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Dynamiques de Glauber, facteurs de mesures produit et mesures de Gibbs

Glauber dynamics, factors of product measures and Gibbs measures.

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Résumé

Cette thèse s'inscrit dans le cadre de la physique statistique. On cherche à comprendre le comportement macroscopique d'un système physique, à partir des interactions locales entre les particules qui le composent. Un phénomène particulièrement intéressant est celui de la *transition de phase* : autour d'un certain paramètre critique (par exemple la température), le modèle change brusquement de comportement à l'échelle macroscopique, alors que la nature microscopique du modèle évolue continûment. Ces systèmes sont modélisés mathématiquement par des mesures de probabilité, dites *mesures de Gibbs*. Lorsque les interactions entre particules sont aux plus proches voisins, ces mesures satisfont une propriété de Markov spatiale. De plus, lorsque des particules voisines ont tendance à adopter le même état, le modèle est positivement corrélé : les mesures de Gibbs vérifient alors les inégalités de corrélation dites « FKG ». Ces deux propriétés sont cruciales pour établir le portrait de phase du modèle ; cela se vérifie typiquement dans le cas du *modèle d'Ising* ferromagnétique, qui représente le comportement d'un aimant en fonction de la température.

Une mesure de Gibbs modélise le comportement à l'équilibre d'un système. On peut toutefois considérer une évolution temporelle du modèle, en définissant une dynamique pour laquelle la mesure de Gibbs est invariante. Le premier à le faire fût Roy J. Glauber, qui introduisait en 1963 une dynamique pour le modèle d'Ising qui porte aujourd'hui son nom. Cette *dynamique de Glauber* est un outil central de la thèse. Un de ses atouts majeurs est qu'on peut la définir sans inégalités de corrélation (l'hypothèse markovienne est nécessaire de prime abord, bien qu'on puisse l'affaiblir).

Dans les chapitres 2 et 3, on s'intéressera à des champs gaussiens markoviens, notamment des champs à dépendance finie ou proche de l'être. On analysera d'abord la compatibilité entre les hypothèses « Markov », « FKG » et « dépendance finie » (chapitre 2). Ensuite, on étudiera un modèle particulier de champ avec des corrélations à décroissance exponentielle (mais non supposées positives ou à portée finie) (chapitre 3). On définira la dynamique de Glauber du modèle et, avec des arguments de couplage par le passé, inventés par Propp et Wilson en 1996, on exprimera le champ comme *facteur d'une mesure produit*. Cette méthode nous permettra notamment de contrôler l'erreur lorsqu'on approxime le modèle par un champ à dépendance finie.

Le chapitre 4 est consacré à un modèle de *percolation*. La percolation s'intéresse aux propriétés de connectivité d'un sous-graphe aléatoire d'un graphe infini donné (typiquement un réseau périodique). Ces modèles sont particulièrement étudiés car ils présentent généralement une transition de phase : lorsque la densité d'arêtes conservées augmente, on observe l'émergence, à partir d'un seuil critique, d'une composante connexe infinie. Dans le chapitre 4, on trouve une condition suffisante pour l'existence d'une transition de phase, pour un modèle de percolation à longue portée : la percolation booléenne.

Enfin, on travaille dans le chapitre 5 sur un modèle à l'interface entre percolation et physique statistique : la percolation FK. Ce modèle illustre parfaitement la difficulté que peut causer l'absence de validité des inégalités FKG. En dimension 2, dans le régime FKG, on sait caractériser la transition de phase du modèle et la valeur du point critique est connue. Dans le régime sans FKG, l'existence même de la transition de phase est encore une question ouverte, et l'on ne connaît les propriétés de percolation du modèle que pour des intervalles très restreints de paramètres. L'objectif du chapitre 5 est de contribuer à l'étude du portrait de phase de la percolation FK « sans FKG », en améliorant légèrement les résultats connus dans ce régime.

Les chapitres 2-3-4-5 peuvent être lus indépendamment, bien que le résultat du chapitre 4 soit un lemme clé dans la preuve du théorème principal du chapitre 3.

Abstract

This thesis falls within the framework of statistical mechanics. We seek to understand the macroscopic behaviour of a physical system based on local interactions between the particles that compose it. A particularly interesting phenomenon is that of *phase transition*: around a certain critical parameter (for example, temperature), the model abruptly changes behaviour at the macroscopic scale, while the microscopic nature of the model evolves continuously. These systems are mathematically modelled by probability distributions, called *Gibbs measures*. When the interactions between particles are nearest-neighbours, these measures satisfy a spatial Markov property. Moreover, when neighbouring particles tend to adopt the same state, the model is positively correlated: the Gibbs measures then satisfy the so-called "FKG" correlation inequalities. These two properties are crucial for establishing the phase diagram of the model; this is typically verified in the case of the ferromagnetic *Ising model*, which represents the behaviour of a magnet as a function of temperature.

A Gibbs measure models the equilibrium behaviour of a system. However, one can consider a temporal evolution of the model by defining a dynamics for which the Gibbs measure is invariant. The first to do so was Roy J. Glauber, who introduced in 1963 a dynamics for the Ising model that today bears his name. This *Glauber dynamics* is a central tool of the thesis. One of its major advantages is that it can be defined without correlation inequalities (the Markovian hypothesis is necessary at first, although it can be weakened).

In Chapters 2 and 3, we will focus on Markovian Gaussian fields, particularly fields with finite dependence or close to being so. We will first analyse the compatibility between the "Markov", "FKG" and "finite dependence" hypotheses (Chapter 2). Next, we will study a particular model with exponentially decaying correlations (but not assumed to be positive or finite range) (Chapter 3). We will define the Glauber dynamics of the model and, with coupling-from-the-past arguments, developed by Propp and Wilson in 1996, we will express the field as a *factor of a product measure*. This method will notably allow us to control the error when approximating the model by a finitely dependent field.

Chapter 4 is devoted to a *percolation* model. Percolation is concerned with the connectivity properties of a random subgraph of a given infinite graph (typically a periodic lattice). These models are particularly studied because they generally exhibit a phase transition: when the density of retained edges increases, we observe the emergence, from a critical threshold, of an infinite connected component. In Chapter 4, we find a sufficient condition for the existence of a phase transition, for a long-range percolation model: the Boolean percolation.

Finally, we work in Chapter 5 on a model at the interface between percolation and statistical physics: the random-cluster model. This model perfectly illustrates the difficulty that can be caused by the absence of validity of FKG inequalities. In dimension 2, in the FKG regime, we know how to characterise the phase transition of the model and the value of the

critical point is known. In the regime without FKG, the very existence of the phase transition is still an open question, and we know the percolation properties of the model only for very restricted intervals of parameters. The objective of Chapter 5 is to contribute to the study of the phase diagram of the random-cluster model "without FKG", by slightly improving the existing results in this regime.

Chapters 2-3-4-5 can be read independently, although the result in Chapter 4 is a key lemma in the proof of the main theorem in Chapter 3.

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Parvenu à l'issue de mon doctorat en mathématiques, et ayant ainsi pratiqué, dans ma quête du savoir, l'exercice d'une recherche scientifique exigeante, je souhaiterais remercier l'ensemble des personnes qui m'ont accompagné dans cette aventure. En premier lieu, je remercie chaleureusement mon directeur de thèse, Vincent Beffara, qui m'a beaucoup apporté durant ces quatre années. Merci de m'avoir fait découvrir tant de belles maths, de m'avoir expliqué, ré-expliqué... Merci surtout pour ta gentillesse, ta confiance et ton enthousiasme permanent.

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Introduction en français[1]

0.1 Champs aléatoires markovien et mesures de Gibbs

0.1.1 Potentiels et mesures de Gibbs

En mécanique statistique, on cherche à étudier le comportement macroscopique de grands systèmes physiques à partir de leur description microscopique. Ces systèmes sont naturellement modélisés par une collection de variables aléatoires $(X_i)_{i\in V}$ où V est un ensemble fini, qui correspond par exemple à l'ensemble des particules d'un système. Les X_i sont à valeurs dans un ensemble fini S, qui représente les états possibles d'une particule. L'ensemble des *configurations* est donné par $\Omega := S^V$, et est muni de la tribu produit, notée $\mathcal{F}_V$ (on utilisera également dans la suite la notation $\mathcal{F}_A$ pour la tribu produit sur un sous-ensemble A de V). Pour chaque site $i \in V$, la variable aléatoire $X_i : \Omega \to S$ définie par $X_i(\omega) = \omega_i$ est appelée le *spin* en i (on fera parfois l'abus d'appeler ω_i le spin en i). Les particules interagissent, donc leurs états ne sont pas indépendants les uns des autres. Pour définir la distribution de $(X_i)_{i\in V}$, on veut suivre l'intuition physique qu'à basse température, le système est presque gelé et favorise les configurations d'énergie minimale, tandis qu'à haute température, les particules sont excitées et se comportent presque indépendamment.

Nous appelons *potentiel* une collection $\{\Phi_A\}_{A\subset V}$ de fonctions telles que pour tout $A \subset V$, $\Phi_A : \Omega \to \mathbf{R}$ est $\mathcal{F}_A-$mesurable (i.e. $\Phi_A(\omega)$ ne dépend que de la restriction $(\omega_i)_{i\in A}$ de la configuration à l'ensemble A). L'énergie d'une configuration $\omega \in \Omega$ sous le potentiel $\{\Phi_A\}_{A\subset V}$ est donnée par

$$H_V(\omega) = \sum_{A\subset V} \Phi_A(\omega).$$

La fonction H_V est appelée le *hamiltonien* du système. À chaque *température inverse* $\beta \in \mathbf{R}$ et couple potentiel-hamiltonien, nous associons une *mesure de Gibbs*, qui est une loi de probabilité sur $(\Omega, \mathcal{F}_V)$ donnée par

$$P_{\beta,V}(\omega) = \frac{\exp(-\beta H_V(\omega))}{Z_{\beta,V}},\ \omega \in \Omega$$

où la fonction de partition $Z_{\beta,V} := \sum_{\omega\in\Omega} \exp(-\beta H_V(\omega))$ est simplement une constante de normalisation qui assure que $P_{\beta,V}$ est une mesure de probabilité. À température nulle (i.e. à la limite $\beta \to +\infty$), la mesure est supportée par les configurations qui minimisent le hamiltonien. Au contraire, quand $\beta = 0$, $P_{\beta,V}$ est simplement la mesure uniforme sur Ω, en particulier c'est une mesure produit.

[1] Je ferais mieux d'aller choisir mon vocabulaire
Pour te plaire dans la langue de Molière

Dans un système physique, toutes les particules n'interagissent pas entre elles de la même façon : une particule interagit davantage avec les particules qui lui sont proches. C'est pourquoi il est naturel de munir l'ensemble V d'une structure de graphe. Désormais, on considère $G = (V, E)$ un graphe simple fini (i.e. sans boucles ni arêtes multiples). Pour toute paire de sommets (i, j), on écrit $i \sim j$ quand i et j sont reliés par une arête dans G, i.e. $\{i, j\} \in E$. Notons $d(\cdot, \cdot)$ la distance de graphe sur G. Le diamètre d'un sous-ensemble $A \subset V$ est défini par $\mathrm{diam}(A) := \sup_{i,j \in A} d(i, j)$.

Définition 1. *La* portée *relativement à G d'un potentiel $\Phi = (\Phi_A)_{A \subset V}$ est définie par*

$$r(\Phi) = \inf\{R \geq 0 : \Phi_A \equiv 0 \text{ pour tout } A \text{ avec } \mathrm{diam}(A) > R\};$$

Φ est dit à plus proches voisins *(relativement à G) si $r(\Phi) = 1$.*

Notons qu'en un certain sens cette dernière définition n'est pas restrictive, puisque tout potentiel peut être vu comme un potentiel à plus proches voisins sur un graphe avec suffisamment d'arêtes. En particulier, l'hypothèse à plus proches voisins est vide si G est le graphe complet.

Le modèle le plus simple et le mieux compris qui entre dans ce cadre est le modèle d'Ising. Il fut introduit par Lenz et Ising dans les années 1920 pour expliquer la *transition de phase* des matériaux magnétiques : lorsqu'il est chauffé au-dessus de sa *température de Curie*, un aimant perd soudainement ses propriétés de magnétisation. On dit qu'il passe d'un comportement ferromagnétique à paramagnétique. On suppose d'abord que nos variables aléatoires prennent leurs valeurs dans $\{-1, +1\}$; l'ensemble des configurations est donc $\{-1, +1\}^V$. On suppose ensuite qu'il n'y a que des interactions par paires entre spins voisins, et que les spins ont tendance à s'aligner (pour une température inverse $\beta > 0$). Le hamiltonien du modèle d'Ising peut alors s'écrire comme

$$H_V(\omega) = -\sum_{i \sim j} \omega(i)\omega(j),$$

pour toute configuration $\omega \in \{\pm 1\}^V$. La mesure de Gibbs associée à la température inverse β est notée $\mu_{\beta,V}$.

Supposons que V est une boîte finie de $\mathbf{Z}^d$. En ajoutant des conditions au bord et en laissant la boîte tendre vers $\mathbf{Z}^d$ tout entier, on obtient des mesures de Gibbs en volume infini (voir la section 0.3). En dimension $d \geq 2$, le modèle présente une transition de phase, ce qui signifie qu'il existe une température inverse critique $\beta_c(d) \in (0, \infty)$ telle que :

- pour tout $0 \leq \beta < \beta_c(d)$ (haute température), il n'existe qu'une seule mesure de Gibbs en volume infini ; en d'autres termes, le système « oublie » la condition au bord ;
- pour tout $\beta > \beta_c(d)$ (basse température), les conditions au bord « $+1$ partout » et « -1 partout » donnent, à la limite, deux mesures de Gibbs différentes μ_β^+ et μ_β^-.

Ceci est cohérent avec le fait qu'à très haute température, $\mu_{\beta,V}$ est proche d'une mesure produit, tandis qu'à très basse température, elle ne charge presque que les configurations d'énergie minimale, qui sont les configurations « tout $+1$ » et « tout -1 » dans ce cas. La description ci-dessus est valide pour le modèle d'Ising *ferromagnétique*, c'est-à-dire pour $\beta \geq 0$. Le modèle d'Ising antiferromagnétique avec « température inverse négative » est beaucoup moins bien compris (en particulier, on ne sait pas si μ_β^+ et μ_β^- sont bien définies).

0.1.2 Propriétés de Markov

Une propriété importante que le modèle d'Ising partage avec tout autre modèle dérivé d'un potentiel à plus proches voisins est la *propriété de Markov spatiale*. Elle exprime le fait que, étant donnée la configuration à l'extérieur d'un site i, la distribution conditionnelle du spin en i ne dépend que des spins voisins. Dans cette section, nous verrons que ceci n'est pas loin d'être une condition suffisante pour qu'une mesure admette une *représentation gibbsienne*.

Pour formaliser cette idée, il y a plusieurs façons de définir la propriété de Markov spatiale. Rappelons que si X, Y, Z sont des variables aléatoires dans un espace de probabilité $(\Omega, \mathcal{F}, P)$, on dit que X et Y sont conditionnellement indépendantes sachant Z si pour tous événements $A \in \sigma(X)$ et $B \in \sigma(Y)$, nous avons p.s. $P(A \cap B \mid \sigma(Z)) = P(A \mid \sigma(Z))P(B \mid \sigma(Z))$[2]. Dans ce cas, on écrit $X \perp Y \mid Z$.

Soit $G = (V, E)$ un graphe simple non orienté et $X = (X_i)_{i \in V}$ une collection de variables aléatoires indexées par les sommets de G. Pour tout $i \in V$ et $r \in \mathbf{N}$, on note $B_r(i) = \{j \in V, d(i,j) \leq r\}$ le $r-$voisinage de i. Si A et B sont des sous-ensembles de V, la distance entre A et B est définie par $d(A,B) = \inf_{i \in A, j \in B} d(i,j)$. Enfin, on écrit X_A pour la restriction $(X_i)_{i \in A}$ de X à l'ensemble A.

On dit que la famille $(X_i)_{i \in V}$ (ou plus formellement sa loi) satisfait, relativement à G,

- la *propriété de Markov par paires d'épaisseur* r si pour toute paire de sommets i et j,
$$d(i,j) > r \implies X_i \perp X_j \mid X_{V-\{i,j\}} \tag{PM}$$
- la *propriété de Markov locale d'épaisseur* r si pour tout $i \in V$,
$$X_i \perp X_{V-B_r(i)} \mid X_{B_r(i)-\{i\}} \tag{LM}$$
- la *propriété de Markov globale d'épaisseur* r si pour tous sous-ensembles $A, B \subset V$,
$$d(A,B) > r \implies X_A \perp X_B \mid X_{V-(A \cup B)} \tag{GM}$$

Ces trois définitions ne coïncident pas en général, et il n'est pas clair laquelle nous devrions choisir pour définir la notion de *champ aléatoire markovien*. Cependant, il n'est pas difficile de voir que la chaîne d'implications suivante est vérifiée :

$$\text{(GM)} \implies \text{(LM)} \implies \text{(PM)}.$$

La réciproque est vraie pour les distributions qui admettent une densité strictement positive par rapport à une mesure produit (c'est le contenu du théorème de Pearl et Paz, voir par exemple [Lau20] pour une présentation complète).

Remarque 2. *En fait,* (GM) *est équivalente à :*

$$\forall A \subset V, X_A \perp X_{V-B_r(A)} \mid X_{B_r(A)-A}$$

où $B_r(A) := \bigcup_{i \in A} B_r(i)$. *C'est évident dans le sens direct, en prenant* $B = V - B_r(A)$, *et la réciproque est aussi vraie : en fait, elle est une conséquence immédiate du fait que pour tout* $B_0 \subset B$,

$$X_A \perp X_B \mid X_{V-(A \cup B)} \implies X_A \perp X_{B_0} \mid X_{V-(A \cup B_0)}$$

[2] $\sigma(X)$ est la tribu engendrée par la variable aléatoire X.

(voir [Lau20] théorème 2.18 (A3)). C'est cette même propriété qui permet de prouver (LM) $\implies$ (PM)*, et l'implication* (GM) $\implies$ (LM) *est maintenant claire.*

Ceci justifie la définition suivante d'un champ aléatoire markovien. Nous nous restreignons pour l'instant aux graphes finis et à un espace d'états fini. Supposons donc que G est fini et S est un ensemble fini. Comme dans le paragraphe précédent, l'ensemble des configurations $\Omega = S^V$ est muni de la tribu produit $\mathcal{F}_V$. Nous discuterons les extensions aux graphes infinis dans la section 0.3 et aux spins à valeurs dans un espace d'états non compact dans la section 0.3.4 (le cas compact se comportant globalement comme le cas fini).

Définition 3. *Une loi de probabilité P sur $(\Omega, \mathcal{F}_V)$ est un* champ aléatoire markovien d'épaisseur r *(ou un champ aléatoire $r-$markovien, ou P est $r-$markovienne) sur G si elle satisfait $P(\omega) > 0$ pour tout $\omega \in \Omega$ et les propriétés équivalentes* (GM)*,* (LM) *et* (PM) *avec épaisseur r.*

Quand l'épaisseur r n'est pas explicitement spécifiée, on considérera alors les propriétés de Markov d'épaisseur 1 et les champs aléatoires $1-$markoviens. Observons que, de la même manière que pour l'hypothèse plus proches voisins sur les potentiels, tout champ aléatoire strictement positif peut être vu comme un champ aléatoire markovien avec une épaisseur (≥ 1) aussi petite qu'on le souhaite sur un graphe avec suffisamment d'arêtes. En particulier, si G est le graphe complet, les conditions (GM), (LM) et (PM) sont vides.

Clairement, les mesures de Gibbs associées aux potentiels à plus proches voisins sont des champs aléatoires markoviens. Le résultat suivant, habituellement nommé d'après Hammersley et Clifford ([CH71]), affirme que tout champ aléatoire markovien admet une représentation gibbsienne, c'est-à-dire qu'il existe un potentiel pour lequel il est une mesure de Gibbs.

Théorème 4 (Hammersley-Clifford)**.** *Une loi de probabilité P sur $(\Omega, \mathcal{F}_V)$ est un champ aléatoire markovien si et seulement si elle est une mesure de Gibbs pour un potentiel à plus proches voisins.*

Ce résultat s'étend directement à des épaisseurs plus grandes : P est $r-$markovienne si et seulement si elle est une mesure de Gibbs pour un potentiel à portée au plus r.

0.2 Dynamique de Glauber et algorithme de simulation exacte

0.2.1 Dynamique de Glauber

Présentons maintenant un outil central de cette thèse, la *dynamique de Glauber*. Elle fut introduite par Roy J. Glauber en 1963 [Gla63], initialement comme une version stochastique du modèle d'Ising. Elle fournit une méthode pour échantillonner les champs aléatoires markoviens (en volume fini), en exprimant la mesure de Gibbs comme la loi invariante d'une chaîne de Markov irréductible et apériodique, suivant l'approche classique de Monte-Carlo. Plusieurs versions de la dynamique de Glauber coexistent dans la littérature, celle présentée ici est aussi appelée la dynamique de *bain thermique*.

On décrit la dynamique pour le modèle d'Ising sur des graphes finis $G = (V, E)$, mais elle s'étend directement à tout champ aléatoire markovien sur un espace d'états fini. L'idée est, à chaque étape, de choisir uniformément au hasard un sommet $i \in V$ et de mettre à jour son spin selon sa loi conditionnelle sachant le reste de la configuration. Pour toute configuration

$\sigma \in \{\pm 1\}^V$, soient σ^i_+ et σ^i_- les configurations qui coïncident avec σ partout sauf en i, telles que $\sigma^i_+(i) = +1$ et $\sigma^i_-(i) = -1$. Observons que

$$\frac{\mu_{\beta,V}(X_i = -1 \mid X_{V-\{i\}} = \sigma_{V-\{i\}})}{\mu_{\beta,V}(X_i = +1 \mid X_{V-\{i\}} = \sigma_{V-\{i\}})} = \exp(H_i(\sigma^i_+) - H_i(\sigma^i_-)) = \exp\left(-2\beta \sum_{j\sim i} \sigma(j)\right).$$

Ainsi, en notant $S = \sum_{j\sim i} \sigma(j)$, on obtient

$$\mu_{\beta,V}(X_i = \pm 1 \mid X_{V-\{i\}} = \sigma_{V-\{i\}}) = \frac{1}{1 + e^{\mp 2\beta S}} = \frac{e^{\pm\beta S}}{e^{\beta S} + e^{-\beta S}}$$

(en résolvant $\frac{x}{1-x} = e^{-2\beta S}$).

Par conséquent, on définit un noyau de transition Q sur $\{\pm 1\}^V$ par

$$Q(\sigma, \sigma^i_\pm) = \frac{1}{|V|} \frac{e^{\pm\beta S}}{e^{\beta S} + e^{-\beta S}} \text{ avec } S = \sum_{j\sim i} \sigma(j)$$

et $Q(\sigma, \sigma') = 0$ si σ' n'est pas de la forme $\sigma^i_\pm$. La chaîne de Markov ayant Q comme noyau de transition est clairement apériodique et irréductible, donc elle admet une unique loi invariante, qui s'avère être la mesure de Gibbs du modèle d'Ising $\mu_{\beta,V}$.

Quand on veut obtenir un échantillon d'une loi de probabilité suivant la méthode de Monte-Carlo, on fait face à deux problèmes. Puisqu'on ne peut pas faire tourner la chaîne de Markov indéfiniment, on n'atteint jamais exactement la loi invariante de la chaîne. De plus, il est difficile de déterminer a priori combien d'étapes sont nécessaires pour s'approcher de la distribution cible, avec un niveau de précision fixé. L'algorithme de *couplage par le passé* introduit en 1996 par Propp et Wilson [PW96] résout ces problèmes, en échantillonnant exactement la distribution invariante en un nombre fini mais aléatoire d'étapes. C'est pourquoi il est aussi appelé algorithme de *simulation parfaite* ou de *simulation exacte*.

0.2.2 Algorithme de couplage par le passé

Soit P une mesure de probabilité sur un espace d'états fini Ω, et considérons une chaîne de Markov irréductible et apériodique dont la mesure invariante est P (par exemple la dynamique de Glauber associée à P, si P est une mesure de Gibbs). Lorsqu'on initie une trajectoire de la chaîne au temps 0, la mesure empirique se rapproche de la loi invariante au cours du temps. Imaginons qu'au lieu de démarrer au temps 0, on démarre la chaîne au temps $-\infty$. Alors, au temps 0, après une infinité d'étapes, elle est distribuée exactement selon la mesure invariante.

Pour formaliser cette idée, on simule des trajectoires de la chaîne de Markov initiées depuis toutes les configurations possibles dans Ω à un grand temps négatif $-T$. Ces trajectoires sont couplées de sorte que dès que deux (ou plus) d'entre elles coïncident à un certain temps, alors elles évoluent ensuite ensemble pour toujours (i.e. jusqu'au temps 0) : on dit qu'elles *coalescent*. Si T est assez grand pour que toutes les trajectoires coalescent avant le (ou au) temps 0, alors la valeur commune au temps 0 est distribuée selon P.

Plus précisément, considérons une famille $(f_n)_{n<0}$ d'applications aléatoires i.i.d. $f_n : \Omega \to \Omega$, distribuées comme un saut de la chaîne de Markov. Les $f_n(\omega)$ pour $\omega \in \Omega$ peuvent être pris indépendants, mais on peut aussi choisir un couplage qui favorise la coalescence. Alors,

pour tout $n \leq 0$, on pose $F_n^0 := f_{-1} \circ f_{-2} \circ \cdots \circ f_n$; la distribution au temps 0 de l'état de la chaîne de Markov partant au temps n avec la configuration initiale $\omega \in \Omega$ est la même que $F_n^0(\omega)$. La coalescence de toutes les trajectoires est donc équivalente au fait que F_{-T}^0 est une application constante. Soit

$$T = \inf\{n > 0 : F_{-n}^0 \text{ est constante}\}.$$

Notons qu'il est crucial de chercher la coalescence depuis le passé et non vers le futur. Partir au temps 0 et s'arrêter quand toutes les trajectoires ont coalescé conduirait à un biais. En effet, si un état $\omega \in \Omega$ n'a qu'un seul antécédent (ce qui signifie que ω ne peut être atteint par un saut de la chaîne que depuis un unique autre état ω'), il est clair que ω ne peut pas être atteint au moment exact où les trajectoires coalescent. Réaliser la chaîne depuis le passé écarte ce problème de biais.

Théorème 5 (Propp-Wilson)**.** *T est presque sûrement fini et l'unique image de F_{-T}^0 a pour distribution P.*

Preuve. Par hypothèse sur la chaîne de Markov, on sait qu'il existe $k \in \mathbf{N}$ tel que pour tout $\omega, \omega' \in \Omega$, il y a une probabilité strictement positive d'aller de ω à ω' en k étapes. Ainsi, pour tout $t \leq 0$, $F_{t-k}^t := f_{t-1} \circ \cdots \circ f_{t-k}$ a aussi une probabilité strictement positive d'être constante[3]. Maintenant, toutes les applications $F_{-k}^0, F_{-2k}^{-k}, \cdots$ ont la même probabilité d'être constantes et ces événements sont indépendants, donc presque sûrement l'un d'eux est réalisé. Dans ce cas, F_{-n}^0 est constante pour n suffisamment grand (observons que $F_{-(n+1)k}^{-nk}$ constante implique $F_{-(n+1)k}^0$ constante), donc T est fini.

On appelle $\overline{F}_{-\infty}^0$ la valeur retournée par l'algorithme, qui est l'unique image de F_{-T}^0, la notation venant du fait que $F_{-T}^0 = F_{-T-1}^0 = \ldots$ presque sûrement. Alors, $\overline{F}_{-\infty}^0$ est distribuée selon P puisqu'elle a la même distribution que $\overline{F}_{-\infty}^{-1}$ (définie de manière évidente) et que p.s. $\overline{F}_{-\infty}^0 = f_{-1} \circ \overline{F}_{-\infty}^{-1}$. □

Remarque 6 (Couplage monotone pour le modèle d'Ising)**.** *Si on applique cet algorithme au modèle d'Ising, ou à un autre champ aléatoire markovien, on va faire face au problème que le cardinal de $\Omega = S^V$ est très grand. En prenant la dynamique de Glauber comme chaîne de Markov sous-jacente dans l'algorithme, on peut définir un couplage qui évite de considérer $|\Omega|$ trajectoires couplées. Cela fonctionne comme suit pour le modèle d'Ising : soit $(U_n, i_n)_{n<0}$ une famille i.i.d. de variables aléatoires telles que U_n est uniformément distribuée sur $[0,1]$ et i_n est un sommet choisi uniformément dans V ; les applications aléatoires (f_n) sont alors définies par $f_n(\omega) = \omega^{i_n}_{\varphi(\sum_{j\sim i_n}\omega(j), U_n)}$ où $\varphi(S,u) = \begin{cases} - & \text{si } u \leq (1+e^{2\beta S})^{-1} \\ + & \text{sinon} \end{cases}$, i.e. à chaque étape, on ré-échantillonne le spin en le site choisi selon sa loi conditionnelle sachant les voisins. Ici, le couplage n'est pas indépendant puisque les mêmes U_n et i_n sont utilisés pour tous les ω, mais néanmoins les $(f_n)_{n<0}$ sont indépendantes. Pour $\beta \geq 0$, φ est croissante en S, donc on peut vérifier que le couplage satisfait la propriété de monotonicité suivante : pour tout $\omega \leq \omega'$[4] et*

[3] Une hypothèse sur les $f_n(\omega)$ est nécessaire ici ; supposer l'indépendance des $(f_{-1}(\omega))_{\omega\in\Omega}$ est clairement suffisant, mais on peut (et on y a intérêt !) considérer des couplages qui favorisent la coalescence, i.e. qui augmentent cette probabilité.

[4] Pour l'ordre partiel canonique sur $\{\pm 1\}^V$, c'est-à-dire $\omega \leq \omega'$ si et seulement si $\omega(j) \leq \omega'(j)$ pour tout $j \in V$.

$n < 0$, $f_n(\omega) \leq f_n(\omega')$. Par conséquent, il suffit de considérer deux chaînes de Markov (au lieu de $|\Omega|$), avec les configurations initiales $\omega \equiv -1$ et $\omega \equiv +1$. Toutes les autres trajectoires étant coincées entre ces deux-là, il suffit d'obtenir la coalescence depuis ces deux configurations initiales extrémales.

Il y a beaucoup d'autres façons d'améliorer l'algorithme à des fins de simulation que nous ne discuterons pas ici.

0.3 Mesures de Gibbs en volume infini

Nous avons défini les mesures de Gibbs et leurs dynamiques de Glauber associées en volume fini, ce qui signifie que le graphe sous-jacent était supposé fini. Les systèmes physiques sont toujours finis, mais le nombre de particules impliquées est généralement énorme, de sorte qu'ils sont bien approximés par leur limite quand le nombre de particules tend vers l'infini. Il est donc naturel de chercher à définir des mesures de Gibbs sur des graphes infinis. De plus, les transitions de phase seront facilement interprétées en termes de mesures en volume infini, puisqu'elles correspondront à la coexistence de plusieurs mesures de Gibbs pour le même modèle.

Il existe deux façons classiques de définir les mesures sur des graphes infinis, qui sont de passer à la limite thermodynamique de systèmes finis, ou de suivre l'approche dite DLR. Nous ne donnerons pas tous les détails topologiques (se référer à [FV18] ou [Geo11] pour une étude complète de la théorie) et décrirons ces deux méthodes en nous focalisant particulièrement sur le modèle d'Ising sur $\mathbf{Z}^d$.

0.3.1 Conditions au bord et limite thermodynamique

Soit $G = (V, E)$ un graphe infini simple, localement fini (c'est-à-dire que chaque sommet a un degré fini), et S un ensemble fini. L'ensemble des configurations $\Omega = S^V$ est équipé de la tribu produit $\mathcal{F}_V$, qui est engendrée par les événements cylindriques, c'est-à-dire les événements de la forme $\{\omega \in \Omega \mid \omega_\Lambda = \sigma\}$ pour un sous-ensemble fini Λ de V (noté $\Lambda \Subset V$) et $\sigma \in S^\Lambda$. Une fonction $f : \Omega \to \mathbf{R}$ est dite *locale* si elle ne dépend que d'un nombre fini de spins, c'est-à-dire si elle est $\mathcal{F}_\Lambda$-mesurable pour un certain $\Lambda \Subset V$. Une suite de mesures de probabilité (P_n) sur Ω converge vers une probabilité P sur Ω si et seulement si $\lim_{n\to\infty} P_n(C) = P(C)$ pour tout événement cylindrique C. Dans ce cas, on écrit $P_n \Rightarrow P$. Il est équivalent de demander que $\lim_{n\to\infty} \int f \mathrm{d}P_n = \int f \mathrm{d}P$ pour toute fonction locale f, cela correspond donc à la convergence faible.

Soit $\Phi = \{\Phi_A\}_{A \Subset V}$ un potentiel et $\Lambda \Subset V$ une boîte finie. Le hamiltonien dans la boîte Λ associé au potentiel Φ est défini par

$$H_\Lambda(\omega) = \sum_{A \Subset V, A \cap \Lambda \neq \emptyset} \Phi_A(\omega). \tag{1}$$

Pour que le hamiltonien soit bien défini, il faut que la somme dans (1) soit convergente ; c'est le cas si on suppose que le potentiel est *absolument sommable*, c'est-à-dire si

$$\forall x \in V, \sum_{A \Subset V, x \in A} \|\Phi_A\|_\infty < \infty \tag{2}$$

(ceci est toujours vérifié quand le potentiel a une portée finie). Soit $\eta \in \Omega$. La mesure de Gibbs sur Λ à température inverse β avec *condition au bord* η est la distribution de probabilité sur S^Λ donnée pour tout $\omega_\Lambda \in S^\Lambda$ par

$$P^\eta_{\beta,\Lambda}(\omega_\Lambda) = \frac{\exp(-\beta H_\Lambda(\omega_\Lambda \eta_{\Lambda^c}))}{Z^\eta_{\beta,\Lambda}},$$

où $\omega_\Lambda \eta_{\Lambda^c}$ est l'élément de Ω qui coïncide avec ω_Λ sur Λ et avec η sur $\Lambda^c := V - \Lambda$, et $Z^\eta_{\beta,\Lambda}$ est la constante de normalisation. Il est aussi commode de considérer $P^\eta_{\beta,\Lambda}$ comme une mesure sur Ω supportée par $\Omega^\eta_\Lambda := \{\omega \in \Omega \mid \omega_{\Lambda^c} = \eta_{\Lambda^c}\}$, l'ensemble des configurations qui coïncident avec η à l'extérieur de Λ. Pour tout $\omega \in \Omega$, nous avons alors

$$P^\eta_{\beta,\Lambda}(\omega) = \frac{\exp(-\beta H_\Lambda(\omega))}{Z^\eta_{\beta,V}} \mathbf{1}_{\{\omega \in \Omega^\eta_\Lambda\}}. \tag{3}$$

Une mesure de probabilité P sur $(\Omega, \mathcal{F})$ est appelée une *mesure de Gibbs limite faible en volume infini* (souvent abrégée limite faible) pour un potentiel et un β fixés si elle est un point d'accumulation de $\{P^\eta_{\beta,\Lambda}\}_{\eta\in\Omega,\Lambda\Subset V}$, c'est-à-dire s'il existe une suite (η_n) de conditions au bord et (Λ_n) une suite croissante de boîtes finies satisfaisant $\Lambda_n \uparrow V$ (i.e. tout $x \in V$ est dans Λ_n pour n assez grand) telle que

$$P^{\eta_n}_{\beta,\Lambda_n} \Rightarrow P \text{ quand } n \to \infty.$$

On peut aussi considérer d'autres types de conditions au bord, comme la condition au bord libre (qui correspond aux mesures $P_{\beta,\Lambda}$ définies dans la section 0.1.1) ou la condition au bord périodique. L'ensemble de telles mesures est noté $\mathcal{W}_\beta(\Phi)$.

Par compacité de Ω et le théorème de Prohorov, l'ensemble des lois de probabilité sur Ω est séquentiellement compact, d'où l'existence de limites faibles. L'unicité cependant n'est pas garantie, comme nous le verrons pour le modèle d'Ising sur $\mathbf{Z}^d$. Dans la littérature, on parle de *transition de phase du premier ordre* quand l'unicité n'est pas atteinte.

Supposons maintenant que $G = (V, E)$ est le réseau hypercubique en dimension d, ce qui signifie que $V = \mathbf{Z}^d$ et $i \sim j$ si et seulement si $\|i - j\|_1 = 1$. Les mesures de Gibbs du modèle d'Ising avec condition au bord sont définies comme dans (3), en prenant

$$\Phi_A(\omega) = \begin{cases} -\omega(i)\omega(j) & \text{si } A = \{i, j\} \in E \\ 0 & \text{sinon} \end{cases}$$

comme potentiel. Il y a deux conditions au bord d'intérêt particulier, qui sont $\eta \equiv +1$ et $\eta \equiv -1$, les deux configurations extrémales de $\Omega = \{\pm 1\}^{\mathbf{Z}^d}$ pour l'ordre partiel canonique. Notons $\mu^+_{\beta,\Lambda}$ et $\mu^-_{\beta,\Lambda}$ respectivement les mesures de Gibbs sur $\Lambda \Subset \mathbf{Z}^d$ associées à ces conditions au bord extrémales. Quand $\beta \geq 0$, on peut montrer qu'il existe deux mesures μ^+_β et μ^-_β telles que pour tout choix de suite $\Lambda_n \uparrow \mathbf{Z}^d$, $\mu^+_{\beta,\Lambda_n} \Rightarrow \mu^+_\beta$ et $\mu^-_{\beta,\Lambda_n} \Rightarrow \mu^-_\beta$. Cela donne une preuve constructive du fait que $\mathcal{W}_\beta \neq \emptyset$ pour le modèle d'Ising d-dimensionnel. Un outil crucial de la preuve est l'inégalité FKG (nommée d'après Fortuin, Kasteleyn et Ginibre [FKG71]), qui énonce que les fonctions locales croissantes sont positivement corrélées, et qui n'est pas valable pour β négatif. Dans cet énoncé, une fonction $f : \Omega \to \mathbf{R}$ est dite *croissante* si $\omega \leq \omega'$ pour l'ordre partiel canonique sur Ω implique $f(\omega) \leq f(\omega')$.

Théorème 7 (Inégalité FKG). *Soit $\beta \geq 0$, $\Lambda \Subset \mathbf{Z}^d$ et $\eta \in \Omega$ une condition au bord. Pour toute paire de fonctions croissantes f et g,*

$$\mu^{\eta}_{\beta,\Lambda}(fg) \geq \mu^{\eta}_{\beta,\Lambda}(f)\mu^{\eta}_{\beta,\Lambda}(g).$$

Il est donc clair que si μ^{+}_{β} et μ^{-}_{β} sont différentes, alors nous sommes dans un régime où le modèle d'Ising présente une transition de phase du premier ordre. Par monotonicité, il s'avère que la réciproque est vraie, donc il suffit de comparer μ^{+}_{β} et μ^{-}_{β} pour déterminer s'il y a ou non unicité de la mesure en volume infini. Nous pouvons maintenant énoncer plus précisément le résultat de transition de phase mentionné précédemment, d'abord obtenu par Peierls [Pei36] en dimension 2 :

Théorème 8 (Transition de phase du premier ordre). *En dimension $d \geq 2$, il existe une température inverse critique $\beta_c(d) \in (0, \infty)$ telle que*

- *Pour tout $\beta \in [0, \beta_c(d)[$, $\mu^{+}_{\beta} = \mu^{-}_{\beta}$ et il y a unicité de la mesure en volume infini ;*
- *Pour tout $\beta \in]\beta_c(d), +\infty[$, $\mu^{+}_{\beta} \neq \mu^{-}_{\beta}$ donc une transition du premier ordre se produit.*

En dimension $d = 1$, il n'y a pas de transition de phase ($\beta_c(1) = +\infty$), cela constituait en fait le résultat principal de la thèse d'Ising. Le théorème affirme que pour $\beta > \beta_c$, l'ensemble des limites faibles $\mathcal{W}_{\beta}$ pour le modèle d'Ising n'est pas réduit à un singleton, mais il ne dit rien sur sa structure. Des élements de réponse à cette question seront donnés dans le paragraphe suivant, où nous décrivons la seconde manière de définir les mesures en volume infini.

0.3.2 Approche DLR

L'approche DLR, pour Dobrushin, Lanford et Ruelle [Dob68] [LR69], considère les probabilités conditionnelles sur des boîtes finies sachant la configuration à l'extérieur de la boîte, plutôt que les lois marginales comme dans le théorème d'extension de Kolmogorov. En fait, la définition même des mesures de Gibbs rend les lois marginales difficiles d'accès. Par exemple, la loi marginale d'un seul spin dans le modèle d'Ising dans $\mathbf{Z}^2$ à basse température ($\beta > \beta_c$) est simplement une variable aléatoire à valeurs dans $\{\pm 1\}$, mais son espérance est différente sous μ^{+}_{β} et μ^{-}_{β}, donc elle ne peut pas être calculée directement à partir du hamiltonien.

Les candidats pour être une collection de lois conditionnelles d'une mesure en volume infini sont appelés des *spécifications*. On définira seulement les spécifications gibbsiennes[5] (encore une fois, se référer à [FV18] ou [Geo11] pour des définitions plus générales).

Soit $\Phi = \{\Phi_A\}_{A \Subset V}$ un potentiel absolument sommable, c'est-à-dire que Φ satisfait (2). La spécification gibbsienne avec potentiel Φ et température inverse β est la collection $\pi^{\Phi,\beta} = (\pi^{\Phi,\beta}_{\Lambda})_{\Lambda \Subset V}$ de fonctions $\pi^{\Phi,\beta}_{\Lambda}(\cdot \mid \cdot) : \mathcal{F}_V \times \Omega \to \mathbf{R}$ telles que pour tout $\Lambda \Subset V$, $\eta \in \Omega$ et $A \in \mathcal{F}_V$,

$$\begin{aligned}\pi^{\Phi,\beta}_{\Lambda}(A \mid \eta) &= \sum_{\omega \in \Omega^{\eta}_{\Lambda}} \frac{\exp(-\beta H_{\Lambda}(\omega))}{Z^{\eta}_{\beta,\Lambda}} \mathbf{1}_{\{\omega \in A\}} \\ &= \sum_{\omega_{\Lambda} \in S^{\Lambda}} \frac{\exp(-\beta H_{\Lambda}(\omega_{\Lambda}\eta_{\Lambda^c}))}{Z^{\eta}_{\beta,\Lambda}} \mathbf{1}_{\{\omega_{\Lambda}\eta_{\Lambda^c} \in A\}}.\end{aligned}$$

[5] De plus, si on regarde la définition 2.9 dans [Geo11], on considérera toujours la mesure de comptage sur S comme « mesure a priori ».

En particulier, si $A \in \mathcal{F}_{\Lambda^c}$, c'est-à-dire que A ne dépend que des spins à l'extérieur de Λ, alors $\pi_\Lambda^{\Phi,\beta}(A \mid \eta) = \mathbf{1}_{\{\eta \in A\}}$. Ainsi, $\pi_\Lambda^{\Phi,\beta}(\Omega_\Lambda^\eta \mid \eta) = 1$ donc $\pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta)$ est supportée par Ω_Λ^η.

Une loi de probabilité μ sur $(\Omega, \mathcal{F})$ est dite compatible avec une spécification π si elle satisfait les équations DLR : pour tout $\Lambda \Subset V$,

$$\mu = \mu\pi_\Lambda, \text{ c'est-à-dire pour tout événement cylindrique } C,\ \mu(C) = \int_\Omega \pi_\Lambda(C \mid \cdot)\mathrm{d}\mu.$$

L'ensemble des lois compatibles avec $\pi^{\Phi,\beta}$ est noté $\mathcal{G}(\pi^{\Phi,\beta}) = \mathcal{G}_\beta(\Phi)$, ou simplement $\mathcal{G}_\beta$ pour le modèle d'Ising, et de telles lois sont appelées *mesures de Gibbs en volume infini* (souvent simplement mesures de Gibbs). Il découle de notre définition que $\pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta)$ coïncide avec la mesure de Gibbs en volume fini dans Λ avec condition au bord η. Par conséquent, une mesure μ est compatible avec $\pi^{\Phi,\beta}$ si et seulement si pour toute boîte finie $\Lambda \Subset V$, sa loi conditionnelle sachant que la configuration à l'extérieur de Λ est η_{Λ^c} coïncide μ-presque sûrement avec la mesure de Gibbs en volume fini sur Λ avec condition au bord η. Les équations DLR peuvent donc être réécrites comme

$$\mu(\cdot \mid \mathcal{F}_{\Lambda^c})(\eta) = P_{\beta,\Lambda}^\eta \text{ pour } \mu\text{-p.s. } \eta \in \Omega. \tag{DLR}$$

La condition (2) assure que le hamiltonien, et donc la spécification gibbsienne, sont bien définis, mais elle n'est pas nécessaire. Dans la section 0.5.2 nous introduirons un exemple important de modèle pour lequel le potentiel n'est pas absolument sommable. Cependant, cette condition garantit l'existence d'une mesure de Gibbs en volume infini.

En effet, si Φ satisfait (2), on peut montrer que $\pi^{\Phi,\beta}$ est *quasilocale*, soit qu'elle est continue en la condition au bord. Plus précisément, une spécification π est dite quasilocale si pour tout $\Lambda \Subset V$ et tout événement cylindrique C, $\eta \mapsto \pi_\Lambda(C \mid \eta)$ est continue[6]. Alors, la quasilocalité implique l'existence d'une mesure de Gibbs en volume infini puisqu'on peut vérifier que toute mesure limite faible en volume infini satisfait les équations DLR (théorème 6.26 dans [FV18]). La même preuve montre que $\mathcal{G}_\beta(\Phi)$ est un sous-ensemble fermé de l'ensemble des mesures de probabilité sur $(\Omega, \mathcal{F})$.

De plus, un avantage du formalisme DLR est qu'il découle de la définition que $\mathcal{G}_\beta(\Phi)$ est un ensemble convexe, ce qui signifie que pour tout $\mu, \mu' \in \mathcal{G}_\beta(\Phi)$ et $\alpha \in [0,1]$, $\alpha\mu + (1-\alpha)\mu' \in \mathcal{G}_\beta(\Phi)$. On peut donc parler de ses éléments extrémaux ; soit $\operatorname{ex}\mathcal{G}_\beta(\Phi)$ l'ensemble des mesures de Gibbs extrémales, qui sont les éléments de $\mathcal{G}_\beta(\Phi)$ qui n'admettent pas de décomposition convexe non triviale. Il y aurait beaucoup à dire sur ce sujet, mais nous nous contenterons d'affirmer que les mesures de Gibbs extrémales peuvent toujours être obtenues comme limites faibles : si $\mu \in \operatorname{ex}\mathcal{G}_\beta(\Phi)$, alors

$$\pi_{\Lambda_n}^{\Phi,\beta}(\cdot \mid \eta) \Rightarrow \mu \text{ quand } \Lambda_n \uparrow V \text{ pour } \mu\text{-p.s. } \eta$$

(théorème 6.63 dans [FV18]). Ainsi, en un sens, elles sont les mesures de Gibbs pertinentes du point de vue physique. On résume la discussion ci-dessus dans le résultat suivant :

Théorème 9. *1. Pour tout potentiel Φ, $\operatorname{ex}\mathcal{G}_\beta(\Phi) \subset \mathcal{W}_\beta(\Phi)$.*

2. Si Φ est absolument sommable, alors $\emptyset \neq \mathcal{W}_\beta(\Phi) \subset \mathcal{G}_\beta(\Phi)$, donc $\mathcal{G}_\beta(\Phi) = \overline{\operatorname{co}\mathcal{W}_\beta(\Phi)}$ où $\overline{\operatorname{co}\mathcal{W}_\beta(\Phi)}$ désigne l'enveloppe convexe fermée de $\mathcal{W}_\beta(\Phi)$.

[6] Pour la topologie induite par les fonctions locales.

Dans les deux items, l'inclusion réciproque n'est pas vraie en général, et le modèle d'Ising fournit des contre-exemples éclairants.

D'abord, les mesures μ_β^+ et μ_β^- définies ci-dessus comme les limites faibles respectivement des conditions au bord « + » et « − » sont extrémales. En dimension 2, le théorème d'Aizenman-Higuchi [Aiz80] [Hig81] nous dit que toutes les limites faibles possibles sont des combinaisons convexes de μ_β^+ et μ_β^-. Par conséquent, elles sont les deux seules mesures de Gibbs extrémales, et $\mathcal{G}_\beta = \{\alpha\mu_\beta^+ + (1-\alpha)\mu_\beta^-, \alpha \in [0,1]\}$. Nous pouvons maintenant énoncer notre premier exemple d'une limite faible qui n'est pas une mesure de Gibbs extrémale. Il n'est pas difficile de montrer que $(\mu_{\beta,\Lambda})_{\Lambda\Subset\mathbf{Z}^2}$ (sans condition au bord, ou prendre $\eta \equiv 0$ bien que ce ne soit pas une configuration) admet un point d'accumulation. Elle est appelée la mesure de Gibbs avec condition au bord libre, notée $\mu_\beta^\varnothing$. Par le théorème d'Aizenman-Higuchi, $\mu_\beta^\varnothing$ doit être une combinaison convexe de μ_β^+ et μ_β^-. Mais $\mu_\beta^\varnothing$ est invariante par retournement des spins (on remarque que $\mu_{\beta,\Lambda}(\omega) = \mu_{\beta,\Lambda}(-\omega)$), donc la seule possibilité est $\mu_\beta^\varnothing = \frac{1}{2}\mu_\beta^+ + \frac{1}{2}\mu_\beta^-$. Elle n'est donc pas extrémale dès que $\mu_\beta^+ \neq \mu_\beta^-$, c'est-à-dire quand $\beta > \beta_c(2)$. De plus, Abraham et Reed [AR76] ont construit une famille $(\eta_\alpha)_{\alpha\in(0,1)}$ de conditions au bord telles que $\mu_{\beta,\Lambda_n}^{\eta_\alpha} \Rightarrow \alpha\mu_\beta^+ + (1-\alpha)\mu_\beta^-$. Cela prouve que pour le modèle d'Ising 2D, $\mathcal{W}_\beta = \mathcal{G}_\beta$, et fournit de nombreux exemples de limites faibles qui ne sont pas extrémales. Ces conditions au bord $(\eta_\alpha)_{\alpha\in(0,1)}$ ont été obtenues en modifiant la condition au bord de Dobrushin que nous présentons dans le paragraphe suivant.

En dimension $d \geq 3$, il est possible de produire de nouvelles mesures extrémales pour β assez grand. Cela est lié à l'existence de mesures de Gibbs non invariantes par translation, d'abord mises en évidence par Dobrushin [Dob72]. Considérons une condition au bord $+1$ sur un demi-plan et -1 sur l'autre, par exemple

$$\eta^{\mathrm{Dob}}(i) = \begin{cases} +1 & \text{si } i = (i_1, \ldots, i_d) \in \mathbf{Z}^d \text{ avec } i_d \geq 0 \\ -1 & \text{sinon} \end{cases}.$$

Ce type de condition au bord est appelé une condition au bord de Dobrushin. Il est prouvé dans [Dob72] que pour une suite bien choisie de boîtes finies $\Lambda_n \uparrow \mathbf{Z}^d$, $\mu_{\beta,\Lambda_n}^{\eta^{\mathrm{Dob}}}$ converge faiblement vers une certaine mesure en volume infini μ_β^{Dob}, qui s'avère être une mesure de Gibbs extrémale pour β assez grand ($\beta > \beta_c(d-1)$ suffit). Cela induit l'existence d'une infinité dénombrable de mesures de Gibbs extrémales, en remplaçant la condition $i_d \geq 0$ dans η^{Dob} par $i_k \geq n$ ou $i_k \leq n$ pour n'importe quels $1 \leq k \leq d$ et $n \in \mathbf{Z}$. Il est conjecturé qu'elles sont, avec μ_β^+ et μ_β^-, les seules mesures de Gibbs extrémales du modèle d'Ising.

En plus de donner de nouvelles mesures de Gibbs extrémales, les conditions au bord de Dobrushin fournissent le premier exemple connu d'une mesure de Gibbs qui n'est pas une limite faible. Considérons la condition au bord $\eta^{-\mathrm{Dob}}$ obtenue en retournant tous les spins dans η^{Dob}, et notons par $\mu_\beta^{-\mathrm{Dob}}$ la mesure limite faible correspondante. Par convexité de $\mathcal{G}_\beta$, $\frac{1}{2}\mu_\beta^{\mathrm{Dob}} + \frac{1}{2}\mu_\beta^{-\mathrm{Dob}} \in \mathcal{G}_\beta$, mais on peut prouver qu'elle ne peut pas être obtenue comme limite faible (pour β assez grand). Cet exemple est apparu d'abord dans le livre de Miyamoto [Miy04] (en japonais) et a ensuite été redécouvert indépendamment par Coquille [Coq15].

0.3.3 Propriétés d'énergie finie et de Markov spatiale

Soit Φ un potentiel absolument sommable. Par (2), pour tout $\Lambda \Subset V$, $\sup_{\omega\in\Omega} H_\Lambda(\omega) < \infty$. Par conséquent, la spécification gibbsienne $\pi^{\Phi,\beta}$ est positive, au sens où

$$\forall \Lambda \Subset V, \forall \eta \in \Omega, \inf_{\omega\in\Omega_\Lambda^\eta} \pi^{\Phi,\beta}(\omega \mid \eta) > 0. \tag{4}$$

Toute mesure de Gibbs $\mu \in \mathcal{G}(\pi^{\Phi,\beta})$ satisfait donc

$$\inf_{A\in\mathcal{F}_\Lambda} \mu(A \mid \mathcal{F}_{\Lambda^c})(\eta) > 0 \text{ pour } \mu-\text{p.s. } \eta \in \Omega \tag{EF}$$

Cette propriété est souvent appelée *énergie finie*, notamment lorsqu'elle est utilisée dans les modèles de percolation. Elle implique notamment que $\mu(C) > 0$ pour tout événement cylindrique C, soit que μ a un support plein.

Supposons que Φ soit un potentiel à plus proches voisins sur un graphe $G = (V, E)$ (on pourrait transposer tout ce qui suit à des potentiels à portée finie). Soit $\Lambda \Subset V$ et $\partial\Lambda := \{i \in \Lambda^c \mid \exists j \in \Lambda, i \sim j\}$. La spécification $\pi^{\Phi,\beta}$ satisfait donc une propriété de Markov spatiale, c'est-à-dire

$$\pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta) = \pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta') \text{ pour tous } \eta, \eta' \in \Omega \text{ tels que } \eta_{\partial\Lambda} = \eta'_{\partial\Lambda}.$$

Cette propriété se transfère à toutes les mesures de Gibbs $\mu \in \mathcal{G}(\pi^{\Phi,\beta})$ de la façon suivante : pour tout $\Lambda \Subset V$

$$\forall A \in \mathcal{F}_\Lambda, \mu(A \mid \mathcal{F}_{\Lambda^c}) = \mu(A \mid \mathcal{F}_{\partial\Lambda}) \ \mu - \text{a.s.} \tag{5a}$$

ou, de manière équivalente,

$$\forall \Delta \subset V, d(\Lambda, \Delta) > 1 \implies X_\Lambda \perp X_\Delta \mid X_{(\Lambda\cup\Delta)^c}, \tag{5b}$$

ou encore

$$X_\Lambda \perp X_{(\partial\Lambda\cup\Lambda)^c} \mid X_{\partial\Lambda}. \tag{5c}$$

L'équivalence entre les deux dernières propriétés découle de la remarque 2 ; l'équivalence entre les deux premières est le théoreme 2.10 dans [Lau20].

Définition 10. *Une mesure de probabilité μ sur $(\Omega, \mathcal{F}_V)$ est appelée un champ aléatoire markovien local (abrégé LMRF pour l'anglais « local Markov random field ») sur $G = (V, E)$ (ou on dit que μ est markovienne) si elle satisfait* (EF) *et l'une des propriétés équivalentes de* (5) *pour toute boîte finie $\Lambda \Subset V$.*

On peut définir de manière similaire une notion de LMRF d'épaisseur $r \geq 1$ (ou distributions $r-$markoviennes). Les propriétés de Markov spatiales définies dans la section 0.1.2 s'étendent telles quelles aux graphes infinis. La propriété de Markov locale (LM) est la même expression que (5c), mais exigée seulement pour Λ réduit à un singleton. Être un LMRF est donc a priori plus fort que satisfaire (LM). Cependant, nous verrons que c'est une équivalence si on suppose en plus la propriété d'énergie finie (EF).

Il découle de la définition qu'une mesure de Gibbs pour un potentiel à plus proches voisins sur G est un LMRF sur G. La réciproque est aussi vraie, et nous obtenons une version en volume infini du théorème 4 (voir [CH71], [Ave72], [Spi71]) :

Théorème 11 (Représentation gibbsienne des LMRF). *Soit μ une mesure de probabilité sur $(\Omega, \mathcal{F}_V)$. Alors pour tout graphe $G = (V, E)$, μ est un champ aléatoire markovien local sur G si et seulement si $\mu \in \mathcal{G}(\pi^{\Phi,\beta})$ pour une certaine spécification gibbsienne $\pi^{\Phi,\beta}$ avec un potentiel plus proches voisins Φ sur G.*

On peut en déduire un énoncé similaire pour les LMRF d'épaisseur r et des potentiels à portée $r(\Phi) \leq r$. Ainsi, sur un graphe donné, la portée $r(\Phi)$ d'un potentiel correspond à l'entier r minimal tel que les mesures de Gibbs dans $\mathcal{G}_\beta(\Phi)$ soient $r-$markoviennes. Le théorème de Kozlov-Sullivan donne en fait un résultat beaucoup plus général, puisqu'il caractérise les spécifications héritées d'un potentiel absolument sommable comme des spécifications quasilocales positives (voir [Geo11] théorème 2.30 et 2.32).

Maintenant, justifions que pour les champs aléatoires à énergie finie, il suffit de vérifier (5) pour $|\Lambda| = 1$ pour être un LMRF. Cela découle du théorème 1.33 dans [Geo11], qui affirme que pour les spécifications positives, la partie singletonne caractérise toute la spécification, mais donnons un esquisse de preuve basée sur la dynamique de Glauber. Supposons que π soit une spécification gibbsienne positive à plus proches voisins, qui satisfait la propriété de Markov locale, c'est-à-dire

$$\pi_i(\cdot \mid \eta) = \pi_i(\cdot \mid \eta') \text{ pour tous } \eta, \eta' \in \Omega \text{ tels que } \forall j \sim i, \eta_j = \eta'_j.$$

Soit $\Lambda \Subset V$ et $\eta^1, \eta^2 \in \Omega$ deux conditions au bord qui coïncident sur $\partial\Lambda$. La dynamique de Glauber sur Λ avec condition au bord η^k (pour $k = 1$ ou 2) est la chaîne de Markov sur S^Λ avec noyau de transition

$$Q^k(\sigma, \sigma^i[x]) = \frac{1}{|\Lambda|}\pi_i(\sigma^i[x] \mid \sigma_\Lambda \eta^k_{\Lambda^c}) \text{ pour tout } i \in \Lambda \text{ et } x \in S,$$

où $\sigma^i[x]$ est la configuration qui coïncide avec σ partout sauf en i, le spin en i étant x ; $Q^k(\sigma, \sigma') = 0$ si σ' n'est pas de la forme $\sigma^i[x]$. Cette dynamique ne dépend que de la partie singletonne de la spécification, et par énergie finie, c'est une chaîne de Markov irréductible apériodique. Par conséquent, elle a une unique loi invariante, qui est par définition $\pi_\Lambda(\cdot \mid \eta^k)$ (par cohérence de la spécification). Mais d'après la propriété de Markov locale, on a en fait $Q^1 = Q^2$, donc $\pi_\Lambda(\cdot \mid \eta^1) = \pi_\Lambda(\cdot \mid \eta^2)$, de sorte que (5) est vérifiée pour Λ.

On conclue cette section par un commentaire sur la propriété de Markov globale (GM). Elle correspond à avoir (5) pour tout $\Lambda \subset V$, pas nécessairement fini. Il est intéressant d'observer qu'une mesure de Gibbs provenant d'un potentiel à plus proches voisins ne satisfait pas toujours (GM). Le contre-exemple suivant, dans le contexte du modèle d'Ising, a été trouvé par Goldstein [Gol80], dans un article où il montre aussi que μ^+_β et μ^-_β satisfont la propriété de Markov globale. Considérons une version décalée de la condition au bord de Dobrushin inversée :

$$\eta^{-\,\mathrm{Dob}\,+1}(i) = \begin{cases} -1 & \text{si } i = (i_1, \ldots, i_d) \in \mathbf{Z}^d \text{ avec } i_d \geq 1 \\ +1 & \text{sinon} \end{cases}$$

et $\mu_\beta^{-\mathrm{Dob}+1}$ la mesure limite faible en volume infini associée. On a alors que le mélange $\mu = \frac{1}{2}\mu_\beta^{\mathrm{Dob}} + \frac{1}{2}\mu_\beta^{-\mathrm{Dob}+1}$ ne satisfait pas la propriété de Markov globale (pour β assez grand en dimension $d \geq 3$). En effet, on peut remarquer que (5) échoue pour la boîte infinie $\Lambda = \{i_d > 0\}$, puisque μ_β^{Dob} et $\mu_\beta^{-\mathrm{Dob}+1}$ coïncident sur $\mathcal{F}_{\partial\Lambda} = \mathcal{F}_{\{i_d=0\}}$ mais sont mutuellement singulières sur $\mathcal{F}_{\Lambda^c}$; révéler la configuration sur Λ^c détermine donc si le système est en phase

μ_β^{Dob} ou $\mu_\beta^{-\mathrm{Dob}+1}$. Ici, μ n'est pas une mesure de Gibbs extrémale, mais Israel [Isr86] a également construit des mesures extrémales pour lesquelles la propriété de Markov globale n'est pas vérifiée (pas pour le modèle d'Ising, mais pour un modèle ferromagnétique avec des interactions à trois spins).

0.3.4 Au-delà du cas compact : le champ libre gaussien

Dans les paragraphes précédents, nous avons défini des mesures de Gibbs en volume infini pour des espaces d'état S finis. On peut chercher à aller plus loin et considérer des ensembles discrets infinis S, ou des spins continus non bornés. Comme mentionné précédemment, lorsque S est compact, tout se passe presque comme dans le cas fini. En particulier, l'existence d'une mesure de Gibbs en volume infini découle de la compacité pour les potentiels absolument sommables, et le théorème 9 est valable.

Le formalisme DLR peut être adapté naturellement pour des espaces d'état de spins plus généraux, mais une *mesure de référence* ou *mesure a priori* $\sigma-$finie λ sur S est nécessaire. Nous considérerons principalement des spins à valeurs réelles, bien que $S = \mathbf{S}^{d-1}$ la sphère unité de $\mathbf{R}^d$ soit aussi un cas largement étudiée dans la littérature. Pour ces deux exemples, un choix naturel est la mesure de Lebesgue (λ n'est pas supposée être une mesure de probabilité !). Pour les espaces d'état de spins dénombrables, on utilise la mesure de comptage comme mesure de référence, comme nous l'avons fait implicitement pour S fini (voir note de bas de page 5).

Pour tout $\Lambda \subset V$, on écrit λ^Λ pour la mesure produit sur S^Λ, habituellement notée $\otimes_{i\in\Lambda}\lambda$. Les spécifications gibbsiennes sont alors définies de manière similaire à la section 0.3.2 : pour tout $\Lambda \Subset V$, $A \in \mathcal{F}_V$ et $\eta \in \Omega$,

$$\begin{aligned}\pi_\Lambda^{\Phi,\beta}(A \mid \eta) &:= \frac{1}{Z_{\beta,\Lambda}^\eta}\int_{\Omega_\Lambda^\eta} \mathbf{1}_{\{\omega\in A\}}\exp(-\beta H_\Lambda(\omega))\lambda^V(\mathrm{d}\omega)\\ &= \frac{1}{Z_{\beta,\Lambda}^\eta}\int_{S^\Lambda} \mathbf{1}_{\{\omega_\Lambda\eta_{\Lambda^c}\in A\}}\exp(-\beta H_\Lambda(\omega_\Lambda\eta_{\Lambda^c}))\lambda^\Lambda(\mathrm{d}\omega_\Lambda)\\ &= \int_{S^\Lambda}\mathbf{1}_{\{\omega_\Lambda\eta_{\Lambda^c}\in A\}}\pi_\Lambda^{\Phi,\beta}(\mathrm{d}\omega_\Lambda \mid \eta)\end{aligned}$$

où la dernière égalité inclut un abus de notation pratique.

Quand on travaille avec des espaces de spins non compacts, l'existence de mesures en volume infini n'est plus garantie, même pour des modèles très raisonnables, comme nous allons le voir avec le *champ libre gaussien*. Le modèle est défini sur le réseau $d-$dimensionnel $\mathbf{Z}^d$, l'espace d'état des spins étant $S = \mathbf{R}$ et l'ensemble des configurations $\Omega = \mathbf{R}^{\mathbf{Z}^d}$ muni de $\mathcal{F} := \mathcal{F}_{\mathbf{Z}^d}$ la tribu produit, engendrée par les cylindres. Le hamiltonien du champ libre gaussien (abrégé GFF, pour l'anglais « Gaussian Free Field ») est donné pour tous $\Lambda \Subset \mathbf{Z}^d$ et $\omega \in \mathbf{R}^{\mathbf{Z}^d}$ par

$$H_{\Lambda,m}(\omega) = \frac{1}{4d}\sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset\\ i\sim j}}(\omega_i-\omega_j)^2 + \frac{m^2}{2}\sum_{i\in\Lambda}\omega_i^2$$

où $m \geq 0$ est la *masse* du modèle. Il n'y a que des interactions par paires, donc cela correspond à un potentiel à plus proches voisins, mais qui n'est pas absolument sommable du fait que les ω_i sont non bornés. Puisque les spins sont à valeurs réelles, à renormalisation près par $\sqrt{\beta}$, on

peut éliminer le paramètre β (ou toujours prendre $\beta = 1$). La mesure de Gibbs pour le GFF dans Λ avec condition au bord $\eta \in \mathbf{R}^{\mathbf{Z}^d}$ est donc la distribution sur $(\Omega, \mathcal{F})$ définie par

$$\forall A \in \mathcal{F},\ \mu^{\eta}_{m,\Lambda}(A) = \frac{1}{Z^{\eta}_{m,\beta}} \int_{\mathbf{R}^{\Lambda}} \mathbf{1}_{\{\omega_\Lambda \eta_{\Lambda^c} \in A\}} \exp(-H_{\Lambda,m}(\omega_\Lambda \eta_{\Lambda^c}))\ \mathrm{d}\omega_\Lambda$$

où nous utilisons la mesure de Lebesgue comme mesure de référence, $\mathrm{d}\omega_\Lambda$ désignant le produit $\otimes_{i\in\Lambda}\mathrm{d}\omega_i$. L'ensemble des lois sur $(\Omega, \mathcal{F})$ satisfaisant les équations (DLR) pour le GFF avec masse $m \geq 0$ est noté $\mathcal{G}_m$.

Avant d'étudier les mesures de Gibbs en volume infini du GFF, justifions sa désignation comme « gaussien ». On réécrit le hamiltonien comme

$$H_{\Lambda,m}(\omega) = -\frac{1}{2d} \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\sim j}} \omega_i\omega_j + \frac{1+m^2}{2} \sum_{i\in\Lambda} \omega_i^2$$

et on remarque que si l'on considère la condition au bord nulle $\eta \equiv 0$, on a

$$H_{\Lambda,m}(\omega_\Lambda 0_{\Lambda^c}) = \frac{1+m^2}{2} \sum_{i,j\in\Lambda} \omega_i(\delta_{i,j} - P(i,j))\omega_j$$

où $\delta_{i,j}$ est le symbole de Kronecker et $P(i,j) = \begin{cases} \frac{1}{1+m^2}\frac{1}{2d} & \text{si } i \sim j \\ 0 & \text{sinon} \end{cases}$. D'après la formule de la densité des vecteurs gaussiens (voir (2.1) dans le chapitre 2), il résulte que $\mu^0_{m,\Lambda}$ est une mesure gaussienne de moyenne 0 et de covariance donnée par l'inverse de la matrice $(1+m^2)(I_\Lambda - P_\Lambda)$ indexée par Λ (I_Λ est l'identité et les éléments de P_Λ sont les $P(i,j)$ pour $i,j \in \Lambda$ définis ci-dessus). Un calcul similaire peut être fait pour une condition au bord arbitraire. On trouve le même noyau de covariance mais la moyenne varie selon la condition au bord.

Le noyau de covariance du GFF en volume fini admet une interprétation élégante en termes de fonction de Green d'une marche aléatoire. Observons que pour $m = 0$, $(P(i,j))_{i,j\in\mathbf{Z}^d}$ est le noyau de transition de la marche aléatoire simple sur $\mathbf{Z}^d$. Pour une masse positive $m > 0$, ce noyau est défectif. On ajoute alors un nouveau site ♣ à $\mathbf{Z}^d$, appelé le cimetière, et on étend la définition de P pour en faire un noyau de transition sur $\mathbf{Z}^d \cup \{\clubsuit\}$:

$$P(i,j) = \begin{cases} \frac{1}{1+m^2}\frac{1}{2d} & \text{si } i \sim j, i,j \in \mathbf{Z}^d \\ 1 - \frac{1}{1+m^2} & \text{si } i \in \mathbf{Z}^d, j = \clubsuit \\ 1 & \text{si } i = j = \clubsuit \\ 0 & \text{sinon} \end{cases}.$$

Soit $Z = (Z_n)_{n\geq 0}$ la chaîne de Markov sur $\mathbf{Z}^d \cup \{\clubsuit\}$ avec noyau de transition P ; Z est appelée la *marche aléatoire tuée* à taux $1 - \frac{1}{1+m^2}$ sur $\mathbf{Z}^d$. On note $\mathbf{P}_i$ la loi de Z partant de i, et $\mathbf{E}_i$ l'espérance correspondante.

Alors, la matrice inverse de $(I_\Lambda - P_\Lambda)$ est simplement donnée par la série de Neumann :

$$(I_\Lambda - P_\Lambda)^{-1} = \sum_{n\geq 0} P^n_\Lambda.$$

Il s'ensuit que la covariance de $\mu^0_{m,\Lambda}$ (et c'est la même pour $\mu^\eta_{m,\Lambda}$ pour tout η) vaut

$$\mu^0_{m,\Lambda}(X_iX_j) = \frac{1}{1+m^2}\mathbf{E}_i\left[\sum_{n=0}^{\tau_\Lambda-1}\mathbf{1}_{\{Z_n=j\}}\right] =: G_\Lambda(i,j) \tag{6}$$

pour tous $i,j\in\Lambda$, où $\tau_\Lambda = \inf\{n\geq 0 \mid Z_n\notin\Lambda\}$. Ainsi, G_Λ est ($(1+m^2)^{-1}$ fois) la fonction de Green de la marche aléatoire simple tuée à taux $1-\frac{1}{1+m^2}$ et au bord de Λ.

Quand $m>0$, la limite dans (6) quand $\Lambda\uparrow\mathbf{Z}^d$ est bien définie (parce que la marche est tuée en temps géométrique, qui est $\mathbf{P}_i-$p.s. fini) et on peut poser

$$G_m(i,j) = \frac{1}{1+m^2}\mathbf{E}_i\left[\sum_{n\geq 0}\mathbf{1}_{\{Z_n=j\}}\right],\ \forall i,j\in\mathbf{Z}^d.$$

On peut alors montrer que le champ gaussien[7] sur $\mathbf{Z}^d$ de moyenne 0 et de covariance G_m, dont la distribution est notée $\mathcal{N}(0,G_m)$, satisfait les équations (DLR) du GFF avec masse m, donc $\mathcal{N}(0,G_m)$ appartient à $\mathcal{G}_m$. On peut aussi montrer que cette mesure est extrémale.

Remarque 12. *En effectuant les calculs avec une condition au bord arbitraire η, on trouve que $\mathcal{N}(u,G_m)\in\operatorname{ex}\mathcal{G}_m$ pour toute fonction $m-$harmonique $u\in\Omega$ (c'est-à-dire satisfaisant $u_i = \frac{1}{1+m^2}\frac{1}{2d}\sum_{j\sim i}u_j$ pour tout $i\in\mathbf{Z}^d$, i.e. $(m^2-\frac{1}{2d}\Delta)u=0$, avec Δ le laplacien discret sur $\mathbf{Z}^d$), donnant lieu à une infinité indénombrable de mesures de Gibbs extrémales.*

Pour le cas sans masse, $m=0$, en dimension $d\geq 3$ la situation est identique. En fait, la limite dans (6) quand $\Lambda\uparrow\mathbf{Z}^d$ est encore bien définie parce que la marche aléatoire simple sur $\mathbf{Z}^d$, $d\geq 3$, est transiente. En dimension 1 ou 2 cependant, la marche aléatoire simple est récurrente et en particulier la variance d'un seul spin diverge :

$$\mathrm{Var}^\eta_{0,\Lambda}(X_0) = G_\Lambda(0,0) \xrightarrow[\Lambda\uparrow\mathbf{Z}^d]{} \sum_{n\geq 0}\mathbf{P}_0(Z_n=0) = +\infty.$$

Cela empêche l'existence d'une mesure de Gibbs en volume infini, conduisant à notre premier exemple d'un modèle pour lequel $\mathcal{G}_0=\emptyset$, bien que le hamiltonien soit à plus proches voisins.

Le chapitre 3 sera consacré à l'étude du GFF avec très grande masse, paramétré d'une façon plus appropriée. Soit $\varepsilon\in(-1,1)$. On considère un champ gaussien centré $(X_i)_{i\in\mathbf{Z}^d}$ sur $\mathbf{Z}^d$, dont la distribution μ_ε est telle que la loi conditionnelle de X_i sachant le reste de la configuration soit gaussienne de variance 1 et de moyenne $\varepsilon\times$ (moyenne des voisins) :

$$\mu_\varepsilon(X_i\in\cdot \mid X_j=\xi_j,\forall j\neq i) = \mathcal{N}\Big(\frac{\varepsilon}{2d}\sum_{j\sim i}\xi_j, 1\Big),\ \text{pour } \mu_\varepsilon-\text{p.s. } \xi\in\mathbf{R}^{\mathbf{Z}^d}. \tag{7}$$

Cela ne décrit que la partie singletonne de la spécification de μ_ε. Comme mentionné plus haut, c'est suffisant pour caractériser la spécification ([Geo11] théorème 1.33) et ici, la formule de la densité gaussienne nous permet d'identifier à partir de (7) quel est le hamiltonien du modèle :

$$\mu_\varepsilon(X_i\in\mathrm{d}x \mid X_j=\xi_j,\forall j\neq i) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\Big(x-\frac{\varepsilon}{2d}\sum_{j\sim i}\xi_j\Big)^2\right)\mathrm{d}x = \frac{1}{Z_i^\xi}\exp[-H_i(x_i\xi_{\{i\}^c})]\mathrm{d}x$$

[7] On reverra proprement la définition des champs gaussiens sur des graphes infinis dans le chapitre 2.

où le hamiltonien ponctuel est donné par $H_i(x) = \frac{1}{2}x_i^2 - \frac{\varepsilon}{2d}\sum_{j\sim i} x_i x_j$ pour tout $x \in \mathbf{R}^{\mathbf{Z}^d}$. Cela détermine de manière unique le potentiel, et le hamiltonien sur toute boîte finie $\Lambda \Subset \mathbf{Z}^d$ associé à μ_ε est donc donné par

$$H_\Lambda(x) = \frac{1}{2}\sum_{i\in\Lambda} x_i^2 - \frac{\varepsilon}{2d}\sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\sim j}} x_i x_j,\ x \in \mathbf{R}^{\mathbf{Z}^d}.$$

Pour $\varepsilon > 0$, avec la renormalisation $\omega_i = \sqrt{\varepsilon}x_i$, nous retrouvons le hamiltonien du GFF avec masse m satisfaisant $(1+m^2)^{-1} = \varepsilon$; pour inclure le cas $\varepsilon < 0$, on a besoin de considérer le GFF avec température négative (i.e. $\beta = -1$) et masse satisfaisant $(1+m^2)^{-1} = |\varepsilon|$.

On montre que pour $|\varepsilon|$ assez petit, la restriction de μ_ε à des boîtes finies est exponentiellement proche d'un processus à dépendance finie (voir définition 2.4). Plus précisément, pour tout $l \in \mathbf{N}$, il existe un champ aléatoire $l-$dépendant $(Y_i^l)_{i\in\mathbf{Z}^d}$ sur $(\Omega, \mathcal{F})$ tel que

$$\forall \Lambda \Subset \mathbf{Z}^d, d_{\mathrm{TV}}(X|_\Lambda, Y^l|_\Lambda) \leq |\Lambda| e^{-cl}$$

où $c > 0$ est une constante universelle. C'est le résultat principal du chapitre 3 (voir théorème 3.3).

Il est obtenu en implémentant un argument de couplage-depuis-le-passé, considérant la dynamique de Glauber associée à μ_ε. Il fournit aussi une construction explicite du fait que μ_ε peut être exprimé comme un facteur d'une mesure produit.

0.4 Facteurs de mesures produits

Cette section traite de l'existence de *facteurs* depuis des processus i.i.d. (indépendants et identiquement distribués) vers des champs aléatoires stationnaires (les définitions précises seront données ci-dessous). Pour simplifier, nous considérons des champs aléatoires sur $\mathbf{Z}^d$ pour $d \geq 1$, bien que la théorie s'étende naturellement à des graphes plus généraux (essentiellement, la plupart des résultats énoncés ici seraient valides pour tout graphe transitif moyennable). À l'origine, l'intérêt pour ces facteurs d'i.i.d. provient de la théorie ergodique. Une classe de processus particulièrement étudiés en théorie ergodique sont les *décalages de Bernoulli*, c'est-à-dire les champs aléatoires (initialement en dimension $d = 1$) qui sont isomorphes à un processus i.i.d. ; Ornstein a prouvé dans [Orn74] le théorème d'isomorphisme pour les décalages de Bernoulli, qui établit que deux processus i.i.d. sur $\mathbf{Z}$ d'entropie égale sont en fait isomorphes ; il a de plus montré que les facteurs de processus i.i.d. sont des décalages de Bernoulli. Ces résultats s'étendent aux dimensions supérieures (voir [KW72] et [OW87]).

0.4.1 Facteurs d'i.i.d. et rayon de codage

Soit $(\mathcal{X}, \mathcal{F})$ un espace mesurable, et μ un champ aléatoire stationnaire avec espace d'états $\mathcal{X}$, c'est-à-dire une mesure de probabilité invariante par translation sur $\mathcal{X}^{\mathbf{Z}^d}$, muni de la tribu produit. On dit que μ est un *facteur d'une mesure produit* ou *facteur d'un processus i.i.d.* (abrégé FIID dans la suite) s'il existe un espace de probabilité $(\mathcal{Y}, \mathcal{G}, \nu)$ et une application mesurable $F : \mathcal{Y}^{\mathbf{Z}^d} \to \mathcal{X}^{\mathbf{Z}^d}$ qui commute avec toutes les translations de $\mathbf{Z}^d$ telle que μ soit l'image de $\nu^{\otimes\mathbf{Z}^d}$ par F. Cela revient à dire qu'il existe un processus i.i.d. à valeurs dans $\mathcal{Y}$

$Y = (Y_i)_{i \in \mathbf{Z}^d}$ tel que $F(Y)$ ait pour distribution μ. L'application F est appelée un facteur ou *codage* de $\nu^{\otimes \mathbf{Z}^d}$ vers μ.

Du point de vue de la mécanique statistique, la notion de FIID n'est pas satisfaisante car elle ne détecte pas les transitions de phase. En effet, il est connu que l'« état + » du modèle d'Ising, c'est-à-dire la mesure μ_β^+ définie dans la section 0.3.1, est un FIID pour toute température inverse $\beta \geq 0$ (prouvé d'abord par Ornstein et Weiss dans un manuscrit non publié, puis par Adams [Ada92]). Van den Berg et Steif [BS99] on trouvé ensuite un moyen de détecter les transitions de phase à travers la notion de codages *finitaires*. Le *rayon de codage* de l'application F est défini comme le rayon aléatoire autour de l'origine qu'il faut révéler dans le processus i.i.d. pour déterminer le spin en 0. Plus précisément, définissons le rayon de codage de F au site $i \in \mathbf{Z}^d$ et en la configuration $y \in \mathcal{Y}^{\mathbf{Z}^d}$ comme

$$R_i(y) := \inf\{r \geq 0 : \forall y' \in \mathcal{Y}^{\mathbf{Z}^d}, y'|_{B_r^i} = y|_{B_r^i} \implies F(y)_i = F(y')_i\}$$

où $B_r^i := \{j \in \mathbf{Z}^d : ||j - i||_1 \leq r\}$ est la boule de rayon r autour de i dans $\mathbf{Z}^d$, et $y|_{B_r^i}$ est la restriction de y à B_r^i. Quand Y est défini comme ci-dessus, les $R_i(Y)$ pour $i \in \mathbf{Z}^d$ sont des variables aléatoires de même loi, par invariance par translation. On pose $R := R_0(Y)$ (le choix de 0 est fait arbitrairement, par invariance par translation de μ) ; R est appelé le rayon de codage de F. Si R est presque sûrement fini, le codage est dit *finitaire*. Dans ce cas, nous disons que μ est un *facteur finitaire* d'un processus i.i.d., abrégé FFIID. De plus, quand le rayon de codage satisfait $\mathbf{P}(R \geq n) \leq Ce^{-cn}$ pour certaines constantes $c, C > 0$ et tout $n \in \mathbf{N}$, on dit que μ est un FFIID avec *queue exponentielle*. Le résultat principal de [BS99] est que la présence d'une transition de phase est « une obstruction fondamentale à l'existence d'un codage finitaire » ; en d'autres termes, s'il existe une mesure différente de μ avec la même spécification, cela empêche μ d'être un FFIID. Cela montre que l'état + du modèle d'Ising n'est *pas* FFIID si $\beta > \beta_c$, malgré le fait qu'il soit un décalage de Bernoulli. Il est aussi prouvé dans [BS99] que dans le cas d'unicité, la mesure de Gibbs du modèle d'Ising est un FFIID, avec queue exponentielle si $\beta < \beta_c$: en revanche, à $\beta = \beta_c$, le moment d'ordre d du rayon de codage est nécessairement infini (ce dernier résultat a été prouvé conjointement avec Peres). Notons que l'existence d'un codage finitaire au point critique découle de la continuité de la transition de phase pour le modèle d'Ising ([Yan52] [AF86] [ADCS15]).

En fait, van den Berg et Steif s'intéressaient principalement aux FFIID *à valeurs finies* (abrégé FV-FFIID), c'est-à-dire qu'ils demandent en plus que l'ensemble $\mathcal{Y}$ de valeurs pour le processus i.i.d. soit fini. En fait, ils ont montré que le modèle d'Ising est un FV-FFIID dans toute la phase sous-critique, sans information sur le rayon de codage (sauf qu'il est p.s. fini). Spinka [Spi20a] a amélioré ce résultat en prouvant que le codage peut être construit de sorte que $\mathbf{E}[R^d]$ soit fini (en fait R a une queue « stretched exponentielle »), et Timár a prouvé dans [Tim23] que le modèle d'Ising critique est un FV-FFIID (seul FFIID peut être déduit de [BS99]). Ces deux articles répondent respectivement à la Question 2 et à la Question 1 de [BS99] ; un problème important (Question 3) qui reste ouvert est de déterminer si en général l'unicité de la mesure de Gibbs implique (au moins) FIID.

Les travaux cités ci-dessus repose sur des arguments de couplage par le passé, et notamment sur des couplages monotones (voir la remarque 6 dans la section 0.2.2) dans leurs constructions de codages finitaires. En particulier, cela nécessite l'inégalité FKG, qui n'est par exemple pas vérifiée pour le modèle d'Ising avec β négatif. Dans le paragraphe suivant, nous présentons une construction, originalement due à Häggström et Steif [HS00], d'un codage finitaire pour

le modèle d'Ising avec β proche de 0, mais possiblement négatif. En fait, les auteurs montrent l'existence d'un codage finitaire pour tous les champs aléatoires markovien qui satisfont une condition « *high-noise* ». C'est une condition sur les spécifications, qui, avec les notations de la section 0.3, équivaut à

$$\sum_{s\in S} \inf_{\eta\in S^{\mathbf{Z}^d}} \pi^{\Phi,\beta}_{\{i\}}(X_i = s \mid \eta) > 1 - \frac{1}{2d},\ i \in \mathbf{Z}^d. \tag{8}$$

Le membre de gauche est appelé *admissibilité multigamma* dans [HS00], suivant la terminologie introduite dans [MG98]. Cette quantité correspond à la *probabilité de couplage maximal* (voir [Tho00] chapitre 1.4) de la collection $(\pi^{\Phi,\beta}_{\{i\}}(X_i \in \cdot \mid \eta))_{\eta\in\Omega}$ de lois sur S (notons que la dépendance en i disparaît pour les spécifications invariantes par translation). L'inégalité des événements de couplage établit que dans tout couplage d'une collection de lois, la probabilité que les réalisations de chaque loi coïncident toutes dans le couplage est majorée par la probabilité de couplage maximal ; de plus, il existe un couplage, qualifié de maximal, où l'égalité a lieu (voir [Tho00] théorèmes 4.1 et 4.2, ou chapitre 3.7 pour des versions plus générales). Les couplages maximaux sont ceux qui favorisent le plus la coalescence, au sens de l'algorithme de Propp et Wilson.

Notons que (8) implique immédiatement la condition d'unicité obtenue par van den Berg et Maes dans [BM94], de sorte que nous sommes nécessairement dans le régime d'unicité de la mesure de Gibbs quand la condition high-noise est satisfaite. Pour le modèle d'Ising, cela correspond à prendre $|\beta| < \frac{1}{4d}\log\left(1 + \frac{2}{2d-1}\right)$. Plus généralement, pour les modèles avec énergie finie, l'hypothèse high-noise est satisfaite quand $|\beta|$ est assez petit (l'admissibilité multigamma vaut 1 pour $\beta = 0$, et est en général continue en β).

0.4.2 Codage finitaire pour le modèle d'Ising avec $|\beta|$ petit

La première étape pour construire un codage finitaire pour le modèle d'Ising sur $\mathbf{Z}^d$ est d'étendre la dynamique de Glauber définie dans la section 0.2.1 en volume infini. Cela fournira un algorithme qui termine presque sûrement quand $|\beta|$ est assez petit, et qui produit un échantillon exact de la distribution $\mu_\beta := \mu^+_\beta = \mu^-_\beta$, la mesure de Gibbs (unique) pour le modèle d'Ising sur $\mathbf{Z}^d$. La preuve du fait que la valeur retournée par l'algorithme est un échantillon non biaisé de μ_β repose sur un argument de couplage par le passé, également étendu au volume infini.

Dynamique de Glauber en temps continu pour le modèle d'Ising sur $\mathbf{Z}^d$. Nous gardons l'idée centrale de la dynamique de Glauber, qui consiste à rééchantillonner les spins selon leur loi conditionnelle sachant les voisins. Cependant, le site mis à jour ne peut pas être choisi uniformément sur $\mathbf{Z}^d$. On attache plutôt une « horloge exponentielle » à chaque site de $\mathbf{Z}^d$, et quand l'horloge au site i sonne, nous mettons à jour son spin conditionnellement à la configuration en dehors de i à cet instant. Formellement, la dynamique de Glauber est un système de particules en interaction (voir [Lig85]) sur $\{\pm 1\}^{\mathbf{Z}^d}$ ayant pour distribution stationnaire μ_β. Ci-dessous, nous décrivons sa *représentation graphique* (voir [Lig85] chapitre III.6 ou [Dur88] section 2)

Soit $\mathcal{P}$ un processus ponctuel de Poisson (PPP) d'intensité $\# \otimes \mathrm{d}t \otimes \mathrm{d}u$, où $\#$, $\mathrm{d}t$, $\mathrm{d}u$ désignent respectivement la mesure de comptage sur $\mathbf{Z}^d$, la mesure de Lebesgue sur $\mathbf{R}$ et la

mesure de Lebesgue sur $[0,1]$. Les points de $\mathcal{P}$ sont appelés les *marques de mise à jour* de la dynamique. Par propriétés des PPP, cela signifie que les sites de $\mathbf{Z}^d$ sont indépendamment dotés d'une suite bi-infinie $(T_n^\cdot, U_n^\cdot)_{n\in\mathbf{Z}}$, satisfaisant

$$\mathcal{P} = \{(i, T_n^i, U_n^i), n \in \mathbf{Z}, i \in \mathbf{Z}^d\},$$

et telle que $(T_n^\cdot)_{n\in\mathbf{Z}}$ soit croissante. Cette observation montre que $\mathcal{P}$ est distribuée selon une mesure produit sur $\mathbf{Z}^d$; on construira un facteur de cette mesure produit vers μ_β. Pour chaque $i \in \mathbf{Z}^d$, les $(T_n^i)_{n\in\mathbf{Z}}$ correspondent aux temps où le site i est mis à jour dans la dynamique, et les incréments $(T_{n+1}^i - T_n^i)_{n\in\mathbf{Z}}$ sont indépendants et ont une distribution exponentielle de paramètre 1 (d'où la terminologie « horloge exponentielle »). Les $(U_n^i)_{i\in\mathbf{Z}^d, n\in\mathbf{Z}}$ sont des variables aléatoires indépendantes distribuées uniformément sur $[0,1]$; la mise à jour de i au temps T_n^i est faite avec l'aide de l'aléa supplémentaire fourni par U_n^i. On définit la *fonction de mise à jour* $\varphi : \mathbf{R} \times [0,1] \to \{\pm 1\}$ (la même que dans la remarque 6) par

$$\varphi(s,u) = \begin{cases} -1 & \text{si } u \le (1+e^{2\beta s})^{-1} \\ +1 & \text{sinon} \end{cases}. \tag{9}$$

Pour chaque marque de mise à jour $(i,t,u) \in \mathcal{P}$, si le système se trouve dans la configuration σ_{t-} avant la mise à jour, alors la nouvelle valeur du spin en i est donnée par $\varphi(\sum_{j\sim i} \sigma_{t-}(j), u)$. Les autres spins restent inchangés, autrement dit on a

$$\forall j \neq i, \sigma_t(j) = \sigma_{t-}(j) \text{ et } \sigma_t(i) = \varphi\left(\sum_{j\sim i} \sigma_{t-}(j), u\right).$$

Notons que nous avons défini l'évolution du système sur une ligne de temps qui est infinie dans les deux directions. Si β est assez petit, le théorème ergodique pour le modèle d'Ising stochastique [Hol72] implique que tous les systèmes convergent vers la distribution stationnaire μ_β. Néanmoins, puisqu'il y a une infinité de sites, on ne peut pas définir la première mise à jour après un temps initial donné $\tau \in \mathbf{R}$. En d'autres termes, pour chaque $\tau \in \mathbf{R}$,

$$\inf\{t \ge \tau : \exists i \in \mathbf{Z}^d, \exists u \in [0,1], (i,t,u) \in \mathcal{P}\} = \tau.$$

Un argument supplémentaire est donc nécessaire pour s'assurer que la dynamique peut effectivement être construite avec la représentation graphique. L'écueil est que la première mise à jour d'un site donné (qui est bien définie) pourrait nécessiter de regarder infiniment loin. En effet, pour chaque mise à jour on doit regarder les voisins, mais il se peut que ces voisins aient eu eux-mêmes besoin de regarder leurs propres voisins avant etc. Un argument de percolation de Harris [Har72] (voir aussi [Dur88] section 2) résout ce problème. L'idée est de prendre un sous-graphe aléatoire de $\mathbf{Z}^d$ défini comme suit : fixons un petit nombre positif t_0 ; pour chaque $i, j \in \mathbf{Z}^d$ tel que $i \sim j$, disons que l'arête $\{i,j\}$ est ouverte s'il existe $n \in \mathbf{Z}$ tel que $\tau < T_n^i < \tau + t_0$ ou $\tau < T_n^j < \tau + t_0$; sinon, l'arête est fermée. Ainsi, une arête ouverte correspond au fait qu'une mise à jour du système nécessite que i regarde j, ou l'inverse. Par conséquent, si i et j ne sont pas dans la même composante connexe dans notre graphe aléatoire, leurs spins au temps $\tau + t_0$ sont indépendants (en supposant qu'ils ont été choisis indépendamment au temps initial τ). Ainsi, on peut calculer séparément l'évolution jusqu'au temps $\tau + t_0$ dans chaque composante connexe. Le résultat de Harris est que pour t_0 assez petit, il n'y a pas de composante connexe infinie pour presque toute réalisation du PPP. Ainsi, chaque composante connexe a un premier temps de mise à jour, donc on peut construire le système jusqu'au temps $\tau + t_0$. On itère l'argument pour construire le système pour tout temps $t \ge \tau$.

Exploration vers le passé de la dynamique. Pour déterminer la valeur du spin en $(i,t) \in \mathbf{Z}^d \times \mathbf{R}$ lors d'une réalisation de la dynamique, on doit explorer la dynamique vers le passé à partir de (i,t). On remonte à la dernière mise à jour du site i avant le temps t, c'est-à-dire qu'on considère la marque de mise à jour (i, T_n^i, U_n^i) où $n = \max\{n \in \mathbf{Z} \mid T_n^i < t\}$. À première vue, la valeur produite par cette marque avec la fonction de mise à jour (voir (9)) dépend des valeurs des spins des voisins au temps T_n^i. Cependant, il se peut que cette marque produise la même valeur pour le nouveau spin en i quelle que soit la configuration autour. C'est le cas quand U_n^i est en dessous de la plus petite valeur possible pour le seuil $(1+e^{2\beta s})^{-1}$ dans (9), ou au-dessus de sa plus grande valeur possible. Le paramètre s représente la somme des spins des voisins dans la configuration courante, donc il ne prend des valeurs qu'entre $-2d$ et $2d$. Les valeurs possibles pour le seuil $(1+e^{2\beta s})^{-1}$ sont donc entre $(1+e^{4|\beta|d})^{-1}$ et $(1+e^{-4|\beta|d})^{-1}$. Ainsi, si $U_n^i \leq (1+e^{4|\beta|d})^{-1}$, la fonction de mise à jour renvoie nécessairement -1, donc on a $\sigma_t(i) = -1$; de même, si $U_n^i > (1+e^{-4|\beta|d})^{-1}$ on obtient $\sigma_t(i) = +1$. Notons que dans les deux cas, $\sigma_t(i)$ est déterminé seulement par U_n^i, et ne dépend pas de l'initialisation de la dynamique. Sinon, quand U_n^i est entre ces deux valeurs extrémales, la valeur retournée dépend vraiment de la configuration courante, donc on doit continuer l'exploration vers le passé pour déterminer le spin en (i,t). On répète alors cette procédure pour tous les voisins de i à partir du temps T_n^i. Dans le cas où pour tout $j \sim i$, la variable uniforme (c'est-à-dire la troisième coordonnée) de la dernière marque de mise à jour portée par j avant le temps T_n^i n'est pas entre $(1+e^{4|\beta|d})^{-1}$ et $(1+e^{-4|\beta|d})^{-1}$, alors tous les spins des voisins de i au temps T_n^i sont déterminés par l'exploration. En notant S leur somme, il suit qu'on peut calculer $\sigma_t(i) = \varphi(S, U_n^i)$. Encore une fois, dans le cas contraire, on continue l'exploration.

Soit

$$\gamma(\beta) := \frac{1}{1+e^{4|\beta|d}} + 1 - \frac{1}{1+e^{-4|\beta|d}} = 1 - \tanh(2d|\beta|).$$

Alors, $\gamma(\beta)$ correspond à la proportion de marques de mise à jour qui renvoient la même valeur quelle que soit la configuration courante ; cette quantité est égale à l'admissibilité multigamma de [HS00] (voir (8) ci-dessus). On déduit alors de la discussion précédente que le nombre de marques de mise à jour qu'on doit révéler dans l'exploration vers le passé est dominé par la population totale d'un processus de Galton-Watson de loi de reproduction $\gamma(\beta)\delta_0 + (1-\gamma(\beta))\delta_{2d}$. Quand ce processus s'éteint, c'est-à-dire quand la population totale est finie, l'exploration vers le passé de la dynamique termine en temps fini. Cela implique que la valeur du système en (i,t) est une fonction mesurable des marques de mise à jour appartenant à une boîte finie de $\mathbf{Z}^d \times (-\infty, t) \times [0,1]$. En particulier cette valeur devient indépendante de la configuration initiale de la dynamique. Quand $2d\tanh(2d|\beta|) < 1$, c'est-à-dire $|\beta| < \frac{1}{4d}\log\left(1+\frac{2}{2d-1}\right)$, le processus de Galton-Watson est sous-critique, donc la situation ci-dessus arrive pour chaque $(i,t) \in \mathbf{Z}^d \times \mathbf{R}$ pour presque toute réalisation du PPP.

Argument de couplage par le passé. La représentation graphique de la dynamique de Glauber fournit un couplage des systèmes partant de n'importe quel temps et n'importe quelle condition initiale, obtenu en utilisant le même processus ponctuel de Poisson pour les construire. Écrivons $\sigma^{\omega,\tau} = (\sigma_t^{\omega,\tau})_{t \geq \tau}$ pour le système démarré au temps $\tau \in \mathbf{R}$ avec configuration initiale $\sigma_\tau^{\omega,\tau} = \omega$, et dont l'évolution est régie par les marques de mise à jour de $\mathcal{P}$. On implémente maintenant les idées de Propp et Wilson pour obtenir une simulation exacte de la distribution stationnaire μ_β, apparaissant comme un facteur finitaire de $\mathcal{P}$.

Pour chaque $i \in \mathbf{Z}^d$, on pose

$$\tau_i = \sup\{\tau \leq 0 \mid \forall \omega, \omega' \in \{\pm 1\}^{\mathbf{Z}^d},\ \sigma_0^{\omega,\tau}(i) = \sigma_0^{\omega',\tau}(i)\},$$

c'est-à-dire, τ_i est le plus grand temps (négatif) tel que la valeur du spin en i au temps 0 soit la même dans toutes les dynamiques initialisées à ce temps depuis n'importe quelle configuration. Remarquons que τ_i est un temps aléatoire, mesurable par rapport à $\mathcal{P}$. La comparaison avec un processus de branchement sous-critique du paragraphe précédent affirme que τ_i est presque sûrement fini pour chaque site $i \in \mathbf{Z}^d$. On peut donc définir un champ aléatoire $\sigma^* = (\sigma_i^*)_{i \in \mathbf{Z}^d}$ par

$$\sigma_i^* = \sigma_0^{\cdot,\tau_i}(i),\ i \in \mathbf{Z}^d$$

(on peut considérer n'importe quelle configuration initiale pour la dynamique, puisque par définition de τ_i, la valeur de σ_i^* sera la même pour n'importe quel choix). Notons que σ^* est une fonction mesurable de $\mathcal{P}$, donc c'est un FIID. Ensuite, le fait que σ^* soit distribuée selon μ_β provient d'un argument classique de couplage par le passé, similaire à la preuve du théorème 1.5 : fixons $\Lambda \Subset \mathbf{Z}^d$, et ω distribuée selon μ_β, indépendante de $\mathcal{P}$; puisque μ_β est invariante pour la dynamique, il s'ensuit que pour tout $t < 0$, $\sigma_0^{\omega,t}$ a pour loi μ_β ; d'autre part, pour tout $t \leq \min_{i \in \Lambda} \tau_i$, on a $\sigma_0^{\omega,t}(i) = \sigma_i^*$ pour tout $i \in \Lambda$; ainsi, $\sigma^*|_\Lambda$ est distribuée selon $\mu_\beta|_\Lambda$; Λ était arbitraire donc le résultat annoncé s'ensuit.

Cette construction donne un codage pour μ_β à partir d'un processus i.i.d., pourvu que $|\beta|$ soit assez petit. Il est immédiat que ce codage est finitaire, et il n'est même pas difficile de voir qu'il a de plus une queue exponentielle. En fait, le rayon de codage est borné par le nombre de marques de mise à jour révélées pendant l'exploration vers le passé depuis $(0, 0)$; de la domination ci-dessus par un processus de branchement sous-critique, il s'ensuit qu'il a une décroissance exponentielle. En déduire un codage à partir d'un processus i.i.d. à valeurs finies nécessite une étape supplémentaire, d'abord effectuée dans [BS99]. Nous ne reproduisons pas cette étape ici, puisque notre objectif principal est ensuite d'étendre ces techniques à des champs gaussiens. Dans ce cas, l'espace d'états étant continu, il est impossible d'obtenir un codage à partir d'un processus à valeurs finies, car l'entropie ne peut pas augmenter via un codage. Dans le chapitre 3, nous construisons d'abord un codage finitaire pour un modèle gaussien conditionné à un compact, pour lequel la construction est très similaire à celle décrite ici rapidement pour le modèle d'Ising. La construction d'un codage pour notre modèle gaussien non conditionné (défini à la fin de la section 0.3.4) est plus compliquée. En effet, comme nous le verrons, l'admissibilité multigamma est dans ce cas réduite à zéro.

0.5 Modèles avec interactions à longue portée

Dans les sections précédentes, nous nous sommes principalement intéressés aux modèles de mécanique statistique avec des interactions à plus proches voisins (ou à portée finie). Ces modèles partagent la propriété agréable d'être markoviens. Cette dernière section de l'introduction est consacrée aux modèles à longue portée. Rappelons que la portée d'un potentiel sur un graphe a été introduite dans la définition 1. Prenons $\mathbf{Z}^d$ comme graphe sous-jacent, afin d'avoir une notion fixe de la portée, bien que les modèles que l'on va définir puissent être étudiés dans un cadre plus général. Un potentiel Φ, ou de manière équivalente le hamiltonien qui lui est ssocié, est dit *à longue portée* si $r(\Phi) = +\infty$. Cela signifie qu'il y a des interactions entre des spins à distance arbitrairement grande les uns des autres. Nous considérerons deux exemples

de modèles à longue portée. D'abord, nous étudierons le modèle d'Ising à longue portée, et montrerons en particulier que la construction ci-dessus d'un codage finitaire pour $|\beta|$ petit peut encore fonctionner, si l'on suppose que l'intensité des interactions décroît suffisamment vite. Ensuite, nous introduirons le modèle « random cluster », ou percolation FK, qui est un modèle de mécanique statistique largement étudié. Ce modèle sera l'objet central du chapitre 5.

0.5.1 Modèle d'Ising à longue portée

On considère un modèle d'Ising à longue portée sur $\mathbf{Z}^d$, $d \geq 1$. L'espace des configurations est $\Omega = \{-1, +1\}^{\mathbf{Z}^d}$ et les interactions sont données par des *constantes de couplage* $(J_{ij})_{i,j\in\mathbf{Z}^d}$ qui satisfont la condition de sommabilité suivante :

$$\forall i \in \mathbf{Z}^d, \sum_{j\in\mathbf{Z}^d} |J_{ij}| < \infty. \tag{10}$$

Ici, nous ne supposerons *pas* que les constantes de couplage sont positives, de sorte que le modèle n'est pas nécessairement ferromagnétique. Le hamiltonien du modèle est défini pour tout sous-ensemble fini Λ de $\mathbf{Z}^d$ et $\sigma \in \Omega$ comme

$$H_\Lambda(\sigma) = -\sum_{\{i,j\}\cap\Lambda\neq\emptyset} J_{ij}\sigma(i)\sigma(j).$$

L'hypothèse (10) assure que le potentiel correspondant est absolument sommable (voir (2)), donc le hamiltonien ci-dessus est bien défini. On peut alors considérer les mesures de Gibbs associées à une certaine température inverse β, comme nous l'avons fait de manière générale dans la section 0.3. Puisqu'on peut mettre un signe dans les constantes de couplage, on peut se restreindre au cas $\beta \geq 0$, sans que cela n'impose d'avoir le même signe pour toutes les interactions.

Soit $J : \mathbf{N} \to \mathbf{R}_+$ une fonction satisfaisant $J(n) = O(n^{-\alpha})$ pour un certain $\alpha > d$. On suppose que les constantes de couplage vérifient $|J_{ij}| \leq J(\|j-i\|_1)$ pour tous $i, j \in \mathbf{Z}^d$, de sorte que (10) est vérifiée. Ainsi, α est un paramètre qui contrôle la décroissance des interactions avec la distance.

Le premier résultat concernant la transition de phase pour le modèle d'Ising à longue portée a été obtenu en 1966 par Ginibre, Grossman et Ruelle [GGR66]. Ils généralisent l'argument de Peierls pour la transition de phase du modèle d'Ising à plus proches voisins, pour montrer que le modèle à longue portée admet plusieurs mesures de Gibbs à température suffisamment basse en dimension $d \geq 2$ quand $\alpha > d + 1$. Dans le cas ferromagnétique, c'est-à-dire $J_{ij} \geq 0$ pour tous i, j, les inégalités de Griffiths [Gri67] (aussi connues sous le nom d'inégalités GKS [KS68]) permettent de prouver une transition de phase à basse température dans le régime complet $\alpha > d$.

Pour le modèle d'Ising à longue portée unidimensionnel, l'unicité de la mesure de Gibbs a été prouvée pour toute température dans [Rue75] quand $\alpha > 2$, généralisant le résultat dans le cas à plus proches voisins de la thèse d'Ising. Kac et Thompson ont conjecturé dans [KT69] que le modèle présente une transition de phase pour $\alpha \in (1, 2]$; cela a été prouvé par Dyson [Dys69] pour $\alpha \in (1, 2)$ puis par Fröhlich et Spencer [FS82] dans le cas $\alpha = 2$.

Harel et Spinka [HS22] ont montré que le modèle d'Ising ferromagnétique à longue portée est FFIID dans tout le régime d'unicité, en toute dimension $d \geq 1$. Cela fournit un énoncé

similaire à celui de Van den Berg et Steif [BS99] pour le modèle d'Ising à portée finie. Sans contrainte sur le signe des constantes de couplage, leur technique, qui repose fortement sur la monotonie, ne peut pas s'appliquer. Comme on l'a vu, dans [HS00], Häggström et Steif exploitent la propriété de Markov spatiale pour construire des codages finitaires (en particulier pour le modèle d'Ising à portée finie avec $|\beta|$ petit). Cela échoue pour le modèle d'Ising à longue portée car il y a des interactions à distance arbitrairement grande. En fait, on pourrait essayer d'imiter la construction esquissée dans la section 0.4.2, mais on se retrouverait avec une exploration vers le passé dominée par un processus de Galton-Watson de loi de reproduction « $\gamma\delta_0 + (1-\gamma)\delta_{+\infty}$ » (γ étant l'admissibilité multigamma, qui peut être définie de la même manière). En effet, à chaque mise à jour où on échoue à coalescer directement, on doit regarder la valeur du spin en tous les autres sites. Cependant, l'influence d'un spin sur la mise à jour d'un site donné décroît avec la distance. L'idée pour obtenir un codage finitaire est donc de révéler la configuration sur une boule aléatoire autour du site mis à jour, de rayon suffisant pour calculer la nouvelle valeur du spin. Alors, avec une version modifiée de l'hypothèse high-noise (8), cela prouvera que le modèle d'Ising à longue portée est FFIID, sans supposer qu'il est ferromagnétique. Cela a été d'abord implémenté par Galves, Löcherbach, et Orlandi dans [GLO10] (incluant des interactions plus générales que simplement par paires).

Le résultat suivant, pour lequel nous avons trouvé une preuve indépendamment, est un cas particulier du théorème 1 de [GLO10]. Une esquisse de preuve est donnée dans l'annexe de la thèse. Elle suit le même plan que la section 0.4.2, mais contient probablement trop de technicités pour cette introduction.

Théorème 13. *Le modèle d'Ising à longue portée avec $\alpha > 2d$ est FFIID si β est suffisamment petit.*

En particulier, dans l'énoncé ci-dessus, β est pris assez petit pour qu'il y ait unicité de la mesure de Gibbs (rappelons que c'est nécessaire pour avoir des codages finitaires) ; on peut toujours le faire, pour toute dimension $d \geq 1$ et $\alpha > d$. Notons cependant qu'on ne prétend pas que cet énoncé est vrai jusqu'au véritable paramètre critique, au-dessus duquel il y a une transition de phase. Notons également que l'on ne sait pas construire un codage finitaire pour le modèle d'Ising à longue portée lorsque $\alpha \in (d, 2d]$.

0.5.2 Percolation FK

Le modèle *« random cluster »* a été introduit par Fortuin et Kasteleyn en 1972 [FK72], dans le but d'unifier plusieurs modèles de mécanique statistique d'intérêt notable. C'est un modèle de *percolation*, dans le sens où il produit un sous-graphe aléatoire d'un graphe donné, ainsi il est souvent nommé percolation de Fortuin-Kasteleyn, ou percolation FK.

Soit $G = (V, E)$ un graphe. Une configuration de percolation $\omega = (\omega_e)_{e \in E}$ sur G est un élément de $\{0,1\}^E$. Une arête $e \in E$ est dite *ouverte* (dans la configuration ω) si $\omega_e = 1$, et *fermée* si $\omega_e = 0$. Une configuration peut ainsi être identifiée avec un sous-graphe de G, défini sur le même ensemble de sommets en ne gardant que les arêtes ouvertes. Les composantes connexes de ce sous-graphe fait d'arêtes ouvertes sont appelées les *clusters* de ω, c'est-à-dire que deux sommets sont dans le même cluster s'ils sont connectés dans G par un chemin d'arêtes ouvertes (ainsi, un cluster peut être réduit à un sommet isolé). L'exemple le plus simple de modèle de percolation est la *percolation de Bernoulli :* elle est caractérisée par un paramètre $p \in [0,1]$, et les arêtes sont indépendamment ouvertes avec probabilité p. La distribution

correspondante sur $\{0,1\}^E$, notée $\mathbb{P}_p^G$, est simplement la mesure produit ayant pour marginale sur chaque arête une loi de Bernoulli de paramètre p. Ce modèle a fait l'objet de nombreuses études, notamment parce qu'il s'agit du modèle le plus simple présentant une transition de phase. En percolation, la transition de phase traite de la question de la présence ou non d'un cluster infini dans une configuration aléatoire de loi $\mathbb{P}_p^G$, pour un graphe infini G (bien sûr, il y a toujours unicité de la mesure de percolation de Bernoulli, puisque c'est une mesure produit). La loi du 0-1 de Kolmogorov nous dit que la probabilité de l'événement d'existence d'un cluster infini vaut 0 ou 1, donc par monotonie, on peut définir un *paramètre critique* $p_c(G) \in [0,1]$ tel que :

- $\forall p < p_c(G), \mathbb{P}_p^G(\exists \text{ un cluster infini}) = 0$; dans ce cas, le modèle est dit *sous-critique*
- $\forall p > p_c(G), \mathbb{P}_p^G(\exists \text{ un cluster infini}) = 1$; dans ce cas, le modèle est dit *surcritique*.

Pour $G = \mathbf{Z}^d$, on a $p_c(\mathbf{Z}^d) \in (0,1)$ pour tout $d \geq 2$, tandis que $p_c(\mathbf{Z}) = 1$. En dimension $d = 2$, Kesten [Kes80] a prouvé que le paramètre critique vaut exactement $1/2$.

On introduit maintenant le modèle de percolation FK. Il est plus complexe que la percolation de Bernoulli puisque l'état d'une arête (ouverte ou fermée) ne sera plus indépendant des autres. Le modèle peut être défini sur n'importe quel graphe fini, mais nous nous concentrons dans cette section sur les boîtes de $\mathbf{Z}^d$, pour $d \geq 1$. Soit $\Lambda \Subset \mathbf{Z}^d$ un sous-ensemble fini de $\mathbf{Z}^d$; on note $E(\Lambda)$ l'ensemble des arêtes ayant leurs deux extrémités dans Λ (c'est-à-dire $(\Lambda, E(\Lambda))$ est un sous-graphe fini de $\mathbf{Z}^d$). Soit $\partial\Lambda$ l'ensemble des sommets dans Λ ayant (au moins) un voisin à l'extérieur. Une *condition au bord* est une partition de $\partial\Lambda$. Pour une configuration $\omega \in \{0,1\}^{E(\Lambda)}$ et une condition au bord α, on définit un graphe à partir de ω en identifiant tous les sommets qui sont dans une même composante dans α ; cette procédure est appelée « *wiring* », et le graphe obtenu est noté $\omega \cup \alpha$. Soit $k(\omega, \alpha)$ le nombre de composantes connexes, ou clusters, dans le graphe $\omega \cup \alpha$. Une configuration ξ sur $\mathbf{Z}^d$ induit toujours une partition de $\partial\Lambda$, en disant que deux sommets sont dans la même composante si et seulement s'ils sont connectés par un chemin ouvert dans ξ qui n'utilise pas d'arête de $E(\Lambda)$. Dans ce cas, nous identifierons dans les notations la partition et la configuration. Cette expression des conditions au bord comme une configuration sur le complémentaire est plus proche de ce que nous avons défini pour le modèle d'Ising ou les modèles de mécanique statistique en général, dans la section 0.3.1. Cependant, toutes les partitions de $\partial\Lambda$ ne peuvent pas être fabriquées de cette façon, donc cette définition est a priori plus riche. Un premier exemple de condition au bord est la partition composée seulement de singletons, de sorte qu'il n'y a aucune identification au bord ; elle est appelée condition au bord « free », et on note $\alpha = 0$. Le cas opposé, appelé condition au bord « wired », est obtenu en prenant comme partition $\{\partial\Lambda\}$, identifiant tous les sommets du bord ; elle est notée $\alpha = 1$.

La *mesure FK* sur Λ avec paramètres $p \in [0,1]$ et $q > 0$, et avec condition au bord α est définie par

$$\phi^\alpha_{\Lambda,p,q}(\{\omega\}) := \frac{1}{Z^\alpha_{\Lambda,p,q}} \left(\prod_{e \in E(\Lambda)} p^{\omega_e}(1-p)^{1-\omega_e} \right) q^{k(\omega,\alpha)}, \ \forall \omega \in \{0,1\}^{E(\Lambda)} \tag{11}$$

où $Z^\alpha_{\Lambda,p,q}$ est la constante de normalisation qui en fait une mesure de probabilité. Le cas $q = 1$ correspond à la percolation de Bernoulli classique. Quand $q \neq 1$, c'est un modèle de percolation dépendant. Il n'est pas possible en général de calculer la marginale sur chaque

arête, mais nous avons une formule très simple pour la probabilité conditionnelle qu'une arête soit ouverte sachant le reste de la configuration : on a p.s. pour tout $e \in E(\Lambda)$,

$$\phi^{\alpha}_{\Lambda,p,q}\left(\omega_e = 1 \mid \omega|_{E\setminus\{e\}}\right) = \begin{cases} p & \text{si } \omega|_{E\setminus\{e\}} \in K^{\alpha}_e \\ \frac{p}{p+q(1-p)} & \text{si } \omega|_{E\setminus\{e\}} \notin K^{\alpha}_e \end{cases}, \tag{12}$$

K^{α}_e étant l'événement que les deux extrémités de e sont connectées par un chemin ouvert sans utiliser l'arête e. Cela découle du fait que $\dfrac{\phi^{\alpha}_{\Lambda,p,q}\left(\omega_e = 1 \mid \omega|_{E\setminus\{e\}}\right)}{\phi^{\alpha}_{\Lambda,p,q}\left(\omega_e = 0 \mid \omega|_{E\setminus\{e\}}\right)} = \dfrac{p}{1-p} q^{-\mathbf{1}_{\{\omega \notin K^{\alpha}_e\}}}$, qui est une conséquence directe de (11). Dans la suite on écrit pour simplifier $p' := \dfrac{p}{p+(1-p)q}$, bien que p' dépende de p et q. Notons que pour $p \in (0,1)$, on a $q < 1$ si et seulement si $p < p'$. Grâce à cela, on peut observer que dans le régime $q < 1$, la mesure FK est biaisée en faveur des configurations avec peu de clusters (en comparaison avec la mesure produit $\mathbb{P}^{\Lambda}_p$), tandis que $q > 1$ favorise les configurations avec davantage de clusters.

Expliquons maintenant les liens entre la percolation FK et d'autres modèles de mécanique statistique bien connus, ce qui était la motivation originale de Fortuin et Kasteleyn pour introduire le modèle. Nous avons déjà vu que le cas $q = 1$ correspond à la percolation de Bernoulli. Dans le cas $q = 2$, il existe un couplage remarquable avec le modèle d'Ising (cela se généralise au modèle de Potts à $q-$états pour des valeurs entières de q supérieures). À partir de $\omega \in \{0,1\}^{E(\Lambda)}$, on peut définir une configuration de spins $\sigma \in \{\pm 1\}^{\Lambda}$ en assignant indépendamment et uniformément un spin $+1$ ou -1 à chaque cluster de ω qui n'intersecte pas le bord de Λ ; cela signifie que tous les sommets d'un même cluster reçoivent le même spin ; ensuite, les sommets qui sont connectés au bord par un chemin ouvert se voient assigner le spin $+1$. Maintenant, on fixe $\beta \geq 0$, et on tire ω selon la mesure FK $\phi^{1}_{\Lambda,p,2}$ avec $p = 1 - e^{-2\beta}$. Alors, la configuration de spins σ construite ci-dessus est distribuée selon la mesure d'Ising $\mu^{+}_{\beta,\Lambda}$. Ce couplage (ω, σ) d'une configuration FK et d'une configuration d'Ising est appelé le couplage d'Edwards-Sokal [ES88]. Dans l'autre sens, on peut aussi obtenir ω à partir de σ avec la procédure suivante : pour tout $e = \{x, y\} \in E(\Lambda)$, si $\sigma_x \neq \sigma_y$, on pose $\omega_e = 0$, sinon on pose

$$\omega_e = \begin{cases} 1 & \text{avec probabilité } 1 - e^{-2\beta} \\ 0 & \text{avec probabilité } e^{-2\beta} \end{cases}.$$

Alors, si σ a pour loi $\mu^{+}_{\beta,\Lambda}$, il s'ensuit que ω est distribué selon $\phi^{1}_{\Lambda,p,2}$ avec $p = 1 - e^{-2\beta}$.

On obtient aussi une propriété intéressante en regardant le comportement du modèle FK quand $q \to 0$ (on se réfère à [Gri06] chapitre 1.5). Pour cela, il est pratique de considérer la paramétrisation du modèle avec p et p', et d'observer que

$$q = \frac{p}{1-p}\frac{1-p'}{p'},$$

de sorte que q tend vers 0 quand $p \to 0$ ou $p' \to 1$ (rappelons aussi que $p \leq p'$ pour $q < 1$). Considérons la condition au bord free et supposons que $(\Lambda, E(\Lambda))$ est connexe. D'abord, si $p > 0$ est fixé et $q \to 0$, ou de manière équivalente $p' \to 1$, on peut voir que les termes dominants dans la fonction de partition

$$Z^{0}_{\Lambda,p,q} = \sum_{\omega \in \{0,1\}^{E(\Lambda)}} \left(\prod_{e \in E(\Lambda)} p^{\omega_e} (1-p)^{1-\omega_e} \right) q^{k(\omega,0)}$$

sont ceux pour lesquels $k(\omega, 0)$ est minimal, donc égal à 1. Cela donne à la limite une distribution concentrée sur les configurations qui génèrent un sous-graphe connexe de Λ. On peut comprendre cela intuitivement à partir de (12) : si $p' \to 1$, la probabilité conditionnelle d'ouvrir une arête sachant que ses deux extrémités ne sont pas dans le même cluster sans cette arête tend vers 1. Par exemple, cela signifie que si on applique une dynamique de Glauber correspondant à ce modèle, dans ce régime il ne serait pas possible de diviser un cluster, et inversement une arête adjacente à deux clusters distincts serait nécessairement ouverte. En fait, la limite quand $q \to 0$ pour $p \in (0, 1)$ fixé de la mesure FK dans Λ avec condition au bord free correspond à la percolation de Bernoulli de paramètre p dans Λ, conditionnée à produire un sous-graphe connexe.

La situation est inversée si « $p' < 1$ est fixé » et $p \to 0$; cela correspond aux cas où p tend vers 0 au moins aussi vite que q. Ici, la probabilité conditionnelle d'ouvrir une arête sachant qu'il y a déjà un chemin ouvert connectant ses extrémités tend vers 0. Cela signifie que dans ce régime, la distribution limite sera concentrée sur les configurations sans cycles ; celles-ci sont appelées des *forêts* (couvrantes) (l'adjectif « couvrant » témoignant du fait qu'il s'agit d'un sous-graphe avec le même ensemble de sommets que le graphe original).

Il reste le cas où p et $1 - p'$ tendent tous deux vers 0. Comme $p' = (1 + (1 - p)q/p)^{-1}$, c'est équivalent à $p \to 0$ et $q/p \to 0$, c'est-à-dire p tend vers 0 plus lentement que q. Dans ce cas, les deux comportements précédents se combinent. La dynamique de Glauber devient déterministe, puisqu'à la limite la probabilité conditionnelle d'ouvrir une arête vaut 0 quand ses extrémités sont dans le même cluster et 1 sinon. Alors, la distribution limite est concentrée sur les configurations qui sont à la fois connexes et sans cycles, c'est-à-dire sur les *arbres couvrants* de Λ. On peut voir de plus que chacun aura la même masse, donc l'objet limite est appelé l'*arbre couvrant uniforme*.

Enfin, la véritable motivation est de définir le modèle sur $\mathbf{Z}^d$ plutôt que sur des sous-graphes finis. Sauf dans le cas $q = 1$, il n'est pas possible de définir directement une mesure FK en volume infini. On peut procéder comme dans la section 0.3.1, en prenant des limites faibles le long d'une suite croissante de boîtes, avec une suite de conditions au bord. Autrement, on peut remarquer que pour tout $\Lambda \subset \Delta \Subset \mathbf{Z}^d$ et tous $\xi, \xi' \in \{0, 1\}^E$ qui coïncident à l'extérieur de $E(\Delta)$,

$$\phi^{\xi'}_{\Delta,p,q}(\cdot \mid \omega|_{E(\Delta)\setminus E(\Lambda)} = \xi|_{E(\Delta)\setminus E(\Lambda)}) = \phi^{\xi}_{\Lambda,p,q}(\cdot).$$

Cela découle de (12), appliqué successivement pour toutes les arêtes de $E(\Delta) \setminus E(\Lambda)$. Cela signifie que $(\phi^{\xi}_{\Lambda,p,q})_{\Lambda \Subset \mathbf{Z}^d, \xi \in \{0,1\}^E}$ se comporte comme une spécification. Nous pouvons alors définir des mesures en volume infini avec l'approche DLR, comme dans la section 0.3.2. Il n'est pas vrai en général que l'ensemble des mesures en volume infini obtenues via les limites faibles ou via les équations DLR coïncident. La situation est encore moins bien comprise dans le cas de la percolation FK. En fait, la spécification pour ce modèle n'est pas quasilocale, c'est-à-dire qu'elle n'est pas continue par rapport à la condition au bord (pour la topologie locale). Cela se voit facilement à partir de (12), car la réalisation de l'événement K^{α}_{e} peut dépendre d'une arête à distance arbitrairement grande de l'arête e. En particulier, cela implique que l'item 2 du théorème 1.9 n'est pas valable. Cependant, on peut montrer que la spécification est presque sûrement quasilocale en un certain sens, voir [PVV95].

Pour la percolation FK sur $\mathbf{Z}^d$, nous pouvons alors étudier la question de la transition de phase à la fois du point de vue de la mécanique statistique (y a-t-il unicité de la mesure en volume infini ?) et de la percolation (y a-t-il un cluster infini ?). Ces deux aspects sont

étroitement liés, et sont mieux compris dans le régime $q \geq 1$. Cela correspond au cas où les *inégalités FKG*, déjà mentionnées pour le modèle d'Ising (théorème 1.7), sont valides. Un événement A est dit *croissant* s'il est préservé par l'ajout d'arêtes ouvertes, c'est-à-dire si pour tous $\omega \leq \omega'$, $\omega \in A$ implique $\omega' \in A$. Dans le contexte de la percolation (FK), les inégalités FKG signifient que les événements croissants sont positivement corrélés. En particulier, elles permettent de montrer que pour $q \geq 1$, les conditions aux bords free et wired induisent deux mesures en volume infini particulières, $\phi^0_{p,q}$ et $\phi^1_{p,q}$, obtenues comme limites faibles le long de n'importe quelle suite de boîtes. De plus, pour toute mesure FK ϕ sur $\mathbf{Z}^d$ (obtenue à partir d'une limite faible ou des équations DLR), on a la propriété de monotonie suivante :

$$\phi^0_{p,q} \preceq \phi \preceq \phi^1_{p,q},$$

où, pour deux mesures μ, μ' sur $\{0,1\}^E$, $\mu \preceq \mu'$ signifie que pour tout événement croissant A, $\mu(A) \leq \mu'(A)$; on dit dans ce cas que μ' *domine stochastiquement* μ. Cela implique que l'unicité de la mesure en volume infini est équivalente à $\phi^0_{p,q} = \phi^1_{p,q}$. Pour ces deux mesures, nous pouvons définir un paramètre critique $p^b_c(q) \in (0,1)$, $b = 0, 1$ pour $q \geq 1$ tel que

$$\phi^b_{p,q}(\exists \text{ un cluster infini}) = \begin{cases} 0 & \text{si } p < p^b_c(q) \\ 1 & \text{si } p > p^b_c(q) \end{cases}, \ b = 0 \text{ ou } 1.$$

Il est connu que pour tout $q \geq 1$, $\phi^0_{p,q} \neq \phi^1_{p,q}$ pour un nombre au plus dénombrable de valeurs de p (théorème 4.63 dans [Gri06]), correspondant aux cas où la probabilité pour l'origine d'appartenir au cluster infini est différente sous $\phi^0_{p,q}$ et sous $\phi^1_{p,q}$ (théorème 5.16 dans [Gri06]). Cela implique que $p^0_c(q) = p^1_c(q) := p_c(q)$, et qu'il y a unicité de la mesure pour $p < p_c(q)$.

En dimension 2, la dualité des graphes planaire permet de calculer le point critique, et en particulier il est montré dans [BDC12] que pour $\mathbf{Z}^2$, le point auto-dual est critique, c'est-à-dire

$$\forall q \geq 1, \ p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

De plus, on connaît précisément le régime où l'unicité a lieu : quand $q \in [1,4]$ il y a une unique mesure en volume infini pour toutes les valeurs de p (voir [DCST17]) ; quand $q > 4$, il y a unicité partout sauf au point critique i.e. $p = p_c(q)$ (voir [DCGH+21]). Un comportement similaire est attendu en dimension $d \geq 3$: il est conjecturé que le seul régime où $\phi^0_{p,q} \neq \phi^1_{p,q}$ est lorsque $p = p_c(q)$ pour q assez grand. La question de l'existence de codages finitaires a été entièrement résolue par Spinka et Harel [HS22], puisqu'ils ont prouvé que dans tout le régime d'unicité, l'unique mesure sur $\mathbf{Z}^d$ (en toute dimension d) est FFIID. De plus, pour $p < p_c(q)$, le rayon de codage a une queue exponentielle.

En raison de l'absence des inégalités FKG, le régime $q < 1$ est plus obscur. Dans ce cas, on ne sait pas définir les mesures $\phi^0_{p,q}$ et $\phi^1_{p,q}$ (l'existence d'au moins une mesure sur $\mathbf{Z}^d$ est encore vraie par un argument de compacité classique). L'existence du point critique est une question ouverte, et le régime où l'unicité est connue est limité à $p' < p_c(1)$ (le paramètre critique de la percolation de Bernoulli) ou p assez proche de 1. En dimension 2, $p' < 1/2$ correspond à $p < \frac{q}{1+q}$, et la planarité permet d'obtenir l'unicité dans le régime dual $p > 1/2$. Cela implique que, s'il existe, le point critique sur $\mathbf{Z}^2$ pour $q < 1$ doit se trouver entre $\frac{q}{1+q}$

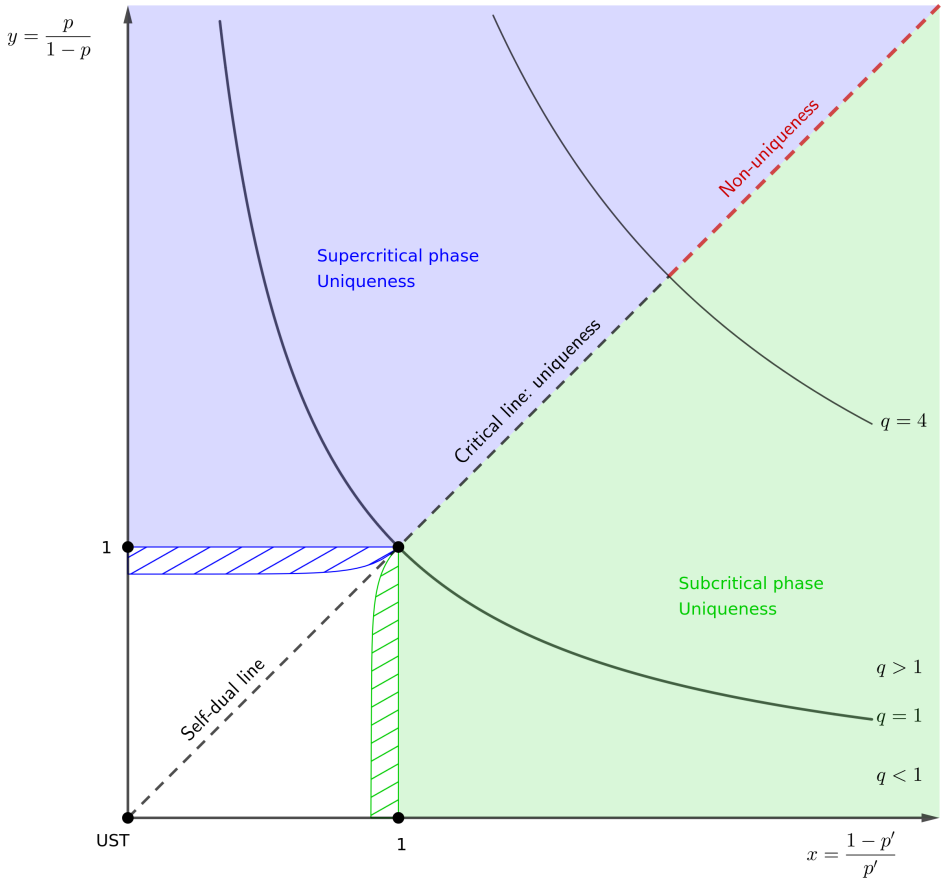


Figure 1: Portrait de phase de la percolation FK sur $\mathbf{Z}^2$.

et 1/2. L'objectif principal du chapitre 5 est d'améliorer légèrement les résultats existants concernant le diagramme de phase de la percolation FK dans le régime $q < 1$.

La figure 1 représente l'état de l'art concernant le portrait de phase du modèle en dimension 2. On place $x = (1-p')/p$ sur l'axe des abscisses et $y = p/(1-p)$ sur l'axe des ordonnées ; ce choix de coordonnées est particulièrement adapté à la dualité planaire, et on notera aussi que $q = xy$. La ligne $x = y$ correspond à $p = 1 - p'$, qui est le point auto-dual, pour toute valeur de q (voir la section 5.4.2 pour une discussion sur la dualité planaire pour le modèle FK) ; on sait qu'il coïncide avec le point critique quand $q \geq 1$. Le long de cette ligne, on a unicité de la mesure FK sur $\mathbf{Z}^2$ pour $q \in [1, 4]$ et une transition de phase pour $q > 4$. La phase surcritique (en bleu) et la phase sous-critique (en vert) correspondent respectivement à la présence ou l'absence d'un cluster infini, et l'unicité y est connue. Rien n'était précédemment connu dans le carré ouvert unité $(0, 1) \times (0, 1)$ (correspondant à $p < 1/2$ et $p' > 1/2$). Dans le chapitre 5, on élargi le régime connu de sur-criticalité et de sous-criticalité (théorème 5.3), et prouvons aussi l'unicité dans les deux cas (théorèmes 5.7 et 5.8) ; ces nouveaux résultats sont représentés par les régions hachurées dans la figure 1.

Chapter 1

Introduction

1.1 Markov random fields and Gibbs measures

1.1.1 Potentials and Gibbs measures

In statistical mechanics, we aim to study the macroscopic behaviour of large physical systems from their microscopic description. These systems are naturally modelled by a collection of random variables $(X_i)_{i\in V}$ where V is a finite set, which corresponds for example to the set of particles of a system. The X_i take values in a finite set S, called the *single-spin state space*, which represents the possible states of a particle. The set of *configurations* is given by $\Omega := S^V$, and is equipped with the product $\sigma-$algebra, denoted $\mathcal{F}_V$ (we will also use the notation $\mathcal{F}_A$ for the product $\sigma-$algebra on a subset A of V). For every site $i \in V$, the random variable $X_i : \Omega \to S$ defined by $X_i(\omega) = \omega_i$ is called the *spin* at i (we will sometimes abuse the notation by calling ω_i the spin of i). The particles are interacting, so their states are not independent from each others. To define the distribution of $(X_i)_{i\in V}$, one wants to follow the physical intuition that at low temperature, the system is almost frozen and favours configurations with minimal energy, whereas at high temperature, particles are excited and behave almost independently.

We call a *potential* a collection $\{\Phi_A\}_{A\subset V}$ of functions such that for every $A \subset V$, $\Phi_A : \Omega \to \mathbf{R}$ is $\mathcal{F}_A-$measurable (i.e. $\Phi_A(\omega)$ only depends on the restriction $(\omega_i)_{i\in A}$ of the configuration to A). The energy of a configuration $\omega \in \Omega$ under the potential $\{\Phi_A\}_{A\subset V}$ is given by

$$H_V(\omega) = \sum_{A\subset V} \Phi_A(\omega).$$

The function H_V is called the *Hamiltonian* of the system. To each *inverse temperature* $\beta \in \mathbf{R}$ and couple potential-Hamiltonian, we associate a *Gibbs measure*, which is a probability distribution on $(\Omega, \mathcal{F}_V)$ given by

$$P_{\beta,V}(\omega) = \frac{\exp(-\beta H_V(\omega))}{Z_{\beta,V}},\ \omega \in \Omega$$

where the partition function $Z_{\beta,V} := \sum_{\omega\in\Omega} \exp(-\beta H_V(\omega))$ is just a normalisation constant that ensures that $P_{\beta,V}$ is a probability measure. At zero-temperature (i.e. in the limit $\beta \to +\infty$), the measure is supported only on configurations that minimise the Hamiltonian. On the

contrary, when $\beta = 0$, $P_{\beta,V}$ is simply the uniform measure on Ω, in particular it is a product measure.

In a physical system, not all particles interact with each other in the same way: a particle interacts more with particles close to it. This is why it is natural to endow the set V with a graph structure. From now on, consider $G = (V, E)$ a finite simple graph (i.e. without loops or multiple edges). For any pair of vertices (i, j), write $i \sim j$ when i and j are connected by an edge in G i.e. $\{i, j\} \in E$. Denote by $d(\cdot, \cdot)$ the graph-distance on G. The diameter of a subset $A \subset V$ is defined by $\mathrm{diam}(A) := \sup_{i,j \in A} d(i, j)$.

Definition 1.1. *The* range *respectively to G of a potential $\Phi = (\Phi_A)_{A \subset V}$ is defined by*

$$r(\Phi) = \inf\{R \geq 0 : \Phi_A \equiv 0 \text{ for all } A \text{ with } \mathrm{diam}(A) > R\};$$

Φ is said to be nearest-neighbours *(with respect to G) if $r(\Phi) = 1$.*

Note that in some sense this last definition is not restrictive, as long as any potential can be seen as a nearest-neighbours one on a graph with sufficiently many edges. In particular, the nearest-neighbours assumption is empty if G is the complete graph.

The simplest and most understood model that fits in this framework is the Ising model. It was introduced by Lenz and Ising in the 1920s to explain the *phase transition* from ferromagnetic to paramagnetic behaviour observed for magnetic materials: when heated above its *Curie temperature*, a material suddenly loses its magnetisation properties. We first assume that our random variables take value in $\{-1, +1\}$; the set of configuration is therefore $\{-1, +1\}^V$. We then assume that there are only pairwise interactions between neighbouring spins, and that the spins tends to align (for positive inverse temperature β). The Hamiltonian of the Ising model can then be written as

$$H_V(\omega) = -\sum_{i \sim j} \omega(i)\omega(j)$$

for all configurations $\omega \in \{\pm 1\}^V$, and the associated Gibbs measure at inverse temperature β is denoted $\mu_{\beta,V}$.

Suppose that V is a finite box of $\mathbf{Z}^d$. By adding boundary conditions and letting the box goes to all of $\mathbf{Z}^d$, one obtains Gibbs measures in infinite-volume (discussed in Section 1.3). In dimension $d \geq 2$, the model exhibits a phase transition, meaning that there exists a critical inverse temperature $\beta_c(d) \in (0, \infty)$ such that:

- for all $0 \leq \beta < \beta_c(d)$ (high temperature), there exists only one infinite-volume Gibbs measure; in other words, the system "forgets" the boundary condition;
- for all $\beta > \beta_c(d)$ (low temperature), the boundary conditions "+1 everywhere" and "−1 everywhere" give, in the limit, two different Gibbs measures μ_β^+ and μ_β^-.

This is consistent with the aforementioned fact that at very high temperature $\mu_{\beta,V}$ is close to a product measure, whereas at very low temperature, it almost charges only configurations of minimal energy, which are the "all +1" and "all −1" configurations in this case. The above description is valid for the *ferromagnetic* Ising model, that is $\beta \geq 0$. The antiferromagnetic Ising model with "negative inverse temperature" is much less well understood (in particular, it is not known whether μ_β^+ and μ_β^- are well defined).

1.1.2 Markov properties

An important property that the Ising model has in common with any other model derived from a nearest-neighbours potential is the *spatial Markov property*. It expresses the fact that given the configuration outside a site i, the conditional distribution of the spin at i actually depends only on the neighbouring spins. In this Section, we will see that this is not far from being a sufficient condition for a measure to admit a *Gibbsian representation*.

To formalise this idea, there are several ways of defining spatial Markov property. Recall that if X, Y, Z are random variables in a probability space $(\Omega, \mathcal{F}, P)$, we say that X and Y are conditionally independent given Z if for any events $A \in \sigma(X)$ and $B \in \sigma(Y)$, we have a.s. $P(A \cap B \mid \sigma(Z)) = P(A \mid \sigma(Z))P(B \mid \sigma(Z))$[1]. In this case, write $X \perp Y \mid Z$.

Let $G = (V, E)$ be an undirected simple graph and $X = (X_i)_{i \in V}$ a collection of random variables indexed by the vertices of G. For all $i \in V$ and $r \in \mathbf{N}$, denote by $B_r(i) = \{j \in V, d(i,j) \leq r\}$ the $r-$neighbourhood of i. If A and B are subsets of V, define the distance between A and B by $d(A,B) = \inf_{i \in A, j \in B} d(i,j)$. Finally, write X_A for the restriction $(X_i)_{i \in A}$ of X to A.

The family $(X_i)_{i \in V}$ (or more formally its distribution) is said to satisfy, relatively to G,

- the *pairwise Markov property of range* r if for any pair of vertices i and j,
$$d(i,j) > r \implies X_i \perp X_j \mid X_{V - \{i,j\}} \tag{PM}$$
- the *local Markov property of range* r if for any $i \in V$,
$$X_i \perp X_{V - B_r(i)} \mid X_{B_r(i) - \{i\}} \tag{LM}$$
- the *global Markov property of range* r if for any subsets $A, B \subset V$,
$$d(A,B) > r \implies X_A \perp X_B \mid X_{V - (A \cup B)} \tag{GM}$$

These three definitions do not coincide in general, and it is not clear which one we should choose to define the notion of *Markov random field*. However, it is not difficult to see that the following chain of implications holds:

$$\text{(GM)} \implies \text{(LM)} \implies \text{(PM)}.$$

The converse is true for distributions which admits a (strictly) positive density with respect to a product measure (this is the content of Pearl and Paz theorem, see e.g. [Lau20] for a complete presentation).

Remark 1.2. *Actually,* (GM) *is equivalent to:*

$$\forall A \subset V, X_A \perp X_{V - B_r(A)} \mid X_{B_r(A) - A}$$

where $B_r(A) := \bigcup_{i \in A} B_r(i)$. *It is obvious in the direct way, by taking* $B = V - B_r(A)$, *and the converse also holds: actually, it is an immediate consequence of the fact that for every* $B_0 \subset B$,

$$X_A \perp X_B \mid X_{V-(A \cup B)} \implies X_A \perp X_{B_0} \mid X_{V-(A \cup B_0)}$$

(see [Lau20] Theorem 2.18 (A3)). This same fact proves that (LM) $\implies$ (PM), *and the implication* (GM) $\implies$ (LM) *is now clear.*

[1] $\sigma(X)$ is the $\sigma-$algebra generated by the random variable X.

This justifies the following definition of a Markov random field. We focus for now on finite graphs and finite state space. Assume therefore that G is finite and S is a finite set. As in the previous paragraph, the set of configurations $\Omega = S^V$ is equipped with the product $\sigma-$algebra $\mathcal{F}_V$. We will discuss extensions to infinite graphs on Section 1.3 and to non-compact single-spin state space in Section 1.3.4 (the compact case behaving mostly like the finite case).

Definition 1.3. *A probability distribution P on $(\Omega, \mathcal{F}_V)$ is a* Markov random field of range r *(or a $r-$Markov random field, or P is $r-$Markovian) on G if it satisfies $P(\omega) > 0$ for all $\omega \in \Omega$ and the equivalent properties* (GM), (LM) *and* (PM) *with range r.*

When the range r is not explicitly specified, we refer to the Markov properties of range 1 and to $1-$Markov random fields. Observe that, in the same way as for the nearest-neighbours assumption on potentials, any positive random field can be seen as a Markov random field with range (≥ 1) as small as one wants on a graph with sufficiently many edges. In particular, if G is the complete graph, the conditions (GM), (LM) and (PM) are empty.

Clearly, Gibbs measures associated with nearest-neighbours potentials are Markov random field. The following result, usually named after Hammersley and Clifford ([CH71]), asserts that every Markov random fields admit a Gibbsian representation, that is there exist a potential for which it is a Gibbs measure.

Theorem 1.4 (Hammersley-Clifford)**.** *A probability distribution P on $(\Omega, \mathcal{F}_V)$ is a Markov random field if and only if it is a Gibbs measure for a nearest-neighbours potential.*

This result directly extends to larger ranges: P is $r-$Markovian if and only if it is a Gibbs measure for a potential of range at most r.

1.2 Glauber dynamics and exact sampling algorithm

1.2.1 Glauber dynamics

We now present a central tool of this thesis, which is *Glauber dynamics*. It was introduced by Roy J. Glauber in 1963 [Gla63], initially as a stochastic version of the Ising model. It provides a method to sample Markov random fields (in finite volume), by expressing the Gibbs measure as the stationary distribution of an irreducible and aperiodic Markov chain, following the classical Monte-Carlo approach. There are several versions of the Glauber dynamics that coexist in the literature, the one presented here is also referred to as the *heat bath* dynamics.

We describe the dynamics for the Ising model on finite graphs $G = (V, E)$, but it extends directly to any Markov random field on a finite state space. The idea is at each step, to choose uniformly at random a vertex $i \in V$ and to update its spin according to its conditional distribution given the configuration outside i. For every configuration $\sigma \in \{\pm 1\}^V$, let σ^i_+ and σ^i_- be the configurations that coincide with σ everywhere but in i, such that $\sigma^i_+(i) = +1$ and $\sigma^i_-(i) = -1$. Observe that

$$\frac{\mu_{\beta,V}(X_i = -1 \mid X_{V-\{i\}} = \sigma_{V-\{i\}})}{\mu_{\beta,V}(X_i = +1 \mid X_{V-\{i\}} = \sigma_{V-\{i\}})} = \exp(H_i(\sigma^i_+) - H_i(\sigma^i_-)) = \exp\left(-2\beta \sum_{j\sim i} \sigma(j)\right)$$

thus if we denote $S = \sum_{j\sim i} \sigma(j)$, we get

$$\mu_{\beta,V}(X_i = \pm 1 \mid X_{V-\{i\}} = \sigma_{V-\{i\}}) = \frac{1}{1+e^{\mp 2\beta S}} = \frac{e^{\pm\beta S}}{e^{\beta S}+e^{-\beta S}}$$

(by solving $\frac{x}{1-x} = e^{-2\beta S}$).

Therefore, define a transition kernel Q on $\{\pm 1\}^V$ by

$$Q(\sigma, \sigma^i_\pm) = \frac{1}{|V|}\frac{e^{\pm\beta S}}{e^{\beta S}+e^{-\beta S}} \text{ with } S = \sum_{j\sim i}\sigma(j)$$

and $Q(\sigma,\sigma') = 0$ if σ' is not of the form $\sigma^i_\pm$. The Markov chain with transition kernel Q is clearly aperiodic and irreducible, so it has a unique invariant distribution, which turns out to be the Gibbs measure of the Ising model $\mu_{\beta,V}$.

When one aims to sample a distribution following the Monte-Carlo method, one is facing two issues. Since one cannot run the Markov chain forever, one never reaches exactly the invariant distribution of the chain. Furthermore, it is hard to determine a priori how many steps does one need to get close to the target distribution, with a fixed level of precision. The *coupling-from-the-past* (CFTP) algorithm introduced in 1996 by Propp and Wilson [PW96] will solve these problems, by sampling exactly the invariant distribution in a finite but random number of steps. This is why it is also called *perfect sampling* or *exact sampling* algorithm.

1.2.2 Coupling-from-the-past algorithm

Let P be a probability measure on a finite state space Ω, and consider an irreducible and aperiodic Markov chain whose invariant measure is P (e.g. the Glauber dynamics associated with P, if P is a Gibbs measure). Imagine that instead of starting a trajectory of the chain at time 0 waiting that it approaches the invariant distribution when time goes to infinity, we start the chain at time $-\infty$ and then, at time 0, it is distributed exactly according to the invariant measure.

To formalise this idea, one runs trajectories of the Markov chain initiated from every possible initial configuration in Ω at a large negative time $-T$. These trajectories are coupled in such a way that whenever two or several of them coincide at a certain time, then they evolve together forever (i.e. up to time 0): one says that they *coalesce*. If T is large enough so that all trajectories coalesce before (or at) time 0, then the common value at time 0 is distributed according to P.

More precisely, consider a family $(f_n)_{n<0}$ of i.i.d. random maps $f_n : \Omega \to \Omega$, distributed as a jump of the Markov chain. The $f_n(\omega)$ for $\omega \in \Omega$ can be independent, but one can also choose a coupling that favours the coalescence. Then, for any $n \le 0$, let $F^0_n := f_{-1} \circ f_{-2} \circ \cdots \circ f_n$; the distribution at time 0 of the state of the Markov chain starting at time n with initial configuration $\omega \in \Omega$ is the same as $F^0_n(\omega)$. Coalescence is therefore equivalent to the fact that F^0_{-T} is a constant map. Let

$$T = \inf\{n > 0 : F^0_{-n} \text{ is constant}\}.$$

Note that it is crucial to seek coalescence from the past and not towards the future. Starting at time 0 and stopping when all the trajectories have coalesced would lead to a bias. In fact,

if some state $\omega \in \Omega$ has only one antecedent (meaning that ω can be reached by a jump of the chain only from a unique other state ω'), it is clear that ω cannot occur at the exact moment when the trajectories coalesce. Running the chain from the past discards this bias issue.

Theorem 1.5 (Propp-Wilson)**.** *T is almost surely finite and the unique image of F^0_{-T} has distribution P.*

Proof. By assumption on the Markov chain, we know that there exists $k \in \mathbf{N}$ such that for all $\omega, \omega' \in \Omega$, there is a positive probability of going from ω to ω' in k steps. Thus, for all $t \leq 0$, $F^t_{t-k} := f_{t-1} \circ \cdots \circ f_{t-k}$ also has some positive probability of being constant[2]. Now, all of the maps $F^0_{-k}, F^{-k}_{-2k}, \cdots$ have the same positive probability of being constant and these events are independent, so almost surely one of them is realised. In this case, F^0_{-n} is constant for sufficiently large n (observe that $F^{-nk}_{-(n+1)k}$ constant implies $F^0_{-(n+1)k}$ constant), so T is finite.

Call $\overline{F}^0_{-\infty}$ the output of the algorithm, which is the unique image of F^0_{-T}, noticing that $F^0_{-T} = F^0_{-T-1} = \dots$ almost surely. Then, $\overline{F}^0_{-\infty}$ is distributed according to P since it has the same distribution as $\overline{F}^{-1}_{-\infty}$ (defined in the obvious way) and since a.s. $\overline{F}^0_{-\infty} = f_{-1} \circ \overline{F}^{-1}_{-\infty}$. □

Remark 1.6 (Monotone coupling for the Ising model)**.** *If one applies this algorithm to the Ising model, or another Markov random field, one will face the problem that the cardinal of $\Omega = S^V$ is huge. Taking the Glauber dynamics as underlying Markov chain in the algorithm, one can define a coupling that avoids considering $|\Omega|$ coupled trajectories. It works as follows for the Ising model: let $(U_n, i_n)_{n<0}$ be an i.i.d family of random variables such that U_n is uniformly distributed on $[0,1]$ and i_n is a uniformly chosen vertex in V; the random maps (f_n) are then defined by $f_n(\omega) = \omega^{i_n}_{\varphi(\sum_{j\sim i_n}\omega(j), U_n)}$ where $\varphi(S,u) = \begin{cases} -1 & \text{if } u \leq (1+e^{2\beta S})^{-1} \\ +1 & \text{otherwise} \end{cases}$, i.e. at each step, resample the chosen spin according to its conditional distribution given the neighbours. Observe that here the coupling is not independent since the same U_n and i_n are used for all ω, but still the $(f_n)_{n<0}$ are independent. For $\beta \geq 0$, φ is non-decreasing in S so one can check that the coupling satisfies the following monotonicity property: for every $\omega \leq \omega'$[3] and $n < 0$, $f_n(\omega) \leq f_n(\omega')$. Therefore, it is sufficient to consider two Markov chains (instead of $|\Omega|$), with initial configurations $\omega \equiv -1$ and $\omega \equiv +1$. All other possible trajectories are stuck between these two, so it is enough to get the coalescence from these two extremal initial configurations.*

There are many other ways to improve the algorithm for simulation purposes that we will not discuss here.

1.3 Infinite-volume Gibbs measures

We have defined Gibbs measures and their associated Glauber dynamics in finite volume, meaning that the underlying graph was assumed to be finite. Physical systems are always finite, but the number of particles involved is usually huge, so much that they are well approximated by their limit when the number of particles goes to infinity. It is therefore natural to

[2] An assumption on the $f_n(\omega)$ is needed here; assuming independence of the $(f_{-1}(\omega))_{\omega\in\Omega}$ is clearly sufficient, but one can (and should!) consider couplings that favour the coalescence, i.e. that increase this probability.

[3] For the canonical partial order on $\{\pm 1\}^V$, that is $\omega \leq \omega'$ if and only if $\omega(j) \leq \omega'(j)$ for every $j \in V$.

aim at defining Gibbs measures on infinite graphs. Moreover, phase transition will be easily interpreted in terms of infinite-volume measures, since it will correspond to a regime where several Gibbs measures coexist for the same model.

They are two classical ways of defining the measures on infinite graphs, which are taking the thermodynamic limit of finite systems, or following the so-called DLR approach. We will not give all the topological details (refer to [FV18] or [Geo11] for a complete survey of the theory) and describe these two methods with a special focus on the Ising model on $\mathbf{Z}^d$.

1.3.1 Boundary conditions and thermodynamic limit

Let $G = (V, E)$ be an infinite locally finite simple graph (i.e. each vertex has finite degree), and S a finite set. The set of configurations $\Omega = S^V$ is equipped with the product $\sigma-$algebra $\mathcal{F}_V$, which is generated by the cylindrical events, i.e. events of the form $\{\omega \in \Omega \mid \omega_\Lambda = \sigma\}$ for some finite subset Λ of V (denoted $\Lambda \Subset V$) and $\sigma \in S^\Lambda$. A function $f : \Omega \to \mathbf{R}$ is said to be *local* if it depends only on finitely many spins, i.e. if it is $\mathcal{F}_\Lambda-$measurable for some $\Lambda \Subset V$. A sequence of probability measures (P_n) on Ω converges to a probability P on Ω if and only if $\lim_{n\to\infty} P_n(C) = P(C)$ for every cylindrical event C. In this case write $P_n \Rightarrow P$. It is equivalent to ask that $\lim_{n\to\infty} \int f \mathrm{d}P_n = \int f \mathrm{d}P$ for every local function f, so it corresponds to the weak convergence.

Let $\Phi = \{\Phi_A\}_{A\Subset V}$ be a potential and $\Lambda \Subset V$ a finite box. The Hamiltonian in the box Λ associated with the potential Φ is defined by

$$H_\Lambda(\omega) = \sum_{A\Subset V, A\cap\Lambda\neq\emptyset} \Phi_A(\omega). \tag{1.1}$$

For the Hamiltonian to be well defined, one needs the sum in (1.1) to be convergent; it is the case if one assumes that the potential is *absolutely summable*, that is

$$\forall x \in V, \sum_{A\Subset V, x\in A} \|\Phi_A\|_\infty < \infty \tag{1.2}$$

(note that this always holds when the potential has finite range). Let $\eta \in \Omega$. The Gibbs measure on Λ at inverse temperature β with *boundary condition* η is the probability distribution on S^Λ given for all $\omega_\Lambda \in S^\Lambda$ by

$$P^\eta_{\beta,\Lambda}(\omega_\Lambda) = \frac{\exp(-\beta H_\Lambda(\omega_\Lambda \eta_{\Lambda^c}))}{Z^\eta_{\beta,\Lambda}},$$

where $\omega_\Lambda\eta_{\Lambda^c}$ is the element of Ω which coincide with ω_Λ on Λ and with η on $\Lambda^c := V - \Lambda$ and $Z^\eta_{\beta,\Lambda}$ is the normalisation constant. It is also convenient to consider $P^\eta_{\beta,\Lambda}$ as a measure on Ω supported by $\Omega^\eta_\Lambda := \{\omega \in \Omega \mid \omega_{\Lambda^c} = \eta_{\Lambda^c}\}$ the set of configurations that coincide with η outside Λ. For all $\omega \in \Omega$, we therefore have

$$P^\eta_{\beta,\Lambda}(\omega) = \frac{\exp(-\beta H_\Lambda(\omega))}{Z^\eta_{\beta,V}} \mathbf{1}_{\{\omega\in\Omega^\eta_\Lambda\}}. \tag{1.3}$$

A probability measure P on $(\Omega, \mathcal{F})$ is called a *weak limit infinite-volume Gibbs measure* (often abbreviate weak limit) for fixed potential and β if it is an accumulation point of

$\{P^{\eta}_{\beta,\Lambda}\}_{\eta\in\Omega,\Lambda\Subset V}$, i.e. if there exists a sequence (η_n) of boundary conditions and (Λ_n) an increasing sequence of finites boxes satisfying $\Lambda_n \uparrow V$ (i.e. any $x \in V$ belongs to Λ_n for n large enough) such that

$$P^{\eta_n}_{\beta,\Lambda_n} \Rightarrow P \text{ as } n \to \infty.$$

One can also consider other kinds of boundary conditions, like free boundary condition (which corresponds to the measures $P_{\beta,\Lambda}$ defined in Section 1.1.1) or periodic boundary condition. The set of such measures is denoted $\mathcal{W}_\beta(\Phi)$.

By compactness of Ω and Prohorov's theorem, the set of probability measures on Ω is sequentially compact, hence the existence of weak limits. Uniqueness however is not guaranteed, as we shall see for the Ising model on $\mathbf{Z}^d$. In the literature, one talks about *first-order phase transition* when uniqueness is not achieved.

We now assume that $G = (V, E)$ is the $d-$dimensional cubic lattice, meaning that $V = \mathbf{Z}^d$ and $i \sim j$ if and only if $\|i - j\|_1 = 1$. Gibbs measures of the Ising model with boundary conditions are defined as in (1.3), taking

$$\Phi_A(\omega) = \begin{cases} -\omega(i)\omega(j) & \text{if } A = \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

as potential. There are two boundary conditions of special interest, which are $\eta \equiv +1$ and $\eta \equiv -1$, the two extremal configurations of $\Omega = \{\pm 1\}^{\mathbf{Z}^d}$ for the canonical partial order. Denote $\mu^+_{\beta,\Lambda}$ and $\mu^-_{\beta,\Lambda}$ respectively the Gibbs measures on $\Lambda \Subset \mathbf{Z}^d$ associated with these extremal boundary conditions. When $\beta \geq 0$, one can show that there exists two measures μ^+_β and μ^-_β such that for any choice of sequence $\Lambda_n \uparrow \mathbf{Z}^d$, $\mu^+_{\beta,\Lambda_n} \Rightarrow \mu^+_\beta$ and $\mu^-_{\beta,\Lambda_n} \Rightarrow \mu^-_\beta$. It gives a constructive proof of the fact that $\mathcal{W}_\beta \neq \emptyset$ for the $d-$dimensional Ising model. A crucial tool of the proof is the FKG inequality (named after Fortuin, Kasteleyn and Ginibre [FKG71]), which states that increasing local functions are positively correlated, and which is not available for negative β. In this statement, a function $f : \Omega \to \mathbf{R}$ is said to be *increasing* if $\omega \leq \omega'$ for the canonical partial order on Ω implies $f(\omega) \leq f(\omega')$.

Theorem 1.7 (FKG inequality)**.** *Let $\beta \geq 0$, $\Lambda \Subset \mathbf{Z}^d$ and $\eta \in \Omega$ be a boundary condition. For any pair of increasing functions f and g,*

$$\mu^\eta_{\beta,\Lambda}(fg) \geq \mu^\eta_{\beta,\Lambda}(f)\mu^\eta_{\beta,\Lambda}(g).$$

It is therefore clear that whenever μ^+_β and μ^-_β are different, we are in a regime where the Ising model exhibits a first-order phase transition. By monotonicity, it actually holds that the converse is true, so one only needs to compare μ^+_β and μ^-_β to determine whether or not there is uniqueness of the infinite-volume measure. We can now state more precisely the aforementioned phase transition result, first obtained by Peierls [Pei36] in dimension 2:

Theorem 1.8 (First-order phase transition)**.** *In dimension $d \geq 2$, there exists a critical inverse temperature $\beta_c(d) \in (0, \infty)$ such that*

- *For all $\beta \in [0, \beta_c(d))$, $\mu^+_\beta = \mu^-_\beta$ and there is uniqueness of the infinite-volume measure;*
- *For all $\beta \in (\beta_c(d), +\infty)$, $\mu^+_\beta \neq \mu^-_\beta$ so a first-order transition occurs.*

In dimension $d = 1$, there is no phase transition ($\beta_c(1) = +\infty$), and this was actually the main result of Ising's thesis. The theorem claims that for $\beta > \beta_c$, the set of weak limits $\mathcal{W}_\beta$ for the Ising model is not reduced to a singleton, but it tells nothing about its structure. Insights about this question will be given in the next paragraph, where we describe the second manner of defining infinite-volume measures.

1.3.2 DLR approach

The DLR approach, for Dobrushin, Lanford and Ruelle [Dob68] [LR69], considers the conditional probabilities on finite boxes given the configuration outside the box, rather than the marginal distributions as in the Kolmogorov's extension theorem. In fact, the definition of Gibbs measures itself makes the marginal distributions difficult to access. For example, the marginal distribution of a single spin in the Ising model in $\mathbf{Z}^2$ at low temperature ($\beta > \beta_c$) is simply a $\{\pm 1\}-$valued random variable, but its expectation is different under μ_β^+ and μ_β^-, so it cannot be computed directly from the Hamiltonian.

Candidates for being a collection of conditional probabilities of an infinite-volume measure are called *specifications*. We will only define Gibbsian specifications[4] (once again, refer to [FV18] or [Geo11] for more general definitions).

Let $\Phi = \{\Phi_A\}_{A \Subset V}$ be an absolutely summable potential, i.e. Φ satisfies (1.2). The Gibbsian specification with potential Φ and inverse temperature β is the collection $\pi^{\Phi,\beta} = (\pi_\Lambda^{\Phi,\beta})_{\Lambda \Subset V}$ of functions $\pi_\Lambda^{\Phi,\beta}(\cdot \mid \cdot) : \mathcal{F}_V \times \Omega \to \mathbf{R}$ such that for all $\Lambda \Subset V$, $\eta \in \Omega$ and $A \in \mathcal{F}_V$,

$$\begin{aligned}\pi_\Lambda^{\Phi,\beta}(A \mid \eta) &= \sum_{\omega \in \Omega_\Lambda^\eta} \frac{\exp(-\beta H_\Lambda(\omega))}{Z_{\beta,\Lambda}^\eta} \mathbf{1}_{\{\omega \in A\}} \\ &= \sum_{\omega_\Lambda \in S^\Lambda} \frac{\exp(-\beta H_\Lambda(\omega_\Lambda \eta_{\Lambda^c}))}{Z_{\beta,\Lambda}^\eta} \mathbf{1}_{\{\omega_\Lambda \eta_{\Lambda^c} \in A\}}.\end{aligned}$$

In particular, if $A \in \mathcal{F}_{\Lambda^c}$, that is A depends only on spins outside Λ, then $\pi_\Lambda^{\Phi,\beta}(A \mid \eta) = \mathbf{1}_{\{\eta \in A\}}$. Then, $\pi_\Lambda^{\Phi,\beta}(\Omega_\Lambda^\eta \mid \eta) = 1$ so $\pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta)$ is supported by Ω_Λ^η.

A probability measure μ on $(\Omega, \mathcal{F})$ is said to be compatible with a specification π if it satisfies the DLR equations: for all $\Lambda \Subset V$,

$$\mu = \mu\pi_\Lambda \text{ i.e. for every cylindrical event } C,\ \mu(C) = \int_\Omega \pi_\Lambda(C \mid \cdot) \mathrm{d}\mu.$$

The set of distributions compatible with $\pi^{\Phi,\beta}$ is denoted $\mathcal{G}(\pi^{\Phi,\beta}) = \mathcal{G}_\beta(\Phi)$, or just $\mathcal{G}_\beta$ for the Ising model, and such distributions are called *infinite-volume Gibbs measures* (often just Gibbs measures). It follows from our definition that $\pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta)$ coincides with the finite-volume Gibbs measure in Λ with boundary condition η. Therefore, a measure μ is compatible with $\pi^{\Phi,\beta}$ if and only if for any finite box $\Lambda \Subset V$, its conditional distribution given that the configuration outside Λ is η_{Λ^c} coincides $\mu-$almost surely with the finite-volume Gibbs measure on Λ with boundary condition η. The DLR equations can thus be rewritten as

$$\mu(\cdot \mid \mathcal{F}_{\Lambda^c})(\eta) = P_{\beta,\Lambda}^\eta \text{ for } \mu\text{-a.e. } \eta \in \Omega. \tag{DLR}$$

[4] Furthermore, if one looks at Definition 2.9 in [Geo11], we always consider the counting measure on S as "a priori measure".

Condition (1.2) ensures that the Hamiltonian and so the Gibbsian specification is well defined, but it is not necessary. In Section 1.5.2 we will introduce an important example of a model for which the potential is not absolutely summable. However, this condition guarantees the existence of an infinite-volume Gibbs measure.

In fact, if Φ satisfies (1.2), one can show that $\pi^{\Phi,\beta}$ is *quasilocal*, in the sense that it is continuous with respect to the boundary condition. More precisely, a specification π is said to be quasilocal if for every $\Lambda \Subset V$ and cylindrical event C, $\eta \mapsto \pi_\Lambda(C \mid \eta)$ is continuous[5]. Then, quasilocality implies the existence of an infinite-volume Gibbs measure since one can check that any weak-limit infinite-volume measure satisfies the DLR equations (Theorem 6.26 in [FV18]). The same proof shows that $\mathcal{G}_\beta(\Phi)$ is a closed subset of the set of probability measures on $(\Omega, \mathcal{F})$.

Furthermore, an advantage of the DLR formalism is that it is clear from the definition that $\mathcal{G}_\beta(\Phi)$ is a convex set, meaning that for any $\mu, \mu' \in \mathcal{G}_\beta(\Phi)$ and $\alpha \in [0,1]$, $\alpha\mu+(1-\alpha)\mu' \in \mathcal{G}_\beta(\Phi)$. One can therefore talk about its extremal elements; let $\operatorname{ex}\mathcal{G}_\beta(\Phi)$ be the set of extremal Gibbs measures, which are elements of $\mathcal{G}_\beta(\Phi)$ that do not admit a non-trivial convex decomposition. There would be a lot more to say about this, but we will only affirm that extremal Gibbs measures can always be obtained as weak limits: if $\mu \in \operatorname{ex}\mathcal{G}_\beta(\Phi)$, then

$$\pi^{\Phi,\beta}_{\Lambda_n}(\cdot \mid \eta) \Rightarrow \mu \text{ as } \Lambda_n \uparrow V \text{ for } \mu\text{−a.e. } \eta$$

(Theorem 6.63 in [FV18]). Thus, in some sense, they are the relevant Gibbs measures from physical point of view. We summarise the above discussion into the following result:

Theorem 1.9. *1. For any potential Φ, $\operatorname{ex}\mathcal{G}_\beta(\Phi) \subset \mathcal{W}_\beta(\Phi)$.*

2. If Φ is absolutely summable, then $\emptyset \neq \mathcal{W}_\beta(\Phi) \subset \mathcal{G}_\beta(\Phi)$, so $\mathcal{G}_\beta(\Phi) = \overline{\operatorname{co}\mathcal{W}_\beta(\Phi)}$ where $\overline{\operatorname{co}\mathcal{W}_\beta(\Phi)}$ denotes the closed convex hull of $\mathcal{W}_\beta(\Phi)$.

In both items, the converse inclusion does not hold in general, and the Ising model will provide enlightening counter-examples.

First, the measures μ^+_β and μ^-_β defined above as the weak limits of respectively '+' and '−' boundary conditions are extremal. In dimension 2, the Aizenman-Higuchi theorem [Aiz80] [Hig81] tells us that all possible weak limits are convex combinations of μ^+_β and μ^-_β. Therefore, they are the only two extremal Gibbs measures, and $\mathcal{G}_\beta = \{\alpha\mu^+_\beta + (1-\alpha)\mu^-_\beta, \alpha \in [0,1]\}$. We can now state our first example of a weak limit which is not an extremal Gibbs measure. It is not difficult to show that $(\mu_{\beta,\Lambda})_{\Lambda\Subset\mathbf{Z}^2}$ (without boundary condition, or take $\eta \equiv 0$ although it is not a configuration) admits an accumulation point. It is called the Gibbs measure with free boundary condition, and denoted $\mu^\varnothing_\beta$. By the Aizenman-Higuchi theorem, $\mu^\varnothing_\beta$ must be a convex combination of μ^+_β and μ^-_β. But $\mu^\varnothing_\beta$ is symmetric under the spin flip (observe that $\mu_{\beta,\Lambda}(\omega) = \mu_{\beta,\Lambda}(-\omega)$), so the only possibility is $\mu^\varnothing_\beta = \frac{1}{2}\mu^+_\beta + \frac{1}{2}\mu^-_\beta$. It is therefore not extremal as soon as $\mu^+_\beta \neq \mu^-_\beta$, that is when $\beta > \beta_c(2)$. Moreover, Abraham and Reed [AR76] constructed a family $(\eta_\alpha)_{\alpha\in(0,1)}$ of boundary conditions such that $\mu^{\eta_\alpha}_{\beta,\Lambda_n} \Rightarrow \alpha\mu^+_\beta + (1-\alpha)\mu^-_\beta$. This proves that for the 2D Ising model, $\mathcal{W}_\beta = \mathcal{G}_\beta$, and provides plenty examples of weak limits that are not extremal. These boundary conditions $(\eta_\alpha)_{\alpha\in(0,1)}$ were obtained by modifying the Dobrushin boundary condition that we present in the next paragraph.

[5] For the topology induced by the local functions.

In dimension $d \geq 3$, it is possible to produce new extremal measures for large enough β. It is linked to the existence of non-translation-invariant Gibbs measures, that have been first highlighted by Dobrushin [Dob72]. Consider a boundary condition which is $+1$ on a half plane and -1 otherwise, for example

$$\eta^{\mathrm{Dob}}(i) = \begin{cases} +1 & \text{if } i = (i_1, \dots, i_d) \in \mathbf{Z}^d \text{ with } i_d \geq 0 \\ -1 & \text{otherwise} \end{cases}.$$

This kind of boundary condition is called a Dobrushin boundary condition. It is proved in [Dob72] that for a well-chosen sequence of finite boxes $\Lambda_n \uparrow \mathbf{Z}^d$, $\mu_{\beta,\Lambda_n}^{\eta^{\mathrm{Dob}}}$ weakly converges to some infinite volume measure μ_β^{Dob} which turns out to be an extremal Gibbs measure for large enough β ($\beta > \beta_c(d-1)$ is enough). This induces the existence of countably many extremal Gibbs measures, by replacing condition $i_d \geq 0$ in η^{Dob} by $i_k \geq n$ or $i_k \leq n$ for any $1 \leq k \leq d$ and $n \in \mathbf{Z}$. These are believed to be, together with μ_β^+ and μ_β^-, the only extremal Gibbs measures of the Ising model.

In addition to giving new extremal Gibbs measures, Dobrushin boundary conditions provide the first known example of a Gibbs measure which is not a weak limit. Consider the boundary condition $\eta^{-\mathrm{Dob}}$ obtained by flipping all spins in η^{Dob}, and denote by $\mu_\beta^{-\mathrm{Dob}}$ the corresponding weak limit measure. By convexity of $\mathcal{G}_\beta$, $\frac{1}{2}\mu_\beta^{\mathrm{Dob}} + \frac{1}{2}\mu_\beta^{-\mathrm{Dob}} \in \mathcal{G}_\beta$, but one can prove that it cannot be obtained as a weak limit (for β large enough). This example first appeared in the book of Miyamoto [Miy04] (in Japanese) and was then rediscovered independently by Coquille [Coq15].

1.3.3 Finite-energy and spatial Markov properties

Let Φ be an absolutely summable potential. By (1.2), for every $\Lambda \Subset V$, $\sup_{\omega\in\Omega} H_\Lambda(\omega) < \infty$. Therefore, the Gibbsian specification $\pi^{\Phi,\beta}$ is positive, in the sense that

$$\forall \Lambda \Subset V, \forall \eta \in \Omega, \inf_{\omega \in \Omega_\Lambda^\eta} \pi^{\Phi,\beta}(\omega \mid \eta) > 0. \tag{1.4}$$

A Gibbs measure $\mu \in \mathcal{G}(\pi^{\Phi,\beta})$ therefore satisfies

$$\inf_{A \in \mathcal{F}_\Lambda} \mu(A \mid \mathcal{F}_{\Lambda^c})(\eta) > 0 \text{ for } \mu\text{−a.e. } \eta \in \Omega \tag{FE}$$

This property is often called *finite-energy*, especially when used in percolation models. Note that it implies that $\mu(C) > 0$ for every cylindrical event C, that is μ has full support.

Suppose that Φ is a nearest-neighbours potential on a graph $G = (V, E)$ (everything would be transposable for finite-range potentials). Let $\Lambda \Subset V$ and $\partial\Lambda := \{i \in \Lambda^c \mid \exists j \in \Lambda, i \sim j\}$. The specification $\pi^{\Phi,\beta}$ therefore satisfies a spatial Markov property, that is

$$\pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta) = \pi_\Lambda^{\Phi,\beta}(\cdot \mid \eta') \text{ for all } \eta, \eta' \in \Omega \text{ such that } \eta_{\partial\Lambda} = \eta'_{\partial\Lambda}.$$

This property transfers to all Gibbs measures $\mu \in \mathcal{G}(\pi^{\Phi,\beta})$ in the following way: for all $\Lambda \Subset V$

$$\forall A \in \mathcal{F}_\Lambda, \mu(A \mid \mathcal{F}_{\Lambda^c}) = \mu(A \mid \mathcal{F}_{\partial\Lambda}) \ \mu - \text{a.s.} \tag{1.5a}$$

or equivalently,

$$\forall \Delta \subset V, d(\Lambda, \Delta) > 1 \implies X_\Lambda \perp X_\Delta \mid X_{(\Lambda\cup\Delta)^c}, \tag{1.5b}$$

or

$$X_\Lambda \perp X_{(\partial\Lambda\cup\Lambda)^c} \mid X_{\partial\Lambda}. \tag{1.5c}$$

The equivalence between the last two properties follows from Remark 1.2; the equivalence between the first two is Theorem 2.10 in [Lau20].

Definition 1.10. *A probability measure μ on $(\Omega, \mathcal{F}_V)$ is called a local Markov random field (LMRF) on $G = (V, E)$ (or say that μ is Markovian) if it satisfies (FE) and one of the equivalent properties in (1.5) for every finite box $\Lambda \Subset V$.*

One can define similarly a notion of LMRF of range $r \geq 1$ (or $r-$Markovian distributions). The spatial Markov properties defined in Section 1.1.2 extend literally to infinite graphs. Note that that the local Markov property (LM) is the same as (1.5c), but required only for Λ reduced to a singleton. Being a LMRF is therefore a priori stronger than satisfying (LM). However, we will see that it is equivalent if we assume the finite energy property (FE).

It follows from the definition that a Gibbs measure for a nearest-neighbours potential on G is a LMRF on G. The converse is also true, and we get an infinite-volume version of Theorem 1.4 (see [CH71], [Ave72], [Spi71]):

Theorem 1.11 (Gibbsian representation of LMRF)**.** *Let μ be a probability measure on $(\Omega, \mathcal{F}_V)$. Then for any graph $G = (V, E)$, μ is a local Markov random field on G if and only if $\mu \in \mathcal{G}(\pi^{\Phi,\beta})$ for some Gibbsian specification $\pi^{\Phi,\beta}$ with nearest-neighbours potential Φ on G.*

One can derive a similar statement for LMRF of range r and potential of range $r(\Phi) \leq r$. Hence, on a given graph, the range $r(\Phi)$ of a potential corresponds to the minimal r such that Gibbs measures in $\mathcal{G}_\beta(\Phi)$ are $r-$Markovian. The Kozlov-Sullivan theorem is actually a much more general result, since it characterises specifications inherited from an absolutely summable potential as positive quasilocal specifications (see [Geo11] Theorem 2.30 and 2.32).

Now, justify that for random fields with finite energy, it is enough to verify (1.5) for $|\Lambda| = 1$ to be a LMRF. It follows from Theorem 1.33 in [Geo11], which claims that for positive specifications, the singleton part characterises the entire specification, but let us give a sketch of proof based on the Glauber dynamics. Suppose that π is a positive nearest-neighbours Gibbsian specification, that satisfies the local Markov property, i.e.

$$\pi_i(\cdot \mid \eta) = \pi_i(\cdot \mid \eta') \text{ for all } \eta, \eta' \in \Omega \text{ such that } \forall j \sim i, \eta_j = \eta'_j.$$

Let $\Lambda \Subset V$ and $\eta^1, \eta^2 \in \Omega$ two boundary conditions that coincide on $\partial\Lambda$. The Glauber dynamics on Λ with boundary condition η^k (for $k = 1$ or 2) is the Markov chain on S^Λ with transition kernel

$$Q^k(\sigma, \sigma^i[x]) = \frac{1}{|\Lambda|}\pi_i(\sigma^i[x] \mid \sigma_\Lambda \eta^k_{\Lambda^c}) \text{ for all } i \in \Lambda \text{ and } x \in S,$$

where $\sigma^i[x]$ is the configuration which coincides with σ everywhere except at i, the spin at i being x; $Q^k(\sigma, \sigma') = 0$ if σ' is not of the form $\sigma^i[x]$. This dynamics only depends on the singleton part of the specification, and by finite-energy it is an irreducible aperiodic Markov chain. Therefore, it has a unique invariant distribution, which is by definition $\pi_\Lambda(\cdot \mid \eta^k)$ (by consistency of the specification). But by local Markov property, one actually has $Q^1 = Q^2$, hence $\pi_\Lambda(\cdot \mid \eta^1) = \pi_\Lambda(\cdot \mid \eta^2)$, so that (1.5) holds for Λ.

End this Section with a comment on the global Markov property (GM). It corresponds to having (1.5) for every $\Lambda \subset V$, not necessarily finite. Interestingly, it is not always the case that a Gibbs measure coming from a nearest-neighbours potential satisfies (GM). The following counter-example in the context of the Ising model was found by Goldstein [Gol80], in a paper in which he also shows that μ_β^+ and μ_β^- satisfy the global Markov property. Consider a shifted version of the reversed Dobrushin boundary condition:

$$\eta^{-\,\mathrm{Dob}+1}(i) = \begin{cases} -1 & \text{if } i = (i_1, \dots, i_d) \in \mathbf{Z}^d \text{ with } i_d \geq 1 \\ +1 & \text{otherwise} \end{cases}$$

and $\mu_\beta^{-\mathrm{Dob}+1}$ the associated weak limit infinite-volume Gibbs measure. Therefore, the mixture $\mu = \frac{1}{2}\mu_\beta^{\mathrm{Dob}} + \frac{1}{2}\mu_\beta^{-\mathrm{Dob}+1}$ does not satisfy the global Markov property (for β large enough in dimension $d \geq 3$). In fact, observe that (1.5) fails for the infinite box $\Lambda = \{i_d > 0\}$, since μ_β^{Dob} and $\mu_\beta^{-\mathrm{Dob}+1}$ agree on $\mathcal{F}_{\partial\Lambda} = \mathcal{F}_{\{i_d=0\}}$ but are mutually singular on $\mathcal{F}_{\Lambda^c}$; revealing the configuration on Λ^c hence determines whether the system is in phase μ_β^{Dob} or $\mu_\beta^{-\mathrm{Dob}+1}$. Here, μ is not an extremal Gibbs measure, but let us mention that Israel [Isr86] have also constructed extremal measures for which the global Markov property do not hold (not for the Ising model, but for a ferromagnetic three-spin interactions model).

1.3.4 Beyond the compact case: the Gaussian Free Field

In the previous paragraphs, we defined infinite-volume Gibbs measures for finite single-spin state spaces S. One may want to go further and consider infinite discrete set S, or continuous spins. As already mentioned, when S is compact, things behave almost like in the finite case. In particular, the existence of an infinite-volume Gibbs measure follows from compactness for absolutely summable potentials, and Theorem 1.9 holds.

The DLR formalism can be adapted naturally for more general single-spin state spaces, but a $\sigma-$finite *reference measure* or *a priori measure* λ on S is needed. We will mostly consider real-valued spins, although $S = \mathbf{S}^{d-1}$ the unit sphere of $\mathbf{R}^d$ is also widely studied in the literature. For these two examples, a natural choice is the Lebesgue measure (λ is not supposed to be a probability measure!). For countable single-spin state spaces, use the counting measure as reference measure, as we implicitly did for finite S (see footnote 4).

For all $\Lambda \subset V$, the product measure on S^Λ, usually denoted $\otimes_{i\in\Lambda}\lambda$, is abbreviated by λ^Λ. Gibbsian specifications are then defined in a similar way to Section 1.3.2: for any $\Lambda \Subset V$, $A \in \mathcal{F}_V$ and $\eta \in \Omega$,

$$\begin{aligned} \pi_\Lambda^{\Phi,\beta}(A \mid \eta) &:= \frac{1}{Z_{\beta,\Lambda}^\eta} \int_{\Omega_\Lambda^\eta} \mathbf{1}_{\{\omega\in A\}} \exp(-\beta H_\Lambda(\omega))\lambda^V(\mathrm{d}\omega) \\ &= \frac{1}{Z_{\beta,\Lambda}^\eta} \int_{S^\Lambda} \mathbf{1}_{\{\omega_\Lambda\eta_{\Lambda^c}\in A\}} \exp(-\beta H_\Lambda(\omega_\Lambda\eta_{\Lambda^c}))\lambda^\Lambda(\mathrm{d}\omega_\Lambda) \\ &= \int_{S^\Lambda} \mathbf{1}_{\{\omega_\Lambda\eta_{\Lambda^c}\in A\}} \pi_\Lambda^{\Phi,\beta}(\mathrm{d}\omega_\Lambda \mid \eta) \end{aligned}$$

where the last equality includes a convenient abuse of notation.

When one turns to non-compact single-spin state spaces, existence is no longer guaranteed even for very reasonable models, as we shall see with the *Gaussian Free Field*. The model is

defined on the $d-$dimensional lattice $\mathbf{Z}^d$, the single-spin state space being $S = \mathbf{R}$ and the set of configurations $\Omega = \mathbf{R}^{\mathbf{Z}^d}$ equipped with $\mathcal{F} := \mathcal{F}_{\mathbf{Z}^d}$ the product $\sigma-$algebra, generated by the cylinders. The Hamiltonian of the Gaussian Free Field (GFF) is given for any $\Lambda \Subset \mathbf{Z}^d$ and $\omega \in \mathbf{R}^{\mathbf{Z}^d}$ by

$$H_{\Lambda,m}(\omega) = \frac{1}{4d} \sum_{\{i,j\}\cap\Lambda\neq\emptyset i\sim j} (\omega_i - \omega_j)^2 + \frac{m^2}{2} \sum_{i\in\Lambda} \omega_i^2$$

where $m \geq 0$ is the *mass* of the model. There are only pairwise interactions, so it corresponds to a nearest-neighbours potential, but is not absolutely summable due to the fact that the ω_i are unbounded. Since the spins are real valued, up to renormalisation by $\sqrt{\beta}$, one can eliminate the parameter β (or always take $\beta = 1$). The Gibbs measure for the GFF in Λ with boundary condition $\eta \in \mathbf{R}^{\mathbf{Z}^d}$ is therefore the distribution on $(\Omega, \mathcal{F})$ defined by

$$\forall A \in \mathcal{F},\ \mu^{\eta}_{m,\Lambda}(A) = \frac{1}{Z^{\eta}_{m,\beta}} \int_{\mathbf{R}^\Lambda} \mathbf{1}_{\{\omega_\Lambda \eta_{\Lambda^c} \in A\}} \exp(-H_{\Lambda,m}(\omega_\Lambda \eta_{\Lambda^c}))\ \mathrm{d}\omega_\Lambda$$

where we use the Lebesgue measure as reference measure, $\mathrm{d}\omega_\Lambda$ denoting the product $\otimes_{i\in\Lambda}\mathrm{d}\omega_i$. The set of distributions on $(\Omega, \mathcal{F})$ satisfying the (DLR) equations for the GFF with mass $m \geq 0$ is denoted $\mathcal{G}_m$.

Before studying the infinite-volume Gibbs measures of the GFF, we justify its designation as "Gaussian". Rewrite the Hamiltonian as

$$H_{\Lambda,m}(\omega) = -\frac{1}{2d} \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\sim j}} \omega_i\omega_j + \frac{1+m^2}{2} \sum_{i\in\Lambda} \omega_i^2$$

and observe that if one considers the null boundary condition $\eta \equiv 0$, one has

$$H_{\Lambda,m}(\omega_\Lambda 0_{\Lambda^c}) = \frac{1+m^2}{2} \sum_{i,j\in\Lambda} \omega_i(\delta_{i,j} - P(i,j))\omega_j$$

where $\delta_{i,j}$ is the Kronecker symbol and $P(i,j) = \begin{cases} \frac{1}{1+m^2}\frac{1}{2d} & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$. Recalling the formula of the density function of Gaussian vectors (see (2.1) in Chapter 2), it results that $\mu^0_{m,\Lambda}$ is a Gaussian measure of mean 0 and covariance given by the inverse of the matrix $(1+m^2)(I_\Lambda - P_\Lambda)$ indexed by Λ (I_Λ is the identity and elements of P_Λ are the $P(i,j)$ for $i,j \in \Lambda$ defined above). A similar computation can be done for an arbitrary boundary condition. One finds the same covariance kernel but the mean varies according to the boundary condition.

The covariance kernel of the finite-volume GFF admits a nice interpretation in terms of the Green function of a random walk. Observe that for $m = 0$, $(P(i,j))_{i,j\in\mathbf{Z}^d}$ is the transition kernel of the simple random walk on $\mathbf{Z}^d$. For positive mass $m > 0$, this kernel is defective. We thus add a new site $\clubsuit$ to $\mathbf{Z}^d$, called the graveyard, and extend the definition of P to make it a transition kernel on $\mathbf{Z}^d \cup \{\clubsuit\}$:

$$P(i,j) = \begin{cases} \frac{1}{1+m^2}\frac{1}{2d} & \text{if } i \sim j, i, j \in \mathbf{Z}^d \\ 1 - \frac{1}{1+m^2} & \text{if } i \in \mathbf{Z}^d, j = \clubsuit \\ 1 & \text{if } i = j = \clubsuit \\ 0 & \text{otherwise} \end{cases}.$$

Let $Z = (Z_n)_{n\geq 0}$ be the Markov chain on $\mathbf{Z}^d \cup \{\clubsuit\}$ with transition kernel P; Z is called the *killed random walk* on $\mathbf{Z}^d$, killed at rate $1 - \frac{1}{1+m^2}$. Denote by $\mathbf{P}_i$ the distribution of Z starting at i, and $\mathbf{E}_i$ the corresponding expectation.

Then, the inverse matrix of $(I_\Lambda - P_\Lambda)$ is simply given by the Neumann series:

$$(I_\Lambda - P_\Lambda)^{-1} = \sum_{n\geq 0} P_\Lambda^n.$$

It follows that the covariance of $\mu^0_{m,\Lambda}$ (and it is the same for $\mu^\eta_{m,\Lambda}$ for every η) is

$$\mu^0_{m,\Lambda}(X_i X_j) = \frac{1}{1+m^2}\mathbf{E}_i\left[\sum_{n=0}^{\tau_\Lambda - 1} \mathbf{1}_{\{Z_n = j\}}\right] =: G_\Lambda(i,j) \tag{1.6}$$

for all $i, j \in \Lambda$, where $\tau_\Lambda = \inf\{n \geq 0 \mid Z_n \notin \Lambda\}$. Thus, G_Λ is ($(1+m^2)^{-1}$ times) the Green function of the simple random walk killed at rate $1 - \frac{1}{1+m^2}$ and at the boundary of Λ.

When $m > 0$, the limit in (1.6) when $\Lambda \uparrow \mathbf{Z}^d$ is well-defined (because the walk is killed in geometric time, which is $\mathbf{P}_i$−a.s. finite) and one can set

$$G_m(i,j) = \frac{1}{1+m^2}\mathbf{E}_i\left[\sum_{n\geq 0} \mathbf{1}_{\{Z_n = j\}}\right], \ \forall i, j \in \mathbf{Z}^d.$$

One can then show that the Gaussian field[6] on $\mathbf{Z}^d$ with mean 0 and covariance G_m, whose distribution is denoted $\mathcal{N}(0, G_m)$, satisfies the (DLR) equations of the GFF with mass m, thus belongs to $\mathcal{G}_m$. It can also be proved that this measure is extremal.

Remark 1.12. *Performing the calculations with an arbitrary boundary condition η, one finds that $\mathcal{N}(u, G_m) \in \operatorname{ex}\mathcal{G}_m$ for any $m-$harmonic function $u \in \Omega$ (that is, satisfying $u_i = \frac{1}{1+m^2}\frac{1}{2d}\sum_{j\sim i} u_j$ for all $i \in \mathbf{Z}^d$, i.e. $(m^2 - \frac{1}{2d}\Delta)u = 0$ if Δ denotes the discrete Laplacian on $\mathbf{Z}^d$), giving rise to uncountably many extremal Gibbs measures.*

For the massless case $m = 0$, in dimension $d \geq 3$ the situation is identical. In fact, the limit in (1.6) as $\Lambda \uparrow \mathbf{Z}^d$ is still well defined because the simple random walk on $\mathbf{Z}^d$, $d \geq 3$, is transient. In dimension 1 or 2 however, the simple random walk is recurrent and in particular the variance of a single spin diverges:

$$\mathrm{Var}^\eta_{0,\Lambda}(X_0) = G_\Lambda(0,0) \xrightarrow[\Lambda\uparrow\mathbf{Z}^d]{} \sum_{n\geq 0} \mathbf{P}_0(Z_n = 0) = +\infty.$$

This prevents the existence of an infinite-volume Gibbs measure, leading to our first example of a model for which $\mathcal{G}_0 = \emptyset$, even though the Hamiltonian is nearest-neighbours.

Chapter 3 will be devoted to the study of the GFF with high mass, parametrised in a more suitable way. Let $\varepsilon \in (-1, 1)$. We will consider a centered Gaussian field $(X_i)_{i\in\mathbf{Z}^d}$ on $\mathbf{Z}^d$ of law μ_ε such that the conditional distribution of X_i given the rest of the configuration is Gaussian of variance 1 and mean $\varepsilon\times$ (average of the neighbours):

[6] We will properly review the definition of Gaussian fields on infinite graphs in Chapter 2.

$$\mu_\varepsilon(X_i \in \cdot \mid X_j = \xi_j, \forall j \neq i) = \mathcal{N}\Big(\frac{\varepsilon}{2d}\sum_{j\sim i}\xi_j, 1\Big), \text{ for } \mu_\varepsilon\text{−a.e. } \xi \in \mathbf{R}^{\mathbf{Z}^d}. \tag{1.7}$$

This only describes the singleton part of the specification of μ_ε. As already mentioned, it is sufficient to characterise the specification ([Geo11] Theorem 1.33) and here, the formula of Gaussian density allows us to identify from (1.7) what is the Hamiltonian of the model:

$$\mu_\varepsilon(X_i \in \mathrm{d}x \mid X_j = \xi_j, \forall j \neq i) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\Big(x - \frac{\varepsilon}{2d}\sum_{j\sim i}\xi_j\Big)^2\right)\mathrm{d}x = \frac{1}{Z_i^\xi}\exp[-H_i(x_i\xi_{\{i\}^c})]\mathrm{d}x$$

where the pointwise Hamiltonian is given by $H_i(x) = \frac{1}{2}x_i^2 - \frac{\varepsilon}{2d}\sum_{j\sim i}x_i x_j$ for every $x \in \mathbf{R}^{\mathbf{Z}^d}$. This uniquely determines the potential, and the Hamiltonian on any finite box $\Lambda \Subset \mathbf{Z}^d$ associated with μ_ε is therefore given by

$$H_\Lambda(x) = \frac{1}{2}\sum_{i\in\Lambda}x_i^2 - \frac{\varepsilon}{2d}\sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset\\ i\sim j}}x_i x_j, \ x \in \mathbf{R}^{\mathbf{Z}^d}.$$

For $\varepsilon > 0$, with the renormalisation $\omega_i = \sqrt{\varepsilon}x_i$, we find again the Hamiltonian of the GFF with mass m satisfying $(1+m^2)^{-1} = \varepsilon$; in order to include the case $\varepsilon < 0$, one need to consider the GFF with negative temperature (i.e. $\beta = -1$) and mass satisfying $(1+m^2)^{-1} = |\varepsilon|$.

We prove that for $|\varepsilon|$ small enough, the restriction of μ_ε to finite boxes is exponentially close to a finitely dependent process (see Definition 2.4). More precisely, for all $l \in \mathbf{N}$, there exists an l−dependent random field $(Y_i^l)_{i\in\mathbf{Z}^d}$ on $(\Omega, \mathcal{F})$ such that

$$\forall \Lambda \Subset \mathbf{Z}^d, d_{\mathrm{TV}}(X|_\Lambda, Y^l|_\Lambda) \leq |\Lambda|e^{-cl}$$

where $c > 0$ is a universal constant. This is the main result of Chapter 3 (see Theorem 3.3).

It is obtained by implementing a coupling-from-the-past argument, considering the Glauber dynamics associated with μ_ε. It also provides an explicit construction of the fact that μ_ε can be expressed as a factor of a product measure.

1.4 Factors of product measures

This section is concerned with the existence of *factor maps* from i.i.d. (independent and identically distributed) processes to stationary random fields (precise definitions will be given below). For simplicity, we consider random fields on $\mathbf{Z}^d$ for some $d \geq 1$, although the theory can be naturally extended to more general graphs (basically, most of the results stated here would hold for any amenable transitive graph). Originally, the interest for factor maps came from ergodic theory. Processes of special interest in ergodic theory are the so-called *Bernoulli shifts*, i.e the random fields (initially in dimension $d = 1$) that are isomorphic to an i.i.d. process. Ornstein proved in [Orn74] the isomorphism theorem for Bernoulli shifts, which states that two i.i.d. processes on $\mathbf{Z}$ of equal entropy are in fact isomorphic; he further showed that factors of i.i.d. processes are Bernoulli shifts. These results extend to higher dimensions (see [KW72] and [OW87]).

1.4.1 Factor maps and coding radius

Let $(\mathcal{X}, \mathcal{F})$ be a Borel space, and μ a stationary random field with single-site state space $\mathcal{X}$, i.e. a translation-invariant probability measure over $\mathcal{X}^{\mathbf{Z}^d}$, equipped with the product σ–algebra. We say that μ is a *factor of a product measure* or *factor of an i.i.d. process* (abbreviated FIID in the following) if there exists a probability space $(\mathcal{Y}, \mathcal{G}, \nu)$ and a measurable map $F : \mathcal{Y}^{\mathbf{Z}^d} \to \mathcal{X}^{\mathbf{Z}^d}$ which commutes with all the translations of $\mathbf{Z}^d$ such that μ is the image of $\nu^{\otimes \mathbf{Z}^d}$ by F. It is equivalent to say that there exists an i.i.d. $\mathcal{Y}-$valued process $Y = (Y_i)_{i \in \mathbf{Z}^d}$ such that $F(Y)$ has distribution μ. The factor map F is called a *coding* from $\nu^{\otimes \mathbf{Z}^d}$ to μ.

From statistical mechanics point of view, the notion of FIID is not satisfying since it does not catch phase transitions. Indeed, it is known that the "plus state" of the Ising model, that is the measure μ_β^+ defined in Section 1.3.1, is an FIID for any inverse temperature $\beta \geq 0$ (proved first by Ornstein and Weiss in an unpublished manuscript, then by Adams [Ada92]). Van den Berg and Steif [BS99] then provide a way of detecting phase transitions through the notion of *finitary* codings. The *coding radius* of the factor map F is defined as the random radius around the origin that one needs to reveal in the i.i.d. process to determine the spin at 0. More precisely, define the coding radius of F at site $i \in \mathbf{Z}^d$ and configuration $y \in \mathcal{Y}^{\mathbf{Z}^d}$ as

$$R_i(y) := \inf\{r \geq 0 : \forall y' \in \mathcal{Y}^{\mathbf{Z}^d}, y'|_{B_r^i} = y|_{B_r^i} \implies F(y)_i = F(y')_i\}$$

where $B_r^i := \{j \in \mathbf{Z}^d : ||j - i||_1 \leq r\}$ is the ball of radius r around i in $\mathbf{Z}^d$, and $y|_{B_r^i}$ is the restriction of y to B_r^i. When Y is as above, the $R_i(Y)$ for $i \in \mathbf{Z}^d$ are random variables that are equally distributed, due to translation invariance. Set $R := R_0(Y)$ (the choice of 0 is made arbitrarily, by translation invariance of μ); R is called the coding radius of F. If R is almost surely finite, the coding is said to be *finitary*. In this case, we say that μ is a *finitary factor* of an i.i.d. process, abbreviated FFIID. Moreover, when the coding radius satisfies $\mathbf{P}(R \geq n) \leq Ce^{-cn}$ for some $c, C > 0$ and all $n \in \mathbf{N}$, we say that μ is an FFIID with *exponential tails*. The main result of [BS99] is that the presence of a phase transition is "a fundamental obstruction for finitary coding"; in words, if there exists a distribution different from μ with the same specification, this prevents μ from being an FFIID. This shows that the plus state of the Ising model is *not* FFIID if $\beta > \beta_c$, despite being a Bernoulli shift. It is also proved in [BS99] that in the case of uniqueness, the Gibbs measure of the Ising model is an FFIID, with exponential tails if $\beta < \beta_c$, but with coding radius's $d-$th moment necessarily infinite at $\beta = \beta_c$ (this last result was jointly proved with Peres). Note that the existence of a finitary coding at the critical point follows from the continuity of the phase transition for the Ising model ([Yan52] [AF86] [ADCS15]).

Actually, van den Berg and Steif were mostly interested in *finite-valued* FFIID (abbreviated FV-FFIID), meaning that they additionally ask that the set $\mathcal{Y}$ in which the i.i.d. process takes its values is finite. In fact, they proved that the Ising model is an FV-FFIID in the whole subcritical phase, with no information on the coding radius (except that it is a.s. finite). Spinka [Spi20a] improved this statement by proving that the coding can be built so that $\mathbf{E}[R^d]$ is finite (even R has stretched-exponential tails), and Timár proved in [Tim23] that the critical Ising model is an FV-FFIID (only FFIID can be deduced from [BS99]). These two papers answer respectively Question 2 and Question 1 of [BS99]; an important problem (Question 3) that is still open is to determine whether in general uniqueness of the Gibbs measure implies (at least) FIID.

The aforementioned works make extensive use of CFTP arguments and monotone couplings (see Remark 1.6 in Section 1.2.2) in their constructions of finitary codings. In particular, it relies on the FKG inequality, which for example is not true for the Ising model with negative β. In the next paragraph, we present a construction, originally due to Häggström and Steif [HS00], of a finitary coding for the Ising model with β close to 0, but possibly negative. In fact, they prove the existence of a finitary coding for all Markov random fields which satisfy a *high-noise* condition. This is a condition on specifications, which, with the notations of Section 1.3, amounts to

$$\sum_{s\in S} \inf_{\eta\in S^{\mathbf{Z}^d}} \pi^{\Phi,\beta}_{\{i\}}(X_i = s \mid \eta) > 1 - \frac{1}{2d},\ i \in \mathbf{Z}^d. \tag{1.8}$$

The left-hand side is called the *multigamma admissibility* in [HS00], following the terminology introduced in [MG98]. It corresponds to the *maximal coupling probability* (see [Tho00] Chapter 1.4) of the collection $(\pi^{\Phi,\beta}_{\{i\}}(X_i \in \cdot \mid \eta))_{\eta\in\Omega}$ of distributions over S (note that the dependence on i disappears for translation-invariant specifications). The coupling event inequality states that in any coupling of a collection of distributions, the probability that their realisation all coincide in the coupling is upper bounded by the maximal coupling probability; moreover, there exists a coupling, qualified as maximal, where equality holds (see [Tho00] Theorems 4.1 and 4.2, or Chapter 3.7 for more general versions). Maximal couplings are the ones that favour the most coalescence, in the sense of the Propp and Wilson algorithm.

Note that (1.8) immediately implies the uniqueness condition obtained by van den Berg and Maes in [BM94], so that we are necessarily in the regime of uniqueness of the Gibbs measure when the high-noise condition is fulfilled. For the Ising model, it corresponds to taking $|\beta| < \frac{1}{4d}\log\left(1 + \frac{2}{2d-1}\right)$. More generally, for models with finite energy, the high noise assumption is fulfilled when $|\beta|$ is small enough (the multigamma admissibility equals 1 at $\beta = 0$, and is in general continuous in β).

1.4.2 Finitary coding for the Ising model with small $|\beta|$

The first step in order to build a finitary coding for the Ising model on $\mathbf{Z}^d$ is to extend the Glauber dynamics defined in Section 1.2.1 to the infinite-volume setting. It will provide an algorithm which terminates almost surely when $|\beta|$ is small enough, and which produces an exact sample of the distribution $\mu_\beta := \mu^+_\beta = \mu^-_\beta$, the (unique) Gibbs measure for the Ising model on $\mathbf{Z}^d$. The proof of the fact that the output is an unbiased sample of μ_β relies on a coupling-from-the-past argument, also extended to an infinite-volume version.

Continuous-time Glauber dynamics for the Ising model on $\mathbf{Z}^d$. We keep the core idea of the Glauber dynamics, which consists in resampling spins according to their conditional distribution given the neighbours. However, the updated site cannot be chosen uniformly over $\mathbf{Z}^d$. We rather attach an "exponential clock" at each site of $\mathbf{Z}^d$, and when the clock at a site i rings, update its spin conditionally on the current configuration outside i. Formally, the Glauber dynamics is an interacting particle system (see [Lig85]) on $\{\pm 1\}^{\mathbf{Z}^d}$ with stationary distribution μ_β. Below, we describe its *graphical representation* (see [Lig85] Chapter III.6 or [Dur88] Section 2)

Let $\mathcal{P}$ be a Poisson point process (PPP) of intensity $\# \otimes \mathrm{d}t \otimes \mathrm{d}u$, where $\#$, $\mathrm{d}t$, $\mathrm{d}u$ denotes the counting measure on $\mathbf{Z}^d$, the Lebesgue measure on $\mathbf{R}$ and the Lebesgue measure on $[0,1]$, respectively. Points of $\mathcal{P}$ are called the *update marks* of the dynamics. By properties of PPP, it means that sites of $\mathbf{Z}^d$ are independently endowed with a bi-infinite sequence $(T_n^\cdot, U_n^\cdot)_{n\in\mathbf{Z}}$, satisfying

$$\mathcal{P} = \{(i, T_n^i, U_n^i), n \in \mathbf{Z}, i \in \mathbf{Z}^d\},$$

and such that $(T_n^\cdot)_{n\in\mathbf{Z}}$ is increasing. This observation shows that $\mathcal{P}$ is distributed according to a product measure over $\mathbf{Z}^d$; we will build a factor map from this product measure to μ_β. For each $i \in \mathbf{Z}^d$, the $(T_n^i)_{n\in\mathbf{Z}}$ correspond to the times when site i is updated in the dynamics, and the increments $(T_{n+1}^i - T_n^i)_{n\in\mathbf{Z}}$ are independent and have exponential distribution of parameter 1 (hence the terminology "exponential clock"). The $(U_n^i)_{i\in\mathbf{Z}^d, n\in\mathbf{Z}}$ are independent random variables distributed uniformly on $[0,1]$; the update of i at time T_n^i is made with the help of the extra randomness provided by U_n^i. Define the *update function* $\varphi : \mathbf{R}\times[0,1] \to \{\pm 1\}$ (the same as in Remark 1.6) by

$$\varphi(s,u) = \begin{cases} -1 & \text{if } u \le (1+e^{2\beta s})^{-1} \\ +1 & \text{otherwise} \end{cases}. \tag{1.9}$$

For every update mark $(i,t,u) \in \mathcal{P}$, if the system is in configuration σ_{t-} before the update, then the new value of the spin at i is given by $\varphi(\sum_{j\sim i}\sigma_{t-}(j), u)$. The other spins remain unchanged, i.e. we have

$$\forall j \neq i, \sigma_t(j) = \sigma_{t-}(j) \text{ and } \sigma_t(i) = \varphi\left(\sum_{j\sim i}\sigma_{t-}(j), u\right).$$

Note that we have defined the evolution of the system on a time line which is infinite in both directions. If β is small enough, the ergodic theorem for the stochastic Ising model [Hol72] implies that all the systems converge to the stationary distribution μ_β. Nevertheless, since there are infinitely many sites, we cannot define the first update after a given initial time $\tau \in \mathbf{R}$. In other words, for every $\tau \in \mathbf{R}$,

$$\inf\{t \ge \tau : \exists i \in \mathbf{Z}^d, \exists u \in [0,1], (i,t,u) \in \mathcal{P}\} = \tau.$$

An additional argument is therefore needed to ensure that the dynamics can indeed be constructed with the graphical representation. The pitfall is that the first update of a given site (which is well defined) could require one to look infinitely far away. Indeed, for each update one need to look at the neighbours, but it may happen that these neighbours needed to look at their own neighbours before etc. A percolation argument of Harris [Har72] (see also [Dur88] Section 2), handles this problem. The idea is to take a random subgraph of $\mathbf{Z}^d$ defined as follows: fix a small positive number t_0; for every $i, j \in \mathbf{Z}^d$ such that $i \sim j$, say that the edge $\{i,j\}$ is open if there exists $n \in \mathbf{Z}$ such that $\tau < T_n^i < \tau + t_0$ or $\tau < T_n^j < \tau + t_0$; otherwise, the edge is closed. Hence, an open edge corresponds to the fact that an update of the system requires i to look at j or the reverse. Consequently, if i and j are not in the same connected component in our random graph, their spin at time $\tau + t_0$ are independent (assuming they were chosen independently at the initial time τ). Hence, one can compute separately the evolution up to time $\tau + t_0$ in each connected component. Harris's result is that for t_0 small enough, there is

no infinite connected component for almost all realisation of the PPP. Thus, each connected component have a first update time, so we can construct the system up to time $\tau + t_0$. Iterate the argument to construct the system for all time $t \geq \tau$.

Backward exploration of the dynamics. To determine the value of the spin at $(i,t) \in \mathbf{Z}^d \times \mathbf{R}$ in a realisation of the dynamics, one needs to explore the dynamics backward starting from (i,t). We go back to the last update of site i before time t, i.e. consider the update mark (i, T_n^i, U_n^i) where $n = \max\{n \in \mathbf{Z} \mid T_n^i < t\}$. At first glance, the output produced during this update through the update function (see (1.9)) depends on the values of the spins of the neighbours at time T_n^i. Actually, it might be the case that this update produces the same value for site i regardless of the current configuration. This is the case when U_n^i is below the smallest possible value for the threshold $(1+e^{2\beta s})^{-1}$ in (1.9), or above its largest possible value. The parameter s represents the sum of the spins of the neighbours in the current configuration, so it takes values only between $-2d$ and $2d$. The possible values for the threshold $(1+e^{2\beta s})^{-1}$ are therefore between $(1+e^{4|\beta|d})^{-1}$ and $(1+e^{-4|\beta|d})^{-1}$. Thus, if $U_n^i \leq (1+e^{4|\beta|d})^{-1}$, the update function necessarily outputs -1, so we have $\sigma_t(i) = -1$; similarly, if $U_n^i > (1+e^{-4|\beta|d})^{-1}$ we get $\sigma_t(i) = +1$. Note that in both cases, $\sigma_t(i)$ is determined only by U_n^i, and does not depend on the initialisation of the dynamics. Otherwise, when U_n^i is between these two extremal values, the output really depends on the current configuration, so one needs to continue the backward exploration to determine the spin at (i,t). It means that we repeat this procedure for all the neighbours of i starting from time T_n^i. If it occurs that for all $j \sim i$, the uniform variable (i.e. the third coordinate) of the last update mark carrying by j before time T_n^i is not between $(1+e^{4|\beta|d})^{-1}$ and $(1+e^{-4|\beta|d})^{-1}$, then all the spins for the neighbours of i at time T_n^i are determined by the exploration. Denoting by S their sum, it follows that one can compute $\sigma_t(i) = \varphi(S, U_n^i)$. Again, otherwise, we continue the exploration.

Let

$$\gamma(\beta) := \frac{1}{1+e^{4|\beta|d}} + 1 - \frac{1}{1+e^{-4|\beta|d}} = 1 - \tanh(2d|\beta|).$$

Then, $\gamma(\beta)$ corresponds to the proportion of updates marks that produces the same output whatever the current configuration; this quantity equals the multigamma admissibility of [HS00] (see (1.8) above). It can be deduced from the above discussion that the number of update marks that one needs to reveal in the backward exploration is dominated by the total population of a Galton-Watson process of reproduction law $\gamma(\beta)\delta_0 + (1-\gamma(\beta))\delta_{2d}$. When this process dies out, that is when the total population is finite, the backward exploration of the dynamics terminates in finite time. This implies that the value of the system at (i,t) is a measurable function of the update marks belonging to a finite box of $\mathbf{Z}^d \times (-\infty, t) \times [0,1]$, and in particular this value becomes independent of the initial configuration of the dynamics. When $2d\tanh(2d|\beta|) < 1$, i.e. $|\beta| < \frac{1}{4d}\log\left(1+\frac{2}{2d-1}\right)$, the Galton-Watson process is subcritical so the above situation happens for every $(i,t) \in \mathbf{Z}^d \times \mathbf{R}$ and almost all realisation of the PPP.

Coupling-from-the-past argument. The graphical representation of the Glauber dynamics provides a coupling of the systems starting from any time and any initial condition, obtained by using the same Poisson point process to construct them. Write $\sigma^{\omega,\tau} = (\sigma_t^{\omega,\tau})_{t\geq\tau}$ for the system started at time $\tau \in \mathbf{R}$ with initial configuration $\sigma_\tau^{\omega,\tau} = \omega$, whose evolution is governed by the update marks of $\mathcal{P}$. We now implement Propp and Wilson ideas to get an exact

sampling of the stationary distribution μ_β, arising as a finitary factor of $\mathcal{P}$.

For every $i \in \mathbf{Z}^d$, let

$$\tau_i = \sup\{\tau \leq 0 \mid \forall \omega, \omega' \in \{\pm 1\}^{\mathbf{Z}^d},\ \sigma_0^{\omega,\tau}(i) = \sigma_0^{\omega',\tau}(i)\},$$

that is, τ_i is the largest (negative) time such that the value of the spin at i at time 0 is the same in all the dynamics initialised at that time from any configuration. Observe that τ_i is a random time, measurable with respect to $\mathcal{P}$. The comparison with a subcritical branching process in the previous paragraph shows that τ_i is almost surely finite for every site $i \in \mathbf{Z}^d$. Therefore, one can define a random field $\sigma^* = (\sigma_i^*)_{i \in \mathbf{Z}^d}$ by

$$\sigma_i^* = \sigma_0^{\cdot,\tau_i}(i),\ i \in \mathbf{Z}^d$$

(one can consider any initial configuration for the dynamics, since by definition of τ_i the value of σ_i^* will be the same for any choice). Note that σ^* is a measurable function of $\mathcal{P}$, so it is an FIID. Then, the fact that σ^* is distributed according to μ_β follows from a classical coupling-from-the-past argument, similar to the proof of Theorem 1.5: fix $\Lambda \Subset \mathbf{Z}^d$ and let ω be distributed according to μ_β, independent of $\mathcal{P}$; since μ_β is invariant for the dynamics, it follows that for all $t < 0$, $\sigma_0^{\omega,t}$ has law μ_β; on the other hand, for all $t \leq \min_{i \in \Lambda} \tau_i$, one has $\sigma_0^{\omega,t}(i) = \sigma_i^*$ for all $i \in \Lambda$; therefore, $\sigma^*|_\Lambda$ is distributed according to $\mu_\beta|_\Lambda$; Λ was arbitrary so the result follows.

This construction yields a coding for μ_β from an i.i.d. process, providing $|\beta|$ is small enough. It is immediate that this coding is finitary, and it is even not difficult to see that it furthermore has exponential tails. In fact, the coding radius is bounded by the number of update marks revealed during the backward exploration from $(0, 0)$; from the above domination by a subcritical branching process, it follows that it has exponential decay. Deriving a coding from a finite-valued i.i.d. process requires an additional step, first carried out in [BS99]. We do not reproduce this step here, since our main goal is then to extend these techniques to Gaussian fields. In this case, the state-space being continuous, it is impossible to get a coding from a finite-valued process, because the entropy cannot increase via a coding. In Chapter 3, we first build a finitary coding for a Gaussian model conditioned to a compact, for which the construction is very similar to the one described here for the Ising model. The construction of a coding for our unconditioned Gaussian model (already defined in the end of Section 1.3.4) is more intricate, since, as we shall see, the multigamma admissibility vanishes in this case.

1.5 Models with long-range interactions

In the previous sections, we were mostly interested in statistical mechanics models with nearest-neighbours (or finite-range) interactions. These models share the nice property of being Markovian. This final section of the Introduction is devoted to long-range models. Recall that the range of a potential on a graph has been introduced in Definition 1.1. We take $\mathbf{Z}^d$ as underlying graph, in order to have a fixed notion of the range, although the following models can be defined in a more general setting. A potential Φ, or equivalently its associated Hamiltonian, is said to be *long-range* if $r(\Phi) = +\infty$. It means that there are interactions between spins at arbitrarily large distance from each others. We will consider two examples of long-range models. First, we will study the long-range Ising model, and especially show

that the above construction of a finitary coding for small $|\beta|$ can be performed, assuming that the strength of the interactions decays sufficiently fast. Then, we will introduce the random-cluster model, or FK-percolation, which is a widely studied statistical mechanics model. This model will be the central object of Chapter 5.

1.5.1 Long-range Ising model

We consider a long-range Ising model on $\mathbf{Z}^d$, $d \geq 1$. The space of configuration is $\Omega = \{-1, +1\}^{\mathbf{Z}^d}$ and the interactions are given by *coupling constants* $(J_{ij})_{i,j\in\mathbf{Z}^d}$ that satisfy the following summability condition:

$$\forall i \in \mathbf{Z}^d, \sum_{j\in\mathbf{Z}^d} |J_{ij}| < \infty. \tag{1.10}$$

Here, we will *not* assume that the coupling constants are non-negative, so that the model is not necessarily ferromagnetic. The Hamiltonian of the model is defined for every finite subset Λ of $\mathbf{Z}^d$ and $\sigma \in \Omega$ as

$$H_\Lambda(\sigma) = - \sum_{\{i,j\}\cap\Lambda\neq\emptyset} J_{ij}\sigma(i)\sigma(j).$$

Assumption (1.10) ensures that the corresponding potential is absolutely summable (see (1.2)), so the above Hamilontian is well defined. We can then consider Gibbs measures associated with a certain inverse temparature β, as we did generally in Section 1.3. Since a sign can be included in the coupling constants, we can restrict ourselves to $\beta \geq 0$. In this setting, it does not imply that all the interactions have the same sign.

We will focus on couplings constants that satisfy $|J_{ij}| \leq J(\|j - i\|_1)$ for all $i, j \in \mathbf{Z}^d$ and some function $J : \mathbf{N} \to \mathbf{R}_+$. We assume that $J(n) = O(n^{-\alpha})$ for some $\alpha > d$, so that (1.10) is satisfied. Thus, α is a parameter which controls the decay of the interactions with the distance.

The first result concerning phase transition for the long-range Ising model was obtained in 1966 by Ginibre, Grossman and Ruelle [GGR66]. They generalise the Peierls argument for the phase transition of the nearest-neighbours Ising model to prove that the long-range model admits several Gibbs measures at low enough temperature in dimension $d \geq 2$ when $\alpha > d+1$. In the ferromagnetic case, i.e. $J_{ij} \geq 0$ for all i, j, the Griffiths inequalities [Gri67] (also known as GKS inequalities [KS68]) enable to prove phase transition at low temperature in the full regime $\alpha > d$.

For one-dimensional long-range Ising model, uniqueness of the Gibbs measure has been proved for any temperature in [Rue75] when $\alpha > 2$, generalising the nearest-neighbours result of Ising thesis. Kac and Thompson conjectured in [KT69] that the model undergoes a phase transition for $\alpha \in (1, 2]$; it has been proved by Dyson [Dys69] for $\alpha \in (1, 2)$ and then by Fröhlich and Spencer [FS82] in the case $\alpha = 2$.

Harel and Spinka [HS22] proved that the ferromagnetic long-range Ising model is FFIID in the full regime of uniqueness, in any dimension $d \geq 1$. This provides a statement similar to that of Van den Berg and Steif [BS99] for the finite-range Ising model. Without constraint on the sign of the coupling constants, their technique, which strongly rely on monotonicity, cannot apply. In [HS00], Häggström and Steif exploit the spatial Markov property to construct finitary codings (in particular for the finite-range Ising model with small $|\beta|$). This fails for the

long-range Ising model as there are interactions at arbitrary large distance. In fact, one could try to imitate the construction sketched in Section 1.4.2, but one would end up with a backward exploration dominated by a Galton-Watson process of reproduction law "$\gamma\delta_0 + (1-\gamma)\delta_{+\infty}$" ($\gamma$ being the multigamma admissibility, which can be defined the same way). Indeed, within each update where we fail to coalesce directly, we need to look at the spin value of every other site. However, the influence of a spin on the update of a given site decays with the distance. The idea to get a finitary coding is therefore to reveal the configuration on a random ball around the update site, so that it is sufficient to compute the new value of the spin. Then, under a modified version of the high-noise assumption (1.8), it will prove that the long-range Ising model is FFIID, without assuming that it is ferromagnetic. This was first implemented by Galves, Löcherbach, and Orlandi in [GLO10] (including non-pairwise interactions).

The following result, for which we independently find a proof, is a special case of Theorem 1 in [GLO10]. A sketch of proof is given in the Appendix of the thesis. It follows the same outline as Section 1.4.2, but we believe it contains too much technicalities for this introduction.

Theorem 1.13. *The long-range Ising model with $\alpha > 2d$ is FFIID if β is small enough.*

In particular, in the above statement, we take β small enough so that we have uniqueness of the Gibbs measure (recall that it is necessary to have finitary codings). This is always possible to do so, for all dimension $d \geq 1$ and $\alpha > d$. Note however that we do not claim that this statement holds up to the actual critical parameter, above which there is a phase transition. Note also that we are unable to construct a finitary coding for the long-range Ising model when $\alpha \in (d, 2d]$.

1.5.2 FK percolation

The *random-cluster* model was introduced by Fortuin and Kasteleyn in 1972 [FK72], to provide a unification of several statistical mechanics models of interest. It is a *percolation* model, in the sense that it produces a random subgraph of a given graph, so it also often called the Fortuin-Kasteleyn percolation or for short FK percolation.

Let $G = (V, E)$ be a graph. A percolation configuration $\omega = (\omega_e)_{e\in E}$ on G is an element of $\{0,1\}^E$. We say that an edge $e \in E$ is *open* (in the configuration ω) if $\omega_e = 1$, and that it is *closed* if $\omega_e = 0$. A configuration can thus be identified with a subgraph of G with the same vertex set keeping only the open edges. The connected components of this subgraph made of open edges are called the *clusters* of ω, that is, two vertices are in the same cluster if they are connected in G by a path formed of open edges (thus, a cluster can be reduced to an isolated vertex). The simplest example of percolation model is the *Bernoulli percolation:* it is characterised by a parameter $p \in [0,1]$, and edges are independently set to be open with probability p. The corresponding distribution on $\{0,1\}^E$, denoted $\mathbb{P}_p^G$, is simply the product measure with marginal Bernoulli law of parameter p on each edge. This model has been the subject of numerous studies, in particular because it is the simplest model exhibiting a phase transition. In percolation theory, phase transition refers to the absence or the presence of an infinite cluster in a random configuration of law $\mathbb{P}_p^G$, for an infinite graph G (of course, there is always uniqueness of the Bernoulli percolation measure, since it is a product measure). The Kolmogorov 0-1's law tells us that the event of existence of an infinite cluster has probability 0 or 1, so by monotonicity, we can define a *critical parameter* $p_c(G) \in [0,1]$ such that:

- $\forall p < p_c(G), \mathbb{P}_p^G(\exists \text{ an infinite cluster}) = 0$; in this case, the model is said to be *subcritical*

- $\forall p > p_c(G), \mathbb{P}_p^G(\exists \text{ an infinite cluster}) = 1$; in this case, the model is said to be *supercritical*.

For $G = \mathbf{Z}^d$, one has $p_c(\mathbf{Z}^d) \in (0,1)$ for all $d \geq 2$, whereas $p_c(\mathbf{Z}) = 1$. In dimension $d = 2$, Kesten [Kes80] proved that the critical parameter is exactly $1/2$.

We now introduce the FK percolation model. It is more complicated than Bernoulli percolation since the state of the edges (open or closed) will no longer be independent. The model can be defined on any finite graph, but we focus in this Section to boxes of $\mathbf{Z}^d$, for $d \geq 1$. Let $\Lambda \Subset \mathbf{Z}^d$ be a finite subset of $\mathbf{Z}^d$ and denoted by $E(\Lambda)$ the set of edges having both endpoints in Λ (i.e. $(\Lambda, E(\Lambda))$ is a finite subgraph of $\mathbf{Z}^d$). Let $\partial\Lambda$ be the set of vertices in Λ having (at least) one neighbour outside it. A *boundary condition* is a partition of $\partial\Lambda$. For a configuration $\omega \in \{0,1\}^{E(\Lambda)}$ and a boundary condition α, we define a graph obtained from ω by identifying all the vertices that are in a same component in α; this procedure is called *wiring*, and the obtained graph is denoted $\omega \cup \alpha$. Let $k(\omega, \alpha)$ be the number of connected component, or clusters, in the graph $\omega \cup \alpha$. A configuration ξ on $\mathbf{Z}^d$ induces a partition of $\partial\Lambda$ by saying that two vertices are in the same component if and only if they are connected by an open path in ξ that does not use an edge in $E(\Lambda)$. In this case, we will identify in the notations the partition with the configuration. This expression of boundary conditions as a configuration on the complement is closer to what we defined for the Ising model or statistical mechanics models in general, in Section 1.3.1. However, not all partitions of $\partial\Lambda$ can be produced this way, so this definition is a priori richer. A first example of boundary condition when the partition is composed of singletons only so that there is no wiring at all; it is called the free boundary condition, and denoted $\alpha = 0$. The opposite case, called wired boundary condition, is obtained by taking as partition $\{\partial\Lambda\}$, wiring all the vertices of the boundary; it is denoted $\alpha = 1$.

The *random cluster measure* or *FK measure* on Λ with parameters $p \in [0,1]$ and $q > 0$ and boundary condition α is defined by

$$\phi^{\alpha}_{\Lambda,p,q}(\{\omega\}) := \frac{1}{Z^{\alpha}_{\Lambda,p,q}} \left(\prod_{e \in E(\Lambda)} p^{\omega_e}(1-p)^{1-\omega_e} \right) q^{k(\omega,\alpha)}, \ \forall \omega \in \{0,1\}^{E(\Lambda)} \tag{1.11}$$

where $Z^{\alpha}_{\Lambda,p,q}$ is the normalising constant that makes it a probability measure. The case $q = 1$ corresponds to the classical Bernoulli percolation. When $q \neq 1$, it is a dependent percolation model. It is not possible in general to compute the marginal on each edge, but we have a very simple formula for the conditional probability for an edge to be open given the rest of the configuration: one has a.s. for every $e \in E(\Lambda)$,

$$\phi^{\alpha}_{\Lambda,p,q}\left(\omega_e = 1 \mid \omega|_{E\setminus\{e\}}\right) = \begin{cases} p & \text{if } \omega|_{E\setminus\{e\}} \in K^{\alpha}_e \\ \frac{p}{p+q(1-p)} & \text{if } \omega|_{E\setminus\{e\}} \notin K^{\alpha}_e \end{cases}, \tag{1.12}$$

K^{α}_e being the event that both endpoints of e are connected by an open path without using edge e. This follows from the fact that $\dfrac{\phi^{\alpha}_{\Lambda,p,q}\left(\omega_e = 1 \mid \omega|_{E\setminus\{e\}}\right)}{\phi^{\alpha}_{\Lambda,p,q}\left(\omega_e = 0 \mid \omega|_{E\setminus\{e\}}\right)} = \dfrac{p}{1-p} q^{-\mathbf{1}_{\{\omega \notin K^{\alpha}_e\}}}$, which is a direct consequence of (1.11). In the following we write $p' := \dfrac{p}{p+(1-p)q}$ for short, although p' depends on both p and q. Note that when $p \in (0,1)$, we have $q < 1$ if and only if $p < p'$.

From this, one can observe that in the regime $q < 1$, the FK measure is biased in favour of configurations with few clusters (in comparison to the product measure $\mathbb{P}_p^\Lambda$), whereas $q > 1$ favours configurations with more clusters.

Let us now mention the connection between the FK percolation and other well-known statistical mechanics models, which was the original motivation of Fortuin and Kasteleyn to introduce the model. We have already seen that the case $q = 1$ corresponds to Bernoulli percolation. In the case $q = 2$, there is beautiful coupling with the Ising model (this generalises to the $q-$states Potts model for larger integer values of q). From $\omega \in \{0,1\}^{E(\Lambda)}$, one can define a spin configuration $\sigma \in \{\pm 1\}^\Lambda$ by independently and uniformly assigning a spin $+1$ or -1 to each cluster of ω that does not intersect the boundary; this means that all the vertices of the same cluster receive the same spin; then, vertices that are connected to the boundary by an open path are assigning spin $+1$. Now, let $\beta \geq 0$ and assume that ω is distributed according to the FK measure $\phi^1_{\Lambda,p,2}$ with $p = 1 - e^{-2\beta}$. Then, it follows that the spin configuration σ constructed above is distributed according to the Ising measure $\mu^+_{\beta,\Lambda}$. This coupling (ω, σ) of an FK configuration and an Ising configuration is called the Edwards-Sokal coupling [ES88]. In the other direction, one can also get ω from σ with the following procedure: for all $e = \{x, y\} \in E(\Lambda)$, if $\sigma_x \neq \sigma_y$, set $\omega_e = 0$, otherwise set

$$\omega_e = \begin{cases} 1 & \text{with probability } 1 - e^{-2\beta} \\ 0 & \text{with probability } e^{-2\beta} \end{cases}.$$

Then, if σ has law $\mu^+_{\beta,\Lambda}$, it follows that ω is distributed according to $\phi^1_{\Lambda,p,2}$ with $p = 1 - e^{-2\beta}$.

One also obtain a nice feature when looking at the behaviour of the FK percolation when $q \to 0$ (we refer to [Gri06] Chapter 1.5). For this, it can be helpful to consider the parametrisation of the model with p and p', and observe that

$$q = \frac{p}{1-p}\frac{1-p'}{p'},$$

so that q goes to 0 when $p \to 0$ or $p' \to 1$ (recall also that $p \leq p'$ for $q < 1$). Suppose that we take the free boundary condition and that $(\Lambda, E(\Lambda))$ is connected. First, if $p > 0$ is fixed and $q \to 0$, or equivalently $p' \to 1$, one can see that the leading terms in the partition function

$$Z^0_{\Lambda,p,q} = \sum_{\omega \in \{0,1\}^{E(\Lambda)}} \left(\prod_{e \in E(\Lambda)} p^{\omega_e}(1-p)^{1-\omega_e} \right) q^{k(\omega,0)}$$

are those for which $k(\omega, 0)$ is minimal, thus equals to 1. This leads in the limit to a distribution concentrated on configurations which generates a connected subgraph of Λ. This can also be understood intuitively from (1.12): if $p' \to 1$, the conditional probability of opening an edge given that its endpoints are not in the same cluster without this edge goes to 1. For example, it means that if one applies a Glauber dynamics corresponding to this model, in this regime it would not be possible to split a cluster, and conversely an edge adjacent to two distinct cluster would necessarily be opened. In fact, the limit when $q \to 0$ for fixed $p \in (0, 1)$ of the FK measure in Λ with free boundary condition is the Bernoulli percolation of parameter p in Λ conditioned to produce a connected subgraph.

The situation is reversed if "$p' < 1$ is fixed" and $p \to 0$; it corresponds to cases when p goes to 0 at least as fast as q does. Here, the conditional probability of opening an edge given

that there is already an open path connecting its endpoints vanishes. It means that in this regime, the limiting distribution will concentrated on configurations without cycles; those are called (spanning) *forest* (the word "spanning" referring to the fact that it is a subgraph with the same vertex set as the original graph).

It remains the case when both p and $1-p'$ go to 0. As $p' = (1+(1-p)q/p)^{-1}$, it is equivalent to $p \to 0$ and $q/p \to 0$, that is, p converges to 0 slower than q does. In this case, the two preceding behaviour combine. The Glauber dynamics becomes deterministic, since at the limit the conditional probability of opening an edge is 1 when its endpoints are not in the same cluster and 0 if they are. Then, it follows that the limiting distribution is concentrated on configuration that are both connected and a spanning forest, i.e. on the *spanning trees* of Λ. One can see moreover that each will have the same mass, so the limiting object is a called the *uniform spanning tree*.

Finally, we aim to consider the model on $\mathbf{Z}^d$ rather than on finite subgraphs. Except in the case $q = 1$, it is not possible define directly an FK measure in infinite-volume. One can proceed as in Section 1.3.1, taking weak-limits along an increasing sequence of boxes, with some sequence of boundary conditions. One can also notice that for all $\Lambda \subset \Delta \Subset \mathbf{Z}^d$ and all $\xi, \xi' \in \{0,1\}^E$ that coincide outside $E(\Delta)$,

$$\phi^{\xi'}_{\Delta,p,q}(\cdot \mid \omega|_{E(\Delta)\setminus E(\Lambda)} = \xi|_{E(\Delta)\setminus E(\Lambda)}) = \phi^{\xi}_{\Lambda,p,q}(\cdot).$$

This follows from (1.12), applied successively for all edges in $E(\Delta) \setminus E(\Lambda)$. This means that $(\phi^{\xi}_{\Lambda,p,q})_{\Lambda\Subset\mathbf{Z}^d,\xi\in\{0,1\}^E}$ behaves like a specification. We can then define infinite-volume measures with the DLR approach, as in Section 1.3.2. It is not true in general that the set of infinite-volume measures obtained via weak limits or via the DLR equations coincide. The situation is even less understood in the case of the FK percolation. In fact, the specification for this model is not quasilocal, that is, it is not continuous with respect to the boundary condition (for the local topology). This is easily seen from (1.12), as the realisation of the event K^{α}_{e} may depend of an edge at arbitrary large distance from the edge e. In particular, this implies that item 2 of Theorem 1.9 does not hold. However, it can be proved that the specification is almost surely quasilocal in some sense, see [PVV95].

For the FK percolation on $\mathbf{Z}^d$, we can then investigate the question of the phase transition from both the statistical mechanics (is there uniqueness of the infinite-volume measure?) and the percolation (is there an infinite cluster?) point of view. These two are highly related, and are better understood in the regime $q \geq 1$. This corresponds to the regime where the *FKG inequalities*, already mentioned for the Ising model (Theorem 1.7), are valid. An event A is said to be *increasing* if it is preserved by the addition of open edges, i.e. if for all $\omega \leq \omega'$, $\omega \in A$ implies $\omega' \in A$. In the context of (FK) percolation, the FKG inequalities means that increasing events are positively correlated. In particular, it enables to show that for $q \geq 1$, the free and the wired boundary conditions give rise to two special infinite volume measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$, obtained as weak limits along any sequence of boxes. Moreover, for every FK measure ϕ on $\mathbf{Z}^d$ (obtained from a weak limit or the DLR equations), we have the following monotonicity property:

$$\phi^0_{p,q} \preceq \phi \preceq \phi^1_{p,q},$$

where for two measures μ, μ' on $\{0,1\}^E$, $\mu \preceq \mu'$ means that for every increasing event A, $\mu(A) \leq \mu'(A)$; we say that μ' *stochastically dominates* μ. It follows that uniqueness of the

infinite-volume measure is equivalent to $\phi^0_{p,q} = \phi^1_{p,q}$. For these two measures, we can define a critical parameter $p_c^b(q) \in (0,1),\ b = 0,1$ for $q \geq 1$ such that

$$\phi^b_{p,q}(\exists \text{ an infinite cluster}) = \begin{cases} 0 & \text{if } p < p_c^b(q) \\ 1 & \text{if } p > p_c^b(q) \end{cases},\ b = 0 \text{ or } 1.$$

It is known that for any $q \geq 1$, $\phi^0_{p,q} \neq \phi^1_{p,q}$ for at most countably many p (Theorem 4.63 in [Gri06]), corresponding to the cases when the probability for the origin to belong to the infinite cluster is different under $\phi^0_{p,q}$ and $\phi^1_{p,q}$ (Theorem 5.16 in [Gri06]). This implies that $p_c^0(q) = p_c^1(q) := p_c(q)$, and that there is uniqueness of the infinite-volume measure for all $p < p_c(q)$.

In dimension 2, the planar duality allows computing the critical point, and in particular it is shown in [BDC12] that for $\mathbf{Z}^2$, the self-dual point is critical, that is

$$\forall q \geq 1,\ p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Moreover, we know precisely the regime where uniqueness holds. Namely, when $q \in [1,4]$ there is a unique infinite volume measure for all values of p (see [DCST17]); when $q > 4$, there is uniqueness everywhere but at $p_c(q)$ (see [DCGH+21]). A similar behaviour is expected in dimension $d \geq 3$: it is conjectured that the only regime when $\phi^0_{p,q} \neq \phi^1_{p,q}$ is at $p = p_c(q)$ for large enough q. Returning to the question of existence of finitary codings, Spinka and Harel [HS22] proved that in the full regime of uniqueness, the unique measure on $\mathbf{Z}^d$ (in any dimension d) is FFIID. Moreover, for all $p < p_c(q)$, the coding radius has exponential tails.

Due to the lack of the FKG inequalities, the regime $q < 1$ is more obscure. In this case, we are not able to define the measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ (existence of some measure still holds by a classical compactness argument). The existence of the critical point is an open question, and the regime where uniqueness is known is limited to $p' < p_c(1)$ (the critical parameter of the Bernoulli percolation) or p close enough to 1. In dimension 2, $p' < 1/2$ corresponds to $p < \frac{q}{1+q}$, and the planarity enables to get uniqueness in the dual regime $p > 1/2$. This implies that if it exists, the critical point on $\mathbf{Z}^2$ for $q < 1$ must be between $\frac{q}{1+q}$ and $1/2$. The main objective of Chapter 5 is to improve slightly the existing results regarding the phase diagram of the FK percolation in the regime $q < 1$.

Figure 1.1 represents the state of the art regarding the phase diagram in dimension 2. We put $(1-p')/p$ on the $x-$axis and $p/(1-p)$ on the $y-$axis; this choice of x and y is particularly suitable regarding the planar duality, and we can also notice that we have $q = xy$. The line $x = y$ corresponds to $p = 1 - p'$, which is the self-dual point for all values of q (see Section 5.4.2 for a discussion about planar duality for the FK model); it is known to coincide with the critical point when $q \geq 1$. Along this line, we have uniqueness of the FK measure on $\mathbf{Z}^2$ for $q \in [1,4]$ and a phase transition for $q > 4$. The supercritical phase (in blue) and the subcritical phase (in green) corresponds respectively to the presence or absence of an infinite cluster, and uniqueness is known. Nothing was previously known in the open unit square $(0,1) \times (0,1)$ (corresponding to $p < 1/2$ and $p' > 1/2$). In Chapter 5, we enlarge the known regime of supercriticality and subcriticality (Theorem 5.3), and also prove uniqueness for both (Theorems 5.7 and 5.8); these new results are represented by the cross-hatched regions in Figure 1.1.

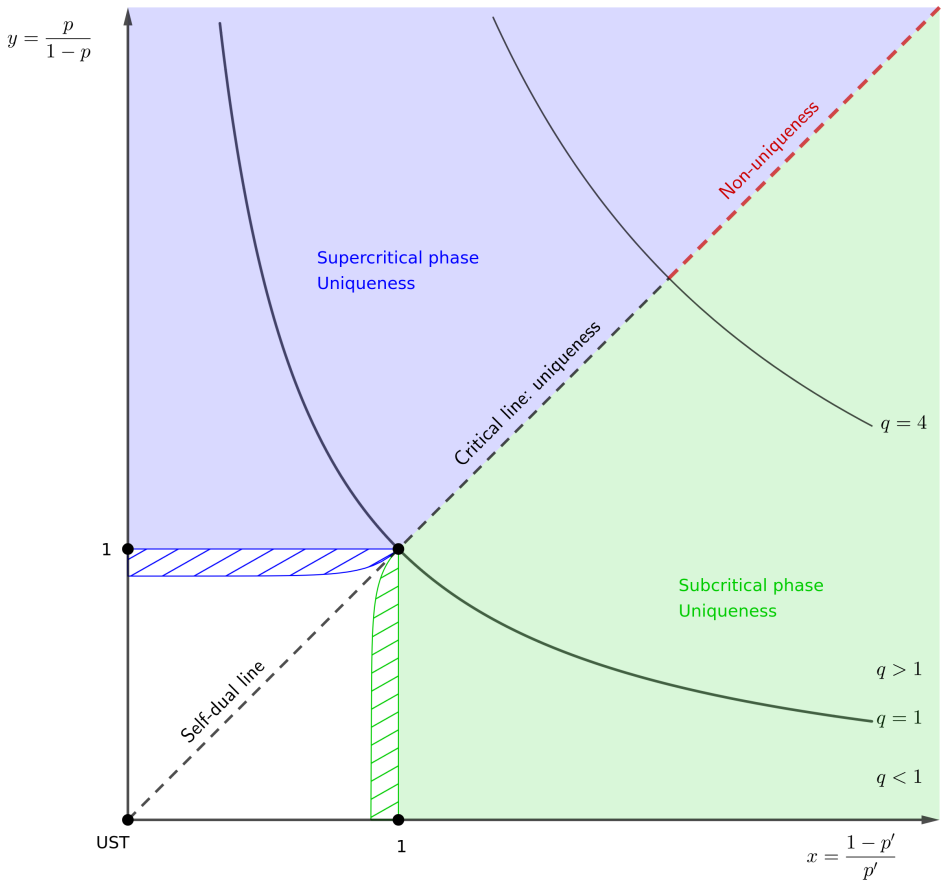


Figure 1.1: Phase diagram of the FK-percolation model on $\mathbf{Z}^2$.

Chapter 2

Gaussian Markov random fields and finite dependence

Résumé. Dans ce chapitre, on s'intéresse à des champs gaussiens indexés par un ensemble discret. On cherchera d'abord à comprendre de quelle façon les champs gaussiens peuvent être vus comme des modèles de physique statistique. On voudra donc exprimer leur distribution comme une mesure de Gibbs pour un certain potentiel. La section 2.1 passe en revue un certain nombre de résultats classique sur le sujet, sans chercher à considérer le cas le plus général possible mais en s'autorisant au contraire quelques hypothèses simplificatrices. Un traitement plus complet se trouve dans le chapitre 13 du livre de Georgii [Geo11], auquel on se référera souvent. Ensuite, notre but est de mettre en lumière que la propriété de Markov spatiale et le fait d'être à dépendance finie sont difficile à combiner pour des champs gaussiens, sauf dans le cas totalement indépendant. On cherchera à montrer que si le rayon de « markovianité » est petit, alors la portée de dépendance est grande, et vice versa. On appellera « principes d'incertitude » les différents résultats originaux de ce chapitre allant dans ce sens. Notre analyse de la dualité entre ces deux propriétés est l'objet de la section 2.2.

Abstract. In this Chapter, we are interested in Gaussian fields indexed by a countable set. First, we aim to understand how Gaussian fields can be viewed as statistical mechanics models, i.e. how a Gaussian distribution can be expressed as a Gibbs measure for a certain potential. In Section 2.1 we review some classical results that can be found in the literature, without considering the most general case possible, but rather making simplifying assumptions. A more complete treatment can be found in Chapter 13 of [Geo11]. Then, our goal is to justify that, except in the independent cases, spatial Markov property and finite dependence are difficult to combine for Gaussian models. We call "uncertainty principles" the different results of this chapter that tend to say that if the range of Markovianity is small, then the range of dependence is big and vice versa. Our analysis of the duality between these two properties is the subject of Section 2.2.

Throughout the chapter, the single-spin state space is thus chosen to be $S = \mathbf{R}$, equipped with the usual Borel $\sigma-$algebra and the Lebesgue measure as reference measure. Consider V a countable set of sites (not yet endowed with a graph structure). The set of configurations $\mathbf{R}^V$ is equipped with the product $\sigma-$algebra $\mathcal{F}_V$.

2.1 Gaussian Gibbs measures

2.1.1 Gaussian fields

Let $\Lambda \Subset V$ be a finite box. A family of real-valued random variables $X_\Lambda = (X_i)_{i\in\Lambda}$ is said to be a *Gaussian vector* indexed by Λ if for any $t_\Lambda \in \mathbf{R}^\Lambda$, the scalar product $t_\Lambda^\top X_\Lambda = \sum_{i\in\Lambda} t_i X_i$ is a real Gaussian variable (possibly of variance 0). Its distribution is entirely determined by its *mean* $m_\Lambda := (\mathbf{E}[X_i])_{i\in\Lambda}$ and *covariance matrix* Γ_Λ, whose coefficients are given by $\Gamma_\Lambda(i,j) := \mathrm{Cov}(X_i X_j) = \mathbf{E}[X_i X_j] - m_i m_j$; thus, we denote $\mathcal{N}(m_\Lambda, \Gamma_\Lambda)$ the distribution of X_Λ. Equivalently, X_Λ has law $\mathcal{N}(m_\Lambda, \Gamma_\Lambda)$ if and only if for all $t_\Lambda \in \mathbf{R}^\Lambda$,

$$\mathbf{E}[\exp(t_\Lambda^\top X_\Lambda)] = \exp\left(\frac{1}{2} t_\Lambda^\top \Gamma_\Lambda t_\Lambda + t_\Lambda^\top m_\Lambda\right).$$

X_Λ is said to be centered if $m_\Lambda \equiv 0$. The covariance matrix is clearly symmetric and positive semi-definite (since $t_\Lambda^\top \Gamma_\Lambda t_\Gamma = \mathrm{Var}(t_\Lambda^\top X_\Lambda) \geq 0$). We say that the Gaussian vector is *non-degenerate* when Γ_Λ is positive definite, and thus invertible. Non-degenerate Gaussian vectors can be characterised by the following: for every $m_\Lambda \in \mathbf{R}^\Lambda$ and positive definite matrix Γ_Λ, X_Λ has law $\mathcal{N}(m_\Lambda, \Gamma_\Lambda)$ if and only if its distribution is absolutely continuous with respect to the product Lebesgue measure on $\mathbf{R}^\Lambda$, with density function proportional to

$$\exp\left(-\frac{1}{2}(x_\Lambda - m_\Lambda)^\top \Gamma_\Lambda^{-1}(x_\Lambda - m_\Lambda)\right),\ x_\Lambda \in \mathbf{R}^\Lambda. \tag{2.1}$$

Definition 2.1. *A* Gaussian field *on V is a collection of random variables $X = (X_i)_{i\in V}$ such that all finite dimensional marginal distributions are Gaussian, that is $(X_i)_{i\in\Lambda}$ is a Gaussian vector for all $\Lambda \Subset V$. In this case, the function $m = (\mathbf{E}[X_i])_{i\in V}$ is called the* mean *and the function $\Gamma : V \times V \to \mathbf{R}$ such that $\Gamma(i,j) = \mathbf{E}[X_i X_j] - m_i m_j$ is the* covariance kernel.

In the rest of the chapter, call a *kernel* any symmetric positive semi-definite[1] function $V \times V \to \mathbf{R}$, which are the candidates for being the covariance kernel of a Gaussian field. By the Kolmogorov's extension theorem, for any function $m : V \to \mathbf{R}$ and kernel $\Gamma : V \times V \to \mathbf{R}$, there exists a unique distribution on $(\mathbf{R}^V, \mathcal{F}_V)$ whose marginals on all $\Lambda \Subset V$ coincide with $\mathcal{N}(m_\Lambda, \Gamma_\Lambda)$. This distribution is denoted by $\mathcal{N}(m, \Gamma)$, and it is the unique Gaussian field on V with mean m and covariance Γ. Gaussian fields are also characterized by the following: $(X_i)_{i\in V}$ has law $\mathcal{N}(m, \Gamma)$ if and only if for every $t \in \mathbf{R}^V$ with finite support,

$$\mathbf{E}\left[\exp\left(\sum_{i\in V} t_i X_i\right)\right] = \exp\left(\frac{1}{2}\sum_{i,j\in V} \Gamma(i,j) t_i t_j + \sum_{i\in V} t_i m_i\right).$$

As for finite vectors, a Gaussian field $(X_i)_{i\in V}$ with covariance Γ is said to be *non-degenerate* if no finite linear combination of the X_i's is almost surely constant. This is equivalent to require that for every $t \in \mathbf{R}^V - \{0\}$ with finite support,

$$\mathrm{Var}\left(\sum_{i\in V} t_i X_i\right) = \sum_{i,j\in V} \Gamma(i,j) t_i t_j > 0 \tag{2.2}$$

that is, Γ is a positive definite kernel. Obtaining an equivalent of (2.1) is a more intricate question that we address in the next paragraph.

[1] For infinite V, say that $\Gamma : V \times V \to \mathbf{R}$ is positive semi-definite if for every $t \in \mathbf{R}^V$ with finite support, $t^\top \Gamma t := \sum_{i,j\in V} \Gamma(i,j) t_i t_j \geq 0$.

2.1.2 Gaussian potential and Gibbs measures

Although the question of existence and uniqueness of infinite-volume Gaussian measures is settled by the Kolmogorov's extension theorem, it is tempting in view of (2.1) to consider non-degenerate Gaussian fields as Gibbs measures with improper Hamiltonian of the form

$$H(x) = \frac{1}{2}(x-m)^\top \Gamma^{-1}(x-m),\ x \in \mathbf{R}^V,$$

considering Γ has an infinite invertible matrix. This is the point of view adopted in Chapter 13 of [Geo11], on which we rely widely in the following presentation.

Assuming that all the sums are convergent, the improper expression for $H(x)$ can be rewritten as

$$\begin{aligned} H(x) &= \frac{1}{2}\sum_{i,j\in V} \Gamma^{-1}(i,j)(x_i-m_i)(x_j-m_j) \\ &= \frac{1}{2}\sum_{i,j\in V} \Gamma^{-1}(i,j)x_i x_j - \sum_{i\in\Lambda}\big(\sum_{j\in V}\Gamma^{-1}(i,j)m_j\big)x_i + \frac{1}{2}\sum_{i,j\in V}\Gamma^{-1}(i,j)m_i m_j. \end{aligned}$$

This informal computation leads to the above definition of the Gaussian Hamiltonian.

Let J be a positive definite kernel and $h \in \mathbf{R}^V$. To ensure that a Hamiltonian can be defined without much concern for the convergence of the series, let us assume that J has finite range, that is

$$\forall i \in V, \exists \partial i \Subset V, \forall j \notin \{i\} \cup \partial i, J(i,j) = 0. \tag{2.3}$$

Therefore, for every $x \in \mathbf{R}^V$ and $\Lambda \Subset V$, the following Hamiltonian

$$H_\Lambda(x) = \frac{1}{2}\sum_{i,j\in\Lambda} J(i,j)x_i x_j + \sum_{i\in\Lambda} h_i x_i + \sum_{i\in\Lambda, k\in\Lambda^c} J(i,k)x_i x_k \tag{2.4}$$

is well defined since all the sums have finitely many terms. The associated *Gaussian potential* $\Phi^{J,h}$ is therefore defined by

$$\Phi^{J,h}_A(x) = \begin{cases} \frac{1}{2}J(i,i)x_i^2 + h_i x_i & \text{if } A = \{i\} \\ J(i,j)x_i x_j & \text{if } A = \{i,j\} \\ 0 & \text{otherwise} \end{cases}.$$

Let μ be a Gibbs measure associated to this potential, that is $\mu \in \mathcal{G}(\Phi^{J,h})$. For every $\Lambda \Subset V$, let $J_\Lambda = (J(i,j))_{i,j\in\Lambda}$; it is a finite matrix, which is invertible by positive definiteness of J. If X is distributed according to μ it follows from the DLR equations (see [Geo11] Proposition 13.13) that for every $\Lambda \Subset V$, the conditional distribution of $(X_i)_{i\in\Lambda}$ knowing the configuration outside Λ is the unique Gaussian vector with mean

$$m_i(X_{\Lambda^c}) = -\sum_{j\in\Lambda} J_\Lambda^{-1}(i,j)\left(h_j + \sum_{k\in\Lambda^c} J(j,k)X_k\right) \tag{2.5}$$

and conditional covariance matrix J_Λ^{-1}.

In particular, the conditional distribution of X_i knowing $(X_j)_{j\neq i}$ is

$$\mathcal{N}\left(-J(i,i)^{-1}\left[h_i+\sum_{k\neq i}J(i,k)X_k\right],J(i,i)^{-1}\right).$$

Therefore, $J(i,i)$ is the inverse of the conditional variance of X_i knowing the rest of the configuration. Moreover, for all $i\neq j$, the conditional covariance between X_i and X_j knowing $X_{V-\{i,j\}}$ is

$$\mathrm{Cov}(X_i,X_j\mid X_{V-\{i,j\}})=J^{-1}_{\{i,j\}}(i,j)=\frac{-J(i,j)}{J(i,i)J(j,j)-J(i,j)^2} \tag{2.6}$$

(the denominator being positive by Cauchy-Schwarz inequality).

Note that returning to improper computations, if one sets $J=\Gamma^{-1}$ and $h=-Jm$ in (2.4), one finds again the improper Hamiltonian $H(x)$, up to a constant (which anyway disappears in the Gibbs measure). Therefore, with those J and h, one should have $\mathcal{N}(m,\Gamma)\in\mathcal{G}(\Phi^{J,h})$. Theorem 13.22 in [Geo11], which we reproduce here, formally states this claim. Set

$$M_{J,h}=\{m\in\mathbf{R}^V\mid\forall i\in V,\ h_i+\sum_{j\in V}J(i,j)m_j=0\}$$

(note the similarity to the harmonic condition for the GFF in Remark 1.12).

Theorem 2.2 (Theorem 13.22 in [Geo11])**.** *Let $\mu=\mathcal{N}(m,\Gamma)$ be the distribution of the Gaussian field on V with mean m and covariance Γ. Let J be a definite positive kernel with finite range, and let $h\in\mathbf{R}^V$. Then, the following conditions are equivalent:*

1. *$\mu\in\mathcal{G}(\Phi^{J,h})$*
2. *For all $i,k\in V$, $\sum_{j\in V}\Gamma(i,j)J(j,k)=\delta_{i,k}$ and $m\in M_{J,h}$.*

In fact, the finite-range condition on J can be weakened, since it is sufficient to ask that the sum in the Hamiltonian is $\mu-$a.s. convergent. This defines an absolutely summable Gaussian potential, rather than a finite range one.

When Γ and J satisfy the relations of item 2 in Theorem 2.2, say that they are *formal inverse* of each other; more precisely, for two symmetric functions $A,B:V\times V\to\mathbf{R}$, say that A is a formal inverse of B (and vice versa) if

$$\forall i,j\in V,\ \sum_{k\in V}A(i,k)B(k,j)=\delta_{i,j} \tag{2.7}$$

(where the above sum is implicitly assumed to be absolutely convergent). Justify the terminology by noticing that (2.7) is just a generalisation of the relations that link the coefficients of a matrix and its inverse. However, it is not true that the formal inverse is unique: if A is a formal inverse of B and $f\in M_{0,B}$, then $A':=(A(i,j)+f(i)f(j))_{i,j\in V}$ is also a formal inverse of B. What remains true is that a given symmetric function admits at most one formal inverse with finite range. Indeed, if A and A' both are formal inverse of B and finite range, then one can write for all $i,j\in V$

$$A(i,j)=\sum_{k\in V}A(i,k)\underbrace{\sum_{l\in V}B(k,l)A'(l,j)}_{=\delta_{k,j}}=\sum_{l\in V}\underbrace{\sum_{k\in V}A(i,k)B(k,l)}_{=\delta_{i,l}}A'(l,j)=A'(i,j)$$

(where we can invert the sums because they are finite, by assumption). In particular, a Gaussian field cannot be a Gibbs measure for two different (finite-range) potentials.

Then, deduce from Theorem 2.2 that for a non-degenerate Gaussian field $(X_i)_{i\in V}$ to admit a Gibbsian representation, its covariance kernel needs to admit a formal inverse with finite range (we will not discuss Gibbsian representation from infinite-range potential). This leads us to the following definition:

Definition 2.3. *Call a* Gaussian Markov random field *(GMRF) any non-degenerate Gaussian field whose covariance kernel admits a formal inverse with finite range (i.e. satisfying (2.3)).*

Let $(X_i)_{i\in V}$ be a GMRF with mean $m \in \mathbf{R}^V$ and covariance kernel Γ. By assumption, Γ is positive definite and admits a formal inverse J having finite range; it implies[2] that J is a positive definite kernel. Then, it suffices to set $h_i = -\sum_{j\in V} J(i,j)m_j$ (which is a finite sum for every $i \in V$) to obtain $\mathcal{N}(m,\Gamma) \in \mathcal{G}(\Phi^{J,h})$. With our definition, a Gaussian Markov random field is therefore always a Gibbs measure derived from a finite-range potential.

Thus, the above formulas for the conditional distribution holds for GMRF, and in particular, according to (2.6), one has

$$J(i,j) = 0 \iff X_i \perp X_j \mid X_{V-\{i,j\}}.$$

It worths comparing this statement with the well-known property of Gaussian variables saying that independence is equivalent to null correlation, that is

$$\Gamma(i,j) = 0 \iff X_i \perp X_j.$$

This crucial observation is generalised in the next Section and is the starting point of the analysis conducted in Section 2.2.

Before this, let us discuss the question of existence of a Gibbs measure for a given Gaussian potential. Let J be a positive definite kernel with finite range, and $h \in \mathbf{R}^V$. Theorem 2.2 provides sufficient conditions for the existence of a Gibbs measure associated with the potential $\Phi^{J,h}$, which are the existence of a formal inverse for J and non-emptiness of $M_{J,h}$. It is not obvious to determine whether a kernel admits a formal inverse or not, but actually a better description of $\mathcal{G}(\Phi^{J,h})$ can be given. In fact, the following alternative holds (see [Geo11] Theorem 13.24): either $\mathcal{G}(\Phi^{J,h}) = \emptyset$ or $\operatorname{ex}\mathcal{G}(\Phi^{J,h}) = \{\mathcal{N}(m,C), m \in M_{J,h}\}$, where C is the kernel defined by

$$C(i,j) := \lim_{\Lambda\uparrow V} J_\Lambda^{-1}(i,j), \tag{2.8}$$

J_Λ being the (finite invertible) matrix $(J(i,j))_{i,j\in\Lambda}$ defined above. Note that the existence of the limit in $C(i,j)$ and non-emptiness of $M_{J,h}$ are (non-trivial) consequences of non-emptiness of $\mathcal{G}(\Phi^{J,h})$.

Then, Theorem 13.26 in [Geo11] provides a sufficient (and obviously necessary) condition for the existence of the limit $\lim_{\Lambda\uparrow V} J_\Lambda^{-1}(i,j)$ and thus the existence of a formal inverse of J, which is

$$\forall i \in V,\ \sup_{\Lambda\Subset V} J_\Lambda^{-1}(i,i) < \infty$$

[2] By observing that for all $x \in \mathbf{R}^V - \{0\}$ with finite support, setting $t_i = \sum_{k\in v} J(i,k)x_k$ for all $i \in V$ gives

$$0 < \sum_{i,j\in V} \Gamma(i,j)t_i t_j = \sum_{k,l\in V} J(k,l)x_k x_l.$$

(in this result, the assumption that J has finite range cannot be relaxed).

2.1.3 Markov property and finite dependence for GMRF

Consider a symmetric function $A : V \times V \to \mathbf{R}$. Define the range of A relatively to a graph $G = (V, E)$ by

$$r_G(A) := \inf\{R \geq 0 : A(i,j) = 0 \text{ for all } i,j \in V \text{ such that } d(i,j) > R\}.$$

Then, the smallest graph $G_A = (V, E_A)$ on which A has range 1 is obtained by setting

$$E_A := \{(i,j) \in V \times V \mid i \neq j \text{ and } A(i,j) \neq 0\} \tag{2.9}$$

(E_A is the off-diagonal part of the support of A).

Recall that for any $r \in \mathbf{N}$, a random field $(Z_i)_{i\in V}$ is said to be $r-$Markovian on G if it satisfies the local Markov property with range r:

$$\forall A \Subset V, \forall B \subset V, d(A,B) > r \implies Z_A \perp Z_B \mid Z_{(A\cup B)^c}.$$

A related but different concept is the one of finitely dependent process:

Definition 2.4. *Let $l \in \mathbf{N}$ and $G = (V, E)$ a graph. A random field $Z = (Z_i)_{i\in V}$ is said to have* finite dependence of range l *or simply that Z is $l-$*dependent *relatively to G if for any $A, B \subset V$ such that $d(A,B) > l$, the restrictions Z_A and Z_B are independent.*

Note that here, independence is required without any conditioning. For Gaussian Markov random fields, these two properties are easily read on the covariance kernel and its formal inverse.

Lemma 2.5. *Let X be a GMRF on V with covariance kernel Γ. The unique formal inverse of Γ with finite range is (improperly) denoted Γ^{-1}.*

1. *X is $r-$Markovian on G if and only if $r_G(\Gamma^{-1}) \leq r$.*
2. *X is $l-$dependent on G if and only if $r_G(\Gamma) \leq l$.*

These two statements are not new, and their proofs are simple generalisation of already observed facts.

Proof of item 1. We have seen in Section 2.1 that the distribution of X belongs to $\mathcal{G}(\Phi^{J,h})$ for $J = \Gamma^{-1}$ and some $h \in \mathbf{R}^V$ (in fact $h = -J\mathbf{E}(X)$). Then, $r_G(\Gamma^{-1})$ corresponds to the range of the Gaussian potential $\Phi^{J,h}$, so X is $r-$Markovian if this range is smaller than r. This can also be observed from the formula for the conditional distribution of the field in a finite box $\Lambda \Subset V$ given X_{Λ^c}, see (2.5). In fact, it follows from (2.5) that a version of (1.5a) with a range r is satisfied, so X is $r-$Markovian. In the other direction, if $r_G(\Gamma^{-1}) > r$, it means that there exist two vertices i and j such that $d(i,j) > r$ and $\Gamma^{-1}_{i,j} \neq 0$. This implies that $\mathrm{Cov}(X_i, X_j \mid X_{V-\{i,j\}}) \neq 0$ (see (2.6)) so the $r-$Markov property is violated for $A = \{i\}$ and $B = \{j\}$. □

Proof of item 2. The "only if" part is easy by applying the definition of $l-$dependent to singletons $A = \{i\}$ and $B = \{j\}$. In the other direction, suppose that $r_G(\Gamma) \leq l$. Let $A, B \subset V$ such that $d(A, B) > l$. Assume for the proof that $\mathbf{E}(X) = 0$ is centered, the computations being similar for non-centered case. Therefore, the joint distribution of (X_A, X_B) is Gaussian of mean 0 and covariance $\Gamma_{A\cup B}$, the restriction of Γ to $A \cup B$. Hence, for any $t \in \mathbf{R}^{A\cup B}$ with finite support,

$$\begin{aligned}\mathbf{E}\left[\exp\left(\sum_{i\in A\cup B} t_i X_i\right)\right] &= \exp\left(\frac{1}{2}\sum_{i,j\in A\cup B} t_i\Gamma(i,j)t_j\right)\\ &= \exp\left(\frac{1}{2}\sum_{i,j\in A} t_i\Gamma(i,j)t_j + \frac{1}{2}\sum_{i,j\in B} t_i\Gamma(i,j)t_j + \sum_{i\in A, j\in B} t_i\underbrace{\Gamma(i,j)}_{=0}t_j\right)\\ &= \exp\left(\frac{1}{2}\sum_{i,j\in A} t_i\Gamma(i,j)t_j\right)\times\exp\left(\frac{1}{2}\sum_{i,j\in B} t_i\Gamma(i,j)t_j\right)\end{aligned}$$

so we recognise the law of independent (centered) Gaussian fields of covariance Γ_A and Γ_B respectively. □

There are therefore two special graphs that one can naturally associate to X: G_Γ is the smallest graph which makes X a $1-$dependent process, and $G_{\Gamma^{-1}}$ is the smallest graph on which X is $1-$Markovian. These two graphs will be important in the next Section, in particular because the connected components of G_Γ form a partition of the vertices according to which X decomposes into independent blocks. We can further notice that if X is a finitely dependent GMRF, then G_Γ and $G_{\Gamma^{-1}}$ share the same connected components. This fact is easily observed if V is finite, by using block matrix computation. For all $A, B \subset V$, let $\Gamma_{A,B} = (\Gamma(i,j))_{i\in A, j\in B}$, $\Gamma_A = \Gamma_{A,A}$ and proceed similarly for Γ^{-1} (NB: Γ_A^{-1} should not be confused with $(\Gamma_A)^{-1}$). Then, we have

$$I = \begin{pmatrix}\Gamma_A^{-1} & \Gamma_{A,A^c}^{-1}\\ \Gamma_{A^c,A}^{-1} & \Gamma_{A^c}^{-1}\end{pmatrix}\begin{pmatrix}\Gamma_A & \Gamma_{A,A^c}\\ \Gamma_{A^c,A} & \Gamma_{A^c}\end{pmatrix} = \begin{pmatrix}\Gamma_A^{-1}\Gamma_A + \Gamma_{A,A^c}^{-1}\Gamma_{A^c,A} & \Gamma_A^{-1}\Gamma_{A,A^c} + \Gamma_{A,A^c}^{-1}\Gamma_{A^c}\\ \Gamma_{A^c,A}^{-1}\Gamma_A + \Gamma_{A^c}^{-1}\Gamma_{A^c,A} & \Gamma_{A^c,A}^{-1}\Gamma_{A,A^c} + \Gamma_{A^c}^{-1}\Gamma_{A^c}\end{pmatrix}$$

so we have $\Gamma_{A,A^c} = 0$ if and only if $\Gamma_{A,A^c}^{-1} = 0$. The connected components of G_Γ, and $G_{\Gamma^{-1}}$ are then the minimal subsets A that satisfy the equivalent properties above. This computation can be generalised to infinite V if both Γ^{-1} and Γ are supposed to be finite range. In fact, assuming that $\Gamma_{A,A^c}^{-1} = 0$, we get $\sum_{l\in A}\Gamma(i,l)\Gamma^{-1}(l,k) = \delta_{i,k}$ for all $i, k \in V$. Then, for all $i \in A$, $j \in A^c$, we have

$$\Gamma(i,j) = \sum_{k\in A}\sum_{l\in A}\Gamma(i,l)\Gamma^{-1}(l,k)\Gamma(k,j) = \sum_{l\in A}\Gamma(i,l)\underbrace{\sum_{k\in A}\Gamma^{-1}(l,k)\Gamma(k,j)}_{=\delta_{l,j}=0} = 0,$$

where we can invert the sum because Γ has finite range. Thus, we get $\Gamma_{A,A^c} = 0$, and the reverse implication is proved the same way.

2.2 Duality between Markov property and finite dependence

For a Gaussian field $X = (X_i)_{i\in V}$, a *decomposition of X into independent blocks* is a partition of V into disjoint subsets such that the restrictions of X to each subset are mutually independent. Our goal will be to study the compatibility between the Markov property, the finite dependence, as well as FKG type assumptions. We will see that in several contexts, the combination of these properties forces the field to be decomposed into small independent blocks. In Section 2.2.2, we prove our first results that point in this direction, with hypothesis on the signs of the coefficients in the covariance or its inverse, and we derive a first version of the "uncertainty principle". In Section 2.2.3, certain limits are placed on this uncertainty principle by constructing an example of $1-$Markovian and $1-$dependent Gaussian field on an infinite connected graph. Finally, in Section 2.2.4, we get a new version of the uncertainty principle without sign assumption for a certain type of graphs, including $\mathbf{Z}^d$. Before we tackle this programme, we will need to see to what extent it is possible to consider covariance kernels as linear operators on a subset of $\mathbf{R}^V$.

2.2.1 Covariance as an operator on $\ell^2(V)$

Let Γ be a positive-definite kernel. By Definition 2.3, Γ is the covariance of a GMRF if it admits a formal inverse with finite range. We introduce a new hypothesis that will already ensure the existence of a formal inverse of Γ. This hypothesis will then provide us with the most favourable case for stating our first versions of the uncertainty principle.

Suppose that there exist two constants $c, c' > 0$ such that for all $x \in \mathbf{R}^V$ with finite support,

$$c\sum_{i\in V} x_i^2 \leq \sum_{i,j\in V} \Gamma(i,j)x_i x_j \leq c'\sum_{i\in V} x_i^2. \tag{2.10}$$

This corresponds to the coercivity and the continuity of the bilinear form whose coefficient are given by the $(\Gamma(i,j))_{i,j\in V}$. Note that (2.10) immediately implies positive definiteness. In fact, we will see that (2.10) amounts to the fact that the kernel Γ can be expressed as an invertible continuous operator on $\ell^2(V)$, the space of square-summable sequences:

$$\ell^2(V) = \{x \in \mathbf{R}^V \mid \|x\| := \Big(\sum_{i\in V} x_i^2\Big)^{1/2} < \infty\}.$$

Denote by $\langle\cdot,\cdot\rangle$ the usual scalar product on $\ell^2(V)$ that make it a Hilbert space. Let $(\delta^i)_{i\in V}$ be the canonical orthonormal basis of $\ell^2(V)$, that is for all $i \in V$, $\delta^i = (\delta_{i,j})_{j\in V}$. Now, recall that the spectrum of a continuous operator T on $\ell^2(V)$ is defined as the set of complex numbers λ for which $T - \lambda I$ is not bijective (I being the identity operator on $\ell^2(V)$). Denote by $\mathrm{Sp}(T)$ the spectrum of T, and let $\rho(T) := \sup_{\lambda\in\mathrm{Sp}(T)} |\lambda|$ be the *spectral radius* of T. It is well known that if T is self-adjoint, the spectrum of T is a compact subset of $\mathbf{R}$ and it holds (see [Bre83] Proposition VI.9) that

$$\mathrm{Sp}(T) \subset \left[\inf_{\|x\|=1} \langle x, Tx\rangle,\ \sup_{\|x\|=1} \langle x, Tx\rangle\right]$$

and both endpoints of this interval belong to $\mathrm{Sp}(T)$. Moreover, the spectral radius coincides with the operator norm, that is

$$\rho(T) = \sup_{\|x\|=1} |\langle x, Tx\rangle| = \sup_{\|x\|=1} \|Tx\|.$$

In the following, call *positive* any operator T on $\ell^2(V)$ that is self-adjoint and satisfies $\mathrm{Sp}(T) \subset (0, +\infty)$; note that sometimes the word "positive" is not limited to self-adjoint operators, and often it simply means that $\langle x, Tx\rangle > 0$ for all $x \in \ell^2(V)$ (allowing $\inf_{\|x\|=1}\langle x, Tx\rangle = 0$); we choose here to include in the definition of "positive" the fact that $0 \notin \mathrm{Sp}(T)$, that is T is invertible. Then, the inverse of a positive operator is clearly positive.

Lemma 2.6. *The application $\psi : T \mapsto (\langle \delta^i, T\delta^j\rangle)_{i,j\in V}$ defines a bijection between the set of positive operators $T : \ell^2(V) \to \ell^2(V)$ and the set of symmetric functions $A : V \times V \to \mathbf{R}$ satisfying (2.10) for some $c' \geq c > 0$.*

Proof of Lemma 2.6. First, for any positive operator T, $\psi(T)$ is symmetric because T is self-adjoint. Let $c = \inf_{\|x\|=1}\langle x, Tx\rangle$ and $c' = \sup_{\|x\|=1}\langle x, Tx\rangle$. Then for all $x \in \ell^2(V)$,

$$\sum_{i,j\in V} \langle \delta^i, T\delta^j\rangle x_i x_j = \langle x, Tx\rangle \in [c\|x\|^2, c'\|x\|^2]$$

and one has $c' \geq c > 0$ by positiveness of T. It induces that ψ is well defined. We prove that ψ is a bijection by constructing the inverse application. Let A be a kernel satisfying (2.10) for some $c' \geq c > 0$, and let

$$\phi : \left| \begin{array}{rcl} \ell^2(V) \times \ell^2(V) & \longrightarrow & \mathbf{R} \\ (x, y) & \longmapsto & \sum_{i,j\in V} A(i,j)x_i y_j \end{array} \right. .$$

It is well defined since for all $x, y \in \ell^2(V)$,

$$\phi(x, y)^2 \leq \phi(x, x)\phi(y, y) \leq c'^2\|x\|^2\|y\|^2 < \infty;$$

the first inequality is Cauchy-Schwarz inequality (which holds due to the lower bound in (2.10), or yet from positive definiteness) and the second one is the upper bound in (2.10); we also used the density of the space of finitely supported sequences to extend these inequalities to $\ell^2(V)$. Then, ϕ is a continuous bilinear form on $\ell^2(V)$, so by the Riesz-Fréchet representation theorem, there exists a unique continuous operator $T_A : \ell^2(V) \to \ell^2(V)$ such that

$$\forall x, y \in \ell^2(V),\ \ \phi(x, y) = \langle x, T_A y\rangle.$$

Moreover, ϕ is symmetric so T_A is self-adjoint. Then, (2.10) is exactly saying that $\mathrm{Sp}(T_A) \subset [c, c']$, in particular T_A is a positive operator. Finally, $\psi(T_A) = A$. □

Let us now see how this implies that (2.10) is a sufficient condition for the existence of a formal inverse of Γ. It is easy to check that for every positive operator T, $\psi(T)$ and $\psi(T^{-1})$ are formal inverses of each other:

$$\begin{aligned} \forall i, j \in V,\ \sum_{k\in V} \langle \delta^i, T\delta^k\rangle\langle \delta^k, T^{-1}\delta^j\rangle &= \sum_{k\in V} (T\delta^i)_k (T^{-1}\delta^j)_k \\ &= \langle T\delta^i, T^{-1}\delta^j\rangle \\ &= \langle \delta^i, \delta^j\rangle. \end{aligned}$$

Therefore, Γ admits a formal inverse $\psi(\psi^{-1}(\Gamma)^{-1})$ that we improperly call Γ^{-1}. It is the only formal inverse of Γ which satisfies (2.10). Indeed, suppose that A is a formal inverse of Γ

satisfying (2.10). Let $T_A = \psi^{-1}(A)$ and $T_\Gamma = \psi^{-1}(\Gamma)$. Then, for all $x, y \in \mathbf{R}^V$ with finite support, one can write

$$\begin{aligned}\langle x, T_A T_\Gamma y\rangle &= \sum_{i,j\in V} A(i,j)(T_\Gamma x)_i y_j \\ &= \sum_{i,j,k\in V} A(i,j)\Gamma(i,k)x_k y_j \\ &= \sum_{k,j\in V} x_k y_j \underbrace{\sum_{i\in V} A(i,j)\Gamma(i,k)}_{=\delta_{k,j}} \\ &= \langle x, y\rangle,\end{aligned}$$

which proves that $T_A T_\Gamma = I$, thus $A = \psi(\psi^{-1}(\Gamma)^{-1})$.

To conclude this section, we point out that coming from a positive operator (i.e. being in the image of ψ) is a sufficient condition for a kernel to admit a formal inverse, but it is not necessary. As an example, one can consider the massless GFF in $\mathbf{Z}^d$ defined in Section 1.3.4. It arises as a Gibbs measure associated with potential $\Phi^{J,0}$, where $J = \frac{1}{2d}\Delta$, Δ being the discrete Laplacian on $\mathbf{Z}^d$, that is the kernel defined for all $i \in \mathbf{Z}^d$ by $\Delta(i,i) = 2d$ and $\Delta(i,j) = -\mathbf{1}_{\{i\sim j\}}$ for all $j \neq i$. We have shown that for all $\Lambda \Subset \mathbf{Z}^d$, J_Λ^{-1} is the Green function of the simple random walk on Λ, killed at the boundary. Hence, its limit as $\Lambda \uparrow \mathbf{Z}^d$ exists when $d \geq 3$, proving the existence of infinite-volume Gibbs measures, as well as providing a formal inverse for the discrete Laplacian (say according to (2.8)). However, neither this formal inverse, which is the Green function of the simple random walk in $\mathbf{Z}^d$, nor J satisfy (2.10).

2.2.2 Uncertainty principle with an FKG type assumption

In this Section, we consider a Gaussian field X with covariance kernel Γ. As in the previous Section, we suppose that Γ satisfies (2.10), which allows us to identify a particular formal inverse of Γ that we denoted Γ^{-1}. According to Lemma 2.6, it means that Γ – and Γ^{-1}– are in the image of the application ψ, that is they both correspond to a positive operator on $\ell^2(V)$.

To obtain our first results, we will also need an "FKG type" assumption: X is said to have the *FKG property* if it has positive correlation, that is if $\Gamma(i,j) \geq 0$ for all $i, j \in V$. In this case, X satisfies a version of the FKG inequality (see Theorem 1.7). Say that X has the *strong FKG property* if $\Gamma^{-1}(i,j) \leq 0$ for all $i \neq j \in V$. According to (2.6), it means that for all $i, j \in V$, X_i and X_j are positively correlated conditionally on the rest of the configuration. By Proposition 2.7, it will imply that $\Gamma(i,j) \geq 0$, so that the strong FKG property is indeed stronger than the FKG property.

In the following, we will always abuse the notation by calling A the kernel associated with the positive operator A, and vice versa. Recall from (2.9) the definition of G_A.

Proposition 2.7 (Abstract uncertainty principle)**.** *Let A be a positive operator on $\ell^2(V)$ (meaning $\mathrm{Sp}(A) \subset (0, +\infty)$). If $A(i,j) \leq 0$ for all $i \neq j \in V$, then*

$$A^{-1}(i,j) \neq 0 \iff i \text{ and } j \text{ are connected in } G_A$$

and in fact in this case one has $A^{-1}(i,j) > 0$.

Proof. Let $\lambda = \max\{\rho(A), \sup_{i\in V} A(i,i)\}$. Consider the operator $T := \lambda I - A$, I being the identity operator on $\ell^2(V)$, and notice that by definition of λ, one has $T(i,j) \geq 0$ for all $i,j \in V$. Moreover, one has $\mathrm{Sp}(A) \subset]0, \rho(A)]$ hence $\mathrm{Sp}(T) \subset [\lambda - \rho(A), \lambda[$, so it follows that $\rho(T) < \lambda$. The inverse of A can therefore be expressed by the Neumann series:

$$A^{-1} = \lambda^{-1}(I - \lambda^{-1}T)^{-1} = \lambda^{-1}\sum_{n\geq 0} \lambda^{-n}T^n.$$

For all $i,j \in V$, by Fubini–Tonelli theorem, one can then decompose $A^{-1}(i,j)$ as a sum over paths from i to j in the graph G_A enlarged with self-loops around each vertex:

$$\begin{aligned} A^{-1}(i,j) &= \sum_{n\geq 0} \lambda^{-n-1}T^n(i,j) \\ &= \sum_{n\geq 0} \lambda^{-n-1} \sum_{\substack{i_0 \cong i_1 \cong \ldots \cong i_n \\ i_0 = i, i_n = j}} T(i_0,i_1) \times \cdots \times T(i_{n-1}, i_n), \end{aligned}$$

where $x \cong y$ means that $x = y$ or $x \sim y$; it follows from the fact that $T(x,y) = 0$ if x and y are different and not neighbours in $G_A = G_T$. This implies that $A^{-1}(i,j) = 0$ if i and j are not in the same connected component in G_A, the sum being empty. Moreover, since $T(x,y) \geq 0$ for all $x,y \in V$, it follows that all the terms in the above sum are non-negative. Then, i and j being connected in G_A exactly means that there is a path from i to j along which all corresponding coefficients in T are non-zero, so here positive. Hence, there is at least one (strictly) positive term in the sum, so $A^{-1}(i,j) > 0$. □

Now, we deduce from Proposition 2.7 our first results that constrain X to decompose itself into small blocks. Proposition 2.7 can be applied to Γ^{-1} if X has the strong FKG property, or to Γ when the field is negatively correlated, which one may call a "reverse FKG" property. This leads to an incompatibility between the strong FKG property and finite dependence, and between the reverse FKG and Markov properties. Recall that in this Section, thus in the following statements, the covariance Γ of X is supposed to be in the image of ψ, so it has a unique formal inverse Γ^{-1} that also belongs to the image of ψ.

Theorem 2.8 (Uncertainty principle with strong FKG)**.** *Let $G = (V,E)$ be a graph, and $r, l \geq 1$ be two integers.*

1. *If X is $l-$dependent on G and $\Gamma^{-1}(i,j) \leq 0$ for all $i \neq j \in V$, then X is decomposable into independent blocks of diameter at most l.*
2. *If X is $r-$Markovian on G and $\Gamma(i,j) \leq 0$ for all $i \neq j \in V$, then X is decomposable into independent blocks of diameter at most r.*

Obviously, the above statements imply that if X is both $l-$dependent and $r-$Markov, and if one of the two assumptions on the signs of the coefficients is fulfilled, then X is decomposable into independent blocks of diameter at most $\max(r,l)$. It is possible to deduce a version of this fact by only assuming the standard FKG property (or, in a dual way, the "reverse strong FKG property —which is weaker than reverse FKG). However, we will need to suppose that the underlying graph is bipartite, so it is not possible to get a result this much general: we only retrieve the result when r or l is equal to 1. A graph $G = (V,E)$ is said to be *bipartite*

if there exists a colouring of G with two colours such that adjacent vertices never share the same colour. Symbolically, it means that there exists $\sigma : V \to \{-1, 1\}$ such that $\sigma(i) \neq \sigma(j)$ for all $i \sim j$.

Corollary 2.9 (Uncertainty principle with FKG on bipartite graphs)**.** *Assume that G is bipartite.*

1. *If X is $1-$dependent and $r-$Markovian on G, and if $\Gamma(i,j) \geq 0$ for all $i, j \in V$, then X is decomposable into independent blocks of diameter at most r.*

2. *If X is $1-$Markovian and $l-$dependent on G, and if $\Gamma^{-1}(i,j) \geq 0$ for all $i, j \in V$, then X is decomposable into independent blocks of diameter at most l.*

Before turning to the proofs of Theorem 2.8 and Corollary 2.9, let us present of simple example of a model which is $1-$Markovian and $1-$dependent on a non-bipartite graph, and that is *not* decomposable into blocks of diameter 1. It also provides an example of a Gaussian field which has the FKG but not the strong FKG property. For $a, b > 0$, let

$$\Gamma = \begin{pmatrix} 1 & a & 0 & 0 & b \\ a & 1 & a & 0 & 0 \\ 0 & a & 1 & a & 0 \\ 0 & 0 & a & 1 & a \\ b & 0 & 0 & a & 1 \end{pmatrix};$$

it is positive definite if a and b are small enough, so it is the covariance matrix of a non-degenerate Gaussian vector X (on $V = \{1, 2, 3, 4, 5\}$) which has the FKG property.

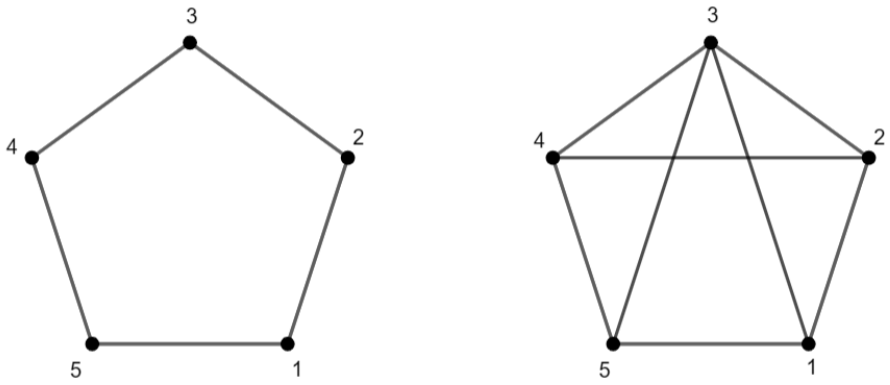


Figure 2.1: Left: the graph G_Γ. Right: the graph $G_{\Gamma^{-1}}$, which coincides with $G_\Gamma \cup G_{\Gamma^{-1}}$. Vertices are labelled in the order in which they appear in the matrices Γ and Γ^{-1}.

Taking $b = a^2/(1-a^2)$, one gets

$$\Gamma^{-1} = \lambda \begin{pmatrix} 1-a^2 & -a & a^2 & 0 & -a^2 \\ -a & 1 & \frac{-a}{1-a^2} & \frac{a^2}{1-a^2} & 0 \\ a^2 & \frac{-a}{1-a^2} & \frac{1-a^2+a^4}{1-a^2} & \frac{-a}{1-a^2} & a^2 \\ 0 & \frac{a^2}{1-a^2} & \frac{-a}{1-a^2} & 1 & -a \\ -a^2 & 0 & a^2 & -a & 1-a^2 \end{pmatrix}$$

with $\lambda = (1-a^2)/(1-3a^2+a^4)$. On the graph $G_\Gamma \cup G_{\Gamma^{-1}}$ (obtained on the same vertex set by taking edges of both graphs), X is $1-$dependent and $1-$Markovian. However, both G_Γ and $G_{\Gamma^{-1}}$ are connected and of diameter 2, so X cannot be decomposed into smaller independent blocks and the conclusion of Corollary 2.9 does not hold. Note also that Γ^{-1} contains negative entries, so X is not strongly FKG (this is consistent since otherwise one could have applied Theorem 2.8!)

Proof of Theorem 2.8. We only prove item 1, since item 2 follows by exchanging the roles of Γ and Γ^{-1} in the proof. By Lemma 2.5, X being $l-$dependent means that $r_G(\Gamma) \leq l$. By the hypothesis on the sign of the coefficients in Γ^{-1} (the strong FKG hypothesis), one can apply Proposition 2.7 to $A = \Gamma^{-1}$ (more precisely to the operator $\psi^{-1}(\Gamma^{-1})$). We deduce that if i and j are connected in $G_{\Gamma^{-1}}$, then one has $\Gamma(i,j) > 0$. In particular, since $r_G(\Gamma) \leq l.$, necessarily $d_G(i,j) \leq l$. This implies that the connected components of $G_{\Gamma^{-1}}$ have diameter at most l in G.

Moreover, Proposition 2.7 also implies that the connected components of $G_{\Gamma^{-1}}$ and G_Γ are the same. Then, the restrictions of X on these connected components are mutually independent, and we have proved that their diameters are bounded by l in G. □

Proof of Corollary 2.9. We only prove item 1 since as in the above proof, item 2 follows by symmetry of the roles of Γ and Γ^{-1}. Consider $\sigma : V \to \{-1, 1\}$ a colouring of G such that $\sigma(i) \neq \sigma(j)$ for all $i \sim j$. Define a new Gaussian field by $\widetilde{X}_i := \sigma(i) X_i$ for all $i \in V$. Its covariance, denoted $\widetilde{\Gamma}$, is given by

$$\widetilde{\Gamma}(i,j) = \sigma(i)\sigma(j)\Gamma(i,j) = \begin{cases} \Gamma(i,j) & \text{if } d_G(i,j) \text{ is even} \\ -\Gamma(i,j) & \text{if } d_G(i,j) \text{ is odd} \end{cases}.$$

In fact, it corresponds to conjugate Γ by the operator D_σ associated with the diagonal kernel $(\sigma(i)\mathbf{1}_{\{i=j\}})_{i,j\in V}$: it satisfies $D_\sigma = D_\sigma^{-1}$ and one has $\widetilde{\Gamma} = D_\sigma \Gamma D_\sigma$, and similarly $\widetilde{\Gamma}^{-1} = D_\sigma \Gamma^{-1} D_\sigma$. If $i \sim j$, one has $\Gamma(i,j) \geq 0$ so $\widetilde{\Gamma}(i,j) \leq 0$. Since X is assumed to be $1-$dependent, one has $\Gamma(i,j) = 0$ if i and j are at distance (strictly) greater than 1, so the same holds for $\widetilde{\Gamma}$. This implies that $\widetilde{\Gamma}(i,j) \leq 0$ for all $i \neq j$, so we can apply Theorem 2.8 (item 2) to $\widetilde{X}$ and the conclusion follows. □

In the proof of the Corollary, one may observe that the very restrictive assumption of being $1-$dependent is used only to get the desired sign for the coefficients in $\widetilde{\Gamma}$; it is thus possible to replace $1-$dependence by an "odd-dependence", that is assuming that sites at even distance are independent.

We will see in the next paragraph that an FKG type assumption is indeed necessary to get a decomposition of X into small blocks, by constructing an infinite graph on which there exists a 1−Markov and 1−dependent Gaussian field.

2.2.3 An example of 1−dependent and 1−Markov Gaussian field

In this section, we construct a non-degenerate covariance kernel Γ which verify $E_\Gamma = E_{\Gamma^{-1}}$. This will imply that both the range of Γ and Γ^{-1} is 1 on $G_\Gamma = G_{\Gamma^{-1}}$ so, according to Lemma 2.5, the associated Gaussian field is 1−Markov and 1−dependent. It is trivially achieved on a complete (possibly infinite) graph, but for such a graph the conclusion of Theorem 2.8 tells you nothing (all subgraphs being of diamater 1). The interest of the following example is that it is built on an infinite locally finite connected graph, whose diameter is thus infinite. We first construct an example on a graph that is not bipartite, and then use a "duplicating trick" to derive an example on a bipartite graph, proving that an FKG type assumption is really necessary in both Theorem 2.8 and Corollary 2.9.

Take as vertex set $V = \mathbf{Z} \times \{-1, 1\}$. To simplify notation, for $i \in \mathbf{Z}$, i_+ (resp. i_-) will denote the vertex $(i, 1)$ (resp. $(i, -1)$). Let $T : V \times V \to \mathbf{R}$ be a symmetric function such that for all $i \in \mathbf{Z}$, $T(i_+, i_-) = a$, $T(i_+, (i+1)_+) = T(i_-, (i+1)_+) = b$, $T(i_+, (i+1)_-) = T(i_-, (i+1)_-) = -b$ where $a, b > 0$ are two small enough parameters. Set $T(i_\pm, j_\pm) = 0$ for all pair (i, j) which is not of the above form (in particular T is null on the diagonal).

One can think of T as an infinite block matrix with block of size two. The block i corresponds to the two vertices i_+ and i_-. In what follows, $T_{\mathbf{i,j}}$ will denote the (i, j)-block of T, i.e.

$$T_{\mathbf{i,j}} = \begin{pmatrix} T(i_+, j_+) & T(i_+, j_-) \\ T(i_-, j_+) & T(i_-, j_-) \end{pmatrix}.$$

The i-th diagonal block $T_{\mathbf{i,i}}$ code the interaction between i_+ and i_-, that is $T_{\mathbf{i,i}} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} := A$. The i-th sup-diagonal block contains the covariance between vertices of block i and $i+1$ and equals $T_{\mathbf{i,i+1}} = \begin{pmatrix} b & -b \\ b & -b \end{pmatrix} := B$. By symmetry, the sub-diagonal is formed of blocks $B^\top$.

$$T = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & A & B & 0 & \\ & B^\top & A & B & \\ & 0 & B^\top & A & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

The natural graph on which T has range 1 is (V, E_T), in which for all $i \in \mathbf{Z}$, an edge is present between i_+ and i_-, and for all $j \neq i$, an edge of $\{i_+, i_-\} \times \{j_+, j_-\}$ is present in the graph if and only if $|i - j| = 1$, see Figure 2.2.

Proposition 2.10. *Let $\Gamma = I + T$. Then, if $a^2 + 4b^2 < 1$, Γ is the covariance kernel of a GMRF on V and $E_T = E_\Gamma = E_{\Gamma^{-1}}$.*

Proof. Observe that it suffices to show that T^2 is of the form λI for some $\lambda \in (0, 1)$. Indeed, this implies that $\mathrm{Sp}(\Gamma) \subset (0, 2)$ and one then easily deduces the powers of T and compute the

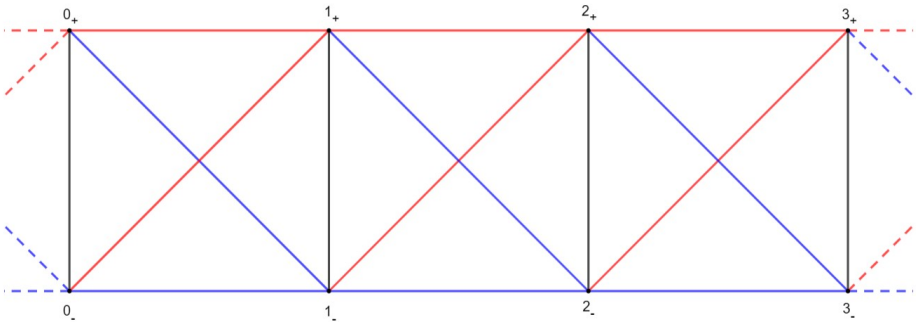


Figure 2.2: A portion of the graph (V, E_T); vertical edges (in black) correspond to entries equal to a in the covariance kernel, upper horizontal and diagonal (parallel to $(0_-, 1_+)$) edges (in red) correspond to b entries, and lower and anti-diagonal edges (in blue) to $-b$ entries.

Neumann series:

$$\Gamma^{-1} = \sum_{n\geq 0}(-1)^n T^n = \sum_{n\geq 0}\lambda^n(I-T) = \frac{1}{1-\lambda}(2I-\Gamma) \tag{2.11}$$

It follows that $\Gamma(i,j) = 0$ if and only if $\Gamma^{-1}(i,j) = 0$, hence $E_\Gamma = E_{\Gamma^{-1}}$.

Let us check that T^2 is of the right form, by block computation. For all $i, j \in \mathbf{Z}$, we have $(T^2)_{\mathbf{i,j}} = \sum_{k\in\mathbf{Z}} T_{\mathbf{i,k}} T_{\mathbf{k,j}} = 0$ when $|i-j| > 2$ because k need to be at distance at most 1 of i and j to have $T_{\mathbf{i,k}} T_{\mathbf{k,j}}$ non zero, by construction of T. Moreover,

$$(T^2)_{\mathbf{i,i+2}} = T_{\mathbf{i,i+1}} T_{\mathbf{i+1,i+2}} = B^2 = 0$$

$$(T^2)_{\mathbf{i,i+1}} = T_{\mathbf{i,i}} T_{\mathbf{i,i+1}} + T_{\mathbf{i,i+1}} T_{\mathbf{i+1,i+1}} = AB + BA = 0$$

$$(T^2)_{\mathbf{i,i}} = T_{\mathbf{i,i+1}} T_{\mathbf{i+1,i}} + T_{\mathbf{i,i-1}} T_{\mathbf{i-1,i}} + T_{\mathbf{i,i}}^2 = BB^\top + B^\top B + A^2 = \begin{pmatrix} a^2+4b^2 & 0 \\ 0 & a^2+4b^2 \end{pmatrix}.$$

With the symmetry of T, one can then conclude that $T^2 = \lambda I$ for $\lambda = a^2 + 4b^2$. □

This construction provides a locally finite connected graph $G_\Gamma = (V, E_\Gamma)$ on which there exists a class of Gaussian fields that are 1−Markov and 1−dependent but cannot be decomposed into independent blocks of diameter 1. None of the FKG or reverse FKG properties are satisfied, since Γ and Γ^{-1} contain entries of both signs. However, G_Γ is not bipartite but one can easily derive an example on a bipartite graph.

Consider a copy V' of V, whose vertices are denoted i'_+, i'_- for $i \in \mathbf{Z}$. Construct similarly as above a non-degenerate covariance kernel Γ on $\mathcal{V} := V \sqcup V'$ which satisfies $E_\Gamma = E_{\Gamma^{-1}}$. It is obtained by replacing every edge $(i_\pm, j_\pm)$ of the previous example by two edges $(i_\pm, j'_\pm)$ and $(i'_\pm, j_\pm)$.

Let $a, b > 0$ and define a symmetric function $T : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$ by setting, for all $i \in \mathbf{Z}$, $T(i_+, i'_-) = a$, $T(i_+, (i+1)'_+) = T(i_-, (i+1)'_+) = b$, $T(i_+, (i+1')_-) = T(i_-, (i+1)'_-) = -b$ and $T(i'_+, i_-) = a$, $T(i'_+, (i+1)_+) = T(i'_-, (i+1)_+) = b$, $T(i'_+, (i+1)_-) = T(i'_-, (i+1)_-) = -b$ and $T(\cdot,\cdot) = 0$ everywhere else. It is more meaningful to consider T as an infinite block matrix,

where the blocks are now of size 4; in fact, for all $i \in \mathbf{Z}$, the $i-$th block corresponds to the vertices i_+, i_-, i'_+ and i'_-. For all $i, j \in \mathbf{Z}$, let

$$T_{\mathbf{i,j}} = \begin{pmatrix} T(i_+, j_+) & T(i_+, j_-) & T(i_+, j'_+) & T(i_+, j'_-) \\ T(i_-, j_+) & T(i_-, j_-) & T(i_-, j'_+) & T(i_-, j'_-) \\ T(i'_+, j_+) & T(i'_+, j_-) & T(i'_+, j'_+) & T(i'_+, j'_-) \\ T(i'_-, j_+) & T(i'_-, j_-) & T(i'_-, j'_+) & T(i'_-, j'_-) \end{pmatrix}.$$

Then, one has $T_{\mathbf{i,i}} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} := \mathcal{A}$ and $T_{\mathbf{i,i+1}} = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} := \mathcal{B}$ (and again $T_{\mathbf{i,i-1}} = \mathcal{B}^\top$) where A and B are the size two matrices defined above. Therefore,

$$T = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & \mathcal{A} & \mathcal{B} & 0 & \\ & \mathcal{B}^\top & \mathcal{A} & \mathcal{B} & \\ & 0 & \mathcal{B}^\top & \mathcal{A} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

The natural graph $\mathcal{G}_T = (\mathcal{V}, E_T)$ on which T has range 1 is bipartite, since edges always join a vertex in V and a vertex in V'. Then, the same computation as in the proof of Proposition 2.10 leads to $T^2 = \lambda I$ with $\lambda = a^2 + 4b^2$. Therefore, if $\lambda < 1$, $\Gamma := I + T$ is a positive operator, so it is the covariance kernel of a GMRF, such that $E_T = E_\Gamma = E_{\Gamma^{-1}}$.

2.2.4 Uncertainty principle without FKG on large girth graphs

As observed in the previous paragraph, in general it is not possible to get an uncertainty principle without FKG assumption. However, we prove a version of the uncertainty principle that prevents the existence of a 1−Markovian and 1−dependent Gaussian field on a certain class of graphs, without further assumption on the field. The proof works especially for 1−Markovian and 1−dependent Gaussian fields, and cannot be adapted directly to $r-$Markovian or $l-$dependent cases for more general r and l. Note, however, that here we do not need the interpretation of the covariance as an operator on $\ell^2(V)$ operator.

In this Section, we thus consider X a GMRF with covariance kernel Γ, meaning that we only suppose that Γ admits a (necessarily unique) formal inverse with finite range, improperly denoted Γ^{-1}. All the results are proved using only the fact that they are a formal inverse of each inverse, i.e. for all $i, j \in V$

$$\delta_{i,j} = \sum_{x \in V} \Gamma(i, x)\Gamma^{-1}(x, j). \tag{2.12}$$

In any graph $G = (V, E)$, a *cycle* is a finite sequence of vertices $(v_1, \dots v_n, v_1)$ such that $v_1, \dots v_n$, are distinct vertices and for all $k \in \mathbf{Z}/n\mathbf{Z}$, $v_k \sim v_{k+1}$ i.e. v_k and v_{k+1} are linked by an edge in G; n is called the length of the cycle. By abuse of language, say that the edges $\{(v_k, v_{k+1})\}_{k \in \mathbf{Z}/n\mathbf{Z}}$ are contained in the cycle. A cycle of length n then includes exactly n distinct edges. The *girth* of a graph is the length of its shortest cycle. In the following, call a *triangle* a cycle of length 3; a triangle-free graph is thus a graph of girth at least 4. Call a *square* (rather than a quadrilateral) a cycle of length 4.

Lemma 2.11. *Suppose that $G = (V, E)$ is triangle-free, and that $E_\Gamma, E_{\Gamma^{-1}} \subset E$. Then, $E_\Gamma = E_{\Gamma^{-1}}$.*

Proof. The proof is a simple application of (2.12) Let $i, j \in V$ be two adjacent vertices in G. Suppose that $x \notin \{i, j\}$. Then, by assumption on E_Γ and $E_{\Gamma^{-1}}$, one has $\Gamma(i, x)\Gamma^{-1}(x, j) = 0$ unless x is a neighbour (in G) of both i and j. But if such a vertex x exists, it forms a cycle (i, x, j, i) of length 3 in G, which is excluded. Therefore, one can restrict the sum in (2.12) to $x \in \{i, j\}$, thus

$$0 = \Gamma(i, i)\Gamma^{-1}(i, j) + \Gamma(i, j)\Gamma^{-1}(j, j).$$

Since $\Gamma(i, i)$ and $\Gamma^{-1}(j, j)$ are both positive, it holds that $\Gamma^{-1}(i, j) = 0$ if and only if $\Gamma(i, j) = 0$. □

Lemma 2.12 (Complete the square's rule)**.** *Suppose that $G = (V, E)$ is triangle-free, and that $E_\Gamma, E_{\Gamma^{-1}} \subset E$. If two adjacent edges (i, j) and (j, k) $(k \neq i)$ are present in E_Γ, then they necessarily belong to a square in G whose four edges are present in E_Γ, that is there exists $x \in V - \{j\}$ such that $(i, x), (k, x) \in E_\Gamma$.*

Proof. The proof is again an application of (2.12). Let $i, j, k \in V$ such that $\Gamma(i, j) \neq 0$ and $\Gamma(j, k) \neq 0$. By Lemma 2.11, one also has $\Gamma^{-1}(i, j) \neq 0$ and $\Gamma^{-1}(j, k) \neq 0$. Then, applying (2.12) to i and k, one gets

$$0 = \underbrace{\Gamma(i, j)\Gamma^{-1}(j, k)}_{\neq 0} + \sum_{x \in V - \{j\}} \Gamma(i, x)\Gamma^{-1}(x, k).$$

There is no edge between i and k, because otherwise it would create a triangle (i, j, k, i) in G, and $E_\Gamma = E_{\Gamma^{-1}} \subset E$, so $\Gamma(i, x)\Gamma^{-1}(x, k) = 0$ if $x \in \{i, k\}$. Therefore,

$$\sum_{x \in V - \{i,j,k\}} \Gamma(i, x)\Gamma^{-1}(x, k) \neq 0.$$

In particular, this sum is non-empty so there is a square containing i, j and k. Furthermore, there is at least one non-zero term, so we find a vertex $x \in V - \{i, j, k\}$ such that $(i, x) \in E_\Gamma$ and $(k, x) \in E_{\Gamma^{-1}} = E_\Gamma$. □

These two lemmas directly imply the following result:

Theorem 2.13 (Uncertainty principle on graphs without triangle or square)**.** *Let $G = (V, E)$ be a graph of girth at least 5. Suppose that X is a $1-$Markovian and $1-$dependent GMRF on G. Then, X is decomposable into independent blocks of diameter at most 1.*

Proof. According to Lemma 2.5, assuming that X is $1-$Markovian and $1-$dependent on G comes down to $r_G(\Gamma) = r_G(\Gamma^{-1}) = 1$, that is E_Γ and $E_{\Gamma^{-1}}$ are included in E. Since G is triangle and square-free, Lemma 2.12 asserts that E_Γ cannot contain two adjacent edges. The connected components of E_Γ are therefore of diameter at most 1. □

The above theorem fails for graphs of girth 4, but the two lemmas do apply and greatly constrain the graph structure of G_Γ — Γ being the covariance kernel of a $1-$Markovian and $1-$dependent GMRF on your favourite graph of girth 4. For example, if one considers the square or square-octagon planar lattice (see Figure 2.3), it follows from Lemma 2.12 that the

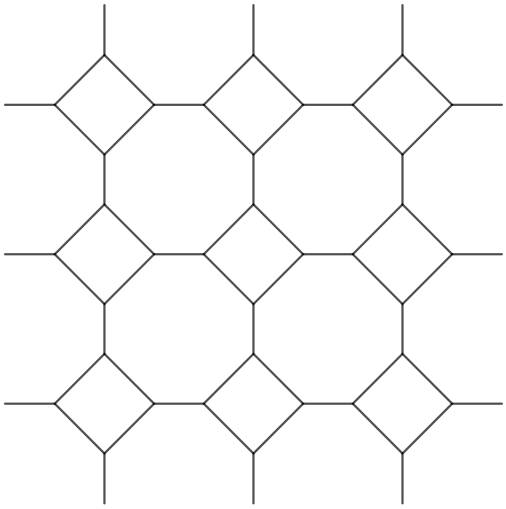


Figure 2.3: A portion of the square-octagon lattice.

connected components of G_Γ can only be reduced to a single point, two adjacent points, or four points forming a square.

Conclude this Chapter by studying the special case of $\mathbf{Z}^d$, endowed with its canonical graph structure, which is a fundamental example of graph of girth 4. The connected components of G_Γ are of diameter bounded by d (instead of 1 in Theorem 2.13). However, it is even more interesting to notice that they necessarily have a very specific shape, that is a hypercube of dimension at most d, generalising the above remark for the $d = 2$ case.

Theorem 2.14 (Uncertainty principle on $\mathbf{Z}^d$)**.** *Let X be a non-degenerate Gaussian field on $\mathbf{Z}^d$, which is $1-$Markovian and $1-$dependent. Then, X is decomposable into independent blocks, which are shaped like hypercubes of dimension at most d.*

Proof. First, note that $\mathbf{Z}^d$ is triangle-free, so that Lemma 2.11 applies: $E_\Gamma = E_{\Gamma^{-1}}$. According to Lemma 2.12, if E_Γ contains two adjacent edges, then they are included in a square whose four edges are in E_Γ. In particular, E_Γ cannot contain two adjacent edges in the same direction, because no cycle of length 4 in $\mathbf{Z}^d$ contains two such edges.

Let us study the connected component of 0 in the graph G_Γ, denoted C_0, and suppose that it contains at least 3 vertices (the case $C_0 = \{0\}$ corresponds to a hypercube of dimension 0 and $|C_0| = 2$ to a $1-$dimensional hypercube). Without loss of generality, one may assume that $e_1 := (0, i_1) \in E_\Gamma$, with $i_1 = (1, 0, \dots, 0) \in \mathbf{Z}^d$.

By assumption on $|C_0|$, E_Γ contains an edge e adjacent to e_1; the possible candidates are of the form $(0, \pm i_k)$ or $(i_1, i_1 + \pm i_k)$ for $k \in \{2, \dots, d\}$ (i_k denote the vertex with $k-$th coordinate 1 and 0 otherwise). Actually, if $e = (0, \pm i_k)$ for some k (say that $\pm i_k = i_2$, without loss of generality), then by Lemma 2.12 the four edges of the (unique) square that includes $(0, i_1)$ and $(0, i_2)$ belong to E_Γ, so $(i_1, i_1 + i_2) \in E_\Gamma$ and $(i_2, i_1 + i_2) \in E_\Gamma$. The situation is identical if e is of the form $e = (i_1, i_1 \pm i_k)$. Hence, if there is no other edge adjacent to the square present in E_Γ, one has $C_0 = \{0, i_1, i_2, i_1 + i_2\}$ and the induced subgraph of G_Γ has a $2-$dimensional hypercube shape.

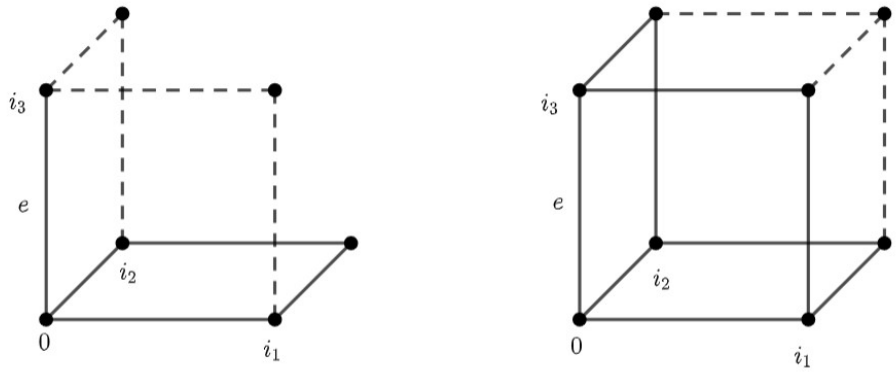


Figure 2.4: Left: Edges of the bottom square and edge e (unbroken lines) are supposed to be in E_Γ; the "complete the square" rule imposes the presence of 4 other edges (in dotted lines) in E_Γ. Right: The rule applies again to the new incomplete squares, so that 3 new edges (in dotted lines) need to be included in E_Γ, completing the whole cube.

Otherwise, since there are no adjacent edges in the same direction in E_Γ, it follows that C_0 cannot contain another vertex in the plane generated by $(0, i_1, i_2)$. Therefore, a new vertex can be added to C_0 only if there is an edge in E_Γ adjacent to the square in an orthogonal direction. Suppose that this edge is $(0, i_3)$; then, the squares $(0, i_1, i_1 + i_3, i_3, 0)$ and $(0, i_2, i_2 + i_3, i_3, 0)$ need to be fully included in C_0, and afterwards the "complete the square" rule also applies to $(i_1 + i_2 + i_3, i_1 + i_3, i_1, i_1 + i_2, i_1 + i_2 + i_3)$, $(i_1 + i_2 + i_3, i_2 + i_3, i_2, i_1 + i_2, i_1 + i_2 + i_3)$ and $(i_1 + i_2 + i_3, i_2 + i_3, i_3, i_1 + i_3, i_1 + i_2 + i_3)$, so that all the edges of the 3−dimensional hypercube generated by $(0, i_1, i_2, i_3)$ are in E_Γ (see Figure 2.4). Then, C_0 can be reduced to this hypercube, or again contain a new vertex in an orthogonal direction, and the same behaviour occurs.

Eventually, C_0 is determined by the choice of edges attached to 0 included in E_Γ, with the rule that one is allowed to take at most one edge in each direction of $\mathbf{Z}^d$. Choosing k edges imposes that the subgraph of E_Γ induced by C_0 is a hypercube of dimension $k \leq d$. □

Construction of hypercubic covariance kernels Finally, observe that this result cannot be improved, since for any $d \geq 0$ it is possible to construct a covariance kernel Γ whose associated graph G_Γ is a d−dimensional hypercube and satisfies $G_\Gamma = G_{\Gamma^{-1}}$. The d−dimensional hypercube $\mathcal{H}^d$ is the graph with vertex set $\{0,1\}^d$ and such that two vertices are linked by an edge if and only if they differ in exactly one coordinate. Denote by E^d the edge-set of $\mathcal{H}^d$. In the following, vertices of the hypercube are ordered with the lexicographical order (it will appear in the writing of matrices indexed by $\{0,1\}^d$). Denote by $\mathscr{C}_d$ the set of definite positive matrices indexed by $\{0,1\}^d$ with only 1's on the diagonal and such that $E_\Gamma = E_{\Gamma^{-1}} = E^d$. Note that we do not lose generality while restricting ourselves to matrices with 1's on the diagonal, since one can multiply Γ on the right and left by a diagonal (positive) matrix to get all the covariance matrices adapted to the hypercube, but actually our goal in this paragraph is only to show that such covariance matrices exist in all dimension.

Start with dimension 2, since dimensions 0 and 1 are trivial: $\mathscr{C}_0 = \{1\}$ and $\mathscr{C}_1 =$

$\left\{\begin{pmatrix}1 & a\\ a & 1\end{pmatrix}, 0<|a|<1\right\}$. A covariance matrix satisfying $E_\Gamma = E^2$ must be of the form

$$\Gamma = \begin{pmatrix}1 & a & b & 0\\ a & 1 & 0 & c\\ b & 0 & 1 & d\\ 0 & c & d & 1\end{pmatrix}$$

with $a, b, c, d \in \mathbf{R} - \{0\}$ small enough (zeros on the anti-diagonal correspond to the absence of edge between $(0,0)$ and $(1,1)$ and between $(0,1)$ and $(1,0)$). Its inverse is given by

$$\Gamma^{-1} = \frac{1}{\det\Gamma}\begin{pmatrix}1-c^2-d^2 & ad^2-bcd-a & bc^2-acd-b & ac+bd\\ ad^2-bcd-a & 1-b^2-c^2 & ab+cd & b^2c-abd-c\\ bc^2-acd-b & ab+cd & 1-a^2-c^2 & a^2d-abc-d\\ ac+bd & b^2c-abd-c & a^2d-abc-d & 1-a^2-b^2\end{pmatrix}$$

where $\det\Gamma = 1-(a^2+b^2+c^2+d^2)+(ad-bc)^2$. Thus, one has

$$E_{\Gamma^{-1}} \subset E^2 \iff \begin{cases}ac+bd=0\\ ab+cd=0\end{cases} \iff \begin{cases}a=d\\ b=-c\end{cases} \text{ or } \begin{cases}a=-d\\ b=c\end{cases}$$

(since a, b, c, d are non-zero). Then in both cases, one has $\det(\Gamma) = (1-a^2-b^2)^2$ and Γ^{-1} actually simplifies to

$$\Gamma^{-1} = \frac{1}{1-a^2-b^2}\begin{pmatrix}1 & -a & -b & 0\\ -a & 1 & 0 & -c\\ -b & 0 & 1 & -d\\ 0 & -c & -d & 1\end{pmatrix}$$

so one actually has $E_\Gamma = E^2$ (this also follows from Lemma 2.11 since $\mathcal{H}^d$ is triangle-free). The eigenvalues of Γ are $1 \pm \sqrt{a^2+b^2}$ (both with multiplicity 2). They are both positive if and only if $a^2+b^2<1$, so we deduce

$$\mathscr{C}_2 = \left\{\begin{pmatrix}1 & \eta_1 a & \eta_2 b & 0\\ \eta_1 a & 1 & 0 & \eta_3 b\\ \eta_2 b & 0 & 1 & \eta_4 a\\ 0 & \eta_3 b & \eta_4 a & 1\end{pmatrix}, a, b>0, \eta_1, \dots, \eta_4 \in \{\pm 1\} \mid a^2+b^2<1, \eta_1\eta_2\eta_3\eta_4 = -1\right\}.$$

In particular, $\mathscr{C}_2$ is not empty. We now provide a recipe for building covariance matrices of $\mathscr{C}_d$ in all dimensions d. We were unable to completely characterise the matrices of $\mathscr{C}_d$, but if the above derivation of $\mathscr{C}_2$ generalises to any dimension, they should be blocks matrices with blocks of size 2^{d-1} of the form

$$\Gamma = \begin{pmatrix}A & \pm\boldsymbol{\lambda}\\ \pm\boldsymbol{\lambda} & A\end{pmatrix}$$

with $A \in \mathscr{C}_{d-1}$ for some $|\lambda|<1$, where the "$\pm\boldsymbol{\lambda}$" should be interpreted as a diagonal matrix (of size 2^{d-1}) with only λ or $-\lambda$ on the diagonal. Nevertheless, we show that matrices of the above form belong to $\mathscr{C}_d$ if the signs of their coefficients satisfy a certain property. This property consists in requiring that the restriction of Γ to any four vertices forming a square in $\mathcal{H}^d$ actually belongs to $\mathscr{C}_2$.

Proposition 2.15. *Let $a_1, \dots, a_d > 0$ such that $a_1^2 + \cdots + a_d^2 < 1$ and $(\eta_e)_{e \in E^d} \in \{\pm 1\}^{E^d}$, such that for every square χ of $\mathcal{H}_d$, one has $\prod_{e \in \chi} \eta_e = -1$. For all $i, j \in \{0,1\}^d$, let*

$$T_{i,j} = \begin{cases} \eta_{(i,j)} a_k & \text{if } i \text{ and } j \text{ differ only in their } k-\text{th coordinate} \\ 0 & \text{if } i \not\sim j \end{cases}.$$

Then $I + T \in \mathscr{C}_d$. In addition, the constraints on the signs $(\eta_e)_{e \in E^d}$ are not incompatible, so in particular $\mathscr{C}_d$ is not empty in all dimensions d.

Proof. Let T be defined as in the above statement, and $\Gamma = I + T$. First, it is clear that $T_{i,j} \neq 0$ if and only if $i \sim j$, thus $E_\Gamma = E_T = E^d$. Compute T^2 to show that Γ is definite positive and that its inverse satisfies $E_{\Gamma^{-1}} = E_\Gamma$.

For all $i, j \in \{0,1\}^d$, one has

$$(T^2)_{i,j} = \sum_{i \sim k \sim j} T_{i,k} T_{k,j}$$

where the sum can be restricted to vertices k that are at distance 1 from both i and j, since otherwise there is a 0 in T. Therefore, the sum is empty if $d(i,j) > 2$ or if $d(i,j) = 1$ (since there are no triangle in the graph). If $d(i,j) = 2$, the sum contains exactly two non-zero terms, since i and j must be on either side of the diagonal of a square (i.e. two non-consecutive vertices of a cycle of length 4). Then, the sum still equals 0 thanks to the sign rules on the squares of $\mathcal{H}^d$, since it is of the form $\eta_1 a_k \eta_2 a_{k'} + \eta_3 a_{k'} \eta_4 a_k$ with $\eta_1\eta_2\eta_3\eta_4 = -1$.

This means that T^2 is supported on the diagonal and furthermore, for every $i \in \{0,1\}^d$,

$$(T_{i,i})^2 = \sum_{k \sim i} T_{i,k} T_{k,i} = a_1^2 + \cdots + a_d^2 := \lambda$$

because every vertex in the hypercube owns exactly one neighbour in each of the d directions. It proves that $T^2 = \lambda I$. Then, T is clearly symmetric and its eigenvalues are $\pm\sqrt{\lambda}$, so $I + T$ is positive definite when $\lambda < 1$.

Finally, one can easily compute the inverse of Γ with the Neumann series, which converges because $\lambda < 1$:

$$\Gamma^{-1} = \sum_{n \geq 0} (-1)^n T^n = \sum_{n \geq 0} \lambda^n (I - T) = \frac{1}{1-\lambda}(I - T).$$

It follows that $E_{\Gamma^{-1}} = E_T = E^d$, hence $\Gamma \in \mathscr{C}_d$.

One still has to show that for all $d \geq 1$, there exists a configuration of signs $\eta \in \{\pm 1\}^{E^d}$ that satisfies the constraint on each square of the $d-$dimensional hypercube. If one denotes by c_d the number of such configuration, one clearly has $c_1 = 2$ (since there is only one edge and no constraint). Let us show by induction that for all $d \geq 1$,

$$c_d = 2^{2^d - 1}.$$

Let $d \geq 1$ and suppose that the formula holds for c_d. Consider that the $d+1-$dimensional hypercube $\mathcal{H}^{d+1}$ is made of two copies of the $d-$dimensional hypercube which are linked by "intermediate" edges. For all $i = (i_1, \dots, i_d) \in \{0,1\}^d$, let $i^+ = (1, i_1, \dots, i_d)$ and $i^- = (0, i_1, \dots, i_d)$. Hence, the vertices of $\mathcal{H}^{d+1}$ are all of the form $i^\pm$ for some $i \in \{0,1\}^d$. Moreover,

$i^{\pm}$ and $j^{\pm}$ are linked by an edge in $\mathcal{H}^{d+1}$ if and only if i and j are in $\mathcal{H}^d$, and i^+ and i^- are linked by an "intermediate" edge in $\mathcal{H}^{d+1}$. Say that the subgraph induced by vertices of the form i^+ (respectively i^-) constitute the "upper" $d-$cube (respectively the "lower" $d-$cube).

Let $\eta^+ \in \{\pm 1\}^{E^d}$ be a configuration of sign on the upper $d-$cube that satisfies the sign constraint on all square included in the upper $d-$cube. One has by definition c_d possible choices. Then, let $\eta^{\text{int}} \in \{+1\}^{\{0,1\}^d}$ be a configuration of sign (without constraint) on the intermediate edges, that is for all $i \in \{0,1\}^d$, η_i will correspond to the sign of the edge (i^+, i^-) in $\mathcal{H}^{d+1}$. One therefore has 2^{2^d} possible choices. Now, show that these choices determine a unique configuration of signs $\eta \in \{\pm 1\}^{E^{d+1}}$ which satisfy the constraint on every square of $\mathcal{H}^{d+1}$ and coincide with η^+ inside the upper cube and with η^{int} on the intermediate edges. This proves that $c_{d+1} = 2^{2^d} c_d = 2^{2^{d+1}-1}$ by induction hypothesis.

Let $i, j \in \{0,1\}^d$ be two neighbours in $\mathcal{H}^d$. Hence, i^+, j^+, j^-, i^- form a square in $\mathcal{H}^{d+1}$, whose edges signs must have a product of -1. Therefore, there is only one possible choice for $\eta_{(i^-,j^-)}$, that is

$$\eta_{(i^-,j^-)} = -\eta_{(i^+,j^+)}\eta_{(i^+,i^-)}\eta_{(j^+,j^-)} = -\eta^+_{(i,j)}\eta_i^{\text{int}}\eta_j^{\text{int}}.$$

It remains to check that the sign constraint is satisfied by η on every square of $\mathcal{H}^{d+1}$ that is included in the lower $d-$cube, since by construction it is already satisfied for any other square.

Let $i, j, k, l \in \{0,1\}^d$ be four vertices forming a square in $\mathcal{H}^d$, and consider the square i^-, j^-, k^-, l^- of $\mathcal{H}^{d+1}$. Observe that the vertices $i^+, j^+, k^+, l^+, i^-, j^-, k^-, l^-$ form a $3-$dimensional cube, made of six square faces. For five of these squares, one already knows that their signs multiply to -1, the other one being i^-, j^-, k^-, l^-. Then, the product over the six squares of the signs products amounts to 1, since each edge belongs to exactly two faces, so that each sign is counted twice. In fact, one has

$$\begin{aligned}
\eta_{(i^-,j^-)}\eta_{(j^-,k^-)}\eta_{(k^-,l^-)}\eta_{(l^-,i^-)} &= \left[\eta_{(i^-,j^-)}\eta_{(i^+,j^+)}\eta_i^{\text{int}}\eta_j^{\text{int}}\right] \cdot \left[\eta_{(j^-,k^-)}\eta_{(j^+,k^+)}\eta_j^{\text{int}}\eta_k^{\text{int}}\right] \cdot \left[\eta_{(k^-,l^-)}\eta_{(k^+,l^+)}\eta_k^{\text{int}}\eta_l^{\text{int}}\right] \\
&\qquad \times \left[\eta_{(l^-,i^-)}\eta_{(l^+,i^+)}\eta_l^{\text{int}}\eta_i^{\text{int}}\right] \cdot \left[\eta_{(i^+,j^+)}\eta_{(j^+,k^+)}\eta_{(k^+,l^+)}\eta_{(l^+,i^+)}\right] \\
&= (-1) \times (-1) \times (-1) \times (-1) \times (-1) \\
&= -1
\end{aligned}$$

and that concludes the proof. □

Alternatively, one could have shown that choosing an admissible configuration of signs on both the upper and lower $d-$cube leads to only two choices of signs for the intermediate edges (which are the opposite of each other). Thus, we get $c_{d+1} = 2c_d^2$ and we find again the same formula for c_d.

Chapter 3

Glauber dynamics and coupling-from-the-past for Gaussian fields

Ce chapitre est la reproduction de l'article [Fai24], disponible sur Hal et arXiv.

Résumé. On étudie l'écriture de champs gaussiens markoviens et stationnaires comme des facteurs de champs i.i.d., notamment en ce qu'elle permet d'approcher le champ par des modèles à dépendance finie. Notre modèle est un champ gaussien sur $\mathbf{Z}^d$ dont la loi conditionnelle en un site sachant le reste de la configuration est une gaussienne centrée en ε fois la moyenne des voisins, et de variance 1. Grâce à des méthodes de couplage par le passé, on prouve que lorsque ε est suffisamment petit, la distribution du champ peut s'exprimer comme l'image par un facteur explicite d'une mesure produit. De plus, on construit des approximations du modèle par des champs à dépendance finie (proche en variation totale), avec une décroissance exponentielle de l'erreur lorsque la portée de la dépendance qu'on autorise croît. On effectue d'abord une preuve pour une version tronquée de notre modèle gaussien, en montrant dans ce cas que le champ admet est facteur d'i.i.d. avec queue exponentielle pour le rayon. Cela donne un nouvel exemple d'application de la condition « high-noise » de Häggström et Steif [HS00] pour un espace d'état indénombrable. La preuve est plus complexe pour le modèle de départ. Notre approche étend les constructions classiques d'algorithmes de couplage par le passé, en introduisant un découpage de l'espace d'état, implémenter pour répondre au caractère non-borné des valeurs prises par le champ gaussien.

This chapter is a reproduction of the preprint [Fai24], available on Hal and arXiv.

Abstract. We study the representation of stationary Gaussian Markov random fields as factors of i.i.d. processes, with a focus on their approximation by finitely dependent distributions. Our model is a Gaussian field on $\mathbf{Z}^d$ such that the conditional law of the field at any site is Gaussian of mean ε times the average of its neighbours, and of variance 1. Building on coupling-from-the-past (CFTP) techniques, we prove that for sufficiently small ε, the distribution of the field can be written as an explicit factor of an i.i.d. process. Furthermore, we construct approximations by finitely dependent fields that are close in total variation to

the original field, with exponential decay when the allowed range of dependence grows. We first do the proof for a truncated version of this Gaussian model, showing in this case that the associated field admits a finitary coding with exponential tails, providing a new application of high-noise condition of [HS00] for an uncountable state space. The proof for the original model is more intricate. Our approach extends classical CFTP-based constructions by developing a stratified coupling method tailored to the continuous and unbounded nature of the Gaussian setting.

3.1 Introduction and main results

This paper is concerned with the existence of *factor maps* from i.i.d. (independent and identically distributed) processes to stationary Markov random fields (precise definitions will be given below). For simplicity, we consider random fields on $\mathbf{Z}^d$ for some $d \geq 1$, although the theory can be naturally extended to more general graphs. Originally, the interest for factor maps came from ergodic theory. Processes of special interest in ergodic theory are the so-called *Bernoulli shifts*, i.e the random fields (initially in dimension $d = 1$) that are isomorphic to an i.i.d. process. Ornstein proved in [Orn74] the isomorphism theorem for Bernoulli shifts, which states that two i.i.d. processes on $\mathbf{Z}$ of equal entropy are in fact isomorphic; he further showed that factors of i.i.d. processes are Bernoulli shifts. These results extend to higher dimensions (see [KW72] and [OW87]).

3.1.1 Markov random fields and Gibbs measures

A random field with state space S is a collection $X = (X_i)_{i\in\mathbf{Z}^d}$ of $S-$valued random variables indexed by $\mathbf{Z}^d$. Equivalently, the distribution of X is a probability measure on $\Omega := S^{\mathbf{Z}^d}$. In this paper, S can be either a finite set, equipped with the discrete $\sigma-$algebra, or a subset of $\mathbf{R}$, equipped with the Borel $\sigma-$algebra. In both cases, Ω is equipped with the product $\sigma-$algebra. Elements of Ω are often called *configurations*. We focus on distributions on Ω that are invariant by the translations of $\mathbf{Z}^d$. Random fields whose distribution is translation-invariant are called *stationary*. We are more specifically interested in stationary random fields that satisfy *local Markov property*: it says that the conditional distribution of the field at a given site i of $\mathbf{Z}^d$ only depends on the values of the field on some finite neighbourhood around i. More precisely,

Definition 3.1. *A random field X is said to be a* Markov random field *with boundary set $B \Subset \mathbf{Z}^d \setminus \{0\}$ (i.e. B is a finite subset of $\mathbf{Z}^d \setminus \{0\}$) if its distribution μ satisfies for all $i \in \mathbf{Z}^d$,*

$$\mu(X_i \in \cdot \mid (X_j)_{j\neq i} = (\xi_j)_{j\neq i}) = \mu(X_i \in \cdot \mid (X_j)_{j\neq i} = (\xi'_j)_{j\neq i}) \tag{3.1}$$

for $\mu-$almost every $\xi, \xi' \in \Omega$ such that $\xi_j = \xi'_j$ for all $j \in i + B$.

When a Markov random field with boundary set B is fixed, we will write $j \sim i$ when $j \in i + B$; since the Markov property remains true by taking larger boundary set, we can always assume that B is symmetric, so that $j \sim i$ is equivalent to $i \sim j$.

Stationary Markov random fields often arise as statistical mechanics models. Below we briefly introduce the notions of specification and Gibbs measure, and refer to [FV18] and [Geo11] for a large overview of the theory. A statistical mechanics model (on $\mathbf{Z}^d$) is defined through a potential Φ, which is a collection of functions $\Phi_\Delta : \Omega \to \mathbf{R}$, for every $\Delta \Subset \mathbf{Z}^d$. The

energy of a configuration is represented by the *Hamiltonian* of the model, defined for every $x \in \Omega$ and $\Lambda \Subset \mathbf{Z}^d$ by

$$H_\Lambda^\Phi(x) = \sum_{\Delta \cap \Lambda \neq \emptyset} \Phi_\Delta(x).$$

To ensure that the above sum is convergent, we assume that the potential satisfies $\Phi_\Delta \equiv 0$ for all Δ with diameter greater that a certain integer r; such potentials are said to have finite range. Since we aim to define stationary random fields, we furthermore assume that the potential is translation-invariant, in the sense that for every $\Delta \Subset \mathbf{Z}^d$, $i \in \mathbf{Z}^d$ and $x \in \Omega$, one has $\Phi_{i+\Delta}(x) = \Phi_\Delta((x_{i+j})_{j \in \mathbf{Z}^d})$. We endow the state space S with an a priori measure λ, which is taken to be the counting measure if S is countable and the Lebesgue measure if S is an uncountable subset of $\mathbf{R}$. The associated product measure over $\Lambda \subset \mathbf{Z}^d$ is denoted λ^Λ. The *specification* of the model is a consistent set of conditional distributions $\pi^\Phi = (\pi_\Lambda^\Phi)_{\Lambda \Subset \mathbf{Z}^d}$: for all $\Lambda \Subset \mathbf{Z}^d$ and $\xi \in \Omega$, $\pi_\Lambda^\Phi(\cdot \mid \xi)$ is the probability measure on S^Λ defined by

$$\pi_\Lambda^\Phi(\cdot \mid \xi) := \frac{1}{Z_\Lambda^\xi} \int_{S^\Lambda} \mathbf{1}_{\{x_\Lambda \in \cdot\}} \exp(-H_\Lambda^\Phi(x_\Lambda \xi_{\Lambda^c}))\, \mathrm{d}\lambda^\Lambda(x_\Lambda)$$

where $Z_\Lambda^\xi := \int_{S^\Lambda} \exp(-H_\Lambda^\Phi(x_\Lambda \xi_{\Lambda^c}))\mathrm{d}\lambda^\Lambda(x_\Lambda)$ is just a renormalising constant, and $x_\Lambda \xi_{\Lambda^c}$ is the configuration that agrees with x_Λ on Λ but with ξ on Λ^c. A probability measure on Ω having the conditional distributions prescribed by π^Φ is called a *Gibbs measure* for π^Φ (or equivalently for the potential Φ). In other words, μ is a Gibbs measure for π^Φ if for all $\Lambda \Subset \mathbf{Z}^d$,

$$\mu\big(X|_\Lambda \in \cdot \mid \forall i \notin \Lambda, X_i = \xi_i\big) = \pi_\Lambda^\Phi(\cdot \mid \xi) \text{ for } \mu\text{−a.e } \xi \in \Omega, \tag{DLR}$$

where $X|_\Lambda := (X_i)_{i \in \Lambda}$ is the restriction of X to Λ. The above relation is called the DLR equation, for Dobrushin, Landford and Ruelle, who introduced this formalism for the definition of statistical mechanics models, see [Dob68] [LR69]. When S is finite or compact, existence of a Gibbs measure for a given potential follows from compactness of Ω (see e.g. Theorem 6.26 in [FV18]). Uniqueness, however, is a more intricate question, which is central in statistical mechanics. We say that the model exhibits a *phase transition* when several Gibbs measures coexist.

By the finite-range assumption, a random field whose distribution is a Gibbs measure is a Markov random field. In fact, it satisfies the local Markov property (3.1) by taking as boundary set the reunion of subsets $\Delta \Subset \mathbf{Z}^d$ that contain 0 and such that $\Phi_\Delta \not\equiv 0$, minus 0. Conversely, it follows from the Hammersley-Clifford theorem [CH71], [Ave72], [Spi71], that the distribution μ of a stationary Markov random field can always be expressed as a Gibbs measure for some (finite range and translation-invariant) potential when the state space is finite and μ is fully supported.

A well-known example of stationary Markov random field, which is a fundamental model in statistical mechanics, is the *Ising model*. Fix $S = \{-1, 1\}$ and let $\beta \in \mathbf{R}$. The potential of the Ising model on $\mathbf{Z}^d$ has the following simple form:

$$\forall \Delta \Subset \mathbf{Z}^d, \forall \omega \in \{\pm 1\}^{\mathbf{Z}^d},\ \Phi_\Delta(\omega) = \begin{cases} -\beta \omega_i \omega_j & \text{if } \Delta = \{i, j\}, i \sim j \\ 0 & \text{otherwise} \end{cases}.$$

Its Hamiltonian is therefore given, for every $\Lambda \Subset \mathbf{Z}^d$ and $\omega \in \{\pm 1\}^{\mathbf{Z}^d}$, by

$$H_\Lambda(\omega) = -\beta \sum_{\substack{i \sim j \\ \{i,j\} \cap \Lambda \neq \emptyset}} \omega_i \omega_j.$$

It is well known that for $d \geq 2$, there exists a critical parameter $\beta_c(d) \in (0, \infty)$ such that for all $\beta \in [0, \beta_c(d))$, there is a unique Gibbs measure μ_β satisfying the (DLR) equation, whereas for $\beta > \beta_c(d)$, the model exhibits a phase transition. In fact, it can be shown that if one considers an increasing sequence of boxes $\Lambda_n \nearrow \mathbf{Z}^d$ and the measures $\mu^+_{\beta,\Lambda_n} := \pi^\Phi_{\Lambda_n}(\cdot \mid +)$ where $+$ denotes the configuration with $+1$ everywhere, then for all $\beta \geq 0$ this sequence of measures converges (for the weak convergence topology) towards a Gibbs measure μ^+_β (independent of the choice of the Λ_n). A Gibbs measure μ^-_β is obtained similarly. One can then prove that the presence of a phase transition is equivalent to $\mu^-_\beta \neq \mu^+_\beta$, and that μ^-_β and μ^+_β indeed differ if β is sufficiently large.

3.1.2 Factors of product measures

Let μ be a translation-invariant distribution on $S^{\mathbf{Z}^d}$ (Markovian assumption is not required for the following general definitions). We say that μ is a *factor of a product measure* or *factor of an i.i.d. process* (abbreviated FIID in the following) if there exists a probability space $(\mathcal{Y}, \mathcal{F}, \nu)$ and a measurable map $F : \mathcal{Y}^{\mathbf{Z}^d} \to S^{\mathbf{Z}^d}$ which commutes with all the translations of $\mathbf{Z}^d$ such that μ is the image of $\nu^{\otimes \mathbf{Z}^d}$ by F. It is equivalent to say that there exists an i.i.d. $\mathcal{Y}-$valued process $Y = (Y_i)_{i \in \mathbf{Z}^d}$ such that $F(Y)$ has distribution μ. The factor map F is called a *coding* from $\nu^{\otimes \mathbf{Z}^d}$ to μ.

From statistical mechanics point of view, the notion of FIID is not satisfying since it does not catch phase transitions. Indeed, it is known that the "plus state" of the Ising model, that is the measure μ^+_β defined in Section 3.1.1, is an FIID for all $\beta \geq 0$ (proved first by Ornstein and Weiss in an unpublished manuscript, then by Adams [Ada92]). Van den Berg and Steif [BS99] then provide a way of detecting phase transitions through the notion of *finitary* codings. The *coding radius* of the factor map F is defined as the random radius around the origin that one needs to reveal in the i.i.d. process to determine the spin at 0. More precisely, define the coding radius of F at site $i \in \mathbf{Z}^d$ and configuration $y \in \mathcal{Y}^{\mathbf{Z}^d}$ as

$$R_i(y) := \inf\{r \geq 0 : \forall y' \in \mathcal{Y}^{\mathbf{Z}^d}, y'|_{B^i_r} = y|_{B^i_r} \implies F(y)_i = F(y')_i\}$$

where $B^i_r := \{j \in \mathbf{Z}^d : ||j - i||_1 \leq r\}$ is the ball of radius r around i in $\mathbf{Z}^d$, and $y|_{B^i_r}$ is the restriction of y to B^i_r. When Y is as above, the $R_i(Y)$ for $i \in \mathbf{Z}^d$ are random variables that are equally distributed, due to translation invariance. Set $R := R_0(Y)$ (the choice of 0 is made arbitrarily, by translation invariance of μ); R is called the coding radius of F. If R is almost surely finite, the coding is said to be *finitary*. In this case, we say that μ is a *finitary factor* of an i.i.d. process, abbreviated FFIID. Moreover, when the coding radius satisfies $\mathbf{P}(R \geq n) \leq Ce^{-cn}$ for some $c, C > 0$ and all $n \in \mathbf{N}$, we say that μ is an FFIID with *exponential tails*. The main result of [BS99] is that the presence of a phase transition is "a fundamental obstruction for finitary coding"; in words, if there exists a distribution different from μ with the same specification, this prevents μ from being an FFIID. This shows that the plus state of the Ising model on $\mathbf{Z}^d$ is *not* FFIID if $\beta > \beta_c(d)$, despite being a Bernoulli shift. It is also proved in [BS99] that in the regime of uniqueness, the Gibbs measure of the Ising model is an FFIID, with exponential tails if $\beta < \beta_c$, but with coding radius's $d-$th moment necessarily infinite at $\beta = \beta_c$ (this last result was jointly proved with Peres). Note that the existence of a finitary coding at the critical point follows from the continuity of the phase transition for the Ising model ([Yan52] [AF86] [ADCS15]).

The work of van den Berg and Steif makes use of coupling-from-the-past (CFTP) arguments, developed by Propp and Wilson [PW96] to build their famous exact sampling algorithm. In particular, to construct a finitary coding for the Ising model up to the critical point, they implement a monotone coupling, which is possible only if $\beta \geq 0$. Another construction due to Häggström and Steif [HS00], also relying on CFTP arguments, enables to get a finitary coding for the Ising model with β close to 0, but possibly negative. Although not specified in their paper, the coding has exponential tails (this is because in [HS00], as in [BS99], the authors are mostly interested in codings from finite-valued i.i.d. processes; this additional restriction makes the tail of the coding radius harder to control). In fact, they prove the existence of a finitary coding for all stationary Markov random fields with finite state space which satisfy a *high-noise* condition. This is a condition on specifications, which, with the notations of Section 3.1.1, amounts to

$$\sum_{s \in S} \inf_{\xi \in \Omega} \pi^{\Phi}_{\{i\}}(\{s\} \mid \xi) > 1 - \frac{1}{2d}, \ i \in \mathbf{Z}^d. \tag{3.2}$$

The left-hand side is called the *multigamma admissibility* in [HS00], following the terminology introduced in [MG98] (it does not depend on i, by translation-invariance). Note that (3.2) immediately implies the uniqueness condition obtained by van den Berg and Maes in [BM94], so that we are necessarily in the regime of uniqueness of the Gibbs measure when the high-noise condition is fulfilled. For the Ising model, it corresponds to taking $|\beta| < \frac{1}{4d} \log\left(1 + \frac{2}{2d-1}\right)$. In [Spi20b], Spinka improved the result of [HS00], proving that stationary Markov random fields (with finite state space) that satisfy exponential strong spatial mixing are FFIID with exponential tails, and this is a weaker condition than (3.2).

We end this paragraph by making the connection between FFIID and finitely dependent processes. For any integer $l \in \mathbf{N}$, we say that a measure μ on Ω has finite dependence of range l (or μ is $l-$dependent) if for any $A, B \subset \mathbf{Z}^d$ such that $d(A, B) > l$, the restrictions $X|_A$ and $X|_B$ are independent, when X has law μ; μ is *finitely dependent* if it is $l-$dependent for some $l \in \mathbf{N}$. If μ is an FFIID with coding radius almost surely bounded by some constant, it is said to be a *block factor* of an i.i.d. process. In this case, it is trivially finitely dependent (take l equal to twice the almost sure bound on the coding radius). More generally, the tail of the coding radius controls the distance (in total variation) of the field to finitely dependent processes. Recall that the *total-variation distance* between two probability measures μ and ν on a measurable space $(\Omega, \mathcal{F})$ is defined by $d_{\mathrm{TV}}(\mu, \nu) = \sup\{|\mu(A) - \nu(A)|, A \in \mathcal{F}\}$. It is well known that we have $d_{\mathrm{TV}}(\mu, \nu) = 1 - \sup \mathbf{P}(X = Y)$ where the supremum is taken over all coupling (X, Y) of μ and ν. If μ is an FIID with coding radius R, for every $l \in \mathbf{N}$, there exists an $l-$dependent distribution ν_l on Ω such that for all $\Lambda \Subset \mathbf{Z}^d$,

$$d_{\mathrm{TV}}(\mu|_\Lambda, \nu_l|_\Lambda) \leq |\Lambda| \mathbf{P}(2R > l). \tag{3.3}$$

This is obtained by truncating the factor map. We give a proof of this fact below (see Proposition 3.24), but we do not claim that this is a new result. Conversely, Spinka proved in [Spi20c] that finitely dependent distributions are FFIID, when the state space is finite (or countable).

3.1.3 Gaussian models and main results

A lot of results regarding finitary codings for Markov random fields have been obtained when the state space S is finite —see [Spi20b] for a comprehensive review of the literature, and for

results on various statistical mechanics models. In this paper, our goal is to study the case when S is an uncountable subset of $\mathbf{R}$. For concreteness, we consider an explicit Gaussian model and a truncated version, although the very same proof would carry over to more general setups under relatively mild assumptions on the marginals and the specification. For example, things should work the same for a Gaussian model conditioned to take integer-values.

Gaussian model. Fix $S = \mathbf{R}$ (thus $\Omega = \mathbf{R}^{\mathbf{Z}^d}$) and a parameter $\varepsilon \in (-1, 1)$. For $m \in \mathbf{R}$, denote by $\mathcal{N}(m, 1)$ the normal distribution of mean m an variance 1. We consider a centered stationary Gaussian field $(X_i)_{i\in\mathbf{Z}^d}$ on $\mathbf{Z}^d$ such that the conditional distribution of X_0 given the rest of the configuration is Gaussian of variance 1 and mean $\varepsilon\times$ (average of the neighbours). It means that its distribution, denoted μ_ε, satisfies for all $i \in \mathbf{Z}^d$

$$\mu_\varepsilon(X_i \in \cdot \mid \forall j \neq i, X_j = \xi_j) = \mathcal{N}\Big(\frac{\varepsilon}{2d} \sum_{j,\|j-i\|_1=1} \xi_j, 1\Big), \text{ for } \mu_\varepsilon\text{−a.e. } \xi \in \mathbf{R}^{\mathbf{Z}^d}. \tag{3.4}$$

By definition, $(X_i)_{i\in\mathbf{Z}^d}$ is a Markov random field, with boundary set $B = B_1^0 \setminus \{0\}$ the unit sphere of $\mathbf{Z}^d$.

Remark 3.2. *The above description of our model does not guarantee that such a Gaussian field is well defined. However, assuming that it exists, we can identify from (3.4) the coefficients of the inverse of the covariance operator. If Γ is the covariance of $(X_i)_{i\in\mathbf{Z}^d}$, then, given the formula for the density of a Gaussian vector, it follows that we must have $\Gamma^{-1} = I - T$ with $T = \Big(\frac{\varepsilon}{2d}\mathbf{1}_{\{i\sim j\}}\Big)_{i,j\in\mathbf{Z}^d}$ and I the identity operator. The inverse of $I - T$ is well defined when $\varepsilon \in (-1, 1)$, because the Neumann series $\sum_{n\geq 0} T^n$ converges in this case. Indeed, for all $i, j \in \mathbf{Z}^d$,*

$$(T^n)_{i,j} = \sum_{i=k_1,k_2,\dots,k_n=j} T_{k_1,k_2} \dots T_{k_{n-1},k_n} = \sum_{i=k_1\sim k_2\sim\cdots\sim k_n=j} \Big(\frac{\varepsilon}{2d}\Big)^n \in [-\varepsilon^n, \varepsilon^n],$$

because the number of paths of length n that join i and j is bounded by $(2d)^n$.

Therefore, for every $\varepsilon \in (-1, 1)$, there exists a unique Gaussian field $(X_i)_{i\in\mathbf{Z}^d}$ with mean 0 and covariance

$$\mathrm{Cov}(X_i, X_j) = \Gamma(i, j) = \sum_{n\geq 0} \sum_{i=k_1\sim\cdots\sim k_n=j} \Big(\frac{\varepsilon}{2d}\Big)^n,$$

and μ_ε denotes its distribution. In the other way, it is now clear that (3.4) holds for μ_ε.

The description of $(X_i)_{i\in\mathbf{Z}^d}$ through its conditional distributions corresponds to the DLR approach in statistical mechanics. The theory of Gaussian fields seen as Gibbs measures has been well developed in [Geo11] (Chapter 13). In fact, (3.4) describes the singleton part of the specification of the model. It is a general fact ([Geo11] Theorem 1.33) that this is sufficient to characterise the whole specification when it has a positive density with respect to some a priori measure (which is the case with the Lebesgue measure in (3.4)). The formula of the Gaussian density allows us to identify from (3.4) what is the Hamiltonian of the model:

$$H_\Lambda(x) = \frac{1}{2}\sum_{i\in\Lambda} x_i^2 - \frac{\varepsilon}{2d} \sum_{\substack{i\sim j \\ \{i,j\}\cap\Lambda\neq\emptyset}} x_i x_j, \ \forall \Lambda \Subset \mathbf{Z}^d, x \in \Omega. \tag{3.5}$$

Theorems 13.24 and 13.26 in [Geo11] characterise the set of Gibbs measures associated with the above Hamiltonian. It corresponds to the convex hull of the set of Gaussian field distributions of covariance Γ and mean $m \in \mathbf{R}^{\mathbf{Z}^d}$ satisfying

$$\forall i \in \mathbf{Z}^d,\ m_i = \frac{\varepsilon}{2d} \sum_{j \sim i} m_j,$$

giving rise to uncountably many Gibbs measures. However, notice that μ_ε is the only Gibbs measure for this model which is translation-invariant. Up to renormalisation, in the case $\varepsilon > 0$, the Hamiltonian defined in (3.5) can be rewritten as the Hamiltonian of the discrete *Gaussian free field* with mass m satisfying $(1+m^2)^{-1} = \varepsilon$ (see e.g. [FV18] Chapter 8). The case $\varepsilon = 1$ would therefore corresponds to the massless ($m = 0$) Gaussian free field, for which it is well known that no Gibbs measure exists in dimensions 1 and 2.

One of the main purposes of this paper is to investigate how close μ_ε is from finitely dependent distributions. To do this, we propose to write $(X_i)_{i \in \mathbf{Z}^d}$ of law μ_ε as a factor of an i.i.d. process, and then to construct a finitely dependent random field that is close to $(X_i)_{i \in \mathbf{Z}^d}$ by truncating the factor map. Since there are infinitely many distributions having the same specification as μ_ε, there is no hope of getting a *finitary* coding for μ_ε. We nevertheless construct a coding for μ_ε, and obtain the same bound on the distance to finitely dependent fields as if we had a coding radius with exponential tails (though it does not directly follow from (3.3)). The following theorem is the main result of the paper.

Theorem 3.3. *For $|\varepsilon|$ small enough,*

1. *μ_ε is a factor of an i.i.d. process.*
2. *There exists $c > 0$ such that for all $l \in \mathbf{N}$, there exists an l–dependent distribution ν_l on $\mathbf{R}^{\mathbf{Z}^d}$ satisfying for every finite subset $\Lambda \subset \mathbf{Z}^d$,*

$$d_{\mathrm{TV}}(\mu_\varepsilon|_\Lambda, \nu_l|_\Lambda) \leq |\Lambda| e^{-cl}.$$

Truncated Gaussian model. For any $L > 0$, one may consider a stationary random field $(X_i)_{i \in \mathbf{Z}^d}$ with state space $[-L, L]$, such that the distribution of X_0 knowing the rest of the configuration is Gaussian of mean $\frac{\varepsilon}{2d} \sum_{j \sim i} X_j$ conditioned to be in $[-L, L]$. This corresponds to a Gibbs measure associated with the same Hamiltonian (3.5) but on a different state space. More precisely, define a specification $\pi^{\varepsilon, L} = (\pi^{\varepsilon, L}_\Lambda)_{\Lambda \Subset \mathbf{Z}^d}$ by

$$\forall \Lambda \Subset \mathbf{Z}^d, \xi \in [-L, L]^{\mathbf{Z}^d},\ \pi^{\varepsilon, L}_\Lambda(\cdot \mid \xi) := \int_{[-L,L]^\Lambda} \mathbf{1}_{\{x_\Lambda \in \cdot\}} \exp(-H_\Lambda(x_\Lambda \xi_{\Lambda^c})) \mathrm{d}\lambda^\Lambda(x_\Lambda).$$

Existence of a Gibbs measure compatible with $\pi^{\varepsilon, L}$ is guaranteed by compactness of the state space (see e.g. Theorem 6.26 in [FV18]). Moreover, for $|\varepsilon|$ small enough ($|\varepsilon| < 1/(2L^2)$ suffices), $\pi^{\varepsilon, L}$ satisfies the Dobrushin's condition of weak dependence (see Theorem 6.35 in [FV18] or Theorem 8.7 in [Geo11]) so there is a unique Gibbs measure specified by $\pi^{\varepsilon, L}$; call it μ^L_ε. We prove a result similar to Theorem 3.3 for μ^L_ε.

Theorem 3.4. *Let $L > 0$. For $|\varepsilon|$ small enough,*

1. *μ^L_ε is an FFIID with exponential tails.*

2. *There exists $c > 0$ such that for all $l \in \mathbf{N}$, there exists an l–dependent distribution ν_l on $[-L, L]^{\mathbf{Z}^d}$ satisfying for every finite subset $\Lambda \subset \mathbf{Z}^d$,*

$$d_{\mathrm{TV}}(\mu_\varepsilon^L|_\Lambda, \nu_l|_\Lambda) \leq |\Lambda| e^{-cl}.$$

In Theorem 3.4, part 2 is a direct consequence of part 1 because of the exponential tails of the coding radius, according to (3.3). Part 1 holds because when $|\varepsilon|$ is sufficiently small, $\pi^{\varepsilon,L}$ satisfies the high-noise condition (3.2) of Häggström and Steif. They proved in [HS00] that high-noise Markov random fields are FFIID with exponential tails. Their result is stated for fields with finite state space, but the proof can be adapted to the compact case. We present a different proof in this article, which can be seen as a preliminary version of the proof of Theorem 3.3. Note that in [FFG07], the authors studied perfect sampling of truncated Gaussian vectors. Our proof of Theorem 3.4 is similar to their method, which they suggest in their final remark to implement for truncated Gaussian fields indexed by $\mathbf{Z}^d$, such as μ_ε^L.

Organisation of the paper. Theorems 3.3 and 3.4 will be derived from more general results, respectively Theorems 3.13 and 3.6. Their proofs rely on a coupling-from-the past argument inspired by Propp and Wilson exact sampling algorithm [PW96]. This is a classical technique to construct coding from i.i.d. processes, as described in [BS99], [HS00], [Spi20a], [Spi20b]. In essence, for a distribution μ on Ω (e.g. $\mu = \mu_\varepsilon$ or μ_ε^L), we couple Markov processes of invariant law μ starting from all possible initial configurations. We say that there is coalescence at a certain space-time position when all the processes coincide at this space-time position. The evolution of the processes is governed by a dynamics analogous to the Glauber dynamics of the Ising model [Gla63], described in Section 3.2. Each transition consists of a new sampling of a single site of $\mathbf{Z}^d$, according to the conditional distribution given the rest of the configuration. We aim to maximise the *coupling probability*, that is, the probability that a transition produces the same output at the updated site regardless of the configuration outside. This makes the processes coalesce, in the sense of Propp and Wilson. This allows us to get an exact sampling of distribution μ, arising as an FIID.

To determine if there is coalescence at a certain space-time position, say $(0, 0)$, we explore the dynamics backwards. The value at $(0, 0)$ in the different coupled processes is determined by the last update of site 0. Say that the update is "good" if it makes all the processes coalesce. In this case, we can stop the exploration. Otherwise, the exploration continues from that time and all the neighbours of site 0. Therefore, the backward exploration is dominated by a Galton-Watson process of mean $|B|\times$(maximal coupling probability). We say that μ satisfies the high noise assumption when the maximal coupling probability is sufficiently large, so that the process extinguish almost surely. In this case, the value of the field at $(0, 0)$ in the dynamics is independent of the initial configuration, if the initial time is chosen small enough. This is discussed in Section 3.3, where we state and prove Theorem 3.6.

Then, we adapt the method to the case where the aforementioned maximal coupling probability is small, or even 0. This is typically the case for μ_ε. We detail the assumption that we will need and prove Theorem 3.13 in Section 3.4. The proposed approach involves stratification of the state space into different levels. Rather than relying on the coalescence of potentially "high" values of the field, which is an unlikely outcome, we aim to decrease the level of the field first. The idea is that during an update, if we know that in all the coupled Markov processes, the field on the neighbourhood of the updated site is below the level n, then it is

likely that the new value of the field at the updated site falls below the level $n-1$; then, if all the values of the neighbours are below the lowest level, it is likely that the update makes the processes coalesce. This translates into an interacting particle system that follows the knowledge about the level of the field we can get independently of the initial configuration, which is precisely described in Section 3.4.1. This particle system admits a dual, which corresponds to the backward exploration of the dynamics, and that encodes the information required to achieve coalescence at the starting point. In contrast to the preceding case, where coalescence was likely within a single update, the backward exploration never stops: it is first necessary to ensure that the last update of site 0 allows to coalesce from all possible values of the field below level 1; then, the second generation of updates seen in the exploration must allow the level to decrease from 2 to 1 etc. Therefore, it will not be true that we can have coalescence at $(0,0)$ with probability 1 by taking an initial time small enough, and we will need additional arguments.

Finally, in Section 3.5, we prove Theorems 3.4 and 3.3 as consequences of Theorems 3.6 and 3.13 respectively.

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3.2 Glauber dynamics and maximal coupling probability

Throughout this Section, as well as Sections 3.3 and 3.4, we fix μ the distribution of a stationary Markov random field $(X_i)_{i\in\mathbf{Z}^d}$, with boundary set $B \Subset \mathbf{Z}^d \setminus \{0\}$. Without loss of generality, we assume that B is symmetric.

3.2.1 Glauber dynamics

In 1963, Roy J. Glauber [Gla63] introduced a stochastic version of the Ising model, known as the *stochastic Ising model* or *Glauber dynamics* (see [Hol74] for an overview on the results for the Ising model). It has now been generalised to a large class of statistical mechanics models. Several versions of the Glauber dynamics coexist in the literature, the one presented and employed in this paper is also named the *heat-bath* algorithm. Roughly speaking, each site of $\mathbf{Z}^d$ is endowed with an "exponential clock", and when the clock at $i \in \mathbf{Z}^d$ rings, we update the value of the field at i according to its conditional distribution given the rest of the configuration. Below, we describe the *graphical representation* (see [Lig85] Chapter III.6 or [Dur88] Section 2) of the Glauber dynamics for μ.

Let $\mathcal{P}$ be a Poisson point process (PPP) of intensity $\# \otimes \mathrm{d}t \otimes \mathrm{d}u$, where $\#$, $\mathrm{d}t$, $\mathrm{d}u$ denotes the counting measure on $\mathbf{Z}^d$, the Lebesgue measure on $\mathbf{R}$ and the Lebesgue measure on $[0,1]$, respectively. Write $\mathbf{P}$ for the law of $\mathcal{P}$. Points of $\mathcal{P}$ are called the *update marks* of the dynamics. By properties of PPP, it means that sites of $\mathbf{Z}^d$ are independently endowed with a bi-infinite sequence $(T_n^{\cdot}, U_n^{\cdot})_{n\in\mathbf{Z}}$, satisfying

$$\mathcal{P} = \{(i, T_n^i, U_n^i), n \in \mathbf{Z}, i \in \mathbf{Z}^d\},$$

and such that $(T_n^{\cdot})_{n\in\mathbf{Z}}$ is increasing. This observation shows that $\mathbf{P}$ is a product measure over $\mathbf{Z}^d$; we will build a factor map from this product measure to μ. For each $i \in \mathbf{Z}^d$, the $(T_n^i)_{n\in\mathbf{Z}}$ correspond to the times when site i is updated in the dynamics, and the increments $(T_{n+1}^i - T_n^i)_{n\in\mathbf{Z}}$ are independent and have exponential distribution of parameter 1 (hence the terminology "exponential clock"). The $(U_n^i)_{i\in\mathbf{Z}^d,n\in\mathbf{Z}}$ are independent random variables distributed uniformly on $[0,1]$; the update of i at time T_n^i is made with the help of the extra randomness provided by U_n^i. Suppose that we have an *update function* $\varphi : S^B \times [0,1] \to S$, with the property that for all $\eta \in S^B$,

$$\int_0^1 \mathbf{1}_{\{\varphi(\eta,u)\in\cdot\}}\mathrm{d}u = \mu(X_0 \in \cdot \mid (X_j)_{j\in B} = \eta),$$

meaning that if U has uniform distribution on $[0,1]$ (denoted $\mathcal{U}[0,1]$), then $\varphi(\eta, U)$ has the same law as X_0 given that the field on B coincide with η; note that by the Markov property, the conditional law of X_0 given $(X_i)_{i\neq 0}$ depends only on the configuration on B. Then, for every update mark (i,t,u), if the dynamics is in configuration $\xi \in \Omega$ before the update, then the new value of the field at i is given by $\varphi(\xi|_{i+B}, u)$; the other values remain unchanged. The property of the update function φ guarantees that μ is preserved by the dyanmics.

Note that we have defined the evolution of the dynamics on a time-line which is infinite in both directions. The graphical representation provides a coupling of processes starting from any time and any initial configuration, obtained by using the same PPP to construct them. For $\tau \in \mathbf{R}$ and $\xi \in \Omega$, write $X^{\xi,\tau} := (X_t^{\xi,\tau})_{t\geq\tau}$ for the process started at time τ with initial configuration $X_\tau^{\xi,\tau} = \xi$, and whose evolution is governed by the graphical representation of the Glauber dynamics. More precisely, we have that for every $(i,t,u) \in \mathcal{P}$ with $t > \tau$, a.s.

$$\forall j \neq i, X_t^{\xi,\tau}(j) = X_{t-}^{\xi,\tau}(j) \text{ and } X_t^{\xi,\tau}(i) = \varphi(X_t^{\xi,\tau}|_{i+B}, u).$$

Nevertheless, since there are infinitely many sites, we cannot define the first update after a given initial time $\tau \in \mathbf{R}$. In other words, for every $\tau \in \mathbf{R}$, a.s.

$$\inf\{t > \tau : \exists i \in \mathbf{Z}^d, \exists u \in [0,1], (i,t,u) \in \mathcal{P}\} = \tau.$$

An additional argument is therefore needed to ensure that the dynamics can indeed be constructed with the graphical representation. The pitfall is that the first update of a given site (which is well defined) could require one to look infinitely far away. Indeed, for each update one need to look at the neighbours, but it may happen that these neighbours needed to look at their own neighbours before etc. A percolation argument of Harris [Har72] (see also [Dur88] Section 2), handles this problem. The idea is to take a random subgraph of $\mathbf{Z}^d$ defined as follows: fix a small positive number t_0; for every $i, j \in \mathbf{Z}^d$ such that $i \sim j$, say that the edge $\{i,j\}$ is open if there exists $n \in \mathbf{Z}$ such that $\tau < T_n^i < \tau + t_0$ or $\tau < T_n^j < \tau + t_0$; otherwise, the edge is closed. Hence, an open edge corresponds to the fact that an update of the system requires i to look at j or the reverse. Consequently, if i and j are not in the same connected component in our random graph, then the values of the field at i and j at time $\tau + t_0$ are independent (assuming they were chosen independently at the initial time τ). Hence, one can compute separately the evolution up to time $\tau + t_0$ in each connected component. Harris's result is that for t_0 small enough, there is no infinite connected component for almost all realisation of the PPP. Thus, each connected component have a first update time, so we can

construct the dynamics up to time $\tau + t_0$. Iterate the argument to construct the dynamics for all time $t \geq \tau$. It follows that for any $t \geq \tau$, $X_t^{\xi,\tau}$ is a well-defined measurable function of the update marks between τ and t and of the initial configuration.

Notations. In the following, we will often consider measurable events of the update marks. For two such events A and B, we will write $A \overset{\text{a.s.}}{\subset} B$ when for almost all realisation of the PPP, one has the inclusion $A \subset B$; in other words, $A \overset{\text{a.s.}}{\subset} B$ means that $\mathbf{P}(B \setminus A) = 0$. Similarly, we write $A \overset{\text{a.s.}}{=} B$ when both inclusion hold, that is when $\mathbf{P}(A \Delta B) = 0$.

3.2.2 Coupling-from-the-past

The main idea to construct a coding for μ from the PPP is to rely on the *coupling-from-the-past* algorithm, introduced by Propp and Wilson [PW96], extended to the infinite-volume setting. It allows to get an exact sampling of distribution μ, by coupling Markov processes with invariant law μ started from all possible initial configuration at a random negative initial time: if the initial time is far enough from 0 so that all the processes take the same value at time 0, it follows that this common value has law μ.

Considering the processes $(X^{\xi,\tau})_{\xi\in\Omega,\tau\in\mathbf{R}}$ defined above, we say that there is *coalescence* from τ at space-time position $(i,t) \in \mathbf{Z}^d \times (\tau, +\infty)$ when

$$\forall \xi, \xi' \in \Omega, X_t^{\xi,\tau}(i) = X_t^{\xi',\tau}(i).$$

This is a measurable event of the update marks between τ and t. Note that if we have coalescence at (i,t) from τ, it follows that we also have coalescence at (i,t) from any earlier time $\tau' < \tau$. For every $i \in \mathbf{Z}^d$, let

$$\widehat{\tau}_i := \sup\{\tau < 0 \mid \forall \xi, \xi' \in \Omega, X_t^{\xi,\tau}(i) = X_t^{\xi',\tau}(i)\}.$$

If $\widehat{\tau}_i$ is finite for all $i \in \mathbf{Z}^d$, one can define a random field $X^* = (X_i^*)_{i\in\mathbf{Z}^d}$ by

$$X_i^* = X_0^{\cdot,\widehat{\tau}_i}(i),\ i \in \mathbf{Z}^d,$$

noticing that one can take any initial configuration for the dynamics, since the value of X_i^* is the same for any choice. Note that X^* is a measurable function of $\mathcal{P}$, so it is an FIID. Then, the fact that X^* is distributed according to μ follows from a classical coupling-from-the-past argument.

Proposition 3.5. *Assuming that a.s. $\widehat{\tau}_i$ is finite for all $i \in \mathbf{Z}^d$, the random field X^* defined above has distribution μ. Moreover, it gives a realisation of μ as an FIID.*

Proof. Fix $\Lambda \Subset \mathbf{Z}^d$ and let ξ be distributed according to μ, independent of $\mathcal{P}$. Since μ is invariant for the dynamics, it follows that for all $\tau < 0$, $X_0^{\xi,\tau}$ has law μ. On the other hand, for all $\tau \leq \min_{i\in\Lambda} \widehat{\tau}_i$, one has a.s. $X_0^{\xi,\tau}(i) = X_i^*$, by definition of the $\widehat{\tau}_i$. Note that $\min_{i\in\Lambda} \widehat{\tau}_i > -\infty$ a.s. because Λ is finite. Therefore, $X^*|_\Lambda$ is distributed according to $\mu|_\Lambda$, and since Λ was arbitrary, the result follows. □

3.2.3 Maximal coupling probability

If we prove that the $\widehat{\tau}_i$ are finite almost surely, according to Proposition 3.5, it yields a coding of μ from the update marks. To achieve that, we aim to have an update function φ which favours coalescence. It means that we want to maximise the probability that for all $\eta, \eta' \in S^B$, $\varphi(\eta, U) = \varphi(\eta', U)$ for $U \sim \mathcal{U}[0,1]$, keeping the property that $\varphi(\eta, U)$ has law $\mu(X_0 \in \cdot \mid (X_j)_{j\in B} = \eta)$.

It is a general fact (see [Tho00] Chapter 3, Theorem 7.1) that for any collection of probability measures π^η, defined on the same measurable space, for η living in an arbitrary index set (e.g. S^B), we can define its greatest common component $\bigwedge_\eta \pi^\eta$ by

$$\text{For any event } A, \Big(\bigwedge_\eta \pi^\eta\Big)(A) = \sup\{\nu(A), \nu \leq \pi^\eta \ \forall \eta\}.$$

In general, $\bigwedge_\eta \pi^\eta$ is not a probability measure. In fact, its total mass is the *maximal coupling probability* ([Tho00] Chapter 3, Theorem 7.2) of laws π^η, that is the maximal probability for realisations of π^η to coincide in a coupling. Theorem 7.3 in [Tho00] (Chapter 3) guarantees that there always exists a coupling that achieves this maximal coupling probability.

Then, returning to our setting, let

$$\gamma = \gamma(\mu) = \gamma_i(\mu) := \max_{\mathbb{P}} \mathbb{P}(\forall \eta, \eta' \in S^B, X_i^\eta = X_i^{\eta'}),$$

where the maximum is taken over all the couplings of random variables $(X_i^\eta)_{\eta\in S^B}$ such that X_i^η has law $\mu(X_i \in \cdot \mid (X_j)_{j\in i+B} = \eta)$. By the Markov property, this distribution is the same as $\mu(X_i \in \cdot \mid \forall j \neq i, X_j = \xi_j)$ for any $\xi \in \Omega$ that coincides with η on $i + B$. Note that since μ is stationary, $\gamma_i(\mu)$ does not depend on i so we can drop the index i, and will often simply write γ for short, since the measure μ is fixed. It follows that γ corresponds to the maximal coupling probability of laws $\pi^\eta := \mu(X_0 \in \cdot \mid (X_j)_{j\in B} = \eta)$, for $\eta \in S^B$. Then, it is clear that if S is finite, γ corresponds to the multigamma admissibility of [HS00] (i.e. the left-hand side of (3.2)).

The best we can hope for the update function is that it returns the same value regardless of the first coordinate (which corresponds to the boundary condition) when the second coordinate belongs to a subset of $[0,1]$ of length γ. An explicit construction of an admissible update function is done in [HS00] for finite S, and see [FFG07] or [MG98] for the case of real-valued random variables, or the Appendix below.

We conclude this Section by computing the maximal coupling probability for our two Gaussian models. For every $x \in \mathbf{R}$, set $g_x(t) := \frac{1}{\sqrt{2\pi}} e^{-(t-x)^2/2}$, the density of $\mathcal{N}(x,1)$. The law $\mathcal{N}(x,1)$ conditioned to $[-L,L]$ (for $L > 0$) is called the *truncated to* $[-L,L]$ *normal distribution* of mean x and variance 1; its density is given by

$$\overline{g_x}^L(t) := \frac{g_x(t)}{\int_{-L}^{L} g_x(s)\mathrm{d}s} \mathbf{1}_{\{t\in[-L,L]\}}.$$

We denote this distribution by $\overline{\mathcal{N}}^L(x,1)$.

Let us start by computing $\gamma(\mu_\varepsilon^L)$, for $L > 0$ and $|\varepsilon| < 1/(2L^2)$. It is the maximal coupling probability of laws $\mu_\varepsilon^L(X_0 \in \cdot \mid (X_j)_{j\in B} = \eta)$ for $\eta \in [-L,L]^B$ (where $B = B_1^0 \setminus \{0\}$). By definition of the model, $\mu_\varepsilon^L(X_0 \in \cdot \mid (X_j)_{j\in B} = \eta)$ coincides with $\overline{\mathcal{N}}^L(\varepsilon \sum_{j\in B} \eta_j, 1)$,

so that $\gamma(\mu_\varepsilon^L)$ is the same as the maximal coupling probability of law $\overline{\mathcal{N}}^L(\varepsilon x, 1)$, for $x \in [-L, L]$. According to [Tho00] (Chapter 3, Corollary 8.1), the maximal coupling probability of a sequence of distributions admitting a density is given by the integral of the infimum of the densities. Although $[-L, L]$ is uncountable, the result still holds for the collection $\overline{\mathcal{N}}^L(\varepsilon x, 1)$, $x \in [-L, L]$, and we have:

$$\gamma(\mu_\varepsilon^L) = \int_{-L}^{L} \inf_{|x| \le L} \overline{g_{\varepsilon x}}^L(t) \mathrm{d}t.$$

The inequality "$\le$" follows from the aforementioned result, applied to the countable collection of law $\overline{\mathcal{N}}^L(\varepsilon x, 1)$, for $x \in \mathbf{Q} \cap [-L, L]$; the fact that it is an equality can be seen by density of $\mathbf{Q}$ and continuity of $\overline{g_{\varepsilon x}}^L$ with respect to x, or because we have an explicit construction of a coupling which matches this coupling probability (see [FFG07], [MG98] or the Appendix). It can be observed (see Lemma 3.23 below) that $\gamma(\mu_\varepsilon^L)$ goes to 1 when ε goes to 0 (L being fixed), so that μ_ε^L satisfies the high noise assumption (3.2) when $|\varepsilon|$ is small enough.

We now compute $\gamma(\mu_\varepsilon)$ for $|\varepsilon| \le 1$. It corresponds to the maximal coupling probability of $\mathcal{N}(\varepsilon x, 1)$, for $x \in \mathbf{R}$. Hence, similarly as above, we have

$$\gamma(\mu_\varepsilon) = \int_{-\infty}^{+\infty} \inf_{x \in \mathbf{R}} g_{\varepsilon x}(t) \mathrm{d}t.$$

It can easily be observed that this quantity equals 0 for all $\varepsilon \neq 0$. This means that there is no chance to reach coalescence within a single update of the dynamics.

3.3 Finitary coding with the high noise assumption

This section is devoted to the proof of the following result, from which we will deduce Theorem 3.4.

Theorem 3.6. *If $\gamma(\mu) > 1 - \frac{1}{|B|}$, then μ is an FFIID with exponential tails.*

We consider the dynamics introduced in Section 3.2 that admits μ as invariant law, and the coupling of processes $(X^{\xi,\tau})_{\xi \in \Omega, \tau \in \mathbf{R}}$ defined with the graphical representation. Recall that $\gamma = \gamma(\mu)$ is the maximal coupling probability of $\mu(X_0 \in \cdot \mid (X_j)_{j \in B} = \eta)$ for $\eta \in S^B$, so we suppose that the update function φ verifies

$$\forall \eta, \eta' \in S^B, \forall u \le \gamma, \varphi(\eta, u) = \varphi(\eta', u) \tag{3.6}$$

It means that for each update mark $(i, t, u) \in \mathcal{P}$ with $u \le \gamma$, the new value of the field at i after the update is determined by u, independently of the values of the neighbours. Hence, one has $X_t^{\xi,\tau}(i) = X_t^{\xi',\tau}(i)$ for any $\tau < t$ and $\xi, \xi' \in \Omega$, so the value of the field at i at time t becomes independent of the initial configuration: we have coalescence from τ at (i, t).

3.3.1 Spin system associated with the dynamics

We associate with the dynamics a *spin system*, that is, following the terminology of [Lig85] (Chapter III), a continuous-time Markov process on $\{0, 1\}^{\mathbf{Z}^d}$, for which only one coordinate changes in each transition. It is constructed with the graphical representation, using the same update marks $\mathcal{P}$ as before. For any time $\tau \in \mathbf{R}$, let $(\omega_t^\tau)_{t \ge \tau}$ be the spin system started at τ

with initial configuration $\omega_\tau^\tau \equiv 1$ (i.e. put spin 1 everywhere), and whose evolution is governed as follows: for all $(i,t,u) \in \mathcal{P}$, with $t > \tau$, the spin at i is set to 1 if $u > \gamma$ and $\omega_{t-}^\tau(j) = 1$ for some $j \sim i$, and it is set to 0 otherwise; the other spins remain unchanged, i.e. we have

$$\forall j \neq i, \omega_t^\tau(j) = \omega_{t-}^\tau(j) \text{ and } \omega_t^\tau(i) = \mathbf{1}_{\{u>\gamma\}}\mathbf{1}_{\{\exists j\sim i, \omega_t^\tau(j)=1\}}. \tag{3.7}$$

Note that a transition occurs at time t only if there is an update mark at that time, but not all the update marks generate an actual transition. Indeed, the value of a spin after an update can be the same as before.

For future purpose, we now give the rate function and the generator of the system. For $\omega \in \{0,1\}^{\mathbf{Z}^d}$ and $i \in \mathbf{Z}^d$, call ω^i the configuration obtained from ω by flipping coordinate i, i.e.

$$\omega^i(j) = \begin{cases} \omega(j) & \text{for } j \neq i \\ 1-\omega(j) & \text{for } j = i \end{cases}.$$

The rate function is defined for all $i \in \mathbf{Z}^d$ and $\omega \in \{0,1\}^{\mathbf{Z}^d}$ by

$$c(i,\omega) = \begin{cases} \gamma + (1-\gamma)\mathbf{1}_{\{\forall j\sim i, \omega(j)=0\}} & \text{if } \omega(i) = 1 \\ (1-\gamma)\mathbf{1}_{\{\exists j\sim i, \omega(j)=1\}} & \text{if } \omega(i) = 0 \end{cases}; \tag{3.8}$$

$c(i,\omega)$ is the rate at which ω jumps to ω^i. Roughly, it means that during an infinitesimal interval of time of length $\mathrm{d}t$, independently for every site i of $\mathbf{Z}^d$, the spin at i is flipped with probability $c(i,\omega)\mathrm{d}t$. Then, the generator $\mathcal{L}$ of the system is defined by

$$\mathcal{L}f(\omega) = \sum_{i\in\mathbf{Z}^d} c(i,\omega)(f(\omega^i) - f(\omega)) \tag{3.9}$$

for every *local* function $f : \{0,1\}^{\mathbf{Z}^d} \to \mathbf{R}$, that is f depends on finitely many coordinates.

The fact that $\mathcal{L}$ is indeed the generator of the systems $(\omega_t^\tau)_{t\geq\tau}$ constructed above with the graphical representation can be seen as follows. The spin at site i can be flipped at time t only if there exists some update mark $(i,t,u) \in \mathcal{P}$. A flip from 1 to 0 is produced by the update mark if $\mathbf{1}_{\{u>\gamma\}}\mathbf{1}_{\{\exists j\sim i, \omega_t^\tau(j)=1\}} = 0$, that is if $u \leq \gamma$ or if $u > \gamma$ and all the neighbours of i have spin 0 at the time of the update. Conversely, a flip from 0 to 1 is produced if $\mathbf{1}_{\{u>\gamma\}}\mathbf{1}_{\{\exists j\sim i, \omega_t^\tau(j)=1\}} = 1$, that is if $u > \gamma$ and at least one neighbour of i has spin 0 at the time of the update. This is consistent with the rate function defined in (3.8).

Observe that $c(\cdot,\cdot)$ is the rate function of an *attractive* spin system, in the sense that when $\omega_1 \leq \omega_2$[1], for every $i \in \mathbf{Z}^d$,

$$c(i,\omega_1) \leq c(i,\omega_2) \text{ if } \omega_1(i) = \omega_2(i) = 0,$$

$$c(i,\omega_1) \geq c(i,\omega_2) \text{ if } \omega_1(i) = \omega_2(i) = 1.$$

This implies that for any $\tau \geq \tau'$, one has

$$\mathbf{P}(\forall t \geq \tau, \omega_t^{\tau'} \leq \omega_t^\tau) = 1.$$

This is because our systems are initialised with the maximal initial condition, so that $\omega_\tau^{\tau'} \leq \omega_\tau^\tau$ almost surely, and this inequality propagates by attractiveness.

[1]For the canonical partial order on $\{0,1\}^{\mathbf{Z}^d}$, i.e. $\omega_1 \leq \omega_2$ if and only if $\omega_1(j) \leq \omega_2(j)$ for every $j \in \mathbf{Z}^d$.

The interpretation of this spin system is that it encapsulates whether or not the value of the field in the dynamics can be computed in a measurable way only from the update marks, and not from the initial configuration; a spin 0 in a certain space-time position will mean that one almost surely has coalescence from τ at this space-time position. More precisely,

Lemma 3.7. *For every $t \geq \tau$ and $i \in \mathbf{Z}^d$, we have*

$$\{\omega_t^\tau(i) = 0\} \overset{a.s.}{\subset} \left\{\forall \xi, \xi' \in \Omega, X_t^{\xi,\tau}(i) = X_t^{\xi',\tau}(i)\right\}.$$

Proof. This is clearly true for all $i \in \mathbf{Z}^d$ at $t = \tau$ since the event on the left-hand side is empty, and one can check that the inclusion is preserved by the updates for almost all realisation of the PPP. Suppose that $(i, t, u) \in \mathcal{P}$ for some $t > \tau$ and that the inclusion holds for all $s \in [\tau, t)$ and all $j \in \mathbf{Z}^d$. By property of the PPP, a.s. there is no other update mark at time t. We have $\omega_t^\tau(i) = 0$ if and only if $u \leq \gamma$ or for all $j \sim i$, $\omega_t^\tau(j) = 0$. If the update mark satisfies $u \leq \gamma$, it follows from (3.6) that we have coalescence at (i, t). In the second case, the fact that $\omega_t^\tau(j) = 0$ for all $j \sim i$ implies, by assumption, that the values of the field on the neighbourhood of i do not depend on the initial configuration of the dynamics; since the output of the update function φ only depends on the values of the field on this neighbourhood, it means that this output itself does not depend on the initial configuration; therefore, we get that $X_t^{\xi,\tau}(i) = X_t^{\xi',\tau}(i)$ for all $\xi, \xi' \in \Omega$. □

For every $i \in \mathbf{Z}^d$, let

$$\tau_i = \sup\{\tau < 0 \mid \omega_0^\tau(i) = 0\}.$$

Observe that τ_i is a random time, measurable with respect to $\mathcal{P}$. By attractiveness, it follows that if $\tau \leq \tau_i$, one has a.s. $\omega_0^\tau(i) = 0$. Moreover, Lemma 3.7 implies that almost surely $\tau_i \leq \widehat{\tau}_i$ for all $i \in \mathbf{Z}^d$. Therefore, it is sufficient to show that $\tau_i > -\infty$ almost surely to be able to define X^* as in Section 3.2.2, and obtain a coding for μ, according to Proposition 3.5.

Our main goal in the rest of this Section is to prove almost sure finiteness of the τ_i. Since they are identically distributed, it is enough to prove it for τ_0 (0 being the origin of $\mathbf{Z}^d$). The proof strategy is to dominate τ_0 by the extinction time of a branching process. For that purpose, we formally define the backward exploration mentioned in the Introduction.

3.3.2 Backward exploration of the dynamics

We introduce the notion of active paths, borrowed from [Lig85], which represents the paths that one follows while exploring the dynamics backward. In the following, for an update mark $\rho \in \mathcal{P}$, write U_ρ for the third coordinate of ρ. Abusing the notation, we will sometimes not distinguish an update mark from its space-time position (i.e. its first two coordinates), as almost surely there is at most one mark at any time t (in this case, we will often drop the word "update" and call a mark the space-time position of an update mark). For every $(i, t) \in \mathbf{Z}^d \times \mathbf{R}$, if there is an update mark at (i, t), put an oriented *arrow* from i to j at time t for all $j \in i + B$. For $i, j \in \mathbf{Z}^d$, $t > s$, say that there is an *active path* of length n from (i, t) to (j, s) if there is a sequence of times $t = t_0 > t_1 > \cdots > t_n \geq t_{n+1} = s$ and a sequence of sites $i = x_0, x_1, \ldots, x_n = j$ such that:

(i) for every $k \in \{1, \ldots n\}$, there is an arrow between x_{k-1} and x_k at time t_k

(ii) for every $k \in \{0, \ldots n\}$, there is no mark in the vertical segment $\{x_i\} \times (t_i, t_{i+1})$.

In other words, active paths go backward in time, and each time they find a mark, they necessarily choose an arrow in the direction of a neighbour of their current location. The length corresponds to the number of crossed arrows, equivalently the number of marks discovered. Note that the active paths do not depend on the third coordinate of the update marks, but only on their space-time position.

For every $(i,t) \in \mathbf{Z}^d \times \mathbf{R}$, let $\rho(i,t)$ be the last mark at site i before time t:

$$\rho(i,t) = (i, \sup\{s \leq t : \exists u \in [0,1], (i,s,u) \in \mathcal{P}\}), \tag{3.10}$$

the supremum being a.s. finite by property of the PPP. From the definition of an active path, we derive a notion of distance[2] on $\mathcal{P}$. For two marks ρ and ρ' in $\mathcal{P}$, there may be several active paths from ρ to ρ'. Define $d(\rho,\rho')$ as the minimal length of such a path. If there is none, set $d(\rho,\rho') := +\infty$.

For each $\rho \in \mathcal{P}$, we define the *active marks* from ρ to be the ones to which there is an active path from ρ. The set $\mathcal{A}_\rho$ of active marks from ρ is therefore given by

$$\mathcal{A}_\rho = \{\rho' \in \mathcal{P} \mid d(\rho,\rho') < +\infty\}. \tag{3.11}$$

In a realisation of the dynamics, the value of the field at (i,t) is therefore a measurable function of the update marks of $\mathcal{A}_{\rho(i,t)}$ and of the initial configuration. Coalescence is achieved when this dependence on the initial configuration disappears.

We consider an oriented Bernoulli site percolation on the marks. For every $\rho \in \mathcal{P}$, say that ρ is *open* if $U_\rho > \gamma$. An *open active path* is an active path that crosses only open marks. It is equivalent to consider that we keep the arrows originating from ρ only if $U_\rho > \gamma$. The orientation of the active paths gives an orientation to the percolation model.

For all $t \leq 0$, let $\hat{\sigma}_t$ be the set of sites $j \in \mathbf{Z}^d$ such that there exists an open active path from $(0,0)$ to (j,t), that is

$$\hat{\sigma}_t = \{j \in \mathbf{Z}^d : (0,0) \xrightarrow{o.a.} (j,t)\},$$

where o.a. stands for open active. Note that $(\hat{\sigma}_t)_{t\leq 0}$ is a measurable function of $\mathcal{P}$ and that, by definition of the active paths, it is a right-continuous Markov process.

The following lemma makes the connection with the spin systems defined in the previous paragraph. In the statement and in the proof, each element of $\{0,1\}^{\mathbf{Z}^d}$ is identified with the subset of $\mathbf{Z}^d$ where it takes value 1.

Lemma 3.8. *For all $\tau < 0$, one has*

$$\{\omega_0^\tau(0) = 0\} \overset{a.s.}{\subset} \{\forall t \in [\tau, 0], \omega_t^\tau \cap \hat{\sigma}_t = \emptyset\}.$$

In particular, $\{\omega_0^\tau(0) = 0\} \overset{a.s.}{\subset} \{\hat{\sigma}_\tau = \emptyset\}$.

Proof. For $t = 0$, we have a.s. $\hat{\sigma}_0 = \{0\}$, so $\{0 \notin \omega_0^\tau\} \overset{\text{a.s.}}{\subset} \{\omega_0^\tau \cap \hat{\sigma}_0 = \emptyset\}$. Thus, we only need to check that the property $\omega_t^\tau \cap \hat{\sigma}_t = \emptyset$ is preserved a.s. by every update of the systems. Actually, since we already have the inclusion for the terminal time 0 (and not the initial time τ), we need to look at the updates backward.

[2] Note that it is not symmetric; it should be understood as a directed graph distance.

Suppose that $(i,t,u) \in \mathcal{P}$ for some $t \in [\tau, 0]$ and that $\omega_t^\tau \cap \hat{\sigma}_t = \emptyset$. Show that $\omega_{t-}^\tau \cap \hat{\sigma}_{t-} = \emptyset$. First, since we have a mark at (i,t), we always have $\hat{\sigma}_{t-}(i) = 0$ (any potential active path from $(0,0)$ to (i,t) will jump to a neighbour of i at time t). Hence $i \notin \omega_{t-}^\tau \cap \hat{\sigma}_{t-}$. Then, transitions on the system $(\omega_t^\tau)_{t\geq\tau}$ can only flip the spin of the site that carries the update mark, so we have $\omega_{t-}^\tau \subset \omega_t^\tau \cup \{i\}$. The update mark (i,t,u) extends an open active path if and only if $i \in \hat{\sigma}_t$ and the mark is open (i.e. $u > \gamma$). If one of these two conditions fails, we have $\hat{\sigma}_{t-} = \hat{\sigma}_t - \{i\}$, so $\omega_{t-}^\tau \cap \hat{\sigma}_{t-} = \emptyset$.

When the mark is open and $i \in \hat{\sigma}_t$, it yields an open active paths from $(0,0)$ to $(j, t-)$ for every neighbour j of i. Hence, every $j \in i + B$ belongs to $\hat{\sigma}_{t-}$. Therefore, we only need to check that we cannot simultaneously have $\omega_{t-}^\tau(j) = 1$. By contradiction, if some j verifies $\omega_{t-}^\tau(j) = 1$, as one has $u > \gamma$, the transition rule (3.7) implies that $\omega_t^\tau(i) = 1$. This contradicts the fact that $\omega_t^\tau \cap \hat{\sigma}_t = \emptyset$, because $i \in \hat{\sigma}_t$.

At time $t = \tau$, the property tells us that $\hat{\sigma}_\tau = \emptyset$, because $\omega_\tau^\tau = \mathbf{Z}^d$. □

3.3.3 Duality between the spin system and the backward exploration

Now, our next step is to prove the reverse inclusion $\{\omega_t^\tau(0) = 0\} \overset{\text{a.s.}}{\supset} \{\hat{\sigma}_\tau = \emptyset\}$, to obtain that τ_0 coincides a.s. with the extinction time of the process $\hat{\sigma}_t$, that is the first (negative) time when it is empty. We will then be able to prove that this extinction time is almost surely finite, by comparison with a branching process. The reverse inclusion is obtained by *duality*, which is a powerful tool in the study of spins systems. The following general definition, that can be found in [Lig85], holds for a larger class of interacting particle systems.

Definition 3.9. *Suppose that $(\omega_t)_{t\geq\tau}$ and $(\sigma_t)_{t\geq\tau}$ are Markov processes on Ω and Σ respectively, started at time $\tau \in \mathbf{R}$. Let H be a bounded measurable function on $\Omega \times \Sigma$. The processes $(\omega_t)_{t\geq\tau}$ and $(\sigma_t)_{t\geq\tau}$ are said to be* dual *with respect to H if for all time $t \geq \tau$,*

$$\mathbf{E}_1^\omega H(\omega_t, \sigma) = \mathbf{E}_2^\sigma H(\omega, \sigma_t) \tag{3.12}$$

for all $\omega \in \Omega$ and $\sigma \in \Sigma$, where $\mathbf{E}_1^\omega$ (resp. $\mathbf{E}_2^\sigma$) denotes the expectation of the process $(\omega_t)_{t\geq\tau}$ (resp. $(\sigma_t)_{t\geq\tau}$) started at time τ from configuration ω (resp. σ).

We will take as duality function the function $H(\omega,\sigma) = \prod_{i\in\mathbf{Z}^d}(1-\omega(i)\sigma(i))$. It is frequently used in the study of spins systems, and is known as the coalescing duality. By identifying ω and σ with the set of sites where they take value one, one can rewrite H as $H(\omega,\sigma) = \mathbf{1}_{\{\omega\cap\sigma=\emptyset\}}$.

We aim to define the dual of the spin system introduced in Section 3.3.1. It will take values in the subset of $\{0,1\}^{\mathbf{Z}^d}$ of configurations containing finitely many 1, denoted Σ. The Hille-Yosida theory on semigroups of operators and their generators enables us to prove duality merely by computation, based on the definitions of the spin systems by their generators. We therefore first abstractly define the generator of a spin system on Σ, and provide some insight into why it is a good candidate for being the generator of the duals systems of $(\omega_t^\tau)_{t\geq\tau}$. We then formally prove the duality relying on Theorem 3.42 in [Lig10] (which is a consequence of the Hille-Yosida theorem), and show that $(\hat{\sigma}_{\tau-t})_{t\geq\tau}$ is a version of the dual of $(\omega_t^\tau)_{t\geq\tau}$.

For $\sigma \in \Sigma$ and $i \in \mathbf{Z}^d$, σ^i still denotes the configuration obtained from σ by flipping the spin at i, and $\sigma^{i,B}$ is the configuration obtained from σ by replacing the spin of each neighbour

of i by 1 and the spin of i by 0, i.e.

$$\sigma^{i,B}(j) = \begin{cases} 1 & \text{if } j \sim i \\ 0 & \text{if } j = i \\ \sigma(j) & \text{otherwise} \end{cases}$$

Let $\mathcal{M}$ be the generator of a spin system on Σ satisfying for every $f : \Sigma \to \mathbf{R}$

$$\mathcal{M}f(\sigma) = \sum_{i \in \mathbf{Z}^d} \sigma(i) \left[(1-\gamma) f(\sigma^{i,B}) + \gamma f(\sigma^i) - f(\sigma) \right]. \tag{3.13}$$

The fact that elements of Σ are 0 almost everywhere ensures that this expression is well-defined for every function f.

Intuitively, the duality can be understood with the graphical representation. Consider a *backward process* $(\sigma_t)_{t \leq 0}$ as, i.e. a process that goes back in time. Thus, it reads the update marks in the other way than the spin systems of Section 3.3.1. It is initialised with configuration $\sigma_0 = \{0\}$ (that is the configuration with spin 0 everywhere except at the origin of $\mathbf{Z}^d$). From a configuration σ, within a single transition, it can only reach configurations of the form σ^i or $\sigma^{i,B}$. A transition can occur at time t only if there exists an update mark $(i,t,u) \in \mathcal{P}$. Note that since we consider $(\sigma_t)_{t \leq 0}$ as a backward process, it will be left-continuous, contrary to the usual. If $(i,t,u) \in \mathcal{P}$ and $\sigma(i)$ denotes the spin at i before the update, that is informally at time $t+$, the transition rules are the following:

- If $\sigma(i) = 0$, nothing happens.
- If $\sigma(i) = 1$ and $u \leq \gamma$, jump to σ^i.
- If $\sigma(i) = 1$ and $u > \gamma$, jump to $\sigma^{i,B}$.

It should be clear that $\mathcal{M}$ is the generator of $(\sigma_t)_{t \leq 0}$. One may think about a contact process: the transition is allowed only if $\sigma(i) = 1$ (equivalent of an infected site) and a jump to σ^i corresponds to the recovery of the particle in i, whereas a jump to $\sigma^{i,B}$ corresponds to the infection of the neighbours (and simultaneously the recovery of the particle).

The interpretation of this process is that σ_t represents what we need at time t to obtain almost sure coalescence at $(0,0)$ from a certain negative time: $\sigma_t(i) = 1$ means that we already need to have coalescence at (i,t) almost surely. This is why the process is initialised at time 0 with spin 0 everywhere except at the origin. Given this interpretation, one can understand the transition rules as follows: going back in time, suppose that we meet an update mark (i,t,u) and that $\sigma_{t+}(i) = 1$. If the update mark is bad, i.e. $u > \gamma$, the output of φ in the dynamics depends on the values of the neighbours at time t; to get that this output is independent of the initial configuration, we therefore need that coalescence occurred for all these neighbours before t; this is why we allocate spin 1 in σ_t to every neighbour of i. If the update mark is good, i.e. $u > \gamma$, the values of the neighbours are unneeded to compute the output of φ, so we only set $\sigma_t(i) = 0$. On the other hand, when the configuration before the update verifies $\sigma_{t+}(i) = 0$, it means that we do not need any information on the value of the field at (i,t) in the dynamics, so we do not have to make a transition in the process $(\sigma_t)_{t \leq 0}$.

We can now forget about the specific process $(\sigma_t)_{t \leq 0}$, which has been introduced only to provide an intuitive understanding of duality. We now prove that $\mathcal{M}$ generates spin systems that are dual to the $(\omega_t^\tau)_{t \geq \tau}$, $\tau \in \mathbf{R}$.

Lemma 3.10. *The spin systems generated by $\mathcal{L}$ and $\mathcal{M}$ are dual with respect to $H(\omega,\sigma) = \mathbf{1}_{\{\omega\cap\sigma=\emptyset\}}$.*

Proof. According to Theorem 3.42 in [Lig10], we only need to check that

$$\mathcal{L}H(\cdot,\sigma)(\omega) = \mathcal{M}H(\omega,\cdot)(\sigma)$$

for every $\omega \in \{0,1\}^{\mathbf{Z}^d}$ and $\sigma \in \Sigma$, providing these two quantities have a meaning.

Let $\omega \in \{0,1\}^{\mathbf{Z}^d}$ and $\sigma \in \Sigma$. The fact that we can apply $\mathcal{L}$ to $H(\cdot,\sigma)$ follows from the fact that σ have finitely many 1, so $H(\cdot,\sigma)$ is a local function; $H(\omega,\cdot)$ can be applied to $\mathcal{M}$, whose action is defined on every function. It is therefore enough to show that for every site $i \in \mathbf{Z}^d$,

$$c(i,\omega)\left[H(\omega^i,\sigma) - H(\omega,\sigma)\right] = \sigma(i)\left[(1-\gamma)H(\omega,\sigma^{i,B}) + \gamma H(\omega,\sigma^i) - H(\omega,\sigma)\right], \tag{3.14}$$

where $c(i,\omega)$ is given by (3.8), since we can then sum over $i \in \mathbf{Z}^d$ to get $\mathcal{L}H(\cdot,\sigma)(\omega) = \mathcal{M}H(\omega,\cdot)(\sigma)$, which is what we want to prove.

For every $i \in \mathbf{Z}^d$, we have

$$\text{(i)}\quad H(\omega^i,\sigma) - H(\omega,\sigma) = \begin{cases} 1 & \text{if } \omega(i)=\sigma(i)=1 \text{ and } \forall j\neq i, \omega(j)\sigma(j)=0 \\ -1 & \text{if } \omega(i)=0, \sigma(i)=1 \text{ and } \forall j, \omega(j)\sigma(j)=0 \end{cases};$$

$$\text{(ii)}\quad H(\omega,\sigma^i) - H(\omega,\sigma) = \begin{cases} 1 & \text{if } \omega(i)=\sigma(i)=1 \text{ and } \forall j\neq i, \omega(j)\sigma(j)=0 \\ -1 & \text{if } \omega(i)=1, \sigma(i)=0 \text{ and } \forall j, \omega(j)\sigma(j)=0 \end{cases};$$

$$\text{(iii)}\quad H(\omega,\sigma^{i,B}) - H(\omega,\sigma) = \begin{cases} 1 & \text{if } \omega(i)=\sigma(i)=1, \forall j\sim i, \omega(j)=0 \text{ and } \forall j\neq i, \omega(j)\sigma(j)=0 \\ -1 & \text{if } \exists j\sim i, \omega(j)=1 \text{ and } \forall j, \omega(j)\sigma(j)=0 \end{cases}.$$

In those three equations, in all other cases, the left-hand side equals zero. Thus, in particular, if there exists $j \neq i$ such that $\omega(j)\sigma(j) = 1$, both sides of (3.14) are null. It is the same if $\sigma(i) = 0$, since (i) equals 0 in this case. We therefore assume that $\sigma(i) = 1$ and that for every $j \neq i$, $\omega(j)\sigma(j) = 0$, and check the equality in (3.14).

- If $\omega(i) = 1$: In this case, with our assumptions, (i) and (ii) equals 1, whereas (iii) equals $\mathbf{1}_{\{\forall j\sim i, \omega(j)=0\}}$. Thus, the left-hand side of (3.14) equals $\gamma + (1-\gamma)\mathbf{1}_{\{\forall j\sim i,\omega(j)=0\}}$ and the right-hand side $\sigma(i)\left[(1-\gamma)\mathbf{1}_{\{\forall j\sim i,\omega(j)=0\}} + \gamma\right]$, so they coincide.
- If $\omega(i) = 0$: Now, (i) is -1, (ii) is 0 and (iii) equals $-\mathbf{1}_{\{\exists j\sim i,\omega(j)=1\}}$, so both sides of (3.14) equal $-\eta\mathbf{1}_{\{\exists j\sim i,\omega(j)=1\}}$. □

We now prove the key result of this paragraph. We do the same identification as before between subsets of $\mathbf{Z}^d$ and elements of $\{0,1\}^{\mathbf{Z}^d}$, this time in the other direction.

Proposition 3.11. *For all $\tau \in \mathbf{R}$, $(\omega_t^\tau)_{t\geq\tau}$ and $(\hat{\sigma}_{\tau-t})_{t\geq\tau}$ are dual with respect to $H(\omega,\sigma) = \mathbf{1}_{\{\omega\cap\sigma=\emptyset\}}$.*

Proof. Fix $\tau \in \mathbf{R}$. According to Lemma 3.10, one only has to verify that the generator of $(\hat{\sigma}_{\tau-t})_{t\geq\tau}$ is $\mathcal{M}$. Recall that this process has been defined through the graphical representation, and note that it is left-continuous. To understand the transition mechanism of $(\hat{\sigma}_{\tau-t})_{t\geq\tau}$, we look at the transition encoded by some update mark $\rho = (i,\tau-t,u) \in \mathcal{P}$, for some $t \geq \tau$. If

$\hat{\sigma}_{\tau-t}(i) = 0$, there is no open active path from $(0,0)$ to $(i, \tau - t)$ so the update mark does not affect the process. Otherwise, when $\hat{\sigma}_{\tau-t}(i) = 1$, there are two cases, depending on whether ρ is open or not: if ρ is closed, the open active path stops at $(i, \tau - t)$ so i does not belong to $\hat{\sigma}_{(\tau-t)-}$; that corresponds to the transition from σ to σ^i when $u \leq \gamma$; if ρ is open, it yields an open active path from $(0,0)$ to $(j, (\tau-t)-)$ for all neighbours j of i (and no more active path from $(0,0)$ to $(i, (\tau - t)-)$); hence $\hat{\sigma}_{(\tau-t)-}(j) = 1$ for every $j \sim i$ (and $\hat{\sigma}_{(\tau-t)-}(i) = 0$); that corresponds to the transition from σ to $\sigma^{i,B}$ when $u > \gamma$.

Therefore, for every $i \in \mathbf{Z}^d$, transitions in $(\hat{\sigma}_{\tau-t})_{t\geq\tau}$ from the current configuration σ to σ^i appear at rate γ, but are discarded if $\sigma(i) = 0$; similarly, transitions from σ to $\sigma^{i,B}$ appear at rate $1 - \gamma$, but are discarded if $\sigma(i) = 0$. These are exactly the rates prescribed by $\mathcal{M}$. □

The duality relation (3.12) applied to $(\omega_t^\tau)_{t\geq\tau}$ and $(\hat{\sigma}_{\tau-t})_{t\geq\tau}$ at time $t = 0$ therefore gives

$$\mathbf{P}(\omega_0^\tau(0) = 0) = \mathbf{P}(\hat{\sigma}_\tau = \emptyset),$$

recalling that $\omega_\tau^\tau \equiv 1$ and $\hat{\sigma}_0 = \{0\}$. Together with the first inclusion, given by Lemma 3.8, we get for all $\tau \in \mathbf{R}$

$$\{\omega_0^\tau(0) = 0\} \overset{\text{a.s.}}{=} \{\hat{\sigma}_\tau = \emptyset\}. \tag{3.15}$$

This characterises τ_0 as the extinction time of the backward exploration process $(\hat{\sigma}_t)_{t\leq 0}$. In the following paragraph, we prove that this process dies out in finite time almost surely, which yields a coding of μ, by Proposition 3.5. We then explain how to control the tail of the coding radius to conclude the proof of Theorem 3.6.

3.3.4 Proof of Theorem 3.6

We start by proving that the backward exploration terminates in finite time. As mentioned in the Introduction, the proof can be summarised as a domination by a Galton-Watson process of reproduction law $\gamma\delta_0 + (1-\gamma)\delta_{|B|}$.

Lemma 3.12. *If $\gamma > 1 - \frac{1}{|B|}$, then almost surely there exists $\tau > -\infty$ such that $\hat{\sigma}_\tau = \emptyset$ (i.e. $(\hat{\sigma}_t)_{t\leq 0}$ dies out).*

Proof. The survival of $(\hat{\sigma}_t)_{t\leq 0}$ is equivalent to the existence of an infinite open active path starting from $(0,0)$. An infinite active path starting from $(0,0)$ is identified with the sequence $(\rho_n)_{n\in\mathbf{N}}$ of marks it crosses (hence one has $\rho_0 = \rho(0,0)$, defined in (3.10)). For any $n \in \mathbf{N}$, there exists an open active path of length n starting from $(0,0)$ if and only if there exist n open marks $(\rho_0, \dots, \rho_{n-1})$ with $\rho_0 = \rho(0,0)$ that correspond to the first n marks of an infinite active path. Since two consecutive marks in the sequence must be carried by neighbours, there are at most $|B|^{n-1}$ such sequences.

Each mark is open with probability $1 - \gamma$. Moreover, the third coordinates of the update marks crossed by an active path are independent, by property of the PPP. Therefore, by a union bound, we get

$$\mathbf{P}\left(\exists \text{ an open active path of length } n \text{ starting at } (0,0)\right) \leq |B|^{n-1}(1-\gamma)^n. \tag{3.16}$$

This quantity tends to 0 as soon as $1 - \gamma < \frac{1}{|B|}$. In this case, the probability of survival of $(\hat{\sigma}_t)_{t\leq 0}$ falls to 0, so almost surely $(\hat{\sigma}_t)_{t\leq 0}$ dies out in finite time. □

Proof of Theorem 3.6. Combining the results of Lemma 3.8 and Proposition 3.11, we already noticed that

$$\{\omega_0^\tau(0) = 0\} \overset{\text{a.s.}}{=} \{\hat{\sigma}_\tau = \emptyset\},$$

see (3.15). By Lemma 3.12, this implies that τ_0 is almost surely finite. Then, Lemma 3.7 and Proposition 3.5 yields a realisation $(X_i^*)_{i\in\mathbf{Z}^d}$ of μ as a measurable function of $\mathcal{P}$, that is we have a coding for μ.

We finish the proof of Theorem 3.6 by studying the coding radius of this coding. The coding radius R is defined as the (random) minimal integer r such that updates marks carried by sites in B_r^0 only are used to generate X_0^*. The update marks used to generate X_0^* are the open ones that are crossed by an open active path from $(0,0)$ and the closed ones at which such a path end up. Indeed, the value of the field at the origin at time 0 in the dynamics started at time $\tau < 0$ is a priori a measurable function of the update marks in $\mathcal{A}_{\rho(0,0)}$ and of the initial configuration. However, if τ is chosen lower than τ_0, then all the open active paths end up at a closed mark before reaching time τ. For these closed marks, the output of the update function φ is independent of the initial configuration. Then, it follows that the values of the field in the dynamics along all these active paths are independent of the initial configuration.

Let $r = \max\{||j||_1, j \in B\}$. A step of length 1 in an active path cannot travel a distance greater that r in $\mathbf{Z}^d$. Thus, an update mark carried by a site outside B_{rn}^0 needs to be revealed to generate X_0^* only if there is an open active path of length at least n starting from $(0,0)$. Then, we derive from (3.16) an upper bound on the tail of R, since

$$\mathbf{P}(R \geq rn) \leq \mathbf{P}\left(\exists \text{ an open active path of length } n \text{ starting at } (0,0)\right) \leq |B|^{n-1}(1-\gamma)^n.$$

It shows the exponential decay of the tail of the coding radius when $1 - \gamma < \frac{1}{|B|}$. □

3.4 Coding without the high noise assumption

As discussed in Section 3.2.3, for $\mu = \mu_\varepsilon$, we have $\gamma(\mu) = 0$, so the previous theorem cannot apply (except for $\varepsilon = 0$ which actually corresponds to the i.i.d. Gaussian field). We therefore need a new strategy to get a non-trivial coupling probability.

Let $(S_n)_{n\geq 1}$ be an increasing sequence of subsets of S such that $S_n \nearrow S$, o a distinguished point in S_1, and $(q_n)_{n\geq 0}$ an increasing to 1 sequence in $[0,1]$. We call $\mathbf{0}$ the configuration of Ω with o everywhere; abusing the notation, we also denote by $\mathbf{0}$ its restriction to B. Let

$$\tilde{\gamma}(\mu) = \max_{\mathbb{P}} \mathbb{P}(\forall \eta \in S_1^B, X^\eta = X^{\mathbf{0}} \text{ and } X^{\mathbf{0}} \in S_1),$$

where the maximum is taken over all the couplings of random variables $(X^\eta)_{\eta\in S_1^B}$ such that X^η has law $\pi^\eta = \mu(X_0 \in \cdot \mid X|_B = \eta)$. Here, $\tilde{\gamma}(\mu)$ corresponds to the maximal probability for realisations of π^η, for $\eta \in S_1^B$, to coincide in a coupling and belong to S_1.

Like in the previous section, we prove a general result of which Theorem 3.3 is a special case.

Theorem 3.13. *Suppose that $(S_n)_{n\geq 1}$ and $(q_n)_{n\geq 0}$ as above satisfy the four following hypotheses:*

$$\tilde{\gamma}(\mu) \geq q_0, \tag{H1}$$

$$\forall n \geq 1,\ \forall \eta \in S_{n+1}^B,\ \mu(X_0 \in S_n \mid X|_B = \eta) \geq q_n, \tag{H2}$$

$$8\sum_{n\geq 0} |B|^{2n+1}(1-q_n) < 1 \text{ and } \exists t > 0, \sum_{n\geq 0} |B|^n e^{t|B|^n}(1-q_n) < \infty, \tag{H3}$$

$$\mu(\{\xi \in \Omega : \forall i \in \mathbf{Z}^d, \xi_j \in S_{\max(n,||i||)}\}) \xrightarrow[n\to+\infty]{} 1. \tag{H4}$$

Then, μ is an FIID and there exists $c > 0$ such that for all $l \in \mathbf{N}$, there exists an l–dependent distribution ν_l on Ω satisfying for every finite subset $\Lambda \subset \mathbf{Z}^d$,

$$d_{\mathrm{TV}}(\mu|_\Lambda, \nu_l|_\Lambda) \leq |\Lambda| e^{-cl}.$$

Note that in the above statement, we say nothing about the tail of the coding radius. The rest of Section 3.4 is devoted to the proof of Theorem 3.13.

3.4.1 Interacting particle system associated with the dynamics

Similarly as for the proof of Theorem 3.6, we consider the dynamics introduced in Section 3.2 that admits μ as invariant law, and the coupling of processes $(X^{\xi,\tau})_{\xi\in\Omega,\tau\in\mathbf{R}}$ defined with the graphical representation. Assume that there exist $(S_n)_{n\geq 1}$, referred to as the *levels*, and $(q_n)_{n\geq 0}$ such that (H1), (H2), (H3) and (H4) hold.

The first two hypotheses result in two properties on the update function φ:

$$\forall \eta \in S_1^B, \forall u \in [0, q_0], \varphi(\eta, u) = \varphi(\mathbf{0}, u) \text{ and } \varphi(\mathbf{0}, u) \in S_1 \tag{3.17}$$

and

$$\forall n \geq 1, \forall \eta \in S_{n+1}^B, \forall u \in [0, q_n], \varphi(\eta, u) \in S_n. \tag{3.18}$$

We explain in the Appendix how to construct an update function that satisfies these properties.

This means that in a realisation of the dynamics, for each update mark $(i, t, u) \in \mathcal{P}$, if we happen to know that the values of the field on the neighbourhood of i belong to S_{n+1} for some $n \geq 1$ at time $t-$, and if $u \leq q_n$, then the new value of the field at i will be in S_n. When the neighbours are in S_1, if $u \leq q_0$, then the new value of the field at i will also belong to S_1 and does not depend on the specific values of the neighbours, provided that they belong to S_1.

We associate with the dynamics an *interacting particle system* which will indicate our knowledge of the coalescence of the algorithm. However, contrary to Section 3.3.1, particles can have many states, not just 0 or 1. It will therefore no longer be a spin system in the sense of [Lig85], but we still call the *spin* the state of a site in a certain configuration.

Let $\mathcal{S} = \mathbf{N} \cup \{+\infty\}$, and let $\mathcal{S}^{\mathbf{Z}^d}$ be the set of spin configurations. For any $\tau \in \mathbf{R}$ and $\kappa \in \mathcal{S}^{\mathbf{Z}^d}$, define a continuous-time Markov process $(\omega_t^{\kappa,\tau})_{t\geq\tau}$ on $\mathcal{S}^{\mathbf{Z}^d}$ started at time τ with initial configuration $\omega_\tau^{\kappa,\tau} = \kappa$, and whose evolution is governed as follows. For all $(i, t, u) \in \mathcal{P}$ with $t > \tau$, the spin at i is updated with the following rules: letting $m = \max_{j\sim i} \omega_{t-}^{\kappa,\tau}(j)$, we have that

1. if $m = 0$, put spin $\inf\{k : u \leq q_k\}$ at site i;
2. if $m \in \mathbf{N}^*$, put spin $\inf\{k \geq m-1 : u \leq q_k\}$ at site i;
3. if $m = +\infty$, put spin $+\infty$ at site i.

We now provide an intuitive interpretation of the different possibilities for a spin value. Spin $+\infty$ at a certain space-time position means that we cannot get any information about the value of the field in the dynamics at that space-time position only from the update marks. For every $k \geq 1$, spin k means that we know in a measurable way from the update marks that the value of the field belongs to S_k. Finally, spin 0 means that the value of the field belongs to S_1 and is independent of the initial configuration (i.e. it is a measurable function from the update marks). With these interpretations, we can understand the transition rules as follows. Rule 3 above implies that if we know nothing about at least one neighbour, i.e. $m = +\infty$, we cannot derive any information on the output of φ from the update mark. Rules 1 and 2 contain the core of the dynamics and translate (3.17) and (3.18). If all the neighbours are in S_m for $m \geq 1$ and if one encounters "good" update mark ρ, i.e. such that $U_\rho \leq q_{m-1}$, then one can reach spin $m-1$, because we know from (3.18) that the output of φ belongs to S_{m-1} (case $m > 1$), and the same holds for $m = 1$ by (3.17); in this case, we moreover know that the output of φ is independent of the initial configuration of the dynamics. If $U_\rho > q_{m-1}$, we look at the first integer $k \geq m$ that satisfies $U_\rho \leq q_k$; the update mark allows us to put spin k (considering (3.18) with $n = k$), because the neighbours are in $S_m \subset S_{k+1}$. The case $m = 0$ is slightly different because there is no spin $m-1$. The assumption $q_n \xrightarrow[n\to\infty]{} 1$ ensures that the infimum is finite.

Remark 3.14. *During an update at site i, if there is coalescence for all the neighbours of i, then the new value at site i does not depend on the initial configuration so there is coalescence at i after the update. However, if there is coalescence for some neighbours but not all of them, this information will not help to get coalescence or a good level for the new value at i. For example, if some neighbours have coalesce and the field takes value in S_9 there, and the other neighbours have a value in S_1, the best we can hope for the new value at i regarding to* (3.17) *and* (3.18) *is that it belongs to S_8. This is why a spin 0 indicates that the value of the field is the same in all the coupled dynamics* and *belongs to S_1. Thus, for some updates, we forget that we have coalescence and only keep information on the level.*

We now give the rate function and the generator of our interacting particle systems. Let $i \in \mathbf{Z}^d$ and $\omega \in \mathcal{S}^{\mathbf{Z}^d}$. For all $k \in \mathcal{S}$, call $\omega^i[k]$ the configuration obtained from ω by replacing the spin at j by k, i.e. $\omega^i[k](j) = \begin{cases} \omega(j) & \text{for } j \neq i \\ k & \text{for } j = i \end{cases}$. Set $m = \max_{j\sim i} \omega(j)$. Then, the rate function $c(\cdot,\cdot,\cdot)$ is defined by

$$c(0,i,\omega) = q_0 \text{ if } m \leq 1, \tag{3.19a}$$

$$c(k,i,\omega) = \begin{cases} q_k & \text{if } m = k+1 \\ q_k - q_{k-1} & \text{if } m \leq k \end{cases} \quad \text{for every } k \in \mathbf{N}^*, \tag{3.19b}$$

$$c(+\infty,i,\omega) = 1 \text{ if } m = +\infty, \tag{3.19c}$$

and $c(\cdot,i,\omega)$ equals zero in other cases. Here, $c(\cdot,i,\omega)$ is the rate at which the system jumps from ω to $\omega^i[\cdot]$. Note that for $k = \omega(i)$, ω and $\omega^i[k]$ coincide so there is not an actual transition. One may observe that for every fixed configuration $\omega \in \mathcal{S}^{\mathbf{Z}^d}$ and site $i \in \mathbf{Z}^d$, we have

$$\sum_{k\in\mathcal{S}} c(k,i,\omega) = 1,$$

that is the total rate at which there is an update of the spin at i. The generator $\mathcal{L}$ of the $(\omega_t^{\kappa,\tau})_{t\geq\tau}$ is then defined by the following relation: for all $\omega \in \mathcal{S}^{\mathbf{Z}^d}$ and every local function f,

$$\mathcal{L}f(\omega) = \sum_{i\in\mathbf{Z}^d}\sum_{k\in\mathcal{S}} c(k,i,\omega)(f(\omega^i[k]) - f(\omega)). \tag{3.20}$$

These rates are consistent with the transition mechanism described above. In fact, when meeting an update mark ρ, a transition to $\omega^i[0]$ takes place if $U_\rho \leq q_0$, only in the case where $m = 0$ or 1; for every $k \in \mathbf{N}^*$, a transition to $\omega^i[k]$ takes place when $m = k+1$ and $U_\rho \leq q_k$, or when $m \leq k$ and $q_{k-1} < U_\rho \leq q_k$; finally, a transition to $\omega^i[+\infty]$ occurs when $m = +\infty$.

The following result formally states the intuitive interpretation of the different values for a spin in the particle systems.

Lemma 3.15. *Let $\tau \in \mathbf{R}$ and $\kappa \in (\mathbf{N}^*)^{\mathbf{Z}^d}$ (i.e. a spin configuration without 0 or $+\infty$). Denote by Ω_κ the set of configurations that are* compatible *with κ, i.e.*

$$\Omega_\kappa := \{\xi \in \Omega \mid \forall i \in \mathbf{Z}^d, \xi_i \in S_{\kappa(i)}\}.$$

Then, for every $t \geq \tau$ and $i \in \mathbf{Z}^d$, we have

$$\{\omega_t^{\kappa,\tau}(i) = 0\} \overset{a.s.}{\subset} \{\forall \xi \in \Omega_\kappa, X_t^{\xi,\tau}(i) = \varphi(\mathbf{0}, U_{\rho(i,t)}) \in S_1\}$$

and for all $n \geq 1$

$$\{\omega_t^{\kappa,\tau}(i) \leq n\} \overset{a.s.}{\subset} \{\forall \xi \in \Omega_\kappa, X_t^{\xi,\tau}(i) \in S_n\}.$$

Proof. The second inclusion is true for all $i \in \mathbf{Z}^d$ and $n \geq 1$ at time $t = \tau$ because by definition of Ω_κ, the event on the right has probability 1; the first one also holds because the event on the left-hand side is empty, as $\kappa(i) \neq 0$. Consider $(i,t,u) \in \mathcal{P}$ for some $t > \tau$, and suppose that both properties hold for all $s \in [\tau, t)$ (this is referred to as the "induction hypothesis"). We check that they are a.s. preserved by the update, so that they hold at time t. We only need to verify the inclusions for site i, since only the spin at i can be affected by the update. Let $n \geq 1$.

Let $m = \max_{j\sim i} \omega_t^{\kappa,\tau}(j)$ and note that since κ does not contain any spin $+\infty$, m cannot be infinite. We first consider the case $m \geq 1$; by induction hypothesis, we know that almost surely for all $\xi \in \Omega_\kappa$, $X_t^{\xi,\tau}(j) \in S_m$ for all $j \sim i$. The value of the spin at i after the update satisfies

$$\omega_t^{\kappa,\tau}(i) = \inf\{k \geq m-1 : u \leq q_k\}.$$

It is smaller than n if and only if $u \leq q_n$ and $n \geq m-1$. In this case, by (3.18) with $\eta = (X_t^{\xi,\tau}(j))_{j\sim i} \in S_m^B \subset S_{n+1}^B$, it follows that $X_t^{\xi,\tau}(i) \in S_n$. Thus, one has $\omega_t^{\kappa,\tau}(i) = 0$ when $m = 1$ and $u \leq q_0$. In this case, (3.17) implies that for all $\xi \in \Omega_\kappa$, one has $X_t^{\xi,\tau}(i) = \varphi(\mathbf{0}, u) \in S_1$, since $(X_t^{\xi,\tau}(j))_{j\sim i} \in S_1^B$.

Turning to the case $m = 0$, the induction hypothesis tells us that almost surely for all $\xi, \xi' \in \Omega_\kappa$, one has $X_t^{\xi,\tau}(j) = X_t^{\xi',\tau}(j) \in S_1$ for all $j \sim i$. We only use the fact that $(X_t^{\xi,\tau}(j))_{j\sim i} \in S_1^B$ and argue as in the case $m \geq 1$. □

If one wanted to pursue the strategy of Section 3.3, one should define for all $i \in \mathbf{Z}^d$ a random time τ_i such that coalescence from τ_i occurs at $(i,0)$ almost surely. In terms of the interacting particle systems defined in the present Section, it would correspond to take

$$\tau_i = \sup\{\tau < 0 \mid \omega_0^{\kappa,\tau}(i) = 0\},$$

with $\kappa \equiv +\infty$, because at the initial time τ, no information about the field can be derived only from the update marks. Since the spin configuration "all $+\infty$" is absorbing for the system ($\omega_t^{i,\cdot} \equiv +\infty$ implies $\omega_s^{i,\cdot} \equiv +\infty$ for all $s \geq t$), τ_i is almost surely infinite. Therefore, we cannot exploit Proposition 3.5 directly. We prove here the following stronger version. The proof still relies on a coupling-from-the-past argument.

We define the following event: there is *weak coalescence* at $(i,t) \in \mathbf{Z}^d \times \mathbf{R}$ if there exists $X^*(i,t) \in S$ measurable of the update marks such that

$$\mu\left(\{\xi \in \Omega : X_t^{\xi,\tau}(i) = X^*(i,t)\}\right) \xrightarrow[\tau \to -\infty]{} 1. \tag{3.21}$$

Proposition 3.16. *Assume that almost surely for every $i \in \mathbf{Z}^d$, there is weak coalescence at $(i,0)$. Then, the random field $X^* := (X^*(i,0))_{i \in \mathbf{Z}^d}$ is distributed according to μ. Moreover, X^* provides a realisation of μ as an FIID.*

Proof. Call ν the distribution of X^*. Fix $\Lambda \Subset \mathbf{Z}^d$ and $\delta > 0$. For every $i \in \Lambda$ and $\tau < 0$, define the random set

$$\Omega_\tau^i := \{\xi \in \Omega : X_0^{\xi,\tau}(i) = X^*(i,0)\}.$$

By assumption, a.s. there exists $\tau_i < 0$ such that a.s. for every $\tau \leq \tau_i$, we have $\mu(\Omega_\tau^i) \geq 1 - \delta$. Let $\tau = \inf_{i \in \Lambda} \tau_i$; since Λ is finite, a.s. $\tau > -\infty$. Then, define $\Omega_\tau = \bigcap_{i \in \Lambda} \Omega_\tau^i$. By a union bound, we have a.s.

$$\mu(\Omega_\tau^c) = \mu\left(\bigcup_{i \in \Lambda} (\Omega_\tau^i)^c\right) \leq |\Lambda|\delta.$$

Then, using the characterization of the total variation distance between two measures as 1–(maximal coupling probability), let us show that $d_{\mathrm{TV}}(\nu|_\Lambda, \mu|_\Lambda) \leq |\Lambda|\delta$. Clearly $X^*|_\Lambda$ has law $\nu|_\Lambda$. Let ξ be a random field of law μ, independent of $\mathcal{P}$. Since μ is invariant for the dynamics, and ξ has law μ, it follows that $X_0^{\xi,\tau}$ is also distributed according to μ. Hence, its restriction to Λ has law $\mu|_\Lambda$. Therefore, $(X^*|_\Lambda, X_0^{\xi,\tau}|_\Lambda)$ yields a coupling of distributions $\nu|_\Lambda$ and $\mu|_\Lambda$. Moreover, $X^*|_\Lambda$ and $X_0^{\xi,\tau}|_\Lambda$ coincide on Ω_τ, so we get

$$d_{\mathrm{TV}}(\nu|_\Lambda, \mu|_\Lambda) \leq 1 - \mu(\Omega_\tau) \leq |\Lambda|\delta.$$

Letting $\delta \to 0$, we get that $\nu|_\Lambda = \mu|_\Lambda$. Then, since Λ was arbitrary, it yields $\nu = \mu$. □

In the following paragraphs, we will prove that there is a.s. weak coalescence at $(i,0)$ for all $i \in \mathbf{Z}^d$. By translation-invariance and because there are countably many sites, it is enough to prove a.s. weak coalescence at $(0,0)$.

3.4.2 Backward exploration of the dynamics

Recall from Section 3.3.2 the notions of active paths and active marks from a given $\rho \in \mathcal{P}$. We introduce a new percolation model on the update marks, which is no more independent, but rather has long-range interactions, namely a *Boolean percolation*. It is defined on a graph, and the concept of an open site is replaced by that of *wet site*. This is in accordance with the terminology used in the original article by Lamperti [Lam70], in which each site is considered to be a fountain that wets all the sites in a ball of random radius around it.

We define the model on the (random) directed graph induced by the space-time positions of the update marks: the vertex set is taken to be $\mathcal{P}$, and we put an edge from ρ' to ρ for each pair $\rho, \rho' \in \mathcal{P}$ satisfying $d(\rho, \rho') = 1$. Note that the edges have the opposite direction of the active paths! For all $\rho, \rho' \in \mathcal{P}$ say that ρ' *wets* ρ if and only if $U_{\rho'} > q_{d(\rho,\rho')}$ (with convention $q_\infty = 1$). Equivalently, ρ' wets all the marks at distance less than $\inf\{k : U_{\rho'} \leq q_k\}$ from it, including ρ' itself (if this infimum is positive). It follows that an update mark ρ can only be wetted by an active mark of A_ρ. The set of *wet marks* is thus defined as

$$W(\mathcal{P}) := \{\rho \in \mathcal{P} \mid \exists \rho' \in \mathcal{A}_\rho, U_{\rho'} > q_{d(\rho,\rho')}\}. \tag{3.22}$$

The update marks belonging to $W(\mathcal{P})$ are called wet, and those of $W(\mathcal{P})^c$ are called *dry*.

We now define the *wet active* paths, which is a more intricate notion than the one of open active paths in Section 3.3.2. For all $(i,t) \in \mathbf{Z}^d \times \mathbf{R}_-$, say that there is a wet active path from $(0,0)$ to (i,t) and write $(0,0) \xrightarrow{w.a.} (i,t)$ if there exists an active path from $(0,0)$ to (i,t) that crosses an update mark that wets $\rho(0,0) := \rho_0$. In particular, a wet active path is *not* an active path that only crosses wet marks. It should be seen as a path of (backward) exploration which reveals the fact that ρ_0 is wet. It is sufficient for our purpose to define the wet active paths starting from $(0,0)$ only, although the definition could naturally be extended to other space-time positions.

We authorise a new value for the spin of a site, namely $-\infty$, and for every $t \leq 0$, define $\hat{\sigma}_t \in (\mathcal{S} \cup \{-\infty\})^{\mathbf{Z}^d}$ by

$$\forall i \in \mathbf{Z}^d, \hat{\sigma}_t(i) = \begin{cases} -\infty & \text{if } (0,0) \xrightarrow{w.a.} (i,t) \\ d(\rho_0, \rho(i,t)) & \text{otherwise} \end{cases}. \tag{3.23}$$

Note that if there is no active path from $(0,0)$ to (i,t), the definition of the distance d (in Section 3.3.2) implies that $\hat{\sigma}_t(i) = +\infty$. It is clear that the events of ρ_0 being wet and the appearance of a spin $-\infty$ in the process $(\hat{\sigma}_t)_{t\leq 0}$ coincide. Moreover, since the distance to ρ_0 is non-decreasing along every active path starting from $(0,0)$, it follows that if ρ_0 is dry, then the process $(\hat{\sigma}_t)_{t\leq 0}$ eventually exceed any $\kappa \in \mathbf{N}^{\mathbf{Z}^d}$.

We prove an analogue of Lemma 3.8 for $(\omega_t^{\kappa,\tau})_{t\geq\tau}$ and $(\hat{\sigma}_t)_{t\leq 0}$. Denote by $\leq$ the natural partial order on $(\mathcal{S} \cup \{-\infty\})^{\mathbf{Z}^d}$, defined by

$$\omega \leq \sigma \iff \forall i \in \mathbf{Z}^d, \omega(i) \leq \sigma(i).$$

Lemma 3.17. *Let $\kappa \in \mathbf{N}^{\mathbf{Z}^d}$. For all $\tau < 0$ it holds that*

$$\{\omega_0^{\kappa,\tau}(0) = 0\} \overset{a.s.}{\subset} \{\forall t \in [\tau, 0], \omega_t^{\kappa,\tau} \leq \hat{\sigma}_t\}.$$

In particular, $\{\omega_0^{\kappa,\tau}(0) = 0\} \overset{a.s.}{\subset} \{\hat{\sigma}_\tau \geq \kappa\}$.

Proof. We first show that on the event that $\omega_0^{\kappa,\tau}(0) = 0$, a.s. no update mark between times τ and 0 wets ρ_0. Since the spin in the process $(\omega_t^{\tau,\kappa})_{t\geq\tau}$ can only increase by at most 1 along the active paths (in the forward sense, an update can decrease the spin by at most 1), $\omega_t^{\tau,\kappa}(0) = 0$ implies that for all $\rho = (i,t,u) \in \mathcal{A}_{\rho_0}$ such that $t \geq \tau$, one has $\omega_t^{\kappa,\tau}(i) \leq d(\rho_0, \rho)$. Arguing by contradiction, suppose that there exists $\rho = (i,t,u) \in \mathcal{P}$ with $t \in [\tau, 0]$ such that $u > q_{d(\rho_0,\rho)}$. This implies that the update encoded by ρ leads to $\omega_t^{\kappa,\tau}(i) > d(\rho_0, \rho)$, which contradicts $\omega_0^{\kappa,\tau}(0) = 0$.

We can now prove the inequality $\omega_t^{\kappa,\tau} \leq \hat{\sigma}_t$ for all $t \in [\tau, 0]$. The outline is the same as in the proof of Lemma 3.8. For $t = 0$, the property is true because $\hat{\sigma}_0$ a.s. equals $+\infty$ everywhere except at 0, where we have $\hat{\sigma}_0(0) = 0$. Therefore, on the event $\omega_0^{\kappa,\tau}(0) = 0$, one has $\omega_0^{\kappa,\tau}(i) \leq \hat{\sigma}_0(i)$ for every $i \in \mathbf{Z}^d$.

Now, we check that the property is a.s. preserved by the (backward) updates. Suppose that $\rho = (i, t, u) \in \mathcal{P}$ for some $t \in [\tau, 0]$ and that $\omega_t^{\kappa,\tau} \leq \hat{\sigma}_t$; let us show that $\omega_{t-}^{\kappa,\tau} \leq \hat{\sigma}_{t-}$. First, since we have a mark at (i, t), we always have $\hat{\sigma}_{t-}(i) = +\infty$ because there is no active path from $(0, 0)$ to (i, t), so $\omega_{t-}^{\kappa,\tau}(i) \leq \hat{\sigma}_{t-}(i)$. The update induced by (i, t, u) can only affect the spin at i in $\omega_t^{\kappa,\tau}$, so $\omega_{t-}^{\kappa,\tau}(j) = \omega_t^{\kappa,\tau}(j)$ for every $j \neq i$. If j is not a neighbour of i, we also have $\hat{\sigma}_{t-}(j) = \hat{\sigma}_t(j)$, so the inequality $\omega_{t-}^{\kappa,\tau}(j) \leq \hat{\sigma}_{t-}(j)$ holds. Therefore, we only need to handle the case where j is a neighbour of i. If $j \sim i$, we have

$$\hat{\sigma}_{t-}(j) = \begin{cases} -\infty & \text{if } \rho \text{ wets } \rho_0 \\ \inf\{\hat{\sigma}_t(i) + 1, \hat{\sigma}_t(j)\} & \text{otherwise} \end{cases}.$$

Indeed, if ρ wets ρ_0, it yields a wet active path from $(0, 0)$ to $(j, t-)$, so $\hat{\sigma}_{t-}(j) = -\infty$. Otherwise, if $\hat{\sigma}_t(i) = -\infty$ or $\hat{\sigma}_t(j) = -\infty$, it also yields a wet active path from $(0, 0)$ to $(j, t-)$, although the fact that it is wet is not due to ρ, so $\hat{\sigma}_{t-}(j) = -\infty$. Assume now that $\hat{\sigma}_t(i) \geq 0$ and $\hat{\sigma}_t(j) \geq 0$. The active paths, if any, from $(0, 0)$ to (j, t) continue to $(j, t-)$, because there is no mark at (j, t). Their minimal length is given by $\hat{\sigma}_t(j)$ (which is $+\infty$ if there is none). On the other hands, the active paths from $(0, 0)$ to (i, t) and the mark ρ create active paths from $(0, 0)$ to $(j, t-)$, whose minimal length is therefore $\hat{\sigma}_t(i) + 1$. Then, the value taken by $\hat{\sigma}_{t-}(j)$ is the minimal length of all these active paths.

We have shown in the beginning of the proof that "ρ wets ρ_0" and $\omega_0^{\kappa,\tau}(0) = 0$ are disjoints events, so we have $\hat{\sigma}_{t-}(j) = \inf\{\hat{\sigma}_t(i) + 1, \hat{\sigma}_t(j)\}$. Furthermore, according to the transition mechanism of the particle system (see Section 3.4.1), we have $\omega_t^{\kappa,\tau}(i) \geq \max_{j \sim i} \omega_{t-}^{\kappa,\tau}(j) - 1$. Hence, we get

$$\hat{\sigma}_t(i) + 1 \geq \omega_t^{\kappa,\tau}(i) + 1 \geq \omega_{t-}^{\kappa,\tau}(j) \text{ and } \hat{\sigma}_t(j) \geq \omega_t^{\kappa,\tau}(j) = \omega_{t-}^{\kappa,\tau}(j).$$

In both statements, the first inequality comes from the "induction hypothesis" $\omega_t^{\kappa,\tau} \leq \hat{\sigma}_t$. It follows that $\hat{\sigma}_{t-}(j) = \inf\{\hat{\sigma}_t(i) + 1, \hat{\sigma}_t(j)\} \geq \omega_{t-}^{\kappa,\tau}(j)$. □

3.4.3 Duality between the particle system and the backward exploration

We now exploit the duality to get the reverse inclusion. The Definition 3.9 of duality holds for the class of particle systems studied here. However, since the state space is no longer $\{0, 1\}^{\mathbf{Z}^d}$, there is no good way to see a spin configuration as a subset of $\mathbf{Z}^d$. Therefore, instead of taking the indicator function of the event that the two configurations do not intersect, we rather choose as duality function

$$H(\omega, \sigma) = \mathbf{1}_{\{\omega \leq \sigma\}}.$$

Let $\Sigma \subset (\mathcal{S} \cup \{-\infty\})^{\mathbf{Z}^d}$ be the set of spin configurations allowing spins $-\infty$, but with finitely many spins different from $+\infty$. For every $\sigma \in \Sigma$, $i \in \mathbf{Z}^d$ and $k \in \mathcal{S}$, let $\sigma^{i,B}[k]$ be the configuration derived from σ by putting spin $+\infty$ to site i and the minimum between k and

the current spin to every neighbour of i, i.e.

$$\sigma^{i,B}[k](j) = \begin{cases} +\infty & \text{if } j = i \\ \min(\sigma(j), k) & \text{if } j \sim i \\ \sigma(j) & \text{otherwise} \end{cases} . \tag{3.24}$$

Then, introduce a generator $\mathcal{M}$ whose action is defined over every function $f : \Sigma \to \mathbf{R}$ by

$$\mathcal{M}f(\sigma) = \sum_{i \in \mathbf{Z}^d} \left[q_{\sigma(i)} f\left(\sigma^{i,B}[\sigma(i)+1]\right) + (1 - q_{\sigma(i)}) f\left(\sigma^{i,B}[-\infty]\right) - f(\sigma) \right] \tag{3.25}$$

with conventions $q_{-\infty} = 0$, $q_{+\infty} = 1$ and $\pm\infty + 1 = \pm\infty$, and observe that the $i-$th term is just $f\left(\sigma^{i,B}[-\infty]\right) - f(\sigma)$ when $\sigma(i) = -\infty$ and 0 when $\sigma(i) = +\infty$, because $\sigma^{i,B}[+\infty] = \sigma$, so in this last case there is not an actual transition. Note that this expression is well defined for every $\sigma \in \Sigma$ without restriction on the function f, since the sum has finitely many non-zero terms.

We first give an intuitive interpretation of this generator in terms of the graphical representation. Consider a backward process $(\sigma_t)_{t \leq 0}$ with state space Σ and initial configuration $+\infty$ everywhere except at the origin, where we set $\sigma_0(0) = 0$. The process is left-continuous and for all $(i, t, u) \in \mathcal{P}$ with $t < 0$, if $\sigma(i) := \sigma_{t+}(i)$, we update the system as follows:

- if $u \leq q_{\sigma(i)}$, transition to $\sigma^{i,B}[\sigma(i)+1]$
- if $u > q_{\sigma(i)}$, transition to $\sigma^{i,B}[-\infty]$.

It should be clear that the generator of $(\sigma_t)_{t \leq 0}$ is $\mathcal{M}$.

The possible values of a spin in σ_t represent, as in Section 3.3.3, what is needed at time t in the dynamics to get almost sure coalescence at $(0, 0)$ from a certain negative time. If $\sigma_t(i) = +\infty$ for some $t \leq 0$ and $i \in \mathbf{Z}^d$, it means that no information on the value of the field at (i, t) is needed. If $\sigma_t(i) = k$ for some $k \geq 1$, it means that we need to know in a measurable way from the update marks that the value of the field in the dynamics belongs to S_k; $\sigma_t(i) = 0$ means that we also need that the value of the field is in S_1, and moreover that it does not depend on the initial configuration of the dynamics. Finally, $\sigma_t(i) = -\infty$ means that we cannot have almost sure coalescence at $(0, 0)$. Transition rules are then understandable in the following way. Consider an update mark (i, t, u) with $t < 0$ and $\sigma := \sigma_{t+}$ the configuration before the update. In order to get the level we want for the dynamics at (i, t), encoded by $\sigma(i)$, we need to know that the field values on the neighbourhood if i at time $t-$ belong to $S_{\sigma(i)+1}$. Furthermore, we need a "good" update mark, that is here $u \leq q_{\sigma(i)}$, so that the update function φ returns a value in $S_{\sigma(i)}$ (see (3.17) and (3.18)). Thus, if $u \leq q_{\sigma(i)}$, jump from configuration σ to $\sigma^{i,B}[\sigma(i)+1]$: the neighbours of i take spin $\sigma(i)+1$ or keep their spins if it is smaller, because a smaller spin means that a more restrictive level is already required. If $u > q_{\sigma(i)}$, the update mark is not good enough to reach the target level. In this case, the coalescence cannot be achieved almost surely at $(0, 0)$, so spins $-\infty$ invade the process (jump to $\sigma^{i,B}[-\infty]$).

We now formally prove the duality.

Lemma 3.18. *The particle systems generated by $\mathcal{L}$ and $\mathcal{M}$ are dual with respect to $H(\omega, \sigma) = \mathbf{1}_{\{\omega \leq \sigma\}}$.*

Proof. The proof is very similar to the one of Lemma 3.10, and relies on the same result about generators (Theorem 3.42 in [Lig10]). In particular, we only prove that

$$\mathcal{L}H(\cdot,\sigma)(\omega) = \mathcal{M}H(\omega,\cdot)(\sigma)$$

for all $\omega \in \mathcal{S}^{\mathbf{Z}^d}$, $\sigma \in \Sigma$, where the left-hand side is well-defined because σ takes spin $+\infty$ almost everywhere, so $H(\cdot,\sigma)$ is a local function.

Fix $i \in \mathbf{Z}^d$, $\omega \in \mathcal{S}^{\mathbf{Z}^d}$, $\sigma \in \Sigma$, and let $m = \max_{j\sim i} \omega(j)$. For all $k \in \mathcal{S}$, set $A_k = H(\omega^i[k],\sigma) - H(\omega,\sigma)$ and $B_k = H(\omega,\sigma^{i,B}[k]) - H(\omega,\sigma)$, and similarly set $B_{-\infty} = H(\omega,\sigma^{i,B}[-\infty]) - H(\omega,\sigma)$ It is sufficient to show that

$$\sum_{k\in\mathcal{S}} c(k,i,\omega)A_k = q_{\sigma(i)}B_{\sigma(i)+1} + (1-q_{\sigma(i)})B_{-\infty}, \tag{3.26}$$

where $c(k,i,\omega)$ is given by (3.19), since then we can sum over $i \in \mathbf{Z}^d$ to get the result.

First, note that if there is some $j \neq i$ such that $\sigma(j) < \omega(j)$, then for every $k \in \mathcal{S}$, $A_k = B_k = B_{-\infty} = 0$. Indeed, in this case the condition $\omega \leq \sigma$ fails at site j so $H(\omega,\sigma) = 0$. Then, $\omega^i[k](j) = \omega(j) > \sigma(j)$ so we also have $H(\omega^i[k],\sigma) = 0$, thus $A_k = 0$. The spin at j may differ between σ and $\sigma^{i,B}[k]$, but $\sigma^{i,B}[k](j) \leq \sigma(j) < \omega(j)$ because $j \neq i$, so we still have $H(\omega,\sigma^{i,B}[k]) = 0$, thus $B_k = 0$ (this sentence is also valid for $k = -\infty$). In this case, (3.26) holds obviously.

Consequently, in the rest of the proof, we handle the case:

$$\forall j \neq i,\ \omega(j) \leq \sigma(j).$$

Under this assumption, one can compute A_k and B_k for all $k \in S$ (including $k = -\infty$ for B_k):

- $A_k = \begin{cases} 1 & \text{if } k \leq \sigma(i) \text{ and } \omega(i) > \sigma(i) \\ -1 & \text{if } k > \sigma(i) \text{ and } \omega(i) \leq \sigma(i) \end{cases}$;
- $B_k = \begin{cases} 1 & \text{if } m \leq k \text{ and } \omega(i) > \sigma(i) \\ -1 & \text{if } m > k \text{ and } \omega(i) \leq \sigma(i) \end{cases}$ and $B_{-\infty} = -\mathbf{1}_{\{\omega(i)\leq\sigma(i)\}}$.

Now, verify the equality in (3.26), considering separately the cases $\omega(i) \leq \sigma(i)$ and $\omega(i) > \sigma(i)$. We will refer to the left-hand side of (3.26) by the generic symbol L and to the right-hand side by M.

1. **$\omega(i) \leq \sigma(i)$:** In this case, we have for all $k \in \mathcal{S}$, $A_k = -\mathbf{1}_{\{k>\sigma(i)\}}$ and $B_k = -\mathbf{1}_{\{m>k\}}$, and $B_{-\infty} = -1$. When $\sigma(i) = +\infty$, one has $M = B_{+\infty} = 0$, and $L = 0$ because all the A_k are null. When $\sigma(i) < +\infty$, one has $-M = -q_{\sigma(i)}B_{\sigma(i)+1} + 1 - q_{\sigma(i)}$, so

$$-M = \begin{cases} 1 - q_{\sigma(i)} & \text{if } m \leq \sigma(i)+1 \\ 1 & \text{if } m > \sigma(i)+1 \end{cases}.$$

On the other hand,

$$-L = \sum_{k=\sigma(i)+1}^{+\infty} c(k,i,\omega).$$

Now, $c(k,i,\omega) = 0$ if $k < m-1$, so if $m \leq \sigma(i)+1$, the sum starts at $\sigma(i)+1$, but if $m > \sigma(i)+1$, one can start the sum at $m-1$. In the first case, we have $-L = q_{\sigma(i)+1} - q_{\sigma(i)} + q_{\sigma(i)+2} - q_{\sigma(i)+1} + \cdots = 1 - q_{\sigma(i)}$ and in the second case we have $-L = q_m + q_{m+1} - q_m + q_{m+2} - q_{m+1} + \cdots = 1$. Hence $L = M$.

2. $\boldsymbol{\omega(i) > \sigma(i)}$ **:** Here, $A_k = \mathbf{1}_{\{k \leq \sigma(i)\}}$, $B_k = \mathbf{1}_{\{m \leq k\}}$ and $B_{-\infty} = 0$, and note that the case $\sigma(i) = +\infty$ is excluded. Thus, $M = q_{\sigma(i)}\mathbf{1}_{\{m \leq \sigma(i)+1\}}$. Then,

$$L = \sum_{k=0}^{\sigma(i)} c(k,i,\omega) = \sum_{k=m-1}^{\sigma(i)} c(k,i,\omega) = \begin{cases} 0 & \text{if } m > \sigma(i)+1 \\ q_{\sigma(i)} & \text{if } m \leq \sigma(i)+1 \end{cases},$$

coinciding with M, where the $q_{\sigma(i)}$ again arises from a telescopic sum. □

We can now deduce from the previous lemma the duality between the particle systems introduced in Section 3.4.1 and the backward exploration.

Proposition 3.19. *For all $\tau \in \mathbf{R}$ and $\kappa \in \mathbf{N}^{\mathbf{Z}^d}$, the processes $(\omega_t^{\kappa,\tau})_{t \geq \tau}$ and $(\hat{\sigma}_{\tau - t})_{t \geq \tau}$ are dual with respect to $H(\omega, \sigma) = \mathbf{1}_{\{\omega \leq \sigma\}}$.*

Proof. Fix $\tau \in \mathbf{R}$. According to Lemma 3.18, one only has to verify that the generator of $(\hat{\sigma}_{\tau-t})_{t \geq \tau}$ is $\mathcal{M}$. Recall that this process has been defined through the graphical representation, and note that it is left-continuous. For some $t \geq \tau$, suppose that there is an update mark $\rho = (i, \tau - t, u) \in \mathcal{P}$. It affects the process as follows:

$$\hat{\sigma}_{(\tau-t)-}(j) = \begin{cases} +\infty & \text{if } j = i \\ \min(\hat{\sigma}_{\tau-t}(j), k) & \text{if } j \sim i \\ \hat{\sigma}_{\tau-t}(j) & \text{otherwise} \end{cases} \quad \text{i.e. } \hat{\sigma}_{(\tau-t)-} = \hat{\sigma}^{i,B}_{\tau-t}[k],$$

where $k = -\infty$ if $u > q_{d(\rho_0,\rho)}$ (i.e. if ρ wets ρ_0) and $\hat{\sigma}_{\tau-t}(i)+1$ otherwise. This is a consequence of (3.23), already observed in the proof of Lemma 3.17. The value of $\hat{\sigma}_{\tau-t}(i)$ can either be $d(\rho_0,\rho)$ or $-\infty$. If it is $-\infty$, one has $\hat{\sigma}_{(\tau-t)-} = \hat{\sigma}^{i,B}_{\tau-t}[-\infty]$. Otherwise, the condition $u > q_{d(\rho_0,\rho)}$ is equivalent to $u > q_{\hat{\sigma}_{\tau-t}(i)}$, and in this case one has $\hat{\sigma}_{(\tau-t)-} = \hat{\sigma}^{i,B}_{\tau-t}[-\infty]$; if $u \leq q_{\hat{\sigma}_{\tau-t}(i)}$, one has $\hat{\sigma}_{(\tau-t)-} = \hat{\sigma}^{i,B}_{\tau-t}[\hat{\sigma}_{\tau-t}(i)+1]$.

Therefore, for every $i \in \mathbf{Z}^d$, transitions of the form σ to $\sigma^{i,B}[-\infty]$ appear at rate $1 - q_{\sigma(i)}$, and transitions of the form σ to $\sigma^{i,B}[\sigma(i)+1]$ appear at rate $q_{\sigma(i)}$. These are the rates prescribed by $\mathcal{M}$. □

The duality relation (3.12) (with function $H(\omega,\sigma) = \mathbf{1}_{\{\omega \leq \sigma\}}$) applied at time $t = 0$ for processes $(\omega_t^{\kappa,\tau})_{t \geq \tau}$ and $(\hat{\sigma}_{\tau-t})_{t \geq \tau}$ gives

$$\mathbf{P}(\omega_0^{\kappa,\tau}(0) = 0) = \mathbf{P}(\kappa \leq \hat{\sigma}_\tau),$$

because $\omega_\tau^{\kappa,\tau} = \kappa$ and $\hat{\sigma}_0$ has spin $+\infty$ everywhere except at the origin, where it is 0. Coupled with the inclusion obtained in Lemma 3.17, it gives

$$\{\omega_0^{\kappa,\tau}(0) = 0\} \overset{\text{a.s.}}{=} \{\hat{\sigma}_\tau \geq \kappa\}. \tag{3.27}$$

Then, if ρ_0 is dry, for every $\kappa \in \mathbf{N}^{\mathbf{Z}^d}$ one can choose $\tau < 0$ such that $\hat{\sigma}_\tau \geq \kappa$. By (3.27) and Lemma 3.15, it follows that we have coalescence at $(0,0)$ of all the processes $(X_t^{\xi,\tau})_{t \geq \tau}$ with initial condition $\xi \in \Omega_\kappa$. By "letting κ go to infinity", we will then obtain a.s. weak coalescence at $(0,0)$ conditionally on the event that ρ_0 is dry. We then explain how to proceed when ρ_0 is wet.

3.4.4 Proof of Theorem 3.13

In this Section, we conclude the proof of Theorem 3.13. We start by two lemmas, the first one formalising the discussion at the end of the previous paragraph, that claims that everything goes well for dry marks, and the second one stating that wet marks do not percolate, which turns out to be sufficient to get a coding for μ. Note that hypotheses (H1) and (H2) have been used to define the update function of the dynamics. The two others hypotheses, (H3) and (H4), will be used, respectively, in the proofs of Lemma 3.21 and Lemma 3.20 below.

Lemma 3.20. *For every $(i,t) \in \mathbf{Z}^d \times \mathbf{R}$, one has*

$$\{\rho(i,t) \text{ is dry}\} \overset{a.s.}{\subset} \{\forall \tau < t_0,\ X_t^{\mathbf{0},\tau}(i) = \varphi(\mathbf{0}, U_{\rho(i,t)})\}$$

and

$$\{\rho(i,t) \text{ is dry}\} \overset{a.s.}{\subset} \left\{ \mu\left(\{\xi \in \Omega : X_t^{\xi,\tau}(i) = \varphi(\mathbf{0}, U_{\rho(i,t)})\} \right) \xrightarrow[\tau \to -\infty]{} 1 \right\}.$$

Proof. It is sufficient to prove the statements for $(0,0)$, by space and time translation-invariance. Consider the process $(\hat{\sigma}_t)_{t \le 0}$ introduced in Section 3.4.2, and let t_0 be the time of the update mark $\rho_0 := \rho(0,0)$, i.e. the only negative number t_0 such that $\rho_0 = (0, t_0, U_{\rho_0})$. On the event of ρ_0 being dry, by definition of $(\hat{\sigma}_t)_{t \le 0}$, one has $\hat{\sigma}_t \ge 1$ for all $t < t_0$ (the update encoded by ρ_0 gives spin 1 to the neighbours of the origin, and spin $+\infty$ everywhere else; in the rest of the evolution of the process, the spin at any site cannot take a value smaller than 1, because ρ_0 is dry). The duality proved in Section 3.4.3 then implies (see (3.27)) that $\omega_0^{\kappa,\tau}(0) = 0$ for all $\tau < t_0$, with $\kappa \equiv 1$. By Lemma 3.15, we get in particular that a.s.

$$\forall \tau < t_0,\ X_0^{\mathbf{0},\tau}(0) = \varphi(\mathbf{0}, U_{\rho_0}),$$

because $\mathbf{0} \in \Omega_\kappa$. We now prove the second statement. Define a sequence $(\kappa_n)_{n \ge 1}$ of spin configurations by $\kappa_n(i) := \max(n, \|i\|_1)$ for all $i \in \mathbf{Z}^d$ and $n \ge 1$. On the event where ρ_0 is dry, it holds that the process $(\hat{\sigma}_t)_{t \le 0}$ eventually exceed every $\kappa \in \mathbf{N}^{\mathbf{Z}^d}$, since there is never the appearance of a spin $-\infty$. It follows that for every $n \ge 1$, there exists $\tau_n > -\infty$ such that $\hat{\sigma}_{\tau_n} \ge \kappa_n$. Therefore, as a consequence of the duality, we get that $\omega_0^{\kappa_n,\tau_n}(0) = 0$, see (3.27). By Lemma 3.15, it then follows that for every $\xi \in \Omega_{\kappa_n}$, we have a.s.

$$X_0^{\xi,\tau_n}(0) = \varphi(\mathbf{0}, U_{\rho_0}).$$

Thus, we get that for every $\tau \le \tau_n$,

$$\Omega_{\kappa_n} \overset{\text{a.s.}}{\subset} \Omega_\tau^0 := \{\xi \in \Omega : X_0^{\xi,\tau}(0) = \varphi(\mathbf{0}, U_{\rho_0})\}.$$

By hypothesis (H4) of Theorem 3.13, one has $\mu(\Omega_{\kappa_n}) \xrightarrow[n \to +\infty]{} 1$, so it gives $\mu(\Omega_\tau^0) \xrightarrow[\tau \to -\infty]{} 1$. □

In words, Lemma 3.20 says that if ρ_0 is dry, then we have weak coalescence at $(0,0)$, and it identify the coalescing value as $\varphi(\mathbf{0}, U_{\rho_0})$ (note that this is clearly measurable of the update marks). Unfortunately, it is not true that ρ_0 is almost surely dry, so it is not sufficient to apply Proposition 3.16 and get a coding for μ,.

To deal with the case of ρ_0 being wet, we prove that the Boolean percolation on the update marks is in some sense subcritical, and will see in the proof of Theorem 3.13 that this

is sufficient for our purpose. A finite subset of $\mathcal{A}_{\rho_0}$ is said to be a *cutset* (above $(0,0)$) if every infinite active path starting from $(0,0)$ passes through a mark of the subset (e.g. the smallest cutset above $(0,0)$ is $\{\rho_0\}$). Let $\mathcal{T} := \sigma((T_n^i)_{i\in\mathbf{Z}^d, n\in\mathbf{Z}})$, that is the $\sigma-$algebra generated by the space-time positions of the update marks.

Lemma 3.21. *There almost surely exists a dry cutset above $(0,0)$, that is a cutset included in $W(\mathcal{P})^c$.*

Moreover, there exists $\lambda > 0$ such that for all $n \geq 0$,

$$\mathbf{P}(\exists \text{ a wet path of length } n \text{ starting from } (0,0) \mid \mathcal{T}) \leq e^{-\lambda n}.$$

In this statement, a wet path should be understood as an active path that only crosses wet marks. Equivalently, a wet path of length n starting from $(0,0)$ is made of n consecutive update marks that are all wet, the first being ρ_0.

This result is a direct consequence of Theorem 2.4 in [Fai25], Theorem 4.4 in the present manuscript, that we reproduce here.

Theorem 3.22 (Theorem 2.4 in [Fai25])**.** *Let G be an infinite directed graph of bounded degree. Let Δ_+ (resp. Δ_-) be the maximal out-degree (resp. in-degree) of a vertex, that is the number of edges emanating from (resp. pointing at) this vertex, and assume that they both are at least 2. We consider the Boolean percolation on G having radius distribution given by $(q_n)_{n\geq 0}$, that is we consider the wet set*

$$W = \bigcup_{v\in G} B_{R_v}(v),$$

where $B_r(v) := \{v' \in G : d(v,v') < r\}$ and the R_v for $v \in V$ are i.i.d. $\mathbf{N}-$valued random variable of distribution $\mathbb{P}(R_v \leq n) = q_n$ for all $n \geq 0$. Although G is directed, we only consider the weak connectivity, that is the connected components of G or W are those of the underlying undirected graph. Let $\rho \in V$ and denote by W_ρ the connected component of ρ in W (with convention $W_\rho = \emptyset$ if $\rho \notin W$).

1. *If $(q_n)_{n\geq 0}$ satisfies*

$$4(\Delta_+ + \Delta_-)\sum_{n\geq 0} \Delta_+^n \Delta_-^n (1-q_n) < 1,$$

then almost surely all connected components of W are finite.

2. *If moreover we have for some $t > 0$*

$$\sum_{n\geq 0} \Delta_-^n e^{t\Delta_+^n}(1-q_n) < \infty,$$

then there exists $\lambda > 0$, depending only on Δ_+, Δ_- and the $(q_n)_{n\geq 0}$, such that for every $n \in \mathbf{N}$,

$$\mathbf{P}(|W_\rho| > n) \leq e^{-\lambda n}.$$

Proof of Lemma 3.21. When conditioning on $\mathcal{T}$, $\mathcal{A}_{\rho_0}$ can be seen as a directed graph (already defined in the beginning of Section 3.4.2): the vertex set is $\mathcal{A}_{\rho_0}$, and we put an oriented edge from ρ' to ρ for each pair $\rho, \rho' \in \mathcal{A}_{\rho_0}$ such that $d(\rho,\rho') = 1$. Therefore, $\mathcal{A}_{\rho_0}$ is a directed graph

of bounded degree, where all the vertices have in-degree exactly $|B|$ and out-degree bounded by $|B|$.

Therefore, we can apply the above Theorem with $\Delta_+ = \Delta_- = |B|$, and we obtain exactly the hypothesis (H3) as a sufficient condition for subcriticality and exponential decay of the Boolean percolation. □

We can now complete the proof of Theorem 3.13.

Proof of Theorem 3.13. The first step of the proof is to show that there is a.s. weak coalescence at $(0,0)$, regardless of the fact that ρ_0 is wet or dry. Let $\mathcal{C}$ be the first dry cutset above $(0,0)$, which is obtained by exploring all the active paths starting from ρ and, for each, take the first dry mark encountered. The fact that $\mathcal{C}$ exists a.s. is a consequence of Lemma 3.21. Then, we consider a starting time τ_0 preceding all the times carrying a mark of $\mathcal{C}$, i.e. τ_0 such that every $\rho = (i,t,u) \in \mathcal{C}$ satisfies $t \in (\tau_0, 0)$ (this is possible to do so because $\mathcal{C}$ is finite). Because they are all dry, according to Lemma 3.20, the marks $(i,t,u) \in \mathcal{C}$ verify a.s.

$$X_t^{\mathbf{0},\tau_0}(i) = \varphi(\mathbf{0}, u)$$

and there is a.s. weak coalescence at (i,t). It follows that there is a.s. weak coalescence at $(0,0)$. Indeed, let $X_0^* := X_0^{\mathbf{0},\tau_0}(0)$, and observe that X_0^* is a measurable function of the update marks (it actually does not depend on τ_0, as soon as it is chosen to be smaller that the infimum of times carrying an update mark of $\mathcal{C}$). For any $\tau < \tau_0$, if the dynamics started at time τ with initial configuration $\xi \in \Omega$ and the one started at τ_0 with initial configuration $\mathbf{0}$ coincide at every mark of $\mathcal{C}$, then they will also coincide at $(0,0)$ almost surely. Thus, we have for all $\tau < \tau_0$, a.s.

$$\{\xi \in \Omega, X_0^{\xi,\tau}(0) = X_0^*)\} \supset \bigcap_{(i,t,u)\in\mathcal{C}} \{\xi \in \Omega, X_t^{\xi,\tau}(i) = \varphi(\mathbf{0}, u)\}.$$

On the event of existence of $\mathcal{C}$, we get by Lemma 3.20 and finiteness of $\mathcal{C}$ that the μ–measure of the right-hand side tends to 1 when τ goes to $-\infty$. Hence $\mu\left(\{\xi \in \Omega, X_0^{\xi,\tau}(0) = X_0^*\}\right) \xrightarrow[\tau\to-\infty]{} 1$, so we get weak coalescence at $(0,0)$.

Everything has been done in a translation-invariant manner, so for all $i \in \mathbf{Z}^d$, we get a similar construction of X_i^*, and a.s. weak coalescence at $(i,0)$. Since there are only countably many sites in $\mathbf{Z}^d$, and according to Proposition 3.16, it yields a realisation of μ, which has been constructed as a factor of the update marks. This proves the first part of Theorem 3.13, that is, μ is an FIID. It remains to show that μ can be well-approximated by finitely dependent fields.

Let $l \geq 1$. Our goal is to construct a field $Y^* = (Y_i^*)_{i\in\mathbf{Z}^d}$ that is $l-$dependent and close to $X^* = (X_i^*)_{i\in\mathbf{Z}^d}$. The strategy is simply to truncate the factor map of X^*, to obtain Y^* as an FIID with coding radius bounded by $\lfloor l/2 \rfloor$, so that it is $l-$dependent. We then use a union bound over all the sites of any finite box of $\mathbf{Z}^d$ to get the result.

We do the construction only for Y_0^*, in a translation-invariant manner. Let $\mathcal{C}_l$ be the cutset (above $(0,0)$) of update marks at distance $\lfloor l/2 \rfloor$ from ρ_0. For all $\rho \in \mathcal{C}_l$, let $Y_\rho := \varphi(\mathbf{0}, U_\rho)$. Then, define Y_ρ recursively for every update mark ρ such that $d(\rho_0, \rho) < \lfloor l/2 \rfloor$. Suppose that all the $Y_{\rho'}$ for ρ' between ρ and $\mathcal{C}_l$ have been defined[3]; denote by i the location (in $\mathbf{Z}^d$) of ρ;

[3]Note that it is possible that such an update mark ρ' satisfies $d(\rho_0, \rho') \leq d(\rho_0, \rho)$, due to an active path from $(0,0)$ to ρ' that does not passes through ρ. Thus, it is not possible to define the Y_ρ by decreasing order of the distance from ρ_0.

let

$$Y_\rho := \varphi(\eta, U_\rho)$$

where for every $j \in B$, $\eta_j = Y_{\rho'}$ if there is a mark ρ' at $i+j$ such that $d(\rho,\rho') = 1$, and $\xi_j = 0$ otherwise. At the end, let $Y_0^* := Y_{\rho_0}$. It follows that Y_0^* is a measurable function of the update marks located at sites at distance less than $\lfloor l/2 \rfloor$ from the origin.

Defining similarly Y_i^* for all $i \in \mathbf{Z}^d$, it yields an $l-$dependent field. It remains to check that it coincides with X^* with high probability. Let us prove that every update mark ρ such that $d(\rho_0,\rho) \leq \lfloor l/2 \rfloor$ satisfies

$$\{\rho \text{ is dry}\} \overset{\text{a.s.}}{\subset} \{Y_\rho = \varphi(\mathbf{0}, U_\rho)\}. \tag{3.28}$$

For this, we first prove that if ρ is dry, then every ρ' between ρ and $\mathcal{C}_l$ satisfies $Y_{\rho'} \in S_{d(\rho,\rho')}$. If $\rho' \in \mathcal{C}_l$, we have $Y_{\rho'} = \varphi(\mathbf{0}, U_{\rho'})$; since ρ is dry, we have that $U_{\rho'} \leq q_{d(\rho,\rho')}$ so $Y_{\rho'} \in S_{d(\rho,\rho')}$ by property (3.18) of the update function. Then, for $\rho' \notin \mathcal{C}_l$, we again argue recursively: assume that the property holds for every ρ'' between ρ' and $\mathcal{C}_l$; we therefore have that all the update marks ρ'' such that $d(\rho',\rho'') = 1$ satisfy

$$Y_{\rho''} \in S_{d(\rho,\rho'')} \subset S_{d(\rho,\rho')+1},$$

because $d(\rho,\rho'') \leq d(\rho,\rho') + 1$. Then, we have $U_{\rho'} \leq q_{d(\rho,\rho')}$ because ρ is dry, so we get $Y_{\rho'} \in S_{d(\rho,\rho')}$ by (3.18). Finally, we have that $Y_{\rho'} \in S_1$ for every ρ' such that $d(\rho,\rho') = 1$. Now, ρ being dry also implies that $U_\rho \leq q_0$, so by (3.17), we deduce that $Y_\rho = \varphi(\mathbf{0}, U_\rho)$, so (3.28) holds.

If ρ_0 itself is dry, we directly obtain $Y_0^* = Y_{\rho_0} = \varphi(\mathbf{0}, U_{\rho_0}) = X_0^*$. More generally, if the first dry cutset $\mathcal{C}$ above $(0,0)$ contains only marks at distance less than $\lfloor l/2 \rfloor$ from ρ_0, it follows that every $\rho = (i,t,u) \in \mathcal{C}$ satisfies $Y_\rho = \varphi(\mathbf{0}, u) = X_t^{\mathbf{0},\tau_0}(i)$, where τ_0 is defined as in the beginning of the proof. Then, the updates encoded by marks between ρ_0 and $\mathcal{C}$ produce the same values in the dynamics $(X_t^{\mathbf{0},\tau_0})_{t \geq \tau_0}$ and in the definition of the Y_ρ's. In the end, we get $Y_0^* = Y_{\rho_0} = X_0^{\mathbf{0},\tau_0}(0) = X_0^*$.

Therefore, it holds that $Y_0^* = X_0^*$ unless there exist (at least) $\lfloor l/2 \rfloor$ consecutive wet marks on an active path starting from $(0,0)$. Thus, according to Lemma 3.21, there exists $\lambda > 0$ such that

$$\mathbf{P}(X_0^* \neq Y_0^*) \leq \mathbf{P}(\exists \text{ a wet path of length } \lfloor l/2 \rfloor \text{ starting from } (0,0)) \leq e^{-\lambda l},$$

Now, by translation-invariance, for every $\Lambda \Subset \mathbf{Z}^d$, it holds that

$$\mathbf{P}(X^*|_\Lambda \neq Y^*|_\Lambda) = \mathbf{P}(\exists i \in \Lambda, X_i^* \neq Y_i^*) \leq |\Lambda| e^{-\lambda l},$$

hence the result. □

3.5 Proofs for the Gaussian models

In this Section, we prove the results stated in the introduction for our Gaussian models. For this, we will use some properties on the Gaussian density that we recall here. First, for all $x \in \mathbf{R}$, we have the following equality of distributions: $\mathcal{N}(x,1) = x + \mathcal{N}(0,1)$. We will use the notation $\mathbb{P}(\mathcal{N}(x,1) \in \cdot)$ to denote the probability that a random variable of law $\mathcal{N}(x,1)$

belongs to some subset of $\mathbf{R}$. Then, for all $L > 0$, the function $x \mapsto \mathbb{P}(\mathcal{N}(x,1) \in [-L,L])$ defined over $\mathbf{R}$ is maximal for $x = 0$, decreasing on $\mathbf{R}^+$ and it is an even function. Moreover, we have the following bound on the Gaussian tail: for every $L > 0$ and $\sigma > 0$,

$$\begin{aligned}\mathbb{P}(|\mathcal{N}(0,\sigma^2)| > L) &= \frac{2}{\sqrt{2\pi\sigma^2}} \int_L^{+\infty} e^{-t^2/(2\sigma^2)} \mathrm{d}t \\ &\leq \frac{2}{\sqrt{2\pi\sigma^2}} \int_L^{+\infty} \frac{t}{L} e^{-t^2/(2\sigma^2)} \mathrm{d}t \\ &\leq \frac{2\sigma}{\sqrt{2\pi}L} e^{-L^2/(2\sigma^2)}. \end{aligned} \tag{3.29}$$

3.5.1 Proof of Theorem 3.4

Theorem 3.4 will be deduced from Theorem 3.6, part 1 by proving that when $|\varepsilon|$ is small enough, μ_ε^L satisfies the high-noise assumption, and part 2 follows by (3.3), of which we give a proof below.

Lemma 3.23. *One has* $\gamma(\mu_\varepsilon^L) \xrightarrow[\varepsilon\to 0]{} 1$.

Proof. We deduce the result from Lipschitz continuity of $\overline{g_x}^L(t)$ with variable x.

Fix $t \in [-L,L]$. First, it is easy to derive from mean value inequality that for all $x, y \in \mathbf{R}$, $|g_x(t) - g_y(t)| \leq c|x-y|$ with $c = \frac{e^{-1/2}}{\sqrt{2\pi}}$. Set

$$\lambda_\varepsilon = \inf_{|x|\leq L} \int_{-L}^{L} g_{\varepsilon x}(s)\mathrm{d}s = \inf_{|x|\leq L} \mathbb{P}(\mathcal{N}(\varepsilon x, 1) \in [-L,L]) = \mathbb{P}(\mathcal{N}(\varepsilon L, 1) \in [-L,L]).$$

Then, for all $x, y \in [L,L]$,

$$\begin{aligned}|\overline{g_{\varepsilon x}}^L(t) - \overline{g_{\varepsilon y}}^L(t)| &= \left| \frac{g_{\varepsilon x}(t)}{\int_{-L}^L g_{\varepsilon x}(s)\mathrm{d}s} - \frac{g_{\varepsilon y}(t)}{\int_{-L}^L g_{\varepsilon y}(s)\mathrm{d}s} \right| \\ &\leq \frac{|g_{\varepsilon x}(t) - g_{\varepsilon y}(t)|}{\int_{-L}^L g_{\varepsilon x}(s)\mathrm{d}s} + g_{\varepsilon y}(t) \left| \frac{1}{\int_{-L}^L g_{\varepsilon x}(s)\mathrm{d}s} - \frac{1}{\int_{-L}^L g_{\varepsilon y}(s)\mathrm{d}s} \right| \\ &\leq \frac{c}{\lambda_\varepsilon}|\varepsilon x - \varepsilon y| + \frac{g_{\varepsilon y}(t)}{\lambda_\varepsilon^2} \left| \int_{-L}^L (g_{\varepsilon y}(s) - g_{\varepsilon x}(s))\mathrm{d}s \right| \\ &\leq \frac{c|\varepsilon|}{\lambda_\varepsilon}|x-y| + \frac{2Lc}{\lambda_\varepsilon^2}|\varepsilon x - \varepsilon y| \leq \frac{3Lc|\varepsilon|}{\lambda_\varepsilon^2}|x-y|, \end{aligned}$$

since $g_{\varepsilon y}(t) \leq 1$, $\lambda_\varepsilon^2 \leq \lambda_\varepsilon$, and assuming $L \geq 1$ (if not, just replace the L by 1 in the last

inequality). Then, let $\eta(\varepsilon, L) := 1 - \gamma(\mu_\varepsilon^L)$ and use Lipschitz continuity to get the result:

$$\begin{aligned}\eta(\varepsilon, L) &= 1 - \int_{-L}^{L} \inf_{|x|\le L} \overline{g_{\varepsilon x}}^L(s)\mathrm{d}s \\ &= \int_{-L}^{L} \overline{g_0}^L(s) - \inf_{|x|\le L} \overline{g_{\varepsilon x}}^L(s)\mathrm{d}s \\ &= \int_{-L}^{L} \sup_{|x|\le L} (\overline{g_0}^L(s) - \overline{g_{\varepsilon x}}^L(s))\mathrm{d}s \\ &\le 2L\frac{3Lc}{\lambda_\varepsilon^2}|\varepsilon|L = \frac{6L^3c}{\lambda_\varepsilon^2}|\varepsilon|.\end{aligned}$$

Finally, we just need to check that λ_ε does not cancel at the limit $\varepsilon \to 0$ to conclude. One can notice that $\lambda_\varepsilon = \mathbb{P}(\mathcal{N}(0,1) \in [(-1+\varepsilon)L, (1+\varepsilon)L]$ remains greater than $\mathbb{P}(\mathcal{N}(0,1) \in [0, 2L])$ for $|\varepsilon| \le 1$, hence the result. □

Proposition 3.24. *Assume that μ is an FIID with coding radius R. Then, for every $l \in \mathbf{N}$, there exists an l–dependent distribution ν_l on Ω such that for all $\Lambda \Subset \mathbf{Z}^d$,*

$$d_{\mathrm{TV}}(\mu|_\Lambda, \nu_l|_\Lambda) \le |\Lambda|\mathbf{P}(2R > l).$$

Proof. By assumption, there exists an i.i.d. process $Y = (Y_i)_{i\in\mathbf{Z}^d}$ and a factor map F such that $F(Y)$ has law μ. Let $l \in \mathbf{N}$ and $r = \lfloor l/2 \rfloor$. For every $i \in \mathbf{Z}^d$, let Y^i be the truncation of Y to the ball B_r^i, that is, $Y_j^i = \mathbf{1}_{\{j\in B_r^i\}}Y_j$ for all $j \in \mathbf{Z}^d$. Set $Z^l = (F(Y^i)_i)_{i\in\mathbf{Z}^d}$ and call ν_l the distribution of Z^l. Then, if two subsets A and B of $\mathbf{Z}^d$ are at distance greater than $l \ge 2r$, it follows that $Z^l|_A$ and $Z^l|_B$ are factors of independent random fields (which are respectively the restriction of Y to $\bigcup_{i\in A} B_r^i$ and $\bigcup_{j\in B} B_r^j$, which are disjoint). Thus, ν_l is l–dependent.

Let $\Lambda \Subset \mathbf{Z}^d$. For every $i \in \mathbf{Z}^d$, on the event $\{R_i(Y) \le r\}$ we have $Z_i^l = F(Y)_i$. Therefore, on $\bigcap_{i\in\Lambda}\{R_i(Y) \le r\}$, $Z^l|_\Lambda$ and $F(Y)|_\Lambda$ coincide. It yields a coupling of the measures $\mu|_\Lambda$ and $\nu_l|_\Lambda$ such that the probability that the two realisations coincide in the coupling is greater than

$$\mathbf{P}\left(\bigcap_{i\in\Lambda}\{R_i(Y) \le r\}\right) \ge 1 - |\Lambda|\mathbf{P}(R > r),$$

by a union bound. With the characterisation of the total variation distance as one minus the maximal coupling probability, it gives the result. □

Proof of Theorem 3.4. For $\mu = \mu_\varepsilon^L$, the local Markov property (3.1) holds with boundary set $B = B_1^0 - \{0\}$, hence $|B| = 2d$. According to Lemma 3.23, $\gamma(\mu_\varepsilon^L)$ tends to 1 when ε goes to 0, so that for $|\varepsilon|$ small enough, this quantity stays above $1 - 1/(2d)$. In this case, it follows from Theorem 3.6 that μ_ε^L is an FFIID with exponential tails. This proves part 1 of Theorem 3.4, and part 2. follows immediately thanks to Prop 3.24, since the coding radius R of the factor map for μ_ε^L satisfies $\mathbf{P}(R > n) \le e^{-cn}$ for some $c > 0$ and all $n \ge 1$ (see the end of the proof of Theorem 3.6). □

3.5.2 Proof of Theorem 3.3

We prove Theorem 3.3 as a special case of Theorem 3.13. For this, the main assignment is to find suitable sequences $(S_n)_{n\geq 1}$ and $(q_n)_{n\geq 0}$ satisfying (H1), (H2), (H3) and (H4). We will search for S_n of the form $[-L_n, L_n]$, where $(L_n)_{n\geq 1}$ is an increasing sequence of real numbers that goes to $+\infty$.

Proof of Theorem 3.3. First, recall that μ_ε is a stationary Markov random field with boundary set $B = B_1 - \{0\}$, so that $|B| = 2d$. Then, for all $\eta \in S^B$, $\mu_\varepsilon(X_0 \in \cdot \mid X|_B = \eta)$ is Gaussian of mean $\frac{\varepsilon}{2d}\sum_{j\sim i}\eta_j$ and variance 1. Therefore, $\tilde{\gamma}(\mu_\varepsilon)$ corresponds to the maximal probability for realisations of $\mathcal{N}(\varepsilon x, 1)$ for $x \in S_1 = [-L_1, L_1]$ to coincide in a coupling and belong to $[-L_1, L_1]$, so we have

$$\tilde{\gamma}(\mu_\varepsilon) = \int_{-L_1}^{L_1} \inf_{|x|\leq L_1} g_{\varepsilon x}(t)\mathrm{d}t.$$

To get (H3), one can choose $q_n = 1 - ae^{-(2d)^n}$ with $a > 0$ small enough. Then, we want to find a sequence $(L_n)_{n\geq 1}$ that respects the links with $(q_n)_{n\geq 0}$ prescribed by (H1) and (H2), taking $S_n = [-L_n, L_n]$. One can rewrite (H2) as

$$\forall n \geq 1, \inf_{|x|\leq L_{n+1}} \mathbb{P}(|\mathcal{N}(\varepsilon x, 1)| \leq L_n) \geq q_n.$$

The infimum is achieved for $x = \pm L_{n+1}$ so we need, for all $n \geq 1$,

$$\mathbb{P}(|\mathcal{N}(\varepsilon L_{n+1}, 1)| > L_n) \leq 1 - q_n = ae^{-(2d)^n}.$$

We have

$$\begin{aligned}\mathbb{P}(|\mathcal{N}(\varepsilon L_{n+1}, 1)| > L_n) &= \mathbb{P}(\mathcal{N}(0,1) > L_n - \varepsilon L_{n+1}) + \mathbb{P}(\mathcal{N}(0,1) < -L_n - \varepsilon L_{n+1}) \\ &\leq 2\mathbb{P}(\mathcal{N}(0,1) > L_n - |\varepsilon| L_{n+1}).\end{aligned}$$

By the Gaussian tail bound (3.29), we get that this is lower than $ae^{-(2d)^n}$ if

$$L_n - |\varepsilon| L_{n+1} \geq \sqrt{2}\sqrt{(2d)^n - \log(a)}. \tag{3.30}$$

Informally, we want L_{n+1} bigger than L_n but $|\varepsilon|L_{n+1}$ smaller that L_n, so we aim to take $L_n \asymp \sqrt{|\varepsilon|}L_{n+1}$. By choosing $L_n = L_1|\varepsilon|^{-(n-1)/2}$ for some $L_1 > 0$, we have

$$L_n - |\varepsilon| L_{n+1} = L_1\sqrt{|\varepsilon|}^{-(n-1)}(1 - \sqrt{|\varepsilon|}),$$

which implies (3.30) so (H2) if L_1 is large enough and $|\varepsilon|$ small enough.

Repeating the last computation of the proof of Lemma 3.23, observe that $\tilde{\gamma}(\mu_\varepsilon) \xrightarrow[\varepsilon\to 0]{} 1$, so (H1) holds for $|\varepsilon|$ small enough.

We finally check that the sequence $(L_n)_{n\geq 1}$ we defined satisfies (H4), that is, we prove that

$$\mu_\varepsilon(\{\xi \in \Omega : \forall i \in \mathbf{Z}^d, |\xi_i| \leq L_{\max(n, \|i\|_1)}\}) \xrightarrow[n\to+\infty]{} 1.$$

Set for all $n \geq 1$, $A_n = \{\xi \in \Omega : \forall i \in \mathbf{Z}^d, |\xi_i| \leq L_{\max(n,\|i\|_1)}\}$. When $(X_i)_{i\in\mathbf{Z}^d}$ is distributed according to μ_ε, for all $i \in \mathbf{Z}^d$ the marginal law of X_i is $\mathcal{N}(0, \sigma^2)$, with $\sigma^2 = \mathrm{Var}(X_i)$ (this does not depend on i by stationarity). One can see, from Remark 3.2, that

$$1 \leq \sigma^2 = \Gamma(0,0) \leq 1 + \sum_{n\geq 1}\varepsilon^{2n} = \frac{1}{1-\varepsilon^2}.$$

Then, by a union bound,

$$\begin{aligned}1-\mu_\varepsilon(A_n) &\le \sum_{i\in B_n^0}\mu_\varepsilon(|X_i|>L_n)+\sum_{i\notin B_n^0}\mu_\varepsilon(|X_i|>L_{\|i\|_1})\\ &\le |B_n^0|\,\mathbb{P}(|\mathcal{N}(0,\sigma^2)|>L_n)+\sum_{k>n}|\partial B_k^0|\,\mathbb{P}(|\mathcal{N}(0,\sigma^2)|>L_k).\end{aligned}$$

The size of the balls and spheres of $\mathbf{Z}^d$ are of order given by $|B_n^0|\asymp n^d$ and $|\partial B_n^0|\asymp n^{d-1}$. Let $\tilde{c}>0$ be a constant such that $|B_n^0|\le \tilde{c}n^d$ and $|\partial B_n^0|\le \tilde{c}n^{d-1}$, and set $c=\dfrac{2\sigma}{\sqrt{2\pi}}\tilde{c}$.

Therefore, by the Gaussian tail bound (3.29),

$$1-\mu_\varepsilon(A_n)\le cn^d\frac{e^{-L_n^2/(2\sigma^2)}}{L_n}+c\sum_{k>n}k^{d-1}\frac{e^{-L_k^2/(2\sigma^2)}}{L_k}.$$

To get that this quantity tends to 0 as n tends to $+\infty$, we need the right-hand side series to converge and $n^de^{-L_n^2/(2\sigma^2)}/L_n\to 0$. This is true when $n^{d-1}e^{-L_n^2/(2\sigma^2)}/L_n\le\frac{1}{n^2}$ for sufficiently large n, which clearly holds for the sequence $L_n=L_1|\varepsilon|^{-(n-1)/2}$. □

3.6 Appendix: construction of the update function

In this Appendix, we present an explicit construction of the update function φ of the dynamics, in the case where $S\subset\mathbf{R}$. Returning to the general setting, μ is a stationary Markov random field distribution on $S^{\mathbf{Z}^d}$, with boundary set $B\Subset\mathbf{Z}^d\backslash\{0\}$. We consider the family of conditional distributions $\pi^\eta=\mu(X_0\in\cdot\mid X|_B=\eta),\eta\in S^B$; $\gamma=\gamma(\mu)$ is the maximal coupling probability of distributions π^η for $\eta\in S^B$.

For all $\eta\in S^B$, let $F(s\mid\eta):=\pi^\eta((-\infty,s])$ for all $s\in\mathbf{R}$ be the cumulative distribution function of π^η. First, the classical (up to slight variations) construction of a maximal coupling of laws π^η works as follows: for all $s\in\mathbf{R}$, let

$$R(s):=\Big(\bigwedge_\eta\pi^\eta\Big)((-\infty,s])\text{ and }\widehat{F}(s\mid\eta):=F(s\mid\eta)-R(s)$$

(notice that $R(s)\xrightarrow[s\to+\infty]{}\gamma$ and $F(s\mid\eta)\ge R(s)$); then, set

$$\forall\eta\in S^B,\forall u\in[0,1],\varphi(\eta,u)=R^{-1}(u)\mathbf{1}_{\{u\le\gamma\}}+\widehat{F}^{-1}(u-\gamma\mid\eta)\mathbf{1}_{\{u>\gamma\}},$$

where generically the notation F^{-1} stands for the generalised inverse of a non-decreasing function F (i.e. $F^{-1}(y)=\inf\{x:F(x)>y\}$). It is clear that φ satisfies (3.6), and easy to check that $\varphi(\eta,U)$ has distribution π^η if $U\sim\mathcal{U}[0,1]$.

We now aim to construct a function φ that satisfies (3.17) and (3.18), which are the conditions imposed on the update function with respect to the levels. We suppose that the levels are of the form $S_n=[-L_n,L_n]$, for all $n\ge1$. Recall that $\widetilde{\gamma}=\widetilde{\gamma}(\mu)$ is the maximal probability for random variables of laws $\pi^\eta,\eta\in S_1^B$ to coincide in a coupling, with the additional requirement that the common value belongs to $[-L_1,L_1]$.

For all $s\in\mathbf{R}$, let

$$\widetilde{R}(s):=\Big(\bigwedge_{\eta\in S_1^B}\pi^\eta\Big)((-L_1,s\wedge L_1]),$$

so that $\widetilde{R}(s) = 0$ for $s \le -L_1$ and $\widetilde{R}(s) = \tilde{\gamma}$ for $s \ge L_1$. Let also for all $\eta \in S^B$ and all $a < b$

$$F(a, b \mid \eta) := \pi^\eta((-a, b]).$$

We introduce a special notation in the case $b = -a = L_n$: denote $q_n(\eta) = F(-L_n, L_n \mid \eta)$. Observe that, then, $q_n \le \inf_{\eta \in S^B_{n+1}} q_n(\eta)$ (by (H2)). Finally, for all $\eta \in S^B$ and $s \in \mathbf{R}$ let

$$\widehat{F}_0(s \mid \eta) := \begin{cases} F(-L_1, s \wedge L_1 \mid \eta) - \widetilde{R}(s) & \text{if } \eta \in S_1^B \\ F(-L_1, s \wedge L_1 \mid \eta) & \text{otherwise} \end{cases}$$

and for all $n \ge 1$, let

$$\widehat{F}_n(s \mid \eta) := F(-L_{n+1}, s \wedge L_{n+1} \mid \eta) - F(-L_n, s \wedge L_n \mid \eta).$$

All these functions are non-decreasing so their generalised inverses are well-defined.

Proposition 3.25. *For all $\eta \in S_1^B$ and $u \in [0,1]$, let*

$$\varphi(\eta, u) = \widetilde{R}^{-1}(u)\mathbf{1}_{\{u \le \tilde{\gamma}\}} + \sum_{n \ge 0} \widehat{F}_n^{-1}(u - q_n(\eta) \mid \eta)\mathbf{1}_{\{q_n(\eta) < u \le q_{n+1}(\eta)\}}$$

and for $\eta \in S^B \setminus S_1^B$, let

$$\varphi(\eta, u) = \sum_{n \ge 0} \widehat{F}_n^{-1}(u - q_n(\eta) \mid \eta)\mathbf{1}_{\{q_n(\eta) < u \le q_{n+1}(\eta)\}}$$

with convention $q_0(\eta) := \widetilde{\gamma}$. Then, if $U \sim \mathcal{U}[0,1]$, it follows that $\varphi(\eta, U)$ is distributed according to π^η. Moreover, for all $\eta \in S^B$ and $u \in [0,1]$, one has

$$\eta \in S_1^B \text{ and } u \le q_0 \implies \varphi(\eta, u) = \varphi(\mathbf{0}, u) \in [-L_1, L_1]$$

and for all $n \ge 1$,

$$\eta \in S_{n+1}^B \text{ and } u \le q_n \implies \varphi(\eta, u) \in [-L_n, L_n].$$

Proof. It is clear that the last two properties on φ, corresponding to (3.17) and (3.18), are verified. This follows from (H1) and (H2) and the fact that for all $n \ge 1$, $\widehat{F}_{n-1}^{-1}$ takes values in $[-L_n, L_n]$. Then, it remains to prove that φ generates the right distribution. Let $U \sim \mathcal{U}[0,1]$, and denote by $\mathbf{P}$ the probability with respect to U. Let $s \in \mathbf{R}$ and $\eta \in S^B$. Suppose that $\eta \in S_1^B$ (the proof being similar if not). One has

$$\begin{aligned}
\mathbf{P}(\varphi(\eta, U) \le s) &= \mathbf{P}(\widetilde{R}^{-1}(U) \le s \text{ and } U \le \widetilde{\gamma}) + \sum_{n \ge 0} \mathbf{P}\left(\widehat{F}_n^{-1}(U - q_n(\eta) \mid \eta) \le s \text{ and } q_n(\eta) < U \le q_{n+1}(\eta)\right) \\
&= \mathbf{P}(U \le \widetilde{R}(s)) + \sum_{n \ge 0} \mathbf{P}\left(0 < U - q_n(\eta) \le \widehat{F}_n(s \mid \eta)\right) \\
&= \widetilde{R}(s) + \sum_{n \ge 0} \widehat{F}_n(s \mid \eta) \\
&= F(-\infty, s \mid \eta),
\end{aligned}$$

where in the second equality, we use the fact that $\widetilde{R}(s) \le \widetilde{\gamma}$ and $\widehat{F}_n(s \mid \eta) \le \widehat{F}_n(L_{n+1} \mid \eta) = q_{n+1}(\eta) - q_n(\eta)$. This shows that $\varphi(\eta, U)$ has distribution π^η. □

Chapter 4

Subcritical Boolean percolation on graphs of bounded degree

Ce chapitre est la reproduction de l'article [Fai25], publié dans Eletronic Communications in Probability *(aussi disponible sur Hal et arXiv).*

Résumé. On étudie un modèle de percolation à longue-portée sur des graphes de degré borné, appelé le modèle de percolation Booléenne. Dans ce modèle, chaque sommet d'un graphe infini et connexe est le centre d'une boule de rayon aléatoire. Les sommets sont indépendamment activés avec probabilité $p \in [0,1]$. On considère W la réunion des boules aléatoires dont le centre est actif. Dans certains cas, le modèle ne présente pas de transition de phase, au sens où pour n'importe quel $p > 0$, W contient une composante connexe infinie, voire même recouvre entièrement le graphe. Dans ce chapitre, on donne une condition suffisante sur la loi des rayons pour qu'il existe une phase sous-critique, c'est-à-dire un intervalle de valeurs du paramètre p suffisamment petites pour que toutes les composantes connexes de W soient finies. De plus, on fournit également une condition pour la décroissance exponentielle de la taille d'une composante typique de W.

This chapter is a reproduction of the paper [Fai25], published in Eletronic Communications in Probability *(also available on Hal and arXiv).*

Abstract. We study a model of long-range site percolation on graphs of bounded degree, namely the Boolean percolation model. In this model, each vertex of an infinite connected graph is the center of a ball of random radius, and vertices are said to be active independently with probability $p \in [0,1]$. We consider W to be the reunion of random balls with an active center. In certain circumstances, the model does not exhibit a phase transition, in the sense that W almost surely contains an infinite component for all $p > 0$, or even W covers the entire graph. In this Chapter, we give a sufficient condition on the radius distribution for the existence of a subcritical phase, namely a regime such that almost surely all the connected components of W are finite. Additionally, we provide a sufficient condition for the exponential decay of the size of a typical component.

4.1 Introduction

We consider a discrete version of Boolean percolation on general graphs of bounded degree. It can be described as follows. Each vertex of an infinite connected graph is considered as the center of a ball with random radius. We assume that the radii are independent and identically distributed (i.i.d.) positive random variables. Each vertex is also set to be *active* with probability $p \in [0,1]$, independently of anything else. Consider W the reunion of random balls with an active center. In the original paper by Lamperti [Lam70], the vertices are considered as fountains that wet their neighbours up to some distance (which is 0 with probability p), this is why we call W the *wet set*. Note that in our setting, an active fountain wets its own site (which is not the case in [Lam70]). Therefore, it is clear that W is empty if $p = 0$ and that W covers the entire graph if $p = 1$. Under weak assumption on the graph, we can construct situations where W almost surely contains an infinite connected component for all $p > 0$. In fact, it is the case as soon as the graph contains an infinite path of vertices and the expected radius of a wet ball diverges. On an undirected graph, we even have that W almost surely covers the entire graph for any $p > 0$ if the expected sizes of the wet balls are infinite. In this paper, we aim to find sufficient conditions on the radius distribution for the existence of a subcritical phase, namely non-trivial values of p such that all the connected components of W are finite almost surely. We also consider the question of exponential decay of the size of a typical connected component.

This model can be seen as a discrete version of the continuum Poisson Boolean percolation, which has been widely studied (e.g. see the book [MR96]). In the continuum, for any $d \geq 1$, we consider random balls centered at the points of a homogeneous Poisson point process on $\mathbf{R}^d$ of intensity $\lambda \in (0, \infty)$. Here W is the subset of $\mathbf{R}^d$ of points covered by the balls, and W_0 denotes the connected component of the origin (possibly empty). Let R be a copy of one of the random radii. In [Hal85], Hall proved that for any $d \geq 1$, if $\mathbf{E}[R^{2d-1}]$ is finite, then W_0 is almost surely bounded for small enough λ. This never happens when $\mathbf{E}[R^d]$ is infinite, because in this case the whole space is almost surely covered for any $\lambda > 0$ (Proposition 3.1 in [MR96]). Gouéré proved in [Gou08] that the finiteness of $\mathbf{E}[R^d]$ is actually a necessary and sufficient condition for the existence of a subcritical phase. Sharpness of the phase transition has been investigated in [ATT18], [DCRT20] and [DT22].

Returning to the discrete setting, Coletti and Grynberg established in [CG14] a result similar to that of Gouéré, proving that Boolean percolation on $\mathbf{Z}^d$ admits a subcritical phase if and only if $\mathbf{E}[R^d]$ is finite, for all $d \geq 1$. Together with Miranda [CMG20], they furthermore prove an analogous statement for doubling graphs, which includes all the graphs with polynomial growth (thus $\mathbf{Z}^d$). In [Bra24], Braun handled the case of the directed d-ary tree, for which the existence of an infinite component in W is equivalent to the survival of a branching process. A related model is the Rumor Percolation model (see the review [JMR19] and the references therein), which can be described as follows. At time 0, only the root (a fixed vertex of the graph) knows about the rumor, and starts to spread it. Then, at each step, a new vertex hears the rumor if it is within a random radius of a previously informed vertex. We point out the paper by Lebensztayn and Rodriguez [LR08] which deals with general graphs of bounded degree (but with geometric distribution for the radii) and shows a non-trivial phase transition. However, contrary to the Boolean setting above, this rumor model (or disk-percolation in [LR08]) is crucially directed. In fact, the rumor necessarily starts from the root and then spreads across the graph, whereas in Boolean percolation the rumor may originate from a

source at infinity and subsequently contaminate the root. In this paper, we prove that for any graph with degree bounded by $\Delta > 1$, the Boolean model admits a subcritical phase if $\mathbf{E}[\Delta^{2R}]$ is finite. In fact, a weaker condition will be given in Theorem 4.1. The graph can be directed or not, and the above condition can be improved for directed graphs. The proof of this result can be seen as a Peierls argument. We then show that the size of the connected component of the root has exponential tails for p small enough if $\mathbf{E}[e^{t\Delta^R}]$ is finite for some $t > 0$. This is obtained by comparison with a subcritical Galton-Watson process, so we do not assert that this result would be sharp.

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4.2 Notations and statement of the main results

Throughout the paper, $\mathbf{N} = \{0, 1, 2 \dots\}$ denotes the set of non-negative integers and $\mathbf{N}^* = \{1, 2 \dots\}$ the set of positive integers. For an event A, $\mathbf{1}_A$ is the indicator function of the event. For two probability measures μ and ν, we write $\mu \preceq \nu$ when ν stochastically dominates μ, i.e. when there exists a coupling (X, Y) of the two measures such that $X \leq Y$ almost surely.

4.2.1 Graph terminology

Let $G = (V, E)$ be an infinite locally finite connected graph. We will consider both directed and undirected graphs. Thus, E is seen as a subset of $V \times V$, and G is undirected if and only if $\forall x, y \in V$, $(x, y) \in E \iff (y, x) \in E$. However, for directed graphs, we will only consider the weak connectivity, i.e. the connectivity of the underlying undirected graph. For all $x, y \in V$, the distance $d(x, y) \in \mathbf{N} \cup \{+\infty\}$ from x to y is defined as the minimal number of (oriented) edges of a path from x to y (and $d(x, y) = +\infty$ if such a path does not exist). Note that it is symmetric if and only if the graph is undirected. For $x \in V$ and $r \in \mathbf{N}$, let $B_r(x) := \{y \in V \mid d(x, y) < r\}$ (so $B_0(x) = \emptyset$, $B_1(x) = \{x\}$ etc.).

We now define three different notions of the boundary of a subset of the graph. For every $B \subset V$, let

$$\partial^+ B := \{y \in V - B \mid \exists x \in B, (x, y) \in E\},$$

$$\partial^- B := \{y \in V - B \mid \exists x \in B, (y, x) \in E\},$$

$$\partial B := \partial^+ B \cup \partial^- B.$$

Note that we only consider outer boundaries, meaning that a vertex in ∂B do not belong to B but it is adjacent (the orientation of the edges between it and the vertices of B determine whether it is on $\partial^+ B$, $\partial^- B$ or both). These three notions coincide for undirected graphs. Associated with that, we define three distinct notions of degree. For every $x \in V$, let

$$d^+(x) := |\partial^+\{x\}|, \quad d^-(x) := |\partial^-\{x\}| \quad \text{and} \quad d(x) := |\partial\{x\}|;$$

$d^+(x)$ and $d^-(x)$ are usually called the out-degree and in-degree of vertex x, and they correspond respectively to the number of oriented edges emanating from x or pointing at x. However, $d(x)$ is not the total number of edges attached to x, but rather the number of neighbours of x without regard to the orientation, i.e. $d(x)$ is the degree of x in the underlying undirected graph. Finally, let

$$\Delta_+ = \Delta_+(G) := \sup\{d_+(x), x \in V\},$$

$$\Delta_- = \Delta_-(G) := \sup\{d_-(x), x \in V\},$$

$$\Delta = \Delta(G) := \sup\{d(x), x \in V\};$$

we clearly have $\Delta_\pm \leq \Delta \leq \Delta_+ + \Delta_-$. If Δ is finite, the graph is said to be of bounded degree. Note again that if the graph is undirected, we have for all $x \in V$, $d^+(x) = d^-(x) = d(x)$ and $\Delta_+ = \Delta_- = \Delta$.

For an oriented graph $G = (V, E)$, we define its *reversed graph* or *transpose graph* $G^\top$ on the same set of vertices, but with reversed orientation of the edges. Therefore, if $(x, y) \in E$ is an edge of G, then (y, x) is an edge of $G^\top$ and vice versa. For $x \in V$ and $r \in \mathbf{N}$, we write $B_r^\top(x)$ for the corresponding ball in $G^\top$, i.e. $B_r^\top(x)$ is the set of vertices $y \in V$ such that $d(y, x) < r$. In other words, for all $x, y \in V$, we have the equivalence

$$y \in B_r(x) \iff x \in B_r^\top(y).$$

Suppose that G is of bounded degree. Define for all $r \in \mathbf{N}$

$$c_r = c_r(G) := \sup_{x \in V} |B_r(x)| \text{ and } c_r^\top = c_r^\top(G) := \sup_{x \in V} |B_r^\top(x)|.$$

Since the graph is of bounded degree, we know that these two quantities are finite. In fact, $c_r \leq 1 + \Delta_+ + \cdots + \Delta_+^{r-1} \leq \max(r, 2\Delta_+^{r-1})$ and similarly $c_r^\top \leq \max(r, 2\Delta_-^{r-1})$. These two quantities are also upper bounded by $2\Delta^{r-1}$ (since $\Delta > 1$ for any infinite connected graph). We also define s_r and $s_r^\top$ respectively as the supremum over $x \in V$ of the number of vertices at distance exactly r from x in G or $G^\top$ respectively:

$$s_r = s_r(G) := \sup_{x \in V} |\partial^+ B_r(x)| \text{ and } s_r^\top = s_r^\top(G) := \sup_{x \in V} |\partial^- B_r^\top(x)|.$$

In the following, we will use the upper bounds $s_r \leq \Delta_+^r$ and $s_r^\top \leq \Delta_-^r$.

4.2.2 Phase transition of the Boolean percolation model

We now precisely define our Boolean model. Let $p \in [0, 1]$ and let ν be a probability distribution on $\mathbf{N}^*$. Let $(R_x)_{x \in V}$ be a family of i.i.d. random variables distributed according to ν, and let $(\sigma_x)_{x \in V}$ be a family of independent Bernoulli random variables of parameter p, independent from $(R_x)_{x \in V}$. A vertex x of the graph should be seen as a fountain that attempts to wet vertices at distance less than R_x, including itself. The attempt is successful if and only if $\sigma_x = 1$. Therefore, the set of wet vertices, or covered set, is defined as

$$W = W(G) := \bigcup_{x \in V, \sigma_x = 1} B_{R_x}(x). \tag{4.1}$$

The balls $B_{R_x}(x)$ with $\sigma_x = 1$ that appear in the union (4.1) are called *wet balls*. For all $x \in V$, write W_x for the connected component of x in W (with the convention that $W_x = \emptyset$ when $x \notin W$). As already observed, the probability of the event $\{|W_x| = \infty\}$ is 0 for $p = 0$ and 1 for $p = 1$, and is clearly non-decreasing in p. The *percolation* event is defined as the reunion $\bigcup_{x\in V}\{|W_x| = \infty\}$. It has probability 0 or 1 by the Kolmogorov's zero–one law. Therefore, we define the critical parameter of the model as

$$p_c(G, \nu) = \inf\{p \in [0, 1] : \mathbf{P}\left(\cup_{x\in V}\{|W_x| = \infty\}\right) = 1\}.$$

As already observed in [CG14] and [LR08], we have $p_c(G, \nu) \leq p_c^{\text{site}}(G)$ where $p_c^{\text{site}}(G)$ is the critical parameter for Bernoulli site percolation on G. In fact, $W(G)$ contains all the sites $x \in V$ such that $\sigma_x = 1$, so the Boolean model stochastically dominates Bernoulli site percolation.

We prove the following result.

Theorem 4.1. *Let G be a graph of bounded degree, and R distributed according to ν. For all $n \geq 1$, let $\varphi(n) := \Delta \sum_{r=1}^{n} c_r c_r^{\top}$. If $\mathbf{E}[\varphi(R)] < \infty$, then*

$$p_c(G, \nu) \geq \frac{1}{\mathbf{E}[\varphi(R)]} > 0.$$

Using the inequalities given at the end of Section 4.2.1, observe that

$$\varphi(n) \leq \Delta \sum_{r=1}^{n} 2\Delta^{r-1} 2\Delta^{r-1} = \frac{4\Delta}{\Delta^2 - 1}(\Delta^{2n} - 1).$$

Therefore, we immediately get from Theorem 4.1 that $p_c(G, \nu) > 0$ if $\mathbf{E}[\Delta^{2R}] < \infty$. This is the best condition we can obtain without further assumption on the graph. On a directed graph, the condition can be replaced by $\mathbf{E}[\max(R, \Delta_+^R) \max(R, \Delta_-^R)] < \infty$.

Remark 4.2. *Consider $\mathbf{T}^d$ the infinite $(d + 1)$–regular tree, and choose an orientation of its edges so that $\Delta_- = d$ and $\Delta_+ = 1$ (this can be done by choosing an end of the tree and orientating all the edges in the direction of this end). For this graph, the condition for having a phase transition given by Theorem 4.1 is $\mathbf{E}[Rd^R] < \infty$. We do not claim that it is necessary, but one can easily observe that the finiteness of $\mathbf{E}[d^R]$ is necessary. It is a general fact in Boolean percolation models that for any site $\rho \in \mathbf{T}^d$, the expected number of wet balls containing ρ is $p \sum_{x\in V} \sum_{r>d(x,\rho)} \nu(r)$. This is exactly $p(\mathbf{E}[d^R] - 1)/(d - 1)$ in the case of $\mathbf{T}^d$ with our choice of orientation. If this expectation diverges, by applying the Borel-Cantelli lemma we find that ρ belongs to infinitely many wet balls, and finally $W = \mathbf{T}^d$ almost surely.*

Let $\rho \in V$ be a specified vertex of the graph that will be called the root. We give a sufficient condition to ensure that the size of W_ρ has an exponential tail for small enough p.

Theorem 4.3. *For all $t > 0$ and $n \geq 1$, let $\psi_t(n) = \sum_{r=1}^{n} c_r^{\top} e^{tc_r}$. Under the same assumption as in Theorem 4.1, if furthermore $\mathbf{E}[\psi_t(R)]$ is finite for some $t > 0$, then for all $p < \mathbf{E}[\varphi(R)]^{-1}$, there exists $\lambda > 0$ such that for every $n \in \mathbf{N}$,*

$$\mathbf{P}(|W_\rho| > n) \leq e^{-\lambda n}.$$

In particular, $\mathbf{E}[\psi_t(R)] < \infty$ holds if $\mathbf{E}\left[e^{2t\Delta^R}\right] < \infty$.

4.2.3 Reformulation of the main results

The wet set defined in (4.1) can be rewritten as

$$W = \bigcup_{x \in V} B_{\sigma_x R_x}(x).$$

To be consistent with this reformulation, for all $n \geq 0$, let $q_n := 1 - p \sum_{k>n} \nu(k)$ be the probability that $\sigma_x R_x$ is at most n.

We derive Theorems 4.1 and 4.3 from proving the following result, stated in this new setting for directed graphs.

Theorem 4.4. *Let G be an infinite directed graph of bounded degree.*

1. *If (q_n) satisfies*
$$\Delta \sum_{n \geq 0} c_{n+1} c_{n+1}^{\top} (1 - q_n) < 1, \tag{4.2}$$
then almost surely all connected components of $W(G)$ are finite.

2. *Moreover, if (4.2) holds and if for some $t > 0$*
$$\sum_{n \geq 0} c_{n+1}^{\top} e^{t c_{n+1}} (1 - q_n) < \infty, \tag{4.3}$$
then there exists $\lambda > 0$ such that for every $n \in \mathbf{N}$,
$$\mathbf{P}(|W_\rho| > n) \leq e^{-\lambda n}.$$

Conditions (4.2) and (4.3) are those that arise from the proof in Section 4.3. We verify that they are implied by $\mathbf{E}[\varphi(R)] < \infty$ and $\mathbf{E}[\psi_t(R)] < \infty$ respectively if p is small enough.

Proof of Theorem 4.1. We use the fact that for any increasing function $f : \mathbf{N} \to \mathbf{R}_+$ such that $\mathbf{E}[f(R)] < \infty$, we have

$$\sum_{n \geq 1} f(n) \nu(n) = f(0) + \frac{1}{p} \sum_{n \geq 0} [f(n+1) - f(n)](1 - q_n). \tag{4.4}$$

This is obtained by noticing that $\nu(n) = \dfrac{(1 - q_{n-1}) - (1 - q_n)}{p}$. Applying (4.4) with $f = \varphi$ gives

$$p\mathbf{E}[\varphi(R)] = \Delta \sum_{n \geq 0} c_{n+1} c_{n+1}^{\top} (1 - q_n).$$

which is less than 1 as soon as $p < \mathbf{E}[\varphi(R)]^{-1}$. Then, part 1 of Theorem 4.4 implies that there is almost surely no percolation for such a p, hence the lower bound on $p_c(G, \nu)$. □

Proof of Theorem 4.3. We use the same identity (4.4) with $f = \psi_t$, where $t > 0$ is such that $\mathbf{E}[\psi_t(R)] < \infty$. It gives

$$p\mathbf{E}[\psi_t(R)] = \sum_{n \geq 0} c_{n+1}^{\top} e^{t c_{n+1}} (1 - q_n),$$

so that part 2 of Theorem 4.4 gives the exponential decay.

Then, since $e^{2t\Delta^{n+1}} - e^{2t\Delta^n} = e^{2t\Delta^n}\left(e^{2t\Delta^n(\Delta-1)} - 1\right) \geq e^{2t\Delta^n}\Delta^n 2t(\Delta-1)$, we obtain

$$\begin{aligned} p\mathbf{E}[\psi_t(R)] &\leq \sum_{n\geq 0} 2\Delta^n e^{2t\Delta^n}(1-q_n) \\ &\leq \frac{1}{t(\Delta-1)} \sum_{n\geq 0} \left(e^{2t\Delta^{n+1}} - e^{2t\Delta^n}\right)(1-q_n) \\ &\leq \frac{p}{t(\Delta-1)} \mathbf{E}\left[e^{2t\Delta^R}\right], \end{aligned}$$

where in the last inequality we apply (4.4) with $f(n) = e^{2t\Delta^n}$. Thus, $\mathbf{E}\left[e^{2t\Delta^R}\right] < \infty$ implies that $\mathbf{E}[\psi_t(R)] < \infty$. □

4.3 Proof of Theorem 4.4

The proof of Theorem 4.4 will be based on an exploration algorithm of the wet connected component of the root, which we describe informally here. First, we discover all the wet balls containing ρ. This forms our first layer of vertices contained in W_ρ. Then, assuming that the n-th layer has been constructed, the $(n+1)$-th layer consists of the union of wet balls whose boundary contains some vertex of the n-th layer. Note that it is not necessary for a wet ball to share a vertex with the previous layer in order to be added to W_ρ, only the existence of an edge (in any direction) between the two is required. If it is finite, W_ρ will be completely revealed after finding an empty layer (otherwise the algorithm never ends). To prove finiteness and exponential decay, we will compare the size of the layers with a subcritical Galton-Watson process.

The randomness of the appearance of the balls in (4.1) is supported by their center. Indeed, a given ball is said to be a wet ball if it is equal to $B_{R_x}(x)$ for some $x \in V$. To obtain a domination by a Galton-Watson process, it requires more independence; more precisely, it would be preferable for the events that a given ball is a wet ball to be independent. This can be achieved by realising the model as a Poisson Point Process (PPP) of balls. Furthermore, since the balls are discovered during the exploration by their boundary, we want to place the randomness of the appearance of a ball at its boundary. In Section 4.3.1, we first realise the model as a PPP on $V \times \mathbf{N}^*$, where the randomness will still come from the centers of the balls. Then, in Section 4.3.2 we transfer the randomness to the boundary by introducing PPP of marked balls. Finally, we conclude the proof of Theorem 4.4 in Section 4.3.3 by showing domination by a subcritical Galton-Watson process.

4.3.1 Realisation of the model as a PPP on $V \times \mathbf{N}^*$

We first define a version of the model by introducing a PPP of a well chosen intensity. Let ν_1 be the measure on $V \times \mathbf{N}^*$ such that for every $(x,n) \in V \times \mathbf{N}^*$, $\nu_1(\{(x,n)\}) = \lambda_n := \log(\frac{q_n}{q_{n-1}})$. Let ω_1 be a PPP of intensity ν_1, i.e. for every $(x,n) \in V \times \mathbf{N}^*$, if we set $\omega_1(x,n) := |\omega_1 \cap \{(x,n)\}|$, then $\omega_1(x,n)$ has Poisson distribution of mean λ_n and the $\omega_1(x,n)$ for $(x,n) \in V \times \mathbf{N}^*$ are independent.

Then, for every $x \in V$, let $Y_x := \sup\{n \geq 1 : \omega_1(x,n) > 0\}$ with the convention $Y_x = 0$ if it is empty. Thus, we have that

$$\begin{aligned}\mathbf{P}(Y_x \leq n) &= \mathbf{P}(\forall k > n, \omega_1(x,k) = 0) \\ &= \prod_{k>n} e^{-\lambda_k} \\ &= \prod_{k>n} \frac{q_{k-1}}{q_k} = q_n.\end{aligned}$$

Moreover, the $(Y_x)_{x\in V}$ are independent because they are functions of disjoint $\omega_1(x,n)$. Note that here the Y_x are distributed as the $\sigma_x R_x$ of the previous section. In particular, we have $\mathbf{P}(Y_x > 0) = 1 - q_0 = p$. Hence, the $(Y_x)_{x\in V}$ define a realisation of the Boolean percolation model we saw above, with wet set

$$W = \bigcup_{x\in V} B_{Y_x}(x),$$

where the union is taken over all the vertices of the graph.

4.3.2 Realisation of the model as PPP of marked balls

Let $\mathcal{B} = \{B_r(x), x \in V, r \in \mathbf{N}^*\}$ be the collection of balls of the graph G. There is a canonical surjection from $V \times \mathbf{N}^*$ to $\mathcal{B}$ given by

$$\tilde{\Phi} : \left| \begin{array}{ccc} V \times \mathbf{N}^* & \longrightarrow & \mathcal{B} \\ (x,r) & \longmapsto & B_r(x) \end{array} \right. .$$

Unfortunately, $\tilde{\Phi}$ is not a bijection, because we could have, for example, $B_n(x) = B_n(y)$ for different x and y. This is why we need to work with balls with marked center, and define $\mathcal{B}^\bullet = \{(B_r(x), x), x \in V, r \in \mathbf{N}^*\}$. For $B^\bullet = (B,x) \in \mathcal{B}^\bullet$, let $\mathrm{rad}(B^\bullet) = \sup_{y\in B} d(x,y) + 1$ be the radius of the ball. We will use the following convention: if $B^\bullet$ is an element of $\mathcal{B}^\bullet$, B will always denote its first coordinate (i.e. the corresponding ball of $\mathcal{B}$).

We now have a canonical surjection from $V \times \mathbf{N}^*$ to $\mathcal{B}^\bullet$ given by

$$\Phi : \left| \begin{array}{ccc} V \times \mathbf{N}^* & \longrightarrow & \mathcal{B}^\bullet \\ (x,r) & \longmapsto & (B_r(x), x) \end{array} \right. .$$

It is still not a bijection because we have $B_r(x) = B_{r+1}(x)$ if $\partial^+ B_r(x) = \emptyset$. However, we can now easily write $\Phi^{-1}(\{B^\bullet\})$ for any $B^\bullet = (x,B) \in \mathcal{B}$: we have $\Phi^{-1}(\{B^\bullet\}) = \{x\} \times [\![\mathrm{rad}(B^\bullet), \infty[\![$ if $\partial^+ B = \emptyset$, and otherwise $\Phi^{-1}(\{B^\bullet\}) = (x, \mathrm{rad}(B^\bullet))$.

Now, we can pushforward ω_1 by Φ, to get a PPP on $\mathcal{B}^\bullet$ of intensity $\nu_2 = \Phi_* \nu_1$, i.e.

$$\nu_2(\{B^\bullet\}) = \begin{cases} \sum_{n \geq \mathrm{rad}(B^\bullet)} \lambda_n & \text{if } \partial^+ B = \emptyset \\ \lambda_{\mathrm{rad}(B^\bullet)} & \text{otherwise} \end{cases} \text{ for all } B^\bullet \in \mathcal{B}^\bullet.$$

As stated before, we aim to put the randomness on the boundary of the balls. This is why we will also consider balls with a marked vertex on their boundary. Let $\mathcal{B}^\bullet_\bullet = \{(B_r(x), x, y), x \in V, r \in \mathbf{N}^*, y \in \partial B_r(x)\}$ and define ν_3 a measure on $\mathcal{B}^\bullet_\bullet$ by

$$\nu_3(\{B^\bullet_\bullet\}) = \begin{cases} \sum_{n \geq \mathrm{rad}(B^\bullet_\bullet)} \lambda_n & \text{if } \partial^+ B = \emptyset \\ \lambda_{\mathrm{rad}(B^\bullet_\bullet)} & \text{otherwise} \end{cases} \text{ for all } B^\bullet_\bullet = (B,x,y) \in \mathcal{B}^\bullet_\bullet,$$

where $\mathrm{rad}(B^\bullet_\bullet)$ is defined in the same way as $\mathrm{rad}(B^\bullet)$.

By defining more precisely the exploration layers mentioned above, we now construct a coupling (ω_2, ω_3) of PPP of intensity respectively ν_2 and ν_3. For this, we will need an auxiliary total order on V, which is chosen arbitrarily.

First, for all $(B, x, y) \in \mathcal{B}^\bullet_\bullet$ such that $\rho \in B$, simply let $\omega_3\big((B, x, y)\big)$ be independent Poisson variables of parameters $\nu_3(\{(B, x, y)\})$, and let

$$\omega_2\big((B, x)\big) = \omega_3\big((B, x, y_{\min})\big),$$

where $y_{\min}$ is the minimal element of ∂B for the auxiliary order of the vertices. Let

$$\mathcal{C}_1 = \bigcup_{\substack{\omega_2(\mathcal{B}^\bullet)>0 \\ \rho\in B}} B.$$

Then, successively for every $n \geq 1$, construct the $\omega_3\big((B, x, y)\big)$ of all $(B, x, y) \in \mathcal{B}^\bullet_\bullet$ such that $\partial B \cap \mathcal{C}_n \neq \emptyset$ and that were not previously defined: independently of the previous draws, let $\omega_3\big((B, x, y)\big)$ be independent Poisson variables of parameters $\nu_3(\{(B, x, y)\})$, and let

$$\omega_2\big((B, x)\big) = \omega_3\big((B, x, y_{\min})\big),$$

where $y_{\min}$ is the minimal element of $\partial B \cap \mathcal{C}_n$ for the auxiliary order. Let

$$\mathcal{C}_{n+1} = W(\mathcal{C}_n) - \bigcup_{k\leq n} \mathcal{C}_k,$$

where for every $C \subset V$,

$$W(C) := \bigcup \{B \mid \omega_3\big((B, x, y)\big) > 0 \text{ for some } x \in V \text{ and } y \in C\}.$$

Finally, for all the $\omega_3\big((B, x, y)\big)$ that have not yet been defined, draw again independently Poisson variables of parameter $\nu_3(\{(B, x, y)\})$, and simply take

$$\omega_2\big((B, x)\big) = \omega_3\big((B, x, y_{\min})\big)$$

as in the first step, $y_{\min}$ being the minimal element of ∂B.

Therefore, it appears that ω_2 and ω_3 are respectively distributed as PPP of intensity ν_2 and ν_3. Since ω_2 has the same distribution as $\Phi(\omega_1)$, it can be coupled with ω_1 in such a way that a.s. $\Phi(\omega_1) = \omega_2$.

Now, rewrite the wet set W in terms of this new PPP of marked balls.

Lemma 4.5. *Almost surely for the coupling $(\omega_1, \omega_2, \omega_3)$, we have*

$$W = \bigcup_{B^\bullet\in\mathcal{B}^\bullet, \omega_2(B^\bullet)>0} B. \tag{4.5}$$

Proof. The coupling have been defined in such a way that a.s. for all $B^\bullet \in \mathcal{B}^\bullet$, we have

$$\sum_{(x,n)\in\Phi^{-1}(\{B^\bullet\})} \omega_1(x, n) = \omega_2(B^\bullet).$$

Let us check both inclusions of (4.5). First, if $y \in W$, then there exists $x \in V$ such that $y \in B_{Y_x}(x)$ i.e. $d(x,y) < Y_x$. By definition of Y_x, it means that there exists $n > d(x,y)$ such that $\omega_1(x,n) > 0$. Let $B^\bullet = \Phi(x,n)$; then $y \in B$ and $\omega_2(B^\bullet) \geq \omega_1(x,n) > 0$.

On the other hand, suppose $y \in B$ for some $B^\bullet$ with $\omega_2(B^\bullet) > 0$; thus, there exists $(x,n) \in \Phi^{-1}(\{B^\bullet\})$ such that $\omega_1(x,n) > 0$. Since $y \in B$, it implies that $d(x,y) < n \leq Y_x$, so $y \in B_{Y_x}(x) \subset W$. □

Then, notice that the wet connected component of the root is entirely discovered by our exploration.

Lemma 4.6. *Almost surely for the coupling $(\omega_1, \omega_2, \omega_3)$, we have*

$$W_\rho \subset \bigcup_{n \geq 1} \mathcal{C}_n.$$

Proof. Let $z \in W_\rho$. According to the identity (4.5), there exists a chain of balls $B_1^\bullet, \dots, B_k^\bullet$ such that $\rho \in B_1$ and $z \in B_k$, and

$$\forall i \in \{1, \dots, k\},\ \omega_2(B_i^\bullet) > 0 \text{ and for } i < k,\ B_i \cap (B_{i+1} \cup \partial B_{i+1}) \neq \emptyset.$$

Furthermore, we can assume that $B_j \not\subset B_i$ for all $i \neq j$, i.e. no ball of the chain is included in a different ball. We argue that this implies $B_i \cap \partial B_{i+1} \neq \emptyset$ for all i. In fact, if there exists $y \in B_i \cap B_{i+1}$, take $x \in B_i - B_{i+1}$; since B_i is connected, there is a path from y to x that stays inside B_i ; but $x \notin B_{i+1}$ so the path necessarily leaves B_{i+1}, so it intersects ∂B_{i+1}; hence, it provides a vertex in $B_i \cap \partial B_{i+1}$.

Assume that $B_k^\bullet$ is not discovered in the exploration, so $B_k \not\subset \bigcup_{n \geq 1} \mathcal{C}_n$. Therefore, there exists a first ball $B_{i+1}^\bullet$ of the chain such that $B_{i+1} \not\subset \bigcup_{n \geq 1} \mathcal{C}_n$. This first ball cannot be $B_1^\bullet$ since $B_1 \subset \mathcal{C}_1$ by definition of $\mathcal{C}_1$. By assumption on the chain, ∂B_{i+1} intersects B_i, so the reunion $\bigcup_{n \geq 1} \mathcal{C}_n$ since it contains B_i. It means that $\mathcal{C}_n \cap \partial B_{i+1} \neq \emptyset$ for some $n \geq 1$ (take n minimal for this property). According to the construction of the coupling, this induces

$$\sum_{y \in \partial B_{i+1} \cap \mathcal{C}_n} \omega_3\big((B_{i+1}^\bullet, y)\big) \geq \omega_2(B_{i+1}^\bullet) > 0.$$

Thus, $\omega_3\big((B_{i+1}^\bullet, y)\big)$ is positive for some $y \in \partial B_{i+1} \cap \mathcal{C}_n$, so $B_{i+1} \subset W(\mathcal{C}_n) \subset \bigcup_{n \geq 1} \mathcal{C}_n$, which yields a contradiction. Hence $z \in B_k \subset \bigcup_{n \geq 1} \mathcal{C}_n$. □

In the last paragraph, we show that $(|\mathcal{C}_n|)$ is dominated by a subcritical Galton-Watson process. Together with the result of Lemma 4.6, it concludes the proof of Theorem 4.4.

4.3.3 Comparison with a Galton-Watson process

For every vertex $y \in V$, define the random variable

$$\xi_y := \sum_{B_\bullet^\bullet = (B,\cdot,y)} |B| \mathbf{1}_{\{\omega_3(B_\bullet^\bullet) > 0\}},$$

which is an upper bound for the number of vertices in $W(\{y\})$. The ξ_y for $y \in V$ are independent since they concern disjoint $\omega_3(\cdot)$. For the root, we also define

$$\tilde{\xi}_\rho := \sum_{B^\bullet, \rho \in B} |B| \mathbf{1}_{\{\omega_2(B^\bullet) > 0\}},$$

which is an upper bound for $|\mathcal{C}_1|$. Both the ξ_y and $\tilde{\xi}_\rho$ are stochastically dominated by the same distribution, that we introduce in the following lemma. Then, we show that $(|\mathcal{C}_n|)$ is dominated by a Galton-Watson process that admits this distribution as reproduction law.

Lemma 4.7. *Let $(Z^{j,r})_{j\geq 1, r\geq 1}$ be a collection of independent Poisson random variables such that for all $j \geq 1$, $r \geq 1$, $Z^{j,r}$ has mean $\sum_{n\geq r} \lambda_n$. Set*

$$\xi := \sum_{r\geq 1} c_r \sum_{j=1}^{\Delta c_r^\top} \mathbf{1}_{\{Z^{j,r}>0\}}. \tag{4.6}$$

Then, for all $y \in V$, we have the stochastic dominations

$$\xi_y \preceq \xi \text{ and } \tilde{\xi}_\rho \preceq \xi.$$

Proof. First, observe that if $B^\bullet_\bullet \in \mathcal{B}^\bullet_\bullet$ has radius $r \geq 1$, we have $|B| \leq c_r$. Then, for every $y \in V$,

$$\xi_y \leq \sum_{r\geq 1} c_r \sum_{B^\bullet_\bullet = (\cdot,\cdot,y)} \mathbf{1}_{\{\omega_3(B^\bullet_\bullet)>0\}} \mathbf{1}_{\{\mathrm{rad}(B^\bullet_\bullet)=r\}}.$$

The $\omega_3(B^\bullet_\bullet)$ that appears in the above sum are independent Poisson variables whose parameters depend on $B^\bullet_\bullet$. Since we are only looking at events of the form $\{\omega_3(B^\bullet_\bullet) > 0\}$, we get an upper bound by considering the largest possible parameter. More precisely, we can construct a coupling of ω_3 with ω_3', where the $\omega_3'(B^\bullet_\bullet)$ are independent Poisson variables of parameters $\sum_{n\geq \mathrm{rad}(B^\bullet_\bullet)} \lambda_n$ and such that a.s. for all $B^\bullet_\bullet \in \mathcal{B}^\bullet_\bullet$, $\omega_3(B^\bullet_\bullet) \leq \omega_3'(B^\bullet_\bullet)$. Then,

$$\xi_y \leq \sum_{r\geq 1} c_r \sum_{B^\bullet_\bullet = (\cdot,\cdot,y)} \mathbf{1}_{\{\omega_3'(B^\bullet_\bullet)>0\}} \mathbf{1}_{\{\mathrm{rad}(B^\bullet_\bullet)=r\}}. \tag{4.7}$$

For every $r \geq 1$, the marked balls $B^\bullet_\bullet \in \mathcal{B}^\bullet_\bullet$ with $\mathrm{rad}(B^\bullet_\bullet) = r$ that contribute in (4.7) are of the form $(B_r(x), x, y)$ for some $x \in V$ such that $y \in \partial B_r(x)$. Then, $y \in \partial B_r(x)$ only if there is a neighbour of y at distance less than r from x; there are at most Δ neighbours of y, and $c_r^\top$ possible x for each of them. Therefore, there are at most $\Delta c_r^\top$ non-zero terms in the second sum in (4.7), hence we get the domination by the law of ξ. We can get an explicit coupling between the two by taking, for every $r \geq 1$, the $(Z^{j,r})_{j\geq 1}$ to be the $\omega_3'(B^\bullet_\bullet)$ of the balls $B^\bullet_\bullet$ with radius $\mathrm{rad}(B^\bullet_\bullet) = r$, arranged in an arbitrary order, and other independent Poisson variables to complete the sequence.

The proof for $\tilde{\xi}_\rho$ works similarly. Introduce ω_2' a collection of independent Poisson variables indexed by $\mathcal{B}^\bullet$ such that for all $B^\bullet \in \mathcal{B}^\bullet$, $\omega_2'(B^\bullet) \geq \omega_2(B^\bullet)$ a.s. and $\omega_2'(B^\bullet)$ has mean $\sum_{n\geq \mathrm{rad}(B^\bullet)} \lambda_n$. Thus,

$$\tilde{\xi}_\rho \leq \sum_{r\geq 1} c_r \sum_{B^\bullet, \rho\in B} \mathbf{1}_{\{\omega_2'(B^\bullet)>0\}} \mathbf{1}_{\{\mathrm{rad}(B^\bullet)=r\}}.$$

The second sum contains at most $c_r^\top$ non-zero terms, since the number of vertices $x \in V$ satisfying $\rho \in B_r(x)$ is bounded by $|B_r^\top(\rho)| \leq c_r^\top \leq \Delta c_r^\top$. Hence, the stochastic domination of the law of $\tilde{\xi}_\rho$ by the law of ξ. □

Let $(\xi_k^{(n)})_{k,n\in\mathbf{N}}$ be a collection of independent copies of ξ, $Z_0 = 1$ and for every $n \in \mathbf{N}$, $Z_{n+1} = \sum_{k=1}^{Z_n} \xi_k^{(n)}$. Then, (Z_n) is a Galton-Watson process of reproduction law the law of ξ. We construct a version of (Z_n) that satisfies $|\mathcal{C}_n| \le Z_n$ almost surely.

First, we know by Lemma 4.7 that we can choose $\xi_1^{(0)}$ so that $|\mathcal{C}_1| \le \xi_1^{(0)}$ a.s., by taking

$$\xi_1^{(0)} = \sum_{r\ge 1} c_r \sum_{j=1}^{\Delta c_r^\top} \mathbf{1}_{\{Z(j,r)_1^{(0)}>0\}},$$

where for every $r \ge 1$, the $Z(j,r)_1^{(0)}$ are taken to be the $\omega_2'(B^\bullet)$ (defined in the proof of Lemma 4.7) in arbitrary order with $\mathrm{rad}(B^\bullet) = r$ as well as others independent Poisson variables of parameter $\sum_{n\ge r}\lambda_n$.

Then, assuming the coupling is defined until step n, i.e. we have $|\mathcal{C}_m| \le Z_m$ for all $m \le n$, construct Z_{n+1} in the following way. Call $y_1, \dots, y_{|\mathcal{C}_n|}$ the vertices of $\mathcal{C}_n$ arranged in arbitrary order. We have $|\mathcal{C}_{n+1}| \le \xi_{y_1} + \cdots + \xi_{y_{|\mathcal{C}_n|}}$.

For all $1 \le k \le |\mathcal{C}_n|$, let

$$\xi_k^{(n)} = \sum_{r\ge 1} c_r \sum_{j=1}^{\Delta c_r^\top} \mathbf{1}_{\{Z(j,r)_k^{(n)}>0\}},$$

where for every $r \ge 1$, the $Z(j,r)_k^{(n)}$ are taken to be the $\omega_3'(B_\bullet^\bullet)$ (see the proof of Lemma 4.7) in arbitrary order for $\mathcal{B}_\bullet^\bullet$ of the form $(B,\cdot,y_k)$ with $\mathrm{rad}(B_\bullet^\bullet) = r$, as well as others independent Poisson variables of parameter $\sum_{n\ge r}\lambda_n$. With this choice of $\xi_k^{(n)}$, we have a.s. $\xi_{y_k} \le \xi_k^{(n)}$. Moreover, it is the case that the $(\xi_k^{(n)})_{k\ge1}$ are independent, and independent of the $\xi_k^{(m)}$ for $m < n$ because the $\omega_3'(B_\bullet^\bullet)$ collected at this step have never been revealed before; here we use the fact that the marked balls of $\mathcal{B}_\bullet^\bullet$ are attached to some vertex of their boundary and that the $\mathcal{C}_m$ are disjoint. Note that they are also independent from the $\omega_2'(B^\bullet)$ present in the first layer, because no ball containing ρ can be present in higher layers.

Therefore, we have

$$|\mathcal{C}_{n+1}| \le \xi_{y_1} + \cdots + \xi_{y_{|\mathcal{C}_n|}} \le \xi_1^{(n)} + \cdots + \xi_{|\mathcal{C}_n|}^{(n)} \le \xi_1^{(n)} + \cdots + \xi_{Z_n}^{(n)} = Z_{n+1},$$

where the $\xi_k^{(n)}$ for $k > |\mathcal{C}_n|$ are i.i.d. copies of ξ.

This yields a coupling such that almost surely $|\mathcal{C}_n| \le Z_n$ for all $n \ge 1$. We conclude the proof of Theorem 4.4 using well-known properties of the Galton-Watson processes.

Recalling the definition of ξ from (4.6), we have

$$\mathbf{P}(Z^{r,j} > 0) = 1 - e^{-\sum_{k\ge r}\lambda_k} = 1 - \prod_{k\ge r} e^{-\lambda_k} = 1 - q_{r-1},$$

hence

$$\mathbf{E}[\xi] = \sum_{r\ge 0} c_{r+1}\Delta c_{r+1}^\top (1 - q_r).$$

It is well known that if $\mathbf{E}[\xi] < 1$, the Galton-Watson process (Z_n) extincts almost surely. Therefore, we also get under that condition that almost surely $\mathcal{C}_n$ is empty for large enough n,

so W_ρ is finite. Since the graph is countable and ρ was chosen arbitrarily, then almost surely the wet set W does not contain any infinite connected component. This proves part 1. of Theorem 4.4.

For the second part, we use the fact (see [LP16] Exercise 5.33) that if the reproduction law of a subcritical Galton-Watson process admits finite exponential moment, then the total size of the process has an exponential tail. For all $t > 0$,

$$\begin{aligned}\mathbf{E}[e^{t\xi}] &= \mathbf{E}\left[\prod_{r\geq 1}\prod_{j=1}^{\Delta c_r^\top} e^{tc_r \mathbf{1}_{\{Z^{r,j}>0\}}}\right] \\ &= \prod_{r\geq 1}\prod_{j=1}^{\Delta c_r^\top} \mathbf{E}[e^{tc_r \mathbf{1}_{\{Z^{r,j}>0\}}}] \\ &= \prod_{r\geq 1}\left(e^{tc_r}(1-q_{r-1})+q_{r-1}\right)^{\Delta c_r^\top}.\end{aligned}$$

Then, taking the logarithm and applying the inequality $\log(1+x) \leq x$, we get

$$\begin{aligned}\log \mathbf{E}[e^{t\xi}] &= \sum_{r\geq 0} \Delta c_{r+1}^\top \log\left(e^{tc_{r+1}}(1-q_r)+q_r\right) \\ &\leq \sum_{r\geq 0} \Delta c_{r+1}^\top e^{tc_{r+1}}(1-q_r).\end{aligned}$$

By assumption (4.3), there exists $t > 0$ such that this sum is finite. Then, by comparison with the total size of (Z_n), we get the exponential decay of the size of W_ρ.

Chapter 5

A new bound for the critical point of the FK percolation with $q < 1$

Ce chapitre est une version préliminaire de l'article [BFO25], encore en préparation. Il s'agit d'un travail en collaboration avec Vincent Beffara et Tejas Oke.

Résumé. Ce dernier chapitre est consacré au modèle de percolation FK, et plus particulièrement au régime « sans FKG ». Ce modèle est caractérisé par deux paramètres $p \in [0,1]$ et $q > 0$; p représente comme en percolation de Bernoulli un poids sur les arêtes, et l'on ajoute un poids q sur les clusters. Le portrait de phase du modèle est bien compris, du moins en dimension 2, lorsque $q \geq 1$. Cela correspond au régime dans lequel les inégalités de corrélations « FKG » sont valides. On connaît alors la valeur du point critique, et la nature de la transition de phase (continue pour $q \in [1,4]$, discontinue pour $q > 4$). La situation est complètement différente lorsque $q < 1$, l'existence même du point critique n'étant pas prouvée. Les inégalités de comparaisons avec des mesures produit donnent les meilleurs bornes connues sur ce point critique, ainsi qu'un régime de valeurs des paramètres pour lequel on a unicité de la mesure en volume infini. Dans ce chapitre, on améliore légèrement ces inégalités de comparaisons, pour élargir le régime connu d'unicité et affiner les bornes sur le point critique. Nous introduirons pour cela la dynamique de Glauber du modèle. Les techniques de preuve ne sont pas spécifiques à la dimension 2, sauf pour la preuve d'unicité dans la phase surcritique.

This chapter is a preliminary version of the article [BFO25], which is still in preparation. It is a joint work with Vincent Beffara and Tejas Oke.

Abstract. This last chapter is devoted to the FK percolation model, and more specifically to the regime "without FKG". This model is characterised by two parameters $p \in [0,1]$ and $q > 0$; p represents, as in Bernoulli percolation, a weight on the edges, and we add a weight q on the clusters. The phase diagram of the model is well understood, at least in dimension 2, when $q \geq 1$. This corresponds to the regime in which the "FKG" correlation inequalities are valid. We then know the value of the critical point, and the nature of the phase transition (continuous for $q \in [1,4]$, discontinuous for $q > 4$). The situation is completely different when $q < 1$, as the very existence of the critical point is not proven. Comparison inequalities with product measures give the best known bounds on this critical point, as well as a regime of

parameter values for which we have uniqueness of the infinite-volume measure. In this chapter, we slightly improve these comparison inequalities, to extend the known regime of uniqueness and refine the bounds on the critical point. To this end, we introduce the Glauber dynamics of the model. The proof techniques are not specific to dimension 2, except for the proof of uniqueness in the supercritical phase.

5.1 Introduction and main results

5.1.1 Introduction

The random cluster model or FK-percolation, named after Fortuin and Kasteleyn, is a statistical mechanics model that covers many other models of interest, including Bernoulli percolation, the Ising model, and the Potts model, as its special cases. It is generally defined on a lattice to produce a random subgraph, where we put weights on the included edges and the number of connected clusters formed. It is characterized by two parameters p and q, where p is the 'edge weight' and q is the 'cluster weight'.

More specifically, the random cluster model on a finite graph is a probability measure on 'edge configurations' (characterizing the subset of edges we include in our random subgraph, also called open edges, and the remaining edges called closed). The measure of a configuration is proportional to

$$p^{\#\text{open edges}} \times (1-p)^{\#\text{closed edges}} \times q^{\#\text{open clusters}},$$

where an open cluster (or simply a cluster) is a maximal connected subgraph formed by open edges. It is 'similar' to percolation except that there is a weight on the number of clusters, making it a dependent percolation model. The case $q = 1$ is exactly the classical Bernoulli percolation.

One of the central concepts studied in statistical mechanics is that of *phase transition*. During a phase transition of a given system, certain properties of the system change as a result of the change in external conditions. We are interested in studying the phase diagram of the random cluster model on $\mathbb{Z}^d$ for $d \geq 1$ (although, as we shall see, many of the results we prove carry over to a wider class of graphs). When we study the random cluster model on a box (a finite subgraph of $\mathbb{Z}^d$), these external conditions correspond to the connectivity properties of the boundary (which might affect the number of open clusters which are weighted by the FK measure).

The model cannot be defined directly on $\mathbb{Z}^d$, but we can consider weak limits of finite-dimensional measures, with some boundary condition and the boxes growing to the whole lattice. Thus, the question of phase transition boils down to identifying the set of parameters for which we have uniqueness of the limiting measure in infinite volume. Having such a uniqueness would be the same as saying that changing external conditions does not affect the (limiting) measure.

From the percolation point of view, phase transition relates to the presence or not of an infinite cluster in the random subgraph formed by open edges. It is generally true for Bernoulli percolation ($q = 1$) that for p small enough, almost surely all the clusters are finite, whereas for p large enough, there is almost surely an infinite cluster. More precisely, on any infinite graph G, we can define a critical parameter $p_c(G) \in [0, 1]$ above which there is almost surely an infinite cluster for the Bernoulli percolation on G, but below which there is almost surely

no infinite cluster. The percolation is said to be supercritical in the first case, and subcritical in the second. It is known that for all $d \geq 2$, $p_c(\mathbb{Z}^d) \in (0,1)$, so that there is a subcritical and a supercritical phase (in contrast, $p_c(\mathbb{Z}) = 1$).

For the random cluster model with $q \geq 1$, the FKG inequalities enable to define an analogous critical parameter, and the exact value is known in dimension 2 (see Section 5.1.4). For $q < 1$, the situation is less understood due to the lack of these FKG inequalities. The definition of the critical point is not rigorous and only crude bounds on its value are known, and uniqueness of the infinite-volume measure is only known for a restricted range of parameters. The main goal of this paper is to contribute to the study of the model in this $q < 1$ regime, improving the existing bounds on the critical parameter and extending the known regime of uniqueness.

5.1.2 Definition of the model and comparison inequalities

In this section, we will give concrete definition of the model on finite graphs with boundary conditions, and see some of its properties. A complete treatment can be found in [Gri06].

Let $G = (V, E)$ be a finite connected graph, and ∂G a subset of vertices, called the *boundary* of G. Intuitively, one should think of G as a finite box of an infinite (connected, locally finite) graph, e.g. the $d-$dimensional lattice, ∂G being the set of vertices of G having at least one neighbor outside G. Let $p \in [0,1], q \in (0,\infty)$ be the two parameters of the model. A configuration ω of the random cluster model on G is an element ω of $\Omega = \{0,1\}^E$. An edge $e \in E$ is said to be open if $\omega(e) = 1$ and closed otherwise. Configurations can naturally be identified with subgraphs of G with the same vertex set, with edge set corresponding to open edges. The inclusion (for the subgraphs) yields a natural partial order on Ω.

Two vertices x and y are said to be connected in a certain configuration if there exists a path of open edges from x to y. The connectivity properties of a configuration will depend on the *boundary condition* that we impose on G. Let $\alpha \in \Pi(\partial G)$ be a partition of ∂G. Two vertices that are in the same block in α are said to be *wired*, which intuitively means that they are connected externally without using edges from G. The graph $G \cup \alpha$ is obtained from G by identifying wired vertices; this identification, or *wiring*, can create self-loops in the graph, which are anyways not excluded from our definition. For any configuration ω, denote by $k(\omega, \alpha)$ the number of connected components in $G \cup \alpha$ — or equivalently, the connected components obtained when starting from those of ω and then recursively identifying any two components sharing vertices in the same block of α; it means that the open clusters of two wired vertices only count as 1 in $k(\omega, \alpha)$. The *random cluster measure* or *FK measure* on G with parameters p and q and boundary condition α is the probability measure on Ω defined by

$$\phi^{\alpha}_{G,p,q}(\omega) := \frac{1}{Z^{\alpha}_{G,p,q}} \left(\prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right) q^{k(\omega,\alpha)}, \ \omega \in \{0,1\}^E \tag{5.1}$$

where $Z^{\alpha}_{G,p,q}$ is the normalizing constant, called the partition function. The case $q = 1$ is the classical Bernoulli bond percolation. There are two extremal boundary conditions: the first, called *free boundary condition*, is obtained by taking the partition $(\{x\})_{x \in \partial G}$, which means that nothing happens outside G; the second corresponds to the opposite situation where all vertices are wired together, i.e. we take $\{\partial G\}$ as partition, and is called the *wired boundary condition*.

An event A is said to be *increasing* if it is preserved by the addition of open edges, i.e. if for all $\omega \leq \omega'$, $\omega \in A$ implies $\omega' \in A$. For two measures μ, μ' on $\{0,1\}^E$, say that μ' *stochastically dominates* μ if for every increasing event A, $\mu(A) \leq \mu'(A)$; in this case, write $\mu \preceq \mu'$. Equivalently, $\mu \preceq \mu'$ if and only if there is a coupling (ω, ω') of μ and μ' such that $\omega \leq \omega'$ almost surely.

For every configuration $\omega \in \{0,1\}^E$ and any edge $e \in E$, denote by $\omega_{\langle e \rangle}$ the restriction of ω to $E \setminus \{e\}$; K_e^α denotes the event that the endpoints of e are connected by an open path not using e, but possibly using an imaginary edge between two wired vertices of the boundary. Note that the event K_e^α is independent of the state of the edge e. From (5.1), one can deduce that for every $e \in E$,

$$\phi^\alpha_{G,p,q}\left(\omega(e) = 1 \mid \omega_{\langle e \rangle}\right) = \begin{cases} p & \text{if } \omega_{\langle e \rangle} \in K_e^\alpha \\ \frac{p}{p+q(1-p)} & \text{if } \omega_{\langle e \rangle} \notin K_e^\alpha \end{cases}. \tag{5.2}$$

For simplicity of notation, we write $p' := \frac{p}{p+q(1-p)}$. It turns out that $\phi^\alpha_{G,p,q}$ is the unique probability measure on Ω having the above conditional probabilities. If G is a tree and α is the free boundary condition, it is clear that K_e^α cannot be achieved, so the FK measure is simply the Bernoulli product measure with parameter p'. In the other way, if all the vertices are wired together (or if V is already reduced to a singleton), K_e^α is always achieved and the FK measure is the Bernoulli product measure with parameter p.

From (5.2), it can be seen that we have the following comparison inequalities of the random cluster measure with product measures:

$$\mathbb{P}^G_{\min(p,p')} \preceq \phi^\alpha_{G,p,q} \preceq \mathbb{P}^G_{\max(p,p')} \tag{5.3}$$

where for $s \in [0,1]$, $\mathbb{P}^G_s$ denotes the Bernoulli bond percolation measure on G of parameter s. Note that $p < p'$ if and only if $q < 1$ (ignoring the trivial cases $p = p' = 0$ and $p = p' = 1$).

More generally, for $(p_e)_{e \in E} \in [0,1]^E$, let $\mathbb{P}^G_{(p_e)}$ be the product measure associated with the inhomogeneous Bernoulli percolation on G, where edge e is open with probability p_e. We prove that the comparison inequalities (5.3) can be enhanced:

Theorem 5.1 (Enhanced comparison inequalities)**.** *Let $G = (V, E)$ be a finite connected graph with boundary ∂G, and $\alpha \in \Pi(\partial G)$ be a boundary condition. Assume that $G \cup \alpha$ is not a tree and contains at least two vertices (after wiring). Then, for all $p \in (0,1)$ and $q \neq 1$, there exist $(\varepsilon_e)_{e \in E} \in [0,1]^E$ and $(\varepsilon'_e)_{e \in E} \in [0,1]^E$, which are both non-identically 0, such that*

$$\mathbb{P}^G_{\min(p,p')+\varepsilon_e} \preceq \phi^\alpha_{G,p,q} \preceq \mathbb{P}^G_{\max(p,p')-\varepsilon'_e}.$$

Remark 5.2. *The condition $G \cup \alpha$ is not a tree is satisfied if G itself is not a tree or if α is not the free boundary condition; the condition $G \cup \alpha$ contains two vertices amounts to the fact that $G \cup \alpha$ is not reduced to a single point with self-loops. For planar graphs, these two conditions are dual of each other.*

We aim to extend this result to infinite graphs, but one should notice before that the definition of the random cluster model on an infinite graph is not immediate. If $G = (V, E)$ is an infinite, connected, locally finite graph, a way to obtain an FK measure on G is by taking the so called *thermodynamic limit*. More precisely, say that a probability measure ϕ

on $\Omega = \{0,1\}^E$ is an *FK limit measure* on G with parameters p, q if there exist an increasing sequence of subgraphs $\Lambda_n \uparrow G$ and a sequence of partitions $\alpha_n \in \Pi(\partial\Lambda_n)$ such that

$$\phi = \lim_{n\to\infty} \phi^{\alpha_n}_{\Lambda_n,p,q} \tag{5.4}$$

(for the weak convergence). By compactness of Ω, there always exists such an FK limit measure for any $p \in [0,1]$ and $q > 0$. It is clear that (5.3) passes to the limit so that the comparison inequalities hold for any FK limit measure. Therefore, if $\max(p,p')$ is less than $p_c(G)$ the critical parameter of the classical Bernoulli percolation on G, it follows that every FK limit measure is dominated by a subcritical percolation, so it is concentrated on configurations without infinite clusters. The dual statement holds for the supercritical case.

The main reason for proving an enhanced version of these comparison inequalities is to enlarge the set of parameters for which we have such a domination by a subcritical percolation. To obtain a version of Theorem 5.1 for infinite graphs, one needs to ensure that the (ε_e) and (ε'_e) do not vanish when the size of the graph goes to infinity. In fact, we prove more quantitative statements in Section 5.2 (see Propositions 5.10 and 5.11), with explicit values for (ε_e) and (ε'_e). Moreover, they will be positive for a positive density of edges. The theory of "essential enhancements" developed by Aizenman and Grimmett [AG91] then states that $\mathbb{P}_{\max(p,p')-\varepsilon'_e}$ is still subcritical if $\max(p,p')$ is slightly above $p_c(G)$, likewise for the supercritical case. We therefore get the following result, which is illustrated by Figure 5.1:

Theorem 5.3 (Extended subcritical and supercritical phases)**.** *Let G be an infinite, locally finite, connected, quasi-transitive graph, and assume that G contains an edge which is on a cycle and on a doubly-infinite path. Then for all $q \neq 1$, there exists $\delta, \delta' > 0$ such that:*

1. *If $\max(p,p') < p_c(G) + \delta'$, then there exists a subcritical percolation which dominates all the FK measures on G; in particular, for each, a.s. there is no infinite cluster.*
2. *If $\min(p,p') > p_c(G) - \delta$, then there exists a supercritical percolation dominated by all the FK measures on G; in particular, for each, a.s. there is an infinite cluster.*

Remark 5.4. *The above result is stated for quasi-transitive graphs, which are those for which the automorphism group partitions the graph into finitely many orbits. We do not believe that this assumption is strictly necessary. The crucial geometric properties on the graph that we are using in the proof are on one hand that it has bounded degree, and on the other hand that it contains special edges which are on a doubly-infinite path and on a cycle of bounded length, and moreover that these special edges are sufficiently spread in the graph (more precisely, we need that each edge of the graph is at bounded distance from such a special edge). We thus believe that the statements remains true for some non-quasi-transitive graphs.*

There is actually another way of defining infinite-volume measures for the random cluster model, given by the DLR approach. The above theorem extends to this new class of FK measures, that we properly define in the next section. Afterwards, we will be in a position to discuss uniqueness of the infinite-volume measure.

5.1.3 Specifications for the Random Cluster model

Before formally stating our main result, a few general considerations on Gibbs measures for the FK model are needed. Let $G = (V,E)$ be an infinite, connected, locally finite graph, and

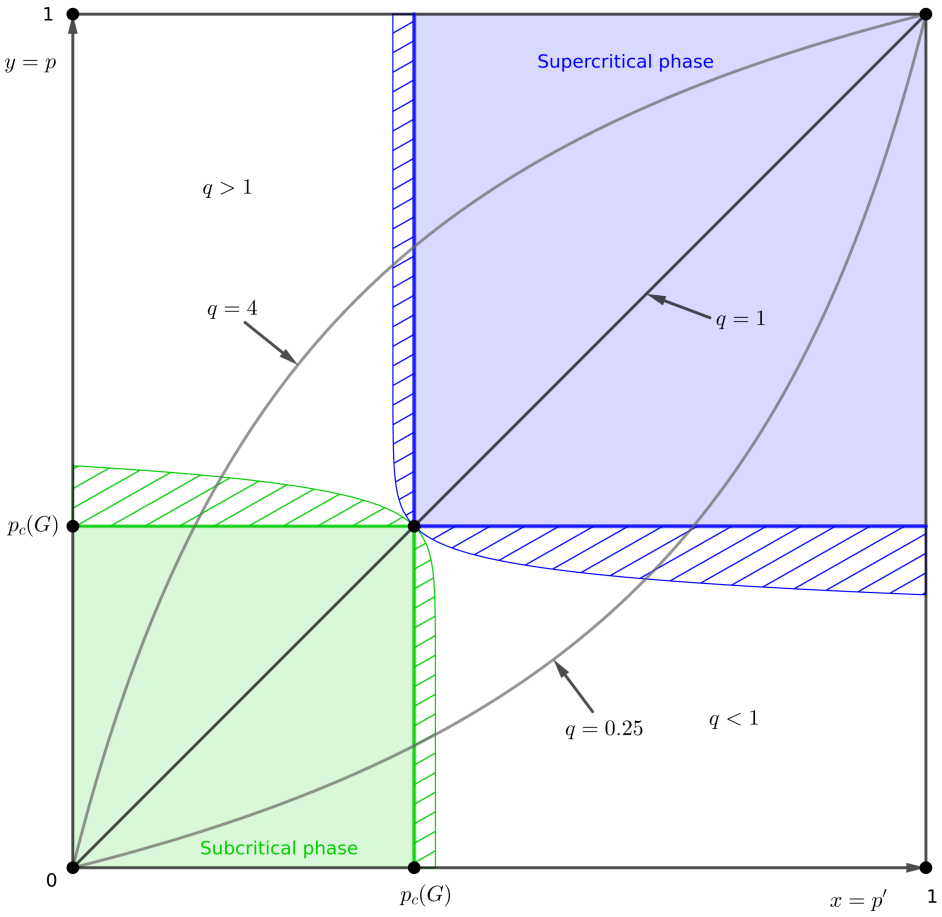


Figure 5.1: Phase diagram of the FK model on a general (quasi-transitive, not tree) graph.

fix $\Lambda \subset G$ a finite subgraph; denote by $\partial\Lambda$ the set of vertices in Λ having at least one neighbor outside it. A configuration of the random-cluster model inside Λ is an element of $\{0,1\}^{E(\Lambda)}$, where $E(\Lambda)$ denotes the set of edges whose both endpoints are in Λ.

By a standard computation, the probability measures $\phi^{\alpha}_{\Lambda,p,q}$ satisfy the following compatibility relation: given $\Lambda \subset \Delta$, a partition α on $\partial\Delta$ and a configuration η on $\Delta\backslash\Lambda$, the conditional distribution, under $\phi^{\alpha}_{\Delta,p,q}$, of $\omega_{|\Lambda}$ given $\omega_{|(\Delta\backslash\Lambda)} = \eta$ is equal to $\phi^{\alpha'}_{\Lambda,p,q}$ where $\alpha' =: \pi(\alpha,\eta)$ is the partition of $\partial\Lambda$ induced by η and α. In other words, the $\phi^{\alpha}_{\Lambda,p,q}$ behave like a specification, with one important difference: it is not the case in general that every partition α on $\partial\Lambda$ can be generated by a configuration outside Λ.

To form a specification in the usual sense, one needs to fix a *convention* to form partitions from configurations, that is, a family of measurable maps

$$\Phi_\Lambda : \{0,1\}^{E(\Lambda)^c} \to \Pi(\partial\Lambda)$$

so that the conditional distribution inside Λ given a configuration ω outside Λ can be defined as $\phi^{\Phi_\Lambda(\omega)}_{\Lambda,p,q}$. To preserve the compatibility relation above, such a convention needs to itself satisfy the following: for every $\Lambda \subset \Delta$ and any configurations ω on Δ^c and η on $\Delta \setminus \Lambda$,

$$\Phi_\Lambda(\omega \cup \eta) = \pi(\Phi_\Delta(\omega), \eta).$$

This implies in particular that

- Any two vertices of $\partial\Lambda$ that are connected by a path in ω outside Λ are in the same block of $\Phi_\Lambda(\omega)$;
- Any two vertices of $\partial\Lambda$ belonging to two disjoint clusters of ω in Λ^c, at least one of which being finite, are in different blocks of $\Phi_\Lambda(\omega)$.

In other words, Φ_Λ only describes which pairs of disjoint infinite clusters outside Λ are "joined at infinity," in a measurable way. We will say that a probability measure ϕ on configurations in G is an *FK Φ-Gibbs measure* with parameters p and q (or shortly is Φ-Gibbs) if it is a Gibbs measure for the specification $\left(\phi^{\Phi_\Lambda(\cdot)}_{\Lambda,p,q}\right)_\Lambda$, that is, ϕ satisfies the DLR equations:

$$\forall \Lambda \Subset G, \forall \xi \in \{0,1\}^E,\ \phi(\cdot \mid \mathcal{F}_{E(\Lambda)^c})(\xi) = \phi^{\Phi_\Lambda(\xi)}_{\Lambda,p,q}(\cdot) \quad \text{for } \phi - \text{a.e. } \xi.$$

It is a general fact that a positive specification is characterized by its singleton part, see [Geo11] Theorem 1.33. Hence, for ϕ to be Φ-Gibbs, it is enough to have for $\phi-$almost every ω,

$$\phi(\omega(e) = 1 \mid \mathcal{F}_{\langle e\rangle})(\omega) = \begin{cases} p & \text{if } \omega \in K^\Phi_e \\ \frac{p}{p+q(1-p)} & \text{if } \omega \notin K^\Phi_e \end{cases}$$

where $\mathcal{F}_{\langle e\rangle}$ is the $\sigma-$algebra generated by the states of all edges except e, and K^Φ_e is the event that both endpoints of e lie in the same block in the partition $\Phi_{\{e\}}(\omega_{\langle e\rangle})$; it means that either they are connected in the configuration $\omega_{\langle e\rangle}$ or that they belong to two distinct infinite clusters that are joined at infinity by the convention Φ.

Two important examples of such are the *free convention* Φ^f for which two vertices of $\partial\Lambda$ are in the same block of $\Phi^f_\Lambda(\omega)$ if and only if they are connected by a path in ω (*i.e.*, nothing happens at infinity) and the *wired convention* Φ^w for which two vertices of $\partial\Lambda$ are in the same

block of $\Phi^w(\omega)$ if and only if they are connected by a path in ω or they are both in infinite clusters of ω (*i.e.*, all infinite clusters are merged). If G is a tree, every measurable partition of the end of G induces a different convention.

In general, there is no reason that the set of all Φ-Gibbs measures is the same for different conventions. For instance, on a tree, the only Φ^f-Gibbs measure is the product measure with parameter p' while for Φ^w, every measure having doubly-infinite clusters will be non-product (and such measures do exist for p close enough to 1). On the other hand, one sees directly from the definition above that if a measure ϕ is Φ-Gibbs and is such that ϕ-a.s. there is at most one infinite cluster, then it is Ψ-Gibbs for every other convention Ψ as well: typically, on an amenable lattice, this is the case for all translation-invariant measures, by an argument of Burton and Keane [BK89]. It is also true (see [Gri06] Theorem 4.31) that if an FK limit measure satisfies that there is at most one infinite cluster a.s., then it is an FK Gibbs measure (for any convention).

We believe that weak limits of configurations on Λ with empty boundary condition (for which all vertices of $\partial\Lambda$ are in different blocks) are Φ^f-Gibbs, and that weak limits of configuration with wired boundary condition (for which all vertices of $\partial\Lambda$ are in the same block) are Φ^w-Gibbs; and also that these two notions are in general different. More generally, we believe that this is a direction worth exploring, where interesting behavior will be found in the case $q < 1$; but it is outside the scope of the present article.

In the following, for a convention Φ, we denote by $\mathcal{G}^{\Phi}_{p,q}$ the set of $\Phi-$Gibbs measures for the random cluster model with parameters p and q on the graph G. Let $\mathcal{W}_{p,q}$ be the set of FK (weak) limit measures, and

$$\mathrm{FK}(p,q) = \bigcup_{\Phi \text{ a convention}} \mathcal{G}^{\Phi}_{p,q} \cup \mathcal{W}_{p,q}$$

the set of all FK measures on G with parameter p and q. It is clear from the definition of a Gibbs measure that $\mathcal{G}^{\Phi}_{p,q}$ is always a convex set (or empty). We write $\operatorname{ex}\mathcal{G}^{\Phi}_{p,q}$ for the set of its extremal elements, which are called extremal Gibbs measure. It is a general fact that extremal Gibbs measures can be obtained as a thermodynamic limit along any sequence of boxes, with some boundary conditions, see [FV18]. We gather this property together with another aforementioned fact about FK measures in the following Proposition.

Proposition 5.5. *For all $p \in [0,1]$ and $q > 0$, we have:*

1. *For any convention Φ, every $\phi \in \operatorname{ex}\mathcal{G}^{\Phi}_{p,q}$ satisfies*

$$\phi = \lim_{n\to\infty} \phi^{\Phi_{\Lambda_n}(\omega)}_{\Lambda_n,p,q} \text{ for } \phi-a.e.\ \omega$$

   for all sequence of boxes $\Lambda_n \uparrow G$; in particular, $\operatorname{ex}\mathcal{G}^{\Phi}_{p,q} \subset \mathcal{W}_{p,q}$.

2. *If $\phi \in \mathrm{FK}(p,q)$ is such that $\phi-$a.s. there is at most one infinite cluster, then $\phi \in \mathcal{G}^{\Phi}_{p,q}$ for all Φ.*

For the purpose of what follows, we will say that *the random-cluster at parameters p and q on the graph G exhibits uniqueness* if for every choice Φ of convention, the set $\mathcal{G}^{\Phi}_{p,q}$ has a single

element, and that it exhibits *strong uniqueness* if $\mathrm{FK}(p,q)$ has a single element. According to item (1) of the above Proposition, strong uniqueness is equivalent to the uniqueness of the FK limit measure, as $|\operatorname{ex}\mathcal{G}^{\Phi}_{p,q}| = 1$ implies $|\mathcal{G}^{\Phi}_{p,q}| = 1$ (this is an application of the stronger fact that $\mathcal{G}^{\Phi}_{p,q}$ is a simplex, see [Geo11] Theorem 7.26).

For instance, the usual Peierls argument shows that if G has bounded degree and $q > 0$ is fixed, strong uniqueness occurs for every p small enough (and the unique Gibbs measure is concentrated on non-percolating configurations). On the other hand on a regular tree of degree at least 3, for p close enough to 1 and $q > 1$, uniqueness occurs but strong uniqueness does not. This is the content of Theorems 6.3 and 6.4 in [GJ05].

In the next Section, we state our main result about the strong uniqueness of the random cluster model. The two properties of Proposition 5.5 will be crucial in the proof of this result.

5.1.4 Phase diagram of the FK model

Let $G = (V, E)$ be an infinite, locally finite, connected graph. When $q \geq 1$, the finite dimensional FK measures $\phi^{\alpha}_{\Lambda,p,q}$ are strongly positively-associated for all finite subgraph $\Lambda \Subset G$ and $\alpha \in \Pi(\partial\Lambda)$, which means in particular that the *FKG inequality* holds: for any two increasing events A and B,

$$\phi^{\alpha}_{G,p,q}(A \cap B) \geq \phi^{\alpha}_{G,p,q}(A)\phi^{\alpha}_{G,p,q}(B).$$

The FKG inequality enables to show the existence of two particular infinite-volume measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$, obtained as weak limits of the free and wired boundary conditions respectively. One can also show that they both are invariant by automorphism of the graph.

The aforementioned Burton-Keane argument then implies that they are $\Phi-$Gibbs for any convention Φ if the graph G is amenable. Furthermore, in this case, one has the following monotonicity property: every $\phi \in \mathrm{FK}(p,q)$ satisfies the stochastic dominations

$$\phi^0_{p,q} \preceq \phi \preceq \phi^1_{p,q}. \tag{5.5}$$

Strong uniqueness and uniqueness are therefore the same, and hold if and only if $\phi^0_{p,q} = \phi^1_{p,q}$. It is known that for any $q \geq 1$, this fails for at most countably many edge-weights p (Theorem 4.63 in [Gri06]). It is then possible to show the existence of a *critical parameter* $p_c(q) \in (0,1)$ for $q \geq 1$ such that every $\phi \in \mathrm{FK}(p,q)$ satisfies

$$\phi(\exists \text{ an infinite cluster}) = \begin{cases} 0 & \text{if } p < p_c(q) \\ 1 & \text{if } p > p_c(q) \end{cases}.$$

In dimension 2, the planar duality allows computing the critical point, and in particular it is shown in [BDC12] that for $G = \mathbb{Z}^2$, the self-dual point is critical, that is

$$\forall q \geq 1,\ p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Moreover, we know precisely the regime where uniqueness holds. Namely, when $q \in [1,4]$ there is a unique infinite volume measure for all values of p (see [DCST17]); when $q > 4$, there is uniqueness everywhere but at $p_c(q)$ (see [DCGH+21]).

For $q < 1$ however, the existence of the critical point is an open question, and the wider regime where uniqueness is known is obtained by the comparison inequalities. The stochastic

dominations (5.3) observed for finite graphs extend directly (we give a proof of this fact in Section 5.3 for our enhanced comparison inequalities) to all infinite volume measures: for every FK measure ϕ on G, one has

$$\mathbb{P}^G_{\min(p,p')} \preceq \phi \preceq \mathbb{P}^G_{\max(p,p')} \tag{5.6}$$

For $q < 1$, assuming that $p_c(q)$ exists, this implies that $\frac{p_c(q)}{p_c(q)+q(1-p_c(q))} \geq p_c(G) \geq p_c(q)$. For $G = \mathbb{Z}^2$, the value of $p_c(G) = p_c(1)$ is known to equal $1/2$ [Kes80], which gives

$$\forall q < 1, \ \frac{q}{1+q} \leq p_c(q) \leq \frac{1}{2}.$$

Theorem 5.3 implies that these two putative inequalities are strict.

When the product measure $\mathbb{P}^G_{\max(p,p')}$ is subcritical, one also obtains uniqueness of the FK measure:

Theorem 5.6 (Theorem 5.119 in [Gri06]). *For all $q > 0$, one has strong uniqueness when $\max(p, p') < p_c(G)$ and the unique FK measure is subcritical.*

The result still holds at $p_c(G)$ if one knows that $\theta(p_c) := \mathbb{P}^G_{p_c(G)}(\exists \text{ an infinite cluster}) = 0$. We prove this without assuming that $\theta(p_c) = 0$, and in fact that it is true slightly above $p_c(G)$, for graphs satisfying the assumptions of Theorem 5.3.

Theorem 5.7 (Strong uniqueness in the subcritical regime). *Let G be a graph satisfying the hypotheses of Theorem 5.3. For all $q \neq 1$, there exists $\delta' > 0$ such that one has strong uniqueness when $\max(p, p') < p_c(G) + \delta'$ and the unique FK measure is subcritical.*

The δ' in the above statement is the same as in Theorem 5.3.

There is no satisfactory analogue to the above theorem for the supercritical regime. In fact, a proof of uniqueness in the case of domination from below by a supercritical product measure is missing. In arbitrary dimension, uniqueness can therefore be proved only for very large p. However, for plane graphs, the duality enables to transpose the subcritical statements of Theorems 5.6 and 5.7 to the supercritical regime (we refer to the last Section 5.4.2 for the definitions regarding planar duality). Strong uniqueness on $\mathbb{Z}^2$ was therefore already known for $\min(p, p') \geq 1/2$, regardless of the value of $q > 0$. As before, we slightly improve this range of parameters, for $\mathbb{Z}^2$ and more general plane graphs:

Theorem 5.8 (Strong uniqueness in the supercritical regime in 2D). *Let G be as in Theorem 5.7; we assume furthermore that G is embedded in $\mathbb{R}^2$, and that its dual G^* is locally finite. For all $q \neq 1$, there exists $\delta'' > 0$ such that one has strong uniqueness when $\min(p, p') > 1 - p_c(G^*) - \delta''$. Moreover, almost surely for the unique FK measure, there is exactly one infinite cluster.*

Remark 5.9. *When $p > 1 - p_c(G^*)$, there is $\mathbb{P}^G_p$−a.s. exactly one infinite cluster, for the Bernoulli percolation of parameter p on G. The fact that there is at most one infinite cluster follows from the fact that the dual configuration is subcritical (as two disjoints infinite clusters would be separated by an infinite dual cluster). The fact that there* is *an infinite cluster is harder to prove. On an amenable graph, since by [BK89] there is always at most one infinite cluster under $\mathbb{P}^G_p$, we have $1 - p_c(G^*) = p_c(G)$.*

Obviously, for $G = \mathbb{Z}^2$, Theorems 5.7 and 5.8 are only new in the regime $q < 1$ (thus $\min(p, p') = p$), but we are unaware of previously known statements with this level of generality on the graph, even for $q > 1$.

In Section 5.2, we prove Theorem 5.1, by introducing the *Glauber dynamics* of the model. In Section 5.3, we extend our new comparison inequalities to the infinite-volume setting, providing a proof of Theorem 5.3. Finally, Theorem 5.7 is proved in Section 5.4.1. Theorem 5.8 follows, by a duality argument developed in Section 5.4.2.

5.2 Enhanced comparison inequalities for finite graphs

Let $G = (V, E)$ be a finite graph, $\partial G \subset V$ and $\alpha \in \Pi(\partial G)$ a boundary condition. Recall that for every $e \in E$,

$$\phi^{\alpha}_{G,p,q}\left(\omega(e) = 1 \mid \omega_{\langle e\rangle}\right) = \begin{cases} p & \text{if } \omega_{\langle e\rangle} \in K^{\alpha}_{e} \\ p' & \text{if } \omega_{\langle e\rangle} \notin K^{\alpha}_{e} \end{cases} \tag{5.7}$$

where $p' = \frac{p}{p+q(1-p)}$ and K^{α}_{e} is the event that the endpoints of e are connected by an open path not using e. We will deduce Theorem 5.1 from the two following propositions.

For all $e \in E$, a subset $\chi \subset E \setminus \{e\}$ of edges is called a *cutset* for e if the endpoints of e are in two different connected components in $G \cup \alpha - (\chi \cup \{e\})$; equivalently, each cycle in $G \cup \alpha$ passing through e contains an edge in χ. Observe that if all the edges of a cutset for e are closed, then the event K^{α}_{e} is not achieved. In fact, the existence of a closed cutset is exactly the complementary event of K^{α}_{e}.

Proposition 5.10. *Let $\widehat{E} \subset E$ be such that for every edge $e \in \widehat{E}$, there exists a cutset for e included in $E \setminus \widehat{E}$. Denote by $Q_{\widehat{E}}(e)$ the collection of such cutsets, and for all $e \in \widehat{E}$, let $\widehat{\varepsilon}_e := |p' - p|\mathbb{P}^{G}_{\max(p,p')}(\exists \chi \in Q_{\widehat{E}}(e), \chi \text{ is closed})$. Then,*

1. *If $p < p'$, then $\mathbb{P}^{G}_{p+\widehat{\varepsilon}_e \mathbf{1}_{\{e\in\widehat{E}\}}} \preceq \phi^{\alpha}_{G,p,q}$.*
2. *If $p > p'$, then $\phi^{\alpha}_{G,p,q} \preceq \mathbb{P}^{G}_{p-\widehat{\varepsilon}_e \mathbf{1}_{\{e\in\widehat{E}\}}}$.*

Proposition 5.11. *Let $\widetilde{E} \subset E$ be such that for every edge $e \in \widetilde{E}$, there exists a path connecting both endpoints of e included in $E \setminus \widetilde{E}$. Denote by $\Gamma_{\widetilde{E}}(e)$ the collection of such paths, and for all $e \in \widetilde{E}$, let $\widetilde{\varepsilon}_e := |p' - p|\mathbb{P}^{G}_{\min(p,p')}(\exists \gamma \in \Gamma_{\widetilde{E}}(e), \gamma \text{ is open})$ Then,*

1. *If $p < p'$, then $\phi^{\alpha}_{G,p,q} \preceq \mathbb{P}^{G}_{p'-\widetilde{\varepsilon}_e \mathbf{1}_{\{e\in\widetilde{E}\}}}$.*
2. *If $p > p'$, then $\mathbb{P}^{G}_{p'+\widetilde{\varepsilon}_e \mathbf{1}_{\{e\in\widetilde{E}\}}} \preceq \phi^{\alpha}_{G,p,q}$.*

Remark 5.12. *Notice that in Proposition 5.10, the condition on $\widehat{E}$ is satisfied if $\widehat{E}$ never contains two consecutive edges. In Proposition 5.11, the condition on $\widetilde{E}$ is satisfied if $(G, E\setminus\widetilde{E})$ is connected.*

Before proving Proposition 5.10 and 5.11, we introduce the main tool of both proofs, that is the Glauber dynamics of the random cluster model. To each edge $e \in E$, we independently assign an i.i.d. sequence $(\xi^{e}_{1}, \xi^{e}_{2}, \ldots)$ of exponential random variables with mean 1, and an

independent i.i.d. sequence $(U_1^e, U_2^e, \ldots)$ of uniform random variables on $[0,1]$. The idea is to update the state of an edge when its 'exponential clock rings' according to its conditional distribution knowing the rest of the configuration. For $e \in E$ and $k = 1, 2, \ldots$, let $\tau_k^e := \xi_1^e + \ldots + \xi_k^e$ so that $(\tau_1^e, \tau_2^e, \ldots)$ are the jump times of a Poisson process of rate 1, and will be the update times of edge e.

For every $\omega \in \{0,1\}^E$, define a continuous-time $\{0,1\}^E$-valued Markov chain $(X_t^\omega)_{t\geq 0}$ with initial configuration $X_0^\omega = \omega$. The evolution of X^ω is governed as follows: the state of an edge e cannot change except possibly at times $\tau_1^e, \tau_2^e, \ldots$ and at time $t = \tau_k^e$ for some $k \geq 1$, let

$$X_t^\omega(e) = \begin{cases} 1 & \text{if } U_k^e < p \text{ and } X_{t-}^\omega \in K_e^\alpha \\ 1 & \text{if } U_k^e < p' \text{ and } X_{t-}^\omega \notin K_e^\alpha \\ 0 & \text{otherwise} \end{cases}. \tag{5.8}$$

It is easy to see that this defines an irreducible and aperiodic Markov chain. According to (5.7), one can observe that $\phi_{p,q}^G$ is invariant for this dynamics. Therefore, we get that X_t^ω converges in distribution towards $\phi_{G,p,q}^\alpha$ as t goes to infinity.

Note that although the dynamics is not local —one might need to explore the configuration far away from e to determine if the event K_e is satisfied or not—, it is well defined because the graph is finite. Also, since we are working with exponential clocks which are continuous, we have that almost surely, $\tau_k^e \neq \tau_j^{e'}$ for all j, k when $e \neq e'$, so that we are never updating two edges at the same time.

Proof of Proposition 5.10. Let $\widehat{E}$ be as in Proposition 5.10; edges of $\widehat{E}$ will be called *enhanced edges*. The other edges are called *auxiliary* edges.

Define two Markov processes $(X_t)_{t\geq 0}$ and $(Y_t, Z_t)_{t\geq 0}$ taking values in the set of configurations and of pairs of configurations respectively; $(X_t)_{t\geq 0}$ will be a modified version of the Glauber dynamics of the random cluster model introduced above, while $(Y_t, Z_t)_{t\geq 0}$ will contain two approximations. They will be coupled in such a way that X_t stays a.s. above Y_t and below Z_t for all $t > 0$ for the usual partial order on $\{0,1\}^E$. Let $\omega \in \{0,1\}^E$ and set $X_0 = Y_0 = Z_0 = \omega$. As in the Glauber dynamics, for process $(X_t)_{t\geq 0}$, edges are resampled at their update times following the rule (5.8). The only difference is that, after each resampling of an enhanced edge, we also resample all the auxiliary edges. Thus, for $e \in \widehat{E}$, in addition to the $(U_k^e)_{k\geq 1}$, let $(U_k^e[1], \ldots U_k^e[A])_{k\geq 1}$ be a collection of i.i.d. random variables, distributed uniformly on $[0,1]$, where $A = |E \setminus \widehat{E}|$ is the number of auxiliary edges. Enumerate by $e_1, \ldots, e_A$ the auxiliary edges.

Therefore, for every $e \in \widehat{E}$ and $k \geq 1$, at time τ_k^e, we execute the following steps:

(1) Let $X_0'(e) = \begin{cases} 1 & \text{if } U_k^e < p \text{ and } X_{\tau_k^e-} \in K_e^\alpha \\ 1 & \text{if } U_k^e < p' \text{ and } X_{\tau_k^e-} \notin K_e^\alpha, \text{ and } X_0' = X_{\tau_k^e-} \text{ outside } e \\ 0 & \text{otherwise} \end{cases}$

(2) Let $X_1'(e_1) = \begin{cases} 1 & \text{if } U_k^e[1] < p \text{ and } X_0' \in K_{e_1}^\alpha \\ 1 & \text{if } U_k^e[1] < p' \text{ and } X_0' \notin K_{e_1}^\alpha, \text{ and } X_1' = X_0' \text{ outside } e_1 \\ 0 & \text{otherwise} \end{cases}$

$\vdots$

$(A+1)$ Let $X'_A(e_A) = \begin{cases} 1 & \text{if } U_k^e[A] < p \text{ and } X'_{A-1} \in K^\alpha_{e_A} \\ 1 & \text{if } U_k^e[A] < p' \text{ and } X'_{A-1} \notin K^\alpha_{e_A}, \text{ and } X'_A = X'_{A-1} \text{ outside } e_A; \\ 0 & \text{otherwise} \end{cases}$

finally let $X_{\tau_k^e} = X'_A$ (whereas for $e \notin \widehat{E}$, we only execute step 1 and set $X_{\tau_k^e} = X'_0$). Since we update only the state of the edges according to their conditional distribution, it remains true that $\phi^\alpha_{G,p,q}$ is the stationary distribution of the process $(X_t)_{t\geq 0}$.

Now, define the evolution of $(Y_t, Z_t)_{t\geq 0}$, and check at the same time that each update preserves the order between the three configurations. Thus, suppose that $t = \tau_k^e$ is an update time of edge e, and we have $Y_{t-} \leq X_{t-} \leq Z_{t-}$.

If e is not an enhanced edge, let $Y_t(e) = \begin{cases} 1 & \text{if } U_k^e < \min(p, p') \\ 0 & \text{otherwise.} \end{cases}$ and $Y_t = Y_{t-}$ outside edge e. Similarly, let $Z_t(e) = \begin{cases} 1 & \text{if } U_k^e < \max(p, p') \\ 0 & \text{otherwise.} \end{cases}$ and $Z_t = Z_{t-}$ outside edge e. The update occurs at the same time as in process X, using the same uniform random variable, hence we get $Y_t(e) \leq X_t(e) \leq Z_t(e)$. Observe that here, if we do nothing different for the enhanced edges, we will find again (5.3) by letting t go to infinity.

Nevertheless, if $e \in \widehat{E}$, we allow to reveal the current configuration on the auxiliary edges to determine the new state of e. The key observation is that if we find a cutset for e comprising only auxiliary edges that are closed in Z_{t-}, then this also holds for X_{t-} and Y_{t-}, and we know that the event K^α_e does not hold for X_{t-}. Thus, e is opened in X_t if and only if $U_k^e < p'$, so we can also use the threshold p' to define $Y_t(e)$ and $Z_t(e)$ while preserving the order.

Thus, if there exists $\chi \in Q_{\widehat{E}}(e)$ such that $Z_{t-}(e') = 0$ for all $e' \in \chi$, let

$$Y'_0(e) = Z'_0(e) = \begin{cases} 1 & \text{if } U_k^e < p' \\ 0 & \text{otherwise} \end{cases};$$

otherwise, as for non-enhanced edges, let

$$Y'_0(e) = \begin{cases} 1 & \text{if } U_k^e < \min(p, p') \\ 0 & \text{otherwise.} \end{cases} \text{ and } Z'_0(e) = \begin{cases} 1 & \text{if } U_k^e < \max(p, p') \\ 0 & \text{otherwise.} \end{cases}$$

and in both cases $Y'_0 = Y_{t-}, Z'_0 = Z_{t-}$ outside e.

Then, in the same way as for process $(X_t)_{t\geq 0}$, resample all the edges of $E \setminus \widehat{E}$ using the $U_k^e[i]$ and the simple rule for non-enhanced edges (that is, open the edge when the uniform random variable is less than $\min(p, p')$ for Y_t or $\max(p, p')$ for Z_t). This defines $Y'_1, \ldots, Y'_A$ and $Z'_1, \ldots, Z'_A$, and finally we keep $Y_t = Y'_A$ and $Z_t = Z'_A$. In each of these operations, the order has been preserved, so in the end we get $Y_t \leq X_t \leq Z_t$.

We conclude the proof by computing the stationary distribution of $(Y_t, Z_t)_{t\geq 0}$, and by letting t tend to infinity. It will differ depending on whether $p < p'$ or $p' < p$. We first handle the case $p < p'$. Let Y and Z be two random configurations, of law $\mathbb{P}^G_{p+\widehat{\varepsilon}_e \mathbf{1}_{\{e\in\widehat{E}\}}}$ and $\mathbb{P}^G_{p'}$ respectively, coupled in such a way that a.s. $Y \leq Z$ (this is possible to do so because $p + \widehat{\varepsilon}_e \leq p'$). We show that if $p < p'$, the stationary distribution of $(Y_t, Z_t)_{t\geq 0}$ is the law of (Y, Z). In light of the coupling with $(X_t)_{t\geq 0}$, which has stationary distribution $\phi^\alpha_{G,p,q}$, it implies

part (1) of the Proposition (note that here we actually do not need to know the marginal law of Z).

Let us prove that the law of (Y, Z) is indeed invariant under the updates of the dynamics. Suppose that t is an update time of edge e (that is $t = \tau_k^e$ for some k) and that (Y_{t-}, Z_{t-}) has the same law as (Y, Z). If e is an auxiliary edge, the update only produces a resampling of the state of e, and sets e to be open in Y_t if $U_k^e < p$, and in Z_t if $U_k^e < p'$, independently of everything else. Then, Y_t and Z_t still have the good marginals, and verify $Y_t \leq Z_t$. Now, if $e \in \widehat{E}$, edge e is open in Y_t if the uniform random variable U_k^e is less than p, or if it is between p and p' and there is a cutset for e of auxiliary edges all closed in Z_{t-}; this happens with probability $p + \widehat{\varepsilon}_e$ (here we use the fact that in Z_{t-}, auxiliary edges are open independently with probability $p' = \max(p, p')$). More directly, e is open in Z_t if $U_k^e < p'$, and note that we still have $Y_t(e) \leq Z_t(e)$. The state of the edge e in Y_t and Z_t is therefore a measurable function of U_k^e and of the configuration on auxiliary edges at time $t-$. Furthermore, the states of the auxiliary edges are also resampled during the update while preserving the ordering, with the uniform variables $U_k^e[1], U_k^e[2], \ldots$, so that the states of all the edges are independent after the update (and clearly the auxiliary edges are open independently with probability p in Y_t and p' in Z_t). It follows that the law of (Y, Z) is indeed preserved by this update. Therefore, in the case $p < p'$, we have identified the stationary distribution of $(Y_t, Z_t)_{t\geq 0}$, so part (1) follows.

The situation is reversed when $p > p'$, but the proof is the same. This time the stationary distribution is the law of (Y, Z) where Y has law $\mathbb{P}_{p'}^G$ and Z has law $\mathbb{P}^G_{p-\widehat{\varepsilon}_e \mathbf{1}_{\{e\in\widehat{E}\}}}$, still coupled so that a.s. $Y \leq Z$. The only notable difference is that after an update, an enhanced edge e is closed in Z_t with probability $1 - p + \widehat{\varepsilon}_e$ (that is, if the uniform variable is above p or between p' and p and the condition on the auxiliary edges holds). This is why we get $\mathbb{P}^G_{p-\widehat{\varepsilon}_e \mathbf{1}_{\{e\in\widehat{E}\}}}$ as second marginal, and part (2) of the Proposition follows. □

Proof of Proposition 5.11. The proof is very similar to the previous one, so we only point out the main differences. Enhanced edges are now the edges in $\widetilde{E}$, and the others are the auxiliary edges. The processes $(X_t)_{t\geq 0}$ and $(Y_t, Z_t)_{t\geq 0}$ are defined similarly, differing only in the update rule of the enhanced edges for $(Y_t, Z_t)_{t\geq 0}$. Here, it should be observed that if it happens that all the edges of a path connecting both endpoints of $e \in \widetilde{E}$ (not using e) are open in Y_{t-}, then this also holds for X_{t-} and Z_{t-}, so in particular K_e^α holds for X_{t-}. This means that e is open in X_t if and only if $U_k^e < p$, so this threshold can also be used for the update of $Y_t(e)$ and $Z_t(e)$ while preserving the order. So in $(Y_t, Z_t)_{t\geq 0}$, when we update an enhanced edge e at time t, we reveal the states of the auxiliary edges; if we find a path of $\Gamma_{\widetilde{E}}(e)$ whose edges are all open in Y_{t-}, e is set to be open if the uniform random variable is less than p; if we do not find such a path, e is updated using the same rule as for non-enhanced edges; then, in any case, all the auxiliary edges are re-sampled, in an arbitrary order.

Therefore, if $p > p'$, an update at time t of an enhanced edge e sets e to be open in Y_t if the uniform random variable is less than p' or if it is between p' and p and there is an open path of $\Gamma_{\widetilde{E}}(e)$ in Y_{t-}. This occurs with probability $p' + \widetilde{\varepsilon}_e$. This update sets e to be open in Z_t with probability p. We deduce that the stationary distribution of $(Y_t, Z_t)_{t\geq 0}$ has marginals $\mathbb{P}^G_{p'+\widetilde{\varepsilon}_e \mathbf{1}_{\{e\in\widetilde{E}\}}}$ and $\mathbb{P}_p^G$ (and we still have the ordering between the two coupled configurations). As before the situation is reversed when $p < p'$. □

Proof of Theorem 5.1. We assume that $G \cup \alpha$ is connected, is not a tree and contains at least two distinct vertices. This last hypothesis and the connectivity implies that there is at least

one edge which is not a self-loop around a vertex, so that we can find a non-empty subset $\widehat{E}$ that satisfies the hypothesis of Proposition 5.10. Then, the fact that $G \cup \alpha$ is not a tree implies that there this at least one cycle in the graph, so that we can find a non-empty subset $\widetilde{E}$ that satisfies the hypothesis of Proposition 5.11. Then, since q is assumed to be different from 1, we have $p \neq p'$ so $\widehat{\varepsilon}_e > 0$ for all $e \in \widehat{E}$ and $\widetilde{\varepsilon}_e > 0$ for all $e \in \widetilde{E}$. Therefore, Propositions 5.10 and 5.11 readily imply Theorem 5.1. □

It is crucial to well choose the subsets $\widehat{E}$ and $\widetilde{E}$ of E to get a non-zero enhancement. As an example let us work out the above proof for finite subgraphs of $\mathbb{Z}^2$. The simplest example of a cutset for an edge e would be to take all the other edges attached to one of the endpoint of e; if all these edges are auxiliary, i.e. not in $\widehat{E}$, then there is a cutset for e included in $E \setminus \widehat{E}$. In fact, it is possible to partition $\mathbb{Z}^2$ with devices (of 4 edges) so that one edge of each device can be included in $\widehat{E}$, and the other edges of the device form a cutset of the first edge. This can be done by taking as devices the 4 edges attached to any even vertex; these are called "star-devices" (see Figure 5.2). On the other hand, the simplest path connecting both endpoints of e in $\mathbb{Z}^2$ would be to take three edges that form a square (i.e. a cycle of length 4 in the graph) together with e. One can therefore partition $\mathbb{Z}^2$ into "square-devices" (see Figure 5.2); from each, one edge can be included in $\widetilde{E}$, whereas the three other edges make a path connecting both endpoints of the first edge, and they are set to be auxiliary. Notice that its easier to think about the devices as partitioning the graph, having exactly one enhanced edge per device, but in fact the devices could share auxiliary edges. Indeed, in the proofs of Proposition 5.10 and 5.11, after a resampling of an enhanced edge, we resample all the auxiliary edges, so that all the auxiliary edges can be used to give a bonus to the enhanced edge. Moreover, it is not necessary that the devices cover the full graph, only that they are sufficiently dense (see in the proof of Theorem 5.3 below).

Therefore, if $G = (V, E)$ is a finite (connected) subgraph of $\mathbb{Z}^2$, one can choose $\widehat{E}$ to be composed of exactly one edge per star-device which is entirely included in G, with the additional requirement to never include in $\widehat{E}$ an edge whose both endpoints are in ∂G. This is done to prevent a connection between the endpoints of an enhanced edge via the boundary condition. One can construct $\widetilde{E}$ similarly with the square-devices (one even not have to worry about a possible connection via the boundary condition, since it could only help to create of path not using enhanced edges). This construction with the devices also yields lower bounds on the $\widehat{\varepsilon}_e$ and $\widetilde{\varepsilon}_e$: for all $e \in \widehat{E}$, there is a closed cutset for e included in $E \setminus \widehat{E}$ in particular if the three auxiliary edges of the star-device of e are closed, so

$$\widehat{\varepsilon}_e \geq |p' - p|(1 - \max(p, p'))^3.$$

Similarly, we get for all $e \in \widetilde{E}$

$$\widetilde{\varepsilon}_e \geq |p' - p| \min(p, p')^3.$$

This devices construction can be generalized to a large class of graphs. In the next section, we build on this to extend the new comparison inequalities to the infinite volume setting and prove Theorem 5.3.

5.3 Extending comparison inequalities to infinite graphs

Let $G = (V, E)$ be an infinite, locally finite, connected graph. As in the statement of Theorem 5.3, we suppose that G contains edges which are on a doubly-infinite path. These

edges are called *pivotal*. Let $\Delta := \max_{v\in V} \deg(v)$ be the maximal degree of G and $L := \min_{\gamma \text{ cycle with pivotal edge}} |\gamma|$ be the minimal length of a cycle containing a pivotal edge. In the following, for any finite subset of vertices $\Lambda \Subset V$, we will identify Λ with the induced subgraph of G, and consider the boundary of Λ to be

$$\partial\Lambda = \{x \in \Lambda \mid \exists y \in \Lambda^c, \{x, y\} \in E\}.$$

Let us start by properly define the devices.

Definition 5.13. *A* device *in G is a finite subset of E that contains a pivotal edge. We call* star-devices *the devices which are formed by all the edges attached to some vertex of G, and* cycle-devices[1] *the devices formed by a cycle of length L in G.*

The restriction on the length of the cycle-devices is added to avoid considering cycles of arbitrary large length around a pivotal edge; in fact, this would cancel the enhancement (see Proposition 5.14 below). Thus, Δ is the maximal size of a star-device, and L is the the size of cycle-devices. We suppose that both Δ and L are finite (this is in particular the case if G satisfies the hypothesis of Theorem 5.3, i.e. if G is quasi-transitive and contains pivotal edges which are located on a cycle). One can therefore prove the following straightforward consequence of Propositions 5.10 and 5.11 for infinite graphs:

Proposition 5.14. *Suppose that there exists $\widehat{E} \subset E$ such that for all $e \in \widehat{E}$, there is a star-device containing e of which all the other edges are in $E \setminus \widehat{E}$. Similarly, suppose that there exists $\widetilde{E} \subset E$ such that for all $e \in \widetilde{E}$, there is a cycle-device containing e of which all the other edges are in $E \setminus \widetilde{E}$. Let $\widehat{\varepsilon} := |p' - p|(1 - \max(p, p'))^{\Delta-1}$ and $\widetilde{\varepsilon} := |p' - p| \min(p, p')^{L-1}$. Then, for any FK measure ϕ on G,*

1. *If $p < p'$, one has $\mathbb{P}^G_{p+\widehat{\varepsilon}\mathbf{1}_{\{e\in\widehat{E}\}}} \preceq \phi \preceq \mathbb{P}^G_{p'-\widetilde{\varepsilon}\mathbf{1}_{\{e\in\widetilde{E}\}}}$.*
2. *If $p > p'$, one has $\mathbb{P}^G_{p'+\widetilde{\varepsilon}\mathbf{1}_{\{e\in\widetilde{E}\}}} \preceq \phi \preceq \mathbb{P}^G_{p-\widehat{\varepsilon}\mathbf{1}_{\{e\in\widehat{E}\}}}$.*

Proof. We do the proof only in the case $p < p'$, the proof being the same in the other case. Let ϕ be an FK measure on G, and first assume that it is a limit measure. This means that there exists an increasing sequence $\Lambda_n \uparrow G$ and a sequence $\alpha_n \in \Pi(\partial\Lambda_n)$ such that

$$\phi = \lim_{n\to\infty} \phi^{\alpha_n}_{\Lambda_n,p,q}.$$

For all $n \geq 0$, we take $\widehat{E}(\Lambda_n)$ to be the set of edges of $\widehat{E}$ whose associated star-device (i.e. the one included in $\{e\} \cup E \setminus \widehat{E}$, which exists by hypothesis) is entirely included in Λ_n, and whose endpoints are not both in $\partial\Lambda_n$. Define $\widetilde{E}(\Lambda_n)$ similarly (we can even relax the assumption regarding $\partial\Lambda$). For all $e \in \widehat{E}(\Lambda_n)$, if all the edges except e of the star-device associated with e are closed, it yields a closed cutset for e included in $E(\Lambda_n) \setminus \widehat{E}(\Lambda_n)$; therefore, $\widehat{\varepsilon}_e$ defined in Proposition 5.10 satisfies

$$\widehat{\varepsilon}_e \geq |p' - p|(1 - \max(p, p'))^{\Delta-1} = \widehat{\varepsilon},$$

so Proposition 5.10 implies that

$$\mathbb{P}^{\Lambda_n}_{p+\widehat{\varepsilon}\mathbf{1}_{\{e\in\widehat{E}(\Lambda_n)\}}} \preceq \phi^{\alpha_n}_{\Lambda_n,p,q}.$$

[1] This terminology replaces the less general notion of square-devices of the previous Section.

Similarly, for all $e \in \widetilde{E}(\Lambda_n)$, if all the edges except e of the cycle-device associated with e are open, it yields an open path connecting both endpoints of e (without using edge e) included in $E(\Lambda_n) \setminus \widetilde{E}(\Lambda_n)$; therefore, we have

$$\widetilde{\varepsilon}_e \geq |p' - p| \min(p, p')^{L-1} = \widetilde{\varepsilon}$$

and Proposition 5.11 gives

$$\phi^{\alpha_n}_{\Lambda_n,p,q} \preceq \mathbb{P}^{\Lambda_n}_{p'-\widetilde{\varepsilon}\mathbf{1}_{\{e\in\widetilde{E}(\Lambda_n)\}}}.$$

Letting n goes to infinity, we obtain $\mathbb{P}^G_{p+\widehat{\varepsilon}\mathbf{1}_{\{e\in\widehat{E}\}}} \preceq \phi \preceq \mathbb{P}^G_{p'-\widetilde{\varepsilon}\mathbf{1}_{\{e\in\widetilde{E}\}}}$.

The proof for FK Gibbs measures (for any convention) can be done directly from the DLR equations, considering large box containing the support of a local test event and integrating over the boundary condition, or simply by noticing that FK Gibbs measures are mixtures of FK limit measures: this is a direct consequence of Prop 5.5, item (1). □

Note that in the above proof, we never use the fact that the devices contain pivotal edges. Nevertheless, this will be crucial for the following proof of Theorem 5.3. We claim that a construction of subsets $\widehat{E}$ and $\widetilde{E}$ satisfying the hypotheses of Proposition 5.14 can be done for all quasi-transitive graphs that contain pivotal edges which belong to a cycle. The quasi-transitivity assumption should not be necessary, but it covers most of the interesting graphs on which the argument would work. Note also that for a given explicit graph, it should not be hard to find appropriate subsets $\widehat{E}$ and $\widetilde{E}$. In the following proof, we present a generic construction, that might not give the optimal enhancement. We then apply the essential enhancement theory of Aizenman and Grimmett [AG91] to deduce Theorem 5.3.

Proof of Theorem 5.3. Suppose that G is quasi-transitive and that it contains a pivotal edge which is located on a cycle; by quasi-transitivity, there are in fact infinitely many such edges. In particular, we have that L is finite, and G has bounded degree (because it is locally finite), so Δ is finite too. The construction of $\widehat{E}$ and $\widetilde{E}$ is done with the same procedure. Thus, in the following, a device refers to a star- or a cycle-device, depending on whether one aims to construct $\widehat{E}$ or $\widetilde{E}$. For all $e \in E$ and for all $k \geq 1$, let $B_k(e)$ be the set of edges at distance less than k from e. Let K be an integer large enough so that for all $e \in E$, $B_K(e)$ contains a device. Such a finite K exists because the graph is quasi-transitive. Fix $o \in E$, and consider a device included in $B_K(o)$, and choose arbitrarily one of its pivotal edges to be included in the set of enhanced edges (either $\widehat{E}$ or $\widetilde{E}$). Then, until it is no longer possible, consider at each step a new device included in $B_K(o)$ disjoint from all the previously discovered devices; choose again one of its edges to be an enhanced edge. Afterwards, consider a new device included in $B_{2K}(o)$ which is not authorized to intersect the previously discovered devices but it can intersect $B_K(o)$. Do this until it is no longer possible, and then repeat the same procedure for $B_{3K}(o)$ etc.

This provides a construction of $\widehat{E}$ and $\widetilde{E}$ satisfying the assumption of Proposition 5.14. Let ϕ be an FK measure and consider the stochastic domination for ϕ given by Proposition 5.14:

$$\phi \preceq \mathbb{P}^G_{\max(p,p')-\varepsilon\mathbf{1}_{\{e\in E'\}}},$$

taking $\varepsilon = \widehat{\varepsilon}$ and $E' = \widetilde{E}$ if $p > p'$, and $\varepsilon = \widetilde{\varepsilon}$ and $E' = \widehat{E}$ if $p < p'$. Each edge $e \in E$ is at distance bounded by K plus the maximal diameter of a device from the set E' of enhanced

edges. Indeed, there is at least one edge in $B_K(e)$ belonging to a device discovered in the construction of E'; otherwise, we could have discover a new device included in $B_K(e)$, disjoint from the other devices of the exploration, and add one of its edges to E'. Since the edges of E' have been chosen so that they are all pivotal, the enhancement is necessarily "essential" in the sense of [AG91] — meaning that for any enhanced edge, there exists a configuration such that there is no doubly-infinite path if the edge is closed, but such a path exists if the edge is open. According to [AG91], this (together with the fact that each edge of the graph is at bounded distance from an enhanced edge) is sufficient to guarantee that the enhancement increases strictly the critical parameter. More precisely, for all $q \neq 1$, there exists $\delta'(q) > 0$ such that if p satisfies $\max(p, p') < p_c(G) + \delta'(q)$, then the percolation associated with the product measure $\mathbb{P}^G_{\max(p,p')-\varepsilon \mathbf{1}_{\{e \in E'\}}}$ is subcritical. This yields item (1) of the theorem, and (2) follows from the same argument with the stochastic domination from below. □

5.4 Proofs of strong uniqueness

5.4.1 Uniqueness in the subcritical phase

In this section, we show strong uniqueness in the regime where the random cluster model is dominated by a subcritical percolation. It differs from the classical proofs of uniqueness as it relies on the structure of the set of Gibbs measures (for a fixed convention). In particular, we do not derive strong uniqueness from uniqueness of the limit measure, as classically done in the regime $q > 1$. Instead, we use the fact that extremal Gibbs measures can be obtained as weak limits along any sequence of boxes, taking as boundary condition a typical realization of the measure. This was mentioned earlier, in item (1) of Proposition 5.5, which is stated in [FV18] (Theorem 6.63). We reproduce the argument in the proof of Theorem 5.15 stated below, and make an appropriate choice of boxes to prove uniqueness. Theorem 5.15 appears to be a general statement regarding uniqueness, when all Gibbs measures for a given specification are dominated by a subcritical percolation. Theorem 5.7 is then a direct consequence of this result plus item (2) of Proposition 5.5.

Theorem 5.15 (Uniqueness through domination by a subcritical percolation)**.** *Fix a convention Φ and parameters p and q. Assume that there exists a probability measure $\mathbb{P}$ on $\{0,1\}^E$ such that $\mathbb{P}(\exists$ an infinite cluster$) = 0$ which dominates every $\Phi-$Gibbs measure, that is for all $\phi \in \mathcal{G}^\Phi_{p,q}$, $\phi \preceq \mathbb{P}$. Then we have uniqueness of the Gibbs measure, that is, $|\mathcal{G}^\Phi_{p,q}| = 1$.*

Proof. Let $\mu_1, \mu_2 \in \operatorname{ex} \mathcal{G}^\Phi_{p,q}$ be two extremal Φ-Gibbs measures. We conclude by showing that $\mu_1 = \mu_2$, this would mean that $|\operatorname{ex} \mathcal{G}^\Phi_{p,q}| = 1$ and consequently $|\mathcal{G}^\Phi_{p,q}| = 1$. Now let $\mathbb{P}$ be as in the statement of the theorem. By stochastic domination, it is possible to find a coupling $(\omega_1, \omega_2, \omega_3)$ such that $\omega_1 \sim \mu_1, \omega_2 \sim \mu_2$ and $\omega_3 \sim \mathbb{P}$, satisfying almost surely $\omega_1 \leq \omega_3$ and $\omega_2 \leq \omega_3$. Let $\tilde{\Omega}$ denote the set of configurations such that we can find an increasing sequence $S_n = S_n(\omega_3) \subset \mathbb{Z}^2$ of connected subgraphs of G such that $S_n \uparrow G$ and $\omega_3|_{\partial S_n} = 0$. By subcriticality of $\mathbb{P}$, it follows that $\mathbb{P}(\tilde{\Omega}) = 1$ (and consequently, considering our coupling, $\mu_i(\tilde{\Omega}) = 1$ as well). Let C be a cylindrical event. By uniqueness of conditional expectation and using the fact that $\mu_i(\cdot \mid \mathcal{F}_{S_n(\omega_3)^c})(\omega_i)$ is the free boundary random cluster measure in the interior of $S_n(\omega_3)$, we have for almost every tuple $(\omega_1, \omega_2, \omega_3)$, for large enough n, that:

$$\mu_1(C|\mathcal{F}_{S_n(\omega_3)^c})(\omega_1) = \mu_2(C|\mathcal{F}_{S_n(\omega_3)^c})(\omega_2),$$

because C is measurable with respect to the interior of $S_n(\omega_3)$ for large enough n. Now we use the backward martingale convergence theorem along with the fact $S_n(\omega_3)^\circ \uparrow \mathbb{Z}^2$ almost surely, to conclude that almost surely,

$$\mu_i(C|\mathcal{F}_{S_n(\omega_3)^c})(\omega_i) \xrightarrow{n\to\infty} \mu_i(C|\mathcal{F}_\infty)(\omega_i) = \mu_i(C) \quad i = 1, 2.$$

In the last equality we have used the fact that extremal measures are tail trivial (see [FV18] Theorem 6.58). This means that μ_1, μ_2 agree on cylindrical events and are thus equal. □

Proof of Theorem 5.7. From Theorem 5.3, we now that every $\phi \in \mathrm{FK}(p, q)$ is dominated by some subcritical percolation $\mathbb{P}$ when $\max(p, p') < p_c(G) + \delta'$. This implies that $\phi-$a.s. there is no infinite cluster, so by Proposition 5.5 item (2), it follows that $\phi \in \mathcal{G}^{\Phi^f}_{p,q}$, so $\mathrm{FK}(p, q) \subset \mathcal{G}^{\Phi^f}_{p,q}$. Then, in particular the domination by $\mathbb{P}$ holds for every $\phi \in \mathcal{G}^{\Phi^f}_{p,q}$, so according to Theorem 5.15, we have $|\mathcal{G}^{\Phi^f}_{p,q}| = 1$. We conclude that $|\mathrm{FK}(p, q)| = 1$ for this range of parameters. □

5.4.2 Uniqueness in the supercritical phase through planar duality

We start by recalling some definitions about plane graphs and their duality. A graph is said to be *plane* if it is[2] embedded in the plane $\mathbb{R}^2$ such that two edges never intersect, except at a common endpoint. Let $G = (V, E)$ be a plane graph. The *faces* of G are defined as the connected component of the complementary of the graph when embedded in $\mathbb{R}^2$. The *dual graph* of G, denoted $G^* = (V^*, E^*)$, is defined as follows: place a dual vertex at each face of G (including possibly infinite faces), and for each edge $e \in E$, put a dual edge between two dual vertices whose corresponding faces of G are separated by the edge e. Obviously, there is a canonical bijection between E and E^*. Therefore, for each configuration $\omega \in \{0, 1\}^E$, one can associate a dual configuration $\omega^* \in \{0, 1\}^{E^*}$, by declaring as open a dual edge if and only if its corresponding (primal) edge is closed. It is straightforward from this definition that if ω is distributed according to $\mathbb{P}^G_p$, then its dual configuration ω^* has law $\mathbb{P}^{G^*}_{1-p}$. This of course extends to inhomogeneous Bernoulli percolation measures. The situation is more intricate for FK measures, but the FK model still have a nice feature with respect to the planar duality.

Let $G = (V, E)$ be a finite plane graph, and take $\partial G = \emptyset$, so that we do not have a boundary condition (thus we also drop it from the notation). It classically follows (see [Gri06] Section 6.1) from the Euler's formula for plane graphs that

$$\forall \omega \in \{0, 1\}^E, \ \phi_{G,p,q}(\omega) = \phi_{G^*,p^*,q}(\omega^*),$$

where p^* is defined by the following relation:

$$\frac{p}{1-p}\frac{p^*}{1-p^*} = q.$$

Note that p^* of course depends on p and also on q, but we drop it from the notation as we did for p'. In fact, there is a strong link between p' and p^*, as we can observe that

$$p' = \frac{p}{p + (1-p)q} = \frac{1}{1 + (1-p)q/p} = \frac{1}{1 + p^*/(1-p^*)} = 1 - p^*.$$

[2] Note the subtle difference with a planar graph, which *can* be embedded in $\mathbb{R}^2$. We choose to consider plane graphs in order to have uniqueness of the dual graph.

Similarly, we have $p'^* = 1 - p$.

Considering the duality relation for FK measures with a boundary condition is more challenging. Suppose now that G is an infinite plane graph, and take a finite subgraph $\Lambda \Subset G$. As before, $\partial\Lambda$ is composed of the vertices of Λ having a neighbor outside. Intuitively, one may believe that the FK measure on Λ with some boundary condition admits a dual measure on Λ^* with a dual boundary condition. If $\alpha \in \Pi(\partial\Lambda)$ is a partition, there is no obvious way to define its dual. Even if the boundary condition comes from an external configuration, —in this case we can define the dual external configuration— the convention at infinity that we choose affect the measure and it is not possible in general to define the dual of a convention. The favorable case is when the external configuration contains at most one infinite cluster: thus, the convention has no effect and we have for any pair of conventions Φ, Φ',

$$\phi^{\Phi_\Lambda(\xi)}_{\Lambda,p,q} = \phi^{\Phi'_\Lambda(\xi)}_{\Lambda,p,q},$$

that we will write for short $\phi^{\xi}_{\Lambda,p,q}$. In this case, we have the following duality relation:

$$\forall \omega \in E(\Lambda),\ \phi^{\xi}_{\Lambda,p,q}(\omega) = \phi^{\xi^*}_{\Lambda^*,p^*,q}(\omega^*). \tag{5.9}$$

Indeed, assuming that ξ has at most one infinite cluster, it follows that the partition induced by ξ at the boundary of Λ can be determined by revealing ξ on a finite box $\Delta \supset \Lambda$, as otherwise it would means that two vertices of $\partial\Lambda$ are connected by a "path" which is not contained in any finite box, i.e. they belong to two disjoint infinite clusters which are merged at infinity by the convention. It follows that we have the duality relations for the FK measures in Δ (and Δ^*) without boundary condition. The relation is then preserved by conditioning the measures on $\Delta \setminus \Lambda$ (resp. $\Delta^* \setminus \Lambda^*$) to coincide with ξ (resp. ξ^*), and this gives the same measure on $\{0,1\}^{E(\Lambda)}$ (resp. $\{0,1\}^{E(\Lambda^*)}$) as $\phi^{\xi}_{\Lambda,p,q}$ (resp. $\phi^{\xi^*}_{\Lambda^*,p^*,q}$).

We can now give the duality argument that allows to get strong uniqueness in the supercritical phase.

Proof of Theorem 5.8. Let $\delta'' > 0$ and take parameters $p \in [0,1]$ and $q \neq 1$ so that $\min(p, p') > 1 - p_c(G^*) - \delta''$. Observe that this condition is equivalent to $\max(p^*, p'^*) < p_c(G^*) + \delta''$. Let $\phi \in \mathrm{FK}(p,q)$ be an FK measure on G; we first prove that ϕ is supported on configurations with exactly one infinite cluster, providing δ'' is small enough. We know from Proposition 5.14 and from our devices construction (Proof of Theorem 5.3) that there exists a subset $E' \subset E$ and $\varepsilon > 0$ such that

$$\mathbb{P}^G_{\min(p,p')+\varepsilon \mathbf{1}_{\{e \in E'\}}} \preceq \phi,$$

and that this product measure is supercritical if $\min(p,p')$ is slightly above $p_c(G)$. This is the case if δ'' is small enough, as $1 - p_c(G^*) \geq p_c(G)$. Call ϕ^* the dual law of ϕ, that is, the distribution of ω^* when ω has law ϕ. The above stochastic domination admits a dual version, namely

$$\phi^* \preceq \mathbb{P}^{G^*}_{1-\min(p,p')-\varepsilon \mathbf{1}_{\{e \in E'\}}} = \mathbb{P}^{G^*}_{\max(p^*,p'^*)-\varepsilon \mathbf{1}_{\{e \in E'\}}}.$$

Again by essential enhancements, applied on the dual graph G^*, this product measure is subcritical if $\max(p^*, p'^*)$ is slightly above $p_c(G^*)$, i.e. if δ'' is small enough. Therefore, ω^* has a.s. no infinite cluster, so ω has a.s. at most one infinite cluster. (Note that the subcriticality of this measure does *not* follow from supercriticality of its dual $\mathbb{P}^G_{\min(p,p')+\varepsilon \mathbf{1}_{\{e \in E'\}}}$ on G, that

we have already obtained in the proof of Theorem 5.3 by essential enhancements; indeed, we have no guarantee that this measure on G has a.s. exactly one infinite cluster! Even in the amenable case, the Burton-Keane argument cannot apply as it is not translation-invariant).

Since we now know that there is $\phi-$a.s. exactly one infinite cluster, for $\delta'' > 0$ small enough, we can now argue as in the proof of Theorem 5.7, and only prove uniqueness of the Gibbs measure for a single fixed convention; according to Proposition 5.5 item (2), it yields strong uniqueness. In fact, for proving strong uniqueness, we only need the fact that there is at most one infinite cluster; in this case, the convention has no effect so for all $\Lambda \Subset G$ and for any pair of conventions Φ, Φ', we have $\phi^{\Phi_\Lambda(\xi)}_{\Lambda,p,q} = \phi^{\Phi'_\Lambda(\xi)}_{\Lambda,p,q} = \phi^{\xi}_{\Lambda,p,q}$.

Let $\phi \in \operatorname{ex} \mathcal{G}^{\Phi^f}_{p,q}$. By extremality of ϕ, we have

$$\phi = \lim_{\Lambda \uparrow G} \phi^{\xi}_{\Lambda,p,q} \text{ for } \phi-\text{a.e. } \xi$$

(Proposition 5.5 item (1)). Now, by Theorem 5.7 applied to G^*, we have strong uniqueness for the FK model with parameters p^* and q on G^*, if δ'' is small enough. Denote by ϕ^* the unique FK measure on G^* with these parameters. Since it is unique it is necessarily extremal, so we have

$$\lim_{\Lambda^* \uparrow G^*} \phi^{\xi^*}_{\Lambda^*,p^*,q} = \phi^* \text{ for } \phi^*-\text{a.e. } \xi^*.$$

By applying the duality relation (5.9) to all finite subgraphs $\Lambda \Subset G$, it follows that ϕ is the (unique) dual measure of ϕ^*, that is the distribution of ω when ω^* has law ϕ^*. Therefore, one has $|\operatorname{ex} \mathcal{G}^{\Phi^f}_{p,q}| = 1$, so $|\mathcal{G}^{\Phi^f}_{p,q}| = 1$ and we get strong uniqueness. □

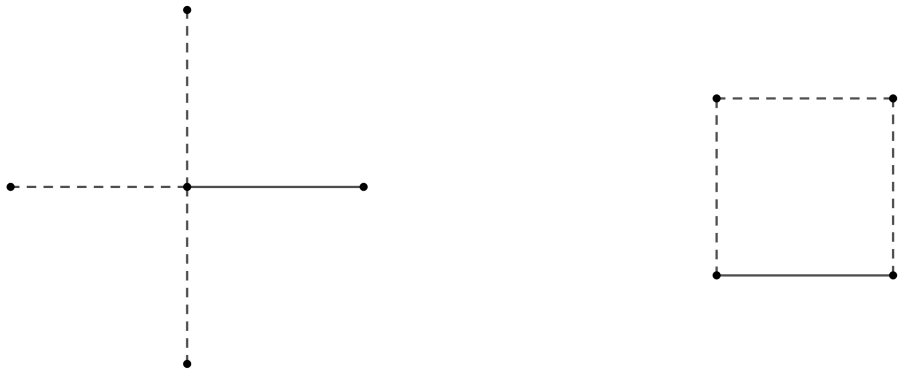


Figure 5.2: Left: a "star-device". Right: a "a square-device". The enhanced edge of the device is represented in unbroken line. In dotted line, we represent the other edges of the device, that ensure the presence of a cutset, or of a path connecting both endpoints, made of auxiliary edges only.

Appendix A

Finitary coding for the long-range Ising model at very high temperature

This Appendix is devoted to a sketch of proof of Theorem 1.13, that is, we construct a finitary coding for the long-range Ising model with $\alpha > 2d$ and β small enough. Recall that β is in particular small enough so that the model has a unique Gibbs measure. Denote it by μ_β^α (although it depends on all the J_{ij} and not only on α), and let $(X_i)_{i\in\mathbf{Z}^d}$ be a random field of law μ_β^α.

We follow the same outline as in Section 1.4.2. In particular, $\mathcal{P}$ is a PPP of intensity $\# \otimes \mathrm{d}t \otimes \mathrm{d}u$ on $\mathbf{Z}^d \times \mathbf{R} \times [0,1]$. For every $(i,t,u) \in \mathcal{P}$, the spin at i is updated at time t according to its conditional probability given the rest of the configuration. One could take the same update function φ as in (1.9), and apply it to $s = \sum_{j\neq i} J_{ij}\sigma(j)$; this gives the right conditional distribution for the spin at i, so it would define a dynamics for which μ_β^α is invariant. However, this choice of update function is far from optimal regarding coalescence, in the sense of Propp and Wilson.

For all $i \in \mathbf{Z}^d$, let

$$p_{i,0}^- := \inf_{\sigma\in\Omega} \mu_\beta^\alpha(X_i = -1 \mid (X_j)_{j\neq i} = (\sigma_j)_{j\neq i})$$

and

$$p_{i,0}^+ := \inf_{\sigma\in\Omega} \mu_\beta^\alpha(X_i = +1 \mid (X_j)_{j\neq i} = (\sigma_j)_{j\neq i});$$

then, $\gamma_{i,0} := p_{i,0}^- + p_{i,0}^+$ is the multigamma admissibility of the model; here it depends on the site i since μ_β^α is not assumed to be translation-invariant. Then, for all $r \geq 1$ and $\sigma \in \Omega$, let

$$p_{i,r}^-(\sigma) := \inf_{\sigma'\in\Omega, \sigma'\equiv\sigma \text{ on } B_r^i} \mu_\beta^\alpha(X_i = -1 \mid (X_j)_{j\neq i} = (\sigma'_j)_{j\neq i})$$

and

$$p_{i,r}^+(\sigma) := \inf_{\sigma'\in\Omega, \sigma'\equiv\sigma \text{ on } B_r^i} \mu_\beta^\alpha(X_i = +1 \mid (X_j)_{j\neq i} = (\sigma'_j)_{j\neq i}).$$

Note that $p_{i,r}^-(\sigma) \underset{r\to\infty}{\nearrow} \mu_\beta^\alpha(X_i = -1 \mid (X_j)_{j\neq i} = (\sigma_j)_{j\neq i}))$ and $p_{i,r}^+(\sigma) \underset{r\to\infty}{\nearrow} \mu_\beta^\alpha(X_i = +1 \mid (X_j)_{j\neq i} = (\sigma_j)_{j\neq i})$.

Then, during an update of site i, if one uses the update function φ of (1.9) and one has $u \leq p_{i,r}^-(\sigma)$ (resp. $u > 1 - p_{i,r}^+(\sigma)$), it suffices to look at the spins at distance less than r to determine

that the new spin at i will be -1 (resp. $+1$). Hence, setting $\gamma_{i,r}(\sigma) := p^-_{i,r}(\sigma) + p^+_{i,r}(\sigma)$, it gives that with probability $\gamma_{i,r}(\sigma)$, the new spin at i is determined without looking at spins outside B^i_r. Let $\gamma_{i,r} = \inf_{\sigma\in\Omega} \gamma_{i,r}(\sigma)$. We would like to say that when u belongs to some subset of $[0,1]$ of measure $\gamma_{i,r}$, whatever the current configuration, the new spin at i can be computed only by revealing the configuration on B^i_r. However, this will be true only when $u \in \bigcap_{\sigma\in\Omega} \left(p^-_{i,r}(\sigma); p^+_{i,r}(\sigma)\right]^c$, which is of lower measure: in fact, $\gamma_{i,r} > \inf_{\sigma\in\Omega} p^-_{i,r}(\sigma) + \inf_{\sigma\in\Omega} p^+_{i,r}(\sigma)$. Therefore, we slightly modify our update function to get what we want. To taking into account the dependence in the update site, it will be easier to define an update function for each $i \in \mathbf{Z}^d$ (in the translation-invariant case, these would be simply translations of each others). Let us define for all $i \in \mathbf{Z}^d$ an update function $\varphi_i : \Omega \times [0,1] \to \{\pm 1\}$ by

$$\varphi_i(\sigma, u) = \begin{cases} -1 & \text{if } u \in [0, p^-_{i,0}] \cup \bigcup_{r\geq 0} \left(\gamma_{i,r}(\sigma), p^+_{i,r}(\sigma) + p^-_{i,r+1}(\sigma)\right] \\ +1 & \text{if } u \in (p^-_{i,0}, \gamma_{i,0}] \cup \bigcup_{r\geq 0} \left(p^+_{i,r}(\sigma) + p^-_{i,r+1}(\sigma), \gamma_{i,r+1}(\sigma)\right] \end{cases} \tag{A.1}$$

(with convention $\gamma_{i,0}(\sigma) = \gamma_{i,0}$ and $p^\pm_{i,0}(\sigma) = p^\pm_{i,0}$ for all $\sigma \in \Omega$). This is well define since $(p^-_{i,r}(\sigma))_{r\geq 0}$ and $(p^+_{i,r}(\sigma))_{r\geq 0}$ are non-decreasing and $\gamma_{i,r}(\sigma) \xrightarrow[r\to\infty]{} 1$. What has been done above is partitioning $[0,1]$ into sub-intervals of length $p^-_{i,0}$, $p^+_{i,0}$, $p^-_{i,1}(\sigma) - p^-_{i,0}$, $p^+_{i,1}(\sigma) - p^+_{i,0}$, $p^-_{i,2}(\sigma) - p^-_{i,1}(\sigma)$ etc. and alternate the sign to determine the output of $\varphi_i(\sigma, u)$. This guarantees two crucial properties for the update functions:

- if U is a uniform random variable on $[0,1]$, then $\varphi_i(\sigma, U)$ is distributed according to the conditional law of X_i given $X_j = \sigma_j$ for all $j \neq i$;
- if $u \leq \gamma_{i,r}$, since $p^-_{i,r}(\sigma)$ and $p^+_{i,r}(\sigma)$, so $\gamma_{i,r}(\sigma)$, depend on σ only through its restriction to B^i_r, one can determine to which of the above interval u belongs – and so the new spin at i – by revealing the current configuration only on B^i_r.

Then, we consider coupled processes $(\sigma^{\omega,\tau})_{\tau\in\mathbf{R},\omega\in\Omega}$ where for all $\tau \in \mathbf{R}$ and $\omega \in \Omega$, $(\sigma^{\omega,\tau}_t)_{t\geq\tau}$ is the process on Ω such that $\sigma^{\omega,\tau}_\tau = \omega$ whose evolution is governed by the update marks: precisely, for all $(i,t,u) \in \mathcal{P}$ with $t \geq \tau$, we have

$$\forall j \neq i, \sigma^{\omega,\tau}_t(j) = \sigma^{\omega,\tau}_{t-}(j) \text{ and } \sigma^{\omega,\tau}_t(j) = \varphi_i(\sigma^{\omega,\tau}_{t-}, u).$$

We explore backwards the dynamics as in Section 1.4.2. The spin at $(0,0)$ in a realisation of the dynamics is a measurable function of the update marks before time 0 and of the initial configuration. If u denotes the third coordinate of the last update mark carried by site 0 before time 0, and r denotes the smallest integer such that $u \leq \gamma_{0,r}$, it follows that the output of the update is a function of u and of the current configuration of the dynamics restricted to the ball B^0_r. We then continue this exploration starting from all sites of this ball. When $r = 0$, the exploration stops. Then, similarly as for the finite-range Ising model, it follows that the number of update marks discovered in the backward exploration is dominated by the total population of a Galton-Watson process of reproduction law

$$\pi := \gamma_0 \delta_0 + \sum_{r\geq 0} (\gamma_{r+1} - \gamma_r) \delta_{|B^*_{r+1}|},$$

where for all $r \geq 0$, $\gamma_r = \inf_{i \in \mathbf{Z}^d} \gamma_{i,r}$ and $B_r^* = B_r^0 \setminus \{0\}$. For all $r \geq 0$, let $\partial B_{r+1} = B_{r+1} \setminus B_r$ be the set of sites in $\mathbf{Z}^d$ at distance exactly $r+1$ from the origin. For ξ a random variable of law π, one has

$$\mathbf{E}[\xi] = \sum_{r \geq 0} (1 - \gamma_r) |\partial B_{r+1}|.$$

If this expectation is less than 1, the Galton-Watson dies out for almost all realisation of $\mathcal{P}$, and it yields a coding of μ_β^α by the same coupling-from-the-past argument as in Section 1.4.2. We prove the following estimates on the decay of $1 - \gamma_r$:

Lemma A.1. *For all $i \in \mathbf{Z}^d$ and $r \geq 0$, one has*

$$1 - \gamma_{i,r} \leq \tanh\left(\beta \sum_{|j-i|>r} |J_{ij}|\right) \underset{r \to \infty}{\sim} \beta \sum_{|j-i|>r} |J_{ij}|.$$

Proof. For $p_{i,r}^-(\sigma)$, the minimum is realized by taking, for j outside B_r^i, $\sigma'(j) = +1$ if $J_{ij} \geq 0$ and $\sigma'(j) = -1$ if $J_{ij} < 0$, so

$$p_{i,r}^-(\sigma) = \left(1 + e^{2\beta \sum_{|j-i|\leq r} J_{ij}\sigma(j) + 2\beta \sum_{|j-i|>r} |J_{ij}|}\right)^{-1}$$

and conversely

$$p_{r,i}^+(\sigma) = \left(1 + e^{-2\beta \sum_{|j-i|\leq r} J_{ij}\sigma(j) + 2\beta \sum_{|j-i|>r} |J_{ij}|}\right)^{-1}.$$

By summing both terms and using the fact[1] that for all $a, b \in \mathbf{R}$, $(1+e^{a+b})^{-1} + (1+e^{-a+b})^{-1} = 1 - \frac{\sinh(b)}{\cosh(a)+\cosh(b)}$, we obtain that

$$\gamma_{i,r}(\sigma) = 1 - \frac{\sinh\left(2\beta \sum_{|j-i|>r} |J_{ij}|\right)}{\cosh\left(2\beta \sum_{|j-i|\leq r} J_{ij}\sigma(j)\right) + \cosh\left(2\beta \sum_{|j-i|>r} |J_{ij}|\right)}.$$

The minimum of this quantity over $\sigma \in \Omega$ is obtained when $\sum_{|j-i|\leq r} J_{ij}\sigma(j)$ is minimal, and since $\cosh(a) \geq 1$ for all $a \in \mathbf{R}$, we get

$$\gamma_{i,r} \geq 1 - \frac{\sinh\left(2\beta \sum_{|j-i|>r} |J_{ij}|\right)}{1 + \cosh\left(2\beta \sum_{|j-i|>r} |J_{ij}|\right)} = 1 - \tanh\left(\beta \sum_{|j-i|>r} |J_{ij}|\right)$$

using the fact[2] that for all $a \in \mathbf{R}$, $\sinh(2a) = (1+\cosh(2a))\tanh(a)$. (This is even an equality when it is possible to choose σ such that $\sum_{|j-i|\leq r} J_{ij}\sigma(j) = 0$; it is e.g. always possible to do so when $|J_{ij}|$ is translation-invariant.) □

It follows that π is integrable if and only if $\sum_{r\geq 0} \sum_{|j-i|>r} |J_{ij}||\partial B_{r+1}| < \infty$ for all $i \in \mathbf{Z}^d$. Due to our hypothesis on the coupling constants, this is achieved when

$$\sum_{r \geq 0} |\partial B_{r+1}| \sum_{n>r} |\partial B_n| J(n) = \sum_{n \geq 0} |\partial B_n||B_n^*| J(n) < \infty.$$

[1] $(1+e^{a+b})^{-1} + (1+e^{-a+b})^{-1} = \frac{2+e^{-a+b}+e^{a+b}}{1+e^{a+b}+e^{-a+b}+e^{2b}} = \frac{e^a+a^{-a}+2e^{-b}}{e^a+e^{-a}+e^b+e^{-b}} = 1 - \frac{e^b-e^{-b}}{e^a+e^{-a}+e^b+e^{-b}}$.

[2] $(1+\cosh(2a))\tanh(a) = 2\cosh(a)^2\tanh(a) = 2\cosh(a)\sinh(a) = \sinh(2a)$.

Given that on $\mathbf{Z}^d$ we have $|B_n| \asymp n^d$ and $|\partial B_n| \asymp n^{d-1}$, and that we take $J(n) = O(n^{-\alpha})$, this is therefore equivalent to $\alpha > 2d$. Then, in this case, taking β small enough ensures that $\mathbf{E}[\xi] < 1$, so we have a coding for μ_β^α. It remains to study the tail of the coding radius, to show that the coding is finitary. Denote by R this coding radius. In fact, as already observed, R itself is dominated by the total population of the Galton-Watson process that dominates the backward exploration. Consider $(\xi_{n,i})_{n,i\in\mathbf{N}}$ be independent random variables of law π. A Galton-Watson process of reproduction law π is classically defined by the following: let $Z_0 = 1$ and recursively for all $n \geq 0$, $Z_{n+1} = \sum_{i=0}^{Z_n - 1} \xi_{n,i}$. When $\mathbf{E}[\xi] < 1$, one has almost surely $Z_n = 0$ for n large enough, and one can naturally associate a finite rooted tree with $(Z_n)_{n\geq 0}$ (the root having $\xi_{0,0}$ children, each of them having respectively $\xi_{1,0}, \xi_{1,1}, \xi_{1,2}$ children etc.). This tree has size $\tau := \sum_{n\geq 0} Z_n$, and we have

$$\mathbf{P}(R > n) \leq \mathbf{P}(\tau > n).$$

Enumerate by $(\xi_k)_{1\leq k\leq \tau}$ the number of children of the vertices of this tree, taken in the lexicographical order Let $(V_n)_{n\geq 0}$ be the associated Lukasiewicz path, defined by $V_0 = 0$ and for all $1 \leq n \leq \tau$, $V_n = V_{n-1} + \xi_n - 1$. It follows that $V_n \geq 0$ for all $n < \tau$ and $V_\tau = -1$. Therefore, one has

$$\mathbf{P}(\tau > n) = \mathbf{P}(V_0, \dots, V_n \geq 0).$$

Thus, τ as the same distribution as the first time a random walk started at 0 with i.i.d. increments distributed as $\xi - 1$ reaches -1. According to Theorem 8.2.4 in [BB08], we have

$$\mathbf{P}(\tau > n) \underset{n\to\infty}{\sim} \mathbf{E}[\tau]\mathbf{P}(\xi - 1 \geq -m_1 n) \text{ where } m_1 = -\mathbf{E}[\xi - 1] < 0. \tag{A.2}$$

In fact, this result cannot directly apply to our random walk, since the increments distribution need to have "regularly varying tail", meaning that it behaves asymptotically like a power-law function[3]. This is not the case of the reproduction law π, because it is supported on a sparse subset of the integers (only those corresponding to the cardinal of a ball in $\mathbf{Z}^d$ have a positive probability). However, we can simply spread the distribution over all the integers to get a distribution with regularly varying tail: take

$$\hat{\pi} := \frac{\gamma_0}{|\partial B_1|} \sum_{i=0}^{|B_1^*|-1} \delta_i + \sum_{r\geq 0} \frac{\gamma_{r+1} - \gamma_r}{|\partial B_{r+2}|} \sum_{i=|B_{r+1}^*|}^{|B_{r+2}^*|-1} \delta_i.$$

Clearly, π is stochastically dominated by $\hat{\pi}$, and we also have that $\hat{\pi}$ is integrable if and only if $\alpha > 2d$. Therefore, if all the random variables denoted by the letter ξ have distribution $\hat{\pi}$ instead of π, the above claim (A.2) is rigorous, and it yields an upper bound on the actual tail of the coding radius. Moreover, by the Doob's convergence theorem for positive supermartingales, applied to $(V_{n\wedge\tau})_{n\geq 0}$ and because τ is finite almost surely, we get that $-1 = \lim_{n\to\infty} \mathbf{E}[V_{n\wedge\tau}] = \lim_{n\to\infty} m_1 \mathbf{E}[\tau\wedge n]$ so $\mathbf{E}[\tau] = -1/m_1$. Moreover, if $\hat{\xi}$ has law $\hat{\pi}$, one has $\mathbf{P}(\hat{\xi} \geq |B_{r+1}^*|) = 1 - \gamma_r$. According to Lemma A.1, one has asymptotically

$$1 - \gamma_r \asymp \beta \sum_{n>r} |\partial B_n| J(n) \asymp c \sum_{n>r} n^{d-1-\alpha} \asymp r^{d-\alpha}$$

[3]More precisely, the function $F : \mathbf{R}_+ \to \mathbf{R}_+$ is regularly varying if $F(t) = t^{-\gamma} L(t)$ for some $\gamma > 0$ and function L such that $\lim_{t\to\infty} \frac{L(at)}{L(t)} = 1 \ \forall a > 0$.

for some constant $c > 0$. Since the tail distribution of $\hat{\xi}$ is regularly varying, we get asymptotically that $\mathbf{P}(\hat{\xi} - 1 \geq n) \asymp n^{1-\alpha/d}$. Therefore, (A.2) implies that the coding radius has a tail which decays at least as fast as $n^{1-\alpha/d}$, so the coding radius is almost surely finite. Moreover, since we already assumed $\alpha > 2d$, we even obtain that the coding radius has finite expectation.

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