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Swann TUBACH

Motifs de Nori relatifs et foncteurs de réalisation.

Relative Nori motives and realisation functors.

Après avis de :

Annette HUBER-KLAWITTER, Professeure, Albert-Ludwigs-Universität Freiburg	Rapporteure
Joseph AYOUB, Professeur, University of Zurich	Rapporteur

Devant le jury composé de :

Annette HUBER-KLAWITTER, Professeure, Albert-Ludwigs-Universität Freiburg	Rapporteure
Joseph AYOUB, Professeur, University of Zurich	Rapporteur
Frédéric DÉGLISE, Directeur de recherche, ENS de Lyon	Examinateur
Javier FRESAN, Professeur des universités, Sorbonne Université	Examinateur
Martin GALLAUER, Professeur, Warwick University	Examinateur
Mattia CAVICCHI, Maître de conférences, Institut de Mathématiques de Bourgogne	Examinateur
Sophie MOREL, Directrice de recherche, ENS de Lyon	Directrice de thèse

Relative Nori motives and realisation functors.

Swann Tubach

Et comme un enfant dans le métro, nous regardons vers le haut, nous nous tournons vers ce ciel transparent et les planètes cachées pour comprendre ce qui nous entoure. Ou pour laisser courir librement notre imagination.
Cela nous apaise comme de caresser un chat.

Ito Naga, *Dans notre libre imagination*, 2020

Notre parole, en archipel, vous offre, après la douleur et le désastre, des fraises qu'elle rapporte des landes de la mort, ainsi que ses doigts chauds de les avoir cherchées.

René Char, *Nous avons*, 1959

À Eugénie et Jean-Pierre.

Résumé

Dans cette thèse nous construisons un foncteur de comparaison entre les motifs de Voevodsky relatifs et la catégorie dérivée de la catégorie abélienne des motifs de Nori pervers. Le foncteur de comparaison est une équivalence lorsque restreint aux 0-motifs. La même construction s'applique aux modules de Hodge, fournissant un foncteur "réalisation de Hodge" compatible aux 6 opération de Grothendieck. Les techniques utilisées sont infini-catégoriques, et cette amélioration permet donc également l'extension des modules de Hodge aux champs algébriques.

Dans une collaboration avec Raphaël Ruimy, nous comparons différentes notions de finitude dans les motifs de Voevodsky étales, répondant positivement à une conjecture de Cisinski-Déglise, et construisons des catégories dérivées de motifs de Nori relatifs à coefficients entiers qui ont les propriétés attendues : six opérations, descente et t-structures.

Abstract

In this thesis, we construct a comparison functor between relative Voevodsky motives and the derived category of the abelian category of perverse Nori motives. The comparison functor is an equivalence when restricted to 0-motives. The same construction applies to Hodge modules, yielding a "Hodge realisation" functor compatible with the Grothendieck six operations. The techniques employed are infinity-categorical, and this refinement also enables the extension of Hodge modules to algebraic stacks.

In collaboration with Raphaël Ruimy, we compare various notions of finiteness in étale Voevodsky motives, providing a positive answer to a conjecture by Cisinski–Déglise. We also construct derived categories of relative Nori motives with integral coefficients, which possess the expected properties : six operations, descent, and t-structures.

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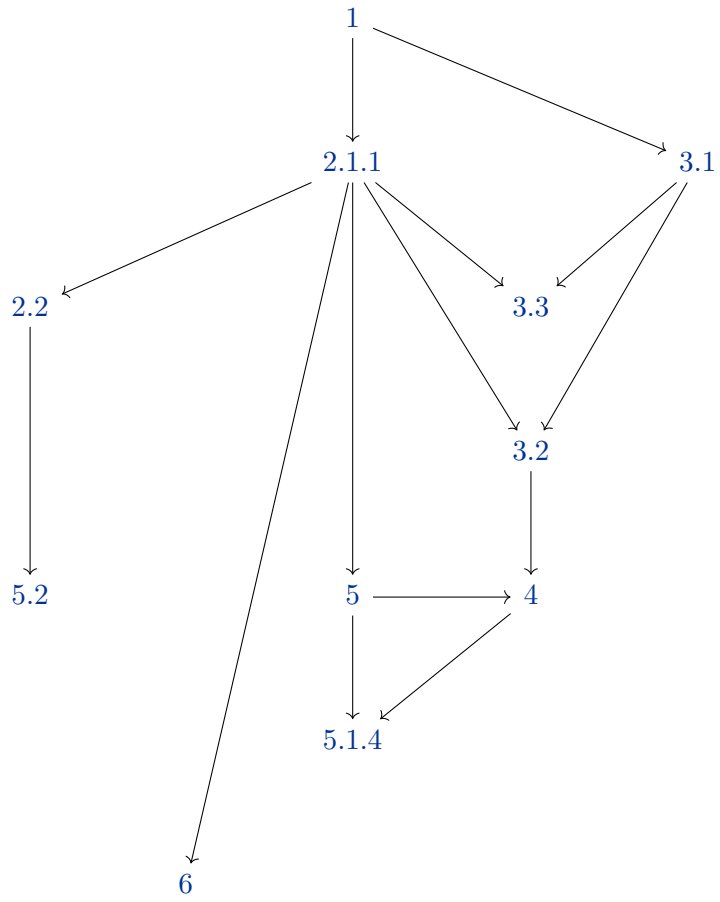
Leitfaden



Introduction en français.

Notations

Tous les anneaux sont commutatifs.

* $\mathbb{Z}$ désigne l'ensemble des entiers relatifs.
* $\mathbb{Q}$ désigne l'ensemble des nombres rationnels.
* $\mathbb{C}$ désigne l'ensemble des nombres complexes.
* Si ℓ est un nombre premier, on désigne par $\mathbb{Q}_\ell$ le corps des nombres ℓ-adiques.
* Si $\mathcal{C}$ est une catégorie, on désigne par $\mathcal{C}^{\mathrm{op}}$ sa catégorie opposée.
* Un schéma X est dit qcqs s'il est quasi-compact et quasi-séparé.

1 Préliminaires.

1.1 L'idée des motifs.

Si k est un sous-corps des nombres complexes et si X est une variété algébrique sur k, on a au moins trois théories cohomologiques associées à X. Il s'agit de la cohomologie de Betti $\mathrm{H}^*_{\mathrm{B}}(X)$, obtenue en prenant la cohomologie singulière des points complexes de X, de la cohomologie étale ℓ-adique $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_\ell)$, définie pour chaque nombre premier ℓ et encodant de l'information arithmétique sur X, et de la cohomologie de de Rham $\mathrm{H}^*_{\mathrm{dR}}(X)$, associée à un complexe de formes différentielles algébriques sur X. Comme ce sont des espaces vectoriels, et qu'ils sont de dimension finie, on obtient alors des suites d'entiers. Par exemple, les nombres de Betti, dont le premier "compte les trous des surfaces", sont associés à la cohomologie de Betti, mais on a aussi des “nombres de Betti étale”. Il se trouve que ces deux suites d'entiers sont égales. En fait on peut montrer plus généralement que si l'on choisit des plongements de corps $\mathbb{Q}_\ell \subset \mathbb{C}$, alors

$$\mathrm{H}^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}.$$

Si l'on connaît les définitions de ces deux espaces vectoriels, le résultat peut être surprenant. Certes, des deux côtés on perd de l'information: du côté Betti, on n'a plus un $\mathbb{Q}$-espace vectoriel mais un $\mathbb{C}$-espace vectoriel, et du côté ℓ-adique l'action du groupe de Galois absolu est perdue. Un isomorphisme similaire existe entre la cohomologie de Betti et la cohomologie de de Rham:

$$\mathrm{H}^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathrm{H}^n_{\mathrm{dR}}(X) \otimes_k \mathbb{C}. \qquad (1.0.1)$$

Cet isomorphisme fait partie une structure de Hodge mixte sur la cohomologie de Betti, et cela donne lieu à la cohomologie de Hodge $\mathrm{H}^*_{\mathrm{H}}(X)$. Une structure de Hodge mixte est la donnée d'un espace vectoriel sur $\mathbb{Q}$ et d'un espace vectoriel sur $\mathbb{C}$, munis de filtrations, et d'un isomorphisme entre les deux espaces vectoriels après extension des scalaires à $\mathbb{C}$, compatible aux filtrations d'une manière que nous n'expliciteront pas ici. La chose à noter est que si X est une variété propre et lisse, alors $\mathrm{H}^i_{\mathrm{H}}(X)$ est une structure de Hodge pure (cela implique que c'est un objet semi-simple), et que si X est une variété algébrique générale, alors $\mathrm{H}^i_{\mathrm{H}}(X)$ est filtrée et les quotients successifs sont des facteurs directs de $\mathrm{H}^j(Y)$ pour Y des variétés projectives lisses.

L'idée de Grothendieck (et de Beilinson et Deligne) est la suivante: si ces théories cohomologiques, pourtant si différentes, contiennent en fait des informations qui se recoupent, ce n'est pas le fruit du hasard. À toute variété algébrique X sur k, on devrait pouvoir associer une suite $\mathrm{H}^n_{\mathcal{M}}(X)$ de *motifs*, qui seraient des $\mathbb{Q}$-espaces vectoriels munis de l'action d'un énorme groupe, appelé le groupe de Galois motivique, et de sorte que toutes les théories cohomologiques ci-dessus puissent être obtenues à partir de la cohomologie motivique de X, en changeant les coefficients et oubliant une partie de l'action du groupe de Galois motivique. Plus précisément, il devrait y avoir une catégorie Tannakienne $\mathcal{M}(k)$ munie de foncteurs $\mathrm{H}^i_{\mathcal{M}}(-)\colon \mathrm{Var}_k^{\mathrm{op}} \to \mathcal{M}(k)$ associant à toute variété algébrique sur k sont i-ème motif, et de "foncteurs de réalisations", qui, lorsque composés avec $\mathrm{H}^i_{\mathcal{M}}$, redonnent les théories cohomologiques connues. Il faut savoir qu'à l'époque de Grothendieck, les motifs existaient déjà pour les variétés de dimension nulle. En effet, une variété algébrique de dimension zéro sur un corps k est une union disjointe de points correspondant aux extensions finies de k, et alors la théorie des motifs n'est autre que la théorie de Galois! La catégorie $\mathcal{M}^0(k)$ des 0-motifs est la catégorie des $\mathbb{Q}$-espaces vectoriels munis d'une action continue du groupe de Galois absolu de k, et des foncteurs de réalisation existent bel et bien. À l'heure actuelle, il n'existe pas de théorie de motifs qui ait toutes les propriétés requises par les rêves de Beilinson, Deligne et Grothendieck. La catégorie des motifs mixtes devrait être munie d'une filtration "par le poids" fonctorielle par une sous-catégorie de motifs purs, qui correspondent à la cohomologie des variétés projectives et lisses. La manière dont les motifs purs devraient s'agencer pour former des motifs mixtes est décrite très précisément par Beilinson à l'aide de cycles algébriques.

1.2 Différentes incarnations des motifs.

À la fin des années 1990, deux théories de motifs mixtes ont été définies. Les motifs de Nori ([HMS17]) sur un sous corps k des nombres complexes forment une catégorie tannakienne $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$, munie de foncteurs

$$\mathrm{H}^i_{\mathcal{M}}\colon \mathrm{Var}_k^{\mathrm{op}} \to \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$$

ainsi que de foncteurs de réalisation de Betti, ℓ-adique, de de Rham et de Hodge, factorisant les cohomologies décrites jusqu'ici. Sa construction est basée sur une théorie de catégories abéliennes universelles. La suite de foncteurs $\mathrm{H}^i_{\mathcal{M}}$ peut en fait être prolongée pour prendre ses valeurs sur des paires de variétés algébriques, à savoir une variété X et un fermé Z de X. La catégorie $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$ a la propriété universelle suivante: pour toute catégorie abélienne $\mathcal{A}$ avec un foncteur fidèle et exact vers les $\mathbb{Q}$-espaces vectoriels, et avec, pour toute paire (X,Z) et tout entier $i \geqslant 0$, un objet $\mathrm{H}^i_{\mathcal{A}}(X,Z) \in \mathcal{A}$ dont l'image dans les $\mathbb{Q}$-espaces vectoriels

est la cohomologie de Betti relative, et qui satisfait une fonctorialité minimale (il faut une fonctorialité par rapport aux paires, mais également une suite exacte longue de cohomologie associée à une suite $Z \subset Y \subset X$ de fermés, et il faut une formule de projection), il existe un foncteur (unique à unique isomorphisme naturel près) $R_{\mathcal{A}} \colon \mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}) \to \mathcal{A}$ qui factorise les $\mathrm{H}^i_{\mathcal{A}}(-,-)$ et tel que la composée avec le foncteur vers les $\mathbb{Q}$-espaces vectoriels de $\mathcal{A}$ soit la réalisation de Betti de $\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q})$. Les motifs de Nori sont munis d'une filtration par le poids, et nous connaissons les objets purs puisqu'il s'agit des motifs purs d'André. Malheureusement les groupes d'extensions entre objets purs sont très difficiles à calculer et n'ont été calculés que dans des cas particuliers. De cette manière, les motifs de Nori ont presque tout pour être une catégorie parfaite de motifs mixtes, mais cette obstruction au calcul des extensions ne permet pas de conclure qu'il s'agit de LA catégorie des motifs mixtes.

Voevodsky (ainsi que Levine et Hanamura, voir [VSF00]) ont défini une catégorie triangulée de motifs mixtes $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$. Cette catégorie est construite pour que ces groupes d'extension soient les bons, décrits par des cycles algébriques comme prédit par Beilinson. On sait depuis les travaux de Bondarko ([Bon11]) qu'une théorie de poids existe sur les motifs de Voevodsky. On a des foncteurs de réalisation ([Hub00]), à valeur dans la catégorie dérivée des $\mathbb{Q}$-espace vectoriels (Betti), des représentation ℓ-adiques (étale), des k-espaces vectoriels (de Rham), ou des structures de Hodge mixtes (de Hodge). Toute variété algébrique X sur k définit un objet $h(X)$ dans $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ dont les réalisations sont des complexes calculant la cohomologie de X. La catégorie $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ a aussi une propriété universelle, mais nous y reviendrons plus tard. La problème de $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$, c'est que ce n'est pas une catégorie abélienne. Il s'agit en fait d'une catégorie candidate pour la catégorie dérivée bornée de la catégorie abélienne des motifs mixtes, mais une première (et en fait, la seule) obstruction est le fait qu'on ne sait pas si $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ admet une t-structure (voir [BBD82] pour une définition) adaptée, qui rendent les foncteurs de réalisations t-exacts. Si nous avions une telle t-structure, alors $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ ressemblerait beaucoup à la catégorie dérivée d'une catégorie abélienne, et les calculs de Ext nous diraient exactement que le coeur de cette t-structure possède toutes les propriétés requises d'une catégorie abélienne de motifs mixtes. L'existence d'une telle t-structure "motivique" semble hors de portée. Par exemple, Beilinson a montré dans [Bei12] que cette existence implique toutes les conjectures standards de Grothendieck. Néanmoins, cette catégorie $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$, de part sa souplesse et ses possibilités calculatoires a été au centre de nombreuses avancées en géométrie algébrique, comme la résolution de la conjecture de Bloch et Kato par Rost et Voevodsky ([Voe11]).

Les catégories de motifs de Voevodsky et de Nori sont reliées par un foncteur de réalisation

$$\rho_{\mathrm{N}} \colon \mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}))$$

compatible aux foncteurs de réalisations. Ce foncteur a été construit par Choudhury-Gallauer ([CGAdS17]), par Harrer ([Har16]), et par Nori lui même. Conjecturalement, ce foncteur est une équivalence, mais nous en sommes encore loin.

Nous venons de décrire la situation sur un corps k. Les deux théories de motifs mixtes ci-dessus ont des versions relatives. En effet, à tout schéma X on peut associer une catégorie $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbb{Q})$ de "faisceaux étales motiviques" sur X. Cette catégorie a été introduite sous différentes formes par Ayoub ([Ayo14a]), Cisinski-Déglise ([CD16]), et Voevodsky ([Voe92]), et (lorsque X varie dans une classe raisonnable de schémas) est munie des 6 opérations de Grothendieck, qui sont les opérations usuelles sur les faisceaux: tiré en arrière, poussé en avant, leurs versions exceptionnelles, un produit tensoriel et un foncteur d'homomorphismes

internes. Par exemple si x est un point de X, alors le tiré en arrière d'un faisceau motivique étale sur X est un motif de Voevodsky sur le corps résiduel $\kappa(x)$. Les objets de $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X,\mathbb{Q})$ ont une forme de descente étale, d'où leur nom de faisceaux motiviques. Lorsque X est une variété quasi-projective sur un corps de caractéristique nulle k il y a aussi une version relative des motifs de Nori, appelée les motifs de Nori pervers, et qu'on notera $\mathcal{M}_{\mathrm{perv}}(X)$. C'est une catégorie définie par Ivorra et Morel ([IM22]). Lorsque X varie parmi les variétés quasi-projectives sur k, la catégorie dérivée bornée $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ est elle aussi munie des 6 opérations de Grothendieck: les quatre premières ont été construites par Ivorra-Morel, et la partie monoïdale a été construite par Terenzi ([Ter24]). Les faisceaux motiviques, tout comme les motifs de Nori pervers, sont munis de foncteurs de réalisation de Betti (à valeurs dans la catégorie dérivées de faisceaux constructibles sur les points complexes de X, lorsque X est une variété sur un sous-corps de $\mathbb{C}$), de réalisations de de Rham (à valeurs dans les $\mathcal{D}$-modules), et de réalisation ℓ-adique (à valeur dans la catégorie des complexes ℓ-adiques constructibles).

2 Ce qui est fait dans cette thèse.

2.1 Foncteurs de réalisations de Hodge et de Nori.

Si X est une variété algébrique sur un corps k qui se plonge dans $\mathbb{C}$, M. Saito a construit dans [Sai90b] une catégorie abélienne $\mathrm{MHM}(X)$ de modules de Hodge mixtes sur X, qui est une interpolation entre les structures de Hodge mixtes et les faisceaux pervers. En effet, tout module de Hodge sur X a un faisceau pervers sous-jacent, mais également un D-module complexe sous-jacent, et ces deux objets sont reliés par l'équivalence de Riemman-Hilbert (l'analogue relatif de (1.0.1)), et sont munis de diverses filtrations. Si $X = \mathrm{Spec}(k)$, alors il y a une équivalence entre les modules de Hodge mixtes sur X et les structure de Hodge mixtes et polarisables. Saito a muni la catégorie dérivée des modules de Hodge mixtes de 6 opérations de Grothendieck. Du fait de la complexité de la construction de Saito, nous ne savions construire un foncteur de réalisation de Hodge

$$\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X,\mathbb{Q}) \to \mathrm{D}^b(\mathrm{MHM}(X))$$

que lorsque X est lisse (cela est du à Ivorra ([Ivo16])), et lorsque X varie nous ne savions pas montrer que ce foncteur est compatible aux opérations. Il s'agit de notre première contribution:

Théorème 1 (Theorem 2.1.38)**.** *Soit k un sous-corps de $\mathbb{C}$. Il existe une réalisation de Hodge*

$$\rho_{\mathrm{H}} \colon \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(-,\mathbb{Q}) \to \mathrm{D}^b(\mathrm{MHM}(-))$$

sur les variétés algébriques sur k, qui commute avec les 6 opérations de Grothendieck.

Du fait de la proximité de la construction des 6 opérations sur les modules de Hodge et sur les motifs de Nori, ce foncteur se factorise en fait par la catégorie dérivée des motifs de Nori:

Théorème 2 (Theorem 2.1.38, Corollary 2.1.41, Corollary 3.2.8)**.** *Les motifs de Nori pervers s'étendent à tous les $\mathbb{Q}$-schémas qcqs de dimension finie. Sur ces schémas, il existe un foncteur*

$$\rho_{\mathrm{N}} \colon \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(-,\mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$$

compatible aux tirés en arrière. Sur les schémas quasi-excellents, ce foncteur commute aux 6 opérations de Grothendieck. Soit k un sous-corps des nombres complexes. Alors sur les variétés algébriques sur k, le foncteur de réalisation ρ_{N} factorise le foncteur de réalisation de Hodge du Théorème 1, et le foncteur induit

$$R_{\mathrm{H}} \colon \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) \to \mathrm{D}^b(\mathrm{MHM}(-))$$

commute également aux 6 opérations de Grothendieck.

La construction de ces foncteurs ρ_{N} et ρ_{H} est basée sur la propriété universelle de $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(-,\mathbb{Q})$, prouvée par Drew et Gallauer ([DG22]). Cette propriété universelle ne peut s'écrire que dans le langage des ∞-catégories, et c'est pour cela que les deux théorèmes Théorème 2 et Théorème 1 découlent du théorème suivant:

Théorème 3 (Proposition 2.1.23)**.** *Soit k un sous-corps de $\mathbb{C}$ et notons, pour toute variété quasi-projective X sur k, la catégorie $\mathrm{D}^b_{\mathcal{M}}(X)$ comme étant ou bien $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ ou bien $\mathrm{D}^b(\mathrm{MHM}(X))$. Alors il existe un foncteur*

$$\mathcal{D}^b_{\mathcal{M}} \colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

à valeurs dans les ∞-catégories symétriques monoïdales, tel que pour X une variété quasi-projective sur k, on ait un isomorphisme de catégories homotopiques $\mathrm{ho}(\mathcal{D}^b_{\mathcal{M}}(X)) \simeq \mathrm{D}^b_{\mathcal{M}}(X)$, compatible aux tirés en arrières.

Ce relèvement ∞-catégorique des modules de Hodge et des motifs de Nori pervers est obtenu de la manière suivante: comme les tirés en arrière sont rarement t-exacts pour la t-structure perverse, on utilise la t-structure constructible (aussi appelée t-structure ordinaire), dont le coeur se réalise dans la catégorie abélienne des faisceaux constructibles. Pour cette t-structure, tous les tirés en arrière sont t-exacts. En adaptant des arguments de Nori, on prouve alors:

Théorème 4 (Corollary 2.1.22)**.** *Soit $\mathcal{M}_{\mathrm{ct}}(X)$ le coeur de la t-structure constructible sur $\mathrm{D}^b_{\mathcal{M}}(X)$. Alors le foncteur naturel*

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \to \mathrm{D}^b_{\mathcal{M}}(X)$$

est une équivalence.

En utilisant les bonnes propriétés fonctorielles de ∞-foncteur associant à une catégorie abélienne son ∞-catégorie dérivée bornée, le Théorème 3 suit facilement.

Une dernière conséquence de ce relèvement ∞-catégorique est l'existence, pour tout $\mathbb{Q}$-schéma qcqs X, d'un objet en algèbre $\mathscr{N}_{X,\mathbb{Q}}$ dans $\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$ la grosse catégorie des faisceaux motiviques étales, telle que

$$\mathrm{Mod}_{\mathscr{N}_{X,\mathbb{Q}}}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})) \simeq \mathrm{DN}(X,\mathbb{Q}) := \mathrm{Ind}\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

fonctoriellement. Un résultat similaire est également obtenu pour les modules de Hodge d'origine géométrique, ce qui montre que la catégorie obtenue par l'approche de Drew aux "modules de Hodge motiviques" est bien une sous-catégorie des modules de Hodge.

2.2 Motifs de Nori à coefficients entiers.

Dans un travail en commun avec Raphaël Ruimy, nous utilisons le relèvement ∞-catégorique de motifs de Nori pervers pour en construire une version à coefficients entiers. La construction est très technique et utilise pleinement les infini-catégories. L'idée est la suivante: le théorème de rigidité de Ayoub–Bachmann–Cisinski-Déglise ([Bac21]) implique que pour tout schémas X et à coefficients de torsion $\mathbb{Z}/n\mathbb{Z}$, la catégorie $\mathrm{DM}^{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$[1] est équivalente aux complexes de faisceaux de $\mathbb{Z}/n\mathbb{Z}$-modules sur le petit site étale de X. Ainsi, si nous avions une grosse catégorie raisonnable $\mathrm{DN}(X, \mathbb{Z})$ de motifs de Nori sur X à coefficients entiers, alors il faudrait d'une part, que ses objets rationnels soient équivalent à $\mathrm{DN}(X, \mathbb{Q})$ la grosse catégorie obtenue à partir de la catégorie dérivée bornée des motifs de Nori pervers, et que ses objets de torsions soient juste des faisceaux sur le petit site étale.

Pour ce faire, pour chaque $\mathbb{Q}$-schémas qcqs X nous construisons un objet en algèbre $\mathscr{N}_X$ dans $\mathrm{DM}(X, \mathbb{Z})$, tel que $\mathscr{N}_X/n\mathscr{N}_X$ soit juste $\mathbb{Z}/n\mathbb{Z}$, et $\mathscr{N}_X \otimes \mathbb{Q}$ soit l'algèbre $\mathscr{N}_{X,\mathbb{Q}}$ décrite ci-dessus. Ainsi, en posant

$$\mathrm{DN}(X, \mathbb{Q}) := \mathrm{Mod}_{\mathscr{N}_X}(\mathrm{DM}(X, \mathbb{Q})),$$

et en se reposant sur les très bonnes propriétés des modules dans les ∞-catégories, nous obtenons une catégorie candidate aux motifs de Nori relatifs sur X à coefficients entiers. Une fois cette ∞-catégorie définie, on peut considérer sa sous-catégorie $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$ des objets géométriques et nous prouvons:

Théorème 5 (Theorem 4.2.1, Theorem 4.3.10, Proposition 4.3.14)**.** *Sur les $\mathbb{Q}$-schémas qcqs de dimension finie, les motifs de Nori entiers ont les 6 opérations et le foncteur de réalisation de Nori $\rho_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}(-, \mathbb{Z}) \to \mathrm{DN}(-, \mathbb{Z})$ est compatible avec les opérations de type f^*, f_*, $f_!$, $f^!$, $\otimes$ et $\underline{\mathrm{Hom}}(M, -)$ pour M géométrique. De plus la catégorie $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$ a les t-structures ordinaire et perverse attendues, et les foncteurs de réalisation sont t-exacts.*

En passant, nous démontrons que plusieurs notions de finitude dans les motifs (que ce soit Nori ou Voevodsky) sont les même:

Théorème 6 (Theorem 3.1.37)**.** *Soit X un schéma qcqs de dimension finie et A un anneau régulier. Alors un objet $M \in \mathrm{DM}(X, A)$ est dans la sous catégorie épaisse engendrée par les twists de motifs de X-schémas lisses si et seulement si pour tout ouvert affine U de X, il existe une stratification finie U_i de U telle que la restriction de M à chaque U_i soit un objet dualisable.*

Comme écrit au dessus, le résultat est aussi vrai pour les motifs de Nori.

2.3 Comparaison des motifs de Nori à d'autres catégories.

Nous avons donc un foncteur de comparaison

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$$

qui commute aux opérations. Il est naturel de se demander si c'est une équivalence, du moins lorsque restreint à des sous-catégories pleines.

Le premier résultat dans cette direction est conditionnel à l'existence d'une t-structure motivique sur les motifs de Voevodsky sur tous les corps de caractéristique zéro:

[1] Il s'agit d'une catégorie contenant les motifs géométriques $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbb{Z}/n\mathbb{Z})$

Théorème 7 (Theorem 2.1.45)**.** *Supposons que pour tout corps k de caractéristique nulle, la catégorie $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ est munie d'une t-structure non dégénérée, compatible avec le produit tensoriel, et telle que les réalisations ℓ-adiques soient t-exactes. Alors pour tout $\mathbb{Q}$-schémas qcqs de dimension finie X, le foncteur de réalisation*

$$\mathrm{DM}(X,\mathbb{Z}) \to \mathrm{DN}(X,\mathbb{Z})$$

est une équivalence.

La preuve se ramène facilement aux coefficients rationnels, puis aux objets constructibles, puis aux variétés sur $\mathbb{Q}$. Dans ce cas, on utilise plusieurs fois les propriétés universelles de $\mathrm{DM}_{\mathrm{gm}}(X,\mathbb{Q})$ et de $\mathcal{M}_{\mathrm{perv}}(X)$.

Nous avons aussi un résultat inconditionnel. Nous rappelons que la sous catégorie $\mathrm{DM}^0(X,\mathbb{Q})$ des motifs d'Artin sur un schéma X est la sous-catégorie engendrée par décalages, extensions et colimites par les motifs de X-schémas étales, dans $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$. Nous montrons:

Théorème 8 (Theorem 5.1.17)**.** *Soit X un $\mathbb{Q}$-schéma qcqs de dimension finie. Alors le foncteur*

$$\mathrm{DM}^0(X,\mathbb{Z}) \to \mathrm{DN}(X,\mathbb{Z})$$

est pleinement fidèle.

La preuve est, après réduction aux coefficients rationnels, une récurrence noetherienne, qui ramène à montrer ce résultat au point générique, où il est déjà connu. Nous calculons également le coeur de la t-structure induite sur les motifs d'Artin lisse sur une base normale, et nous en déduisons que sur une base normale, un motif d'Artin est dualisable si et seulement si il est dans la sous-catégorie épaisse engendrée par les motifs de schémas finis et étales.

Dans une autre direction, et à coefficients rationnels, nous savons maintenant que la catégorie abélienne $\mathcal{M}_{\mathrm{perv}}(X)$ possède tous les foncteurs de réalisation désirés. En particulier, en mettant tous ces foncteurs de réalisation ensemble, on obtient un foncteur de réalisation

$$\mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{MR}_{\mathrm{perv}}(X)$$

à valeurs dans la catégorie des faisceaux pervers de réalisations mixtes: un objet de la catégorie des réalisation mixte est la donnée d'un faisceau pervers $\mathcal{F}_B$ sur la variété des points complexes de X, d'un module de Hodge $\mathcal{F}_H$ sur X dont le faisceau perverse sous-jacent est $\mathcal{F}_B$, un $\mathcal{D}$-module filtré $\mathcal{F}_{dR}$ holonôme sur X dont le changement de base à $\mathbb{C}$ est le $\mathcal{D}$-module sous-jacent à $\mathcal{F}_H$, et pour chaque nombre premier ℓ un faisceau perverse ℓ-adique $\mathcal{F}$ sur X, dont le changement de base à $\mathbb{C}$ est $\mathcal{F}_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Soit $\mathcal{MT}^1(X)$ la sous catégorie tannakienne de la catégorie $\mathcal{M}^{\mathrm{liss}}(X)$ systèmes locaux motivique sur X (qui sont les objets de $\mathcal{M}_{\mathrm{perv}}(X)$ dont la réalisation de Betti est un système local) engendré par les H^i des poussés en avant $p_*\mathbb{Q}_C$ où $p\colon C \to X$ est une courbe projective lisse relative. Nous montrons:

Théorème 9 (Theorem 5.2.12)**.** *Supposons que X est une variété lisse sur un sous-corps k de $\mathbb{C}$. Le foncteur $\mathcal{MT}^1(X) \to \mathcal{MR}_{\mathrm{perv}}(X)$ est pleinement fidèle, et son image est stable par sous-quotients.*

En passant, nous montrons que la catégorie $\mathcal{M}^1_{\mathrm{Del}}(X)$ des 1-motifs de Deligne relatifs est une sous catégorie pleine de $\mathcal{MT}^1(X)$, et cette sous catégorie l'engendre en tant que catégorie tannakienne.

La preuve est basée sur un résultat de Yves André ([And21]), qui a montré que le foncteur $\mathcal{MT}^1(\bar{k}) \to \mathrm{MHS}$ est pleinement fidèle et son image est stable par sous-quotients. Nous montrons que son résultat se généralise à tout sous corps de $\mathbb{C}$ si l'on change le but du foncteur, en remplaçant MHS par les réalisations mixtes, puis sur une base lisse le résultat découle du résultat au point générique.

2.4 Extension des modules de Hodge aux champs.

C'est un fait folklorique que les catégories triangulées ne se recollent pas. Par exemple, si (U_i) est un recouvrement ouvert d'une variété algébrique X sur un sous-corps de $\mathbb{C}$, il n'est pas vrai que $\mathrm{D}^b_c(X,\mathbb{Q})$ est obtenue par recollement à partir des $\mathrm{D}^b_c(U_i,\mathbb{Q})$ et des diverses intersections. En tous cas, ce n'est pas vrai au niveau triangulé. Par contre, si l'on considère ces catégories comme des ∞-catégories stables, le résultat devient vrai! Plus généralement, c'est un fait général que si $\mathrm{D}(X,\mathscr{O}_X)$ est l'∞-catégorie dérivée des faisceaux de $\mathscr{O}_X$-modules sur un topos annelé $(X,\mathscr{O}_X)$, alors l'∞-foncteur

$$(U \in X) \mapsto \mathrm{D}(X_{/U}, (\mathscr{O}_X)_{|U})$$

est un faisceau. Une autre conséquence du relèvement ∞-catégorique des modules de Hodge est le suivant:

Théorème 10 (Theorem 2.1.33)**.** *Si X est un schéma de type fini sur $\mathbb{C}$, notons $\mathrm{D_H}(X) := \mathrm{D}(\mathrm{IndMHM}(X))$ la catégorie dérivée non bornée des ind-modules de Hodge. Alors le foncteur*

$$\mathrm{D_H}\colon \mathrm{Sch}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$$

induit par la f^*-fonctorialité, où l'on a noté Pr^L l'∞-catégorie des ∞-catégories présentables, est un h-hyperfaisceau.

En particulier, si $\mathfrak{X}$ est un champs algébrique localement de type fini sur $\mathbb{C}$, avec une diagonale représentable par des schémas, alors si $X_i \to \mathfrak{X}$ est une famille jointement surjective de morphismes lisses, avec X_i des $\mathbb{C}$-schémas de type fini, la limite

$$\lim(\ \textstyle\prod_i \mathrm{D_H}(X_i) \rightrightarrows \prod_{i,j} \mathrm{D}(X_i \times_{\mathfrak{X}} X_j) \ \substack{\rightarrow\\ \rightarrow\\ \rightarrow}\ \dots\ )$$

ne dépend par du recouvrement (X_i) choisi. En particulier, il devient raisonnable d'étendre le foncteur $\mathrm{D_H}$ par extension de Kan à droite des $\mathbb{C}$-schémas de type finis aux champs algébriques localement de type fini sur $\mathbb{C}$. En se basant sur des résultats d'extension des opérations aux champs dûs à Liu et Zheng ([LZ17]), on montre:

Théorème 11 (Theorem 6.1.6, Theorem 6.1.13, Corollary 6.1.28, Theorem 6.1.40)**.** *Les 6 opérations de Grothendieck, ainsi que les t-structures, et les poids existent pour l'extension de $\mathrm{D_H}(-)$ aux champs. C'est aussi le cas des cycles proches.*

Lorsque $\mathfrak{X} = [X/G]$ on retrouve ainsi les modules de Hodge G-équivariants d'Achar ([Ach13]).

3 Articles contenus dans cette thèse

Cette thèse contient la plus grande partie des articles suivants:

[Tub23] Swann Tubach. On the Nori and Hodge realisations of Voevodsky étale motives. Preprint, available at https://arxiv.org/abs/2309.11999, 2023.

[RT24a] Raphaël Ruimy and Swann Tubach. Étale motives of geometric origin. Preprint, available at https://arxiv.org/abs/2405.07095, 2024.

[RT24b] Raphaël Ruimy and Swann Tubach. Nori motives (and mixed hodge modules) with integral coefficients. Preprint, available at https://arxiv.org/abs/2407.01462, 2024.

[Tub24a] Swann Tubach. Artin motives in relative Nori and Voevodsky motives. Preprint, available at https://arxiv.org/abs/2409.12841, 2024.

[Tub24b] Swann Tubach. Mixed hodge modules on stacks. Preprint, available at https://arxiv.org/abs/2407.02256, 2024.

4 Organisation du manuscrit

Le manuscrit est composé de six chapitres et d'un appendice.

1 Dans le premier chapitre nous faisons quelques rappels. Il s'agit de parler des ∞-catégories stables, de différentes formes de motifs, et des modules de Hodge. Quelques preuves sont déjà contenues dans cette partie.

2 Dans le second chapitre, nous avons consigné la plus grande partie de l'article [Tub23]: Nous montrons comment construire un relèvement ∞-catégorique cohérent des 6 opérations sur les motifs de Nori pervers et les modules de Hodge. Pour ce faire nous montrons que la catégorie dérivée bornée des motifs de Nori pervers s'identifie à la catégorie dérivée bornée de son coeur constructible. Nous terminons le chapitre avec quelques rappels et nouveautés sur les catégories de réalisations mixtes.

3 Dans le troisième chapitre, nous discutons de modules dans les motifs étales. Après avoir défini les catégories de motifs enrichies de Drew, nous exposons le contenu de l'article [RT24a] joint avec Raphaël Ruimy, traduit dans ce langage. Ceci fait, nous montrons que les motifs de Nori pervers et les modules de Hodge d'origine géométrique sont des exemples de modules dans les motifs étales, comme cela a été montré dans [Tub23] et [Tub24b].

4 Le quatrième chapitre est issu intégralement de la collaboration [RT24b] avec Raphaël Ruimy. Il s'agit de construire des catégories de motifs de Nori entiers sur une base relativement générale. La construction, bien que très technique, peut se résumer par la phrase suivante: nous recollons les motifs de Nori pervers à coefficients rationnels avec les faisceaux étales de torsion. Nous montrons également qu'il existe les t-structures attendues.

5 Dans le cinquième chapitre, nous comparons des sous-catégories de motifs de Nori à d'autres catégories. Dans une première partie, tirée de [Tub24a], nous montrons que sur une base générale de caractéristique nulle, les motifs d'Artin sont les mêmes dans les motifs étales de Voevodsky et dans les motifs de Nori. Une sous-section est en collaboration avec Raphaël Ruimy: il s'agit de généraliser des propriétés d'annulation d'Artin qu'il a montré sur les motifs d'Artin à des bases de caractéristique zéro. Dans une seconde partie, nous étudions ce que nous avons nommé les 1-motifs de Tate, qui forment la sous-catégorie Tannakienne des motifs de Nori lisse engendrée par les motifs de Deligne. Sur une base lisse, nous montrons que le foncteur "réalisation mixtes" des motifs de Nori pervers est pleinement fidèle sur cette sous-catégorie. Comme corollaire nous obtenons que la construction canonique sur les variétés de Shimura se factorise par les motifs de Nori.

6 Dans le sixième et dernier chapitre, qui contient l'article [Tub24b], nous expliquons comment, à l'aide du relèvement des modules de Hodge dans le monde ∞-catégorique, il est possible de les étendre canoniquement aux champs algébriques. En s'appuyant sur des résultats de Liu et de Zheng, nous montrons que cette extension a toujours les 6 opérations de Grothendieck, ainsi que des poids et des t-structures. Dans la dernière section, en s'inspirant de l'interprétation de la construction des cycles proches d'Ayoub à l'aide de la dualité de Kozsul présentée dans [CHS24], nous montrons que les modules de Hodge sur les champs ont également des cycles proches satisfaisant de bonnes propriétés.

A Enfin, dans l'appendice nous avons écrit un bréviaire sur les ∞-catégories, qui se veut être fonctionnel.

Introduction in English.

Notations

All rings are commutative.

* $\mathbb{Z}$ denotes the set of integers.
* $\mathbb{Q}$ denotes the set of rational numbers.
* $\mathbb{C}$ denotes the set of complex numbers.
* If ℓ is a prime number, $\mathbb{Q}_\ell$ denotes the field of ℓ-adic numbers.
* If $\mathcal{C}$ is a category, $\mathcal{C}^{\mathrm{op}}$ denotes its opposite category.
* A scheme is called qcqs if it is quasi-compact and quasi-separated.

1 Preliminaries

1.1 The idea of motives

If k is a subfield of the complex numbers and X is an algebraic variety over k, we have at least three cohomological theories associated to X. We have the Betti cohomology $\mathrm{H}^*_{\mathrm{B}}(X)$, obtained by taking the singular cohomology of the complex points of $\mathbb{C}$, the étale ℓ-adic cohomology $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_\ell)$, defined for each prime number ℓ and encoding arithmetic information on X, and the de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X)$, associated to a complex of algebraic differential forms on X. Since these are vector spaces and they are finite-dimensional, every cohomology theory gives a sequence of integers by taking dimensions. For example, the Betti numbers, the first one "counting holes on surfaces", are associated to Betti cohomology and we also have "étale Betti numbers" obtained from ℓ-adic cohomology. It turns out that these two sequences are equal. In fact, we can show more generally that if we choose embeddings of fields $\mathbb{Q}_\ell \subset \mathbb{C}$, then

$$\mathrm{H}^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}.$$

If we know the definitions of these two vector spaces, the result can be surprising. Certainly, on both sides, we lose information: on the Betti side, we no longer have a $\mathbb{Q}$-vector space but a $\mathbb{C}$-vector space, and on the ℓ-adic side, the action of the absolute Galois group is lost. A similar isomorphism exists between Betti cohomology and de Rham cohomology:

$$\mathrm{H}^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathrm{H}^n_{\mathrm{dR}}(X) \otimes_k \mathbb{C}. \tag{1}$$

This isomorphism creates a mixed Hodge structure on Betti cohomology. We denote by $\mathrm{H}^*_{\mathrm{H}}(X)$ the associated Hodge cohomology. A mixed Hodge structure is the data of a vector space over $\mathbb{Q}$ and a vector space over $\mathbb{C}$, equipped with filtrations, and an isomorphism between the two vector spaces after extending scalars to $\mathbb{C}$, compatible with the filtrations in a way that we will not specify here. The thing to note is that if X is a proper and smooth variety, then $\mathrm{H}^i_{\mathrm{H}}(X)$ is a pure Hodge structure (this implies that it is a semisimple object), and if X is a general algebraic variety, then $\mathrm{H}^i_{\mathrm{H}}(X)$ is filtered, and the successive quotients are direct factors of $\mathrm{H}^j(Y)$ for Y smooth projective varieties.

The idea of Grothendieck (and Beilinson and Deligne) is the following: if these cohomology theories, although so different, contain information that overlaps, it is not by chance. To every algebraic variety X over k, we should be able to associate a sequence $\mathrm{H}^n_{\mathcal{M}}(X)$ of *motives*, which would be $\mathbb{Q}$-vector spaces equipped with the action of a huge group, called the motivic Galois group, and such that all the cohomological theories above can be obtained from the motivic cohomology of X, by changing coefficients and forgetting part of the action of the motivic Galois group. More precisely, there should be a Tannakian category $\mathcal{M}(k)$ equipped with functors $\mathrm{H}^i_{\mathcal{M}}(-)\colon \mathrm{Var}_k^{\mathrm{op}} \to \mathcal{M}(k)$ associating to every algebraic variety over k its i-th motive, and "realisation functors", that is functors from the category of motives to various Tannakian categories which, when composed with $\mathrm{H}^i_{\mathcal{M}}$, give back the known cohomological theories. It should be noted that at the time of Grothendieck, motives already existed for zero-dimensional varieties. Indeed, a zero-dimensional algebraic variety over a field k is a disjoint union of points corresponding to finite extensions of k, and then the theory of motives is nothing other than Galois theory! The category $\mathcal{M}^0(k)$ of 0-motives is the category of $\mathbb{Q}$-vector spaces equipped with a continuous action of the absolute Galois group of k, and realisation functors do exist. At present, there is no theory of motives that has all the properties required by the dreams of Beilinson, Deligne, and Grothendieck. The category of mixed motives should be equipped with a functorial "weight filtration" by a subcategory of pure motives, which correspond to the cohomology of smooth projective varieties. The way pure motives should arrange themselves to form mixed motives is described very precisely by Beilinson using algebraic cycles.

1.2 Different incarnations of motives

At the end of the 1990s, two theories of mixed motives were defined. Nori motives ([HMS17]) over a subfield k of the complex numbers form a Tannakian category $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$, equipped with functors

$$\mathrm{H}^i_{\mathcal{M}}\colon \mathrm{Var}_k^{\mathrm{op}} \to \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$$

as well as realisation functors for Betti, ℓ-adic, de Rham, and Hodge cohomology, factoring the cohomologies described so far. Its construction is based on a theory of universal abelian categories. The sequence of functors $\mathrm{H}^i_{\mathcal{M}}$ can in fact be extended to take values on pairs of algebraic varieties, namely a variety X and a closed subset Z of X. The category $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$ has the following universal property: for every abelian category $\mathcal{A}$ with a faithful and exact functor to $\mathbb{Q}$-vector spaces, and with, for every pair (X,Z) and every integer $i \geq 0$, an object $\mathrm{H}^i_{\mathcal{A}}(X,Z) \in \mathcal{A}$ whose image in $\mathbb{Q}$-vector spaces is the relative Betti cohomology, and which satisfies a minimal amount of functoriality (there must be functoriality with respect to pairs, but also a long exact cohomology sequence associated with a sequence $Z \subset Y \subset X$ of closed subsets, and a projection formula), there exists a functor (unique up to unique

natural isomorphism) $R_{\mathcal{A}}\colon \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}) \to \mathcal{A}$ that factors the $\mathrm{H}^i_{\mathcal{A}}(-,-)$ and such that the composition with the functor to $\mathbb{Q}$-vector spaces of $\mathcal{A}$ is the Betti realisation of $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$. Nori motives are equipped with a weight filtration, and we know the pure objects, which are André pure motives. Unfortunately, the extension groups between pure objects are very difficult to calculate and have only been calculated in particular cases. In this way, Nori motives have almost everything to be a perfect category of mixed motives, but this obstruction to calculating extensions does not allow us to conclude that it is THE category of mixed motives.

Voevodsky (as well as Levine and Hanamura, see [VSF00]) defined a triangulated category of mixed motives $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$. This category is constructed so that the extension groups are the right ones, described by algebraic cycles as predicted by Beilinson. It has been known since Bondarko's work ([Bon11]) that a weight theory exists on Voevodsky motives. We have realisation functors ([Hub00]), with values in the derived category of $\mathbb{Q}$-vector spaces (Betti), ℓ-adic representations (étale), k-vector spaces (de Rham), or mixed Hodge structures (Hodge). Every algebraic variety X over k defines an object $h(X)$ in $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ whose realisations are complexes calculating the different cohomologies of X. The category $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ also has a universal property, but we will come back to this later. The problem with $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ is that it is not an abelian category. It is in fact a candidate category for the bounded derived category of the abelian category of mixed motives, but a first (and in fact, the only) obstruction is the fact that we do not know if $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ have a suitable t-structure (see [BBD82] for a definition) that makes the realisation functors t-exact. If we had a t-structure, then $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ would closely resemble the derived category of an abelian category, and the Ext calculations would tell us exactly that the heart of this t-structure has all the required properties of an abelian category of mixed motives. The existence of such a "motivic" t-structure seems to be out of reach. For example, Beilinson has shown ([Bei12]) that this existence would imply all of Grothendieck's standard conjectures. Nevertheless, this category $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$, due to its flexibility and computational possibilities, has been at the center of many advances in algebraic geometry, such as the resolution of the Bloch-Kato conjecture by Rost and Voevodsky ([Voe11]).

The categories of Voevodsky motives and Nori motives are related by a realisation functor

$$\rho_{\mathrm{N}}\colon \mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})) \tag{2}$$

compatible with the realisation functors. This functor was constructed by Choudhury-Gallauer ([CGAdS17]), by Harrer ([Har16]), and by Nori himself. Conjecturally, this functor is an equivalence, but we are still far from it.

We have just described the situation over a field k. The two theories of mixed motives above have relative versions. Indeed, to every scheme X, we can associate a category $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Q})$ of "étale motivic sheaves" on X. This category has been introduced in different forms by Ayoub ([Ayo14a]), Cisinski-Déglise ([CD16]), and Voevodsky ([Voe92]), and (when varying X in a reasonable class of schemes) is equipped with Grothendieck's 6 operations, which are the usual operations on sheaves: pullback, pushforward, their exceptional versions, a tensor product, and an internal homomorphism functor. For example, if x is a point of X, then the pullback of an étale motivic sheaf on X is a Voevodsky motive over the residue field $\kappa(x)$. The objects of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Q})$ have a form of étale descent, hence their name of motivic sheaves. When X is a quasi-projective variety over a field k of characteristic zero, there is also a relative version of Nori's motives, called perverse Nori motives, and denoted

$\mathcal{M}_{\mathrm{perv}}(X)$. This is a category defined by Ivorra and Morel ([IM22]). When X varies among quasi-projective varieties over k, the bounded derived category $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is also equipped with Grothendieck's 6 operations: the first four were constructed by Ivorra-Morel, and the monoidal part was constructed by Terenzi ([Ter24]). Motivic sheaves, like perverse Nori motives, are equipped with Betti realisation functors (with values in the derived category of constructible sheaves on the complex points of X, when X is a variety over a subfield of $\mathbb{C}$), de Rham realisations (with values in $\mathcal{D}$-modules), and ℓ-adic realisations (with values in the category of constructible ℓ-adic complexes).

2 What is done in this thesis

2.1 Hodge and Nori realisation functors

If X is an algebraic variety over a field k that embeds into $\mathbb{C}$, M. Saito constructed in [Sai90b] an abelian category $\mathrm{MHM}(X)$ of mixed Hodge modules on X, which is an interpolation between mixed Hodge structures and perverse sheaves. Indeed, every Hodge module on X has an underlying perverse sheaf, but also an underlying complex D-module, and these two objects are related by the Riemann-Hilbert equivalence (the relative analogue of (1)), and are equipped with various filtrations. If $X = \mathrm{Spec}(k)$, then there is an equivalence between mixed Hodge modules on X and polarizable mixed Hodge structures. Saito equipped the derived category of mixed Hodge modules with Grothendieck's 6 operations. Due to the complexity of Saito's construction, we did not know how to construct a Hodge realisation functor

$$\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbb{Q}) \to \mathrm{D}^b(\mathrm{MHM}(X))$$

except when X is smooth (this is due to Ivorra ([Ivo16])), and when X varies, we did not know how to show that this functor (when it exists) is compatible with the operations. This is our first contribution:

Theorem 1 (Theorem 2.1.38)**.** *Let k be a subfield of $\mathbb{C}$. There exists a Hodge realisation*

$$\rho_{\mathrm{H}} \colon \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(-, \mathbb{Q}) \to \mathrm{D}^b(\mathrm{MHM}(-))$$

on algebraic varieties over k, which commutes with Grothendieck's 6 operations.

Due to the similarity of the construction of the 6 operations on Hodge modules and on Nori motives, this functor actually factors through the derived category of Nori motives, providing a relative version of (2):

Theorem 2 (Theorem 2.1.38, Corollary 2.1.41, Corollary 3.2.8)**.** *Perverse Nori motives extend to all finite-dimensional qcqs $\mathbb{Q}$-schemes. On these schemes, there exists a functor*

$$\rho_{\mathrm{N}} \colon \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(-, \mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$$

compatible with pullbacks. Let k be a subfield of the complex numbers. Then on algebraic varieties over k, the realisation functor ρ_{N} commutes with the 6 operations and factors the Hodge realisation functor of Theorem 1. The induced functor

$$R_{\mathrm{H}} \colon \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) \to \mathrm{D}^b(\mathrm{MHM}(-))$$

also commutes with Grothendieck's 6 operations.

The construction of these functors ρ_{N} and ρ_{H} is based on the universal property of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\mathbb{Q})$, proved by Drew and Gallauer. This universal property can only be written in the language of ∞-categories, and that is why the two theorems Theorem 2 and Theorem 1 follow from the following theorem:

Theorem 3 (Proposition 2.1.23)**.** *Let k be a subfield of $\mathbb{C}$ and denote, for every quasi-projective variety X over k, the category $\mathrm{D}^b_{\mathcal{M}}(X)$ as either $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ or $\mathrm{D}^b(\mathrm{MHM}(X))$. Then there exists a functor*

$$\mathcal{D}^b_{\mathcal{M}} \colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

with values in symmetric monoidal ∞-categories, such that for X a quasi-projective variety over k, we have an isomorphism of homotopy categories $\mathrm{ho}(\mathcal{D}^b_{\mathcal{M}}(X)) \simeq \mathrm{D}^b_{\mathcal{M}}(X)$, compatible with pullbacks.

This ∞-categorical lifting of Hodge modules and perverse Nori motives is obtained as follows: since pullbacks are rarely t-exact for the perverse t-structure, we use the constructible t-structure (also called the ordinary t-structure), whose heart is realised in the abelian category of constructible sheaves through the conservative functor to constructible complexes on the complex points. For this t-structure, all pullbacks are t-exact. By adapting arguments from Nori, we then prove:

Theorem 4 (Corollary 2.1.22)**.** *Let $\mathcal{M}_{\mathrm{ct}}(X)$ be the heart of the constructible t-structure on $\mathrm{D}^b_{\mathcal{M}}(X)$. Then the natural functor*

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \to \mathrm{D}^b_{\mathcal{M}}(X)$$

is an equivalence.

Using the good functorial properties of the ∞-functor associating to an abelian category its bounded derived ∞-category (Proposition A.6.2), Theorem 3 follows easily.

A final consequence of this ∞-categorical lifting is the existence, for every qcqs $\mathbb{Q}$-scheme X, of an algebra object $\mathscr{N}_{X,\mathbb{Q}}$ in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$, the ∞-category of possibly non constructible étale motivic sheaves, such that

$$\mathrm{Mod}_{\mathscr{N}_{X,\mathbb{Q}}}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})) \simeq \mathrm{DN}(X,\mathbb{Q}) := \mathrm{Ind}\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

functorially in X. A similar result is also obtained for Hodge modules of geometric origin, which shows that the category obtained by Drew's approach to "motivic Hodge modules" is indeed a subcategory of Hodge modules.

2.2 Nori motives with integer coefficients

In a joint work with Raphaël Ruimy, we use the ∞-categorical lifting of perverse Nori motives to construct a version with integer coefficients. The construction is very technical and fully uses ∞-categories. The idea is as follows: the rigidity theorem of Ayoub–Bachmann–Cisinski-Déglise ([BH21]) implies that for every scheme X and with torsion coefficients $\mathbb{Z}/n\mathbb{Z}$, the category $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z})$ is equivalent to complexes of sheaves of $\mathbb{Z}/n\mathbb{Z}$-module on the small étale site of X. Thus, if we had a reasonable category $\mathrm{DN}(X,\mathbb{Z})$ of Nori motives on X with integer coefficients, then on one hand, its rational objects should be equivalent to $\mathrm{DN}(X,\mathbb{Q})$,

the large category obtained by indization from the bounded derived category of perverse Nori motives, and its torsion objects should just be sheaves on the small étale site.

To do this, for every qcqs $\mathbb{Q}$-scheme X, we construct an algebra object $\mathscr{N}_X$ in $\mathrm{DM}(X,\mathbb{Z})$, such that $\mathscr{N}_X/n\mathscr{N}_X$ is just $\mathbb{Z}/n\mathbb{Z}$, and $\mathscr{N}_X \otimes \mathbb{Q}$ is the algebra $\mathscr{N}_{X,\mathbb{Q}}$ described above. Thus, by setting

$$\mathrm{DN}(X,\mathbb{Z}) := \mathrm{Mod}_{\mathscr{N}_X}(\mathrm{DM}(X,\mathbb{Z})),$$

and relying on the very good properties of modules in ∞-categories, we obtain a candidate category for relative Nori motives on X with integer coefficients. Once this ∞-category is defined, we can consider its subcategory $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ of geometric objects, and we prove:

Theorem 5 (Theorem 4.2.1, Theorem 4.3.10, Proposition 4.3.14)**.** *On finite-dimensional qcqs $\mathbb{Q}$-schemes, integral Nori motives have the 6 operations, and the Nori realisation functor $\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(-,\mathbb{Z}) \to \mathrm{DN}(-,\mathbb{Z})$ is compatible with the operations of type f^*, f_*, $f_!$, $f^!$, $\otimes$, and $\underline{\mathrm{Hom}}(M,-)$ for M geometric. Over quasi-excellent schemes the operations restrict to geometric motives. Moreover, $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ has the expected perverse and ordinary t-structures, and the realisation functors are t-exact.*

Along the way, we prove that several notions of finiteness in motives (whether Nori or Voevodsky) are the same:

Theorem 6 (Theorem 3.1.37)**.** *Let X be a finite-dimensional qcqs scheme and A a regular ring. Then an object $M \in \mathrm{DM}(X,A)$ is in the thick subcategory generated by twists of motives of smooth X-schemes if and only if for every affine open U of X, there exists a finite stratification U_i of X such that the restriction of M to each U_i is a dualisable object.*

As written above, the result is also true for Nori motives.

2.3 Comparison of Nori motives with other categories

We therefore have a comparison functor

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

that commutes with the operations. It is natural to ask whether this is an equivalence, at least when restricted to full subcategories.

The first result in this direction is conditional on the existence of a motivic t-structure on Voevodsky's motives over all fields of characteristic zero:

Theorem 7 (Theorem 2.1.45)**.** *Suppose that for every field k of characteristic zero, the category $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ is equipped with a non-degenerate t-structure, compatible with the tensor product, and such that the ℓ-adic realisations are t-exact. Then for every finite-dimensional qcqs $\mathbb{Q}$-scheme X, the realisation functor*

$$\mathrm{DM}(X,\mathbb{Z}) \to \mathrm{DN}(X,\mathbb{Z})$$

is an equivalence.

The proof easily reduces to rational coefficients, then to constructible objects, and then to varieties over $\mathbb{Q}$. In this case, we use several times the universal properties of $\mathrm{DM}_{\mathrm{gm}}(X,\mathbb{Q})$ and $\mathcal{M}_{\mathrm{perv}}(X)$.

We also have an unconditional result. We recall that the subcategory $\mathrm{DM}^0(X,\mathbb{Q})$ of Artin motives on a scheme X is the subcategory generated by shifts, extensions, and colimits by the motives of étale X-schemes, in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$. We show:

Theorem 8 (Theorem 5.1.17)**.** *Let X be a finite-dimensional qcqs $\mathbb{Q}$-scheme. Then the functor*

$$\mathrm{DM}^0(X,\mathbb{Z}) \to \mathrm{DN}(X,\mathbb{Z})$$

is fully faithful.

The proof, after reducing to rational coefficients, is a Noetherian induction, which reduces to showing this result at the generic point, where it is already known. We also calculate the heart of the induced t-structure on smooth Artin motives over a normal base, and we deduce that on a normal base, an Artin motive is dualisable if and only if it is in the thick subcategory generated by the motives of finite étale schemes.

In another direction, and with rational coefficients, we now know that the abelian category $\mathcal{M}_{\mathrm{perv}}(X)$ has all the desired realisation functors. In particular, by putting all these realisation functors together, we obtain a realisation functor

$$\mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{MR}_{\mathrm{perv}}(X)$$

with values in the category of perverse sheaves of mixed realisations: an object of the category of mixed realisations is the data of a perverse sheaf $\mathcal{F}_B$ on the complex points of X, a Hodge module $\mathcal{F}_H$ on X whose underlying perverse sheaf is $\mathcal{F}_B$, a filtered holonomic $\mathcal{D}$-module $\mathcal{F}_{dR}$ on X whose base change to $\mathbb{C}$ is the $\mathcal{D}$-module underlying $\mathcal{F}_H$, and for each prime ℓ, a perverse ℓ-adic sheaf $\mathcal{F}$ on X, whose base change to $\mathbb{C}$ is $\mathcal{F}_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Let $\mathcal{MT}^1(X)$ be the Tannakian subcategory of the category $\mathcal{M}^{\mathrm{liss}}(X)$ of motivic local systems on X (which are the objects of $\mathcal{M}_{\mathrm{perv}}(X)$ whose Betti realisation is a local system) generated by the H^i of the pushforwards $p_*\mathbb{Q}_C$ where $p\colon C \to X$ is a smooth projective relative curve. We show:

Theorem 9 (Theorem 5.2.12)**.** *Suppose X is a smooth variety over a subfield k of $\mathbb{C}$. The functor $\mathcal{MT}^1(X) \to \mathcal{MR}_{\mathrm{perv}}(X)$ is fully faithful, and its image is stable under subquotients.*

Along the way, we show that the category $\mathcal{M}^1_{\mathrm{Del}}(X)$ of relative Deligne 1-motives is a full subcategory of $\mathcal{MT}^1(X)$, and this subcategory generates it as a Tannakian category.

The proof is based on a result of Yves André, who showed ([And21]) that the Hodge realisation functor $\mathcal{MT}^1(\overline{k}) \to \mathrm{MHS}$ is fully faithful and its image is stable under subquotients. We show that his result generalises to every subfield of $\mathbb{C}$ if we change the target of the functor, replacing MHS with mixed realisations, and then on a smooth base, the result follows from the result at the generic point. A consequence of this result is that the canonical construction for Shimura varieties of Hodge type factors through perverse Nori motives (Proposition 5.2.14).

2.4 Extension of Hodge modules to stacks

It is folklore that triangulated categories do not glue. For example, if (U_i) is an open cover of an algebraic variety X over a subfield of $\mathbb{C}$, it is not true that $\mathrm{D}^b_c(X,\mathbb{Q})$ is obtained by gluing from the $\mathrm{D}^b_c(U_i,\mathbb{Q})$ and the various intersections. In any case, it is not true at the triangulated level. However, if we consider these categories as stable ∞-categories, the result becomes true! More generally, it is a general fact that if $\mathrm{D}(X,\mathscr{O}_X)$ is the bounded derived ∞-category of $\mathscr{O}_X$-module sheaves on a ringed topos $(X,\mathscr{O}_X)$, then the ∞-functor

$$(U \in X) \mapsto \mathrm{D}(U,(\mathscr{O}_X)_{|U})$$

is a sheaf. Another consequence of the ∞-categorical lifting of Hodge modules is the following:

Theorem 10 (Theorem 2.1.33)**.** *If X is a finite-type scheme over $\mathbb{C}$, let $\mathrm{D}_{\mathrm{H}}(X) := \mathrm{D}(\mathrm{IndMHM}(X))$ be the unbounded derived category of ind-Hodge modules. Then the functor*

$$\mathrm{D}_{\mathrm{H}} \colon \mathrm{Sch}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$$

induced by the f^*-functoriality, where Pr^L is the ∞-category of presentable ∞-categories, is an h-hypersheaf.

In particular, if $\mathfrak{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$, with a diagonal representable by schemes, then if $X_i \to \mathfrak{X}$ is a jointly surjective family of smooth morphisms, with X_i finite-type $\mathbb{C}$-schemes, the limit

$$\lim\Big(\ \prod_i \mathrm{D}_{\mathrm{H}}(X_i) \rightrightarrows \prod_{i,j} \mathrm{D}(X_i \times_{\mathfrak{X}} X_j) \ \substack{\rightarrow\\ \rightarrow\\ \rightarrow}\ \cdots\ \Big)$$

does not depend on the chosen cover (X_i). In particular, it becomes reasonable to extend the functor D_{H} by right Kan extension from finite-type $\mathbb{C}$-schemes to algebraic stacks locally of finite type over $\mathbb{C}$. Based on results by Liu and Zheng ([LZ17]) on extending operations to stacks, we show:

Theorem 11 (Theorem 6.1.6, Theorem 6.1.13, Corollary 6.1.28, Theorem 6.1.40)**.** *Grothendieck's 6 operations, as well as t-structures and weights, exist for the extension of $\mathrm{D}_{\mathrm{H}}(-)$ to stacks. This is also the case for nearby cycles.*

When $\mathfrak{X} = [X/G]$, we thus recover the G-equivariant Hodge modules of Achar ([Ach13]).

3 Articles contained in this thesis

This thesis contains most of the following articles:

[Tub23] Swann Tubach. On the Nori and Hodge realisations of Voevodsky étale motives. Preprint, available at https://arxiv.org/abs/2309.11999, 2023.

[RT24a] Raphaël Ruimy and Swann Tubach. Étale motives of geometric origin. Preprint, available at https://arxiv.org/abs/2405.07095, 2024.

[RT24b] Raphaël Ruimy and Swann Tubach. Nori motives (and mixed Hodge modules) with integral coefficients. Preprint, available at https://arxiv.org/abs/2407.01462, 2024.

[Tub24a] Swann Tubach. Artin motives in relative Nori and Voevodsky motives. Preprint, available at https://arxiv.org/abs/2409.12841, 2024.

[Tub24b] Swann Tubach. Mixed Hodge modules on stacks. Preprint, available at https://arxiv.org/abs/2407.02256, 2024.

4 Organisation of the manuscript

The manuscript consists of six chapters and an appendix.

1 In the first chapter, we provide some recolletions. We discuss stable ∞-categories, different forms of motives, and Hodge modules. Some proofs are already contained in this part.

2 In the second chapter, we include most of the article [Tub23]: We show how to construct a coherent ∞-categorical lifting of the 6 operations on perverse Nori motives and Hodge modules. This goes by proving that the bounded derived category of the heart of the constructible t-structure on the derived category of perverse Nori motives agrees with the derived category of perverse Nori motives. We end the chapter with some reminders and novelties on mixed realisations categories.

3 In the third chapter, we discuss modules in étale motives. After defining the enriched categories of motives of Drew, we present the content of the article [RT24a] joint with Raphaël Ruimy, translated into this language. Next, we show that perverse Nori motives and Hodge modules of geometric origin are examples of modules in étale motives, as shown in [Tub23] and [Tub24b].

4 The fourth chapter is entirely from the collaboration [RT24b] with Raphaël Ruimy. It is about constructing categories of integer Nori motives on a relatively general base. The construction, although very technical, can be summarised by the following sentence: we glue perverse Nori motives with rational coefficients with torsion étale sheaves. We also show that the expected t-structures exist.

5 In the fifth chapter, we compare subcategories of Nori motives with other categories. In a first part, taken from [Tub24a], we show that on a general base of characteristic zero, Artin motives are the same in Voevodsky's étale motives and in Voevodsky's motives. A subsection is in collaboration with Raphaël Ruimy: it is about generalizing Artin's vanishing properties he showed on Artin motives to bases of characteristic zero. In a second part, we study what we have called Tate 1-motives, which form the Tannakian subcategory of smooth Nori motives generated by Deligne's motives, on a smooth base. We show that the "mixed realisations" functor of perverse Nori motives is fully faithful on this subcategory. As a corollary, we obtain that the canonical construction on Shimura varieties factors through Nori motives.

6 In the sixth and final chapter, which contains the article [Tub24b], we explain how, using the lifting of Hodge modules into the ∞-categorical world, it is possible to canonically extend them to algebraic stacks. Relying on results by Liu and Zheng, we show that this extension always has Grothendieck's 6 operations, as well as weights and t-structures.

In the last section, inspired by the interpretation of Ayoub's nearby cycles construction using Kozsul duality presented in [CHS24], we show that Hodge modules on stacks also have nearby cycles satisfying good properties.

A Finally, in the appendix, we have written a breviary on ∞-categories, which is intended to be functional.

Chapter 1

Recollections

We begin with some recollections: stable ∞-categories, motives and categories of coefficients. Although most of this chapter is probably well known to experts, we also provided some proofs of results we could not find references for.

1.1 Stable ∞-categories

Stable ∞-categories are the natural generalisation of the notion of triangulated categories to ∞-categories. It is much better behaved, as we will see below. We give here only the definition and basic results. More results and some demonstrations are in the last section of the Appendix A.

If $\mathcal{C}$ is an ∞-category admitting finite limits and colimits, recall that, given an object $c \in \mathcal{C}$, we can form its loop space and its suspension:

Definition 1.1.1. Let $\mathcal{C}$ be an ∞-category admitting finite limits and colimits. Let $c \in \mathcal{C}$.

1. The suspension Σc of c is the pushout

$$\begin{array}{ccc} c & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \Sigma c \end{array}$$

   where $*$ is a terminal object of $\mathcal{C}$.

2. The loop space Ωc of c is the pullback

$$\begin{array}{ccc} \Omega c & \longrightarrow & \emptyset \\ \downarrow & \lrcorner & \downarrow \\ \emptyset & \longrightarrow & c \end{array}$$

   where $\emptyset$ an initial object of $\mathcal{C}$.

It is not hard to show that Σ and Ω are endofunctors of $\mathcal{C}$. We say that $\mathcal{C}$ is pointed if any initial object is also terminal. In that case, we have that Ω is a right adjoint to Σ.

In the ∞-category $\mathcal{S}_*$ of pointed spaces, they literally are the suspension and loop spaces of a topological space. There are other examples.

We now give the definition of stable ∞-categories. We emphasize that being a stable ∞-category is a property and not a structure!

Definition 1.1.2. Let $\mathcal{C}$ be an ∞-category admitting finite limits and colimits. We say that $\mathcal{C}$ is *stable* if it is pointed and if for any square

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

of objects in $\mathcal{C}$, the square is cartesian if and only if it is cocartesian. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between stable ∞-categories is *exact* if it preserves finite colimits. If $\mathcal{C}$ is a stable ∞-category, its initial object will be denoted by 0.

In particular by taking the squares defining the suspension and the loop space, we see that if $\mathcal{C}$ is a stable ∞-category, the functors Ω and Σ are equivalences, inverses of each other. We will denote $\Sigma = (-)[1]$ and $\Omega = (-)[-1]$. It is clear that if $\mathcal{C}$ is stable, then $\mathcal{C}^{\mathrm{op}}$ is also stable. Moreover, a functor between stable ∞-categories is exact if and only if it preserves finite limits. In fact, it is exact if and only if it preserves finite coproducts and exact triangles:

Definition 1.1.3. Let $\mathcal{C}$ be a stable ∞-category. A sequence $X \to Y \to Z$ in $\mathcal{C}$ is called an *exact triangle* if the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

is cartesian. In this case, we say that X is a fiber of g, and that Z is a cofiber of f. We use the notation $X = \mathrm{fib}(g)$ and $Z = \mathrm{cof}(f)$.

Sometimes, exact triangles are called cofiber sequences or fiber sequences. We will also use the notation M/N for the cofiber of a map $N \to M$.

The following proposition is the fundamental fact that give the good behaviour of stable ∞-categories, as opposed to triangulated categories.

Proposition 1.1.4. *Let $\mathcal{C}$ be a stable ∞-category. There exist functors*

$$\mathrm{fib}, \mathrm{cofib}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

whose value at any arrow f is $\mathrm{fib}(f)$ *and* $\mathrm{cofib}(f)$.

Proof. We do the case of fibers. We can consider the composition

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{F} \mathrm{Fun}(\lrcorner, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

where the functor F sends an arrow $f\colon X \to Y$ to the diagram

$$\begin{array}{ccc} & & X \\ & & {\scriptstyle f}\downarrow \\ 0 & \longrightarrow & Y \end{array}.$$

□

Thus, in a stable ∞-category, the cone is functorial! The link with triangulated categories is the following:

Theorem 1.1.5 (Lurie [Lur22, Theorem 1.1.2.14.])**.** *Let $\mathcal{C}$ be a stable ∞-category. The shift functor $(-)[1]$, together with the class of exact triangles, define a structure of a triangulated category on $\mathrm{ho}(\mathcal{C})$.*

Definition 1.1.6. Let $\mathcal{C}$ be a stable ∞-category. Let $S \subset \mathcal{C}$ be a collection of objects. The *thick* subcategory of $\mathcal{C}$ spanned by S is the smallest stable subcategory that contains S and which is stable under taking direct factors.

Definition 1.1.7. Let $\mathcal{C}$ be a stable ∞-category. A t-structure on $\mathcal{C}$ is a t-structure on the triangulated category $\mathrm{ho}(\mathcal{C})$ ([BBD82]): it is a pair a of full subcategories $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 1})$ satisfying the usual axioms.

It turns out that this implies that the truncations functors are ∞-functors ([Lur22, Proposition 1.2.1.5]). Moreover, if $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 1})$ is a t-structure on a stable ∞-category $\mathcal{C}$, then the truncations functors actually provide truncated spaces: If $x, y \in \mathcal{C}^{[-n,n]}$, then the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is $2n+1$-truncated: its homotopy groups vanish above $2n+1$. This will be really useful later, as many limits simplify in a truncated category.

Denote by Cat_∞ the ∞-category of small ∞-categories (Theorem A.1.11). Stable ∞-categories enjoy very good stability properties under limits and colimits:

Proposition 1.1.8 ([Lur22, Theorem 1.1.4.4 and Proposition 1.1.4.6.])**.** *Let $\mathrm{Cat}_\infty^{\mathrm{st}}$ be the ∞-category of small stable ∞-categories and exact functors between them. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$ admits all small limits and colimits, and the forgetful functor*

$$\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty$$

commutes with small limits and small filtered colimits.

The forgetful functor cannot commute with all colimits: in $\mathrm{Cat}_\infty^{\mathrm{st}}$, finite products and finite coproducts are isomorphic.

One can glue t-structures along a limit and a filtered colimit:

Proposition 1.1.9. *Let $\mathcal{C} \colon I \to \mathrm{Cat}_\infty^{\mathrm{st}}$ be a diagram of stable ∞-categories and exact functors. We assume that for each $i \in I$ the ∞-category $\mathcal{C}(i)$ is endowed with a t-structure and that all transition functors are t-exact. Then:*

1. *The limit $\lim_I \mathcal{C}(i)$ is endowed with a t-structure such that all projection functors are t-exact.*
2. *If I is filtered, then the colimit $\mathrm{colim}_i \mathcal{C}(i)$ in Cat_∞ is endowed with a t-structure such that the inclusion functors are t-exact.*

The obtained hearts are the limits and the colimit of the hearts, respectively.

Proof. The first point is Lemma 3.2.18 of [RS20]. The second part is easy as everything is explicit: mapping spaces are colimits of mapping spaces, and the truncation triangle exists already in some $\mathcal{C}(i)$. □

We might have to use the above proposition for limit of stable presentable ∞-categories endowed with t-structures. The proposition still applies, because limits in Pr^L are computed in $\widehat{\mathrm{Cat}_\infty}$ the ∞-category of large ∞-categories, thus the same proof works, up to a change of universe.

Proposition 1.1.10. *Let $\mathcal{C}$ be a stable ∞-category. Then any full subcategory of $\mathcal{C}$ stable under cofibers and translations is itself stable, and the inclusion functor is then exact. For any simplicial set K, the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ is stable, and any evaluation functor is exact.*

A property which is false for triangulated categories but which holds for stable ∞-categories is the following:

Proposition 1.1.11 ([Lur22, Proposition 1.1.3.6.] and [Lurb, Lemma C.2.4.3.])**.** *Let $\mathcal{C}$ be a small stable ∞-category. The ∞-category $\mathrm{Ind}(\mathcal{C})$ is also stable. The idempotent completion of $\mathcal{C}$ is also stable. Moreover, if $\mathcal{C}$ has a t-structure, then $\mathrm{Ind}(\mathcal{C})$ is endowed with a t-structure and the inclusion functor*

$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$$

is t-exact. Any exact and t-exact functor between small stable ∞-categories with t-structures induces a t-exact functor between indizations. In fact, we have $(\mathrm{Ind}\mathcal{C})^{\leqslant 0} = \mathrm{Ind}(\mathcal{C}^{\leqslant 0})$ and $(\mathrm{Ind}\mathcal{C})^{\geqslant 1} = \mathrm{Ind}(\mathcal{C}^{\geqslant 1})$. Furthermore, if the t-structure on $\mathcal{C}$ is right bounded, then the t-structure on $\mathrm{Ind}(\mathcal{C})$ is right complete, which mean in this case that

$$\cap_n \mathrm{Ind}(\mathcal{C})^{\geqslant n} = \{0\}.$$

There is a way of producing a stable ∞-category out of any ∞-category with finite limits. Indeed, one may remark ([Lur22, Proposition 1.4.2.11]) that an ∞-category $\mathcal{C}$ with finite limits and colimits is stable if and only if the loop space functor Ω is an equivalence. Thus, given an pointed ∞-category $\mathcal{C}$ admitting finite limits, there is a universal way of producing a stable ∞-category with a map to $\mathcal{C}$:

Proposition 1.1.12. *Let $\mathcal{C}$ be a pointed ∞-category with finite limits. Then the limit*

$$\mathrm{Sp}(\mathcal{C}) := \lim(\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C})$$

is a stable ∞-category. Moreover, composition with the canonical functor $\Omega^\infty \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ projecting on the last term yields for every stable ∞-category $\mathcal{D}$, an equivalence of ∞-categories between exact functors $\mathcal{D} \to \mathrm{Sp}(\mathcal{C})$ and functors $\mathcal{D} \to \mathcal{C}$ that commute with finite limits. We call $\mathrm{Sp}(\mathcal{C})$ the stabilisation of $\mathcal{C}$.

The definition of a symmetric monoidal triangulated category is unfortunately too long to be written here. We give the definition of a symmetric monoidal stable ∞-category.

Definition 1.1.13. A stable ∞-category $\mathcal{C}$ is a stable symmetric monoidal ∞-category if $\mathcal{C}$ is a symmetric monoidal ∞-category and if the tensor product $- \otimes -$ is exact in both variables: if $X \in \mathcal{C}$ then $X \otimes -$ is an exact functor.

We will come back to this notion in the next sections and in the Appendix.
Before giving examples, we mention the interaction between stability and presentability.

Proposition 1.1.14 ([Lur22, Corollary 1.4.4.2.])**.** *Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is presentable if and only if the following conditions are verified:*

1. *The category $\mathcal{C}$ admits small coproducts.*
2. *The homotopy category $\mathrm{ho}(\mathcal{C})$ is locally small (which means that the* Hom*-groups are small).*
3. *There exists a regular cardinal κ and a κ-compact generator $X \in \mathcal{C}$.*

If $\mathcal{C}$ is presentable and $\mathcal{D}$ is a stable ∞-category admitting small coproducts, then $\mathcal{D}$ admits small colimits and an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves small colimits if and only if it preserves small coproducts.

Definition 1.1.15. Let $\mathcal{C}$ be a presentable stable ∞-category. Let $S \subset \mathcal{C}$ be a collection of objects. Then the *localising* full subcategory of $\mathcal{C}$ generated by S is the smallest full subcategory of $\mathcal{C}$ that contains S and is stable under small colimits and finite limits. It is stable.

Proposition 1.1.10 implies that any exact Bousfield localisation of a presentable stable ∞-category is again stable. In particular, if $\mathcal{C}$ is a stable presentable ∞-category and X is a site endowed with a topology τ, then the category of sheaves $\mathrm{Sh}_\tau(X, \mathcal{C})$ (see Definition 1.2.14) is again stable.

The Bousfield localisation of a stable presentable ∞-category is easy to compute:

Proposition 1.1.16 ([BGT13, Proposition 5.6])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving functor between presentable stable ∞-categories. Let S be the collection of maps in $\mathcal{D}$ whose cone lie in the essential image of $\mathcal{C}$. Then the Bousfield localisation $S^{-1}\mathcal{D}$ is naturally isomorphic to the cofiber in $\mathrm{Pr}^L_{\mathrm{st}}$ of F.*

In the case of presentable ∞-categories, the stabilisation construction is more similar to the usual localisation of rings where we invert the multiplication under one element:

Proposition 1.1.17. *Let $\mathcal{C}$ be a pointed presentable ∞-category. Then $\mathrm{Sp}(\mathcal{C})$ is presentable, and can be written as the colimit in Pr^L of the diagram*

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots .$$

Moreover, the inclusion of the full subcategory $\mathrm{Pr}^L_{\mathrm{st}} \subset \mathrm{Pr}^L$ of presentable and stable ∞-categories admits the functor Sp *as a left adjoint: It is a Bousfield localisation.*

We now give examples: The initial example is the ∞-category of spectra: it is the stabilisation of pointed spaces. To the category $\mathcal{S}$ of spaces one can associate pointed spaces $\mathcal{S}_* = \mathcal{S}_{\setminus *}$, which is pointed (by the point). The stabilisation $\mathrm{Sp}(\mathcal{S}_*)$ is the ∞-category Sp of spectra. It is presentable, compactly generated, and symmetric monoidal. Moreover, it has a t-structure whose heart identifies with the abelian category of abelian groups.

Proposition 1.1.18. *Let $\mathcal{C}$ be a stable ∞-category. Then the functor*

$$\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

factors through the functor $\Omega^\infty\colon \mathrm{Sp} \to \mathcal{S}$. In other words, for any $x, y \in \mathcal{C}$, there is a mapping spectrum $\mathrm{map}_{\mathcal{C}}(x, y)$ *that satisfies $\Omega^\infty \mathrm{map}_{\mathcal{C}}(x, y) = \mathrm{Map}_{\mathcal{C}}(x, y)$ functorially.*

We will give a conceptual explanation of the above proposition using the Lurie tensor product (Construction A.5.5).

In this thesis, the fundamental example of a stable ∞-category is the bounded ∞-category $\mathrm{D}^b(\mathcal{A})$ of an abelian category $\mathcal{A}$.

Definition 1.1.19. Let $\mathcal{A}$ be an abelian category. The bounded derived ∞-category $\mathrm{D}^b(\mathcal{A})$ is the localisation of the category $\mathrm{Ch}^b(\mathcal{A})$ of complexes with respect to the collection of quasi-isomorphism.

Proposition 1.1.20. *The ∞-category $\mathrm{D}^b(\mathcal{A})$ is stable and admits a t-structure whose heart is $\mathcal{A}$. It can also be obtained by first inverting chain homotopies to get $\mathcal{K}^b(\mathcal{A})$ the ∞-category of complexes (which is also the dg-nerve of the 1-category $\mathrm{Ch}^b(\mathcal{A})$ with its simplicial structure), and then inverting quasi-isomorphisms.*

$\mathrm{D}^b(\mathcal{A})$ is universal among stable ∞-categories having an exact functor from $\mathcal{A}$. By an exact functor from an abelian category to a stable ∞-category, we mean a functor that commutes with finite coproducts and sends short exact sequences to exact triangles.

Theorem 1.1.21 ([BCKW19, Corollary 7.4.12])**.** *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{A}$ be an abelian category. Then restriction to the heart gives an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{D}^b(\mathcal{A}),\mathcal{C}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A},\mathcal{C})$$

between ∞-categories of exact functors.

1.2 Overview of motives

1.2.1 Motives over a field

Let k be a field of characteristic zero. Following the vision of Beilinson, Deligne, Grothendieck and others there should exist an abelian category of mixed motives $\mathcal{MM}(k)$, target of the universal cohomology theory

$$M^* : \mathrm{Var}_k^{\mathrm{op}} \to \mathcal{MM}(k)$$

on k-varieties in the sense that any other reasonable cohomology theory on k-varieties would factor uniquely through M^*. Moreover, the homomorphism and extension groups in this category should be given by algebraic cycles following very precise formulas conjectured by Grothendieck and Beilinson. This is out of reach of the current technology. However since the late 90s there exist two constructions of motives. Both have their upsides, and their downsides, and if put together they would check all the boxes for a category of mixed motives. The two approaches start by choosing common properties of cohomology theories, and then consider the "free category" that has those properties.

Let us begin with the category constructed by Hanamura, Levine, Voevodsky and others, that now has the name *geometric motives*. This is ought to be the bounded derived category of motives. We give here a sketch of the classical construction, but we will give a modern and precise definition later (Definition 1.2.14). We know from the work of Deligne on mixed Hodge structures that many cohomological theories of varieties over k can be recovered from the cohomology of smooth varieties. Thus, one starts by contemplating smooth varieties

over k. Next, because many cohomology theories (singular, ℓ-adic...) on smooth varieties admits more functoriality than just pullbacks along morphisms of schemes, on consider finite correspondences, that emulate finite pushforward on cohomology classes. Out of this, one can construct a triangulated category $K^b(\mathrm{SmCor}_k)$ of complexes of smooth varieties up to chain homotopies, and force two natural conditions: $\mathbb{A}^1$-invariance and Mayer-Vietoris excision. This way, we get a triangulated category $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k,\mathbb{Z})$ of *effective geometric motives*, together with a functor

$$M\colon \mathrm{SmCor}_k \to \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k,\mathbb{Z})$$

sending to a smooth scheme its motive $M(X) = X[0]$. It comes naturally with a tensor product, given on smooth varieties by the product, and in order to make Poincaré duality work, we need to add negative Tate twists: the object $M(\mathbb{P}^1_k)$ is naturally split as

$$M(\mathbb{P}^1_k) = \mathbb{Z} \oplus \mathbb{Z}(1)[2],$$

and we invert the object $\mathbb{Z}(1)$ for the tensor product giving us the category $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Z})$ of Voevodsky geometric motives. In this category, every object is dualisable for the tensor product, and the extension groups have the wanted formulae conjectured by Beilinson: they are Bloch's higher Chow groups. Because the properties we forced on geometric motives are shared by many cohomological varieties, we can construct ([Hub00]) realisation functors from the category of geometric motives to several categories of coefficients, such as complexes of $\mathbb{Q}$-vector spaces (the so-called Betti realisation, modelled on singular cohomology), complexes of ℓ-adic representations of the absolute Galois group of k (modelled on ℓ-adic cohomology), or complexes of k-vector spaces (modelled on de Rham cohomology). Conjecturally, the rational version of $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Z})$ is the derived category of mixed motives, but no reasonable motivic t-structure has been constructed on $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ yet, thus we cannot have an abelian category of motives from this construction yet.

This is where Nori's construction comes handy. Indeed, Nori category of motives is a rigid tensor abelian category. It is constructed from the following observation: in characteristic zero, all cohomology theories compare to singular cohomology, and moreover the minimal set of axioms that a cohomology theory should have is the following:

1. For every morphism $f\colon Y \to X$ of varieties, a morphism $f^*\colon \mathrm{H}^*(X) \to \mathrm{H}^*(Y)$.
2. For every triple of closed subsets $Z \subset Y \subset X$ of varieties, a long exact sequence of cohomology

$$\cdots \longrightarrow \mathrm{H}^i(X,Y) \longrightarrow \mathrm{H}^i(X,Z) \longrightarrow \mathrm{H}^i(Y,Z) \xrightarrow{\partial} \mathrm{H}^{i+1}(X,Y) \longrightarrow \cdots$$

3. The object $\mathbb{Z}(-1) := \mathrm{H}^2(\mathbb{P}^1_k)$ is invertible for the tensor product.

All of this functoriality can be made into an oriented graph, or quiver Pairs_k, which has a representation

$$\mathrm{H}^*_B\colon \mathrm{Pairs}_k \to \mathrm{Vect}_k$$

creating singular cohomology, and in a precise sense that we will give later (Theorem 1.2.22), the category $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$ of *Nori motives* is the closest abelian category which receives a representation of Pairs_k factoring H^*_B, and in which $\mathbb{Z}(-1)$ is invertible. This universal property

allows one to construct several realisations of Nori motives: the Betti realisation of course, but also the de Rham realisation, the ℓ-adic realisation, and the Hodge realisation. Again conjecturally, the category $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$ should be *the* category of mixed motives. Unfortunately due to the very formal definition of Nori motives using a universal property, extension groups between Nori motives are hard, if not impossible, to compute, thus the link with algebraic cycles and the conjectural description of Beilinson is far from being clear.

However, the work of Nori, and then the work of Harrer and Choudhury-Gallauer, enables one to construct a comparison functor

$$\rho_{\mathrm{N}} \colon \mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Z}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$$

to the derived category of Nori motives, compatible with the realisations, and which is conjecturally an equivalence with rational coefficients.

This functor cannot be an equivalence with integral coefficients: a famous counterexample of Voevodsky ([VSF00, Chap. 5, Proposition 4.3.8]) shows that there cannot be any motivic t-structure on $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Z})$. The argument relies on the vanishing of some group of zero cycles. It turns out that étale-locally, every variety has a zero cycle, thus that this argument does not work anymore. There is a modification of the category $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Z})$, using the étale topology, such that the functor ρ_{N} above factors through "étale sheafification", and gives an enhanced realisation

$$\rho_{\mathrm{N}}^{\text{ét}} \colon \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,\mathbb{Z}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$$

which is conjectured to be an equivalence. We will define $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,\mathbb{Z})$ later, but one thing to know is that the description of extension groups using cycles still holds, if one sheafify finite correspondences in the right way. For example, when algebraic K-theory groups where distinguishable in $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Z})$, in the category $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,\mathbb{Z})$ we only get étale K-theory groups.

In this manuscript, we talk of realisations and t-structures, thus we will mostly care about étale motives and étale sheaves.

1.2.2 Grothendieck 6 operations

In this section, we give an overview of what can be a 6 functors formalism. We begin with the general idea and finish by some precise definitions.

General idea and a definition

Let S be a Noetherian base scheme, in this section all schemes will be of finite type over S. Say, for each finite type S-scheme X we are given a stable ∞-category $\mathrm{D}(X)$ (we could do everything here with triangulated categories, but the language of higher categories is more convenient and one recovers the triangulated definitions after taking homotopy categories). We want to see $\mathrm{D}(X)$ as some derived category of sheaves over X. For example, $\mathrm{D}(X)$ could be the derived category of étale sheaves on X with coefficients in some ring R, or maybe the derived category of D-modules on X, or even something weirder such as Zariski sheaves of spectra on X. Given a morphism of schemes $f \colon Y \to X$ it is usually possible to restrict any sheaf $\mathcal{F}$ on X to a sheaf $f^*\mathcal{F}$ on Y. We want, for any such morphism f, an exact ∞-functor

$$f^* \colon \mathrm{D}(X) \to \mathrm{D}(Y)$$

called the *pullback functor*. Moreover, any diagram of S-schemes should provide a diagram of categories: all pullback functors f^* should assemble to give a functor

$$\mathrm{D}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

For each morphism $f\colon Y \to X$, this functor f^* should have a right adjoint

$$f_*\colon \mathrm{D}(Y) \to \mathrm{D}(X),$$

called the pushforward functor, which has to be seen as a relative version of cohomology: in most cases, when X is the spectrum of a field, the pushforward f_* is identified with the hypercohomology functor $\mathrm{R}\Gamma(Y, -)$. This already contains a lot of information: given any commutative square

$$\begin{array}{ccc} Y' & \xrightarrow{g} & X' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow[f]{} & X \end{array}$$

of S-schemes, there is a canonical exchange morphism

$$g_* b^* \to a^* f_*$$

obtained by a play of adjunctions. It is common that if the square is cartesian and if f is proper or a is smooth, this exchange morphism is an equivalence, giving a categorical interpretation of the proper base change and smooth base change theorems: the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{D}(Y) & \xrightarrow{f_*} & \mathrm{D}(X) \\ {\scriptstyle b^*}\downarrow & & \downarrow{\scriptstyle a^*} \\ \mathrm{D}(Y') & \xrightarrow[g_*]{} & \mathrm{D}(X') \end{array},$$

a priori only lax-commutative, commutes!

We also want a tensor structure, that will play nicely with the operations to recover the Künneth formula. The ∞-category $\mathrm{D}(X)$ should be symmetric monoidal, and all pullback functors f^* should be symmetric monoidal functors. The best is to have a functor

$$\mathrm{D}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

that encompasses both the fact that each $\mathrm{D}(X)$ is symmetric monoidal (it is a commutative algebra in the ∞-category of ∞-categories) and each f^* is symmetric monoidal (it is a morphism of algebras), and even more: every diagram of categories now gives a diagram of symmetric monoidal ∞-categories and symmetric monoidal functors. The tensor product can be closed, so that we may have internal homomorphism $\underline{\mathrm{Hom}}(-,-)$. Playing with adjunctions again, we obtain for each morphism $f\colon Y \to X$ an exchange map

$$(-) \otimes f_*(-) \to f_*(f^*(-) \otimes (-))$$

which are often isomorphisms when f is proper, giving a categorical interpretation of the proper projection formula: The category $\mathrm{D}(X)$ acts on $\mathrm{D}(Y)$ through the formula $M \cdot N =$

$f^*(M) \otimes N$, and on itself using the tensor product, and the proper projection formula asserts that the functor f_* is $\mathrm{D}(X)$-linear. The Künneth formula can easily (at least when one of the two factors is proper but one can reduce to this, see below) be recovered from this projection formula and the proper base change: let $f : X \to \mathrm{Spec}(k)$ be a proper variety and let $g : Y \to \mathrm{Spec}(k)$ be any algebraic variety. Denote by $\pi\colon X \times_k Y \to \mathrm{Spec}(k)$ the structural morphism, and $p\colon X \times_k Y \to X$, $q\colon X \times_k Y \to Y$ the projections. Then we have

$$\begin{aligned} f_*\mathbf{1}_{\mathrm{D}(X)} \otimes g_*\mathbf{1}_{\mathrm{D}(Y)} &\simeq f_*f^*\mathbf{1}_k \otimes f_*\mathbf{1}_{\mathrm{D}(Y)} \\ &\simeq f_*(\mathbf{1}_{\mathrm{D}(X)} \otimes f^*g_*\mathbf{1}_{\mathrm{D}(Y)}) \\ &\simeq f_*(p_*q^*\mathbf{1}_{\mathrm{D}(Y)}) \\ &\simeq \pi_*\mathbf{1}_{\mathrm{D}(X\times_k Y)}, \end{aligned}$$

where we pass from the first to the second line using the projection formula, and from the third to the last using the proper base change theorem. Using the hypercohomology notation, this gives:

$$\mathrm{R}\Gamma(X) \otimes \mathrm{R}\Gamma(Y) \xrightarrow{\sim} \mathrm{R}\Gamma(X \times_k Y).$$

Let $i\colon Z \to X$ be a closed subset, and let $j\colon U \to X$ be the open immersion of the complement. We want the functor j^* to be itself a right adjoint, with left adjoint a functor $j_!$ the *extension by zero*. Using the adjunctions unit and co-units, we may write a sequence

$$j_!j^* \to \mathrm{Id}_{\mathrm{D}(X)} \to i_*i^*$$

of endofunctors of $\mathrm{D}(X)$. We say that D has the *localisation property* if for any such i, the above triangle is an exact triangle. This can be rewritten categorically as the fact that the square

$$\begin{array}{ccc} \mathrm{D}(Z) & \xrightarrow{i_*} & \mathrm{D}(X) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ * & \longrightarrow & \mathrm{D}(U) \end{array}$$

is cartesian. In terms of hypercohomology, after applying hypercohomology with compact supports $\mathrm{R}\Gamma_c(X, -)$, this implies that the sequence

$$\mathrm{R}\Gamma_c(U) \to \mathrm{R}\Gamma_c(X) \to \mathrm{R}\Gamma_c(Z)$$

gives an exact triangle. Another property on may ask is $\mathbb{A}^1$-invariance: indeed, many cohomology theories are $\mathbb{A}^1$-invariant, meaning that the cohomology of a scheme X is the same as the cohomology of the scheme $\mathbb{A}^1_X$. This is expressed in categorical terms by asking that pullback along the projection map

$$\mathrm{D}(X) \to \mathrm{D}(\mathbb{A}^1_X)$$

is fully faithful.

We have seen that pushforward is a relative version of cohomology. But what about cohomology with compact support? In [Del77], Deligne defines cohomology with compact support in the following way: if a morphism $f\colon Y \to X$ is separated, we may use a Nagata compactification ([Con07]) to factor f as a composition

$$Y \xrightarrow{j} \overline{Y} \xrightarrow{p} X$$

where j is an open immersion and p is proper. Now, we may define relative cohomology with compact supports

$$f_! \colon \mathrm{D}(Y) \to \mathrm{D}(X)$$

as the composition $p_* j_!$. Because the set of possible compactification is filtered, and using proper base change, one can show (the ∞-categorical proof of this (together with all the coherences) is insanely hard ([LZ15])) that the functor $f_!$ does not depend on the choice, providing us with an *exceptional direct image* functor. This functor often has a right adjoint $f^!$. Moreover, for each cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{g} & X' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow[f]{} & X \end{array}$$

as above with f separated, there exists a canonical morphism

$$g_! b^* \to a^* f_!$$

which is always an isomorphism! There is also a projection formula

$$(-) \otimes f_!(-) \simeq f_!(f^*(-) \otimes (-)).$$

This exceptional functoriality enable us to define relative homology as $f_! f^!$. If f is proper, we see from the definition that $f_! \simeq f_*$. When f is étale, we have a canonical isomorphism $f^! \simeq f^*$. When X is the spectrum of a field, we are tempted to denote $f_!$ by $\mathrm{R}\Gamma_c(Y, -)$. Moreover, when D satisfy Zariski descent, one can construct the functor $f_!$ for any finite type morphism.

More generally, there is a way to compare f^* and $f^!$ when f is only smooth. For this we need to introduce Thom twists. As in algebraic topology, Thom spaces are associated to vector bundles. If $p \colon V \to X$ is a vector bundle with zero section s, the (inverse) Thom twist of V is the functor $\mathrm{Th}^{-1}(V) := s^! p^*$. In the cases that interest us, this functor will be invertible, and moreover, there will be a canonical isomorphism

$$\mathrm{Th}(T_f)(\mathbf{1}_{\mathrm{D}(X)}) \otimes f^* \simeq f^!$$

called the purity isomorphism, with T_f the tangent bundle of the smooth morphism f. Furthermore, our categories of choice D will be *oriented* ([Dé18]) which implies that all Thom twists are isomorphic to Tate twists, that is Thom twists of trivial bundles. In this case, if f is pure of relative dimension d, the purity isomorphism reads

$$f^*(d)[2d] \simeq f^!.$$

In particular, the functor f^* has a left adjoint $f_\sharp$, given by the formula

$$f_\sharp \simeq f_!(-d)[-2d].$$

Let us recall what we saw in a somewhat vague definition:

Definition 1.2.1. A 6 functor formalism D on S-schemes is a functor

$$\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$$

with values in symmetric monoidal stable ∞-categories, such that:

1. For each morphism $f\colon Y \to X$, we have an adjunction $f^*\colon \mathrm{D}(X) \leftrightarrow \mathrm{D}(Y)\colon f_*$.
2. Proper base change and smooth base change hold.
3. For each morphism $f\colon Y \to X$, we have an adjunction $f_!\colon \mathrm{D}(Y) \leftrightarrow \mathrm{D}(X)\colon f^!$.
4. Proper base change holds for $f_!$.
5. If f is proper $f_! \simeq f_*$, if f is étale, $f^! \simeq f^*$.
6. Proper projection formula holds.

All those point above are subject to many coherences conditions. Moreover, the following properties may be wanted:

A. D has the localisation property.

B. D satisfies $\mathbb{A}^1$-invariance.

C. D has a purity isomorphism for smooth morphisms.

D. D is oriented.

The whole start of the relative theory of motives, as began by Voevodsky and studied extensively by Ayoub in his PhD, is that the whole 6 functors formalism can be recovered from the pullbacks if one has left adjoints $f_\sharp$ for smooth morphisms. The ∞-categorical version of this definition is due to Drew and Gallauer ([DG22])

Definition 1.2.2 (Ayoub, Drew-Gallauer)**.** A coefficient system on S-schemes is a functor

$$\mathrm{D}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$$

such that if one denote by $f^* = D(f)$, we have:

1. (a) For each morphism f, the functor f^* has a right adjoint f_*.
   (b) For each S-scheme X, the tensor structure on $\mathrm{D}(X)$ is closed: there exist internal homomorphisms.
2. **(Smooth Base Change and projection formula)** For each smooth morphism $p\colon Y \to X$ of S-schemes the functor f^* admits a left adjoint $f_\sharp$ and for each cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{g} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array}$$

   the natural transformation $q_\sharp g^* \to f^* p_\sharp$ is an equivalence. Moreover, the natural transformation $p_\sharp(p^*(-) \otimes (-)) \to (-) \otimes p_\sharp(-)$ is an equivalence.

3. **(Localisation)** For each closed immersion $i\colon Z \to X$ with open complement $j\colon U \to X$ the square

$$\begin{array}{ccc} \mathrm{D}(Z) & \xrightarrow{i_*} & \mathrm{D}(X) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ * & \longrightarrow & \mathrm{D}(U) \end{array}$$

is cartesian.

4. For each $X \in \mathrm{Sch}_S$, if $\pi_{\mathbb{A}^1}\colon \mathbb{A}^1_X \to X$ is the projection with zero section $s\colon X \to \mathbb{A}^1_X$, then

   (a) **($\mathbb{A}^1$-invariance)** The functor $\pi^*_{\mathbb{A}^1}\colon \mathrm{D}(X) \to \mathrm{D}(\mathbb{A}^1_X)$ is fully faithful.

   (b) **Tate stability** The composite $\pi_{\mathbb{A}^1\sharp}s_*\colon \mathrm{D}(X) \to \mathrm{D}(X)$ is an equivalence.

A morphism of coefficient systems is a natural transformation $\phi : \mathrm{D} \to \mathrm{D}'$ such that for each smooth morphism $p : Y \to X$ in Sch_k, the exchange transformation

$$p_\sharp \phi_Y \to \phi_X p_\sharp$$

is an equivalence. This defines a sub ∞-category $\mathrm{CoSy}_X \subset \mathrm{Fun}(\mathrm{Sch}_X^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}}))$. We say that a coefficient system C is *cocomplete* if it takes values in symmetric monoidal cocomplete ∞-categories and colimit preserving functors. We denote by CoSy_X^c their ∞-category.

The miracle here is that all points of the previous proposition are properties! Not additional data. The only data is the functor D. Because adjunctions between stable ∞-categories are detected on the homotopy category, a functor D is a coefficient system if and only if its homotopy functor is a stable homotopical 2-functor in the sense of Ayoub ([Ayo07a]). The following theorem summarises the above discussion:

Theorem 1.2.3 (Ayoub–Cisinski-Déglise–Röndings–Voeveodsky)**.** *Any coefficient system gives a 6 functors formalism.*

We end this formal section with a result about extending coefficient systems from quasi-projective schemes to all finite type schemes.

Definition 1.2.4. We let QPCoSy_k the ∞-category defined in the same terms as Definition 1.2.2 but with functors

$$\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$$

with source quasi-projective varieties instead of finite type k-schemes.

Proposition 1.2.5. *The restriction functor*

$$\mathrm{CoSy}_k \to \mathrm{QPCoSy}_k$$

is an equivalence of ∞-categories with quasi-inverse the right Kan extension.

Proof. By [Gal22, Corollary 2.19] or [DG22, Proposition 7.13], any coefficient system (on all finite type k-schemes or varieties) affords Nisnevich descent, thus Zariski descent. Restriction and right Kan extension give an equivalence of categories between Zariski sheaves

$$\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Var}_k, \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})) \simeq \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sch}_k, \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}}))$$

because any finite type k-scheme admits a Zariski covering by affine schemes. Therefore, it suffices to check that if $C \in \mathrm{QPCoSy}_k$, its right Kan extension to $\mathrm{Sch}^{\mathrm{op}}$ is an object of CoSy_k, and that a morphism in QPCoSy_k gives a morphism in CoSy_k after Kan extension.

We denote by $\mathcal{D}$ the right Kan extension of the functor

$$C : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

along the inclusion $\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{Sch}_k^{\mathrm{op}}$. Because C is a Zariski sheaf, for $X \in \mathrm{Sch}_k$ we may choose a Zariski covering $p : U \to X$ with U affine and as p is quasi-projective, all $U_n := U^{\times_X n+1}$ are quasi-projective over k and we have

$$\mathcal{D}(X) \xrightarrow{\sim} \lim_{\Delta} C(U_n)$$

in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. But then if $f : Y \to X$ is a smooth morphism, base change induces a map of simplicial schemes $f_\bullet : V_\bullet \to U_\bullet$ with $V_n = U_n \times_X Y$ and we also have

$$\mathcal{D}(Y) \xrightarrow{\sim} \lim_{\Delta} C(V_n)$$

in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. In fact, the functor

$$f^* \colon \mathcal{D}(X) \to \mathcal{D}(Y)$$

is the limit of the functors

$$f_n^* \colon C(U_n) \to C(V_n)$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$. Because each f_n^* has a left adjoint $(f_n)_\sharp$, and for every map $[n] \to [m]$ in Δ the square

$$\begin{array}{ccc} C(V_n) & \xrightarrow{(f_n)_\sharp} & C(U_n) \\ \downarrow & & \downarrow \\ C(V_m) & \xrightarrow{(f_m)_\sharp} & C(U_m) \end{array} \tag{1.2.5.1}$$

commutes thanks to the canonical Beck-Chevalley morphism and the smooth base change property, the diagram $(f_n^*)_{[n]\in\Delta}$ in fact takes values in $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_\infty)$ the ∞-category of left adjointable functors (see [Lur22, Definition 4.7.4.16]). Thus by [Lur22, Corollary 4.7.4.18] the limit f^* is also a left adjointable functor: the left adjoint $f_\sharp$ exists and has a base change property with respect to all the functors $\mathcal{D}(X) \to C(U_n)$. Moreover, because f^* is symmetric monoidal for $M \in \mathcal{D}(X)$ and $N \in \mathcal{D}(Y)$ there exists a canonical map

$$f_\sharp(N \otimes f^* M) \to f_\sharp N \otimes M.$$

This map is an equivalence. Indeed, if $M_n = M_{|U_n}$, we have a map of diagrams $(f_n^*)_n \to (f_n^*)_n$ given by the commutative squares

$$\begin{array}{ccc} \mathcal{D}(U_n) & \xrightarrow{f_n^*} & \mathcal{D}(V_n) \\ {\scriptstyle -\otimes M_n}\downarrow & & \downarrow{\scriptstyle -\otimes f_n^* M_n} \\ \mathcal{D}(U_n) & \xrightarrow{f_n^*} & \mathcal{D}(V_n) \end{array}$$

which, because of the projection formula for each f_n and the base change for $(f_m)_\sharp$, is a map of diagrams with values in $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_\infty)$. Thus at the limit the map $f^* \to f^*$ in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ given by

$$f^*(-) \otimes M \Rightarrow f^*(- \otimes M)$$

is left adjointable, giving the projection formula.

Moreover, if $g\colon X' \to X$ is a map in Sch_k we can form the pullback

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

and doing a base change to $U_\bullet$ we obtain a pullback square of simplicial objects in Var_k of the form

$$\begin{array}{ccc} V'_\bullet & \xrightarrow{b'} & U'_\bullet \\ {\scriptstyle a'}\downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ V_\bullet & \xrightarrow{b} & U_\bullet \end{array}.$$

Then as above, the commutative square

$$\begin{array}{ccc} \mathcal{D}(X) & \xrightarrow{f^*} & \mathcal{D}(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle (g')^*} \\ \mathcal{D}(X') & \xrightarrow{(f')^*} & \mathcal{D}(Y') \end{array}$$

can be seen as a map $f^* \to (f')^*$ in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$, which itself is a limit of maps $b'_n \to b_n$ in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$. As this last simplicial diagram of maps takes values in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ because of the smooth base change holds for maps in Var_k, the limit is left adjointable, giving the smooth base for the pair (f, g).

The existence of pushforwards and internal $\mathscr{Hom}$ is proven in a similar way to the existence of the $f_\sharp$ (using right adjointable squares instead of left adjointable). The only thing to check in order to obtain those functors as functors to a limit of ∞-categories is that over quasi-projective schemes these functors commute with pullbacks by Zariski covering so that they define a compatible system as in (1.2.5.1). This is the case because these pullbacks π^* are right adjoints to $\pi_\sharp$ and the smooth base change and the projection formula ensures the commutativity of $\pi_\sharp$ with the left adjoints of internal $\mathscr{Hom}$ and pushforwards. Localisation follows from the fact that the category $\mathrm{Sq}^{\mathrm{cart}}(\mathrm{Cat}_\infty)$ of cartesian squares in Cat_∞ is closed

under limits in the category of commutative squares of ∞-categories. The properties of $\mathbb{A}^1$-invariance and T-stability can be checked Zariski locally thanks to the smooth projection formula and the smooth base change, hence they hold. The same is true for extensions of morphisms of coefficient systems. □

Remark 1.2.6. The above proposition on extension of coefficient systems from quasi-projective to finite type objects works *verbatim* if one replaces $\operatorname{Spec} k$ with any noetherian finite dimensional scheme B.

Two examples

Let X be an algebraic variety over k, a subfield of the complex number. Then if R is a regular ring one can classically construct a triangulated category of complexes of sheaves on $X(\mathbb{C})$ with constructible cohomology. In fact this definition naturally gives a coefficient system.

Definition 1.2.7. A sheaf $\mathcal{F}$ on $X(\mathbb{C})$ with values in the abelian category of R-modules is called *constructible* if there exists a finite stratification (X_i) of X into locally closed smooth subschemes such that the restriction of $\mathcal{F}$ to each stratum X_i is a local system, that is a representation of the fundamental group in a finite type R-module. We denote by $\mathrm{Constr}(X, R)$ the category of constructible sheaves on X. It is a weak Serre full sub abelian category of $\mathrm{Shv}(X(\mathbb{C}), R)$. We also denote by $\mathrm{D}^b_c(X, \mathbb{C})$ the full subcategory of the derived ∞-category $\mathrm{D}(\mathrm{Shv}(X(\mathbb{C}), R))$ of sheaves on $X(\mathbb{C})$ which consists of bounded complexes with constructible cohomology.

Proposition 1.2.8. *The subcategory $\mathrm{D}^b_c(X, \mathbb{C}) \subset \mathrm{D}(\mathrm{Shv}(X(\mathbb{C}), R))$ consists exactly of complexes K such that there exists a finite stratification (X_i) of X into locally closed smooth subschemes such that the restriction of K to each X_i is dualisable for the tensor structure on $\mathrm{D}(\mathrm{Shv}(X(\mathbb{C}), R))$. In particular, $\mathrm{D}^b_c(X, R)$ is a symmetric monoidal ∞-category.*

Proof. By choosing a fine enough stratification, it suffices to prove that a complex K is dualisable if and only if it is bounded with local systems as cohomology sheaves. The "if" direction follows from the facts that the full subcategory of dualisable objects is thick in any stable symmetric monoidal ∞-category, that any bounded complex can be obtained by doing finitely many extensions of shifts of its cohomology sheaves, and that a local system with fibres that are finite type R-modules is dualisable. For the other direction we can use [MW22, Corollary 7.4.12] together with the fact that $\mathrm{D}(\mathrm{Shv}(X(\mathbb{C}), R)) \simeq \mathrm{Shv}_{\mathrm{hyp}}(X(\mathbb{C}), \mathrm{D}(R))$ is the ∞-category of sheaves of complexes of R-modules on $X(\mathbb{C})$ to conclude that an object K is dualisable if and only if there exists a perfect complex $P \in \mathrm{Perf}_R$ and an open covering U_i of X such that the restriction of K to each U_i is isomorphic to the constant sheaf $\underline{P}$. We then see that the n-th cohomology sheaf of K is, when restricted to each U_i, isomorphic to the i-th cohomology module of P, which is of finite presentation because R is regular. □

Theorem 1.2.9 ([Dim04, Theorem 4.1.5])**.** *The functor*

$$\mathrm{D}^b_c(-, R) \colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

is a coefficient system. In particular, it has the 6 operations. Moreover, pullbacks, pushforwards, tensor products and internal homomorphisms are computed in $\mathrm{D}(\mathrm{Shv}(-(\mathbb{C}), R))$.

Now if X is any qcqs scheme, we can always construct the derived category of étale ℓ-adic sheaves on X. We give here the modern treatment of this construction, as found in [HRS23], that uses condensed mathematics. We consider the proétale site ([BS15a]) $X_{\text{proét}}$ associated to X by Bhatt and Scholze. By definition, a condensed ring is a sheaf of ring Λ on the proétale site of a point $*_{\text{proét}}$. Using the map of sites $X_{\text{proét}} \to *_{\text{proét}}$, we obtain a sheaf of rings Λ_X on $X_{\text{proét}}$, and we may consider $\mathrm{D}(X_{\text{proét}}, \Lambda)$ to be the derived category of proétale sheaves of Λ_X-modules on X. It is a stable symmetric monoidal ∞-category. Examples that we have in mind are the ∞-categories associated as above when $\Lambda = \mathbb{Z}_\ell$ or $\Lambda = \mathbb{Q}_\ell$ is an ℓ-adic ring or finite extensions of those, seen as a topological ring thus a condensed ring.

Definition 1.2.10. The derived category of constructible sheaves $\mathrm{D}^b_c(X_{\text{ét}}, \Lambda)$ it the full subcategory of $\mathrm{D}(X_{\text{proét}}, \Lambda)$ of complexes K such that for each affine open subset U of X, there exists a finite stratification U_i by locally closed subschemes of U such that the restriction of K to each U_i is dualisable.

Theorem 1.2.11 ([BS15a, Section 6.7])**.** *Let S be a quasi-excellent scheme over which ℓ is invertible. The functor*

$$\mathrm{D}^b_c(-, \mathbb{Z}_\ell) \colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

is a coefficient system. It also works for $\mathbb{Q}_\ell$, $\widehat{\mathbb{Z}}$, or integral extensions thereof.

Remark 1.2.12. If Λ is any discrete ring, the above definition still works, and we recover the usual category of constructible étale sheaves of Λ-modules. When $\Lambda = \mathbb{Z}_\ell$, the homotopy category of $\mathrm{D}^b_c(X, \mathbb{Z}_\ell)$ is the usual bounded derived category of étale ℓ-adic sheaves.

Proposition 1.2.13. *Let Λ be a discrete ring, let Λ' be a perfect derived Λ-algebra and let X be a qcqs scheme. Assume that $X = \lim_{i \in I} X_i$ is a limit of qcqs schemes with affine transitions. Then the natural functor*

$$\mathrm{colim}_i \mathrm{D}_{\mathrm{cons}}((X_i)_{\text{ét}}, \Lambda') \to \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \Lambda')$$

is an equivalence, where the colimit is taken in Cat_∞.

Proof. This is an adaptation of [BM21, Proposition 5.10]. Up to writing each X_i as a limit of Noetherian schemes with affine transition, we can assume the X_i to be Noetherian. We first show that the functor is fully faithful. Let $i_0 \in I$. It is enough to prove that for any two complexes K and L in $\mathrm{D}_{\mathrm{cons}}((X_{i_0})_{\text{ét}}, \Lambda')$ the map

$$\mathrm{colim}_{i \geqslant i_0} \mathrm{Map}_{\mathrm{D}_{\mathrm{cons}}((X_i)_{\text{ét}}, \Lambda')}(K_{|X_i}, L_{|X_i}) \to \mathrm{Map}_{\mathrm{D}_{\mathrm{cons}}(X, \Lambda)}(K_{|X}, L_{|\mathrm{X}})$$

is an equivalence. By dévissage and localisation, we may assume that K is dualisable and then reduce to the case where $K = \Lambda'$. By adjunction, we thus have to show that the map

$$\mathrm{colim}_{i \geqslant i_0} \mathrm{Map}_{\mathrm{Mod}_\Lambda}(\Lambda, \mathrm{R}\Gamma(X_i, L)) \to \mathrm{Map}_{\mathrm{Mod}_\Lambda}(\Lambda, \mathrm{R}\Gamma(X, L))$$

is an equivalence. Now, as perfect Λ'-modules are perfect Λ-modules because Λ' is perfect over Λ, the object L, seen as an object of $\mathrm{Shv}((X_0)_{\text{ét}}, \Lambda)$ is constructible, in particular bounded so that up to a dévissage, it is concentrated in one degree. The result then follows from the commutation of étale cohomology with colimits over the (X_i).

For the essential surjectivity, by dévissage it is enough to prove that any dualisable object lies in the image. By Lemma 3.1.27, if K is a dualisable object of $\mathrm{Shv}(X_{\text{ét}}, \Lambda')$ there exists an étale covering $f\colon U \to X$ and a perfect complex of Λ'-modules P such that $f^*(K)$ is the constant sheaf with value P. By Zariski descent we may even assume that X is affine and up to cover the source of f by an affine Zariski cover, we may also assume that the source of f is affine. Then by spreading out ([Gro66b, Théorème 8.10.5, Proposition 17.7.8]) the morphism f is the pullback of some étale f^i with target X_i. Let (f_n) and (f_n^i) be the Čech nerves of f and f^i. As the image of $(f_n^i)_\sharp P$ in $\mathrm{D}_{\mathrm{cons}}(X, \Lambda')$ is $(f_n)_\sharp P$, full faithfulness ensures that the simplicial object $((f_n)_\sharp P)_{n\in\Delta}$ is the image of a simplicial object $((f_n^i)_\sharp P)_{n\in\Delta}$ in $\mathrm{D}_{\mathrm{cons}}(X_i, \Lambda')$. By étale descent, we have that $K = \mathrm{colim}_n (f_n)_\sharp P$ in $\mathrm{D}_{\mathrm{cons}}(X, \Lambda')$. Moreover, if we denote by $L = \mathrm{colim}_n (f_n^i)_\sharp P$ in $\mathrm{Shv}((X_i)_{\text{ét}}, \Lambda')$, we have that $(f^i)^* L = P$ hence L is dualisable. As the pullback functor $p_i^*\colon \mathrm{Shv}((X_i)_{\text{ét}}, \Lambda') \to \mathrm{Shv}(X_{\text{ét}}, \Lambda')$ commutes with colimits, we obtain that K is the image of L, which is a constructible object over X_i, finishing the proof.

□

When Λ is regular (if it is a topological ring, one has to be careful with what "regular" means. By [HRS23, Corollary 6.11] T1-topological Prüfer domains such as $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$ have properties similar to regular rings.) and X is a scheme with Zariski locally a finite number of connected components, there is a natural t-structure on $\mathrm{D}_c^b(X_{\text{ét}}, \Lambda)$ by [HRS23, Theorem 6.2]. The t-structure further reduces to dualisable sheaves. We will denote by $\mathrm{Rep}_\Lambda(\mathrm{Gal}(\overline{k}/k))$ the heart of constructible étale sheaves of Λ-modules on $\mathrm{Spec}(k)$.

1.2.3 Motivic sheaves

1.2.3.1 Definition

In a previous section we sketched constructions of categories of motives over a field. Now is the time to give proper definitions, and constructions, over a general scheme. We give the ∞-categorical construction provided by Robalo in [Rob14].

Definition 1.2.14. Let X be a scheme and let R be a $\mathbb{E}_\infty$-ring. Let τ be either the étale or the Nisnevich topology. The ∞-category $\mathrm{DM}^\tau(X, R)$ is the universal recipient of a functor M_X from smooth X-schemes such that:

1. The ∞-category $\mathrm{DM}^\tau(X, R)$ is an R-linear symmetric monoidal presentable ∞-category.
2. **(Künneth)** The functor
$$M_X\colon \mathrm{Sm}_X \to \mathrm{DM}^\tau(X, R)$$
is symmetric monoidal, when Sm_X is endowed with the cartesian monoidal structure.
3. **($\mathbb{A}^1$-invariance)** For each $Y \in \mathrm{Sm}_X$, the natural map $M_X(\mathbb{A}^1_Y) \to M_X(Y)$ is an equivalence.
4. **(τ-hyperdescent)** For each hyper τ-covering (U_n) of a smooth X-scheme Y, the augmented (semi-)simplicial diagram
$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} M_X(U_2) \substack{\longrightarrow\\ \longrightarrow} M_X(U_1) \longrightarrow M_X(Y)$$
is a colimit diagram.

5. **(Tate-invertibility)** The section at ∞ of $\mathbb{P}^1_X$ induces a decomposition

$$M_X(\mathbb{P}^1_X) \simeq M_X(\infty_X) \oplus \mathbb{T}[-2]$$

and the canonical object $\mathbb{T}$ is invertible for the tensor product.

In other words, the ∞-category $\mathrm{DM}^\tau(X,R)$ is given by the formula

$$\mathrm{DM}^\tau(X,R) := \mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_\tau, \mathrm{Mod}_R)[\mathbb{T}^{\otimes -1}] :$$

one takes $\mathbb{A}^1$-invariant τ-hypersheaves of R-modules on Sm_X, and then inverts $\mathbb{T}$ for the tensor product.

For $R = \mathbb{S}$ the sphere spectrum and τ the Nisnevich topology, we will denote by $\mathcal{SH}(X)$ the obtained ∞-category.

Remark 1.2.15. The ∞-category $\mathrm{DM}^\tau(X,R)$ is also given other names in the literature: In Ayoub's thesis ([Ayo07a]), the underlying triangulated category is denoted by $\mathbf{SH}^\tau_{\mathfrak{M},\tau}(X)$ for $\mathfrak{M}$ a combinatorial model category of Mod_R. In his papers ([Ayo14a],[Ayo10]), this category would be denoted by $\mathbf{DA}^\tau(X,R)$. In the book [CD19] by Cisinski and Déglise this would be denoted by $D_{\mathbb{A}^1,\tau}(X,R)$, but they show that this is equivalent to $\mathrm{DM}_h(X,R)$ ([CD16, Corollary 5.5.7]). In the recent paper [Ayo22] the category is denoted by $\mathrm{MSh}_\tau(X,R)$, and in [AGV22] this is denoted by $\mathbf{SH}^\wedge_\tau(X,R)$.

Note also that this category a priori differs from the category $\mathrm{DM}^\tau_{\mathrm{tr}}(X,R)$ of motives *with transfers* considered in [CD16], and there denoted by $\mathrm{DM}^\tau(X,R)$. So far, we only know that those two constructions (with or without transfers) agree when X is a geometrically unibranch noetherian scheme ([CD16, Corollary 5.5.5]).

Theorem 1.2.16 (Ayoub,Cisinski-Déglise,Ayoub-Gallauer-Vezzani)**.** *On schemes with locally of finite type morphism between them the functor* $\mathrm{DM}^{\text{ét}}(-,R)$ *has the 6 operations.*

Proof. For quasi-projective S-schemes this is [Ayo07a] and [Ayo14a]. The quasi-projectivity is removed in [CD16]. The most general version is [AGV22, Theorem 4.6.1]. □

1.2.3.2 Rigidity

A useful property of étale motives is its rigidity property. Let X be a qcqs scheme, and denote by $X_{\text{ét}}$ the small étale site of X. For any ring of coefficients R, the change of site from $X_{\text{ét}}$ to $(\mathrm{Sm}_X)_{\text{ét}}$ creates a colimit preserving symmetric monoidal functor

$$\rho_! \colon \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{DM}^{\text{ét}}(X,R).$$

This functor sends the representable sheaf $R[Y]$, for Y an étale X-scheme, to its motive $M_X(Y)$, which is an Artin motive (a notion to be defined later). The rigidity theorem asserts that this functor is an equivalence when the ring R is torsion. The theorem has first be proven by Ayoub ([Ayo14a]), then generalised by Cisinski-Déglise ([CD16]), and its final form follows from Bachmann ([Bac21]) rigidity theorem, as explained by Bachmann-Hoyois ([BH21]). The theorem is the following:

Theorem 1.2.17. *Let X be a scheme with $n \in \mathscr{O}_X^\times$, and let R be a n-torsion ring. Then the canonical functor*

$$\rho_! \colon \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{DM}^{\text{ét}}(X, R)$$

is an equivalence.

This theorem allows one to see the ℓ-adic realisation functor ρ_ℓ as the ℓ-completion functor on étale motives. The Betti realisation ρ_B can also be obtained by a similar rigidity property. The latter is not a rigidity with respect to coefficients, but with respect to topology. In fact in [Ayo10] Ayoub proves that if one redo the definition of $\mathrm{DM}^{\text{ét}}(X, R)$ but with smooth analytic X^{an}-spaces instead of smooth X-schemes (here X^{an} is the analytification of X, with X a finite type $\mathbb{C}$-scheme), and the usual topology instead of the étale topology, then the inclusion of the small site of open subsets of X^{an} inside this bigger site induces an equivalence

$$\mathrm{D}(X^{\text{an}}, R) \simeq \mathrm{DM}^{\text{usu}}(X^{\text{an}}, R).$$

The analytification functor

$$\mathrm{DM}^{\text{ét}}(X, R) \to \mathrm{DM}^{\text{usu}}(X^{\text{an}}, R) \simeq \mathrm{D}(X^{\text{an}}, R)$$

thus gives an explicit Betti realisation functor. The two realisation functors can also be obtained through the universal property of $\mathrm{DM}^{\text{ét}}(X, R)$, but it is then much less explicit.

1.2.3.3 The universal property of motivic sheaves

We defined above what is a coefficient system. It turns out that the stable motivic homotopy category $\mathcal{SH}$, seen as a coefficient system, is the initial coefficient system. This is the main theorem of [DG22].

Theorem 1.2.18 (Drew-Gallauer)**.** *Let S be a finite dimensional noetherian scheme. Then on finite type S-schemes, $\mathcal{SH}$ is the initial coefficient system.*

This means that any coefficient system $\mathcal{D}$ receives a unique morphism of coefficient systems from $\mathcal{SH}$.

From the definition of $\mathcal{SH}(X)$, when X is a scheme, it is clear, as we mentioned above, that the ∞-category $\mathcal{SH}(X)$ has a universal property. This universal property comes from the several constructions we apply to the category Sm_X, that are all the "free" construction having some property: we want a presentable stable ∞-category, we take presheaves of spectra, we want Nisnevich descent we take Nisnevich sheaves, and we want $\mathbb{A}^1$-invariance we take $\mathbb{A}^1$-invariant sheaves. Even Tate-stabilisation is a universal property. The work of Drew and Gallauer gives that the functor $X \mapsto \mathrm{Sm}_X$ on finite type S-schemes, is universal among functors that have the smooth base change. Then they prove that the steps we described before are actually witnesses of operations we can apply to functors on finite type S-schemes that have smooth base change. All those operations are left adjoints thus the stable motivic homotopy category is initial in the category of functors satisfying the categorification of the definition of motivic sheaves, a category whose objects are named "pullback formalisms". The last step is to prove that Nisnevich descent follows from the axioms of a coefficient system, so that coefficient systems embeds fully faithfully in pullback formalisms.

1.2.3.4 From motivic spectra to rational étale motives

In the construction of realisations functors from étale motives, we will need to know how to pass from the stable $\mathbb{A}^1$-homotopy category to rational étale motives. Let k be a field of characteristic zero (this would work in greater generality but we only need this case). We thus recall a construction originally due to Fabien Morel, and we will use [CD19, Section 16.2] as a reference. Let X be a finite type k-scheme. There is a canonical decomposition

$$\mathcal{SH}_{\mathbb{Q}}(X) \simeq \mathcal{SH}^{+}_{\mathbb{Q}}(X) \times \mathcal{SH}^{-}_{\mathbb{Q}}(X)$$

of the rational part of the motivic homotopy category $\mathcal{SH}(X)$ over X. It comes from the decomposition in $\mathcal{SH}_{\mathbb{Q}}(X)$ of the unit $\mathbb{Q}_X$ as a direct sum

$$\mathbb{Q}_X \simeq \mathbb{Q}^{+}_X \oplus \mathbb{Q}^{-}_X.$$

Moreover, the object $\mathbb{Q}^{+}_X$ has a commutative algebra structure (because it is idempotent) in $\mathcal{SH}(X)_{\mathbb{Q}}$ such that the projection map $\mathbb{Q}_X \to \mathbb{Q}^{+}_X$ is a map of algebras. By [CD19, Theorem 16.2.13], there is a canonical identification of $\mathcal{SH}^{+}_{\mathbb{Q}}(X) \simeq \mathrm{Mod}_{\mathbb{Q}^{+}}(\mathcal{SH}_{\mathbb{Q}}(X))$ with étale motives $\mathrm{DM}^{\text{ét}}(X)$. Thus, we have:

Proposition 1.2.19. *A symmetric monoidal colimit preserving ∞-functor*

$$F \colon \mathcal{SH}_{\mathbb{Q}}(X) \to \mathcal{D}$$

to a stable presentably symmetric monoidal ∞-category $\mathcal{D}$ factors through $\mathrm{DM}^{\text{ét}}(X)$ if and only if $F(\mathbb{Q}^{-}_X) = 0$.

By [CD19, Corollary 16.2.14] the object $\mathbb{Q}^{-}_X \in \mathcal{SH}_{\mathbb{Q}}(X)$ vanishes whenever -1 is a sum of squares in the residual fields of X. We will need the following lemma:

Lemma 1.2.20. *The maps $\mathcal{SH}_{\mathbb{Q}}(X) \to \mathrm{DM}^{\text{ét}}(X)$ and $\mathcal{SH}(X) \to \mathcal{SH}_{\mathbb{Q}}(X)$ are part of morphisms of coefficient systems $\mathcal{SH}_{\mathbb{Q}} \to \mathrm{DM}^{\text{ét}}$ and $\mathcal{SH} \to \mathcal{SH}_{\mathbb{Q}}$.*

Proof. This is a particular case of a theorem of Brad Drew (although in the second case of rationalisation a simpler proof could have been given see [Ayo14a, Lemma A.11]): [Dre18, Theorem 8.10], where for example in the first case we take (with his notations) $\mathcal{V} = \mathcal{SH}_{\mathbb{Q}}(\mathrm{Spec}(k))$, $\mathcal{M}^* = \mathcal{SH}_{\mathbb{Q}}$ and $A = \mathbb{Q}^{+}_{\mathrm{Spec}(k)}$. □

This lemma is probably well known to experts but the author could not find a reference.

Lemma 1.2.21. *The morphisms $\rho_{\mathbb{Q}} \colon \mathcal{SH}_c \to \mathcal{SH}_{\mathbb{Q},c}$ and $a_{\text{ét}} \colon \mathcal{SH}_{\mathbb{Q},c} \to \mathrm{DM}^{\text{ét}}_c$ commute with the 6 operations.*

Proof. The case of $\rho_{\mathbb{Q}}$ follows from [Ayo14a, Théorème A.15], thus we focus on $a_{\text{ét}}$. We know that $a_{\text{ét}}$ is a morphism of coefficient systems generated by objects of the form $g_\sharp \mathbf{1}_Y(n)$ for $g \colon Y \to X$ a smooth map and $n \in \mathbb{Z}$. Thus to prove that it commutes with the 6 operations it suffices to check that it commutes with pushforwards (see the proof of [CD19, Theorem 4.4.25]). Let $f \colon Y \to X$ be a morphism of schemes. As the functor $a_{\text{ét}}$ commutes with f^* its right adjoint $\mathrm{ff}^{\text{ét}}$ commutes with f_*, and because $\mathrm{ff}^{\text{ét}}$ is conservative, it suffices to check that the natural transformation

$$\mathrm{ff}^{\text{ét}} \circ a_{\text{ét}} \circ f_* \to f_* \circ \mathrm{ff}^{\text{ét}} \circ a_{\text{ét}}$$

is an equivalence. Moreover, by [Lur22, Corollary 4.2.4.8] there is a canonical identification of functors

$$\mathrm{ff}^{\text{ét}} \circ a_{\text{ét}} \simeq - \otimes \mathbb{Q}^+.$$

Thus, we have to prove that the natural map

$$f_*(-) \otimes \mathbb{Q}^+ \to f_*(- \otimes \mathbb{Q}^+) \tag{1.2.21.1}$$

is an equivalence. This is the case because as $\mathbb{Q}^+$ is a direct factor of $\mathbb{Q}$, it is a dualisable object of $\mathcal{SH}_{\mathbb{Q}}$ so that we can apply [Ayo14b, Lemma 2.8], using that $\mathbb{Q}^+_Y \simeq f^*\mathbb{Q}^+_X$.

□

1.2.4 Perverse Nori motives

1.2.4.1 Nori motives over a field

Let k be a subfield of $\mathbb{C}$ and let R be a commutative ring. The category of Nori motives $\mathcal{M}_{\text{Nori}}(k, R)$ is an tensor abelian category, equipped with a tensor faithful exact functor

$$R_B \colon \mathcal{M}_{\text{Nori}}(k, R) \to R\text{-mod}$$

to the abelian category of R-modules. It is obtained by inverting for the tensor product an object $\mathbb{L} := R(1) \in \mathcal{M}^{\text{eff}}_{\text{Nori}}(k, R)$ in the effective category of Nori motives. We describe the latter.

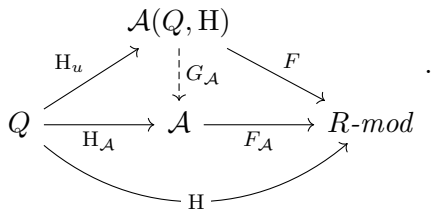
A *quiver* Q (or diagram, or oriented graph) is the data of a set of objects V and for each pair of objects $(x, y) \in V^2$ a set $\mathrm{Hom}_Q(x, y)$. We do not require any composition to exist. Any category can be seen as a quiver in the obvious way (by forgetting that composition exists), and a morphism of quivers can be defined in the obvious manner.

Let Q be a quiver. A *representation* of the quiver Q in a category $\mathcal{C}$ is a morphism of quivers $\mathrm{H} \colon Q \to \mathcal{C}$. The main theorem of Nori's theory of diagram categories is the following:

Theorem 1.2.22 (Nori, [HMS17, Theorem 7.1.13])**.** *Let R be a commutative noetherian ring and let $\mathrm{H} \colon Q \to R$-mod be a representation to the abelian category of finite type R-modules. There exists a R-linear abelian category $\mathcal{A}(Q, \mathrm{H})$ together with a representation $\mathrm{H}_u \colon Q \to \mathcal{A}(Q, \mathrm{H})$ and a R-linear faithful exact functor $F \colon \mathcal{A}(Q, \mathrm{H}) \to R$-mod, together with an isomorphism*

$$\mathrm{H} \simeq F \circ \mathrm{H}_u$$

such that for any R-linear abelian category $\mathcal{A}$ with a representation $\mathrm{H}_{\mathcal{A}} \colon Q \to \mathcal{A}$ and a R-linear faithful exact functor $F_{\mathcal{A}} \colon \mathcal{A} \to R$-mod there exists a R-linear faithful exact functor $G_{\mathcal{A}} \colon \mathcal{A}(Q, \mathrm{H}) \to \mathcal{A}$, unique up to unique natural isomorphism, such that $\mathrm{H}_{\mathcal{A}} = G_{\mathcal{A}} \circ \mathrm{H}_u$ and $F_{\mathcal{A}} = F \circ G_{\mathcal{A}}$. As a diagram:



The construction is simple: if Q is a finite quiver (that is, the set of objects and the sets of morphisms are finite sets), then $\mathcal{A}(Q, H)$ is just the category of left E_Q-modules with $E_Q = \mathrm{End}(\mathrm{H})$ the algebra of endomorphisms of the representation. When Q is non necessarily finite, we can write is as a filtered colimit of its finite subquivers and define

$$\mathcal{A}(Q, \mathrm{H}) := \mathrm{colim}_F \mathcal{A}(F, \mathrm{H}_{|F})$$

where the colimit is taken in the 2-category of categories.

The construction has refinements when given more structure, such a multiplicative structure on the quiver, compatible with the tensor structure on modules. See [HMS17] for details.

Definition 1.2.23. Let k be a subfield of the complex numbers. The quiver $\mathrm{Pairs}_k^{\mathrm{eff}}$ of effective pairs is the quiver whose objects are the triples (X, Y, i) where X is an algebraic variety over k, Y is a closed algebraic subvariety and $i \in \mathbb{N}$ is an integer. Morphisms are described as follows: Let $x = (X, Y, i)$ and $x' = (X', Y', i')$ be two objects of $\mathrm{Pairs}_k^{\mathrm{eff}}$. Then we have the following:

1. If $i = i'$ then $\mathrm{Hom}(x, x')$ is the set of morphisms of varieties $X' \to X$ that send Y' inside Y.
2. If $i = i' - 1$ then if $X = Y'$ there is one morphism $x \to x'$ that we denote by ∂, else $\mathrm{Hom}(x, y) = \emptyset$.
3. Else, $\mathrm{Hom}(x, x') = \emptyset$.

Let R be a commutative ring. We denote by

$$\mathrm{H}\colon \mathrm{Pairs}_k^{\mathrm{eff}} \to R\text{-mod}$$

the representation that sends a triple (X, Y, i) to $\mathrm{H}_B^i(X, Y) := \mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}}, Y^{\mathrm{an}})$ the singular relative cohomology group of the analytification of the algebraic varieties. It sends a morphism f^* of type 1. to the induced pullback $f^*\colon \mathrm{H}^{i'}(X', Y') \to \mathrm{H}^i(X, Y)$ on singular cohomology, and morphisms of type 2. to the boundary morphism $\partial\colon \mathrm{H}^{i-1}(X, Y) \to \mathrm{H}^i(Y', X)$ in the long exact sequence of relative cohomology.

Finally, the category of *effective Nori motives* $\mathcal{M}_{\mathrm{Nori}}(k, R)$ is the universal factorisation $\mathcal{A}(\mathrm{Pairs}_k^{\mathrm{eff}}, \mathrm{H})$ of the representation H defined above. We have a morphism

$$\mathrm{H}^*\colon (X, Y, i) \mapsto \mathrm{H}^i(X, Y)$$

from $\mathbb{Q}$ to $\mathcal{M}^{\mathrm{eff}}_{\mathrm{Nori}}(k, R)$, and a *Betti realisation functor*

$$R_B\colon \mathcal{M}_{\mathrm{Nori}}(k, R) \to R\text{−mod}.$$

Their composition is the representation H and the triple $(\mathcal{M}^{\mathrm{eff}}_{\mathrm{Nori}}(k, R), \mathrm{H}^*, R_B)$ has the universal property described above in Theorem 1.2.22.

Using Čech resolutions associated to affine coverings, one can see that if in the definition of the diagram pairs one replaces algebraic varieties by *affine* algebraic varieties, the result category is the same ([HMS17, Theorem 9.2.22]). Using a geometric lemma (the so-called

Beilinson's basic lemma, or its version due to Nori), one can prove that for any closed subvariety Y of an affine variety X there exist a filtration X_i of X by closed subvarieties such that for each i the singular cohomology of $(X_i, X_i \cap Y)$ is concentrated in degree i, and free. This implies that one could further replace affines varieties by such triplets $(X_i, Y \cap X_i, i)$, thus enabling one to have a multiplicative structure on the representation H (because then the Kunneth formula is just saying that H is multiplicative). In particular, the category $\mathcal{M}_{\mathrm{Nori}}(k, R)$ is a tensor category, the functor R_B is a tensor structure and the Künneth formula holds in $\mathcal{M}_{\mathrm{Nori}}(k, R)$ (see [HMS17, Section 9.3]).

Definition 1.2.24. Let $\mathbb{L} := \mathrm{H}^2(\mathbb{P}^1, \emptyset) \in \mathcal{M}_{\mathrm{Nori}}(k, R)$. The category of Nori motives is the category

$$\mathcal{M}_{\mathrm{Nori}}(k, R) := \mathcal{M}^{\mathrm{eff}}_{\mathrm{Nori}}(k, R)[\mathbb{L}^{\otimes -1}].$$

This means that there is a tensor functor

$$\mathcal{M}^{\mathrm{eff}}_{\mathrm{Nori}}(k, R) \to \mathcal{M}_{\mathrm{Nori}}(k, R)$$

that sends $\mathbb{L}$ to an invertible object $R(-1)$, and this functor is universal for this property. In particular, there is a Betti realisation functor

$$R_B \colon \mathcal{M}_{\mathrm{Nori}}(k, R) \to R\text{-mod}$$

and any object of $\mathcal{M}_{\mathrm{Nori}}(k, R)$ is a subquotient of an object of the form $\mathrm{H}^i(X, Y)(j)$ where $(X, Y, i) \in \mathrm{Pairs}^{\mathrm{eff}}_k$ and we denote by $-(j)$ the functor $-\otimes\mathbb{L}^{\otimes -j}$. See [HMS17] for more details.

Using the universal property of Nori motives, it is not hard to see that $\mathcal{M}_{\mathrm{Nori}}(k, R)$ has a Hodge realisation functor, a de Rham realisation functor and ℓ-adic realisation functors (see [HMS17, Examples 9.1.11, 9.1.12 and 9.1.13]). If R is a number field or the ring of integers of a number field, we will denote by R_ℓ the ℓ-adic realisation functor, with values in $\mathrm{Rep}_{R_\ell}(\mathrm{Gal}(\overline{k}/k))$ the abelian category of ℓ-adic representations of the absolute Galois group of k.

1.2.4.2 From Voevodsky motives to Nori motives

In this subsection, we do a quick recollection of realisation functors from the triangulated category of Voevodsky motives to the derived category of Nori motives. This construction goes back to Nori, but it has been written down by Harrer ([Har16]) and Choudhury-Gallauer ([CGAdS17]). We describe Choudhury and Gallauer' construction because it is more easily seen to provide an ∞-functor whose existence will be needed later in Chapter 4.

Let k be a subfield of $\mathbb{C}$ and let R be a Dedekind ring which is flat over $\mathbb{Z}$. We will construct a realisation functor

$$\rho'_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}(k, R) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k, R)),$$

compatible with the Betti realisation.

By [CGAdS17, Proposition 7.1 and its proof], there is a lax symmetric monoidal 1-functor

$$\mathrm{SmAff}_k \to \mathrm{Ch}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k, R)^{\mathrm{eff}})$$

from the category of affine k-schemes to the category of bounded complexes of effective Nori motives ([HMS17, Definition 9.1.3]). We can compose this functor with the functor

$$\mathrm{Ch}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})^{\mathrm{eff}}) \to \mathrm{Ch}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$$

given by the map $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})^{\mathrm{eff}} \to \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$.

The construction roughly goes as follows: for each smooth affine variety X, one can construct (using Nori's Basic Lemma) a finite *cellular filtration* $\mathcal{F}$ on X by closed subschemes which computes the cohomology of X (this is similar to how the cellular filtration of a CW-complex computes its cohomology). This gives a bounded complex $C^*_{\mathcal{F}}(X)$ in $\mathrm{Ch}^b(\mathcal{M}_{\mathrm{Nori}}(k,R))$ whose Betti realisation computes the singular cohomology of the complex points of X. To make a functor out of it, one has to make choices to have cellular filtrations compatible with morphisms; to that end, one considers the homotopy colimit $C^*(X)$ (in $\mathrm{Ch}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,R))$) over all possible cellular filtrations. Moreover, the composition $R_{\mathrm{B}} \circ C^*$ with the Betti realisation functor R_{B} is naturally isomorphic (as a monoidal functor) to the functor $C^*_{\mathrm{sing}} \colon X \mapsto C^*_{\mathrm{sing}}(X^{\mathrm{an}})$ sending a k-variety to the complex of singular chains over the analytic variety of its complex points.

We now use Blumberg, Gepner and Tabuda's [BGT14, Proposition 3.5]. First notice that

$$C^* \colon \mathrm{SmAff}_k \to \mathrm{Ch}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,R))$$

is a lax symmetric monoidal functor between symmetric monoidal 1-categories. This means that $C^*(\mathrm{Spec}(k)) = R[0]$ and if X, Y are smooth and affine algebraic varieties, there is a map $C^*(X) \otimes C^*(Y) \to C^*(X \times_k Y)$ that satisfies compatibility with respect to the unit and associativity of the tensor product. Moreover, by construction of the functor C^*, the canonical map

$$C^*(X_1) \otimes C^*(X_2) \to C^*(X_1 \times X_2)$$

is a quasi-isomorphism, because this is true when C^* is replaced by C^*_{sing}. Setting the weak equivalences on SmAff_k to be the isomorphisms of schemes, and the class of weak equivalences W_{qiso} in $\mathrm{Ch}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,R))$ to be the class of quasi-isomorphisms, the functor C^* preserves weak equivalences. Therefore, the proposition quoted above yields a symmetric monoidal structure on the ∞-category $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$ and a symmetric monoidal functor

$$\mathrm{SmAff}_k^{\times} \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,R))^{\otimes}$$

whose underlying functor is

$$\mathrm{SmAff}_k \xrightarrow{C^*} \mathrm{Ch}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,R)) \xrightarrow{W_{\mathrm{qiso}}^{-1}} \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,R)).$$

Now, by [CGAdS17, Proposition 7.1] the above functor factor through étale motives (for this we use the universal property). If K/k is an extension of subfields of $\mathbb{C}$, then as any cellular filtration of a variety over k provides a cellular filtration of its base change to K, this functor is compatible with change of fields. Thus we obtained:

Theorem 1.2.25 (Choudhury-Gallauer, Harrer, Nori)**.** *Let k be a subfield of $\mathbb{C}$. There exists a symmetric monoidal functor*

$$\rho'_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}({}^{\text{ét}}k,R) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,R))$$

that gives back the Betti realisation when composed with the Betti realisation of Nori motives, and sends $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,R)$ *to* $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,R))$*. Moreover, the construction is compatible with pullback of fields.*

We denote by ρ_{N} the restriction to $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,R)$ of ρ'_{N}. Using the operations on étale motives, it is not hard to show that there is a morphism of quivers $\mathrm{Pairs}_k^{\mathrm{eff}} \to \mathrm{ho}(\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,R))$ to the triangulated category of étale motives over k, such that the composition with the Betti realisation is the representation H of Definition 1.2.23. This implies easily that we have a equivalence of categories

$$\mathcal{A}(\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,R), \mathrm{H}^0 \circ \rho_{\mathrm{B}}) \simeq \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$$

compatible with the Betti realisation and the universal representation of $\mathrm{Pairs}_k^{\mathrm{eff}}$. This could have been another definition of Nori motives ([HMS17, Corollary 10.1.6]). Moreover, when the quiver is underlying an *additive* category, there are other constructions of the universal abelian category associated to it, and this allows ([Fre66, Pre11, BVP18, BVHP20, Ivo17]) the definition of universal factorisation of representation categories with values in more general categories than just R-modules. This is the way Ivorra and Morel chose.

Before going into the construction of perverse Nori motives, we recall a result that we will need about torsion Nori motives. As Artin motives map to $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,\Lambda)$ and the composition with the Betti realisation conservative and t-exact, the functor

$$\mathrm{H}^0_{\mathrm{N}} : \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(k,\Lambda) \to \mathcal{M}_{\mathrm{Nori}}(k,\Lambda)$$

induces a faithful exact functor

$$\iota\colon \mathrm{Rep}(\mathrm{Gal}(\bar{k}/k),\Lambda) \to \mathcal{M}_{\mathrm{Nori}}(k,\Lambda) \tag{1.2.25.1}$$

where $\mathrm{Rep}(\mathrm{Gal}(\bar{k}/k),\Lambda)$ is the abelian category of Λ-linear continuous representations of $\mathrm{Gal}(\bar{k}/k)$ which are finitely generated as Λ-modules.

Theorem 1.2.26 (Nori [Fak00])**.** *The functor* ι *defined above induces an equivalence*

$$\iota\colon \mathrm{Rep}(\mathrm{Gal}(\bar{k}/k),\mathbb{Z})_{\mathrm{tors}} \xrightarrow{\sim} \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})_{\mathrm{tors}}$$

on the full subcategories of torsion objects.

Proof. The proof can be found in [Fak00, Theorem 6.1], but see also the proof of Proposition 4.3.22 which is a generalisation. □

1.2.4.3 The six operations on perverse Nori motives

Let X be a quasi-projective k-scheme with k a field of characteristic zero. In [IM22], Ivorra and S. Morel introduced an abelian category of perverse Nori motives $\mathcal{M}_{\mathrm{perv}}(X)$. The construction of the category $\mathcal{M}_{\mathrm{perv}}(X)$ goes as follows:

Let X be a quasi-projective k-scheme. Pick your favourite prime number ℓ. We can compose the ℓ-adic realisation functor

$$\rho_\ell : \mathrm{DM}_c^{\text{ét}}(X) \to \mathcal{D}^b(X_{ét},\mathbb{Q}_\ell)$$

with the perverse cohomology functor

$$^{\mathrm{p}}\mathrm{H}^0 : \mathrm{D}^b(X_{\acute{e}t}, \mathbb{Q}_\ell) \to \mathrm{Perv}(X, \mathbb{Q}_\ell)$$

to obtain a homological functor $^{\mathrm{p}}\mathrm{H}^0_\ell \colon \mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X) \to \mathrm{Perv}(X, \mathbb{Q}_\ell)$. Let $\mathcal{R}(\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X))$ be the abelian category of coherent modules over $\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X)$: it is the full subcategory of additive presheaves $\mathcal{F} : \mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X)^{\mathrm{op}} \to \mathrm{Ab}$ that fit in exact sequences

$$y_M \to y_N \to \mathcal{F} \to 0$$

where $M \to N$ is a morphism in $\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X)$.

The functor $^{p}\mathrm{H}^0_\ell$ factors through $\mathcal{R}(\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X))$, giving a functor

$$\widetilde{^{\mathrm{p}}\mathrm{H}^0_\ell} : \mathcal{R}(\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X)) \to \mathrm{Perv}(X, \mathbb{Q}_\ell).$$

Denote by Z_ℓ the kernel of $\widetilde{^{\mathrm{p}}\mathrm{H}^0_\ell}$, that is the objects mapping to zero. The category of perverse Nori motives $\mathcal{M}_{\mathrm{perv}}(X)$ is the quotient $\mathcal{R}(\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X))/Z_\ell$. It comes with a universal functor $\mathrm{H}_{\mathrm{univ}} \colon \mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X) \to \mathcal{M}_{\mathrm{perv}}(X)$.

By construction, $\mathcal{M}_{\mathrm{perv}}(X)$ has the following universal property: any homological functor $H : \mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X) \to \mathcal{A}$ to an abelian category $\mathcal{A}$ such for all $M \in \mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X)$ we have $H(M) = 0$ whenever $^{\mathrm{p}}\mathrm{H}^0_\ell(M) = 0$ factors uniquely through the universal functor

$$\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X) \xrightarrow{\mathrm{H}_{\mathrm{univ}}} \mathcal{M}_{\mathrm{perv}}(X).$$

A priori this category depends on the chosen ℓ, but one can show, using a continuity argument, that this is not the case. Indeed, for finite type fields (or more generally, for fields embeddable into $\mathbb{C}$), the Betti realisation can also be used to define $\mathcal{M}_{\mathrm{perv}}(X)$ and the comparison between the ℓ-adic realisation functor and the Betti realisation functor (see for example the proof of [Ayo24, Proposition 6.10.] that works over any finite type $\mathbb{C}$-scheme) gives that the two constructions of perverse Nori motives agree.

In fact, every realisation functor existing on $\mathrm{DM}_c^{\acute{\mathrm{e}}\mathrm{t}}(X)$ gives us a realisation functor on $\mathcal{M}_{\mathrm{perv}}(X)$: we have the ℓ-adic realisation R_ℓ and the Betti realisation R_B. Ivorra and S. Morel have proven that $\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}\, k)$ coincides with the category of cohomological motives constructed by Nori, and thus also has an Hodge realisation functor. More generally, one can prove (see for example [Ter24, Proposition 1.15] for a proof using only the Betti realisation) that for any finite type extension of fields L/k the functor $\mathcal{M}_{\mathrm{Nori}}(L, \mathbb{Q}) \to \mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}(L))$ is an equivalence. The right hand side can be defined as the colimit of the $\mathcal{M}_{\mathrm{perv}}(U)$ for U ranging over the nonempty open subset of a smooth connected variety with function field L. This uses some sheaf operations, that exists:

Ivorra and S. Morel show in [IM22, Theorem 5.1] that the triangulated derived category of the abelian category of perverse motives $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$ is part of a homotopical 2-functor in the sense of Ayoub [Ayo07a, Definition 1.4.1]. This means that they constructed four of the six operations, namely for f a morphism of quasi-projective varieties we have $f^*, f_*, f_!, f^!$, together with Verdier duality $\mathbb{D}$, verifying $f^! \simeq \mathbb{D} \circ f^* \circ \mathbb{D}$ and $f_! \simeq \mathbb{D} \circ f_* \circ \mathbb{D}$. In his PhD thesis Terenzi [Ter24] constructed the remaining two operations: tensor product and internal $\mathscr{H}om$. This proves that $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$ is part of a full 6 functor formalism. The 6 operations commute with the realisation functors R_ℓ and R_B making them morphisms of symmetric

monoidal stable homotopical 2-functors. The construction of the operations on perverse Nori motives is very similar to Saito's construction of mixed Hodge modules. We will not give the full construction, be we can give the main steps:

1. First we have all the perverse t-exact functors in one variable: if f is smooth of relative dimension d we have $f^*[d]$, if i is a closed immersion we have i_*, and if j is an affine open immersion we have $j_!$. Moreover, we also have the unipotent nearby cycle functor Ψ_f, for any map f to $\mathbb{A}^1$. Those are obtained from the universal property of Nori motives.

2. Now, using ideas due to Beilinson, we know we can glue together perverse sheaves on a open subset and its closed complement, and the proof of this result gives, in the setting of perverse Nori motives, that the functor i_* is fully faithful on the bounded derived category. Its image is exactly the kernel of the pullback along the immersion of the open complement.

3. Thus, to construct a left adjoint of i_*, it suffices to construct a left adjoint to the inclusion of sheaves supported on the closed subset. This can be done by hand: if the inclusion j of the open complement is affine, then the right adjoint is just the cone of the map $j_!j^* \to \mathrm{Id}$. In the general case one uses an affine covering and a Čech complex.

4. As any map of quasi-projective varieties factors as a closed immersion followed by a smooth map, one can use a formal result of Ayoub to glue inverse images and obtain pullback functors along any morphism.

5. The pullback along the projection $\mathbb{A}^1_X \to X$ has a right adjoint. This is proven using unipotent nearby cycles and an explicit construction. This is a long and technical step.

6. The pullback along a vector bundle $f\colon E \to X$ has a right adjoint. For this, the original proof in [IM22] has a mistake. Indeed, the strategy was to consider a formal right adjoint R to $f^*\colon \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{perv}}(X)) \to \mathrm{D}(\mathrm{Ind}(\mathcal{M}_{\mathrm{perv}}(E)))$ (which exists by the adjoint functor theorem), and then showing that R sends $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(E))$ to $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ (which are indeed full subcategories of $\mathrm{D}(\mathrm{Ind}(\mathcal{M}_{\mathrm{perv}}(-)))$ because the category $\mathcal{M}_{\mathrm{perv}}(-)$ is Noetherian thus we may apply [Hub93, Proposition 2.2]). For this, because for an affine open immersion j, the functor j^* has a left adjoint $j_!$ that commutes with any f^*, the functor R commutes with j^*, so that the above condition may be tested for E a trivial vector bundle. In that case, we unfortunately cannot apply the previous point because we do not know that the restriction of R to $D^b(\mathcal{M}_{\mathrm{perv}}(E))$ is the functor f_* constructed in the previous point. We propose an alternative strategy in Remark 2.1.34. The idea is to construct f_* Zariski locally on X. This has to use ∞-categories and some theory of adjointable squares developped by Lurie. As Joseph Ayoub pointed out to me, the issue could also maybe be fixed by considering $\mathrm{Ind}\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$.

7. The pullback under any affine map has a right adjoint: one can write it as the composition of a closed immersion and a vector bundle.

8. Using Jouanoulou's trick and the case of vector bundles again, one concludes in general: the pullback under any map has a right adjoint.

9. From this we get easily the four operations.

10. Now we go into Terenzi's work. The issue here is that although the external tensor product is perverse t-exact in two variables, it is not true that if $M, N \in \mathrm{DM}_c^{\text{ét}}(X, \mathbb{Q})$ are motives, the map

$$ {}^{\mathrm{p}}\mathrm{H}^0(\rho_B(M)) \boxtimes {}^{\mathrm{p}}\mathrm{H}^0(\rho_B(N)) \to {}^{\mathrm{p}}\mathrm{H}^0(\rho_B(M \boxtimes N)) $$

is an equivalence. However, this property holds whenever M and N have Betti realisation concentrated in degree 0. If $\mathrm{DM}_c^{\text{ét}}(X, \mathbb{Q})^0$ denotes the full subcategory of motives having their Betti realisation concentrated in degree zero, then if one redo the construction of perverse Nori motives with this smaller category, the induced map ${\mathcal{M}_{\mathrm{perv}}}^0(X) \to \mathcal{M}_{\mathrm{perv}}(X)$ is an equivalence. This is a hard result and relies ultimately on the gluing formalism of Beilinson.

11. Using that ${\mathcal{M}_{\mathrm{perv}}}^0(X) \simeq \mathcal{M}_{\mathrm{perv}}(X)$, one obtains a functor

$$ - \boxtimes - \colon \mathcal{M}_{\mathrm{perv}}(X) \times \mathcal{M}_{\mathrm{perv}}(Y) \to \mathcal{M}_{\mathrm{perv}}(X \times Y) $$

for every pair of quasi-projective varieties X and Y.

12. It is then a lot of work to prove that the functor $\Delta^*((p_1^*(-)) \boxtimes (p_2^*(-)))$ has all the structure of a tensor product of triangulated categories. Here Δ is the diagonal $X \to X \times X$ and p_i are the projections $X \times X \to X$.

13. Now if $M \in \mathcal{M}_{\mathrm{perv}}(X)$ has a local system as Betti realisation, we have a formula for the internal homomorphism from M: it is a dualisable object.

14. Finally, in general the functor $- \otimes M$ has a right adjoint when seen as a functor on $\mathrm{D}(\mathrm{Ind}(\mathcal{M}_{\mathrm{perv}}(X)))$ by the adjoint functor theorem. On can prove that this right adjoint $\mathscr{H}om(M, -)$ is compatible with the localisation triangle in M and with the adjunctions (i^*, i_*) and $(j_!, j^*)$ to reduce to the case where M is lisse. In that case the two functors $\mathscr{H}om(M, -)$ and $- \otimes M$ are perverse t-exact thus the abstract right adjoint can be computed easily and we conclude using the previous point.

With this we have the 6 operations at the level of triangulated categories.

1.2.5 Mixed Hodge modules

Let X be a separated finite type $\mathbb{C}$-scheme. In [Sai90b], M. Saito constructed an abelian category $\mathrm{MHM}(X)$ of mixed Hodge modules. He also constructed the six operations on $\mathrm{D}^b(\mathrm{MHM}(-))$. There is a faithful exact functor $\mathrm{MHM}(X) \to \mathrm{Perv}(X^{\mathrm{an}}, \mathbb{Q})$ to the abelian category of perverse sheaves on the analytification of X. This induces a functor $\mathrm{D}^b(\mathrm{MHM}(X)) \to \mathrm{D}^b(\mathrm{Perv}(X^{\mathrm{an}}, \mathbb{Q})) \to \mathrm{D}_c^b(X^{\mathrm{an}}, \mathbb{Q})$ and gives a natural transformation $\mathrm{D}^b(\mathrm{MHM}(-)) \to \mathrm{D}_c^b(-, \mathbb{Q})$ that commutes with the 6 operations. If $X = \operatorname{Spec} \mathbb{C}$, then there is a natural equivalence $\mathrm{MHM}(\operatorname{Spec} \mathbb{C}) \simeq \mathrm{MHS}_{\mathbb{Q}}^p$ between polarisable mixed Hodge structures and mixed Hodge modules. One of the most interesting properties of the category of mixed Hodge modules is that its *lisse* objects, that is the objects such that the underlying perverse sheaf is a shift of a locally constant sheaf are exactly the polarisable variations of mixed Hodge structures. We will not go into details about the construction of Saito's category, and we will mostly need to know that the functor taking a complex of mixed Hodge modules to its underlying complex

of perverse sheaves is conservative and commutes with the operations. We note that by definition, if X is a smooth scheme, then the abelian category of mixed Hodge modules on X is a full subcategory of the category of filtered objects in the pullback

$$\begin{array}{ccc} \mathrm{MF}_h(\mathcal{D}_X, \mathbb{Q}) & \longrightarrow & \mathrm{Perv}(X^{\mathrm{an}}, \mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Fil}\mathcal{D}_X\text{-}\mathrm{mod}^{\mathrm{hol}} & \longrightarrow & \mathrm{Perv}(X^{\mathrm{an}}, \mathbb{C}) \end{array}$$

where $\mathrm{Fil}\mathcal{D}_X\text{-}\mathrm{mod}^{\mathrm{hol}} \to \mathrm{Perv}(X^{\mathrm{an}}, \mathbb{C})$ it the Riemann-Hilbert functor applied to the underlying holonomic $\mathcal{D}_X$-module of a filtered holonomic $\mathcal{D}_X$-module, and $\mathrm{Perv}(X^{\mathrm{an}}, \mathbb{Q})$ is the abelian category of perverse sheaves.

Chapter 2

Realisation functors.

In this chapter we construct realisation functors on the category of étale motives. Section 2.1.1 is separated in two sub-subsections. In [Nor02] Nori proves that the canonical functor

$$\mathrm{D}^b(\mathrm{Cons}(X,\mathbb{Q})) \to \mathrm{D}^b_c(X,\mathbb{Q})$$

from the bounded derived category of constructible sheaves to the triangulated category of constructible complexes of sheaves on the variety of complex points of X, is an equivalence. In the first sub-subsection, we prove that the argument of Nori can be adapted to work in the settings of mixed Hodge modules and perverse Nori motives. In the second sub-subsection, we use the good ∞-categorical functoriality of the functor sending an abelian category to its bounded derived ∞-category proved in Proposition A.6.2 to construct the ∞-categorical enhancement of the functoriality of mixed Hodge modules and perverse Nori motives as derived from the functoriality of its constructible heart, that is the abelian full subcategory of complexes sent to $\mathrm{Cons}(X,\mathbb{Q})$ by the forgetful functor (or in the motivic case, by the Betti realisation functor).

In Section 2.1.2 we use the universal property of étale motives recalled in Section 1.2.3.3 to construct the Hodge and Nori realisation functors, and in Section 2.1.3 we prove that if étale motives have a motivic t-structure, then the Nori realisation functor is an equivalence.

We finish this chapter with Section 2.2 in which we recall some definitions of mixed realisations and show that the Hodge realisation we constructed in Section 2.1.2 is part of a functor to an ∞-category of mixed realisations over a base.

2.1 The Hodge and Nori realisation functors.

In this section and this section only, we will use different symbols for the triangulated bounded derived category $\mathrm{D}^b(\mathcal{A})$ of an abelian category, and its ∞-categorical enhancement $\mathcal{D}^b(\mathcal{A})$, because this subtlety is the core of the chapter. If X is a scheme we will denote by $\mathcal{DM}_c^{\text{ét}}(X)$ the ∞-category $\mathrm{DM}_c^{\text{ét}}(X,\mathbb{Q})$ of étale motives with rational coefficients, and by $\mathcal{SH}(X)$ the ∞-category of motivic spectra.

2.1.1 Enhancement of Nori motives and Hodge modules.

2.1.1.1 Cœur constructible cœur pervers.

Hypothesis on the coefficient system.

Let k be a field of characteristic zero. For this section we will denote by Var_k either the category of quasi-projective k-varieties or of separated reduced k-schemes of finite type, and we will call objects of Var_k *varieties*. Choose $\mathrm{D} : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{Triang}$ your favourite $\mathbb{Q}$-linear triangulated category of coefficients with a 6 functors formalism (by this we mean a symmetric monoidal stable homotopy 2-functor as in Ayoub's [Ayo07a]) and a conservative realisation functor $R : \mathrm{D} \to \mathrm{D}_c^b$ compatible with the operations, where D_c^b is either the derived category of constructible sheaves on the analytification (if k is a subfield of $\mathbb{C}$), of the derived category of ℓ-adic constructible sheaves. For $X \in \mathrm{Var}_k$ we will call motives the elements of $\mathrm{D}(X)$. For example D could be the derived category of perverse Nori motives ([IM22, Theorem 5.1] and [Ter24, Main theorem], see Section 1.2.4.3) or of mixed Hodge modules ([Sai90b, Theorem 0.1], see Section 1.2.5). Note that in the mixed Hodge modules situation the realisation is nothing more that the functor forgetting the structure of a mixed Hodge module, and is called 'taking the $\mathbb{Q}$-structure" by Saito. We assume that for each variety X the category $\mathrm{D}(X)$ has a perverse t-structure for which R is t-exact when $\mathrm{D}_c^b(X)$ is endowed with its perverse t-structure of heart the abelian category of perverse sheaves $\mathrm{Perv}(X)$.

Recall the following definition:

Definition 2.1.1. The *constructible t-structure* on $\mathrm{D}(X)$ is the unique t-structure on $\mathrm{D}(X)$ that makes all pullback functors t-exact. Explicitly it is the t-structure given by

$$ {}^{\mathrm{ct}}\mathrm{D}^{\leqslant 0}(X) := \{M \in \mathrm{D}(X) \mid \forall x \in X,\ \text{closed},\ x^*M \in \mathrm{D}^{p\leqslant 0}(\mathrm{Spec}\,\kappa(x))\} $$

and

$$ {}^{\mathrm{ct}}\mathrm{D}^{\geqslant 0}(X) := \{M \in \mathrm{D}(X) \mid \forall x \in X,\ \text{closed},\ x^*M \in \mathrm{D}^{p\geqslant 0}(\mathrm{Spec}\,\kappa(x))\}. $$

That this indeed gives a t-structure can be proven by gluing, using that over a smooth variety, a lisse object (see Definition 2.1.3) is positive (*resp.* negative) for the constructible t-structure if and only it is positive (*resp.* negative) for the perverse t-structure when shifted by the dimension. We give more details in Proposition 2.1.6 below.

Denote by $\mathcal{M}_{\mathrm{ct}}(X)$ the heart of the constructible t-structure on $\mathrm{D}(X)$. As R is t-exact and conservative, all known t-exactness results about the 6 functors are true in D. We will call elements of $\mathcal{M}_{\mathrm{ct}}(X)$ *constructible motives*. Denote by $\mathscr{H}om(-,-)$ and $\mathbb{Q}_X$ the internal Hom and unit object of $\mathrm{D}(X)$.

We assume that $\mathrm{D}(X)$ has enough structure so that there exists a natural triangulated functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \to \mathrm{D}(X)$ extending the inclusion of the constructible heart (for example if $\mathrm{D}(X)$ is a derived category as shown by Beilinson, Bernstein, Deligne and Gabber in [BBD82, section 3.1] or more generally if $\mathrm{D}(X)$ is the homotopy category of some stable ∞-category $\mathcal{D}(X)$ as proven in [BCKW19, Remark 7.4.13]), and we assume that the natural functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(\mathrm{Spec}\,k)) \to \mathrm{D}(\mathrm{Spec}\,k)$ is an equivalence. This is true for perverse Nori motives and mixed Hodge modules.

From now on in this section by the t-structure we mean the constructible t-structure, unless specified otherwise.

Remark 2.1.2. In this manuscript two names will be given to the constructible t-structure, the other being the ordinary t-structure.

Definition 2.1.3. Let X be a k-variety. A motive L is a *lisse* object if its realisation $R(L)$ is a local system, that is a locally constant bounded complex of finite dimensional $\mathbb{Q}$-vector spaces. We denote by $\mathrm{D}^{liss}(X)$ the full subcategory of lisse objects in $\mathrm{D}(X)$.

We will denote by $\mathbb{Q}_X \in \mathcal{M}_{\mathrm{ct}}(X)$ the unit object for the tensor structure. Its realisation is the constant sheaf.

Remark 2.1.4. Note that as $\mathrm{D}(X)$ is closed as a tensor category, given an object $M \in \mathrm{D}(X)$, we have a natural candidate for the strong dual of M : it is $\mathscr{H}om(M, \mathbb{Q}_X)$. Therefore, an object M is dualisable if and only if the natural map

$$\mathscr{H}om(M, \mathbb{Q}_X) \otimes N \to \mathscr{H}om(M, N) \tag{2.1.4.1}$$

is an isomorphism for every N (see [Del90, Proposition 2.3]). We will denote by $M^\vee = \mathscr{H}om(M, \mathbb{Q}_X)$ the dual of M when M is dualisable.

Lemma 2.1.5. *Let X be a k-variety. An object $K \in \mathrm{D}(X)$ is lisse if and only if it is dualisable. In particular, for every object $N \in \mathrm{D}(X)$, there exists a dense open on which N is dualisable. If an object $L \in \mathcal{M}_{\mathrm{ct}}(X)$ is lisse, then its dual $L^\vee$ is also an object of $\mathcal{M}_{\mathrm{ct}}(X)$.*

Proof. For the first point, dualisable objects in $\mathrm{D}^b_c(X, \mathbb{Q})$ are exactly local systems by Ayoub's [Ayo22, Lemma 1.24] in the analytic case, and by [HRS23, Proposition 7.6] together with [MW22, Corollary 7.4.12.] in the ℓ-adic case. As the realisation functor is conservative, we can test if the map (2.1.4.1) is an isomorphism after applying the realisation functor. If $L \in \mathcal{M}_{\mathrm{ct}}(X)$ is dualisable, the functor $\mathscr{H}om(L, -) = - \otimes L^\vee$ is t-exact because it is left and right adjoint to the exact functor $- \otimes L$. Therefore, $L^\vee = \mathscr{H}om(L, \mathbb{Q}_X) \in \mathcal{M}_{\mathrm{ct}}(X)$ lies in the heart.

The second point follows from the fact that after realisation any constructible motive is locally constant on a stratification, hence on some dense open subset. □

Proposition 2.1.6. *The two subcategories of Definition 2.1.1 define a t-structure.*

Proof. By Lemma 2.1.5 we have that

$$\mathrm{D}(X) \simeq \mathrm{colim}_{\mathcal{S}} \mathrm{D}_{\mathcal{S}}(X)$$

where the colimit is over all finite stratifications by smooth locally closed sub-schemes, and $\mathrm{D}_{\mathcal{S}}(X)$ is the full subcategory of $\mathrm{D}(X)$ whose objects are dualisable on the stratification $\mathcal{S}$. Moreover, by [BBD82, Proposition 2.1.14] and because functor $\mathrm{D}(X) \to \mathrm{D}^b_c(X)$ is t-exact for the perverse t-structure, for each such stratification $\mathcal{S}$, the perverse t-structure restricts to $\mathrm{D}_{\mathcal{S}}(X)$. It suffices to prove that the intersection of the subcategories of Definition 2.1.1 with each $\mathrm{D}_{\mathcal{S}}(X)$ define a t-structure, as then all inclusions $\mathrm{D}_{\mathcal{S}}(X) \to \mathrm{D}_{\mathcal{S}'}(X)$ will be t-exact for the constructible t-structure, and this will show that we have a constructible t-structure on $\mathrm{D}(X)$. If the stratification has only one element, then the perverse t-structure restricts to dualisable objects and the induced t-structure on satisfy the conditions of Definition 2.1.1, shifted by the dimension of X ([BBD82, Definition 2.1.2]). Assume that $\mathcal{S}$ has for open stratum U, and denote by Z the complement of U in X, and by $\mathcal{T}$ the stratification on Z

induced by $\mathcal{S}$. By doing a noetherian induction on X, we may assume that Definition 2.1.1 produces a t-structure on $\mathrm{D}_{\mathcal{T}}(Z)$. Because U is smooth, Definition 2.1.1 also induces a t-structure on $\mathrm{D}_{\{U\}}(U)$. Denote by $j\colon U \to X$ and $i\colon Z \to X$ the immersions. By [BBD82, Théorème 1.4.10], the category ${}^{\mathrm{ct}}\mathrm{D}^{\leqslant 0}(X) \cap \mathrm{D}_{\mathcal{S}}(X)$ is the left part of a t-structure, whose positive side is given by the formula

$$\{K \in \mathrm{D}_{\mathcal{S}}(X) \mid j^*K \in {}^{\mathrm{ct}}\mathrm{D}^{\geqslant 0}(U) \cap \mathrm{D}_{\{U\}}(U) \text{ and } i^!K \in {}^{\mathrm{ct}}\mathrm{D}^{\geqslant 0}(Z) \cap \mathrm{D}_{\mathcal{T}}(Z)\}.$$

We have to show that an element $K \in \mathrm{D}_{\mathcal{S}}(X)$ satisfies the above condition if and only if for every closed point x of X we have $x^*K \in \mathrm{D}^{p\geqslant 0}(\mathrm{Spec}(\kappa(x)))$. This can be checked after applying th functor $\mathrm{D}(X) \to \mathrm{D}^b_c(X)$, where this is true because, by [BBD82, Proposition 1.4.12], the constructible t-strcuture on $\mathrm{D}^b_{c\mathcal{S}}(X)$ is obtained by recollement: the functor $j_!j^*$ is right t-exact (and even t-exact). □

Cohomology of motives over an affine variety.

Definition 2.1.7.

1. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called *effaceable* if for any $M \in \mathcal{A}$ there is an injection $M \to N$ in $\mathcal{A}$ such that the induced map $F(M) \to F(N)$ is the zero map.
2. An object $M \in \mathrm{D}(X)$ is called *admissible* if the functor $\mathrm{Hom}_{\mathrm{D}(X)}(M, -[q]) : \mathcal{M}_{\mathrm{ct}}(X) \to \mathrm{Vect}_{\mathbb{Q}}$ is effaceable for every $q > 0$.

Lemma 2.1.8. *Let X be a variety and M be a constructible motive on $\mathbb{A}^1_X$. Assume that there is a smooth open subset $U \subset \mathbb{A}^1_X$ on which M is lisse and such that the restriction of $\pi : \mathbb{A}^1_X \to X$ to $Z = \mathbb{A}^1_X \setminus U$ is finite and surjective. Assume that $M_{|Z} = 0$. Then there exists $N \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^1_X)$ such that $\pi_*N = 0$ and that M injects into N.*

Proof. Assume that k is a subfield of $\mathbb{C}$. Then the following holds : Denote by $p_i : \mathbb{A}^2_X \to \mathbb{A}^1_X$ the projections, and by $\Delta : \mathbb{A}^1_X \to \mathbb{A}^2_X$ the diagonal. The map $\alpha : p_1^*M \to \Delta_*M$ in $\mathcal{M}_{\mathrm{ct}}(X)$ (both functors are t-exact) obtained by adjunction from $\mathrm{id} : \Delta^*p_1^*M \simeq M \to M$ is surjective because it is surjective after realisation. We therefore have an exact sequence in $\mathcal{M}_{\mathrm{ct}}(\mathbb{A}^1_X)$

$$0 \to \ker \alpha \to p_1^*M \to \Delta_*M \to 0.$$

Let $N := \mathrm{H}^1(p_2)_* \ker \alpha$. Then the long exact sequence of cohomology obtained after applying $(p_2)_*$ to the above sequence gives a map $M \simeq (p_2)_*\Delta_*M \simeq \mathrm{H}^0((p_2)_*\Delta_*M) \to N$, which is injective by [Nor02, Proposition 2.2].

Moreover, Nori proves in *loc cit.* that for all $p \geqslant 0$ we have $\mathrm{H}^p(\pi_*N) = 0$, which implies that $\pi_*N = 0$.

In the case k is too big to be embedded in $\mathbb{C}$, on may use [Bar21, Proposition 3.2] and the ℓ-adic realisation, noting that the construction of N is the same. □

Nori's method gives in our setting a slightly less good result than in the case of constructible sheaves (see Remark 2.1.10). It will thankfully be sufficient for us.

Theorem 2.1.9 (*cf.* [Nor02] Theorem 1)**.** *For every $n \in \mathbb{N}$, the motive $\mathbb{Q}_{\mathbb{A}^n_k} \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^n_k)$ is admissible.*

Proof. We argue as in Nori's paper.
We proceed by induction on n to prove the theorem. First the case of a point. If $n = 0$ and $M \in \mathcal{M}_{\mathrm{ct}}(\operatorname{Spec} k)$, as $\mathrm{D}(\operatorname{Spec} k) = \mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(\operatorname{Spec} k))$, the functors $\mathrm{Hom}_{\mathrm{D}(\operatorname{Spec} k)}(M, -[q])$ are effaceable by [BBD82, Proposition 3.1.16].

Suppose that $n \geqslant 1$ and that the result is known for $n-1$. Let $M \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^n_k)$ and $q > 0$. Let $f \in k[T_1, \ldots, T_n]$ be a polynomial such that M is lisse when restricted to the open subset $U = D(f) \subset \mathbb{A}^n_k$. Up to a change of coordinates, we may assume that the projection $\pi : \mathbb{A}^n_k \to \mathbb{A}^{n-1}_k$ on the $n-1$ first coordinates gives a finite map $\pi_{|V(f)} : V(f) \to \mathbb{A}^{n-1}_k$ when restricted to $Z = V(f)$. Denote by j and i the inclusions of U and Z in $\mathbb{A}^n_k$.
By Lemma 2.1.8, $j_! M_{|U}$ is a submotive of a constructible motive N' such that $\pi_* N' = 0$. Taking N to be the push-out of the maps $j_! M_{|U} \to M$ and $j_! M_{|U} \to N'$ we get a morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_! M_{|U} & \longrightarrow & M & \longrightarrow & i_* M_{|Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & i_* M_{|Z} & \longrightarrow & 0 \end{array}$$

with γ also injective.

Also, as $\pi_* N' = 0$, we see that $\pi_* N \to \pi_* i_* M_{|Z} = (\pi_{|Z})_* M_{|Z}$ is an isomorphism because of the distinguished triangle $N' \to N \to i_* M_Z \xrightarrow{+1}$. As $\pi_{|Z}$ is finite, its pushforward $(\pi_{|Z})_*$ is t-exact, hence $\pi_* N \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^{n-1}_k)$.

By the induction hypothesis, we can find an injection $g : \pi_* N \to K$ with K a constructible motive such that $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^{n-1}_k)}(\mathbb{Q}_{\mathbb{A}^{n-1}_k}, g[q]) = 0$. Take L the pushout of the counit $\pi^* \pi_* N \to N$ and the injection $\pi^* g : \pi^* \pi_* N \to \pi^* K$ (note that π^* is t-exact). We get a morphism of exact sequences :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^* \pi_* N & \xrightarrow{\pi^* g} & \pi^* K & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & N & \xrightarrow{h} & L & \longrightarrow & C & \longrightarrow & 0 \end{array} . \tag{2.1.9.1}$$

The unit map $\pi_* N \to \pi_* \pi^* \pi_* N$ is an isomorphism by $\mathbb{A}^1$-invariance. Therefore, in the diagram above the left and the right vertical maps become isomorphisms after applying π_*, thus this is also the case for the middle map $\iota : \pi^* K \to L$.

Using the adjunction (π^*, π_*) and the fact that $\pi_* N \to \pi_* \pi^* \pi_* N$ is an isomorphism one gets that the map

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, \pi^* \pi_* N[q]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^{n-1}_k)}(\mathbb{Q}_{\mathbb{A}^{n-1}_k}, \pi_* \pi^* \pi_* N[q]) \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, N[q])$$

is an isomorphism. Moreover, the map

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, \pi^* \pi_* N[q]) \xrightarrow{\pi^* g} \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, \pi^* K[q])$$

induced by $\pi^* g$ vanishes as it was already the case before applying π^*.

Therefore, when applying $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, -[q])$ to the left square of (2.1.9.1), we see that the map

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, N[q]) \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, L[q])$$

induced by h factors through the zero map hence is zero.

Hence, we see that the injection $M \xrightarrow{\gamma} N \xrightarrow{h} L$ induces the zero map after applying the functor $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, -[q])$, finishing the induction. □

Remark 2.1.10. In his paper [Nor02] proves a stronger statement than Corollary 2.1.11. He proves that any constructible sheaf M on $\mathbb{A}^n_k$ can be embedded in a constructible sheaf N such that $\mathrm{H}^q(\mathbb{A}^n, N) = 0$ for all $q > 0$. The reason for which our result is weaker is the starting point of the induction: for Nori, if $n = 0$ then $\mathrm{D}(\mathrm{Spec}(k)) = \mathrm{D}^b(\mathbb{Q})$ has for heart $\mathrm{vect}_{\mathbb{Q}}$, which has cohomological dimension 0. In particular, the identity morphism $M \to M$ is a monomorphism with $\mathrm{H}^q(\mathbb{A}^n, M) = 0$ for all $q > 0$. In our case of perverse Nori motives or mixed Hodge modules, the cohomological dimension of $\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}(k))$ is strictly positive: it is 1 for mixed Hodge modules, and is conjectured to be $1 + \mathrm{trdeg}_{\mathbb{Q}}(k)$ for motives. Thus we cannot start the induction as Nori does, but [BBD82, Proposition 3.1.16] saves us.

Corollary 2.1.11. *If X is affine, the object $\mathbb{Q}_X$ is admissible.*

Proof. Let $q > 0$. Take $i : X \to \mathbb{A}^n_k$ a closed immersion and $M \in \mathcal{M}_{\mathrm{ct}}(X)$. By Theorem 2.1.9, there is an injection $i_*M \to K$ with K a constructible motive such that $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n)}(\mathbb{Q}_{\mathbb{A}^n_k}, i_*M[q]) \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n)}(\mathbb{Q}_{\mathbb{A}^n_k}, K[q])$ is the zero map. We have an inclusion $f : i^*i_*M \simeq M \to N := i^*K$, and a commutative diagram :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, M[q]) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, i^*K[q]) \\ \sim\uparrow & & \sim\uparrow \\ \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, i_*M[q]) & \xrightarrow{0} & \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, K[q]) \end{array}$$

giving that $f \circ -$ is zero, thus finishing the proof. □

Admissibility of constant motives on any variety.

Notation 2.1.12. *If $j : U \hookrightarrow X$ is an open subset of a variety X, and $M \in \mathcal{M}_{\mathrm{ct}}(X)$ is a constructible motive, we will use the notation $M[U] := j_!M_{|U}$. Note that by localisation we have $M/M[U] \simeq i_*M_{|Z}$ with $i : Z = X \setminus U \to X$ the closed complement.*

Lemma 2.1.13 (Stability of admissibility)**.** *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of constructible motives on X.

1. *Assume that M'' is admissible. Then M' is admissible if and only if M is admissible.*
2. *Assume that M' and M are admissible. If the functor*

$$\mathrm{coker}\ (\mathrm{Hom}_{\mathrm{D}(X)}(M, -) \to \mathrm{Hom}_{\mathrm{D}(X)}(M', -))$$

is effaceable, then M'' is admissible.

Proof. These are formal properties of abelian categories, proven in [Nor02, Lemma 3.2]. □

Corollary 2.1.14 ([Nor02] Lemma 3.5)**.** *Let $M \in \mathcal{M}_{\mathrm{ct}}(X)$.*

1. *If $U \subset X$ is open, and if M and $M[U]$ are admissible, then so is $M/M[U]$.*
2. *If V, W are open subsets of X, and if $M[V], M[W], M[V \cap W]$ are admissible, then the same is true for $M[V \cup W]$*

Proof. We use part 2. of Lemma 2.1.13. For 1., it suffices to show that any $P \in \mathcal{M}_{\mathrm{ct}}(X)$ is a subsheaf of a $Q \in \mathcal{M}_{\mathrm{ct}}(X)$ such that the map $f : \mathrm{Hom}_{\mathrm{D}(X)}(M, Q) \to \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q)$ is surjective. Denote by $j : U \to X$ the open immersion and by $i : Z \to X$ its closed complement. Then define $Q = i_* i^* P \oplus j_* j^* P$. Localisation ensures that the map $P \to Q$ is injective,

$$\mathrm{Hom}_{\mathrm{D}(X)}(M, i_* P_{|Z}) \oplus \mathrm{Hom}_{\mathrm{D}(U)}(M_{|U}, P_{|U}) \simeq \mathrm{Hom}_{\mathrm{D}(X)}(M, Q)$$

and

$$\mathrm{Hom}_{\mathrm{D}(U)}(M_{|U}, P_{|U}) \simeq 0 \oplus \mathrm{Hom}_{\mathrm{D}(X)}(j_! M_{|U}, j_* P_{|U}) \simeq \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q)$$

thus f is surjective as it is just the projection on the second factor.
For 2. first one has the Mayer-Vietoris exact sequence

$$0 \to M[V \cap W] \to M[V] \oplus M[W] \to M[V \cup W] \to 0. \tag{2.1.14.1}$$

We let $U = V \cap W$. For a given P we take the same Q as above for U that gives surjectivity of $\mathrm{Hom}_{\mathrm{D}(X)}(M, Q) \to \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q)$. Now the map $a : M[U] \to M$ factors through $M[V]$ thus the map

$$\mathrm{Hom}_{\mathrm{D}(X)}(M[V], Q) \xrightarrow{a^*} \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q) \tag{2.1.14.2}$$

is surjective. Finally, the map induced by $M[U] \to M[V] \oplus M[W]$ is also surjective as one of its direct summands is. □

Proposition 2.1.15 (Avoiding the use of injective objects, after Nori [Nor02] Remark 3.8)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories. Let $G : \mathcal{B} \to \mathcal{A}$ be a functor which has an exact left adjoint. Let $H : \mathcal{A} \to \mathcal{C}$ be an effaceable functor. Then HG is also effaceable.*

Proof. Let F be a left adjoint of G. Take B an object of $\mathcal{B}$. By assumption on H, there is an injection $u : G(B) \to A$ such that the induced map $H(u) : HG(B) \to H(A)$ vanishes. By exactness of F, the morphism $F(u) : FG(B) \to F(A)$ is a monomorphism. We also have the counit of the adjunction $\varepsilon : FG(B) \to B$. Take B' the pushout of these two maps :

$$\begin{array}{ccc} FG(B) & \xrightarrow{F(u)} & F(A) \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle t} \\ B & \xrightarrow[v]{} & B' \end{array}$$

The map $v : B \to B'$ is a monomorphism and we have a commutative diagram (with η the unit of the adjunction) :

$$\begin{array}{ccc} G(B) & \xrightarrow{u} & A \\ {\scriptstyle\eta_{G(B)}}\downarrow & & \downarrow{\scriptstyle\eta_A} \\ GFG(B) & \xrightarrow{GF(u)} & GF(A) \\ {\scriptstyle G(\varepsilon)}\downarrow & & \downarrow{\scriptstyle G(t)} \\ G(B) & \xrightarrow{G(v)} & G(B') \end{array} \qquad \mathrm{Id}$$

The composition $G(\varepsilon) \circ \eta_{G(B)}$ is the identity, therefore, the map $G(v)$ factors as $w \circ u$ with $w = G(t) \circ \eta_A$. Now, $HG(v) = H(w) \circ H(u) = 0$ hence HG is effaceable. □

Corollary 2.1.16. *Let X and Y be varieties. Let $F : \mathrm{D}(X) \to \mathrm{D}(Y)$ be a t-exact exact functor which is left adjoint to a t-exact functor. If $M \in \mathcal{M}_{\mathrm{ct}}(X)$ is admissible, then so is $F(M) \in \mathcal{M}_{\mathrm{ct}}(Y)$.*

Proof. Let $q > 0$ and let G be a right adjoint of F. The functor $H = \mathrm{Hom}_{\mathrm{D}(X)}(M, -[q])$ is effaceable by definition. By Proposition 2.1.15, $H \circ G$ is therefore effaceable. We have a natural isomorphism $H \circ G \simeq \mathrm{Hom}_{\mathrm{D}(Y)}(F(M), -[q])$ because G is t-exact, which gives the claim. □

Corollary 2.1.17. *Let X be a variety. Let $M \in \mathcal{M}_{\mathrm{ct}}(X)$ be an admissible motive.*

1. *Let $j : X \to Y$ be an étale morphism. Then $j_!M$ is admissible.*
2. *Let $L \in \mathcal{M}_{\mathrm{ct}}(X)$ be a lisse motive. Then $M \otimes L$ is admissible.*

Proof. We use Corollary 2.1.16. For the first point, we have the adjunction $(j_!, j^*)$ where $j_!$ and j^* are t-exact. Let $L \in \mathcal{M}_{\mathrm{ct}}(X)$ be a lisse motive. The functor $- \otimes L$ is left adjoint to $\mathscr{H}om(L, -)$ and right adjoint to $\mathscr{H}om(L^\vee, -)$ hence is t-exact. The same can be said about $\mathscr{H}om(L, -) = - \otimes L^\vee$. □

Proposition 2.1.18 ([Nor02] Proposition 3.6)**.** *Let $U \subset X$ an open of a variety X. Then the constructible motive $\mathbb{Q}_X[U]$ is admissible.*

Proof. Let $j : U \to X$ be the open immersion. We prove the theorem by induction on the number n of affines needed to cover U. If U is affine, Corollary 2.1.11 ensures that $\mathbb{Q}_U$ is admissible hence by Corollary 2.1.17, $j_!\mathbb{Q}_U = \mathbb{Q}_X[U]$ is admissible. For the induction step, one can write $U = V \cup W$ with V affine and W covered by $n-1$ affines, and separateness of X gives that $V \cap W$ is covered by $n-1$ affines. Therefore by induction $\mathbb{Q}_X[V]$, $\mathbb{Q}_X[V \cap W]$ and $\mathbb{Q}_X[W]$ are admissible. Then, Corollary 2.1.14 ensures that $\mathbb{Q}_X[U]$ is admissible. □

Corollary 2.1.19 ([Nor02] Theorem 2)**.** *Let X be a variety over k. Then the constant motive $\mathbb{Q}_X$ is admissible, that is, for every $q > 0$, every constructible motive $M \in \mathcal{M}_{\mathrm{ct}}(X)$ can be embedded in a constructible motive $N \in \mathcal{M}_{\mathrm{ct}}(X)$ such that the map*

$$\mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, M[q]) \to \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, N[q])$$

vanishes.

Proof. One takes $U = X$ in the previous proposition. □

Lisse motives enter the party.

Proposition 2.1.20. *Let X be a variety. Any lisse motive in $\mathcal{M}_{\mathrm{ct}}(X)$ is admissible.*

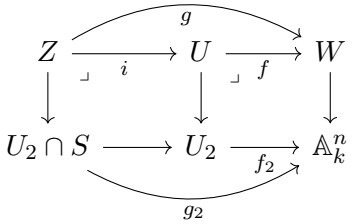
Proof. By Corollary 2.1.19 the unit object $\mathbb{Q}_X$ on X is admissible. By Corollary 2.1.17 $L \simeq \mathbb{Q}_X \otimes L$ is admissible. □

Theorem 2.1.21. *Let X be a variety. Then any $M \in \mathcal{M}_{\mathrm{ct}}(X)$ is admissible, that is, for every $q > 0$, the functor $\mathrm{Hom}_{\mathrm{D}(X)}(M, -[q]) : \mathcal{M}_{\mathrm{ct}}(X) \to \mathrm{Vect}_{\mathbb{Q}}$ is effaceable.*

Proof. We prove the result by Noetherian induction on the support S of M in X, that is the complement of the largest open subset V of X such that $M_{|V} = 0$. We follow the proof of [Nor02, Proposition 3.10].

Let S^d be the union of the irreducible components of maximal dimension of S. We denoted by d the dimension of S. Let U_{liss} be the largest open subset of S^d on which M is lisse. Then there is an affine open subset U_1 of X such that the intersection $U_1 \cap S^d$ is contained in U_{liss} and is nonempty. Let U_2' be the open subset of X obtained by removing from U_1 the irreducible components of S that are not of maximal dimension. We have that $U_2' \cap S$ is nonempty and contained in U_{liss}. We choose a nonempty affine open subset U_2 of U_2' such that $U_2 \cap S$ is nonempty and a Noether normalisation $g_2 \colon U_2 \cap S \to \mathbb{A}_k^d$. It extends to a map $f_2 \colon U_2 \to \mathbb{A}_k^d$. Indeed, because $U_2 \cap S$ is closed in U_2 and the three schemes U_2, $U_2 \cap S$ and $\mathbb{A}_k^d$ are affine thus we may choose in $\mathscr{O}_{U_2}(U_2)$ lifts of the images of the coordinates of $\mathbb{A}_k^d$ in $\mathscr{O}_{U_2 \cap S}(U_2 \cap S)$. Let $W \subset \mathbb{A}_k^d$ be a nonempty open subset over which g_2 is smooth. As g_2 is of relative dimension 0 in fact g_2 is étale over W. We set $U := f_2^{-1}(W)$.

Let $Z = U \cap S$ and $g \colon Z \to W$ be the restriction of g_2 to Z. By construction we have $Z = g_2^{-1}(W)$ thus the map $g \colon Z \to W$ is finite and étale. We also let $f : U \to W$ be the restriction of f_2 to U. The situation may be summarised in the following commutative diagram:



where we have denoted by i the inclusion of Z inside U.

Note that because the restriction of M to Z is lisse and g is finite étale, the object $g_* M_{|Z}$ is still lisse, so that $L := f^* g_* M_{|Z}$ is lisse. The restriction $i^* L$ to Z of L is $g^* g_* M_{|Z}$ which has $M_{|Z}$ as a direct factor because g is étale.

Because the support of $M_{|U}$ is contained in Z we have that $i_* M_{|Z} = i_* i^* M_{|U} \simeq M_{|U}$. By the above, this shows that $M_{|U}$ is a direct factor of $i_* L_{|Z}$. Let j be the inclusion of the open complement of Z in U. By localisation, we have that $i_* L_{|Z} \simeq L / j_! j^* L$. By Proposition 2.1.20 the objects L and $j^* L$ are admissible, thus as by Corollary 2.1.17 the object $j_! j^* L$ is then admissible, we deduce by Corollary 2.1.14 that the object $i_* L_{|Z}$ is admissible. Thus, the restriction $M_{|U}$ of M to U is admissible as a direct factor of an admissible object.

If $\iota \colon U \to X$ is the open immersion then $\iota_! M_{|U}$ is admissible by Corollary 2.1.17. We have an exact sequence

$$0 \to \iota_! M_{|U} \to M \to M / \iota_! M_{|U} \to 0$$

in which the support of $M / \iota_! M_{|U}$ is strictly smaller than the support of M. By Noetherian induction $M / \iota_! M_{|U}$ is admissible, thus by Lemma 2.1.13 we finally obtain that M is admissible. □

Corollary 2.1.22. *Let X be a variety. The natural functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \to \mathrm{D}(X)$ is an equivalence.*

Proof. We apply [BBD82, Proposition 3.1.16]. The conditions are verified because of the previous Theorem 2.1.21. □

2.1.1.2 Higher categorical lifting of the operations.

In this subsection, we will work with perverse Nori motives or mixed Hodge modules identically. Therefore, we denote by Var_k the category of quasi-projective k-schemes when dealing with perverse Nori motives or the category of separated and reduced finite type $\mathbb{C}$-schemes when dealing with mixed Hodge structures. We will call objects of Var_k varieties or k-varieties. For $X \in \mathrm{Var}_k$, we will denote by $\mathcal{M}_{\mathrm{perv}}(X)$ the category of perverse Nori motives constructed in [IM22] or the category of Saito's mixed Hodge modules constructed in [Sai90b]. We will have to use realisation functors hence we denote by $R : \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathrm{D}^b_c(X)$ either the Betti realisation of perverse Nori motives when $k \subset \mathbb{C}$, the underlying $\mathbb{Q}$-structure functor of mixed Hodge modules, or the ℓ-adic realisation of perverse Nori motives. Here $\mathrm{D}^b_c(X)$ is the derived category of bounded cohomologically constructible sheaves on X^{an} in the two first cases, and the derived category of constructible ℓ-adic étale sheaves on X. The triangulated category $\mathrm{D}^b_c(X)$ is endowed with a perverse and a constructible t-structures and the functor R is t-exact (for both t-structures) and conservative.

We begin with a recollection on the ∞-categorical enhancement of derived categories of constructible sheaves.

Proposition 2.1.23. *There exists a functor*

$$\mathcal{D}^b_c \colon \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

such that for each map $f : Y \to X$ in Var_k, the symmetric monoidal functor induced by $\mathcal{D}^b_c(f)$ on the homotopy categories is the classical pullback of constructible sheaves.

Proof. In both analytic sheaves and étale sheaves, for $X \in \mathrm{Var}_k$, the category $\mathrm{D}^b_c(X)$ is embedded in a bigger category $\mathcal{D}(X)$ which is of the form $\mathcal{D}(\mathrm{Shv}(\mathfrak{X}, \mathscr{O}))$ with $\mathfrak{X}$ a topos and $\mathscr{O}$ a sheaf of rings on $\mathfrak{X}$ and is naturally a symmetric monoidal stable ∞-category. Indeed, in the case of analytic sheaves one can take $\mathfrak{X}$ to be the categories of analytic opens of the variety of complex points of X and $\mathscr{O}$ is the constant sheaf $\mathbb{Q}_X$. In the case of ℓ-adic sheaves, following [HRS23] one can take $\mathfrak{X}$ to be the proétale topos of X, and $\mathscr{O}$ to be the sheaf of rings induced by the condensed ring $\mathbb{Q}_\ell = \mathbb{Z}_\ell[1/\ell]$. This construction has a canonical lift to the world of symmetric monoidal ∞-categories thanks to [Lurb, Remark 2.1.0.5]. As the constructions $X \mapsto \mathfrak{X}$ and $\mathfrak{X} \mapsto \mathcal{D}(\mathrm{Shv}(\mathfrak{X}, \mathscr{O}))$ are functorial, we obtain a functor

$$\mathcal{D} : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Now as for each f in Var_k, the functor $f^* \colon \mathcal{D}(Y) \to \mathcal{D}(X)$ preserves constructible objects, this induces a functor

$$\mathcal{D}^b_c : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

which gives the usual pullback on homotopy categories by definition in the case of analytic sheaves, and by [HRS23, Theorem 7.7] for ℓ-adic sheaves.

□

Construction 2.1.24. We have a symmetric monoidal homotopy 2-functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_1).$$

By restricting to the constructible hearts, this induces a functor

$$\mathcal{M}_{\mathrm{ct}}(-) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_1)$$

such that the tensor product is exact in each variable and the pullback functors are exact (this can be checked after realisation). By applying Proposition A.6.2 we obtain a functor

$$\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(-)) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Moreover, the equivalence in Corollary 2.1.22 lifts to an equivalence of ∞-categories thanks to Theorem A.1.17. The Proposition A.1.29 then gives a functor

$$\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty),$$

canonically equivalent to $\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(-))$.

Proposition 2.1.25. *Let $f : Y \to X$ be a map in Var_k. Then the functor induced by $\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(f))$ on the homotopy categories is the pullback functor constructed by Ivorra and Morel or by Saito, depending on which case we are.*

Proof. We claim that if we know that the pullback functors constructed by Ivorra, Morel and Saito admit ∞-categorical lifts f^*, then they will be canonically equivalent to $f^*_{\mathrm{new}} := \mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(f))$. Indeed, in that case the ∞-functors f^* and f^*_{new} coincide on the constructible heart by definition, hence they are equivalent by Theorem 1.1.21.

We prove this first for mixed Hodge modules: For any map $f : Y \to X$ of varieties, there is the factorisation

$$Y \xrightarrow{i} X \times Y \xrightarrow{p} X$$

where i is the closed immersion given by the inclusion of the graph of f (our varieties are separated) and p is the projection. By definition, we have $f^* \simeq p^* \circ i^*$. The functor p^* is given by the external product with the constant object: $p^*(M) = M \boxtimes \mathbb{Q}_Y$, hence is an ∞-functor because the external product is defined as the derived functor on the perverse hearts. The functor i^* is the left adjoint of the functor i_*, which is obtained ([Sai90b, (4.2.4) and (4.2.10)]) by deriving the functor i_* on the perverse heart and is therefore an ∞-functor. Thus, by Theorem A.1.16 the functor i^* is an ∞-functor. Finally, the functor f^* admits an ∞-categorical lift.

Now for perverse Nori motives, any map $f : Y \to X$ being a map of quasi-projective varieties, factors as

$$Y \xrightarrow{i} Z \xrightarrow{g} X$$

with i a closed immersion and g a smooth map. Hence $f^* \simeq i^* g^*$. The case of i^* is the same as for mixed Hodge modules, and thus we have an ∞-functor. For the smooth map g, we can assume that X is connected (as a direct sum of ∞-functors is an ∞-functor), so that g is smooth of pure relative dimension d for some $d \in \mathbb{N}$. Now by definition $f^* = (f^*[d])[-d]$ where $f^*[d]$ is the derived functor of the exact functor $f^*[d]$ on the perverse heart, hence we have an ∞-functor. □

Corollary 2.1.26. *Let $f : Y \to X$ be a map in $\mathrm{Var}_k^{\mathrm{op}}$, then the ∞-functor*

$$f^* : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(Y))$$

admits a right adjoint f_*. If f is moreover smooth, f^* admits a left adjoint $f_\sharp$.

Proof. These adjoints exist on the homotopy categories, thus they have ∞-categorical lifts by Theorem A.1.16. □

Proposition 2.1.27. *For each $X \in \mathrm{Var}_k$, the monoidal structure obtained in Construction 2.1.24 induces on the homotopy categories the same monoidal structure as the one constructed by M. Saito for mixed Hodge modules and by Terenzi ([Ter24]) for perverse Nori motives.*

Moreover, the projection formula for a smooth morphism holds in $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$.

Proof. Note that in both [Ter24] and [Sai90b] the tensor product is built in the following way : there is an external tensor product $\boxtimes : \mathcal{M}_{\mathrm{perv}}(X) \times \mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{M}_{\mathrm{perv}}(X \times X)$ that is exact in both variable hence induces by Corollary A.6.1 an ∞-functor $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X \times X))$ which we compose with the pullback Δ^* under the diagonal $\Delta : X \to X \times X$ to obtain the tensor product, thus given by $M \otimes N = \Delta^*(M \boxtimes N)$. The other structural morphisms of the monoidal structure are obtained similarly. Denote by $\otimes^T : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ Saito's or Terenzi's tensor product and denote by $\otimes^\infty : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ the one we just constructed. We want to check that $\mathrm{ho}(\otimes^\infty) = \mathrm{ho}(\otimes^T)$. For this there are several isomorphisms to check (see [ML98, Section XI.1]) :

1. A functorial isomorphism $\otimes^T \Rightarrow \otimes^\infty$.
2. Modulo point 1. isomorphism, an identification of the unit isomorphism $\rho : M \otimes 1 \to M$ and $\lambda : 1 \otimes M \to M$.
3. Modulo point 1. isomorphism, an identification of the associativity isomorphism $c : M \otimes (N \otimes P) \to (M \otimes N) \otimes P$.
4. Modulo point 1. isomorphism, an identification of the commutativity isomorphism $\gamma : M \otimes N \to N \otimes M$.

By definition, $\otimes^\infty$ coincides with $\otimes^T$ when restricted to $\mathcal{M}_{\mathrm{ct}}(X) \times \mathcal{M}_{\mathrm{ct}}(X)$. By Corollary A.6.1 this gives 1. We explain how to obtain 3. and the other checks will be left to the reader as they are very similar. Denote by $t_1^\infty : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ the functor sending M, N, P to $M \otimes^\infty (N \otimes^\infty P)$ and by $t_2^\infty : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ the functor sending M, N, P to $(M \otimes^\infty N) \otimes^\infty P$. The associativity isomorphism is a natural equivalence $c^\infty : t_1^\infty \Rightarrow t_2^\infty$. We give the same definition for t_1^T, t_2^T and c^T. Again by definition, the image by the restriction functor of the morphisms c^∞ and c^T are the same, hence up to the equivalence of 1., we obtain 3.

For the projection formula, the arrow exists by a play of adjunctions, and is an equivalence on the homotopy categories. □

Corollary 2.1.28. *Let X be a variety. The tensor structure on $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is closed.*

Proof. Again this is true on the homotopy categories and we can apply Theorem A.1.16. □

Proposition 2.1.29. *Let X be a variety. The realisation functor*

$$R : \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathrm{D}^b_c(X)$$

admits a canonical lift to a map in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$.

Moreover, it induces a natural transformation

$$\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) \Rightarrow \mathcal{D}^b_c(-)$$

on functors

$$\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Proof. In the case of the Betti realisation this is easy as we have an equivalence

$$\mathcal{D}^b(\mathrm{Cons}(X)) \simeq \mathcal{D}^b_c(X)$$

thanks to Nori's theorem. This enables us to consider the functor $\Delta^1 \times \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_1)$ that send $(0, f)$ to $f^* : \mathcal{M}_{\mathrm{ct}}(Y) \to \mathcal{M}_{\mathrm{ct}}(X)$, the edge $(1, f)$ to $f^* : \mathrm{Cons}(Y) \to \mathrm{Cons}(X)$ and $(0 \to 1, X)$ to the realisation functor. Therefore by Proposition A.6.2 we obtain the result.

For the ℓ-adic realisation it is not true (if the field k has cohomological dimension $\geqslant 2$, see [Bar21]) that the canonical functor

$$\mathrm{real}_X : \mathcal{D}^b(\mathrm{Cons}(X, \mathbb{Q}_\ell)) \to \mathcal{D}^b_c(X, \mathbb{Q}_\ell)$$

is an equivalence. What we did in the case of the Betti realisation gives a symmetric monoidal natural transformation

$$\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(-)) \Rightarrow \mathcal{D}^b(\mathrm{Cons}(-, \mathbb{Q}_\ell)).$$

Hence, we have to check that the real_X assemble to give a symmetric monoidal natural transformation

$$\mathcal{D}^b(\mathrm{Cons}(-, \mathbb{Q}_\ell)) \Rightarrow \mathcal{D}^b_c(-).$$

This is Proposition A.6.3. □

Corollary 2.1.30. *The realisation functor*

$$\mathcal{R} \colon \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) \to \mathcal{D}^b_c(-)$$

commutes with all the operations constructed above. This includes: pullbacks f^*, pushforwards f_*, left adjoints $f_\sharp$ when f is a smooth map, tensor product $- \otimes -$ and internal homomorphism $\mathscr{H}om(-,-)$.

Proof. Indeed, this is true on the homotopy categories, and the exchange morphisms exist because $\mathcal{R}$ commutes with pullbacks and tensor products by definition. □

Verdier duality also lifts:

Proposition 2.1.31. *Let X be a k-variety. Then the Verdier duality functor $\mathbb{D}_X$ has a ∞-categorical lift $\mathbb{D}_X \colon \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))^{\mathrm{op}}$ which commutes with the Betti realisation. Moreover, there is a canonical isomorphism $\mathbb{D}_X \circ \mathbb{D}_X \simeq \mathrm{Id}_{\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))}$.*

Proof. Indeed the functor $\mathbb{D}_X$ both in Ivorra-Morel and in Saito, is constructed as the trivial derived functor of the exact functor $\mathbb{D}_X \colon \mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{M}_{\mathrm{perv}}(X)^{\mathrm{op}}$ thus it is an ∞-functor. Thus the functor also commutes with the Betti realisation as an ∞-functor (as in fact we apply the functor $\mathcal{D}^b(-)$ to the commutative square involving Verdier duality of $\mathcal{M}_{\mathrm{perv}}(X)$, $\mathrm{Perv}(X)$ and R). The fact that $\mathbb{D}_X \circ \mathbb{D}_X \simeq \mathrm{Id}_{\mathcal{D}^b_{\mathcal{M}}(X)}$ holds can be checked on the homotopy category as well. □

For free, we also obtain the exceptional functoriality:

Corollary 2.1.32. *The exceptional functors $f^!$ and $f_!$ constructed by Ivorra, Morel and Saito have ∞-categorical lifts that assemble to give functors*

$$\mathrm{Sch}_k^{(\mathrm{op})} \to \mathrm{Cat}_\infty.$$

Moreover, Verdier duality is a natural isomorphism between the star functoriality (f_* or f^*) and the shriek functoriality of the opposite category ($f_!$ or $f^!$).

Proof. The last statement of this corollary gives a construction. Indeed the formula $\mathbb{D} \circ f_* \simeq f_! \circ \mathbb{D}$ tells us that if we apply Proposition A.1.29 to the isomorphisms $\mathbb{D}_X : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))^{\mathrm{op}}$, the obtained functor $\mathrm{Sch}^{(\mathrm{op})} \to \mathrm{Cat}_\infty$ encode exactly the exceptional functoriality. □

Theorem 2.1.33. *The functor $X \mapsto \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$, with respect to the $(-)^*$-functoriality, is an hypersheaf with respect to h-topology.*

Proof. Denote by $\mathcal{D}^b_{\mathcal{M}}(X) := \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$. For this proof, we will assume that D^b_c is the ∞-category of ℓ-adic sheaves on the base change of X to the algebraic closed $\overline{k}$ of k. This is possible because both ℓ-adic constructible sheaves on X, and constructible complexes on X^{an}, have a conservative functor to $\mathrm{D}^b_c(X \times_k \overline{k}, \mathbb{Q}_\ell)$, compatible with the operations. By [HS23, Theorem 2.2], D^b_c has h-hyperdescent. Denote by $B \colon \mathrm{D} \to \mathrm{D}^b_c$ the conservative realisaiton. We will use Corollary A.1.35, thus we need to make sure that the limits ensuring descent exist in $\mathcal{D}^b_{\mathcal{M}}$. This is not obvious. However for any variety X and $m \in \mathbb{N}$, if we denote by $\mathcal{D}^{[-m,m]}_{\mathcal{M}}(X)$ the full subcategory of $\mathcal{D}^b_{\mathcal{M}}(X)$ of objects concentrated, for the constructible t-structure, in degrees $[-m, m]$, we have

$$\bigcup_m \mathcal{D}^{[-m,m]}_{\mathcal{M}}(X) = \mathcal{D}^b_{\mathcal{M}}(X).$$

Moreover, the functor $\mathcal{D}^b_{\mathcal{M}}$ is a hyper sheaf if and only if for each m, the functor $\mathcal{D}^{[-m,m]}_{\mathcal{M}}$ is a hyper sheaf, and the same holds for D^b_c. Indeed, if $\mathcal{D}^b_{\mathcal{M}}$ is a hyper sheaf, then as being concentrated in degrees $[-m, m]$ can be tested after pulling back by any surjective map of schemes, it turns out that each $\mathcal{D}^{[-m,m]}_{\mathcal{M}}$ is also a hyper sheaf. Conversely, if each $\mathcal{D}^{[-m,m]}_{\mathcal{M}}$ is a hyper sheaf, then because pullback functors f^* are t-exact functors, in turns out that the conditions to test that $\mathcal{D}^b_{\mathcal{M}}$ is a hypersheaf only takes place in $\mathcal{D}^{[-m,m]}_{\mathcal{M}}$: if $U_\bullet \to X$ is a hyper covering, full faithfulness of the sheaf condition is computed in $\mathcal{D}^{[-m,m]}_{\mathcal{M}}(X)$ for some m, and then essential surjectivity holds because given an element $K = (K_n)_n$ of the limit there exists a fixed m such that K has its K_0 hence also all its factors K_n in $\mathcal{D}^{[-m,m]}_{\mathcal{M}}(U_n)$.

Thus because $\mathrm{D}_c^{[-m,m]}$ is an h-hyper sheaf, it suffices to check the conditions of Corollary A.1.35 for the realisation functor

$$B\colon \mathcal{D}_{\mathcal{M}}^{[-m,m]} \to \mathrm{D}_c^{[-m,m]}.$$

Let $f_\bullet\colon Y_\bullet \to X$ be an h-hyper covering. As the realisation functor is conservative, it suffices to check that:

1. For each complex M in $\mathcal{D}_{\mathcal{M}}^{[-m,m]}(X)$, the limit

$$\lim_{\Delta} \tau^{\leqslant m}(f_n)_* f_n^* M$$

exists, where Δ is the simplicial category and $\tau^{\leqslant m}$ is the truncation functor for the t-structure.

2. The realisation functor

$$\mathcal{D}_{\mathcal{M}}^{[-m,m]}(X) \to \mathrm{D}_c^{[-m,m]}(X)$$

commutes with such a limit.

In fact as $\mathcal{D}_{\mathcal{M}}^{[-m,m]}(X)$ is a $(2m+1)$-category, the above totalisation in (1) is finite thus exists (because the finite limit $\lim(f_n)_* f_n^* M$ exists in $\mathcal{D}_{\mathcal{M}}^{\geqslant -m}$ because it exists in $\mathcal{D}_{\mathcal{M}}^{b}$, and then we apply the right adjoint $\tau^{\leqslant m}$), and the realisation functor clearly commutes with such a finite limit because the realisation functor

$$\mathcal{D}_{\mathcal{M}}^{\geqslant -m}(X) \to \mathrm{D}_c^{\geqslant -m}(X)$$

commutes with finite limits (the inclusion $\mathrm{D}^{\geqslant -m} \to \mathrm{D}^b$ commutes with limits) and the image by the realisation functor of the above limit in (1) is the truncation by $\tau^{\leqslant m}$, which is a right adjoint, of a limit in $\mathrm{D}_c^{\geqslant -m}(X)$.

□

Remark 2.1.34. Assume we are in the setting of perverse Nori motives. Using a similar argument to the previous theorem and Construction 2.1.24, one can show that if X is a quasi-projective k-scheme, there is a functor $\mathcal{D}_X\colon \mathrm{AffOp}(X)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending U to $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(U))$ and any inclusion to the associated restriction functor, and that this functor is a sheaf. In fact, the argument is simpler here because for Zariski coverings, the limit of the sheaf condition is already finite: one can count intersections, so that if the open covering is made of n affines subsets, the limit of the sheaf condition stabilizes at n. Thus all the machinery used to truncated our categories is not needed here. However, because we want to use this fact to construct pushforward under affine morphisms, we do not want to use any j_* as in the proof of the theorem above. Fortunately, we can pass to the opposite categories and use $j_!$ which are easy to construct.

Now let $f\colon E \to X$ be a vector bundle. We want to show that if for any $n \in \mathbb{N}$ and any affine scheme U we know that the pullback functor $p^*\colon \mathcal{D}(\mathcal{M}_{\mathrm{perv}}(U)) \to \mathcal{D}(\mathcal{M}_{\mathrm{perv}}(\mathbb{A}^n_U))$ has a right adjoint p_*, compatible with the Betti realisation and such that for any affine open immersion j, we have a base change for (j^*, p_*), then the pullback functor $f^*\colon \mathcal{D}(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}(\mathcal{M}_{\mathrm{perv}}(E))$ has a right adjoint, compatible with the Betti realisation, and such that the base change with respect to affine open immersion also holds.

For this, we may use Lurie theory of adjointable functors (see the proof of Proposition 1.2.5). Recall that there is a subcategory $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_\infty)$ of the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ of functors between ∞-categories, that can be described unformally as follows: its objects are the functors $F\colon \mathcal{C} \to \mathcal{D}$ that are left adjoints, and a morphism $F \to G$ is a commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle V} \\ \mathcal{C}' & \xrightarrow{G} & \mathcal{D}' \end{array}$$

such that if we denote by F_* and G_* the right adjoints of F and G, the natural map $UF_* \to G_*V$ is an equivalence. Lurie proves in [Lur22, Corollary 4.7.4.18] that this subcategory admits all limits, and that the inclusion functor $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_\infty) \subset \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ preserves those limits.

Let (U_i) be a finite affine covering of X that trivialises E: $E_{|U_i} \simeq \mathbb{A}^n_{U_i}$ as vector bundles. We set $U = \coprod_i U_i$. Then for any $n \in \mathbb{N}$, the n-th fibre product $U_n := U \times_X U \times \cdots \times_X U$ of U is still an affine open subset of X, over which E is trivial. In particular, the pullback under the map

$$p_n\colon E_{|U_n} \to U_n$$

has a right adjoint on perverse Nori motives, and this right adjoint commutes with the restrictions to any U_m. This proves that the data of $(p_n^*)_{n\in\Delta}$ can be put together to obtain a semi-simplicial diagram

$$\Delta^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_\infty).$$

By Zariski descent and the fact that limits are computed term wise in this ∞-category of left adjoints functors, we see that the limit of this diagram is the functor f^*, which thus has a right adjoint compatible with pullbacks along inclusions of affine open subsets. The compatibility with the Betti realisation can then be checked Zariski locally. Thus we obtain point 6 of the construction of the operations sketched in Section 1.2.4.3.

2.1.2 Construction of the realisation functor.

In this section, we construct realisation functors from the category of étale motives to the derived categories of mixed Hodge modules and perverse Nori motives that commute with the 6 operations. Denote by Sch_k the category of finite type k-schemes. We use Drew and Gallauer' result in [DG22] (see Section 1.2.3.3) in which they prove that the stable motivic $\mathbb{A}^1$-invariant ∞-category $\mathcal{SH}$ has a universal property among *coefficient systems*: these are functors

$$\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

that have the minimal functoriality needed for the 6 operations to exist thanks to Ayoub's thesis (see Definition 1.2.2 for a precise definition and some results on coefficient systems). Any coefficient system $C \in \mathrm{CoSys}_k$ with values in idempotent complete ∞-categories receives an unique realisation functor

$$\mathcal{SH}_c \to C$$

from the compact objects of the motivic homotopy category that commutes with enough operations to ensures that it commutes with the 6 operations (if C has some good properties, see below).

Theorem 2.1.35. *Denote by $\mathcal{D}^b_{\mathcal{M}}$ the functor $\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ of Construction 2.1.24. The functor $\mathcal{D}^b_{\mathcal{M}}$ defines an object of CoSys_k.*

Proof. This is the content of Section 2.1.1.2 and Proposition 1.2.5. See Definition 1.2.2 for the precise definition of coefficient systems. □

The above theorem contains the statement that perverse Nori motives and mixed Hodge modules extend to every finite type k-scheme, together with all the operations and h-hyperdescent (the only non-trivial statement here is the h-hyperdescent, but this follows from the fact that the Kan extension of an hypersheaf on a basis of a site to the full site is still an hypersheaf, see for example [Aok23, Theorem A.6]).

Remark 2.1.36. Using that the mapping spectra in $\mathcal{D}^b_{\mathcal{M}}$, $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}})$ and $\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}})$ are étale sheaves (for the perverse heart, one has to use affine étale morphisms) and the fact that a limit of a diagram of stable ∞-categories with t-structures and t-exact exact transition functors is endowed with a t-structure (see [RS20, Lemma 3.2.18]), one can show by descent that after extension to finite type k-schemes, we still have

$$\mathcal{D}^b_{\mathcal{M}}(X) \simeq \mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \simeq \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

and in the case of perverse Nori motives, the category $\mathcal{M}_{\mathrm{perv}}(X)$ is the universal category as for quasi-projective schemes. The weights also extend to that case.

Notation 2.1.37. *For X a finite type k-scheme, we will denote by $\mathcal{D}^b(\mathrm{MHM}(X))$ (*resp. *by $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$) the value at X of the functor $\mathcal{D}^b_{\mathcal{M}}$ in the mixed Hodge modules case (*resp. *in the Nori motives case).*

Theorem 2.1.38. *There exists an (essentially unique) morphism of coefficients systems $\mathcal{DM}^{\text{ét}} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ on the category Sch_k. It restricts to a functor*

$$\nu : \mathcal{DM}^{\text{ét}}_c \to \mathcal{D}^b_{\mathcal{M}}. \tag{2.1.38.1}$$

The functor (2.1.38.1) commutes with the 6 operations, that is all the exchange transformations are isomorphisms.

In the case of mixed Hodge modules we will denote by Hdg^* the realisation functor, and in the case of Nori motives we will denote it by Nor^*.

Proof. Thanks to Theorem 2.1.35 we know that $\mathcal{D}^b_{\mathcal{M}}$ is an object of CoSys_k, hence its indization $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ is an object of CoSys^c_k. Together with Section 1.2.3.3, this gives a morphism of coefficient systems

$$\rho_{\mathcal{SH}} \colon \mathcal{SH} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}.$$

Note that the functor $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ naturally takes values in $\mathrm{CAlg}(\mathrm{Pr}^L)_{\backslash \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k))}$ the co-slice ∞-category, because $\mathrm{Spec}(k)$ is the initial object of $\mathrm{Sch}_k^{\mathrm{op}}$. Using the functor $\rho_{\mathcal{SH}}$ over $\mathrm{Spec}(k)$ we see that $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ takes values in $\mathrm{CAlg}(\mathrm{Pr}^L)_{\backslash \mathcal{SH}(\mathrm{Spec}(k))}$ the ∞-category of $\mathcal{SH}(\mathrm{Spec}(k))$-algebras in Pr^L, where we endowed Pr^L with the Lurie tensor product constructed in [Lur22, Section 4.8.1]. We claim that $\rho_{\mathcal{SH},k} \colon \mathcal{SH}(\mathrm{Spec}(k)) \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k))$ factors through $\mathcal{DM}^{\acute{e}t}(\mathrm{Spec}(k))$.

First, note that because $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k))$ is $\mathbb{Q}$-linear, the functor $\rho_{\mathcal{SH},k}$ factors through $\mathcal{SH}_{\mathbb{Q}}(\mathrm{Spec}(k))$ to give a symmetric monoidal functor

$$\rho_{\mathcal{SH}_{\mathbb{Q}},k}\colon \mathcal{SH}_{\mathbb{Q}}(\mathrm{Spec}(k)) \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k)).$$

Moreover by the universal property of the localisation, if K is a finite extension of k in which we added a square root of -1, the square

$$\begin{array}{ccc} \mathcal{SH}_{\mathbb{Q}}(\mathrm{Spec}(k)) & \xrightarrow{\rho_{\mathcal{SH}_{\mathbb{Q}},k}} & \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k)) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\ \mathcal{SH}_{\mathbb{Q}}(\mathrm{Spec}(K)) & \xrightarrow{\rho_{\mathcal{SH}_{\mathbb{Q}},K}} & \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(K)) \end{array}$$

commutes, where $g\colon \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ it the structural morphism. Now because $\mathbb{Q}^-_{\mathrm{Spec}(k)}$ is a direct factor of $\mathbb{Q}_{\mathrm{Spec}(k)}$, its image under $\rho_{\mathcal{SH}_{\mathbb{Q}},k}$ lands in $\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k))$. By étale descent of $\mathcal{D}^b_{\mathcal{M}}$ proven in Theorem 2.1.33 the functor g^* is conservative on $\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k))$, and as $\mathbb{Q}^-_{\mathrm{Spec}(K)} = 0$, we see that the image of $\mathbb{Q}^-_{\mathrm{Spec}(k)}$ in $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(\mathrm{Spec}(k))$ vanishes. By Proposition 1.2.19 this proves that the functor $\rho_{\mathcal{SH}_{\mathbb{Q}},k}$ factors through $\mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k))$ to give a functor that we will denote by $\widetilde{\nu_k}$. Through $\widetilde{\nu_k}$ we see that our functor $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ takes values in $\mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k))$-algebras in Pr^L. Using [Lur22, Corollary 4.3.3.14] the forgetful functor

$$\mathrm{ff}\colon \mathrm{CAlg}(\mathrm{Pr}^L)_{\backslash \mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k))} \to \mathrm{CAlg}(\mathrm{Pr}^L)_{\backslash \mathcal{SH}(\mathrm{Spec}(k))}$$

is right adjoint to the functor

$$- \otimes_{\mathcal{SH}(\mathrm{Spec}(k))} \mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k)).$$

Applying this adjunction term-wise we obtain by [Cis19, Theorem 6.1.22] an adjunction between functors on $\mathrm{Sch}_k^{\mathrm{op}}$. In particular, the natural transformation $\rho_{\mathcal{SH}}$ may be seen as a map $\mathcal{SH} \to \mathrm{ff}(\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}})$ in the ∞-category

$$\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^L)_{\backslash \mathcal{SH}(\mathrm{Spec}(k))}).$$

By adjunction, we obtain a natural transformation

$$\mathcal{SH} \otimes_{\mathcal{SH}(\mathrm{Spec}(k))} \mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k)) \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$$

of functors on $\mathrm{Sch}_k^{\mathrm{op}}$ and with values in $\mathrm{CAlg}(\mathrm{Pr}^L)_{\backslash \mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k))}$. The left-hand side sends a scheme X to the ∞-category

$$\mathcal{SH}(X) \otimes_{\mathcal{SH}(\mathrm{Spec}(k))} \mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k)).$$

By using that

$$\mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k)) \simeq \mathrm{Mod}_{\mathbb{Q}_k^+}(\mathcal{SH}(\mathrm{Spec}(k)))$$

and Lurie's [Lur22, Theorem 4.8.4.6] we see that

$$\mathcal{SH}(X) \otimes_{\mathcal{SH}(\mathrm{Spec}(k))} \mathcal{DM}^{\text{ét}}(\mathrm{Spec}(k)) \simeq \mathrm{Mod}_{\mathbb{Q}_X^+}(\mathcal{SH}(X)) \simeq \mathcal{DM}^{\text{ét}}(X).$$

This proves that $\rho_{\mathcal{SH}}$ factors through the functor $\mathcal{DM}^{\text{ét}}$, giving us a natural transformation

$$\widetilde{\nu}\colon \mathcal{DM}^{\text{ét}} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$$

of functors on $\mathrm{Sch}_k^{\mathrm{op}}$ with values in $\mathrm{CAlg}(\mathrm{Pr}^L)$. The factorisation is given by the unit of the adjunction between functor categories described above.

We claim that we obtained a morphism of coefficient systems. Let $f\colon X \to Y$ be a smooth map. Because $\mathcal{DM}^{\text{ét}}(X)$ is generated under colimits by objects of the form $a^{\mathbb{Q}}_{\text{ét}}(M)$ with $M \in \mathcal{SH}(X)$ and $a^{\mathbb{Q}}_{\text{ét}}$ the morphism $\mathcal{SH} \to \mathcal{DM}^{\text{ét}}$, it suffices to prove that the natural exchange morphism

$$f_\sharp(\widetilde{\nu}(a_{\text{ét}}(M))) \to \widetilde{\nu}(f_\sharp^{\mathcal{DM}^{\text{ét}}}(a_{\text{ét}}(M)))$$

is an equivalence. This follows from the following sequence of equivalences:

$$\begin{aligned} f_\sharp(\widetilde{\nu}(a_{\text{ét}}(M))) &\simeq f_\sharp(\rho(M)) \\ &\overset{(1)}{\simeq} \rho(f_\sharp^{\mathcal{SH}}(M)) \\ &\simeq \widetilde{\nu}(a^{\mathbb{Q}}_{\text{ét}}(f_\sharp^{\mathcal{SH}}(M))) \\ &\overset{(2)}{\simeq} \widetilde{\nu}(f_\sharp^{\mathcal{DM}^{\text{ét}}}(a^{\mathbb{Q}}_{\text{ét}}(M))), \end{aligned}$$

where we used in (1) the fact that $\rho_{\mathcal{SH}}$ is a morphism of coefficient systems and in (2) the fact that $a^{\mathbb{Q}}_{\text{ét}}$ is a morphism of coefficient systems by Lemma 1.2.20.

Denote by $\mathcal{DM}^{\text{ét}}_c(X)$ (*resp.* by $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}(X)$) the thick full subcategory of $\mathcal{DM}^{\text{ét}}(X)$ (*resp.* of $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(X)$) spanned by the $f_\sharp\mathbb{Q}_Y(i)$ for $f : Y \to X$ smooth and $i \in \mathbb{Z}$. As $\mathcal{D}^b_{\mathcal{M}}(X) \subset \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(X)$ is thick and each $f_\sharp\mathbb{Q}_Y(i)$ is in $\mathcal{D}^b_{\mathcal{M}}(X)$, we have $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}(X) \subset \mathcal{D}^b_{\mathcal{M}}(X)$. Because $\widetilde{\nu}$ is a morphism of coefficient systems, it induces a morphism of coefficient systems

$$\nu^{\mathrm{gm}} : \mathcal{DM}^{\text{ét}}_c \to \mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}. \tag{2.1.38.2}$$

In particular, at the level of triangulated categories, we obtain a premotivic morphism that verifies all the hypotheses of [CD19, Theorem 4.4.25] : $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}$ is τ-generated for τ the Tate twist, $\mathcal{DM}^{\text{ét}}_c$ is τ-dualisable by absolute purity, $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}$ is separated because it is a sub pullback formalism of $\mathcal{D}^b_{\mathcal{M}}$ which satisfy étale descent by Theorem 2.1.33 (which is equivalent to separateness in the $\mathbb{Q}$-linear case by [CD19, Corollairy 3.3.34]) and all Tate twists are invertible in $\mathcal{DM}^{\text{ét}}_c$ by construction.

Therefore, the functor ν^{gm} commutes with the 6 operations. One has to be a little careful here: a priori, [CD19, Theorem 4.4.25] implies only that ν^{gm} commutes with all the operations when $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}$ is endowed with the operations they construct in their book, and not necessarily with the constructions of Saito, Ivorra, Morel as in Corollary 2.1.32. We know that for $f : Y \to X$ a morphism, f^* is the same for both constructions, hence by adjunction this is also the case for f_*. For $j : U \to X$ an open immersion the $j_\sharp$ are the same, but $j_\sharp = j_!$ hence by Nagata compactification (and the fact that for a proper p, we have $p_* \simeq p_!$) and composition both $f_!$ are isomorphic for any $f : Y \to X$ separated. By adjunction we also have an isomorphism between the $f^!$. This defines a isomorphism of étale sheaves of ∞-categories, hence we also have compatibility of operations for non separated morphisms (when they are defined for $\mathcal{DM}^{\text{ét}}$). The tensor product is the same, hence also the internal $\mathscr{Hom}$ by adjunction.

We have obtained that the functor

$$\nu : \mathcal{DM}_c^{\text{ét}} \to \mathcal{D}_{\mathcal{M}}^b \tag{2.1.38.3}$$

commute with the 6 operations at the triangulated categories level. But all exchange morphisms exist at the level of ∞-categories, and the functor from an ∞-category to its homotopy category is conservative, hence ν commutes with all the operations. □

Remark 2.1.39. One could also probably show that the functor $\mathcal{SH} \to \mathrm{IndD}_{\mathcal{M}}^b$ factors through $\mathcal{DM}^{\text{ét}}$ by showing that Drew-Gallauer's universal property extends to any topology and has a linear version (see [Ayo22, 2.1.13]), so that the ∞-category of "rational étale local pullback formalisms" has the coefficient system $\mathcal{DM}^{\text{ét}}$ has a initial object. To use this modified universal property, one would also have to prove that $\mathrm{IndD}_{\mathcal{M}}^b$ has non-effective étale descent, which seems not trivial when looking at the formula for mapping spaces in the ind-category.

Corollary 2.1.40. *The realisation functor*

$$\mathcal{SH}_c \to \mathcal{D}_{\mathcal{M}}^b$$

commutes with the 6 operations.

Proof. Indeed, it suffices to check that the functor $\mathcal{SH}_c \to \mathcal{DM}_c^{\text{ét}}$ commutes with the operations. This is the content of Lemma 1.2.21. □

Corollary 2.1.41. *There exists a Hodge realisation functor $R_H : \mathcal{D}^b(\mathcal{M}_{\text{perv}}) \to \mathcal{D}^b(\mathrm{MHM})$ compatible with the six operations. It factors the Hodge realisation that we have constructed in Theorem 2.1.38.*

Proof. By Theorem 2.1.38, there exists a Hodge realisation from $\mathcal{DM}_c^{\text{ét}}(-, \mathbb{Q})$ to the derived category of mixed Hodge modules. Therefore, taking the underlying $\mathbb{Q}$-structure and then perverse ${}^{\mathrm{p}}\mathrm{H}^0$, we obtain a factorisation of the perverse ${}^{\mathrm{p}}\mathrm{H}^0$ of the Betti realisation through $\mathrm{MHM}(X)$. The universal property of perverse Nori motives then gives a faithful exact functor $\mathcal{M}_{\text{perv}}(X) \to \mathrm{MHM}(X)$, which induces a functor $\mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \to \mathcal{D}^b(\mathrm{MHM}(X))$. Now all the constructions of [IM22] mimicks the constructions of the operations for mixed Hodge modules, hence this functor is compatible with the operations. The fact that this functor factors the Hodge realisation we constructed above follows from the universal property of $\mathcal{DM}_c^{\text{ét}}$ and the fact that R_H can be promoted to a natural transformation of ∞-functors on $\mathrm{Sch}_k^{\mathrm{op}}$, using that it is also the derived functor of its restriction to the constructible heart. □

Recall that by [IM22, Corollary 6.18], [Bon11, Proposition 2.3.1] and [Hé11, Theorem 3.3] the categories $\mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))$, $\mathcal{D}^b(\mathrm{MHM}(X))$ and $\mathcal{DM}_c^{\text{ét}}(X)$ admit weight structures (for the last one, note that Beilinson motives and étale motives are equivalent by [CD19, Theorem 16.1.2]).

Proposition 2.1.42. *Let X be a finite type k-scheme. Then the functors*

$$R_H : \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \to \mathcal{D}^b(\mathrm{MHM}(X)),$$

$$\mathrm{Nor}^* : \mathcal{DM}_c^{\text{ét}}(X) \to \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))$$

and

$$\mathrm{Hdg}^* : \mathcal{DM}_c^{\text{ét}}(X) \to \mathcal{D}^b(\mathrm{MHM}(X))$$

are compatible with weights.

Proof. First note that by definition of the weights on $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$, the ℓ-adic realisation $R_\ell : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b_c(X, \mathbb{Q}_\ell)$ is compatible with weights and reflects weights (*i.e.* if a complex $K \in \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ satisfies that $R_\ell(K)$ is of weights $\leqslant w$, then K is of weights $\leqslant w$). As all weight filtrations and structures considered are bounded, it suffices to show that our functors map weight zero objects to weight zero objects. Let us first consider functors with source $\mathcal{DM}_c^{\text{ét}}(X)$. For those, is suffices to check that the image of the relative motive of a smooth proper X-scheme is pure of weight 0. For the ℓ-adic realisation this is [Del80] and [Mor19]. Therefore, the functor $\mathrm{Nor}^* : \mathcal{DM}_c^{\text{ét}}(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is also weight exact. The fact that Hdg^* is exact is the fact that the mixed Hodge modules associated to a smooth proper X-scheme is pure of weight 0, see [Sai06, Theorem 6.7 and Proposition 6.6].

Now is only left $R_H : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathrm{MHM}(X))$. By [IM22, Corollary 6.27], any pure complex of perverse Nori motives is the direct sum of shifts of its ${}^{\mathrm{p}}\mathrm{H}^i$, hence to check that R_H is weight exact it suffices to show that it sends pure objects of $\mathcal{M}_{\mathrm{perv}}(X)$ to pure objects of $\mathrm{MHM}(X)$. By construction of R_H, we have a commutative diagram:

$$\mathcal{DM}_c^{\text{ét}}(X) \xrightarrow{\mathrm{H}_u} \mathcal{M}_{\mathrm{perv}}(X) \xrightarrow{R_H} \mathrm{MHM}(X) \ , \qquad \overset{{}^{\mathrm{p}}\mathrm{H}^0\mathrm{Hdg}^*}{\frown} \tag{2.1.42.1}$$

With H_u the universal functor defining $\mathcal{M}_{\mathrm{perv}}(X)$. But by [IM22, Corollary 6.27], H_u sends pure objects to pure objects, and any motive $M \in \mathcal{M}_{\mathrm{perv}}(X)$ is a quotient of some $\mathrm{H}_u(N)$ for $N \in \mathcal{DM}_c^{\text{ét}}(X)$. Assume that M is pure of some weight w. Then M is a direct factor of a quotient of the weight filtration of $\mathrm{H}^0(N)$. As Hdg^* is weight exact, it follows that $R_H(M)$ is a direct factor of a weight w mixed Hodge module, hence R_H is weight exact. □

Remark 2.1.43. Our construction of the realisation functor $\mathcal{DM}_c^{\text{ét}}(X)$ is very similar to that of Ivorra in [Ivo16] (which works for smooth varieties). Indeed, given a smooth variety X, in both definitions one first gives the image of the smooth X-schemes in $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$. To be more precise, Ivorra only gives the image of smooth affine X-schemes and then extend to smooth X-schemes by colimits but as the functor $(f : Y \to X) \mapsto f_! f^! \mathbb{Q}_X$ is a Nisnevich cosheaf, its value on smooth X-schemes is indeed determined by its value on smooth affine X-schemes. Once one has the value of the functor on smooth X-schemes, the rest of the construction is just applying the universal property of $\mathcal{DM}_c^{\text{ét}}(X)$: one has to check that the functor on smooth X-schemes is an $\mathbb{A}^1$-invariant étale sheaf and $\mathbb{P}^1$-stable. Therefore, it is easy to see that if both definitions on smooth affine schemes (ours with the 6-operations and Ivorra's using cellular complexes and Beilinson basic lemma) coincide, then our functor $\mathcal{DM}_c^{\text{ét}}(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ and Ivorra's are equivalent. In particular, this would prove that Ivorra's construction is compatible with the Betti realisation constructed by Ayoub on Voevodsky motives, hence that the category of perverse Nori motives as in [IM22] is equivalent as the diagram category of relative pairs studied in [Ivo17] and [IM22, Section 2.10].

This can be reduced to check a simple fact, that the author did not manage to do: in [Ivo16, Proposition 4.15] Ivorra proves that for a given smooth affine map $f : Y \to X$ with X a smooth variety, there is a specific cellular stratification $Y^\bullet$ of Y such that the cellular complex induced on Y is an $\mathrm{H}^0 f_!$-acyclic resolution of $f^! \mathbb{Q}_X$, hence that there is an isomorphism in $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ between Ivorra's functor at Y and the homology of Y, that is $\mathrm{ra}_X^{\mathcal{M}_{\mathrm{perv}}}(Y, Y_\bullet) \simeq f_! f^! \mathbb{Q}_X$. This isomorphism induces a canonical and functorial isomorphism

on each ${}^{\mathrm{p}}\mathrm{H}^i$. The question is whether one can construct such an isomorphism in a functorial way, as this would give the wanted isomorphism of functors. This does not seem easy, and it is unlikely that one can arrange those specific stratification so that they are compatible with a given morphism of smooth affine X-schemes.

2.1.3 Relations with the t-structure conjecture.

Let k be a field of characteristic zero. Recall the following conjecture ([And04, Section 21.1.7] and [Bei12]):

Conjecture 2.1.44. *There exists a non degenerate t-structure on $\mathcal{DM}_c^{\text{ét}}(k)$ compatible with the tensor product and such that the ℓ-adic realisation functor is t-exact.*

In [Bei12] Beilinson proves that if the characteristic of k is zero the conjecture implies that the realisation functors of $\mathcal{DM}_c^{\text{ét}}(k)$ are conservative. In [Bon15, Theorem 3.1.4], Bondarko proves that if such a t-structure exists for all fields of characteristic 0, then there exists a perverse t-structure on $\mathcal{DM}_c^{\text{ét}}(X)$ for each finite type k-scheme X. The joint conservativity of the family of x^* for $x \in X$ and the conservativity of the realisation functor over fields implies that in that case, the ℓ-adic realisation over X is conservative and t-exact, when its target is endowed with the perverse t-structure.

Theorem 2.1.45. *Assume Conjecture 2.1.44 for all fields of characteristic 0. Let X be a finite type k-scheme. Then the heart of the perverse t-structure of $\mathcal{DM}_c^{\text{ét}}(X)$ is canonically equivalent to the category $\mathcal{M}_{\text{perv}}(X)$ of perverse Nori motives. Moreover, the functor $\mathrm{Nor}^* : \mathcal{DM}_c^{\text{ét}}(X) \to \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))$ is an equivalence of stable ∞-categories. This implies that $\mathcal{DM}_c^{\text{ét}}(X)$ is the derived category of the abelian category of both its perverse and constructible heart.*

From now on, we assume that Conjecture 2.1.44 is true. We begin by two lemmas, in which we denote by Var_k the category of quasi-projective k-varieties.

Lemma 2.1.46. *Let X be quasi-projective scheme. Denote by $\mathcal{M}(X)$ the heart of the perverse t-structure on $\mathcal{DM}_c^{\text{ét}}(X)$. There is a faithful exact functor $F_X : \mathcal{M}_{\text{perv}}(X) \to \mathcal{M}(X)$ that commutes with the ℓ-adic realisation. Denote by*

$$\gamma_X : \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \to \mathcal{DM}_c^{\text{ét}}(X)$$

the canonical functor induced by the universal property of the bounded derived category applied to the functor

$$\mathcal{M}_{\text{perv}}(X) \xrightarrow{F_X} \mathcal{M}(X) \to \mathcal{DM}_c^{\text{ét}}(X).$$

The composition $\mathrm{Nor}_X^* \circ \gamma_X$ is equivalent to the identity functor $\mathrm{Id}_{\mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))}$. Moreover the $\mathrm{ho}(\gamma_X)$ fit in a morphism γ of functors $\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{SymMono}_1$ with values in symmetric monoidal categories, using the $(-)^*$-functoriality.

Proof. By assumption the restriction to the heart of the ℓ-adic realisation functor

$$\rho_\ell \colon \mathcal{M}(X) \to \mathrm{Perv}(X, \mathbb{Q}_\ell)$$

is faithful exact and the composition $\rho_\ell \circ {}^{\mathrm{p}}\mathrm{H}^0$ with the perverse ${}^{\mathrm{p}}\mathrm{H}^0$ on $\mathcal{DM}_c^{\text{ét}}(X)$ is isomorphic to the perverse H^0 of the ℓ-adic realisation. By the universal property of perverse Nori motives [IM22, Proposition 6.10] this provides a faithful exact functor $F_X \colon \mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{M}(X)$ such the composition $\rho_\ell \circ F_X$ is isomorphic to the ℓ-adic realisation R_ℓ of perverse Nori motives and the composition $F_X \circ \mathrm{H}_{\mathrm{univ}}$ with the universal functor $\mathrm{H}_{\mathrm{univ}} \colon \mathcal{DM}_c^{\text{ét}}(X) \to \mathcal{M}_{\mathrm{perv}}(X)$ is the perverse ${}^{\mathrm{p}}\mathrm{H}^0$ on $\mathcal{DM}_c^{\text{ét}}(X)$: we have a commutative (up to natural isomorphism) diagram

$$\begin{array}{ccccc} \mathcal{DM}_c^{\text{ét}}(X) & \xrightarrow{\mathrm{H}_{\mathrm{univ}}} & \mathcal{M}_{\mathrm{perv}}(X) & & \\ & \searrow^{{}^{\mathrm{p}}\mathrm{H}^0} & \downarrow{\scriptstyle F_X} & \searrow^{\mathcal{R}_\ell} & \\ & & \mathcal{M}(X) & \xrightarrow[\rho_\ell]{} & \mathrm{Perv}(X, \mathbb{Q}_\ell) \end{array} .$$

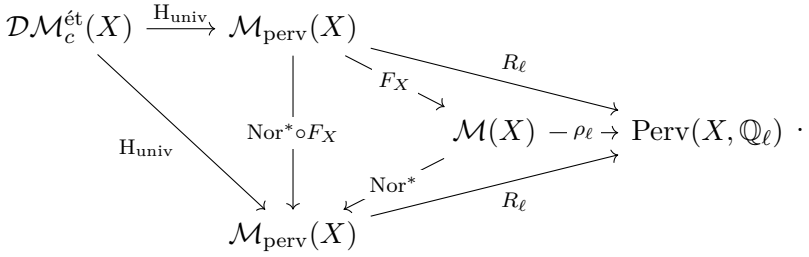
Now consider the endofunctor

$$\mathcal{M}_{\mathrm{perv}}(X) \xrightarrow{F_X} \mathcal{M}(X) \xrightarrow{\mathrm{Nor}^*} \mathcal{M}_{\mathrm{perv}}(X)$$

of $\mathcal{M}_{\mathrm{perv}}(X)$. Because Nor^* is perverse t-exact (this can be checked after ℓ-adic realisation), if we compose F_X with the universal $\mathrm{H}_{\mathrm{univ}}$ we obtain

$$\mathrm{Nor}^* \circ F_X \circ \mathrm{H}_{\mathrm{univ}} \simeq \mathrm{Nor}^* \circ {}^{\mathrm{p}}\mathrm{H}^0 \simeq {}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^*$$

as functors $\mathcal{DM}_c^{\text{ét}}(X) \to \mathcal{M}_{\mathrm{perv}}(X)$. But by Corollary 3.2.10 the functor ${}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^*$ can canonically be identified with $\mathrm{H}_{\mathrm{univ}}$. Thus we have a commutative diagram



By the universal property of $\mathcal{M}_{\mathrm{perv}}(X)$, the only vertical faithful exact functor fitting in the above diagram and making the two outer triangles commute is (up to natural isomorphism) the identity functor: this implies that there exists a natural equivalence $\mathrm{Nor}^* \circ F_X \simeq \mathrm{Id}_{\mathcal{M}_{\mathrm{perv}}(X)}$.

Now by the universal property of the bounded derived category we have a functor $\mathcal{D}^b(\mathcal{M}(X)) \to \mathcal{DM}_c^{\text{ét}}(X)$, thus by composition with the derived functor of F_X there is a functor

$$\gamma_X \colon \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{DM}_c^{\text{ét}}(X).$$

By definition, the restriction to $\mathcal{M}_{\mathrm{perv}}(X)$ of γ_X is exactly the functor F_X composed with the inclusion $\mathcal{M}(X) \to \mathcal{DM}_c^{\text{ét}}(X)$. In particular, the composition $\mathrm{Nor}_X^* \circ \gamma_X$ induces the functor $\mathrm{Nor}_X^* \circ F_X$ between the hearts. By the universal property of $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ and the above paragraph, this implies that there exists a natural equivalence $\mathrm{Nor}_X^* \circ \gamma_X \simeq \mathrm{Id}_{\mathcal{D}(\mathcal{M}_{\mathrm{perv}}(X))}$.

Using [IM22, Section 2.2] and the t-exactness of the ℓ-adic realisation of étale motives we see that there exist natural isomorphisms $i_* \circ \gamma \simeq \gamma \circ i_*$ for i a closed immersion, $f^*[d] \circ \gamma \simeq \gamma \circ f^*[d]$ for f a smooth morphism of relative dimension d and $\gamma(-) \boxtimes \gamma(-) \simeq \gamma(- \boxtimes -)$ with

$\boxtimes$ the external tensor product. Indeed for the pullback by a smooth map $f\colon Y \to X$ we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{DM}_c^{\text{ét}}(X) & \xrightarrow{\mathrm{H}_{\text{univ}}} & \mathcal{M}_{\text{perv}}(X) & \xrightarrow{F_X} & \mathcal{M}(X) \\ {\scriptstyle f^*[d]}\downarrow & & {\scriptstyle f^*[d]}\downarrow & & {\scriptstyle f^*[d]}\downarrow \\ \mathcal{DM}_c^{\text{ét}}(Y) & \xrightarrow{\mathrm{H}_{\text{univ}}} & \mathcal{M}_{\text{perv}}(Y) & \xrightarrow[F_Y]{} & \mathcal{M}(Y) \end{array}$$

obtained by the universal property of $\mathcal{M}_{\text{perv}}(X)$, which induces a commutative square

$$\begin{array}{ccc} \mathcal{M}_{\text{perv}}(X) & \longrightarrow & \mathcal{DM}_c^{\text{ét}}(X) \\ {\scriptstyle f^*[d]}\downarrow & & \downarrow{\scriptstyle f^*[d]} \\ \mathcal{M}_{\text{perv}}(Y) & \longrightarrow & \mathcal{DM}_c^{\text{ét}}(Y) \end{array}$$

that gives the wanted natural isomorphism by applying the universal property of the bounded derived category. The cases of i_* and $\boxtimes$ are similar. By adjunction there is a natural transformation $i^* \circ \gamma \Rightarrow \gamma \circ i^*$, which is an isomorphism because Nor^* is conservative, commutes with i^* and is a retraction of γ. Using the fact that any morphism of quasi-projective varieties can be written as a composition of a closed immersion and a smooth map, together with the formula for the tensor product using the external tensor product and pullbacks, we see that $\mathrm{ho}(\gamma)$ commutes with pullbacks and tensor products, giving the last claim of the lemma. □

By the lemma above we know that we have a natural transformation

$$\gamma\colon \mathrm{D}^b(\mathcal{M}_{\text{perv}}(-)) \to \mathrm{ho}(\mathcal{DM}_c^{\text{ét}}(-))$$

of contravariant functors on quasi-projective with values in symmetric monoidal 1-categories. We need an ∞-categorical enhancement of this functor. This is the next lemma. Note that as for perverse Nori motives, the existence of a perverse t-structure on étale motives $\mathcal{DM}_c^{\text{ét}}(X)$ implies that there is also a constructible t-structure, whose heart we denote by $\mathcal{CM}(X)$. The functors γ_X and Nor_X^* are t-exact for the constructible t-structure, as this can be checked after ℓ-adic realisation.

Lemma 2.1.47. *Let X be a quasi-projective scheme. The functor $\mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \to \mathcal{DM}_c^{\text{ét}}(X)$ induces a symmetric monoidal functor $\mathcal{M}_{\text{ct}}(X) \to \mathcal{CM}(X)$. This gives a natural transformation*

$$\delta\colon \mathcal{D}^b(\mathcal{M}_{\text{ct}}(-)) \Rightarrow \mathcal{DM}_c^{\text{ét}}(-)$$

in $\mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat}_\infty))$, where we use the $(-)^*$-functoriality. Moreover δ is a morphism of coefficient systems and satisfies $\mathrm{Nor}^* \circ \delta \simeq \mathrm{Id}_{\mathcal{D}^b(\mathcal{M}_{\text{ct}}(-))}$. In fact, we have $\mathrm{ho}(\delta) \simeq \gamma$.

Proof. Because $\mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \xrightarrow{\gamma_X} \mathcal{DM}_c^{\text{ét}}(X)$ and $\mathcal{DM}_c^{\text{ét}}(X) \xrightarrow{\mathrm{Nor}^*} \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))$ are t-exact for the constructible t-structure, we have faithful exact functors $\mathcal{M}_{\text{ct}}(X) \to \mathcal{CM}(X)$ and $\mathcal{CM}(X) \to \mathcal{M}_{\text{ct}}(X)$ whose composition is the identity of $\mathcal{M}_{\text{ct}}(X)$ by Lemma 2.1.46.

Now by Proposition A.6.3 there is a natural transformation

$$\mathcal{D}^b(\mathcal{CM}(-)) \to \mathcal{DM}_c^{\text{ét}}$$

of functors $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$. By the last part of Lemma 2.1.46 we have a functor $\Delta^1 \times \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{SymMono}_1$ sending, for $f\colon Y \to X$ a map of quasi-projective varieties, the map $(0<1, f)$ to $\mathcal{M}_{\mathrm{ct}}(Y) \xrightarrow{f^*} \mathcal{M}_{\mathrm{ct}}(X) \xrightarrow{\gamma_X} \mathcal{CM}(X)$, which is also isomorphic to $\mathcal{M}_{\mathrm{ct}}(Y) \xrightarrow{\gamma_Y} \mathcal{CM}(Y) \xrightarrow{f^*} \mathcal{CM}(X)$. Thus by Proposition A.6.3 there is a natural transformation

$$\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(-)) \to \mathcal{D}^b(\mathcal{CM}(-))$$

of functors $\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$. Composing the two we obtain a natural transformation

$$\delta\colon \mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(-)) \Rightarrow \mathcal{DM}_c^{\text{ét}}$$

of functors with values in symmetric monoidal stable ∞-categories. The restriction to $\mathcal{M}_{\mathrm{ct}}(X)$ of δ_X is, by definition, the restriction to $\mathcal{M}_{\mathrm{ct}}(X)$ of γ_X. Thus $\delta_X \simeq \gamma_X$ by Theorem 1.1.21.

To prove that δ is a morphism of coefficient systems, it suffices to prove that if $f\colon Y \to X$ is a smooth morphism of varieties, the natural transformation $f_\sharp \circ \delta_Y \to \delta_X \circ f_\sharp$ is an equivalence. For this, we may apply the conservative functor

$$\rho_\ell\colon \mathcal{DM}_c^{\text{ét}}(X) \to \mathrm{D}_c^b(X_{\text{ét}}, \mathbb{Q}_\ell)$$

so that the exchange morphism above becomes

$$f_\sharp \circ \rho_\ell \circ \delta_Y \to \rho_\ell \circ \delta_X \circ f_\sharp.$$

Now remark that for S a variety, the composition $\mathrm{Nor}_S^* \circ \delta_S$ is the identity functor and the functor Nor^* commutes with the ℓ-adic realisation thus we have an isomorphism

$$\rho_\ell \circ \delta_S \simeq R_\ell \circ \mathrm{Nor}^* \circ \delta_S \simeq R_\ell.$$

As the functor R_ℓ commutes with $f_\sharp$, the proof is finished. □

Proof. (of Theorem 2.1.45) Because both $\mathcal{DM}_c^{\text{ét}}$ and $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$ are Zariski hypersheaves and Nor^* is a morphism of sheaves, we may assume that X is quasi-projective.

Let δ_X be the functor of Lemma 2.1.47. We know that $\mathrm{Nor}_X^* \circ \delta_X \simeq \mathrm{Id}_{\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))}$. We can now show that $\delta_X \circ \mathrm{Nor}_X^* \simeq \mathrm{Id}_{\mathcal{DM}_c^{\text{ét}}(X)}$. By Lemma 2.1.47 and Theorem 2.1.38 the functor $\delta_X \circ \mathrm{Nor}_X^*$ is a morphism of coefficient systems. In particular, its composition with the canonical symmetric monoidal functor

$$M_X\colon \mathrm{Sm}_X \to \mathcal{DM}_c^{\text{ét}}(X)$$

is isomorphic to the canonical symmetric monoidal functor M_X. Using the universal property of $\mathcal{DM}_c^{\text{ét}}(X)$ proved by Robalo in [Rob15, Corollary 2.29] (or rather, its étale and $\mathbb{Q}$-linear counterpart which can be deduced from it) this implies:

$$\delta_X \circ \mathrm{Nor}_X^* \simeq \mathrm{Id}_{\mathcal{DM}_c^{\text{ét}}(X)}.$$

This finishes the proof as now we have that Nor_X^* and δ_X are inverses of each other.

□

2.2 Mixed realisations.

We finish this chapter on realisation functors by using the ∞-categorical enhancement of mixed Hodge modules that we constructed to define a derived category of mixed realisations and the Hodge realisation functor to construct a realisation functor from étale motives to mixed realisations. As a consequence of the existence of a Hodge realisation of étale motives, we saw that we could obtain a Hodge realisation of perverse Nori motives, hence a mixed realisation functor. We will use this in Chapter 5

2.2.1 Mixed realisations over a field.

Let k be a subfield of the complex numbers. In this manuscript we choose the most naive definition of a category of mixed realisations. Moreover, we only take one embedding of k in $\mathbb{C}$ at a time.

Definition 2.2.1. Let $\sigma\colon k \to \mathbb{C}$ be an embedding. We choose an algebraic closure of k in $\mathbb{C}$. An object of the category of mixed realisations $\mathcal{MR}^{\sigma}(k)$ is described as follows:

* A finite dimension k-vector space V_{dR} with a decreasing filtration $F^{\bullet}$ on $V_{\mathrm{dR}} \otimes_{k,\sigma} \mathbb{C}$.
* For each prime number ℓ a $\mathbb{Q}_\ell$-vector space V_ℓ together with a continuous action ρ of $\mathrm{Gal}(\overline{k}/k)$, such that (V_ℓ, ρ) is a constructible ℓ-adic sheaf on $k_{\text{ét}}$.
* A finite dimensional $\mathbb{Q}$-vector space V_B with an increasing filtration $W_{\bullet}$.
* An isomorphism $M_B \otimes_{\mathbb{Q}} \mathbb{C} \simeq M_{\mathrm{dR}} \otimes_{k,\sigma} \mathbb{C}$ such that $V_H := (V_B, V_{\mathrm{dR}} \otimes_{k,\sigma} \mathbb{C}, W_{\bullet}, F^{\bullet})$ is a greded polarisable mixed Hodge structure.
* For each prime number ℓ, an isomorphism $V_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq V_\ell$.

Morphisms are morphism at each level that are compatible with each data.

Note that we do not ask for the weight filtration to exist on the ℓ-adic representation. As our subfield k of $\mathbb{C}$ is arbitrary this would be unreasonable. Also, we do not ask for the Hodge filtration on the de Rham part to exist with coefficients k but only with complex coefficients. This is because we want only abelian categories to appear in the next remark. For other similar definitions, see [Hub95, Del89, Wil97].

Remark 2.2.2. With this definition, it is clear that $\mathcal{MR}^{\sigma}(k)$ is a limit of abelian categories, for example it fits in the equaliser diagram

$$\begin{array}{c} \mathcal{MR}^{\sigma}(k) \\ \downarrow \\ \mathrm{Vect}_k \times \mathrm{MHS}_{\mathbb{Q}} \times \prod_\ell \mathrm{Constr}(k_{\text{ét}}, \mathbb{Q}_\ell) \\ \overset{u}{\downarrow} \quad \overset{v}{\downarrow} \\ \mathrm{Vect}_{\mathbb{C}} \times \prod_\ell \mathrm{Vect}_{\mathbb{Q}_\ell} \end{array}$$

where the two parallel arrows u and v send a tuple $(V_{\mathrm{dR}}, H, \mathcal{F}_\ell, \dots)$ to $(V_{\mathrm{dR}} \otimes_{k,\sigma} \mathbb{C}, \mathrm{rat}(H) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, \dots)$ and to $(H_{\mathrm{dR}}, \mathcal{F}_\ell(\overline{k}), \dots)$ respectively, where for a mixed Hodge structure H we denoted

by H_{dR} the underlying $\mathbb{C}$-vector space, and by $\mathrm{rat}(H)$ the underlying $\mathbb{Q}$-vector space. The limit is taken in the (2,1)-category of categories.

Remark 2.2.3. Let L/k be a field extension, and let Σ be an extension of the embedding σ to L. Denote by $f\colon \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ the associated morphism of schemes. Then tensoring by L on the de Rham part of the mixed Hodge structure and doing the pullback f^* on the ℓ-adic parts provides us with a functor

$$f^*\colon \mathrm{MR}^{\sigma}(k) \to \mathrm{MR}^{\Sigma}(L).$$

Moreover, if L/k is a finite extension the functor f^* has a right adjoint f_* given by the pushforward on ℓ-adic representations, and as the forgetful functor on the de Rham side.

Using Remark 2.2.2, knowing that all factors in this limit diagram are abelian categories, heart of a t-structure on a stable ∞-category, we can easily give a definition of a derived category of mixed realisations:

Definition 2.2.4. The ∞-category of mixed realisations is defined as the limit of equaliser diagram

$$\begin{array}{c} \mathrm{D}^b_{\mathcal{MR}^{\sigma}}(k) \\ \downarrow \\ \mathrm{D}^b(k) \times \mathrm{D}^b(\mathrm{MHS}_{\mathbb{Q}}) \times \prod_{\ell} \mathrm{D}^b_c(k_{\text{ét}}, \mathbb{Q}_{\ell}) \\ \overset{u}{\downarrow} \quad \overset{v}{\downarrow} \\ \mathrm{D}^b(\mathbb{C}) \times \prod_{\ell} \mathrm{D}^b_c(\overline{k}_{\text{ét}}, \mathbb{Q}_{\ell}) \end{array}$$

where the two parallel arrows u and v send a tuple $(V_{\mathrm{dR}}, H, (V_{\ell}, \rho), \dots)$ to $(V_{\mathrm{dR}} \otimes_{k,\sigma} \mathbb{C}, \mathrm{rat}(H) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}, \dots)$ and to $(H_{\mathrm{dR}}, V_{\ell}, \dots)$ respectively, where for a complex of mixed Hodge structures H we denoted by H_{dR} the underlying complex $\mathbb{C}$-vector spaces, and by $\mathrm{rat}(H)$ the underlying complex $\mathbb{Q}$-vector spaces. The limit is taken in the ∞-category of ∞-categories.

Proposition 2.2.5. *The ∞-category $\mathrm{D}^b_{\mathcal{MR}^{\sigma}}(k)$ is stable and is endowed with a bounded t-structure of heart $\mathcal{MR}^{\sigma}(k)$. The projection functors to $\mathrm{D}^b(k)$, $\mathrm{D}^b(\mathrm{MHS}_{\mathbb{Q}})$ and $\mathrm{D}^b_c(k_{\text{ét}}, \mathbb{Q}_{\ell})$ are exact t-exact functors. In particular, its homotopy is a triangulated category.*

Proof. All functors in the definition are exact t-exact functors, hence the limit is a stable ∞-category endowed with a t-structure and the heart is the limit of the hearts by Proposition 1.1.9. □

Remark 2.2.6. Although we will not try to do a complete comparison, the homotopy category of $\mathrm{D}^b_{\mathcal{MR}^{\sigma}}(k)$ is very similar to the category defined by Huber in [Hub95]. In fact, if one would do the same definition as Huber but without the filtrations of the de Rham and ℓ-adic parts, one would obtain a category equivalent to $\mathrm{ho}(\mathrm{D}^b_{\mathcal{MR}^{\sigma}}(k))$. Indeed one should be able to construct a functor from Huber's modified category to $\mathrm{ho}(\mathrm{D}^b_{\mathcal{MR}^{\sigma}}(k))$, and this functor would then be fully faithful by the computations of the RHom in [Hub95, 11.1.7] and the following proposition.

Proposition 2.2.7. *Let $M, N \in \mathrm{D}^b_{\mathcal{MR}^\sigma}(k)$. Then the mapping spectra can be computed as the limit in $\mathrm{D}(\mathbb{Q})$ of the equaliser diagram*

$$\begin{array}{c} \mathrm{map}_{\mathrm{D}^b_{\mathcal{MR}^\sigma}(k)}(M,N) \\ \downarrow \\ \mathrm{map}_{\mathrm{D}^b(k)}(M_{\mathrm{dR}},N_{\mathrm{dR}}) \times \mathrm{map}_{\mathrm{D}^b(\mathrm{MHS}_{\mathbb{Q}})}(M_H,N_H) \times \prod_\ell \mathrm{map}_{D^b_c(k_{\text{ét}},\mathbb{Q}_\ell)}(M_\ell,N_\ell) \\ \downarrow\downarrow \\ \mathrm{map}_{\mathrm{D}^b(\mathbb{C})}(M_{\mathrm{dR}}\otimes_k \mathbb{C}, N_{\mathrm{dR}}\otimes_k \mathbb{C}) \times \prod_\ell \mathrm{map}_{D^b(\mathbb{Q}_\ell)}(M_\ell(\overline{k}),N_\ell(\overline{k})) \end{array}$$

In other words, by abbreviating the above diagram as

$$K_{\mathcal{MR}} \longrightarrow K_{\mathrm{dR}} \times K_H \times \prod_\ell K_\ell \underset{v}{\overset{u}{\rightrightarrows}} K_{\mathbb{C}} \times \prod_\ell \overline{K}_\ell$$

we have an exact triangle

$$K_{\mathcal{MR}} \to K_{\mathrm{dR}} \oplus K_H \oplus \prod_\ell K_\ell \xrightarrow{u-v} K_{\mathbb{C}} \oplus \prod_\ell \overline{K}_\ell$$

in the ∞-category $\mathrm{D}(\mathbb{Q})$.

Proof. This follows from the computation of mapping spectra in a limit of stable ∞-categories. □

2.2.2 Mixed realisations over a base.

In this section, we introduce the derived categories of mixed realisations over a basis. This generalises categories of Huber ([Hub95]) and Wildeshaus ([Wil97]). The only novelty is that mixed Hodge modules being ∞-categorical now, we can have a nice definition of a derived category which is the target of a realisation functor from $\mathrm{DM}^{\text{ét}}$.

Definition 2.2.8. Let X be a scheme of finite type over k, and let $\sigma\colon k \to \mathbb{C}$ be an embedding. The category of perverse mixed realisation is the category $\mathcal{MR}^\sigma_{\mathrm{perv}}(X)$ is the category whose objects are tuples consisting of

1. A mixed Hodge module P_H on $X_{\mathbb{C}}$.
2. A regular holonomic D-module P_{dR} on X.
3. A perverse sheaf P_B on X^{an}.
4. For each prime number ℓ, a $\mathbb{Q}_\ell$-adic perverse sheaf P_ℓ on $X_{\text{ét}}$.
5. An isomorphism between the underlying perverse sheaf of P_H and P_B.
6. An isomorphism between the underlying D-module of P_H and the base change to $X_{\mathbb{C}}$ of P_{dR}.
7. An isomorphism between the analytification of the base change to $\mathbb{C}$ of P_ℓ and $P_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, for all prime numbers ℓ.

We also define a weaker version $\mathcal{MR}^{\sigma}_{\mathrm{perv,wk}}(X)$ where points 2 and 6 are removed.

The above categories are naturally the heart of perverse t-structure on stable ∞-categories:

Definition 2.2.9. Let X be a smooth k-scheme. The stable ∞-category of mixed realisation $\mathrm{D}^{\sigma}_{\mathcal{MR}}(X)$ is defined as the following equaliser diagram:

$$\begin{array}{c} \mathrm{D}^{\sigma}_{\mathcal{MR}}(X) \\ \downarrow \\ \mathrm{D}^b(\mathrm{MHM}(X_{\mathbb{C}})) \times \mathrm{D}^b_c(X^{\mathrm{an}}, \mathbb{Q}) \times \text{D-Mod}_{\mathrm{reg,hol}}(X) \times \prod_\ell \mathrm{D}^b_c(X_{\text{ét}}, \mathbb{Q}_\ell) \\ \overset{u}{\downarrow} \quad \overset{v}{\downarrow} \\ D^b_c(X^{\mathrm{an}}, \mathbb{Q}) \times \text{D-Mod}_{\mathrm{reg,hol}}(X_{\mathbb{C}}) \times \prod_\ell \mathrm{D}^b_c(X^{\mathrm{an}}, \mathbb{Q}_\ell) \end{array}$$

with

$$u \colon (K_H, K_B, K_{\mathrm{dR}}, (K_\ell)_\ell) \mapsto (\mathrm{rat}(K_H), K_{\mathrm{dR}} \otimes_k \mathbb{C}, ((K_\ell)_{|X^{\mathrm{an}}})_\ell)$$

and

$$v \colon (K_H, K_B, K_{\mathrm{dR}}, (K_\ell)_\ell) \mapsto (K_B, \mathrm{f}_{\mathrm{dR}}(K_H), (P_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)_\ell)$$

where we have denoted by f_{dR} the underlying complex of D-modules of a mixed Hodge module. When X is not necessarily smooth, we still have a derived category of mixed realisations $\mathrm{D}^{\sigma}_{\mathcal{MR}_{\mathrm{wk}}}(X)$ by taking the same definition but removing the D-module part.

Remark 2.2.10.

1. It is certainly possible to define a category $\mathrm{D}^{\sigma}_{\mathcal{MR}}(X)$ for every finite type k-scheme but one has to check that the forgetful functor f_{dR} is well-defined as an ∞-functor for a non smooth basis, a verification that the author did not attempt to do.
2. We note that there is also a version with C_2-action when k is in fact a subfield of the reals, and also with integral coefficients. Indeed, one can replace $\mathrm{D}^b(\mathrm{MHM}(X_{\mathbb{C}}))$ with the enhancement provided in the last section of [RT24b]. We choose not to pursue this here.

Proposition 2.2.11. *Let X be a smooth variety over k. The category $\mathrm{D}^{\sigma}_{\mathcal{MR}}(X)$ has a canonical perverse t-structure, whose heart identifies with $\mathcal{MR}^{\sigma}_{\mathrm{perv}}(X)$. The same statement is true for $\mathrm{D}^{\sigma}_{\mathcal{MR}_{\mathrm{wk}}}$ and $\mathcal{MR}^{\sigma}_{\mathrm{perv,wk}}(X)$, for any variety X.*

Proof. This is because the functors u and v, composed with any of the projections to a factor of the product, are perverse t-exact, thus we are looking at a limit of a diagram of stable ∞-categories with t-structure and t-exact transitions and we can apply [RS20, Lemma 3.2.18]. □

Corollary 2.2.12. *All projection functors from $\mathrm{D}^{\sigma}_{\mathcal{MR}}(X) \to \mathrm{D}(X)$ with $\mathrm{D}(X)$ one of $\mathrm{D}^b(\mathrm{MHM}(X_{\mathbb{C}}))$, $\mathrm{D}^b_c(X^{\mathrm{an}}, \mathbb{Q})$, $\text{D-Mod}_{\mathrm{reg,hol}}(X)$ or $\mathrm{D}^b_c(X_{\text{ét}}, \mathbb{Q}_\ell)$ are conservative and t-exact.*

Proof. We just saw the t-exactness. For the conservativity, first note that the cohomological amplitude of any object $M \in \mathrm{D}^{\sigma}_{\mathcal{MR}}$ is contained in the cohomological amplitude of M_B its Betti part. Thus the t-structure is bounded. As on the heart the functors are faithful exact, the claim follows. □

Proposition 2.2.13. *The system of categories $X \mapsto \mathrm{D}^{\sigma}_{\mathcal{MR}_{\mathrm{wk}}}(X)$ affords the 6 operations and satisfy h-hyperdescent.*

Proof. This holds because it is true on each factor and the comparison functors are compatible with the 6 operations. Any descent property verified by all factors is verified on mixed realisations because the limit diagram of descent commute with the products and the equaliser in the definition of the category. □

Remark 2.2.14. This has to be compared with [Hub95, Lemma 11.3.5].

Theorem 2.2.15. *There exists a mixed realisation of motives*

$$\rho^{\sigma}_{\mathcal{MR}} \colon \mathrm{DM}_{\mathrm{gm}}(X, \mathbb{Q}) \to \mathrm{D}^{\sigma}_{\mathcal{MR}_{\mathrm{wk}}}(X)$$

that commutes with the 6 operations and factors through the derived category $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ of perverse Nori motives.

Proof. Indeed, we can define a realisation from $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ by putting together all realisations of perverse Nori motives, and then we can compose with the Nori realisation functor

$$\rho_{\mathrm{N}} \colon \mathrm{DM}_{\mathrm{gm}}(X, \mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

constructed in Theorem 2.1.38. □

Remark 2.2.16. When X is smooth, the above realisation factors through the better version $\mathrm{D}^{\sigma}_{\mathcal{MR}}(X)$ of mixed realisations with rational D-module part. Indeed, the there is a forgetful functor

$$\mathrm{MHM}(X_{\mathbb{C}}) \to \text{D-mod}_{\mathrm{reg,hom}}(X_{\mathbb{C}})$$

that when derived provide the wanted realisation functor.

Definition 2.2.17. Let X be a smooth connected scheme. The category $\mathcal{MR}^{\sigma}_{\mathrm{liss}}(X)$ is the Tannakian category of perverse mixed realisations over X that have underlying sheaves local systems. To avoid unnecessary shifts, we see this category as lying in cohomological degree $-d = -\dim(X)$.

Proposition 2.2.18. *Let U be a smooth connected scheme over k and let η be the generic point of U. Denote by L the field of sections of η. We choose an extension $\Sigma \colon L \to \mathbb{C}$ of our embedding σ of k in $\mathbb{C}$. Then there exists a faithful exact tensor functor*

$$\psi^{\Sigma} \colon \mathcal{MR}^{\sigma}_{\mathrm{liss}}(U) \to \mathcal{MR}(L)$$

that fits in the commutating diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{liss}}(U) & \xrightarrow{\eta^*} & \mathcal{M}(L) \\ \downarrow{\scriptstyle R^{\sigma}_{\mathcal{MR}}} & & \downarrow{\scriptstyle R^{\Sigma}_{\mathcal{MR}}} \\ \mathcal{MR}^{\sigma}_{\mathrm{liss}}(U) & \xrightarrow[\psi^{\Sigma}]{} & \mathcal{MR}^{\Sigma}(L) \end{array}.$$

Proof. We proceed as in Terenzi's [Ter24, Proposition 1.15]. The L-point η of U induces a morphism $\mathscr{O}_U \to L \xrightarrow{\Sigma} \mathbb{C}$ that gives a point $x^{\mathrm{an}}\colon \mathrm{Spec}(\mathbb{C}) \to U$ and thus a point $x\colon \mathrm{Spec}(\mathbb{C})^{\mathrm{an}} \to U^{\mathrm{an}}$ that avoids every analytification of closed sub algebraic varieties of U. This provides functors

$$x^{\mathrm{an}*}\colon \mathrm{LS}(U^{\mathrm{an}}) \to \mathrm{Vect}_{\mathbb{Q}}$$

and

$$x^{\mathrm{an}*}_{\mathrm{MHM}}\colon \mathrm{VMHS}(U^{\mathrm{an}}) \to \mathrm{MHS}$$

that are compatible, meaning that if rat is the functor giving the underlying local system of a variation of mixed Hodge structures, we have

$$\mathrm{rat}_{\mathrm{pt}} \circ x^{\mathrm{an}*}_{\mathrm{MHM}} \simeq x^{\mathrm{an}*} \circ \mathrm{rat}_U.$$

Now, Saito proved in [Sai90b, Remark after 3.27] that lisse objects of $\mathrm{MHM}(U)$ are exactly algebraic admissible variations of mixed Hodge modules. Analytification thus provides a functor

$$\mathrm{MHM}_{\mathrm{liss}}(U) \to \mathrm{VMHS}(U^{\mathrm{an}}),$$

compatible with the analytification functor $\mathrm{LS}(U) \simeq \mathrm{LS}(U^{\mathrm{an}})$. In short, we have a commutative square

$$\begin{array}{ccc} \mathrm{MHM}_{\mathrm{liss}}(U) & \longrightarrow & \mathrm{MHS} \\ {\scriptstyle \mathrm{rat}}\downarrow & & \downarrow \\ \mathrm{LS}(U) & \xrightarrow[x^{\mathrm{an}*}]{} & \mathrm{Vect}_{\mathbb{Q}} \end{array}.$$

Now on the ℓ-adic side, the functor η^* is well-defined, and we have a commutative diagram

$$\begin{array}{ccccccc} \mathrm{LS}(U_{\text{ét}}, \mathbb{Q}_\ell) & \longrightarrow & \mathrm{LS}((U_{\overline{k}})_{\text{ét}}, \mathbb{Q}_\ell) & \longrightarrow & \mathrm{LS}((U_{\mathbb{C}})_{\text{ét}}, \mathbb{Q}_\ell) & \xrightarrow{\sim} & \mathrm{LS}(U^{\mathrm{an}}, \mathbb{Q}_\ell) \\ {\scriptstyle \eta^*}\downarrow & \searrow^{x^*} & & & \downarrow{\scriptstyle (x, \mathrm{Id}_{\mathrm{Spec}(\mathbb{C})})^*} & & \downarrow{\scriptstyle x^{\mathrm{an}*}} \\ \mathrm{LS}(\eta_{\text{ét}}, \mathbb{Q}_\ell) & & \xrightarrow[\Sigma^*]{} & & \mathrm{LS}(\mathrm{Spec}(\mathbb{C})_{\text{ét}}, \mathbb{Q}_\ell) & \xrightarrow{\sim} & \mathrm{Vect}_{\mathbb{Q}_\ell} \end{array}.$$

Now on the D-module side, we can do as in the Betti side: we first analytify and then pullback along the point.

Thus, the functor having as Hodge component $x^{\mathrm{an}*}_{\mathrm{MHM}}$, ℓ-adic component η^*, D-module component the pullback along the point x^{an}, and compatibilities given by the above two commutative diagrams, defines the wanted tensor functor

$$\Psi^{\Sigma}\colon \mathcal{MR}^{\sigma}_{\mathrm{liss}}(U) \to \mathcal{MR}^{\Sigma}(L)$$

compatible with η^* on motives. The faithfulness is a consequence of faithfulness on the ℓ-adic sides. □

Chapter 3

Modules in étale motives.

In this chapter, we prove some properties of modules in étale motives, and we prove that mixed Hodge modules of geometric origin and perverse Nori motives are examples of modules over an algebra in étale motives. The result for Nori motives will be used extensively for construct the integral version of Nori motives in the next chapter. In Section 3.1, after giving some general definitions, we prove in Section 3.1.2 that two notions of constructibility agree for a particular family of enriched motivic theories, and the latter section, aswell as the following Section 3.1.3 where we prove h-descent for geometric étale motives, is based on the joint work [RT24a] with Raphaël Ruimy.

In Section 3.2 and Section 3.3 we prove that the indization of the bounded derived categories of both perverse Nori motives and mixed Hodge modules of geometric origin are examples of modules in étale motives.

3.1 Enriched motivic theories.

3.1.1 Definitions and first properties.

The goal of this section is to introduce enriched motivic theories after Ayoub ([Ayo07a]) and Drew ([Dre18]). Let $\mathbf{M}$ be a presentably symmetric monoidal ∞-category (Definition A.3.7). Assume that there exists an $\mathbb{E}_\infty$-ring spectum R such that $\mathbf{M}$ is R-linear. In this case we have a canonical unit functor $\mathrm{Mod}_R \to \mathbf{M}$, where Mod_R is introduced in Notation A.4.7.

Definition 3.1.1 ([Dre18, Section 3])**.** Let X be a qcqs scheme. The ∞-category of $\mathbf{M}$-*enriched étale motivic sheaves* over X is the relative tensor product

$$\mathrm{DM}^{\text{ét}}(X, \mathbf{M}) := \mathrm{DM}^{\text{ét}}(X, R) \otimes_{\mathrm{Mod}_R} \mathbf{M}$$

in Pr^L.

There is another possible definition of $\mathbf{M}$-enriched motives, using construction as Tate-spectra of $\mathbb{A}^1$-invariant sheaves. It turns out that both definitions are equivalent, essentially because tensoring with $\mathbf{M}$ commutes with colimits:

Proposition 3.1.2. *Let X be qcqs scheme. Then canonical functor $\mathrm{Mod}_R \to \mathbf{M}$ provides a functor*

$$\mathrm{DM}^{\text{ét}}(X, R) \to \mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathbf{M})[(\mathbb{P}^1_X/\infty)^{\otimes -1}]$$

which induces an equivalence

$$\mathrm{DM}^{\text{ét}}(X, \mathbf{M}) \to \mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathbf{M})[(\mathbb{P}^1_X/\infty)^{\otimes -1}]$$

of symmetric monoidal $\mathbf{M}$*-linear* ∞*-categories.*

Proof. That the functor $\mathrm{DM}^{\text{ét}}(X, \mathbf{M}) \to \mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathbf{M})[(\mathbb{P}^1_X/\infty^{\otimes -1})]$ exists follows from the existence of the functor written above it, together with the fact that the right-hand side is $\mathbf{M}$-linear (using Theorem A.4.5). To prove that it is an equivalence, it suffices to prove that at each step of the construction of $\mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathbf{M})[(\mathbb{P}^1_X/\infty)^{\otimes -1}]$ we can pass the $- \otimes_R \mathbf{M}$ outside.

1. The canonical functor

   $$\mathrm{Fun}(\mathrm{Sm}_X^{\mathrm{op}}, \mathrm{Mod}_R) \otimes_R \mathbf{M} \to \mathrm{Fun}(\mathrm{Sm}_X^{\mathrm{op}}, \mathbf{M})$$

   is an equivalence by Corollary A.5.4.

2. The canonical functor

   $$\mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathrm{Mod}_R) \otimes_R \mathbf{M} \to \mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathbf{M})$$

   is an equivalence. This holds by [Vol23, Corollary 2.24] (his result is only for sheaves but the proof applies to $\mathbb{A}^1$-local objects aswell).

3. The canonical functor

   $$\mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathrm{Mod}_R)[(\mathbb{P}^1_X/\infty)^{\otimes -1}] \otimes_R \mathbf{M} \to \mathrm{Shv}_{\mathbb{A}^1,\mathrm{hyp}}((\mathrm{Sm}_X)_{\text{ét}}, \mathbf{M})[(\mathbb{P}^1_X/\infty)^{\otimes -1}]$$

   is an equivalence because inverting the object $\mathbb{P}^1_X/\infty$ for the tensor product is a colimit because this object is 3-symmetric (see [Voe98, Theorem 5.6]) thus we may apply Proposition A.5.11, and again tensoring by $\otimes_R \mathbf{M}$ commutes with colimits.

□

Remark 3.1.3. In particular we have that if $A \to R$ is a map of $\mathbb{E}_\infty$-ring spectra, then $\mathrm{Mod}_R(\mathrm{DM}^{\text{ét}}(X, A)) \simeq \mathrm{DM}^{\text{ét}}(X, R)$. Also, the above result has nothing to do with the choice of the site or the étale topology, and would work in any similar situation, such as h-sheaves on finite type X-schemes.

Remark 3.1.4. This also proves that when $\mathbf{M}$ underlies a combinatorial model category $\mathfrak{M}$, the homotopy category of $\mathrm{DM}^{\text{ét}}(X, \mathbf{M})$ is the category $\mathbf{SH}^{\text{ét}}_{\mathfrak{M}}(X)$ defined by Ayoub in [Ayo07a].

Note that for all schemes X there is a canonical R-linear functor

$$\rho^{\mathbf{M}}_X \colon \mathrm{DM}^{\text{ét}}(X, R) \to \mathrm{DM}^{\text{ét}}(X, \mathbf{M}).$$

The above construction works well in family, in fact, we have the following theorem originally due to Ayoub in the triangulated form, and to Drew in the ∞-categorical form:

Theorem 3.1.5 (Drew). *The functors $\rho_X^{\mathbf{M}}$ assemble to give a natural transformation*

$$\rho^{\mathbf{M}} \colon \mathrm{DM}^{\text{ét}}(-, R) \to \mathrm{DM}^{\text{ét}}(-, \mathbf{M})$$

of functors from the opposite category of all qcqs schemes to $\mathrm{CAlg}(\mathrm{Pr}^L)$. *Moreover, for any scheme S, the restriction of $\rho^{\mathbf{M}}$ to finite type S-schemes is a morphism of coefficient systems as in Definition 1.2.2.*

Proof. This is [Dre18, Theorem 8.10]. □

Following Drew, we want to apply this construction to $\mathbf{M} = \mathrm{D}(\mathrm{IndMHS}^p_{\mathbb{Q}})$ the derived category of ind mixed Hodge structures. Indeed, this enables one to have more "basic objects": while the ∞-category $\mathrm{DM}^{\text{ét}}(X, R)$ is generated under colimits by objects of the form $f_\sharp R_Y(n)$ for $f \colon Y \to X$ smooth and $n \in \mathbb{Z}$, the ∞-category $\mathrm{DM}^{\text{ét}}(X, \mathbf{M})$ is generated under colimits by objects of the form $f_\sharp M(n)$ where $M \in \mathbf{M}$ is seen in $\mathrm{DM}^{\text{ét}}(Y, \mathbf{M})$ as $\mathbf{1}_Y \otimes_R M$. This allows to introduce non geometric objects in the building blocks of our motivic categories. For example in $\mathrm{DM}^{\text{ét}}(X, \mathrm{D}(\mathrm{IndMHS}^p_{\mathbb{Q}}))$ every mixed Hodge structure, not necessarily of geometric origin, provides a natural object.

Fix a base scheme S. We remark that the above construction is a particular case of an even more general construction. Indeed, for any S-scheme X we have, by simplification of tensor products, a natural isomorphism

$$\mathrm{DM}^{\text{ét}}(X, \mathbf{M}) \simeq \mathrm{DM}^{\text{ét}}(X, R) \otimes_{\mathrm{DM}^{\text{ét}}(S,R)} \mathrm{DM}^{\text{ét}}(S, \mathbf{M}).$$

Here we see $\mathrm{DM}^{\text{ét}}(X, R)$ as a $\mathrm{DM}^{\text{ét}}(S, R)$-algebra through the pullback functor under the projection map $X \to S$. Thus, we may make the following definition:

Definition 3.1.6. Let $\mathbf{D}$ be a presentably symmetric monoidal ∞-category which has a colimit preserving symmetric monoidal functor $\mathrm{DM}^{\text{ét}}(S, R) \to \mathbf{D}$ and let X be a S-scheme. The ∞-category of $\mathbf{D}$-enriched étale motives is the category

$$\mathrm{DM}^{\text{ét}}(X, \mathbf{D}) := \mathrm{DM}^{\text{ét}}(X, R) \otimes_{\mathrm{DM}^{\text{ét}}(S,R)} \mathbf{D}.$$

We are particularly interested in the case where $\mathbf{D}$ is of the form

$$\mathbf{D} \simeq \mathrm{Mod}_A(\mathrm{DM}^{\text{ét}}(S, \mathbf{M}))$$

for some ∞-category $\mathbf{M}$ as above and $A \in \mathrm{CAlg}(\mathrm{DM}^{\text{ét}}(S, \mathbf{M}))$ an algebra object. In this case, the formula simplifies. Indeed, if $f \colon X \to S$ is the structural morphism, we have by Proposition A.5.6 an equivalence

$$\mathrm{DM}^{\text{ét}}(X, \mathrm{Mod}_A(\mathrm{DM}^{\text{ét}}(S, \mathbf{M}))) \simeq \mathrm{Mod}_{f^*A}(\mathrm{DM}^{\text{ét}}(X, \mathbf{M})).$$

In the more general setting, Drew's theorem still applies:

Theorem 3.1.7 (Drew). *Let S be a base scheme, and let $\mathbf{D}$ be a $\mathrm{DM}^{\text{ét}}(S, R)$-algebra in Pr^L. The assignment $X \mapsto \mathrm{DM}^{\text{ét}}(X, \mathbf{D})$ is a coefficient system on finite type S-schemes. Moreover, the canonical map*

$$\rho_{\mathbf{D}} \colon \mathrm{DM}^{\text{ét}}(-, R) \to \mathrm{DM}^{\text{ét}}(-, \mathbf{D})$$

is a map of coefficient systems.

Remark 3.1.8. Although we will not provide a rigorous proof, we can explain an alternative proof of the above theorem. Let R be a $\mathbb{E}_\infty$-ring spectra. Then one can define an ∞-category $\mathrm{CoSys}_S^{\mathrm{Pr}(R),\text{ét}}$ of presentable, R-linear coefficient systems that have non-effective étale hyper descent. Then as in [DG22] one can prove that this ∞-category has an initial object, given by $\mathrm{DM}^{\text{ét}}(-,R)$. More generally for any $\mathbf{D}$ as in the theorem, one can ask for coefficient systems that have the structure of a $\mathbf{D}$-modules.They form an ∞-category $\mathrm{CoSys}_S^{\mathrm{Pr}(R),\text{ét}}(\mathbf{D})$. There is a natural functor

$$\mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{CAlg}_{\mathrm{DM}^{\text{ét}}(S,R)}(\mathrm{Pr}^L)) \to \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{CAlg}_{\mathbf{D}}(\mathrm{Pr}^L)) \tag{3.1.8.1}$$

given by tensoring by $\mathbf{D}$ over $\mathrm{DM}^{\text{ét}}(S,R)$, which has a conservative right adjoint. Using [HSS17, Section 4.4] one can prove that this functor is in fact underlying an $(\infty,2)$-functor

$$\mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{CAlg}_{\mathrm{DM}^{\text{ét}}(S,R)}(\mathbf{Pr}^L)) \to \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{CAlg}_{\mathbf{D}}(\mathbf{Pr}^L)).$$

The projection formula for smooth morphisms implies that for any smooth morphism f the functor $f_\sharp$ on any R-linear coefficient system is actually $\mathrm{DM}^{\text{ét}}(S,R)$-linear: it is internal to the $(\infty,2)$-category $\mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{CAlg}_{\mathrm{DM}^{\text{ét}}(S,R)}(\mathbf{Pr}^L))$. Moreover, all conditions in the definition of a coefficient system *that is presentable* can be rephrased using only tensor products, pullback, and $f_\sharp$ for f a smooth morphism (the only maybe less clear condition is the localisation sequence, but the fact that the square is cartesian is equivalent to the fact that the square of left adjoint being cocartesian, and this uses only i^* and $j_\sharp$.) In particular, all formulae defining a coefficient system are $\mathrm{DM}^{\text{ét}}(S,R)$-linear, hence are internal to the $(\infty,2)$-category of functors $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{DM}^{\text{ét}}(S,R)}(\mathrm{Pr}^L)$. This proves that $\mathrm{CoSys}_S^{\mathrm{Pr}(R),\text{ét}}(\mathbf{D})$ has a natural 2-categorical enhancement, and that the functor (3.1.8.1) induces a 2-functor between ∞-categories of coefficient systems, because the image of an internal adjunction by a 2-functor is again an adjunction.

Corollary 3.1.9. *Let S be a base scheme, and let $\mathbf{D}$ be a $\mathrm{DM}^{\text{ét}}(S,R)$-algebra in Pr^L. Then $\mathrm{DM}^{\text{ét}}(-,\mathbf{D})$ has the 6 operations.*

Remark 3.1.10. Although $\mathrm{DM}^{\text{ét}}(-,\mathbf{D})$ has the 6 operations, it is unclear in which generality one can claim that the morphism

$$\mathrm{DM}^{\text{ét}}(-,R) \to \mathrm{DM}^{\text{ét}}(-,\mathbf{D})$$

commutes with the operations. We know that it commutes with pullbacks, the left adjoint of the pullback functor under a smooth morphism, and tensor products.

Proposition 3.1.11. *Let S be a scheme, let R be a ring and let A be a commutative algebra in $\mathrm{DM}(S,R)$. We assume that A is a filtered colimit of dualisable objects and that for any integer n the map $\mathbb{Z}/n\mathbb{Z} \to A/nA$ is an equivalence or the zero map. Then the morphism*

$$\rho_A \colon \mathrm{DM}^{\text{ét}}(-,R) \to \mathrm{DM}^{\text{ét}}(-,A) := \mathrm{Mod}_A(\mathrm{DM}(-,R))$$

commutes with the 6 operations on qcqs finite dimensional S-schemes.

Proof. We know that it is a morphism of coefficient systems. It turns out that to prove that it commutes with the operations, it suffices to prove that it commutes with pushforward (see the proof of [CD19, Theorem 4.4.25]). Let X be a S-scheme and denote by $p\colon X \to S$ the structural morphism. Let $f\colon Y \to X$ be a morphism. Let $M \in \mathrm{DM}^{\text{ét}}(Y, R)$. We want to prove that the natural transformation

$$\rho_A \circ f_*(M) \to f_* \circ \rho_A(M)$$

is an equivalence. Because the right adjoint G of ρ_A is conservative and commutes with f_* (its left adjoint commute with f^*), using that the composite $G \circ \rho_A$ is the endofunctor $- \otimes_R A$ (Theorem A.4.5 (2)), it suffices even to prove that the map

$$f_*(M) \otimes_R A \to f_*(M \otimes_R A)$$

is an equivalence. This can be checked (Lemma A.6.16) after being tensored by $\mathbb{Z}/p$ for all prime numbers p and by $\mathbb{Q}$. The $- \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ statement is clear by our assumption on A, and the fact that all functors in the equation commute with finite colimits such as

$$- \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = \mathrm{cofib}(- \xrightarrow{\times p} -).$$

Thus it suffices to prove the rationalised version. Using Proposition 3.1.25 that we prove below, the map tensored by $\mathbb{Q}$ can be rewritten

$$f_*(M \otimes \mathbb{Q}) \otimes A \to f_*(M \otimes (A \otimes \mathbb{Q})).$$

Thus we may replace R by $R \otimes \mathbb{Q}$ and A by $A \otimes \mathbb{Q}$. Now using Lemma 3.1.21, we see that both $\mathrm{DM}^{\text{ét}}(-, R)$ is compactly generated and f^* preserve compact objects. Thus the functor f_* preserve colimits. Using that A is a colimit of dualisable objects, we reduce to proving that the map

$$f_*(M) \otimes P \to f_*(M \otimes f^*P)$$

with P a dualisable object. This follows from Lemma A.5.8.

□

A frequent natural apparition of enriched motivic theories is in the following setting:

Construction 3.1.12. Let $S = \mathrm{Spec}(k)$ be the spectrum of a field and let D be a six functors formalism on finite dimensional S-schemes that take values in presentable ∞-categories. Let

$$\rho\colon \mathrm{DM}^{\text{ét}}(-, R) \to \mathrm{D}$$

be a colimit preserving realisation functor that commutes with the 6 operations. Let $A = \rho_* \mathbf{1}_S$ be the image of the unit object of $\mathrm{D}(S)$ by the right adjoint functor ρ_* of ρ. It is a commutative algebra in $\mathrm{DM}^{\text{ét}}(S, R)$. Then there is a factorisation of ρ as

$$\mathrm{DM}^{\text{ét}}(-, R) \to \mathrm{DM}^{\text{ét}}(-, A) \to \mathrm{D}$$

where the second functor is given by

$$\mathrm{DM}^{\text{ét}}(-, A) \to \mathrm{Mod}_{\rho(A)}(\mathrm{D}) \to \mathrm{D}$$

using, for each $p\colon X \to S$, the map of algebra

$$\rho(f^* \rho_* \mathbf{1}_S) \to f^* \mathbf{1}_S \simeq \mathbf{1}_X.$$

Proposition 3.1.13. *Assume that R is a $\mathbb{Q}$-algebra and that pushforwards of* D *commute with small colimits. The natural morphism*

$$\overline{\rho}\colon \mathrm{DM}^{\text{ét}}(-,A) \to \mathrm{D}$$

commutes with the operations on finite dimensional S-schemes.

Proof. That it commutes with pullback functors and tensor products comes with the definition. Let $f\colon Y \to X$ be a smooth morphism. By Proposition A.6.15 and Lemma 3.1.21 below, we know that $\mathrm{DM}^{\text{ét}}(Y,A)$ and $\mathrm{DM}^{\text{ét}}(Y,R)$ are compactly generated. Moreover, the compact objects of $\mathrm{DM}^{\text{ét}}(Y,A)$ are of the form $M \otimes A$ with M a compact object of $\mathrm{DM}^{\text{ét}}(Y,R)$. In particular, we have that

$$\begin{aligned} f_\sharp(\overline{\rho}(\rho_A(M))) &\simeq f_\sharp\rho(M) \\ &\simeq \rho(f_\sharp(M)) \\ &\simeq \overline{\rho}(\rho_A(f_\sharp(M))) \end{aligned}$$

so that using that ρ_A commutes with $f_\sharp$, we conclude that $\overline{\rho}$ commutes with $f_\sharp$ as well. The same can be done for the pushforwards under a general morphism (using that A is a filtered colimit of dualisable objects, because over a field, every constructible motive is dualisable), finishing the proof.

□

Example 3.1.14. For example, Ayoub applied in [Ayo22] the above construction to the Betti realisation. As a consequence there exists an algebra $\mathscr{B}_X \in \mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ for every finite type k-scheme, where k is a subfield of $\mathbb{C}$. The Betti realisation factors through $\mathrm{Mod}_{\mathscr{B}}(\mathrm{DM}^{\text{ét}}(-,\mathbb{Z}))$, and the functor

$$\mathrm{Mod}_{\mathscr{B}}(\mathrm{DM}^{\text{ét}}(-,\mathbb{Z})) \to \mathrm{IndD}^b_c((-)^{\mathrm{an}},\mathbb{Z})$$

is fully faithful. The t-structures restrict to the full subcategory spanned by the image.

3.1.2 Geometric origin and constructibility.

In this section, we write the proof that geometric étale motives and constructible étale motives are the same, proven in [RT24a], in the slightly more general setting of enriched motives. This way the proof will apply directly to integral Nori motives in Chapter 4.

Definition 3.1.15. Fix S a base scheme. Let R be a $\mathbb{E}_\infty$-ring spectral and let $\mathbf{D}$ be a $\mathrm{DM}^{\text{ét}}(S,R)$-algebra in Pr^L. Let X be a S-scheme.

1. We say that a motive $M \in \mathrm{DM}^{\text{ét}}(X,\mathbf{D})$ is geometric if it lies in the thick full subcategory generated by the $f_\sharp\mathbf{1}_Y(n)$ for $f\colon Y \to X$ smooth and $n \in \mathbb{Z}$. Denote by $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbf{D})$ the full subcategory of geometric motives.

2. We say that a motive $M \in \mathrm{DM}^{\text{ét}}(X,\mathbf{D})$ is constructible if for any affine open subset U there exists a finite stratification (U_i) of U by locally closed subset such that each restriction $M_{|U_i}$ is a dualisable object. Denote by $\mathrm{DM}^{\text{ét}}_c(X,\mathbf{D})$ the full subcategory of constructible motives.

Remark 3.1.16. If X is qcqs, then if K is an object of $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X, \mathbf{D})$, we can find a finite stratification (X_i) of X such that each $K_{|X_i}$ is dualisable.

Remark 3.1.17. The functors of type f^*, $f_\sharp$ and $\otimes$ readily preserve geometric objects. The functor i_*, for i a closed immersion, preserves geometric objects by using the localisation triangle.

On the other hand, the functors of type f^* and $\otimes$ preserve constructible objects. The functors $j_\sharp$ and i_*, for j an open immersion and i a closed immersion, also preserve constructible objects.

Remark 3.1.18. If $\mathbf{D} \to \mathbf{E}$ is any symmetric monoidal colimit preserving functor to a presentably symmetric monoidal ∞-category $\mathbf{E}$, the induced functor

$$\rho_{\mathbf{D},\mathbf{E}} \colon \mathrm{DM}^{\text{ét}}(X, \mathbf{D}) \to \mathrm{DM}^{\text{ét}}(X, \mathbf{E})$$

preserves both geometric and constructible objects.

We restrict ourselves to a restrictive class of enriched motivic theories (but many of the result probably hold in greater generality).

Hypothesis 3.1.19. *For simplicity we assume that $S = \mathrm{Spec}(\mathbb{Z}[1/N])$ for some $N \geqslant 1$, $S = \mathrm{Spec}(\mathbb{Q})$ or $S = \mathrm{Spec}(\mathbb{C})$. Let A be an algebra object in $\mathrm{DM}^{\text{ét}}(S, \mathbb{Z})$ such that for all nonzero integers n the natural map $\mathbb{Z} \to A$, tensored by $\mathbb{Z}/n\mathbb{Z}$, is either an isomorphism or the zero map. If R is a ring we define $A_R = A \otimes_{\mathbb{Z}} R$. Finally, we let $\mathbf{D}_R := \mathrm{Mod}_{A_R}(\mathrm{DM}^{\text{ét}}(S, R))$.* ***From now on, we assume that the Hypothesis is satisfied.***

Remark 3.1.20. One could probably extend the hypothesis to A an algebra object in $\mathrm{DM}^{\text{ét}}(S, \mathbf{M})$ where $\mathbf{M}$ is a compactly generated symmetric monoidal stable ∞-category whose compact generators are dualisable, and such that for all $n \in \mathbb{Z}$, the canonical functor $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}(\mathbf{M})$ is either an equivalence or the zero functor. The object A also has to satisfy the same hypothesis as above. We do not pursue this here.

The following lemma is very useful:

Lemma 3.1.21. *Let X be a S-scheme which is finite dimensional and qcqs. Assume that Hypothesis 3.1.19 holds. Then the canonical functor*

$$\mathrm{Ind}(\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_{\mathbb{Q}})) \to \mathrm{DM}^{\text{ét}}(X, \mathbf{D}_{\mathbb{Q}})$$

is an equivalence.

Proof. Indeed, if $\mathbf{D} = \mathrm{DM}^{\text{ét}}(S, \mathbb{Z})$ this is [EHIK21b, Lemma 5.1 (i)]. Now thanks to our hypothesis, we know that there exists an algebra $A_{\mathbb{Q}}$ in $\mathrm{DM}^{\text{ét}}(S, \mathbb{Q})$ such that $\mathrm{DM}^{\text{ét}}(X, \mathbf{D}_{\mathbb{Q}}) = \mathrm{Mod}_{A_{\mathbb{Q}|X}}(\mathrm{DM}^{\text{ét}}(X, \mathbb{Q}))$. Using Proposition A.6.15 we can conclude. □

We first prove descent properties for geometric objects and for constructible objects. This will follow from a descent result for the big category:

Proposition 3.1.22. *Let R be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}(-, \mathbf{D}_R)$ is an étale hypersheaf on qcqs S-schemes.*

Proof. This follows from étale hyper descent of $\mathrm{DM}^{\text{ét}}(-,R)$, together with [AC23, Proposition 2.12] that asserts that if $\mathcal{C}$ is a symmetric monoidal presentable stable ∞-category, written as a limit $\mathcal{C} \simeq \lim_i \mathcal{C}_i$ of symmetric monoidal presentable stable ∞-categories, and if $A \in \mathcal{C}$ is an algebra object whose image in $\mathcal{C}_i$ we denote by A_i, then the canonical map

$$\mathrm{Mod}_A(\mathcal{C}) \to \lim \mathrm{Mod}_{A_i}(\mathcal{C}_i)$$

is an equivalence. □

Lemma 3.1.23. *Let R be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\mathbf{D}_R)$ is a Nisnevich hypersheaf over qcqs S-schemes.*

Proof. It suffices to show that $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\mathbf{D}_R)$ it is a subsheaf of $\mathrm{DM}^{\text{ét}}(-,\mathbf{D}_R)$ (a subsheaf of a hypersheaf is itself a hypersheaf for qcqs sites). Assume that (U,V) is a Nisnevich cover of X with U open and $V \to X$ étale and let M be a motive over X. Then if $M|_U$ and $M|_V$ are geometric, so is $M|_{U\times_X V}$, so that by Nisnevich excision, the motive M itself is geometric. □

Lemma 3.1.24. *Let R be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{\mathrm{c}}(-,\mathbf{D}_R)$ is an étale hypersheaf over qcqs S-schemes.*

Proof. It suffices to show that $\mathrm{DM}^{\text{ét}}_{\mathrm{c}}(-,\mathbf{D}_R)$ is a subsheaf of $\mathrm{DM}^{\text{ét}}(-,\mathbf{D}_R)$. The proof is very similar to the proof of [HRS23, Lemma 4.5], so we only give a sketch.

Take a qcqs scheme X and an étale-locally constructible motive M over X. We can assume X to be affine and to have a surjective étale map $f\colon Y \to X$ such that $f^*(M)$ is constructible. We then reduce to f being finite étale using [Aut18, Tag 03S0] and then to the case where f is a torsor under some finite group G: writing X as a limit of Noetherian schemes, the map f is the pullback of some étale map defined over a Noetherian scheme which we can then replace by a torsor under some finite group.

Finally, we can construct a G-equivariant stratification of Y from the stratification $\mathcal{Y}$ that witnesses the constructibility of $f^*(M)$. This yields a stratification of X whose pullback to Y is $\mathcal{Y}$. Hence, replacing X with one of its strata, we can assume that $f^*(M)$ is dualisable; as being dualisable is étale-local, the motive M is also dualisable which finishes the proof. □

We now prove the continuity property for geometric motives. We first need some compatibilities with rationalisation:

Proposition 3.1.25. *Let R be a commutative ring. Assume that Hypothesis 3.1.19 holds. Let $f\colon Y \to X$ be a morphism of qcqs finite-dimensional S-schemes, let M and N in $\mathrm{DM}^{\text{ét}}(X,\mathbf{D}_R)$ be motives over X with M constructible or geometric and let P be a motive over Y. Then, the natural transformations below are equivalences:*

1. $\mathrm{map}(M,N)\otimes\mathbb{Q} \to \mathrm{map}(M, N\otimes\mathbb{Q})$.
2. $\underline{\mathrm{Hom}}(M,N)\otimes\mathbb{Q} \to \underline{\mathrm{Hom}}(M, N\otimes\mathbb{Q})$.
3. $f_*(P)\otimes\mathbb{Q} \to f_*(P\otimes\mathbb{Q})$.
4. $i^!(N)\otimes\mathbb{Q} \to i^!(N\otimes\mathbb{Q})$ *(for i a closed immersion).*

Proof. We begin with the case where M is geometric. In this proof, we say that M is rationalisable if

$$\mathrm{map}(M, N) \otimes \mathbb{Q} \to \mathrm{map}(M, N \otimes \mathbb{Q})$$

is an equivalence. Rationalisable objects are closed under the functors of type $g_\sharp$. Therefore, to prove (1), we can assume that $M = \mathbf{1}_X$. To do so, we first deal with the case of $\mathbf{D}_R = \mathrm{DM}^{\text{ét}}(S, R)$ with R a commutative ring. Note first that if K is a complex of étale sheaves of abelian groups, the canonical map:

$$\mathrm{H}^i_{\text{ét}}(X, K) \otimes \mathbb{Q} \to \mathrm{H}^i_{\text{ét}}(X, K \otimes \mathbb{Q})$$

is an isomorphism for any integer i. Indeed, [CM21, Corollary 3.29] gives a bound to the étale cohomological dimension with rational coefficients over qcqs finite-dimensional schemes, and the above assertion then follows from [CD16, Lemma 1.1.10]. By the same argument as [CD16, Corollary 5.4.8] this gives (1) in this case. Now for a general $\mathbf{D}$, we have by adjunction the following commutative square:

$$\begin{array}{ccc} \mathrm{map}_{\mathrm{DM}(X,\mathbf{D}_R)}(\mathbf{1}_X, N) \otimes \mathbb{Q} & \longrightarrow & \mathrm{map}_{\mathrm{DM}(X,\mathbf{D}_R)}(\mathbf{1}_X, N \otimes \mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathrm{DM}(X,R)}(R_X, \rho_*^{\mathbf{D}} N) \otimes \mathbb{Q} & \longrightarrow & \mathrm{map}_{\mathrm{DM}(X,R)}(R_X, \rho_*^{\mathbf{D}}(N \otimes \mathbb{Q})) \end{array},$$

where we have denoted by $\rho_*^{\mathbf{D}}$ the right adjoint of the functor $\rho_{\mathbf{D}} \colon \mathrm{DM}^{\text{ét}}(X, R) \to \mathrm{DM}^{\text{ét}}(X, \mathbf{D}_R)$. Note that by our hypothesis the functor $\rho_*^{\mathbf{D}}$ is the forgetful functor from a module category, hence preserves colimits by Theorem A.4.5. In particular, the canonical map

$$\rho_*^{\mathbf{D}}(N) \otimes \mathbb{Q} \to \rho_*^{\mathbf{D}}(N \otimes \mathbb{Q})$$

is an equivalence, and thus the bottom horizontal map is an equivalence by the already dealt with case of $\mathbf{D} = \mathrm{DM}^{\text{ét}}(S, R)$.

Geometric objects generate $\mathrm{DM}^{\text{ét}}(X, \mathbf{D}_R)$ as a localising subcategory of itself. Hence to show that the maps of (2), (3) and (4) are equivalences, it suffices to show that they become equivalences after applying $\mathrm{map}(C, -)$ for C geometric. Since the functors of type f^*, i_* and $\otimes$ preserve geometric objects, this follows by adjunction from the equivalence of (1).

We now tackle the case where M is constructible. The third equivalence implies that the functors of type g^* preserve rationalisable objects. Hence, being rationalisable is Zariski-local: if we have a finite Zariski cover $\{U_i \to X\}$ and a motive M over X such that the $M|_{U_i}$ are rationalisable, so are the restrictions of M over the intersections of the U_i; as we can write M as a finite colimit with terms of the form $(V \to X)_\sharp(M|_V)$ with V an intersection of the U_i, the motive M is also rationalisable. Hence, we can assume X to be affine.

Note now that (4) implies that rationalisable objects are preserved by the functor i_* for i a closed immersion. By dévissage and localisation, we are therefore reduced to the case where M is dualisable. In that case, replacing N by $N \otimes M^\vee$ we can further reduce to the case where $M = \mathbf{1}_X$ which we already proved.

Assertion (2) in the constructible setting can be proved in the same way as in the geometric setting. □

For R a $\mathbb{E}_\infty$-ring object in $\mathrm{D}(\mathbb{Z})$, denote by $\mathrm{Shv}_{\mathrm{hyp}}(X_{\text{ét}}, \mathrm{Mod}_R)$ the ∞-category of étale hypersheaves of R-modules. The above proposition gives a control on the rationalised part of our categories, and the following theorem will be the control on the torsion part.

Theorem 3.1.26 (Bachmann)**.** *Let $n \in \mathbb{N}^*$ be an integer and let X be a qcqs scheme such that n is invertible on X. Assume that $\pi_0(R)$ is killed by n. The natural functor*

$$\rho_! \colon \mathrm{Shv}_{\mathrm{hyp}}(X_{\text{ét}}, \mathrm{Mod}_R) \to \mathrm{DM}^{\text{ét}}(X, \mathbf{D}_R) \simeq \mathrm{DM}^{\text{ét}}(X, \mathbf{D}_R)$$

is an equivalence. It induces an equivalence between $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, R)$ (Remark 1.2.12) and $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X, \mathbf{D}_R)$.

Proof. Thanks to Theorem 1.2.17 the above functor is an equivalence if $\mathbf{D}_R = \mathrm{DM}^{\text{ét}}(S, R)$, and then this form of the theorem follows by taking A-modules on both sides (because of our hypothesis, $A = R$). □

Lemma 3.1.27 (Martini-Wolf [MW22, Corollary 7.4.12])**.** *Let Λ be an $\mathbb{E}_\infty$ ring spectra and let X be a qcqs scheme. An object K of $\mathrm{Shv}_{\mathrm{hyp}}(X_{\text{ét}}, \mathrm{Mod}_\Lambda)$ is dualisable if and only if there exists an étale covering $f \colon U \to X$ such that f^*K is the constant sheaf associated to a perfect complex.*

Lemma 3.1.28. *Let R be a $\mathbb{Q}$-algebra. Let $X = \lim_i X_i$ be a limit of qcqs S-schemes with affine transitions. Then the canonical functor*

$$\mathrm{colim}_i \mathrm{DM}^{\text{ét}}(X_i, \mathbf{D}_R) \to \mathrm{DM}^{\text{ét}}(X, \mathbf{D}_R)$$

is an equivalence, where the colimit is computed in Pr^L.

Proof. First, note that the result holds if $\mathbf{D} = \mathrm{DM}^{\text{ét}}(S, \mathbb{Q})$ as in [AGV22, Lemma 2.5.11]. Now there are two things to observe (Theorem A.4.5):

1. If $\mathcal{E}$ is any presentably symmetric monoidal stable ∞-category the forgetful functor

   $$\mathrm{Mod}_{\mathcal{E}}(\mathrm{Pr}^L) \to \mathrm{Pr}^L$$

   creates sifted colimits.

2. If $\mathcal{E} \to \mathcal{F}$ is any presentably symmetric monoidal stable ∞-category the functor

   $$- \otimes_{\mathcal{E}} \mathcal{F} \colon \mathrm{Mod}_{\mathcal{E}}(\mathrm{Pr}^L) \to \mathrm{Mod}_{\mathcal{F}}(\mathrm{Pr}^L)$$

   preserves colimits.

With this, observe that because the result holds for $\mathbf{D} = \mathrm{DM}^{\text{ét}}(S, \mathbb{Q})$ and the colimit computed in Pr^L, the first point ensures that the colimit is also computed in $\mathrm{Mod}_{\mathrm{Mod}_{\mathbb{Q}}}(\mathrm{Pr}^L)$, and even in $\mathrm{Mod}_{\mathrm{DM}^{\text{ét}}(S,\mathbb{Q})}(\mathrm{Pr}^L)$. Using point two, we see that the result holds for a general $\mathbf{D}$ if we compute the colimit in $\mathrm{Mod}_{\mathbf{D}_R}(\mathrm{Pr}^L)$, but using point 1 again finishes the proof. □

Theorem 3.1.29. *(Continuity) Let R be a commutative ring. Assume that Hypothesis 3.1.19 holds. Consider a qcqs finite-dimensional S-scheme X that is the limit of a projective system of qcqs finite-dimensional schemes (X_i) with affine transition maps. Then, the natural map*

$$\mathrm{colim}\ \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X_i, \mathbf{D}_R) \to \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_R)$$

is an equivalence.

Proof. We can again assume that each X_i is Noetherian. Full faithfulness can be checked after tensorisation of the mapping spectra by $\mathbb{Q}$ and by $\mathbb{Z}/p\mathbb{Z}$. By Proposition 3.1.25, the $- \otimes \mathbb{Q}$ passes inside the mapping spectra (so that the map we want to be an equivalence is now between objects of the form $\mathrm{map}(M \otimes \mathbb{Q}, N \otimes \mathbb{Q})$), and then our objects become compact by Lemma 3.1.21. The restul then follows from Corollary A.2.17 and continuity of the presentable category as in Lemma 3.1.28. Modding out by p also passes in the mapping spectra; hence, noting that $R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is a perfect R-algebra, the result will then follow from the rigidity theorem in the form of Theorem 3.1.26 and Proposition 1.2.13, once we know that geometric objects are constructible with torsion coefficients. This last assertion is true with $\mathbb{Z}/p\mathbb{Z}$ coefficients by [CD16, Theorem 6.3.26]. By Remark 3.1.18 this implies the assertion with coefficients $\mathbf{D} \otimes_{\mathbb{Z}} \mathrm{Mod}_{\mathbb{Z}/p\mathbb{Z}}$ because the generators of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbf{D} \otimes_{\mathbb{Z}} \mathrm{Mod}_{\mathbb{Z}/p\mathbb{Z}})$ are then constructible.

To prove the essential surjectivity, first using that the Čech nerve of a covering coming from a finite open covering is finite, we can reduce to X affine using Zariski descent. Then Zariski descent again (but this time on the functor that sends $g\colon Y \to X$ smooth to $g_\sharp g^* \mathbf{1}_X$) implies that it suffices to reach $f_\sharp(\mathbf{1}_Y)(n)$ with Y and X affine and $f\colon Y \to X$ smooth. The usual spreading out properties of [Gro66b, Théorème 8.10.5, Proposition 17.7.8] then give the result. □

Corollary 3.1.30. *Let R be a commutative ring. Assume that Hypothesis 3.1.19 holds. The six functors preserve the category $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-, \mathbf{D}_R)$ when restricted to quasi-excellent finite-dimensional S-schemes and to morphisms of finite type between these schemes. More precisely*

1. *If X is a qcqs S-scheme, then $\otimes$ induces a monoidal structure on $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbf{D})$. It is closed when X is quasi-excellent and finite-dimensional.*

2. *If f is a morphism between qcqs S-schemes, then f^* preserves geometric objects. If f is furthermore of finite type, then $f_!$ also preserves geometric objects.*

3. *If f is a morphism of finite type between quasi-excellent finite-dimensional S-schemes, then f_* and $f^!$ preserve geometric objects.*

Proof. The first point is clear. For point (2), the stability under f^* is also clear. The stability under $f_!$ follows from continuity and proper base change: Let M be an object of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(Y, \mathbf{D}_R)$, and write $X = \lim X_i$ as a limit of schemes that are of finite type over the integers. Let $p\colon Y \to X$ be proper. By [Gro66b, Théorème 8.10.5.(xii)] one can find a $p_{i_0}\colon Y_{i_0} \to X_{i_0}$ proper such that the pullback to X is p. In particular

$$Y = \lim_{i \geqslant i_0} Y_{i_0} \times_{X_{i_0}} X_i$$

and there is some i such that M is the pull back of a motive N over Y_i by continuity. Then by [CD19, Proposition 4.2.11] the object $(p_i)_* M_i$ is geometric in $\mathrm{DM}^{\text{ét}}(X_i, \Lambda)$. Thus, its pullback to X is geometric, but by the proper base change theorem this is $p_* M$. Finally, if f is separated of finite type, choosing a Nagata compactification of f yields the result. An easier proof could be given if f was quasi-projective, using the known cases of i_* and $f_\sharp$ for i a closed immersion and f a smooth morphism.

For the rest of the proposition, we can work over quasi-excellent finite-dimensional $\mathbb{Q}$-schemes which are Noetherian by assumption, so that the results of [CD16] apply. Let X

be a quasi-excellent finite-dimensional $\mathbb{Q}$-scheme. The motives of type $\rho_{\mathbf{D}}(M)$ with M in $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,R)$ generate $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbf{D}_R)$ as a thick subcategory of $\mathrm{DM}^{\text{ét}}(X,\mathbf{D}_R)$. Therefore, the result follows from the same result for $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}$ which is [CD16, Corollary 6.2.14] and from the compatibility of the six functors with the functor $\rho_{\mathbf{D}}$, which is Proposition 3.1.11. □

As a consequence of continuity, we see that geometric motives are constructible.

Corollary 3.1.31. *Let R be a commutative ring. Assume that Hypothesis 3.1.19 holds. There is an inclusion*

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbf{D}_R) \subset \mathrm{DM}^{\text{ét}}_{c}(X,\mathbf{D}_R)$$

as full subcategories of $\mathrm{DM}^{\text{ét}}(X,\mathbf{D}_R)$.

Proof. Let X a qcqs scheme. Write X as a limit of Noetherian finite-dimensional schemes X_i with affine transition maps. Any geometric motive with coefficients $\mathbb{Z}$, over X_i is constructible using [CD16, Theorem 6.3.26] then this is also true with coefficients R by tensoring by R over $\mathbb{Z}$. Using Remark 3.1.18 this result is also true over each X_i with coefficients $\mathbf{D}_R$. If M is a geometric object, it is the pullback of some M_i over some X_i. As the motive M_i is constructible, so is M. □

We now prove the converse when X is R-finite[1] by linking both notions to compactness:

Proposition 3.1.32. *Let R be a commutative ring. Let X be a qcqs R-finite scheme and let M be an object of* $\mathrm{DM}^{\text{ét}}(X,\mathbf{D}_R)$. *Then, the following are equivalent:*

1. *M is of geometric origin.*
2. *M is constructible.*
3. *The functor $\underline{\mathrm{Hom}}(M,-)$ is compatible with colimits.*
4. *M is a compact object of* $\mathrm{DM}^{\text{ét}}(X,\mathbf{D}_R)$.

Proof. The equivalence between geometric objects and compact objects is Lemma 3.1.21. On the other hand, the formula

$$\mathrm{Map}(f_\sharp(\mathbf{1}_Y)(n)\otimes M,-) = \mathrm{Map}(f_\sharp(\mathbf{1}_Y)(n),\underline{\mathrm{Hom}}(M,-))$$

for $f\colon Y\to X$ smooth and n an integer, combined with the fact that $f_\sharp(\mathbf{1}_Y)(n)$ is compact implies that conditions (3) and (4) are equivalent.

We proved in Corollary 3.1.31 that motives of geometric origin are constructible. We now prove the converse. Let M be constructible over X. We can assume that X is affine by Zariski hyperdescent of geometric motives. By dévissage and localisation, we can further assume that M is dualisable. In that case $\underline{\mathrm{Hom}}(M,-)$ is compatible with colimits which finishes the proof. □

From this, we deduce continuity for constructible motives. We begin with a general lemma:

[1] See the beginning of [EHIK21b, Section 5] for the precise definition. This hypothesis implies that $\mathrm{DM}^{\text{ét}}(X,R)$ is compactly generated and any finite dimensional scheme is R-finite if R is a $\mathbb{Q}$-algebra.

Lemma 3.1.33. *Let R be a commutative ring and let M be an object of $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X, \mathbf{D}_R)$ with X a qcqs finite-dimensional scheme. Then, there is a geometric motive P as well as a map $P \to M \oplus M[1]$ whose cone C is such that $C/n \simeq C \oplus C[1]$ for some integer n (more precisely, we have $n \cdot \mathrm{Id}_C = 0$).*

Proof. Take M a constructible motive. Proposition 3.1.25 implies that $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_{R\otimes\mathbb{Q}})$ is the idempotent-completion in $\mathrm{DM}^{\text{ét}}(X, \mathbf{D}_{R\otimes\mathbb{Q}})$ of the rationalisation of $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_R)$. The characterisation of the idempotent completion [BG23, Recollection 2.13] as well as Proposition 3.1.25 yields that $(M \oplus M[1]) \otimes \mathbb{Q}$ belongs to the rationalisation of $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_R)$. This gives a geometric motive P and a map $P \to M \oplus M[1]$ which becomes an isomorphism after tensoring with $\mathbb{Q}$. Let C be the cone of this map, then $C \otimes \mathbb{Q} = 0$. Using Proposition 3.1.25 once again, we see that the multiplication by n seen as a map $C \to C$ vanishes for n large enough. This together with the exact triangle $C \xrightarrow{\times n} C \to C/n$ implies that $C/n \simeq C \oplus C[1]$. □

Corollary 3.1.34. *Let R be a commutative ring. Assume that Hypothesis 3.1.19 holds. Consider a qcqs finite-dimensional S-scheme X that is the limit of a projective system of qcqs finite-dimensional schemes (X_i) with affine transition maps. Then, the natural map*

$$\mathrm{colim}_i \, \mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X_i, \mathbf{D}_R) \to \mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X, \mathbf{D}_R)$$

is an equivalence.

Proof. We first prove that the functor above is fully faithful. This can be checked by tensoring the mapping spectra with $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$, and then it follows from Proposition 3.1.25, Proposition 3.1.32 and Proposition 1.2.13.

We now prove the essential surjectivity of the functor. Let M be a constructible motive. We want to prove that M is in the essential image of the functor

$$\mathrm{colim} \, \mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X_i, \mathbf{D}_R) \to \mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X, \mathbf{D}_R).$$

Write the exact triangle $P \to M \oplus M[1] \to C$ of the above lemma. Theorem 3.1.29 implies that P belongs to the essential image. As constructible motives are idempotent-complete, we can assume that M is of the form C/n. But then, this follows from continuity with torsion coefficients which is a combination of Theorem 3.1.26 and Proposition 1.2.13. □

We can now prove that, under some assumptions on $\mathbf{D}$, constructible motives are of geometric origin.

Hypothesis 3.1.35. *We restrict furthermore Hypothesis 3.1.19. Recall that we are given an algebra object $A \in \mathrm{CAlg}(\mathrm{DM}^{\text{ét}}(S, \mathbb{Z}))$. We assume that for each field k over S the cone of the map $\mathbb{Z}_k \to A_{|\mathrm{Spec}(k)}$ lives in the right orthogonal of the ∞-category of Artin motives (see Chapter 5). More concretely, we ask that for any such field k and any finite étale morphism $f\colon X \to \mathrm{Spec}(k)$, the map*

$$\mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})}(f_\sharp \mathbb{Z}_X, \mathbb{Z}_k) \to \mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})}(f_\sharp \mathbb{Z}_X, A_{|\mathrm{Spec}(k)})$$

is an equivalence of spectra. ***Until the end of this section we assume this more restrictive hypothesis to hold.***

Remark 3.1.36.

1. Note that because of our rigidity assumption that implies Theorem 3.1.26, together with Proposition 3.1.25, the hypothesis holds for $\mathbf{D}$ whenever it holds for $\mathbf{D}_{\mathbb{Q}}$.
2. The hypothesis holds for $\mathbf{D} = \mathrm{DM}^{\text{ét}}(S, \mathbb{Z})$ by [Rui22].
3. The hypothesis holds for $\mathbf{D} = \mathrm{DN}(S, \mathbb{Z})$ that is integral Nori motives (that will be defined in Chapter 4), because it holds rationally, as proven in Chapter 5. Note that the Chapter 5 does not use any results of this section on enriched étale motives.

Theorem 3.1.37. *Let R be a commutative ring which is a filtered colimit of regular rings and let X be a qcqs finite-dimensional scheme. Assume that Hypothesis 3.1.35 holds. Then*

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbf{D}_R) = \mathrm{DM}^{\text{ét}}_{\mathrm{c}}(X, \mathbf{D}_R).$$

Lemma 3.1.38. *Let R be a regular ring. Let k be a field of characteristic exponent p. Then, the Artin motive functor*

$$\rho_! \colon \mathrm{D}(k_{\text{ét}}, R[1/p]) \to \mathrm{DM}^{\text{ét}}(k, \mathbf{D}_{R[1/p]}) \xleftarrow{\sim} \mathrm{DM}^{\text{ét}}(k, \mathbf{D}_R)$$

is a fully faithful monoidal functor. Furthermore any torsion motive (that is an object on which the multiplication by some nonzero integer vanishes) lies in its essential image.

Proof. The result is known for $\mathbf{D} = \mathrm{DM}^{\text{ét}}(k, \mathbb{Z})$, for example by [Rui22]. Now under the Hypothesis 3.1.35, the proof is easy. Indeed, because the ∞-category $\mathrm{D}(k_{\text{ét}}, R[1/p))$ is generated under colimit by its full subcategory of dualisable objects (this follows from the description as $\mathrm{Gal}(\overline{k}/k)$)-modules, see [Rui22, 3.1.5.13], it suffices for the full faithfulness to check that for any finite étale map $f \colon X \to \mathrm{Spec}(k)$, the morphism of spectra

$$\mathrm{map}_{\mathrm{D}(k_{\text{ét}}, R[1/p])}(f_\sharp \mathbf{1}_X, M) \to \mathrm{map}_{\mathrm{DM}^{\text{ét}}(k, \mathbf{D}_R)}(f_\sharp \mathbf{1}_X, \rho_!(M))$$

is an equivalence. To prove that such a map is an equivalence, it suffices to check that it is an equivalence after being tensored (over $\mathbb{Z}$) by $\mathbb{Q}$ and by $\mathbb{Z}/p\mathbb{Z}$ for all prime numbers p. For $\mathbb{Z}/p\mathbb{Z}$ the result follows from Theorem 3.1.26, and in the rational case we can put the $- \otimes \mathbb{Q}$ inside the mapping spectra using Proposition 3.1.25 and [CD16, Lemma 1.1.11]. In particular the object $f_\sharp \mathbf{1}_X \otimes \mathbb{Q}$ becomes compact on both sides, thus we may assume that M is also of the form $g_\sharp \mathbf{1}_Y$ for some finite étale map $g \colon Y \to \mathrm{Spec}(k)$. Using the $(f_\sharp, f^*)$ adjunctions and the étale base change, we can assume replace g by the identity map and f by the map $X \times_k Y \to \mathrm{Spec}(k)$. Using the case

$$\mathbf{D} = \mathrm{DM}^{\text{ét}}(k, \mathbb{Z})$$

we can then rewrite the morphism of spectra as

$$\mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,R)}(f_\sharp R_X, R_k) \to \mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,R)}(f_\sharp R_X, A_{R|\mathrm{Spec}(k)})$$

which is an equivalence by assumption. The fact that any $\mathbb{Z}$-torsion object of $\mathrm{DM}^{\text{ét}}(k, \mathbf{D}_R)$ is in the image of $\rho_!$ follows from rigidity, as proven in [Rui22]. □

Lemma 3.1.39. *Let $R = \mathrm{colim} R_i$ be a filtered colimit of rings and let X be a Noetherian scheme. Then the natural functor*

$$\mathrm{colim}_i \mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X, \mathbf{D}_{R_i}) \to \mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X, \mathbf{D}_R)$$

is an equivalence.

Proof. Full faithfulness can be checked after tensorisation by $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$. The rationalised statement follows from Proposition 3.1.32,Corollary A.2.17 and the fact that the functor $\Lambda \mapsto \mathrm{Mod}_\Lambda(\mathrm{DM}^{\text{ét}}(X, \mathbf{D}))$ commutes with filtered colimits ([Lur22, Theorem 4.8.5.11]). The mod p statement can be proven as follows: a dévissage reduces to showing the statement for dualisable objects, and then for maps out of the constant sheaf, and even from maps out of the constant sheaf $\mathbb{Z}$, and it is now a consequence of commutativity of étale cohomology with uniformly bounded below complexes as in [HRS23, Lemma 5.3].

Now for the essential surjectivity first using Lemma 3.1.33, as it is clear that geometric objects are in the image, we reduce to reaching an object of the form C/n which is of the form $\rho_! K$ with K in $\mathrm{D}_{\mathrm{cons}}(X, R \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$. Then, a dévissage reduces to a dualisable object for which by Lemma 3.1.27 there exists an étale covering $f : U \to X$ such that $f^*(K)$ is constant with value a complex P in $\mathrm{Perf}_{R \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}}$. As in the proof of Proposition 1.2.13 we see that $K = \mathrm{colim}_{n \in \Delta}(f_n)_\sharp P$ with (f_n) the Čech nerve of f. As P is of the form $Q \otimes_{R_i \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}} (R \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$ (this follows from the fact that the objects of $\mathrm{Perf}_{(-)}$ are the compact objects of $\mathrm{Mod}_{(-)}$ which commutes with colimts), we see that $K = (\mathrm{colim}_{n \in \Delta}(f_n)_\sharp Q) \otimes_{R_i \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}} R \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is the image of a dualisable object. □

Lemma 3.1.40. *Let R be a commutative which is a filtered colimit of regular rings. The theorem holds when X is the spectrum of a field k.*

Proof. Let M be a constructible motive. We have to show that M is geometric. Lemma 3.1.33 yields a geometric motive P and a map $P \to M \oplus M[1]$ whose cone C is such that $C \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Furthermore, as C is dualisable, we have

$$\underline{\mathrm{Hom}}(C, \mathbf{1}_k) \otimes_{\mathbb{Z}} \mathbb{Q} = \underline{\mathrm{Hom}}(C, \mathbf{1}_k \otimes_{\mathbb{Z}} \mathbb{Q}) = \underline{\mathrm{Hom}}(C \otimes \mathbb{Q}, \mathbf{1}_k \otimes_{\mathbb{Z}} \mathbb{Q}) = 0.$$

Hence, the dual of C is also a torsion motive. It suffices to show that any such C is geometric.

Assume first that R is a regular ring. Writing $C = \rho_! D$, then D is a dualisable object of $\mathrm{D}(k_{\text{ét}}, R[1/p])$. In particular, it belongs to the thick stable subcategory generated by the $f_\sharp \underline{R[1/p]}$ for f finite étale according to [Rui25, Proposition 3.1.10]. Thus, its image C through $\rho_!$ is geometric as $\rho_!$ is compatible with the functors $f_\sharp$ and f^* for f finite étale using [CD16, §4.4.2].

Now we let R be an ind-regular commutative ring, we write it as $R = \mathrm{colim}_i R_i$ with each R_i regular. By Lemma 3.1.39 any object of $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(k, R)$ comes from a object in $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(k, R_i)$ for some i, hence is geometric by the above paragraph. □

Proof. (of Theorem 3.1.37) Using Theorem 3.1.29 and Corollary 3.1.34, we can assume X to be Noetherian. This is now an adaptation of the arguments given in [CD16, Lemma 6.3.25]. Let M be a constructible motive on X. By Noetherian induction, using the localisation property and the stability of geometric motives under pushforwards by closed immersions it suffices to show that there is an open immersion $j : U \to X$ such that $j^*(M)$ is geometric, *i.e.*

we can replace X with any non-empty open subscheme. Therefore, we can assume X to be irreducible. Let η be the generic point of X. Consider the commutative diagram:

$$\begin{array}{ccc} \operatorname*{colim}_{U \text{ open},\, \eta\in U} \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(U, \mathbf{D}_R) & \xrightarrow{u} & \operatorname*{colim}_{U \text{ open},\, \eta\in U} \mathrm{DM}^{\text{ét}}_{\mathrm{c}}(U, \mathbf{D}_R) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle v} \\ \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\eta, \mathbf{D}_R) & \xrightarrow{\beta} & \mathrm{DM}^{\text{ét}}_{\mathrm{c}}(\eta, \mathbf{D}_R). \end{array}$$

Then, Lemma 3.1.40 above implies that β is an equivalence and Theorem 3.1.29 implies that α is an equivalence. On the other hand u is fully faithful and v is an equivalence by Corollary 3.1.34. Hence, they are both equivalences. The fact that u is an equivalence yields the result. □

Example 3.1.41. This result fails without the assumption that R is regular: take $R = \mathbb{F}_2[x]/(x^2-1)$. Then, by rigidity, we have

$$\mathrm{DM}^{\text{ét}}(\mathrm{Spec}(\mathbb{R}), R) \underset{\rho_!}{\overset{\sim}{\leftarrow}} \mathrm{Shv}(\mathbb{R}_{\text{ét}}, R) \simeq \mathrm{D}(R[C_2])$$

where C_2 is the cyclic group with 2 elements. Taking dualisable objects, we get an equivalence between $\mathrm{DM}^{\text{ét}}_{\mathrm{c}}(\mathrm{Spec}(\mathbb{R}), R)$ and the subcategory $\mathrm{D}_{R\text{-perf}}(R[C_2])$ of $\mathrm{D}(R[C_2])$ made of those complexes with a perfect underlying complex of R-modules.

If $f\colon X \to \mathbb{R}$ is smooth, this equivalence sends $M_{\mathbb{R}}(X)(n)$ to the complex $f_\sharp(R)(n)$. The latter is isomorphic to $(f_\sharp(\mathbb{F}_2)(n)) \otimes_{\mathbb{F}_2} R$. As by [BG23, Corollary 4.5], $f_\sharp(\mathbb{F}_2)(n)$ belongs to the thick subcategory generated by $\mathbb{F}_2$ and $\mathbb{F}_2[C_2]$, the complex $f_\sharp(R)(n)$ therefore belongs to the thick subcategory $\mathrm{D}_{\mathrm{perm}}(R[C_2])$ generated by R and $R[C_2]$. Hence, through this equivalence $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{R}), R)$ corresponds to $\mathrm{D}_{\mathrm{perm}}(R[C_2])$.

Consider the complex R_x of $R[C_2]$-modules whose underlying complex is R placed in degree 0 and with the action of the generator of C_2 given by multiplication by x. The complex R_x belongs to $\mathrm{D}_{R\text{-perf}}(R[C_2])$ but by [BG22, Example 3.15] it does not belong to $\mathrm{D}_{\mathrm{perm}}(R[C_2])$. Hence, the inclusion $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{R}), R) \to \mathrm{DM}^{\text{ét}}_{\mathrm{c}}(\mathrm{Spec}(\mathbb{R}), R)$ is strict.

In fact, the result fails even for a characteristic 0 non-regular ring: take $R' = \mathbb{Z}[x]/(x^2-1)$, and consider the $R'[C_2]$-module R'_x, defined in the same way. It can be seen as an étale motive through the Artin motive functor $\rho_!\colon \mathrm{Shv}(\mathbb{R}_{\text{ét}}, R') \to \mathrm{DM}^{\text{ét}}(\mathrm{Spec}(\mathbb{R}), R')$. This object is dualisable, and if it were geometric then its image under the functor $- \otimes_{R'} R$, which is R_x would also be geometric, which is not.

Corollary 3.1.42. *Let $R \to R'$ be a morphism between commutative rings. The canonical map*

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbf{D}_R) \otimes_{\mathrm{Perf}_R} \mathrm{Perf}_{R'} \to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbf{D}_{R'})$$

is an equivalence.

Proof. We claim that it is sufficient to deal with the case $R = \mathbb{Z}$. Indeed, assume that

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbf{D}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_B \to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbf{D}_B)$$

is an equivalence for all rings B, then we have

$$\begin{aligned}\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_R) \otimes_{\mathrm{Perf}_R} \mathrm{Perf}_{R'} &\simeq \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_R \otimes_{\mathrm{Perf}_R} \mathrm{Perf}_{R'} \\ &\simeq \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{R'} \\ &\simeq \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_{R'}).\end{aligned}$$

It suffices to prove full faithfulness as generators of $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, R')$ clearly are in the image. Let B be a ring and let M, N be objects of $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_B)$. We will use that for any perfect complex of B-modules P, the canonical map

$$\mathrm{map}_{\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_B)}(M, N) \otimes_B P \to \mathrm{map}_{\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_B)}(M, N \otimes_B P) \tag{3.1.42.1}$$

is an equivalence: this follows from the fact that P is a finite colimit of free B-modules. By Lemma 3.1.39 we can assume that R' is a ring of finite type over $\mathbb{Z}$, that is a quotient $\mathbb{Z}[T_1, \dots, T_r]/I$ of a polynomial ring by an ideal. Say we knew the result for $R' = \mathbb{Z}[T_1, \dots, T_r]$. Then as $\mathbb{Z}[T_1, \dots, T_r]/I$ is a perfect $\mathbb{Z}[T_1, \dots, T_r]$-module, the proof would be finished by (3.1.42.1). By induction, it suffices to prove that if the result is known for a Noetherian ring B, then we can show that it holds for $B[T]$. But then writing

$$B[T] = \mathrm{colim}_n B[X]_{\leqslant n}$$

with $B[X]_{\leqslant n}$ the polynomials of degree $\leqslant n$, using (3.1.42.1) we see that is suffices to prove that for M, N in $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_B)$, the map

$$\mathrm{colim}_n \mathrm{map}_{\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbf{D}_B)}(M, N \otimes_B B[X]_{\leqslant n}) \to \mathrm{map}_{\mathrm{DM}^{\text{ét}}(X, \mathbf{D}_B)}(M, \mathrm{colim}_n N \otimes_B B[X]_{\leqslant n})$$

is an equivalence. Finally, this can be checked after tensorisation by $\mathbb{Q}$ (where this holds because M becomes compact), and after tensorisation by $\mathbb{Z}/p$ where this follows from [HRS23, Lemma 5.3] (note that *loc. cit.* is stated for a ring and not a derived ring, but the proof is the same for our derived ring $B \otimes_{\mathbb{Z}} \mathbb{Z}/p$: pushforwards commutes with uniformly bounded below colimits by [BGH20, Corollary 3.10.5] and then one has to prove the statement for the internal Hom, which is obvious is M is dualisable, and follows in general by dévissage.) □

Remark 3.1.43. The Hypothesis 3.1.19 ends here.

3.1.3 h-descent for étale motives.

We finish this section with result that we apply only in the classical case $\mathrm{DM}^{\text{ét}}(-, \Lambda)$ when Λ is a usual ring, because we do not feel like they are needed in another setting (because in every other setting we know, t-structures exist thus descent results are much stronger).

Corollary 3.1.44. *Let Λ be a commutative ring and let X be a finite-dimensional qcqs scheme.*

1. *Let $f\colon Y \to X$ be a separated finite type morphism. Then the functor $f_!$ restricts to a functor*
$$f_!\colon \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(Y, \Lambda) \to \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \Lambda)$$
between geometric motives.

2. *The ∞-category $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$ is the thick subcategory of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ generated by the $p_*(\Lambda_Y)(n)$ for $p\colon Y\to X$ projective and $n\in\mathbb{Z}$.*

3. *The ∞-category $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ is the localising subcategory of itself generated by the $p_*(\Lambda_Y)(n)$ for $p\colon Y\to X$ projective and $n\in\mathbb{Z}$.*

Proof. For the first point, assume first that $f=p$ is proper. Let M be an object of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(Y,\Lambda)$, and write $X=\lim X_i$ as a limit of schemes that are of finite type over the integers. By [Gro66b, Théorème 8.10.5.(xii)] one can find a $p_{i_0}\colon Y_{i_0}\to X_{i_0}$ proper such that the pullback to X is p. In particular

$$Y=\lim_{i\geqslant i_0} Y_{i_0}\times_{X_{i_0}} X_i$$

and there is some i such that M is the pull back of a motive N over Y_i by continuity. Then by [CD19, Proposition 4.2.11] the object $(p_i)_*M_i$ is geometric in $\mathrm{DM}^{\text{ét}}(X_i,\Lambda)$. Thus, its pullback to X is geometric, but by the proper base change theorem this is p_*M. Finally, if f is separated of finite type, choosing a Nagata compactification of f yields the result.

Now for the second point, write $X=\lim_i X_i$ as a limit of schemes that are of finite type over the integers. Let M be an object of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$, that comes by continuity above from an object M_i of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_i,\Lambda)$ for some index i. Thanks to [Ayo07a, Lemme 2.2.23] we know that the result is true for $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_i,\Lambda)$, so that there exists a finite diagram

$$F\colon J\to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_i,\Lambda)$$

such that $F(j)$ is of the form $p^j_*\Lambda(n_j)[m_j]$ for p^j projective and m_j and n_j integers, and such that M_i is a direct factor of $\mathrm{colim}_J F$. Thus, the object M is a direct factor of $(\mathrm{colim}_J F)_{|X}$, which by proper base change is the colimit of the objects $q^j_*\Lambda(n_j)[m_j]$, with q^j the base change of p^j to X. This finishes the proof, as the third point readily follows from the second one. □

Proposition 3.1.45. *Let Λ be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ satisfies cdh-descent over qcqs finite-dimensional schemes. Over qcqs schemes of finite valuative dimension, it also satisfies cdh hyperdescent.*

Proof. Since the presentable ∞-category $\mathrm{DM}^{\text{ét}}(-,\Lambda)$ satisfies Nisnevich descent, the localisation property and proper base change, it satisfies cdh-descent as the proof of [CD19, Theorem 3.3.10] applies in this level of generality. Thus $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ satisfies non-effective cdh descent. The effectivity of descent follows from cdh-excision: a descent data of geometric motives gives us a motive as a finite limit of proper pushforwards, which is itself geometric by Corollary 3.1.44 (1). For hyperdescent, this follows from the fact that the cdh topos is hypercomplete over qcqs schemes of finite valuative dimension by [EHIK21a, Corollary 2.4.16]. □

We now prove that geometric étale motives satisfy h-descent in the sense of Rydh.

Theorem 3.1.46. *Let Λ be a commutative ind-regular ring. The functor $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ satisfies h-descent in the sense of [Aut18, Tag 0ETQ] over qcqs finite-dimensional schemes.*

Proof. By [BS17, Theorem 2.9], h-descent is equivalent to cdh-descent combined with fppf-descent. Also, fppf-descent is the same as finite flat descent together with Nisnevich descent, see [GK15, Definition 2.5(15)]. Therefore it suffices to prove that $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(-,\Lambda)$ has descent for the finite flat topology. Let $f\colon T \to S$ be a finite morphism.

By writing $S = \lim_i S_i$ where the S_i are affine $\mathbb{Z}$-schemes of finite type, we can find an index i_0 such that f is the pullback of a finite map $f_{i_0}\colon T_{i_0} \to S_{i_0}$ using [Gro66b, Théorème 8.10.5]. By Noetherian induction, we can find a finite stratification $\{S_{i_0}^1, \dots, S_{i_0}^p\}$ of S_{i_0} such that $S_{i_0}^k$ is open in $\bigcup_{k' \geqslant k} S_{i_0}^{k'}$ for each k and f_{i_0} is the composition of a finite étale map and a purely inseparable map over each $S_{i_0}^k$. Hence, pulling back this stratification to S, we get a stratification $\{S^1, \dots, S^p\}$ of S with the same properties.

First we prove non-effective descent. This means the full faithfulness of the functor

$$\alpha\colon \mathrm{DM}^{\text{ét}}(S) \to \lim_{\Delta} \mathrm{DM}^{\text{ét}}(T_n)$$

where (T_n) is he Čech nerve of f, when restricted to $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(S)$. For this, using the theory of left adjointable squares of Lurie [Lur22, Corollary 4.7.4.18] and Proposition A.1.34, the functor α has a right adjoint, given informally by sending a compatilble family (K_n) to $\lim_n f_*^n K_n$, where $f^n\colon T^n \to S$ is the structural map. Thus we want to prove that if $K \in \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(S)$ is a constructible motive, the map

$$K \to \lim_{\Delta} f_*^n (f^n)^* K$$

is an equivalence. Using verdier duality to write K as the dual $\mathbb{D}(L)$ of a certain constructible motive L, we see that the above map is the functor $\mathscr{H}om(-, \omega_S)$ applied to the co-unit

$$\mathrm{colim}_{\Delta} f_!^n (f^n)^! L \to L,$$

where ω_S is a dualising object. Thus is suffices to prove that the latter map is an equivalence. Because the pair of functors $(i^!, j^*)$ is conservative, it suffices to check this after appying either of the two. By doing an induction on the number of strata, we may assume that f is (the composition of an inseparable map with a map which is) finite étale over U, and that we know the result for the restriction of f to Z. By étale descent (and invariance under unviersal homeomorphisms), the case of j^* is then dealt with. Using that $i^!$ commutes with colimits (this follows from the localisation triangle and the fact that j_* commutes with colimits, a fact that can be checked mod p and $\otimes\mathbb{Q}$. The mod p part follows from [Ayo22, Proposition 1.1.26] and continuity. The rational part follows from Proposition 3.1.25), we can use the induction hypothesis and conclude.

For the effectivity of descent, we also prove the result by induction. Keeping the same notations as before, we have the following morphism of fiber sequences of ∞-categories:

$$\begin{array}{ccccc}
\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(U,\Lambda) & \xrightarrow{i^*} & \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X,\Lambda) & \xrightarrow{j_!} & \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(Z,\Lambda) \\
\downarrow\sim & & \downarrow & & \downarrow\sim \\
\lim_{n\in\Delta} \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(U_n,\Lambda) & \xrightarrow{\tilde{i^*}} & \lim_{n\in\Delta} \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X_n,\Lambda) & \xrightarrow{\tilde{j_!}} & \lim_{n\in\Delta} \mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(Z_n,\Lambda)
\end{array}$$

in which the left vertical map is an equivalence by induction, and the right vertical map is an equivalence by étale descent. The functors $\tilde{i^*}$ and $\tilde{j_!}$ have right adjoint $\tilde{i_*}$ and $\tilde{j^*}$ that are

computed term wise. In particular as we have a cofiber sequence of stable ∞-categories, in $\lim_\Delta \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_n,\Lambda)$ we have an exact sequence

$$\tilde{j}_\sharp \tilde{j}^* \to \mathrm{Id} \to \tilde{i}_* \tilde{i}^*,$$

which is compatible with the same exact sequence in $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$. Thus, if (M_n) is some descent data in $\lim_\Delta \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_n,\Lambda)$, there are objects N_U and N_Y in $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(U,\Lambda)$ and $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(Y,\Lambda)$ such that they correspond to the restrictions of (M_n) to (U_n) and (Y_n). Thus in $\lim_\Delta \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_n,\Lambda)$ there is a map $\tilde{i}_* N_Y \to \tilde{j}_\sharp N_U[1]$ of fiber (M_n), which by full faithfulness gives a similar map in $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$, whose fiber has image (M_n). □

3.2 Nori motives are modules in Voevodsky motives.

Notation 3.2.1. *For k a field of characteristic zero and X a finite type k-scheme, we will denote by $\mathrm{DN}(X,\mathbb{Q})$ the ∞-category $\mathrm{IndD}^b(\mathcal{M}_{\mathrm{perv}}(X))$. We will denote by DN_c its full subcategory of compact objects. We will also denote by ρ_{N} the realisation functor*

$$\rho_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}(X,\mathbb{Q}) \to \mathrm{DN}(X,\mathbb{Q})$$

that we constructed in Theorem 2.1.38.

Recall Construction 3.1.12: the functor ρ_{N} preserves colimits, hence it has a right adjoint $\rho_{\mathrm{N}*} : \mathrm{DN}(-,\mathbb{Q}) \to \mathrm{DM}^{\text{ét}}(-,\mathbb{Q})$. For k a field a characteristic zero, denote by

$$\mathscr{N}_k = \rho_{\mathrm{N}*}\rho_{\mathrm{N}}\mathbb{Q}_{\mathrm{Spec}\, k} \in \mathrm{DM}^{\text{ét}}(\mathrm{Spec}\, k,\mathbb{Q}). \tag{3.2.1.1}$$

This is a $\mathbb{E}_\infty$-ring object in $\mathrm{DM}^{\text{ét}}(\mathrm{Spec}(k),\mathbb{Q})$. For X a finite type k-scheme, denote by $\mathscr{N}_{|X} \in \mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$ the restriction of $\mathscr{N}_k$ to X. As in Ayoub's [Ayo22, Construction 1.91] (see Construction 3.1.12), ρ_{N} factors canonically through the functor

$$\mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})) : X \mapsto \mathrm{Mod}_{\mathscr{N}_{|X}}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})) \tag{3.2.1.2}$$

as

$$\rho_{\mathrm{N}} : \mathrm{DM}^{\text{ét}}(-,\mathbb{Q}) \xrightarrow{\mathscr{N}\otimes -} \mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})) \xrightarrow{\widetilde{\rho_{\mathrm{N}}}} \mathrm{DN}(-,\mathbb{Q}). \tag{3.2.1.3}$$

Lemma 3.2.2. *Let k be a field of characteristic zero and let $\pi_X : X \to \mathrm{Spec}\, k$ be a finite type k-scheme. The functor*

$$\widetilde{\rho_{\mathrm{N}}} : \mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(X),\mathbb{Q}) \to \mathrm{DN}(X,\mathbb{Q})$$

is fully faithful. Moreover the natural map $\pi_X^*\rho_{\mathrm{N}*}\mathbb{Q}_{\mathrm{Spec}\, k} \to \rho_{\mathrm{N}*}\mathbb{Q}_X$ is an equivalence.

Proof. The proof adds nothing new to [Ayo22, Remark 1.97]. Denote by $\widetilde{\rho_{\mathrm{N}*}}$ the right adjoint of $\widetilde{\rho_{\mathrm{N}}}$. Assume first that $X = \mathrm{Spec}\, k$. We have to show that the unit map

$$\mathrm{Id} \to \widetilde{\rho_{\mathrm{N}*}}\widetilde{\rho_{\mathrm{N}}}$$

is an equivalence. By [Iwa18, Lemma 2.6], the ∞-category $\mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Q}))$ is compactly generated by the $f_\sharp\mathbb{Q}_Y(i)\otimes\mathscr{N}_k$ for $f : Y \to \mathrm{Spec}\, k$ smooth and $i \in \mathbb{Z}$. As $\widetilde{\rho_{\mathrm{N}}}$ of those compact

generators are compact objects of $\mathrm{DN}(k,\mathbb{Q})$, the right adjoint $\widetilde{\rho_{\mathrm{N}*}}$ preserves colimits. Thus it suffices to check that for any compact object $C \otimes \mathscr{N}_k$ of the above form, the map

$$C \otimes \mathscr{N}_k \to \widetilde{\rho_{\mathrm{N}*}}\widetilde{\rho_{\mathrm{N}}}(C \otimes \mathscr{N}_k) \simeq \widetilde{\rho_{\mathrm{N}*}}\rho_{\mathrm{N}}C$$

is an equivalence. As the forgetful functor $\mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})) \to \mathrm{DM}^{\text{ét}}(k,\mathbb{Q})$ is conservative it is even sufficient to prove that

$$C \otimes \mathscr{N}_k \to \rho_{\mathrm{N}*}\rho_{\mathrm{N}}C$$

is an equivalence in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})$. This is true because over a field, compact objects in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})$ are dualisable and for dualisable objects this map is an equivalence by Lemma A.5.8 (see [Ayo14b, Lemma 2.8]).

Now for a general X we use that the functor $\widetilde{\rho_{\mathrm{N}}}$ commutes with the 6 operations (this is Proposition 3.1.11). For compact $M, N \in \mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q}))$ we have:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q}))}(M,N) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q}))}(\mathbb{Q}_X, \mathscr{Hom}(M,N)) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Q}))}(\mathbb{Q}_k, (\pi_X)_*\mathscr{Hom}(M,N)) \\
&\simeq \mathrm{Map}_{\mathrm{DN}(k,\mathbb{Q})}(\mathbb{Q}_k, \widetilde{\rho_{\mathrm{N}}}(\pi_X)_*\mathscr{Hom}(M,N)) \\
&\simeq \mathrm{Map}_{\mathrm{DN}(k,\mathbb{Q})}(\mathbb{Q}_k, (\pi_X)_*\mathscr{Hom}(\widetilde{\rho_{\mathrm{N}}}M, \widetilde{\rho_{\mathrm{N}}}N)) \\
&\simeq \mathrm{Map}_{\mathrm{DN}(X,\mathbb{Q})}(\widetilde{\rho_{\mathrm{N}}}M, \widetilde{\rho_{\mathrm{N}}}N)
\end{aligned}$$

which finishes the proof of full faithfulness. Now that we know that the unit

$$\mathrm{Id} \to \widetilde{\rho_{\mathrm{N}*}}\widetilde{\rho_{\mathrm{N}}}$$

is an equivalence in $\mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q}))$, by plugging in $\pi_X^*\mathscr{N}_k$ we obtain

$$\pi_X^*\mathscr{N}_k \xrightarrow{\sim} \widetilde{\rho_{\mathrm{N}*}}\widetilde{\rho_{\mathrm{N}}}(\pi_X^*\mathscr{N}_k)$$

and we have

$$\widetilde{\rho_{\mathrm{N}}}(\pi_X^*\mathscr{N}_k) \simeq \pi_X^*\widetilde{\rho_{\mathrm{N}}}(\mathscr{N}_k) \simeq \mathbb{Q}_X.$$

Therefore the canonical map

$$\pi_X^*\mathscr{N}_k \to \widetilde{\rho_{\mathrm{N}*}}\mathbb{Q}_X$$

is an equivalence, thus it stays an equivalence after applying the forgetful functor, leaving us with the equivalence

$$\pi_X^*\mathscr{N}_k \xrightarrow{\sim} \rho_{\mathrm{N}*}\mathbb{Q}_X$$

in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$.

□

From the previous lemma there is no more ambiguity for $\mathscr{N}_X = \pi_X^*\mathscr{N}_k \simeq \rho_{\mathrm{N}*}\mathbb{Q}_X \in \mathrm{DM}^{\text{ét}}(X)$.

Proposition 3.2.3. *Let $k = \mathrm{colim}_i k_i$ be a filtered colimit of fields of characteristic zero. Then the natural functors*

$$\mathrm{colim}_i \mathrm{DN}_c(\mathrm{Spec}\, k_i, \mathbb{Q}) \to \mathrm{DN}_c(\mathrm{Spec}\, k, \mathbb{Q})$$

and

$$\mathrm{colim}_i \mathrm{DN}(\mathrm{Spec}\, k_i, \mathbb{Q}) \to \mathrm{DN}(\mathrm{Spec}\, k, \mathbb{Q})$$

are equivalences in Cat_∞ and Pr^L respectively.

The same is true for $\eta = \lim_i U_i$ with $\eta \in X$ the generic point of an integral finite type k-scheme and the U_i are the nonempty open subsets of X:

$$\mathrm{colim}_i \mathrm{DN}_{(c)}(U_i, \mathbb{Q}) \to \mathrm{DN}_{(c)}(\eta, \mathbb{Q}).$$

Proof. As the functor Ind, from small idempotent complete stable ∞-categories to compactly generated stable ∞-categories is an equivalence and that the inclusion functor $\mathrm{Pr}^L_\omega \to \mathrm{Pr}^L$ preserves colimits, the second statement follows from the first. By [IM22, Proposition 6.8] (or [IM22, Corollary 6.10] for the generic point case), the diagram $\mathcal{M} : I^\triangleright \to \mathrm{AbCAt}$ from I to the $(2,1)$-category of small abelian categories with exact functors sending $i \in I$ to $\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}\, k_i)$ and the cone point v to $\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}\, k)$ is a colimit diagram. By [BCKW19, Theorem 7.4.9] the functor sending an Abelian category to the ∞-category of bounded complexes up to homotopy is a left adjoint, hence commutes with colimits, so that we obtain a colimit diagram

$$\mathcal{K}^b \circ \mathcal{M} : I^\triangleright \to \mathrm{Cat}^{\mathrm{st}}_\infty.$$

Now as $\mathcal{D}^b(\mathcal{A})$ is a localisation of $\mathcal{K}^b(\mathcal{A})$ for any abelian category $\mathcal{A}$, and as the transitions of $\mathcal{K}^b \circ \mathcal{M}$ preserve quasi-isomorphisms, $\mathcal{D}^b \circ \mathcal{M}$ is a colimit diagram. □

Corollary 3.2.4. *Let $f : \mathrm{Spec}\, k \to \mathrm{Spec}\, \mathbb{Q}$ be a map of fields. Then the natural map $f^* \mathscr{N}_{\mathbb{Q}} \to \mathscr{N}_k$ is an equivalence. Consequently, for any scheme X of finite type over k if we denote by $p_X : X \to \mathrm{Spec}\, \mathbb{Q}$ the unique map (which has to factor through f), then $\mathscr{N}_X \simeq p_X^* \mathscr{N}_{\mathbb{Q}}$.*

Proof. The second claim follows from the first. Write $k = \mathrm{colim}_i k_i$ with $k_i/\mathbb{Q}$ of finite type. Then the same proof as [AGV22, Lemma 3.5.7], using continuity of both $\mathrm{DM}^{\text{ét}}$ and DN for fields (Proposition 3.2.3) and that these categories are compactly generated gives that the natural map

$$\mathrm{colim}_i f_i^* \mathscr{N}_{\mathbb{Q}} \to \mathscr{N}_k$$

is an equivalence so that this reduces to the case where $k/\mathbb{Q}$ is of finite type. Then k is the generic point of an integral $\mathbb{Q}$-scheme, and again the same proof as [AGV22, Lemma 3.5.7], using continuity for the generic point of perverse Nori motives, reduces to show that $\pi_U^* \mathscr{N}_{\mathbb{Q}} \to \mathscr{N}_U$ is an equivalence for U of finite type over $\mathbb{Q}$, which is Lemma 3.2.2.

□

Therefore now there is even less ambiguity for $\mathscr{N}_X = (X \to \mathrm{Spec}\, \mathbb{Q})^* \mathscr{N}_{\mathbb{Q}}$.

Proposition 3.2.5. *Let $X = \lim_i X_i$ be a limit of $\mathbb{Q}$-schemes with affine transitions. The natural morphism*

$$\mathrm{colim}_i \mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(X_i, \mathbb{Q})) \to \mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})) \tag{3.2.5.1}$$

is an equivalence where the colimit is computed in Pr^L.

Proof. By [EHIK21b, Lemma 5.1 (ii)], the functor

$$\mathrm{colim}_i \mathrm{DM}^{\text{ét}}(X_i, \mathbb{Q}) \to \mathrm{DM}^{\text{ét}}(X, \mathbb{Q}) \tag{3.2.5.2}$$

is an equivalence. Therefore, the assignment $i \mapsto (\mathrm{DM}^{\text{ét}}(X_i, \mathbb{Q}), \mathscr{N}_{|X_i})$ is a colimit diagram in the category $\mathrm{Pr}^{\mathrm{Alg}}$ of [Lur22, Notation 4.8.5.10]. By [Lur22, Theorem 4.8.5.11], the functor $(\mathcal{C}, A)$ that sends a presentable symmetric monoidal category $\mathcal{C}$ together with an algebra object $A \in \mathcal{C}$ to the category $\mathrm{Mod}_A(\mathcal{C})$ has a right adjoint, hence preserves colimits. This is why the assignment $i \mapsto \mathrm{Mod}_{\mathscr{N}_{X_i}}(\mathrm{DM}^{\text{ét}}(X_i, \mathbb{Q}))$ is a colimit diagram. We found a precise statement giving the symmetric monoidal structure in the paper by Ayoub, Gallauer and Vezzani. The precise statement is [AGV22, Lemma 3.5.6]. □

Proposition 3.2.6. *The functor*

$$\widetilde{\rho_{\mathrm{N}}} : \mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})) \to \mathrm{DN}(X, \mathbb{Q}) \tag{3.2.6.1}$$

is an equivalence for every finite type k-scheme X.

Proof. The functor is fully faithful by Lemma 3.2.2. We first prove that it is essential surjective for $X = \operatorname{Spec} K$ the spectrum of a field, not necessarily finite over k. As $\mathrm{DN}(\operatorname{Spec} K, \mathbb{Q})$ is compactly generated it suffices to show that $\widetilde{\rho_{\mathrm{N}}}$ reaches all compact objects $M \in \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(K))$. As the functor is already fully faithful, by dévissage it suffices to check that any object of the heart $M \in \mathcal{M}_{\mathrm{perv}}(K)$ is in the image of $\widetilde{\rho_{\mathrm{N}}}$. Any such object has a weight filtration whose graded pieces are direct factors of $\mathrm{H}^{n+2j}(\pi_X)_*\mathbb{Q}_X(i)$ with $\pi_X : X \to \operatorname{Spec} K$ a smooth projective morphism by Arapura's [HMS17, Theorem 10.2.5]. Note that here we used that if K is a finite type extension of k, then the category $\mathcal{M}_{\mathrm{perv}}(\operatorname{Spec} K)$ obtained by colimit as motives over the generic point of a variety over k coincides with Nori motives over K. This holds with a beautiful proof in [Ter24, Proposition 1.15] or can be proven by using that the ℓ-adic realisation is absolute, *i.e.* does not depend on the base field. Therefore by dévissage again it suffices to check that any such $\mathrm{H}^{n+2j}(\pi_X)_*\mathbb{Q}_X(i)$ is in the image. But as $(\pi_X)_*\mathbb{Q}_X(i)$ is pure of weight $-2i$, it is the direct sum of its $\mathrm{H}^n[-n]$, thus is suffices to show that each $(\pi_X)_*\mathbb{Q}_X(i)$ is in the image, which is obvious by compatibility with the 6 operations.

Now for a finite type k-scheme, the proposition is proven by Noetherian induction. Assume now that X is such that the statement is true for any proper closed subset of X. We take an object $M \in \mathrm{DN}(X)$ that we may assume to be compact. Let η be a generic point of X. By the case of a field, there exists $N \in \mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(\eta, \mathbb{Q}))$, compact, such that $\widetilde{\rho_{\mathrm{N}}}(N) = M_\eta$. By Proposition 3.2.5 there exists an open subset U of X and an object $N' \in \mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(U, \mathbb{Q}))$, compact, such that $(N')_\eta = N$. The functor $\mathrm{colim}_{\eta \in U}\mathrm{DN}(U, \mathbb{Q}) \to \mathrm{DN}(\eta, \mathbb{Q})$ is fully faithful by proposition 3.2.3. Thus the isomorphism $\widetilde{\rho_{\mathrm{N}}}(N) \to M_\eta$ lifts to a smaller open subset V of U : we have $\widetilde{\rho_{\mathrm{N}}}((N')_{|V}) \simeq M_{|V}$. Denote by Z the reduced closed complement of V in X. By induction hypothesis, there exists a $N'' \in \mathrm{Mod}_{\mathscr{N}}(\mathrm{DM}^{\text{ét}}(Z, \mathbb{Q}))$ such that $\widetilde{\rho_{\mathrm{N}}}(N'') = M_{|Z}$. The cofiber sequence $\widetilde{\rho_{\mathrm{N}}}(j_!(N')_{|V}) \to M \to \widetilde{\rho_{\mathrm{N}}}(i_*N'')$ with $j : U \to X$ and $i : Z \to X$ the immersion then ensures that M is the image of $\widetilde{\rho_{\mathrm{N}}}$, which finishes the proof.

□

Corollary 3.2.7. *The ∞-functor $X \mapsto \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is of the form $\mathrm{DM}^{\text{ét}}(-, \mathbf{D})$ as in Definition 3.1.6 with $\mathbf{D} = \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}(\mathbb{Q})))$.*

Corollary 3.2.8. *The ∞-functor $X \mapsto \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ has an unique extension $\mathrm{DN}_c(X)$ to quasi-compact and quasi-separated schemes of characteristic zero such that for all limit $X = \lim_i X_i$ of such schemes with affines transitions the natural functor*

$$\mathrm{colim}_i \mathrm{DN}_c(X_i, \mathbb{Q}) \to \mathrm{DN}_c(X, \mathbb{Q})$$

is an equivalence of ∞-categories, where we compute the colimit in Pr^L.

Proof. Let $\mathcal{C}$ be category of schemes that are of finite type over a field of characteristic zero. Let $\mathrm{Sch}_{\mathbb{Q}}$ be the category of quasi-compact and quasi-separated (qcqs) $\mathbb{Q}$-schemes, and $\iota : \mathcal{C} \to \mathrm{Sch}_{\mathbb{Q}}$ the inclusion. Nori motives, together with the pullback functoriality, define a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ which sends limits of schemes with affine transitions to colimits of ∞-categories by Proposition 3.2.5 together with the fact that the colimit is taken in Pr^L hence restricts to compact objects. We can denote by $\mathrm{DN}_c(-, \mathbb{Q})$ its left Kan extension along ι. □

Corollary 3.2.9. *Let X be a finite type k-scheme. The ∞-category $\mathrm{DN}(X, \mathbb{Q})$ is compactly generated by the $f_\sharp \mathbb{Q}_Y(i)$ for $f : Y \to X$ smooth and $i \in \mathbb{Z}$.*

Proof. Indeed, this is the case for $\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})$, thus for $\mathrm{Mod}_{\mathcal{M}_X}(\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})) \simeq \mathrm{DN}(X, \mathbb{Q})$ by [Iwa18, Lemma 2.6]. □

We finish with the proof of the following natural corollary, whose proof was surprisingly harder than expected for a result that the author took for granted for a long time.

Corollary 3.2.10. *Let X be a quasi-projective k-scheme. The two functors*

$$\mathrm{DM}_c^{\text{ét}}(X, \mathbb{Q}) \to \mathcal{M}_{\mathrm{perv}}(X)$$

given by ${}^{\mathrm{p}}\mathrm{H}^0 \circ \rho_{\mathrm{N}}$ and the universal functor $\mathrm{H}_{\mathrm{univ}}$ are canonically equivalent.

Proof. First, note that the functor ${}^{\mathrm{p}}\mathrm{H}^0 \circ \rho_{\mathrm{N}}$ gives a factorisation of the ${}^{\mathrm{p}}\mathrm{H}^0$ of the Betti realisation of étale motives

$$\mathrm{DM}_c^{\text{ét}}(X, \mathbb{Q}) \to \mathrm{Perv}(X).$$

By the universal property of perverse Nori motives, this induces a canonical functor

$$F : \mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{M}_{\mathrm{perv}}(X)$$

such that $R_B \circ F \simeq R_B$ and ${}^{\mathrm{p}}\mathrm{H}^0 \circ \rho_{\mathrm{N}} \simeq F \circ \mathrm{H}_{\mathrm{univ}}$. Now the compatibility of ρ_{N} with the operations ensures that for any perverse t-exact operation h the functor F commutes with the operation h. Hence F commutes with i_* for i a closed immersion and for $f^*[d]$ for f a smooth map of relative dimension d. We will still denote by F the functor induced by it on $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$. As F commutes with i_*, there exists an exchange morphism $i^* \circ F \to F \circ i^*$. Applying the Betti (or ℓ-adic in the case the field k is too big) realisation where this exchange morphism is an equivalence, we see that F commutes with i^*. The same argument works to show that F commutes with $f_\sharp$ for smooth f: because F commutes with f^* there is a natural transformation $f_\sharp \circ F \to F \circ f_\sharp$ which is an equivalence after realisation, hence F also commutes with the $f_\sharp$ functors. It also commutes with external tensor product because is it

perverse t-exact, hence with the internal tensor product which is obtained from the external one by pulling back along the diagonal.

The functor F induces a compact and colimit preserving functor

$$F : \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

which commutes with all pullbacks, pushforwards and left adjoint to pushforward by smooth maps. Denote by G its right adjoint. We claim that the natural transformation

$$\mathrm{Id} \to GF$$

is an equivalence. Indeed, by Corollary 3.2.9 we have that $\mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is generated by colimits by the $f_\sharp\mathbb{Q}_Y(i)$ for $f : Y \to X$ a smooth map and $i \in \mathbb{Z}$. This means that it suffices to check that for $M \in \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$, the map

$$\mathrm{Map}(f_\sharp\mathbb{Q}_Y(i), M) \to \mathrm{Map}(f_\sharp\mathbb{Q}_Y(i), GF(M))$$

is an equivalence. As G and F commute with filtered colimits and $f_\sharp\mathbb{Q}_Y(i)$ is compact, we can assume that M is as well compact. Moreover, compact objects are constructed by extensions, finite limits, finite colimits and direct factors from the same $f_\sharp\mathbb{Q}_Y(i)$, thus by dévissage we can assume that $M = g_\sharp\mathbb{Q}_Z(j)$ for some smooth map $g : Z \to X$ and $j \in \mathbb{Z}$. Using that G is a right adjoint and that F commutes with the operations, the above map is equivalent to the map

$$\mathrm{Map}(f_\sharp\mathbb{Q}_Y(i), g_\sharp\mathbb{Q}_Z(j)) \to \mathrm{Map}(f_\sharp F(\mathbb{Q}_Y)(i), g_\sharp F(\mathbb{Q}_Z)(j))$$

and as $F(\mathbb{Q}) = \mathbb{Q}$, this finishes the proof.

Now that F is fully faithful, we note that it is an equivalence whenever $X = \mathrm{Spec}\, K$ is the spectrum of a field by [HMS17, Corollary 10.1.7]. Thus an easy dévissage using the localisation sequence and continuity proves that F is essentially surjective, finishing the proof that F is an isomorphism. Now, as F commutes with the operations, we have by universality of $\mathrm{DM}^{\text{ét}}(-, \mathbb{Q})$ that $F \circ {}^{\mathrm{p}}\mathrm{H}^0 \circ \rho_{\mathrm{N}} \simeq {}^{\mathrm{p}}\mathrm{H}^0 \circ \rho_{\mathrm{N}}$, thus $F \circ F \circ \mathrm{H}_{\mathrm{univ}} \simeq {}^{\mathrm{p}}\mathrm{H}^0 \circ \rho_{\mathrm{N}}$ and the universal property of $\mathcal{M}_{\mathrm{perv}}(X)$ tells us that $F \circ F \simeq F$, so that $F \simeq \mathrm{Id}$. □

3.3 Motivic Hodge modules (after Drew).

In this section, we prove that Drew's approach in [Dre18] to motivic Hodge modules gives a full subcategory of the derived category of mixed Hodge modules, stable under truncation, and thus has a t-structure. If X is a finite type $\mathbb{C}$-scheme, we will denote by $\mathrm{D}_H(X) := \mathrm{Ind}\mathrm{D}^b(\mathrm{MHM}(X))$ the indization of the bounded derived ∞-category of mixed Hodge modules on X. As recalled above, we have a Hodge realisation

$$\rho_H \colon \mathrm{DM}^{\text{ét}}(-, \mathbb{Q}) \to \mathrm{D}_H$$

compatible with all the operations. This functor is $\mathrm{Mod}_{\mathbb{Q}}$-linear. The target D_H is naturally valued in $\mathrm{D}_H(\mathbb{C}) = \mathrm{Ind}\mathrm{D}^b(\mathrm{MHS}^p_{\mathbb{C}})$-linear presentable ∞-categories, where $\mathrm{MHS}^p_{\mathbb{C}}$ is the abelian category of graded polarisable mixed Hodge structures over $\mathbb{C}$ (with rational coefficients). Thus the above realisation has a natural enrichment

$$\rho_{\mathbf{H}} \colon \mathrm{DM}(-, \mathrm{MHS}^p_{\mathbb{C}}) \to \mathrm{D}_H$$

where $\mathrm{DM}(-,\mathrm{MHS}^p_{\mathbb{C}}) := \mathrm{DM}^{\text{ét}}(-,\mathbb{Q}) \otimes_{\mathrm{Mod}_{\mathbb{Q}}} \mathrm{D}_H(\mathbb{C})$ is the $\mathrm{D}_H(\mathbb{C})$-linearisation of $\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})$. It can be computed as

$$\mathrm{DM}(X,\mathrm{MHS}^p_{\mathbb{Q}}) = \mathrm{DM}^{\text{ét}}(X,\mathbb{Q}) \otimes_{\mathrm{Mod}_{\mathbb{Q}}} \mathrm{D}_H(\mathbb{C}),$$

but can also be put inside the definition:

$$\mathrm{DM}(X,\mathrm{MHS}^p_{\mathbb{Q}}) \simeq \mathrm{Shv}^{\wedge}_{\mathbb{A}^1,\text{ét}}(\mathrm{Sm}_X, \mathrm{D}_H(\mathbb{C}))[\mathbb{Q}(1)^{\otimes -1}]$$

by Proposition 3.1.2, where we can take $\mathbf{M} = \mathrm{IndD}^b(\mathrm{MHS}^p_{\mathbb{C}})$. In particular, it also affords the 6 operations and the canonical morphism $\mathrm{DM}^{\text{ét}}(-,\mathbb{Q}) \to \mathrm{DM}(-,\mathrm{MHS}^p_{\mathbb{Q}})$ commutes with them. For each finite type $\mathbb{C}$-scheme X, the functors

$$\rho_{H,X} \colon \mathrm{DM}^{\text{ét}}(X) \to \mathrm{D}_H(X)$$

and

$$\rho_{\mathbf{H},X} \colon \mathrm{DM}(X,\mathrm{MHS}^p_{\mathbb{Q}}) \to \mathrm{D}_H(X)$$

are colimit preserving symmetric monoidal functors, hence they have lax symmetric monoidal right adjoints $\rho_*^{H,X}$ and $\rho_*^{\mathbf{H},X}$ that create commutative algebras

$$\mathcal{H}_X := \rho_*^{H,X}\mathbb{Q}_X \in \mathrm{CAlg}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q}))$$

and

$$\boldsymbol{\mathcal{H}}_X := \rho_*^{\mathbf{H},X}\mathbb{Q}_X \in \mathrm{CAlg}(\mathrm{DM}(X,\mathrm{MHS}^p_{\mathbb{Q}}))$$

in étale motives. Because the right adjoints commute with pushforwards, the counit maps induces algebra maps

$$\mathcal{H}_{X/\mathbb{C}} := \pi_X^*\mathcal{H}_{\mathbb{C}} \to \mathcal{H}_X$$

and

$$\boldsymbol{\mathcal{H}}_{X/\mathbb{C}} := \pi_X^*\boldsymbol{\mathcal{H}}_{\mathbb{C}} \to \boldsymbol{\mathcal{H}}_X,$$

where $\pi_X : X \to \mathrm{Spec}\,\mathbb{C}$ is the structural map. We will show below that those maps are in fact equivalences.

Definition 3.3.1 (Drew)**.** The ∞-category of motivic Hodge modules over X is

$$\mathrm{DH}(X) := \mathrm{Mod}_{\mathcal{H}_{X/\mathbb{C}}}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})).$$

There is an enriched version, that we will call the ∞-category of enriched mixed Hodge modules over X, defined as

$$\mathbf{DH}(X) := \mathrm{Mod}_{\boldsymbol{\mathcal{H}}_{X/\mathbb{C}}}(\mathrm{DM}(X,\mathrm{MHS}^p_{\mathbb{Q}})).$$

Notation 3.3.2. *As the reader begins to guess, both situations, motivic Hodge modules and enriched motivic Hodge modules are parallel. Thus from now on except for the important results we will only deal with motivic Hodge modules, the proofs in the enriched case being the same. This is only to avoid doubling the size of this section, and the extensive use of bold.*

By Theorem 3.1.7 the functor

$$\mathrm{DH}\colon \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Pr}^{L}_{\mathrm{St}}$$

is naturally a coefficient system in the sense of Drew and Gallauer, and therefore has the 6 operations in a way that is compatible to the functor $\mathrm{DM}^{\text{ét}} \to \mathrm{DH}$ (by [Ayo07a] and [CD19]), and has h-descent (by [CD19, Theorem 3.3.37]).

The Hodge realisation naturally factors as

$$\mathrm{DM}^{\text{ét}}(-,\mathbb{Q}) \xrightarrow{\otimes \mathcal{H}_{\mathbb{C}}} \mathrm{DH} \xrightarrow{\rho_H} \mathrm{D}_H$$

where all functors commute with the operations and all categories are compactly generated on schemes. This is Construction 3.1.12.

The first observation is a consequence of the commutation with the operations:

Lemma 3.3.3. *For each finite type $\mathbb{C}$-scheme, the functor*

$$\underline{\rho_H}\colon \mathrm{DH} \to \mathrm{D}_H$$

is fully faithful. Moreover the map

$$\mathcal{H}_{X/\mathbb{C}} \to \mathcal{H}_X$$

is an equivalence.

Proof. The proof is originally due to Cisinski and Déglise. We recall it here. First over $\operatorname{Spec}\mathbb{C}$ the functor is fully faithful by [Dre18, Lemma 4.11], because $\mathrm{DM}^{\text{ét}}(\mathbb{C})$ is rigid and ρ_{H} preserves compact objects. Over a general base it suffices to prove that the functor is fully faithful on compact objects DH^{ω}. Using that

$$\mathrm{Map}_{\mathrm{D}_{\mathrm{H}}(X)}(\underline{\rho_{\mathrm{H}}}(M),\underline{\rho_{\mathrm{H}}}(N)) \simeq \mathrm{Map}_{\mathrm{D}_{\mathrm{H}}(\mathbb{C})}(\underline{\rho_{\mathrm{H}}}(\mathbb{Q}), p_*\,\mathscr{H}\!\mathit{om}(\underline{\rho_{\mathrm{H}}}(M),\underline{\rho_{\mathrm{H}}}(N))),$$

where $p\colon X \to \operatorname{Spec}(\mathbb{C})$ is the structural morphism, and that the same formula holds in DH, we see that the lemma would follow from the commutation of $\underline{\rho_{\mathrm{H}}}$ with pushforwards and internal homomorphisms. Because the image of the compact preserving functor $-\otimes\mathcal{H}\colon \mathrm{DM}^{\text{ét}} \to \mathrm{DH}$ generates DH, it suffices even to prove that $\underline{\rho_{\mathrm{H}}}$ commutes with pushforwards and internal homomorphisms when restricted to the essential image of $-\otimes\mathcal{H}\colon (\mathrm{DM}^{\text{ét}})^{\omega} \to \mathrm{DH}^{\omega}$. This finishes the proof as $\rho_{\mathrm{H}} = \underline{\rho_{\mathrm{H}}} \circ (-\otimes\mathcal{H})$ and $-\otimes\mathcal{H}$ commute with the 6 operations. □

Denote by ${}^{\mathrm{ct}}\mathrm{H}^n$ the cohomology functor for the ordinary t-structure.

Definition 3.3.4 (Ayoub)**.** Let X be a finite type $\mathbb{C}$-scheme.

1. We let $\mathrm{MHM}_{\mathrm{geo}}(X)$ (*resp.* $\mathrm{MHM}_{hod}(X)$) be the full subcategory of $\mathrm{IndMHM}_{\mathrm{ord}}(X)$ generated under kernels, cokernels, extensions and filtered colimits by objects of the form ${}^{\mathrm{ct}}\mathrm{H}^n(f_*\pi_Y^*K)(m)$ with $K \in \mathrm{Mod}_{\mathbb{Q}}$ (*resp.* with $K \in \mathrm{D}_H(\mathbb{C})$), $f : Y \to X$ a proper morphism, $\pi_Y : Y \to \operatorname{Spec}\mathbb{C}$ the structural morphism and $n, m \in \mathbb{Z}$. These are Grothendieck abelian categories that we called Mixed Hodge modules of geometric origin (*resp.* of Hodge origin).

2. We let $\mathrm{D}_{H,\mathrm{geo}}(X)$ (*resp.* $\mathrm{D}_{H,\mathrm{hod}}(X)$) be the full subcategory of complexes $K \in \mathrm{D}_H(X)$ such that for all $n \in \mathbb{Z}$ we have ${}^{\mathrm{ct}}\mathrm{H}^n(K) \in \mathrm{MHM}_{\mathrm{geo}}(X)$ (*resp.* we have ${}^{\mathrm{ct}}\mathrm{H}^n(K) \in \mathrm{MHM}_{\mathrm{hod}}(X)$). Those are stable ∞-categories on which the canonical t-structure on $\mathrm{D}_H(X) = \mathrm{D}(\mathrm{IndMHM}_{\mathrm{ord}}(X))$ restricts. Moreover they are stable under pullbacks by proper base change.

Lemma 3.3.5. *The functors*

$$\underline{\rho_H} \colon \mathrm{DH} \to \mathrm{D}_H$$

and

$$\underline{\rho_{\mathbf{H}}} \colon \mathbf{DH} \to \mathrm{D}_H$$

land in $\mathrm{D}_{H,\mathrm{geo}}$ *and* $\mathrm{D}_{H,\mathrm{hod}}$.

Proof. This results from the commutation with the operations and the fact that $\mathrm{DH}(X)$ (*resp.* $\mathbf{DH}(X)$) is generated under colimits by the $p_*\pi_X^* K(m)$ for $K \in \mathrm{Mod}_{\mathbb{Q}}$ (*resp.* $K \in \mathrm{D}_H(\mathbb{C})$), thanks to [Ayo07a, Lemme 2.2.23]. □

Proposition 3.3.6. *Let η be the generic point of an irreducible finite type $\mathbb{C}$-scheme X, and consider the functor*

$$\mathrm{colim}_{\eta \in U} \mathrm{DH}(U) \to \mathrm{colim}_{\eta \in U} (\mathrm{D}_{H,\mathrm{geo}}(U)) \tag{3.3.6.1}$$

induced by $\underline{\rho_{\mathrm{H}}}$, where the colimit (taken in Pr^L) runs over all the smooth open subsets U of X that contain η. It is an equivalence.

Proof. Note that by [Lur09, 4.4.5.21, 5.5.7.8], the colimit is in fact computed in Pr^L_ω the ∞-category of compactly generated ∞-categories, and we can rewrite it as

$$\mathrm{Ind}\Big(\mathrm{colim}_{\eta \in U} \mathrm{DH}(U)^\omega \to \mathrm{colim}_{\eta \in U} (\mathrm{D}_{H,\mathrm{geo}}(U))^\omega\Big)$$

where now the colimit inside the $\mathrm{Ind}(-)$ is taken in Cat_∞. By [HRS25, Proposition 2.1] the functor between compact objects is fully faithful, using Lemma 3.3.3. It remains to show that it is essentially surjective. It suffices to reach compact objects.

Denote by

$$\mathcal{D} := \mathrm{colim}_{\eta \in U} (\mathrm{D}_{H,\mathrm{geo}}(U)^\omega)$$

the colimit of the compact objects, computed in Cat_∞.

For each U, the compact objects of $\mathrm{D}_{H,\mathrm{geo}}(U)$ have an standard t-structure and its heart is the category defined in the same way as $\mathrm{MHM}^{\mathrm{std}}_{\mathrm{geo}}(U)$, but only allowing only direct factors instead of all filtered colimits. As all transitions in the diagram are t-exact, the colimit $\mathcal{D}$ of the compact objects (which are the compact objects of the colimit) has a bounded t-structure. Moreover, the category $\mathcal{D}$ is generated under finite colimits, finite limits, extensions and truncations by images in the colimit of objects of the form $g_*\pi_Z^* K(n)$ where $g \colon Z \to U$ is a proper morphism and $n \in \mathbb{Z}$ is an integer.

By continuity of motives [EHIK21b, Lemma 5.1 (ii)] and [Lur22, Theorem 4.8.5.11], the left hand side of (3.3.6.1) is canonically equivalent to $\mathrm{DH}(\eta) := \mathrm{Mod}_{u^*\mathcal{H}_{\mathbb{C}}}(\mathrm{DM}^{\text{ét}}(\eta))$, where $u : \eta \to \mathrm{Spec}\,\mathbb{C}$ is the structural morphism. Moreover, by [Ayo07a, Lemma 2.2.27], the ∞-category $\mathrm{DH}(\eta)$ is generated under colimits by the $f_*\pi_Y^* K(n)$ where $f : Y \to \eta$ is a smooth and projective morphism, $n \in \mathbb{Z}$ and $K \in \mathrm{Mod}_{\mathbb{Q}}$. By spreading out ([Gro66a, 8.10.5(xiii)], [Gro67,

17.7.8(ii)]) and proper base change, such an object is the restriction to η of a $g_*\pi_Z^*K(n)$ with $g : Z \to U$ a smooth projective morphism, and U an open subset of X.

Now consider an object of the form $g_*\pi_Z^*K(n)$ in $\mathcal{D}$, where $g\colon Z \to U$ is a proper morphism and $n \in \mathbb{Z}$ is an integer. This object is the image by (3.3.6.1) of $(g_*\pi_Z^*K(n))_\eta \in \mathrm{DH}(\eta)$. By the above paragraph, there exists an open subset V of U, a finite category I and a functor $F\colon I \to \mathrm{DH}(V)$ such that for each $i \in I$, the object $F(i)$ is of the form $(f_i)_*\pi_{Z_i}^*K(n_i)$ for $f_i\colon Z_i \to U$ a *smooth and projective* morphism and $n_i \in \mathbb{Z}$, and such that in $\mathrm{DH}(\eta)$, we have

$$(g_*\pi_Z^*K(n))_\eta \simeq (\mathrm{colim}_I F(i))_\eta.$$

In particular, in $\mathcal{D}$, the object $g_*\pi_Z^*K(n)$ is isomorphic to the image in the colimit over all open subsets, of the finite colimit $\mathrm{colim}_I F'(i)$, where F' is the composition of the functor F with (3.3.6.1), which has the same formula for $F'(i)$.

This gives that the category $\mathcal{D}$ is generated under finite colimits, finite limits, direct factors, extensions and truncations by images in the colimit of objects of the form $g_*\pi_Z^*K(n)$ with $g : Z \to U$ smooth and projective. We may also assume U to be smooth.

It turns out that to prove that the functor of the proposition is essentially surjective, it suffices to check that it reaches all objects of the heart of the compact objects. Thanks to [Ayo22, Lemma 1.6.22], we see that it suffices to show that all subquotients of the image in the colimit of all ${}^{\mathrm{std}}\mathrm{H}^n(f_*\pi_Y^*K)$, for $K \in \mathrm{Mod}_{\mathbb{Q}}$ compact and $f : Y \to U$ projective and smooth, are in the image. Now by dévissage we can assume K to be pure, so that by the conservation of weights under pushforwards by proper maps, and because U is smooth hence Y is also smooth so that π_Y^*K is pure, the object $f_*\pi_Y^*K$ is pure. Using the decomposition theorem for pure complexes, we see that there is a decomposition

$$f_*\pi_Y^*K \simeq \bigoplus_n {}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K)[-n].$$

In particular, each ${}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K)$ lies in the image. As f is smooth and proper and π_Y^*K is a dualisable coming from $\mathrm{Spec}(\mathbb{C})$, we see that (as in [Rio05, Theorem 2.2], the dual is $f_*\pi_Y^*K^\vee(d)[2d]$ where d is the relative dimension of f) the complex $f_*\pi_Y^*K$ is dualisable, so that ${}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K)$ are local systems. Because duality datum consists of finitely many maps, and because compact objects in the colimit are dualisable, we see that any subquotient of the image of ${}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K)$ in the colimit will lift as a subquotient in the category of local systems, of the restriction of ${}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K)$ to a smaller open subset. Moreover, as ${}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K)$ is lisse we have ${}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K[d])[-d] \simeq {}^{\mathrm{std}}\mathrm{H}^n(f_*\pi_Y^*K)$ where $d = \dim U$ (this is because over a smooth scheme, and for lisse complexes, the standard t-structure is the perverse t-structure, shifted by the dimension). This proves that any subquotient of the image of ${}^{\mathrm{std}}\mathrm{H}^n(f_*\pi_Y^*K)$ in the colimit is in fact a subquotient in perverse sheaves of a suitable restriction of ${}^{\mathrm{p}}\mathrm{H}^n(f_*\pi_Y^*K[d])$ to an open subset of U. As this object is a pure perverse sheaf, any subquotient is in fact a direct factor, thus is in the image, and the proof is finished. □

This theorem is the same as Ayoub's theorem for Betti sheaves, and is proved the same way.

Theorem 3.3.7. *Let X be a finite type $\mathbb{C}$-scheme. The functors*

$$\underline{\rho_H}\colon \mathrm{DH}(X) \to \mathrm{D}_{H,\mathrm{geo}}(X)$$

and

$$\underline{\rho_{\mathbf{H}}}\colon \mathbf{DH}(X) \to \mathrm{D}_{H,\mathrm{hod}}(X)$$

are equivalences of categories.

Proof. This is now a simple Noetherian induction: if $K \in \mathrm{D}_{H,\mathrm{geo}}(X)$, we may find a nonempty irreducible open subset U of X with generic point η and Proposition 3.3.6 ensures that up to reducing the size of U, the restriction $K_{|U}$ of K to U is in the image, but the localisation sequence

$$j_! K_{|U} \to K \to i_* i^* K$$

with $j : U \to X$ and $i : Z \to X$ the complement, together with the full faithfulness and a Noetherian induction, give that K is in the image. □

Corollary 3.3.8. *All desiderata of Drew in [Dre18] are fulfilled.*

Proof. Only was missing the t-structure, which we now have, and the compatibility with the weight structure, which is clear when looking at the pure objects. □

Although Corollary 3.3.8 ensures that the goals of Drew are achieved, it may still seem unsatisfactory: another motivation for Drew's work was probably to give an alternative construction of mixed Hodge modules. Here our proof uses mixed Hodge modules hence it is not quite right. It should be possible to prove that motivic Hodge modules afford a t-structure without comparing them to M. Saito's mixed Hodge modules.

We give here a weak reason for which one could possibly define the t-structure only by geometric means (a t-structure on the generic points is enough to have a t-structure for every variety by gluing, see [Bon15, Theorem 3.1.4]):

Proposition 3.3.9. *Let $K = \mathbb{C}(X)$ be the fraction field of an integral complex algebraic variety. Then the t-structure on*

$$\mathrm{DH}(K/\mathbb{C}) := \mathrm{colim}_U \mathrm{DH}(U),$$

where the colimit runs over nonempty open subsets of X, is characterised by the fact that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is the smallest subcategory of $\mathrm{DH}(K/\mathbb{C})$ stable under colimits, twists and extensions that contains the objects $f_\sharp \mathbb{Q}_X$ for $f : X \to \mathrm{Spec}\, K$ smooth and affine.

Proof. We have that $\mathrm{DH}(K/\mathbb{C})$ is compactly generated by $\mathrm{colim}_U \mathrm{DH}_c(U)$ where this time the colimit is taken in Cat_∞. Thus we know that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is the prestable ∞-category

$$\mathrm{DH}(K/\mathbb{C})^{\leqslant 0} \simeq \mathrm{Ind}\big(\mathrm{colim}_U \mathrm{DH}_c(U)^{\leqslant 0}\big).$$

In particular, as the heart of each $\mathrm{DH}(U)$ is generated under kernels, cokernels, extensions, twists and filtered colimits by objects of the form ${}^{\mathrm{ct}}\mathrm{H}^n(f_* \mathbb{Q}_Y)$ with $f\colon Y \to U$ proper, the proper base change theorem ensures that the heart of $\mathrm{DH}(K/\mathbb{C})$ is generated under the same operations by the ${}^{\mathrm{ct}}\mathrm{H}^n(f_* \mathbb{Q}_Y)$ with $f : Y \to \mathrm{Spec}\, K$ proper and $n \in \mathbb{N}$. In fact we have seen in the proof of Proposition 3.3.6 that we can even assume that f is smooth and projective. Thus the prestable ∞-category $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is generated under colimits, extensions and twists by objects of the form ${}^{\mathrm{ct}}\mathrm{H}^n(f_* \mathbb{Q})$ for $f\colon Y \to \mathrm{Spec}\, K$ smooth and projective. Moreover as

this is true over a small open subset U of X, we have a decomposition in $\mathrm{DH}(K/\mathbb{C})$ of the form

$$f_*\mathbb{Q}_Y \simeq \bigoplus_n {}^{\mathrm{ct}}\mathrm{H}^n(f_*\mathbb{Q}_Y)[-n].$$

This implies that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is generated by the $f_*\mathbb{Q}_Y[2\dim Y]$ for $f : Y \to \operatorname{Spec} K$ projective and smooth. By covering Y with smooth and affine schemes $(U_i)_{i\leqslant n}$, if one denote by $U_J = \cap_{i\in J} U_J$ for $J \subset \{1,\dots,n\}$, as we have a colimit diagram

$$\cdots \substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow} \bigoplus_{|J|=2} f_!^{U_J}\mathbb{Q}_{U_J}[2d] \substack{\longleftarrow\\\longleftarrow\\\longleftarrow} \bigoplus_{|J|=1} f_!^{U_J}\mathbb{Q}_{U_J}[2d] \longrightarrow f_*\mathbb{Q}_Y[2d]$$

by Zariski descent, with $f^{U_J} : U_J \to Y \to \operatorname{Spec} K$ and $d = \dim Y$, we see that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is generated by the $f_!\mathbb{Q}_X[2d]$ for $f : X \to \operatorname{Spec} K$ smooth and affine of dimension d. As $f_\sharp\mathbb{Q}_X \simeq f_!\mathbb{Q}_X(d)[2d]$, this finishes the proof.

$\square$

Chapter 4

Integral Nori motives.

In this chapter, we define relative Nori motives with integral coefficients. This is joint work with Raphaël Ruimy, and most results can be found in [RT24b]. The idea is the following: thanks to Ivorra, S. Morel, Terenzi and Chapter 2, we have a reasonable theory of relative Nori motives with rational coefficients. Conjecturally if X is a qcqs finite dimensional scheme of characteristic zero the ∞-category $\mathrm{DN}(X,\mathbb{Q})$ of Nori motives with rational coefficients over X, defined as the indization of the bounded derived category of perverse Nori motives (for the extension of perverse Nori motives to this generality of schemes, see Corollary 3.2.8), is equivalent to $\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$ the ∞-category of étale motives with rational coefficients. Integrally, we now that $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ has the following property, by rigidity Theorem 1.2.17: for every nonzero integer n, the $\mathbb{Z}/n\mathbb{Z}$-modules in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ agree with complexes of sheaves on the small étale site of X with $\mathbb{Z}/n\mathbb{Z}$-coefficients.

Hence a good theory of integral Nori motives should also have this property. Thanks to Proposition 3.2.6 we know that, rationally, the ∞-category $\mathrm{DN}(X,\mathbb{Q})$ is a category of modules in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$ over some algebra $\mathscr{N}_{X,\mathbb{Q}}$, pulled back from $\mathrm{Spec}(\mathbb{Q})$. The idea in this section is the following: we define an algebra $\mathscr{N}_{\mathrm{Spec}(\mathbb{Q}),\mathbb{Z}}$ in $\mathrm{DM}^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}),\mathbb{Z})$ such that:

1. $\mathscr{N}_{\mathrm{Spec}(\mathbb{Q}),\mathbb{Z}} \otimes \mathbb{Q} \simeq \mathscr{N}_{\mathrm{Spec}(\mathbb{Q}),\mathbb{Q}}$.
2. $\mathscr{N}_{\mathrm{Spec}(\mathbb{Q}),\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z}$ for every nonzero integer n.

This way, the ∞-category $\mathrm{DN}(X,\mathbb{Z}) := \mathrm{Mod}_{\mathscr{N}_{X,\mathbb{Z}}}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Z}))$ will have the following properties:

1. $\mathrm{DN}(X,\mathbb{Z}) \otimes \mathrm{Mod}_{\mathbb{Q}} \simeq \mathrm{DN}(X,\mathbb{Q})$.
2. $\mathrm{DN}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}} \simeq \mathrm{DM}^{\text{ét}}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}} \simeq \mathrm{D}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$ for every nonzero integer n.

Thus we will have a well-behaved category of integral Nori motives. We then explore the category of its constructible objects and prove that they have all expected properties: t-structures, operations, descent.

4.1 Definition of integral Nori motives.

4.1.1 A technical result on the rationalisation of the Nori realisation functor.

Recall (Theorem 1.2.25) that over a field there is a theory of Nori motives with integral coefficients, and there is a realisation functor from Voevodsky motives:

Theorem 4.1.1 (Choudhury-Gallauer, Harrer, Nori)**.** *Let k be a subfield of $\mathbb{C}$. There exists a symmetric monoidal functor*

$$\rho'_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}(k,\mathbb{Z}) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$$

that gives back the Betti realisation when composed with the Betti realisation of Nori motives, and sends $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,R)$ to $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$.

It will be crucial for us that, rationally, the realisation functor is in some sense compatible with the realisation functor to $\mathrm{IndD}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$ obtained by taking the indisation

$$\rho_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}(k,\mathbb{Q}) \to \mathrm{IndD}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$$

of the restriction $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$ of ρ'_{N}. The functor ρ_{N} is also the realisation functor constructed in Theorem 2.1.38. To that end, we first have to compare the rationalisation of $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$ with $\mathrm{IndD}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$. We begin with a lemma.

Lemma 4.1.2. *Let k be a subfield of $\mathbb{C}$. The canonical functor*

$$\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$$

induces an equivalence $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$ where the tensor product is taken in Pr^{L}.

Proof. The ∞-category $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}}$ is a localisation of $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$ at the map $K \xrightarrow{\times n} K$ for all nonzero integers n. In fact, it is the full subcategory of $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$ that consists of objects K on which the multiplication by any nonzero integer is an equivalence. Because the t-structure on a derived category is compatible with filtered colimits, we see that the t-structure on $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$ restricts to $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}}$, and its heart is identified with $\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}) \otimes \mathbb{Q}$, which is the full subcategory of $\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$ spanned by objects over which each non zero integer is invertible. Because $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}) = \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}) \otimes \mathbb{Q}$ (here, as abelian category thus this means the idempotent completion of the category obtained by tensoring all Hom-groups with $\mathbb{Q}$), there is a canonical functor $\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}) \to \mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}) \otimes \mathbb{Q}$. It is fully faithful because $-\otimes\mathbb{Q}$ is idempotent and objects of $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$ are compact in $\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$. Using this it is easy to check that the induced t-structre on $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}}$, with heart $\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$, satisfies all the points of the universal property [Lurb, Remark C.5.4.11] of the unbounded derived category of a Grothendieck abelan category, and this finishes the proof. □

With rational coefficients, this category $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$ is closely related to the ind-category $\mathrm{IndD}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$. Recall that with rational coefficients, we denote by $\mathrm{DN}(X,\mathbb{Q})$ the indization of the bounded derived category of perverse Nori motives over a scheme X.

Proposition 4.1.3. *Let k be a subfield of $\mathbb{C}$. Denote by $\mathrm{DN}(k,\mathbb{Q})^\flat$ the Verdier quotient of $\mathrm{DN}(k,\mathbb{Q})$ by the kernel of the Betti realisation. Then the presentable ∞-category $\mathrm{DN}(k,\mathbb{Q})^\flat$ admits a separated t-structure whose heart is canonically equivalent to $\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$, and the natural functor*

$$\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})) \to \mathrm{DN}(k,\mathbb{Q})^\flat$$

is an equivalence.

Proof. As $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$ is Noetherian, we can apply [Kra05, Proposition 3.6] to its indisation. This implies that the canonical functor

$$J\colon \mathrm{IndD}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$$

is a Bousfield localisation (note that what is called $\mathrm{K}(\mathrm{Inj}(-))$ in *loc. cit.* is nothing but $\mathrm{IndD}^b((-)^\omega)$ by [Kra05, Proposition 2.3]).

We now only have to show that J and the Betti realisation have the same kernel. This amounts to the conservativity of the functor

$$R'_{\mathrm{B}}\colon \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})) \to \mathrm{Mod}_{\mathbb{Q}}$$

obtained by deriving the Betti realisation. Take an object M to be killed by R'_{B}. As the t-structure on $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$ is separated, we can assume M to be in the heart. By [Hub93, Lemma 1.3] we can write $M = \mathrm{colim} M_i$ as a filtered colimit of objects M_i of $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$ with injective transition maps. In particular, because $R'_{\mathrm{B}}(M)$ vanishes, so does each $R'_{\mathrm{B}}(M_i)$ and therefore M_i by conservativity of the Betti realisation on Nori motives. □

Remark 4.1.4. Using this, we can show that a conjecture of Ayoub ([Ayo17, Conjecture 2.8]) holds for Nori motives: Let k be a subfield of $\mathbb{C}$. Let $M \in \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$ be a complex of Nori motives and let $\mathcal{F} \in \mathrm{DN}(k,\mathbb{Q})$ be killed by the Betti realisation. Then the group

$$\mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})}(\mathcal{F}, M) = 0$$

vanishes.

Indeed because $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$ is Noetherian the above Proposition 4.1.3 and [Hub93, Proposition 2.2] imply that the functor

$$\pi\colon \mathrm{DN}(k,\mathbb{Q}) \to \mathrm{DN}(k,\mathbb{Q})^\flat$$

is fully faithful when restricted to $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$. We can write $\mathcal{F} = \mathrm{colim}_i N_i$ as a filtered colimit of objects in $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$, and we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})}(\mathcal{F}, M) &\simeq \lim_i \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})}(N_i, M) \\
&\simeq \lim_i \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})^\flat}(\pi(N_i), \pi(M)) \\
&\simeq \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})^\flat}(\pi(\mathcal{F}), \pi(M)) \\
&\simeq \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})^\flat}(0, \pi(M)) = 0.
\end{aligned}$$

Proposition 4.1.3 implies that we have a canonical functor

$$\mathrm{DN}(k,\mathbb{Q}) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})) \simeq \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}}$$

which is fully faithful when restricted to $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$: we have some control on the rational part of $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$.

Corollary 4.1.5. *Let k be a subfield of $\mathbb{C}$. Then the composition*

$$\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})\otimes \mathrm{Mod}_{\mathbb{Q}} \simeq \mathrm{DM}^{\text{ét}}(k,\mathbb{Q}) \xrightarrow{\rho_{\mathrm{N}}} \mathrm{IndD}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))\otimes \mathrm{Mod}_{\mathbb{Q}}$$

is naturally isomorphic to the functor ρ'_{N} constructed in Theorem 1.2.25 tensored by $\mathrm{Id}_{\mathrm{Mod}_{\mathbb{Q}}}$.

Proof. Indeed, the restriction of both functors to geometric motives are isomorphic: they are the realisation functor

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})).$$

Hence both functors coincide on compact objects and are therefore naturally equivalent. □

4.1.2 The algebra representing Nori motivic cohomology.

Let k be a subfield of $\mathbb{C}$. Thanks to Theorem 4.1.1 we have at hand a symmetric monoidal functor

$$\rho'_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(k,\mathbb{Z}) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$$

compatible with the Betti realisation.

Definition 4.1.6. Let F be the right adjoint of ρ'_{N}. We denote by $\mathscr{N}_k$ the $\mathbb{E}_\infty$-ring object in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ given by applying the lax symmetric monoidal (Proposition A.3.4) functor F to the unit object $\mathbb{Z}_k$ of $\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$.

Lemma 4.1.7. *Let $M \in \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$. Then the canonical map*

$$F(M)\otimes\mathbb{Q} \to F(M\otimes\mathbb{Q})$$

is an equivalence.

Proof. It suffices to prove that for any smooth k-scheme X and any integer $i\in\mathbb{Z}$, the map

$$\mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})}(M_k(X)(i), F(M)\otimes\mathbb{Q}) \to \mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})}(M_k(X)(i), F(M\otimes\mathbb{Q}))$$

is an equivalence. Using that, for constructible étale motives, tensorisation by the rationals can be set out of the mapping spectra (see for example [CD16, Corollary 5.4.9]), and using the adjunction, we see that it suffices to prove that the map

$$\mathrm{map}_{\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)), M)\otimes\mathbb{Q} \to \mathrm{map}_{\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)), M\otimes\mathbb{Q})$$

is an equivalence. The right-hand side can be rewritten as

$$\mathrm{map}_{\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))\otimes\mathrm{Mod}_{\mathbb{Q}}}(\rho'_{\mathrm{N}}(M_k(X)(i))\otimes\mathbb{Q}, M\otimes\mathbb{Q})$$

thus using Proposition 4.1.3, the above map is in fact

$$\mathrm{map}_{\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)), M)\otimes\mathbb{Q} \to \mathrm{map}_{\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))}(\rho'_{\mathrm{N}}(M_k(X)(i))\otimes\mathbb{Q}, M\otimes\mathbb{Q})$$

which is an equivalence by [BVK16, Proposition B.4.1]. □

Let $k \subset K$ be subfields of $\mathbb{C}$ and denote by $f\colon \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ the associated map of schemes. By construction we have a commutative square

$$\begin{array}{ccc} \mathrm{DM}^{\text{ét}}(k,\mathbb{Z}) & \xrightarrow{\rho'_{\mathrm{N}}} & \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{DM}^{\text{ét}}(K,\mathbb{Z}) & \xrightarrow[\rho'_{\mathrm{N}}]{} & \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(K,\mathbb{Z})) \end{array}$$

in Pr^{L}. It induces an exchange transformation between f^* and F. In particular, we have a map

$$f^*\mathscr{N}_k \to \mathscr{N}_K \tag{4.1.7.1}$$

of algebras in $\mathrm{DM}^{\text{ét}}(K,\mathbb{Z})$. We will prove that this map is an equivalence, but first we need two important propositions:

Proposition 4.1.8. *Let $\mathscr{N}_{k,\mathbb{Q}}$ be the algebra constructed in Section 3.2, that is the algebra obtained by applying the right adjoint of the functor ρ_{N} to the unit object. By Corollary 4.1.5 and Lemma 4.1.7 we have a natural ring map*

$$\mathscr{N}_{k,\mathbb{Q}} \to \mathscr{N}_k \otimes \mathbb{Q}$$

in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})$. This map is an equivalence.

Proof. It suffices to prove that the map is an equivalence after applying the functors $\mathrm{map}_{\mathrm{DM}(k,\mathbb{Q})}(M_k(X)(i),-)$ for X a smooth k-scheme and i an integer. Using adjunctions and Lemma 4.1.7, we see that it suffices to check that the map

$$\mathrm{map}_{\mathrm{DN}(k,\mathbb{Q})}(\rho_{\mathrm{N}}(M_k(X)(i)),\mathbb{Q}_k) \to \mathrm{map}_{\mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))}(\rho'_{\mathrm{N}}(M_k(X)(i)),\mathbb{Q}_k)$$

is an equivalence, which is the case because on $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$ the functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})) \to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$$

is fully faithful. □

Recall that the change of site from the small étale site to the smooth étale site induces a functor

$$\iota\colon \mathrm{D}(X_{\text{ét}},R) \to \mathrm{DM}^{\text{ét}}(X,R)$$

for any commutative ring R and any scheme X. By the rigidity theorem (see [Bac21] for the most general statement), this functor is an equivalence when R is a torsion ring.

Proposition 4.1.9. *The ring map $\mathbb{Z}_k \to \mathscr{N}_k$ in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ becomes an equivalence after passing mod n for any nonzero integer $n \in \mathbb{N}^*$: $\mathbb{Z}_k/n\mathbb{Z}_k \simeq \mathscr{N}_k/n\mathscr{N}_k$.*

Proof. Again to prove this, it suffices to prove that for every smooth k-scheme X and any integer $i \in \mathbb{Z}$, the map

$$\mathrm{map}_{\mathrm{DM}(k,\mathbb{Z})}(M_k(X)(i),\mathbb{Z}/n\mathbb{Z}) \to \mathrm{map}_{\mathrm{DM}(k,\mathbb{Z})}(M_k(X)(i),\mathscr{N}_k \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$$

is an equivalence. Using adjunctions and rigidity for étale motives, together with the fact that tensoring with $\mathbb{Z}/n\mathbb{Z}$ is a finite colimit thus goes out of the mapping spectra, it suffices to prove that the map

$$\mathrm{map}_{\mathrm{D}^b_c(k_{\text{ét}},\mathbb{Z}/n\mathbb{Z})}(\iota^{-1}(M_k(X)(i)),\mathbb{Z}/n\mathbb{Z}) \to \mathrm{map}_{\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)),\mathbb{Z}_k)\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$$

is an equivalence. The right-hand side is in fact by Proposition A.6.20 (that we can use because $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$ has enough torsion-free objects) the mapping complex in the category $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})[n])$, the bounded derived ∞-category of the abelian category of n-torsion objects in $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$. By Theorem 1.2.26 the category $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})[n]$ is the category of finite type representations of the absolute Galois group of k with $\mathbb{Z}/n\mathbb{Z}$ coefficients. In particular, $\mathrm{D}^b(k_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$ embeds in $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})[n])$, its image being exactly complexes of representations with perfect underlying complex of $\mathbb{Z}/n\mathbb{Z}$-modules. This finishes the proof. □

Corollary 4.1.10. *Let $k\subset K$ be a map of subfield of the complex numbers. The natural map $(\mathscr{N}_k)_{|K}\to\mathscr{N}_K$ of (4.1.7.1) is an equivalence.*

Proof. Indeed, it suffices to prove this after rationalisation and passing mod n for all nonzero integers n. The rational statement is proven in Corollary 3.2.4, and the torsion statement follows from the fact that $f^*(\mathbb{Z}/n\mathbb{Z})\simeq\mathbb{Z}/n\mathbb{Z}$ for all f, as any exact functor commutes with cofibers. □

We finish this section with the following result:

Theorem 4.1.11. *Let k be a subfield of the complex numbers. Consider the factorisation of ρ'_{N} through the modules category*

$$\overline{\rho'_{\mathrm{N}}}\colon \mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Z}))\to \mathrm{D}(\mathrm{Ind}\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})).$$

This functor is fully faithful when restricted to dualisable objects and induces an equivalence between those and $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$.

Proof. We denote by $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z}):=\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\mathrm{Spec}(k),\mathscr{N})$ the category of dualisable objects in $\mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Z}))$. It is an idempotent complete stable ∞-category. We first prove that $\overline{\rho'_{\mathrm{N}}}$ is fully faithful when restricted to $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$. To do so, it suffices to check that it is the case after being tensored over $\mathrm{Perf}_{\mathbb{Z}}$ by $\mathrm{Perf}_{\mathbb{Q}}$ and $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$ for all nonzero integers $n\in\mathbb{N}^*$. When tensored by $\mathrm{Perf}_{\mathbb{Q}}$, using Proposition 4.1.8 and Proposition 3.2.6 we see that the functor is fully faithful. When tensored by $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$, this follows from Theorem 1.2.26, Proposition A.6.20, Proposition 4.1.9 and rigidity for étale motives. We now prove essential surjectivity. In fact, the proof of full faithfulness rationally gives that the functor

$$\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})\to\mathrm{D}^b(k,\mathbb{Z})$$

is an equivalence when tensored by $\mathrm{Perf}_{\mathbb{Q}}$. In particular, if $K\in\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z}))$ is a complex of Nori motives, we see that there exist an object $P\in\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ and an map $\overline{\rho'_{\mathrm{N}}}(P)\to M$ that becomes an equivalence after being tensored by $\mathbb{Q}$. Said in an other way, this map $\overline{\rho'_{\mathrm{N}}}(P)\to M$ as torsion cofibre, thus we may assume that M is of torsion: there exists an integer $n\in\mathbb{N}^*$ such that the multiplication by n on M is zero. In this case we have that $M\oplus M[1]\simeq M\otimes\mathbb{Z}/n\mathbb{Z}$, so it suffices to reach $M\otimes\mathbb{Z}/n\mathbb{Z}$. But because M is dualisable, the image $M\otimes^L_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$ of M in $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})[n])$ has perfect underlying complex of $\mathbb{Z}/n\mathbb{Z}$-modules, thus lies in $\mathrm{D}^b_c(k_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$. By rigidity of étale motives again, we see that $M\otimes\mathbb{Z}/n\mathbb{Z}$ is in the image of $\overline{\rho'_{\mathrm{N}}}$, finishing the proof. □

4.1.3 Relative Nori motives.

In Proposition 3.2.6 it was proven that the construction of Ivorra and Morel in [IM22] has a natural interpretation in terms of modules in étale motives. Indeed, if for X a finite type k-scheme, one denote by

$$\mathrm{DN}(X,\mathbb{Q}) := \mathrm{IndD}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

the indization of the bounded derived category of perverse motives, the realisation functor

$$\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(X,\mathbb{Q}) \to \mathrm{DN}(X,\mathbb{Q})$$

factors through the ∞-category of modules in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$ over the algebra $\mathscr{N}_{X,\mathbb{Q}} := \pi_X^*\mathscr{N}_{k,\mathbb{Q}}$, with $\pi_X\colon X \to \mathrm{Spec}(k)$ the structural morphism, and induces an equivalence with the latter and $\mathrm{DN}(X,\mathbb{Q})$: there is a natural equivalence of functors

$$\mathrm{Mod}_{\mathscr{N}_{(-)}}(\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})) \xrightarrow{\sim} \mathrm{DN}(-,\mathbb{Q}).$$

In the previous section, we constructed a commutative algebra $\mathscr{N}_k$ in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ which is equivalent to $\mathscr{N}_{k,\mathbb{Q}}$ when rationalised, and to $\mathbb{1}_k/n$ when tensored with $\mathbb{Z}/n\mathbb{Z}$. For X a finite type k-scheme, we define $\mathscr{N}_X = \pi_X^*\mathscr{N}_k$ to be the pullback of $\mathscr{N}_k$ to X. Clearly, this algebra also has the property that when rationalised, it is $\mathscr{N}_{X,\mathbb{Q}}$, and that the natural map $\mathbb{1}_X/n \to \mathscr{N}_X/n$ is an equivalence for n a nonzero integer. Furthermore, if $f\colon X \to \mathrm{Spec}(\mathbb{Q})$ is the structural map, Corollary 4.1.10 implies that the natural transformation $f^*\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})} \to \mathscr{N}_X$ is an equivalence. We therefore extend the definition of $\mathscr{N}_X$ to any $\mathbb{Q}$-scheme X by setting $\mathscr{N}_X := f^*\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})}$.

Definition 4.1.12. Let X be a $\mathbb{Q}$-scheme. The *presentable ∞-category of Nori motives over X* is the ∞-category

$$\mathrm{DN}(X,\mathbb{Z}) := \mathrm{Mod}_{\mathscr{N}_X}\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$$

of modules in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ over the algebra $\mathscr{N}_X$.

Let Λ be a commutative ring. We then define

$$\mathrm{DN}(X,\Lambda) := \mathrm{Mod}_\Lambda \mathrm{DN}(X,\mathbb{Z}).$$

We denote by Λ_X the unit object of $\mathrm{DN}(X,\Lambda)$

Using Theorem 3.1.7 we can promote this construction into a functor

$$\mathrm{DN}(-,\Lambda)\colon \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\Lambda),$$

and there is a natural transformation $\mathrm{DM}^{\text{ét}}(-,\Lambda) \to \mathrm{DN}(-,\Lambda)$. By Corollary 3.1.9 we thus have:

Proposition 4.1.13. *The functor*

$$\mathrm{DN}(-,\Lambda)\colon \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\Lambda)$$

is naturally part of a six functors formalism such that the functors of type f^*, $f_!$ and $\otimes$ are compatible with the forgetful functor to $\mathrm{DM}^{\text{ét}}$. We denote the tensor structure by $\otimes_{\mathrm{N}}$ and the internal Hom by $\underline{\mathrm{Hom}}_{\mathrm{N}}$.

Furthermore, DN also has some continuity properties when $\mathrm{DM}^{\text{ét}}$ has.

Definition 4.1.14. Let X be a qcqs scheme and let Λ be a commutative ring. We say that X is Λ-finite if it has finite Krull dimension and

$$\sup_{x\in X, p\notin\Lambda^\times} \mathrm{cd}_p(\kappa(x)) < \infty$$

where $\mathrm{cd}_p(k)$ is the Galois cohomological dimension of a field k with $\mathbb{F}_p$-coefficients.

Proposition 4.1.15. *Let $X = \lim X_i$ be a limit of qcqs étale-locally Λ-finite $\mathbb{Q}$-schemes with affine transition maps. Assume that Λ is a $\mathbb{Q}$-algebra. The natural map*

$$\mathrm{colim}_i \mathrm{DN}(X_i, \Lambda) \to \mathrm{DN}(X, \Lambda)$$

in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *is an equivalence.*

Proof. This follows from Lemma 3.1.28 □

Proposition 4.1.16. *The functor* $\mathrm{DN}(-,\Lambda)$ *has the same descent properties as* $\mathrm{DM}^{\text{ét}}(-,\Lambda)$. *In particular:*

1. *It is an étale hypersheaf and a cdh-sheaf.*
2. *It is a cdh-hypersheaf over qcqs $\mathbb{Q}$-schemes of finite valuative dimension.*

Proof. This is Proposition 3.1.22, the same proof applies for the cdh topology if one knows that $\mathrm{DM}^{\text{ét}}$ is a cdh-sheaf (the proof of [CD19, Theorem 3.3.10] works in that generality). □

Remark 4.1.17 (Universal property of Nori motives). Let k be a subfield of $\mathbb{C}$. By Drew and Gallauer's work [DG22], we know that $\mathrm{DM}^{\text{ét}}(-,\mathbb{Z})$ is universal among functors

$$\mathrm{Sch}_k^{\mathrm{ft}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}})$$

that satisfy $\mathbb{A}^1$-invariance, $\mathbb{P}^1$-stability, smooth base change and étale hyperdescent. We claim that $\mathrm{DN}(-,\mathbb{Z})$ is universal among the above class of functors $\mathcal{C}$ that verify the following additional conditions: the ∞-category $\mathcal{C}_{\mathrm{liss}}(k)$ of dualisable objects in $\mathcal{C}(k)$ affords a t-structure and the induced functor $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z}) \to \mathcal{C}_{\mathrm{liss}}(k)$ factors the Betti realisation in a way that the functor $\mathcal{C}_{\mathrm{liss}}(k) \to \mathrm{Perf}_{\mathbb{Z}}$ is t-exact and conservative. Indeed for such a $\mathcal{C}$, the universal property of Nori motives over k gives us a faithful exact functor $\mathcal{M}(k) \to \mathcal{C}_{\mathrm{liss}}(k)^\heartsuit$ such that the induced functor $\mathrm{D}^b(\mathcal{M}(k)) \to \mathcal{C}_{\mathrm{liss}}(k) \to \mathrm{Perf}_{\mathbb{Z}}$ is the Betti realisation of Nori motives. In particular we have a commutative diagram

$$\begin{array}{ccc} \mathrm{DM}^{\text{ét}}(k,\mathbb{Z}) & \longrightarrow & \mathcal{C}(k) \\ \downarrow & \nearrow & \\ \mathrm{DN}(k,\mathbb{Z}) & & \end{array}$$

which induces, if one denotes by C the commutative algebra representing cohomology with values in $\mathcal{C}(k)$, a map of algebras $\mathscr{M}_k \to C$ in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$. By pulling back C over finite type k-schemes we obtain morphisms of functors

$$\mathrm{DM}^{\text{ét}}(-,\mathbb{Z}) \to \mathrm{DN}(-,\mathbb{Z}) \to \mathrm{Mod}_C(\mathrm{DM}^{\text{ét}}(-,\mathbb{Z})) \to \mathcal{C},$$

thus the promised morphism of functors $\mathrm{DN} \to \mathcal{C}$.

4.2 Constructible motives: derived categories.

4.2.1 Compatibilities of the realisation with the six functors

We now study the compatibility of the Nori realisation with the six functors. We also show that the six functors induce functors on constructible motives when restricted to quasi-excellent schemes. Recall that by [AGV22, Theorem 4.6.1] this is also the case for $\mathrm{DM}^{\text{ét}}$.

Theorem 4.2.1. *Let Λ be a commutative ring. Over qcqs $\mathbb{Q}$-schemes of finite dimension, the Nori realisation:*

$$\rho_{\mathrm{N}} := - \otimes \mathscr{N} \colon \mathrm{DM}^{\text{ét}}(-, \Lambda) \to \mathrm{DN}(-, \Lambda)$$

is compatible with the operations of type f^*, f_*, $f_!$, $f^!$, $\otimes$ and $\underline{\mathrm{Hom}}(M, -)$ for M geometric.

Proof. The theorem amounts to proving that the exchange transformations are equivalences, which can be tested after tensoring those maps with $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for any prime number p. After tensoring with $\mathbb{Z}/p\mathbb{Z}$, the Nori realisation becomes an equivalence so the exchange maps are also equivalences. After tensoring with $\mathbb{Q}$, those maps can be interpreted as the same maps but with coefficients $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ since the operations of type f^*, f_*, $f_!$, $f^!$ and $\underline{\mathrm{Hom}}(M, -)$ for M geometric are compatible with rationalisation by Proposition 3.1.25. Hence, we can assume that Λ is a $\mathbb{Q}$-algebra. In this case, this is Proposition 3.1.13. □

Recall (Definition 3.1.15) that if X is a qcqs $\mathbb{Q}$-scheme of finite dimension and Λ is a commutative ring, $\mathrm{DN}_{\mathrm{gm}}(X, \Lambda)$ is the thick full subcategory of $\mathrm{DN}(X, \Lambda)$ spanned by twists of motives of smooth X-schemes.

Proposition 4.2.2. *Let Λ be a commutative ring. The six functors preserve the category $\mathrm{DN}_{\mathrm{gm}}(-, \Lambda)$ when restricted to quasi-excellent finite-dimensional $\mathbb{Q}$-schemes and to morphisms of finite type between these schemes. More precisely*

1. *If X is a qcqs $\mathbb{Q}$-scheme, then $\otimes_{\mathrm{N}}$ induces a monoidal structure on $\mathrm{DN}_{\mathrm{gm}}(X, \Lambda)$. It is closed when X is quasi-excellent and finite-dimensional.*
2. *If f is a morphism between qcqs $\mathbb{Q}$-schemes, then f^* preserves geometric objects. If f is furthermore of finite type, then $f_!$ also preserves geometric objects.*
3. *If f is a morphism of finite type between quasi-excellent finite-dimensional $\mathbb{Q}$-schemes, then f_* and $f^!$ preserve geometric objects.*

Proof. This is Corollary 3.1.30. □

Proposition 4.2.3. *Let Λ be a commutative ring. Consider an excellent scheme B of dimension 2 or less as well as a regular separated B-scheme of finite type S, endowed with a geometric and $\otimes$- invertible object ω_S in $\mathrm{DN}(S, \Lambda)$. For any separated morphism of finite type $f \colon X \to S$, we define the* Verdier duality functor $\mathbb{D}_X$ *by the formula*

$$\mathbb{D}_X(M) := \underline{\mathrm{Hom}}_{\mathrm{N}}(M, f^!(\omega_S)).$$

Then, for any geometric Nori motive M, the canonical map $M \to \mathbb{D}_X(\mathbb{D}_X(M))$ is an isomorphism.

Proof. From the compatibility of the Nori realisation with the six functors (Theorem 4.2.1) and the same result for étale motives [CD16, Theorem 6.2.17], we deduce the result when $\omega_S = \Lambda_S$. This implies the result in general using [CD19, Proposition 4.4.22]. □

Remark 4.2.4. A priori with rational coefficients we have two Verdier duality functors: the one constructed using the above formula, and the one obtained using the universal property of Nori motives. Thankfully, these two functors are canonically isomorphic by Terenzi's [Ter24, Proposition 5.19]. The proof uses the existence of a Nori realisation of rational étale motives.

4.2.2 The ℓ-adic realisation

In this paragraph, we define the ℓ-adic realisation on Nori motives.

Proposition 4.2.5. *Let Λ be a commutative ring, let X be a qcqs $\mathbb{Q}$-scheme and let ℓ be a prime number. The maps*

$$\mathrm{D}(X_{\text{ét}}, \Lambda)_{\text{tors}} \xrightarrow{\rho_!} \mathrm{DM}^{\text{ét}}(X, \Lambda)_{\text{tors}} \xrightarrow{\rho_{\mathrm{N}}} \mathrm{DN}(X, \Lambda)_{\text{tors}}$$

$$\mathrm{D}(X_{\text{ét}}, \Lambda)^\wedge_\ell \xrightarrow{(\rho_!)^\wedge_\ell} \mathrm{DM}^{\text{ét}}(X, \Lambda)^\wedge_\ell \xrightarrow{(\rho_{\mathrm{N}})^\wedge_\ell} \mathrm{DN}(X, \Lambda)^\wedge_\ell$$

are equivalences.

Proof. The second row of equivalences follows from the first and from the identification of p-torsion objects with p-complete objects. By Lemma A.6.16, it suffices to prove the first identification after tensoring with $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}$. The proposition then follows from the rigidity theorem [BH21, Theorem 3.1] (see Theorem 1.2.17) and from the fact that the canonical map $\mathbb{1}_X/n \to \mathscr{N}_X/n$ is an equivalence. □

Definition 4.2.6. Let X be a qcqs $\mathbb{Q}$-scheme. The big ℓ-adic realisation of Nori motives is the functor

$$R^{\text{big}}_\ell \colon \mathrm{DN}(X, \mathbb{Z}) \xrightarrow{(-)^\wedge_\ell} \mathrm{DN}(X, \mathbb{Z})^\wedge_\ell \xrightarrow{\sim} \mathrm{D}(X_{\text{ét}}, \mathbb{Z})^\wedge_\ell.$$

Remark 4.2.7. As in [CD16, Section 7.2], we can endow $\mathrm{D}(-_{\text{ét}}, \mathbb{Z})^\wedge_\ell$ with the six functors formalism and show that the big ℓ-adic realisation is compatible with the six functors.

We now construct the ℓ-adic realisation for constructible motives. It will land into the category of constructible sheaves of [BS15b, HRS23].

Construction 4.2.8. If X is a qcqs finite-dimensional $\mathbb{Q}$-scheme, we have maps:

$$\mathrm{DN}(X, \mathbb{Z}) \xrightarrow{R^{\text{big}}_\ell} \mathrm{D}(X_{\text{ét}}, \mathbb{Z})^\wedge_\ell \xrightarrow{-\otimes \mathbb{Z}/\ell^n\mathbb{Z}} \mathrm{D}(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

The image of $\mathrm{DN}_{\text{gm}}(X, \mathbb{Z})$ through the latter is contained in the subcategory of constructible sheaves $\mathrm{D}_{\text{cons}}(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ (that is those complexes whose restriction to any open affine subset are dualisable on a finite stratification). The change of sites from the étale to the pro-étale site then induces an equivalence

$$\mathrm{D}_{\text{cons}}(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to \mathrm{D}_{\text{cons}}(X_{\text{proét}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

using [HRS23, Proposition 7.1]. We therefore get a map

$$\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \lim \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z}/\ell^n\mathbb{Z}).$$

[HRS23, Proposition 5.1] ensures that the right-hand side is equivalent to $\mathrm{D}_{\mathrm{cons}}(X_{\text{proét}},\mathbb{Z}_\ell)$ the ∞-category of constructible proétale sheaves (again, those complexes whose restriction to any open affine subset are dualisable on a finite stratification but in the pro-étale world). The categories $\mathrm{D}_{\mathrm{cons}}((-)_{\text{proét}},\mathbb{Z}_\ell)$ are endowed with the six functors over quasi-excellent base schemes using [BS15b, Section 6.7]; these are defined by passing to the limit from the functors over $\mathrm{D}_{\mathrm{cons}}((-)_{\text{ét}},\mathbb{Z}/\ell^n\mathbb{Z})$ which are induced from those over $\mathrm{D}(-_{\text{ét}},\mathbb{Z}/\ell^n\mathbb{Z})$. This yields a map

$$R_\ell\colon \mathrm{DN}_{\mathrm{gm}}(-,\mathbb{Z}) \to \mathrm{D}_{\mathrm{cons}}((-)_{\text{ét}},\mathbb{Z}_\ell)$$

compatible with the six functors over quasi-excellent schemes which we call the *ℓ-adic realisation.*

Finally, if Λ is a localisation of a ring integral and flat over $\mathbb{Z}$, we can tensor the above functor by Perf_Λ over $\mathrm{Perf}_\mathbb{Z}$ to get the ℓ-adic realisation

$$R_\ell\colon \mathrm{DN}_{\mathrm{gm}}(-,\Lambda) \to \mathrm{D}_{\mathrm{cons}}((-)_{\text{proét}},\mathbb{Z}_\ell) \otimes_{\mathrm{Perf}_\mathbb{Z}} \mathrm{Perf}_\Lambda \xrightarrow{\sim} \mathrm{D}_{\mathrm{cons}}((-)_{\text{proét}},\Lambda_\ell),$$

where the last equivalence is [HS23, Corollary 2.3].

This functor is compatible with the functors of type $\otimes$, f^* and $f_!$ and when we restrict to quasi-excellent base schemes and to finite type morphisms, it is compatible with the functors of type f_* and $f^!$. Indeed, the same is true for the ℓ-adic realisation of étale motives and therefore this follows from Theorem 4.2.1.

Proposition 4.2.9. *Let Λ be a localisation of a flat integral extension of $\mathbb{Z}$. Let X be a qcqs $\mathbb{Q}$-scheme of finite dimension.*

1. *If Λ is a $\mathbb{Q}$-algebra, and ℓ is a prime number, then the ℓ-adic realisation functor*

$$R_\ell\colon \mathrm{DN}_{\mathrm{gm}}(X,\Lambda) \to \mathrm{D}_{\mathrm{cons}}(X_{\text{proét}},\Lambda_\ell)$$

   is conservative.

2. *Otherwise, the family of the R_ℓ where $\ell \notin \Lambda^\times$ is conservative.*

Proof. We first show the first assertion. By continuity (Theorem 3.1.29), we can reduce to the case where X is a finite type $\mathbb{Q}$-scheme. As Λ is a $\mathbb{Q}$-algebra, we have by an equivalence

$$\mathrm{DN}_{\mathrm{gm}}(X,\Lambda) \xrightarrow{\sim} \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q})) \otimes_{\mathrm{Perf}_\mathbb{Q}} \mathrm{Perf}_\Lambda.$$

Moreover using the description of mapping spectra in tensor products provided by [HL23, Proposition 3.5.5] we see that if the functor

$$R_\ell\colon \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Q}) \to \mathrm{D}_{\mathrm{cons}}(X_{\text{proét}},\mathbb{Q}_\ell)$$

is conservative, so is the functor of the proposition. Thus this follows from [IM22, Proposition 6.11].

We now prove the second assertion. As the family $\{-\otimes\mathbb{Q},\ -\otimes\mathbb{Z}/\ell\mathbb{Z} \mid \ell \notin \Lambda^\times\}$ is conservative, it suffices to show that if $R_\ell(M/\ell)$ vanishes, then so does M/ℓ for M a geometric motive. This amounts to the rigidity theorem. □

4.3 Constructible motives: abelian categories.

In this section we define t-structure on integral Nori motives. Although the construction could be done in the more general setting of motives with coefficients in a regular ring, we choose in this manuscript to only mention integral coefficients.

4.3.1 Relative Nori motives as a pullback and the ordinary t-structure

Until further notice, k is an algebraically closed subfield of $\mathbb{C}$. Recall that for any finite type k-scheme X Ayoub constructed a Betti realisation ρ_{B} of $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ with values in the derived category $\mathrm{D}(\mathrm{Shv}(X^{\mathrm{an}},\mathbb{Z}))$ of sheaves of abelian groups on the analytification X^{an}. It factors through the indization $\mathrm{IndD}^b_c(X^{\mathrm{an}},\mathbb{Z})$ of the category of cohomologically constructible sheaves. Moreover as in Construction 3.1.12, it further factors through modules over an algebra $\mathscr{B}_X$ in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$, pulled back from $\mathrm{Spec}(k)$. We will denote by

$$\mathrm{DB}(X,\mathbb{Z}) := \mathrm{Mod}_{\mathscr{B}_X}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Z}))$$

the category of modules. It has a map to $\mathrm{D}(\mathrm{Shv}(X^{\mathrm{an}},\mathbb{Z}))$, and we still denote by

$$\rho_{\mathrm{B}}\colon \mathrm{DM}^{\text{ét}}(X,\mathbb{Z}) \to \mathrm{DB}(X,\mathbb{Z})$$

the induced realisation functor, that we may call the *refined realisation functor*. Ayoub proves that the ∞-category $\mathrm{DB}(X,\mathbb{Z})$ is endowed with the t-structure restricted from the ordinary t-structure on $\mathrm{IndD}^b_c(X^{\mathrm{an}},\mathbb{Z})$.

The realisation to Nori motives

$$\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(-,\mathbb{Q}) \to \mathrm{DN}(-,\mathbb{Q})$$

factors the refined Betti realisation of étale motives, hence there exists a map of algebra

$$\varphi_X\colon \mathscr{N}_{X,\mathbb{Q}} \to \mathscr{B}_{X,\mathbb{Q}}$$

compatible with pullback by finite type morphisms.

Proposition 4.3.1. *There exist a natural map of algebras $\mathscr{N}_X \to \mathscr{B}_X$ giving back φ_X after rationalisation. Moreover, it induces a pullback diagram*

$$\begin{array}{ccc} \mathscr{N}_X & \longrightarrow & \mathscr{B}_X \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{N}_{X,\mathbb{Q}} & \longrightarrow & \mathscr{B}_{X,\mathbb{Q}} \end{array}.$$

Proof. Let A be the pullback

$$\begin{array}{ccc} A & \longrightarrow & \mathscr{B}_X \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{N}_{X,\mathbb{Q}} & \longrightarrow & \mathscr{B}_{X,\mathbb{Q}} \end{array}.$$

Thanks to Artin's Comparison Theorem [SGA72, Exposé XI, Théorème 4.4 & Exposé XVI, Théorème 4.1] (see also Step 1 of the proof of [Ayo22, Lemma 1.107]), the Hasse arithmetic fracture square for $\mathscr{B}_X$ is of the form:

$$\begin{array}{ccc} \mathscr{B}_X & \longrightarrow & \lim_n \mathbb{1}_X/n \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{B}_{X,\mathbb{Q}} & \longrightarrow & (\lim_n \mathbb{1}_X/n) \otimes \mathbb{Q} \end{array},$$

so that by pasting pullbacks, A also fits in the pullback square

$$\begin{array}{ccc} A & \longrightarrow & \lim_n \mathbb{1}_X/n \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{N}_{X,\mathbb{Q}} & \longrightarrow & (\lim_n \mathbb{1}_X/n) \otimes \mathbb{Q} \end{array},$$

where we recognise the Hasse fracture square of $\mathscr{N}_{\mathbb{Z}}$ (the arrow are the correct ones because the equivalence $\mathscr{N}_X/n \xrightarrow{\sim} \mathscr{B}_X/n$ factors the equivalence $\mathbb{1}_X/n \xrightarrow{\sim} \mathscr{B}_X/n$.) □

Therefore we have a canonical functor

$$\mathrm{DN}(-,\mathbb{Z}) \to \mathrm{IndD}^b_c(-,\mathbb{Z}).$$

Corollary 4.3.2. *Relative Nori motives with integral coefficients (even relative to a non algebraically closed field) have a Betti realisation*

$$\mathrm{DN}(-,\mathbb{Z}) \to \mathrm{D}((-)^{\mathrm{an}},\mathbb{Z})$$

which is compatible with the six functors.

Proposition 4.3.3. *The square*

$$\begin{array}{ccc} \mathrm{DN}(-,\mathbb{Z}) & \longrightarrow & \mathrm{DB}(-,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{DN}(-,\mathbb{Q}) & \longrightarrow & \mathrm{DB}(-,\mathbb{Q}) \end{array}$$

is cartesian in $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}}))$.

Proof. It suffices to check the proposition after evaluating at a finite type k-scheme X. If we let $P(X)$ be the pullback of the proposition, there is a functor $\mathrm{DN}(X,\mathbb{Z}) \to P(X)$. This functor preserves compact objects, thus its right adjoint commutes with colimits, whence commutes with $-\otimes\mathbb{Q}$ so that we may use Lemma A.6.16. □

Corollary 4.3.4. *Let k be any subfield of $\mathbb{C}$ and let X be a finite type k-scheme. The Betti realisation*

$$\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{D}^b_c(X^{\mathrm{an}},\mathbb{Z})$$

is conservative.

Proof. Indeed, the functor $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X \times_k \overline{k},\mathbb{Z})$ is conservative by continuity and Galois descent (if $M_{\overline{k}} = 0$, there exists a finite extension L of k such that M_L is zero, and the by Galois descent the functor $(-)_L$ is conservative, thus $M = 0$). Then by Proposition 4.3.3 the functor $\mathrm{DN}_{\mathrm{gm}}(X_{\overline{k}},\mathbb{Z}) \to \mathrm{D}^b_c(X^{\mathrm{an}},\mathbb{Z})$ is conservative: if $R_{\mathrm{B}}(M) = 0$, then by conservativity of the realisation rationally we have that $M \otimes \mathbb{Q} = 0$, thus both $R_{\mathrm{B}}(M)$ and $M \otimes \mathbb{Q}$ are zero thus $M = 0$ in the pullback. □

The pullback square of Proposition 4.3.3 allows us to endow the $\mathrm{DN}(X,\mathbb{Z})$ with a t-structure.

On the other hand, we have $\mathrm{DN}(X,\mathbb{Q}) = \mathrm{Ind}\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q}))$. In Definition 1.2.23, we defined a t-structure on $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q}))$ that is called the *constructible t-structure* in Chapter 2 but that we will call the *ordinary t-structure* in this chapter for consistency. Denote its heart by $\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q})$, then by Corollary 2.1.22, the natural map

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q})) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q}))$$

is an equivalence. Furthermore, the functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q})) \to \mathrm{DB}(X,\mathbb{Q})$ is t-exact with the ordinary t-structure on both sides. Therefore, using Proposition 1.1.9, we get

Proposition 4.3.5. *Let X be a finite type k-scheme, there is a t-structure on $\mathrm{DN}(X,\mathbb{Z})$ such that the rationalisation, endowed with its ordinary t-structure, and the Betti realisation are t-exact. Furthermore, we have a pullback square:*

$$\begin{array}{ccc} \mathrm{DN}(X,\mathbb{Z})^{\heartsuit} & \longrightarrow & \mathrm{DB}(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Ind}\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q}) & \longrightarrow & \mathrm{DB}(X,\mathbb{Q}). \end{array}$$

Remark 4.3.6. Using the ideas of this section, we can define a category of Nori motivic spectra which is akin to the category $\mathcal{SH}^{\text{ét}}(X)$ of étale motivic spectra when X is a scheme of finite type over an *algebraically closed* field of characteristic 0. First, let $\mathscr{B}^{\mathbb{S}}_X$ be the algebra in $\mathcal{SH}^{\text{ét}}(X)$ representing Betti cohomology with $\mathbb{S}$-coefficients. Once again, Ayoub constructed in [Ayo22, Theorem 1.93] an equivalence $\mathrm{Mod}_{\mathscr{B}^{\mathbb{S}}_X}(\mathcal{SH}^{\text{ét}}(X)) \to \mathrm{Ind}\mathrm{D}_{\mathrm{geo}}(X^{\mathrm{an}},\mathbb{S})$, where $\mathrm{D}_{\mathrm{geo}}(X^{\mathrm{an}},\mathbb{S})$ is the full subcategory of the category $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\mathbb{S})$ of constructible analytic spectral sheaves, made of objects of geometric origin and is endowed with an induced t-structure.

By analogy with Proposition 4.3.1, we can define $\mathscr{N}^{\mathbb{S}}_X$ to fit in the pullback

$$\begin{array}{ccc} \mathscr{N}^{\mathbb{S}}_X & \longrightarrow & \mathscr{B}^{\mathbb{S}}_X \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{N}_{X,\mathbb{Q}} & \longrightarrow & \mathscr{B}_{X,\mathbb{Q}} \end{array}$$

and then define $\mathcal{SH}^{\text{ét}}_{\mathrm{Nori}}(X) := \mathrm{Mod}_{\mathscr{N}^{\mathbb{S}}_X}(\mathcal{SH}^{\text{ét}}(X))$. We now claim that the square

$$\begin{array}{ccc} \mathcal{SH}^{\text{ét}}_{\mathrm{Nori}}(X) & \longrightarrow & \mathrm{Mod}_{\mathscr{B}^{\mathbb{S}}_X}(\mathcal{SH}^{\text{ét}}(X)) \\ \downarrow & & \downarrow \\ \mathrm{DN}(-,\mathbb{Q}) & \longrightarrow & \mathrm{DB}(X,\mathbb{Q}) \end{array}$$

is cartesian. The full faithfulness of the map $\Psi_X \colon \mathcal{SH}^{\text{ét}}_{\text{Nori}}(X) \to P(X)$ (where $P(X)$ denotes the pullback) follows from the general claim that if $A = B \times_D C$ is a pullback of commutative algebra objects in a stable ∞-category $\mathcal{C}$, the map $\mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_B(\mathcal{C}) \times_{\mathrm{Mod}_D(\mathcal{C})} \mathrm{Mod}_C(\mathcal{C})$ is fully faithful which can be seen using a direct computation of mapping spectra. To show that the map Ψ_X is essentially surjective, note that $P(X)$ is endowed with an ordinary t-structure by Proposition 1.1.9 so that it suffices to reach the objects of its heart. We claim that it is the case. Indeed the functor

$$- \otimes_{\mathbb{S}} \mathbb{Z} \colon \mathrm{IndD}_{\text{geo}}(X^{\text{an}}, \mathbb{S}) \to \mathrm{DB}(X, \mathbb{Z})$$

yields the equivalence $\tau^{\geqslant 0}(- \otimes_{\mathbb{S}} \mathbb{Z})$ between the hearts, therefore we have an equivalence $P(X)^{\heartsuit} \to \mathrm{DN}(X, \mathbb{Z})^{\heartsuit}$. Using that at the generic point η for a projective smooth morphism f, the object $f_*\mathbb{Z}$ is the sum in $\mathrm{DN}(\eta, \mathbb{Z})$ of its cohomology objects (by Deligne's criterion for degeneracy of spectral sequences [Del68, Théorème 1.5]), one can prove the claim at the generic point, and then localisation enables one to prove that Ψ_X is an equivalence in general. Hence we have an ordinary motivic t-structure on $\mathcal{SH}^{\text{ét}}_{\text{Nori}}(X)$ whose heart is the same as that of $\mathrm{DN}(X, \mathbb{Z})$.

4.3.2 The ordinary t-structure on geometric Nori motives

Let X be a finite dimensional qcqs $\mathbb{Q}$-scheme. In this section we construct an ordinary t-structure on $\mathrm{DN}_{\text{gm}}(X, \mathbb{Z})$. By continuity and because a filtered colimit of stable ∞-categories with t-structures and t-exact functors has a natural t-structure, we can assume that X is a finite type $\mathbb{Q}$-scheme. In this case we may use the punctual gluing of t-structures due to Vaish [Vai20, Proposition 3.8].

First, we note that there is a t-structure on $\mathrm{DN}_{\text{gm}}(k, \mathbb{Z})$ for any subfield k of $\mathbb{C}$.

Proposition 4.3.7. *Let k be a subfield of $\mathbb{C}$. There exists a t-structure on $\mathrm{DN}_{\text{gm}}(k, \mathbb{Z})$ such that the Betti realisation is t-exact.*

Proof. Indeed by Theorem 4.1.11 we know that there is an equivalence $\mathrm{DN}_{\text{gm}}(k, \mathbb{Z}) \simeq \mathrm{D}^b(\mathcal{M}_{\text{Nori}}(k, \mathbb{Z}))$ compatible with the Betti realisation. □

In order to use Vaish's result, we have to verify the condition he calls "continuity of the t-structure", which is the purpose of the next proposition. Let us first recall how the t-structure is defined:

Definition 4.3.8. For each $x \in X$, denote by $i_x \colon \mathrm{Spec}(\kappa(x)) \to X$ the inclusion of x in X. An object $M \in \mathrm{DN}_{\text{gm}}(X, \mathbb{Z})$ is called

1. *punctually connective* if for each $x \in X$ we have $i_x^* M \in \mathrm{D}^b(\mathcal{M}_{\text{Nori}}(\kappa(x), \mathbb{Z}))^{\leqslant 0}$
2. *punctually 1-coconnective* if for each $x \in X$ we have $i_x^! M \in \mathrm{D}^b(\mathcal{M}_{\text{Nori}}(\kappa(x), \mathbb{Z}))^{\geqslant 1}$.

We denote by $\mathrm{DN}_{\text{gm}}(X, \mathbb{Z})^{\leqslant 0}$ and $\mathrm{DN}_{\text{gm}}(X, \mathbb{Z})^{\geqslant 1}$ the associated full subcategories.

Proposition 4.3.9. *Let X be a finite type $\mathbb{Q}$-scheme. Let $x \in X$ and let $M_x \in \mathrm{D}^b(\mathcal{M}(\kappa(x), \mathbb{Z}))$.*

1. *If $M \in \mathrm{D}^b(\mathcal{M}(\kappa(x), \mathbb{Z}))^{\leqslant 0}$ there exists a nonempty open subset U of $\overline{\{x\}}$ and an object $M \in \mathrm{DN}_{\text{gm}}(U, \mathbb{Z})^{\leqslant 0}$ such that $i_x^* M = M_x$.*

2. *If* $M \in \mathrm{D}^b(\mathcal{M}(\kappa(x),\mathbb{Z}))^{\geqslant 1}$ *there exists a nonempty open subset* U *of* $\overline{\{x\}}$ *and an object* $M \in \mathrm{DN}_{\mathrm{gm}}(U,\mathbb{Z})^{\geqslant 1}$ *such that* $i_x^! M = M_x$.

Proof. In both cases we can assume that x is a generic point of X and that X is irreducible, and even smooth up to replacing X by one of its dense open subsets. In this case both i_x^* and $i_x^!$ agree. Thus the proof of the two cases are almost the same and we only deal with the first point. By continuity, we know that there exist some $U_0 \subset X$ nonempty and open, and $M_0 \in \mathrm{DN}_{\mathrm{gm}}(U_0,\mathbb{Z})$ such that $i_x^* M_0 = M_x$. Now the Betti realisation being conservative, we know that there exists some integer $C \in \mathbb{N}^*$ such that if p is a prime verifying that the multiplication by p is zero on M_0, then $p \mid C$. Moreover, there is some other integer $a > 0$ such that the Betti realisation of M_0 lives in degrees $[-a,a]$. Because $\mathbb{Z}_\ell$ is flat over $\mathbb{Z}$ and the functors $\mathrm{D}_c^b(X_{\text{ét}},\mathbb{Z}_\ell) \to \mathrm{D}_c^b(X^{\mathrm{an}},\mathbb{Z}_\ell)$ are conservative and t-exact, we see that for each prime number ℓ, the ℓ-adic realisation of M_0 is concentrated in degrees $[-a,a]$. Let U_1 be a nonempty open subset of U_0 on which M_0 is dualisable. For each $j \in [1,a]$ and each prime number ℓ dividing C, because $i_x^* \mathrm{H}^j(R_\ell(M_1))$ is zero and $\mathrm{H}^j(R_\ell(M_1))$ is a local system on $(U_1)_{\text{ét}}$, there exists a nonempty open subset $U_{j,\ell}$ of U_1 such that the restriction to $U_{j,\ell}$ of $\mathrm{H}^j(R_\ell(M_1))$ is zero. We let U be the intersection of all of those $U_{j,\ell}$ and M be the restriction of M_1 to U. Then we see that $R_{\mathrm{B}}(M)$ is connective, thus M is punctually connective, as wanted. □

Theorem 4.3.10. *The conditions of Definition 4.3.8 define a t-structure on* $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$. *Furthermore for any finite dimensional qcqs* $\mathbb{Q}$*-scheme* X *this induces a t-structure on the stable* ∞*-category* $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ *such that all* ℓ*-adic realisations are t-exact.*

Proof. The t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is given by the above proposition together with [Vai20, Proposition 3.8]. For a general X one uses Noetherian approximation and a filtered colimit of t-structures. □

Definition 4.3.11. The category of *ordinary Nori motives* $\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Z})$ is the heart of the t-structure given by Theorem 4.3.10. We call the latter the *ordinary t-structure*.

Remark 4.3.12. As the ℓ-adic realisation are jointly conservative by Proposition 4.2.9, any t-exactness property of the ordinary t-structure on ℓ-adic sheaves transfers to the ordinary t-structure on Nori motives.

Remark 4.3.13. Let k be a subfield of $\mathbb{C}$ and let X be a k-variety. Arapura considered in [Ara13] an abelian category of motivic sheaves modelled on constructible sheaves. By construction, his category $\mathcal{M}_{\mathrm{Arapura}}(X,\mathbb{Z})$ has an universal property, so that there exists a faithful exact functor $\mathcal{M}_{\mathrm{Arapura}}(X,\mathbb{Z}) \to \mathcal{M}_{\mathrm{ord}}(X,\mathbb{Z})$ compatible with the Betti realisations. We do not know much about this functor, except that it is an equivalence for $X = \mathrm{Spec}(k)$.

4.3.3 The perverse t-structure

In this subsection, we define the perverse t-structure on Nori motives. The definition is very similar to that of the perverse t-structure on constructible complexes of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules given in [BBD82, Section 2.2]. We also prove that under some assumptions the derived category of perverse Nori motives is $\mathrm{DN}_{\mathrm{gm}}$, which is in the spirit of [Bei87b].

Proposition 4.3.14. *Let X be an excellent finite-dimensional $\mathbb{Q}$-scheme endowed with a dimension function δ*[1]*. We let*

$$ {}^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \mid \forall x \in X, i_x^*(M) \leqslant -\delta(x)\},$$

$$ {}^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \mid \forall x \in X, i_x^!(M) \geqslant -\delta(x)\}.$$

Then, the pair $({}^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0}, {}^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0})$ defines a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ such that the ℓ-adic realisation functor

$$R_\ell \colon \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{proét}},\mathbb{Z}_\ell)$$

is t-exact when the right-hand side is endowed with its perverse t-structure.

Proof. The proof is the same as in [BBD82, Section 2.2] but we include a sketch for the reader's convenience. Consider couples $(\mathcal{S},\mathcal{L})$ where $\mathcal{S}$ is a finite stratification of X by locally closed connected regular subschemes, and $\mathcal{L}$ is the data, for each stratum Z of $\mathcal{S}$ of a finite set $\mathcal{L}(Z)$ of dualisable Nori motives on Z. We denote by $\mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$ the full subcategory of $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ made of $(\mathcal{S},\mathcal{L})$-constructible Nori motives, that is the motives M such that, for each stratum Z of $\mathcal{S}$ and each integer n, $\mathrm{H}^n(M|_Z)$ is isomorphic to an iterated extension of elements of $\mathcal{L}(Z)$.

Any such couple can be refined into a couple $(\mathcal{S},\mathcal{L})$ such that if Z is a stratum in $\mathcal{S}$ and N belongs to $\mathcal{L}(Z)$, letting $j\colon Z \to X$ be the inclusion, the Nori motive $j_*(N)$ is $(\mathcal{S},\mathcal{L})$-constructible.

This assumption ensures that we can define a t-structure on $\mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$ by gluing (in the sense of[BBD82, Section 1.4])the t-structure on each stratum Z by $-\delta(Z)$. If $(\mathcal{S},\mathcal{L})$ refines $(\mathcal{S}',\mathcal{L}')$, the t-structure on $\mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$ induces the t-structure on $\mathrm{DN}_{(\mathcal{S}',\mathcal{L}')}(X,\mathbb{Z})$; this is a consequence of the absolute purity property for Nori motives:

Lemma 4.3.15. *Let $i\colon Z \to X$ be a closed immersion everywhere of codimension c between regular schemes and let M be a dualisable object of $\mathrm{DN}(Z,\mathbb{Z})$, then the natural map*

$$i^*(M)(-c)[-2c] \to i^!(M)$$

is an equivalence.

Proof. This follows from the same statement for ℓ-adic sheaves [ILO14, Exposé XVI, Théorème 3.1.1] and from the conservativity of the ℓ-adic realisation Proposition 4.2.9 and its compatibility with the six functors over quasi-excellent schemes Construction 4.2.8. □

To finish the proof of the proposition, note that

$$\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) = \mathrm{colim}_{(\mathcal{S},\mathcal{L})}\ \mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$$

and therefore we get a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$ by Proposition 1.1.9. The t-exactness of the realisation follows from the fact that the perverse t-structure is defined in the same way by gluing along stratifications in [BBD82, Section 2.2]. The characterisation of the positive and negative parts of the t-structure then follows from the t-exactness of the realisation combined with its conservativity and the analogous characterisation of the positive and negative parts of the perverse t-structure on $\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{proét}},\mathbb{Z}_\ell)$ which is [BBD82, 2.2.12]. □

[1] See [ILO14, Exposé XIV, Définition 2.1.8] for the definition of a dimension function.

Remark 4.3.16. Along the way we proved that if X is regular, the perverse t-structure on the subcategory $\mathrm{DN}_{\mathrm{lisse}}(X,\mathbb{Z})$ of $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ made of dualisable objects is the ordinary t-structure shifted by $-\delta(X)$.

This result also implies that any t-exactness property of the perverse t-structure on ℓ-adic sheaves is true for the perverse t-structure on Nori motives. In particular, the Lefschetz Hyperplane Theorem (also known as Artin Vanishing) is true for Nori motives. The Hard Lefschetz theorem for Nori motives is also true using similar arguments.

Definition 4.3.17. Let X be an excellent finite-dimensional $\mathbb{Q}$-scheme endowed with a dimension function. The *abelian category of perverse Nori motives* is the heart of the perverse t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$. We denote it by $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$.

Remark 4.3.18. One of the main features of the perverse t-structure with rational coefficients is that it is preserved by Verdier duality. This is famously not the case with integral coefficients (note for instance that $\mathrm{map}_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}[-1]$). However, if ℓ is a prime number there is a t-structure on $\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{proét}},\mathbb{Z}_\ell)$ that is exchanged with the perverse t-structure by duality (see [BBD82, 4.0 (a)]).

We can mimic its definition to get a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ as follows.Define

$$ {}^{\mathrm{ord}^+}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \mid M \leqslant_{\mathrm{ord}} 1 \text{ and } \mathrm{H}^1(M)\otimes_{\mathbb{Z}}\mathbb{Q} = 0\}$$

$$ {}^{\mathrm{ord}^+}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \mid M \geqslant_{\mathrm{ord}} 0 \text{ and } \mathrm{H}^0(M) \text{ is torsion-free}\}.$$

This is a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$; then by gluing as before this t-structure along stratifications (using the same shifts) we get a t-structure $({}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0}, {}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0})$ whose heart we denote by $\mathcal{M}^+_{\mathrm{perv}}(X,\mathbb{Z})$.

Because the six functors are compatible with the ℓ-adic realisation and because the same statement is true after realisation by [BBD82, 3.3.4], the Verdier duality functor $\mathbb{D}_X$ of Proposition 4.2.3 exchanges the subcategory ${}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0}$ (resp. ${}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0}$) with ${}^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0}$ (resp. ${}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0}$) and therefore it exchanges $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ with $\mathcal{M}^+_{\mathrm{perv}}(X,\mathbb{Z})$.

4.3.4 Universal motives and the derived category of perverse Nori motives

By the universal property of the bounded derived category, there is a canonical realisation functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}). \tag{4.3.18.1}$$

This functor is an equivalence after tensorisation by $\mathrm{Perf}_{\mathbb{Q}}$. We have the following rigidity result to compute its torsion:

Proposition 4.3.19. *Let X be an excellent finite-dimensional $\mathbb{Q}$-scheme endowed with a dimension function δ. Let n be a prime number. Consider the composition*

$$\iota_n \colon \mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z}) \subseteq \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z}) \xrightarrow{f} \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z})_{tors} \xrightarrow{\iota} \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})_{tors}$$

where $\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$ is the category of étale perverse sheaves and f is the forgetful functor. It lands in the abelian category $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})[n]$ of n-torsion objects in $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ and induces an equivalence

$$\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})[n].$$

Proof. By Proposition 4.2.5 the last functor is an equivalence. Moreover as $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})_{tors}$ is the kernel of the perverse t-exact functor $-\otimes_{\mathbb{Z}}\mathbb{Q}$, it is endowed with a perverse t-structure, whose heart is the abelian category of torsion perverse motives.

First note that the forgetful functor f is t-exact, because if commutes with the gluing functors, and because any perfect complex of $\mathbb{Z}/n\mathbb{Z}$-modules is a perfect complex of $\mathbb{Z}$-modules.

This implies that the functor ι_n lands in $\mathcal{M}_{\mathrm{perv}}(\mathbb{Z})[n]$. Furthermore, the restriction of the forgetful functor to the heart affords a left adjoint, namely ${}^{\mathrm{p}}\mathrm{H}^0(-\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z})$; it is in fact the underived tensor product with $\mathbb{Z}/n\mathbb{Z}$ that sends a perverse sheaf K to the cokernel of the map $K\xrightarrow{\times n}K$. With this, it is clear that if M belongs to $\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$, we have

$$\mathrm{H}^0(f(M)\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z})=M$$

and for any object of $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z})^{\heartsuit}_{tors}$ which is of n-torsion, we have

$$f(\mathrm{H}^0(N\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}))=N.$$

Whence, the functor ι_n indeed induces an equivalence. □

Unfortunately, we do not know if the abelian category $\mathcal{M}_{\mathrm{perv}}$ has enough torsion-free sheaves. If rhis was true, we could use Proposition A.6.20 to conclude that the functor (4.3.18.1) is fully faithful thanks to Proposition 4.3.19 and Beilinson's Theorem on the derived category of perverse sheaves [Bei87b]; this would then imply that this functor is an equivalence. We have a partial answer in that direction, that requires to introduce the universal category of perverse motives. Let k be a subfield of the complex numbers (a more general field can be considered, but the following results for those fields can be reduced to subfields of $\mathbb{C}$ by the continuity properties of perverse categories considered along fields).

Construction 4.3.20. Let X be a finite type k-scheme. Consider the cohomological functor

$${}^{\mathrm{p}}\mathrm{H}^0\circ\rho_{\mathrm{B}}\colon \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})\to\mathrm{Perv}(X^{\mathrm{an}},\mathbb{Z})$$

where $\mathrm{Perv}(X^{\mathrm{an}},\mathbb{Z})$ is the category of analytic perverse sheaves and ${}^{\mathrm{p}}\mathrm{H}^0$ is the 0-th perverse cohomology functor. As in Definition 1.2.23, we can consider the universal abelian factorisation $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ of this functor through a faithful exact functor $r_{\mathrm{B}}\colon\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})\to\mathrm{Perv}(X^{\mathrm{an}},\mathbb{Z})$. We thus have a universal homological functor

$$\mathrm{H}_{\mathrm{univ}}\colon \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})\to\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}).$$

Because the Betti realisation of perverse Nori motives is faithful exact (as the realisation from $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is conservative), we actually have a finer factorisation:

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})\xrightarrow{\mathrm{H}_{\mathrm{univ}}}\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})\xrightarrow{\mathrm{comp}}\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$$
$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})\xrightarrow[{}^{\mathrm{p}}\mathrm{H}^0\circ\rho_{\mathrm{N}}]{}\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$$

with comp a faithful exact functor.

Lemma 4.3.21. *Let X be a finite type k-scheme. The functor*

$$\mathrm{comp}\otimes_{\mathbb{Z}}\mathbb{Q}\colon\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$$

is an equivalence.

Proof. The definition of $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q})$ is $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Q})$. Thus we have to show that $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is the universal category for ${}^{\mathrm{p}}\mathrm{H}^0\circ\rho_{\mathrm{B}}\otimes_{\mathbb{Z}}\mathbb{Q}$. This follows easily from the construction of the universal category together with the fact that the rationalisation of the abelian hull of an additive category if the abelian hull of the rationalisation. □

For each nonzero integer n, we have an Artin motive functor

$$\iota\colon \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})$$

such that the composition with the realisation is perverse t-exact and conservative. Thus it induces a faithful exact functor

$$i\colon \mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\to \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})[n].$$

The following result is an adaptation of a proof due to Nori ([Fak00]) in the case of fields.

Proposition 4.3.22. *Lt X be a finite type k-scheme. Let n be an integer. The functor*

$$i\colon \mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\to \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})[n]$$

is an equivalence of quasi-inverse the functor

$$\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})[n]\xrightarrow{\mathrm{comp}}\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})[n]\simeq \mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n).$$

Proof. We begin by noting three things:

1. The composite functor $\mathrm{comp}\circ i$ lands in $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})[n]=\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$ and induces the identity of the latter, because we have that $\mathrm{comp}\circ i = {}^{\mathrm{p}}\mathrm{H}^0\circ\rho_{\mathrm{N}}\circ\iota$.

2. The functors comp and i are faithful; thus it suffices to prove that i is essentially surjective. Indeed, if i is essentially surjective, this implies that comp is fully faithful, thus i is also fully faithful.

3. The image of i is stable under subquotients. Indeed, if M belongs to $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})[n]$ and is a subobject of something of the form $i(N)$, then $\mathrm{comp}(M)$ is a subobject of $\mathrm{comp}\circ i(N)=N$. This implies that $i\circ\mathrm{comp}(M)$ and M are both subobjects of $i(N)$ that are isomorphic after applying the faithful functor comp thus they are isomorphic. For the quotients, one uses that if there is a surjection $i(N)\to M$ then the kernel is in the image of i thus the cokernel also.

Note also that if N belongs to $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})$, we can write $N/n=\iota(K)$ for some K in $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$, thus

$$\mathrm{H}_{\mathrm{univ}}(N/n)=\mathrm{H}_{\mathrm{univ}}(\iota(K))\overset{(*)}{=}\mathrm{H}_{\mathrm{univ}}(\iota({}^{\mathrm{p}}\mathrm{H}^0(K)))=i({}^{\mathrm{p}}\mathrm{H}^0(K))$$

is in the image of i, where $(*)$ follows from the perverse t-exactness of the functor

$$\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}).$$

It suffices to show that any object of the form M/nM is in the image of i, because any n-torsion object is of this form. Thus, let M be in $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ and let $\mathrm{H}_{\mathrm{univ}}(N)\to M$ be

a surjective map (this exists by definition of the universal category see Theorem 1.2.22). As there is an exact triangle

$$N \xrightarrow{\times n} N \to N/n$$

in $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbb{Z})$, we have a short exact sequence

$$\mathrm{H}_{\mathrm{univ}}(N) \xrightarrow{\times n} \mathrm{H}_{\mathrm{univ}}(N) \to \mathrm{H}_{\mathrm{univ}}(N/n).$$

Denote by t the image of the map $\mathrm{H}_{\mathrm{univ}}(N) \to \mathrm{H}_{\mathrm{univ}}(N/n)$. Because $\mathrm{H}_{\mathrm{univ}}(N/n)$ is in the image of i we see that t is also in the image of i as the latter is closed under subquotients. Moreover we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{H}_{\mathrm{univ}}(N) & \xrightarrow{\times n} & \mathrm{H}_{\mathrm{univ}}(N) & \longrightarrow & T & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \vdots & & \\
M & \xrightarrow[\times n]{} & M & \longrightarrow & M/nM & \longrightarrow & 0
\end{array}$$

where the map $T \to M/nM$ exists because the composite of the map $M \to M/nM$ with the map $\mathrm{H}_{\mathrm{univ}}(N) \to M$ vanishes on the kernel of the quotient map $\mathrm{H}_{\mathrm{univ}}(N) \to T$. As $\mathrm{H}_{\mathrm{univ}}(N) \to M$ is surjective, this is also the case of the map $T \to M/nM$ thus M/nM is in the image, finishing the proof. □

Thus the functor

$$\mathrm{comp}\colon \mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z}) \to \mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})$$

is an equivalence after being tensored by $\mathbb{Q}$ and induces an equivalence on n-torsion objects for all n. To conclude that it is an equivalence, one could compute some Ext^1 in $\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})$, but this seems to be beyond the scope of our current technology. However, the following conjecture seems more reachable:

Conjecture 4.3.23. *Let X be an algebraic variety over $\mathbb{Q}$. Then the abelian category $\mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z})$ has enough torsion-free object: for every M in $\mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z})$ there exists an epimorphism $N \twoheadrightarrow M$ in $\mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z})$ with N a torsion-free object.*

This conjecture would follow from the equivalence

$$\mathcal{M}_{\mathrm{pairs}}(X, \mathbb{Z}) \xrightarrow{\sim} \mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z}),$$

where $\mathcal{M}_{\mathrm{pairs}}(X, \zeta)$ is the universal category associated to a diagram of very good pairs as in Nori's original construction, as considered by F. Ivorra in [Ivo17] (he was considering rational motives but his construction remains valid for integral sheaves). This equivalence would follows from the fact that Ivorra's realisation functor, the main construction in [Ivo16] is the ${}^{\mathrm{p}}\mathrm{H}^0$ of our functor ρ_{N}. Note that in [Ivo16] the author was only considering smooth varieties but it is not hard to see that a slight modification of his functor (namely, replace the constant sheaf $\mathbb{Q}[-\dim X]$ on X by $\pi_X^! \mathbb{Q}_{\mathrm{Spec}(\mathbb{Q})}$) would give a functor for any variety. Such a conjecture is related to [IM22, Conjecture 2.12], and might be doable with the current technology.

We now explain how Conjecture 4.3.23 implies that both $\mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z}) \simeq \mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})$ for all X but also that $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})) \simeq \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$.

Proposition 4.3.24. *Assume that Conjecture 4.3.23 holds. Then for every field k of characteristic zero and X a finite type k-scheme, the functor* $\mathrm{comp}\colon \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}) \to \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ *is an equivalence, and the functor*

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

is an equivalence.

Proof. By continuity along change of fields, it suffices to prove the theorem when X is of finite type over $\mathbb{Q}$. It suffices to prove that the functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

induced by universal property from the functor

$$\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}) \to \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$$

is fully faithful. Indeed, once it is fully faithful, it will be an equivalence by dévissage because the essential image is then stable under extension. Indeed if $M \in \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$, then there is an short exact sequence

$$0 \to M_{\mathrm{tors}} \to M \to M_{\mathrm{fr}} \to 0$$

in $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ with M_{tors} of torsion and M_{fr} torsion-free. By Proposition 4.3.22 and Proposition 4.3.19 the object M_{tors} is in the image. Moreover by Lemma 4.3.21 there exists $P \in \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ without torsion and a map $P \to M_{\mathrm{fr}}$ which becomes an equivalence after tensorisation by $\mathbb{Q}$. In particular, this map has to be injective, and the cokernel is torsion, thus in the image. By full faithfulness, the object M_{fr} is in the image, whence M is in the image.

The functor is an equivalence after tensorisation by $\mathbb{Q}$, and if we tensor the mapping spectra by $\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}$, using Proposition A.6.20 (we have enough torsion-free objects by hypothesis!) together with [Bei87b] and rigidity, we conclude. Note that this also proves that comp is an equivalence. □

4.3.5 Motivic intersection complex.

Consider an excellent scheme B of dimension 2 or less as well as a regular separated B-scheme of finite type S, endowed with a geometric and $\otimes$-invertible object ω_S in $\mathrm{DN}(S,\mathbb{Z})$ (note that $\omega_S = \mathbb{Z}$ works). We fix X a finite type S-scheme.

From now on, we assume that X is endowed with a bounded below dimension function δ; by shifting, we can assume 0 to be its lower bound. Any non-increasing function $\overline{q}\colon \mathbb{N} \to \mathbb{Z}$, defines a perversity function $q = \overline{q} \circ \delta$ on X along any stratification of X by connected regular and locally closed strata (in the sense of [BBD82, Section 2.1.1]). The construction given in Proposition 4.3.14 for $\overline{q} = -\mathrm{Id}$ works whenever the condition

$$(*)\colon \forall n \in \mathbb{N},\ q(n) - q(n+1) \in \{0,1,2\}$$

is satisfied. It then yields a q-perverse t-structure ${}^q\mathrm{t}$ on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$, whose heart we will denote by $\mathcal{M}_{q-\mathrm{perv}}(X,\mathbb{Z})$ and whose cohomology functors are will be denoted by ${}^q\mathrm{H}^n(-)$.

Definition 4.3.25. Let $\overline{q}\colon \mathbb{N} \to \mathbb{Z}$ be a non-increasing function satisfying condition $(*)$, let $q = \overline{q} \circ \delta$ and let $j\colon U \to X$ be an open immersion. The *q-intermediate extension functor* is the functor

$$^{q}j_{!*}\colon \mathcal{M}_{q-\mathrm{perv}}(U,\mathbb{Z}) \to \mathcal{M}_{q-\mathrm{perv}}(X,\mathbb{Z})$$

defined by

$$^{q}j_{!*}M = \mathrm{Im}(^{q}\mathrm{H}^0(j_!M) \to {}^{q}\mathrm{H}^0(j_*M)).$$

The *q-motivic intersection complex* is the q-perverse Nori motive

$$^{q}\mathrm{IC}_{X,\mathbb{Z}} := {}^{q}j_{!*}(\mathbb{Z}[-q(U)])[q(U)].$$

When X is endowed with a dimension function δ and $q = -\delta$ is the self-dual perversity, we simply refer to those as the intermediate extension functor $j_{!*}$ and the motivic intersection complex $\mathrm{IC}_{X,\mathbb{Z}}$.

Remark 4.3.26. In [Wil12, Wil19] Jörg Wildeshaus constructed a motivic intersection complex in $\mathrm{DM}(X,\mathbb{Q})$ using weights, in several cases (mainly when X is a Shimura variety). In those cases, we expect the complex $\mathrm{IC}_{X,\mathbb{Q}}$ to be the Nori realisation of Wildeshaus' motivic intersection complex up to a shift of $-\dim(X)$. Indeed assuming the existence of the motivic t-structure, this would be true by uniqueness of Wildeshaus' complex [Wil12, Theorem 3.1].

We fix a stratification (X_i) of X by regular and connected locally closed subschemes. Given an integer n, we set U_n to be the union of those strata X_i such that $\delta(X_i) \geqslant n$ and $j_n\colon U_n \to U_{n-1}$ to be the open immersion.

Proposition 4.3.27. *Let $\overline{q}\colon \mathbb{N} \to \mathbb{Z}$ be a non-increasing function satisfying condition $(*)$ and let $q = \overline{q} \circ \delta$. Let $k \geqslant 0$ be an integer and let $j\colon U_k \to X = U_0$ be the open immersion. Then there is a canonical isomorphism of functors*

$$^{q}j_{!*} \simeq \left(\tau^{\leqslant \overline{q}(0)-1}(j_1)_*\right) \circ \cdots \circ \left(\tau^{\leqslant \overline{q}(k-2)-1}(j_{k-1})_*\right) \circ \left(\tau^{\leqslant \overline{q}(k-1)-1}(j_k)_*\right)$$

on $\mathcal{M}_{q-\mathrm{perv}}(U_k)$, with $\tau^{\leqslant i}$ the truncations functors for the ordinary t-structure.

Proof. This is a consequence of [BBD82, Proposition 1.4.23] (see the proof of [BBD82, Proposition 2.1.11]). □

Corollary 4.3.28. *Let $\overline{q},\overline{r}\colon \mathbb{N} \to \mathbb{Z}$ be non-increasing functions and $q = \overline{q} \circ \delta$ and $r = \overline{r} \circ \delta$. Let $k \geqslant 0$ be an integer and let $j\colon U_k \to X = U_0$ be the open immersion. There exists a canonical pairing in $\mathrm{DN}_c(X,\mathbb{Z})$*

$$^{q}j_{!*}(M[-q(U_k)]) \otimes {}^{r}j_{!*}(N[-r[U_k]]) \to {}^{q+r}j_{!*}(M \otimes N[-q(U_k) - r(U_k)])$$

for $M, N \in \mathcal{M}^{\mathrm{rig}}_{\mathrm{ord}}(U_k)$ motivic local systems (that is, objects in the ordinary heart which are dualisable).

Proof. This is a direct consequence of the formula for the intermediate extension given in Proposition 4.3.27, knowing that the functors of the form γ_* are lax-monoidal as right adjoints of monoidal functors and that for any M and N, we have a map

$$\tau^{\leqslant n}(M) \otimes \tau^{\leqslant m}(N) \to \tau^{\leqslant n+m}(M \otimes N)$$

given by the right t-exactness of the tensor product and adjunction. □

This yields a pairing

$$\mathrm{IC}^q_{X,\mathbb{Z}} \otimes \mathrm{IC}^r_{X,\mathbb{Z}} \to \mathrm{IC}^{q+r}_{X,\mathbb{Z}}$$

in $\mathrm{DN}(X,\mathbb{Z})$. Now consider the *top perversity* $\bar{t} = -2\mathrm{Id}_{\mathbb{Z}}$. Gluing on the stratification as in [GM83, Section 5.1], whenever $\bar{q} \leqslant \bar{t}$, there exists a map $\mathrm{IC}^q_{X,\mathbb{Z}} \to \omega_X[\delta(X)]$ where $\omega_X = \mathbb{D}_X(\mathbb{Z})$ is the dualising complex. Whence a map

$$\mathrm{IC}_{X,\mathbb{Z}} \otimes \mathrm{IC}_{X,\mathbb{Z}} \to \omega_X[\delta(X)]. \tag{4.3.28.1}$$

With rational coefficients, this pairing induces an equivalence

$$\mathrm{IC}_{X,\mathbb{Q}} \xrightarrow{\sim} \mathbb{D}_X(\mathrm{IC}_{X,\mathbb{Q}})[-\delta(X)] \tag{4.3.28.2}$$

by [GM83, Section 5.3]. Let k be a field of characteristic zero.

Lemma 4.3.29. *Let $K \in \mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ and $M \in \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$. For each $i \in \mathbb{Z}$ there exists an short exact sequence*

$$0 \to \mathrm{H}^1(\underline{\mathrm{Hom}}(\mathrm{H}^{-i+1}(K), M)) \to \mathrm{H}^i(\underline{\mathrm{Hom}}(K, M)) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{H}^{-i}(K), M)) \to 0$$

in $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$.

Proof. This realises to the classical universal coefficient theorem of abelian groups (or $\mathbb{Z}_\ell$-modules if one takes ℓ-adic realisations). □

Definition 4.3.30. Assume that X is a finite type k-scheme. The intersection homology complex of X is the object

$$\mathrm{IH}(X,\mathbb{Z}) := p_*\mathrm{IC}_{X,\mathbb{Z}}$$

in $\mathrm{DN}_c(k,\mathbb{Z})$ where $p\colon X \to \mathrm{Spec}(k)$ is the structural morphism. Denote by $\mathrm{IH}_i(X,\mathbb{Z}) = \mathrm{H}^{-i}(\mathrm{IH}(X,\mathbb{Z}))$ the i-th intersection homology motive. There is also a version with compact support

$$\mathrm{IH}_c(X,\mathbb{Z}) := p_!\mathrm{IC}_{X,\mathbb{Z}}$$

and its homology motive $\mathrm{IH}_{i,c}(X,\mathbb{Z}) = \mathrm{H}^{-i}(\mathrm{IH}_c(X,\mathbb{Z}))$

Rationally, we obtain from (4.3.28.2) a perfect pairing

$$\mathrm{IH}_{i,c}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{IH}_{d-i}(X,\mathbb{Q}) \to \mathbb{Q}_k(-d),$$

where d is the dimension of X.

We finish with a motivic lift of the results in [GS83, Section 3] providing a motivic Seifert linking pairing. The proof of the next proposition is the same as in *loc. cit.*

Proposition 4.3.31. *For $i \in \mathbb{Z}$ denote by $T_i(X)$ (*resp. *by $T_{i,c}(X)$) the maximal torsion subobject of $\mathrm{IH}_i(X,\mathbb{Z})$ (*resp. *of $\mathrm{IH}_{i,c}(X,\mathbb{Z})$). There is a pairing*

$$T_{i,c}(X) \times T_{d-i-1}(X) \to \mathbb{Q}/\mathbb{Z}(-d)$$

in $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$.

Proof. We apply Lemma 4.3.29 and obtain a short exact sequence

$$0 \to \mathrm{H}^1(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z}),\mathbb{Z})) \to \mathrm{H}^{-i}(\underline{\mathrm{Hom}}(\mathrm{IH}(X,\mathbb{Z})[d](d),\mathbb{Z})) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i}(X,\mathbb{Z}),\mathbb{Z})) \to 0.$$

Using the exact triangle $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ and the fact that for any $M \in \mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$ the object $\mathrm{H}^1(\underline{\mathrm{Hom}}(M,\mathbb{Q}))$ vanishes we obtain that

$$\mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z})(d),\mathbb{Q}/\mathbb{Z})) \xrightarrow{\sim} \mathrm{H}^1(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z})(d),\mathbb{Z}))$$

is an equivalence. Moreover because $\mathbb{Q}/\mathbb{Z}$ is torsion, the map

$$\mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{T}_{d-i-1}(X)(d),\mathbb{Q}/\mathbb{Z})) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z})(d),\mathbb{Q}/\mathbb{Z}))$$

is also an equivalence (this is true after realisation). Also any map $M \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i}(X,\mathbb{Z})(d),\mathbb{Z}))$ with M torsion vanishes because it can be checked after realisation. Thus the composed map $T_{i,c}(X) \to \mathrm{IH}_{i,c}(X,\mathbb{Z}) \to \mathrm{H}^{-i}(\underline{\mathrm{Hom}}(\mathrm{IH}(X,\mathbb{Z})[d](d),\mathbb{Z}))$ obtained from applying $\mathrm{H}^{-i}p_!$ to (4.3.28.1) factors through the inclusion

$$\mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{T}_{d-i-1}(X)(d),\mathbb{Q}/\mathbb{Z})) \to \mathrm{H}^{-i}(\underline{\mathrm{Hom}}(\mathrm{IH}(X,\mathbb{Z})[d](d),\mathbb{Z})),$$

yielding

$$T_{i,c}(X) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{T}_{d-i-1}(X)(d),\mathbb{Q}/\mathbb{Z})).$$

By adjunction (here we use that $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Z})$ has enough flat objects) this provides the desired pairing. □

Remark 4.3.32. The above pairing is non-degenerate when X is smooth. It is not non-degenerate in general, there is a topological obstruction introduced in [GS83] when X is singular and proper.

Chapter 5

Artin motives and mixed abelian motives.

In this chapter, we study Artin motives (that is, motives of étale schemes) and mixed abelian motives (that is, motives that can be obtained by taking tensor products, duals and subquotients, from 1-motives). In the first part we show that over a general basis of characteristic zero, the derived category of Artin motives is the same in the realm of étale motives and of Nori motives. In the second section we show that smooth mixed abelian motives in Nori motives over a smooth basis embeds fully faithfully in the category of mixed realisations. This is used to show that the canonical construction for Shimura varieties of Hodge type factors through perverse Nori motives.

5.1 Artin motives.

Let k be a field. The theory of Artin motives over k, also known as 0-motives, is well understood. Indeed, one has a Tannakian category $\mathcal{M}^0(k,\mathbb{Q}) := \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))$ of Artin motives over k with rational coefficients, that consists of Artin representations of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k. Those are the finite dimensional representations $\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\mathbb{Q})$ that factor through a finite quotient. This category fits well in the general framework of motives.

Indeed, as shown by Orgogozo in [Org04] following Voevodsky [VSF00] the bounded derived category $\mathrm{D}^b(\mathcal{M}^0(k,\mathbb{Q}))$ embeds in geometric motives $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Q})$ and its image is the thick subcategory spanned by motives of finite étale k-schemes. Moreover, it is shown in Huber and Müller-Stachs' [HMS17, Section 9.4] (see also[ABV15, Section 4]) that $\mathcal{M}^0(k,\mathbb{Q})$ is a Tannakian full subcategory of Nori motives $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$, and consists exactly of subquotients of objects of the form $\mathrm{H}^0(X,\mathbb{Q})$ with X a finite étale k-scheme. We know that $\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})$ has weights, and that pure objects form a semisimple subcategory. As Artin motives are pure, this implies easily that the natural functor

$$\mathrm{D}^b(\mathcal{M}^0(k,\mathbb{Q})) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$$

is fully faithful. As this functor agree with the composition

$$\mathrm{D}^b(\mathcal{M}^0(k,\mathbb{Q})) \to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Q}) \xrightarrow{\rho_{\mathrm{N}}} \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})),$$

this shows that the realisation functor ρ_{N} is an equivalence when restricted to Artin motives. In this section we extend this result to general schemes of characteristic zero, and give some consequences. We begin with a definition:

Definition 5.1.1. Let k be a field, let X be a finite type k-scheme and let R be a ring. For this definition, $\mathrm{D}(X)$ denotes either $\mathrm{DM}^{\text{ét}}(X,R)$ or $\mathrm{DN}(X,R)$. In the $\mathrm{DM}^{\text{ét}}$ case we will drop the "ét" from the notations.

1. The ∞-category $\mathrm{D}^0(X)$ of Artin motives in $\mathrm{D}(X)$ is the localising[1] subcategory of $\mathrm{D}(X)$ spanned by the objects f_*R_Y where $f : Y \to X$ is finite. We denote by $\mathrm{D}^0_c(X)$ its full subcategory of constructible objects, which is the thick full subcategory spanned by the f_*R_Y for $f : Y \to X$ finite.

2. The ∞-category $\mathrm{D}^{0,\mathrm{sm}}(X)$ of smooth Artin motives in $\mathrm{D}(X)$ is the localising subcategory of $\mathrm{D}(X)$ spanned by the objects f_*R_Y where $f : Y \to X$ is finite and étale. We denote by $\mathrm{D}^{0,\mathrm{sm}}_c(X)$ its full subcategory of constructible objects, which is the thick full subcategory spanned by the f_*R_Y for $f : Y \to X$ finite and étale.

3. The ∞-category $\mathrm{D}^{\mathrm{coh}}(X)$ of cohomological motives in $\mathrm{D}(X)$ is the localising subcategory of $\mathrm{D}(X)$ spanned by the objects f_*R_Y where $f : Y \to X$ is proper. We denote by $\mathrm{D}^{\mathrm{coh}}_c(X)$ its full subcategory of constructible objects, which is the thick full subcategory spanned by the f_*R_Y for $f : Y \to X$ proper.

4. The ∞-category $\mathrm{D}^{0,\mathrm{rig}}(X)$ of rigid Artin motives in $\mathrm{D}(X)$ is the full subcategory of $\mathrm{D}^0(X)$ spanned by the dualisable objects (seen as objects of $\mathrm{D}(X)$).

Proposition 5.1.2. *If $R = \mathbb{Q}$ we have the following sequence of inclusions:*

$$\mathrm{D}^{0,\mathrm{sm}}_c(X) \subset \mathrm{D}^{0,\mathrm{rig}}(X) \subset \mathrm{D}^0_c(X) \subset \mathrm{D}^{\mathrm{coh}}_c(X) \subset \mathrm{D}_c(X).$$

If $X = \mathrm{Spec}(k)$ is the spectrum of a field of characteristic zero, then the reverse inclusion $\mathrm{D}^0_c(X) \subset \mathrm{D}^{0,\mathrm{sm}}_c(X)$ holds.

Proof. There are two non-trivial inclusions, and they are the first two. The first holds because for $f : Y \to X$ finite and étale, the object f_*R_Y is dualisable with dual itself (this follows from the projection formula). The second is a consequence of the fact that with rational coefficients, compact and constructible objects agree, and the unit object is compact hence also dualisable objects. □

By the work of S. Pépin-Lehalleur [PL19b] and [PL19a] the category $\mathrm{DM}^0(X,\mathbb{Q})$ affords an ordinary motivic t-structure which restricts to compact objects $\mathrm{DM}^0_c(X,\mathbb{Q})$, and for which each ℓ-adic realisation

$$\mathrm{DM}^0_c(X,\mathbb{Q}) \to \mathrm{D}^b_c(X_{\acute{e}t},\mathbb{Q}_\ell)$$

is t-exact (and conservative) when $\mathrm{D}^b_c(X_{\acute{e}t},\mathbb{Q}_\ell)$ is endowed with the ordinary t-structure whose heart is the abelian category of constructible sheaves.

Because the functor ρ_{N} commutes with the 6 operations, it sends Artin motives to Artin motives, cohomological motives to cohomological motives, rigid objects to rigid objects *etc.*

[1] That is, the smallest full subcategory that contains the generators and is stable under all colimits.

Moreover, the conservativity and t-exactness of the ℓ-adic realisation on Nori motives implies that the functor

$$\rho_{\mathrm{N}} \colon \mathrm{DM}^0(X, \mathbb{Q}) \to \mathrm{DN}(X, \mathbb{Q})$$

is t-exact. We will study the full faithfulness of this functor, and then deduce the full faithfulness of the functor with any coefficient ring.

5.1.1 Artin motives over a field.

Let k be a field of characteristic zero. As reminded above we have:

Theorem 5.1.3 (Ayoub – Barbieri-Viale – Huber – Müller-Stachs – Orgogozo – Pepin-Lehalleur)**.** *Let k be a field a characteristic zero. The functor*

$$\rho^0_{\mathrm{N}} : \mathrm{D}^b(\mathrm{Rep}^A_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))) \simeq \mathrm{DM}^0_c(k, \mathbb{Q}) \to \mathrm{DN}^0_c(k, \mathbb{Q})$$

is an equivalence of stable ∞-categories. Moreover, composition of this equivalence with the inclusion

$$\mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}))$$

is t-exact.

We begin with a statement about cohomological Nori motives.

Proposition 5.1.4. *The t-structure on $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}))$ restricts to $\mathrm{DN}^{\mathrm{coh}}_c(k, \mathbb{Q})$. We denote by $\mathcal{M}^{\mathrm{coh}}_{\mathrm{Nori}}(k, \mathbb{Q})$ its heart.*

Proof. Let $\mathcal{A}$ be the full subcategory of $\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q})$ whose objects are the $M \in \mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q})$ satisfying the following property $(*)$: For all $w \in \mathbb{Z}$ the graded piece $\mathrm{Gr}^W_w M$ is a direct sum of subquotients of objects of the form $\mathrm{H}^i(f_* \mathbb{Q}_X)$ for $i \in \mathbb{Z}$ and $f \colon X \to \mathrm{Spec}(k)$ smooth and projective. Because for such an f the object $\mathrm{H}^i(f_* \mathbb{Q}_X)$ is pure of weigh i, it turns out that only $i = w$ appears in the property. Moreover, because the weight truncation functors are exact and the categories of objects pure of a certain weight $w \in \mathbb{Z}$ are semi-simple, the category $\mathcal{A}$ is a Serre abelian subcategory of $\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q})$, that is it is stable under extensions and subquotients. In particular, the t-structure on $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}))$ restricts to the stable ∞-category $\mathcal{C}$ consisting of complexes having cohomology sheaves in $\mathcal{A}$. By the decomposition theorem we have $\mathrm{DN}^{\mathrm{coh}}_c(k, \mathbb{Q}) \subset \mathcal{C}$, and by dévissage and the decomposition theorem again, the reverse inclusion is also true, finishing the proof. □

Remark 5.1.5. The above proposition works verbatim if one replaces cohomological motives $\mathrm{DN}^{\mathrm{coh}}_c(k, \mathbb{Q})$ with n-motives $\mathrm{DN}^n_c(k, \mathbb{Q})$, the thick category spanned by the $f_* \mathbb{Q}_X$ for f a smooth proper morphism of relative dimension $\leqslant n$, providing abelian categories of cohomological n-motives $\mathcal{M}^n(k, \mathbb{Q})$ for all $n \in \mathbb{N}$. It is conjecturally the opposite of the category introduced in [ABV15].

We use the ideas of Morel [Mor08], Nair and Vaish [NV15], and Ruimy [Rui23] to have a well-behaved Artin truncation functor on $\mathrm{DN}^{\mathrm{coh}}_c(k, \mathbb{Q})$. Note that at first, there are *a priori* two ways of producing an Artin motive out of a cohomological motive. The first one is tautological: the inclusion $\mathrm{DN}^0(k, \mathbb{Q}) \to \mathrm{DN}^{\mathrm{coh}}(k)$ preserves colimits by definition, hence admits a right adjoint ω^0 by Lurie's adjoint functor theorem. The second one is inspired by [Mor08]. Recall (Definition 3.1.15) that $\mathrm{DN}_{\mathrm{gm}}(k, \mathbb{Q})$ is the thick full subcategory of $\mathrm{DN}(k, \mathbb{Q})$ generated by twists of motives of smooth varieties over k. It is equivalent to $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}))$.

Construction 5.1.6. Denote by

$$\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q})^{\leqslant \mathrm{Id}} := \{M \in \mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q}) \mid \forall i \in \mathbb{Z}, \mathrm{H}^i(M) \text{ is of weights } \leqslant 0\}$$

and by

$$\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q})^{>\mathrm{Id}} := \{M \in \mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q}) \mid \forall i \in \mathbb{Z}, \mathrm{H}^i(M) \text{ is of weights } > 0\}.$$

By [Mor08, Proposition 3.1.1], this forms a t-structure t^{Id} on $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q})$. Denote by $\tau_c^{\leqslant \mathrm{Id}}$ be the truncation functor right adjoint to the inclusion $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q})^{\leqslant \mathrm{Id}} \to \mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q})$. By taking indization (Proposition 1.1.11) this t-structure induces one on $\mathrm{DN}(k,\mathbb{Q})$ and the associated functor

$$\tau^{\leqslant \mathrm{Id}} \colon \mathrm{DN}(k,\mathbb{Q}) \to \mathrm{DN}(k,\mathbb{Q})^{\leqslant \mathrm{Id}}$$

preserves compact objects, giving back the above $\tau_c^{\leqslant \mathrm{Id}}$. Moreover, $\tau^{\leqslant \mathrm{Id}}$ is an exact functor of stable ∞-categories, because $\mathrm{DN}(k,\mathbb{Q})^{\leqslant \mathrm{Id}}$ is itself a stable ∞-category.

Proposition 5.1.7. *We have an equality*

$$\mathrm{DN}^0(k,\mathbb{Q}) = \mathrm{DN}^{\mathrm{coh}}(k,\mathbb{Q}) \cap \mathrm{DN}(k,\mathbb{Q})^{\leqslant \mathrm{Id}}$$

of full subcategories of $\mathrm{DN}(k,\mathbb{Q})$*. Moreover, the restriction of* $\tau^{\leqslant \mathrm{Id}}$ *to* $\mathrm{DN}^{\mathrm{coh}}(k,\mathbb{Q})$ *coincides with* ω^0*, so that the latter preserves compact objects and is t-exact.*

Proof. Because the generators of $\mathrm{DN}^0(k,\mathbb{Q})$ are in $\mathrm{DN}(k,\mathbb{Q})^{\leqslant \mathrm{Id}}$, we have a trivial inclusion

$$\mathrm{DN}^0(k,\mathbb{Q}) \subset \mathrm{DN}^{\mathrm{coh}}(k,\mathbb{Q}) \cap \mathrm{DN}(k,\mathbb{Q})^{\leqslant \mathrm{Id}}.$$

To prove the reverse inclusion, it suffices to prove that if $M = f_*\mathbb{Q}_X$ is a generator of $\mathrm{DN}^{\mathrm{coh}}(k,\mathbb{Q})$, meaning that $f\colon X \to \mathrm{Spec}(k)$ is smooth and proper, then the object $\tau^{\leqslant \mathrm{Id}} f_*\mathbb{Q}_X$ is an Artin motive. Denote by $X \xrightarrow{\alpha} \pi_0(X) \xrightarrow{\pi} \mathrm{Spec}\, k$ the Stein factorisation of f (see [Gro61, Corollaire 4.3.3]). There is a natural map $\pi_*\mathbb{Q}_{\pi_0(X)} \to f_*\mathbb{Q}_X$, which is an isomorphism on H^0, as it can be seen after realisation ($\mathrm{H}^0(\pi_0(X)) \simeq \mathrm{H}^0(X)$ is of rank the number of connected components of X). It is also an isomorphism on H^i for $i < 0$ because pushforward are left exact. As $\mathrm{H}^i(\pi_*\mathbb{Q}_{\pi_0(X)}) = 0$ if $i > 0$ because π is finite étale hence π_* is t-exact, we have in fact that the cofiber C of the map $\pi_*\mathbb{Q}_{\pi_0(X)} \to f_*\mathbb{Q}_X$ verifies $\mathrm{H}^i(C) = 0$ if $i \leqslant 0$ and $\mathrm{H}^i(C) \simeq \mathrm{H}^i(f_*\mathbb{Q}_X)$ for $i > 0$. The cofiber sequence

$$\pi_*\mathbb{Q}_{\pi_0(X)} \to f_*\mathbb{Q}_X \to C$$

is therefore equivalent to the truncation cofiber sequence

$$\tau^{\leqslant 0} f_*\mathbb{Q}_X \to f_*\mathbb{Q}_X \to \tau^{>0} f_*\mathbb{Q}_X$$

for the canonical t-structure on $\mathrm{D}^b(\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q}))$. For $i \geqslant 1$, the weight of $\mathrm{H}^i(f_*\mathbb{Q}_X)$ is i and the weight of $\pi_*\mathbb{Q}_{\pi_0(X)}$ is zero, hence this cofiber sequence is also the truncation cofiber sequence for the Morel t-structure: the map $\pi_*\mathbb{Q}_{\pi_0(X)} \to f_*\mathbb{Q}_X$ coincides with the map $\tau^{\leqslant \mathrm{Id}} f_*\mathbb{Q}_X \to f_*\mathbb{Q}_X$. In particular, $\tau^{\leqslant \mathrm{Id}} M$ is an Artin motive. The fact that

$$\omega^0 \simeq (\tau^{\leqslant \mathrm{Id}})_{|\mathrm{DN}^{\mathrm{coh}}}$$

can be seen by a direct application of the Yoneda lemma. Finally, the t-exactness of ω^0 follows from the t-exactness of the weight truncation functors. □

Corollary 5.1.8. *Let $\omega^0 : \mathrm{DM}^{\mathrm{coh}}(k,\mathbb{Q}) \to \mathrm{DM}^0(k,\mathbb{Q})$ be the right adjoint to the inclusion constructed by [PL19b], [Vai20] and [ABV09]. Then the natural map $\rho_{\mathrm{N}} \circ \omega^0 \to \omega^0 \circ \rho_{\mathrm{N}}$ is an equivalence.*

Proof. As ρ_{N} sends Artin motives to Artin motives we have a natural transformation $\rho_{\mathrm{N}} \circ \omega^0 \to \omega^0 \circ \rho_{\mathrm{N}}$. To show that this map is an equivalence it suffices to check that this natural transformation is an equivalence when evaluated at the generators $f_*\mathbb{Q}_X$ of cohomological motives, where $f : X \to \operatorname{Spec} k$ is proper and smooth. This is the case by commutation of ρ_{N} with the operations and because of the computation $\omega^0 f_* \mathbb{Q}_X \simeq \pi_* \mathbb{Q}_{\pi_0(X)}$ with $\pi : \pi_0(X) \to \operatorname{Spec}(k)$ the Stein factorisation of X. Indeed for Nori motives this has been proven in the previous proposition, and for Voevodsky motives this is [PL19b, Proposition 3.7]. □

5.1.2 General permanence properties.

In order to do efficient dévissages, we will need our categories to have a good functorial behaviour. We start with stability by the operations:

Proposition 5.1.9. *Let k be a field of characteristic zero and let $f\colon Y \to X$ be a map of finite type k-schemes. Then*

1. *The functor f^* preserves all categories defined in Definition 5.1.1.*
2. *The functors f_* and $f_!$ preserve $\mathrm{DN}^{\mathrm{coh}}(-,\mathbb{Q})$ and $\mathrm{DM}^{\mathrm{coh}}(-,\mathbb{Q})$.*
3. *If f is quasi-finite the functor $f_!$ preserves $\mathrm{DN}^0(-,\mathbb{Q})$ and $\mathrm{DM}^0(-,\mathbb{Q})$*
4. *If f is quasi-finite, the functor $f^!$ preserves $\mathrm{DN}^{\mathrm{coh}}(-,\mathbb{Q})$ and $\mathrm{DM}^{\mathrm{coh}}(-,\mathbb{Q})$.*

Proof. In each case, it suffices to check that the functor sends generators to the correct category. In the case of $\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})$ this is [PL19b, Propositions 1.12 and 1.14], if one notes that we work in characteristic zero so that we even have resolutions by singularities. By commutation of ρ_{N} with the operations, the result for $\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})$ implies the result for $\mathrm{DN}(-,\mathbb{Q})$. Pépin-Lehalleur results are for separated morphisms but we can do a Zariski covering to reduce to this case. □

To make the statement of the next proposition clearer we have to use that DN can be extended to all finite dimensional qcqs $\mathbb{Q}$-schemes as a functor that sends limit of schemes with affine transitions to colimits of presentable ∞-categories. This follows from the presentations of DN as modules in DM over an algebra $\mathscr{N} \in \mathrm{CAlg}(\mathrm{DM}(\mathbb{Q}))$, proven in Proposition 3.2.6, and continuity of DM as proven in [EHIK21b, Lemma 5.1]. In this setting, the definitions of Definition 5.1.1 make sense and we use the same notations.

Proposition 5.1.10. *Let $X = \lim_i X_i$ be a limit of finite dimension qcqs $\mathbb{Q}$-schemes with affine transitions. Let $\mathcal{C}$ be one of the categories defined in Definition 5.1.1, with rational coefficients, and denote by $\mathcal{C}_c$ its compact version. Then the natural functor*

$$\operatorname{colim}_i \mathcal{C}(X_i) \to \mathcal{C}(X)$$

*(*resp.

$$\operatorname{colim}_i \mathcal{C}_c(X_i) \to \mathcal{C}_c(X_i))$$

*is an equivalence in Pr^{L} (*resp. *in Cat_∞.)*

Proof. We first deal with the case of rigid objects. In that case the colimit is in Cat_∞. Full faithfulness follows from the full faithfulness for the ambient category $\mathrm{DM}_c^{\text{ét}}(-,\mathbb{Q})$ and $\mathrm{DN}_c(-,\mathbb{Q}) = \mathrm{DN}_{\mathrm{gm}}(-,\mathbb{Q})$. Essential surjectivity follows from the fact that the dualisability data is a finite data, so that each part of the data (the dual, the evaluation, the co-evaluation *etc.*) can be reach at a finite level X_i by the essential surjectivity on the ambient category.

For the other categories, it suffices to deal with the compact versions. Full faithfulness is as above, a consequence of full faithfulness of the ambient category. Essential surjectivity follows from the approximation properties of finite type properties, as in [PL19b, Lemma 1.24]. □

Corollary 5.1.11. *Let k be a field of characteristic zero and let X be a finite type k-scheme. Let $M \in \mathrm{DN}_c^0(X,\mathbb{Q})$ (*resp.* $M \in \mathrm{DM}_c^0(X,\mathbb{Q})$). Then there exists a dense open subset $U \subset X$ such that the restriction $M_{|U}$ is in $\mathrm{DN}_c^{0,\mathrm{sm}}(U,\mathbb{Q})$ (*resp.* in $\mathrm{DM}_c^{0,\mathrm{sm}}(U,\mathbb{Q})$).*

Proof. Indeed because over a field we have that all 0-motives are smooth, the continuity of smooth motives around the scheme of generic points of X provides the wanted U. □

The same proofs as [PL19b, Proposition 1.19 and Proposition 1.25] give :

Proposition 5.1.12. *Let k be a field of characteristic zero and let X be a finite type k-scheme and let $M \in \mathrm{DN}_c(X,\mathbb{Q})$. The following are equivalent :*

1. *$M \in \mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q})$ (*resp.* in $\mathrm{DN}_c^0(X,\mathbb{Q})$)*
2. *There exists a closed subset $i : Z \to X$ with open complement $j : U \to X$ such that $j^*M \in \mathrm{DN}_c^{\mathrm{coh}}(U,\mathbb{Q})$ (*resp.* in $\mathrm{DN}_c^0(U,\mathbb{Q})$) and $i^*M \in \mathrm{DN}_c^{\mathrm{coh}}(Z,\mathbb{Q})$ (*resp.* in $\mathrm{DN}_c^0(Z,\mathbb{Q})$).*
3. *For all $x \in X$, $x^*M \in \mathrm{DN}_c^{\mathrm{coh}}(\kappa(x),\mathbb{Q})$ (*resp.* in $\mathrm{DN}_c^0(\kappa(x),\mathbb{Q})$)*

The same result holds with $\mathrm{DN}(-,\mathbb{Q})$ replaced by $\mathrm{DM}(-,\mathbb{Q})$.

Recall that the ordinary t-structure on X is the unique t-structure on $\mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q})$ such that all pullback functors are t-exact, and which agrees with the perverse t-structure on fields. We denote by $\mathcal{M}_{\mathrm{ct}}(X)$ its heart.

Corollary 5.1.13. *Let k be a field of characteristic zero and let X be a finite type k-scheme. Then the ordinary t-structure on $\mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q})$ restricts to $\mathrm{DN}_c^0(X,\mathbb{Q})$, $\mathrm{DN}^{0,\mathrm{rig}}(X,\mathbb{Q})$ and $\mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q})$.*

Proof. It restricts to $\mathrm{DN}^{\mathrm{rig}}(X,\mathbb{Q})$ because it is true after any realisation functor, as a dualisable complex of sheaves has locally constant cohomology sheaves. We prove that the t-structure restricts to $\mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q})$, the case of Artin motives being similar. Let $M \in \mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q})$. We want to show that $\mathrm{H}^0(M)$ is still a cohomological motive. By the above Proposition 5.1.12 it suffices to prove that for all $x \in X$ the object $x^*\mathrm{H}^0(M)$ is cohomological. This follows from t-exactness of x and the case of fields Proposition 5.1.4. □

5.1.3 The equivalence of Artin motives.

In this section we prove that the functor ρ_{N} induces an equivalence on 0-motives. We begin by the following computation:

Proposition 5.1.14. *Let k be a field of characteristic zero and let X be a smooth k-scheme. Let $f\colon Y \to X$ be a finite étale map and $n \in \mathbb{Z}$ be an integer. Then the maps*

$$\mathrm{H}^n(Y_{\acute{e}t}, \mathbb{Q}) \to \mathrm{Hom}_{\mathrm{DM}^{\text{ét}}(X)}(\mathbb{Q}_X, f_*\mathbb{Q}_Y[n]) \to \mathrm{Hom}_{\mathrm{DN}(X)}(\mathbb{Q}_X, f_*\mathbb{Q}_Y[n])$$

induced by the functors

$$\mathrm{D}(X_{\acute{e}t}, \mathbb{Q}) \to \mathrm{DM}^{\text{ét}}(X, \mathbb{Q}) \to \mathrm{DN}(X, \mathbb{Q})$$

are both equivalences.

Proof. We can assume X and Y connected. The computation for Voevodsky motives follows from [Rui22, Proposition 1.7.2 and Corollary 2.1.4]. For Nori motives, the cases $m \neq 0$ are easy: both the étale cohomology and the Hom group are vanishing, because of t-structure ($m < 0$) and weight ($m > 0$) reasons. For $m = 0$ by adjunction we reduce to the case $f = \mathrm{Id}$ and it follows from the fact that the group of endomorphisms of the unit object is $\mathbb{Q}$ (this is clear if k is a subfield of $\mathbb{C}$ because it is a nonzero $\mathbb{Q}$-vector space that embeds in $\mathbb{Q} = \mathrm{End}_{\mathrm{Shv}(X(\mathbb{C}))}(\mathbb{Q}_X)$, and then follows by continuity for general fields). □

Corollary 5.1.15. *Let k be a field of characteristic zero and let X be a smooth k-scheme. The functor*

$$\mathrm{DM}^{0,\mathrm{sm}}(X, \mathbb{Q}) \to \mathrm{DN}^{0,\mathrm{sm}}(X, \mathbb{Q})$$

induced by ρ_{N} is an equivalence.

Proof. It suffices to show that it is fully faithful on the compact objects, and even, as compact objects are the idempotent completion of the stable category generated by the $f_*\mathbb{Q}_Y$ for $f\colon Y \to X$ finite and étale, it suffices to prove that it is fully faithful on the smallest stable subcategory of $\mathrm{DM}(X)$ that contains the $f_*\mathbb{Q}_Y$ for $f\colon Y \to X$. By dévissage, it suffices to prove that for all $m \in \mathbb{Z}$, $f\colon Y \to X$ and $g\colon Z \to X$ both finite étale, the map

$$\mathrm{Hom}_{\mathrm{DM}^{\text{ét}}(X)}(g_*\mathbb{Q}_Z, f_*\mathbb{Q}_Y[n]) \to \mathrm{Hom}_{\mathrm{DN}(X)}(g_*\mathbb{Q}_Z, f_*\mathbb{Q}_Y[n])$$

is an equivalence. Because $g_* = g_!$ and $g^! = g^*$, by adjunction and proper base change we can assume that $g = \mathrm{Id}_X$ and replace Y by $Y \times_X Z$. In this case, the result is the above computation done in Proposition 5.1.14. □

We now deal with non necessarily smooth 0-motives. For this, we will need the truncation functor of Vaish (see also [PL19a]) over a general basis.

Proposition 5.1.16. *Let k be a field of characteristic zero and let X be a finite type k-scheme. The inclusion $\mathrm{DN}^0_c(X, \mathbb{Q}) \subset \mathrm{DN}^{\mathrm{coh}}_c(X, \mathbb{Q})$ admits a right adjoint ω^0. Moreover, ρ_{N} commutes with ω^0.*

Proof. The existence of ω^0 for Nori motives is proven exactly in the same way as for étale motives in [Vai20, Theorem 5.2.2]. This uses the existence of a truncation functor over a field, a fact ensured by Proposition 5.1.7. By definition, ω^0 is a truncation functor for a m-structure ([Vai20, section 2.5]), obtained by punctual gluing. As ρ_{N}^K is m-exact for all fields K by Corollary 5.1.8, we have that ρ_{N}^X is also m-exact, hence commutes with truncations. □

We can now prove the main result of the section.

Theorem 5.1.17. *Let k be a field of characteristic zero and let X be a finite type k-scheme. The functor*

$$\mathrm{DM}^0(X,\mathbb{Q}) \to \mathrm{DN}^0(X,\mathbb{Q})$$

is an equivalence.

Proof. As the functor clearly reaches the generators and preserves compact objects, it suffices to prove that it is fully faithful on $\mathrm{DM}_c^0(X,\mathbb{Q})$. We do a Noetherian induction on X. Let $M,N \in \mathrm{DM}_c^0(X,\mathbb{Q})$. By Corollary 5.1.11 and generic smoothness, there exists a nonempty smooth open subset $j\colon U \to X$ such that both j^*M and j^*N are smooth. Denote by $i\colon Z \to X$ the inclusion of the reduced closed complement. By localisation, we have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{map}_{\mathrm{DM}^{\text{ét}}(X)}(i_*i^*N, M) & \longrightarrow & \mathrm{map}_{\mathrm{DM}^{\text{ét}}(X)}(N, M) & \longrightarrow & \mathrm{map}_{\mathrm{DM}^{\text{ét}}(X)}(j_!j^*N, M) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{map}_{\mathrm{DN}(X)}(i_*i^*\rho_{\mathrm{N}}(N), \rho_{\mathrm{N}}(M)) & \longrightarrow & \mathrm{map}_{\mathrm{DN}(X)}(\rho_{\mathrm{N}}(N), \rho_{\mathrm{N}}(M)) & \longrightarrow & \mathrm{map}_{\mathrm{DN}(X)}(j_!j^*\rho_{\mathrm{N}}(N), \rho_{\mathrm{N}}(M))
\end{array}$$

where the lines are cofiber sequences of spectra. By the $(j_!, j^*)$ adjunction, the compatibility of ρ_{N} with the 6 functors, and the smooth case Corollary 5.1.15, the right vertical map is an equivalence. Using the $(i_*, i^!)$ adjunction, the fact that i being quasi-finite, the functor $i^!$ preserves cohomological motives by Proposition 5.1.9 and the fact that the truncation functor ω^0 commutes with ρ_{N} by Proposition 5.1.16, we see that the left vertical map is equivalent to the map

$$\mathrm{map}_{\mathrm{DM}(Z)}(i^*N, \omega^0 i^! M) \to \mathrm{map}_{\mathrm{DN}(Z)}(\rho_{\mathrm{N}}(i^*N), \rho_{\mathrm{N}}(\omega^0 i^! M))$$

which is an equivalence by Noetherian induction, finishing the proof. □

Corollary 5.1.18. *Let X be any finite dimensional qcqs $\mathbb{Q}$-scheme and let R be a ring. The natural functor*

$$\mathrm{DM}^0(X,R) \to \mathrm{DN}^0(X,R)$$

is an equivalence.

Proof. First assume $R = \mathbb{Q}$. Then the result from continuity on both sides: we know that if X is a finite type $\mathbb{Q}$-scheme the functor is an equivalence, and by noetherian approximation and continuity the general case holds. Now if $R = \mathbb{Z}$, it suffices to prove that the functor $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z}) \to \mathrm{DN}(X,\mathbb{Z})$ is fully faithful when restricted to Artin motives. Denote by ρ_*^{N} its right adjoint. We want to prove that if $M \in \mathrm{DM}^0(X,\mathbb{Z})$, the map

$$M \to \rho_*^{\mathrm{N}}\rho_{\mathrm{N}}(M)$$

is an equivalence. As all functors commute with rationalisation and passing mod p for p a prime number, the result follows from the rational case, and from rigidity with torsion coefficients. Now, for a general ring R, we can tensor the equivalence with integral coefficients with Mod_R to obtain the result. □

Smooth and rigid Artin motives.

In this section we prove that smooth and rigid Artin motives coincide on a normal scheme, and we identify the heart of the t-structure. We fix a field k of characteristic zero.

Lemma 5.1.19. *Let $i : Z \to S$ be a closed immersion of finite type k-schemes with S regular, such that the codimension of Z in S is $\geqslant 1$. Then for any $M \in \mathrm{DN}_c^{0,\mathrm{rig}}(S, \mathbb{Q})$ we have $\omega^0(i^! M) = 0$.*

Proof. We argue as in [PL19a, Lemma 2.4]: If i was a regular immersion of codimension $c > 0$, then by absolute purity as in [PL19b, Proposition 1.7] we would have an isomorphism $i^! M \simeq i^* M(-c)[-2c]$. Now, by Theorem 5.1.17, $M = \rho_{\mathrm{N}}(M')$ for some $M' \in \mathrm{DM}_c^0(S, \mathbb{Q})$. By [PL19b, Corollary 3.9 (iii)], $\omega^0(i^* M'(-c)[2c]) = 0$ (note that $i^! M' \in \mathrm{DM}^{\mathrm{coh}}(Z, \mathbb{Q})$ by [PL19b, Proposition 1.12 (iv)]) hence $\omega^0(i^! M) = \rho_{\mathrm{N}}(0) = 0$. Now if i is not a regular immersion, we can find a dense open subset $u : U \to Z$, such that U is regular with reduced complement $i_1 : Z_1 \to Z$. As U and S are regular, [Aut18, Tag 0E9J] ensures that $i \circ u$ is a regular immersion of codimension $c \geqslant 1$. We have a cofiber sequence

$$(i_1)_*(i \circ i_1)^! M \to i^! M \to u_*(i \circ u)^! M \tag{5.1.19.1}$$

which gives $\omega^0((i_1)_*(i \circ i_1)^! M) \simeq \omega^0(i^! M)$, because $(i \circ u)^! M \simeq (i \circ u)^* M[-2c](-c)$ thus $\omega^0(u_*(i \circ u)^! M) = 0$ by the same argument as above using [PL19b, Corollary 3.9 (iii)]. As i_1 is quasi-finite and $(i_1)_* \simeq (i_1)_!$, the functors $(i_1)_*$ and ω^0 commute, giving finally that $(i_1)_* \omega^0((i \circ i_1)^! M) \simeq \omega^0(i^! M)$. A Noetherian induction then implies that $\omega^0(i^! M) = 0$. □

Lemma 5.1.20. *Let S be a normal and connected finite type k-scheme with generic point η. Let $M \in \mathrm{Rep}_{\mathbb{Q}}(\pi_1^{\acute{e}t}(\eta))$ be an continuous representation such that there exists $N \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1^{\acute{e}t}(S))$ verifying $N_\eta \simeq M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then there exists an object $M' \in \mathrm{Rep}_{\mathbb{Q}}(\pi_1^{\acute{e}t}(S))$ such that $M'_\eta \simeq M$.*

Proof. The object M corresponds to a morphism $\pi_1^{\acute{e}t}(\eta) \to \mathrm{GL}_n(\mathbb{Q})$. The hypothesis may be translated as the existence of a commutative square

$$\begin{array}{ccc} \pi_1^{\acute{e}t}(\eta) & \longrightarrow & \mathrm{GL}_n(\mathbb{Q}) \\ \downarrow & & \downarrow \\ \pi_1^{\acute{e}t}(S) & \longrightarrow & \mathrm{GL}_n(\mathbb{Q}_\ell) \end{array} .$$

Now the right vertical map is injective, and the left vertical map is surjective (because S is normal, by [SGA71, Exposé V. Proposition 8.2]). Thus in fact the image of the lower horizontal map lands in $\mathrm{GL}_n(\mathbb{Q})$, providing the seeked M'. □

Recall that by Cavicchi, Déglise and Nagel [CDN23, Proposition 2.3.4] the canonical functor

$$\mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}(\pi_1^{\acute{e}t}(S))) \to \mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$$

is an equivalence when S is a regular connected scheme. In particular the stable ∞-category $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$ have a t-structure.

Proposition 5.1.21 (Haas)**.** *Let S be a connected smooth k-scheme of finite type. Let $M \in \mathrm{DM}_c^0(S,\mathbb{Q})^\heartsuit$. The following are equivalent :*

1. *$M \in \mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})^\heartsuit \simeq \mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S))$.*
2. *M is dualisable as an object of $\mathrm{DM}^{\text{ét}}(S,\mathbb{Q})$.*
3. *M is dualisable as an object of $\mathrm{DN}(S,\mathbb{Q})$.*
4. *The ℓ-adic realisation of M is a lisse sheaf.*

That is, the natural functors

$$\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S)) \to \mathrm{DM}_c^{\mathrm{rig}}(S,\mathbb{Q}) \cap \mathrm{DM}_c^0(S,\mathbb{Q})^\heartsuit \to \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit \to \mathrm{LS}(S_{\acute{e}t},\mathbb{Q}_\ell) \times_{\mathrm{Shv}_{\acute{e}t}(S,\mathbb{Q}_\ell)} \mathrm{DN}_c^0(S,\mathbb{Q})^\heartsuit$$

are equivalences, where $\mathrm{LS}(S_{\acute{e}t},\mathbb{Q}_\ell)$ consists of $\mathbb{Q}_\ell$-adic étale local systems on S.

Proof. Of course, 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. because of Poincaré duality for finite étale morphisms and because a symmetric monoidal functor preserves dualisable objects.

We prove that 4. $\Rightarrow$ 1. by adapting Johann Haas' [Haa19, Lemma 6.12] to the simpler case of 0-motives. Let η be the generic point of S. We first claim that the functor $\mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit \to \mathrm{DN}_c^{0,\mathrm{rig}}(\eta,\mathbb{Q})^\heartsuit$ is fully faithful. Indeed for any dense open subset $j : U \to S$, the combination of Lemma 5.1.19 and of the localisation sequence gives that for $M \in \mathrm{DM}^{0,\mathrm{rig}}(S,\mathbb{Q})$, $M \to \omega^0(j_*j^*M)$ is an equivalence as $\omega^0(i^!M) = 0$. Therefore the unit of the adjunction $(j^*, \tau^{\leqslant 0}\omega^0 j_*)$ on $\mathrm{DN}_c^0(S,\mathbb{Q})^\heartsuit$ is an equivalence on $\mathrm{DN}^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit$, this means that j^* is fully faithful on $\mathrm{DN}^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit$. Passing to the colimit, η^* is fully faithful.

Therefore there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S)) & \longrightarrow & \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit \\ \downarrow & & \downarrow \\ \mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(\eta)) & \longrightarrow & \mathrm{DN}_c^0(\eta,\mathbb{Q})^\heartsuit \end{array} \tag{5.1.21.1}$$

in which the lower horizontal functor is an equivalence of categories, the top functor is fully faithful by Theorem 5.1.17 and the right vertical functor is fully faithful by what we've just seen. The left vertical functor is then also fully faithful. But then by Lemma 5.1.20 if $M \in \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit$, we can lift to S its pre-image in $\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(\eta))$ and the top functor is also essentially surjective.

Therefore if $M \in \mathrm{DM}_c^0(S,\mathbb{Q})^\heartsuit$ is such that its ℓ-adic realisation is lisse (*i.e.* dualisable), then its image in the category of perverse Nori motives is in $\mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit$, which is equivalent to $\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S))$: our object M is in $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})^\heartsuit$, and the proof is finished. □

Corollary 5.1.22. *Let S be a regular k-scheme of finite type. Let $M \in \mathrm{DM}_c^0(S,\mathbb{Q}) = \mathrm{DN}_c^0(S,\mathbb{Q})$. Then the following are equivalent :*

1. *$M \in \mathrm{DM}^{0,\mathrm{sm}}(S,\mathbb{Q}) = \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S)))$.*
2. *$M \in \mathrm{DM}^{\acute{\mathrm{e}}\mathrm{t}}(S,\mathbb{Q})$ is dualisable.*
3. *The ℓ-adic realisation of M is dualisable.*

Moreover, in that case, the dual of M is also a 0-motive. That is, the inclusion $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q}) \to \mathrm{DM}_c^{0,\mathrm{rig}}(S,\mathbb{Q})$ is an equivalence of categories, with $\mathrm{DM}_c^{0,\mathrm{rig}}(S,\mathbb{Q})$ the full subcategory of $M \in \mathrm{DM}_c^0(S,\mathbb{Q})$ that are dualisable in $\mathrm{DM}(S,\mathbb{Q})$.

Proof. 2. $\Rightarrow$ 3. is obvious, and the proof of 1. $\Rightarrow$ 2. follows by dévissage, using that if 1., each $\mathrm{H}^i(M)$ are dualisable, that M is an iterated extension of its $\mathrm{H}^i(M)$ and that the full subcategory of dualisable objects in a symmetric monoidal stable ∞-category is thick. The proof of 3. $\Rightarrow$ 1. goes as follows : the hypothesis on M implies that M, seen as an object of $\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(S,\mathbb{Q}))$, is dualisable, where we have denoted by $\mathcal{M}_{\mathrm{ct}}(S)$ the ordinary heart of $\mathrm{DN}_c^{\mathrm{coh}}(S,\mathbb{Q})$. Then because the t-structure restricts to dualisable objects each $\mathrm{H}^i(M)$ is dualisable hence by the previous proposition, $\mathrm{H}^i(M) \in \mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$. As the latter category is thick and M is an iterated extension of its $\mathrm{H}^i(M)$, we obtain that $M \in \mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$. □

In the case S is only normal, following Haas we also have a result in that direction, but not as strong as the preceding corollary that would be false by [Rui22, Remark 2.1.2]. However this enables us to endow $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$ with a t-structure.

Corollary 5.1.23. *Let S be a normal finite type k-scheme. The natural functors $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q}) \to \mathrm{DM}^{0,\mathrm{rig}}(S,\mathbb{Q}) \to \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})$ are equivalences of ∞-categories. If we transport the natural ordinary t-structure from $\mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})$ to $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$, the inclusion $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q}) \to \mathrm{DM}_c^0(S,\mathbb{Q})$ is t-exact, and the heart of $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$ is the category $\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S))$.*

Proof. We can assume S to be connected. By Theorem 5.1.17 the two functors are fully faithful. Therefore it suffices to show that the composition functor is essentially surjective. By dévissage it suffices to show that any object $M \in \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit$ is in the image of $\mathrm{DM}_c^{0,\mathrm{sm}}(S) \to \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})$. For this, we are going to show that the natural functor $\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S)) \to \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit$ is an equivalence of categories. Pick a dense regular open subset U of S, with immersion $j : U \to S$. We have the following commutative diagram :

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S)) & \longrightarrow & \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit \\ \downarrow & & \downarrow \\ \mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(U)) & \longrightarrow & \mathrm{DN}_c^{0,\mathrm{rig}}(U,\mathbb{Q})^\heartsuit \end{array} . \tag{5.1.23.1}$$

In this diagram, the lower horizontal functor is an equivalence of categories. The left vertical functor is fully faithful because both U and S are normal and j is dominant. The top horizontal functor is fully faithful because the right vertical functor is faithful (it can be checked on the ℓ-adic realisation). Now, the top functor is also essentially surjective hence an equivalence thanks to Lemma 5.1.20. This shows $\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S)) \to \mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$ is fully faithful and that $\mathrm{Rep}_{\mathbb{Q}}^A(\pi_{\acute{e}t}^1(S)) \to \mathrm{DN}_c^{0,\mathrm{rig}}(S,\mathbb{Q})^\heartsuit$ is an equivalence of categories. Therefore, $\mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$ has

a t-structure whose heart is $\mathrm{Rep}^A_{\mathbb{Q}}(\pi^1_{\acute{e}t}(S))$. Now, if $M \in \mathrm{DM}_c^{0,\mathrm{rig}}(S,\mathbb{Q})$, then $\rho_{\mathrm{N}}(M)$ has all its cohomology sheaves in $\mathrm{Rep}^A_{\mathbb{Q}}(\pi^1_{\acute{e}t}(S))$, hence $M \in \mathrm{DM}_c^{0,\mathrm{sm}}(S,\mathbb{Q})$. This finishes the proof. □

We now apply this computation of 0-motives to motivic Galois groups.

Notation 5.1.24. *Let $\sigma : \overline{k} \to \mathbb{C}$ be an embedding. Let X be a finite type k-scheme. We denote by $\mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ the subcategory of $\mathcal{M}_{\mathrm{ct}}(X,\mathbb{Q})$ whose objects are the dualisable objects. This means that for $M \in \mathcal{M}_{\mathrm{ct}}(X,\mathbb{Q})$, we have $M \in \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ if and only if M is dualisable and its dual is in $\mathcal{M}_{\mathrm{ct}}(X,\mathbb{Q})$ (that last condition is automatic).*

Definition 5.1.25. Let X be a connected and normal finite type k-scheme with a geometric point $\overline{x} : \mathrm{Spec}(\overline{k}) \to X$. We define $p : \widetilde{X} \to X$ to be the limit of all finite étale $f : Y \to X$ that factor $\overline{x}$. The limit exists as a scheme because the transitions morphisms have to be affine. There is a morphism $\pi_{\widetilde{X}} : \widetilde{X} \to \mathrm{Spec}(\overline{k})$ compatible with p, which has a section $\widetilde{x}$. Therefore there is also a map $q : \widetilde{X} \to X_{\overline{k}}$. By construction, the point $\overline{x}$ factors through $\widetilde{X}$ to give a geometric point $\tilde{x}\colon \mathrm{Spec}(\overline{k}) \to \widetilde{X}$.

Lemma 5.1.26. *The scheme $\widetilde{X}$ is connected.*

Proof. If $U \subset \widetilde{X}$ is a clopen subset such that $\overline{x}$ does not factor through U, then there exists $Y \to X$ finite étale factorising $\overline{x}$ and a clopen V of Y whose pullback to $\widetilde{X}$ is U. Let Z be the open complement of V in Y. Then $\overline{x}$ factors through V and $Z \to X$ is finite étale, so that $\widetilde{X} = \lim_i Y_i$ with $Y_i \to Z$ finite étale factoring $\overline{x}$, and we see that $U = \emptyset$. □

Base change gives a monoidal functor $p^* : \mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q}) \to \mathrm{DN}_c(\widetilde{X},\mathbb{Q})$ which sends $\mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ to $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$.

Lemma 5.1.27. *Let $x : \mathrm{Spec}(\overline{k}) \to X$ be a geometric point of a connected finite type k scheme. Let $\omega_x : \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q}) \to \mathrm{Vect}_{\mathbb{Q}}$ be the composition of x^* with the Betti realisation (through σ). Then $\mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ is a Tannakian category with fiber functor ω_x.*

Proof. Every object of $\mathcal{M}(X,\mathbb{Q})$ is dualisable, the endomorphisms of the unit objects are one dimensional and the functor ω_x is monoidal. □

Lemma 5.1.28. *Let X and $\widetilde{X}$ as in Definition 5.1.25. The category $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$ together with the fiber functor*

$$\omega_{\tilde{x}}\colon \mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q}) \to \mathrm{Vect}_{\mathbb{Q}}$$

is neutral Tannakian.

Proof. The category $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$ is the colimit of all $\mathcal{M}^{\mathrm{rig}}(Y,\mathbb{Q})$ with Y finite étale over X. Hence all of its objects are dualisable. As $\widetilde{X}$ is connected, the category $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$ is Tannakian. It is neutralised by $\tilde{x}$. □

Notation 5.1.29. *Let X be a connected normal finite type k-scheme with a geometric point x. We denote by $\mathrm{G}_{\mathrm{mot}}(X,x,\sigma)$ the Tannakian group of $\mathcal{M}^{\mathrm{rig}}(X)$ and by $\mathrm{G}_{\mathrm{mot}}(\widetilde{X},\tilde{x},\sigma)$ the Tannakian group of $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$*

Let $s : X_{\overline{k}} \to X$ be the projection.

Proposition 5.1.30. *Let X be a normal connected finite type k-scheme. The commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{G}_{\mathrm{mot}}(\widetilde{X},\widetilde{x},\sigma) & \longrightarrow & \mathrm{G}_{\mathrm{mot}}(\widetilde{X},\widetilde{x},\sigma) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{G}_{\mathrm{mot}}(X_{\overline{k}},\overline{x},\sigma) & \longrightarrow & \mathrm{G}_{\mathrm{mot}}(X,x,\sigma) & \longrightarrow & \mathrm{Gal}(\overline{k}/k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1^{\acute{e}t}(X_{\overline{k}},\overline{x}) & \longrightarrow & \pi_1^{\acute{e}t}(X,x) & \longrightarrow & \mathrm{Gal}(\overline{k}/k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.1.30.1}$$

has exact rows and columns.

Proof. The diagram is obtained by Tannaka duality from the categories of lisse motives and lisse Artin motives over X, $X_{\overline{k}}$, $\widetilde{X}$, k and $\overline{k}$. The facts that the second maps in each column and each row is faithfully flat is equivalent to the fact that the maps between Tannakian categories are fully faithful and the image is closed under sub-quotients. Therefore it suffices to check that $\mathrm{Rep}^A_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k)) \to \mathrm{Rep}^A_{\mathbb{Q}}(\pi^1_{\acute{e}t}(X))$ and $\mathrm{Rep}^A_{\mathbb{Q}}(\pi^1_{\acute{e}t}(S)) \to \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ are fully faithful with image closed under sub-quotients. For the first map, this is [SGA71]. The second is fully faithful, we have to check that it is closed under sub-quotients. Working dually it suffices to check that the category $\mathrm{Rep}^A_{\mathbb{Q}}(\pi^1_{\acute{e}t}(X))$ is stable under subobjects in $\mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$. Let $M \to N$ be a monomorphism in $\mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ with N a 0-motive. By Proposition 5.1.12 to show that M is a 0-motive it suffices to check that for all $x \in X$ the motive M_x is a 0-motive. But by [HMS17, Theorem 9.1.16] the category of 0-motives is closed under sub-quotients in the categories of motives over a field.

We prove the middle exactitude and injectivity at the same time. As the proofs are the same, we only deal with the sequence

$$\mathrm{DN}^{0,\mathrm{sm}\heartsuit}(X,\mathbb{Q}) \to \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q}) \to \mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q}). \tag{5.1.30.2}$$

As in the proof of [HMS17, Theorem 9.1.16 and Erratum p. 234], we first prove that any motive $M \in \mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$ is a direct factor of the image of a motive $N \in \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$. Indeed, if $M \in \mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$, there is a finite étale morphism $f : Y \to X$ (that we can assume to be Galois) such that M is defined over Y that is $M \in \mathcal{M}^{\mathrm{rig}}(Y,\mathbb{Q})$. But as the composition $M \to f^*f_*M = f^!f_!M \to M$ is the multiplication by $\deg f$, M is a direct factor of f^*N with $N = f_*N \in \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$.

Let $\mathcal{U}$ be the Tannakian full subcategory of $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$ generated by the unit object $\mathbb{Q}$. We say that elements of $\mathcal{U}$ are *trivial*. To finish the proof we just have to prove that $M \in \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ has trivial image in $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$ if and only if it is in $\mathrm{DN}^{0,\mathrm{sm}\heartsuit}(X,\mathbb{Q})$. Of course, as any representation of $\pi_1^{\acute{e}t}(X)$ is trivialised after a base change to a finite étale cover of X, the image of Artin motives in $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$ is trivial. Conversely, let $M \in \mathcal{M}^{\mathrm{rig}}(X,\mathbb{Q})$ with trivial image in $\mathcal{M}^{\mathrm{rig}}(\widetilde{X},\mathbb{Q})$. This means that there is a finite étale morphism $f : Y \to X$

such that f^*M is trivial. We can assume that f is Galois. We have that f_*f^*M is in the category generated by $f_*\mathbb{Q}_Y$, which is inside $\mathrm{DN}^{0,\mathrm{sm}\heartsuit}(X,\mathbb{Q})$. As M is a direct factor of f_*f^*M, we get that M is an Artin motive.

□

5.1.4 Integral smooth Artin motives and Artin vanishing.

This section is joint work with Raphaël Ruimy, and can be found in [RT24b].

Lemma 5.1.31. *Let X be a qcqs $\mathbb{Q}$-scheme. Then, the functor $\mathrm{DM}_c^{\text{ét},0}(X,\mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is t-exact when $\mathrm{DM}_c^{\text{ét},0}(X,\mathbb{Z})$ is endowed with the ordinary homotopy t-structure (see [Rui22, Definition 3.1.1.1 and Theorem 3.1.2.7]).*

Proof. The t-exactness follows from the t-exactness of the ℓ-adic realisations functors on Artin motives ([Rui22, Theorem 3.1.2.7]) and their conservativity and t-exactness on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$. □

We will now use Nori motives to give a characterisation of dualisable objects of $\mathrm{DM}_c^{\text{ét},0}(X,\mathbb{Z})$.

Lemma 5.1.32. *For X a qcqs $\mathbb{Q}$-scheme denote by $\mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Z})$ the full subcategory of $\mathrm{DM}^{\text{ét},0}(X,\mathbb{Z})$ made of those objects which are dualisable in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$. The functor $\mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(-,\mathbb{Z})$ is a finitary étale sheaf.*

Proof. As $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(-,\mathbb{Z})$ is finitary by Theorem 3.1.29, the functor $\mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(-,\mathbb{Z})$ is also finitary: being Artin and rigid are both finitary conditions. We have to show that $\mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(-,\mathbb{Z})$ is a subsheaf of $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(-,\mathbb{Z})$, the latter being a sheaf by Lemma 3.1.24, meaning that if $f\colon Y \to X$ is an étale covering and M is an object of $\mathrm{DM}_{\mathrm{c}}^{\text{ét}}(X,\mathbb{Z})$ such that f^*M is dualisable and Artin, then so is M. But dualisability is étale local and so is being a 0-motive. □

Proposition 5.1.33. *Let X be a normal $\mathbb{Q}$-scheme. The inclusion*

$$\mathrm{DM}_c^{\text{ét},\mathrm{sm}0}(X,\mathbb{Z}) \subseteq \mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Z})$$

is an equivalence. The t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ restricts to the full subcategory $\mathrm{DM}_c^{\text{ét},\mathrm{sm}0}(X,\mathbb{Z})$ and the heart of the restricted t-structure is equivalent to $\mathrm{Rep}(\pi_1^{\text{ét}}(X),\mathbb{Z})$.

Proof. Using étale descent, continuity and Lemma 5.1.32, it suffices to prove the statements if X is of finite type over an algebraically closed field.

First note that the inclusion $\mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Z}) \to \mathrm{DN}_{\mathrm{rig}}^{0}(X,\mathbb{Z})$ is an equivalence, where the right-hand side consists of the objects of $\mathrm{DN}^0(X,\mathbb{Z})$ which are dualisable in $\mathrm{DN}(X,\mathbb{Z})$. Indeed, if M belongs to $\mathrm{DN}_{\mathrm{rig}}^{0}(X,\mathbb{Z})$ then its dual is also a 0-motive, because being a constructible 0-motive can be tested on points and all constructible 0-motives over a point are dualisable, with 0-motivic duals. But now, the t-structure of $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ restricts to $\mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Z})$ because it can be restricted to both dualisable objects and 0-motives.

By Proposition 4.3.1 we have a pullback square

$$\begin{array}{ccc} \mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Z})^{\heartsuit} & \longrightarrow & \mathrm{Loc}(X^{\mathrm{an}},\mathbb{Z}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Q})^{\heartsuit} & \longrightarrow & \mathrm{Loc}(X^{\mathrm{an}},\mathbb{Q}) \end{array}$$

where $\mathrm{Loc}(X^{\mathrm{an}},\mathbb{Z})$ is the abelian category of local systems on X^{an} with $\mathbb{Z}$ coefficients. By Corollary 5.1.23 the category $\mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Q})^{\heartsuit}$ is equivalent (through $\rho_!$) to $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Q})$. As the category $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Z})$ also fits in the corner of this pullback diagram, this proves that the functor $\rho_!$ induces an equivalence $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Z}) \to \mathrm{DM}_{\mathrm{rig}}^{\text{ét},0}(X,\mathbb{Z})^{\heartsuit}$. Finally, by dévissage, because any object of $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Z})$ defines an object of $\mathrm{DM}_c^{\text{ét},\mathrm{sm}0}(X,\mathbb{Z})$, any rigid 0-motive is smooth Artin. □

Using the framework of Nori motives, we can prove new cases of [Rui22, Theorem 3.2.3.7] which is one of the main results of *loc. cit.* and is an analogue of the Artin Vanishing Theorem in the setting of Artin motives. We begin by the following lemma:

Lemma 5.1.34. *Let X be a qcqs $\mathbb{Q}$-scheme. The ordinary and perverse t-structures on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ restrict to the category $\mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Z})$ defined as the thick subcategory of $\mathrm{DN}(X,\mathbb{Z})$ generated by the $f_*\mathbb{Z}_Y$ where $f\colon Y \to X$ is proper.*

Proof. By Corollary 5.1.13, the result holds with $\mathbb{Q}$-coefficients. Moreover it suffices to prove the result over the spectrum of a field k. Because the map

$$\mathrm{DN}_c^{\mathrm{coh}}(k,\mathbb{Z}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Q}} \to \mathrm{DN}_c^{\mathrm{coh}}(k,\mathbb{Q})$$

is an equivalence (this means that the right hand side is obtained from the ∞-category with integral coefficiens by tensoring the Hom's by $\mathbb{Q}$ and then taking the idempotent completion) and if M is in $\mathrm{DN}_c^{\mathrm{coh}}(k,\mathbb{Z})$ then $\mathrm{H}^i(M\otimes_{\mathbb{Z}}\mathbb{Q}) \simeq \mathrm{H}^i(M)\otimes_{\mathbb{Z}}\mathbb{Q}$ is cohomological, there is a cohomological motive P in $\mathrm{DN}_c^{\mathrm{coh}}(k,\mathbb{Z})$ and a map $P \to \mathrm{H}^i(M)$ whose cone is torsion and dualisable, thus a constructible 0-motive, which are cohomological. The claim about the perverse t-structure is true by gluing. □

Recall that a smooth Artin motive M is $\mathbb{Q}$-constructible if $M\otimes_{\mathbb{Z}}\mathbb{Q}$ is constructible as a smooth Artin motive. We denote by $\mathrm{DM}_{\mathbb{Q}-c}^{\text{ét},sm0}(X,\mathbb{Z})$ the category of $\mathbb{Q}$-constructible smooth Artin motives. We can now prove new cases of [Rui22, Theorem 3.2.3.7].

Proposition 5.1.35. *Let S be an excellent scheme $\mathbb{Q}$-scheme, let $f\colon X \to S$ be an affine quasi-finite morphism. Assume that X is regular. Then, the functor*

$$f_!\colon \mathrm{DM}_{\mathbb{Q}-c}^{\text{ét},sm0}(X,\mathbb{Z}) \to \mathrm{DM}^{\text{ét},0}(S,\mathbb{Z})$$

is t-exact with respect to the perverse homotopy t-structure (see [Rui22, Definition 3.2.1.1]).

Proof. The same proof as in [Rui22, Theorem 3.2.3.7] allows reducing to case with rational coefficients. We now have to prove that $f_!\colon \mathrm{DN}_c^{sm0}(X,\mathbb{Q}) \to \mathrm{DN}^0(S,\mathbb{Q})$ is t-exact with respect to the perverse homotopy t-structure (defined in the same fashion as in [Rui22, Definition 3.2.1.1]).

The rest of the proof is an adaptation of the arguments of [Rui23, Section 4.8]. We can write the functor

$$f_!\colon \mathrm{DM}_c^{\text{ét},sm0}(X,\mathbb{Q}) \to \mathrm{DM}^{\text{ét},0}(S,\mathbb{Q})$$

as the composition

$$\mathrm{DM}_c^{\text{ét},sm0}(X,\mathbb{Q}) \simeq \mathrm{DN}_c^{sm0}(X,\mathbb{Q}) \xrightarrow{\iota_0} \mathrm{DN}_c^{\mathrm{coh}}(X,\mathbb{Q}) \xrightarrow{f_!} \mathrm{DN}_c^{\mathrm{coh}}(S,\mathbb{Q}) \xrightarrow{\omega^0} \mathrm{DN}_c^0(S,\mathbb{Q}) \simeq \mathrm{DM}_c^{\text{ét},0}(S,\mathbb{Q}),$$

where the Artin truncation functor

$$\omega^0 \colon \mathrm{DN}^{\mathrm{coh}}(X, \mathbb{Q}) \to \mathrm{DN}^0(X, \mathbb{Q})$$

defined as the right adjoint to the inclusion ι_0 of Artin motives into cohomological motives, lands in $\mathrm{DN}^0_c(X, \mathbb{Q})$ by [Tub24a]. Thus it suffices to check that all functors are t-exact. The functor ω^0 is t-exact with respect to the perverse t-structure on the left-hand side (which exists by the above Lemma 5.1.34) and the perverse homotopy t-structure on the right-hand side: this is because an object of $\mathrm{DN}^0(X, \mathbb{Q})$ is perverse homotopy t-negative if and only if it is perverse t-negative (this can be shown as in [Rui23, Proposition 4.5.1]) and because the functor ω^0 is perverse right t-exact as per [NV15, Lemma 3.2.6]. But as ι_0 is t-exact when restricted to smooth Artin objects because it is t-exact for the ordinary t-structure, the result reduces to the usual Artin Vanishing Theorem. □

5.2 Mixed abelian motives.

In this section we consider Tannakian categories of lisse Nori motives. More specifically, we consider what we call mixed abelian motives, the Tannakian category spanned by H^i's of relative curves. We wish we had a derived statement in Theorem 5.2.12, be we could not compute the Ext^i.

5.2.1 Mixed abelian motives over a field.

If k is a field, we know (by Ayoub-Barbieri–Viale [ABV15]) that the abelian category of Deligne 1-motives over k embed fully faithfully in Nori motives $\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q})$ and that its image is stable under subquotients (we recall their theorem below in Theorem 5.2.6). We denote by $\mathcal{MT}^1(k)$ the Tannakian sub category of $\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q})$ generated by Deligne 1-motives, and call its objects *mixed abelian motives*.

André proved in [And21]:

Theorem 5.2.1 (André)**.** *Let k be an algebraically closed subfield of $\mathbb{C}$, then the Hodge realisation functor*

$$R_H \colon \mathcal{MT}^1(k) \to \mathrm{MHS}$$

is fully faithful and its image is stable under subquotients.

Let k be a field that we embed in $\mathbb{C}$ through a morphism $\sigma \colon k \to \mathbb{C}$. We consider (see Section 2.2) the category $\mathcal{MR}^\sigma(k)$ of mixed realisations. We have a mixed realisation functor $R^\sigma_{\mathcal{MR}} \colon \mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}) \to \mathcal{MR}^\sigma(k)$ on Nori motives over k.

Proposition 5.2.2. *Let $k \xrightarrow{\sigma} \mathbb{C}$ be a subfield of $\mathbb{C}$. The mixed realisation functor*

$$R^\sigma_{\mathcal{MR}} \colon \mathcal{MT}^1(k) \to \mathcal{MR}^\sigma(k)$$

is fully faithful and its image is stable under subquotients.

Proof. Let $\overline{\sigma}$ be an embedding of $\overline{k}$ in $\mathbb{C}$ that extends σ. Because the composite

$$\mathcal{MT}^1(\overline{k}) \to \mathcal{MR}^{\overline{\sigma}}(\overline{k}) \to \mathrm{MHS}$$

is fully faithful by Theorem 5.2.1 and all functors are faithful, we already know the result for $\overline{k}$ (the stability under subquotients follows from the case of mixed Hodge structures and the faithfulness of the functor $\mathcal{MR}^\sigma(k) \to \mathrm{MHS}$). Now, let $M, N \in \mathcal{MT}^1(k)$ and denote by G the absolute Galois group of k. We can argue exactly as for 1-motives in [ABV15, Proposition 6.5]: if ℓ is a prime number we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{MT}^1(k)}(M,N) & \hookrightarrow & \mathrm{Hom}_{\mathcal{MT}^1(\overline{k})}(M_{\overline{k}}, N_{\overline{k}}) \\
\downarrow f & & \downarrow \sim \\
\mathrm{Hom}_{\mathcal{MR}^\sigma(k)}(R^\sigma_{\mathcal{MR}}(M), R^\sigma_{\mathcal{MR}}(N)) & \longrightarrow & \mathrm{Hom}_{\mathcal{MR}^{\overline{\sigma}}(\overline{k})}(R^{\overline{\sigma}}_{\mathcal{MR}}(M_{\overline{k}}), R^{\overline{\sigma}}_{\mathcal{MR}}(N_{\overline{k}})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Rep}_{\mathbb{Q}_\ell}(G)}(R_\ell(M), R_\ell(N)) & \hookrightarrow & \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{Q}_\ell}}(R_\ell(M_{\overline{k}}), R_\ell(N_{\overline{k}}))
\end{array},$$

and to prove that the map f is surjective, it suffices to prove that the outer square is cartesian. Here a hooked arrow indicates an injective map.

Moreover, the group G acts on $\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{Q}_\ell}}(R_\ell(M_{\overline{k}}), R_\ell(N_{\overline{k}}))$ and the vector space

$$\mathrm{Hom}_{\mathrm{Rep}_{\mathbb{Q}_\ell}(G)}(R_\ell(M), R_\ell(N)) = \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{Q}_\ell}}(R_\ell(M_{\overline{k}}), R_\ell(N_{\overline{k}}))^G$$

identifies with the invariants under this action (also known as G-equivariant maps). As the right vertical maps in the above diagram are G-equivariant, it suffices to prove that the canonical map

$$\mathrm{Hom}_{\mathcal{MT}^1(k)}(M,N) \to \mathrm{Hom}_{\mathcal{MT}^1(\overline{k})}(M_{\overline{k}}, N_{\overline{k}})$$

identifies the source with the invariants of the target. This is a particular case of a general fact about motives that we state in Lemma 5.2.3 below. This proves the full faithfulness.

Now let $M \in \mathcal{MT}^1(k)$ be a mixed abelian motive, and let $V \subset R^\sigma_{\mathcal{MR}}(M)$ be a subobject. Then we know that there exists $\tilde{N} \subset M_{\overline{k}}$ such that the mixed realisation of $\tilde{N}$ is $V_{\overline{k}}$. Now by continuity of motives there is finite Galois extension L of k and a $N^L \subset M_L$ in $\mathcal{MT}^1(L)$ such that the mixed realisation of N^L is the base change to L of V. Let $f : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ the induced map of schemes. Because we work with rational coefficients the motive M is a direct factor of $f_* f^* M = f_* M_L$, and we may define N as the intersection in $f_* f^* M$ of M and $f_* N^L$. Then the mixed realisation of N is the intersection of $R^\sigma_{\mathcal{MR}(k)}(M)$ and $f_* f^* V$ in $f_* f^* R^\sigma_{\mathcal{MR}(k)}(M)$, which is V, finishing the proof because quotients are obtained by taking duals. □

Lemma 5.2.3. *Let k be a field. Then if $M, N \in \mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q})$ are Nori motives, the map*

$$\mathrm{Hom}_{\mathcal{M}(k)}(M,N) \to \mathrm{Hom}_{\mathcal{M}(\overline{k})}(M_{\overline{k}}, N_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$$

is an isomorphism.

Proof. Indeed we have first that

$$\mathrm{Hom}_{\mathcal{M}(\overline{k})}(M_{\overline{k}}, N_{\overline{k}}) = \mathrm{colim}_L \mathrm{Hom}_{\mathcal{M}(L)}(M_L, N_L)$$

by continuity, the colimit ranging over finite Galois extensions of k, and then the Galois invariants on the left-hand side correspond to the colimit of the Galois invariants on the

right-hand side. But the action of $\mathrm{Gal}(\overline{k}/k)$ on $\mathrm{Hom}_{\mathcal{M}(L)}(M_L, N_L)$ factors through the finite quotient $\mathrm{Gal}(L/k)$. Étale descent implies (see [RT24b, Corollary 2.1.9]) that

$$\mathcal{M}_{\mathrm{Nori}}(k, \mathbb{Q}) \simeq \mathrm{Fun}(\mathrm{BGal}(L/k), \mathcal{M}(L)) \simeq \mathrm{Rep}(\mathrm{Gal}(L/k), \mathcal{M}(L)),$$

so that in particular, the map

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{Nori}}(k,\mathbb{Q})}(M, N) \to \mathrm{Hom}_{\mathcal{M}(L)}(M_L, N_L)^{\mathrm{Gal}(L/k)}$$

is an isomorphism. □

5.2.2 Lisse mixed abelian motives.

Let k be a subfield of $\mathbb{C}$, embedded though a map $\sigma\colon k \to \mathbb{C}$. Recall the definition of relative Deligne 1-motives ([PL19b, Definition A.1]).

Definition 5.2.4. Let X be a scheme. We denote by $\mathcal{M}^1_{\mathrm{Del}}(X)$ the category of rational relative Deligne 1-motives. It is the idempotent completion of the rationalisation of the category $\mathcal{M}^1_{\mathrm{Del}}(X, \mathbb{Z})$ of relative Deligne 1-motives, which consists of 2-term complexes

$$M = [L \to G]$$

with L a étale locally constant finitely generated and torsion free commutative group scheme over X and G a semi-abelian scheme over X.

By seeing a Deligne 1-motive as a complex of étale sheaves of $\mathbb{Q}$-vector spaces on Sm_X, one can construct a functor

$$\mathcal{M}^1_{\mathrm{Del}}(X) \to \mathrm{DM}^{\text{ét}}(X, \mathbb{Q}). \tag{5.2.4.1}$$

Theorem 5.2.5 (Pépin-Lehalleur [PL19b, Theorem 4.31] and Haas [Haa19, Theorem E])**.** *If X is normal and of finite type over k, then $\mathcal{M}^1_{\mathrm{Del}}(X)$ is an abelian category. In fact the functor (5.2.4.1) is fully faithful and its image consists exactly of dualisable (in $\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})$) objects of $\mathcal{M}^1(X)$ where $\mathcal{M}^1(X)$ is the heart of Pépin-Lehalleur 1-motivic t-structure.*

Theorem 5.2.6 (Ayoub–Barbieri-Viale [ABV15, Proposition 7.1])**.** *The functor*

$$\mathcal{M}^1_{\mathrm{Del}}(k) \to \mathcal{MR}^{\sigma}(k),$$

obtained by composing the inclusion of Deligne 1-motives in étale motives with the mixed realisation functor of étale motives, is fully faithful and its image is closed under subquotients.

Remark 5.2.7. The category of mixed realisations that the authors use in *loc. cit.* is not exactly the same we use, as for them the Hodge filtration is defined over k. However a quick look at their proof can convince the reader that it also proves that the above functor is fully faithful and that its image is closed under subquotients.

Proposition 5.2.8. *Let S be a finite type smooth k-scheme. Then the functor*

$$\mathcal{M}^1_{\mathrm{Del}}(S) \to \mathcal{M}_{\mathrm{perv}}(S)$$

is fully faithful and its image is stable under subquotients. Here the functor is obtained as

$$\mathcal{M}^1_{\mathrm{Del}}(S) \subset \mathrm{DM}^{\text{ét}}_c(S, \mathbb{Q}) \xrightarrow{\rho_{\mathrm{N}}} \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(S)) \xrightarrow{-[d]} \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(S))$$

where $d = \dim X$.

Proof. We can assume that S is connected. Full faithfulness follows from continuity on both sides, the full faithfulness of the functors restricting to the generic point, and the full faithfulness of the functor $\mathcal{M}^1_{\mathrm{Del}}(\eta) \to \mathcal{M}_{\mathrm{perv}}(\eta)$ at the generic point (see the proof of Theorem 5.2.12). Indeed, for Nori motives this is Ivorra-Morel [IM22, Corollary 6.10], and for Deligne 1-motives this is proven by Pépin-Lehalleur in [PL19b, Proposition A.10]. The stability under subquotients follows from Lemma 5.2.11 below, to reduce to the case of the generic point, and Theorem 5.2.6. □

Definition 5.2.9. Let S be a smooth k-scheme. The category $\mathcal{MT}^1(S)$ of mixed abelian motives is the Tannakian subcategory spanned by $\mathcal{M}^1_{\mathrm{Del}}(S)$ in $\mathcal{M}_{\mathrm{liss}}(S)$ the Tannakian category of Nori motivic local systems.

Example 5.2.10. The following motives are mixed abelian motives: for any semi-abelian scheme $p : A \to S$, all ${}^{\mathrm{p}}\mathrm{H}^i(p_*\mathbb{Q}_A)$ for $i \in \mathbb{Z}$, for any curve $p : C \to S$ and any integers $n, m \in \mathbb{Z}$, ${}^{\mathrm{p}}\mathrm{H}^1(p_*\mathbb{Q}_C)^{\otimes m}(n)$. Every Tate motive is a mixed abelian motive.

Lemma 5.2.11. *Let S be a finite type smooth k-scheme. Let η be the generic point of S. Then the pullback functor*

$$\eta^* \colon \mathcal{MT}^1(S) \to \mathcal{MT}^1(\eta)$$

is fully faithful and its image is stable under subquotients.

Proof. Full faithfulness was proven in [IM22]. In fact, for any $j \colon U \to S$ a nonempty open subset, the restriction map $j^* \colon \mathcal{MT}^1(S) \to \mathcal{MT}^1(U)$ is fully faithful. Moreover, if $\tilde{N} \subset \eta^* M$ is a subobject, there exists a $U \subset S$ and some $N^U \subset j^*M$ such that $\eta^* N^U = \tilde{N}$. Thus to prove the lemma it suffices to prove that the image of j^* is stable under subquotients. By the proof of [BBD82, Lemme 4.3.2] we see that on (motivic) local systems the functor j^* has a left-inverse, given by $j_{!*}$. Now, if M is a motivic local system on S and $N \subset j^*M$ is a sub motivic local system, then using [FL81, 0.7 (B)] that says that the topological fundamental group of U surjects onto that of S, we see that the Betti realisation $R_B(N)$ of N extends to a sub local system L of $R_B(M)$. We thus have $L = j_{!*}R_B(N)$, and this proves that the perverse Nori motive $j_{!*}N$ (which is a subobject of M) is in fact a motivic local system as its Betti realisation is L. Thus the image of j^* is stable under sub objects. The stability under quotients is obtained by taking duals. □

Theorem 5.2.12. *Let S be a finite type smooth k-scheme. The mixed realisation functor*

$$R_{\mathcal{MR}^\sigma} \colon \mathcal{MT}^1(S) \to \mathcal{MR}^\sigma(S)$$

is fully faithful and its image is stable under subquotients.

Proof. We can assume S connected. Now as in Proposition 2.2.18 we have the commutative diagram

$$\begin{array}{ccc} \mathcal{MT}^1(S) & \longrightarrow & \mathcal{MT}^1(\eta) \\ \downarrow & & \downarrow \\ \mathcal{MR}^\sigma_{\mathrm{liss}}(S) & \longrightarrow & \mathcal{MR}^\Sigma(\eta) \end{array}$$

where the lower horizontal functor and the left vertical functor are fully faithful, and the other two are faithful. This implies that the upper horizontal functor is fully faithful.

Now let $M \in \mathcal{MT}^1(S)$ and let $F \subset R^\sigma_{\mathcal{MR}}(S)$ be a subobject. By the case of a field (Proposition 5.2.2), we know that there exists a motive $\tilde{N} \in \mathcal{MR}^1(\eta)$ such that the mixed realisation of $\tilde{N}$ is the restriction to η of F. Because the left vertical functor in the above commutative square has its image stable under subquotients, there exists a subobject $N \subset M$ such that the restriction to the generic point of $R^\sigma_{\mathcal{MR}}(N)$ is F_η. To prove that the mixed realisation of N is F, it suffices to check this on the ℓ-adic part, where this follows from the fact that $\pi_1^{\text{ét}}(\eta) \to \pi_1^{\text{ét}}(S)$ is surjective because S is normal. □

5.2.3 Application: the canonical construction for Shimura varieties.

Recall that if $(G, \mathcal{X})$ is a Shimura datum ([Del79, 2.1.1]) and $K \subset G(\mathbb{A}^f_{\mathbb{Q}})$ is an open compact subgroup, we have (if the level K is small enough) the associated Shimura variety $\mathrm{M}^K(G, \mathcal{X})$, which is an algebraic variety over a finite extension k of $\mathbb{Q}$ in $\mathbb{C}$ (called the reflex field) whose complex points are $G(\mathbb{Q}) \setminus \mathcal{X} \times (G(\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}})/K)$. A classical construction ([Pin92, Wil12]), called the canonical construction on the Shimura variety $\mathrm{M}^K(G, \mathcal{X})$, provides a tensor functor

$$\mu_K \colon \mathrm{Rep}(G) \to \mathrm{LS}(\mathrm{M}^K(G, \mathcal{X}))$$

where $\mathrm{LS}(\mathrm{M}^K(G, \mathcal{X}))$ is a category of local systems (ℓ-adic, topological local systems or variations of mixed Hodge structures) on the Shimura variety.

Theorem 5.2.13 (Pink, Burgos-Wildeshaus)**.** *Let $\mathrm{M}^K(G, \mathcal{X})$ be a Shimura variety and let σ be the embedding of the reflex field into $\mathbb{C}$. Then there exists a tensor functor*

$$\mu_K \colon \mathrm{Rep}_{\mathbb{Q}}(G) \to \mathcal{MR}^\sigma(\mathrm{M}^K(G, \mathcal{X}))$$

compatible with the canonical constructions with values in ℓ-adic local systems, topological local systems or variations of mixed Hodge structures.

Proof. Indeed Pink constructs in [Pin92] a functor to ℓ-adic local systems, Wildeshaus constructs a functor to variations of mixed Hodge structures, and one checks as in [Mor08, Section 2.1.4] (see also Wildeshaus's [Wil12]) that they are compatible and define a functor from $\mathrm{Rep}_{\mathbb{Q}}(G)$. □

We will now use Shimura varieties of Hodge type ([Mil13, Definition 11.13]). For those, the group G embeds in GSp_{2g} for some $g > 0$ and moreover there is a closed immersion ([Del71, Proposition 1.15]) in a Siegel modular variety $\mathcal{A}_g^{K'}$ (which is a Shimura variety associated to GSp_{2g}) for some open compact subgroup $K' \subset \mathrm{GSp}_{2g}(\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}})$. In particular there exists an abelian scheme $\pi \colon A \to \mathrm{M}^K(G, \mathcal{X})$, pulled back from the universal abelian scheme on the Siegel modular variety.

Proposition 5.2.14. *Let $\mathrm{M}^K(G, \mathcal{X})$ be a Shimura variety of Hodge type. The canonical construction factors through perverse Nori motives.*

Proof. There is a faithful representation V of G obtained from the embedding $G \to \mathrm{GSp}_{2g}$. Then the functor μ_K sends V to ${}^{\mathrm{p}}\mathrm{H}^1(\pi_* \mathbb{Q}_A)$, where $\pi \colon A \to \mathrm{M}^K(G, \mathcal{X})$ is the abelian scheme described above. In the case we are dealing with a Siegel modular variety this is as in the proof of [Pin92, Lemma 5.6.6], and then it follows by compatibility with pullbacks of the

canonical construction. All objects of $\mathrm{Rep}_{\mathbb{Q}}(G)$ are direct factors of tensor products of V and $V^\vee$ ([DMOS82, Chapter I, Proposition 3.1 (a)]), thus the functor μ_K lands in the Tannakian category generated by ${}^{\mathrm{p}}\mathrm{H}^1(\pi_*\mathbb{Q})$ which is contained in the image of the fully faithful functor

$$\mathcal{MT}^1(\mathrm{M}^K(G,\mathcal{X})) \to \mathcal{MR}^\sigma(\mathrm{M}^K(G,\mathcal{X})),$$

thus this defines a functor

$$\mathrm{Rep}_{\mathbb{Q}}(G) \to \mathcal{M}_{\mathrm{perv}}(\mathrm{M}^K(G,\mathcal{X})).$$

□

Chapter 6

Mixed Hodge modules on stacks.

In this chapter, we explain how to extend Mixed Hodge modules on stacks. The idea is simple but the realisation is complicated as we have to use the machinery developped by Liu and Zheng. In the first section we construct the extension and compare it to existing constructions in particular cases, and in the second section we prove that nearby cycles extend also well to algebraic stacks.

6.1 Extension to stacks of Hodge modules.

6.1.1 Operations and t-structures.

We begin by explaining the idea: In Section 2.1.1.2 we proved that the 6 operations on mixed Hodge modules could be lifted at the level of ∞-categories with a lot of coherence. In particular, we have at hand a h-hypersheaf

$$\mathrm{D_H}\colon \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$$

that takes a finite type $\mathbb{C}$-scheme X to $\mathrm{IndD}^b(\mathrm{MHM}(X))$ the indization of the bounded derived category of mixed Hodge modules, and a map of scheme is sent to the pullback functor. From this, we can extend mixed Hodge modules to algebraic spaces, because of étale descent: if Y is an algebraic space locally of finite type over $\mathbb{C}$, there exists an étale covering $X_i \to Y$ of Y by finite type $\mathbb{C}$-schemes, and such that, because the diagonal of Y is representable by schemes, any higher intersection $X_J = X_{i_1} \times_Y X_{i_2} \times_Y \cdots X_{i_n}$ with $J = \{i_1, \ldots, i_n\}$ is still a finite type $\mathbb{C}$-scheme. In particular, the limit of the diagram

$$\prod_{i\in I} \mathrm{D_H}(X_i) \rightrightarrows \prod_{\substack{\mathrm{Card}(J)=2\\ J\subset I}} \mathrm{D_H}(X_J) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots \tag{6.1.0.1}$$

makes sense, and does not depend (by étale descent) on the chosen covering, thus defines a well-defined presentable ∞-category $\mathrm{D_H}(X)$. To make it more clear that it is choice free, we can make the following definition:

Definition 6.1.1. We define $\mathrm{D_H}\colon \mathrm{AlgSpc}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ to be the right Kan extension of the above functor along the inclusion of finite type $\mathbb{C}$-schemes into locally of finite type algebraic spaces over $\mathbb{C}$.

Proposition 6.1.2. *The functor* $\mathrm{D_H}\colon \mathrm{AlgSpc}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ *obtained as above is a h-hypersheaf. In particular, its value at an algebraic space* Y *can be computed as in* (6.1.0.1).

Proof. This follows from the fact that a right Kan extension of an hypersheaf along the inclusion of a basis of a site is still an hypersheaf, by Proposition A.1.38. □

Now if $\mathfrak{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$, there is a smooth surjection $\pi\colon X \to \mathfrak{X}$ with X a scheme locally of finite type over $\mathbb{C}$. All self intersections $X \times_{\mathfrak{X}} X$ are algebraic spaces, hence we may do as above, setting

$$\mathrm{D_H}(\mathfrak{X}) \xrightarrow{\pi^*} \mathrm{D_H}(X) \rightrightarrows \mathrm{D_H}(X \times_{\mathfrak{X}} X) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \dots \tag{6.1.2.1}$$

as a limit in $\mathrm{CAlg}(\mathrm{Pr}^L)$. As above, by descent along smooth morphisms, the above definition of $\mathrm{D_H}(\mathfrak{X})$ does not depend on the choice of $\pi\colon X \to \mathfrak{X}$.

Definition 6.1.3. The functor $\mathrm{D_H}\colon \mathrm{Stk}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ is the right Kan extension of the functor defined in Definition 6.1.1 along the inclusion of algebraic spaces locally of finite type to stacks locally of finite type over $\mathbb{C}$.

As before, we have:

Proposition 6.1.4. *The functor* $\mathrm{D_H}\colon \mathrm{Stk}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ *obtained as above is a h-hypersheaf. In particular, its value at an algebraic stack* $\mathfrak{X}$ *can be computed as in* (6.1.2.1).

Remark 6.1.5. One could continue this further and define the value of $\mathrm{D_H}$ at any derived stack locally of finite type over $\mathbb{C}$. We choose not to pursue this here. Moreover, the localisation sequence shows that the ∞-category obtained this way will only depend on the underlying stack in groupoids.

Using Liu and Zheng gluing technique ([LZ17]) as Khan in [Kha19, Appendix A.], one can prove that the extension of $\mathrm{D_H}$ to (higher) Artin stacks still have the six operations. Unless mentioned otherwise, all stacks and schemes considered will be locally of finite type over $\mathbb{C}$. This subsection is more or less book keeping of the work of Liu, Zheng and Khan. More precisely, we have:

Theorem 6.1.6 (Liu-Zheng, Khan)**.**

1. *For every Artin stack* $\mathfrak{X}$ *there is a closed symmetric monoidal structure on* $\mathrm{D_H}(\mathfrak{X})$.
2. *For any morphism* $f : \mathcal{Y} \to \mathfrak{X}$ *there is an adjunction*

$$\mathrm{D_H}(\mathfrak{X}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{D_H}(\mathcal{Y}) \qquad (\perp)$$

   with f^* *a symmetric monoidal functor.*
3. *For any locally of finite type morphism of Artin stacks* $f : \mathfrak{X} \to \mathcal{Y}$ *there is an adjunction (functorial in* f*)*

$$\mathrm{D_H}(\mathfrak{X}) \underset{f^!}{\overset{f_!}{\rightleftarrows}} \mathrm{D_H}(\mathcal{Y}) \quad . \qquad (\perp)$$

4. *The operations $f_!$ satisfy base change and projection formula against g^*, and the operations $f^!$ satisfy base change against g_*. If f is representable by Deligne-Mumford stacks, then there is a natural transformation $\alpha_f : f_! \to f_*$, which is an isomorphism if f is proper and representable by algebraic spaces.*

5. *For any closed immersion $i : \mathfrak{Z} \to \mathfrak{X}$ of Artin stacks with $j : \mathfrak{U} \to \mathfrak{X}$ the inclusion of the open complement, we have a pullback diagram*

$$\begin{array}{ccc} \mathrm{D_H}(\mathfrak{Z}) & \xrightarrow{i_*} & \mathrm{D_H}(\mathfrak{X}) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ * & \longrightarrow & \mathrm{D_H}(\mathfrak{U}) \end{array} .$$

6. *For any Artin stack $\mathfrak{X}$, the functor*

$$\pi^* : \mathrm{D_H}(\mathfrak{X}) \to \mathrm{D_H}(\mathbb{A}^1_{\mathfrak{X}})$$

is fully faithful.

7. *For a smooth morphism $f : \mathfrak{Y} \to \mathfrak{X}$ of relative dimension d, we have a purity isomorphism*

$$\mathfrak{p}_f : f^*(d)[2d] \simeq f^!.$$

8. *The $!$-functoriality $\mathrm{D_H}(-)^!$ is also an étale hypersheaf.*

Proof. The proofs of [Kha19, Appendix A] hold for any motivic coefficient system, so that 1,2,3 and 4 are [Kha19, Theorem A.5], 5. is [Kha19, Theorem A.9], 6. is [Kha19, Proposition A.10] and 7. is [Kha19, Theorem A.13]. The last point 8. follows from [LZ17, Proposition 4.3.5].

□

Remark 6.1.7. All results above hold more generally for higher Artin stacks, except for 4. where one need f to be 0-truncated on top of being proper for $f_!$ to be isomorphic to f_*.

Remark 6.1.8. One could state the same result for mixed Hodge modules of geometric or Hodge origin to stacks, and all the results we stated above and below also hold in this more restrictive setting. Note that because the construction is a right Kan extension from the category of correspondences of schemes to the category of correspondences of algebraic stacks, the functor $\mathrm{DH} \to \mathrm{D_H}$ will automatically commute with the left adjoints f^*, $f_!$ and $\otimes$, but the commutation with the right adjoints is unclear. For internal homomorphisms this can be checked on a smooth atlas, but the cases of f_* and $f^!$ for a general non-representable morphism seems subtle.

Now, as $\mathrm{D}^b(\mathrm{MHM}(-))$ also satisfies h-descent, we can also right Kan extend it and have the same formula as (6.1.2.1). In particular as limits in Cat_∞ are computed term wise, if one denote by $\mathrm{D}^b_{\mathrm{H},c}(-)$ the extension of $\mathrm{D}^b(\mathrm{MHM}(-))$ to (higher) Artin stacks, we have a fully faithful natural transformation

$$\iota \colon \mathrm{D}^b_{\mathrm{H},c}(-) \to \mathrm{D_H}(-).$$

The question whether for a general morphism f of Artin stacks, the operations f_* and $f_!$ preserve constructibility is a bit subtle. Of course, it is false in general that $f_*\mathbb{Q}$ is cohomologically bounded.

For a finite type $\mathbb{C}$-scheme X, the category $\mathrm{D_H}(X) = \mathrm{IndD}^b(\mathrm{MHM}(X))$ admits a t-structure by Proposition 1.1.11. The t-structure on $\mathrm{D_H}(X)$ is right complete and compatible with filtered colimits.

Proposition 6.1.9. *Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$. The ∞-category $\mathrm{D_H}(\mathfrak{X})$ admits a t-structure such that for any smooth presentation $\pi : X \to \mathfrak{X}$ the conservative functor π^* is t-exact up to a shift. Moreover, the t-structure is right complete and compatible with filtered colimits. It restricts to a t-structure on $\mathrm{D}^b_{\mathrm{H},c}(-)$.*

Proof. We have to first deal with the case $\mathfrak{X}$ is an algebraic space. In this case, the map π can be chosen étale, and as the diagonal of $\mathfrak{X}$ is representable by schemes, the limit diagram

$$\mathrm{D_H}(\mathfrak{X}) \to \lim_{\Delta}(X^{n/X})$$

has t-exact transitions whence the limit $\mathrm{D_H}(\mathfrak{X})$ has a perverse t-structure. The left separation and compatibility with colimits can be checked after applying π^*, hence hold. The right completeness is checked exactly as in the case of stacks, that we deal with below.

We can assume that $\mathfrak{X}$ is a connected algebraic stack. Choose a presentation $\pi : X \to \mathfrak{X}$ which is smooth of relative dimension d for some integer d. Then all the projections

$$p_n : X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X \to X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$$

are also smooth of relative dimension d, so that each $p_n^*[d]$ are t-exact for the perverse t-structure. As (6.1.2.1) can be done with the shifts $p_n^*[d]$, this creates a t-structure on the limit of the Čech nerve of π such that π^* is t-exact. It does not depend on π as one can see by taking another presentation $p : Y \to \mathfrak{X}$ and then $Y \times_{\mathfrak{X}} X \to \mathfrak{X}$. The t-structure obviously restricts to $\mathrm{D}^b_{\mathrm{H},c}(\mathfrak{X})$. For the right completeness, we want to show that the natural functor

$$\mathrm{D_H}(\mathfrak{X}) \to \lim \left(\cdots \xrightarrow{\tau^{\geqslant 1}} \mathrm{D}^{\geqslant 1}_{\mathrm{H}}(\mathfrak{X}) \xrightarrow{\tau^{\geqslant 2}} \mathrm{D}^{\geqslant 2}_{\mathrm{H}}(\mathfrak{X}) \xrightarrow{\tau^{\geqslant 3}} \mathrm{D}^{\geqslant 3}_{\mathrm{H}}(\mathfrak{X}) \xrightarrow{\tau^{\geqslant 4}} \cdots \right)$$

is an equivalence. As (shifts of) pullbacks by smooth maps are t-exact, this can be checked locally on $\mathfrak{X}$, hence holds. □

Definition 6.1.10. Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$. The ∞-category of cohomologically constructible mixed Hodge modules over $\mathfrak{X}$ is

$$\mathrm{D}_{\mathrm{H},c}(\mathfrak{X}) := \{K \in \mathrm{D_H}(\mathfrak{X}) \mid \forall n \in \mathbb{Z}, \mathrm{H}^n(K) \in \mathrm{D}^b_{\mathrm{H},c}(\mathfrak{X})\}.$$

By definition the t-structure restricts to $\mathrm{D}_{\mathrm{H},c}(\mathfrak{X})$, and the heart is the same as the heart of $\mathrm{D}^b_{\mathrm{H},c}(\mathfrak{X})$.

For a finite type $\mathbb{C}$-scheme X, the stable ∞-category $\mathrm{D}^b(\mathrm{MHM}(X))$ also have a *ordinary* t-structure (as in Definition 2.1.1), and in Section 2.1 (where the t-structure had the unfortunate name 'constructible') we proved that it is the derived category of the constructible heart. In fact, all the considerations above about the perverse t-structure are true for the constructible t-structure. We mention them because the ordinary t-structure will happen to be handy when proving the constructibility of the operations.

Proposition 6.1.11. *Let $\mathfrak{X}$ be an Artin stack over $\mathbb{C}$. The ∞-category $\mathrm{D}_{\mathrm{H}}(\mathfrak{X})$ admits a ordinary t-structure such that for any presentation $\pi : X \to \mathfrak{X}$ the conservative functor π^* is t-exact if we endow $\mathrm{D}_{\mathrm{H}}(X)$ with the t-structure induced by the canonical t-structure on $\mathrm{D}^b(\mathrm{MHM}_{\mathrm{ord}})$. Moreover, the t-structure is right complete and compatible with filtered colimits. It restricts to a t-structure on $\mathrm{D}^b_{\mathrm{H},c}(-)$. All pullbacks by all morphisms of Artin stacks are t-exact for this t-structure.*

Proof. The proof is exactly the same as for Proposition 6.1.9, except that we do not need to shift the pullback functors. □

We will denote by ${}^{\mathrm{ct}}\mathrm{H}^n$ the cohomology objects for the ordinary t-structure.

Proposition 6.1.12. *Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$ and let $K \in \mathrm{D}_{\mathrm{H}}(\mathfrak{X})$. Then $K \in \mathrm{D}_{\mathrm{H},c}(\mathfrak{X})$ if and only if for all $n \in \mathbb{Z}$ the object ${}^{\mathrm{ct}}\mathrm{H}^n(K)$ is in $\mathrm{D}^b_{\mathrm{H},c}(\mathfrak{X})$.*

Proof. Both conditions on K are local on $\mathfrak{X}$, hence we can assume that $\mathfrak{X}$ is a finite type $\mathbb{C}$-scheme X. If K is bounded the result is trivial, and if not, we can reduce to the bounded case by noting the following: $\tau^{p\leqslant n}$ preserves $\mathrm{D}_{\mathrm{H}}(X)^{\mathrm{ord}\leqslant n}$, $\tau^{\mathrm{ord}\geqslant n}$ preserves $\mathrm{D}_{\mathrm{H}}(X)^{p\geqslant 0}$, the restriction of $\tau^{p\geqslant n}$ to $\mathrm{D}_{\mathrm{H}}(X)^{\mathrm{ord}\geqslant n}$ is the identity and the restriction of $\tau^{\mathrm{ord}\leqslant n}$ to $\mathrm{D}_{\mathrm{H}}(X)^{p\leqslant n}$ is the identity. Here τ^{ord} and τ^p are the truncation functors for the ordinary and perverse t-structures, respectively. □

Theorem 6.1.13 (Liu-Zheng)**.** *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of Artin stacks locally of finite type over $\mathbb{C}$. Then we have the following:*

1. *The ∞-category $\mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})$ is stable under tensor products, and f^* restricts to a functor*
$$f^* \colon \mathrm{D}_{\mathrm{H},c}(\mathfrak{X}) \to \mathrm{D}_{\mathrm{H},c}(\mathfrak{Y}).$$
2. *The functor f_* restricts to a functor*
$$f_* \colon \mathrm{D}^+_{\mathrm{H},c}(\mathfrak{Y}) \to \mathrm{D}^+_{\mathrm{H},c}(\mathfrak{Y})$$
and even to
$$f_* \colon \mathrm{D}_{\mathrm{H},c}(\mathfrak{Y}) \to \mathrm{D}_{\mathrm{H},c}(\mathfrak{Y})$$
if f is representable by algebraic spaces.
3. *The functor $f_!$ restricts to a functor*
$$f_! \colon \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{Y}) \to \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{Y})$$
and even to
$$f_! \colon \mathrm{D}_{\mathrm{H},c}(\mathfrak{Y}) \to \mathrm{D}_{\mathrm{H},c}(\mathfrak{Y})$$
if f is representable by algebraic spaces.
4. *The functor $f^!$ restricts to a functor*
$$f^! \colon \mathrm{D}_{\mathrm{H},c}(\mathfrak{X}) \to \mathrm{D}_{\mathrm{H},c}(\mathfrak{Y}).$$

5. *The internal Hom functor restricts to a functor*

$$\mathrm{D}^{-}_{\mathrm{H},c}(\mathfrak{X})^{\mathrm{op}} \times \mathrm{D}^{+}_{\mathrm{H},c}(\mathfrak{X}) \to \mathrm{D}^{+}_{\mathrm{H},c}(\mathfrak{X}).$$

Proof. The proof is essentially the same as the proofs of [LZ17, 6.4.4 and 6.4.5]. The point 1. is easy. We show how to obtain 2. by hand, for example: Take $\pi : X \to \mathfrak{X}$ a presentation of $\mathfrak{X}$ and form the pullback

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{g} & X \\ \downarrow{\scriptstyle q} & \lrcorner & \downarrow{\scriptstyle \pi} \\ \mathfrak{Y} & \xrightarrow{f} & \mathfrak{X} \end{array}.$$

Then to show that each ${}^{\mathrm{ct}}\mathrm{H}^n(f_! K)$ are constructible, we may check it locally, whence as π^* is t-exact it suffices to show that ${}^{\mathrm{ct}}\mathrm{H}^n(\pi^* f_! K)$ is constructible. By base change, it suffices to show that ${}^{\mathrm{ct}}\mathrm{H}^n(g_! q^* K)$ is constructible: we reduced to the case where $\mathfrak{X}$ is a scheme. Now, choose a presentation $r : Z \to \mathfrak{Z}$ of $\mathfrak{Z}$. Descent for !-functoriality implies that $q^* K = \mathrm{colim}_{\Delta}(g_n)_! r_n^! q^* K$ where (g_n) is the composition of the Čech nerve (r_n) of r with g. The spectral sequence induced by this geometric realisation reads

$$E_1^{p,q} = {}^{\mathrm{ct}}\mathrm{H}^{-q}((g_p)_! r_p^! q^* K) \Rightarrow {}^{\mathrm{ct}}\mathrm{H}^{p-q}(g_! q^* K).$$

Now the each $r_p^! q^* K$ is cohomologically constructible by purity (7. of Theorem 6.1.6) and each g_p is a morphism of algebraic spaces, of relative dimension the sum of the relative dimensions of g and of r. In particular, all g_p have the same cohomological amplitude thus our spectral sequence is concentrated in a shifted quadrant: it vanishes if $p < 0$ and if q is smaller than some bound depending on the cohomological amplitude of $g_!$ and of K. This implies that the spectral sequence converges and we have that $\mathrm{H}^n(g_! q^* K)$ is constructible for all n. In the case f is representable by algebraic spaces, we can reduce to the case of schemes where it follows from finite cohomological amplitude. The case of f_* is similar. □

Remark 6.1.14. We may have to use the same extension to stack for others systems of coefficients such as analytic sheaves or étale motives. Because the functor between those and mixed Hodge modules commute with all the operations on schemes, this is also the case for their extensions to stacks.

6.1.1.1 Duality

We want a working Verdier duality on algebraic stacks locally of finite type over the complex numbers.

Definition 6.1.15. Let $\mathfrak{X}$ be an algebraic stack locally of finite type over the complex numbers. The dualising object on $\mathfrak{X}$ is the object $\omega_{\mathfrak{X}} := \pi_{\mathfrak{X}}^! \mathbb{Q}(0) \in \mathrm{D}_{\mathrm{H}}(\mathfrak{X})$ with $\pi_{\mathfrak{X}} \colon \mathfrak{X} \to \mathrm{Spec}\,\mathbb{C}$. The Verdier duality functor is

$$\mathbb{D}_{\mathfrak{X}} := \mathscr{H}om(-, \omega_{\mathfrak{X}}).$$

If follows directly from the projection formula between $g_!$ and g^*, for g a morphism of algebraic stacks, that there are natural equivalences

$$g^! \circ \mathbb{D}_{\mathfrak{X}} \simeq \mathbb{D}_{\mathfrak{Y}} \circ g^*$$

and

$$g_* \circ \mathbb{D}_{\mathfrak{Y}} \simeq \mathbb{D}_{\mathfrak{X}} \circ g_!$$

for any morphism $g : \mathfrak{Y} \to \mathfrak{X}$ between algebraic stacks locally of finite type over $\mathbb{C}$.

Lemma 6.1.16. *Let $g : \mathfrak{Y} \to \mathfrak{X}$ be a* smooth *morphism between algebraic stacks locally of finite type over $\mathbb{C}$. Then there is also a natural equivalence*

$$g^* \circ \mathbb{D}_{\mathfrak{X}} \simeq \mathbb{D}_{\mathfrak{Y}} \circ g^!$$

of functors $\mathrm{D_H}(\mathfrak{X}) \to \mathrm{D_H}(\mathfrak{Y})$.

Proof. The smooth projection formula implies that the canonical map

$$g^* \mathscr{H}om(-, \omega_X) \to \mathscr{H}om(g^*(-), g^*\omega_{\mathfrak{X}})$$

is an equivalence. Now, using the purity isomorphism twice $g^* \simeq g^!(-d)[-2d]$ we see that the right hand side is equivalent to $\mathscr{H}om(g^!(-), g^!\omega_{\mathfrak{X}})$, and because $g^!\omega_{\mathfrak{X}} \simeq \omega_{\mathfrak{Y}}$, the proof is finished. □

Proposition 6.1.17. *Verdier duality is an anti auto-equivalence when restricted to $\mathrm{D}^b_H(\mathfrak{X})$. It swaps $!$ and $*$ when they preserve $\mathrm{D}^b_{\mathrm{H},c}$ and it is perverse t-exact.*

Proof. First note that duality preserves $\mathrm{D}^b_{\mathrm{H},c}$ because this can be checked locally. There is a canonical map

$$\mathrm{Id} \to \mathbb{D}_{\mathfrak{X}} \circ \mathbb{D}_{\mathfrak{X}}.$$

For a given $M \in \mathrm{D}^b_{\mathrm{H},c}(\mathfrak{X})$ and a presentation $g : \mathfrak{X} \to X$, it suffices to check that the map

$$g^*M \to g^*(\mathbb{D}_{\mathfrak{X}} \circ \mathbb{D}_{\mathfrak{X}}(M))$$

is an equivalence. But now by Lemma 6.1.16 the above map is an equivalence because it is the map

$$g^*M \to \mathbb{D}_X \circ \mathbb{D}_X(g^*M).$$

The remaining assertions follows from the autoduality over schemes ([Sai90b, Theorem 0.1]). □

Corollary 6.1.18. *Verdier duality in fact extends to an equivalence of $\mathrm{D}^+_{\mathrm{H},c}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})^{\mathrm{op}}$. It swaps $!$ and $*$.*

Proof. By the previous assertion, the functor $\mathbb{D}_{\mathfrak{X}}$ is perverse t-exact, and an isomorphism on the heart. As the perverse t-structure is left separated when restricted to $\mathrm{D}^+_{\mathrm{H},c}(\mathfrak{X})$, this implies the claim. □

We now give an amelioration of Theorem 6.1.6(4) whose proof in the ℓ-adic setting is due to Olsson [Ols15, Corollary 5.17], see also [Kha19, Theorem A.7]:

Proposition 6.1.19. *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a proper morphism represented by Deligne-Mumford stacks. Then the canonical morphism $f_! \to f_*$ is an isomorphism on $\mathrm{D}^+_{\mathrm{H},c}(\mathfrak{Y})$.*

Proof. By Theorem 6.1.6 we know the result if f is representable by algebraic spaces. Moreover because the diagonal is finite, we can use [AOV11, Theorem 3.1] to obtain a factorisation of f as $f = g \circ \pi$ with g proper representable and π proper quasi-finite. Thus we may assume f to be proper and quasi-finite. We start by two facts:

1. Both functors f_* and $f_!$ send $\mathrm{D}^b_{\mathrm{H,c}}(\mathfrak{Y})$ to $\mathrm{D}^b_{\mathrm{H,c}}(\mathfrak{X})$. Indeed, by duality (Proposition 6.1.17), it suffices to prove the result for $f_!$. We know by Theorem 6.1.13 that $f_!$ sends bounded below complexes to bounded below complexes. But using that f is quasi-finite, one proves that $f^!$ is left t-exact for the ordinary t-structure thus $f_!$ is right t-exact (using [BBD82, Proposition 1.3.17]), and in particular it also preserves bounded above complexes. The proof that $f^!$ is left t-exact is the same as see [Ols15, Lemma 4.3] (*loc. cit.* reduces to showing that is g is a morphism of pure dimension d between schemes, then $g^![-2d]$ is left t-exact, this can be reduced to quasi-projective morphisms where this is clear).

2. If f is a proper morphism of algebraic stacks, then the formation of f_* is compatible with base change on $\mathrm{D}^+_{\mathrm{H,c}}$. Indeed using [Ols05, Theorem 1.1] the proof is the same as [LMB00, Théorème 18.5.1]: one uses the h-hyperdescent spectral sequence to reduce to the case of representable morphisms.

By proper and smooth base change we may assume that $\mathfrak{X}$ is a scheme X to prove the proposition. By dévissage, it suffices to prove the result for a $M \in \mathrm{D}^b_{\mathrm{H,c}}(\mathfrak{Y})^\heartsuit$. By the first point, it suffices to check the isomorphism after after applying x^* to $f_!M \to f_*M$ for every closed point x of X (indeed, on $\mathrm{D}^b_{\mathrm{H,c}}(X)$ the functor rat is conservative, and for constructible sheaves, the family of x^* is conservative). By the second point and the functoriality of the map $f_! \to f_*$, we have that

$$x^*(f_!M \to f_*M) \simeq (\overline{f}_!M_{|\mathfrak{Y}_x} \to \overline{f}_*M_{|\mathfrak{Y}_x})$$

where $\overline{f}$ is the base change of f by $x \to X$, and $\mathfrak{Y}_x$ is the fiber of f at x. In particular, we may assume that the target of f is $\mathrm{Spec}(\mathbb{C})$. By definition, the underlying topological space of $\mathfrak{Y}$ is then a finite discrete space. In particular, we may assume that $\mathfrak{Y}$ is connected: it has only one point. This implies that we have a nil-immersion $BG \to \mathfrak{Y}$ where G is the (finite) group of automorphisms of this point. Using that localisation makes mixed Hodge modules invariant under universal homeomorphisms, we may assume that the target $\mathfrak{Y}$ of f is of the form BG for a finite group G. Then we have a finite covering $\mathrm{Spec}(\mathbb{C}) \to \mathfrak{Y}$ and we may apply [Kha19, Theorem A.7.]. □

Proposition 6.1.20. *All result above about the extension of the operations work for the category* $\mathrm{D}_B := \mathrm{Ind}\mathrm{D}^b_c((-)^{\mathrm{an}}, \mathbb{Q})$ *of Ind-constructible complexes, the forgetful functor* $\mathrm{rat}\colon \mathrm{D}_{\mathrm{H}} \to \mathrm{D}_B$ *extends to schemes, and commutes with the following functors:*

1. $- \otimes -$, $\mathscr{H}om(-,-)$, *and Verdier duality* $\mathbb{D}_{\mathfrak{X}}$.
2. f^* *for any morphism.*
3. $f_!$ *for any finite type morphism.*
4. f_* *for any separated representable morphism, or proper morphism representable by Deligne-Mumford stacks.*

When restricted to $\mathrm{D}^+_{\mathrm{H,c}}$*, it also commutes with:*

1. *Any pushforward* f_**.*
2. *Any exceptional pullback* $f^!$ *if* f *is a finite type morphism.*

Proof. As both functors on stacks are defined using the right Kan extension on the category of correspondences (see the proof of [Man22, Proposition A.5.16]), it is clear that the map $\mathrm{D}_{\mathrm{H}} \to \mathrm{D}_B$ commutes with tensor products, $*$-pullbacks and !-pushforwards. We prove point 4: using smooth base change, by choosing a smooth atlas, we see that it suffices to prove point 4 for a morphism $f\colon Z \to X$ where X is a scheme and Z is an algebraic space. Because the morphism is separated we can take a Nagata compatification ([CLO12]), which reduces to j_* for j an open immersion (this is easy because the morphism is representable by schemes) and to p_* for p a proper morphism of algebraic spaces (which we know is $p_!$). For $\mathscr{H}om(-,-)$ and Verdier duality it is simpler as one can reduce to schemes directly using that rat commutes with twists and that both functors $\mathscr{H}om$ and $\mathbb{D}_{\mathfrak{X}}$ commute with π^* for π a smooth morphism, up to a twist and a shift (by purity).

Now on constructibles, Verdier duality is an equivalence an swaps $*$ and !'s, thus points 1 and 2 follow from 2 and 3 of the above list.

□

6.1.1.2 Weights

In this subsection we introduce weights for mixed Hodge modules on stacks. We will work only with stack that are of finite type over $\mathbb{C}$ in this section. The abelian category of mixed Hodge modules over a scheme has a functorial and exact *weight filtration*. Using ideas of the PhD of Sophie Morel, for each integer $w \in \mathbb{Z}$ we can define a t-structure "$\omega \leqslant w$" on $\mathrm{D}^b_{\mathrm{H,c}}(X)$, that we call the *weight-t-structure* (Construction 6.1.21). It has the particularity that both its right and left sides are stable subcategories of $\mathrm{D}^b_{\mathrm{H,c}}(X)$, so that the heart is zero, but the truncations functors are exact functors. This provides the canonical extension of the weight filtration to the bounded derived category: we obtain exact functors $\tau^{(\omega\leqslant w)\leqslant 0} =: \omega_{\leqslant w}\colon \mathrm{D}^b_{\mathrm{H,c}}(X) \to \mathrm{D}^b_{\mathrm{H,c}}(X)$ that are perverse t-exact, and whose restriction to the perverse heart gives back the weight filtration of mixed Hodge modules. By indization we obtain the weight filtration on the Ind-category. For this t-structure, all pullbacks by smooth morphisms are t-exact, so that we can extend the weight filtration to any algebraic stack by descent (Proposition 6.1.22). In particular, the perverse heart of $\mathrm{D}_{\mathrm{H}}(\mathfrak{X})$ is endowed with an exact weight filtration. This gives the possibility to define *punctual weights* (we chose this name because this is the correct analogue of punctual weights in the ℓ-adic setting) in Definition 6.1.23.

However, this does not quite exactly gives a weight structure *à la Bondarko*. Indeed, the weight filtration may fail to have sufficiently many orthogonality property, so that the property that a pure perverse objects is semi-simple may not be satisfied (see Remark 6.1.29). However, if $\mathfrak{X}$ is an algebraic stack *with affine stabilisers*, then the objects that are of punctual weight 0 are indeed the heart of a weight structure, as we show in Corollary 6.1.28.

Construction 6.1.21. Let $w \in \mathbb{Z}$ and X be a finite type $\mathbb{C}$-scheme. Recall that following [Mor08] we can construct a weight-t-structure on $\mathrm{D}^b_{\mathrm{H},c}(X)$ by setting

$$\mathrm{D}^b_{\mathrm{H},c}(X)^{\omega\leqslant w} := \{K \in \mathrm{D}^b_{\mathrm{H},c}(X) \mid \forall n \in \mathbb{Z}, \forall m > w,\ \mathrm{gr}^W_m({}^{\mathrm{p}}\mathrm{H}^n(K)) = 0\}$$

and

$$\mathrm{D}^b_{\mathrm{H},c}(X)^{\omega\geqslant w} := \{K \in \mathrm{D}^b_{\mathrm{H},c}(X) \mid \forall n \in \mathbb{Z}, \forall m < w,\ \mathrm{gr}^W_m({}^{\mathrm{p}}\mathrm{H}^n(K)) = 0\},$$

where the gr^W_m are the graded pieces of the weight filtration on mixed Hodge modules ([Sai89, Proposition 1.5]). The pair $(\mathrm{D}^b_{\mathrm{H},c}(X)^{\omega\leqslant w}, \mathrm{D}^b_{\mathrm{H},c}(X)^{\omega\geqslant w+1})$ form a t-structure (of trivial heart) on $\mathrm{D}^b_{\mathrm{H},c}(X)$, such that the truncations functors $\omega_{\leqslant w}$ and $\omega_{\geqslant w+1}$ are exact functors, t-exact for the perverse t-structure ([Mor08, Proposition 3.1.1.]). They give a filtration $(\omega_{\leqslant w}K)_w$ on each complex K, that give back the weight filtration on the heart. For any smooth map $f : Y \to X$ of finite type $\mathbb{C}$-schemes, and any integer d, the functor $f^*[d]$ is t-exact for the weight t-structures (indeed, if d is the relative dimension of f we see that $f^*[d]$ is perverse t-exact and preserves weights, thus is t-exact for the weight-t-structure. The shift functor being weight-t-exact this proves the claim). This induces a weight t-structure on $\mathrm{D}_{\mathrm{H}}(X)$ by indization and it has a similar description.

Proposition 6.1.22. *Let $\mathfrak{X}$ be an Artin stack of finite type over $\mathbb{C}$. Then the ∞-categories $\mathrm{D}_{\mathrm{H}}(\mathfrak{X})$ and $\mathrm{D}_{\mathrm{H},c}(\mathfrak{X})$ admit weight-t-structures such that for every smooth map $f : X \to \mathfrak{X}$ with X a finite type $\mathbb{C}$-scheme and every integer d, the functor f^* is weight-t-exact. Moreover, the weight truncations functors are perverse t-exact. The inclusion functor $\mathrm{D}_{\mathrm{H},c}(\mathfrak{X}) \to \mathrm{D}_{\mathrm{H}}(\mathfrak{X})$ is weight-t-exact.*

Proof. Once again assume that $\mathfrak{X}$ is connected, choose $\pi : X \to \mathfrak{X}$ a presentation of $\mathfrak{X}$, of relative dimension d. Then if f_n is a part of the Čech nerve of π, the functors $f_n^*[d]$ are t-exact for the weight t-structures, thus this induces a weight t-structure on the limit as in (6.1.2.1) (as in the proof of Proposition 6.1.9 we have to first deal with algebraic spaces but everything works the same). Of course it restricts to $\mathrm{D}^b_{\mathrm{H},c}(\mathfrak{X})$, and then to $\mathrm{D}_{\mathrm{H},c}(\mathfrak{X})$. The perverse t-exactness of the weight truncation follows from the fact that each $f_n^*[d]$ are also perverse t-exact so that this follows from the same property over schemes. □

Definition 6.1.23. Let $w \in \mathbb{Z}$. A object $K \in \mathrm{D}_{\mathrm{H}}(\mathfrak{X})$ is of *punctual weights* $\leqslant w$ if for each $i \in \mathbb{Z}$, the map ${}^{\mathrm{p}}\mathrm{H}^i(\omega_{\leqslant w+i}K) \to {}^{\mathrm{p}}\mathrm{H}^i(K)$ is an equivalence (or equivalently, if ${}^{\mathrm{p}}\mathrm{H}^i(\omega_{\geqslant w+i+1}K) = 0$). We say that K is of punctual weights $\geqslant w$ if its Verdier dual $\mathbb{D}_{\mathfrak{X}}(K)$ is of punctual weights $\leqslant -w$. We shall denote by $\mathrm{D}^?_{\mathrm{H},(c)}(\mathfrak{X})_{\leqslant w}$ and $\mathrm{D}^?_{\mathrm{H},(c)}(\mathfrak{X})_{\geqslant w}$ the corresponding full subcategories, where $? \in \{\emptyset, b, +, -\}$.

Remark 6.1.24. The above definition *differs* from the weight t-structure we used to define weights.

Proposition 6.1.25. *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of algebraic stacks of finite type over $\mathbb{C}$ and let $w, w' \in \mathbb{Z}$.*

1. *Verdier duality on $\mathfrak{X}$ swaps $\mathrm{D}^+_{\mathrm{H},c}(\mathfrak{X})_{\leqslant w}$ and $\mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})_{\geqslant -w}$.*
2. *The pullback f^* sends $\mathrm{D}_{\mathrm{H},c}(\mathfrak{X})_{\leqslant w}$ to $\mathrm{D}_{\mathrm{H},c}(\mathfrak{Y})_{\leqslant w}$ and the exceptional pullback $f^!$ sends $\mathrm{D}_{\mathrm{H},c}(\mathfrak{X})_{\geqslant w}$ to $\mathrm{D}_{\mathrm{H},c}(\mathfrak{Y})_{\geqslant w}$. If f is smooth, the pullback functor f^* preserves weights.*
3. *The tensor product restricts to*

$$- \otimes - : \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})_{\leqslant w} \times \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})_{\leqslant w'} \to \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})_{\leqslant w+w'}.$$

4. The internal homomorphism functor restricts to

$$\mathscr{H}om(-,-)\colon \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})_{\leqslant w} \times \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})_{\geqslant w'} \to \mathrm{D}^-_{\mathrm{H},c}(\mathfrak{X})_{\geqslant w'-w}.$$

Proof. By definition, if $\pi : X \to \mathfrak{X}$ is a smooth presentation, the functor π^* is conservative and detects weights. Thus all results follows from the usual results over schemes (see [Sai89, Propositions 1.7 and 1.9]). □

Proposition 6.1.26. *Let $\mathfrak{X}$ be an algebraic stack with affine stabilisers, of finite type over $\mathbb{C}$. Let $f : \mathfrak{X} \to \operatorname{Spec}\mathbb{C}$ be the structure map. If K is an object of $\mathrm{D}_{\mathrm{H}}(\mathfrak{X})$ of punctual weights $\leqslant w$, then $f_!K$ has punctual weights $\leqslant w$.*

Proof. First note that for any stack $\mathfrak{X}$, we have the formula

$$\mathrm{D}_{\mathrm{H}}(\mathfrak{X})_{\leqslant w} = \{K \in \mathrm{D}_{\mathrm{H}}(\mathfrak{X}) \mid \forall i \in \mathbb{Z},\ {}^{\mathrm{p}}\mathrm{H}^i(\omega^{w+i+1}K) = 0\}.$$

In particular, as the functors ω^{w+i+1} and ${}^{\mathrm{p}}\mathrm{H}^i$ commute with filtered colimits, we see that $\mathrm{D}_{\mathrm{H}}(\mathfrak{X})_{\leqslant w}$ is stable under filtered colimits in $\mathrm{D}_{\mathrm{H}}(\mathfrak{X})$.

By [Kre99, Proposition 3.5.2 and Proposition 3.5.9] and the fact that $\mathfrak{X}$ is of finite type over k, there exists a finite stratification $\mathfrak{X}_i$ of $\mathfrak{X}$ by locally closed substacks such that each $\mathfrak{X}_i \simeq [X_i/G_i]$ is a global quotient stack of a quasi-projective variety X_i by a smooth connected algebraic group G_i acting linearly on X_i. As $\mathrm{D}_{\mathrm{H}}(\operatorname{Spec}(\mathbb{C}))_{\leqslant w}$ is closed under extensions, using the localisation triangle we see that we may assume that $\mathfrak{X} \simeq [X/G]$ is a global quotient stack with X quasi-projective.

By [Tot99, Remark 1.4] we may find a specific increasing sequence (U_n) of representations of G (hence vector bundles on BG), called a Borel resolution, that helps compute the invariants of $[X/G]$ as follows: Let $X \overset{G}{\times} U_n := [X/G] \times_{BG} U_n$, then for any $\mathbb{A}^1$-invariant étale sheaf $F\colon \mathrm{Sm}^{\mathrm{op}}_{[X/G]} \to \mathcal{C}$ on the category of schemes that are smooth over $[X/G]$ with values in an ∞-category $\mathcal{C}$ admitting all small limits, the canonical map

$$F^{\triangleleft}([X/G]) \to \lim_n F^{\triangleleft}(X \overset{G}{\times} U_n)$$

is an equivalence, where $F^{\triangleleft}$ is the right Kan extension of F to algebraic stacks. This is [KR24, Theorem 3.6]. Note that by [KR24, Remark 3.4], each $X \overset{G}{\times} U_n$ is a quasi-projective variety.

Let $K \in \mathrm{D}_{\mathrm{H}}([X/G])_{\leqslant w}$. We apply the above to the functor F, opposite to the functor that sends a smooth map $g\colon Y \to [X/G]$, with Y a variety, to $\pi^Y_! g^! K$. By !-descent for the étale topology and $\mathbb{A}^1$-invariance of D_{H} this functor F satisfies the conditions stated in the previous paragraph. Moreover, its right Kan extension to stacks over $[X/G]$ has the same formula, again by !-descent. In particular, we have that the map

$$\operatorname{colim}_n (fp_n)_! p_n^! K \to f_! K$$

is an equivalence, where $p_n\colon X \times^G U_n \to [X/G]$ is the projection. Note that this is a filtered colimit, so that it suffices to prove that each $(fp_n)^! p_n^! K$ is of punctual weights $\leqslant w$. But as p_n is smooth, the weights of $p_n^! K$ are the weights of K, and the map fp_n is a map of schemes, so that we have reduced to the case of schemes, where this is true by [Sai89, Proposition 1.7]. □

Corollary 6.1.27. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks of finite type over $\mathbb{C}$. Assume that $\mathcal{Y}$ has affine stabilisers. Then if $K \in \mathrm{D}^-_{\mathrm{H},c}(\mathcal{Y})$ (resp. $K \in \mathrm{D}^+_{\mathrm{H},c}(\mathcal{Y})$) is of weights $\leqslant w$ (resp. $\geqslant w$) the object $f_!K$ (resp. f_*K) also has weights $\leqslant w$ (resp. $\geqslant w$).*

Proof. By duality it suffices to deal with $f_!$. Let $\pi : X \to \mathcal{X}$ be a presentation. Then it suffices to show that $\pi^* f_! K$ has weights $\leqslant w$, thus it suffices to check, for each closed point x of X, that the restriction to x of $f_!K$ has weights $\leqslant w$. Indeed the family of x^* is conservative (because we work with $\mathrm{D}^-_{\mathrm{H},c}(\mathcal{Y})$) and jointly detects the fact of having weights $\leqslant w$ (see for example [BBD82, Proposition 5.1.9]). But then the above proposition, combined with the $(x^*, f_!)$ base change, gives the result. □

Corollary 6.1.28. *Let $\mathcal{X}$ be an algebraic stack with affine stabilisers, locally of finite type over $\mathbb{C}$. Then there is a weight structure* a la *Bondarko on $\mathrm{D}^b_H(\mathcal{X})$ whose heart consist of objects of weight 0 as in Definition 6.1.23.*

Proof. We want to use [Bon10, Theorem 4.3.2] to construct a weight structure from its heart of pure objects of weight 0. Thus we want to prove that pure objects of weight 0 generated $\mathrm{D}^b_{\mathrm{H},c}(\mathcal{X})$ as a thick subcategory, and that it is negative (for us, semi-simple will be enough). Thus if $K, L \in \mathrm{D}^b_H(\mathcal{X})$ are pure of weight zero, we want to show that $\mathrm{Hom}_{\mathrm{D}^b_H(\mathcal{X})}(K, L[1])$ vanishes. Let $f : \mathcal{X} \to \mathrm{Spec}\,\mathbb{C}$ be the structural morphism. As we have

$$\mathrm{Hom}_{\mathrm{D}^b_H(\mathcal{X})}(K, L[1]) \simeq \mathrm{Hom}_{\mathrm{D_H}(\mathrm{Spec}\,\mathbb{C})}(\mathbb{Q}_{\mathrm{Spec}\,\mathbb{C}}, f_*\,\mathscr{H}om(K, L)[1]),$$

by Corollary 6.1.27 and the usual orthogonality in the derived category of mixed Hodge structures, it suffices to show that $\mathscr{H}om(K, L)$ has weights $\geqslant 0$, which results from Proposition 6.1.25.

Now objects of weight 0 generate the ∞-category under finite colimits. Indeed by dévissage it suffice to show that objects of the perverse heart which are pure are in the ∞-category generated by the complexes of weight 0, but now this is only a shift. □

Remark 6.1.29. The above Corollary 6.1.28 is *false* for general algebraic stacks, as shown by Sun (following an example that goes back to Drinfeld) in [Sun12]. Indeed one can show that the pushforward of the constant sheaf by the quotient map $\pi\colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{B}E$ with $\mathrm{B}E$ the classifying space of an elliptic curve is not semi-simple although π is smooth and proper.

6.1.1.3 Nearby cycles.

Let $\mathcal{X}$ be an algebraic stack locally of finite type over $\mathbb{C}$ and let $f : \mathcal{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function. We have the following diagram:

$$\begin{array}{ccccc} \mathfrak{U} & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & \mathfrak{Z} \\ \downarrow{\scriptstyle f_\eta} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_s}\,. \\ \mathbb{G}_{m,\mathbb{C}} & \xrightarrow{j} & \mathbb{A}^1_{\mathbb{C}} & \xleftarrow{i} & \{0\} \end{array} \tag{6.1.29.1}$$

We wish to construct a nearby cycle functor

$$\Psi_f\colon \mathrm{D_H}(\mathfrak{U}) \to \mathrm{D_H}(\mathfrak{Z})$$

with good properties. We begin with the unipotent part of the nearby cycle following [CHS24], whose authors give a wonderful interpretation of Ayoub's unipotent nearby cycle functor (as in [Ayo07b]).

In [CHS24, Definition 2.22], Cass, van den Hove and Scholbach define a $\mathbb{Q}$-linear stable ∞-category Nilp (denoted by $\mathrm{Nilp}_{\mathbb{Q}}$ in *loc. cit.*) whose objects can be interpreted as pairs (K, N) with $K \in \mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}$ a complex of graded $\mathbb{Q}$-vector spaces and $N : K(1) \to K$ is a locally nilpotent map of graded complexes, where $K(1) := K \otimes_{\mathbb{Q}} \mathbb{Q}(1)$ and $\mathbb{Q}(1)$ is the graded $\mathbb{Q}$-vector space placed in degree -1. This ∞-category is compactly generated by objects $(\mathbb{Q}(k), \mathbb{Q}(k+1) \xrightarrow{0} \mathbb{Q}(k))$, and if $(K, N) \in \mathrm{Nilp}$ is compact, its underlying complex K is compact and the operator N is nilpotent. Now for any stably presentable ∞-category $\mathcal{C}$ with and action of $\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}$, on can define

$$\mathrm{Nilp}\mathcal{C} := \mathcal{C} \otimes_{\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}} \mathrm{Nilp},$$

which has a similar description as pairs (c, N) with $c \in \mathcal{C}$ and $N : c(1) \to c$, which is nilpotent if c is compact.

The relation of this construction with unipotent nearby cycles Υ_f comes from the desire to have a monodromy operator on Υ_f, thus a lift of Υ as a functor

$$\mathrm{D_H}(\mathfrak{U}) \to \mathrm{Nilp}\mathrm{D_H}(\mathfrak{Z}).$$

This is made possible thanks to the following observation.

Proposition 6.1.30. *Let $p : \mathbb{G}_{m\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ be the projection. Then the pushforward*

$$p_* \colon \mathrm{D}_{HT}(\mathbb{G}_m) \to \mathrm{D}_{HT}(\mathbb{C})$$

exhibits the ∞-category $\mathrm{D}_{HT}(\mathbb{G}_m)$ as $\mathrm{Nilp}\mathrm{D}_{HT}(\mathbb{C})$. Here we have denote by $\mathrm{D}_{HT}(X)$ the full subcategory of $\mathrm{D_H}(X)$ of mixed Hodge Tate modules over X, that is the ∞-category generated under colimits and shifts by the $\mathbb{Q}(i)$ for $i \in \mathbb{Z}$.

Proof. This is the combination of [CHS24, Corollary 2.20 and Lemma 2.17], that we reproduce for the reader's convenience. First we remark that the functor p_* indeed preserves Hodge Tate objects because $p_*\mathbb{Q}(n) \simeq \mathbb{Q}(n) \oplus \mathbb{Q}(n+1)[1]$. Now, it is clear that p_* preserves colimits as its left adjoint p^* preserves compact objects, and it is also conservative because if $p_*M = 0$ then for each $n \in \mathbb{Z}$ the mapping spectra $\mathrm{Map}_{\mathrm{D}_{HT}(\mathbb{G}_{m\mathbb{C}})}(\mathbb{Q}(n), M) \simeq \mathrm{Map}_{\mathrm{D}_{HT}(\mathbb{C})}(\mathbb{Q}(n), p_*M)$ vanishes, so that $M = 0$. Thus by Barr-Beck's theorem, the functor p_* upgrades to an equivalence

$$\mathrm{D_H}(\mathbb{G}_{m\mathbb{C}}) \xrightarrow{\sim} \mathrm{Mod}_{p_*\mathbb{Q}}(\mathrm{D_H}(\mathbb{C})).$$

There is a unique colimit preserving symmetric monoidal functor

$$i \colon \mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}} \to \mathrm{D}_{HT}(\mathbb{C})$$

that sends $\mathbb{Q}(1)$ to $\mathbb{Q}(1)$. The image under the Hodge realisation of [CHS24, Lemma 2.18] tells us that $p_*\mathbb{Q}$ is, as a commutative algebra object, the object $i(\Lambda)$ where Λ is the split square zero extension $\Lambda := \mathbb{Q} \oplus \mathbb{Q}(-1)[-1]$. Thus we have that

$$\mathrm{D}_{HT}(\mathbb{G}_{m\mathbb{C}}) \simeq \mathrm{Mod}_{p_*\mathbb{Q}}(\mathrm{D}_{HT}(\mathbb{C})) \simeq \mathrm{D}_{HT}(\mathbb{C}) \otimes_{\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}} \mathrm{Mod}_{\Lambda}(\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}).$$

and the equivalence $\mathrm{Mod}_{\Lambda}(\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}) \simeq \mathrm{Nilp}$ of [CHS24, Lemma 2.9] finishes the proof. □

We come back to the setup (6.1.29.1), and define, as [CHS24, Definition 3.1]:

Definition 6.1.31. The unipotent nearby cycle functor is the composite

$$\Upsilon_f \colon \mathrm{D_H}(\mathfrak{U}) \to \mathrm{Mod}_{f_\eta^* p^* p_* \mathbb{Q}}(\mathrm{D_H}(\mathfrak{U})) \simeq \mathrm{NilpD_H}(\mathfrak{U}) \xrightarrow{i^* j_*} \mathrm{NilpD_H}(\mathfrak{Z})$$

where the first functor is the way one can see any object $K \in \mathrm{D_H}(\mathfrak{U})$ as a $f_\eta^* p^* p_* \mathbb{Q}$-module with the counit $f_\eta^* p^* p_* \mathbb{Q} \otimes K \to K$, and the second is expressing that $i^* j_*$ preserves the nilpotent structure.

In fact as they show in [CHS24, Section 3.1], there is a better way to look at the above definition. Indeed the construction of the operations for stacks in Theorem 6.1.6 goes by extending the functor $\mathrm{D_H}$ on schemes to a lax monoidal functor

$$\mathrm{D}_H \colon \mathrm{Corr}_{\mathbb{C}} \to \mathrm{Pr}^L_{\mathrm{St}},$$

where $\mathrm{Corr}_{\mathbb{C}}$ is the ∞-category of of correspondences of stacks over $\mathbb{C}$: its objects are higher algebraic stacks locally of finite type over $\mathbb{C}$ and its morphisms are informally diagrams

$$Y \xleftarrow{g} Z \xrightarrow{f} X$$

of algebraic stacks locally of finite type over $\mathbb{C}$. The functor sends X to $\mathrm{D_H}(X)$, and the above morphism to

$$g_! f^* \colon \mathrm{D_H}(Y) \to \mathrm{D_H}(X).$$

In fact, the projection formulae even allow us to have a functor with values in $\mathrm{Mod}_{\mathrm{D}_H(\mathbb{C})}(\mathrm{Pr}^L_{\mathrm{St}})$.

Now, unipotent nearby cycles can be upgraded to a highly coherent natural transformation $\Upsilon \colon \mathrm{D}_{H\eta} \to \mathrm{NilpD}_{Hs}$ of lax monoidal functors

$$\mathrm{Corr}^{\mathrm{pr,sm}}_{\mathbb{A}^1_{\mathbb{C}}} \to \mathrm{Pr}^L_{\mathrm{gr}},$$

where $\mathrm{Cor}^{\mathrm{pr,sm}}_{\mathbb{A}^1_{\mathbb{C}}}$ is the plain subcategory of $\mathrm{Corr}_{\mathbb{A}^1_{\mathbb{C}}} = (\mathrm{Corr}_{\mathbb{C}})_{/\mathbb{A}^1_{\mathbb{C}}}$ whose morphisms are the roofs $Y \xleftarrow{g} Z \xrightarrow{f} X$ over $\mathbb{A}^1$ with g proper and f smooth, and $\mathrm{Pr}^L_{\mathrm{gr}} := \mathrm{Mod}_{\mathrm{Mod}^{\mathbb{Z}}_{\mathbb{Q}}}(\mathrm{Pr}^L_{\mathrm{St}})$ is the ∞-category of presentable ∞-categories with an action of graded complexes of vector spaces.

Here, $\mathrm{D}_{H\eta}$ is the restriction of the composition

$$\mathrm{D}_{H\eta} \colon \mathrm{Corr}_{\mathbb{A}^1} \xrightarrow{-\times_{\mathbb{A}^1}\mathbb{G}_m} \mathrm{Corr}_{\mathbb{G}_m} \xrightarrow{\mathrm{D_H}} \mathrm{Mod}_{\mathrm{D}_{HT}(\mathbb{G}_{m\mathbb{C}})}(\mathrm{Pr}^L_{\mathrm{St}}) \xrightarrow{U} \mathrm{Mod}_{\mathrm{D}_{HT}(\mathbb{C})}(\mathrm{Pr}^L_{\mathrm{St}}) \to \mathrm{Pr}^L_{\mathrm{gr}}$$

with $U : \mathrm{D}_{HT}(\mathbb{G}_m) \to \mathrm{D}_{Ht}(\mathbb{C})$ the forgetful functor, and D_{Hs} is the restriction of the composition

$$\mathrm{D}_{Hs} \colon \mathrm{Corr}_{\mathbb{A}^1_{\mathbb{C}}} \xrightarrow{-\times_{\mathbb{A}^1}\{0\}} \mathrm{Corr}_{\mathbb{C}} \xrightarrow{\mathrm{D_H}} \mathrm{Pr}^L_{\mathrm{gr}} \xrightarrow{\mathrm{Nilp}} \mathrm{Pr}^L_{\mathrm{gr}}.$$

Theorem 3.2 in *loc. cit.* is:

Theorem 6.1.32 (Cass, van den Hove, Scholbach)**.** *There is a natural transformation of lax monoidal functors* $\mathrm{Corr}^{\mathrm{pr,sm}}_{\mathbb{A}^1_{\mathbb{C}}} \to \mathrm{Pr}^L_{\mathrm{gr}}$

$$\Upsilon \colon \mathrm{D}_{H\eta} \to \mathrm{NilpD}_{Hs},$$

whose evaluation at a stack $X/\mathbb{A}^1$ *is the functor defined in Definition 6.1.31*

Proof. The proof is *verbatim* the same as their proof in [CHS24, Section 3.3]. The only thing to check is that over stacks, where our categories are not compactly generated (but over schemes everything is compactly generated hence for example mixed Hodge Tate categories are rigid), all functors in the play indeed preserve colimits. Only j_* could be a problem, but the preservation of colimits can be checked on a presentation and it thus follows from smooth base change. Also, NilpD_{Hs} stays a h-sheaf because Nilp is compactly generated, hence the functor $\otimes_{\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}}\mathrm{Nilp}$ commutes with limits. □

We will denote by $\Upsilon_f\colon \mathrm{D_H}(\mathfrak{U}) \to \mathrm{D_H}(\mathfrak{Z})$ the functor obtained after applying Υ to $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ and forgetting the monodromy locally nilpotent operator N. The fact that Υ above is a lax monoidal natural transformation expresses the usual compatibilities of Υ_f with proper pushforward and smooth pullbacks, together with a natural transformation

$$\Upsilon_{f_1} \boxtimes \Upsilon_{f_2} \to \Upsilon_{f\times g}$$

compatible with the nilpotent operators when one has two functions $f_i\colon \mathfrak{X}_i \to \mathbb{A}^1_{\mathbb{C}}$.

Proposition 6.1.33. *Let $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function on a algebraic stack locally of finite type over $\mathbb{C}$. The functor $\Upsilon_f[-1]$ preserves $\mathrm{D}^b_{\mathrm{H},c}$, is perverse t-exact and commutes with the external tensor product.*

Proof. The functor

$$\mathrm{rat}\colon \mathrm{D_H} \to \mathrm{D}_B$$

commutes with Υ_f because it commutes with the 6-operations (see Proposition 6.1.20 and [CHS24, Lemma 3.18]). On top of this, it detects constructibility and is conservative on constructible objects. Moreover, by taking a presentation $\pi : X \to \mathfrak{X}$ of $\mathfrak{X}$, it suffices to prove the proposition for Υ_f with $f : X \to \mathbb{A}^1_{\mathbb{C}}$ a function on a scheme. Thus the proof of the proposition reduces to the case of a function of a scheme, on analytic sheaves. The comparison [CHS24, Example 3.19] of Υ in this case, with Beilinson's unipotent nearby cycles functor [Bei87a] imply the result, because for analytic sheaves the functor $\Upsilon_f[-1]$ is perverse t-exact and commutes with the external tensor product.

□

Proposition 6.1.34. *Let $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function on a algebraic stack locally of finite type over $\mathbb{C}$. There is an exact triangle*

$$\Upsilon_f[-1] \xrightarrow{N} \Upsilon_f(-1)[-1] \to i^*j_*.$$

Proof. This is [CHS24, Proposition 3.9], shifted by -1. □

Proposition 6.1.35. *Let X be a reduced and separated finite type $\mathbb{C}$-scheme and let $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a function. Then there is a natural isomorphism of functors*

$$\Upsilon_f[-1] \simeq \psi_{f,1}$$

on the triangulated category $\mathrm{D}^b(\mathrm{MHM}(U))$, where $U = f^{-1}(\mathbb{G}_m)$ and $\psi_{f,1}$ is the unipotent nearby cycle defined by Saito in [Sai90b].

Proof. As both functors are t-exact it suffices to construct an isomorphism on $\mathrm{MHM}(U)$. By [Sai90a, Proposition 1.3] for $K \in \mathrm{MHM}(U)$ there is a canonical isomorphism

$$\mathrm{colim}_n \, {}^{\mathrm{p}}\mathrm{H}^{-1} i^* j_*(K \otimes f^* E_n) \xrightarrow{\sim} \psi_{f,1} K.$$

The proof of the comparison with Beilinson construction [CHS24, Example 3.19] gives in fact that

$$\Upsilon_f(K) \simeq \mathrm{colim}_n i^* j_*(K \otimes f^* E_n),$$

where $E_n = \mathbb{Q} \oplus \cdots \mathbb{Q}(-n) \in \mathrm{D}_{\mathrm{H}}(\mathbb{G}_m)$, so that the compatibility with colimits of the t-structure gives the result. □

Corollary 6.1.36. *On* $\mathrm{D}^b_{\mathrm{H},c}$*, the natural map*

$$\Upsilon_f \circ \mathbb{D}_{\mathfrak{U}}(-) \to \mathbb{D}_{\mathfrak{Z}} \circ \Upsilon_f(-)(1)[2].$$

is an equivalence.

Proof. The map is defined as follows: Υ_f is lax monoidal hence there is a canonical map

$$\Upsilon_f \circ \mathscr{H}om(-, f_\eta^! \mathbb{Q}) \otimes \Upsilon_f(-) \to \Upsilon_f(\mathscr{H}om(-, f_\eta^! \mathbb{Q}) \otimes (-))$$

that we can compose with the evaluation map

$$\mathscr{H}om(-, f_\eta^! \mathbb{Q}) \otimes (-) \to \Upsilon_f(f_\eta^! \mathbb{Q})$$

to obtain a map

$$\Upsilon_f \circ \mathscr{H}om(-, f_\eta^! \mathbb{Q}) \otimes \Upsilon_f(-) \to f_\eta^! \mathbb{Q}.$$

By adjunction, this provides a map

$$\Upsilon_f \circ \mathscr{H}om(-, f_\eta^! \mathbb{Q}) \to \mathscr{H}om(\Upsilon_f(-), \Upsilon_f(f_\eta^! \mathbb{Q})).$$

Now, the commutation of Υ with !-pushforwards implies that there is a exchange map

$$\Upsilon_f \circ f_\eta^! \to f_\sigma^! \circ \Upsilon_{\mathrm{Id}},$$

and by [CHS24, Lemma 3.12] (to which we apply the Hodge realisation), we have that $\Upsilon_{\mathrm{Id}}(\mathbb{Q}) \simeq \mathbb{Q}$. This gives a map

$$\Upsilon_f \circ \mathscr{H}om(-, f_\eta^! \mathbb{Q}) \to \mathscr{H}om(\Upsilon_f(-), f_\sigma^! \mathbb{Q})$$

in which in the right hand side we recognise (because f_σ is the structural map of $\mathfrak{Z}$) the functor $\mathbb{D}_{\mathfrak{Z}} \circ \Upsilon_f$. The dualising object on $\mathfrak{U}$ is $\omega_{\mathfrak{U}} = f_\eta^! \pi_{\mathbb{G}_m}^! \mathbb{Q} \simeq f_\eta^! \mathbb{Q}(1)[2]$. Using that Υ_f commutes with Tate twists (this follows easily from the projection formula and the fact that it commutes with $p_!$, for $p \colon \mathfrak{X} \times \mathbb{P}^1 \to \mathfrak{X}$), we finally have a map

$$\Upsilon_f \circ \mathbb{D}_{\mathfrak{U}}(-) \to \mathbb{D}_{\mathfrak{Z}} \circ \Upsilon_f(-)(1)[2].$$

This comparison map is an isomorphism on $\mathrm{D}^b_{\mathrm{H}}$ because this can be checked locally hence once finite type $\mathbb{C}$-schemes, where it has been proven by Saito (alternatively, one can apply the functor to analytic sheaves and use the result of Beilinson). □

Now, for each $n \in \mathbb{N}^*$ we may consider π_n the elevation to the n-th power in $\mathbb{A}^1_{\mathbb{C}}$, and form the cartesian square

$$\begin{array}{ccc} \mathfrak{X}_n & \xrightarrow{e_n} & \mathfrak{X} \\ \downarrow{\scriptstyle f_n} & \lrcorner & \downarrow{\scriptstyle f} \\ \mathbb{A}^1_{\mathbb{C}} & \xrightarrow{\pi_n} & \mathbb{A}^1_{\mathbb{C}} \end{array}.$$

Denote by $i_n \colon f_n^{-1}(\{0\}) \to f^{-1}(\{0\})$ be the nil-immersion obtained by restricting e_n (the functor $(i_n)_*$ is an equivalence). Then by functoriality of Υ obtain a system of functors $((i_n)_* \circ \Upsilon_{f_n} \circ e_n^*)_{n \in (\mathbb{N}^*)^{\mathrm{op}}}$, where we still denote by e_n the restriction of e_n to the inverse image of $\mathbb{G}_m$ and we can set

Definition 6.1.37. The total nearby cycle functor is the functor

$$\Psi_f \colon \mathrm{D}_{\mathrm{H}}(\mathfrak{U}) \to \mathrm{D}_{\mathrm{H}}(\mathfrak{Z})$$

defined as

$$\Psi_f := \mathrm{colim}_{n \in (\mathbb{N}^*)^{\mathrm{op}}} (i_n)_* \Upsilon_{f_n} \circ e_n^*.$$

Remark 6.1.38. This definition in the case of motives is due to Ayoub ([Ayo07b]), but its formal description as a colimit can be found in Preis' [Pre23].

Proposition 6.1.39. *Let $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a function on a reduced and separated finite type $\mathbb{C}$-scheme. Then there is a natural equivalence of triangulated functors*

$$\Psi_f[-1] \simeq \Psi_f^S$$

on $\mathrm{D}^b(\mathrm{MHM}(U))$, where Ψ_f^S is Saito's nearby cycle functor. In particular, Ψ_f preserves constructible objects.

Proof. First, because the t-structure is compatible with colimits and each e_n^* are perverse t-exact, it is clear that $\Psi_f[-1]$ is perverse t-exact on $\mathrm{D}(\mathrm{IndMHM}(U))$. Thus by dévissage it suffices to construct an isomorphism $\Psi_f[-1] \simeq \Psi_f^S$ of exact functors on the heart $\mathrm{MHM}(U)$.

By definition ([Sai06, Proposition 5.7]) together with the comparison of Proposition 6.1.35, for $M \in \mathrm{MHM}(U)$ we have

$$\Psi_f^S(M) = \Upsilon_f(M)[-1] = \Upsilon_f(M \otimes f_n^*(\pi_n)_* \mathbb{Q})[-1]$$

for n divisible enough, in other terms,

$$\Psi_f^S = \mathrm{colim}_n \Upsilon_f(- \otimes f_n^*(\pi_n)_* \mathbb{Q})[-1].$$

For each $n \in \mathbb{N}^*$ because e_n is proper we have $(i_n)_* \Upsilon_{f_n} \simeq \Upsilon_f \circ (e_n)_*$ and now the proper projection formula and proper base change give $(e_n)_* \simeq (- \otimes f_n^*(\pi_n)_* \mathbb{Q})$, finishing the proof. □

Theorem 6.1.40. *Let $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function on an algebraic stack locally of finite type over $\mathbb{C}$.*

1. *The functor*
$$\Psi_f[-1]\colon \mathrm{D}_{\mathrm{H}}(\mathfrak{U}) \to \mathrm{D}_{\mathrm{H}}(\mathfrak{Z})$$
preserves the full subcategory $\mathrm{D}^b_{\mathrm{H},c}$*, is perverse t-exact and lax monoidal.*

2. *Given another function* $g : \mathfrak{Y} \to \mathbb{A}^1_{\mathbb{C}}$*, the natural map*
$$\Psi_f(-) \boxtimes \Psi_g(-) \to \Psi_{f\times g}(- \boxtimes -)$$
is an equivalence.

3. *On* $\mathrm{D}^+_{\mathrm{H},c}(\mathfrak{U})$ *the natural transformation*
$$\Psi_f \circ \mathbb{D}_{\mathfrak{U}}(-) \to \mathbb{D}_{\mathfrak{Z}} \circ \Psi_f(-)(1)[2]$$
is an equivalence.

Proof. Thanks to the same property for Υ the functor Ψ commutes with smooth pullback, hence all those properties are local for the smooth topology and can be checked on a separated finite type $\mathbb{C}$-scheme, where they follow from the results on the underlying perverse sheaves and the comparison with Saito's functor. The last assertion follows from the result on the bounded category and t-exactness. □

We finish this section with the remark that because taking cones is functorial in the world of stable ∞-categories, we have a vanishing cycles functors for free:

Definition 6.1.41. Let $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function on an algebraic stack locally of finite type over $\mathbb{C}$. We define the vanishing cycle functor
$$\Phi_f\colon \mathrm{D}_{\mathrm{H}}(\mathfrak{X}) \to \mathrm{D}_{\mathrm{H}}(\mathfrak{Z})$$
as the cone of the natural map $i^* \to \Psi_f j^*$. We thus have an exact triangle
$$i^* \to \Psi_f j^* \xrightarrow{\mathrm{can}} \Phi_f.$$

Proposition 6.1.42. *The functor*
$$\Phi_f[-1]\colon \mathrm{D}_{\mathrm{H}}(\mathfrak{X}) \to \mathrm{D}_{\mathrm{H}}(\mathfrak{Z})$$
preserves $\mathrm{D}^b_{\mathrm{H},c}$*, is perverse t-exact and commutes with Verdier duality up to a twist* $-(1)$*.*

Proof. This can be checked on a smooth atlas, thus on schemes where this holds for in constructible sheaves $\mathrm{D}^b_c(-,\mathbb{Q})$. □

6.1.1.4 Comparison with existing constructions.

In this section, we compare our construction to the construction of Achar [Ach13] that dealt with equivariant mixed Hodge modules, as well as with Davison's [Dav24] in which they constructed pushforwards of the constant object under a morphism from a stack.

In Achar's [Ach13], the construction goes as follows: G is an affine algebraic group acting on a complex algebraic variety X, and he defines a triangulated category $\mathrm{D}_G(X)$ of

G-equivariant mixed Hodge modules on X. This triangulated has a perverse t-structure, whose heart is indeed the abelian category of G-equivariant mixed Hodge modules. The definition is the following: for a special Grothendieck topology called the *acyclic topology* in which covering are smooth G-equivariant morphism satisfying some cohomological vanishing (it is asked that for some n and universally, f satisfies that $\tau^{\leqslant n} f_* f^*$ is the identity when restricted to the abelian category of mixed Hodge modules), it turns out that the presheaf of triangulated categories $U \mapsto \mathrm{hoD}^b(\mathrm{MHM}(X))$ is a sheaf, and as X admits a covering in the acyclic topology by a U on which G acts freely (so that $[U/G]$ is a scheme), this sheaf canonically extends to G-equivariant varieties.

Proposition 6.1.43. *Let X be a G-variety. Then the homotopy category of $\mathrm{D}^b_{\mathrm{H},c}([X/G])$ coincides with Achar's category $\mathrm{D}^b_G(X)$.*

Proof. First, if G acts freely on X (for the definition of a free action, see [Ach13, Definition 6.4]), then $[X/G]$ is a scheme and the result is trivial. Then exactly for any n-acyclic map $U \to X$ such that the action of G on U is free and $a \leqslant b$ integers such that $b - a < n$, the same proof as [Ach13, Lemma 8.1] implies that the functor

$$\mathrm{D}^{[a,b]}_{\mathrm{H}}([X/G]) \to \mathrm{D}^{[a,b]}_{\mathrm{H}}([U/G])$$

is fully faithful. Thus the functor

$$\mathrm{hoD}^b_{\mathrm{H},c}([X/G]) \to \mathrm{D}_G(X)$$

is fully faithful. It is not hard to check that the heart of $\mathrm{D}^b_{\mathrm{H},c}([X/G])$ is also the abelian category of G-equivariant mixed Hodge modules (because they satisfy descent), so that this finishes the proof. □

Similarly one can check that the 6 operations defined by Achar coincide with ours.

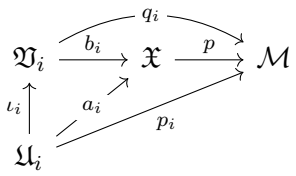
We now compare the pushforward we constructed with the one considered in [Dav24]. Let X be a smooth algebraic variety over $\mathbb{C}$, and let G be an affine algebraic groups acting on it. We are interested to the perverse cohomology groups of the object $p_!\mathbb{Q}$, where

$$p\colon \mathfrak{X} := [X/G] \to \mathcal{M}$$

is a morphism of stacks, with $\mathcal{M}$ an algebraic variety. Here is how Davison and Meinhardt proceed: the construction is very similar to the construction of the compactly supported motive of a classifying space by Totaro in [Tot16], and our proof of the comparison is inspired from the proof of [HPL21, Proposition A.7] by Hoskins and Pépin-Lehalleur and goes back to Borel. We will denote by X/G the quotient stacks instead of $[X/G]$.

First, they choose an increasing family $V_1 \subset V_2 \subset \cdots \subset V_i \subset \cdots$ of representations of G, and a subsystem $U_1 \subset U_2 \subset \cdots \subset U_i \subset \cdots$ of representations on which G acts freely (this can be done by choosing a closed embedding $G \subset \mathrm{GL}_r(\mathbb{C})$ and then setting $V_i = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^i, \mathbb{C}^r)$, and U_i is the subset of surjective linear applications). Then U_i/G is an algebraic variety, and in the case they deal with (they ask for the group G to be *special*), the quotient stack

$(X \times U_i)/G$ is also a scheme by [EG98, Proposition 23] of Edidin and Graham. We denote by $\mathfrak{V}_i := (V_i \times X)/G$, $\mathfrak{U}_i := (U_i \times X)/G$. We have a commutative diagram:



Moreover the maps a_i, b_i and ι_i are smooth. Davison and Meinhardt prove that for a given $n \in \mathbb{Z}$, the object

$$^{\mathrm{p}}\mathrm{H}^n((p_i)_!\mathbb{Q}_{\mathfrak{U}_i}(-ir)[-2ir])$$

is independent of the choices for i and M large depending on n, and they denote by $^{\mathrm{p}}\mathrm{H}^n(p_!\mathbb{Q}_{\mathfrak{X}})$ the common value. We will show that indeed the canonical map

$$^{\mathrm{p}}\mathrm{H}^n((p_i)_!\mathbb{Q}_{\mathfrak{U}_i}\{-ir\}) \to {}^{\mathrm{p}}\mathrm{H}^n(p_!\mathbb{Q}_{\mathfrak{X}})$$

is an isomorphism in $\mathrm{MHM}(\mathcal{M})$, where we have denoted by $(-)\{n\} := (-)(n)[2n]$ for $n \in \mathbb{Z}$. In fact we will show better, as the result (which is classical, and see [KR24] for a vast generalisation, and Proposition 6.1.26 for another use of this method) holds universally in motives:

Proposition 6.1.44. *The natural map*

$$\mathrm{colim}_i(p_i)_!\mathbb{Q}_{\mathfrak{U}_i}\{-ir\} \to p_!\mathbb{Q}_{\mathfrak{X}}$$

is an equivalence in $\mathrm{DM}^{\text{ét}}(\mathcal{M}, \mathbb{Q})$*, where* $\mathrm{DM}^{\text{ét}}(\mathcal{M}, \mathbb{Q})$ *is the* ∞*-category of étale motives with rational coefficients.*

Proof. First note that the counit map

$$(b_i)_!(b_i)^!\mathbb{Q}_{\mathfrak{X}} \to \mathbb{Q}_{\mathfrak{X}}$$

is an equivalence by $\mathbb{A}^1$-invariance, because b_i is a vector bundle on $\mathfrak{X}$. As b_i is smooth of relative dimension ir, purity gives that the canonical map

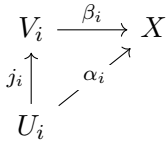
$$(q_i)_!\mathbb{Q}_{\mathfrak{V}_i}\{-ir\} \to p_!\mathbb{Q}_{\mathfrak{X}}$$

is an equivalence for each $i \in \mathbb{N}$. Thus it suffices to prove that the map

$$\mathrm{colim}_i(a_i)_!\mathbb{Q}_{\mathfrak{U}_i}\{-ir\} \to \mathrm{colim}_i(b_i)_!\mathbb{Q}_{\mathfrak{V}_i}\{-ir\}$$

on $\mathfrak{X}$ induced by the counits $(\iota_i)_!\iota_i^* \to \mathrm{Id}$ is an equivalence, because applying $p_!$ would produce the sought isomorphism.

Denote by $\pi : X \to \mathfrak{X}$ the projection. Recall that π^* is conservative, proper base change ensures that it suffices to deal with the analogous situation over X



where every thing has been pulled back to X, and the map is now

$$\mathrm{colim}_i (\alpha_i)_! \mathbb{Q}_{U_i}\{-ir\} \to \mathrm{colim}_i (\beta_i)_! \mathbb{Q}_{V_i}\{-ir\}.$$

Everything is now a scheme. The family of pullbacks $(x^*)_{x\in X}$ is conservative on $\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})$ by localisation, thus by proper base change we reduce further to the case where $X = \operatorname{Spec} k$ is the spectrum of a field, and each V_i is a smooth k-scheme (in fact, a vector space). Furthermore, the codimension of $W_i := V_i \setminus U_i$ is V_i goes to ∞ when $i \to \infty$, and the smoothness of α_i and β_i ensure that

$$(\alpha_i)_! \alpha_i^* \mathbb{Q}\{-ir\} \simeq (\alpha_i)_\sharp \alpha_i^* \mathbb{Q} \simeq M(U_i) \in \mathrm{DM}^{\text{ét}}(k, \mathbb{Q}),$$

and the same for $(\beta_i)_! \mathbb{Q}\{-ir\} = M(V_i)$, where for Y a smooth k-scheme, the object $M(Y)$ is the motive of Y. We are looking at

$$\mathrm{colim}_i M(U_i) \to \mathrm{colim}_i M(V_i)$$

in $\mathrm{DM}^{\text{ét}}(k, \mathbb{Q})$, with $\mathrm{codim}_{V_i}(V_i \setminus U_i) \underset{i\to\infty}{\longrightarrow} \infty$. By [HPL21, Proposition 2.13] this is an equivalence.

□

Corollary 6.1.45. *In* $\mathrm{D_H}(\mathcal{M}) = \mathrm{IndD}^b(\mathrm{MHM}(\mathcal{M}))$ *the natural map*

$$\mathrm{colim}_i (p_i)_! \mathbb{Q}_{\mathfrak{U}_i}\{-ir\} \to p_! \mathbb{Q}_{\mathfrak{X}}$$

is an equivalence. In particular, for each $n \in \mathbb{N}$*, the natural maps*

$${}^{\mathrm{p}}\mathrm{H}^n((p_i)_! \mathbb{Q}_{\mathfrak{U}_i})\{-ir\} \to \mathrm{H}^p(p_! \mathbb{Q}_{\mathfrak{X}})$$

are equivalences for i *big enough.*

Proof. The Hodge realisation

$$\rho_{\mathrm{H}} \colon \mathrm{DM}^{\text{ét}} \to \mathrm{D_H}$$

extends naturally to stacks, in a way that commutes with the operations and colimits, thus the first statement is just the Hodge realisation applied to Proposition 6.1.44. The second statement follows from the fact that the t-structure on $\mathrm{D_H}(\mathcal{M})$ is compatible with filtered colimits, and that each ${}^{\mathrm{p}}\mathrm{H}^n(p_! \mathbb{Q}_{\mathfrak{X}})$ is constructible thus Noetherian. □

Appendix A

Quick review of higher categories

In this section we try to give an introducton to ∞-categories and higher algebra. We shall give almost no proof but provide as many references as possible. The path we choose is less a logical path (meaning that we do not give *constructions* and proofs of everything using simplicial sets as done by Lurie) than a "∞-categories for the working mathematician" path in which after giving the basic definitions, we assume that we have a working ∞-category theory and use it to define the objects of interest: mapping spaces, limits and so on. The general idea is that any construction for 1-categories exists for ∞-categories, and any property that holds for 1-categories holds for ∞-categories. The expert is encouraged to skip this.

A.1 ∞-categories.

In this manuscript we will have to use ∞-categories. Although we will never use the definition of an ∞-category, when writing "$\mathcal{C}$ is an ∞-category" we will be thinking about a specific kind of simplicial set.

Disclaimer A.1.1. When mentioning a simplicial sets, or ∞-categories, or ∞-groupoids, one should fix an universe $\mathbf{U}$ and only consider simplicial $\mathbf{U}$-sets, and call such objects *small*, as in *small ∞-categories*, as opposed to large objects (living in a bigger universe $\mathbf{V}$) or very large objects (living in a even bigger universe $\mathbf{W}$). Although we will use the term *small* ∞-categories quite regularly, it is highly probable that this adjective disappears in some statements of the manuscript, as in the following definition:

Definition A.1.2. (Joyal, Lurie) A simplicial set X is an ∞-category if for any map

$$\Lambda^n_k \to X$$

for $0 < k < n$ and $n \in \mathbb{N}$ there exists an arrow

$$\Delta^n \to X$$

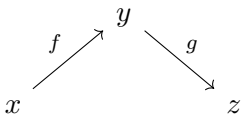
making the following triangle commutes:

$$\begin{array}{ccc} \Delta^n & \dashrightarrow & X \\ \uparrow & \nearrow & \\ \Lambda^n_k & & \end{array} .$$

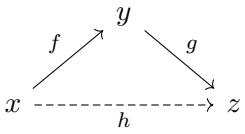
Here we denoted by Λ^n_k the k-th inner horn, which can be thought of as the standard n-dimensional simplex Δ^n where we removed the interior and the face opposite to the k-th vertex.

A functor is just a map of simplicial sets that are ∞-categories. One can show ([Cis19, Corollary 3.2.10.]) that if $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, the functor simplicial set $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ whose n-simplices are given by $\mathrm{Hom}_{\mathrm{sSets}}(\mathcal{C}\times\Delta^n,\mathcal{D})$, is an ∞-category. Seen this way, an natural transformation between two functors $F,G\colon\mathcal{C}\to\mathcal{D}$ is really a homotopy $\mathcal{C}\times\Delta^1\to\mathcal{D}$ between F and G.

When seen as a simplicial set X, an ∞-category $\mathcal{C}$ has objects X_0, and if $x,y\in X_0$, the set of morphisms from x to y is the subset consisting of $f\in X_1$ such that $d_0(f)=x$ and $d_1(f)=y$. By the Yoneda lemma, a map $\Lambda^2_1\to X$ is the same as giving two *composable morphisms*



and the condition of being an ∞-category implies that the diagram can be filled



so that *given two composable morphisms, there always exists a composition.*

Any usual (or 1) category $\mathcal{C}$ can be seen naturally as an ∞-category through the nerve construction ([Cis19, Example 1.5.3]). The nerve $N(\mathcal{C})$ is a simplicial set whose n-th set of vertices consists of compositions of n maps in $\mathcal{C}$.

Proposition A.1.3 ([Lur09, Proposition 1.1.2.2])**.** *A simplicial set X is the nerve of an usual category if an only if in the condition of Definition A.1.2 the dashed arrow is unique.*

In other terms, a 1-category is an ∞-category in which composition is unique: $h=g\circ f$. In an ∞-category, the composition is unique only up to contractile choice ([Lura, Proposition 0048]).

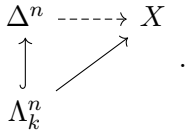
Definition A.1.4. (Joyal, Lurie) A simplicial set X is an ∞-groupoid if for any map

$$\Lambda^n_k\to X$$

for $0\leqslant k\leqslant n$ and $n\in\mathbb{N}$ there exists an arrow

$$\Delta^n\to X$$

making the following triangle commutes:



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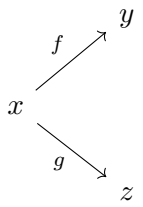
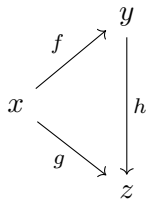
In particular, any ∞-groupoid X is an ∞-category, but there is more: the meaning of a map

$$\Lambda^0_2 \to X$$

is the same as a diagram



in X, and the condition of filling the diagram means that there exists $h\colon y \to z$ such that the diagram



commutes: $g \simeq h \circ f$. When applied to $g = \mathrm{Id}_x$, the map induced by the composition

$$\Delta^1 \xrightarrow{s^0_0} \Delta^0 \xrightarrow{x} X$$

we see that every map is invertible. In fact, this is enough:

Theorem A.1.5 (Joyal, [Cis19, Theorem 3.5.1])**.** *Let X be an ∞-category. Then X is an ∞-groupoid if and only if each arrow in X is invertible.*

All ∞-groupoids assemble to an ∞-category:

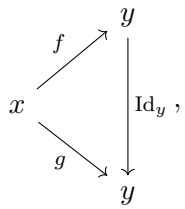
Theorem A.1.6 ([Cis19, Theorem 5.2.10.],[Lur09, Remark 1.2.16.3])**.** *There exists a ∞-category $\mathcal{S}$ of ∞-groupoids. It is sometimes denoted by* Grpd_∞ *for ∞-groupoids, or* An *for anima. The $\mathcal{S}$ stands for spaces, as a model for ∞-groupoids is homotopy types of CW-complexes.*

This ∞-category can be constructed from a model structure on Kan complexes, as model structures always provide ∞-categories. Given any nonempty ∞-groupoid $X \in \mathcal{S}$, there is a notion of homotopy groups $\pi_n(X)$, which are sets (in fact if $n = 1$ we have a group, and if $n \geqslant 2$ we have a commutative group). If one sees ∞-groupoids as homotopy types of CW-complexes though the geometric realisation, those are exactly the homotopy groups in the topological sense. A functor $F\colon X \to Y$ between ∞-groupoids is an equivalence if an only if for each $n \in \mathbb{N}$ the induced map $\pi_n(X) \to \pi_n(Y)$ is a bijection ([Cis19, Corollary 3.8.14.]).

The ∞-groupoids are the good higher analogue of sets. In fact any set is an ∞-groupoid (for example, through the nerve construction). Moreover, given an ∞-category $\mathcal{C}$ and two objects $x, y \in \mathcal{C}$, we have a *space of morphisms* $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathcal{S}$, called the mapping space, whose π_0 is the set of morphisms $x \to y$ as described above, up to homotopy, where a homotopy $f \to g$ is a diagram

$$\Delta^2 \to \mathcal{C}$$

of the form



or equivalently with Id_x, see [Cis19, Section 1.6.3]. In fact we can do better ([Lura, Proposition 01L5,Corollary 03K1 and Proposition 03JQ]): there is a functor

$$\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

such that for any pair (x, y) of objects in $\mathcal{C}$, the space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is equivalent the mapping space. Here by $\mathcal{C}^{\mathrm{op}}$ we mean the opposite ∞-category, which can be constructed ([Cis19, 1.5.7]) using the explicit model of simplicial sets but whose existence and obvious properties are the only things we will use. Any functor between ∞-categories induces maps between mapping spaces, so that we can define a fully faithful functor:

Definition A.1.7. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is fully faithful if for any pair of objects x, y of $\mathcal{C}$ the map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(F(x), F(y))$$

is an equivalence.

The universal example of a fully faithful functor is the Yoneda functor:

Theorem A.1.8 ([Lura, Proposition 03M5])**.** *Let $\mathcal{C}$ be an ∞-category. The Yoneda functor*

$$y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) =: \mathrm{PSh}(\mathcal{C})$$

sending an object $x \in \mathcal{C}$ to $\mathrm{Map}_{\mathcal{C}}(-, x)$ is fully faithful.

We take this occasion to define two other useful properties of functors:

Definition A.1.9. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is

1. essentially surjective if for any object d of $\mathcal{D}$ there is an object c of $\mathcal{C}$ and an isomorphism $d \simeq F(c)$.
2. an equivalence of categories if it is fully faithful and essentially surjective.
3. conservative if for any map g in $\mathcal{C}$, the fact that g is invertible is equivalent to the fact that $F(g)$ is invertible.

Remark A.1.10. It is true that a functor F is an equivalence of ∞-categories if and only if there exists a quasi-inverse G with the obvious properties, see [Cis19, Theorem 3.9.7.].

As a corollary or the Yoneda lemma, we see that if $\mathcal{C}$ is an ∞-category, the family of functors $\mathrm{Map}_{\mathcal{C}}(-, c)$ for c ranging through the objects of $\mathcal{C}$, is conservative.

It is true [Cis19, Corollary 3.5.12] that a natural transformation is invertible as an arrow of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ if an only if it is point wise invertible, which means that for each object c of $\mathcal{C}$, the induced map $F(c) \to G(c)$ is invertible in $\mathcal{D}$. There is also an ∞-groupoid $\mathrm{Fun}^{\simeq}(\mathcal{C}, \mathcal{D})$ of functors and invertible natural transformations between them. This in mind Theorem A.1.6 is not particular to ∞-groupoids:

Theorem A.1.11 ([Lura, Proposition 020B, Remark 020G])**.** *There exists an ∞-category Cat_∞ of small ∞-categories, whose objects are small ∞-categories, and the mapping spaces are given by the ∞-groupoid $\mathrm{Fun}^{\simeq}(-,-)$ of functors between them. There is a natural fully faithful functor*

$$\mathcal{S} \to \mathrm{Cat}_\infty.$$

The construction of $\pi_0(\mathrm{Map}_{\mathcal{C}}(x, y))$ as a quotient by an universal property sketched above is part of a bigger picture:

Theorem A.1.12. *The nerve functor*

$$N \colon \mathrm{Cat}_1 \to \mathrm{Cat}_\infty$$

sending a 1-category to its associated ∞-category has a left adjoint, denoted $\mathrm{ho}(-)$, sending an ∞-category to its homotopy category*.*

We did not explain what is an adjunction of ∞-category yet, but we can describe the functor $\mathrm{ho}(-)$ more explicitly: It sends an ∞-category $\mathcal{C}$ to the category $\mathrm{ho}(\mathcal{C})$ whose objects are the same as objects of $\mathcal{C}$, and whose morphisms are sent to their class in the π_0 of the mapping spaces. In particular there is a natural functor

$$\mathcal{C} \to N(\mathrm{ho}(\mathcal{C}))$$

that will be the co-unit of the adjunction.

Notation A.1.13. *From now on, we will see every 1-category as an ∞-category with the nerve functor (which is fully faithful), and forget the $N(-)$ from the notation.*

We know talk about adjoints. In the world of 1-categories, a pair of functors is an adjunction if there exists a unit and a co-unit satisfying the triangle relations. It turns out that although the initial definition (and the one used to prove theoretical results on ∞-categories) of an adjunction of ∞-categories is quite involved ([Lur09, Definition 5.2.2.1]), it is ([Lur09, Proposition 5.2.2.8]) equivalent to the following definition:

Definition A.1.14 ([Lura, Definition 02EL])**.** Let

$$F \colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G$$

be a pair of functors between ∞-categories. We say that they form an adjunction if there exists a natural transformations $\eta \colon \mathrm{Id}_{\mathcal{C}} \to G \circ F$ and $\varepsilon \colon F \circ G \to \mathrm{Id}_{\mathcal{D}}$ such that

1. The identity morphism Id_F of F is a composition of

$$F \simeq F \circ \mathrm{Id}_{\mathcal{C}} \xrightarrow{\mathrm{Id}_F \circ \eta} F \circ G \circ F$$

and

$$F \circ G \circ F \xrightarrow{\varepsilon \circ \mathrm{Id}_F} \mathrm{Id}_{\mathcal{D}} \circ F \simeq F.$$

2. The identity morphism Id_G of G is a composition of

$$G \simeq \mathrm{Id}_{\mathcal{C}} \circ G \xrightarrow{\eta \circ \mathrm{Id}_G} G \circ F \circ G$$

and

$$G \circ F \circ G \xrightarrow{\mathrm{Id}_G \circ \varepsilon} G \circ \mathrm{Id}_{\mathcal{D}} \simeq G.$$

In that case F is a left adjoint to G and G is a right adjoint to F.

We have the following:

Proposition A.1.15 ([Lura, Corollary 02FX])**.** *Let*

$$F \colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G$$

be a pair of functors between ∞-categories. Then F is left adjoint to G if and only if there exists an invertible natural transformation

$$\mathrm{Map}_{\mathcal{D}}(F(-), -) \to \mathrm{Map}_{\mathcal{C}}(-, G(-))$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$.

It is true ([Lura, Remark 02F3]) that if an adjoint to a functor exists, it is unique up to a contractile choice. Moreover, any adjunction induces an adjunction between the homotopy categories. The converse is true under moderate assumption (we use the notion of limits which is defined just below).

Theorem A.1.16 (Theorem 3.3.1 of [NRS20])**.** *Let $\mathcal{D}$ be an ∞-category which admits finite limits and let $\mathcal{C}$ be an ∞-category. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor which preserves finite limits. Then G admits a left adjoint if and only if* $\mathrm{ho}(G) : \mathrm{ho}(\mathcal{D}) \to \mathrm{ho}(\mathcal{C})$ *does.*

In this setting, another property that can be checked on the homotopy categories is being an equivalence:

Theorem A.1.17 (Theorem 7.6.10 of [Cis19])**.** *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories having finite limits. Assume that F preserves finite limits. Then F is an equivalence of ∞-categories if and only if* $\mathrm{ho}(F) : \mathrm{ho}(\mathcal{D}) \to \mathrm{ho}(\mathcal{C})$ *is an equivalence of categories.*

One can prove [HSS17, 5.2.4.2] that given a functor $F \colon \mathcal{C} \to \mathcal{D}$ between ∞-categories, the fact that F has an adjoint is just a property: the ∞-category $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ (or $\mathrm{RFun}(\mathcal{C}, \mathcal{D})$) of left (or right) adjoint functors between $\mathcal{C}$ and $\mathcal{D}$ is a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

For 1-categories we can talk about limits and colimits, and we know that, for example, any left adjoint functor preserves all colimits that exists in the source category. This is also true for higher categories, once that we know what is a limit in this setting. First, note that

it suffices to define limits, as colimits are just limits in the opposite ∞-category. Moreover, as one can construct generalised slices ∞-categories ([Cis19, Section 3.4]) (which means, given any functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories, there exists a slice ∞-category $\mathcal{D}/F$ whose objects are informally given by pairs (d, η) where $d \in \mathcal{D}$ is an object and $\eta\colon \underline{d} \Rightarrow F$ is a natural transformation between the constant functor with value d and F), and a limit is just a final object in the slice category (called the category of co-cones in this setting), it suffices to define what is a final object in an ∞-category.

Definition A.1.18. A object x in an ∞-category $\mathcal{C}$ is *final* if and only if for every object $y \in \mathcal{C}$ the mapping space $\mathrm{Map}_{\mathcal{C}}(y, x)$ is contractile, meaning that it is not the empty space and all its homotopy groups are trivial.

We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories has a limit if the slice category $\mathcal{D}/F$ as a final object. This means the following:

Definition A.1.19. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories has a limit if there exists an object

$$\lim_{c \in \mathcal{C}} F(c) \in \mathcal{D}$$

and a natural transformation

$$\underline{\lim_{c \in \mathcal{C}} F(c)} \to F$$

between the constant functor and F such that there exists an invertible natural transformation

$$\mathrm{Map}_{\mathcal{D}}(-, \lim_{c \in \mathcal{C}} F(c)) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\underline{(-)}, F)$$

where $\underline{(-)}$ is the constant functor.

There is a similar description for colimits. Given ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we say that all (co)limits of shape $\mathcal{C}$ exists in $\mathcal{D}$ if all functors $\mathcal{C} \to \mathcal{D}$ admit a (co)limit. In this case, we have:

Proposition A.1.20 ([Cis19, Proposition 6.2.9])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Assume that all limits (*resp. *all colimits) of shape $\mathcal{C}$ exist in $\mathcal{D}$. Then the constant functor $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a right adjoint (*resp. *a left adjoint), which sends a diagram $F\colon \mathcal{C} \to \mathcal{D}$ to its limit $\lim_{\mathcal{C}} F$ (*resp. *its colimit $\mathrm{colim}_{\mathcal{C}} F$).*

In particular, as a composition of right adjoints is a right adjoint, if $R\colon \mathcal{D} \to \mathcal{E}$ is right adjoint to a functor $L\colon \mathcal{E} \to \mathcal{D}$, because it induces an adjunction between $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ in the obvious way, wee see that R commutes with the limit functor. The same can be said about colimits and left adjoints. If a limit of F exists, it is not hard to see that it gives an extended diagram $\overline{F}\colon \mathcal{C}^{\triangleleft} \to \mathcal{D}$ where $\mathcal{C}^{\triangleleft}$ is the ∞-category $\mathcal{C}$ to which we added a final object. The same can be said about colimits, and $\mathcal{C}^{\triangleright}$.

The following theorem is one of the most important results of ∞-category theory.

Theorem A.1.21 ([Lura, Corollary 05QP,Corollary 02VF,Corollary 02TK and Corollary 02V0])**.** *The ∞-categories $\mathcal{S}$ and Cat_∞ admits all small limits and colimits. Moreover, the inclusion $\mathcal{S} \to \mathrm{Cat}_\infty$ preserves limits and colimits.*

In fact, the inclusion $\mathcal{S} \to \mathrm{Cat}_\infty$ has a left adjoint $(-)^\simeq$ sending an ∞-category $\mathcal{C}$ to the ∞-category $\mathcal{C}^\simeq$ in which we inverted all the maps, and a right adjoint $k(-)$ called the core, and that sends an ∞-category to the wide subcategory having the same objects but only invertible morphisms.

Remark A.1.22. For 1-categories, there also exist a notion of 2-limit and 2-colimit. It turns out that they are computed in Cat_∞ (all limits, and filtered colimits), see https://mathoverflow.net/questions/466588/co-limits-and-2-co-limits-of-categories-in-the-infty-and-infty-2-catego.

Assume we are given a functor $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$. How can one compute its limit ? There are several ways. First, we did not give a precise definition of the functor $\mathrm{Map}_{\mathcal{C}}(-,-)$, this will come later when mentioning fibered ∞-categories. However, if one is only interested in the functoriality in *one variable*, the following construction works: if c and c' are objects of an ∞-category $\mathcal{C}$, this defines a functor $* \simeq * \times * \xrightarrow{(c,c')} \mathcal{C} \times \mathcal{C}$ by the Yoneda lemma. Now the mapping space fits in the following pullback square in Cat_∞:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(c,c') & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\Delta^1,\mathcal{C}) & \longrightarrow & \mathcal{C}\times\mathcal{C} \end{array} \tag{A.1.22.1}$$

where the lower horizontal map is the (source, target) functor.

Remark A.1.23. In fact, this square is an actual pullback in the 1-category sSets of simplicial sets, and this can be a definition of $\mathrm{Map}_{\mathcal{C}}(c,c')$. To see that we have a pullback in Cat_∞, one has to combine the following facts using terms that we will not use in this manuscript: given any monomorphism $(K \to L)$ of simplicial sets and given an ∞-category $\mathcal{C}$, the functor

$$\mathrm{Fun}(L,\mathcal{C}) \to \mathrm{Fun}(K,\mathcal{C})$$

is an inner fibration (one can see this as a fibration in ∞-categories) by [Cis19, Corollary 3.2.8]. Thus the functor

$$\mathrm{Fun}(\Delta^1,\mathcal{C}) \to \mathcal{C}\times\mathcal{C}$$

obtained this way from the monomorphism $\partial\Delta^1 = * \coprod * \to \Delta^1$, is an inner fibration. Then one has to use that given a pullback of simplicial sets, if one of the two arrows is an isofibration (that is, an inner fibration that lifts isomorphisms), then the pullback is in fact a homotopy pullback ([Lura, Corollary 033T]), thus computes the limit in Cat_∞. The functor

$$\mathrm{Fun}(\Delta^1,\mathcal{C}) \to \mathcal{C}\times\mathcal{C}$$

is indeed an isofibration because we just saw it is an inner fibration, and if $c \simeq c'$ and $d \simeq d'$ in $\mathcal{C}\times\mathcal{C}$, and if (c,d) is the image of a map $c \to d$, then it is clear that there is an isomorphism map $(c \to d') \simeq (c' \to d')$ mapping to the isomorphism $(c \simeq c', d \simeq d')$.

Fixing c or c', this provides functoriality of the mapping space in one variable. What really interests us is the following: say now that $\mathcal{C} \simeq \lim_{i\in I} \mathcal{C}_i$ is a limit of ∞-category. Then by the universal property of limits, giving an object c of the limit $\mathcal{C}$ is the same as giving a compatible family $(c_i)_{i\in I}$ of objects in $\mathcal{C}_i$ (this follows from Proposition A.1.20, because

by compatible family we mean a natural transformation of functors $I \to \mathrm{Cat}_\infty$ between the constant functor $*$ and $\mathcal{C}_{(-)}$). Now the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ *is* a limit, more precisely the limit of the constant functor

$$\Delta^1 \to * \xrightarrow{\mathcal{C}} \mathrm{Cat}_\infty.$$

Because limits commute with limits, this implies that our diagram (A.1.22.1) above is the limit of the squares

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(c_i, c_i') & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\Delta^1, \mathcal{C}_i) & \longrightarrow & \mathcal{C}_i \times \mathcal{C}_i \end{array}.$$

We showed that:

Proposition A.1.24. *Let $F\colon I \to \mathrm{Cat}_\infty$ be a functor and let $c, c' \in \lim_I F$. Denote by $\mathcal{C}_i = F(i)$ and $\mathcal{C} = \lim_i \mathcal{C}_i$. Then (c, c') corresponds to compatible families (c_i, c_i') in $\mathcal{C}_i \times \mathcal{C}_i$ and the canonical map*

$$\mathrm{Map}_{\mathcal{C}}(c, c') \to \lim_i \mathrm{Map}_{\mathcal{C}_i}(c_i, c_i')$$

is an equivalence.

In particular, limits in Cat_∞ are "computed term wise": objects are limits of objects, and mapping spaces are limits of mapping spaces.

Remark A.1.25. Because Δ^1 is a finite groupoid, and filtered colimits commute with finite limits, the same argument shows that in Cat_∞, filtered colimits are also computed term wise. To be a tiny bit more precise, given a filtered colimit $\mathcal{C} \simeq \mathrm{colim}_i \mathcal{C}_i$, one can show that all finite ∞-categories are compact objects of Cat_∞ thus any object c of $\mathcal{C}$ comes from some object c_{i_0} of some $\mathcal{C}_{i_0}$, and we have a natural map

$$\mathrm{colim}_{i \geqslant i_0} \mathrm{Map}_{\mathcal{C}_i}(c_i, c_i') \to \mathrm{Map}_{\mathcal{C}}(c, c')$$

where c_i (*resp* c_i') is the image in $\mathcal{C}_i$ of c_{i_0} (*resp* of c_{i_0}'). This map is an equivalence.

The above computation gives a way to have a grasp of what is the limit in Cat_∞. But really, is there an explicit construction? The answer is yes, and as for 2-limits of 1-categories, this uses fibered categories and the Grothendieck construction.

First we have to define cartesian maps:

Definition A.1.26. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Let $f\colon c \to c'$ be a map in $\mathcal{C}$. We see that f is p-cartesian (or just cartesian if p is obvious) if for any $n \geqslant 2$ and any commutative square

$$\begin{array}{ccc} \Lambda^n_n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \mathcal{D} \end{array}$$

such that the composite

$$\Delta^1 \simeq \{n-1 < n\} \hookrightarrow \Lambda^n_n \to \mathcal{C}$$

corresponds to f, there exists a dashed arrow filling the diagram. If one replaces $\{n-1 < n\}$ by $\{0 < 1\}$ we say that f is p-cocartesian.

Unravelling the definition, for $n = 2$ we find the following: an arrow $f\colon c \to c'$ is cartesian in given any diagram

$$\begin{array}{ccc} & & p(c'') \\ & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle p(g)} \\ p(c) & \xrightarrow[p(f)]{} & p(c') \end{array}$$

in $\mathcal{D}$, with $g\colon c'' \to c'$ a map in $\mathcal{C}$, there exists an arrow $a\colon c \to c''$ such that $p(a) = \alpha$. Up to the unicity of the lift, we recover the classical notion of cartesian morphism invented by Grothendieck in [SGA71].

Definition A.1.27 ([Lura, 01UA])**.** Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that p is a Cartesian fibration if the following lifting properties hold:

1. For any $n \in \mathbb{N}$ and any $0 < k < n$, for any commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \dashrightarrow & \mathcal{D} \end{array}$$

   there exists a dashed arrow filling the diagram.

2. For every map $f\colon d \to d'$ in $\mathcal{D}$ and any object c' of $\mathcal{C}$ satisfying $p(c') = d'$ there exists a p-cartesian morphism $g\colon c \to c'$ such that $p(g) = f$.

We say that p is a cocartesian fibration if it verifies 1. and the dual of 2., or equivalently if $p^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a cocartesian fibration.

If p is a functor between 1-categories we recover the classical notion of fibered categories in [SGA71].

Remark A.1.28. A functor $p\colon \mathcal{C} \to \mathcal{D}$ satisfying part 1 of the above definition is called an inner fibration of ∞-categories. This notion could have been defined more generally for a map of simplicial sets. Using this notion one can define a subcategory $\mathcal{C}'$ of an ∞-category $\mathcal{C}$: it is a sub-simplicial set $\mathcal{C}' \subset \mathcal{C}$ such that the inclusion morphism is an inner fibration [Lura, Definition 01CD].

Using fibrations one can prove the following result:

Proposition A.1.29. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume that for a collection $A \subset \mathcal{C}$ of objects of $\mathcal{C}$ is given the data, for each $a \in A$ of an isomorphism $f_a : F(a) \to d_a$ with $d_a \in \mathcal{D}$. Then there exist a functor $G : \mathcal{C} \to \mathcal{D}$ and a natural equivalence $g : F \Rightarrow G$ such that for every $a \in A$, we have $g_a = f_a$ (and this is really an equality).*

Proof. Indeed, denote by $\iota : A \to \mathcal{C}$ the canonical monomorphism of simplicial sets. Then by [Lura, Tag 01F1], the restriction

$$\iota^* : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(A, \mathcal{D}) \tag{A.1.29.1}$$

is an isofibration of simplicial sets (*cf.* [Cis19, 3.3.15]). The data of the f_a defines an isomorphism γ in $\mathrm{Fun}(A, \mathcal{D})$ between $F_{|A}$ and the map of simplicial sets $A \to \mathcal{D}$ defined by $a \mapsto d_a$.

By the lifting property of isofibrations, there is a functor $G : \mathcal{C} \to \mathcal{D}$ and an isomorphism $F \simeq G$, that is a natural equivalence $g : F \Rightarrow G$ such that $\iota^* g = \gamma$, which is exactly the statement of the proposition. □

In [SGA71], Grothendieck proves that the 2-category of cocartesian fibrations above a 1-category $\mathcal{C}$ is isomorphic to the 2-category of functors $\mathcal{C} \to \mathrm{Cat}_1$ to the 2-category of categories. The same is true for ∞-categories:

Theorem A.1.30 ([Lura, Theorem 0293])**.** *There is a natural equivalence*

$$(\mathrm{Cat}_\infty)^{\mathrm{cocart}}_{/\mathcal{D}} \simeq \mathrm{Fun}(\mathcal{D}, \mathrm{Cat}_\infty)$$

of ∞-categories, where $(\mathrm{Cat}_\infty)^{\mathrm{cocart}}_{/\mathcal{D}}$ is the subcategory of the slice $(\mathrm{Cat}_\infty)_{/\mathcal{D}}$ consisting of cocartesian fibrations.

Remark A.1.31. There is an universal cocartesian fibration $\mathcal{C}^{\mathrm{univ}} \to \mathrm{Cat}_\infty$ and the functor is obtained by taking base change above this universal fibration.

This theorem, together with the Yoneda lemma, provide an explicit construction of Cat_∞ as a simplicial set: its n-simplices are the cocartesian fibrations above Δ^n. This can be made into a definition ([CN22]). If one thinks about it, the ∞-category $\mathcal{C}^{\mathrm{univ}}$ has to be the ∞-category of pairs $(\mathcal{C}, X)$ consisting of a ∞-category $\mathcal{C}$ and an object X of $\mathcal{C}$, and the cocartesian fibration has to be the functor forgetting the object. Indeed, for any small ∞-category $\mathcal{D}$, we can consider the functor $* \to \mathrm{Cat}_\infty$ pointing to $\mathcal{D}$. The associated cocartesian fibration above $*$ is $\mathcal{D}$ itself, so that we obtain a functor $V \colon \mathcal{D} \to \mathcal{C}^{\mathrm{univ}}$ such that for any object $d \in \mathcal{D}$, its image under the universal cocartesian fibration is $\mathcal{D} \in \mathrm{Cat}_\infty$. We thus obtain a map

$$\mathrm{Map}_{\mathcal{D}}(d, d') \to \mathrm{Map}_{\mathcal{C}^{\mathrm{univ}}}(V(d), V(d')),$$

and one can show using the lifting property that this map is an equivalence.

To be more precise we should have defined *essentially small* cocartesian fibration to avoid any possible set theoretical contradiction. We avoid those problems here.

Given a functor $F \colon \mathcal{D} \to \mathrm{Cat}_\infty$, the associated cocartesian fibration is said to classify F, and is called the straightening of F, and the converse operation is called unstraightening.

Remark A.1.32. With this, we can give a more precise definition of the functor

$$\mathrm{Map}_{\mathcal{C}}(-,-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

mentioned above, when $\mathcal{C}$ is an ∞-category: there exists an ∞-category $\mathrm{Tw}(\mathcal{C})$ called the *twisted arrow category*, defined explicitly in terms of simplicial sets (see [Lura, Tag 03JG]), together with a cocartesian fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ tat classifies the functor $\mathrm{Map}_{\mathcal{C}}(-,-)$.

We come back to the computations of limits. Let $F \colon I \to \mathrm{Cat}_\infty$ be a functor whose limit we want to catch a grasp of. We just saw that we can straighten our functor to obtain a cocartesian fibration $p \colon \mathcal{D} \to I$. Informally, objects of $\mathcal{D}$ are objects of some $F(i)$, and a map $a_i \to b_j$ in $\mathcal{D}$, where a_i is above i and b_j is above j, is a map between the image of a_i in $F(j)$ and b_j. We want the limit to be compatible systems of (c_i) in $F(i)$. It turns out that unravelling what we would want to be a compatible system, one comes out with the definition of cocartesian sections:

Definition A.1.33. Let $p\colon \mathcal{D} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. A cocartesian section of p is a functor $s\colon \mathcal{C} \to \mathcal{D}$ such that $p \circ s \simeq \mathrm{Id}_{\mathcal{C}}$ and s sends any morphism in $\mathcal{C}$ to a p-cartesian morphism in $\mathcal{D}$.

In the situation above, it turns out that the limit of the functor $F\colon\ I \to \mathrm{Cat}_\infty$ consists exactly of cocartesian sections of its classifying fibration p. One can even classify this way the functors that are limit diagrams, this is Lurie's diffraction criterion [Lura, Tag 02T8]: given a functor $F\colon I \to \mathrm{Cat}_\infty$ to which one has added an edge to obtain $\overline{F}\colon I^{\triangleleft} \to \mathrm{Cat}_\infty$ where $I^{\triangleleft}$ is the ∞-category I to which we added an initial object, there this is a criterion to say whether $\overline{F}$ is a limit diagram. We can use it to give the first original proof of this manuscript, of a fact we believe is well known but we could not find a reference. We had the help of Robin Carlier in writing up this proof. The reader should skip the proof of the proposition.

Proposition A.1.34. *Let $\chi : (\mathcal{C}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}_\infty$ be a functor. For any $f : c \to c'$ in $(\mathcal{C}^{\mathrm{op}})^{\triangleleft}$, denote $f^* = \chi(f)$, and assume that any such f^* has a right adjoint f_*. For each $c \in \mathcal{C}$, denote $f_c : c \to v$ the unique map from c to the end point v (we have that $(\mathcal{C}^{\mathrm{op}})^{\triangleleft} \simeq (\mathcal{C}^{\triangleright})^{\mathrm{op}}$.) Let $\overline{\pi} : \overline{\mathcal{D}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$ be the cocartesian fibration that classifies χ, and let $\pi : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$ be its pullback by the inclusion $\mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$.*
Then χ is a limit diagram if and only if the following two conditions are verified :

1. *The family $(f_c^*)_{c\in\mathcal{C}}$ is conservative.*
2. *For any cocartesian section $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ of π, the limit $X(v) := \lim_{c\in\mathcal{C}}(f_c)_*X(c)$ exists in $\chi(v)$ and the map $f_c^*X(v) \to X(c)$ adjoint to the canonical map*

$$\lim_c (f_c)_*X(c) \to (f_c)_*X(c)$$

is an equivalence.

Proof. By [Lura, Construction 02T0], there is a diffraction functor

$$\mathrm{Df} : \overline{\mathcal{D}}_v \to \mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \tag{A.1.34.1}$$

that fits in a commutative triangle :

$$\begin{array}{ccc} \overline{\mathcal{D}}_v & \xrightarrow{\quad\mathrm{Df}\quad} & \mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \\ & {\scriptstyle \mathrm{ev}_v} \swarrow \quad \searrow {\scriptstyle \mathrm{res}} & \\ & \mathrm{Fun}^{\mathrm{Cocart}}_{/(\mathcal{C}^{\mathrm{op}})^{\triangleleft}}((\mathcal{C}^{\mathrm{op}})^{\triangleleft}, \overline{\mathcal{D}}) & \end{array} \quad . \tag{A.1.34.2}$$

By the diffraction criterion [Lura, Theorem 02T8], χ is a limit diagram if and only if the restriction morphism in (A.1.34.2) is an equivalence. In that case by [Lura, Remark 02TF], the functor Df is an equivalence.

Assume that for any cocartesian section $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ of π, the functor $c \mapsto (f_c)_*X(c)$ admits a limit in $\overline{D}_v$. By the preservation of limits of any right adjoint in $(\mathcal{C}^{\mathrm{op}})^{\triangleleft}$ and [Lura, Corollary 0311 and Corollary 02KY], this is equivalent to saying that for any cocartesian section $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ of π, the lifting problem

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\quad X\quad} & \overline{\mathcal{D}} \\ \downarrow{\scriptstyle i} & \overset{\overline{X}}{\nearrow} & \downarrow{\scriptstyle \overline{\pi}} \\ (\mathcal{C}^{\mathrm{op}})^{\triangleleft} & = & (\mathcal{C}^{\mathrm{op}})^{\triangleleft} \end{array} \tag{A.1.34.3}$$

admits a solution $\overline{X}$ which is a $\overline{\pi}$-limit ([Lura, Definition 02KG].) By [Lura, Example 02ZA], in that case $\overline{X}$ is the right Kan extension functor along the inclusion $i : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$. This easily implies that the restriction functor $\mathrm{res} : \mathrm{Fun}^{\mathrm{Cocart}}_{/(\mathcal{C}^{\mathrm{op}})^{\triangleleft}}((\mathcal{C}^{\mathrm{op}})^{\triangleleft}, \overline{\mathcal{D}}) \to \mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is fully faithful as the left adjoint of the right Kan extension along $i : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$.

Therefore, assuming point 2. of the proposition, χ is a limit diagram if and only if the restriction functor is conservative by [Lura, Corollary 03UZ]. But as the family of evaluations $\mathrm{ev}_c : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \to \mathcal{D}$ is conservative, this is equivalent to the family of functors $\mathrm{ev}_c \circ \mathrm{res} = \mathrm{ev}_c \circ \mathrm{Df} \circ \mathrm{ev}_v = f_c^*$ being conservative, which is point 1. of the proposition.

By (A.1.34.2), if χ is a limit diagram, Df is an equivalence which implies that 1. holds. To finish the proof of the proposition, we therefore only have to show that if χ is a limit diagram, then 2. holds. Let $\overline{X}$ be a cocartesian section of $\overline{\pi}$. Assume first that $\overline{X}$ is a $\overline{\pi}$-limit diagram. By [Lura, Example 03ZA] this is equivalent to suppose that $\overline{X}$ is the $\overline{\pi}$-right Kan extension of its restriction X to $\mathcal{C}^{\mathrm{op}}$. Then by [Lura, Corollary 0307], the limit of $c \mapsto (f_c)_* X(c)$ exists and the fact that $\overline{X}$ is cocartesian is equivalent to the fact that for every $c \in \mathcal{C}$, the canonical map

$$f_c^*(\lim_{c'}(f_{c'})_* X(c)) \to X(c) \tag{A.1.34.4}$$

is an equivalence.

Therefore, to show that 2. holds, we only have to show that any cocartesian section $\overline{X}$ of $\overline{\pi}$ is a $\overline{\pi}$-limit. This is exactly what is proved in the second paragraph of the proof of [Lur11, Theorem 5.17].

□

Corollary A.1.35 (See also [Lur22, Corollary 5.2.2.37])**.** *Let $\chi, \chi' : (\mathcal{C}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}_\infty$ be two functors as in Proposition A.1.34, and assume one has a natural transformation $B : \chi \to \chi'$ which commutes with the rights adjoints. Suppose also that the limits of point 2. in Proposition A.1.34 exist for χ, that $B_v : \chi(v) \to \chi'(v)$ preserves them and that each B_c for $c \in \mathcal{C}$ is conservative. Then if χ' is a limit diagram, so is χ.*

We just saw one way of constructing new functors and ∞-categories out of old one: colimits. Another fundamental construction is the Kan extension, which works exactly as in usual category theory.

Theorem A.1.36 ([Lura, Section 02Y1])**.** *Let $\iota \colon \mathcal{C} \to \mathcal{C}'$ be a fully faithful functor of ∞-categories. Let $\mathcal{D}$ be an ∞-category. Assume that $\mathcal{D}$ admits all small limits (*resp *all small colimits). Then the restriction functor*

$$\mathrm{res} \colon \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

*admits a right adjoint (*resp *admits a left adjoint), called the right Kan extension (*resp *the left Kan extension).*

If $F \colon \mathcal{C} \to \mathcal{D}$ is a functor with right Kan extension $\overline{F}$, and c' is an object of $\mathcal{C}'$, then we have

$$\overline{F}(c') = \lim_{c \in \mathcal{C}^{\backslash c'}} F(c)$$

where $\mathcal{C}^{\backslash c'}$ is the co-slice category consisting of objects $c \in \mathcal{C}$ with a map $c' \to c$ in $\mathcal{C}'$.

A crucial example of right Kan extension is when extending sheaves:

Definition A.1.37. Let $\mathcal{D}$ be an ∞-category admitting all small limits and let $\mathcal{C}$ be a small 1-category having fiber products and which is equipped with a Grothendieck topology τ. A functor

$$F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$$

is called a (hyper)sheaf if given any (hyper)covering $c_\bullet \to c$ in $\mathcal{C}$ the natural map

$$F(c) \to \lim_{[n]\in\Delta} F(c_n)$$

is an equivalence. Here by covering we mean: $p\colon c_0 \to c$ is a covering for the Grothendieck topology on $\mathcal{C}$, and $c_\bullet$ is the Čech nerve of p. We denote by $\mathrm{Shv}_\tau^{(\mathrm{hyp})}(\mathcal{C},\mathcal{D})$ the ∞-category of $\mathcal{D}$-valued sheaves on $\mathcal{C}$.

The fact that F is a sheaf implies that if $U \to X$ is a covering in $\mathcal{C}$, the natural diagram

$$F(X) \longrightarrow F(u) \rightrightarrows F(U \times_X U) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \dots$$

is a limit diagram. We recover the usual definition of sheaves when $\mathcal{D}$ is a 1-category thanks to the fact in a n-category, a simplicial limit stops at the step $n+1$ (see https://math.stackexchange.com/questions/4565984/can-we-ignore-higher-dimensional-information-when-computing-the-geometric-realis).

Now we can use right Kan extension to extend sheaves from one site to the other:

Proposition A.1.38. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small sites. Assume that $\mathcal{C}$ is a full subcategory of $\mathcal{C}'$ and that the topology on $\mathcal{C}$ is induced from the topology on $\mathcal{C}$. Assume that any object in $\mathcal{C}'$ can be covered by object in $\mathcal{C}$. Let $\mathcal{D}$ be an ∞-category with small limits. Then the restriction functor*

$$\mathrm{Shv}(\mathcal{C}',\mathcal{D}) \to \mathrm{Shv}(\mathcal{C},\mathcal{D})$$

is an equivalence. A quasi-inverse is given by right Kan extension along $\mathcal{C} \to \mathcal{C}'$.

A proof can be found in [Hoy14, Appendix C], and the case of hypercoverings is in [Aok23]. The way we defined sheaves was as full subcategory of presheaves, namely functors that have a descent property. It turns out that the inclusion functor is the right adjoint of a Bousfield localisation:

Definition A.1.39. A functor $L\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is called a

1. *Bousfield Localisation functor* if it has a right adjoint that is fully faithful.
2. *Dyer-Kan Localisation functor* if when we denote by S the collection of maps in $\mathcal{C}$ that are sent to isomorphisms in $\mathcal{D}$, for every ∞-category $\mathcal{E}$ the restriction along L gives a fully faithful functor
$$\mathrm{Fun}(\mathcal{D},\mathcal{E}) \to \mathrm{Fun}(\mathcal{C},\mathcal{E})$$
whose image is exactly spanned by the functors that send arrows in S to isomorphisms.

It is not hard to check that any Bousfield localisation is a Dyer-Kan localisation, namely because the restriction along L has then a fully faithful right adjoint, given by restriction along the right adjoint of L.

Using ∞-categorical localisation, one can jump into the realm of ∞-category from the world of model categories:

Proposition A.1.40 ([Cis19, Theorem 7.5.30]). *Let $(\mathcal{C}, \mathrm{Fib}, \mathrm{Cofib}, W)$ be a model category. Then $\mathcal{C}_\infty := \mathcal{C}[W^{-1}]$ is an ∞-category whose homotopy category is the model-theoretic homotopy category of $\mathcal{C}$. Moreover, given any Quillen adjunction*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

between model categories we obtain an adjunction

$$\mathcal{C}_\infty \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}_\infty$$

giving the correct adjunction on homotopy categories.

A.2 Presentability

Compact objects.

We will use a lot the fact that some categories and ∞-categories are compactly generated. We begin by recalling the definition of a compact object:

Definition A.2.1. Let $\mathcal{C}$ be an ∞-category admitting small colimits. An object $X \in \mathcal{C}$ is compact if the functor $\mathrm{Map}_{\mathcal{C}}(X, -) \colon \mathcal{C} \to \mathcal{S}$ commutes with filtered colimits.

As an example, if $\mathcal{C} = \mathrm{Ab}$ is the category of abelian groups, an object A is compact if and only if it is a finite type abelian group.

More generally, given any regular cardinal κ, one can define κ-compact objects. First one has to define κ-filtered diagram.

Definition A.2.2. Let κ be a regular cardinal and let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is κ-filtered if for any κ-small simplicial set K and any map $K \to \mathcal{C}$ of simplicial sets, there exists an extension $K^{\triangleright} \to \mathcal{C}$.

Remark A.2.3. In the case $\kappa = \omega$ is the countable cardinal, then a κ-small simplicial set is a finite simplicial set, and the condition exactly says that for any finite diagram K in $\mathcal{C}$, there exists $c \in \mathcal{C}$ "above" the diagram K. A particular case is when $K = \Delta^0 \amalg \Delta^0$ is two points, so that the condition becomes: for any two objects $x, y \in \mathcal{C}$, there exist $z \in \mathcal{C}$ with maps $x \to z$ and $y \to z$, so that we recover the usual definition of filtered.

If $\kappa' > \kappa$ is another regular cardinal, it is clear that κ'-filtered diagrams are κ-filtered.

Another similar fashion is the following, in which we say that a limit is κ-small if the index category is κ-small.

Proposition A.2.4 ([Lur09, Proposition 5.3.3.3]). *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is κ-filtered if and only if the colimit functor*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}) \to \mathcal{S}$$

commutes with κ-small limits.

The above proposition implies the same result in any ∞-category $\mathcal{D}$ admitting κ-small limits and small limits, by the Yoneda lemma. Thus we recover the usual slogan "filtered colimits commutes with finite limits".

Remark A.2.5. We may have to use the notion of sifted colimit. This is a slight generalisation of filtered colimits. Roughly speaking, a functor preserves sifted colimits if and only if preserves filtered colimits an simplicial colimits. The precise definition of a sifted simplicial set is the following: the diagonal $K \to K \times K$ is cofinal. The analogue of the above proposition is that "sifted colimits commute with finite products".

We can now give the definition of κ-compact.

Definition A.2.6. Let $\mathcal{C}$ be a ∞-category admitting small colimits. An object $X \in \mathcal{C}$ is κ-compact if the functor

$$\mathrm{Map}_{\mathcal{C}}(X, -) \colon \mathcal{C} \to \mathcal{S}$$

commutes with κ-filtered colimits.

If $\mathcal{C}$ is a small 1-category, there exists a universal way of adjoining to $\mathcal{C}$ filtered colimits: the ind-completion. This construction exists in the world of ∞-categories.

Proposition A.2.7 ([Lur09, Proposition 5.3.5.10])**.** *Let $\mathcal{C}$ be an ∞-category and let κ be a regular cardinal. There exists an ∞-category $\mathrm{Ind}_\kappa(\mathcal{C})$ admitting κ-filtered colimits together with a fully faithful functor $j \colon \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ such that for any ∞-category $\mathcal{D}$ admitting κ-filtered colimits, composition with j induces an equivalence between functors $\mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ that commute with κ-filtered colimits and functors $\mathcal{C} \to \mathcal{D}$.*

It is true that for any object $C \in \mathcal{C}$, the object $j(C)$ is κ-compact ([Lur09, Proposition 5.3.55]). If $\kappa = \omega$ we denote $\mathrm{Ind}_\omega =: \mathrm{Ind}$. We have the following:

Proposition A.2.8. *Let $\mathcal{C}$ be a small ∞-category and let κ be a regular cardinal. The compact objects of $\mathrm{Ind}_\kappa(\mathcal{C})$ identify with the idempotent completion of $\mathcal{C}$.*

We did not define what is an idempotent in a ∞-category. We do not do in details but we emphasise on the fact that to give an idempotent map $p \colon X \to X$ if does not suffices to give the map p, one also have to provide an equivalence $p \circ p \simeq p$. For further details, see [Lur09, Section 5.1.4].

Accessibility.

Accessibility is a technical condition that asserts that an ∞-category $\mathcal{C}$ is "not too big", or at least, big in a controlled sense.

Definition A.2.9. Let κ be a regular cardinal. An ∞-category $\mathcal{C}$ is κ-accessible if there exists a small ∞-category $\mathcal{C}_0$ and an equivalence

$$\mathrm{Ind}_\kappa(\mathcal{C}_0) \to \mathcal{C}.$$

We say that $\mathcal{C}$ is accessible if it is κ-accessible for some κ.

One can prove ([Lur09, Proposition 5.4.2.2]) that an ∞-category $\mathcal{C}$ is κ-accessible if an only if $\mathcal{C}$ is generated under small κ-filtered by a small full subcategory of κ-compact objects.

The fundamental example of an accessible ∞-category is a category of presheaves on a small ∞-category.

Definition A.2.10. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor between ∞-categories, with $\mathcal{C}$ accessible. We say that F is accessible if it preserves some κ-filtered colimits for some regular cardinal κ.

Presentability

Definition A.2.11. An ∞-category $\mathcal{C}$ is presentable if it is accessible and admits all small colimits.

One can show ([Lur09, Theorem 5.5.1.1]) an ∞-category is presentable if and only for some regular cardinal κ it is κ-accessible and the fullsubcategory of κ-compact objects admits κ-colimits, if and only if $\mathcal{C}$ is an accessible localisation of a category of presheaves (by accessible localisation we mean a Bousfield localisation for which the localisation functor is an accessible functor). The main theorem is the following:

Theorem A.2.12 ([Lur09, Corollary 5.5.2.9])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.*

1. *If F preserves small colimits, then F admits a right adjoint.*
2. *If F preserves small limits and F is accessible, then F admits a left adjoint.*

Using this, one can construct an ∞-category Pr^L of presentable ∞-categories with colimit preserving functors, and an ∞-category Pr^R of presentable ∞-categories with accessible limit preserving functor between them, and prove ([Lur09, Corollary 5.5.3.4]) that there is a canonical equivalence

$$\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}. \tag{A.2.12.1}$$

Moreover, the following result is extremely useful:

Theorem A.2.13 ([Lur09, Proposition 5.5.3.13 and Theorem 5.5.3.18])**.** *The ∞-categories Pr^L and Pr^R admit all small limits and colimits. The forgetful functors*

$$\mathrm{Pr}^L \to \mathrm{Cat}_\infty$$

and

$$\mathrm{Pr}^R \to \mathrm{Cat}_\infty$$

commute with small limits.

In other words, if one wants to compute a colimit in Pr^L, one passes to right adjoints and then compute the limit in Cat_∞.

Definition A.2.14. An ∞-category $\mathcal{C}$ is said to be κ-compactly generated if it is presentable, and κ-accessible. If $\kappa = \omega$, we will just write compactly generated.

Theorem A.2.15 ([Lur09, Proposition 5.5.7.6]). *Let κ be a regular cardinal. There exists an ∞-category Pr^L_ω of κ-compactly generated ∞-categories and functors preserving small colimits and κ-compact objects. The forgetful functor*

$$\mathrm{Pr}^L_\kappa \to \mathrm{Cat}_\infty$$

commutes with small limits the functor

$$\mathrm{Pr}^L_\kappa \to \mathrm{Pr}^L$$

commutes with small colimits.

There is a way of computing κ-filtered colimits of κ-compactly generated categories.

Proposition A.2.16 ([Lur09, Proposition 5.5.7.10, Corollary 4.4.5.16, Proposition 5.5.7.11]). *Let $\mathrm{Cat}_\infty^{\mathrm{Rex}(\kappa),(\mathrm{perf})}$ be the subcategory of (idempotent complete) Cat_∞ whose objects are the ∞-categories admitting κ-small colimits and whose morphisms are functors that preserve κ-small colimits. The functor*

$$\mathrm{Ind}_\kappa \colon \mathrm{Cat}_\infty^{\mathrm{Rex}(\kappa),\mathrm{perf}} \to \mathrm{Pr}^L_\kappa$$

is an equivalence of ∞-categories.

If $\kappa > \omega$ the idempotent completeness is automatic for ∞-categories admitting κ-small colimits.

The functor $\mathrm{Cat}_\infty^{\mathrm{Rex}(\kappa)} \to \mathrm{Cat}_\infty$ preserves κ-filtered colimits.

Corollary A.2.17. *To compute a filtered colimit in Pr^L_ω one has to use the following rule:*

1. *Compute the colimit of the compact objects in Cat_∞ as in Remark A.1.25.*
2. *Take idempotent completion.*
3. *Take $\mathrm{Ind}(-)$.*

Of course, step two can be avoided.

A.3 Symmetric monoidal structures.

Let $\mathcal{C}$ be a category. A symmetric monoidal structure on $\mathcal{C}$ is already a lot of data: one needs a unit object $\mathbf{1} \in \mathcal{C}$, a tensor product $- \otimes - \colon \mathcal{C} \to \mathcal{C}$ and natural equivalences such as symmetry

$$s_{X,Y} \colon X \otimes Y \simeq Y \otimes X,$$

associativity

$$c_{X,Y,Z} \colon X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$$

and neutralisation of the unit

$$\eta \colon X \otimes \mathbf{1} \simeq X$$

satisfying some coherences. By the coherence theorem ([ML98, Theorem 1 in chapter XI]), we now that any two equations with the same n entries, using only the above data will be isomorphic. This is no longer true in the realm of ∞-categories. Moreover, the data s, c and

η are only defined up to equivalence, and those equivalences have to be part of the data, and the coherences with those higher equivalences. This cannot be written by hand. Thankfully, the Grothendieck construction can be used to bypass this difficulty. The main observation is that all this data can be encoded by a functor

$$\mathbb{F}\colon (\mathrm{Fin}^{\leqslant 3})_* \to \mathrm{Cat}_1$$

where $(\mathrm{Fin}^{\leqslant}3)_*$ is the category of finite sets of cardinality less than 3, pointed, such that if for $n \in \{0,1,2,3\}$ we denote by $\langle n\rangle$ the pointed set $\{1,2,\dots,n,*\}$, the natural map

$$\mathbb{F}(\langle n\rangle) \to \prod_{i=1}^{n} \mathbb{F}(\langle 1\rangle) \tag{A.3.0.1}$$

is an equivalence induced by the maps $\langle n\rangle \to \langle 1\rangle$ sending i to 1 and any $j \neq i$ to $*$. In fact, if $\mathcal{C} := \mathbb{F}(\langle 1\rangle)$, we can construct a symmetric monoidal structure on $\mathcal{C}$ out of F: Because $F(\langle 0\rangle) = *$, the unique map $\langle 0\rangle \to \langle 1\rangle$ gives a distinguished object $\mathbf{1} \in \mathcal{C}$. For this exposition, if $n \in \mathbb{N}$ and $\{m_1,\dots,m_n\} \subset \langle m\rangle$ for some integer m, denote by $(m_1,\dots,m_n)$ the map $\langle n\rangle \to \langle m\rangle$ sending i to m_i and $*$ to $*$. The map $(1,1)\colon \langle 2\rangle \to \langle 1\rangle$ sending 1 and 2 to 1 gives us a map $-\otimes -\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$. Note that the projections $\mathcal{C}\times\mathcal{C} \to \mathcal{C}$ are induces by the two maps $(1,*)$ and $(*,1)$. Now this implies that the map induced by $\langle 1\rangle \xrightarrow{1} \langle 1\rangle$ is the functor $(\mathrm{Id},\mathbf{1})\colon \mathcal{C} \to \mathcal{C}\times\mathcal{C}$. Thus the commutation of

$$\begin{array}{ccc} \langle 1\rangle & & \\ \downarrow{\scriptstyle (1)} & \searrow^{(1)=\mathrm{Id}_{\langle 1\rangle}} & \\ \langle 2\rangle & \xrightarrow[(1,1)]{} & \langle 1\rangle \end{array}$$

gives that for any $X \in \mathcal{C}$, $X\otimes\mathbf{1} \simeq X$. The symmetry comes from the involution $(2,1)$ of $\langle 2\rangle$. The associativity comes from the commutation of the diagram

$$\begin{array}{ccc} \langle 3\rangle & \xrightarrow{(1,1,2)} & \langle 2\rangle \\ {\scriptstyle (1,2,2)}\downarrow & & \downarrow{\scriptstyle (1,1)} \\ \langle 2\rangle & \xrightarrow[(1,1)]{} & \langle 1\rangle \end{array}.$$

The coherences needed in the axioms of a symmetric monoidal category are all encoded by the category $(\mathrm{Fun}^{\leqslant 3})_*$. If we remove the "3" in the notation, we obtain all higher coherences. The coherence theorem [Lur22, Corollary 5.1.1.7] asserts that restriction along $(\mathrm{Fin}^{\leqslant 3})_* \to \mathrm{Fin}_*$ induces an equivalence between functors satisfying the condition that (A.3.0.1) is an equivalence for all n to functors satisfying the condition (A.3.0.1) is an equivalence for all $n \leqslant 3$.

Definition A.3.1. The ∞-category $\mathrm{CAlg}(\mathrm{Cat}_\infty)$ of symmetric monoidal ∞-categories is the ∞-category of functors

$$\mathbb{F}\colon \mathrm{Fin}_* \to \mathrm{Cat}_\infty$$

such that for every $n \in \mathbb{N}$, the map $\mathbb{F}(\langle n\rangle) \to \prod_{i=1}^{n}\mathbb{F}(\langle 1\rangle)$ is an equivalence.

By Lurie straightening theorem, this ∞-category is equivalent to cocartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ such that the maps $(*, *, \dots, *, 1, *, \dots) \colon \langle n \rangle \to \langle 1 \rangle$ induces equivalences between the fibers

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \to \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{\langle 1 \rangle}.$$

Remark A.3.2. If one weakens the definition of symmetric monoidal ∞-category, one obtains the definition of ∞-operad. Except in a technical proof (Proposition A.6.3) we shall not need this notion, thus we will not extend this discussion to ∞-operads.

The notation "CAlg" will be explained in the section on the Lurie tensor product.

Definition A.3.3. Let $\mathcal{C}, \mathcal{D}$ be two symmetric monoidal ∞-categories. A symmetric monoidal functor $F \colon \mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ that makes the diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{F^{\otimes}} & \mathcal{D}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{Fin}_* & \end{array}$$

commute.

Proposition A.3.4 ([Lur22, Corollary 7.3.2.7.])**.** *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories. Assume that the underlying functor of F admits a right adjoint G. Then G is naturally a map of operads above Fin_*. In other words, G is naturally lax-symmetric monoidal.*

The above proposition asserts that if G is a right adjoint of a symmetric moinoidal functor, then we have map $G(-) \otimes G(-) \to G(- \otimes -)$ in a very coherent way.

Some constructions preserve symmetric monoidality:

Proposition A.3.5 ([Lur22, Corollary 4.8.1.14])**.** *Let $\mathcal{C}$ be a symmetric monoidal small ∞-category. Then $\mathrm{Ind}(\mathcal{C})$ is symmetric monoidal and the map $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is a symmetric monoidal functor.*

Proposition A.3.6 ([Lur22, Corollary 3.2.2.5 and Corollary 3.2.3.2])**.** *The forgetful functor*

$$\mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$$

commutes with small limits and small sifted colimits.

We shall also need presentably symmetric monoidal ∞-categories.

Definition A.3.7. Let $\mathcal{C}$ be a presentable ∞-category. We say that $\mathcal{C}$ is presentably symmetric monoidal if $\mathcal{C}$ is symmetric monoidal and the tensor functor

$$- \otimes - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

commutes with small colimits in both variables. We denote by $\mathrm{CAlg}(\mathrm{Pr}^L)$ the ∞-category of presentably symmetric monoidal ∞-categories and symmetric monoidal functors that preserves colimits. We will also say "presentable symmetric monoidal ∞-category".

Proposition A.3.8 ([Lur22, Corollary 3.2.2.5 and Corollary 3.2.3.2])**.** *The forgetful functor*

$$\mathrm{CAlg}(\mathrm{Pr}^L) \to \mathrm{Pr}^L$$

commutes with small limits and small sifted colimits.

A.4 Modules and algebras.

Already for 1-categories, the notion of algebra can be categorified: if R is a commutative ring, a commutative R-algebra is an algebra object in the category of R-modules. This is used, for example, by Deligne in [Del89] in which he defines an affine scheme internal to a Tannakian category $\mathcal{T}$: it is just an object of the opposite category of algebra objects in $\mathcal{T}$. For ∞-categories, the definitions of algebras is simple:

Definition A.4.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\pi\colon \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ be the associated cocartesian fibration. An algebra object in $\mathcal{C}$ is a section of π.

If one sees Fin_* as a symmetric monoidal structure on the 1-object category $*$, an algebra object in $\mathcal{C}$ is just a symmetric monoidal functor $* \to \mathcal{C}$. A map of algebra is then just a morphism in the ∞-category of symmetric monoidal functors $* \to \mathcal{C}$.

Notation A.4.2. *If $\mathcal{C}$ is a symmetric monoidal ∞-category, we denote by $\mathrm{CAlg}(\mathcal{C})$ the ∞-category of algebras in $\mathcal{C}$. We sometimes call $\mathbb{E}_\infty$-ring object the commutative algebra in a symmetric monoidal ∞-category.*

Remark A.4.3. If G is a lax-symmetric monoidal functor between symmetric monoidal ∞-categories, then G preserves algebras objets. In particular, if F is a symmetric monoidal ∞-functor with right adjoint G, then (F, G) induces an adjunction on algebra objects ([Lur22, Remark 7.3.2.13.]).

Proposition A.4.4 ([Lur22, Corollary 3.2.2.5 and Corollary 3.2.3.2])**.** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The forgetful functor*

$$\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$$

commutes with small limits and small sifted colimits that exist in $\mathcal{C}$.

The next step is the notion of modules over an algebra. Because it uses all the technicity of ∞-operads, we cannot give a definition here, and we will use modules as a black box.

Theorem A.4.5 ([Lur22, Corollary 4.2.4.8. and Theorem 4.7.3.5])**.** *Let $\mathcal{C}$ be an ∞-category and let $A \in \mathrm{CAlg}(\mathcal{C})$. There exists a symmetric monoidal ∞-category $\mathrm{Mod}_A(\mathcal{C})$ and an adjunction*

$$\mathcal{C} \underset{\mathrm{ff}_A}{\overset{F_A}{\rightleftarrows}} \mathrm{Mod}_A(\mathcal{C})$$

such that $\mathrm{Mod}_A(\mathcal{C})$ has small limits that exist in $\mathcal{C}$, as well as small filtered colimits that exist in $\mathcal{C}$.

1. *The functor $F_A\colon \mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ is symmetric monoidal.*
2. *The composition $\mathrm{ff}_A \circ F_A$ is the endofunctor $- \otimes A$ of $\mathcal{C}$.*
3. *The functor $\mathrm{ff}_A\colon \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ commutes with small limits and sifted colimits that exist in $\mathcal{C}$.*

Objects of $\mathrm{Mod}_A(\mathcal{C})$ can be heuristically described as objects X of $\mathcal{C}$, together with a map $X \otimes A \to X$ that satisfies the usual axioms of a module, together with a lot of coherences. If $\mathcal{C}$ is any symmetric monoidal ∞-category, then $\mathrm{Mod}_{\mathbf{1}}(\mathcal{C}) \simeq \mathcal{C}$.

We will use modules only in a presentable ∞-categories.

Theorem A.4.6. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and let $A \in \mathrm{CAlg}(\mathcal{C})$. Then Mod_A is presentably symmetric monoidal and the functor F_A, that we also denote by $- \otimes A$ when no confusion can arise, preserves colimits. Moreover, if $\mathrm{Pr}^{\mathrm{Alg}}$ denote the ∞-category of pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a presentably symmetric monoidal ∞-category and $A \in \mathrm{CAlg}(\mathcal{C})$, which is presentable, there is a functor*

$$\mathrm{Pr}^{\mathrm{Alg}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$$

that sends $(\mathcal{C}, A)$ to Mod_A. This functor commutes with all small colimits.

Examples of categories of modules are the following: Let R be a commutative ring. Then R defines a commutative algebra in $\mathrm{Mod}_{\mathbb{Z}} := \mathrm{D}(\mathbb{Z})$ the unbounded derived ∞-category of abelian groups. Then the Schwede-Shipley theorem ([Lur22, Theorem 7.1.2.1]) implies that there is a natural equivalence

$$\mathrm{Mod}_R(\mathrm{D}(\mathbb{Z})) \simeq \mathrm{D}(R)$$

between modules over R in $\mathrm{D}(\mathbb{Z})$ and the unbounded derived ∞-category of R-modules. In this setting, the functor F_R is the derived tensor product, and ff_R is the forgetful functor.

Notation A.4.7. *We will denote by Mod_R the ∞-category $\mathrm{Mod}_R(\mathrm{D}(\mathbb{Z}))$ discussed above, for any algebra object $R \in \mathrm{CAlg}(\mathrm{D}(\mathbb{Z}))$. More generally, if R is an algebra object in Sp, we will still denote by Mod_R the ∞-category $\mathrm{Mod}_R(\mathrm{Sp})$ of modules over R.*

We finish with the notion of idempotent algebra. There shall be a reminiscence of the fact that if A is a commutative ring and $S \subset A$ is a multiplicative subset, then $S^{-1}A \otimes_A S^{-1}A \to \S^{-1}A$ is bijective.

Definition A.4.8. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. An object $X \in \mathcal{C}$ is idempotent if there is a map $\mathbf{1} \to X$ inducing an equivalence $X \otimes X \simeq X$.

Proposition A.4.9 ([Lur22, Proposition 4.8.2.4 and Proposition 4.8.2.7])**.** *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Let $X \in \mathcal{C}$ be an idempotent object. Then:*

1. *The object X has a natural structure of commutative algebra in $\mathcal{C}$ such that the isomorphism $X \otimes X \simeq X$ is the algebra structure and the map $\mathbf{1} \to X$ is the unit map.*
2. *The forgetful functor $\mathrm{ff}_X \colon \mathrm{Mod}_X(\mathcal{C}) \to \mathcal{C}$ is fully faithful. The image consist of object $M \in \mathcal{C}$ such that the map $M \simeq M \otimes \mathbf{1} \to M \otimes X$ is an equivalence.*

In other words, if X is an idempotent object, it is automatically an algebra and being a X-module is a property, not a structure. If R is a commutative ring, then any localisation of R is an idempotent algebra in R-modules.

A.5 The Lurie tensor product

If R is a commutative ring and M, N are R-modules, then the tensor product $M \otimes_R N$ is the universal recipient of a bilinear map from $M \times N$. Lurie categorified this notion:

Theorem A.5.1 ([Lur22, Section 4.8.1]). *The ∞-category Pr^L is naturally symmetric monoidal. If $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, their tensor product $\mathcal{C} \otimes \mathcal{D}$ is the universal recipient of a functor preserving small colimits in each variable, with source $\mathcal{C} \times \mathcal{D}$. In other words there is an ∞-category $\mathcal{C} \otimes \mathcal{D}$ and a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ preserving colimits in both variables such that for every presentable ∞-category $\mathcal{E}$, composing with it induces an equivalence of ∞-categories between functors $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ preserving small colimits and functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserving small colimits in both variables.*

Commutative algebras in Pr^L for this tensor product are exactly presentably symmetric monoidal ∞-categories.

The above theorem is very powerful and give a lot of conceptual ways of think about symmetric monoidal ∞-categories.

Example A.5.2.

1. Any presentable ∞-category is a module over spaces. The ∞-category of spaces is presentably symmetric monoidal (the tensor structure being given by the cartesian product), and for any presentable ∞-category $\mathcal{C}$, there is an action functor
$$\mathcal{C} \otimes \mathcal{S} \to \mathcal{C}$$
obtained from the functor
$$\mathcal{C} \times \mathcal{S} \to \mathcal{C}$$
sending a pair (C, X) to the colimit in $\mathcal{C}$ of the constant diagram C indexed by X. This can be upgraded to a module structure. We note that this also says that the unit object of Pr^L is $\mathcal{S}$, as one could have guessed: if $\mathcal{C}$ is presentably symmetric monoidal, then the unit of $\mathcal{C}$ is a map $* \to \mathcal{C}$, that can be left Kan extended to $\mathcal{S}$.
2. Let k be a ring. Then the category of dg-categories over k is naturally an ∞-category, given by $\mathrm{Mod}_{\mathrm{D}(k)}(\mathrm{Pr}^L)$.

One can also prove ([Lur22, Remark 4.81.18]) that the tensor product is closed: the integral homomorphism functor is given by $\mathrm{LFun}(-,-)$ the ∞-category of colimit preserving functors.

There is a useful computation of the tensor product:

Proposition A.5.3 ([Lur22, Proposition 4.8.1.7]). *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. Then there is a natural equivalence*
$$\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$$
where RFun *denotes accessible limit preserving functors.*

Corollary A.5.4. *Let $\mathcal{C}$ be a small ∞-category and let $\mathcal{D} \to \mathcal{M}$ be a symmetric monoidal colimit preserving functor between presentably symmetric monoidal ∞-categories. Then the canonical functor*
$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{M} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})$$
is an equivalence.

Proof. Indeed, we have (using the universal property of presheaves of spaces):

$$\begin{aligned}\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{D})\otimes_{\mathcal{D}}\mathcal{M} &\simeq \mathrm{LFun}(\mathrm{PSh}(\mathcal{C}),\mathcal{D})\otimes_{\mathcal{D}}\mathcal{M}\\ &\simeq \mathrm{RFun}(\mathrm{PSh}(\mathcal{C})^{\mathrm{op}},\mathcal{D})\otimes_{\mathcal{D}}\mathcal{M}\\ &\simeq (\mathrm{PSh}(\mathcal{C})\otimes\mathcal{D})\otimes_{\mathcal{D}}\mathcal{M}\\ &\simeq \mathrm{PSh}(\mathcal{C})\otimes\mathcal{M}\\ &\simeq \mathrm{RFun}(\mathcal{P}(\mathcal{C})^{\mathrm{op}},\mathcal{M})\\ &\simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{M}).\end{aligned}$$

□

One can consider the following construction:

Construction A.5.5. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. We denote by $\mathrm{Pr}^L_{\mathcal{C}}$ the ∞-category of modules over $\mathcal{C}$ in Pr^L. Then for every object $\mathcal{D}\in\mathrm{Pr}^L_{\mathcal{C}}$, the mapping space can naturally be enriched to take values in $\mathcal{C}$. Indeed, we give a construction, without all the coherences: we have a functor $\mathcal{C}\otimes\mathcal{D}\to\mathcal{D}$. Using that the tensor product on Pr^L is closed, we obtain by adjunction a colimit preserving map $\mathcal{D}\to\mathrm{LFun}(\mathcal{C},\mathcal{C})$. Now, note that there is a natural equivalence of ∞-categories

$$\mathrm{LFun}(\mathcal{C},\mathcal{C})\simeq\mathrm{RFun}(\mathcal{C}^{\mathrm{op}},\mathcal{C}).$$

And passing to right adjoint gives an accessible limit preserving functor

$$\mathrm{RFun}(\mathcal{C}^{\mathrm{op}},\mathcal{C})\to\mathcal{D}.$$

Finally, using Proposition A.5.3, we obtain an accessible limit preserving functor

$$\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{D}.$$

One can check that the composition of this functor with the limit preserving functor $\mathcal{D}\to\mathcal{S}$, adjoint to the unit map $\mathcal{S}\to\mathcal{D}$, is the mapping space in $\mathcal{C}$. For a precise discussion, see Gepner and Hausgeng' [AEWH15, 7.4.10 and 5.7.6].

The following result is also very useful:

Proposition A.5.6 ([Lur22, Theorem 4.8.4.6])**.** *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and let $A\in\mathrm{CAlg}(\mathcal{C})$. Then the functor*

$$-\otimes_{\mathcal{C}}\mathrm{Mod}_A(\mathcal{C})\colon \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L))\to\mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Mod}_A(\mathcal{C})}(\mathrm{Pr}^L))$$

is naturally equivalent to the functor

$$\mathrm{Mod}_{\iota_{(-)}(A)}(-)\colon \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L))\to\mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Mod}_A(\mathcal{C})}(\mathrm{Pr}^L))$$

where if $\mathcal{D}\in\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L))$, we denote by $\iota_{\mathcal{D}}$ the unit map $\mathcal{C}\to\mathcal{D}$.

For example, if $F\colon\mathcal{C}\to\mathcal{D}$ is a symmetric monoidal functor between presentably symmetric monoidal ∞-categories and $A\in\mathrm{CAlg}(\mathcal{C})$, then there is a natural equivalence

$$\mathcal{D}\otimes_{\mathcal{C}}\mathrm{Mod}_A(\mathcal{C})\simeq\mathrm{Mod}_{F(A)}(\mathcal{D}).$$

We finish this section with the notions of dualisable and invertible objects.

Definition A.5.7. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We say that an object $X \in \mathcal{C}$ is dualisable if it is dualisable when seen in $\mathrm{ho}(\mathcal{C})$. The same definition works for invertible objects.

The following lemma is simple but will be used several times:

Lemma A.5.8. *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be an adjunction between symmetric monoidal* ∞*-categories, and assume that the functor* F *is symmetric monoidal. Then if* $X \in \mathcal{C}$ *is dualisable, for any* $M \in \mathcal{D}$ *the natural map*

$$G(M) \otimes A \to G(M \otimes F(A))$$

is an equivalence.

Proof. Use the Yoneda lemma, it is then an easy game with adjunctions. See also [Ayo14b, Lemma 2.8]. □

Proposition A.5.9 ([Rob14])**.** *Let* $\mathcal{C}$ *be a presentably symmetric monoidal* ∞*-category and let* $X \in \mathcal{C}$*. Then there exists a symmetric monoidal* ∞*-category* $\mathcal{C}[X^{\otimes -1}]$ *together with a colimit preserving symmetric monoidal functor* $\iota \colon \mathcal{C} \to \mathcal{C}[X^{\otimes -1}]$ *such that:*

1. *The image of* X *by* ι *is invertible for the tensor product.*
2. *For any presentably symmetric monoidal* ∞*-category* $\mathcal{D}$*, composing with* ι *defines an equivalence of* ∞*-categories between symmetric monoidal colimit preserving functors* $\mathcal{C}[X^{\otimes -1}] \to \mathcal{D}$ *and symmetric monoidal colimit preserving functors* $\mathcal{C} \to \mathcal{D}$ *that send* X *to an invertible object.*

We may have to use this last proposition:

Proposition A.5.10 ([Lurb, Theorem D.7.0.7.])**.** *Let* $\mathcal{C}$ *be a presentable stable* ∞*-category which is compactly generated. Then* $\mathcal{C}$ *is a dualisable object of* Pr^L*. In particular, the functor* $- \otimes \mathcal{C}$ *commutes with all small limits.*

When the object X is nice, the ∞-category $\mathcal{C}[X^{\otimes -1}]$ has a nice description.

Proposition A.5.11 ([AI23, Proposition 1.6.3])**.** *Assume that in the setting of Proposition A.5.10 the cyclic permutation on* $X^{\otimes 3}$ *is homotopic to the identity. Then there is a canonical equivalence*

$$\mathrm{colim}(\mathcal{C} \xrightarrow{-\otimes X} \mathcal{C} \xrightarrow{-\otimes X} \mathcal{C} \to \cdots) \to \mathcal{C}[X^{\otimes -1}].$$

A.6 More on stable ∞-categories.

We finish the appendix with some more results on stable ∞-categories.

There is an easy generalisation of Theorem 1.1.21 with multiple variables:

Corollary A.6.1 (of Theorem 1.1.21)**.** *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be abelian categories. Denote by $\mathrm{Fun}^{\mathrm{nex}}(\prod_i \mathrm{D}^b(\mathcal{A}_i), \mathcal{C})$ the ∞-category of n-multi-exact functors $\prod_i \mathrm{D}^b(\mathcal{A}_i) \to \mathcal{C}$ that is functors that are exact in each variable. Denote also similarly by $\mathrm{Fun}^{\mathrm{nex}}(\prod_i \mathcal{A}_i, \mathcal{C})$ the ∞-category of n-multi-exact functors. Then the restriction to the hearts functor*

$$\mathrm{Fun}^{\mathrm{nex}}(\prod_i \mathrm{D}^b(\mathcal{A}_i), \mathcal{C}) \to \mathrm{Fun}^{\mathrm{nex}}(\prod_i \mathcal{A}_i, \mathcal{C}) \tag{A.6.1.1}$$

is an equivalence of ∞-categories.

Proof. We give the proof for $n = 2$, the general case being similar and can be reduced to the case $n = 2$ by induction. We have canonical equivalence of ∞-categories $\mathrm{Fun}(\mathrm{D}^b(\mathcal{A}_1) \times \mathrm{D}^b(\mathcal{A}_2), \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{D}^b(\mathcal{A}_1), \mathrm{Fun}(\mathrm{D}^b(\mathcal{A}_2), \mathcal{C}))$. This equivalence induces

$$\mathrm{Fun}^{2\mathrm{ex}}(\mathrm{D}^b(\mathcal{A}_1) \times \mathrm{D}^b(\mathcal{A}_2), \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}(\mathcal{A}_1), \mathrm{Fun}^{\mathrm{ex}}(\mathrm{D}^b(\mathcal{A}_2), \mathcal{C})).$$

By Theorem 1.1.21 the latter category is exactly $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}_1, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}_2, \mathcal{C}))$, which again is equivalent to $\mathrm{Fun}^{2\mathrm{ex}}(\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{C})$. □

We will also need a result on the monoidal structure of the bounded derived category. We denote by $\mathrm{SymMono}_1$ the 2-category of symmetric monoidal 1-categories and symmetric monoidal functors.

Proposition A.6.2. *Let $\mathcal{B}$ be a symmetric monoidal abelian category such that the tensor product which is exact in each variable. Then there is a canonical symmetric monoidal structure on the stable ∞-category $\mathrm{D}^b(\mathcal{B})$ such that the inclusion functor $\mathcal{B} \to \mathrm{D}^b(\mathcal{B})$ is symmetric monoidal. Moreover, for any category I and any functor $\mathcal{A} : I \to \mathrm{SymMono}_1$ such that for each $i \in I$ the category $\mathcal{A}(i)$ is a symmetric monoidal abelian category with exact tensor product as above, and such that the transition functors are exact, there is a lift of the functor $\mathrm{D}^b \circ \mathcal{A} \colon I \to \mathrm{Cat}_\infty$ to a functor $I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$.*

Proof. The first case follows from the second with $I = *$. Let $\mathcal{A} : I \to \mathrm{SymMono}_1$ be a diagram of symmetric monoidal abelian categories and symmetric monoidal exact functors, and such that for each $i \in I$ the tensor product on $\mathcal{A}(i)$ is exact in both variables. We can compose this functor with $\mathrm{Ch}^b(-)$ to obtain a diagram of symmetric monoidal additive categories (the monoidal structure on $\mathrm{Ch}^b(\mathcal{A}(i))$ is induced by that of $\mathcal{A}(i)$, using the sign trick). By [ML98, Chapter XI, Section 1, Theorem 1] we know that any symmetric monoidal 1-category has all higher coherences. Written differently as in Lurie's [Lur22, Corollary 5.1.1.7], the forgetful functor induces an equivalence of 2-categories

$$\mathrm{CAlg}(\mathrm{Cat}_1) \to \mathrm{SymMono}_1.$$

Thus we can see $\mathrm{Ch}^b \circ \mathcal{A}$ as a functor

$$I \to \mathrm{CAlg}(\mathrm{Cat}_1) \subset \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Now by [Lur22, Theorem 2.4.3.18 and Proposition 2.4.2.5] the data of this functor is equivalent to the data of a $I^{\amalg}$-monoid, which is classified by a cocartesian fibration $p \colon \mathfrak{C} \to I^{\amalg}$ (we denote by $I^{\amalg}$ the ∞-operad constructed in [Lur22, Construction 2.4.3.1]). By [Hin16, Proposition 2.1.4] we can fiberwise invert quasi-isomorphisms to obtain a cocartesian fibration $q \colon \mathfrak{D} \to I^{\amalg}$

which classifies (using unstraightening[1] and [Lur22, Theorem 2.4.3.18, Proposition 2.4.2.5] again) a functor $I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ sending an object $i \in I$ to $\mathrm{D}^b(\mathcal{A}(i))$. The fact that the functor $\mathcal{A}(i) \to \mathrm{D}^b(\mathcal{A}(i))$ is symmetric monoidal comes from the exactness of the tensor product on $\mathcal{A}(i)$. □

We also have a version of the universal property of $\mathrm{D}^b(\mathcal{A})$ (see Theorem 1.1.21) that works in families.

Proposition A.6.3. *Let $\mathcal{C} : I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ be a diagram of stably symmetric monoidal ∞-categories such that for each $i \in I$ the ∞-category $\mathcal{C}(i)$ has a t-structure such that the tensor product is t-exact in each variable and every arrow $i \to j$ in I induces a t-exact functor. Then the canonical functors $\mathrm{D}^b(\mathcal{C}(i)^\heartsuit) \to \mathcal{C}(i)$ assemble to give a natural transformation $\mathrm{D}^b(\mathcal{C}^\heartsuit) \Rightarrow \mathcal{C}$ of functors $I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$.*

Proof. For $\mathcal{B}$ an abelian category, denote by $\mathcal{K}^b(\mathcal{B})$ the ∞-category of bounded complexes in $\mathcal{B}$, that is the ∞-categorical localisation of the additive category of bounded chain complexes $\mathrm{Ch}^b(\mathcal{B})$ with respect to chain homotopies. By the work of Bunke, Cisinski, Kasprowski and Winges, we know ([BCKW19, Theorem 7.4.0]) that $\mathcal{K}^b(\mathcal{B})$ is the value at $\mathcal{B}$ of a functor

$$\mathcal{K}^b(-) \colon \mathrm{Cat}_\infty^{\mathrm{add}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$$

from the ∞-category of additive ∞-categories and coproduct preserving functors to the ∞-category $\mathrm{Cat}_\infty^{\mathrm{ex}}$ of small stable ∞-categories and exact functors. Moreover, this functor is left adjoint to the forgetful functor

$$\mathrm{ff} \colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_\infty^{\mathrm{add}}.$$

By Cisinski's [Cis19, Theorem 6.1.22] this induces an adjunction

$$\mathrm{Fun}(I^{\amalg}, \mathrm{Cat}_\infty^{\mathrm{add}}) \underset{\circ\mathrm{ff}}{\overset{\circ\mathcal{K}^b}{\rightleftarrows}} \mathrm{Fun}(I^{\amalg}, \mathrm{Cat}_\infty^{\mathrm{ex}}) \ , \qquad \text{(A.6.3.1)}$$

where the category $I^{\amalg}$ is as introduced in the proof of the above proposition. Our assumptions on $\mathcal{C}$ imply that if we denote by $\mathcal{A}(i)$ the heart of the t-structure on $\mathcal{C}(i)$, we have a natural transformation $\mathcal{A} \Rightarrow \mathcal{C}$ of functors $I \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{add}})$. Now, as in the proof of Proposition A.6.2, Lurie's [Lur22, Theorem 2.4.3.18 and Proposition 2.4.2.5] give that the data of this natural transformation is equivalent to the data of a natural transformation $\mathfrak{A} \Rightarrow \mathfrak{C}$ of $I^{\amalg}$-monoids $I^{\amalg} \to \mathrm{Cat}_\infty^{\mathrm{add}}$ (see [Lur22, Definition 2.4.2.1] for a definition). By (A.6.3.1), this gives a natural transformation

$$\mathrm{can} \colon \mathcal{K}^b \circ \mathfrak{A} \Rightarrow \mathfrak{C}$$

of functors $I^{\amalg} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$. We claim that $\mathcal{K}^b \circ \mathfrak{A}$ is still a $I^{\amalg}$-monoid. Indeed, it suffices to prove that $\mathcal{K}^b$ preserves finite products, but both in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ and in $\mathrm{Cat}_\infty^{\mathrm{add}}$, finite product and coproduct agree (this can be proven exactly as in [BCKW19, Lemma 2.1.38]), so that this follows from $\mathcal{K}^b$ being a left adjoint which thus preserves colimits.

[1] One has to verify that the functor $I^{\amalg} \to \mathrm{Cat}_\infty$ obtained by unstraightening is still a $I^{\amalg}$-monoid, but this is clear because $\mathrm{D}^b(-)$ commutes with finite products: this is the case for $\mathrm{Ch}^b(-) \subset \mathrm{Fun}(\mathbb{Z}, -)$ and then we invert products of maps.

Unravelling the definitions, we have proven that the natural transformation $\mathcal{K}^b(\mathcal{A}) \Rightarrow \mathcal{C}$ is well defined and is symmetric monoidal. We will now show that this factors through $\mathrm{D}^b(\mathcal{A})$, and that everything stays symmetric monoidal. For this, we will see our map of $I^{\amalg}$-monoidal $\mathcal{K}^b \circ \mathfrak{A} \Rightarrow \mathfrak{C}$ through straightening, thus as a commutative triangle

$$\begin{array}{ccc} \int_{I^{\amalg}} \mathcal{K}^b \circ \mathfrak{A} & \xrightarrow{\mathfrak{r}} & \int_{I^{\amalg}} \mathfrak{C} \\ \downarrow{\scriptstyle \kappa} & \swarrow{\scriptstyle \gamma} & \\ I^{\amalg} & & \end{array}$$

where κ and γ are the cocartesian fibrations classifying the functors $\mathcal{K}^b \circ \mathfrak{A}$ and $\mathfrak{C}$. We consider a marking W on $\int_{I^{\amalg}} \mathcal{K}^b \circ \mathfrak{A} =: \mathscr{K}$ to be the set of arrows that are products of quasi-isomorphisms. That is an arrow f in $\mathscr{K}$ is marked if there exists an integer n and an object $\underline{i} = (i_1, \dots, i_n) \in I^{\amalg}_{\langle n \rangle}$ such that f is a map in the fiber $\prod_{j=1}^n \mathcal{K}^b(\mathcal{A}(i_j))$ above $\underline{i}$ and is of the form $f_1 \times \cdots \times f_n$ with each f_j being a quasi-isomorphism. Then the cocartesian fibration κ, together with the data of W and the marking of isomorphisms in $I^{\amalg}$, is a marked cocartesian fibration in the sense of Hinich [Hin16, Definition 2.1.1]. Because each $\mathcal{A}(i)$ is the heart of a t-structure on $\mathcal{C}(i)$, all maps in W are sent to isomorphisms in $\int_{I^{\amalg}} \mathfrak{C} =: \mathscr{C}$. Indeed, over a multi-index $(i_1, \dots, i_n)$ in $I^{\amalg}$, the functor $\mathfrak{r}$ becomes the functor

$$\prod_{j=1}^n \mathcal{K}^b(\mathcal{A}(i_j)) \to \prod_{j=1}^n \mathcal{C}(i_j)$$

which sends marked maps to isomorphisms by [BCKW19, Proposition 7.4.11]. Thus the map

$$\mathscr{K} \to \mathscr{C}$$

factors through

$$\mathscr{K} \to \mathscr{K}[W^{-1}].$$

By [Hin16, Proposition 2.1.4], the map $\mathscr{K}[W^{-1}] \to I^{\amalg}$ is a cocartesian fibration, and moreover, the fiber above a $\underline{i} = (i_1, \dots, i_n)$ is the localisation of $\prod_{j=1}^n \mathcal{K}^b(\mathcal{A}(i_j))$ with respect to products of quasi-isomorphisms, which is $\prod_{j=1}^n \mathrm{D}^b(\mathcal{A}(i_j))$. Because $I^{\amalg}$ is a 1-category, to check that the diagram

$$\begin{array}{ccc} \mathscr{K}[W^{-1}] & \longrightarrow & \mathfrak{C} \\ \downarrow{\scriptstyle \kappa} & \swarrow{\scriptstyle \gamma} & \\ I^{\amalg} & & \end{array}$$

commutes it suffices by adjunction to check that $\mathrm{ho}(-)$ of it commutes, which is clear because the horizontal map preserves fibers. Thus after unstraightening we obtain a map of $I^{\amalg}$-monoids

$$\mathrm{D}^b \circ \mathfrak{A} \to \mathfrak{C}$$

which express exactly the symmetric monoidality of a natural transformation $\mathrm{D}^b \circ \mathcal{A} \to \mathcal{C}$, finishing the proof.

□

Remark A.6.4. The proof of Proposition A.6.3 also give the same result with $\mathcal{K}^b(-)$ instead of $\mathrm{D}^b(-)$, and the fact that $\mathcal{K}^b(-) \Rightarrow \mathrm{D}^b(-)$ is symmetric monoidal on symmetric monoidal abelian categories and symmetric monoidal exact functors.

Lemma A.6.5. *Let $\mathcal{A} = \mathrm{colim}_i \mathcal{A}_i$ be a filtered colimit of small abelian categories with exact transitions. Then the canonical functor*

$$\mathrm{colim}_i \mathrm{D}^b(\mathcal{A}_i) \to \mathrm{D}^b(\mathcal{A})$$

is an equivalence.

Proof. Denote by

$$F \colon \mathrm{colim}_i \mathrm{D}^b(\mathcal{A}_i) \to \mathrm{D}^b(\mathcal{A})$$

the canonical functor induced by the functors $\mathrm{D}^b(\mathcal{A}_i) \to \mathrm{D}^b(\mathcal{A})$. By Proposition 1.1.9 the ∞-category $\mathrm{colim}_i \mathrm{D}^b(\mathcal{A}_i)$ has a t-structure, of heart $\mathrm{colim}_i \mathcal{A}_i = \mathcal{A}$. Thus by the universal property of $\mathrm{D}^b(\mathcal{A})$ proved in [BCKW19, Corollary 7.4.12], there exists a canonical functor

$$G \colon \mathrm{D}^b(\mathcal{A}) \to \mathrm{colim}_i \mathrm{D}^b(\mathcal{A}_i)$$

determined by its restriction to $\mathcal{A}$. The restriction of $F \circ G$ to $\mathcal{A}$ is the identity of $\mathcal{A}$, thus $F \circ G \simeq \mathrm{Id}_{\mathrm{D}^b(\mathcal{A})}$. Conversely, the functor $G \circ F$ is determined by its composition with the natural functors $\iota_j \colon \mathrm{D}^b(\mathcal{A}_j) \to \mathrm{colim}_i \mathrm{D}^b(\mathcal{A}_i)$ which are themselves determined by their restriction to $\mathcal{A}_j$. The restriction to $\mathcal{A}_j$ of $G \circ F \circ \iota_j$ lands in

$$\mathcal{A}_j \subset \mathrm{colim}_i \mathcal{A}_i \subset \mathrm{colim}_i \mathrm{D}^b(\mathcal{A}_i)$$

and is just the canonical functor $\mathcal{A}_j \to \mathcal{A}$, thus we have that $G \circ F \simeq \mathrm{Id}_{\mathrm{colim}_i \mathrm{D}^b(\mathcal{A}_i)}$. □

In the case the abelian category $\mathcal{A}$ is a Grothendieck abelian category, there is also good functoriality for the unbounded derived ∞-category.

Definition A.6.6 ([Lur22, Proposition 1.3.5.15., Proposition 1.3.5.21])**.** Let $\mathcal{A}$ be a Grothendieck abelian category. The unbounded derived ∞-category $\mathrm{D}(\mathcal{A})$ is the localisation of the category $\mathrm{Ch}(\mathcal{A})$ of unbounded chain complexes with respect to the class of quasi-isomorphisms. It is presentable, stable, endowed with a t-structure compatible with filtered limit, which is right complete. The heart of the t-structure is $\mathcal{A}$.

An example is the following:

Proposition A.6.7 ([Lurb, Corollary 2.1.2.3.])**.** *Let $\mathcal{C}$ be a site endowed with a topology τ and let $\mathscr{O}$ be a sheaf of commutative rings on $\mathcal{C}$. Then the derived ∞-category $\mathrm{D}(\mathrm{Sh}_\tau(\mathcal{C}, \mathscr{O}))$ of the abelian category $\mathrm{Sh}_\tau(\mathcal{C}, \mathscr{O})$ of sheaves of $\mathscr{O}$-modules is equivalent to the ∞-category of hypersheaves of complexes of $\mathscr{O}$-modules.*

Corollary A.6.8 ([Lurb, Remark 2.1.0.5.])**.** *Let $\mathcal{X}$ be a topos and let $\mathscr{O}$ be a sheaf of rings on $\mathcal{X}$. There exists a functor*

$$\mathcal{X} \to \mathrm{Pr}^L$$

sending an object U of the topos to $\mathrm{D}(\mathrm{Sh}_\tau(\mathcal{X}_{/U}, \mathscr{O}_{|U}))$, and this functor is a sheaf.

Tensor products of stable ∞-categories.

We introduced above the Lurie tensor product. Here we recollect some interaction between stable ∞-categories and the Lurie tensor product. The first one is the following:

Theorem A.6.9 ([Lur22, Proposition 4.8.2.18.]). *The ∞-category of spectra is presentably symmetric monoidal. Moreover, it is idempotent: the functor $\mathrm{Sp} \otimes \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence. As a consequence, the forgetful functor*

$$\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^L) \to \mathrm{Pr}^L$$

is fully faithful, and its image can be identified with the ∞-category of presentable stable ∞-categories.

In fact, the symmetric monoidal structure on spectra is constructed by proving that it is idempotent. In particular, using Construction A.5.5 this explains the mapping spectra.

As a consequence of the above theorem, any presentable stable ∞-category is tensored over spectra. We will use the following proposition:

Proposition A.6.10. *Let $\mathcal{C}$ be a presentable stable ∞-category. Then we can compute the action of $\mathbb{Q}$ as spectra on $\mathcal{C}$: For any $M \in \mathcal{C}$, the object $M \otimes \mathbb{Q}$ is equivalent to the colimit the multiplications by all nonzero integers on M, ordered by divisibility: $M \otimes \mathbb{Q} = \{\mathbb{N}^*\}^{-1}(M)$.*

Corollary A.6.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories with $\mathcal{C}$ stable. Then their tensor product $\mathcal{C} \otimes \mathcal{D}$ is again stable.*

Definition A.6.12. Algebras in Sp are called $\mathbb{E}_\infty$-rings. If R is an $\mathbb{E}_\infty$-ring the ∞-category $\mathrm{Mod}_R(\mathrm{Sp})$ will be denoted by Mod_R.

Each ∞-category Mod_R is stable:

Proposition A.6.13 ([Lur22, Proposition 7.1.1.4.]). *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and let $A \in \mathrm{CAlg}(\mathcal{C})$. Then $\mathrm{Mod}_A(\mathcal{C})$ is stable.*

For Λ a commutative ring, we denote by $\mathrm{Pr}^{\mathrm{L}}_\Lambda := \mathrm{Mod}_{\mathrm{Mod}_\Lambda}(\mathrm{Pr}^{\mathrm{L}})$ the ∞-category of Λ-linear presentable ∞-categories.

Proposition A.6.14. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear presentable ∞-category. Then the action of $\mathbb{Z}/n\mathbb{Z} \in \mathrm{D}(\mathbb{Z})$ on a object $M \in \mathcal{C}$ can be computed: there is an equivalence $M \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathrm{cofib}(M \xrightarrow{\times n} M)$. There is also a cofiber sequence*

$$M \to M \otimes \mathbb{Q} \to M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$$

where $\mathbb{Q}/\mathbb{Z} = \mathrm{colim}_n \mathbb{Z}/n\mathbb{Z}$.

Moreover, when $\mathcal{C}$ is compactly generated we have:

Proposition A.6.15 ([Iwa18, Lemma 2.6]). *Let $\mathcal{C}$ be a symmetric monoidal compactly generated ∞-category and let $A \in \mathrm{CAlg}(\mathcal{C})$. Then $\mathrm{Mod}_A(\mathcal{C})$ is compactly generated by the image of the compact objects of $\mathcal{C}$ under the functor $\mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$.*

Lemma A.6.16. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a map in $\mathrm{CAlg}\left(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}}\right)$ and let G be the right adjoint to F. Assume that for any object N of $\mathcal{D}$, the natural map*

$$G(N) \otimes_{\mathbb{Z}} \mathbb{Q} \to G(N \otimes_{\mathbb{Z}} \mathbb{Q})$$

is an equivalence. If A is a commutative algebra in $\mathrm{Mod}_{\mathbb{Z}}$, denote by F_A the functor $F \otimes_{\mathrm{Mod}_{\mathbb{Z}}} \mathrm{Mod}_A$.

*Then, the functor F is fully faithful (*resp. *an equivalence) if and only if the functors $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ are fully faithful (*resp. *an equivalence) for any nonzero integer n.*

Proof. If F is fully faithful, then [Hai22, Lemma 2.14] ensures that F_A is for any commutative algebra in $\mathrm{Mod}_{\mathbb{Z}}$. The same assertion about equivalences is obvious. This proves the "only if" part of the statement.

We now prove the converse. Assume first that the functors $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ are fully faithful. Let M be an object of $\mathcal{C}$. We want to prove that the natural map

$$M \to GF(M)$$

is an equivalence. This can be checked after tensoring with $\mathbb{Q}$ and with $\mathbb{Z}/n\mathbb{Z}$. Since tensoring with $\mathbb{Z}/n\mathbb{Z}$ commutes with any exact functor, the map $M/n \to GF(M)/n$ being an equivalence amounts to the full faithfulness of $F_{\mathbb{Z}/n\mathbb{Z}}$.

On the other hand, our assumption on G implies that the right adjoint of $F_{\mathbb{Q}}$ sends an object of the form $N \otimes \mathbb{Q}$ to $G(N \otimes \mathbb{Q})$. Therefore, the full faithfulness of $F_{\mathbb{Q}}$ implies that the map

$$M \otimes_{\mathbb{Z}} \mathbb{Q} \to GF(M) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an equivalence. Hence, the functor F is fully faithful.

Assume now further that $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ are essentially surjective. Let N be an object of $\mathcal{D}$. Then, we have an exact triangle

$$N \to N \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{colim}_n\ N \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z};$$

the essential surjectivity of $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ imply that $N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ belong to the image of F. As F is fully faithful, the latter is closed under colimits and finite limits, so that N belongs to it. This finishes the proof □

Remark A.6.17. An obvious case where the previous Lemma A.6.16 applies is when $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between compactly generated presentable $\mathrm{Mod}_{\mathbb{Z}}$-linear ∞-categories that preserves compact objects. Indeed, in that case the right adjoint to F commutes with colimits hence also with

$$(\mathrm{Id} \to - \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{colim}_n(\mathrm{Id} \xrightarrow{n} \mathrm{Id}).$$

However we will apply this lemma in a case where $\mathcal{C}$ is *not* compactly generated.

Definition A.6.18. If $\mathcal{C}$,$\mathcal{D}$ and $\mathcal{E}$ are idempotent complete small stable ∞-categories, we say that $\mathcal{C}$ is a $\mathcal{D}$-module if $\mathrm{Ind}\mathcal{C}$ is a $\mathrm{Ind}\mathcal{D}$-module. In this setting, if $\mathcal{E}$ is also a $\mathcal{D}$-module, we denote by $\mathcal{C} \otimes_{\mathcal{D}} \mathcal{E} := (\mathrm{Ind}\mathcal{C} \otimes_{\mathrm{Ind}\mathcal{D}} \mathrm{Ind}\mathcal{E})^{\omega}$ the compact objects of the tensor product of the indisation.

Recall that the compact objects of Mor_R for R any ring, is the stable ∞-category Perf_R of perfect complexes.

Definition A.6.19. Let $\mathcal{A}$ be a small abelian category and let n be an integer. An object A of $\mathcal{A}$ is

1. torsion-free if for all nonzero integers $m \in \mathbb{Z}\setminus\{0\}$ the map $A \xrightarrow{\times m} A$ is a monomorphism.
2. of n-torsion if the map $A \xrightarrow{\times n} A$ is the zero map.

We denote by $\mathcal{A}[n]$ the full subcategory of $\mathcal{A}$ that consists of n-torsion objects.

Finally, we say that $\mathcal{A}$ has *enough torsion-free objects* if for any object A in $\mathcal{A}$, there exists an epimorphism $B \twoheadrightarrow A$ with B a torsion-free object.

For A a commutative ring, we denote by Perf_A the ∞-category of perfect complexes of A-modules. It consists of dualisable objects of Mod_A.

Proposition A.6.20. *Let $\mathcal{A}$ be a small abelian category and let n be a positive integer. Assume that $\mathcal{A}$ has enough torsion-free objects. Then the left derived functor*

$$- \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \colon \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A}[n])$$

of the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \colon A \mapsto A/nA$ exists, factors through the canonical functor

$$\mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$$

and induces an fully faithful functor

$$\mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{D}^b(\mathcal{A}[n]).$$

Proof. The proof goes with several steps.

Step 1: The functor

$$- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \colon \ \mathcal{A} \to \mathcal{A}[n]$$

is the left adjoint of the inclusion functor

$$\iota \colon \mathcal{A}[n] \to \mathcal{A},$$

and we can derive this adjunction to obtain an adjunction

$$- \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \colon \mathrm{D}^n(\mathcal{A}) \leftrightarrows \mathrm{D}^b(\mathcal{A}[n]) \colon \iota.$$

Indeed, the adjunction induces an adjunction of functors between $\mathrm{Ch}^b(\mathcal{A})$ and $\mathrm{Ch}^b(\mathcal{A}[n])$ the categories of bounded complexes. Now we use Cisinski's formalism of derived functors: We endow $\mathrm{Ch}^b(\mathcal{A}[n])$ with the trivial structure of category with equivalences and fibrations (so that equivalences are quasi-isomorphisms and all maps are fibrations), and $\mathrm{Ch}^b(\mathcal{A})$ with the following structure of a category with equivalences and cofibrations: equivalences are quasi-isomorphisms, and cofibrations are monomorphism with term wise torsion-free cokernels, this is indeed a structure of category with weak equivalences and cofibrations because there are enough torsion-free objects and that they are acyclic for $- \otimes_{\mathbb{Z}} \mathbb{Z}/n$. By [Cis19, Theorem

7.5.30], using that $\mathrm{D}^b(\mathcal{A}) \simeq \mathrm{Ch}^b(\mathcal{A})[W_{qiso}^{-1}]$, we obtain the existence of the adjunction on the derived categories.

Step 2: The functor $- \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ factors trough $\mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$. This is because of the universal property of the tensor product of idempotent complete categories: It is computed as the compact objects of the tensor product between indisation, and the latter has a universal property by definition of Lurie's tensor product of presentable ∞-categories. As we have a $\mathrm{Perf}_{\mathbb{Z}}$-linear functor $\mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A}[n])$ and $\mathrm{D}^b(\mathcal{A}[n])$ is $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$-linear, this step is clear.

Step 3: The functor

$$\alpha \colon \mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{D}^b(\mathcal{A}[n])$$

is fully faithful. Given two complexes K, L in $\mathrm{D}^b(\mathcal{A})$ and two complexes P, Q in $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$, we have

$$\mathrm{map}_{\mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}}(K \boxtimes P, L \boxtimes Q) \simeq \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(K, L) \otimes_{\mathbb{Z}} \mathrm{map}_{\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}}(P, Q),$$

as it is proven in [HL23, Proposition 3.5.5], where $K \boxtimes P$ is the image of (K, P) under the canonical functor

$$\mathrm{D}^b(\mathcal{A}) \times \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}.$$

To prove that α is fully faithful, by dévissage it suffices to prove that it is fully faithful on objects of the form $A \boxtimes \mathbb{Z}/n\mathbb{Z}$ with A an object of $\mathcal{A}$. The image of such an object in $\mathrm{D}^b(\mathcal{A}[n])$ is $A \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Thus, given the above formula for the mapping spectra in the tensor product, we have to show that for two objects A and B of $\mathcal{A}$, the map

$$\mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}/n \to \mathrm{map}_{\mathrm{D}^b(\mathcal{A}[n])}(A \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}, B \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$$

is an equivalence. This is true because

$$\begin{aligned}
\mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}/n &\simeq \mathrm{cofib}(\mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B) \xrightarrow{\times n} \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B)) \\
&\simeq \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, \mathrm{cofib}(B \xrightarrow{\times n} B)) \\
&\overset{(*)}{\simeq} \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, \iota(B \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})) \\
&\simeq \mathrm{map}_{\mathrm{D}^b(\mathcal{A}[n])}(a \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}, b \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}),
\end{aligned}$$

where $(*)$ can be checked by taking a torsion-free resolution of b. This finishes the proof. □

Remark A.6.21. The above functor in proposition is not essentially surjective in general, as it can be seen by taking $\mathcal{A} = \mathrm{Ab}^{\mathrm{ft}}$ the abelian category of finitely generated abelian groups, where we get the difference between $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$ and $\mathrm{D}^b(\mathbb{Z}/n\mathbb{Z}\text{-mod}^{\mathrm{ft}})$, where $\mathbb{Z}/n\mathbb{Z}\text{-mod}^{\mathrm{ft}}$ is the abelian category of finite type $\mathbb{Z}/n\mathbb{Z}$-modules.

We finish with the stability of the ∞-category where we inverted an object:

Proposition A.6.22. *Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category and let $X \in \mathcal{C}$. Then $\mathcal{C}[X^{\otimes -1}]$ is stable.*

Proof. Indeed $\mathcal{C}[X^{\otimes -1}]$ is a $\mathcal{C}$-algebra, hence in particular a Sp-algebra, thus is stable. □

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Swann Tubach swann.tubach@ens-lyon.fr
E.N.S Lyon, UMPA, 46 Allée d'Italie,
69364 Lyon Cedex 07, France

Dans cette thèse nous construisons un foncteur de comparaison entre les motifs de Voevodsky relatifs et la catégorie dérivée de la catégorie abélienne des motifs de Nori pervers. Le foncteur de comparaison est une équivalence lorsque restreint aux 0-motifs. La même construction s'applique aux modules de Hodge, fournissant un foncteur "réalisation de Hodge" compatible aux 6 opération de Grothendieck. Les techniques utilisées sont infini-catégoriques, et cette amélioration permet donc également l'extension des modules de Hodge aux champs algébriques.

Dans une collaboration avec Raphaël Ruimy, nous comparons différentes notions de finitude dans les motifs de Voevodsky étales, répondant positivement à une conjecture de Cisinski-Déglise, et construisons des catégories dérivées de motifs de Nori relatifs à coefficients entiers qui ont les propriétés attendues : six opérations, descente et t-structures.

In this thesis, we construct a comparison functor between relative Voevodsky motives and the derived category of the abelian category of perverse Nori motives. The comparison functor is an equivalence when restricted to 0-motives. The same construction applies to Hodge modules, yielding a "Hodge realisation" functor compatible with the Grothendieck six operations. The techniques employed are infinity-categorical, and this refinement also enables the extension of Hodge modules to algebraic stacks.

In collaboration with Raphaël Ruimy, we compare various notions of finiteness in étale Voevodsky motives, providing a positive answer to a conjecture by Cisinski–Déglise. We also construct derived categories of relative Nori motives with integral coefficients, which possess the expected properties : six operations, descent, and t-structures.