-1 biophysical processes, especially for greenhouse application. The various objectives discussed previously lead to the formulation of three optimization problems (according to three different couples of criterion and target, all fulfilling a state constraint) that we study in a unified procedure. In order to look for a common structure of the optimal irrigation strategies in terms of feedback, we shall establish and derive the optimality necessary conditions. There exists various ways to deal with state constraints ([78, 152]), but in our study we introduce an original "compensation technique", and we work out an equivalent problem formulation without state constraint on which we can apply a standard unconstrained Pontryagin Maximum Principle [35].

In Section 3.2, we provide a description of the CCI model and its hypotheses, then we present the different criteria and targets that we will study and we formalize their associated optimal control problems. After explaining the compensation technique, that we use, leading to equivalent problem formulations freed from state constraints, we show in Section 3.3 some necessary properties of optimal solutions. Then we apply in Section 3.4 the non-smooth Pontryagin Maximum Principle and we derive the optimality necessary conditions to finally get optimal structures of irrigation strategies.

3.2 Model description and hypotheses

3.2.1 Basis model description

We consider the Controlled Crop Irrigation model developed by [91], inspired from [133] and simplified for control theory purposes. A first analysis of these dynamics was made in [19] and we extend the study with targets, criteria and constraints as discussed below. The CCI model relies on a soil moisture balance incorporating plant transpiration, soil evaporation and water irrigation input. It also tracks biomass production and water consumed through irrigation, as follows:

$$\begin{cases} \dot{S}(t) = k_1\Big(-\varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t)\Big), \\ \dot{B}(t) = \varphi(t)K_S(S(t)), \\ \dot{V}(t) = u(t). \end{cases} \tag{3.1}$$

The system is supposed to progress over a fixed time period $[0, T]$ corresponding to the crop season from sowing to harvest. The soil moisture is represented by $S \in [0, 1]$, B is the production of biomass and V denotes the water consumption. Irrigation is represented by the control $u(\cdot)$ which is a measurable function defined on $[0, T]$ and taking values in $[0, 1]$. The parameters k_1 and k_2 are constants depending on agronomic and operational characteristics. The value k_1 is the ratio of the daily reference evapotranspiration over the volume of water reservoir, and the value k_2 is the ratio of the maximal irrigation rate over the daily reference evapotranspiration.

Assumption 3.2.1. *The constants k_1 and k_2 satisfy*

$$k_1 > 0\,, \quad k_2 > 1.$$

The function φ models the radiation interception efficiency, related to the growth of plant cover.

Assumption 3.2.2. *Assume that $\varphi : [0,T] \to [0,1]$ is increasing and of class C^1.*

The functions K_S and K_R respectively describe the variations of soil water content through plant transpiration and soil evaporation.

Assumption 3.2.3. *The functions K_S and K_R are defined for $S \in [0,1]$, and are non-decreasing, piecewise affine, continuous. Typically, K_S and K_R are found in the literature ([19, 133]) as*

$$K_S(S) = \begin{cases} 0 & , \ S \in [0, S_w), \\ \frac{S - S_w}{S^* - S_w} & , \ S \in [S_w, S^*), \\ 1 & , \ S \in [S^*, 1]; \end{cases} \tag{3.2a}$$

$$K_R(S) = \begin{cases} 0 & , \ S \in [0, S_h), \\ \frac{S - S_h}{1 - S_h} & , \ S \in [S_h, 1], \end{cases} \tag{3.2b}$$

where $0 < S_h < S_w < S^* < 1$ represent respectively soil water levels depending on soil and plant characteristics (residual water point, wilting point, point of incipient water stress).

Remark 3.2.1. *It is possible to consider more generic shapes than Equations* (3.2a) *and* (3.2b) *(with more non-differentiable points for example), as long as they remain non-decreasing, piecewise affine, continuous, and verify K_S concave over $[S_w, 1]$ and K_R convex over $[0,1]$. Notice that K_S and K_R are Lipschitz, and denote $L_S > 0$ and $L_R > 0$ their respective Lipschitz constant.*

Assumption 3.2.4. *The system has the following initial conditions:*

$$S(0) = S_0 \in (S^*, 1] \,, \quad B(0) = 0 \,, \quad V(0) = 0.$$

Remark 3.2.2. *It is reasonable to consider a quite high initial soil moisture, since the crop season begins after the year period with usually wet weather.*

Remark 3.2.3. *The condition $k_2 > 1$ ensures local controllability of the S variable, in the sense that it is possible to select controls in order to chose the sign of $\dot{S}$ over $[0,T]$. Moerover, the control keeping soil moisture constant at a fixed value is locally admissible.*

Definition 3.2.1. Under Assumption 3.2.1, the control defined by

$$u_{sing}(t, \tilde{S}) = \frac{\varphi(t) K_S(\tilde{S}) + (1 - \varphi(t)) K_R(\tilde{S})}{k_2} \tag{3.3}$$

belongs to $[0,1]$, for any $\tilde{S} \in [0,1]$.

3.2.2 Framework and constraints

According to the context of deficit irrigation, we restrict our study to (optimal) trajectories inducing water stress. In other words, avoiding water stress means keeping the term $K_S(S(t))$ equal to 1, i.e. maintaining the soil moisture S above the threshold S^*. Denote by $S_{t_i, S_i, u(\cdot)}(\cdot)$ the trajectory of S that takes the value S_i at the time t_i and is controlled by $u(\cdot)$. Let us introduce $t^* = \sup\{t \in [0,T], \ S_{0,S_0,0}(t) > S^*\}$, and define

$$u^*(t) = \begin{cases} 0 & , \ t \in [0, t^*), \\ u_{sing}(t, S^*) & , \ t \in [t^*, T], \end{cases} \tag{3.4}$$

where u_{sing} is defined in Equation (3.3). This control thus induces a limit trajectory where no water stress is encountered. We introduce the corresponding limit values for biomass production and water consumption in this context.

Definition 3.2.2. Using the control u^* defined above, the associated biomass production and water consumption are

$$B^* = \int_0^T \varphi(t) dt, \tag{3.5a}$$

$$V^* = \int_0^T u^*(t) dt. \tag{3.5b}$$

In addition to the context of deficit irrigation, we consider a state constraint on the soil moisture, representing the biological need to prevent critically low levels of soil water content. More precisely, the system is subject to the following state constraint:

$$\forall t \in [0,T],\ S_{ope} \leq S(t) \leq 1, \tag{3.6}$$

where S_{ope} is called tolerance threshold, and verifies $S_w < S_{ope} < S^*$.

Similarly as above, we introduce $t_{ope} = \sup\{t \in [0,T],\ S_{0,S_0,0}(t) > S_{ope}\}$, and we define the control inducing the limit trajectory where soil moisture merely fulfills the tolerance threshold constraint, as follows:

$$u_{ope}(t) = \begin{cases} 0 & ,\ t \in [0, t_{ope}), \\ u_{sing}(t, S_{ope}) & ,\ t \in [t_{ope}, T]. \end{cases} \tag{3.7}$$

As previously we introduce the corresponding limit values for biomass production and water consumption.

Definition 3.2.3. Using the control u_{ope} defined above, the associated biomass production and water consumption are

$$B_{ope} = \int_0^T \varphi(t) K_S(S_{0,S_0,u_{ope}(\cdot)}(t))dt, \tag{3.8a}$$

$$V_{ope} = \int_0^T u_{ope}(t)dt. \tag{3.8b}$$

As a consequence, we set our study gathering deficit irrigation and biological constraint : each biomass target Y considered will satisfy $B_{ope} < Y < B^*$, and each water consumption target Q considered must fulfill $V_{ope} < Q < V^*$.

3.2.3 Problems formulation

The problem of optimizing crop irrigation might be seen in various ways according to classical agronomic objectives. We introduce three optimization problems corresponding to three natural points of view about the operational targets and the criteria to be optimized.

- $(P1)$: maximizing the biomass production at the end of the season, when using a limited quota of water, i.e. maximizing $B(T)$ under the terminal condition $V(T) \leq Q$ where Q is the initial available water.
- $(P2)$: minimizing the water consumption to reach a certain target of biomass, i.e. minimizing $V(T)$ under the terminal condition $B(T) \geq Y$ where Y is the objective of biomass production at the end of the season.
- $(P3)$: maximizing a financial balance between biomass sales and water consumption expenses, i.e. maximizing the value $g(B(T)) - c(V(T))$ where g and c are functions representing the earnings and spendings, with $g(0) = c(0) = 0$, $g' > 0$ and $c' > 0$.

We introduce the associated sets of admissible controls for each problem.

Definition 3.2.4. For any control $u(\cdot)$, consider its associated solution to Equation (3.1) denoted by $(S(\cdot), B(\cdot), V(\cdot))$, and define

$$\mathcal{U}^{P1} = \Big\{u \in L^1\big([0,T],[0,1]\big);\ S(\cdot) \text{ verifies Equation (3.6)},\ V(T) \leq Q,\ V_{ope} < Q < V^*\Big\},$$
$$\mathcal{U}^{P2} = \Big\{u \in L^1\big([0,T],[0,1]\big);\ S(\cdot) \text{ verifies Equation (3.6)},\ B(T) \geq Y,\ B_{ope} < Y < B^*\Big\},$$
$$\mathcal{U}^{P3} = \Big\{u \in L^1\big([0,T],[0,1]\big);\ S(\cdot) \text{ verifies Equation (3.6)}\Big\}.$$

Consequently, the three considered problems are formally written as the following optimal control problems.

Definition 3.2.5. Let

$$(P1): \sup_{u(\cdot)\in\mathcal{U}^{P1}} B(T),$$
$$(P2): \inf_{u(\cdot)\in\mathcal{U}^{P2}} V(T),$$
$$(P3): \sup_{u(\cdot)\in\mathcal{U}^{P3}} g(B(T)) - c(V(T)).$$

Remark 3.2.4. *From the theory of optimal control (see [98, 190] for instance), our problems satisfy the usual assumptions that guarantee that the above supremum and infimum are reached.*

As mentioned in the introduction, these optimal control problems present several levels of difficulty. Indeed the model dynamics are non-autonomous and non-smooth, trajectories are subject to running state constraints and admissible controls are constrained to additional terminal conditions.

3.2.4 Preliminaries about trajectories of soil moisture

3.2.4.1 First lemmas

Lemma 3.2.5. *Consider a control $u \in \mathcal{U}^{Pj}$ and denote $S(\cdot)$ its associated solution of Equation* (3.1)*. Then*

$$\forall t \in [0,T],\ S(t) > S_h.$$

Proof. Denote $S^0(\cdot)$ the solution of the system Equation (3.1) using the constant control $u^0 = 0$. Since $u \geq 0$, we have $\dot{S}(t) \geq \dot{S}^0(t)$ for all $t \in [t_0, T]$, and then $S(t) \geq S^0(t)$ for all $t \in [t_0, T]$.

Moreover the trajectory $S_c(\cdot)$ such that $S_c(t) = S_h$ for all $t \in [0,T]$ is a solution of the differential equation of the state variable S using the constant control u^0. By Cauchy-Lipschitz Theorem, we know that two trajectories of the same equation with different initial conditions don't coincide at any time. Thus for all $t \in [0,T]$ one has $S^0(t) > S_c(t)$. As a consequence, one gets $S(t) > S_h$ for all $t \in [0,T]$. □

Lemma 3.2.6. *Let $S_1(\cdot)$ and $S_2(\cdot)$ be two solutions of the system Equation* (3.1) *and $0 \leq t_1 < t_2 \leq T$ such that*

$$S_1(t_1) = S_2(t_1) \quad \text{and} \quad S_1(t) \geq S_2(t) \ \text{ for all } t \in [t_1, t_2].$$

Let u_1, u_2 be their corresponding controls. Then

$$\forall t \in [t_1, t_2]\ , \quad \int_{t_1}^{t} u_1(\tau)d\tau \geq \int_{t_1}^{t} u_2(\tau)d\tau.$$

In addition, if the controls u_1 and u_2 belong to $\mathcal{U}^{Pj}$, then this inequality is strict unless the trajectories of S_1 and S_2 are identical.

Proof. Let $t \in [t_1, t_2]$. One has

$$S_1(t) - S_2(t) = k_1 \int_{t_1}^{t} h(\tau, S_2(\tau)) - h(\tau, S_1(\tau)) + k_1 k_2 \int_{t_1}^{t} u_1(\tau) - u_2(\tau)d\tau, \tag{3.9}$$

where $h(t,S) = \varphi(t)K_S(S) + (1-\varphi(t))K_R(S)$ is an non-decreasing function with respect to the variable S. Since $S_1 \geq S_2$ over the interval $[t_1, t_2]$, from Equation (3.9), we get

$$\forall t \in [t_1, t_2],\ 0 \leq S_1(t) - S_2(t) \leq k_1 k_2 \int_{t_1}^{t} u_1(\tau) - u_2(\tau)d\tau.$$

Hence one has

$$\forall t \in [t_1, t_2],\ \int_{t_1}^{t} u_1(\tau)d\tau \geq \int_{t_1}^{t} u_2(\tau)d\tau. \tag{3.10}$$

Moreover, if $u_1, u_2 \in \mathcal{U}^{Pj}$, we know from Lemma 3.2.5 that their associated trajectories are above S_h. Since the function h is increasing w.r.t. $S \in [S_h, 1]$, then the inequality Equation (3.10) is strict unless $S_1(\cdot)$ and $S_2(\cdot)$ are identical. □

3.2.4.2 Most Rapid Approach Path

In this section, we introduce the notion of Most Rapid Approach Path (MRAP, as in [76, 77, 79]), which entail particular trajectories similar to exact turnpike ([178, 179]). They are useful when building a control to be compared to another, as in Propositions 3.3.2 and 3.3.5. We illustrate their definition in Figure 3.1.

Definition 3.2.6. Let $0 < \overline{S} < 1$.
For any $(\theta, \sigma) \in [0, T) \times [\overline{S}, 1]$, we define

$$t_\vee^+(\overline{S}, \theta, \sigma) = \begin{cases} T, & \text{if } S_{\theta,\sigma,0}(t) > \overline{S} \text{ for all } t \in [\theta, T], \\ \inf\left\{t > \theta,\ S_{\theta,\sigma,0}(t) = \overline{S}\right\}, & \text{otherwise.} \end{cases}$$

For any $(\theta, \sigma) \in (0, T] \times [\overline{S}, 1]$, we define

$$t_\vee^-(\overline{S}, \theta, \sigma) = \begin{cases} 0, & \text{if } S_{\theta,\sigma,1}(t) > \overline{S} \text{ for all } t \in [0, \theta], \\ \sup\left\{t < \theta,\ S_{\theta,\sigma,1}(t) = \overline{S}\right\}, & \text{otherwise.} \end{cases}$$

Definition 3.2.7 (MRAP from above)**.** Let $0 < \overline{S} < 1$.
For any $(\theta_1, \sigma_1) \in [0, T) \times [\overline{S}, 1]$ and $(\theta_2, \sigma_2) \in (0, T] \times [\overline{S}, 1]$ such that σ_2 is attainable at the time θ_2 from (θ_1, σ_1) with some admissible control $u(\cdot)$, we call "MRAP to the value $\overline{S}$ from above" the control $\tilde{u}_\vee^{\overline{S}}(\cdot)$ defined on $[\theta_1, \theta_2]$ by the following:

(i) If $t_\vee^+(\overline{S}, \theta_1, \sigma_1) \leq t_\vee^-(\overline{S}, \theta_2, \sigma_2)$, then

$$\tilde{u}_\vee^{\overline{S}}(t) = \begin{cases} 0, & \text{if } t \in [\theta_1, t_\vee^+(\overline{S}, \theta_1, \sigma_1)), \\ u_{sing}(t, \overline{S}), & \text{if } t \in [t_\vee^+(\overline{S}, \theta_1, \sigma_1), t_\vee^-(\overline{S}, \theta_2, \sigma_2)], \\ 1, & \text{if } t \in (t_\vee^-(\overline{S}, \theta_2, \sigma_2), \theta_2]. \end{cases}$$

(ii) If $t_\vee^-(\overline{S}, \theta_2, \sigma_2) < t_\vee^+(\overline{S}, \theta_1, \sigma_1)$, then

$$\tilde{u}_\vee^{\overline{S}}(t) = \begin{cases} 0, & \text{if } t \in [\theta_1, \bar{t}_\vee(\theta_1, \sigma_1, \theta_2, \sigma_2)), \\ 1, & \text{if } t \in [\bar{t}_\vee(\theta_1, \sigma_1, \theta_2, \sigma_2), \theta_2], \end{cases}$$

where the time $\bar{t}_\vee(\theta_1, \sigma_1, \theta_2, \sigma_2)$ is defined as the unique $\bar{t} \in [\theta_1, \theta_2]$ such that $S_{\theta_1,\sigma_1,0}(\bar{t}) = S_{\theta_2,\sigma_2,1}(\bar{t}) > \overline{S}$. Indeed the function $I(t) = S_{\theta_1,\sigma_1,0}(t) - S_{\theta_2,\sigma_2,1}(t)$ is decreasing on $[\theta_1, \theta_2]$, and one has $I(\theta_1) \geq 0$ and $I(\theta_2) \leq 0$, hence the existence and uniqueness of $\bar{t}_\vee(\theta_1, \sigma_1, \theta_2, \sigma_2)$.

Definition 3.2.8. Let $0 < \overline{S} < 1$.
For any $(\theta, \sigma) \in [0, T) \times [0, \overline{S}]$, we define

$$t_\wedge^+(\overline{S}, \theta, \sigma) = \begin{cases} T, & \text{if } S_{\theta,\sigma,1}(t) < \overline{S} \text{ for all } t \in [\theta, T], \\ \inf\left\{t > \theta,\ S_{\theta,\sigma,1}(t) = \overline{S}\right\}, & \text{otherwise.} \end{cases}$$

For any $(\theta, \sigma) \in (0, T] \times [0, \overline{S}]$, we define

$$t_\wedge^-(\overline{S}, \theta, \sigma) = \begin{cases} 0, & \text{if } S_{\theta,\sigma,0}(t) < \overline{S} \text{ for all } t \in [0, \theta], \\ \sup\left\{t < \theta,\ S_{\theta,\sigma,0}(t) = \overline{S}\right\}, & \text{otherwise.} \end{cases}$$

Definition 3.2.9 (MRAP from below)**.** Let $0 < \overline{S} < 1$.
For any $(\theta_1, \sigma_1) \in [0, T) \times [0, \overline{S}]$ and $(\theta_2, \sigma_2) \in (0, T] \times [0, \overline{S}]$ such that σ_2 is attainable at the time θ_2 from (θ_1, σ_1) with some admissible control $u(\cdot)$, we call "MRAP to the value $\overline{S}$ from below" the control $\tilde{u}_\wedge^{\overline{S}}(\cdot)$ defined on $[\theta_1, \theta_2]$ by the following:

(i) If $t_\wedge^+(\overline{S}, \theta_1, \sigma_1) \leq t_\wedge^-(\overline{S}, \theta_2, \sigma_2)$, then

$$\tilde{u}_\wedge^{\overline{S}}(t) = \begin{cases} 1, & \text{if } t \in [\theta_1, t_\wedge^+(\overline{S}, \theta_1, \sigma_1)), \\ u_{sing}(t, \overline{S}), & \text{if } t \in [t_\wedge^+(\overline{S}, \theta_1, \sigma_1), t_\wedge^-(\overline{S}, \theta_2, \sigma_2)], \\ 0, & \text{if } t \in (t_\wedge^-(\overline{S}, \theta_2, \sigma_2), \theta_2]. \end{cases}$$

(ii) If $t^-_\wedge(\overline{S},\theta_2,\sigma_2) < t^+_\wedge(\overline{S},\theta_1,\sigma_1)$, then

$$\tilde{u}^{\overline{S}}_\wedge(t) = \begin{cases} 1, & \text{if } t \in [\theta_1, \bar{t}_\wedge(\theta_1,\sigma_1,\theta_2,\sigma_2)), \\ 0, & \text{if } t \in [\bar{t}_\wedge(\theta_1,\sigma_1,\theta_2,\sigma_2), \theta_2], \end{cases}$$

where the time $\bar{t}_\wedge(\theta_1,\sigma_1,\theta_2,\sigma_2)$ is defined as the unique $\bar{t} \in [\theta_1,\theta_2]$ such that $S_{\theta_1,\sigma_1,1}(\bar{t}) = S_{\theta_2,\sigma_2,0}(\bar{t}) < \overline{S}$. Indeed the function $I(t) = S_{\theta_1,\sigma_1,1}(t) - S_{\theta_2,\sigma_2,0}(t)$ is increasing on $[\theta_1,\theta_2]$, and one has $I(\theta_1) \leq 0$ and $I(\theta_2) \geq 0$, hence the existence and uniqueness of $\bar{t}_\wedge(\theta_1,\sigma_1,\theta_2,\sigma_2)$.

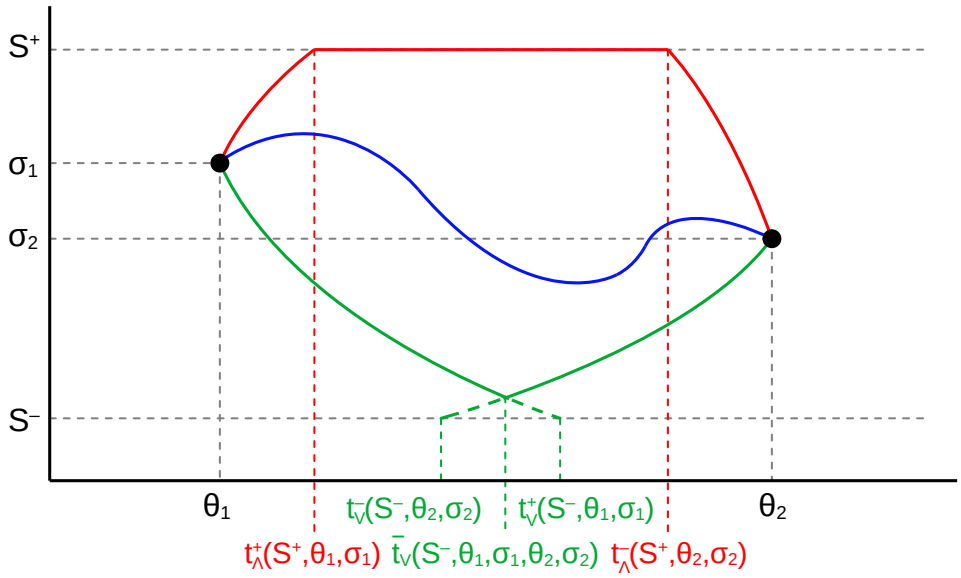


Figure 3.1: Example of a reference $S(\cdot)$ trajectory (in blue), with its induced MRAP trajectories associated to the MRAP controls "from below" $\tilde{u}^{S^+}_\wedge$ (case (i), in red), and "from above" $\tilde{u}^{S^-}_\vee$ (case (ii), in green).

With the above definitions and Lemma 3.2.6, we easily deduce the following property about comparing trajectories associated to MRAP control, and their respective water consumption.

Proposition 3.2.7. *Let $S_h < \underline{S} < \overline{S} \leq 1$.*
Let $S(\cdot)$ be a solution to Equation (3.11) on $[t_1,t_2]$ (with $0 \leq t_1 < t_2 \leq T$) for an admissible control $u(\cdot)$ such that $\underline{S} \leq S(t) \leq \overline{S}$ for all $t \in [t_1,t_2]$. We denote $S_1 = S(t_1)$ and $S_2 = S(t_2)$. Then, the respective MRAP controls $\tilde{u}^{\underline{S}}_\vee(\cdot)$ and $\tilde{u}^{\overline{S}}_\wedge(\cdot)$ over $[t_1,t_2]$, and their respective trajectory $\tilde{S}^{\underline{S}}_\vee(\cdot)$ and $\tilde{S}^{\overline{S}}_\wedge(\cdot)$ with initial conditions $\tilde{S}^{\underline{S}}_\vee(t_1) = \tilde{S}^{\overline{S}}_\wedge(t_1) = S_1$, satisfy the following properties:

$$\tilde{S}^{\underline{S}}_\vee(t_2) = \tilde{S}^{\overline{S}}_\wedge(t_2) = S_2,$$

$$\forall t \in [t_1,t_2],\ \tilde{S}^{\underline{S}}_\vee(t) \leq S(t) \leq \tilde{S}^{\overline{S}}_\wedge(t),$$

$$\int_{t_1}^{t_2} \tilde{u}^{\underline{S}}_\vee(t)dt \leq \int_{t_1}^{t_2} u(t)dt \leq \int_{t_1}^{t_2} \tilde{u}^{\overline{S}}_\wedge(t)dt.$$

Moreover the last inequalities are strict unless $S(\cdot)$ is identical to $\tilde{S}^{\underline{S}}_\vee(\cdot)$, or identical to $\tilde{S}^{\overline{S}}_\wedge(\cdot)$.

3.3 Equivalent formulation and preliminary properties

3.3.1 Budget compensation technique

We aim at deriving optimality necessary conditions through the use of a Pontryagin Maximum Principle, and we need to overcome the difficulty of analysis resulting from the presence of running state constraints Equation (3.6). To this end, we deal with the lower inequality $S_{ope} \leq S(t)$ by

introducing in this section a slightly modified formulation of the dynamical system Equation (3.1). We replace this running state constraint by a mere terminal condition, and more importantly we follow an idea discussed in [78] by defining the function $f(S) = \max(0, S_{ope} - S)$, that we add in the dynamics of the water consumption, with some $C > 0$, as follows:

$$\begin{cases} \dot{S}(t) = k_1\Big(- \varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + k_2 u(t)\Big), \\ \dot{B}(t) = \varphi(t)K_S(S(t)), \\ \dot{W}(t) = u(t) + Cf(S(t)). \end{cases} \tag{3.11}$$

This system keeps the same initial conditions, adapted to the variable W, as follows:

$$S(0) = S_0 \in (S^*, 1]\,, \quad B(0) = 0\,, \quad W(0) = 0.$$

Let us define the three new problems corresponding to this formulation, recalling that this formulation now allows a wider range of trajectories and bigger sets of admissible controls.

Definition 3.3.1. For any control $u(\cdot)$, denote its associated solution to Equation (3.11) by ($S(\cdot)$, $B(\cdot)$, $W(\cdot)$), and define

$$\begin{aligned} \mathcal{U}^{\widetilde{P1}} &= \Big\{ u \in L^1\big([0,T],[0,1]\big);\ S(\cdot) \le 1,\ S(T) \ge S_{ope},\ W(T) \le Q,\ V_{ope} < Q < V^* \Big\}, \\ \mathcal{U}^{\widetilde{P2}} &= \Big\{ u \in L^1\big([0,T],[0,1]\big);\ S(\cdot) \le 1,\ S(T) \ge S_{ope},\ B(T) \ge Y,\ B_{ope} < Y < B^* \Big\}, \\ \mathcal{U}^{\widetilde{P3}} &= \Big\{ u \in L^1\big([0,T],[0,1]\big);\ S(\cdot) \le 1,\ S(T) \ge S_{ope} \Big\}. \end{aligned}$$

Definition 3.3.2. Let

$$\begin{aligned} (\widetilde{P1}): &\max_{u(\cdot)\in\mathcal{U}^{\widetilde{P1}}} B(T), \\ (\widetilde{P2}): &\min_{u(\cdot)\in\mathcal{U}^{\widetilde{P2}}} W(T), \\ (\widetilde{P3}): &\max_{u(\cdot)\in\mathcal{U}^{\widetilde{P3}}} g(B(T)) - c(W(T)). \end{aligned}$$

Remark 3.3.1. *We evidently have $\mathcal{U}^{P1} \subset \mathcal{U}^{\widetilde{P1}}$, $\mathcal{U}^{P2} \subset \mathcal{U}^{\widetilde{P2}}$ and $\mathcal{U}^{P3} \subset \mathcal{U}^{\widetilde{P3}}$, thus any admissible control of a problem (Pj) is admissible for $(\widetilde{Pj})$.*

The additional term $Cf(S(t))$ plays the role of a cost penalization for the problems $(\widetilde{P2})$ and $(\widetilde{P3})$. However for the problem $(\widetilde{P1})$ it may be seen as some room of maneuver for the input $u(\cdot)$ which is restricted by the terminal target. The proposed technique consists in determining a well-suited (large enough) constant $C > 0$ such that a variation of a trajectory of S travelling below S_{ope} over some time interval I, induces a variation of the quantity $Cf(S(t))$ on the interval I that matches with the variation of corresponding input $u(t)$ over I. This way, we will be able to claim that the optimization of the new problems $(\widetilde{P1})$, $(\widetilde{P2})$ and $(\widetilde{P3})$ is equivalent to the optimization of the initial ones $(P1)$, $(P2)$ and $(P3)$. That is the purpose of the following proposition.

Proposition 3.3.2. *Let $C = \frac{L_S + L_R}{k_2} > 0$. Suppose that $u_{\widetilde{Pj}} \in \mathcal{U}^{\widetilde{Pj}}$ is an optimal solution of the problem $(\widetilde{Pj})$. Denote $(S_{\widetilde{Pj}}(\cdot), B_{\widetilde{Pj}}(\cdot), W_{\widetilde{Pj}}(\cdot))$ its associated trajectory. Then*

$$\forall t \in [0,T],\ f(S_{\widetilde{Pj}}(t)) = 0.$$

As a consequence, one has $u_{\widetilde{Pj}} \in \mathcal{U}^{Pj}$, and $u_{\widetilde{Pj}}$ is an optimal solution of the initial problem (Pj).

Proof. First note that the function f is non-negative. Consider a feasible control $u(\cdot)$, and denote $S_u(\cdot)$, $B_u(\cdot)$, $W_u(\cdot)$ the corresponding solutions. Suppose that u is optimal for the problem $(\widetilde{Pj})$, and suppose that there exists an interval $[t_1, t_2]$ with $0 \le t_1 < t_2 \le T$ such that $f(S_u(t)) > 0$ for all $t \in (t_1, t_2)$ and $f(S_u(t_1)) = f(S_u(t_2)) = 0$. In other words, one has $S_u(t_1) = S_u(t_2) = S_{ope}$ and $S_u(t) < S_{ope}$ for all $t \in (t_1, t_2)$. Denote $\underline{S}$ the local minimum of S_u over $[t_1, t_2]$, which is

reached at the time $\underline{t} \in (t_1, t_2)$. By continuity there exists $t^- \in (t_1, \underline{t})$ and $t^+ \in (\underline{t}, t_2)$ such that $S_u(t^-) = S_u(t^+) = (S_{ope} + \underline{S})/2$.

We consider the control $v(\cdot)$, and its corresponding solutions $S_v(\cdot)$, $B_v(\cdot)$, $W_v(\cdot)$, defined by

$$\begin{cases} v(t) = u(t), & \text{if } t \in [0, t^-), \\ v(t) = \tilde{u}_\wedge^{S_{ope}}(t), & \text{if } t \in [t^-, t^+], \\ v(t) = u(t), & \text{if } t \in (t^+, T], \end{cases} \tag{3.12}$$

where $\tilde{u}_\wedge^{S_{ope}}$ is the MRAP to the value S_{ope} from below, over the time interval $[t^-, t^+]$. The associated trajectory $S_v(\cdot)$ satisfies

$$\forall t \in [0,T] \setminus (t^-, t^+),\ S_v(t) = S_u(t) \quad \text{and} \quad \forall t \in (t^-, t^+),\ S_v(t) \geq S_u(t),$$

with

$$\int_{t^-}^{t^+} S_v(t)dt > \int_{t^-}^{t^+} S_u(t)dt.$$

Moreover, from the dynamics of S, we get

$$\begin{aligned} 0 &= \int_{t^-}^{t^+} \dot{S}_v(t) - \dot{S}_u(t)dt \\ &= k_1 \int_{t^-}^{t^+} -\varphi(t)\Big(K_S(S_v(t)) - K_S(S_u(t))\Big) \\ &\qquad - (1-\varphi(t))\Big(K_R(S_v(t)) - K_R(S_u(t))\Big) + k_2(v(t) - u(t))dt. \end{aligned}$$

Using the definition of K_S and K_R and the fact that $\varphi(t) \in (0,1)$ for all $t \in (t^-, t^+)$, one obtains

$$\int_{t^-}^{t^+} v(t) - u(t)dt < \frac{L_S + L_R}{k_2} \int_{t^-}^{t^+} S_v(t) - S_u(t)dt.$$

Then with $C = \frac{L_S + L_R}{k_2}$ we have

$$\begin{aligned} W_v(T) &= \int_{[0,T]\setminus[t^-,t^+]} \dot{W}_v(t)dt + \int_{t^-}^{t^+} v(t) + C(S_{ope} - S_v(t))dt \\ &< \int_{[0,T]\setminus[t^-,t^+]} \dot{W}_u(t)dt + \int_{t^-}^{t^+} u(t) + C(S_{ope} - S_u(t))dt. \end{aligned}$$

Furthermore the comparison between $S_v(\cdot)$ and $S_u(\cdot)$ leads to

$$\begin{aligned} B_v(T) &= \int_{[0,T]\setminus[t^-,t^+]} \dot{B}_v(t)dt + \int_{t^-}^{t^+} \varphi(t) K_S(S_v(t))dt \\ &> \int_{[0,T]\setminus[t^-,t^+]} \dot{B}_u(t)dt + \int_{t^-}^{t^+} \varphi(t) K_S(S_u(t))dt. \end{aligned}$$

To conclude, we have

$$B_v(T) > B_u(T) \quad \text{and} \quad W_v(T) < W_u(T).$$

Thus the control $v(\cdot)$ is admissible for $(\widetilde{Pj})$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{Pj})$. Hence one has

$$\forall t \in [0,T],\ f(S_{\widetilde{Pj}}(t)) = 0.$$

As a consequence, the trajectory $S_{\widetilde{Pj}}$ satisfies $S_{\widetilde{Pj}}(t) \geq S_{ope}$ for all $t \in [0,T]$, and the control $u_{\widetilde{Pj}}$ is thus admissible for the initial problem (Pj). □

3.3.2 Properties of optimal solutions

In this section, we aim at providing several characteristic features of optimal solutions to the optimization problems $(\widetilde{Pj})$.

Proposition 3.3.3. *Let $u_{\widetilde{P1}}(\cdot), u_{\widetilde{P2}}(\cdot), u_{\widetilde{P3}}(\cdot)$ be optimal controls of their respective problems. Denote the corresponding solutions by $(S_{\widetilde{P1}}(\cdot), B_{\widetilde{P1}}(\cdot), W_{\widetilde{P1}}(\cdot))$, $(S_{\widetilde{P2}}(\cdot), B_{\widetilde{P2}}(\cdot), W_{\widetilde{P2}}(\cdot))$ and $(S_{\widetilde{P3}}(\cdot), B_{\widetilde{P3}}(\cdot), W_{\widetilde{P3}}(\cdot))$. Then one has*

$$\begin{aligned} &(\text{I}): \quad u_{\widetilde{Pj}}(t) = 0\ , \quad \text{a.e. } t \in [0, t^*], \\ &(\text{II}): \quad S_{\widetilde{Pj}}(t) \leq S^*\ , \quad \forall t \in [t^*, T]. \end{aligned}$$

Proof. (I) For any feasible control $u(\cdot)$, consider $S_u(\cdot), B_u(\cdot), W_u(\cdot)$ the corresponding solutions. Suppose that u is optimal for the problem $(\widetilde{Pj})$, and denote $E = \{t \in [0,T],\ S_u(t) \leq S^*\}$ and $\hat{t} = \inf E \geq t^*$. One has

$$\int_0^{\hat{t}} u^*(t)dt \leq \int_0^{\hat{t}} u(t)dt \quad \text{and} \quad K_S(S_{0,S_0,u^*(\cdot)}(t)) = 1 \quad \text{for all } t \in [0,T],$$

where u^* is defined in Equation (3.4).

The inequality $\int_{\hat{t}}^T u(t)dt \geq \int_{\hat{t}}^T u^*(t)dt$ cannot be fulfilled, since it would imply

$$W_u(T) \geq V^* \quad , \quad B_u(T) = B^* \quad \text{and} \quad g(B_u(T)) - c(W_u(T)) \leq g(B^*) - c(V^*),$$

which contradict the definition of the problems $(\widetilde{P1})$ and $(\widetilde{P2})$ and mean that u would not be optimal for $(\widetilde{P3})$.

Then the inequality $\int_{\hat{t}}^T u(t)dt < \int_{\hat{t}}^T u^*(t)dt$ holds, and one necessarily has $\hat{t} < T$. We set

$$E_1 = \big\{t \in [\hat{t}, T],\ u(t) < 1\big\}.$$

The set E_1 has non null measure, and the intersection $E \cap E_1$ has also non null measure (otherwise we would have $u = 1$ a.e. in E, which would imply S_u increasing over E, that prevents S_u to reach the domain below S^*).

Assume that $\hat{t} > t^*$. Then $\int_0^{\hat{t}} u^*(t)dt < \int_0^{\hat{t}} u(t)dt$. We consider the control $u_{\text{I}}^\dagger(\cdot)$, and its corresponding solutions $S_{u_{\text{I}}^\dagger}(\cdot)$, $B_{u_{\text{I}}^\dagger}(\cdot)$, $W_{u_{\text{I}}^\dagger}(\cdot)$, defined by

$$\begin{cases} u_{\text{I}}^\dagger(t) = u^*(t), & t \in [0, \hat{t}), \\ u_{\text{I}}^\dagger(t) = u(t), & t \in [\hat{t}, T] \setminus (E \cap E_1) \text{ and } S_{u_{\text{I}}^\dagger}(t) < 1, \\ u_{\text{I}}^\dagger(t) = \min(u(t), 1/k_2), & t \in [\hat{t}, T] \setminus (E \cap E_1) \text{ and } S_{u_{\text{I}}^\dagger}(t) = 1, \\ u_{\text{I}}^\dagger(t) \in [u(t), 1], & t \in E \cap E_1, \end{cases} \tag{3.13}$$

with

$$0 < \int_{E \cap E_1} u_{\text{I}}^\dagger(t) - u(t)dt < \int_0^{\hat{t}} u(t) - u^*(t)dt.$$

Then we have

$$\begin{aligned} W_{u_{\text{I}}^\dagger}(T) &= \int_0^{\hat{t}} u^*(t)dt + \int_{E \cap E_1} u_{\text{I}}^\dagger(t)dt + \int_{[\hat{t},T] \setminus (E \cap E_1)} u_{\text{I}}^\dagger(t)dt \\ &< \int_0^{\hat{t}} u(t)dt + \int_{E \cap E_1} u(t)dt + \int_{[\hat{t},T] \setminus (E \cap E_1)} u(t)dt. \end{aligned}$$

The associated solution $S_{u_{\text{I}}^\dagger}(\cdot)$ satisfies

$$\forall t \in [0,T],\ S_{u_{\text{I}}^\dagger}(t) \leq 1 \quad \text{and} \quad \forall t \in [\hat{t}, T],\ S_{u_{\text{I}}^\dagger}(t) \geq S_u(t),$$

with

$$\int_{E\cap E_1} S_{u_{\text{I}}^\dagger}(t)dt > \int_{E\cap E_1} S_u(t)dt.$$

Moreover one has $S_u(t) \leq S^*$ over $E\cap E_1$, and there exists $J \subset E\cap E_1$ of non null measure such that $S_u(t) < S^*$ over J. Then

$$\int_{E\cap E_1} \varphi(t)K_S(S_{u_{\text{I}}^\dagger}(t))dt > \int_{E\cap E_1} \varphi(t)K_S(S_u(t))dt.$$

This leads to

$$B_{u_{\text{I}}^\dagger}(T) > \int_0^{\hat{t}} \varphi(t)K_S(S_u(t))dt + \int_{E\cap E_1} \varphi(t)K_S(S_u(t))dt + \int_{[\hat{t},T]\setminus(E\cap E_1)} \varphi(t)K_S(S_u(t))dt.$$

To conclude, we have

$$B_{u_{\text{I}}^\dagger}(T) > B_u(T) \quad \text{and} \quad W_{u_{\text{I}}^\dagger}(T) < W_u(T).$$

Thus the control $u_{\text{I}}^\dagger(\cdot)$ is admissible for $(\widetilde{Pj})$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{Pj})$. Hence one has

$$\hat{t} = t^*, \quad \text{i.e.} \quad u_{\widetilde{Pj}}(t) = 0 \quad \text{a.e. } t \in [0, t^*].$$

(II) For any feasible control $u(\cdot)$ such that $u(t) = 0$ over $[0, t^*]$, we denote $S_u(\cdot)$, $B_u(\cdot)$, $W_u(\cdot)$ the corresponding trajectories. Suppose that u is optimal for the problem $(\widetilde{Pj})$ and consider the sets E and E_1 as above. We set

$$G = \{t \in [t^*, T],\ S_u(t) > S^*\}.$$

Assume that $G \neq \emptyset$. Then we have

$$\int_G u_{sing}(t, S^*)dt < \int_G u(t)dt.$$

Let us consider the following control $u_{\text{II}}^\dagger(\cdot)$, and its associated solution $S_{u_{\text{II}}^\dagger}(\cdot)$, $B_{u_{\text{II}}^\dagger}(\cdot)$, $W_{u_{\text{II}}^\dagger}(\cdot)$, defined by

$$\begin{cases} u_{\text{II}}^\dagger(t) = 0, & t \in [0, t^*], \\ u_{\text{II}}^\dagger(t) = u_{sing}(t, S^*), & t \in G, \\ u_{\text{II}}^\dagger(t) = u(t), & t \in E \setminus E_1 \text{ and } S_{u_{\text{II}}^\dagger}(t) < 1, \\ u_{\text{II}}^\dagger(t) = \min(u(t), 1/k_2), & t \in E \setminus E_1 \text{ and } S_{u_{\text{II}}^\dagger}(t) = 1 \\ u_{\text{II}}^\dagger(t) \in [u(t), 1], & t \in E \cap E_1, \end{cases} \tag{3.14}$$

with

$$0 < \int_{E\cap E_1} u_{\text{II}}^\dagger(t) - u(t)dt < \int_G u(t) - u_{sing}(t, S^*)dt.$$

Then one has

$$\begin{aligned} W_{u_{\text{II}}^\dagger}(T) &= \int_G u_{sing}(t, S^*)dt + \int_{E\setminus E_1} u_{\text{II}}^\dagger(t)dt + \int_{E\cap E_1} u_{\text{II}}^\dagger(t)dt \\ &< \int_G u(t)dt + \int_{E\setminus E_1} u(t)dt + \int_{E\cap E_1} u(t)dt. \end{aligned}$$

The associated solution $S_{u_{\text{II}}^\dagger}(\cdot)$ satisfies

$$\forall t \in [0, T],\ S_{u_{\text{II}}^\dagger}(t) \leq 1\ ; \quad \forall t \in G,\ S_{u_{\text{II}}^\dagger}(t) \geq S^*\ ; \quad \forall t \in [0, T] \setminus G,\ S_{u_{\text{II}}^\dagger}(t) \geq S_u(t),$$

with

$$\int_{E\cap E_1} S_{u_{\text{II}}^\dagger}(t)dt > \int_{E\cap E_1} S_u(t)dt.$$

Moreover one has $S_u(t) \leq S^*$ over $E \cap E_1$, and there exists $J \subset E \cap E_1$ of non null measure such that $S_u(t) < S^*$ over J. Then

$$\int_{E\cap E_1} \varphi(t) K_S(S_{u_{\text{II}}^\dagger}(t))dt > \int_{E\cap E_1} \varphi(t) K_S(S_u(t))dt.$$

This leads to

$$B_{u_{\text{II}}^\dagger}(T) > \int_{[0,t^*]\cup G} \varphi(t) K_S(S_u(t))dt + \int_{E\cap E_1} \varphi(t) K_S(S_u(t))dt + \int_{E\setminus E_1} \varphi(t) K_S(S_u(t))dt.$$

To conclude, we have

$$B_{u_{\text{II}}^\dagger}(T) > B_u(T) \quad \text{and} \quad W_{u_{\text{II}}^\dagger}(T) < W_u(T).$$

Thus the control $u_{\text{II}}^\dagger(\cdot)$ is admissible for $(\widetilde{Pj})$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{Pj})$. Hence one has

$$G = \emptyset\ , \quad \text{i.e.} \quad S_{\widetilde{Pj}}(t) \leq S^*\ , \quad \forall t \in [t^*, T].$$

□

Remark 3.3.4. *The upper state constraint $S \leq 1$ is then never saturated by an optimal trajectory $S_{\widetilde{Pj}}(\cdot)$ under water scarcity assumption, except from initial time if the initial condition is $S(0) = 1$.*

Proposition 3.3.5. *Let $u_{\widetilde{P1}}(\cdot), u_{\widetilde{P2}}(\cdot)$ be the optimal controls of their respective problems and we denote their corresponding solutions by $(S_{\widetilde{P1}}(\cdot), B_{\widetilde{P1}}(\cdot), W_{\widetilde{P1}}(\cdot))$ and $(S_{\widetilde{P2}}(\cdot), B_{\widetilde{P2}}(\cdot), W_{\widetilde{P2}}(\cdot))$. Then*

$$\begin{aligned} &(\text{III}): \quad W_{\widetilde{P1}}(T) = Q, \\ &(\text{IV}): \quad B_{\widetilde{P2}}(T) = Y. \end{aligned}$$

Proof. (III) Consider a feasible control $u(\cdot)$ such that $u(t) = 0$ on $[0, t^*]$ and that the corresponding solution $S_u(\cdot), B_u(\cdot), W_u(\cdot)$ satisfies $S_u(t) \leq S^*$ for all $t \in [t^*, T]$. Suppose that u is optimal for the problem $(\widetilde{P1})$, and consider the set E_1 as previously.

Assume that $W_u(T) < Q$. There exists $J \subset [t^*, T]$ of non null measure such that $S_u(t) < S^*$ over J, and $J \cap E_1$ has non null measure. Let us consider the control $u_{\text{III}}^\dagger(\cdot)$, and its associated solution $S_{u_{\text{III}}^\dagger}(\cdot)$, $B_{u_{\text{III}}^\dagger}(\cdot)$, $W_{u_{\text{III}}^\dagger}(\cdot)$, defined by

$$\begin{cases} u_{\text{III}}^\dagger(t) = u(t), & t \in [0,T] \setminus (J \cap E_1), \\ u_{\text{III}}^\dagger(t) = \min(u(t), 1/k_2), & t \in J \cap E_1 \text{ and } S_{u_{\text{III}}^\dagger}(t) = 1, \\ u_{\text{III}}^\dagger(t) \in [u(t), 1], & t \in J \cap E_1 \text{ and } S_{u_{\text{III}}^\dagger}(t) < 1, \end{cases} \tag{3.15}$$

with

$$0 < \int_{(J\cap E_1)\cap\{S_{u_{\text{III}}^\dagger} < 1\}} u_{\text{III}}^\dagger(t) - u(t)dt < Q - W_u(T).$$

Then one has

$$\begin{aligned} W_{u_{\text{III}}^\dagger}(T) &= \int_{(J\cap E_1)\cap\{S_{u_{\text{III}}^\dagger} < 1\}} u_{\text{III}}^\dagger(t)dt + \int_{(J\cap E_1)\cap\{S_{u_{\text{III}}^\dagger} = 1\}} u_{\text{III}}^\dagger(t)dt + \int_{[0,T]\setminus(J\cap E_1)} u(t)dt \\ &< Q - W_u(T) + \int_{[0,T]} u(t)dt. \end{aligned}$$

The associated solution $S_{u_{\text{III}}^\dagger}(\cdot)$ satisfies

$$\forall t \in [0,T],\ S_{u_{\text{III}}^\dagger}(t) \leq 1 \quad \text{and} \quad \forall t \in [0,T],\ S_{u_{\text{III}}^\dagger}(t) \geq S_u(t),$$

with

$$\int_{J\cap E_1} S_{u_{\text{III}}^\dagger}(t)dt > \int_{J\cap E_1} S_u(t)dt.$$

Moreover one has $S_u(t) < S^*$ over $J \cap E_1$, then

$$\int_{J\cap E_1} \varphi(t) K_S(S_{u^\dagger_{\text{III}}}(t))dt > \int_{J\cap E_1} \varphi(t) K_S(S_u(t))dt.$$

This leads to

$$B_{u^\dagger_{\text{III}}}(T) > \int_{J\cap E_1} \varphi(t) K_S(S_u(t))dt + \int_{[0,T]\setminus(J\cap E_1)} \varphi(t) K_S(S_u(t))dt.$$

To conclude, we have

$$B_{u^\dagger_{\text{III}}}(T) > B_u(T) \quad \text{and} \quad W_{u^\dagger_{\text{III}}}(T) < Q.$$

Thus the control $u^\dagger_{\text{III}}(\cdot)$ is admissible for $(\widetilde{P1})$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{P1})$. Hence one has

$$W_{u_{\widetilde{P1}}}(T) = Q.$$

(IV) Consider a feasible control $u(\cdot)$ such that $u(t) = 0$ on $[0, t^*]$ and that the corresponding solution $S_u(\cdot)$, $B_u(\cdot)$, $W_u(\cdot)$ satisfies $S_u(t) \le S^*$ for all $t \in [t^*, T]$. Suppose that u is optimal for the problem $(\widetilde{P2})$.

Assume that $B_u(T) > Y$. We set

$$E_2 = \{t \in [t^*, T],\ u(t) > 0\}.$$

The set E_2 has non null measure, and let $t_2 = \operatorname{ess\,sup} E_2$.

If $t_2 = T$, set $I_\varepsilon = [T - \varepsilon, T]$, for $\varepsilon > 0$ small enough, and consider the control $u^\dagger_{\text{IV}}(\cdot)$, with its associated solution $S_{u^\dagger_{\text{IV}}}(\cdot)$, $B_{u^\dagger_{\text{IV}}}(\cdot)$, $W_{u^\dagger_{\text{IV}}}(\cdot)$, defined by

$$\begin{cases} u^\dagger_{\text{IV}}(t) = u(t), & t \in [0,T] \setminus I_\varepsilon, \\ u^\dagger_{\text{IV}}(t) = 0, & t \in I_\varepsilon. \end{cases}$$

If $t_2 < T$, set $I_\varepsilon = \left[t_2 - \frac{\varepsilon}{2}, t_2 + \frac{\varepsilon}{2}\right]$, for $\varepsilon > 0$ small enough, and consider the control $u^\dagger_{\text{IV}}(\cdot)$, with its associated solution $S_{u^\dagger_{\text{IV}}}(\cdot)$, $B_{u^\dagger_{\text{IV}}}(\cdot)$, $W_{u^\dagger_{\text{IV}}}(\cdot)$, defined by

$$\begin{cases} u^\dagger_{\text{IV}}(t) = u(t), & t \in [0,T] \setminus I_\varepsilon, \\ u^\dagger_{\text{IV}}(t) = \tilde{u}^{S_{ope}}_\vee(t), & t \in I_\varepsilon, \end{cases}$$

where $\tilde{u}^{S_{ope}}_\vee$ is the MRAP to the value S_{ope} from above, over the interval I_ε.

The following arguments hold in both cases. One has

$$0 < \int_{I_\varepsilon} u(t) - u^\dagger_{\text{IV}}(t)dt \le \varepsilon. \tag{3.16}$$

The associated solution $S_{u^\dagger_{\text{IV}}}(\cdot)$ satisfies

$$\forall t \in [0,T] \setminus I_\varepsilon,\ S_{u^\dagger_{\text{IV}}}(t) = S_u(t) \quad \text{and} \quad \forall t \in I_\varepsilon,\ S_{u^\dagger_{\text{IV}}}(t) \le S_u(t),$$

with

$$\int_{I_\varepsilon} S_u(t) - S_{u^\dagger_{\text{IV}}}(t)dt > 0.$$

Moreover one has $S_u(t) \le S^*$ over I_ε. Then

$$\int_{I_\varepsilon} \varphi(t) K_S(S_{u^\dagger_{\text{IV}}}(t))dt < \int_{I_\varepsilon} \varphi(t) K_S(S_u(t))dt,$$

and

$$0 < \int_0^T \varphi(t)\Big(K_S(S_u(t)) - K_S(S_{u_{\text{IV}}^\dagger}(t))\Big)dt.$$

Let $t \in I_\varepsilon$. From the dynamics of S, one has

$$\begin{aligned} S_u(t) - S_{u_{\text{IV}}^\dagger}(t) = k_1 k_2 \int_{t^*}^t u(\tau) - u_{\text{IV}}^\dagger(\tau) d\tau \\ + k_1 \int_{t^*}^t \varphi(\tau)\Big(K_S(S_{u_{\text{IV}}^\dagger}(\tau)) - K_S(S_u(\tau))\Big)d\tau \\ + k_1 \int_{t^*}^t (1-\varphi(\tau))\Big(K_R(S_{u_{\text{IV}}^\dagger}(\tau)) - K_R(S_u(\tau))\Big)d\tau. \end{aligned}$$

As a consequence, one has

$$\big|S_u(t) - S_{u_{\text{IV}}^\dagger}(t)\big| \leq k_1 k_2 \varepsilon + k_1 \int_{t^*}^t \Big(\varphi(\tau)L_S + (1-\varphi(\tau))L_R\Big)\big|S_u(\tau) - S_{u_{\text{IV}}^\dagger}(\tau)\big|d\tau.$$

By the Grönwall Lemma, we deduce that

$$\big|S_u(t) - S_{u_{\text{IV}}^\dagger}(t)\big| \leq \varepsilon k_1 k_2 \exp\left(\int_0^T k_1\Big(\varphi(\tau)L_S + (1-\varphi(\tau))L_R\Big)d\tau\right) := \varepsilon K.$$

Then

$$0 < B_u(T) - B_{u_{\text{IV}}^\dagger}(T) \leq \varepsilon \int_{I_\varepsilon} \varphi(t) L_S K dt \leq \varepsilon^2 L_S K.$$

For $\varepsilon > 0$ small enough, we have

$$0 < B_u(T) - B_{u_{\text{IV}}^\dagger}(T) \leq \varepsilon^2 L_S K < B_u(T) - Y.$$

With Equation (3.16), it leads to

$$Y < B_{u_{\text{IV}}^\dagger}(T) \quad \text{and} \quad W_{u_{\text{IV}}^\dagger}(T) < W_u(T).$$

To conclude, the control $u_{\text{IV}}^\dagger(\cdot)$ is admissible for $(\widetilde{P2})$, and the control $u(\cdot)$ is therefore non optimal for $(\widetilde{P2})$. Hence one has

$$B_{\widetilde{P2}}(T) = Y.$$

□

Remark 3.3.6. *Consider an optimal solution $u_{\widetilde{P3}}$ to the problem $(\widetilde{P3})$, with its associated trajectory $(S_{\widetilde{P3}}(\cdot), B_{\widetilde{P3}}(\cdot), W_{\widetilde{P3}}(\cdot))$ and its associated optimal cost $J_{\widetilde{P3}} = g(B_{\widetilde{P3}}(T)) - c(W_{\widetilde{P3}}(T))$. From properties above and Lemma 3.2.6 we deduce that $W_{\widetilde{P3}}(T) \in [V_{ope}, V^*]$.*

If $W_{\widetilde{P3}}(T) = V_{ope}$ then $u_{\widetilde{P3}} = u_{ope}$, and if $W_{\widetilde{P3}}(T) = V^*$ then $u_{\widetilde{P3}} = u^*$. They correspond to trivial trajectories and these situations will then not be considered in the rest of our analysis.

If $W_{\widetilde{P3}}(T) \in (V_{ope}, V^*)$, we can set $Q = W_{\widetilde{P3}}(T) \in (V_{ope}, V^*)$ and solve the problem $(\widetilde{P1})$ associated to the quota Q. The optimal solution $u_{\widetilde{P1}}$ and its corresponding trajectory $(S_{\widetilde{P1}}(\cdot)$, $B_{\widetilde{P1}}(\cdot)$, $W_{\widetilde{P1}}(\cdot))$ satisfy $W_{\widetilde{P1}}(T) = Q$ and $B_{\widetilde{P1}}(T) \geq B_{\widetilde{P3}}(T)$. To conclude, we have $J_{\widetilde{P3}} = g(B_{\widetilde{P1}}(T)) - c(W_{\widetilde{P1}}(T))$, i.e. there exists an optimal control of the problem $(\widetilde{P3})$ which is an optimal control of the problem $(\widetilde{P1})$.

3.4 Optimal synthesis

3.4.1 Application of the Pontryagin Maximum Principle

In this section, we apply the non-smooth Pontryagin Maximum Principle (see for instance [35, 78]) for the optimal control problems $(\widetilde{Pj})$ (without state constraint, due to Propositions 3.3.2 and 3.3.3). We write the optimality necessary conditions, and we show further properties about the signs of adjoint states.

3.4.1.1 Optimality necessary conditions

The Hamiltonian of the problem $(\widetilde{Pj})$ is written as follows:

$$\begin{aligned} H_{\widetilde{Pj}}(t,S,\lambda_S^{\widetilde{Pj}},\lambda_B^{\widetilde{Pj}},\lambda_W^{\widetilde{Pj}},u) = \lambda_S^{\widetilde{Pj}} k_1 \Big(k_2 u - (\varphi(t)K_S(S) + (1-\varphi(t))K_R(S)) \Big) \\ + \lambda_B^{\widetilde{Pj}} \varphi(t) K_S(S) + \lambda_W^{\widetilde{Pj}} (u + Cf(S)). \end{aligned} \tag{3.17}$$

We also write the associated adjoint equations, as follows:

$$\begin{cases} \dot{\lambda}_S^{\widetilde{Pj}}(t) \in \varphi(t)(\lambda_S^{\widetilde{Pj}}(t)k_1 - \lambda_B^{\widetilde{Pj}})\partial_C K_S(S_{\widetilde{Pj}}(t)) \\ \qquad\qquad +\lambda_S^{\widetilde{Pj}}(t)(1-\varphi(t))k_1 \partial_C K_R(S_{\widetilde{Pj}}(t)) - \lambda_W^{\widetilde{Pj}} C \partial_C f(S_{\widetilde{Pj}}(t)), \\ \dot{\lambda}_B^{\widetilde{Pj}} = 0, \\ \dot{\lambda}_W^{\widetilde{Pj}} = 0, \end{cases} \tag{3.18}$$

where ∂_C denotes the Clarke generalized gradient [35].

Denoting $X = (S,B,W)^\top$, for any optimal solution $u_{\widetilde{Pj}}, X_{\widetilde{Pj}}$ there exists an absolutely continuous solution of Equation (3.18) denoted by $\Lambda^{\widetilde{Pj}} = (\lambda_S^{\widetilde{Pj}}, \lambda_B^{\widetilde{Pj}}, \lambda_W^{\widetilde{Pj}})$ that maximizes Equation (3.17) and satisfies

$$\forall t \in [0,T],\ \left|\lambda_S^{\widetilde{Pj}}(t)\right| + \left|\lambda_B^{\widetilde{Pj}}\right| + \left|\lambda_W^{\widetilde{Pj}}\right| \neq 0, \tag{3.19}$$

and the following necessary transversality condition:

$$\Lambda^{\widetilde{Pj}}(T) \in -N_{K^{\widetilde{Pj}}}(X_{\widetilde{Pj}}(T)) + \mathrm{sgn}(\widetilde{Pj})\lambda_0^{\widetilde{Pj}} \nabla_X L^{\widetilde{Pj}}(X_{\widetilde{Pj}}), \tag{3.20}$$

with $\lambda_0^{\widetilde{Pj}} \geq 0$, and where $\mathrm{sgn}(\widetilde{Pj}) = 1$ if the cost is to be maximized, and $\mathrm{sgn}(\widetilde{Pj}) = -1$ if it is to be minimized. We used the notations N_K for the normal cone to the set K, $K^{\widetilde{Pj}}$ for the target sets and $L^{\widetilde{Pj}}$ for the cost functions.

More precisely, from the optimality necessary conditions, one has

$$(\widetilde{Pj}): \quad \lambda_S^{\widetilde{Pj}}(T) \geq 0, \tag{3.21a}$$

$$(\widetilde{P1}): \quad \lambda_B^{\widetilde{P1}} = \lambda_0^{\widetilde{P1}}, \quad \lambda_W^{\widetilde{P1}} \leq 0, \tag{3.21b}$$

$$(\widetilde{P2}): \quad \lambda_B^{\widetilde{P2}} \geq 0, \quad \lambda_W^{\widetilde{P2}} = -\lambda_0^{\widetilde{P2}}, \tag{3.21c}$$

$$(\widetilde{P3}): \quad \lambda_B^{\widetilde{P3}} = \lambda_0^{\widetilde{P3}} g'(B_{\widetilde{P3}}(T)), \quad \lambda_W^{\widetilde{P3}} = -\lambda_0^{\widetilde{P3}} c'(W_{\widetilde{P3}}(T)). \tag{3.21d}$$

We define the switching function as follows:

$$\psi^{\widetilde{Pj}}(t) = \lambda_S^{\widetilde{Pj}}(t)k_1k_2 + \lambda_W^{\widetilde{Pj}}. \tag{3.22}$$

Then maximizing the Hamiltonian Equation (3.17) provides for an optimal solution and for a.e. $t \in [0,T]$ the following conditions:

$$\begin{cases} u_{\widetilde{Pj}}(t) = 0, & \text{if } \psi^{\widetilde{Pj}}(t) < 0, \\ u_{\widetilde{Pj}}(t) \in [0,1], & \text{if } \psi^{\widetilde{Pj}}(t) = 0, \\ u_{\widetilde{Pj}}(t) = 1, & \text{if } \psi^{\widetilde{Pj}}(t) > 0. \end{cases} \tag{3.23}$$

Remark 3.4.1. *From the dynamics of S and Assumption 3.2.1, we get that the function $S_{\widetilde{Pj}}(\cdot)$ is decreasing when $\psi^{\widetilde{Pj}} < 0$ and increasing when $\psi^{\widetilde{Pj}} > 0$.*

3.4.1.2 Properties of adjoints

Proposition 3.4.2. *For any optimal solution to the problem $(\widetilde{Pj})$, one necessarily has $\lambda_S^{\widetilde{Pj}}(t) \geq 0$ for all $t \in [0,T]$.*

Proof. Suppose that there exists $\hat{t} \in [0,T]$ such that $\lambda_S^{\widetilde{Pj}}(\hat{t}) < 0$.

There exists $\varepsilon > 0$ such that $\lambda_S^{\widetilde{Pj}}(t) < 0$ for all $t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)$. Since $\lambda_W^{\widetilde{Pj}} \leq 0$ we deduce that the switching function $\psi^{\widetilde{Pj}}$ is negative over $(\hat{t}-\varepsilon, \hat{t}+\varepsilon)$. Therefore $S_{\widetilde{Pj}}(\cdot)$ is decreasing on $(\hat{t}-\varepsilon, \hat{t}+\varepsilon)$, and then $S_{\widetilde{Pj}}(t) \in (S_{ope}, 1)$ for all $t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)$. Consequently, for all $t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)$, one has

$$\dot{\lambda}_S^{\widetilde{Pj}}(t) \in \varphi(t)\Big(\lambda_S^{\widetilde{Pj}}(t)k_1 - \lambda_B^{\widetilde{Pj}}\Big)\partial_C K_S(S_{\widetilde{Pj}}(t)) + \lambda_S^{\widetilde{Pj}}(t)k_1(1-\varphi(t))\partial_C K_R(S_{\widetilde{Pj}}(t)).$$

Since $0 \leq \varphi \leq 1$, $k_1 > 0$, $\lambda_B^{\widetilde{Pj}} \geq 0$, $\partial_C K_S \subset \mathbb{R}^+$ and $\partial_C K_R \subset \mathbb{R}^+$, then one has

$$\forall t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon),\ \dot{\lambda}_S^{\widetilde{Pj}}(t) \in \mathbb{R}^-.$$

Let $A = \{t \in [0,T],\ \lambda_S^{\widetilde{Pj}}(t) < 0\}$. The function $\lambda_S^{\widetilde{Pj})}(\cdot)$ is then non increasing over A, which leads to a contradiction regarding the necessary terminal condition $\lambda_S^{\widetilde{Pj}}(T) \geq 0$. Hence one has

$$\forall t \in [0,T],\ \lambda_S^{\widetilde{Pj}}(t) \geq 0.$$

□

Proposition 3.4.3. *For any optimal solution to the problem $(\widetilde{Pj})$, one necessarily has $\lambda_W^{\widetilde{Pj}} < 0$.*

Proof. Assume that $\lambda_W^{\widetilde{Pj}} = 0$. Then, for all $t \in [0,T]$, one has

$$\dot{\lambda}_S^{\widetilde{Pj}}(t) \in \varphi(t)(\lambda_S^{\widetilde{Pj}}(t)k_1 - \lambda_B^{\widetilde{Pj}})\partial_C K_S(S_{\widetilde{Pj}}(t)) + \lambda_S^{\widetilde{Pj}}(t)k_1(1-\varphi(t))\partial_C K_R(S_{\widetilde{Pj}}(t)).$$

The function $\lambda_S^{\widetilde{Pj}}(\cdot)$ is non negative, and can vanish on a time interval of non null measure only if $0 \in \partial_C K_S(S_{\widetilde{Pj}}(t))$ on this interval. As a result, for any t such that $S_{\widetilde{Pj}}(t) \in (S_{ope}, S^*)$ we have $\lambda_S^{\widetilde{Pj}}(t) > 0$. Moreover one has $\psi^{\widetilde{Pj}}(t) = \lambda_S^{\widetilde{Pj}}(t)k_1k_2$, then for any t such that $S_{\widetilde{Pj}}(t) \in (S_{ope}, S^*)$ one has $\psi^{\widetilde{Pj}}(t) > 0$ thus $u_{\widetilde{Pj}}(t) = 1$, i.e. $S_{\widetilde{Pj}}(\cdot)$ increases when belonging to (S_{ope}, S^*).

We deduce that the trajectory $S_{\widetilde{Pj}}(\cdot)$ can not reach below S^*, thus from the comparison of trajectories in Lemma 3.2.6 we get $W_{\widetilde{Pj}}(T) \geq V^*$, which contradicts the framework hypothesis. Hence one has

$$\lambda_W^{\widetilde{Pj}} < 0.$$

□

Proposition 3.4.4. *For any optimal solution to the problem $(\widetilde{Pj})$, one necessarily has $\lambda_B^{\widetilde{Pj}} > 0$.*

Proof. Assume that $\lambda_B^{\widetilde{Pj}} = 0$. Then, for all $t \in [0,T]$, one has

$$\dot{\lambda}_S^{\widetilde{Pj}}(t) \in \lambda_S^{\widetilde{Pj}}(t)k_1\Big(\varphi(t)\partial_C K_S(S_{\widetilde{Pj}}(t)) + (1-\varphi(t))\partial_C K_R(S_{\widetilde{Pj}}(t))\Big) - \lambda_W^{\widetilde{Pj}} C\partial_C f(S_{\widetilde{Pj}}(t)).$$

We know from Proposition 3.3.3 that the trajectory of $S_{\widetilde{Pj}}(\cdot)$ is decreasing over $[0,t^*)$ and $S_{ope} \leq S_{\widetilde{Pj}}(t) \leq S^*$ for all $t \in [t^*,T]$. As a consequence, for any t, one has either $S_{\widetilde{Pj}}(t) = S_{ope}$ or $S_{\widetilde{Pj}}(t) > S_{ope}$.

Suppose that the set $N = \{t \in [0,T],\ S_{\widetilde{Pj}}(t) = S_{ope}\}$ has non null measure. Define $t^- = \operatorname{ess\,inf} N$ and $t^+ = \operatorname{ess\,sup} N$. The function $S_{\widetilde{Pj}}$ being neither decreasing nor increasing over N implies that

$\psi^{\widetilde{Pj}}(t) = 0$ for a.e. $t \in N$. Thus $\lambda_S^{\widetilde{Pj}}$ is equal to $\gamma^{\widetilde{Pj}} = -\frac{\lambda_W^{\widetilde{Pj}}}{k_1 k_2} > 0$ for a.e. $t \in N$.

First suppose that $t^+ < T$. Then one has $S_{\widetilde{Pj}}(t) > S_{ope}$ for all $t \in (t^+, T]$, as a consequence the function $\lambda_S^{\widetilde{Pj}}$ satisfies

$$\forall t \in (t^+, T],\ \dot{\lambda}_S^{\widetilde{Pj}}(t) \in \lambda_S^{\widetilde{Pj}}(t) k_1 \Big(\varphi(t)\partial_C K_S(S_{\widetilde{Pj}}(t)) + (1-\varphi(t))\partial_C K_R(S_{\widetilde{Pj}}(t))\Big). \tag{3.24}$$

Since $\lambda_S^{\widetilde{Pj}}(t^+) = \gamma^{\widetilde{Pj}} > 0$, then $\lambda_S^{\widetilde{Pj}}$ is increasing over $(t^+, T]$, implying that $\psi^{\widetilde{Pj}}(t) > 0$ for all $t \in (t^+, T]$, and then $S_{\widetilde{Pj}}$ is increasing over $(t^+, T]$. As a result, $S_{\widetilde{Pj}}(T) > S_{ope}$, then the transversality condition Equation (3.20) brings $\lambda_S^{\widetilde{Pj}}(T) = 0$, which is a contradiction.

We then have $t^+ = T$, and a similar argument as above brings that $\psi^{\widetilde{Pj}}(t) < 0$ for all $t \in [0, t^-)$. We deduce that $S_{\widetilde{Pj}}(t) = S_{0,S_0,u_{ope}(\cdot)}(t)$ for all $t \in [0, T]$. Then $B_{\widetilde{Pj}}(T) = B_{ope}$ which contradicts the framework hypothesis.

Finally, the set N has null measure, and $S_{\widetilde{Pj}}(t) > S_{ope}$ for a.e. $t \in [0, T]$. The equation Equation (3.24) is thus satisfied for a.e. $t \in [0, T]$, and the function $\lambda_S^{\widetilde{Pj}}$ is either increasing or constant equal to 0 on $[0, T]$. If there exists $\hat{t} \in (0, T)$ such that $\lambda_S^{\widetilde{Pj}}(t) > \gamma^{\widetilde{Pj}}$ for all $t \in (\hat{t}, T]$, then the trajectory $S_{\widetilde{Pj}}$ is increasing on $(\hat{t}, T]$. As previously, this would imply $\lambda_S^{\widetilde{Pj}}(T) = 0$ contradicting the fact that it is increasing. As a consequence, $\lambda_S^{\widetilde{Pj}}(t) \in [0, \gamma^{\widetilde{Pj}})$ for all $t \in [0, T]$, and the switching function thus verifies

$$\forall t \in [0, T],\ \psi^{\widetilde{Pj}}(t) < 0.$$

We deduce that $u_{\widetilde{Pj}}(t) = 0$ for a.e. $t \in [0, T]$, and then $B_{\widetilde{Pj}}(T) \leq B_{ope}$ which contradicts the framework hypothesis. Hence one has

$$\lambda_B^{\widetilde{Pj}} > 0.$$

□

Remark 3.4.5. *Finally, the constants $\lambda_B^{\widetilde{Pj}}$, $\lambda_W^{\widetilde{Pj}}$ both have strict sign (either positive or negative). As a result, all $\lambda_0^{\widetilde{Pj}} \neq 0$ and there is no abnormal extremal.*

In addition we deduce from previous arguments that for an optimal control of a problem $(\widetilde{Pj})$, there necessarily exists $\alpha > 0$, such that $u_{\widetilde{Pj}}(t) = 0$ for a.e. $t \in (T - \alpha, T]$.

3.4.2 Structures of optimal solutions

In this section, we aim at characterizing the structures of optimal solutions from the necessary optimality conditions. To this end, we first discuss about the possible existence of singular arcs and the loci where they can exist, and then we investigate the switching function in order to extract information about the number of switches.

3.4.2.1 Singular arcs

Proposition 3.4.6. *For any optimal solution to the problem $(\widetilde{Pj})$, a singular arc can only occur at $S_{\widetilde{Pj}} = S_{ope}$ or $S_{\widetilde{Pj}} = S^*$.*

Proof. The problems $(\widetilde{P1})$ and $(\widetilde{P2})$ are written with piecewise affine and continuous dynamics with respect to the state variables, and their cost can be rewritten as an integral cost with a piecewise affine continuous integrand function. Moreover, following Remark 3.3.6, either the problem $(\widetilde{P3})$ is trivial (and presents singular arcs at S_{ope} or S^*), or it can be solved with an optimal control of the problem $(\widetilde{P1})$.

As a consequence, the optimal control problems $(\widetilde{Pj})$ fulfill the framework of [27], and we deduce that there exists an optimal solution of $(\widetilde{Pj})$ composed of a succession of bang-bang arcs over time interval when the trajectory belongs to a region where the dynamics Equation (3.11) is differentiable, and possible singular arcs located where the dynamics Equation (3.11) is non-differentiable, or when a state constraint is active.

By definition of the functions K_S and K_R, and using optimality properties of Section 3.3, we deduce that a singular arc can only occur at the non-differentiable points S_{ope} and S^*. □

Remark 3.4.7. *We see that the loci of singular arcs are the non-differentiable points of the dynamics, therefore the use of more generic shapes for the functions K_S and K_R as discussed in Remark 3.2.1 is compatible with our methodological process. In particular, more non-differentiable points in K_S and K_R would simply generate a higher number of possible values of singular arcs.*

3.4.2.2 Analysis of switching function

Proposition 3.4.8. *For any optimal solution to the problem $(\widetilde{Pj})$, the set $I_{\widetilde{Pj}}$ defined by*

$$I_{\widetilde{Pj}} = \{t \in [t^*, T],\ \psi^{\widetilde{Pj}}(t) \geq 0\}$$

is non-empty and connected.

Proof. First notice that $I_{\widetilde{Pj}}$ has non null measure, otherwise we would have $u_{\widetilde{Pj}}(t) = 0$ for a.e. $t \in [0, T]$, which cannot be optimal.

Suppose that $I_{\widetilde{Pj}}$ is not connected. In other words, assume that there exists an interval $(t_1, t_2) \subset [t^*, T]$ such that $\psi^{\widetilde{Pj}}(t_1) = \psi^{\widetilde{Pj}}(t_2) = 0$ and $\psi^{\widetilde{Pj}}(t) < 0$ for all $t \in (t_1, t_2)$. This implies $u(t) = 0$ for a.e. $t \in (t_1, t_2)$ and thus $S_{\widetilde{Pj}}(\cdot)$ is decreasing on (t_1, t_2), which means that $S_{ope} < S_{\widetilde{Pj}}(t) < S^*$ for all $t \in (t_1, t_2)$. Therefore the function $\lambda_S^{\widetilde{Pj}}(\cdot)$ satisfies

$$\forall t \in (t_1, t_2),\ \dot{\lambda}_S^{\widetilde{Pj}}(t) \in \varphi(t)(\lambda_S^{\widetilde{Pj}}(t)k_1 - \lambda_B^{\widetilde{Pj}})\partial_C K_S(S_{\widetilde{Pj}}(t)) + \lambda_S^{\widetilde{Pj}}(t)k_1(1-\varphi(t))\partial_C K_R(S_{\widetilde{Pj}}(t))$$

Moreover, the function $\psi^{\widetilde{Pj}}$ attains a local minimum over (t_1, t_2), at a time denoted $\hat{t} \in (t_1, t_2)$. Since $\psi^{\widetilde{Pj}}(t) = \lambda_S^{\widetilde{Pj}}(t)k_1k_2 + \lambda_W^{\widetilde{Pj}}$ then the function $\lambda_S^{\widetilde{Pj}}$ also attains its local minimum over (t_1, t_2) at the time $\hat{t}$. At this local minimum, one has

$$0 \in \varphi(\hat{t})(\lambda_S^{\widetilde{Pj}}(\hat{t})k_1 - \lambda_B^{\widetilde{Pj}})\partial_C K_S(S_{\widetilde{Pj}}(\hat{t})) + \lambda_S^{\widetilde{Pj}}(\hat{t})k_1(1-\varphi(\hat{t}))\partial_C K_R(S_{\widetilde{Pj}}(\hat{t})). \tag{3.25}$$

First note that Equation (3.25) can be fulfilled only if $\lambda_S^{\widetilde{Pj}}(\hat{t})k_1 - \lambda_B^{\widetilde{Pj}} < 0$. Since we have $S_{ope} < S_{\widetilde{Pj}}(t) < S^*$ for all $t \in (t_1, t_2)$, the quantities $K'_S(S_{\widetilde{Pj}}(t))$ and $K'_R(S_{\widetilde{Pj}}(t))$ are well defined and constant on (t_1, t_2), respectively equal to $K'_S(S_{\widetilde{Pj}}(\hat{t})) > 0$ and $K'_R(S_{\widetilde{Pj}}(\hat{t})) > 0$. Then, for all $t \in (t_1, \hat{t})$, one has

$$\begin{aligned}
\dot{\lambda}_S^{\widetilde{Pj}}(t) &= \varphi(t)\big(\lambda_S^{\widetilde{Pj}}(t)k_1 - \lambda_B^{\widetilde{Pj}}\big)K'_S(S_{\widetilde{Pj}}(t)) + \lambda_S^{\widetilde{Pj}}(t)k_1(1-\varphi(t))K'_R(S_{\widetilde{Pj}}(t)) \\
&\geq \varphi(t)\big(\lambda_S^{\widetilde{Pj}}(\hat{t})k_1 - \lambda_B^{\widetilde{Pj}}\big)K'_S(S_{\widetilde{Pj}}(\hat{t})) + \lambda_S^{\widetilde{Pj}}(\hat{t})k_1(1-\varphi(t))K'_R(S_{\widetilde{Pj}}(\hat{t})) \\
&> \varphi(\hat{t})\big(\lambda_S^{\widetilde{Pj}}(\hat{t})k_1 - \lambda_B^{\widetilde{Pj}}\big)K'_S(S_{\widetilde{Pj}}(\hat{t})) + \lambda_S^{\widetilde{Pj}}(\hat{t})k_1(1-\varphi(\hat{t}))K'_R(S_{\widetilde{Pj}}(\hat{t})).
\end{aligned} \tag{3.26}$$

The last inequality holds since φ takes values in $[0, 1]$ and is increasing.

We deduce from Equation (3.26) that $\dot{\lambda}_S^{\widetilde{Pj}}(t) > 0$ for all $t \in (t_1, \hat{t})$, which contradicts the local minimality of $\lambda_S^{\widetilde{Pj}}(\hat{t})$ over (t_1, t_2). Finally we conclude that the set $I_{\widetilde{Pj}}$ is connected. □

3.4.3 Main result

We now define a class of time-varying feedback controls, and we prove that the optimal solutions to the problems $(\widetilde{Pj})$ belong to this class of strategies.

Definition 3.4.1. For $t_d \in [t^*, T]$, $W_c \in [V_{ope}, V^*]$, $t_c \in [t_d, T]$, we define the following time-varying feedback control:

$$U^{\widetilde{Pj}}_{t_d,W_c,t_c}(t,S,W) = \begin{cases} 0, & \big(S \geq S_{ope} \wedge t \leq t_d\big) \vee W = W_c \vee t \geq t_c, \\ u_{sing}(t, S_{ope}), & S = S_{ope} \text{ and } t \leq t_d, \\ u_{sing}(t, S^*), & S = S^* \text{ and } \big(W < W_c \wedge t < t_c\big), \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3.4.9. *In the framework of Definition 3.3.1 for the optimal control problems given in Definition 3.3.2 with the dynamics Equation* (3.11)*, there exists $t_d \in [t^*, T]$, $W_c \in [V_{ope}, V^*]$, $t_c \in [t_d, T]$ such that the feedback control $U^{\widetilde{Pj}}_{t_d,W_c,t_c}$ is optimal for the problem $(\widetilde{Pj})$.*

Proof. Let $u_{\widetilde{Pj}}(\cdot)$ be an optimal control for the problem $(\widetilde{Pj})$. As seen in Proposition 3.4.8, the set $I_{\widetilde{Pj}}$ is non-empty and connected. Since the function $\psi^{\widetilde{Pj}}(\cdot)$ is continuous, we deduce that $I_{\widetilde{Pj}}$ is an interval $[t_a, t_c]$ where $t^* \leq t_a < t_c \leq T$, and the control $u_{\widetilde{Pj}}(\cdot)$ is null outside this interval.

At any $t \in I_{\widetilde{Pj}}$ the switching function $\psi^{\widetilde{Pj}}(t)$ is non-negative, then for a.e. $t \in I_{\widetilde{Pj}}$ one has either $u_{\widetilde{Pj}}(t) = 1$ or $u_{\widetilde{Pj}}(t) = u_{sing}(t, \tilde{S})$ where $\tilde{S} \in \{S_{ope}, S^*\}$. As a consequence, the quantity $W_{\widetilde{Pj}}(\cdot)$ is increasing on $I_{\widetilde{Pj}}$ and $S_{\widetilde{Pj}}(\cdot)$ is non-decreasing on $I_{\widetilde{Pj}}$. More precisely the trajectory $S_{\widetilde{Pj}}(\cdot)$ is composed of one increasing part (with $u_{\widetilde{Pj}} = 1$) and possible constant parts (at $S_{\widetilde{Pj}} = S_{ope}$ before the increasing part, or at $S_{\widetilde{Pj}} = S^*$ after the increasing part). The increasing part is thus over an interval $[t_d, t_f]$ with $t_a \leq t_d < t_f \leq t_c$.

At $t = t_c$, we know that $S_{\widetilde{Pj}}(t_c) \leq S^*$ and since $W_{\widetilde{Pj}}$ is increasing on $I_{\widetilde{Pj}}$ we deduce that t_c and $W_{\widetilde{Pj}}(t_c)$ are equivalently defined, and we set $W_c = W_{\widetilde{Pj}}(t_c)$. We have either $t_a = t_d$ or t_a fully determined by the trajectory $S_{0,S_0,0}(\cdot)$ such that $S_{0,S_0,0}(t_a) = S_{ope}$. Moreover, the time t_f is either equal to t_c or fully determined by the trajectory $S_{t_d,S_{\widetilde{Pj}}(t_d),1}(\cdot)$ such that $S_{t_d,S_{\widetilde{Pj}}(t_d),1}(t_f) = S^*$.

Finally, the control $u_{\widetilde{Pj}}(\cdot)$ is fully determined by the two parameters t_d and t_c (or equivalently t_d and W_c). We check that it fulfills the feedback formulation $u_{\widetilde{Pj}}(t) = U^{\widetilde{Pj}}_{t_d,W_c,t_c}(t, S_{\widetilde{Pj}}(t), W_{\widetilde{Pj}}(t))$ for a.e. $t \in [0, T]$. □

Remark 3.4.10. *It may seem that the optimal parameterized feedback in Definition 3.4.1 is defined with three parameters (t_d, W_c, t_c). The proof of Theorem 3.4.9 hints that we only need two of these parameters. In fact we can be more precise for each problem $(\widetilde{Pj})$.*

For the problem $(\widetilde{P1})$, we set $W_c = Q$, and we can posit $t_c = T$ as convention. This way, the control $U^{\widetilde{P1}}_{t_d,Q,T}(t,S,W)$ belongs to the admissibility set $\mathcal{U}^{\widetilde{P1}}$, and it suffices to perform a one-dimensional optimization over $t_d \in [t^*, T]$ to find the optimal solution $u_{\widetilde{P1}}$.

For the problem $(\widetilde{P2})$, we can posit $W_c = V^*$ as convention, and for each t_d we can compute t_c such that the associated trajectory $(S(\cdot), B(\cdot), W(\cdot))$ reaches exactly $B(T) = Y$. This way, the control $U^{\widetilde{P2}}_{t_d,V^*,t_c}(t,S,W)$ belongs to the admissibility set $\mathcal{U}^{\widetilde{P2}}$, and it suffices to perform a one-dimensional optimization over $t_d \in [t^*, T]$ to find the optimal solution $u_{\widetilde{P2}}$.

For the problem $(\widetilde{P3})$, we don't have more information from its settings to help determining W_c or t_c, so we can just posit $W_c = V^*$ as convention. This way, the control $U^{\widetilde{P3}}_{t_d,V^*,t_c}(t,S,W)$ belongs to the admissibility set $\mathcal{U}^{\widetilde{P3}}$, and it will be needed perform a two-dimensional optimization over $t_d \in [t^*, T]$ and $t_c \in [t_d, T]$ to find the optimal solution $u_{\widetilde{P3}}$.

We illustrate this result in Figure 3.2 with an optimal control obtained for a typical set of parameters, and its corresponding optimal soil moisture trajectory.

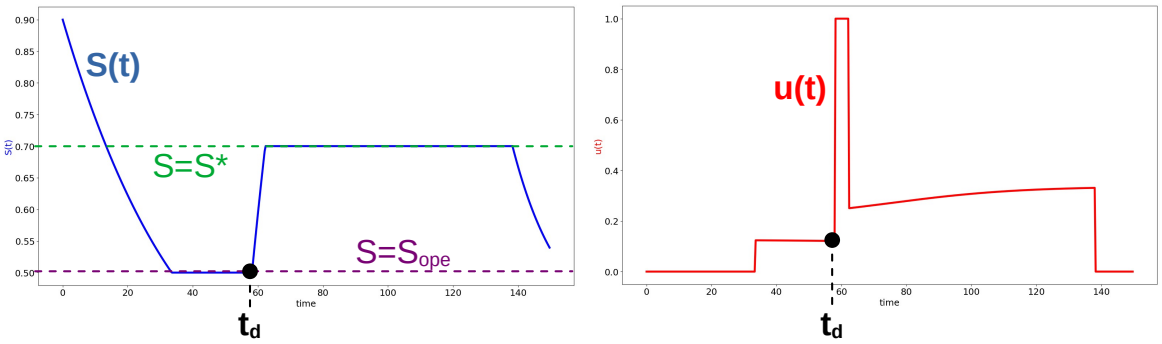


Figure 3.2: Graphical representations of a typical optimal control (on the right) with its associated optimal soil moisture trajectory (on the left), computed from the parameterization of the class of optimal strategies.

3.5 Conclusion

We dealt with a family of optimal control problems, and we showed, using dedicated tools (in particular an application of the extended bang-bang principle), that optimal irrigation strategies for these problems share the same structure of optimal feedback. These strategies can be interpreted as threshold-based strategies, where the values of thresholds correspond to non-differentiable points of the dynamics. This result allows to compute numerically the optimal solutions as a simple low dimensional optimization of the parameters of the structured optimal policies, hence enabling large numerical model exploration. Such threshold-based irrigation strategies can also be tested on more details models (for which the analytic study of the optimal feedback strategies would not be possible), and are suited to closed-loop experimental implementation, typically using soil moisture sensors.

Chapter 4

Exploring the optimality of Threshold-Based crop Irrigation feedback Strategies

This chapter corresponds to Article [26], which in preparation for an upcoming submission. This work is based on the properties of the Controlled Crop Irrigation (CCI) model, for which it has already been proven (see Chapter 3) that optimal policies are Threshold-Based Irrigation Strategies. We aim at studying both the optimal TBIS (the number of thresholds, their values and the most influential parameters which dictate the shape of optimal irrigation scenarios), and their sub-optimal declensions (the effects of errors made on their determination). Much attention is drawn to the crucial "decision time" at which the operator decides to increase soil water content from the low soil moisture threshold (corresponding to the risk aversion level) to the high soil moisture threshold (corresponding to the lowest level of maximal biomass production rate). To these ends, we couple known results from optimal control theory and numerical methods for sensitivity analysis. We show that the optimal decision time is mostly driven by parameters related to plant growth dynamics and the evaluation of stress effects on biomass production, and that suboptimal decisions induce a trade-off between biomass production, irrigation water use and plant water stress. We provide visuals and we explain their qualitative interpretation for operating purposes of efficient water management.

4.1 Introduction

Threshold-Based Irrigation Strategies (TBIS) produce irrigation scenarios, i.e. chronicles of irrigation events, from the application of decision rules in which irrigation is triggered when selected values of one or several variables of the soil-plant-atmosphere system are reached, or assumed to be reached, via observations, measurements or modeling. This leaves the possibility of considering different irrigation triggering criteria, associated with different irrigation amounts, across the different stages of the cropping season. However, the settings, setpoints and objectives of irrigation are most often predefined thus neither reconsidered nor adjusted during the course of the season. Such ambitions are rather left for ex-post analyses or ex-ante planning purposes, facing the issue of relevant if not optimal resource management. The classical problem is to reduce water consumption under a certain limit (quota) while still ensuring enough biomass production (objective) and economic viability, for various operational contexts and environmental constraints.

TBIS are common practice in irrigation management, either (i) implicitly, in an informal way, throughout expert irrigation decisions [38, 93, 105] aiming to counterbalance the observed detrimental effects of unobserved resource depletion, or (ii) explicitly via indications on soil water status drawn from soil water content or matric potential measurements [53, 202, 203], whereas (iii) modeling tools and their decision support declensions [17, 60, 40, 65, 92, 106, 162] remain surprisingly rarely used in practice. And yet, apart from costly and risky trial-and-error attempts, this is the only approach that allows generating, comparing and improving scenarios of water consumption,

biomass production and economic yield, as controlled by the tested decision rules, thresholds and associated parameter sets.

TBIS require choosing threshold values endowed with physical sense from the points of view of soil physics, crop physiology and root water uptake properties and performance, while also playing on the explored ranges of values for the sake of optimization purposes. Chosen threshold values are expected to be related to characteristic values of soil water content. These values are θ_{sat} the saturation value, when all soil pores are filled with water, θ_{fc} the field capacity value, above which drainage occurs as water contents between θ_{sat} and θ_{fc} are non-equilibrium conditions, θ_x a quite lower value, whose definition and evaluation are uncertain, under which water uptake by plant roots becomes difficult and incomplete, hampering crop transpiration process as the result of incipient water stress, θ_{wp} a far lower value, known as the permanent wilting point, at which water uptake by the roots becomes impossible, and θ_r the residual water content, not even accessible to evaporation [3]. The objective assigned to optimization is thus to maximize biomass (in this paper) or benefit (in general) for known constraints on system performances and total water consumption afforded, while staying within realistic admissible limits in terms of soil water content.

Ancestral habits, field-based knowledge and common sense lead to an intuitive but fuzzy compromise between favoring crop production by avoiding too low water contents, staying above θ_x (if possible), and sparing water by staying under θ_{fc} but only in situations in which water costs are significant, water availability is really limited and irrigation amounts may be properly adjusted. More often than not, the irrigators' behaviour is quite different between the initial part of the cropping season and the quick growth, most productive part of it. A current practice, in the frame of deficit irrigation, is indeed to delay the first irrigation and provide limited water amounts in the first series of irrigations, both assuming limited early crop demands and aiming to favor root deepening, thus tolerating lower water contents and lower irrigation triggering thresholds than in the productive stage [25, 94, 144]. In the latter, the rate of biomass production is deemed strongly correlated to the absence of any significant water stress, while the cumulative biomass produced will benefit from low cumulative water stresses [32, 66, 107]. On the other hand, even for models having proven their ability to reproduce crop development and production in various mild stress conditions, literature shows a lack of frameworks, methodologies or formal approaches for exploring the merits and drawbacks of different TBIS [172].

As regards mathematical formalisms and methods to be employed, irrigation planning can be seen as a "control problem", which consists in "finding a control" (an irrigation chronicle following decision rules) for a dynamical system (coupled model equations describing soil filling and biomass production in dynamic interaction) over a period of time (the full cropping season) to handle the evolution of the system according to a certain objective. Taking advantage of advanced control theory in agriculture presents a high potential to improve operating efficiency [48, 145]. Some studies investigated the irrigation problem through the model predictive control point of view [47, 149, 156], targeting adaptive and robust strategies regarding unknown entries like weather or model uncertainties. In a purely deterministic approach, other studies addressed the problem of optimizing irrigation planning, using optimal control tools where a certain objective function is optimized (e.g. final biomass production). To this end, several analytical formalisms have been used to investigate the optimal irrigation problem with various points of view [91, 112, 154, 157]. The choice made in this paper is to address the generic, methodological objective by resorting to a parsimonious Controlled Crop Irrigation (CCI) model [19, 28], prone to analytical optimization techniques but thought to possess enough physical sense to describe the order-1 biophysical processes at play. The CCI model is suited to fully controlled conditions, assuming all obtained effects are due to irrigation (greenhouse conditions, associated crops, well-gauged contexts). In addition, the mentioned context (greenhouses) certainly allows feedback loop (process where the output observation can drive the input action), allowing to stick to the predefined plan, within closed-loop modeling. So TBIS typically are control feedback strategies that naturally follow the CCI model approach.

The originality of our work is to bring, thanks to the CCI model framework, a coupled analysis of optimal TBIS strategies gathering results from the optimal control theory and statistical methods of sensitivity analyses. Indeed, the CCI model has the key property to present optimal solution

in terms of TBIS, and is suited to a large numerical exploration. A first objective is to find what characterizes the structure of optimal TBIS, for a wide variety of contexts (soil capacity and initial filling, climatic demand, crop type) and constraints (water quota, irrigation rate). A closely-related objective, of applied interest, is to assess the effects (on water consumption, biomass production and overall irrigation efficiency) of mistakes committed on the various thresholds. In coherence, the present work now explores the optimality or sub-optimality of various sets of irrigation-triggering thresholds.

4.2 Material and methods

4.2.1 Threshold-Based Irrigation Strategies

A TBIS is composed of time periods when the farmer does not irrigate, time periods when the farmer irrigates at maximum rate, and time periods when the flow of irrigation is adapted to have the soil moisture remaining constant at or near a chosen target (threshold) value, possibly oscillating around it across repeated cycles of water input then consumption, via transpiration and evaporation. To be general, let us renormalize the irrigation rate by its maximal value. That is, if the irrigation rate is called Ψ, then let us consider that no irrigation means setting $\Psi = 0$, an irrigation rate based on different threshold values of the soil moisture across time will be denoted $\widetilde{\Psi}$ and is supposed to be valued between 0 and 1, and maximum irrigation rate means setting $\Psi = 1$. Then a TBIS can be characterized by corresponding "decision rules", denoted R_0 (for no irrigation), R_{S_k} (for an irrigation based on a soil moisture threshold value S_k) and R_1 (for maximum irrigation). These rules can depend on time, soil moisture S and a vector of model decision variables D. The set of decision rules is supposed to be complete and mutually compatible, i.e. at any time (or situation) there is exactly one decision rule that holds true. This description of TBIS may be written as a time-dependent feedback instruction, as follows:

$$\Psi(t,S,V) = \begin{cases} 0, & \text{if } R_0(t,S,D) \text{ is true,} \\ \widetilde{\Psi}(t,S_k), & \text{if } R_{S_k}(t,S,D) \text{ is true,} \\ 1, & \text{if } R_1(t,S,D) \text{ is true.} \end{cases} \tag{4.1}$$

4.2.2 The Controlled Crop-Irrigation model

4.2.2.1 Model description

We denote here as CCI the Controlled Crop Irrigation model introduced in [19, 28, 91] and based on in the initial formulation [133] related to the FAO crop water simulation model Aquacrop [170]. Basically, the CCI model is a parsimonious dynamical system of ODEs suitable for applying the optimal control theory [91]. The CCI model has only three state variables : soil moisture (S), crop biomass (B) and water budget consumption (V).

As in [133], soil properties are assumed to be spatially homogeneous with an even plant spacing, not needing to consider and spatial pattern or heterogeneity within the modeled area. With characteristics seen in such a spatially-averaged way, the CCI model consists in a small set of ordinary differential equations representing soil moisture, biomass production and water budget consumption, during a growing season denoted by a time interval $[0, T]$. In the present work, contents of soil water S will be normalized between 0 and 1, rescaling $\theta = \theta_{fc}$ to $S = 1$ (and keeping $\theta = 0$ to $S = 0$). Taking $S = 1$ at θ_{fc} means not considering drainage, following the framework of controlled greenhouse conditions, justifying a deterministic model. The CCI model is written as a continuous-time dynamical system, that can be understood as a system evolving with a certain variation per unit of time (for example per day), and parameters representing rates can thus be seen as quantities per unit of time (for example daily).

The equations describing the variations of S, B and V over time write

$$\dot{S}(t) = \frac{ET_0}{Vol}\Big(- \varphi(t)K_S(S(t)) - (1-\varphi(t))K_R(S(t)) + \frac{F_{max}}{ET_0}u(t)\Big), \tag{4.2}$$

$$\dot{B}(t) = k_3 \varphi(t) K_S(S(t)), \tag{4.3}$$

$$\dot{V}(t) = F_{max} u(t), \tag{4.4}$$

where ET_0 represents the average reference evapotranspiration rate over the growing season (climatic demand), Vol is the size of the reservoir, F_{max} corresponds to the maximum irrigation rate (and is likely greater than ET_0 in the controlled conditions of this study), and $u(t) \in [0,1]$ is the normalized irrigation chronicle at the time t (in particular $u(\cdot)$ may be generated by the feedback Ψ given in Equation (4.1)).

The terms $\varphi(t)K_S(S(t))$ and $(1-\varphi(t))K_R(S(t))$ are respectively for plant transpiration and soil evaporation, where K_S and K_R account for the decreasing efficiency of the transpiration and evaporation processes for reduced water contents, denoted as levels of S. Hence, the last term in Equation (4.2) somehow denotes the strength of anthropic controls (irrigation decisions) relative to the climatic demand (ET_0) and soil water loss via transpiration and evaporation (the K_S and K_R terms).

The crop coefficients K_S and K_R are piecewise affine and continuous functions of the soil moisture S, and are parameterized by characteristic values of soil water content [3], following the rescaling $\theta = \theta_{fc}$ to $S = 1$, in ascending order : S_{rw} the residual water content, S_{pw} the permanent wilting point, S^* the point of incipient water stress. Their shape and their formulas can be found in Figure 4.1.

$$K_S(S) = \begin{cases} 0 & , S \in [0, S_{pw}], \\ \frac{S - S_{pw}}{S^* - S_{pw}} & , S \in (S_{pw}, S^*], \\ 1 & , S \in (S^*, 1]. \end{cases}$$

$$K_R(S) = \begin{cases} 0 & , S \in [0, S_{rw}], \\ \frac{S - S_{rw}}{1 - S_{rw}} & , S \in (S_{rw}, 1]. \end{cases}$$

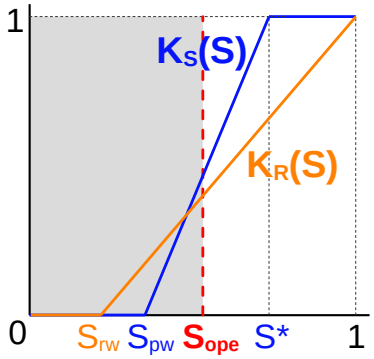


Figure 4.1: Parameter function representing plant transpiration (K_S, in blue) and soil evaporation (K_R, in orange), with the operating constraint S_{ope} (its position with respect to the crossing point of K_S and K_R is arbitrary)

The fraction of intercepted radiation φ is an increasing function of time and taking values in $[0,1]$, related to the canopy growth throughout the season, hence giving more weight to evaporation early in the season then to transpiration later in the season. Reliable hypotheses on the shape of φ will be made later for numerical purposes. Moreover k_3 is a potential biomass growth rate, which acts as a normalizing constant.

At the beginning of the cropping season, it is assumed that soil has been partly refilled during the inter-cropping period, and the initial soil moisture is then supposed to be sufficiently high. As a consequence, as in [19], we consider initial conditions such that

$$S(0) = S_0 \in (S^*, 1], \quad B(0) = 0, \quad V(0) = 0. \tag{4.5}$$

4.2.2.2 Optimization criteria and constraints

An optimal control problem consists in optimizing an objective function related to the evolution of the associated dynamical system. As in [28], in this work we aim at maximizing the final biomass production $B(T)$, among all the admissible irrigation controls $u(\cdot)$. A control u is said to be admissible if it takes values in $[0,1]$ for all $t \in [0,T]$, and if the corresponding state variables

fulfill the model constraints. To this end, the system is endowed with biological and operating constraints as follows:

- $S(t) \geq S_{ope}$ for all $t \in [0, T]$: water stress is tolerated to some extend, but it is assumed by the operator that soil moisture should not go below a value S_{ope} in order to avoid critical stress. In the following, this so-called operating constraint $S_{ope} \in (S_{pw}, S^*)$ is assumed chosen by the operator and denotes their risk aversion or inclination: the lower S_{ope} is, the more water is saved but the riskier the strategy is for biomass production.
- $V(T) \leq Q$: the total amount of irrigation water used during the season fulfills a quota constraint (following the initial condition $V(0) = 0$). The value of water quota Q might depend on the availability and cost of irrigation water, or intentional water savings in a context of deficit irrigation, hence either reflects the irrigator's choice in examining various scenarios, or an strict, external constraint.

4.2.2.3 Biomass renormalization

In the deficit irrigation context there is not enough water to keep soil filling at or above S^* throughout the whole cropping season, meaning that the maximum biomass production will not be achieved. The present work deals with various limiting quotas for which the optimum irrigation dynamics, thresholds and setpoints are sought. For any given problem parameterization, the lowest cumulative irrigation amount needed to maintain the above condition is identified as the limit Q^* quota. In accordance, any quota $Q < Q^*$ will result in periods of stress, hence in losses in maximum possible final biomass, i.e. $\overline{B}_{(S_0,Q)} < \overline{B}_{Q^*}$, where $\overline{B}_{(S_0,Q)}$ is called the "contextual optimum" and $\overline{B}_{Q^*}$ is called the "overall optimum". In fact, in absence of any water stress, that overall optimum is only dependent on plant growth characteristics, depicted by the φ function and its specific "shape parameters" (δ, γ) detailed later. Hence, for a given φ function, there is only one value of $\overline{B}_{Q^*}$, common to all possible combinations of all other parameters. Following Equation (4.3), one has $\overline{B}_{Q^*} = k_3 \int_0^T \varphi(t)\mathrm{d}t$. In this work, for each study case and numerical analysis, we will deal with normalized biomass (thus cancelling the factor k_3), typically in the following way:

$$\beta(t) = \frac{B(t)}{\overline{B}_{Q^*}}. \tag{4.6}$$

4.2.3 Methodological tools for analysis

4.2.3.1 Optimal control

We will make a link between TBIS and results of optimal control, that has already been successfully applied to the CCI model [28]. As mentioned previously, the optimal control problem, given initial conditions $(S_0, 0, Q)$, is formulated as follows:

$$\overline{B}_{(S_0,Q)} = \max_{u(\cdot)} B(T)\,, \quad \text{such that } V(T) \leq Q, \tag{4.7}$$

where u is admissible and (S, B, V) are the associated solutions to Equations (4.2) to (4.4).

The CCI model is written as a non-autonomous dynamical system under constraints, and is piecewise affine and continuous with respect to the state variables. This framework is then suited to apply theoretical control theory results to characterize optimal solutions. In the first place, a theoretical result about piecewise affine systems [27] has been involved in order to extract properties about the candidates structures of optimal solutions. Secondly, a complete analysis based on the derivation of the non-smooth Pontryagin Maximum Principle [35] has been conducted [28] and led to a formulation of optimal irrigation strategies in term of a simple parameterized time-varying feedback. In the following we present these properties and give their interpretation in terms of modeling.

An optimal irrigation strategy subject to Equation (4.7) is necessarily seen as a TBIS with two threshold values, in which an optimal irrigation chronicle can be written as a time-varying feedback strategy. This optimal irrigation strategy for the CCI model is a particular case of the TBIS general

Type	Notation	Definition	Role	Domain
Dynamics	$S(\cdot)$	Soil moisture	State variable	$[0,1]$
	$B(\cdot)$	Crop biomass	"	$\mathbb{R}^+$
	$\beta(\cdot)$	Normalized biomass	"	$[0,1]$
	$V(\cdot)$	Water budget consumption	"	$\mathbb{R}^+$
Control	$u(\cdot)$	Irrigation chronicle	Control variable	$[0,1]$
	S_{bot}	Bottom threshold of a TBIS_{CCI}	TBIS parameter	$[0,1]$
	S_{top}	Top threshold of a TBIS_{CCI}	"	$[0,1]$
	t_d	Decision time	Decision parameter	$[0,T]$
	$t_d^{(opti)}$	Optimal decision time	"	$[0,T]$
Model output	$\overline{B}_{Q^*}$	Potential biomass optimum		$\mathbb{R}^+$
	$\overline{B}_{(S_0,Q)}$	Contextual biomass optimum		$\mathbb{R}^+$
	$B(T)$	Final biomass production	Optimization criterion	$\mathbb{R}^+$
	Q^*	Minimum quota to obtain $\overline{B}_{Q^*}$		$\mathbb{R}^+$
Model parameter	F_{max}	Maximum irrigation rate	Context	$\mathbb{R}^+$
	k_3	Potential biomass growth rate	"	$\mathbb{R}^+$
	Q	Initial water quota	"	$[0,Q^*)$
	$Q^{(low)}$	Low initial water quota	"	$0.55 \times Q^*$
	$Q^{(med)}$	Medium initial water quota	"	$0.7 \times Q^*$
	$Q^{(high)}$	High initial water quota	"	$0.85 \times Q^*$
	S_{pw}	Permanent wilting point	"	0.35
	S_{rw}	Point of residual water content	"	0.2
	S_{ope}	Operating constraint	"	(S_{pw}, S^*)
	$S_{ope}^{(low)}$	Low operating boundary (high risk)	"	0.37
	$S_{ope}^{(high)}$	High operating boundary (low risk)	"	0.45
	T	Duration of crop season	"	$\mathbb{R}^+$
	$\varphi(\cdot)$	Fraction of intercepted radiation	Time-varying parameter	$[0,1]$
	δ	$\varphi(\cdot)$ characteristic growth time	Exploration	$(0,1)$
	γ	$\varphi(\cdot)$ characteristic growth rate	"	$\mathbb{R}^+$
	ET_0	Average reference evapotranspiration	"	$\mathbb{R}^+$
	k_2	Ratio F_{max}/ET_0	"	$\mathbb{R}^+$
	S_0	Initial soil moisture	"	$(S^*,1]$
	S^*	Point of incipient water stress	"	$(S_{pw},1)$
	Vol	Average water reservoir	"	$\mathbb{R}^+$

Table 4.1: List of model notations required to describe the CCI model. We decide to distinguish the model parameters investigated through our global exploration from those associated to contextual information, in the current study.

framework, thus they will be referred in the following as "TBIS_{CCI}". The strategy will consist in a succession of time intervals with no irrigation, time intervals with maximal flow irrigation, and time intervals where the amount of irrigation matches the water demand from soil evaporation and plant transpiration in order to keep the soil moisture constant, either at a bottom threshold (denoted S_{bot}) or at a top threshold (denoted S_{top}).

Regarding the number of such time intervals where optimal instruction switches, it has been proven optimal to consider active irrigation (i.e. $u \neq 0$) in a single time interval. This implies that if both bottom and top thresholds are reached, then the TBIS_{CCI} will induce, in that order : one time interval at the bottom threshold, one time interval at maximal flow of irrigation, and one time interval at the top threshold. Another order of these three intervals cannot be part of an optimal solution, and outside of this concatenated interval there is no irrigation.

Moreover the values of the optimal bottom and top thresholds are in fact known. The optimal bottom threshold, denoted S_{bot}^{opti}, corresponds to the operating boundary S_{ope}. The optimal top threshold, denoted S_{top}^{opti}, is the point of incipient water stress S^*.

The initial quota of available water Q, recalled to be such that our work belongs to a context of deficit irrigation, has to be entirely consumed to ensure an optimal use of water. More precisely, it is optimal to finish consuming all the water during the season, then switching off irrigation (i.e. $u = 0$) for the end of the cropping season.

Finally, the optimal strategy belongs to a class of strategies, summarized by the following parameterized control, where time and state variable "soil moisture" (S) are involved, as well as the decision variables "available irrigation water" (V) and "decision time" (t_d), in accordance to the general TBIS framework, as follows:

$$U_{t_d}(t,S,V) = \begin{cases} 0 & \text{, if } \big[V = Q\big] \text{ or } \big[(S > S_{ope}) \text{ and } (t < t_d)\big], \\ \tilde{u}(t, S_{ope}) & \text{, if } \big[(S = S_{ope}) \text{ and } (t < t_d)\big], \\ \tilde{u}(t, S^*) & \text{, if } \big[(S = S^*) \text{ and } (V < Q) \text{ and } (t > t_d)\big], \\ 1 & \text{, otherwise,} \end{cases} \tag{4.8}$$

where S and V are the state variables of our dynamical system, t_d is a decision parameter, referred to as a "decision time" (which therefore may or may not be optimal) and the quantity $\tilde{u}$ represents a time-varying irrigation keeping the soil moisture constant at some threshold value $\tilde{S}$. This irrigation can in fact be expressed analytically by

$$\tilde{u}(t,\tilde{S}) = \frac{\varphi(t)K_S(\tilde{S}) + (1-\varphi(t))K_R(\tilde{S})}{k_2}. \tag{4.9}$$

Notice that with $k_2 = F_{max}/ET_0 > 1$ (as mentioned in Section 4.2.2.1), this expression fulfills the condition $\tilde{u}(t) \in [0,1]$.

The "decision time" is in fact not a proper "triggering time" since the irrigation can start before reaching this time t_d (if the soil moisture reaches S_{ope} before the time t_d). This "decision time" can be understood as a tipping point between a period where irrigation was kept at a relatively low rate (and soil moisture was kept at a relatively low value), and a period where we irrigate at full rate until reaching a sufficiently high soil moisture and then maintain that soil moisture value with relatively high irrigation rate. This behaviour intuitively follows the needs of the plant, since crop's development is slower at early stages, and the water demand is higher when the crop's development is faster.

The problem of finding the optimal solution is thus reduced to a one-dimensional direct optimization over the class of feedback controls as above, by exploring the range of possible values for the decision time t_d. Its optimal value is denoted t_d^{opti}.

4.2.3.2 Sensitivity analyses

Since the analytical results do not provide a full characterization of optimal TBIS_{CCI}, in particular the optimal decision time t_d^{opti}, we propose to study its drivers by means of numerical exploration

using global sensitivity analysis (GSA) techniques [41]. Indeed the use of GSA can lead to relating the variability of t_d^{opti} to those of model inputs, and thus to point out in particular which input drives early or late irrigation triggering. Moreover, this global exploration is complemented by a multi-local perturbation analysis that will allow to understand the consequence of sub-optimal choices of soil moisture threshold and decision time.

4.2.3.2.1 Model inputs and outputs In order to perform a global exploration through numerical simulation, let us precise the nature of inputs and outputs. First, the duration of crop season T can be seen as a silent variable, since the model is built in a way that it is always possible to rescale the time span to $[0, 1]$ for numerical solving, without altering the structure of the policies (it simply consists in rescaling the time, including in the definition of φ). The values of the operating (Q, F_{max}) constraints are seen as external forcings, related to water availability and rights (Q) or to equipment performance and characteristics (F_{max}), deemed sufficient here as in most real-life cases. By contrast, the assumed "operating constraint" (S_{ope}) relates altogether to the irrigator's risk aversion, perception of water stress harmfulness, and willingness to use as few water as possible, resulting in a self-imposed constraint. We will then consider different study cases in the form of different (S_{ope}, Q) couples, in combination with fixed S_{rw} and S_{pw} values, allegedly known from soil and plant characteristics.

The remaining model entries are the exploration parameters listed in Table 4.1. They consist in the initial soil moisture value S_0, the values of ET_0, Vol and S^*, the value k_2 corresponding to the quotient between the maximal irrigation rate F_{max} and the reference evapotranspiration ET_0 (it might be seen as the capacity of the irrigation material), and the fraction of intercepted radiation φ. For this function φ related to the canopy cover, we will follow most crop models and consider a function built from a logistic shape with two parameters, rescaled over $[0, 1]$, as follows:

$$\forall \tau \in [0,1],\ L_{a,b}(\tau) = \frac{1}{1+\exp(a-2a\tau^b)} \quad \text{, with } a>0,\ b>0.$$

Then we normalize it to be 0 at $\tau = 0$ and 1 at $\tau = 1$, as follows:

$$\forall \tau \in [0,1],\ F_{a,b}(\tau) = \frac{L_{a,b}(\tau) - L_{a,b}(0)}{L_{a,b}(1) - L_{a,b}(0)}.$$

This leads to a parameterized logistic-like function $F_{a,b}$ whose parameters (a, b) may then be used to shape the φ function, as follows. Consider a parameterization (δ, γ), where δ represent the time of half growth (i.e. such that $F_{a,b}(\delta) = 0.5$) and γ is related to the characteristic growth rate (i.e. the derivative of $F_{a,b}$ at the time δ), with the following bijective transformations:

$$\delta = f(b) = \frac{1}{2^{1/b}}\,, \quad \gamma = g(a) = a \cdot \frac{\exp(a)+1}{\exp(a)-1}.$$

By computing the inverse transformations f^{-1} and g^{-1}, we can finally describe the (rescaled) function used for the fraction of intercepted radiation $\widetilde{\varphi}$ as follows:

$$\forall \tau \in [0,1],\ \widetilde{\varphi}^{\delta,\gamma}(\tau) = F_{g^{-1}(\gamma), f^{-1}(\delta)}(\tau). \tag{4.10}$$

This way, the time-rescaling of φ writes $\varphi(t) = \widetilde{\varphi}^{\delta,\gamma}(t/T)$ for all $t \in [0, T]$.

We refer to Table 4.2 for a list of the model inputs considered in the numerical exploration.

The outputs of a simulation of the CCI model are the trajectories of soil moisture (S), biomass production (B) and water budget consumption (V). These trajectories are computed by integrating Equations (4.2) to (4.4) over the interval of crop season, thus require a last crucial ingredient : the irrigation strategy u. As shown in [19, 28], in order to compute an optimal strategy, it is sufficient to perform a simulation of CCI for a given grid of possible values of each of the parameterized controls Equation (4.8). From the process of optimization, we extract the optimal decision time t_d^{opti}, which is the subject of the sensitivity analysis in Section 4.3.2.1.

4.2.3.2.2 Model parameters ranges As mentioned before, the CCI model has been implemented in such a way that the crop season can be rescaled to a normalized time interval $[0, 1]$. The soil water content values (S_{rw}, S_{pw}) are considered fixed and follow [3, 143] with the aforementioned scaling of the soil moisture θ_{fc} to $S = 1$.

As discussed previously, we consider several cases as declensions of different contexts. We set two values of operating constraint (greater than the permanent wilting point S_{pw} and less than the incipient water stress point S^*). We chose $S_{ope}^{(low)} = 0.37$ for the riskier behaviour and $S_{ope}^{(high)} = 0.45$ as a safer approach. The water quantity Q^* (quota needed for absence of stress) is computed numerically. Then we introduce three contexts of water availability : a "low initial quota" defined by $Q^{(low)} = 0.55 \times Q^*$, a "medium initial quota" defined by $Q^{(med)} = 0.7 \times Q^*$, and a "high initial quota" defined by $Q^{(high)} = 0.85 \times Q^*$.

Regarding model inputs investigated through the numerical exploration, we consider wide ranges of values for reference evapotranspiration ET_0, water reservoir Vol and point of incipient water stress S^*, following [3, 143]. The initial soil moisture S_0 is supposed to be sufficiently high at the beginning of the season, therefore S_0 is greater than any value of S^*, and up to the field capacity $S = 1$. The remaining non-documented exploration parameters are : k_2 the capacity of irrigation material to bring water to the soil, δ the (rescaled) date of half plant growth, and γ a characteristic value of growth rate. For these exploration parameters, we chose to consider sufficiently large intervals of values allowing to deal with many types of crop growths and operating systems. The exploration settings are summarized in Table 4.2.

Notation	Unit	Range of values
ET_0	mm/d	$[2, 7]$
k_2	-	$[2, 6]$
S_0	mm/mm	$[0.88, 1]$
S^*	mm/mm	$[0.58, 0.87]$
Vol	mm	$[70, 270]$
δ	d/d	$[0.2, 0.8]$
γ	-	$[2, 7]$

Table 4.2: List of exploration parameters with their range of values for the numerical exploration.

4.2.3.2.3 Global sensitivity analysis on optimal controls A global sensitivity analysis was conducted in order to obtain insight into the model parameters and better understand their influence on the shape of optimal TBIS_{CCI}. This analysis was repeated to provide a broader view of model responses to model inputs in various typical situations of targeted study cases, represented by the pair of contextual parameters (S_{ope}, Q).

We considered a set of 6 typical situations : two values for the tolerance setpoint S_{ope} (a lower one representing riskier irrigation strategies allowing more water stress, and a higher one allowing less water stress corresponding to a behaviour looking for production guarantees), combined with three values for the initial water quota (a low one, a medium one and a high one, for exploring the range of possibilities in terms of water availability).

For each situation, the sensitivity analysis was based on a global exploration of the space of model parameters, each of these parameters belonging to an interval with a uniform distribution according to the range of variation given in Table 4.2. Thanks to the low computational time of the simulation, we can conduct a variance-based sensitivity analysis over our global exploration settings, providing first-order Sobol Indices (SI) and Total order Sobol Indices (TSI) [166] which quantify the contribution of input variance (either alone or including all interactions effects) to the variance of model outputs.

If X denotes the exploration parameters, and Y denotes the model output of interest, we denote X_i the i-th component of X, and $X_{\sim i}$ for the components of X except the i-th. The formulas for the first-order and Total Sobol Indices corresponding to each exploration parameter X_i are defined with the expectancy $\mathbb{E}$ and the variance Var as follows:

$$SI_i = \frac{\mathrm{Var}(\mathbb{E}_{X_{\sim i}}(Y|X_i))}{\mathrm{Var}(Y)}\,, \quad TSI_i = \frac{\mathbb{E}(\mathrm{Var}_{X_i}(Y|X_{\sim i}))}{\mathrm{Var}(Y)}. \tag{4.11}$$

4.2.3.2.4 Multi-local perturbation analysis Once an optimal solution has been identified, in which irrigation dynamics is dictated by an optimal set of control parameters, a complementary analysis investigates the effects of mistakes committed on the most important control parameters. This analysis consists in a perturbation around the optimal control parameters values to account for either experimental limitations (difficult-to-measure parameter values), conceptual flaws (the problem has not been perfectly understood) or other incidents (the correct decisions were known but have not been taken, or not in due time).

The multi-local perturbation analysis was conducted as follows. For each exploration parameter set drawn from the ranges described in Table 4.2, we determined the pair $(S_{top}^{opti}, t_d^{opti})$ that corresponds to the optimal $\mathrm{TBIS}_{\mathrm{CCI}}$, S_{top}^{opti} being given by S^* and t_d^{opti} being numerically computed. At this point, the final biomass is thus what we called the "contextual biomass optimum" $\overline{B}_{(S_0,Q)}$ associated to the initial soil moisture S_0 and water quota Q. We then built a grid of pairs (S_{top}, t_d) around that optimal point, using a prescribed grid size: $\Delta S_{top} = S_{top} - S^* = \pm 10\%$ of relative soil moisture, and $\Delta t_d = t_d - t_d^{opti} = \pm 10\%$ of the overall duration of the cropping season. At each point $P = (S_{top}, t_d)$ of the grid, we numerically compute the corresponding sub-optimal terminal biomass $B_P(T)$. Then we determine the relative loss of biomass production compared to the optimal one, expressed in percentage the following way:

$$\Delta_{sub-opti}(P) = \frac{\overline{B}_{(S_0,Q)} - B_P(T)}{\overline{B}_{(S_0,Q)}}. \tag{4.12}$$

Finally, at each grid point P, we compute $\mathrm{mLoss}(P)$ the mean value of relative biomass loss $\Delta_{sub-opti}(P)$ averaged over the exploration parameter set.

Besides percentage losses in final biomass values, in comparison to the contextual optimum, we provide complementary indications as regards situations in which soil water content goes below the chosen operating constraint S_{ope}, or conversely, situations in which the water quota Q is not entirely consumed. Justifications for this are that biomass sub-optimality may likely be explained by cumulative water stress, and that reaching high water use efficiency, relating biomass with cumulative irrigation amounts ("more crops per drop"), may also be a relevant management criterion. To do so, we considered 6 "typical" situations combining riskier or safer operating constraints, with limiting or comfortable initial water quotas, to explore various near-optimal situations strategies.

4.2.3.2.5 Implementation details To run a simulation, the model requires to solve Equations (4.2) to (4.4), then the code relies on the classical Euler explicit method for discretization and integration of systems of ODEs. The time horizon was discretized using 400 time steps. This has been implemented with the Python language.

The first order Sobol Indices and Total order Sobol Indices were computed through a Sobol-Jansen methodology [88, 150] using the R package "Sensitivity" for implementation [82]. Our simulation was done using a sample of 6000 points, with 7 model parameters, leading to an exploration plan of 54000 points.

For the multi-local sensitivity analysis, we used a sample of 3000 random points combined with a (S_{top}, t_d) grid of size 21×29 to produce a simulation output.

4.3 Results

4.3.1 Soil filling from possible irrigation strategies

4.3.1.1 Reference optimal structure

For the indicated context, connecting the CCI model (and its theoretical analysis) to the TBIS_{CCI} (and its physical interpretation) provides first results. We can conclude from theory (Section 4.2.3.1) that five successive stages are present, at most, in the irrigation planing, in the general case, and always in the same order, as listed below and illustrated in Figure 4.2. Considering a bottom threshold S_{bot} for the first stage of the season and a top threshold S_{top} for the productive phase, a corresponding TBIS_{CCI} can be written as the following comprehensive set of successive instructions, containing guidelines about the amount of irrigation to employ during each stage and a "switching condition" to move from one stage to the next, throughout an intermediate period starting at the decision time t_d.

- Stage (i) : $u(t) = 0$
  At the beginning of the cropping season, the initial soil moisture S_0 is supposed to be high, thus the irrigator lets the system evolve only by relying on soil resources, until it reaches the aimed value S_{bot} for the first stage of the cropping season.
  - The switching condition (i) $\rightarrow$ (ii) is $S = S_{bot}$.
- Stage (ii) : $u(t) = \tilde{u}(t, S_{bot})$
  During this stage, the operator irrigates with limited water amounts in order to keep the soil moisture at the chosen value S_{bot}. This irrigation is kept until the irrigator decides at the time t_d to not tolerate high stress anymore and move on to the production phase.
  - The switching condition (ii) $\rightarrow$ (iii) occurs at the decision time t_d to trigger maximal flow of irrigation.
- Stage (iii) : $u(t) = 1$
  Once the irrigator decides to advance towards the production phase, the irrigation needs to be applied at its maximal capacity, in order to reach the desired level S_{top} as soon as possible.
  - The switching condition (iii) $\rightarrow$ (iv) is $S = S_{top}$.
- Stage (iv) : $u(t) = \tilde{u}(t, S_{top})$
  When the soil moisture reaches the value corresponding to the production phase, the operator irrigates with the appropriate rate to keep the soil moisture at this chosen value S_{top}, until the whole water quota is entirely consumed (i.e. $V = Q$).
  - The switching condition (iv) $\rightarrow$ (v) is $V = Q$.
- Stage (v) : $u(t) = 0$
  When the entire quota of water has been consumed, then there is no choice but let the crop finishing its growth only using the remaining soil resources.

4.3.1.2 Declensions in particular cases

Particular cases, still fulfilling necessary conditions listed in Section 4.2.3.1, will arise from specific combinations of the contextual parameters so that some of these five stages may be absent (Figure 4.3). In much more details, each of the eight different possible shapes of soil moisture dynamics, all shown in Figure 4.3, appears for certain combinations of values of the contextual parameters. Those exhibit general as well as specific features, listed below.

A very general feature is that the slope of the falling limbs in the dynamics of soil moisture (corresponding to the "$u = 0$" irrigation periods) is dictated both by the capacity of soil reservoir (Vol) and the transpiration and evaporation rates, calculated from crop status (K_S, K_R, φ), in response to the climatic demand (ET_0). Smaller soil reservoirs and higher demands will result in steeper downward slopes, although the relative contributions of transpiration and evaporation will

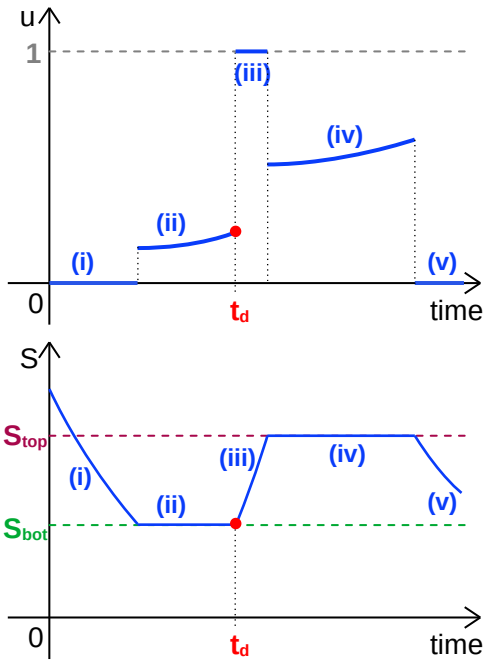


Figure 4.2: General case of a parameterized TBIS_{CCI} showing the irrigation chronicle $u(\cdot)$ and associated soil moisture dynamics in time $S(\cdot)$, across the five successive stages, (i) to (v), known from theory (see text for details). Note that the t_d value, indicating the "decision time" at which the bottom threshold should be left and the top threshold should be aimed at, is an outcome of problem parameterization and not a prescribed value.

differ for different canopy covers thus crop development stages. Conversely, the slope of the rising limbs (corresponding to the "$u = 1$" irrigation periods, initiated at t_d) is controlled by the F_{max} characteristic of the irrigation system, as well as by Vol (the relative filling of smaller reservoirs will increase faster) and the transpiration and evaporation rates, acting against F_{max}. In complement, the greater the quota, the longer soil moisture should stay at the top threshold S_{top} during the corresponding stage, and the shorter the bottom stage S_{bot} should last.

By contrast, specific features arise from specific cases. Insufficient irrigation rates and (or) too limited quotas will thus suppress stage (iv) and prevent reaching the setpoint S_{top} value (b). High initial S_0 values, when combined with high quotas, will suppress stages (ii-iii) thus the necessity to stay a certain time at the S_{bot} level (c), low initial S_0 values, combined with limited irrigation rates, will suppress stages (i-ii) in which water contents are allowed to drop before initiating irrigation (d), the large difference between high setpoint S_{top} values and low S_{bot} values, with a sufficient quota, will suppress stage (ii) as well as the risk to approach those low S_{bot} values (e). High initial S_0 values, combined with a low reference evapotranspiration rate (limited water losses) and a low quota will suppress stages (ii) and (iv) thus hold S values in an intermediate region between S_{bot} and S_{top} (f). Initial S_0 values close to the setpoint S_{top} values and large quotas will suppress stages (i, ii, iii) thus the necessity to temporarily tolerate low water contents (g). Low initial S_0 values, combined with low maximal irrigation rate, will suppress stages (i, ii) and (iv) and suggest to progressively approach the setpoint S_{top} value, in line with the canopy growth rate, before ceasing irrigation (h). Examples of sets of model parameters for some of these particular cases are given in Table 4.3. Situations (a) to (h) might also be reached by other non-trivial combinations of effects following less easily interpretable sets of model parameters.

Notice that (d) (g) and (h) are not part of the global exploration since we have $S_0 > S^*$ and $S_{top}^{opti} = S^*$, but these cases might occur with particular settings in the multi-local perturbation analysis.

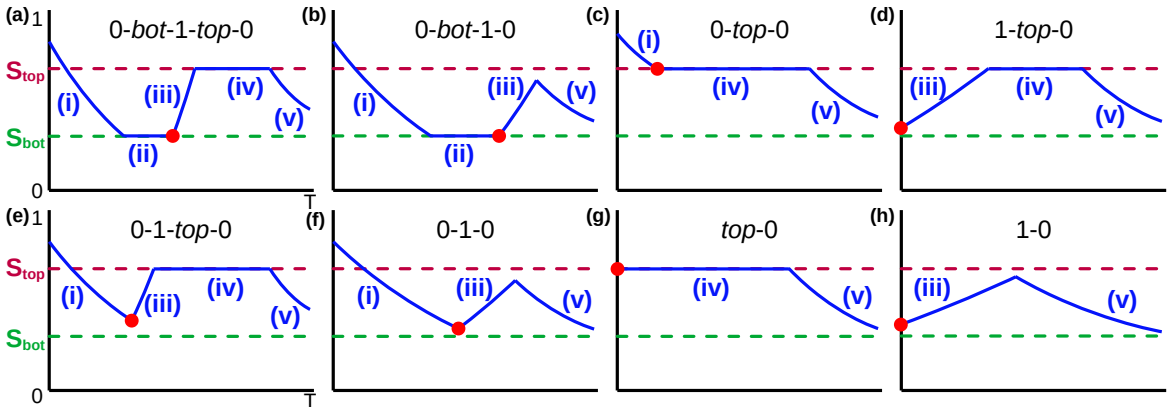


Figure 4.3: Families of possible soil moisture trajectories fulfilling the TBIS_{CCI} framework. According to the different model parameters values, some of the (i)-(ii)-(iii)-(iv)-(v) stages listed in Figure 4.2 may be absent while stage succession remains the same. Also indicated is the point corresponding to the decision time t_d (red dot).

Shape	S_0	S_{bot}	S_{top}	t_d	ET_0	k_2	S^*	Vol	δ	γ	Q
(a)	0.9	0.4	0.75	$0.47 \times T$	4	3	0.7	150	0.5	4	$0.7 \times Q^*$
(b)	0.9	0.45	0.85	$0.67 \times T$	3	2	0.7	220	0.5	4	$0.55 \times Q^*$
(c)	0.9	0.4	0.75	$0.07 \times T$	4	3	0.7	220	0.5	4	$0.85 \times Q^*$
(d)	0.5	0.4	0.75	0	4	2	0.7	180	0.3	2.5	$0.85 \times Q^*$
(e)	0.9	0.37	0.75	$0.27 \times T$	4	3	0.7	150	0.5	4	$0.8 \times Q^*$
(f)	0.9	0.37	0.8	$0.5 \times T$	2	2	0.7	250	0.3	4	$0.55 \times Q^*$
(g)	0.8	0.4	0.8	0	4	3	0.7	150	0.5	4	$0.85 \times Q^*$
(h)	0.41	0.4	0.85	0	2	2	0.7	250	0.3	2.5	$0.55 \times Q^*$

Table 4.3: Examples of particular model parameter sets corresponding to the different cases of shape for the TBIS_{CCI}.

4.3.2 Numerical exploration

4.3.2.1 Sensitivity of the optimal decision time to the explored parameters

We present the results of the global sensitivity analysis conducted with the CCI model on 6 study cases following the possible pairs (S_{ope}, Q). The values and ranges for each exploration parameter are those defined in Table 4.2. Results are summarized in Figure 4.4.

4.3.2.1.1 Variability of the optimal decision time First notice that the optimal decision time t_d^{opti} has a quite significant variability (globally, its possible values practically cover the whole time span) over the global exploration. This variability is reduced when the quota increases and when the operating constraint becomes lower. The latter seems quite intuitive, as lower thresholds mean stronger water stress which expectedly encourages leaving low moisture levels during an earlier (thus smaller) window of time. The former may seem less intuitive, as it means that lower quotas enhance the diversity of admissible solutions whereas one would probably expect that larger quotas would open way for a diversity of optimal solutions. In fact, greater quotas lead to more admissible trajectories, i.e. wider windows for the choice of optimal decision time. This way, admissible trajectories are able to generate optimal values for different exploration parameter settings that tend to be closer. In contrast, lower quotas induce smaller windows for the choice of optimal decision time for each parameter settings, then optimal choices are a bit more spread over time.

In addition the median optimal decision time tends to be slightly more precocious in situations with low S_{ope} compared to high S_{ope}. Even without any explicit description of water stress effects, other than a reduction of transpiration and evaporation, this tends to indicate that more risky situations (lower operating boundary) should not last long while less risky situations can last a little longer (higher operating boundary). This suggests the existence of a "hidden effect" by which the overall impact on biomass production is the integral of water deficit over time, as usually found only in more complex models. Conversely, lesser quotas tend to delay the optimal decision time so that increased impacts over the plant stress (lower S_{ope}, lower quota), when considered individually, have opposite effects on the optimal decision time values. Finally, high and almost non-limiting quotas lead to significantly earlier and clearly identified optimal decision times, with little variability.

In summary, this analysis of the t_d^{opti} values and variability allows identifying the roles played by the assumed key external constraint (Q) and the allegedly associated irrigator's decision (S_{ope}) so as to frame the overall context in which irrigation management and optimization will take place. For a complementary and thorough view, next section details which of the "internal" problem exploration parameters (soil, crop, climate, performance of the irrigation system) control those general insights.

4.3.2.1.2 Explored parameters that dictate the optimal decision time For a low operating constraint, the t_d^{opti} value mostly depends on one exploration parameter (δ), while it depends on the pair of exploration parameters (S^*, δ) for a high operating constraint. More precisely, with a low operating constraint, the TSI of each exploration parameter except δ does not exceed 0.14 for any initial water quota, while the SI of δ is always above 0.65. With a high operating constraint, the TSI of each exploration parameter except S^* and δ does not exceed 0.15 for any initial water quota, while the SI of S^* varies from 0.19 to 0.48 and the SI of δ varies from 0.19 to 0.59. This motivates further investigations, wondering about the nature of the dependency of t_d^{opti} with respect to these most influential exploration parameters.

We illustrate, in Figure 4.5 (low operating constraint $S_{ope}^{(low)}$) and Figure 4.6 (high operating constraint $S_{ope}^{(high)}$), the nature of the response with respect to their corresponding most influential exploration parameters. For lower S_{ope} values, apart from limited dispersion around a clearly visible trend, the t_d^{opti} values tend to increase with δ values, in a roughly linear and identical way, whatever the ($Q^{(low)}$, $Q^{(med)}$, $Q^{(high)}$) quota values (Figure 4.5). For higher S_{ope} values, the trend to increasing t_d^{opti} values with increasing δ values remains whatever the S^* values (reading the plots from left to right) but the larger gradients of t_d^{opti} values are obtained for jointly increasing δ values and decreasing S^* values (Figure 4.6). Finally, the range of t_d^{opti} values found for higher S_{ope} values is larger, which denotes the influence of S^* values, or rather that of a reduced distance between S^*

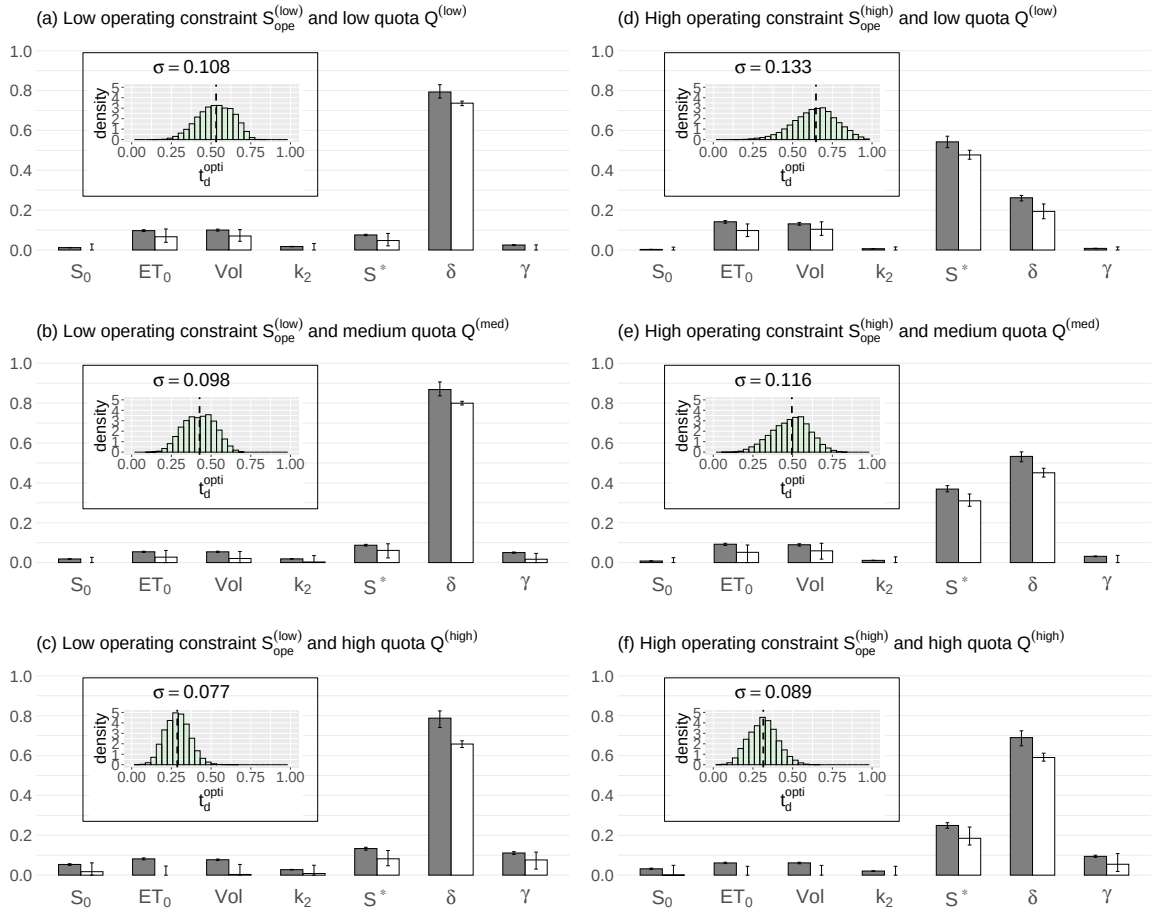


Figure 4.4: Variability (small sketches) and sensitivity analysis of the optimal decision time t_d^{opti} aiming to identify the most influential model exploration parameters. Attached to each exploration parameter are first-order Sobol indices in white and total-effect indices in grey, also indicating the associated confidence interval (vertical segments). This representation combines two levels of the operating constraint S_{ope} (increasing from the first to the second column), with three levels of the initial water quota Q (increasing from the first to the third row: $Q^{(low)}$, $Q^{(med)}$, $Q^{(high)}$).

and S_{ope}. A likely implication, in operational terms, is that high risk aversion, signaled by high S_{ope} values close to the assumed S^* stress initiation threshold, introduces a stronger dependence of the optimal decision time t_d^{opti} on crop growth characteristics. Overall, crops characterized by a rather late quick growth phase (high δ values, which is the case for most crops) will need delayed t_d^{opti} values to obtain the best possible biomass production, all the more for reduced quotas and high risk aversion.

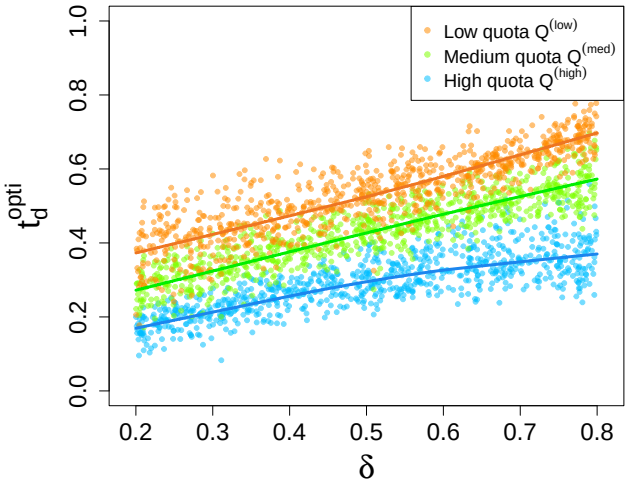


Figure 4.5: Points of model responses t_d^{opti} with respect to δ, over an exploration sample for the situation with a low operating boundary $S_{ope}^{(low)}$. Lines drawn represent the respective conditional expectation of response over a sliding window of input, for each cloud of points.

4.3.2.2 Biomass losses with sub-optimal strategies

We present the results of the multi-local perturbation analysis conducted with the CCI model on 6 study cases following the possible pairs (S_{ope}, Q). The values and ranges for each exploration parameter are those defined in Table 4.2. Results are summarized in Figure 4.8.

4.3.2.2.1 Possible consequences of small mistakes Figure 4.7 shows the principles and outcomes of this exploration. In accordance, biomass production was computed at each point of the tested grid, i.e., in the tested configurations of Figure 4.7, so as to assess the loss in percentage compared to the optimal biomass value obtained with (S^*, t_d^{opti}) (known as the "contextual optimum"). Some pairs (S_{top}, t_d) in the exploration grid might lead to non-admissible values of S in light of the constraints defined in Section 4.2.2.2, here $S < S_{ope}$ in some cases, late in the season. Non-admissible trajectories are discarded from the computation of the average relative biomass losses, but will be counted to indicate their ratio of occurrence. Other couples (S_{top}, t_d) in the grid lead to trajectories consuming less than the total amount of available irrigation water Q. These two characteristics of some of the irrigation scenarios (non-admissibility, under quota thus water savings) are illustrated, among others, in Figure 4.7. As also possibly anticipated from Figure 4.7, a too early decision time t_d combined with a too high S_{top} value might lead to too severe final stress conditions (S values below S_{ope}) while a too late decision time t_d combined with a too low S_{top} value might lead to water savings (water consumed for irrigation is below the initial quota) but necessarily sub-optimal biomass production. Those situations are described in much more details in Figure 4.8.

4.3.2.2.2 Performances of sub-optimal strategies Overall, the most striking feature of Figure 4.8 is the difference in biomass loss between cases $S_{top} > S^*$ (upper half of each plot) and $S_{top} < S^*$ (lower half). Indeed, slightly (less than 10% of the optimal soil moisture) overestimating the setpoint value during the productive stage leads to less than 5% of biomass loss in almost all

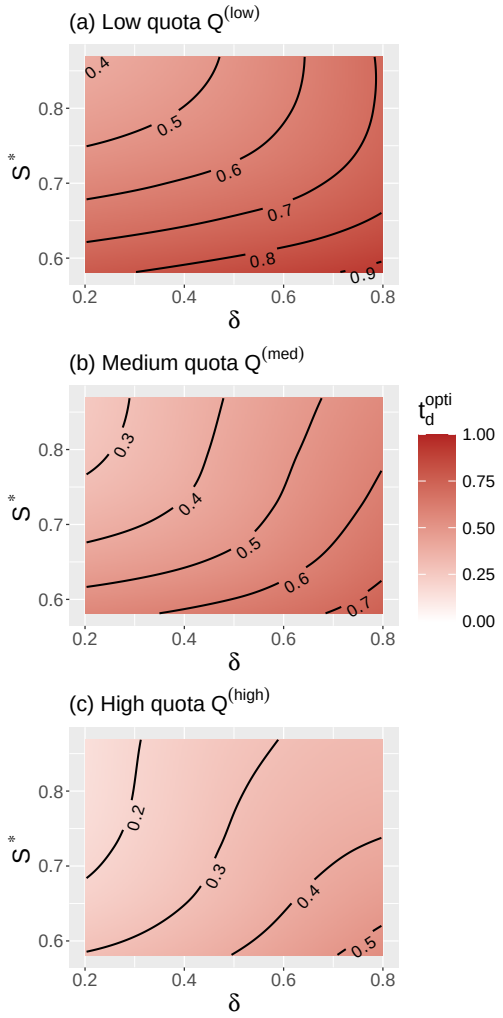
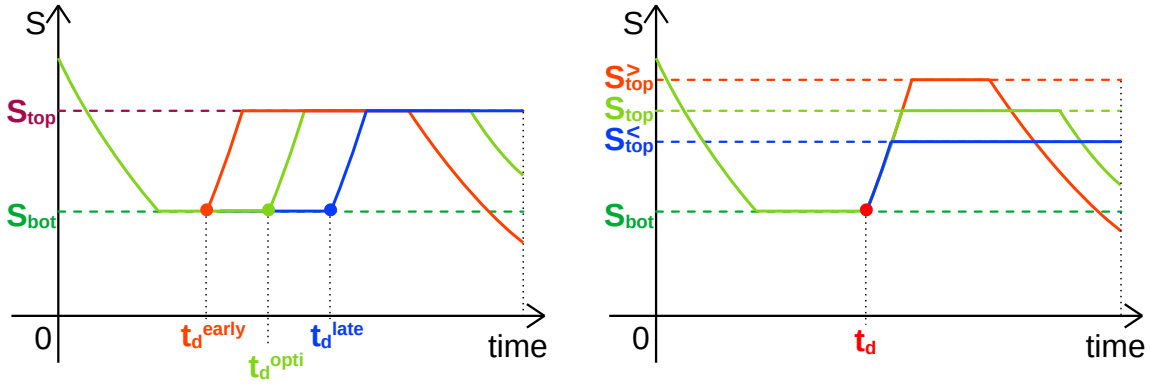


Figure 4.6: The plotted values are the average model outputs t_d^{opti}, gathered with respect to their (δ, S^*) coordinates, over the exploration settings.



(a) Examples of issues from different decision times

(b) Examples of issues from different top thresholds

Figure 4.7: Trajectories associated with variability on the choices of the decision time t_d and the threshold S_{top} of productive phase. On the left, we depict how a too early decision time t_d^{early} (in orange), an optimal decision time t_d^{opti} (in green), and a too late decision time t_d^{late} (in blue), for strategies based on the same thresholds, can lead to different trajectories (possibly non-admissible or not entirely consuming the water quota). On the right, we schematize possible consequences (in terms of constraint violation and water savings) of trajectories based on a higher (in orange) or a lower (in blue) threshold than a reference one (in green), using the same decision time.

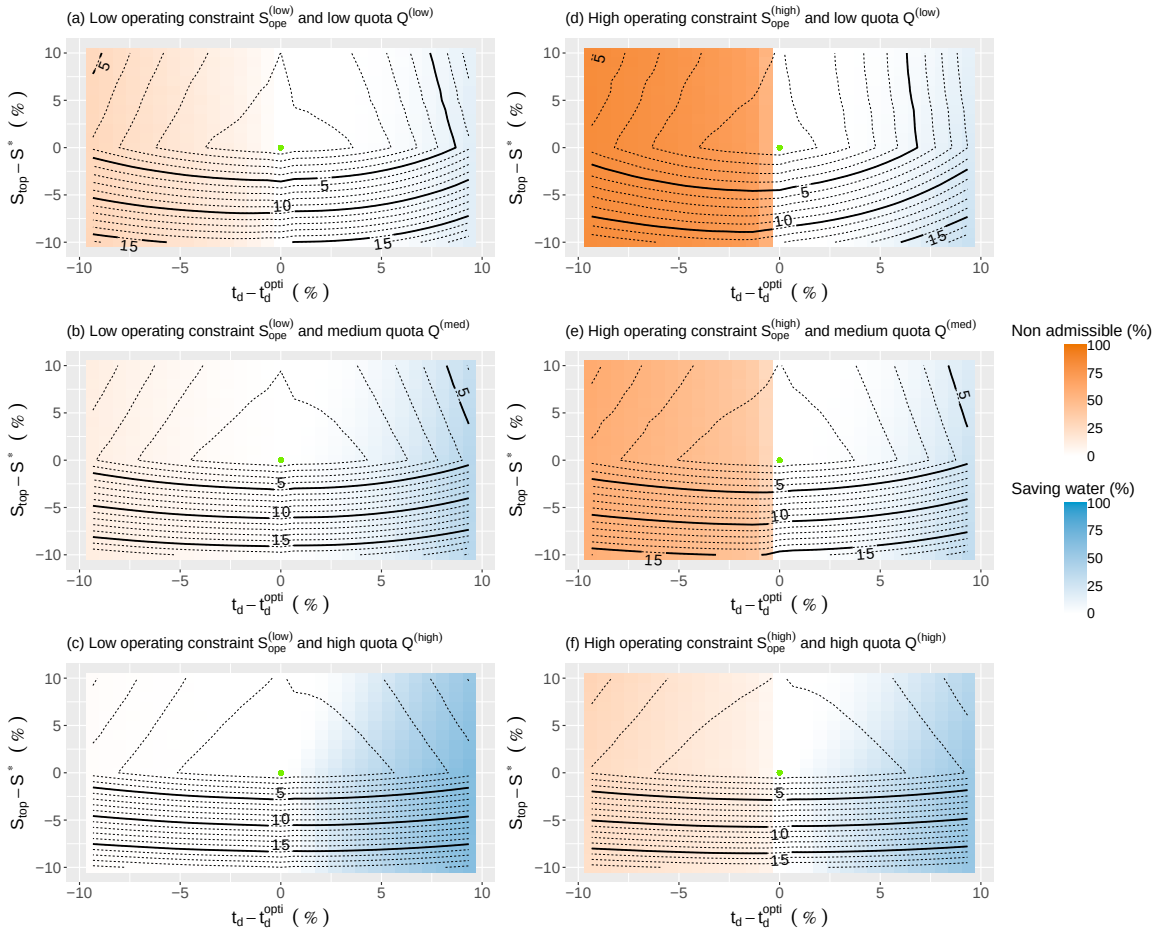


Figure 4.8: Performance of sub-optimal strategies, evaluated as mean percentage losses in final biomass (i.e. the values mLoss computed from the formula defined in Equation (4.12)), combining soil moisture setpoint values (S_{top}) in the vicinity of the optimal level (S^*) and irrigation decision time values (t_d^{opti}) in the vicinity of the optimal decision time (t_d^{opti}). The isovalues in each sketch correspond to the mean percentage loss of final biomass relative to the "contextual optimum" $\overline{B}_{(S_0,Q)}$ (for the indicated parameterization), located in the center of each plot (green dot). In complement, orange shades represent the percentage of exploration parameter sets leading to non-admissible trajectories of soil moisture values (too low final S values), and blue shades represent the percentage of exploration parameter sets leading to water saving trajectories (under quota), i.e. for which $V(T) < Q$.

cases (with both low and high operating constraints, and any initial water quota), while underestimating it (by up to 10%) may cause up to at least 15% of biomass loss. In complement, in the lower half of the plots, the isovalues tend to be approximately horizontal, especially for high quotas: gradients of biomass loss are therefore due to mistakes in S_{top} threshold values (rather than mistakes in the decision time t_d). Conversely, in the upper half, and to a lesser extent, it is rather mistakes on the decision time t_d values that cause losses in final biomass (rather than mistakes in S_{top} values), especially for low initial quotas, where percentages of biomass losses are about twice as large as for high water quotas. Notice that there is no significant overall difference in terms of relative biomass losses between the two situations of risk inclination (left column) or aversion (right column).

4.3.2.2.3 Complementary operating indications As regards non-admissible or water-saving parameterization and trajectories (Table 4.4), the greater the initial water quota, the less probability there is to violate the stress constraint near the end of the cropping season ($S < S_{ope}$) when triggering maximal rate of irrigation too early, and the more probability there is to save a fraction of the quota when delaying the decision time. In contrast to the analysis over biomass losses, we observe quite significant dissimilarities between risk inclination (left column) and risk aversion (right column), especially regarding the occurrence of non-admissible trajectories. Indeed it is definitely more likely to not fulfill the operating constraint when the productive threshold is set at a higher value, and it is intuitive that riskier behaviour allows to consider a wider variety of trajectories. We also remark a non-intuitive phenomenon, that there is slightly more chances to save water with a higher operating constraint in the case of a low water quota, but there is slightly more chances to save water with a lower operating constraint in the case of a high water quota. It might be explained by the fact that a lower operating constraint leads to more water stress then reduces the possibility to allocate irrigation water at will, which is particularly affected when the water quota is low. Conversely, the increase of water quota leads to more choices of trajectories, thus greater possibility to modulate the allocation of water, which is even amplified when the operating constraint is low.

	Low operating constraint		**High operating constraint**	
	Non-admissible	Saving water	Non-admissible	Saving water
Low quota	up to 29.6%	up to 16.6%	up to 83.9%	up to 29.0%
Medium quota	up to 12.5%	up to 38.7%	up to 61.2%	up to 34.4%
High quota	up to 2.1%	up to 64.7%	up to 30.8%	up to 55.5%

Table 4.4: Complementary indications about suboptimal trajectories.

4.4 Discussion

This paper offers the possibility of handling very different cases (initial, tolerance and setpoint water content levels, soil volume, crop characteristics, maximum irrigation rate, climatic demand) with the same framework, deriving the optimal, associated chronicles of S (water content, soil filling) and u (irrigation chronicle) in the general case (Figure 4.2) broken down into different declensions for the different particular cases (Figure 4.3). This may have interesting operational applications, as knowledge of the contextual and exploration parameters will decide which of the strategies to follow, either with maximal rate of irrigation scheme (Figure 4.2) or with shortcuts and missing stages, in the relevant sketch of Figure 4.3.

For the irrigation operator, the emerging part of the previous developments, their outcome, certainly is the so called "decision time" (t_d), depending on all previously mentioned elements. Noticeably, those elements include the risk aversion (high S_{ope} values) or inclination (low S_{ope} values) of the irrigator, which also show throughout the chosen values for the S_{ope} parameter denoting the lower limit of admissible water contents. One step further, the proposed framework altogether accounts for those irrigator's choices, seeking the optimal solution ($S(t)$, $u(t)$, t_d) for the assigned context, and allows questioning those choices, comparing the final biomass production in other para-

metric configurations. Investigating the effects of mistakes committed on the essential TBIS_{CCI} and decision parameters (S^*, S_{ope}, t_d), whether those mistakes arise from imperfect measurements and (or) conceptualization, is making another step towards future operational applications.

Regarding the implementation of TBIS_{CCI} in real-world irrigation systems, the capacity to irrigate at given rates depends on the irrigation equipment. In general, an irrigation event consists in providing in a certain time frame some amount of water. The irrigator may define a setpoint level to trigger this irrigation, which may consist in either a fixed amount of water, or watering up to a target setpoint level. A TBIS_{CCI} can be achieved by splitting in two successive time periods to keep the humidity level close to the desired setpoint value on each period. This will require to perform up (when irrigating) and down (when not irrigating) fluctuations around the setpoint value, and the "decision time" t_d corresponds to the moment of switching from the first period to the second one. Such implementation can be seen as a time discretization of the theoretical TBIS_{CCI}.

Since the optimal strategies of the CCI model are written as TBIS_{CCI}, thus are consistent with usual irrigation practices, it is reasonable to foresee putting to the test the scenarios generated by those optima. Thanks to their low-dimensional structure, implementing TBIS_{CCI} in experimental process or into complex numerical crop irrigation models can be considered. Such studies should bring more reliability on the validity of qualitative results presented in this work, and help understanding the potential and limitations of TBIS_{CCI} for improving water management.

The previous sections illustrate the merits and drawbacks of different strategies, either optimal or sub-optimal, in maximizing biomass production from a given cumulative irrigation amount. Hence, a given irrigation strategy may be characterized by those two values: the final biomass and the irrigation amount. An additional, elaborate variable may be introduced to describe irrigation efficiency, relating the biomass produced to the irrigation amount needed for it. This will expectedly not replace the separate criteria in terms of quota (using less than a certain quota) and yield (producing more than a contractual yield) but rather provide an additional criteria, allowing to sort the eligible strategies according to their efficiency.

The proposed framework may be enriched with the adjunction of economic variables of three kinds: fixed costs accounting for system operation, variables costs linked to water and energy consumption and the expected selling price (or valorization) of the produced biomass. From the economic point of view, the eligible strategies would still respect the quota and ensure sufficient, if not maximal income. Biomass production and economic income (say E) are expected to be as closely related as variables costs are limited and irrigation efficiency is high but still the optimisation problem should likely be relaxed, as “joint optimization” (both the B and E variables at their maximum possible values) is not achievable in the general case. Due to its genericity, the CCI model is compatible with such considerations of various criteria ([28]).

Fertilization is not addressed here, meaning that non-limiting nutrient availability is hypothesized. The proposed framework may also be widened by the introduction of nitrogen requirements evolving with plant growth, likely following the dynamics of the φ function with a slight delay (e.g. the nitrogen demand sharply increases in the phase of quick canopy growth) (see [73]). Introducing nitrogen concentration as an additional state would bring more complexity for the methodology, and the decision policy would also depend on a more diversified set of rules. Here, nutrients could certainly be brought via fertigation (the irrigation system also delivers nutrients) in complement of an initial amount. This would consequently affect the economic budget.

4.5 Conclusion

In this work, we dealt with the problem of improving crop irrigation strategies, in various contexts suffering limited resources. For this purpose, we considered a class of crop irrigation strategies whose structure is based on assumed known or operator-defined setpoint levels of soil moisture, in succession throughout the cropping season. It has been established that these Threshold-Based Irrigation Strategies emerge from solving an optimal control problem formulated with the Controlled Crop Irrigation model, which is an infinite dimensional optimization problem. More precisely, op-

timal policies are described as TBIS_{CCI} with two soil moisture thresholds, and a time of decision to trigger maximal irrigation rate leading to a one-dimensional optimization. This optimization is thus suited to efficient numerical simulations, and we implemented a global model exploration in order to provide properties about the nature of the crucial time of decision, and the consequences of near-optimal decisions in terms of biomass losses and water savings.

Our first numerical exploration aimed at assessing Sobol Indices giving valuable insights on the global sensitivity of the decision time with respect to a wide range of soil, plant, operational characteristics. We highlighted that in both alternatives of risk aversion or inclination, the optimal time of triggering maximal irrigation rate is strongly related to plant characteristics, regardless of the initial irrigation water quota. In particular, the values of optimal decision time globally follow an increasingly monotonous behaviour with respect to a characteristic growth time, and a decreasingly monotonous behaviour with respect to the soil water content of incipient stress. A second numerical experiment investigates the sub-optimality of TBIS_{CCI} based on small errors around the optimal values of decision time (in the window of admissible decision times) and top soil moisture threshold. We observed that in both alternatives as above and in any case of initial irrigation water quota, mistakes over the top soil moisture threshold may undermine the biomass production by up to 15% compared to the maximal potential biomass. Conversely, mistakes over the decision time tend to have moderate effects over the biomass production. However, early and late decisions lead to noteworthy different outcomes in terms of chances to violate the operating constraint or saving a fraction of irrigation water. The combination of low irrigation quotas, high risk aversion and early decisions, creates strong operational constraints on the use of water resources. It makes more difficult (or unlikely) the identification (or the existence) of admissible soil moisture trajectories. The opposite extreme combination of high irrigation quotas, high risk inclination and late decisions, provides favorable conditions for unhindered and efficient operational use of water.

Our results demonstrate the capabilities of the coupled optimal control and global exploration methodology, to point out some principles of enhancing the management of irrigation scheduling. With our in-depth analysis of TBIS_{CCI}, we opened a perspective to assess their potential in terms of water use efficiency. This could lead to suggest a multi-criteria approach for a decision making based on a compromise between producing more biomass, saving more water, and preventing prolonged periods of water stress. The present study case was limited to greenhouse systems. Future work might extend the methodology to more general systems including rainfall events. Further perspectives might additionally consider identifying sets of parameters corresponding to particular contexts (soil type thus properties, crop growth dynamics), in order to put the TBIS to the test on experimental studies.

Part II

On-going works towards an application to the Optirrig model and an extension to open-field conditions

Chapter 5

The Optirrig and CCI models

This chapter corresponds to a work in progress, included in both the thesis objectives and the collaborative multidisciplinary project conducted for the development of Optirrig. Indeed we tackle here the subject of double modeling, involving the Optirrig model formalisms and the simple CCI model that we studied in the previous chapters (in the greenhouse context). This work aims for providing a correspondence framework between those two models in order to conduct a shared investigation of irrigation strategies. In particular, the Threshold-Based Irrigation Strategies (that are proven to be optimal for the CCI formulation) present a promising candidate scenario for practical and operational efficiency purposes. As a consequence, it is of great interest to adapt these irrigation strategies for an implementation into Optirrig, in order to evaluate their performances in study cases closer to real-world conditions. Furthermore, a thorough comparison between the physical processes and the conceptual formalisms of both models allows us to notice where the approximations in the simple CCI model may have to be completed in order to improve its biophysical viability. Conversely, the genericity of the CCI model enables to point out the most essential intrinsic roles and effects of the system, and may help to elaborate new versions of Optirrig where we concentrate the description on these essential features.

5.1 Models comparison guidelines

Following the conceptual scheme of the double modeling framework Figure 1.1 that we intend to carry out between Optirrig and the CCI model, several directions and elements of development may be considered. First we may wonder if the theoretical results obtained in the greenhouse context with the CCI formulation (see Chapter 3 and Chapter 4) are still valid, or at least provide interesting material, when trying to transpose them into operational situations with Optirrig. Then, the standard loop of double modeling involves calibrating the two models at some operating point, and running parallel simulations in order to conduct, for example, a joint optimization. In any case, the first step for comparing models is to evaluate their compatibility, and establish a set of connections as complete as possible.

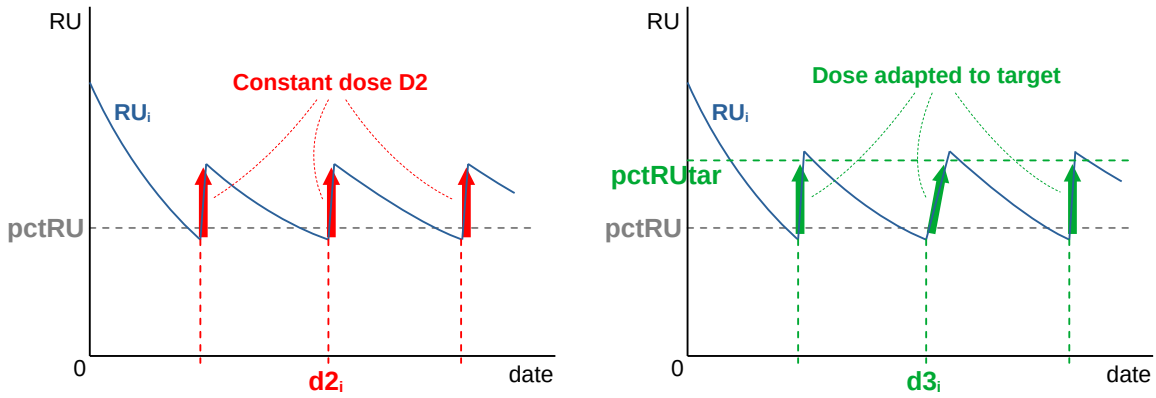
As previously mentioned, Optirrig is focused on implementing how the different irrigation strategies (or water management) may have different impacts on the crop production (or financial benefits). In order to cover a wide variety of farming types, objectives, needs or constraints, we present in Figure 5.1 a list of possible strategies for the operator when using Optirrig. The specificity of Optirrig is that it allows the operator to evaluate and compare different classes of strategies which are an integral part of the implementation for the simulation runs. Moreover, it is expected that an operator may want to define several successive time periods where they will conduct the irrigation following different strategies, or even they may decide during the season to shift their scenario to another strategy.

Moreover, the CCI model formulation precisely relies on the control framework, where the decision variable (the irrigation rate) is the key feature of analysis. As a result, the compatibility discussion between Optirrig and the CCI model should begin with a work plan explaining how irri-

Irrigation strategy			Decision rules			
#	Family	Variable	Triggering criterion	Date	Dose	Parameters
1	Scheduling	d	date	$d1_i$	$D1_i(d1_i)$	($d1_i$, $D1_i$) i=1..n
2	Threshold	RU	$\%RU(d2_i) < pctRU$	$d2_i$	D2	pctRU, D2
3	Threshold	RU	$\%RU(d3_i) < pctRU$	$d3_i$	$D3_i$: $\%RU(d3_i) \rightarrow pctRUtar$	pctRU, pctRUtar
4	Threshold	RU	$RU(d4_i) < mmRU$	$d4_i$	D4	mmRU, D4
5	Threshold	RU	$RU(d5_i) < mmRU$	$d5_i$	$D5_i$: $RU(d5_i) \rightarrow mmRUtar$	mmRU, mmRUtar
6	Threshold	Sw	$Sw(d6_i) < lowSw$	$d6_i$	D6	lowSw, D6
7	Threshold	P, I	$\sum_{i-nj}^{i}(P_k + I_k) < SigmaPI$	$d7_i$	D7	SigmaPI, D7, nj
8	Threshold	T	$\frac{1}{nj+1}\sum_{i-nj}^{i} T_k > Thot$	$d8_i$	D8	Thot, D8, nj
9	Water turn + Threshold	ETM	date, $\sum_{i-N}^{i} \frac{(P_k+I_k)}{ETM_k} < \frac{pctETM}{100}$	$d9_i$	D9	pctETM, D9, N, $d9_1$
10	Water turn + Threshold	RU	date, $\%RU(d10_i) < pctRU$	$d10_i$	D10	pctRU, D10, N, $d10_1$
11	Water turn + Threshold	RU	date, $\%RU(d11_i) < pctRU$	$d11_i$	$D11_i$: $\%RU(d11_i) \rightarrow pctRUtar$	pctRU, pctRUtar, N, $d11_1$
12	Water turn + Threshold	RU	date, $RU(d12_i) < mmRU$	$d12_i$	D12	mmRU, D12, N, $d12_1$
13	Water turn + Threshold	RU	date, RU(di) < mmRU	d13i	$D13_i$: $RU(d13_i) \rightarrow mmRUtar$	mmRU, mmRUtar, N, $d13_1$
14	Water turn + Threshold	Sw	date, Sw(d14i) < lowSw	$d14_i$	$D14_i(d14_i)$	lowSw, D14, N, $d14_1$
15	Water turn + Threshold	P, I	date, $\sum_{i-nj}^{i}(P_k + I_k) < SigmaPI$ nj<N	$d15_i$	D15	pctRU, D15, N, nj, $d15_1$
16	Water turn + Threshold	T	date, $\frac{1}{nj+1}\sum_{i-nj}^{i} T_k > Thot$ nj<N	$d16_i$	D16	Thot, D16, N, nj, $d16_1$

Figure 5.1: List of strategies that are possible to operate in practice in the Optirrig model. These strategies come in different characteristics, from their scheduling type (open-loop, closed-loop, closed-loop with a constrained calendar), their decision rules (triggering criterion, watering amount) and their corresponding parameters for the implementation.

gation strategies (i.e. the main object of study for both models) can be transferred from one model to the other. Since the mathematical framework allows complete flexibility for the nature of irrigation chronicles (measurable functions in continuous time), then any irrigation scenario produced by Optirrig with any strategy can at least be expressed as a time-varying function suited to CCI control framework. Conversely, we do not aim for representing every measurable function in the Optirrig formalism, but only the theoretical strategies of interest. In particular, the 2-TBIS and their declensions detailed in Chapter 4 are strategies that we should be able to test in Optirrig. Notice that the strategies in Figure 5.1 may be formulated in open-loop (n°1), closed-loop (from n°2 to n°8), or closed-loop with scheduling constraints (from n°9 to n°16). Moreover, the criteria of irrigation triggering may be based on soil water levels, stress levels, water accumulation or temperature accumulation over a sliding window time. We illustrate in Figure 5.2 two typical ways to base a strategy on a soil moisture threshold (denoted *pctRU* for "percentage of available water capacity" in these cases), that only differ in the type of action: strategy n°2 consists in giving always the same amount of water (regardless to the resulting soil moisture level), and strategy n°3 consists in giving the amount (regardless of the dose value) of water to reach a given soil moisture level.



(a) Graphical representation of the strategy n°2.

(b) Graphical representation of the strategy n°3.

Figure 5.2: Visual comparison of the strategies n°2 and n°3, which are based on the same irrigation triggering criteria, but differ in the decision of irrigation: while strategy n°2 consists in giving always the same amount of water (regardless to the resulting soil moisture level), strategy n°3 consists in giving the amount (regardless of the dose value) of water to reach a given soil moisture level.

The objective of the following sections is to provide a detailed correspondence between Optirrig and the CCI model, in order to prepare in particular the implementation and evaluation of the 2-TBIS in Optirrig. We may suggest here that we identified the strategy n°3 as a candidate for such purposes. Indeed, an irrigation strategy based on two thresholds appears to require a succession of two periods associated to two different thresholds (of triggering, thus also two of target), on which we could conduct the two corresponding strategies n°3 successively. The previously called "decision time" would correspond here to the date of shifting from the first to the second strategy n°3.

5.2 Presentation of the Optirrig model

5.2.1 A wide-purpose irrigation optimization tool

The Optirrig model [34, 33] was built to perform the generation, analysis and optimization of irrigation scenarios, assuming the simplest possible description of plant physiology while focusing on the identification of relevant resources allocation, in ex-post (analysis), ex-ante (prospective) or real-time (decision support) contexts. Model development relies on collaborative, multidisciplinary contributions. Its logic of development is to initiate collaborative developments regarding the description of soil and crop-related physical processes, the specification of variables of interest or the exploration of case-studies in envisioned model applications. Moreover, the purpose of Optirrig is to be used for interactions with policy makers, to propose, demonstrate and promote new ways to tackle the water allocation issue.

The model is formulated with discrete time and a fixed step, and relies on a parsimonious description of soil water budget, plant growth, properties and biomass production to rather place emphasis on plot-scale water management, via the identification of the most appropriate (constrained) irrigation strategies. Being a 1-D vertical model, Optirrig neither allows any explicit description of lateral fluxes nor any indication as regards the area of its domain of application. The latter only has to be a domain in which heterogeneities are not too strong, hence supporting spatially-averaged descriptions. Some key physical variables are mandatory components of the model, such as the time-varying content (or percentage filling) of soil water reservoirs, the evolving Leaf Area Index (LAI) of the crop and the daily rate of biomass production.

In the Optirrig model, water stress is the driver of losses in crop growth, and the issue addressed here is about reducing such losses as much as possible, in various so called "deficit irrigation" situations. Here, piloting irrigation thus means acting upon water stress, described as deficit of plant transpiration, over certain period of time. In other words, stress terms are the emerging, accessible part of the conceptual scheme, irrigation being the lever of action upon them (notice that it is however difficult to conduct direct measurements of stress). Water stresses are visible through their short-term ans long-term effects on the other variables, which are observable for most of them. Providing efficient irrigation schedules therefore consists in managing water stresses for given constraints and objectives.

The piloting of irrigation and its optimization purpose may differ according to various points of view (water management, crop production, financial aspects) with their associated metrics (water use efficiency, water productivity, economic viability, societal expectations, contractual crop yields, water or environmental footprints). To this end, the Optirrig model focuses on order-1 processes and levers of action, i.e. how the manipulated variable (which is irrigation) spreads out in the model conceptual scheme, seeking compromise between being right for the right biophysical reasons and providing meaningful evaluation of the operational variables (biomass production, harvest index, crop yield) regarding harmfulness and overall effect due to water stress.

Obtaining irrigation strategies consists in the difficult task of finding irrigation decision rules able to cope at best with (i) contradictory objectives (in the "more crop per drop" view), (ii) operational constraints (insufficient water rights or quota, limited water availability or shared access to water, characteristics of the irrigation equipment), (iii) regulatory constraints (restrictions during periods of tension on the shared resource), (iv) poor availability of supervision data (ungauged or poorly-gauged contexts), (v) multiple uncertainties (biases and shortcuts in the model, errors affecting model forcings, parameters and observation data), (vi) unexpected difficulties (equipment defaults, personal issues, irrigations missed) and (vii) events that are unpredictable by nature, such as variations in the climatic forcings, especially in rain chronicles, to which irrigation must react and adapt.

Different irrigation decision rules, i.e. irrigation settings and parameters, will likely translate into different irrigation scenarios, i.e. chronicles of irrigation dates and amounts. In other words, different sets of irrigation parameters (triggering criteria, setpoints and controls) will result in different dynamics for the filling of soil reservoirs, with cascading effects throughout the entire conceptual scheme. This will produce different dynamics for the main state variables (reserves, leaf are index, biomass) and, likely, different final values for the operational variables (cumulative irrigation, crop yield) which opens way for a wide and multi-criteria case comparison. The latter strongly depends on the model ability to handle and analyse multiple runs. Put together, all components take place in a software (manual, graphical interface, versioning...), not described here but being the vehicle for model transfer, which is the actual development object that we call Optirrig.

5.2.2 Model rationale

This section provides an introduction of the Optirrig model concepts, variables and parameters that are relevant for both the understanding of the Optirrig model and the correspondence with the CCI model. This presentation is not exhaustive but is as detailed as needed for the current purposes.

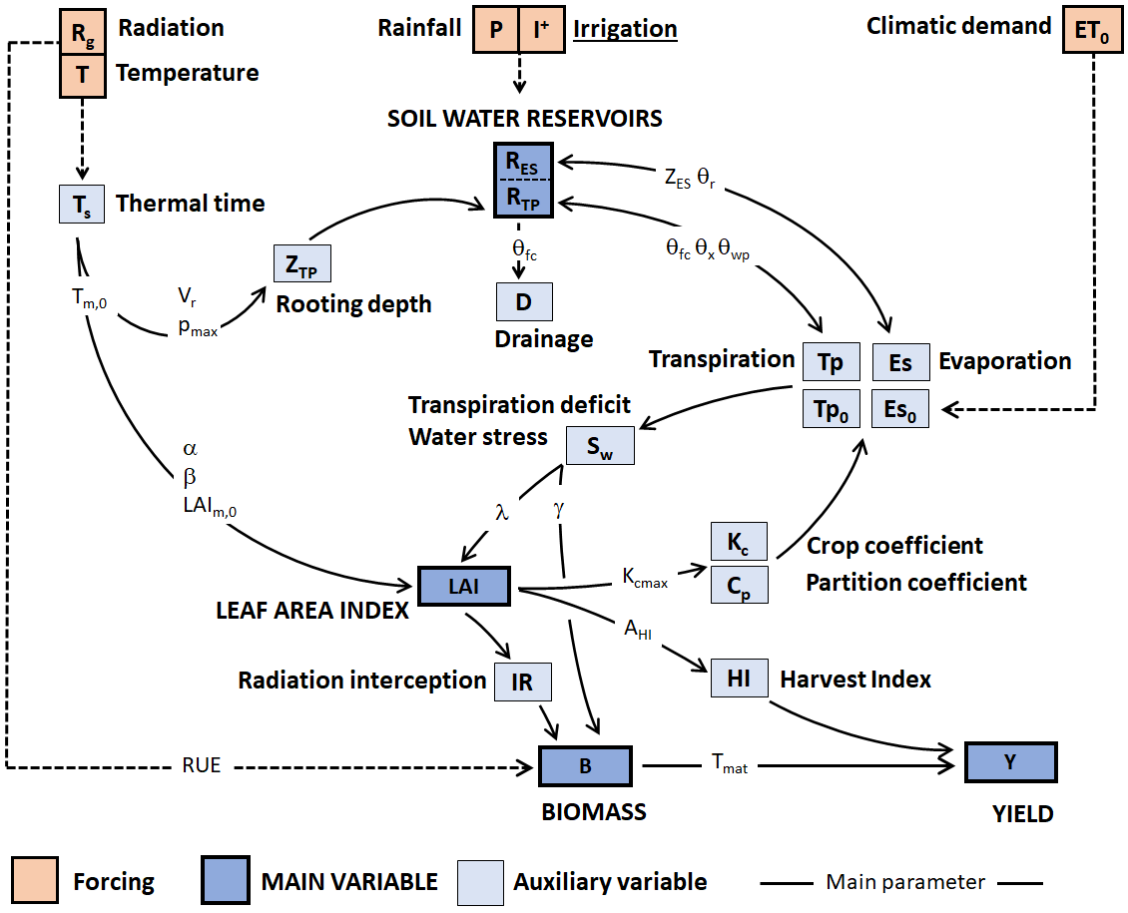


Figure 5.3: Reference conceptual scheme of Optirrig, briefly detailed in the current section, showing the causal links between model variables (coloured boxes, see Table 5.1) as well as the position and limited number of key model parameters (see Table 5.2). This scheme pertains to field crops modelling in classical conditions. For some families of species, this reference scheme will need some adaptations, but will not be discussed here.

Name	Description	Variable type
T	Daily-averaged air temperature	Forcing
$ET0$	Climatic demand	"
PAR	Photosynthetically Active Radiation	"
P	Rain (precipitation)	"
I	Irrigation	"
Rg	Global radiation	"
LAI	Leaf Area Index value	Main
R_{ES}	Reserve in the evaporation reservoir	"
R_{TP}	Reserve in the transpiration reservoir	"
B	Biomass	"
Y	Crop yield	"
T_s	Temperature sum	Auxiliary
S_w	Stress term	"
S_c	Cumulative stress	"
TP	Transpiration	"
$TP0$	Transpiration demand	"
Kc	Crop coefficient	"
Cp	Partition coefficient	"
$ES0$	Evaporation demand	"
Z_{TP}	Depth of the transpiration reservoir	"
ES	Evaporation	"
D	Drainage	"
IR	Radiation interception	"
HI	Harvest Index	"
T_e	Normalized temperature sum	Calculation
LAI_p	Potential LAI value	"
TL	Logistic temperature	"
LAI_m	Maximal reachable LAI value	"
LAI_o	Origin of the LAI_p curve	"
T_m	End of crop expansion phase	"
θ_{ES}	Water content in the evaporation reservoir	"
θ_{TP}	Water content in the transpiration reservoir	"
K_{ES}	Efficiency of the evaporation process	"
K_{TP}	Efficiency of the transpiration process	"
θ	Water content	"
Z_D	Vertical extension of the drainage reservoir	"
K_D	Drainage rate, hydraulic conductivity	"
R_D	Drainage reservoir	"
I_N	Normalized irrigation	"

Table 5.1: List of Optirrig model variables. Variable types include : Forcing, Main (observable), Auxiliary (directly controlling the main variables) and Calculation (internal, intermediate calculation variables).

Name	Description	Source
T_b	Base temperature	Literature
$T_{m,0}$	Thermal time to reach maximal LAI value	Literature
$LAI_{o,0}$	Initial value of LAI	Field
$LAI_{m,0}$	Maximal possible LAI value	Literature
α_1	Shape parameter of the LAI_p curve	Database/User
β	Shape parameter of the LAI_p curve	Database/User
λ	Harmfulness of transpiration deficit for LAI	Database/User
A_0	Amplitude of the potential LAI curve	Literature
n_d	Number of days to calculate transpiration deficit	Fixed
Kc_{max}	Maximal value of the crop coefficient	Literature
k	Coefficient in the dependence of Kc on LAI	Fixed
c_1	Coefficient in the dependence of Cp on LAI	Fixed
c_2	Coefficient in the dependence of Cp on LAI	Fixed
Z_{ES}	Depth of the evaporation reservoir	Soil texture
θ_{fc}	Field capacity	Soil texture
RUE	Radiation use efficiency	Literature
γ	Harmfulness of transpiration deficit for B	Database/User
k_{BL}	Extinction coefficient	Fixed
θ_r	Residual water content	Soil texture
θ_{wp}	Permanent wilting point	Soil texture
θ_x	Lower limit of the Readily Available Water	Soil texture
I_{max}	Maximal irrigation rate	Database/User

Table 5.2: Model parameters. Most parameter values are supposedly known (literature) or easily taken from contextual data (soil texture, field). Some others can theoretically be taken from the model database, if documented experiments have been conducted in similar conditions, but may otherwise need to be adjusted (database/user) if observations are available, or assumed imperfectly known.

Window of time The Optirrig model is written as a discrete-time evolution model, and any variable or time-dependent quantity $a(t)$ wil be denoted a_i, where i represents the current time step. Moreover, according to its conceptual scheme (Figure 5.3), the Optirrig model is a thermal time model, which means the driving force behind crop development is the sum of daily-averaged air temperature values. More precisely, crop development is calculated on a daily basis from the sum of the differences between the daily-averaged air temperature value T_i and a crop-specific base temperature T_b under which crop growth is supposed to halt, so that the actual variable representing time is

$$T_{s,i} = \sum_{k=1}^{i} T_k - T_b, \tag{5.1}$$

with an initial cumulative thermal time set at $T_{s,0} = 0$ (at sowing).

The phase of crop expansion is assumed to end when $T_{s,i}$ reaches the crop-specific T_m value, associated with the maximum development of plant foliage and canopy, i.e. the maximum Leaf Area Index (LAI) value. This way, the thermal time interval of system evolution (from sowing to maximal LAI value) is $[0, T_m]$. Let us thus define the normalized elapsed thermal time $T_{e,i}$ at day i as the following fraction:

$$T_{e,i} = \frac{T_{s,i}}{T_m}, \tag{5.2}$$

where $T_{e,i}$ ranges between 0 and 1 from the initiation to the end of crop expansion phase.

Potential Leaf Area Index The potential LAI growth, in absence of any stress, is calculated as

$$LAI_{p,i} = LAI_m TL_i(T_{e,i}, \alpha_1, \beta), \tag{5.3}$$

where LAI_m is the magnitude term and the maximal possible LAI value, while TL is a pseudo-logistic, S-shaped function of thermal time (T_e) and shape parameters (α_1, β). The expected property of S-shape is present in the chosen expression for TL, as follows:

$$TL_i = T_{e,i}^{\beta} \exp\Big(\frac{\beta}{\alpha_1}(1 - T_{e,i}^{\alpha_1})\Big), \tag{5.4}$$

where only $\beta > \alpha_1 > 1$ will ensure the TL and LAI curves have the required S-shape, which means an inflection point is present in the curve.

Evaporation and transpiration Let us begin with the transpiration ($TP0$) and evaporation ($ES0$) demands, which are evaluated from the crop coefficient Kc, the partition coefficient Cp, and the climatic demand $ET0$, as follows:

$$TP0_i = Kc_i Cp_i ET0_i, \tag{5.5}$$

$$ES0_i = Kc_i(1 - Cp_i)ET0_i, \tag{5.6}$$

where both the crop coefficient and the partition coefficient (also called canopy coefficient) are functions of LAI.

The crop coefficient writes

$$Kc_i = Kc_{max}\big(1 - \exp(-kLAI_i)\big), \tag{5.7}$$

where Kc_{max} is the maximal possible value of Kc for a given crop. In previous studies, the k coefficient has always been set to $k = 1$. Smaller k values would rather affect the plateau values of Kc than its dynamics, but might enable some modeling flexibility, to handle situations in which the link between LAI and Kc could otherwise not be correctly modeled.

The partition coefficient writes

$$Cp_i = c_1 - \exp(-c_2 LAI_i), \tag{5.8}$$

where, again, c_1 and c_2 have always taken fixed values ($c_1 = 1$, $c_2 = 0.7$) but offer potentially interesting degrees of freedom, unmentioned so far, with associated difficulties in parameter estimation. In fact, as Cp aims to depict how the transpiration demand takes over the evaporation demand with foliage development and canopy growth, Cp is expected to start with $Cp = 0$ for $LAI = 0$, which requires $c_1 = 1$.

Once the demands ($ES0$, $TP0$) are known, the next step is to estimate the answers (ES, TP). First define the evaporation reservoir as follows:

$$R_{ES,i} = Z_{ES}\theta_{ES,i}, \tag{5.9}$$

where Z_{ES} is a characteristic depth of soil from where the evaporation is possible (it strongly depends on soil texture), and θ_{ES} is the average soil water content in the evaporation reservoir (of depth Z_{ES}). Then define the transpiration reservoir as follows:

$$R_{TP,i} = Z_{TP,i}\theta_{TP,i}, \tag{5.10}$$

where Z_{TP} represents the root growth, and θ_{TP} is the average soil water content in the transpiration reservoir (of depth Z_{TP}).

The ES term is then taken as the most limiting between the content of the evaporation reservoir and soil ability to respond to the evaporation demand. The latter is represented by the crop coefficient K_{ES} (called efficiency of the evaporation process) describing the way soil evaporation occurs with respect to the current soil water content. The general expression of ES is the following:

$$ES_i = \min(R_{ES,i}, K_{ES,i}ES0_i). \tag{5.11}$$

Consequently, if the product $Z_{ES}\theta_{ES}$ is low, the soil evaporation can not match the evaporation demand and it only releases the low water content that is contained in the evaporation reservoir. Conversely, when the soil capacity matches the evaporation demand, its evaporation response is still weighted by the coefficient K_{ES} which depends on soil water content.

Next comes the necessity to evaluate the actual transpiration TP from the comparison between the content of the transpiration reservoir and soil ability to respond to the transpiration demand. The latter is represented by the crop coefficient K_{TP} (called efficiency of the transpiration process) describing the way plant transpiration occurs with respect to the current soil water content. The general expression of TP is the following:

$$TP_i = \min(R_{TP,i}, K_{TP,i}TP0_i). \tag{5.12}$$

As the plant root is progressively deepening, the transpiration reservoir is growing. As a consequence, transpiration only releases low water contents ($Z_{TP}\theta_{TP}$) at the beginning of the cropping season. Conversely, when the transpiration reservoir can match the transpiration demand, its transpiration response is still weighted by the coefficient K_{TP} which depends on soil water content.

Water stress and its effects on LAI We define the relative deficit of transpiration as follows:

$$S_{w,i} = \frac{\sum_{k=i-n_d}^{i-1} TP_k}{\sum_{k=i-n_d}^{i-1} TP0_k}, \tag{5.13}$$

where $S_{w,i}$ corresponds to the coefficient of effect of water stress over a sliding window of the last n_d days. Typically, for an instantaneous stress affecting the daily increment of LAI, it may be suitable to set $n_d = 1$. When $S_w = 1$, it means that the crop transpiration matches the transpiration demand, i.e. crop is not suffering from any water stress. Moreover the effect of transpiration deficit over plant development depends on the harmfulness $\lambda > 0$ associated to the crop, enabling us to depict the way crop development is limited under the different levels of water stress.

It is possible to consider a cumulative effect of stress, either implicitely by adjusting the value n_d in the previous S_w formula, or explicitely by adding a multiplicative term S_c that acts on the

current LAI to represent the eventual early senescence induced by long periods of stress.

As a result, the actual development of LAI taking into account the incremental stress (generating the effect S_w) and cumulative stress (generating the effect S_c, which is for the moment a facultative feature) is formulated dynamically as follows:

$$LAI_i = LAI_{i-1} S_{c,i} + (LAI_{p,i} - LAI_{p,i-1}) S_{w,i}^{\lambda}. \tag{5.14}$$

We verify that if the crop does not suffer from any stress S_c or S_w, then the LAI develops at its potential rate, following the shape and magnitude of LAI_p.

Drainage As reviewed by [39] there is a large consensus in the literature on the definition of field capacity as the threshold water content value above which significant drainage occurs, in the form of gravity flow, resulting from non-equilibrium conditions and aiming at returning to the equilibrium conditions. Denote D_i the drainage occuring at the date i, then $D_i = 0$ if the soil water content is below the field capacity, and the chosen formulation for daily drainage when soil moisture exceeds the field capacity is

$$D_i = \min(Z_D(\theta_i - \theta_{fc}), K_D), \tag{5.15}$$

where $\theta_i - \theta_{fc}$ is the excess water content to be drained, evaluated the current day, and Z_D is the vertical extension of the considered soil reservoir (thus their product indicates the water amount to be drained, say R_D) while K_D is the current drainage rate, representing the near-saturation hydraulic conductivity, calculated from both soil structure and current filling.

It should be noted that this expression of drainage only holds for vertically-averaged water contents above θ_{fc}, taking the average over the full extension (Z_D) of a given reservoir, to be taken as the deepest reservoir between the evaporation one (Z_{ES}) and the transpiration one (Z_{TP}).

Soil water content variations All crop related variables involved in the variation of soil moisture have been introduced, there only remains the water inputs from both rainfall events (denoted P) and irrigation (denoted I). The model Optirrig does not directly compute the soil water content balance between all these contributions, but rather proceeds in successive steps, priorising evaporation from the upper reservoir, then considering transpiration (if the soil water content is sufficient and the transpiration demand has yet to be matched) occuring in the dedicated reservoir, then adding the positive terms of rainfall and irrigation. However, we may write a conceptual balance corresponding to the result of these steps as if they happened simultaneously, as follows:

$$R_i = R_{i-1} - TP_i - ES_i + I_i + P_i - D_i, \tag{5.16}$$

where R is the deepest reservoir between R_{ES} and R_{TP}.

Biomass, water stress effect and yield The lower part of the conceptual scheme of the model (Figure 5.3) shows that biomass, here biomass accumulation, is calculated from LAI. Biomass increment between successive time steps writes

$$B_i = B_{i-1} + PAR_i IR_i(LAI_i) RUE S_{w,i}^{\gamma}, \tag{5.17}$$

where B_i is the biomass at day i, PAR_i is the photosynthetically active radiation, taken as half the global daily radiation Rg_i, IR_i is the fraction of intercepted radiation, calculated from the current LAI value, RUE is radiation use efficiency, considered a fixed parameter for a given species, and $S_{w,i}$ is the transpiration deficit affecting biomass production, with its specific harmfulness coefficient γ. In the latest, currently-developped versions of the model, γ is tested as an increasing function of time (thermal time) to account for the known, increased sensitivity of most crops to water stresses occurring during their (final) reproductive stages, while stresses occurring during their (initial) vegetative stages are assumed of lesser impact.

Finally, radiation interception is calculated by means of the Beer-Lambert law as follows:

$$IR_i = 1 - \exp(-k_{BL} LAI_i), \tag{5.18}$$

where k_{BL} is the so called "extinction coefficient".

Crop yield is then evaluated as follows:

$$Y = B_h HI_h, \tag{5.19}$$

where Y is crop yield, B_h is the total biomass at harvest (date h) and HI_h is the harvest index the same day, representing the useful fraction of the total plant weight.

5.3 Towards a correspondence with the CCI model

In this section, we present the similarities and differences between the Optirrig and CCI models, in order to settle a framework of double modeling. The comparison will take place in two steps : a conceptual one (trying to replicate the Optirrig scheme into a CCI adapted structure), and a more concrete one (trying to introduce the correspondence in terms of values and formulas of the respective models).

5.3.1 Comparison of structures

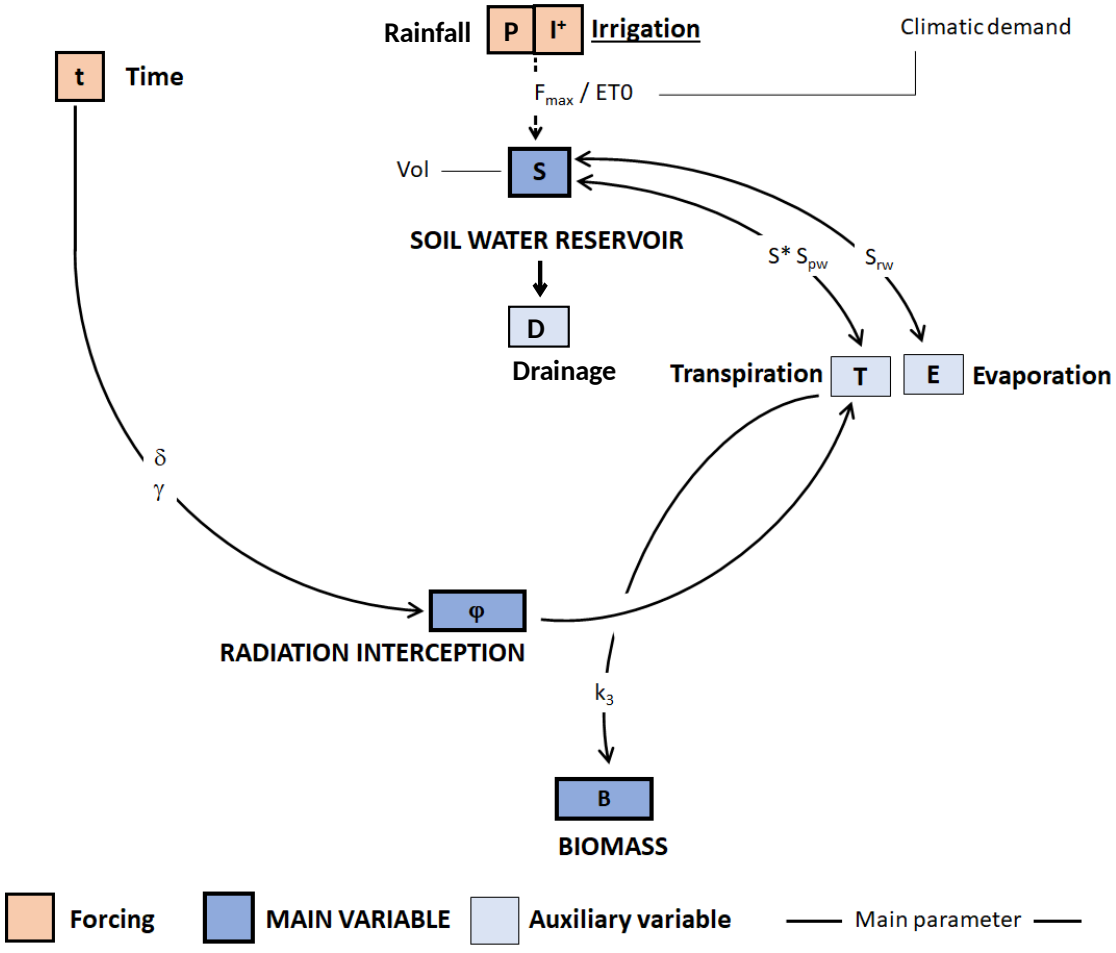


Figure 5.4: Conceptual scheme of the CCI model, showing the causal links between model variables (coloured boxes, see Table 5.3) as well as the position and limited number of key model parameters (see Table 5.4). This scheme includes strong hypotheses and physical approximations, but aims for a more generic modeling of various crops and field conditions.

Time span In both Optirrig and the CCI model, the variables of interest evolve throughout the cropping season, in evolution systems depending on time (CCI) or on thermal time (Optirrig) thus

Name	Description	Variable type
t	Time	Forcing
$ET0$	Climatic demand	"
RA	Rainfall contribution	"
r	Rain (precipitation)	"
IR	Irrigation	"
φ	Fraction of intercepted radiation	Main
S	Soil water content	"
B	Biomass	"
V	Irrigation water budget	"
TR	Transpiration	Auxiliary
EV	Evaporation	"
DR	Drainage	"
K_S	Efficiency of the transpiration process	Calculation
K_R	Efficiency of the evaporation process	"
K_D	Efficiency of the drainage process	"
u	Normalized irrigation	"

Table 5.3: List of CCI model variables. Variable types include : Forcing, Main (observable, accessible), Auxiliary (directly controlling the main variables) and Calculation (internal, intermediate calculation variables).

Name	Description	Source
T	Duration of the cropping season	Literature
δ	Shape parameter of the φ curve	Database/User
γ	Shape parameter of the φ curve	Database/User
Vol	Depth of the evapotranspiration reservoir	Soil texture
S_{fc}	Field capacity	Soil texture
k_3	Potential biomass production rate	Database/User
S_{rw}	Point of residual water content	Soil texture
S_{pw}	Permanent wilting point	Soil texture
S_{ope}	Operating constraint	Database/User
S^*	Point of incipient water stress	Literature
F_{max}	Maximal irrigation rate	Database/User
Q	Initial water quota	Database/User
α	Rain infiltration capacity	Soil texture

Table 5.4: Model parameters. Most parameter values are supposedly known (literature) or easily taken from contextual data (soil texture, field). Some others can theoretically be taken from the model database, if documented experiments have been conducted in similar conditions, but may otherwise need to be adjusted (database/user) if observations are available, or assumed imperfectly known.

also, indirectly, on time. For the CCI model, we directly use the real time denoted t, and the system is supposed to evolve over a time interval $[0, T]$ of cropping season. By contrast, Optirrig is written as a thermal time system, meaning that it is a variable accounting for the accumulation of temperature that plays the role of the time component. In the particular case where the temperature would be controlled and kept constant throughout the whole season, then the thermal time is simply a dilatation of the real time. In the general case, the thermal time consists in a monotonous change of time variable that slightly accelerates when the warm season arrives. The analogous "thermal time" interval representing the cropping season is $[0, T_m]$, and in the following we will speak about time-dependent variables and thermal time-dependent variables without distinction, since they play the same role in similar ways.

Water components Regarding soil water budget, several terms deal with external contributions to the soil processes : irrigation, rainfall and drainage. In turns out that the Optirrig and CCI models contain very similar concepts of these contributions. Indeed, both present the action of irrigation as a single external chronicle that adds water directly into the soil (in a discrete or continuous time, using discrete or continuous watering amounts). Rainfall is generally seen as a daily contribution in Optirrig, and acts on the soil water content the same way as the irrigation contribution. In the CCI model framework, we allow a more generic formalism of rainfall since we wrote the contribution of rainfall with a time-dependent measure, enabling both continuous and instantaneous precipitation. However, in both cases the contribution of rainfall plays the same role, similar to the irrigation contribution, in the soil water contents. Then, in both Optirrig and CCI in the open-field conditions, drainage is assumed to be possible when the soil moisture exceeds the physical equilibrium point of the field capacity. Moreover, in both cases drainage is bounded even in extreme flooding situations. It is thus affordable to consider that the Optirrig and CCI models have quite direct analogous external water components.

Soil components One of the main differences between Optirrig and the CCI model is that Optirrig provides rather explicit formulations of the order-1 physical processes while the CCI model contains rather abstract formulations. These are possibly endowed with physical sense, although most often combining processes treated separately in the companion model. More precisely, Optirrig calculates the responses in evaporation and transpiration as logical consequences of a demand, and their calculation is explicitely formulated as the combined effects of the climatic demand, the current soil filling and the radiation interception (due to plant covering the soil surface). Furthermore, the level of soil filling might induce explicit stress that reduce the development of LAI, which bounces back in the expression of the evapotranspiration capacity. In the CCI formulation, this combined effect is implicitely contained in a single formula partitioning evaporation and transpiration, which are moreover weighted by efficiency crop coefficients. These coefficients have to solely carry the effect of water stress, that is not concretely calculated in the CCI formalism (due to the absence of a loop of processes in its conceptual scheme). As a result, the terms of evaporation and transpiration respectively may be seen as similar contributions for Optirrig and the CCI model into the soil water content, but we have to be conscious that they come from different descriptions.

Plant components The common purpose for Optirrig and the CCI model is making crops grow with the best possible irrigation, for assigned objectives, in presence of external constraints. Consequently, both formulate in their respective way a time-dependent variable that represents the growth function of the crop. Optirrig jointly describes root growth and canopy development (LAI) in function of thermal time. More precisely, the depth of the transpiration reservoir is dictated by root deepening, which allows a progressively larger access to soil water, for transpiration, throughout the season. Conversely, the CCI model assumes that the available volume of soil water reservoir is fixed during the whole cropping season, implying in particular that early transpiration effects are quite reduced compared to those in Optirrig. Regarding the evolution of leaves, the Optirrig formulation involves the concrete Leaf Area Index, quantifying how the plant covers the surface with its canopy, and it then enables to assess the intercepted radiation IR and the surface partition Cp. In the CCI model, one may consider that the time-varying function φ represents the LAI growth (with its pseudo-logistic shape), but this function is rather accounting for the partition of surface receiving radiation. As a consequence, the function φ in CCI plays a double role of both IR and Cp in Optirrig, knowing that these two variables are calculated from LAI using a similar formula.

5.3.2 Term-to-term correspondence

Irrigation, rain and drainage The daily amount of water introduced into the soil through irrigation in the Optirrig model is denoted by $I_i = I_{max} I_{N,i}$. In the CCI model, irrigation is a fraction of the maximum rate, identified as $k_2 = F_{max}/ET0$, which represents the capacity of the irrigation equipment to deliver several times the daily climatic demand, thus to exert a strong control on the soil-plant system. The irrigation rate thus writes $\frac{F_{max}}{ET0} u(t)$, recalling that in this expression the climatic demand $ET0$ is a constant, taken as the average value of the climatic demand throughout the season. Similarly, in Optirrig, the ratio of irrigation to the climatic demand writes $\frac{I_{max}}{ET0_i} I_{N,i}$, this time with time variable $ET0_i$ values. Consequently, F_{max} and I_{max} being considered known, given some data as $ET0$ and $ET0_i$, it is possible to compute the irrigation chronicle u in CCI from the one I_N in Optirrig, or conversely. This relationship will be useful in terms of transfering the irrigation scenarios from a model to the other. We can translate the rainfall contributions in an analogous way, since the respective daily or instantaneous amounts of rainwater are written as P_i (for Optirrig) and $\alpha r(t)$ (for the CCI model). Finally, the drainage terms match (at 0) in Optirrig and CCI when the soil moisture is below the field capacity, then they are written by $D_i = \min(Z_D(\theta_i - \theta_{fc}), K_D)$ (for Optirrig) and $DR = \min(k_{sat}(S-1)/(S_{sat}-1), k_{sat})$ (for CCI). Given that θ_{fc} corresponds to $S = 1$, we observe that the parameters (Z_D,K_D) of one side can be deduced from the parameters (k_{sat},S_{sat}) of the other side, in both directions.

Evapotranspiration rates The transpiration rate is calculated in Optirrig with the formula $TP = \min(R_{TP}, KcCpET0K_{TP})$ and as $TR = ET0\varphi K_S$ for the CCI model. This allows both identifying the terms playing similar roles in the two models, and identifying structural differences between models. As for the latter, the min term in Optirrig specifies that transpiration should only be taken from soil water contained in the soil region accessible to the transpiration demand. This region is taken as the rooting depth of the crop, starting from almost zero then increasing up to a maximum value, assumed to be reached at the same thermal time that is needed to reach the maximum development of the aerial part of the plant (LAI). This limitation, via the min term, does not exist in the CCI model which calculates TR over "a certain soil depth" that one is tempted to adjust to the full soil profile of Optirrig - but other options are likely possible when needing to specify the depth of the soil profile in the CCI model. Apart from this discussion on the depth over which TP is calculated and on the subsequent limitations of the avaliable water amount, the comparison between formalisms shows that φ in the CCI model plays the same role as $KcCp$ in Optirrig, once Kc normalized by its Kc_{max} value. Both Kc and Cp are exponential functions of LAI which is a sigmoid of thermal time. This also makes the $KcCp$ product a sigmoid of thermal time, while φ is a sigmoid of time. There is therefore a strong parallel between formalisms.

Evaporation is calculated as $ES = \min(R_{ES}, Kc(1-Cp)ET0K_{ES})$ in Optirrig and $EV = ET0(1-\varphi)K_R$ for the CCI model. The min term in Optirrig specifies that evaporation should only be calculated from the soil region accessible to evaporation. This is a region of fixed and limited depth, known from soil structure : sandy soils allow deeper evaporation uptake than silty or clayey ones. This is a region which does not cover the full rooting depth : in most cases, plant roots largely overcome the evaporation depth during crop growth phase. This limitation, via the min term, does not exist in the CCI model which calculates EV over "a certain soil depth", with the same elements of discussion as for the TR term. Then if φ in the CCI model plays exactly the same role as $KcCp$ in Optirrig, as previously voiced, $Kc(1-Cp)$ is not stricly equivalent to $(1-\varphi)$ and one likely has to admit the link between Kc, Cp and φ is a bit loose and relies on the shape of the functions with time. Another option would have been to involve the efficiency terms K_{TP}, K_{ES}, K_S, K_R in the comparison between Kc, Cp and φ, but this option may likely be discarded because of the obvious, strong analogy between the efficiency terms of the two models : in both models the efficiency of the evaporation and transpiration processes is linked to threshold values of soil water content (in θ for Optirrig) or saturation level (in S for CCI).

Plant growth, radiation interception and biomass production The previous argument on the similar shapes of the canopy-related terms ($KcCp$ and φ) appears rather crucial for model comparison. It calls for further investigation of the roles played by those terms, the biophysical processes they actually describe and the overlaps between the reduced fomulations of the CCI model and the developed ones of Optirrig. An interesting example is the representation, in the models' conceptual

schemes, of the causal link between plant growth, canopy development, radiation interception and biomass production. In Optirrig, plant growth with thermal time is signaled by increasing LAI values, accompanied by increasing radiation interception $IR(LAI)$ values taking place in the equation used to describe biomass production. There are two parameters (α_1, β) controlling the S-shape of the LAI curve and one parameter k_{BL} controlling that of the IR curve, an exponential that saturates for high-enough LAI values, which finally makes IR another S-shaped function of thermal time.

The next step is to compare the expressions used for biomass production, which are for Optirrig $PAR_i IR_i(LAI_i) RUE S_{w,i}^{\gamma}$, and $k_3 \varphi(t) K_S(S(t))$ for the CCI model.

- Radiation interception IR in Optirrig has a S-shape, as φ in the CCI model, which tends to indicate a link between those terms. However, from previous arguments, φ has already been linked to Kc and Cp. Put together, the two parameters (δ, γ) controlling the shape of φ would have to be selected in coherence with the three parameters (α_1, β, k_{BL}) present in Kc, Cp and IR. A good starting point is that the match is good between the shape of the different functions.
- The k_3 term (parameter) being the scaling term for the CCI model, it somehow describes the potential daily biomass production for Optirrig, combining PAR_i and RUE, that is the amount of photosynthetically active radiation (forcing variable) and radiation use efficiency (parameter) denoting the plant's ability to convert solar radiation into biomass.
- From the two points above, there remains the equivalence between $S_{w,i}^{\gamma}$ in Optirrig and K_S in the CCI model, which questions the equivalence previously assumed between K_{TP} and K_S, when comparing transpiration efficiencies. In Optirrig, K_{TP} indicates how transpiration efficiency is reduced for too low water contents, while $S_{w,i}$ converts the effect of this reduction in a stress term (comparing the effective transpiration to the transpiration demand over a certain number of days), and finally $S_{w,i}^{\gamma}$ denotes the effect of the transpiration reduction (and deficit) on biomass production.

5.4 Discussion and perspectives

Follow up implementation As hinted in Section 5.1, the framework of the Optirrig model allows us to consider irrigation strategies as an entry of a simulation run. We noticed that the strategy n°3 was suitable for an attempt of implementing the so-called 2-TBIS, and that the "decision time" could be identified as the time of switching from the first period to the second period applying strategy n°3 (using different thresholds). While the identification of triggering thresholds (in Optirrig, in terms of percentage of available water capacity) appears to be directly related to the values of soil moisture thresholds (in the CCI model, in terms of S variable), the definition of the threshold target *pctRUtar* is far from trivial. Indeed, we may choose any threshold that is above the triggering threshold, while staying reasonably close to be easily reachable in one irrigation. This flexibility suggests that different practices may emerge, depending on the material constraints or the desires of the operator. Indeed, selecting a target threshold relatively far above the triggering threshold implies that the operator would irrigate less often with larger amounts, and conversely selecting a target slightly above the triggering threshold implies irrigating quite consistently with small amounts. The first case is easier to carry out in practice, but moves away from the theoretical singular arc. The second case requires a higher frequency of management, and relies on the accuracy of soil moisture sensors to support an effective irrigation design. The closer we define the triggering and the target thresholds, the closer we approximate a singular arc, thus the closer we are of obtaining an additional correspondence between theoretical values of singular controls (u_{sing}) and "singular" irrigation amounts in practice. Such implementation perspectives consolidate the capacity to transfer properties and informations through the double modeling loops in Figure 1.1.

Calibration issues As mentioned in the previous section, and as expected since we are comparing two models whose formulations differ in terms of dimensionality, some phenomena described in Optirrig are challenging to identify from the approximated formalism of the CCI model. As a result, when the correspondence is not a direct transfer of terms, it may seem reasonable to adjust the value(s) of the CCI parameter(s) in order to choose one that allows to match the main variables (soil

moisture, biomass production) produced by Optirrig and the CCI model. However, doing so means accepting that the physical sense endowed in the CCI formulation becomes more difficult to interpret accurately. Furthermore, both term-to-term correspondence and parameter fitting will likely not be complete, in the sense that the validity of such connections will depend on the operational context, and there may exist situations that the CCI model would not be able to represent (for example the case of early senescence induces a temporary decrease of LAI, which makes it difficult to reach with the CCI formulation). That being said, Optirrig is an irrigation model, which also has its own domain of validity, and its purpose is to generate and optimize irrigation scenarios. That means considering situations where the available irrigation water is sufficient to foresee potential significant benefits from an optimization. This suggests that its domain of validity should contain (or focus) a certain range of classical situations, which are likely to be in the range of application of the CCI model too.

Further research directions Another perspective would be to use this previous discussion about the term-to-term correspondence, in order to point out which components of the CCI model show a potential lack of description. It would thus be natural to wonder how the CCI formulations may be improved or supplemented. For example, the fact that every process in the CCI formalism assumes that they are carried out over a fixed soil volume is one of the main limitations mentioned above. Some issues of correspondence might then result from such an approximation, given that considering a fixed volume alienates the description of evaporation and transpiration especially at the beginning of the cropping season. In order to bypass this limitation, we might suggest different modeling options, such as writing the reservoir volume as the soil porosity ρ multiplied by the root depth $z(t)$. This way, we would explicitely provide the time dependence of the reservoir size, and the root depth function z might be considered as a parameterized time-varying function (the same way as φ). Consequently, it would be easier to adjust the correspondence between Optirrig and the CCI model, but it would certainly come along with additional difficulties for the theoretical analysis of the optimal control problem based on the CCI model. However, it is possible that with reasonable hypotheses over such a function z, we might still be able to work out some properties of optimal solutions. This is at least an interesting perspective for future works.

Chapter 6

Formulation and study of optimal averaged cost control problems in view of operational irrigation management

This chapter corresponds to works in progress, in collaboration with Nathalie Khalil, Maria do Rosario de Pinho (researchers at Faculdade de Engenharia da Universidade do Porto, Portugal) and Sofia Lopes (assistant professor in the Mathematics Department at the School of Science of Minho University, Portugal), during a three month stay at University of Porto, from January to April 2025. The aim of this chapter is to study an approach based on the so-called "averaged cost control" which is a way to formulate optimal control problems under some uncertainties, described as a set of all possible scenarios with their known probability of occurrence. This approach consists in looking for a control which is common to all scenarios, and which is optimal in the sense that it optimizes the averaged cost of all these scenarios. The goal of introducing this tool is to take into consideration uncertainties, typically climatic ones, closer to real-world conditions. In this chapter, we present the frameworks in which such situations may be formalized and are intended to be more deeply investigated in the near future.

6.1 Robustness of optimal strategies with respect to the crop growth dynamics variability

This section is devoted the application of the averaged cost control framework to the irrigation problem, in order to look for irrigation strategies that optimize the expected value of biomass production, with respect to uncertainties over the function φ (which has been proven to be the main driver of the shape of optimal solutions). We consider that the function φ belongs to a finite set of possible profiles φ_i, associated to a probability μ_i. Then we formalize our irrigation problem by duplicating the known dynamics as many times as the number of φ_i. We obtain a high dimensional optimal control problem for which we write optimality necessary conditions as available in the literature for this kind of problems. Deriving these conditions will provide properties for characterization of optimal solutions.

6.1.1 Problem formulation

We consider the CCI model, in its version in greenhouse context, and we suppose that there exists N possible profiles of functions φ, that we denote $(\varphi_i)_{1\leq i\leq N}$, associated to their probability of occurrence $(\mu_i)_{1\leq i\leq N}$. Writing the augmented dynamical system consists in copying N times the originate dynamical system, and indexing them according to their corresponding function φ_i. Then,

we write for $t \in [0,T]$ the following system:

$$\begin{cases} \dot{S}_i(t) = k_1 \left(-\varphi_i(t) K_S(S_i(t)) - (1-\varphi_i(t)) K_R(S_i(t)) + k_2 u(t)\right), \\ \dot{B}_i(t) = \varphi_i(t) K_S(S_i(t)), \qquad \text{(for } 1 \leq i \leq N\text{)} \\ \dot{V}_i(t) = u(t). \end{cases} \tag{6.1}$$

We keep all the assumptions of the greenhouse context valid, and we precise that all the functions φ_i play the same role as expected.

Assumption 6.1.1. *Assume that $\varphi_i : [0,T] \to [0,1]$ is increasing and of class C^1 for every $1 \leq i \leq N$.*

We also indicate that the initial condition is taken as uniform with respect to the scenarios.

Assumption 6.1.2. *The system has the following initial conditions:*

$$S_i(0) = S_0 \in (S^*, 1], \quad B_i(0) = 0, \quad V_i(0) = 0, \qquad \text{for } 1 \leq i \leq N.$$

Remark 6.1.1. *Following the definition of the dynamics and the initial conditions, notice that all the V_i are in fact equal. Indeed, this problem considers a unique control for piloting all the dynamics, then their variable V_i representing the water consumption (which is thus common) has no dependence on the scenario i, but we keep the notation V_i instead of V for convenience.*

In order to define the terminal target Q of irrigation water quota in a suitable range of initial water quotas, we would like to set a framework of deficit irrigation as before, but having several trajectories of soil moisture controlled simultaneously brings supplementary difficulty. To this aim, we recall that avoiding water stress (for any scenario) means keeping the soil moisture S_i above the level S^* (for all i). Then, for all $t \in [0,T]$, we define

$$E_{S^*}(t) = \{i \in [\![1,N]\!],\ S_i(t) \geq S^*\}, \quad I_{S^*}(t) = \{i \in [\![1,N]\!],\ S_i(t) = S^*\}$$

In the case of no water stress, we would obtain $E_{S^*}(t) = [\![1,N]\!]$ for all $t \in [0,T]$. Following the initial condition of soil moisture, this equality holds at time 0, and notice that we have $I_{s^*}(0) = \emptyset$. Let us define the following function:

$$\forall t \in [0,T],\ \varphi^+_{S^*}(t) = \max_{i \in I_{S^*}(t)} \varphi_i(t).$$

Then define the following control:

$$u^*(t) = \begin{cases} 0, & \text{if } I_{S^*}(t) = \emptyset, \\ \frac{\varphi^+_{S^*}(t) + (1-\varphi^+_{S^*}(t)) K_R(S^*)}{k_2}, & \text{if } I_{S^*}(t) \neq \emptyset. \end{cases} \tag{6.2}$$

This control enables the whole set of soil moisture trajectories to stay above the level S^*, and it is the least consuming control that possesses such a feature. As a consequence, we can introduce the corresponding limit value for the irrigation water quota under which stress will occur.

Definition 6.1.1. Using the control u^* defined above, we obtain

$$\overline{Q} = \int_0^T u^*(t) dt. \tag{6.3}$$

In addition to the context of deficit irrigation, we consider as before the state constraint corresponding to the greenhouse context, as follows:

$$\forall t \in [0,T],\ S_{ope} \leq S_i(t) \leq 1, \qquad \text{for } 1 \leq i \leq N.$$

Similarly as above, this state constraint is fulfilled if the simultaneously controlled soil moisture trajectories are all above S_{ope} throughout the cropping season. Then, for all $t \in [0,T]$, we define

$$E_{S_{ope}}(t) = \{i \in [\![1,N]\!],\ S_i(t) \geq S_{ope}\}, \quad I_{S_{ope}}(t) = \{i \in [\![1,N]\!],\ S_i(t) = S_{ope}\}.$$

Following the initial condition of soil moisture, this equality holds at time 0, and notice that we have $I_{S_{ope}}(0) = \emptyset$. Let us define the following control:

$$u_{ope}(t) = \begin{cases} 0, & \text{if } I_{S_{ope}}(t) = \emptyset, \\ \max\limits_{i \in I_{S_{ope}}(t)} \frac{\varphi_i(t)K_S(S_{ope}) + (1-\varphi_i(t))K_R(S_{ope})}{k_2}, & \text{if } I_{S_{ope}}(t) \neq \emptyset. \end{cases} \tag{6.4}$$

This control enables the whole set of soil moisture trajectories to stay above the level S_{ope}, and it is the least consuming control that possesses such a feature. As a consequence, we can introduce the corresponding limit value for the irrigation water quota under which the state constraint $S_{ope} \leq S_i(t)$ cannot be fulfilled for all i and for all $t \in [0, T]$.

Definition 6.1.2. Using the control u_{ope} defined above, we obtain

$$\underline{Q} = \int_0^T u_{ope}(t)dt. \tag{6.5}$$

Finally, we set our study in such a way that we will consider controls which consume water quotas between these two values defined above, i.e. the terminal target of water consumption will be $V_i(T) \leq Q$ (which are the same for each V_i) where $\underline{Q} < Q < \overline{Q}$ is given.

Remark 6.1.2. *Let us notice that the use of either u^* or u_{ope} does not allow any trajectory S_i to reach the soil moisture level $S_i = 1$, since the control to keep it constant at $S_i = 1$ would be $u = 1/k_2$, which is strictly greater than any possible value of u^* or u_{ope}. As a consequence, it would only be possible to reach the upper state constraint $S_i \leq 1$ when using a larger irrigation amount. However, the purpose of this section is to represent a function φ that is not known, but still around an expected profile. Then it is reasonable to suppose that the resulting trajectories S_i are not too far from each other in practice, and that the upper state constraint is likely to never be reached in a situation of not wasting irrigation water. Such assumption would require further investigation for clearly stating the conditions of feasibility, which is not cover in this work for the sake of simplicity.*

Let us now introduce the sets of admissible controls.

Definition 6.1.3. For any control $u(\cdot)$, consider its associated solution to Equation (6.1) denoted by $(S_i(\cdot), B_i(\cdot), V_i(\cdot))_{1 \leq i \leq N}$, and define

$$\mathcal{U}^{AVG} = \Big\{ u \in L^1\big([0,T],[0,1]\big);\ \forall i \in [\![1, N]\!] :\ S_{ope} \leq S_i(t),\ V_i(T) \leq Q,\ \underline{Q} < Q < \overline{Q} \Big\}.$$

Consequently, let us now formulate the optimization problem associated to our study.

Definition 6.1.4. We define the following problem with its cost to be optimized (which represents the expectation of the biomass production):

$$(P_{AVG}) : \sup_{u(\cdot) \in \mathcal{U}^{AVG}} \sum_{i=1}^{N} \mu_i B_i(T),$$

where $\mu_i > 0$ for all $i \in [\![1, N]\!]$, and $\sum_{i=1}^{N} \mu_i = 1$.

Remark 6.1.3. *Assuming that the admissibility set $\mathcal{U}_{AVG}$ is non-empty, then our problem formulation satisfies the usual assumptions which guarantee that the above supremum is reached.*

6.1.2 Properties of the optimal solutions

First notice that the budget compensation technique presented in Chapter 3 does not work anymore in this framework, since it was strongly relying on the fact that the irrigation could control a single trajectory, which is not the case in the current work. More precisely, having one trajectory S_i below S_{ope} over a certain time interval would make us want to "compensate" this state constraint violation by adding this contribution in the water budget consumption. Notice that this additional term depends on S_i, thus the variables V_i and V_j would become different. In this case we may have another trajectory S_j that does not violate the state constraint during the same interval as S_i, so that when defining a new control that uses slightly more water in order to reduce the state violation

of S_i, it won't affect the variables V_i and V_j the same way, which might prevent the terminal target $V_j(T) \leq Q$ to hold true. As a consequence, in this work we keep the state constraint and the initial formulation for the analysis.

Let us now provide some properties that are similar to the case of Chapter 3 and still hold in the current case.

Proposition 6.1.4. *Let $u_{AVG}(\cdot)$ be an optimal control of the problem (P_{AVG}). Denote its corresponding solution with $(S_i(\cdot), B_i(\cdot), V_i(\cdot))_{1\leq i\leq N}$. Then one has*

$$\begin{aligned} &(\textsc{i}): \quad u_{AVG}(t) = 0, \quad \text{a.e. } t \in [0, t^*_{min}], \\ &(\textsc{ii}): \quad \forall t \in [t^*_{min}, T],\ \exists i \in [\![1, N]\!],\ S_i(t) \leq S^*, \end{aligned}$$

where t^*_{min} is the smallest time t such that the set $I_{S^*}(t)$ is non-empty.

Remark 6.1.5. *The property (II) consolidates the assumption mentioned in Remark 6.1.2 in the case when the set of φ_i represents a family of functions φ_i which are close to each other, thus generating soil moisture trajectories which remain relatively close to each other as well. Consequently, it will be unlikely to have one trajectory reaching $S_i = 1$ while another is still below S^*.*

Proposition 6.1.6. *Let $u_{AVG}(\cdot)$ be an optimal control of the problem (P_{AVG}). Denote its corresponding solution with $(S_i(\cdot), B_i(\cdot), V_i(\cdot))_{1\leq i\leq N}$. Then one has*

$$(\textsc{iii}): \quad V_i(T) = Q \quad , \forall i \in [\![1, N]\!].$$

Remark 6.1.7. *These two previous properties may be proven by following the corresponding properties given in Chapter 3, by simply adapting the arguments for a finite set of trajectories sharing the same type of dynamics.*

6.1.3 Towards an optimal synthesis

6.1.3.1 Optimality necessary conditions of the averaged cost problem

Applying the given result in [13] to our irrigation problem, we obtain the following optimality necessary conditions.

There exists $\lambda \geq 0$, functions $p_i(\cdot) \in W^{1,1}([0,T], \mathbb{R}^3$, that we denote $p_i = (p_i^S, p_i^B, p_i^V)^\top$, measurable functions $q_i(\cdot) : [0,T] \to \mathbb{R}^3$, that we denote $q_i = (q_i^S, q_i^B, q_i^V)^\top$, and a positive measures ν_i, such that

$$\big(\lambda, \{p_i(\cdot), \nu_i\}_{1\leq i\leq N}\big) \neq (0,0,0). \tag{6.6}$$

Moreover these functions verify for a.e. $t \in [0,T]$ the following conditions:

$$\begin{cases} \dot{p}_i^S(t) \in \varphi_i(t)\big(q_i^S(t)k_1 - q_i^B(t)\big)\partial_C K_S(S_i(t)) + q_i^S(t)(1-\varphi_i(t))k_1\partial_C K_R(S_i(t)), \\ \dot{p}_i^B(t) = 0, \\ \dot{p}_i^V(t) = 0, \end{cases}$$

with the following terminal conditions:

$$\begin{cases} q_i^S(T) = 0, \\ q_i^B(T) = \lambda, \\ q_i^V(T) \in \{-1, 0\}. \end{cases}$$

Furthermore, we write the averaged Hamiltonian as

$$\begin{aligned} H_{AVG}\big(t, \{S_i, B_i, V_i\}, \{q_i\}, u\big) = \sum_{i=1}^{N} \mu_i \Big(& q_i^S k_1 \big(k_2 u - (\varphi_i(t)K_S(S_i) + (1-\varphi_i(t))K_R(S_i))\big) \\ & + q_i^B \varphi_i(t) K_S(S_i) + q_i^V u \Big), \end{aligned}$$

which enables to write the maximization condition for an optimal control u_{AVG} as

$$H_{AVG}\big(t,\{S_i,B_i,V_i\},\{q_i\},u_{AVG}\big)=\max_{u\in[0,1]}H_{AVG}\big(t,\{S_i,B_i,V_i\},\{q_i\},u\big),$$

where for all $i\in[\![1,N]\!]$, one has

$$q_i^S(t)=\begin{cases}p_i^S(0), & t=0,\\ p_i^S(t)+\int_{[0,t]}\gamma_i(s)\nu_i(ds), & t\in(0,T],\end{cases}$$

with $\gamma_i(t)\in\partial_S^{\geq}h(t,S_i(t))$, ν_i-a.e. $t\in[0,T]$, where the function h is defined from the state constraint $h(t,S_i(t))=S_{ope}-S_i(t)$, and $\partial_S^{\geq}$ denotes the partial hybrid limiting subdifferential (see [35]). Moreover, the state constraint only involves the variables S_i, then we have

$$\forall 1\leq i\leq N,\ q_i^B=p_i^B \text{ and } q_i^V=p_i^V.$$

In addition, we write the corresponding switching function resulting from the maximization condition on the averaged Hamiltonian, as follows:

$$\Psi_{AVG}(t)=\sum_{i=1}^{N}\mu_i\big(q_i^S(t)k_1k_2+q_i^V\big).$$

This way, all the ingredients are set and ready to start deriving the necessary conditions in order to look for properties about the adjoint components, then about the switching function, to finally deduce some properties about the optimal control u_{AVG}.

First let us list here the properties. We only prove the first property since the components q_i^S are directly impacted by the additional term from the state constraint, while other components q_i^B and q_i^V are constant as in Chapter 3. These results can be obtained by adapting the proofs from the "single dynamics" case investigated in Chapter 3.

Proposition 6.1.8. *For any optimal solution to the problem (P_{AVG}), one necessarily has $q_i^S(t)\geq 0$ for all $t\in[0,T]$ and for all $1\leq i\leq N$.*

Proof. Suppose that there exists $i\in[\![1,N]\!]$, and $\hat{t}\in[0,T]$ such that $q_i^S(\hat{t})<0$.

Noticing that, by definition of the state constraint, the function γ_i is non-positive, we deduce that the possible jumps in q_i^S are necessarily negative jumps. Consequently there exists $\varepsilon>0$ such that for a.e. $t\in[\hat{t},\hat{t}+\varepsilon)$, one has

$$\dot{q}_i^S(t)\in\varphi_i(t)\big(q_i^S(t)k_1-q_i^B(t)\big)\partial_C K_S(S_i(t))+q_i^S(t)(1-\varphi_i(t))k_1\partial_C K_R(S_i(t))+\gamma_i(t)\nu_i(dt).$$

Since $0\leq\varphi\leq 1$, $k_1>0$, $q_i^B\geq 0$, $\partial_C K_S\subset\mathbb{R}^+$, $\partial_C K_R\subset\mathbb{R}^+$ and $\gamma_i(t)\nu_i(dt)\leq 0$, then one has

$$\forall t\in[\hat{t},\hat{t}+\varepsilon),\ \dot{q}_i^S(t)\in\mathbb{R}^-.$$

Let $\mathcal{A}=\{t\in[0,T],\ q_i^S(t)<0\}$. The function $q_i^S(\cdot)$ is then non increasing over $\mathcal{A}$, which leads to a contradiction regarding the necessary terminal condition $q_i^S(T)=0$. Hence one has

$$\forall i\in[\![1,N]\!],\ \forall t\in[0,T],\ q_i^S(t)\geq 0.$$

□

Proposition 6.1.9. *For any optimal solution to the problem (P_{AVG}), one necessarily has $q_i^V<0$ for all $1\leq i\leq N$.*

Proposition 6.1.10. *For any optimal solution to the problem (P_{AVG}), one necessarily has $q_i^B>0$ for all $1\leq i\leq N$.*

Remark 6.1.11. *We deduce from these properties that for an optimal solution to the problem (P_{AVG}), there exists $\beta>0$ such that $\Psi_{AVG}(t)<0$ for all $t\in(T-\beta,T]$, then the optimal control u_{AVG} is necessarily equal to 0 on this terminal interval.*

6.1.3.2 Singular arcs and analysis of the switching function

Notice that our problem formulation involves a dynamical system in finite dimension, where the dynamics satisfies as before the hypotheses of being piecewise affine and continuous with respect to the state. This allows us to once again invoke the exented bang-bang principle for optimal control problems with such dynamics. We obtain the same following result about the existence of singular arcs, adapted to the multiple trajectories situation.

Proposition 6.1.12. *For any optimal solution to the problem (P_{AVG}), a singular arc can only occur when there exists i such that one trajectory S_i that satisfies $S_i = S_{ope}$ or $S_i = S^*$.*

The only major difference in this section compared to Chapter 3 relies in the condition of maximization of the Hamiltonian. Indeed, in the current section, we are dealing with an averaged Hamiltonian, thus the switching function is also averaged. As a result, the previous argument consisting in showing that the set where the switching function is non-negative is connected, relying on a proof by contradiction, does not hold in our present case. In fact, replicating the argument by contradiction in Proposition 3.4.8 would imply the existence of a local minimum of the new averaged switching function Ψ_{AVG}, which is no longer directly related to an adjoint function q_i^S, thus we can no longer claim that there also exists a local minimum for one of the adjoint functions q_i^S. Though the argument is no longer valid in this case, we still believe that a similar (yet more complicated) argument could provide a similar result. For instance, the demonstration process in Proposition 3.4.8 seems to only exploit a local minimality, which is far from all the underlying properties that may arise in our problem, especially some local convexity (or concavity) properties.

Such a successful argument would allow us to deduce the following result yet to be proven.

Conjecture 6.1.1. For any optimal solution to the problem (P_{AVG}), the set I_{AVG} defined by

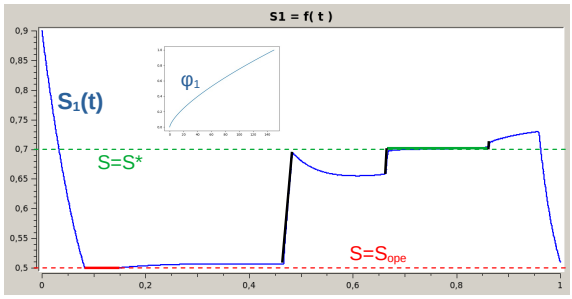
$$I_{AVG} = \{t \in [t^*_{min}, T],\ \Psi_{AVG}(t) \geq 0\}$$

is non-empty and connected.

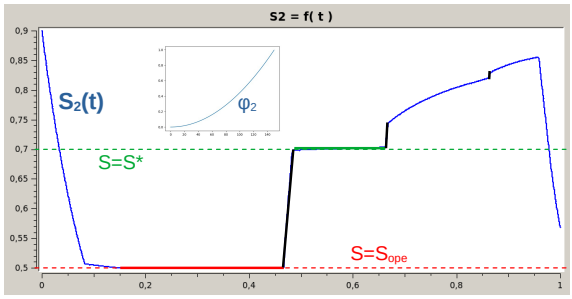
6.1.4 Conclusion

The above Conjecture 6.1.1, if proven, would enable a conclusion about the structure of optimal solutions to the problem (P_{AVG}), which would in fact be the same as the "single-dynamics" situation. Such a result would have significant consequences for the operational point of view, since it would mean that the Threshold-Based Irrigation Strategies based on two thresholds (S_{ope} and S^*) that we studied in detail in Chapter 4 are robust with respect to the variability of their most sensible parameter (the function φ).

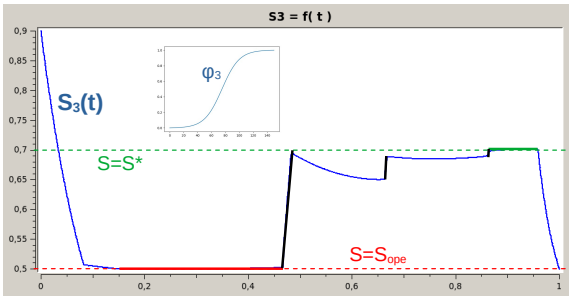
Furthermore, some numerical simulations were conducted to test this conjecture, and we did not find yet any counter-example, i.e. any example of parameterization of a set of functions φ_i associated to some occurrence probabilities μ_i for which our simulations would provide a different structure of optimal solution. As a consequence, this strengthens our belief that the 2-TBIS are indeed optimal for the averaged cost control problem formulation of this section. However, since there is now multiple soil moisture trajectories evolving simultaneously, and since we mentioned that a singular arc can occur if one of these trajectories is located at S_{ope} or S^*, then it is not easy to anticipate about the exact structure of the optimal solution, especially the number of singular arcs and the number of switches from singular arcs to bang ($u = 1$) arcs or conversely (as shown in Figure 6.1).



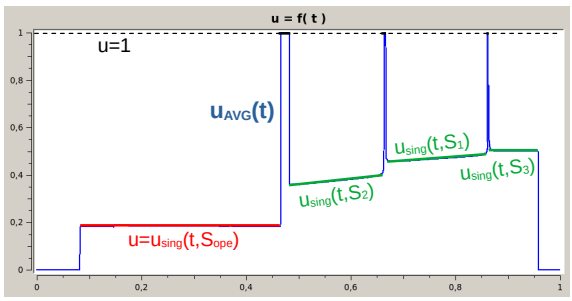
(a) Soil moisture trajectory associated to φ_1 and u_{AVG}



(b) Soil moisture trajectory associated to φ_2 and u_{AVG}



(c) Soil moisture trajectory associated to φ_3 and u_{AVG}



(d) Optimal control u_{AVG} that maximizes the averaged biomass.

Figure 6.1: Example of numerical resolution where we use three possible profiles for the function φ (a concave one, a convex one, and a pseudo-logistic one), with probabilities of occurrence $\mu_1 = 0.3, \mu_2 = 0.4, \mu_3 = 0.3$. We can observe that the optimal irrigation occurs on a single interval, with several switches between bang ($u = 1$) arcs and singular arcs (corresponding to the successive arcs $S_i = S^*$).

6.2 Integrating rainfall uncertainties and soil moisture jumps

This section focuses on the irrigation problem in open-field conditions, representing rainfall as instantaneous events, i.e. Dirac leading to dynamics with jumps. We consider that an annual sequence of rainfall belongs to a finite set of possible sequences, associated to some probabilities. As a consequence, we once again formulate the irrigation problem using the averaged cost control framework, aiming for optimizing the expected value of biomass production, with respect to the rainfall uncertainties. However, such systems with discontinuous state do not belong to the standard application framework. As a result, we seek to employ a well-known technique of time variable change, based on a graph completion, but we need to formalize it in such a way that the new time variable is the same for every considered scenario. This way, we will be able to apply the dedicated optimality necessary conditions in order to search and characterize optimal solutions.

6.2.1 Framework of rainfed irrigation

We consider the CCI model, in its version in open-field context, and we represent the rainfall events by jumps in the soil moisture trajectory. More precisely, let us consider one annual rainfall sequence (denoted $R^{(i)}$, associated to its probability of occurrence μ_i) defined by its timings ($t_k \in [0, T]$) and depths ($h_k > 0$) of rainfall events, as follows:

$$R^{(i)} = \left\{ (t_1^{(i)}, ..., t_{n_i}^{(i)}),\ (h_1^{(i)}, ..., h_{n_i}^{(i)}) \right\}, \tag{6.7}$$

where n_i is the number of rainfall events for the sequence $R^{(i)}$. Then, assuming that there exists N possible profiles of annual rainfall sequences, we write the following system representing our crop

irrigation problem for $t \in [0,T]$ as

$$\begin{cases} \dot{S}^{(i)}(t) = k_1\big(-\varphi(t)K_S(S^{(i)}(t)) - (1-\varphi(t))K_R(S^{(i)}(t)) + k_2 u(t) - k_{sat}K_D(S^{(i)}(t))\big), \\ \dot{B}^{(i)}(t) = \varphi(t)K_S(S^{(i)}(t)), \qquad \text{(for } 1 \le i \le N\text{)} \\ \dot{V}^{(i)}(t) = u(t), \\ \forall k \in [\![1, n_i]\!],\ S^{(i)}(t_k^{(i)+}) = S^{(i)}(t_k^{(i)-}) + h_k^{(i)}. \end{cases} \tag{6.8}$$

Remark 6.2.1. *Notice that the notion of solution to the dynamical system must be revisited, since it will involve discontinuous trajectories. We refer for example to [160, 161] for the notion of "robust solution" that emerge from measure-driven differential equations, which is a more general case than our present problem using only state jumps, that may be seen here as Dirac terms in the left side of the dynamics.*

We keep all the usual assumptions on the parameter functions φ, K_S, K_R, K_D, the constant parameters k_1, k_2, k_{sat}, and the initial conditions.

Assumption 6.2.1. *The system has the following initial conditions:*

$$S^{(i)}(0) = S_0 \in (S^*, 1]\,, \quad B^{(i)}(0) = 0\,, \quad V^{(i)}(0) = 0, \qquad \text{for } 1 \le i \le N.$$

Remark 6.2.2. *Following the definition of the dynamics and the initial conditions, notice as before that all the $V^{(i)}$ are in fact equal. Indeed, this problem considers a unique control for piloting different dynamics, then their variable $V^{(i)}$ representing the water consumption (which is thus common) has no dependence on the scenario i.*

As the rainfall sequences are unknown, the terminal target Q of irrigation water quota should include any possible occurrence of rainfall scenario. In particular, the scenario with zero annual rainfall has to be considered, and the water quota needed in this case to prevent any stress would exactly be the quantity V^* defined in Chapter 3, that we recall here:

$$V^* = \int_0^T u^*(t)dt,$$

where

$$u^*(t) = \begin{cases} 0, & t \in [0, t^*), \\ u_{sing}(t, S^*), & t \in [t^*, T], \end{cases}$$

with $t^* = \inf\{t \in [0,T],\ S^{(0)}(t) = S^*\}$, where we denoted by $S^{(0)}$ the trajectory of soil moisture following the irrigation $u = 0$ and subject to zero rainfall events.

In addition, we recall that we consider the lower state constraint $S_{ope} \le S^{(i)}(t)$ for all $t \in [0,T]$ and for any $1 \le i \le N$. As a result, we similarly define the least amount of water quota that allows every possible soil moisture trajectory to remain above the threshold S_{ope} by

$$V_{ope} = \int_0^T u_{ope}(t)dt,$$

where

$$u_{ope}(t) = \begin{cases} 0, & t \in [0, t_{ope}), \\ u_{sing}(t, S_{ope}), & t \in [t_{ope}, T], \end{cases}$$

with $t_{ope} = \inf\{t \in [0,T],\ S^{(0)}(t) = S_{ope}\}$, where we denoted by $S^{(0)}$ the trajectory of soil moisture following the irrigation $u = 0$ and subject to zero rainfall events.

Finally, we set our study in such a way that we will consider controls which consume water quotas between these two values defined previously, i.e. the terminal target of water consumption will be $V^{(i)}(T) \le Q$ (whatever is i) where $V_{ope} < Q < V^*$ is given.

Remark 6.2.3. *Let us notice that in the situation where the considered annual rainfall sequences would all contribute very significantly to soil water contents, then it might be possible that we wouldn't need to consume the entire water quota to ensure no stress at all during the cropping season. However, since we aim for an operating context of arid or semi-arid areas, it is reasonable to consider that this situation will likely not happen, and that we shall rather have annual rainfall sequences that contain relatively scarce rainfall events, where irrigation water is a key element for optimizing production.*

Let us now define the sets of admissible controls.

Definition 6.2.1. For any control $u(\cdot)$, consider its associated solution to Equation (6.8) denoted by $(S^{(i)}(\cdot), B^{(i)}(\cdot), V^{(i)}(\cdot))_{1\leq i\leq N}$, and define

$$\mathcal{U}_{rain}^{AVG} = \Big\{ u \in L^1\big([0,T],[0,1]\big);\ \forall i \in [\![1,N]\!] :\ S_{ope} \leq S^{(i)}(t),\ V^{(i)}(T) \leq Q,\ V_{ope} < Q < V^* \Big\}.$$

Then, we can now formulate the optimization problem associated to our study.

Definition 6.2.2. Let

$$(P_{rain}^{AVG}) : \sup_{u(\cdot)\in\mathcal{U}_{rain}^{AVG}} \sum_{i=1}^{N} \mu_i B^{(i)}(T),$$

where $\mu_i > 0$ for all $i \in [\![1,N]\!]$, and $\sum_{i=1}^{N} \mu_i = 1$.

6.2.2 Problem reformulation with an abstract time

Following the time reparameterization technique explained in [139, 129] and generalized in [124, 160], we aim for applying such technique for transforming our non-standard dynamical system into a standard one. However, if we considered each equation on $S^{(i)}$ presenting state jumps individually, we would obtain, through the so-called "graph completion" technique, new abstract time variable that depends on the scenario i. As a consequence, this would not be suitable for our problem since we need to control simultaneously (thus using the same abstract time variable) all the different scenarios i.

Therefore we would like to build a unique time reparameterization for all the scenarios, thus based on a unique graph completion technique. To this aim, and for the sake of simplicity, we formulate the following additional hypothesis.

Assumption 6.2.2. *Denoting $E^{(i)} = \{t_1^{(i)}, ..., t_{n_i}^{(i)}\}$ the set of dates of rainfall events for each scenario i, we suppose that the timings of rainfall events in different scenarios do not coincide, i.e.*

$$\forall i \neq j,\ E^{(i)} \cap E^{(j)} = \emptyset.$$

Remark 6.2.4. *Notice that if two dates of rainfall events in two different scenarios coincide, then it does not prevent the following development, but it would induce slightly more complicated indices notations.*

This way, let us define the set of all dates of rainfall events, without redundancy.

Definition 6.2.3. Denote E_{rain} the set of all dates of rainfall events of each scenario i, defined by

$$E_{rain} = \bigcup_{i=1}^{N} E^{(i)}.$$

Any time belonging to this set will be denoted by $t_k^{(j)}$ where $j \in [\![1,N]\!]$ and $k \in [\![1,n_j]\!]$.

The idea now is to define a uniform time reparameterization, following the graph completion technique, which means that we consider "jumps" for all the state variables $S^{(i)}$ at all the times belonging to E_{rain}. As a consequence, some "jumps" will be artificial, in the sense that they do not correspond to their scenario, but we simply fix it of size 0. Let us redefine then the amplitude of the rainfall events in a uniform way.

Definition 6.2.4. For any scenario i, and any time $t_k^{(j)} \in E_{rain}$, we posit

$$h_k^{(i,j)} = \begin{cases} 0, & \text{if } i \neq j, \\ h_k^{(i)}, & \text{if } i = j. \end{cases}$$

Consequently, we can write in a uniform way the following:

$$\forall i \in [\![1, N]\!],\ \forall t_k^{(j)} \in E_{rain},\ S^{(i)}(t_k^{(j)+}) = S^{(i)}(t_k^{(j)-}) + h_k^{(i,j)}.$$

Now we can employ the uniform graph completion technique for the common time reparameterization. It consists in rescaling the time variable $t \in [0, T]$ into a time variable $\tau \in [0, 1]$ by "inserting" artificial time interval at each occurrence of a rainfall event. Outside these artificial time intervals, the system should evolve accordingly to the usual time and its normal dynamics, and inside these artificial time intervals, we "freeze" the normal dynamics, enabling the "jump" to happen in a continuous way, with respect to the new abstract time.

Let us first define some auxiliary functions, starting with the function F representing the distribution of the rainfall events, as follows:

$$F(t) = \begin{cases} 0, & \text{if } t = 0, \\ \displaystyle\sum_{\substack{j,k \\ t_k^{(j)} \in E_{rain} \\ t_k^{(j)} \leq t}} h_k^{(j)}, & \text{if } t \in (0, T]. \end{cases} \tag{6.9}$$

Then we define the corresponding reparameterization function η as

$$\eta(t) = \begin{cases} 0, & \text{if } t = 0, \\ \dfrac{t + F(t)}{T + \displaystyle\sum_{\substack{j,k \\ t_k^{(j)} \in E_{rain}}} h_k^{(j)}}, & \text{if } t \in (0, T]. \end{cases} \tag{6.10}$$

This function η is increasing, from $[0, T]$ into $[0, 1]$, and then we can define the following non-decreasing function as

$$\theta(\tau) = \sup\{t \in [0, T],\ \eta(t) \leq \tau\}, \quad \forall \tau \in [0, 1]. \tag{6.11}$$

This function θ is defined from $[0, 1]$ into $[0, T]$, presents constant parts, at values corresponding to the dates of jumps $t_k(j)$, and increasing parts elsewhere (with a uniform slope, inverse of the slope of η). An illustration of the functions η and θ can be found in Figure 6.2.

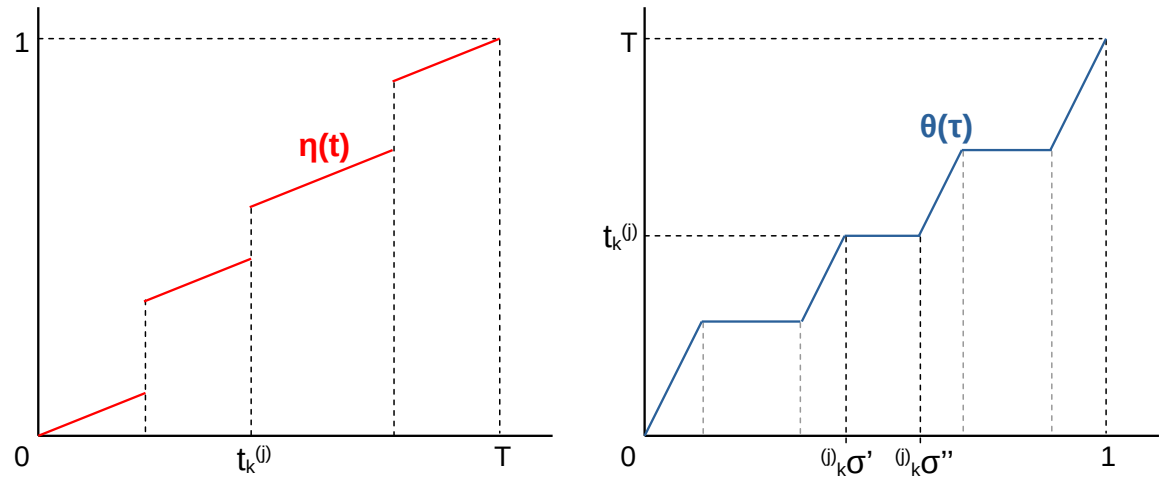
For any $t_k^{(j)} \in E_{rain}$, denote $\left[{}_k^{(j)}\sigma',\ {}_k^{(j)}\sigma'' \right] = \theta^{-1}(\{t_k^{(j)}\})$, and define the function ξ, which will play the complementary role of $\frac{d\theta}{d\tau}$, as follows:

$$\xi(\tau) = \begin{cases} 0, & \text{if } \tau \notin \displaystyle\bigcup_{j,k} \left[{}_k^{(j)}\sigma',\ {}_k^{(j)}\sigma'' \right], \\ \frac{1}{{}_k^{(j)}\sigma'' - {}_k^{(j)}\sigma'}, & \text{if } \tau \in \left[{}_k^{(j)}\sigma',\ {}_k^{(j)}\sigma'' \right]. \end{cases} \tag{6.12}$$

Indeed, denoting $w = \frac{d\theta}{d\tau}$, we have $w = 0$ when $\xi \neq 0$ and $w \neq 0$ when $\xi = 0$. They are the key to make us "freeze" the dynamics as we mentioned earlier, and their slope is well adjusted to ensure that the resulting jumps will have the correct size.

Let us finally write the transformed dynamics, denoting the new state variables by $(\widetilde{S}, \widetilde{B}, \widetilde{V})$, using the new time variable τ and adding the term of jumps. We have, for $1 \leq i \leq N$, the following:

$$\begin{cases} \frac{d\widetilde{S}^{(i)}}{d\tau}(\tau) = k_1\Big(-\varphi(\theta(\tau))K_S(\widetilde{S}^{(i)}(\tau)) - (1 - \varphi(\theta(\tau)))K_R(\widetilde{S}^{(i)}(\tau)) + k_2 v(\tau) \\ \qquad\qquad -k_{sat}K_D(\widetilde{S}^{(i)}(\tau))\Big)w(\tau) + \xi(\tau)H^{(i)}(\tau), \\ \frac{d\widetilde{B}^{(i)}}{d\tau}(\tau) = \varphi(\theta(\tau))K_S(\widetilde{S}^{(i)}(\tau))w(\tau), \\ \frac{d\widetilde{V}^{(i)}}{d\tau}(\tau) = v(\tau)w(\tau), \end{cases} \tag{6.13}$$



(a) Graphical representation of the function $\eta(t)$.

(b) Graphical representation of the function $\theta(\tau)$.

Figure 6.2: Visual representation of the functions $\eta(t)$ and $\theta(\tau)$, that allow the reparameterization of the time. We observe that the new time τ progresses as normal rate when the initial time t does not encounter any state jump, and the new time τ keeps progressing while the initial time t is "frozen" when it reaches a time $t_k^{(j)}$ of state jump.

where we wrote $u(\theta(\tau)) = v(\tau)$, and with

$$H^{(i)}(\tau) = \begin{cases} h_k^{(i,j)}, & \text{if } \tau \in \left[{}^{(j)}_k\sigma', {}^{(j)}_k\sigma'' \right], \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the optimization problem (P_{rain}^{AVG}) can be rewritten in this new framework as

$$(P_{rain}^{AVG})_{NEW} : \sup_{v(\cdot)\in\mathcal{V}_{NEW}} \sum_{i=1}^{N} \mu_i \widetilde{B}^{(i)}(1), \tag{6.14}$$

where the admissibility set is defined by

$$\mathcal{V}_{NEW} = \Big\{ v \in L^1\big([0,1],[0,1]\big);\ \forall i \in [\![1,N]\!] :\ S_{ope} \leq \widetilde{S}^{(i)}(\tau),\ \widetilde{V}^{(i)}(1) \leq Q,\ V_{ope} < Q < V^* \Big\}.$$

6.2.3 Perspectives

We may now use the results from the averaged cost control problem and apply them on our new problem formulation $(P_{rain}^{AVG})_{NEW}$, which is written as a "standard" system of differential equations. We then project to derive the optimality necessary conditions associated to this problem, and, even if the construction was not straightforward, it only brings additional terms or factors that are time-dependent, then we expect that the derivation of the necessary conditions that we already investigated in a more simple case in Chapter 3 might be adapted here. In particular, the state dependence remains the same, and the time-dependent terms will vanish as soon as we compute the derivative with respect to the state, in the construction of the adjoint system. Once the analysis is done and we obtain properties about optimal controls $v(\cdot)$ and trajectories $(\widetilde{S}, \widetilde{B}, \widetilde{V})$, we will be able to write them back in the originate time variable using the following relations:

$$S(t) = \widetilde{S}(\eta(t)), \quad B(t) = \widetilde{B}(\eta(t)), \quad V(t) = \widetilde{V}(\eta(t)), \quad u(t) = v(\eta(t)), \qquad \text{for a.e. } t \in [0,T].$$

Our perspectives will also include using direct methods to obtain numerical simulations for this problem reformulation and get insights on the structure of optimal solutions. Provided that such analysis process detailed above can result in properties able to characterize a structure of optimal solutions, then we may deduce a formulation of irrigation strategies that will be efficient in the context of open-field conditions, in the sense that they account for the possible various outcomes from uncertain weather, not only in the short-term, but at the cropping season time-scale. Such strategies would be a strong support for crop irrigation management, especially in the regions where sustainable production crucially relies on contribution from irrigation to supplement scarce rainfalls.

Conclusions and perspectives

In this thesis, we investigated the optimization of crop irrigation from the point of view of optimal control theory. The study was conducted at the plot scale, in greenhouse and in open-field conditions, and in a context of limited water resources. The main motivation was to integrate a double modeling approach, thus to develop optimal control tools derived from a low dimensional ODE model in order to help designing efficient irrigation strategies for an operational model. As a result, we based our theoretical work on the designed-on-purpose Controlled Crop Irrigation (CCI) model, improving its formulation and investigating its structural features, and we considered the Optirrig model as the destination model.

Summary of contributions

The first contribution of this thesis corresponds to the extensions of the Bang-Bang Principle detailed in Chapter 2. These studies were motivated by the simple (yet non-linear) dynamics of the CCI model: in its complete form, the dynamics are piecewise affine and continuous with respect to the state, and contain a non-deterministic term. We first revisited the classical linear bang-bang principle for minimum-time problems in a piecewise affine continuous form, under state constraints. We showed in Section 2.1 that in this framework there exists optimal trajectories that are generated by "bang" controls at the loci where the dynamics is differentiable, and possible "singular" controls when the state is located exactly at the non-differentiable loci of the dynamics. Then we revisited the classical bang-bang principle in a linear form with additional brownian noise. We showed in Section 2.2 that the general result stating the equality of the reachable sets (with any control or with only "bang" controls) does not hold in such settings with stochastic controls, but we showed (under some conditions) that the optimal cost of a Mayer problem with a compact control set coincides with the optimal cost of the same problem with controls taking values in the closed convex hull of the compact set. These two theoretical results provide structural properties about the nature of optimal solutions and are expected to be useful for concrete applications.

The resolution of the optimal control problem based on the CCI model formulation in the context of greenhouse crops detailed in Chapter 3 constitutes the second contribution of this thesis. We introduced a family of optimal control problems adapted to different operational objectives, and we showed, using dedicated tools (the home-made budget compensation technique allowing to consider an equivalent problem removing a state constraint as detailed in Section 3.3, and an application of the extended bang-bang principle for piecewise affine continuous dynamics taken from Section 2.1), and applying the non-differentiable Pontryagin Maximum Principle, that optimal irrigation strategies for this family of problems share the same structure. More precisely, optimal strategies belong to a class of simple parameterized time-varying feedback controls, which means that the initial infinite dimensional optimization problem is thus reduced to a small dimensional (one or two) optimization problem suitable for numerical purposes. Moreover, the obtained optimal trajectories may actually be interpreted in terms of real-world irrigation practices that can be called Threshold-Based Irrigation Strategies (TBIS).

The third main contribution of this thesis, presented in Chapter 4, corresponds to the in-depth analysis of such Threshold-Based Irrigation Strategies. The computationally affordable features of the class of optimal strategies introduced in Chapter 3 allowed us to look for the optimal values of these parameterized controls in a large class of model settings. Moreover, in Chapter 4, we studied the sensitivity of the shape of optimal strategies with respect to the variability of the model settings

(contextual parameters) through a global exploration and variance-based computational methods, and we showed that the optimal shape (in particular the optimal decision time) is mostly driven by plant characteristics (their growth dynamics and the effect of water stress on their biomass production). We also investigated the sub-optimality of irrigation chronicles obtained by making small mistakes on the determination of decision time and soil moisture thresholds, and we highlighted that mistakes on the top soil moisture threshold may significantly undermine the biomass production, while mistakes on the decision time have moderate effects on the biomass. However, we showed that anticipated or delayed decisions, combined with some specific contexts, may either hinder the healthy crop development or provide favorable conditions for an efficient use of water.

On-going works

As initially stated as the motivation of this thesis, we aim for applying the theoretical irrigation strategies developed in the contributions in the Optirrig model. To this end, we introduced in Chapter 5 the formalism of Optirrig, and we presented the compatibility between Optirrig and the CCI model. We highlighted their conceptual similarities, enabling to settle a term-to-term correspondence for a majority of the models main components, and we described their formulation differences that result in compatibility difficulties. With some mild assumptions allowing us to precise the correspondence of some slightly challenging terms, we showed that it is possible to implement the theoretical Threshold-Based Irrigation Strategies in the Optirrig framework, at least for testing purposes in a first step. The next stage of the study is to conduct a performance analysis of TBIS in a wide range of operational conditions, and especially by providing quantitative comparisons with other existing practices regarding various criteria (agronomic viability, economical outcomes, robustness to variable conditions). A further perspective involves contributing to the CCI model formulation with modeling improvements about the main obstacles identified for the correspondence (as the soil reservoir), without bringing too much complexity in the CCI model.

We presented in Chapter 3 a complete resolution of the optimal control problem corresponding to the context of greenhouse crops and with fully known conditions. As a result, the natural extension of this work is to either consider that the conditions (i.e. parameter values) are no longer exactly known, or take into account the weather conditions from open-field agriculture. In Chapter 6, we introduced a theoretical tool called optimal "averaged cost control" allowing us to address such optimal control problems involving uncertainties. We presented in Section 6.1 how the investigation of Chapter 3 can be adapted when considering uncertainties on the most sensible model parameter (the crop growth function φ). Some numerical experiments suggest that a similar result might be derived, and that a more general (yet very similar) structure of optimal control is expected to be robust to parameter uncertainties. In Section 6.2, we considered that rainfall events generate jumps in the soil moisture state variable, then we explained how a set of possible rainfall scenarios can be included in the crop irrigation problem. To this end, we employed a suited graph completion technique to transform our problem into a formulation adapted to the averaged cost control framework. We aim to further study both these averaged cost control problems, in view of the development of analytical results about optimal irrigation strategies in these contexts.

Further perspectives

In addition to the above perspectives that are direct follow up studies of the preliminary work introduced in Chapter 5 and Chapter 6, we can mention here several research questions related to biological, operational or methodological aspects that were not investigated in this thesis. First the lower state constraint on the soil moisture was considered constant, but we may deal with a time-dependent state constraint that is likely higher at specific stages of plant development (as flowering). This situation was observed numerically but the search for analytical resolution may provide supplementary difficulties. Then we always considered that the irrigation material could face the climatic demand and refill the soil with maximal irrigation rate, but it may be worth of interest to examine the opposite situation (where the maximal irrigation rate cannot match the daily evapotranspiration demand, i.e. when $k_2 < 1$). We observed numerically that this local non-controllability generates non-intuitive irrigation strategies whose structures can no longer be formulated with the time-varying feedback presented in our study. Furthermore, the optimization

problem may be formulated not only by maximizing the biomass production or the financial benefits, but also minimizing a cumulative stress that may be related to the production quality. Such consideration would end up in optimizing a compromise between the quantity and the quality of production.

Taking one step back and integrating the crop irrigation problem in a larger view, we may reconsider the problem formulation in several directions. First we never addressed the question of water acquisition, which may be part of a preliminary problem of resource allocation at the basin scale. Indeed, water availability is a prerequisite for water management, and operational issues about optimal water allocation beyond the plot scale is non-trivial. Then, we supposed in this thesis that the nutrients were not a limiting factor of the crop development. However, we may consider the formulation of the CCI model with an additional state variable for the soil nitrogen content (as in [42]), whose dynamics is highly non-linear (not even piecewise affine) in the situation where nutrients are considered as a limited resource. We can complement such considerations by integrating different sources of irrigation water, especially reuse water which contain a certain concentration of nitrogen, and which may further address the wastewater treatment. A problem combining stress and limitations in both water and nutrients has not been addressed with an analytical optimal control framework to our knowledge. Furthermore, the studies in this thesis were conducted under a spatial homogeneity assumption, but operational situations might ask for a spatialization (thus the emergence of partial differential equations), or a formal homogenization, as for example soil texture heterogeneity or irregular topography.

Finally, all these considerations and problem extensions are certainly of interest for the application point of view, but they also present the great advantage of giving rise to interesting and technical mathematical problems or open theoretical questions. As hinted by Chapter 2, a further perspective of investigation about the bang-bang principle would be to extend the framework combining piecewise affine and stochastic dynamics, or consider a more general notion of solutions with discontinuous dynamics. Moreover, the uniform procedure of change of time variable in Chapter 6 indicates that we may be able to generalize the averaged cost control results to dynamical systems containing state discontinuities, provided that we find general and suitable hypotheses on the nature of such discontinuities. At last, a great variety of control problems features arise from the further perspectives in the diverse research directions about crop irrigation and water management mentioned previously, such as non-linear and non-smooth dynamics, viability theory, stochastic dynamics, multi-criteria optimization, switched dynamical systems, networks, partial differential equations, homogenization theory. There is still plenty of room in crop irrigation and control theory. What this thesis has done is a drop, what opens ahead is an ocean.

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