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Étude théorique et numérique de la stabilisation de certains problèmes de transmission d'interface avec contrôles frontières de type dynamique

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Résumé

Cette thèse porte sur la stabilisation de problèmes de transmission d'interface impliquant les équations onde-plaque et les modèles de poutre/onde d'Euler-Bernoulli avec des contrôles frontières dynamiques, en utilisant la théorie des semi-groupes et l'analyse numérique. Tout d'abord, nous étudions un modèle de transmission bidimensionnel composé d'une équation des ondes et d'une équation de plaque de Kirchhoff, couplées par des conditions de transmission le long d'une interface fixe et contrôlées par trois contrôles frontières dynamiques sous certaines conditions géométriques. Bien que ce système ne soit pas exponentiellement stable, nous établissons une décroissance polynomiale de l'énergie de type $1/t$ pour des données initiales régulières en utilisant une approche fréquentielle combinant un raisonnement par contradiction avec la technique des multiplicateurs. Cette méthode impose des conditions géométriques particulières sur les domaines de l'onde et de la plaque. Ensuite, nous généralisons ces résultats à un modèle onde-plaque avec seulement deux contrôles frontières dynamiques au lieu des trois habituels. Notre analyse montre que cette réduction ne modifie pas le taux de décroissance polynomiale de l'énergie, qui reste de l'ordre de $1/t$. Cette approche impose également des conditions géométriques spécifiques au domaine. Enfin, nous étudions la solution numérique d'un problème de transmission onde/poutre d'Euler-Bernoulli en une dimension avec contrôles frontières dynamiques. Le couplage s'effectue via des connexions aux frontières, où les équations d'onde et de poutre évoluent sur des intervalles adjacents. En utilisant la méthode des volumes finis, nous obtenons des estimations de stabilité et prouvons la convergence de la solution numérique vers la solution faible du système continu. De plus, nous présentons une expérimentation numérique validant l'étude théorique, notamment en ce qui concerne le taux de décroissance de la solution sous l'effet d'un unique contrôle frontière dynamique.

Mots-clés: C_0-semi-groupe, stabilité, problèmes de transmission, approche fréquentielle, méthode des multiplicateurs, conditions géométriques, contrôles frontières dynamiques, méthode des volumes finis, analyse numérique.

Abstract

This thesis focuses on the stabilization of interface transmission problems involving wave-plate equations and Euler-Bernoulli beam/wave models with dynamical boundary controls, using semigroup theory and numerical analysis. First, we investigate a two-dimensional transmission model consisting of a wave equation and a Kirchhoff plate equation, coupled through transmission conditions along a steady interface and controlled by three dynamical boundary controls under geometric conditions. While the system lacks exponential stability, we establish a polynomial energy decay estimate of type $1/t$ for smooth initial data using a frequency domain approach which combines a contradiction argument with the multiplier technique. This method leads to particular geometric conditions on the wave's and the plate's domains. Next, we extend these results to a wave-plate model with only two dynamical boundary controls instead of the conventional three. Our analysis confirms that this reduction does not affect the polynomial energy decay rate, which remains of order $1/t$. This method also imposes particular geometric conditions on the domain. Finally, we investigate the numerical solution of a one-dimensional Euler-Bernoulli beam/wave transmission problem with dynamical boundary controls. The coupling occurs through boundary connections, where the wave and beam equations evolve across adjacent intervals. Using the Finite Volume Method, we derive stability estimates and prove the convergence of the numerical solution to the weak solution of the continuous system. Additionally, we present a numerical experiment to validate the theoretical study, particularly concerning the decay rate of the solution under the influence of a single dynamical boundary control.

Keywords: C_0-semigroup, stability, transmission problems, frequency domain approach, multiplier method, geometric conditions, dynamical boundary controls, finite volume method, numerical analysis.

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List of Symbols

This section provides definitions for the main symbols and notations used in this thesis.

$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}^*$	The set of non-zero natural numbers.
$\mathbb{Z}$	The set of integer numbers.
$\mathbb{Z}^*$	The set of non-zero integer numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^+$	The set of non-negative real numbers.
$\mathbb{R}^*$	The set of non-zero real numbers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Q}^+$	The set of non-negative rational numbers.
$\mathbb{Q}^*$	The set of non-zero rational numbers.
$\mathbb{C}$	The set of complex numbers.
i	The imaginary unit.
$\Re$	The real part of a complex number.
Im	The imaginary part of a complex number.
L^p	The Lebesgue space.
H^m	The Sobolev space.
C^0	The space of continuous functions.
C^k	The space of functions with k-th order continuous derivatives.
C_0^∞	The space of smooth functions with compact support.
E'	The dual space of E.
$\langle \cdot, \cdot \rangle_{E' \times E}$	The duality pairing between elements of E' and E.
$\lvert \cdot \rvert$	The modulus.
$\lVert \cdot \rVert$	The norm.
max	The maximum.
min	The minimum.
sup	The supreme.
inf	The infimum.
$f_x = \partial_x f$	The partial derivative of f with respect to x.
$f_{xx} = \partial_{xx} f$	The second partial derivative of f with respect to x.
$g'(s)$	The derivative of g with respect to s.

Asymptotic Notation

Let f and g be two functions defined on some subset of the real numbers. The following notations are used:

- $f(x) = O(g(x))$ as $x \to \infty$ to state that there exist a positive number M and a real number x_0 such that
$$|f(x)| \leq M|g(x)|, \quad \forall x \geq x_0.$$
- $f(x) = o(g(x))$ as $x \to \infty$ to state that
$$\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = 0.$$

Chapter 1

Introduction

This thesis includes an introduction in both English and French to reflect the international nature of the research conducted under joint supervision within a cotutelle agreement.

1.1 Introduction in English

Control theory, a branch of applied mathematics, is concerned with designing inputs to influence the behavior of dynamical systems. The controller, often referred to as feedback or damping, is applied to specific outputs to achieve desired goals and enhance system performance. In recent years, the field of control theory for systems governed by partial differential equations (PDEs) has expanded significantly as more realistic models have been introduced. This expansion has had wide-ranging applications in areas such as modern mathematical physics, engineering, biology, computer science, and biomedical research. The field has been further advanced by the availability of sophisticated numerical tools and almost unlimited computational power, which have been used to develop and implement numerical methods for solving complex scientific and engineering problems. Fundamental concepts such as stability and controllability are essential in modern mathematical control theory, as they define the qualitative properties of control systems. In this thesis, we use semigroup theory in conjunction with numerical analysis to investigate the stabilization of interface transmission problems involving wave-plate equations and Euler-Bernoulli beam/wave models with dynamical boundary controls.

We will provide a brief overview of the thesis structure and the key results obtained. The dissertation is organized into four chapters.

Taking into consideration that the majority of studies in this thesis are based on semigroup theory, Chapter 2 outlines several well-known results on semigroups, particularly the results and theorems related to the strong, exponential, and polynomial stability of a C_0-semigroup. Additionally, theorems related to the numerical aspects that will be used in later chapters are introduced.

Chapter 3 investigates a two-dimensional transmission model consisting of a wave equation and a Kirchhoff plate equation with dynamical boundary controls under geometric conditions. The two equations are coupled through transmission conditions along a steady interface between the domains in which the wave and plate equations evolve, respectively. Our primary concern is the stability analysis of the system employing a set of three dynamical boundary controls, which has not appeared in the literature. More precisely, we consider $\Omega \subset \mathbb{R}^2$ an open bounded set with Lipschitz boundary $\partial\Omega$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where Ω_i is a bounded set with Lipschitz boundary $\partial\Omega_i$, $i = 1, 2$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. We denote by I the interior of $\overline{\Omega}_1 \cap \overline{\Omega}_2$, which is called the interface between Ω_1 and Ω_2, and $\Gamma_i = \partial\Omega_i \backslash \overline{I}$ represents the exterior boundary of Ω_i with positive measure, $i = 1, 2$. We

consider the following interface transmission wave-plate model:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega_1 \times \mathbb{R}_+^*, \\ w_{tt} + \Delta^2 w = 0, & \text{in } \Omega_2 \times \mathbb{R}_+^*, \\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I \times \mathbb{R}_+^*, \\ \partial_{\nu_1} u + \eta = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ \mathcal{B}_1 w + \xi = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*, \\ \mathcal{B}_2 w - \zeta = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*, \end{cases} \tag{1.1}$$

with the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega_1, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1, \tag{1.2}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in \Omega_2, \quad \xi(x,0) = \xi_0(x), \quad \zeta(x,0) = \zeta_0(x), \quad x \in \Gamma_2, \tag{1.3}$$

where η, ξ and ζ denote the dynamical boundary controls, $\nu_i = (\nu_{i1}, \nu_{i2})$ is the unit outward normal vector along $\partial\Omega_i$, and $\tau_i = (-\nu_{i2}, \nu_{i1})$ is the unit tangent vector along $\partial\Omega_i$, $i = 1, 2$. Additionally, in this problem's context, we introduce the constant $\mu \in \left(0, \frac{1}{2}\right)$ as the Poisson coefficient. Moreover, we define the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ on $\partial\Omega_2$ as follows:

$$\mathcal{B}_1 f = \Delta f + (1-\mu)\mathcal{C}_1 f$$

and

$$\mathcal{B}_2 f = \partial_{\nu_2} \Delta f + (1-\mu)\partial_{\tau_2} \mathcal{C}_2 f,$$

where

$$\mathcal{C}_1 f = 2\nu_{21}\nu_{22} f_{x_1x_2} - \nu_{21}^2 f_{x_2x_2} - \nu_{22}^2 f_{x_1x_1} \quad \text{and} \quad \mathcal{C}_2 f = (\nu_{21}^2 - \nu_{22}^2) f_{x_1x_2} - \nu_{21}\nu_{22}\left(f_{x_1x_1} - f_{x_2x_2}\right).$$

The damping of the system is made via the indirect damping mechanism that involves the following first-order differential equations:

$$\begin{cases} \eta_t - u_t + \eta = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ \xi_t - \partial_{\nu_2} w_t + \xi = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*, \\ \zeta_t - w_t + \zeta = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*. \end{cases} \tag{1.4}$$

We define the associated system energy as

$$E(t) = \frac{1}{2}\left\{\int_{\Omega_1} \left(|\nabla u|^2 + |u_t|^2\right) dx + \int_{\Gamma_1} |\eta|^2 d\Gamma\right\} + \frac{1}{2}\left\{a(w,\overline{w}) + \int_{\Omega_2} |w_t|^2 dx + \int_{\Gamma_2} \left(|\xi|^2 + |\zeta|^2\right) d\Gamma\right\},$$

where the sesquilinear form $a : H^2(\Omega_2) \times H^2(\Omega_2) \longmapsto \mathbb{C}$ is defined by

$$a(f,\overline{g}) = \int_{\Omega_2} \left[f_{x_1x_1}\overline{g}_{x_1x_1} + f_{x_2x_2}\overline{g}_{x_2x_2} + \mu\left(f_{x_1x_1}\overline{g}_{x_2x_2} + f_{x_2x_2}\overline{g}_{x_1x_1}\right) + 2(1-\mu) f_{x_1x_2}\overline{g}_{x_1x_2}\right] dx.$$

If $U = (u, u_t, \eta, w, w_t, \xi, \zeta)$ is a regular solution of system (1.1)-(1.4), then the energy $E(t)$ satisfies the following estimation

$$E'(t) = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} \left(|\xi|^2 + |\zeta|^2\right) d\Gamma \leq 0,$$

indicating that the system is dissipative in the sense that the energy $E(t)$ is non-increasing with respect to time variable t. The primary focus in this chapter is deriving the decay rate of the energy $E(t)$ under specific geometric conditions.

We demonstrate the well-posedness of the system through the semigroup theory. To begin, we introduce the following functional spaces:

$$H^1_*(\Omega_1) = \left\{ u \in H^1(\Omega_1) \mid \int_{\Omega_1} u dx = 0 \right\},$$

$$H^2_*(\Omega_2) = \left\{ w \in H^2(\Omega_2) \mid \int_{\Omega_2} w dx = \int_{\Omega_2} \nabla w dx = 0 \right\}$$

and the energy space $\mathcal{H}$ by

$$\mathcal{H} = \left\{ (u,v,\eta,w,z,\xi,\zeta) \in H^1_*(\Omega_1) \times L^2(\Omega_1) \times L^2(\Gamma_1) \times H^2_*(\Omega_2) \times L^2(\Omega_2) \times L^2(\Gamma_2) \times L^2(\Gamma_2) \mid \ u = w \ \text{ on } \ I \right\},$$

which is endowed with the following usual inner product

$$\left(U, \tilde{U}\right)_{\mathcal{H}} = \int_{\Omega_1} \left(\nabla u \cdot \nabla \overline{\tilde{u}} + v\overline{\tilde{v}}\right) dx + \int_{\Gamma_1} \eta \overline{\tilde{\eta}} d\Gamma + a(w, \overline{\tilde{w}}) + \int_{\Omega_2} z\overline{\tilde{z}} dx + \int_{\Gamma_2} \left(\xi \overline{\tilde{\xi}} + \zeta \overline{\tilde{\zeta}}\right) d\Gamma,$$

where $U = (u,v,\eta,w,z,\xi,\zeta)$, $\tilde{U} = (\tilde{u},\tilde{v},\tilde{\eta},\tilde{w},\tilde{z},\tilde{\xi},\tilde{\zeta}) \in \mathcal{H}$. We next define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u,v,\eta,w,z,\xi,\zeta) \in \mathcal{H} \mid \Delta u \in L^2(\Omega_1), \quad v \in H^1_*(\Omega_1), \quad \Delta^2 w \in L^2(\Omega_2), \quad z \in H^2_*(\Omega_2), \\ \partial_{\nu_1} u + \eta = 0 \ \text{ on } \ \Gamma_1, \quad \mathcal{B}_1 w + \xi = 0 \ \text{ on } \ \Gamma_2, \quad \mathcal{B}_2 w - \zeta = 0 \ \text{ on } \ \Gamma_2, \\ v = z, \quad \mathcal{B}_1 w = 0 \ \text{ and } \ \mathcal{B}_2 w = \partial_{\nu_1} u \ \text{ on } \ I \end{array} \right\}$$

and

$$\mathcal{A}U = \left(v, \Delta u, \gamma_1(v) - \eta, z, -\Delta^2 w, \gamma_{2,2}(z) - \xi, \gamma_{2,1}(z) - \zeta\right), \qquad \forall U = (u,v,\eta,w,z,\xi,\zeta) \in D(\mathcal{A}),$$

where $\gamma_1 : H^1(\Omega_1) \longmapsto L^2(\Gamma_1)$ and $\gamma_2 : H^2(\Omega_2) \longmapsto L^2(\Gamma_2) \times L^2(\Gamma_2)$ are the trace operators such that $\gamma_1(\varphi) = \varphi|_{\Gamma_1}$ and $\gamma_2(\psi) = (\gamma_{2,1}(\psi), \gamma_{2,2}(\psi)) = (\psi|_{\Gamma_2}, \partial_{\nu_2}\psi|_{\Gamma_2})$. Setting $U = (u, u_t, \eta, w, w_t, \xi, \zeta)$ as the state of system (1.1)-(1.4), we rewrite the problem into a first-order evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{1.5}$$

where the initial value $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \xi_0, \zeta_0) \in \mathcal{H}$ is given by the equations (1.2) and (1.3).

We prove that the unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$. Then, according to Lumer-Philips theorem (see [44]), we deduce that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ in $\mathcal{H}$ which gives the well-posedness of (1.5). Therefore, we have the following result:

Theorem 1.1.1. *For all $U_0 \in \mathcal{H}$, system (1.5) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the system (1.5) admits a unique strong solution

$$U(t) \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

In the following, our goal is to establish energy estimates of the system (1.5). Our stability results are based on a frequency domain approach combined with a multiplier technique and hence require some Green's formulas in Ω_1 and Ω_2. In order to justify these formulas, we need some regularity results proved in [11] and [16]. Hence, from now on, we assume that Ω_1 and Ω_2 have a polygonal boundary in the sense that their boundary is piecewise smooth. For $i = 1, 2$, we denote by $\omega_{i,j}$, $j = 1, \ldots, N_i$ the interior angles at the corners of Ω_i enclosed between two consecutive curves. These angles may vary within the range $0 < \omega_{i,j} \leq 2\pi$. To obtain the needed regularity, we require the following assumptions concerning the angles at the corners of Ω_1 and Ω_2:

Assumption (A1) *The inner angles $\omega_{1,j} < \pi$, for all $j = 1, \ldots, N_1$.*

Assumption (A2) *There exists a minimal angle ω_0 where ω_0 depends on the Poisson coefficient μ (for instance, $\omega_0 \simeq 77.753311\ldots^\circ$ when $\mu = 0.3$) such that $\omega_{2,j} < \omega_0$, for all $j = 1, \ldots, N_2$.*

The following proposition summarizes the regularity results:

Proposition 1.1.1. *Let $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$. Assume that Ω_1 and Ω_2 are polygonal domains as described above, satisfying Assumptions (A1) and (A2), respectively. Then, there exists $\varepsilon \in (0, \frac{1}{2})$ such that u belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ and there exists a sequence $\{w_k\}_{k\geq 0} \subset H^4(\Omega_2)$ such that w_k converges to w in $H^2(\Omega_2)$ and $\Delta^2 w_k$ converges to $\Delta^2 w$ in $L^2(\Omega_2)$.*

Building on these regularity results, we prove the strong stability of the system (1.1)-(1.4) in the sense that its energy $E(t)$ converges to zero as t goes to infinity for all initial data in $\mathcal{H}$. It is important to note that, since the resolvent of $\mathcal{A}$ is not compact, classical methods such as Lasalle's invariance principle [55] or the spectrum decomposition theory of Benchimol [9] are not applicable in this case. Instead, we employ a more general criteria of Arendt-Batty [7] which states that in a reflexive Banach space, a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable if $\mathcal{A}$ has no eigenvalues on the imaginary axis, and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$. In summary, our main result is the following theorem:

Theorem 1.1.2. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Assume also that Assumptions (A1) and (A2) hold. Then, the semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable in the energy space $\mathcal{H}$ in the sense that*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

Moreover, we demonstrate that the system (1.1)-(1.4) is not exponentially stable. To characterize the exponential stability of a C_0-semigroup of contractions, we rely on the results established by Huang [28] and Prüss [45]. Specifically, according to their findings, we can prove that the resolvent of the operator $\mathcal{A}$ is not uniformly bounded on the imaginary axis.

Given that the system (1.1)-(1.4) lacks uniform stability, we turn our attention to studying its polynomial decay for smooth solutions by employing a multiplier method. One of the main ingredients is to use the frequency domain approach from semigroup theory. This method combines a contradiction argument with the multiplier technique. Notably, the key ingredient in proving polynomial stability is Theorem 2.4 of [12] (see also [8] and [38]). To proceed with our polynomial energy decay result, we require the following additional geometric assumption on the wave and the plate, which is due to the requirements of integration in Ω_1 and Ω_2:

Assumption (A3) Assume that there exists a fixed point $x_0 \in \mathbb{R}^2$ such that, putting $m(x) = x - x_0$, we have

$$(m \cdot \nu_1) = 0, \quad \forall x \in I,$$

and

$$(m \cdot \nu) \geq \delta, \quad \forall x \in \partial\Omega,$$

where δ is a positive real number and $(\cdot)$ designates the scalar product in $\mathbb{R}^2$.

With these assumptions, we can now present our result concerning the energy decay rate:

Theorem 1.1.3. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Also, assume that Assumptions (A1), (A2) and (A3) hold. Then, there exists a constant $\mathcal{C} > 0$ such that for all initial data $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \xi_0, \zeta_0) \in D(\mathcal{A})$, the energy of system (1.1)-(1.4) satisfies the following estimate:*

$$E(t) \leq \frac{\mathcal{C}}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

The results presented in this chapter have been published in the *Acta Applicandae Mathematicae* journal in an article titled "Stability for an Interface Transmission Problem of Wave-Plate Equations with Dynamical Boundary Controls" (see [1]).

In Chapter 4, we extend the results obtained in the previous chapter to a wave-plate model subject to only two dynamical boundary feedbacks at the exterior boundaries of the wave's and the plate's domains. The primary aim of this chapter is to estimate the decay rate of the energy under the condition where the dynamical control ξ is omitted, thereby subjecting the system to the influence of two dynamical controls, namely η and ζ. We consider a similar geometric domain as described previously. The following interface transmission wave-plate model is considered:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega_1 \times \mathbb{R}^*_+, \\ w_{tt} + \Delta^2 w = 0, & \text{in } \Omega_2 \times \mathbb{R}^*_+, \\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I \times \mathbb{R}^*_+, \\ \partial_{\nu_1} u + \eta = 0, & \text{on } \Gamma_1 \times \mathbb{R}^*_+, \\ \mathcal{B}_1 w = 0, & \text{on } \Gamma_2 \times \mathbb{R}^*_+, \\ \mathcal{B}_2 w - \zeta = 0, & \text{on } \Gamma_2 \times \mathbb{R}^*_+, \end{cases} \tag{1.6}$$

with the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega_1, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1, \tag{1.7}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in \Omega_2, \quad \zeta(x,0) = \zeta_0(x), \quad x \in \Gamma_2, \tag{1.8}$$

where η and ζ denote the dynamical boundary controls, $\nu_i = (\nu_{i1}, \nu_{i2})$ is the unit outward normal vector along $\partial\Omega_i$, and $\tau_i = (-\nu_{i2}, \nu_{i1})$ is the unit tangent vector along $\partial\Omega_i$, $i = 1, 2$. Moreover, the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined on $\partial\Omega_2$ as described above. The damping of the system is made via the indirect damping mechanism that involves the following first-order differential equations:

$$\begin{cases} \eta_t - u_t + \eta = 0, & \text{on } \Gamma_1 \times \mathbb{R}^*_+, \\ \zeta_t - w_t + \zeta = 0, & \text{on } \Gamma_2 \times \mathbb{R}^*_+. \end{cases} \tag{1.9}$$

For smooth solutions u, η, w and ζ of the system (1.6)-(1.9), we define the associated energy as follows

$$E(t) = \frac{1}{2}\left\{\int_{\Omega_1}\left(|\nabla u|^2 + |u_t|^2\right)dx + \int_{\Gamma_1}|\eta|^2 d\Gamma\right\} + \frac{1}{2}\left\{a(w,\overline{w}) + \int_{\Omega_2}|w_t|^2 dx + \int_{\Gamma_2}|\zeta|^2 d\Gamma\right\},$$

where the sesquilinear form $a : H^2(\Omega_2) \times H^2(\Omega_2) \longmapsto \mathbb{C}$ is defined as described previously. A straightforward computation shows that the energy $E(t)$ satisfies the following estimate

$$E'(t) = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} |\zeta|^2 d\Gamma \leq 0,$$

indicating that the system (1.6)-(1.9) is dissipative in the sense that the energy $E(t)$ is non-increasing with respect to time variable t.

To investigate the well-posedness of the system (1.6)-(1.9), we introduce the spaces $H^1_*(\Omega_1)$ and $H^2_*(\Omega_2)$ as previously defined, along with the energy space $\mathcal{H}$ given by:

$$\mathcal{H} = \left\{(u, v, \eta, w, z, \zeta) \in H^1_*(\Omega_1) \times L^2(\Omega_1) \times L^2(\Gamma_1) \times H^2_*(\Omega_2) \times L^2(\Omega_2) \times L^2(\Gamma_2) \mid \ u = w \ \text{ on } \ I\right\},$$

which is endowed with the following usual inner product

$$\left(U, \tilde{U}\right)_{\mathcal{H}} = \int_{\Omega_1} \left(\nabla u \cdot \nabla \overline{\tilde{u}} + v\overline{\tilde{v}}\right) dx + \int_{\Gamma_1} \eta\overline{\tilde{\eta}} d\Gamma + a(w, \overline{\tilde{w}}) + \int_{\Omega_2} z\overline{\tilde{z}} dx + \int_{\Gamma_2} \zeta\overline{\tilde{\zeta}} d\Gamma,$$

where $U = (u, v, \eta, w, z, \zeta)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{w}, \tilde{z}, \tilde{\zeta}) \in \mathcal{H}$. We next define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, \eta, w, z, \zeta) \in \mathcal{H} \mid \Delta u \in L^2(\Omega_1), \quad v \in H^1_*(\Omega_1), \quad \Delta^2 w \in L^2(\Omega_2), \quad z \in H^2_*(\Omega_2), \\ \partial_{\nu_1} u + \eta = 0 \ \text{ on } \ \Gamma_1, \quad \mathcal{B}_1 w = 0 \ \text{ on } \ \Gamma_2, \quad \mathcal{B}_2 w - \zeta = 0 \ \text{ on } \ \Gamma_2, \\ v = z, \quad \mathcal{B}_1 w = 0 \ \text{ and } \ \mathcal{B}_2 w = \partial_{\nu_1} u \ \text{ on } \ I \end{array} \right\}$$

and

$$\mathcal{A}U = \left(v, \Delta u, \gamma_1(v) - \eta, z, -\Delta^2 w, \gamma_{2,1}(z) - \zeta\right), \qquad \forall U = (u, v, \eta, w, z, \zeta) \in D(\mathcal{A}),$$

where $\gamma_1 : H^1(\Omega_1) \longmapsto L^2(\Gamma_1)$ and $\gamma_2 : H^2(\Omega_2) \longmapsto L^2(\Gamma_2) \times L^2(\Gamma_2)$ are the trace operators such that $\gamma_1(\varphi) = \varphi|_{\Gamma_1}$ and $\gamma_2(\psi) = (\gamma_{2,1}(\psi), \gamma_{2,2}(\psi)) = (\psi|_{\Gamma_2}, \partial_{\nu_2}\psi|_{\Gamma_2})$. Now, setting $U = (u, u_t, \eta, w, w_t, \zeta)$ as the state of system (1.6)-(1.9), we rewrite the problem into a first-order evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{1.10}$$

where the initial value $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \zeta_0) \in \mathcal{H}$ is given by the equations (1.7) and (1.8).

We prove that the unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$. Then, according to Lumer-Philips theorem (see [44]), we deduce that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ in $\mathcal{H}$. This result guarantees the well-posedness of the system (1.10), leading to the following theorem:

Theorem 1.1.4. *For all $U_0 \in \mathcal{H}$, system (1.10) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the system (1.10) admits a unique strong solution

$$U(t) \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

Next, we impose sufficient geometric conditions on the domains $\Omega_i, i = 1, 2$ (refer to Assumptions (A1) and (A2) above) to ensure the regularity of the solution to this system. To establish the strong stability of the system (1.6)-(1.9) and guarantee the strict decay of the energy, i.e., $E(t)$ tends to zero as t goes to infinity, we introduce the following geometric assumption on the plate, which is due to the requirements of integration in Ω_2:

Assumption (A4) Assume that there exists a fixed point $x_0 \in \mathbb{R}^2$ such that, putting $m(x) = x - x_0$, we have

$$(m \cdot \nu_2) = 0, \quad \forall x \in I,$$

and

$$(m \cdot \nu_2) \geq 0, \quad \forall x \in \Gamma_2,$$

where $(\cdot)$ designates the scalar product in $\mathbb{R}^2$.

Our main result is summarized by the following theorem:

Theorem 1.1.5. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Assume also that Assumptions (A1), (A2) and (A4) hold. Then, the semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable in the energy space $\mathcal{H}$ in the sense that*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

It is evident that exponential decay is not achievable when three dynamical controls are applied. Consequently, the lack of exponential stability persists within the system (1.6)-(1.9) even when only two dynamical controls are applied. Therefore, it is natural to hope for polynomial stability of the system (1.6)-(1.9) for smooth solutions. To proceed with our polynomial energy decay result, we need to impose an additional geometric assumption on the wave, which is due to the requirements of integration in Ω_1:

Assumption (A5) Assume that there exists a fixed point $x_0 \in \mathbb{R}^2$ such that, putting $m(x) = x - x_0$, we have

$$(m \cdot \nu_1) \geq \delta, \quad \forall x \in \Gamma_1,$$

where δ is a positive real number and $(\cdot)$ designates the scalar product in $\mathbb{R}^2$.

We can now present our energy decay rate result:

Theorem 1.1.6. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Also, assume that Assumptions (A1), (A2), (A4) and (A5) hold. Then, there exists a constant $\mathcal{C} > 0$ such that for all initial data $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \zeta_0) \in D(\mathcal{A})$, the energy of system (1.6)-(1.9) satisfies the following estimate:*

$$E(t) \leq \frac{\mathcal{C}}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

The results presented in this chapter have been published in the *Advances in Partial Differential Equations and Control* conference paper in chapter titled "Energy Decay Estimate for a Wave-Plate Interface Transmission Problem with Only Two Dynamical Boundary Controls" (see [2]).

In Chapter 5, we focus on studying the numerical solution for a transmission problem of a wave equation and an Euler-Bernoulli beam with dynamical boundary controls in one spatial dimension. To achieve this, we use the Finite Volume Method (FVM) for spatial discretization. The two equations

are coupled via boundary connections between the intervals where the wave and the Euler-Bernoulli beam evolve. The system that describes this model is as follows:

$$\begin{cases} u_{tt} - au_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ w_{tt} + bw_{xxxx} = 0, & (x,t) \in (-L,0) \times (0,\infty), \\ u(0,t) = w(0,t), \quad w_x(0,t) = 0, & t \in (0,\infty), \\ au_x(0,t) + bw_{xxx}(0,t) = 0, & t \in (0,\infty), \\ u_x(L,t) + \eta(t) = 0, & t \in (0,\infty), \\ w_{xx}(-L,t) - \xi(t) = 0, & t \in (0,\infty), \\ w_{xxx}(-L,t) + \zeta(t) = 0, & t \in (0,\infty), \end{cases} \tag{1.11}$$

with the initial conditions of the form:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,L), \quad \eta(0) = \alpha, \tag{1.12}$$

$$w(x,0) = \phi(x), \quad w_t(x,0) = \theta(x), \quad x \in (-L,0), \quad \xi(0) = \beta, \quad \zeta(0) = \gamma, \tag{1.13}$$

where η, ξ and ζ denote the dynamical boundary controls. Here, the coefficients a and b are strictly positive constant numbers representing the speed of propagation of the wave equation and the Euler-Bernoulli beam respectively. The damping of the system is made via the indirect damping mechanism that involves the following first-order differential equations:

$$\begin{cases} \eta_t(t) - u_t(L,t) + \eta(t) = 0, & t \in (0,\infty), \\ \xi_t(t) - w_{xt}(-L,t) + \xi(t) = 0, & t \in (0,\infty), \\ \zeta_t(t) - w_t(-L,t) + \zeta(t) = 0, & t \in (0,\infty). \end{cases} \tag{1.14}$$

The energy of the system described by (1.11)-(1.14) is defined as

$$\begin{aligned} E(t) = \frac{1}{2}\left\{\int_0^L \left(|u_t(x,t)|^2 + a|u_x(x,t)|^2\right) dx + a|\eta(t)|^2\right\} \\ + \frac{1}{2}\left\{\int_{-L}^0 \left(|w_t(x,t)|^2 + b|w_{xx}(x,t)|^2\right) dx + b|\xi(t)|^2 + b|\zeta(t)|^2\right\}. \end{aligned}$$

Finding the derivative of $E(t)$ along the solution of (1.11)-(1.14) gives

$$\frac{dE(t)}{dt} = -\left(a|\eta(t)|^2 + b|\xi(t)|^2 + b|\zeta(t)|^2\right) \leq 0,$$

which implies that the energy of the system decreases as time goes on. Our objective is to study this system numerically.

We begin by defining the weak solution for the problem (1.11)-(1.14). Throughout this text, we will employ the following notation for function spaces:

Definition of the Functional Spaces. *Let us introduce the following spaces:*

$$H^1_*(\ell_1,\ell_2) = \left\{u \in H^1(\ell_1,\ell_2) \;\; | \;\; \int_{\ell_1}^{\ell_2} u dx = 0\right\},$$

$$H^2_*(\ell_1,\ell_2) = \left\{w \in H^2(\ell_1,\ell_2) \;\; | \;\; \int_{\ell_1}^{\ell_2} w dx = w_x(\ell_2) = 0\right\},$$

$$H^2_{**}(\ell_1,\ell_2)=\left\{w\in H^2(\ell_1,\ell_2)\ \ |\ \ w_x(\ell_2)=0\right\},$$
$$\mathbb{H}=\left\{(u,w)\in H^1_*(0,L)\times H^2_*(-L,0)\ \ |\ \ u(0)=w(0)\right\},$$
$$\mathbb{H}_*=\left\{(u,w)\in H^1(0,L)\times H^2_{**}(-L,0)\ \ |\ \ u(0)=w(0)\right\},$$

and

$$\mathbb{L}^2=L^2(0,L)\times L^2(-L,0).$$

The weak formulation of the problem (1.11)-(1.14) is defined as follows:

Definition 1.1.1. *We say that the function (u,w,η,ξ,ζ) is a weak solution of (1.11)-(1.14) if, for all $T>0$, the following conditions hold:*

1. *$(u,w)\in L^2([0,T];\mathbb{H})$ with $(u_t,w_t)\in L^2([0,T];\mathbb{L}^2)$ and $(\eta,\xi,\zeta)\in L^2([0,T];\mathbb{R}^3)$,*

2. *$(u(\cdot,0),w(\cdot,0))=(\varphi(\cdot),\phi(\cdot))$,*

3. *(u,w,η,ξ,ζ) is such that*

$$\begin{aligned}&-\int_0^T\int_0^L u_tv_tdxdt+a\int_0^T\int_0^L u_xv_xdxdt-\int_0^L\psi(x)v(x,0)dx\\&-\int_0^T\int_{-L}^0 w_tz_tdxdt+b\int_0^T\int_{-L}^0 w_{xx}z_{xx}dxdt-\int_{-L}^0\theta(x)z(x,0)dx\\&+a\int_0^T\eta(t)v(L,t)dt+b\int_0^T\xi(t)z_x(-L,t)dt+b\int_0^T\zeta(t)z(-L,t)dt=0,\end{aligned}\tag{1.15}$$

   for all $(v,z)\in L^2([0,T];\mathbb{H}_*)$ with $(v_t,z_t)\in L^2([0,T];\mathbb{L}^2)$, such that $v(T,x)=z(T,x)=0$.

4. *(u,w,η,ξ,ζ) is such that*

$$-\int_0^T\eta p_tdt-\alpha p(0)+\int_0^T u(L,t)p_t(t)dt+u(L,0)p(0)+\int_0^T\eta pdt=0,\tag{1.16}$$
$$-\int_0^T\xi q_tdt-\beta q(0)+\int_0^T w_x(-L,t)q_t(t)dt+w_x(-L,0)q(0)+\int_0^T\xi qdt=0,\tag{1.17}$$

   and

$$-\int_0^T\zeta r_tdt-\gamma r(0)+\int_0^T w(-L,t)r_t(t)dt+w(-L,0)r(0)+\int_0^T\zeta rdt=0,\tag{1.18}$$

   for all $(p,q,r)\in L^2([0,T];\mathbb{R}^3)$ with $(p_t,q_t,r_t)\in L^2([0,T];\mathbb{R}^3)$, such that $p(T)=q(T)=r(T)=0$.

We mention here that, according to the Lumer-Philips theorem (see [44]), we can prove the existence and uniqueness of a strong solution $(u,w)\in L^2([0,T];\mathbb{H})$ with $(u_t,w_t)\in L^2([0,T];\mathbb{L}^2)$ and $(\eta,\xi,\zeta)\in L^2([0,T];\mathbb{R}^3)$ to the system (1.11)-(1.14). This result also confirms the existence and uniqueness of the weak solution to problem (1.11)-(1.14).

To perform a space finite volume discretization of the problem, a mesh $\mathcal{T}$ of $(-L,0)\cup(0,L)$, over which (1.11)-(1.14) to be studied, is introduced as follows:

Definition 1.1.2 (**One-dimensional admissible mesh**). *An admissible mesh $\mathcal{T}$ of the spatial domain $(-L,0)\cup(0,L)$, is given by an increasing sequence of real values $\left(x_{i+\frac{1}{2}}\right)_{i=0,\dots,N_{\max}}$ such that $[-L,L]=\bigcup_{i=1}^{N_{\max}}\left[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right]$. The mesh $\mathcal{T}$ is the set $\{K_i:\ i\in\{1,\dots,N_{\max}\};\quad N_{\max}\in\mathbb{N}^*\}$ of control volumes defined by $K_i=\left(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right)$ for all $i\in\{1,\dots,N_{\max}\}$. The family $(x_i)_{i=1,\dots,N_{\max}}$ is assumed to be the center of $(K_i)_{i=1,\dots,N_{\max}}$ such that*

$$-L=x_0=x_{\frac{1}{2}}<x_1<x_{\frac{3}{2}}<\cdots<x_{i-\frac{1}{2}}<x_i<x_{i+\frac{1}{2}}<\cdots<x_{N_{\max}-\frac{1}{2}}<x_{N_{\max}}<x_{N_{\max}+\frac{1}{2}}=x_{N_{\max}+1}=L.$$

We discretize the intervals $[-L,0]$ and $[0,L]$ into N_1 and N_2 internal points respectively such that N_1 and $N_2\in\mathbb{N}^*$. The relation is $N_{\max}=N_1+N_2$. To be clear, let the spatial mesh size $h_1=L/N_1$ and $h_2=L/N_2$ and discretize as the following:

We discretize the interval $[-L,0]$ such that:

- *For $i=0,\dots,N_1$, $x_{i+\frac{1}{2}}=-L+ih_1$, which yields, $x_{\frac{1}{2}}=-L$ and $x_{N_1+\frac{1}{2}}=0$.*
- *For $i=1,\dots,N_1$, $x_i=-L+\left(i-\frac{1}{2}\right)h_1$, which yields, $x_1=-L+\dfrac{h_1}{2}$ and $x_{N_1}=-\dfrac{h_1}{2}$.*

We discretize the interval $[0,L]$ such that:

- *For $i=N_1+1,\dots,N_{\max}$, $x_{i+\frac{1}{2}}=(i-N_1)h_2$, which yields, $x_{N_1+\frac{3}{2}}=h_2$ and $x_{N_{\max}+\frac{1}{2}}=L$.*
- *For $i=N_1+1,\dots,N_{\max}$, $x_i=\left(i-N_1-\frac{1}{2}\right)h_2$, which yields, $x_{N_1+1}=\dfrac{h_2}{2}$ and $x_{N_{\max}}=L-\dfrac{h_2}{2}$.*

The length of each control volume K_i is denoted as:

$$h_i=m(K_i)=\begin{cases}h_1, & i=1,\dots,N_1,\\ h_2, & i=N_1+1,\dots,N_{\max}.\end{cases}$$

We also define:

$$h_{i+\frac{1}{2}}=\begin{cases}x_{i+1}-x_i, & i=0,\dots N_1-1,\ N_1+1,\dots,N_{\max},\\ x_{N_1+1}-x_{N_1+\frac{1}{2}}, & i=N_1,\end{cases}$$

which implies that

$$h_{i+\frac{1}{2}}=\begin{cases}\dfrac{h_1}{2}, & i=0,\\ h_1, & i=1,\dots,N_1-1,\\ h_2, & i=N_1+1,\dots,N_{\max}-1,\\ \dfrac{h_2}{2}, & i=N_1,\ N_{\max}.\end{cases}$$

Further, we define the discrete unknowns for the (finite volume) space discretization as $\{u_i(t);\ i = N_1+1,\dots,N_{\max},\ t\in[0,T]\}$ and $\{w_i(t);\ i = 1,\dots,N_1,\ t\in[0,T]\}$, which stand for the approximation of the mean values of u and w over the control volumes K_i, respectively. However, this data is used to construct the approximations of u and w within the cells. In fact, we have:

$$\begin{aligned} u_i(t) &= \frac{1}{h_i}\int_{K_i} u(x,t)dx, \quad \text{for } i = N_1+1,\dots,N_{\max},\\ w_i(t) &= \frac{1}{h_i}\int_{K_i} w(x,t)dx, \quad \text{for } i = 1,\dots,N_1. \end{aligned}$$

We also denote

$$U_i = \begin{cases} u_i, & i = N_1+1,\dots,N_{\max},\\ w_i, & i = 1,\dots,N_1, \end{cases}$$

and let

$$\Xi = (\eta,\xi,\zeta).$$

To describe the schemes in the most general way, we introduce a few auxiliary unknowns. Specifically, we denote the values of u and w at $x_{N_1+\frac{1}{2}}$ by $u_{N_1+\frac{1}{2}}$ and $w_{N_1+\frac{1}{2}}$, respectively. Furthermore, let $u_{N_{\max}+1}$ represent the value of u at $x_{N_{\max}+1}$ and w_0 denote the value of w at x_0. Using these definitions, we can establish the difference operators required for discretizing the spatial derivatives. These operators are expressed in terms of the discrete unknowns $(u_i)_{i=N_1+1,\dots,N_{\max}}$, $(w_i)_{i=1,\dots,N_1}$ and other data.

Based on these definitions, we construct a semi-discrete numerical scheme for System (1.11)-(1.14) and subsequently estimate the discrete energy using finite volume discretization in space. To begin, we address the discretization of initial conditions. For example, we start the solution with $\Phi_i = (\varphi_i,\phi_i)$ such that

$$\begin{aligned} &u_i(0) = \varphi_i \quad \text{where} \quad \varphi_i = \frac{1}{h_i}\int_{K_i}\varphi(x)dx, \qquad \text{for} \quad i = N_1+1,\dots,N_{\max},\\ &w_i(0) = \phi_i \quad \text{where} \quad \phi_i = \frac{1}{h_i}\int_{K_i}\phi(x)dx, \qquad \text{for} \quad i = 1,\cdots,N_1. \end{aligned}$$

As a result, we have

$$U_i(0) = \Phi_i, \quad \text{for} \quad i = 1,\dots,N_{\max}.$$

In addition, let us define $\Psi_i = (\psi_i,\theta_i)$ by setting

$$\begin{aligned} \psi_i &= \frac{1}{h_i}\int_{K_i}\psi(x)dx, \qquad \text{for} \quad i = N_1+1,\dots,N_{\max},\\ \theta_i &= \frac{1}{h_i}\int_{K_i}\theta(x)dx, \qquad \text{for} \quad i = 1,\dots,N_1. \end{aligned}$$

Thus, using the above notations, we approximate the initial conditions of u_t and w_t in the following way

$$\begin{aligned} &u_{i,t}(0) = \psi_i, \qquad \text{for} \quad i = N_1+1,\dots,N_{\max},\\ &w_{i,t}(0) = \theta_i, \qquad \text{for} \quad i = 1,\dots,N_1. \end{aligned}$$

Consequently, we have

$$U_{i,t}(0) = \Psi_i, \quad \text{for} \quad i = 1, \cdots, N_{\max},$$

and we set

$$\Xi(0) = \Theta.$$

From these definitions, we derive the following semi-discretized numerical scheme for System (1.11)-(1.14):

$$\begin{cases} h_i U_{i,tt} + F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} = 0, & i = 1, \ldots, N_{\max}, \\ u_{N_1+\frac{1}{2}} = w_{N_1+\frac{1}{2}}, \quad \delta_x w_{N_1+\frac{1}{2}} = 0, \quad \delta_x^2 w_0 - \xi = 0, \end{cases} \tag{1.19}$$

where the numerical flux $F_{i+\frac{1}{2}}$ is defined as follows:

$$F_{i+\frac{1}{2}} = \begin{cases} b\delta_x^3 w_{i+\frac{1}{2}}, & i = 0, \ldots, N_1 - 1, \\ -a\delta_x u_{i+\frac{1}{2}}, & i = N_1, \ldots, N_{\max}, \end{cases} \tag{1.20}$$

and the boundary conditions are given by

$$\delta_x u_{N_{\max}+\frac{1}{2}} + \eta = 0, \qquad \delta_x^3 w_{\frac{1}{2}} + \zeta = 0, \tag{1.21}$$

along with the semi-discretized differential equations:

$$\begin{cases} \eta_t - u_{N_{\max}+1,t} + \eta = 0, \\ \xi_t - \delta_x w_{\frac{1}{2},t} + \xi = 0, \\ \zeta_t - w_{0,t} + \zeta = 0. \end{cases} \tag{1.22}$$

The discrete operators δ_x , δ_x^2 , δ_x^3 represent a discrete approximation of the first, second and third derivative of the unknowns at the interface points. Their rigorous definition are detailed in the section 5.3 of the chapter 5 dedicated to the numerical approximation of our transmission problem in 1D. For this system, we define:

- The discrete kinetic energy for $U_{\mathcal{T}}$ as:

$$\mathcal{E}_k(U_{\mathcal{T}}) = \frac{1}{2} \sum_{i=1}^{N_{\max}} h_i \left(U_{i,t}\right)^2 .$$

- The discrete potential energy for $U_{\mathcal{T}}$ as:

$$\mathcal{E}_p(U_{\mathcal{T}}) = \frac{1}{2} \left(b \sum_{i=1}^{N_1} h_i (\delta_x^2 w_i)^2 + a \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} (\delta_x u_{i+\frac{1}{2}})^2 \right) .$$

- The dynamical boundary energy for Ξ as:

$$\mathcal{E}_b(\Xi) = \frac{1}{2} \left(a\eta^2 + b\xi^2 + b\zeta^2 \right) .$$

The total discrete energy $\mathcal{E}(t)$ of the semi-discretized system (1.19)-(1.22) is given by

$$\mathcal{E}(t) = \mathcal{E}_k(U_\mathcal{T}) + \mathcal{E}_p(U_\mathcal{T}) + \mathcal{E}_b(\Xi), \qquad \forall t \in [0,T].$$

The time derivative of this discrete energy, $\mathcal{E}(t)$, satisfies the relation:

$$\frac{d\mathcal{E}(t)}{dt} = -\left(a|\eta|^2 + b|\xi|^2 + b|\zeta|^2\right).$$

This implies that the discretized energy $\mathcal{E}(t)$ of the system (1.19)-(1.22) is conserved when no dynamical controls are applied. However, the energy is non-increasing over time as long as at least one dynamical boundary control is active, either on the wave or the Euler-Bernoulli beam.

Next, we establish stability estimates for the approximate solution and prove its convergence towards the weak solution of the problem (1.15)-(1.18) in an adequate sense, as the mesh size tends to zero. In order to give a stability result and prove the convergence for this scheme, we set $\Phi = (\varphi, \phi)$, $\Psi = (\psi, \theta)$, $\Theta = (\alpha, \beta, \gamma)$ and study our problem under a certain hypothesis, which we call **Hypothesis (H1)**:

$$\Phi \in \mathbb{H}, \quad \Psi \in \mathbb{L}^2 \quad \text{and} \quad \Theta \in \mathbb{R}^3. \qquad \textbf{Hypothesis (H1)}$$

For arbitrary $U_\mathcal{T}$, we define the following $\mathbb{L}^2$ norm:

$$\|U_\mathcal{T}\| = \left(\sum_{i=1}^{N_{\max}} h_i (U_i)^2\right)^{\frac{1}{2}}.$$

We also define the discrete $\mathbb{H}$ norm of the function $U_\mathcal{T} = (u_\mathcal{T}, w_\mathcal{T})$ as

$$\|U_\mathcal{T}\|_{\mathbb{H},\mathcal{T}} = \left(\|u_\mathcal{T}\|_{1,\mathcal{T}}^2 + \|w_\mathcal{T}\|_{2,\mathcal{T}}^2\right)^{\frac{1}{2}},$$

where

$$\|u_\mathcal{T}\|_{1,\mathcal{T}}^2 = a \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} (\delta_x u_{i+\frac{1}{2}})^2$$

and

$$\|w_\mathcal{T}\|_{2,\mathcal{T}}^2 = b \sum_{i=1}^{N_1} h_i (\delta_x^2 w_i)^2$$

represent the discrete norms in $H_*^1(0,L)$ and $H_*^2(-L,0)$ for the functions $u_\mathcal{T}$ and $w_\mathcal{T}$, respectively. For the function $\Xi = (\eta, \xi, \zeta)$, we define the $\mathbb{R}^3$ norm as

$$\|\Xi\|_{\mathbb{R}^3} = \left(a\eta^2 + b\xi^2 + b\zeta^2\right)^{\frac{1}{2}}.$$

Theorem 1.1.7. *Let the **Hypothesis (H1)** hold. Then, the numerical solution $U_\mathcal{T}$ of the numerical scheme (1.19)-(1.22) is bounded independently on discretization parameters in $L^\infty([0,T];\mathbb{L}^2) \cap L^\infty([0,T];\mathbb{H})$. Moreover, the first time derivation is bounded in $L^\infty([0,T];\mathbb{L}^2)$ and the function Ξ is bounded in $L^\infty([0,T];\mathbb{R}^3)$ as well. Therefore, the following estimation holds*

$$\|U_{\mathcal{T},t}\|^2 + \|U_\mathcal{T}\|^2 + \|U_\mathcal{T}\|_{\mathbb{H},\mathcal{T}}^2 + \|\Xi\|_{\mathbb{R}^3}^2 \leq C,$$

where C is a strictly positive constant independent of the space discretization parameter.

We then prove that, as the mesh size tends to zero, the numerical solution obtained by the numerical scheme (1.19)-(1.22) converges to the weak solution. The strategy is based on the stability estimates obtained and passing through the limit, we obtain the convergence to the weak solution stated in the following theorem:

Theorem 1.1.8. *Assume that **Hypothesis (H1)** holds. For $m \in \mathbb{N}$, let $\mathcal{T}_m$ be an admissible mesh described in the sense of Definition 1.1.2. Let $(u_{\mathcal{T}_m}, w_{\mathcal{T}_m}, \eta, \xi, \zeta)$ be the discrete solution of (1.19)-(1.22). Assume that size$(\mathcal{T}_m) \longrightarrow 0$ as $m \longrightarrow \infty$. Then, there exists a subsequence of the sequence of approximate solutions, still denoted by $(u_{\mathcal{T}_m}, w_{\mathcal{T}_m}, \eta, \xi, \zeta)$, which converges to the weak solution (u, w, η, ξ, ζ) in $L^\infty([0,T]; L^2(0,L)) \times L^\infty([0,T]; L^2(-L,0)) \times L^\infty([0,T]; \mathbb{R}^3)$ of the problem (1.15)-(1.18), as m tends to ∞, for the weak-$\star$ topology.*

For the time discretization, let $\mathcal{N} \in \mathbb{N}^*$ and define a uniform discrete time step $\tau = T/\mathcal{N}$ where we denote $t_n = n\tau$, for $n = 0, 1, \ldots \mathcal{N}$. The discrete unknowns are denoted by $\{u_i^n; \quad i = N_1 + 1, \ldots, N_{\max}, \quad n = 0, 1, \ldots \mathcal{N}\}$ and $\{w_i^n; \quad i = 1, \ldots, N_1, \quad n = 0, 1, \ldots \mathcal{N}\}$, representing approximations of the mean values of u and w at time t_n over each control volume K_i.

We impose initial conditions as follows. Starting with $\Phi_i = (\varphi_i, \phi_i)$, we set

$$u_i^0 = \varphi_i \quad \text{where} \quad \varphi_i = \frac{1}{h_i} \int_{K_i} \varphi(x) dx, \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$w_i^0 = \phi_i \quad \text{where} \quad \phi_i = \frac{1}{h_i} \int_{K_i} \phi(x) dx, \qquad \text{for} \quad i = 1, \cdots, N_1.$$

Thus,

$$U_i^0 = \Phi_i, \quad \text{for} \quad i = 1, \ldots, N_{\max}.$$

For the first time step, we use an artificial time level -1 denoting the values of our approximation functions at this time step by

$$u_i^{-1} \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$w_i^{-1} \qquad \text{for} \quad i = 1, \ldots, N_1.$$

We introduce $\Psi_i = (\psi_i, \theta_i)$ such that

$$\psi_i = \frac{1}{h_i} \int_{K_i} \psi(x) dx, \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$\theta_i = \frac{1}{h_i} \int_{K_i} \theta(x) dx, \qquad \text{for} \quad i = 1, \ldots, N_1.$$

Using the above notations, the initial conditions for u_t and w_t can then be approximated using central differences:

$$\frac{u_i^1 - u_i^{-1}}{2\tau} = \psi_i, \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$\frac{w_i^1 - w_i^{-1}}{2\tau} = \theta_i, \qquad \text{for} \quad i = 1, \ldots, N_1.$$

Thus,

$$u_i^{-1} = u_i^1 - 2\tau\psi_i, \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$w_i^{-1} = w_i^1 - 2\tau\theta_i, \qquad \text{for} \quad i = 1, \ldots, N_1.$$

Therefore,

$$U_i^{-1} = U_i^1 - 2\tau\Psi_i, \quad \text{for} \quad i = 1, \cdots, N_{\max}.$$

Finally, we define $\Theta = (\alpha, \beta, \gamma)$ so that

$$\Xi^0 = \Theta.$$

We construct the implicit numerical scheme for System (1.11)-(1.14) using the finite volume method in space and an alternative implicit formulation that follows from time-centering the derivative term. Time derivatives are discretized with the finite difference method. The implicit scheme for the unknown U_i^{n+1} is given by

$$\begin{cases} h_i \dfrac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\tau^2} + \dfrac{1}{2}\left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1}\right) + \dfrac{1}{2}\left(F_{i+\frac{1}{2}}^{n-1} - F_{i-\frac{1}{2}}^{n-1}\right) = 0, & i = 1, \ldots, N_{\max},\ n = 1, \ldots, \mathcal{N} - 1, \\ u_{N_1+\frac{1}{2}}^n = w_{N_1+\frac{1}{2}}^n, \quad \delta_x w_{N_1+\frac{1}{2}}^n = 0, \quad \delta_x^2 w_0^n - \xi^n = 0, & n = 0, 1, \ldots \mathcal{N}, \end{cases} \tag{1.23}$$

where the numerical flux at point $x_{i+\frac{1}{2}}$ and time t_n is defined as follows

$$F_{i+\frac{1}{2}}^n = \begin{cases} b\delta_x^3 w_{i+\frac{1}{2}}^n, & i = 0, \ldots, N_1 - 1, \\ -a\delta_x u_{i+\frac{1}{2}}^n, & i = N_1, \ldots, N_{\max}, \end{cases} \tag{1.24}$$

with the boundary conditions

$$\delta_x u_{N_{\max}+\frac{1}{2}}^n + \eta^n = 0, \qquad \text{for} \quad n = 0, 1, \ldots \mathcal{N},$$

and

$$\delta_x^3 w_{\frac{1}{2}}^n + \zeta^n = 0, \qquad \text{for} \quad n = 0, 1, \ldots \mathcal{N}.$$

The discretized form of the equations in (1.14) is then given by:

$$\begin{cases} \eta^{n+1} = \eta^{n-1} + u_{N_{\max}+1}^{n+1} - u_{N_{\max}+1}^{n-1} - \tau\left(\eta^{n+1} + \eta^{n-1}\right), & n = 1, \ldots, \mathcal{N} - 1, \\ \xi^{n+1} = \xi^{n-1} + \delta_x w_{\frac{1}{2}}^{n+1} - \delta_x w_{\frac{1}{2}}^{n-1} - \tau\left(\xi^{n+1} + \xi^{n-1}\right), & n = 1, \ldots, \mathcal{N} - 1, \\ \zeta^{n+1} = \zeta^{n-1} + w_0^{n+1} - w_0^{n-1} - \tau\left(\zeta^{n+1} + \zeta^{n-1}\right), & n = 1, \ldots, \mathcal{N} - 1. \end{cases} \tag{1.25}$$

Following that, we present an estimate of the discrete energy. To this end, we define:

- The discrete kinetic energy for $U_{\mathcal{T}}^n$ as:

$$\mathcal{E}_k\left(U_{\mathcal{T}}^n\right) = \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^{n+1} - U_i^n}{\tau}\right)^2.$$

- The discrete potential energy for $U_{\mathcal{T}}^n$ as:

$$\mathcal{E}_p(U_{\mathcal{T}}^n) = \frac{1}{4}\left(b\sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^{n+1}\right)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^{n+1}\right)^2\right) + \frac{1}{4}\left(b\sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^n\right)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^n\right)^2\right).$$

- The discrete boundary control energy for Ξ^n as:

$$\mathcal{E}_b(\Xi^n) = \frac{1}{4}\left(a\left(\eta^{n+1}\right)^2 + b\left(\xi^{n+1}\right)^2 + b\left(\zeta^{n+1}\right)^2\right) + \frac{1}{4}\left(a\left(\eta^{n}\right)^2 + b\left(\xi^{n}\right)^2 + b\left(\zeta^{n}\right)^2\right).$$

The total discrete energy $\mathcal{E}^n$ of the implicit numerical scheme (1.23)-(1.25) is expressed as

$$\mathcal{E}^n = \mathcal{E}_k(U_{\mathcal{T}}^n) + \mathcal{E}_p(U_{\mathcal{T}}^n) + \mathcal{E}_b(\Xi^n).$$

This energy satisfies the following estimate:

$$\mathcal{E}^n - \mathcal{E}^{n-1} = -\frac{\tau}{4}\left(a|\eta^{n+1} + \eta^{n-1}|^2 + b|\xi^{n+1} + \xi^{n-1}|^2 + b|\zeta^{n+1} + \zeta^{n-1}|^2\right).$$

This implies that the total discrete energy $\mathcal{E}^n$ is preserved when there are no dynamical boundary controls, and it is non-increasing over time when at least one dynamical boundary control is applied to either the wave or the Euler-Bernoulli beam.

The stability estimates of this scheme are established under specific initial data conditions and a suitable choice of time step. For stability reasons, we define $\Phi = (\varphi, \phi)$, $\Psi = (\psi, \theta)$, $\Theta = (\alpha, \beta, \gamma)$ and study our problem under a certain hypothesis, which we call **Hypothesis (H2)** of the following form:

$$\Phi \in \mathbb{H}, \quad \Psi \in H^2(0,L) \times H^4(-L,0) \quad \text{and} \quad \Theta \in \mathbb{R}^3. \qquad \textbf{Hypothesis (H2)}$$

We also assume the following for the time step τ :

$$\text{There exists a real number } \tau_0 \in [0,T] \text{ such that } \tau \leq \tau_0. \qquad \textbf{Assumption (TA)}$$

With these conditions, we derive necessary stability estimates for the implicit numerical scheme using discrete norms and the discrete Poincaré inequality. These estimates lead to the convergence of the scheme. These results are formalized in the following theorem:

Theorem 1.1.9. *Let **Hypothesis (H2)** and **Assumption (TA)** hold. Then, the numerical solution $U_{\mathcal{T},\tau}$ of the numerical scheme (1.23)-(1.25) is bounded independently on discretization parameters in $L^\infty\left([0,T];\mathbb{L}^2\right) \cap L^\infty\left([0,T];\mathbb{H}\right)$. Moreover, the first time derivative approximation is bounded in $L^\infty\left([0,T];\mathbb{L}^2\right)$ and the function Ξ_τ is bounded in $L^\infty\left([0,T];\mathbb{R}^3\right)$ as well. Therefore, the following estimation holds:*

$$\|\delta_t U_{\mathcal{T}}^m\|^2 + \|U_{\mathcal{T}}^m\|^2 + \frac{1}{2}\|U_{\mathcal{T}}^m\|_{\mathbb{H},\mathcal{T}}^2 + \frac{1}{2}\|\Xi^m\|_{\mathbb{R}^3}^2 \leq C, \qquad \text{for} \qquad m = 1, \ldots, \mathcal{N},$$

where C is a strictly positive constant independent of space and time discretization parameters.

A detailed proof of convergence of the approximate solution towards the weak solution is developed as the mesh size and time step tend to zero. This is summarized in the following convergence theorem:

Theorem 1.1.10. *Assume that **Hypothesis (H2)** holds. For $m \in \mathbb{N}$, let $\mathcal{T}_m$ be an admissible mesh described in the sense of Definition 1.1.2. Let τ_m be a time step satisfying **Assumption (TA)**. Let $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m}, \eta_{\tau_m}, \xi_{\tau_m}, \zeta_{\tau_m})$ be the solution of the system described by (1.23)-(1.25). Assume that size$(\mathcal{T}_m) \longrightarrow 0$ and $\tau_m \longrightarrow 0$ as $m \longrightarrow \infty$. Then, there exists a subsequence of the sequence of approximate solutions, still denoted by $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m}, \eta_{\tau_m}, \xi_{\tau_m}, \zeta_{\tau_m})$, which converges, in the weak-$\star$ topology, to the weak solution (u, w, η, ξ, ζ) in $L^\infty([0,T];L^2(0,L)) \times L^\infty([0,T];L^2(-L,0)) \times \left(L^\infty([0,T])\right)^3$ of the problem (1.15)-(1.18), as m tends to ∞.*

Furthermore, we investigate the asymptotic behavior of the energy of the system (1.11)-(1.14) with a single dynamical boundary control η, obtaining an optimal polynomial energy decay of type $1/\sqrt{t}$ for smooth initial data. However, the full proof is not included due to its technical nature and length, which would divert from the main focus of the chapter. Numerical examples confirm this decay rate and verify the theoretical results through graphical representation.

Finally, in Chapter 6, we summarize our conclusions and provide insights into future developments.

1.2 Introduction en Français

La théorie du contrôle, une branche des mathématiques appliquées, s'intéresse à la conception d'entrées permettant d'influencer le comportement des systèmes dynamiques. Le contrôleur, souvent appelé rétroaction ou amortissement, est appliqué à des sorties spécifiques afin d'atteindre des objectifs souhaités et d'améliorer les performances du système. Ces dernières années, le domaine de la théorie du contrôle pour les systèmes régis par des équations aux dérivées partielles (EDP) a connu un développement significatif, en raison de l'introduction de modèles de plus en plus réalistes. Cette expansion a donné lieu à de nombreuses applications dans des domaines variés tels que la physique mathématique moderne, l'ingénierie, la biologie, l'informatique et la recherche biomédicale. L'essor de ce domaine a également été favorisé par la disponibilité d'outils numériques sophistiqués et d'une puissance de calcul presque illimitée, permettant ainsi le développement et la mise en œuvre de méthodes numériques pour la résolution de problèmes scientifiques et d'ingénierie complexes. Des concepts fondamentaux comme la stabilité et la contrôlabilité jouent un rôle essentiel en théorie moderne du contrôle mathématique, car ils définissent les propriétés qualitatives des systèmes de contrôle. Dans cette thèse, nous utilisons la théorie des semi-groupes en conjonction avec l'analyse numérique afin d'étudier la stabilisation de problèmes de transmission d'interface impliquant des équations d'ondes-plaque ainsi que des modèles de poutre d'Euler-Bernoulli couplés à des équations d'ondes avec des contrôles dynamiques au bord.

Nous présenterons un bref aperçu de la structure de la thèse ainsi que des principaux résultats obtenus. Cet mansucrit de thèse est organisé en quatre chapitres.

En tenant compte du fait que la majorité des études présentées dans cette thèse sont basées sur la théorie des semi-groupes, le Chapitre 2 expose plusieurs résultats bien connus sur les semi-groupes, en particulier les résultats et théorèmes relatifs à la stabilité forte, exponentielle et polynomiale d'un C_0-semi-groupe. De plus, les théorèmes liés aux aspects numériques, qui seront utilisés dans les chapitres suivants, sont introduits.

Le Chapitre 3 étudie un modèle de transmission bidimensionnel constitué d'une équation des ondes et d'une équation de plaque de Kirchhoff avec des contrôles dynamiques au bord sous certaines conditions géométriques. Ces deux équations sont couplées par des conditions de transmission le long d'une interface fixe séparant les domaines où évoluent respectivement l'équation des ondes et l'équation de plaque. Notre principal objectif est l'analyse de la stabilité du système en utilisant un ensemble de trois contrôles dynamiques au bord, ce qui n'a pas été étudié auparavant dans la littérature. Plus précisément, nous considérons $\Omega \subset \mathbb{R}^2$ comme un ensemble ouvert borné avec une frontière de Lipschitz $\partial\Omega$ tel que $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, où Ω_i est un ensemble borné avec une frontière de Lipschitz $\partial\Omega_i$, $i = 1, 2$, tel que $\Omega_1 \cap \Omega_2 = \emptyset$. Nous notons par I l'intérieur de $\overline{\Omega}_1 \cap \overline{\Omega}_2$, qui est appelé l'interface entre Ω_1 et Ω_2, et par $\Gamma_i = \partial\Omega_i \backslash \overline{I}$ la frontière extérieure de Ω_i de mesure positive, $i = 1, 2$.

Nous considérons alors le modèle de transmission onde-plaque suivant:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{dans } \Omega_1 \times \mathbb{R}_+^*, \\ w_{tt} + \Delta^2 w = 0, & \text{dans } \Omega_2 \times \mathbb{R}_+^*, \\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{sur } I \times \mathbb{R}_+^*, \\ \partial_{\nu_1} u + \eta = 0, & \text{sur } \Gamma_1 \times \mathbb{R}_+^*, \\ \mathcal{B}_1 w + \xi = 0, & \text{sur } \Gamma_2 \times \mathbb{R}_+^*, \\ \mathcal{B}_2 w - \zeta = 0, & \text{sur } \Gamma_2 \times \mathbb{R}_+^*, \end{cases} \tag{1.26}$$

avec les conditions initiales

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega_1, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1, \tag{1.27}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in \Omega_2, \quad \xi(x,0) = \xi_0(x), \quad \zeta(x,0) = \zeta_0(x), \quad x \in \Gamma_2, \tag{1.28}$$

où η, ξ et ζ désignent les contrôles dynamiques de frontière, $\nu_i = (\nu_{i1}, \nu_{i2})$ est le vecteur normal unitaire sortant le long de $\partial\Omega_i$, et $\tau_i = (-\nu_{i2}, \nu_{i1})$ est le vecteur tangent unitaire le long de $\partial\Omega_i$, $i = 1, 2$. De plus, dans le contexte de ce problème, nous introduisons la constante $\mu \in \left(0, \frac{1}{2}\right)$ comme le coefficient de Poisson. Nous définissons également les opérateurs de frontière $\mathcal{B}_1$ et $\mathcal{B}_2$ sur $\partial\Omega_2$ comme suit:

$$\mathcal{B}_1 f = \Delta f + (1-\mu)\mathcal{C}_1 f$$

et

$$\mathcal{B}_2 f = \partial_{\nu_2} \Delta f + (1-\mu)\partial_{\tau_2} \mathcal{C}_2 f,$$

où

$$\mathcal{C}_1 f = 2\nu_{21}\nu_{22} f_{x_1x_2} - \nu_{21}^2 f_{x_2x_2} - \nu_{22}^2 f_{x_1x_1} \quad \text{et} \quad \mathcal{C}_2 f = (\nu_{21}^2 - \nu_{22}^2) f_{x_1x_2} - \nu_{21}\nu_{22}\left(f_{x_1x_1} - f_{x_2x_2}\right).$$

L'amortissement du système est réalisé par le mécanisme d'amortissement indirect qui implique les équations différentielles d'ordre un suivantes:

$$\begin{cases} \eta_t - u_t + \eta = 0, & \text{sur } \Gamma_1 \times \mathbb{R}_+^*, \\ \xi_t - \partial_{\nu_2} w_t + \xi = 0, & \text{sur } \Gamma_2 \times \mathbb{R}_+^*, \\ \zeta_t - w_t + \zeta = 0, & \text{sur } \Gamma_2 \times \mathbb{R}_+^*. \end{cases} \tag{1.29}$$

Nous définissons l'énergie du système associée comme

$$E(t) = \frac{1}{2}\left\{\int_{\Omega_1} \left(|\nabla u|^2 + |u_t|^2\right) dx + \int_{\Gamma_1} |\eta|^2 d\Gamma\right\} + \frac{1}{2}\left\{a(w,\overline{w}) + \int_{\Omega_2} |w_t|^2 dx + \int_{\Gamma_2} \left(|\xi|^2 + |\zeta|^2\right) d\Gamma\right\},$$

où la forme sesquilinéaire $a : H^2(\Omega_2) \times H^2(\Omega_2) \longmapsto \mathbb{C}$ est définie par

$$a(f,\overline{g}) = \int_{\Omega_2} \left[f_{x_1x_1}\overline{g}_{x_1x_1} + f_{x_2x_2}\overline{g}_{x_2x_2} + \mu\left(f_{x_1x_1}\overline{g}_{x_2x_2} + f_{x_2x_2}\overline{g}_{x_1x_1}\right) + 2(1-\mu) f_{x_1x_2}\overline{g}_{x_1x_2}\right] dx.$$

Si $U = (u, u_t, \eta, w, w_t, \xi, \zeta)$ est une solution régulière du système (1.26)-(1.29), alors l'énergie $E(t)$ satisfait l'estimation suivante

$$E'(t) = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} \left(|\xi|^2 + |\zeta|^2\right) d\Gamma \leq 0,$$

indiquant que le système est dissipatif dans le sens où l'énergie $E(t)$ est non croissante par rapport à la variable temporelle t. L'objectif principal de ce chapitre est de dériver le taux de décroissance de l'énergie $E(t)$ sous des conditions géométriques spécifiques.

Nous démontrons le caractère bien posé du système à travers la théorie des semi-groupes. Pour commencer, nous introduisons les espaces fonctionnels suivants:

$$H^1_*(\Omega_1) = \left\{ u \in H^1(\Omega_1) \mid \int_{\Omega_1} u dx = 0 \right\},$$

$$H^2_*(\Omega_2) = \left\{ w \in H^2(\Omega_2) \mid \int_{\Omega_2} w dx = \int_{\Omega_2} \nabla w dx = 0 \right\}$$

et l'espace d'énergie $\mathcal{H}$ défini par

$$\mathcal{H} = \left\{ (u, v, \eta, w, z, \xi, \zeta) \in H^1_*(\Omega_1) \times L^2(\Omega_1) \times L^2(\Gamma_1) \times H^2_*(\Omega_2) \times L^2(\Omega_2) \times L^2(\Gamma_2) \times L^2(\Gamma_2) \mid u = w \text{ sur } I \right\},$$

qui est doté du produit scalaire usuel suivant

$$\left(U, \tilde{U}\right)_{\mathcal{H}} = \int_{\Omega_1} \left(\nabla u \cdot \nabla \overline{\tilde{u}} + v \overline{\tilde{v}}\right) dx + \int_{\Gamma_1} \eta \overline{\tilde{\eta}} d\Gamma + a(w, \overline{\tilde{w}}) + \int_{\Omega_2} z \overline{\tilde{z}} dx + \int_{\Gamma_2} \left(\xi \overline{\tilde{\xi}} + \zeta \overline{\tilde{\zeta}}\right) d\Gamma,$$

où $U = (u, v, \eta, w, z, \xi, \zeta)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{w}, \tilde{z}, \tilde{\xi}, \tilde{\zeta}) \in \mathcal{H}$. Nous définissons ensuite l'opérateur linéaire non borné $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ par

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, \eta, w, z, \xi, \zeta) \in \mathcal{H} \mid \Delta u \in L^2(\Omega_1), \quad v \in H^1_*(\Omega_1), \quad \Delta^2 w \in L^2(\Omega_2), \quad z \in H^2_*(\Omega_2), \\ \partial_{\nu_1} u + \eta = 0 \text{ sur } \Gamma_1, \quad \mathcal{B}_1 w + \xi = 0 \text{ sur } \Gamma_2, \quad \mathcal{B}_2 w - \zeta = 0 \text{ sur } \Gamma_2, \\ v = z, \quad \mathcal{B}_1 w = 0 \text{ et } \mathcal{B}_2 w = \partial_{\nu_1} u \text{ sur } I \end{array} \right\}$$

et

$$\mathcal{A}U = \left(v, \Delta u, \gamma_1(v) - \eta, z, -\Delta^2 w, \gamma_{2,2}(z) - \xi, \gamma_{2,1}(z) - \zeta\right), \qquad \forall U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A}),$$

où $\gamma_1 : H^1(\Omega_1) \longmapsto L^2(\Gamma_1)$ et $\gamma_2 : H^2(\Omega_2) \longmapsto L^2(\Gamma_2) \times L^2(\Gamma_2)$ sont les opérateurs de trace définis par $\gamma_1(\varphi) = \varphi|_{\Gamma_1}$ et $\gamma_2(\psi) = (\gamma_{2,1}(\psi), \gamma_{2,2}(\psi)) = (\psi|_{\Gamma_2}, \partial_{\nu_2}\psi|_{\Gamma_2})$. En posant $U = (u, u_t, \eta, w, w_t, \xi, \zeta)$ comme l'état du système (1.26)-(1.29), nous reformulons le problème sous la forme d'une équation d'évolution du premier ordre:

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{1.30}$$

où la condition initiale $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \xi_0, \zeta_0) \in \mathcal{H}$ est donnée par les équations (1.27) et (1.28).

Nous prouvons que l'opérateur linéaire non borné $\mathcal{A}$ est m-dissipatif dans l'espace d'énergie $\mathcal{H}$. Ensuite, d'après le théorème de Lumer-Phillips (voir [44]), nous en déduisons que l'opérateur $\mathcal{A}$ génère un C_0-semi-groupe de contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ dans $\mathcal{H}$, ce qui garantit le caractère bien posé de (1.30). Par conséquent, nous obtenons le résultat suivant:

Théorème 1.2.1. *Pour tout $U_0 \in \mathcal{H}$, le système (1.30) admet une solution faible unique*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

De plus, si $U_0 \in D(\mathcal{A})$, alors le système (1.30) admet une solution forte unique

$$U(t) \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

Dans ce qui suit, notre objectif est d'établir des estimations d'énergie pour le système (1.30). Nos résultats de stabilité reposent sur une approche en domaine fréquentiel combinée à une technique de multiplicateurs et nécessitent donc certaines formules de Green dans Ω_1 et Ω_2. Afin de justifier ces formules, nous avons besoin de quelques résultats de régularité démontrés dans [11] et [16]. Ainsi, à partir de maintenant, nous supposons que Ω_1 et Ω_2 ont une frontière polygonale, au sens où leur frontière est lisse par morceaux. Pour $i = 1, 2$, nous notons par $\omega_{i,j}$, $j = 1, \dots, N_i$ les angles intérieurs aux sommets de Ω_i formés entre deux courbes consécutives. Ces angles peuvent varier dans l'intervalle $0 < \omega_{i,j} \leq 2\pi$. Pour obtenir la régularité nécessaire, nous imposons les hypothèses suivantes sur les angles aux sommets de Ω_1 et Ω_2 :

Hypothèse (A1) *Les angles intérieurs $\omega_{1,j}$ satisfont $\omega_{1,j} < \pi$, pour tout $j = 1, \dots, N_1$.*

Hypothèse (A2) *Il existe un angle minimal ω_0 dépendant du coefficient de Poisson μ (par exemple, $\omega_0 \simeq 77.753311\dots^\circ$ lorsque $\mu = 0.3$) tel que $\omega_{2,j} < \omega_0$, pour tout $j = 1, \dots, N_2$.*

La proposition suivante résume les résultats de régularité:

Proposition 1.2.1. *Soit $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$. Supposons que Ω_1 et Ω_2 soient des domaines polygonaux comme décrits précédemment, satisfaisant respectivement les Hypothèses (A1) et (A2). Alors, il existe $\varepsilon \in (0, \frac{1}{2})$ tel que u appartienne à $H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ et il existe une suite $\{w_k\}_{k\geq 0} \subset H^4(\Omega_2)$ telle que w_k converge vers w dans $H^2(\Omega_2)$ et $\Delta^2 w_k$ converge vers $\Delta^2 w$ dans $L^2(\Omega_2)$.*

En s'appuyant sur ces résultats de régularité, nous prouvons la stabilité forte du système (1.26)-(1.29) au sens où son énergie $E(t)$ converge vers zéro lorsque t tend vers l'infini pour toute donnée initiale dans $\mathcal{H}$. Il est important de noter que, puisque la résolvante de $\mathcal{A}$ n'est pas compact, des méthodes classiques telles que le principe d'invariance de Lasalle [55] ou la théorie de décomposition spectrale de Benchimol [9] ne sont pas applicables dans ce cas. À la place, nous employons un critère plus général d'Arendt-Batty [7], qui stipule que dans un espace de Banach réflexif, un C_0-semi-groupe de contractions, $(e^{t\mathcal{A}})_{t\geq 0}$, est fortement stable si $\mathcal{A}$ n'admet aucune valeur propre sur l'axe imaginaire et si $\sigma(\mathcal{A}) \cap i\mathbb{R}$ est dénombrable, où $\sigma(\mathcal{A})$ désigne le spectre de $\mathcal{A}$. En résumé, notre principal résultat est le théorème suivant:

Théorème 1.2.2. *Supposons que Ω_1 et Ω_2 soient des domaines polygonaux comme décrit ci-dessus. Supposons également que les hypothèses (A1) et (A2) soient vérifiées. Alors, le semi-groupe de contractions $e^{t\mathcal{A}}$ est fortement stable dans l'espace d'énergie $\mathcal{H}$ en ce sens que*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

De plus, nous démontrons que le système (1.26)-(1.29) n'est pas exponentiellement stable. Pour caractériser la stabilité exponentielle d'un C_0-semi-groupe de contractions, nous nous appuyons sur les résultats établis par Huang [28] et Prüss [45]. Plus précisément, selon leurs résultats, nous pouvons prouver que la résolvante de l'opérateur $\mathcal{A}$ n'est pas uniformément borné sur l'axe imaginaire.

Étant donné que le système (1.26)-(1.29) manque de stabilité uniforme, nous nous tournons vers l'étude de son amortissement polynomial pour des solutions régulières en utilisant une méthode des multiplicateurs. L'un des éléments principaux est d'utiliser l'approche du domaine de fréquence provenant de la théorie des semi-groupes. Cette méthode combine un argument par contradiction avec la technique des multiplicateurs. Il est à noter que l'élément clé pour prouver la stabilité polynomiale est le Théorème 2.4 de [12] (voir aussi [8] et [38]). Pour poursuivre notre résultat sur l'amortissement polynomial de l'énergie, nous avons besoin de l'hypothèse géométrique supplémentaire suivante concernant l'onde et la plaque, qui découle des exigences d'intégration dans Ω_1 et Ω_2:

Hypothèse (A3) Supposons qu'il existe un point fixe $x_0 \in \mathbb{R}^2$ tel que, en posant $m(x) = x - x_0$, nous ayons

$$(m \cdot \nu_1) = 0, \quad \forall x \in I,$$

et

$$(m \cdot \nu) \geq \delta, \quad \forall x \in \partial\Omega,$$

où δ est un réel strictement positif et $(\cdot)$ désigne le produit scalaire dans $\mathbb{R}^2$.

Avec ces hypothèses, nous pouvons maintenant présenter notre résultat concernant le taux de décroissance de l'énergie:

Théorème 1.2.3. *Supposons que Ω_1 et Ω_2 soient des domaines polygonaux comme décrit précédemment. Supposons également que les hypothèses (A1), (A2) et (A3) soient vérifiées. Alors, il existe une constante $\mathcal{C} > 0$ telle que, pour toute donnée initiale $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \xi_0, \zeta_0) \in D(\mathcal{A})$, l'énergie du système (1.26)-(1.29) satisfait l'estimation suivante:*

$$E(t) \leq \frac{\mathcal{C}}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

Les résultats présentés dans ce chapitre ont été publiés dans la revue *Acta Applicandae Mathematicae* dans un article intitulé "Stability for an Interface Transmission Problem of Wave-Plate Equations with Dynamical Boundary Controls" (voir [1]).

Dans le Chapitre 4, nous étendons les résultats obtenus dans le chapitre précédent à un modèle de plaque et d'onde soumis à seulement deux rétroactions dynamiques aux frontières extérieures des domaines de l'onde et de la plaque. L'objectif principal de ce chapitre est d'estimer le taux de décroissance de l'énergie sous la condition où le contrôle dynamique ξ est omis, soumettant ainsi le système à l'influence de deux contrôles dynamiques, à savoir η et ζ. Nous considérons un domaine géométrique similaire à celui décrit précédemment. Le modèle suivant de transmission d'interface onde-plaques est envisagé:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{dans } \Omega_1 \times \mathbb{R}^*_+, \\ w_{tt} + \Delta^2 w = 0, & \text{dans } \Omega_2 \times \mathbb{R}^*_+, \\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{sur } I \times \mathbb{R}^*_+, \\ \partial_{\nu_1} u + \eta = 0, & \text{sur } \Gamma_1 \times \mathbb{R}^*_+, \\ \mathcal{B}_1 w = 0, & \text{sur } \Gamma_2 \times \mathbb{R}^*_+, \\ \mathcal{B}_2 w - \zeta = 0, & \text{sur } \Gamma_2 \times \mathbb{R}^*_+, \end{cases} \tag{1.31}$$

avec les conditions initiales:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega_1, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1, \tag{1.32}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in \Omega_2, \quad \zeta(x,0) = \zeta_0(x), \quad x \in \Gamma_2, \tag{1.33}$$

où η et ζ désignent les contrôles frontaliers dynamiques, $\nu_i = (\nu_{i1}, \nu_{i2})$ est le vecteur unitaire normal sortant le long de $\partial\Omega_i$, et $\tau_i = (-\nu_{i2}, \nu_{i1})$ est le vecteur unitaire tangent le long de $\partial\Omega_i$, pour $i = 1, 2$. De plus, les opérateurs frontaliers $\mathcal{B}_1$ et $\mathcal{B}_2$ sont définis sur $\partial\Omega_2$ comme décrit précédemment. L'amortissement du système se fait via un mécanisme d'amortissement indirect impliquant les équations différentielles d'ordre un suivantes:

$$\begin{cases} \eta_t - u_t + \eta = 0, & \text{sur } \Gamma_1 \times \mathbb{R}^*_+, \\ \zeta_t - w_t + \zeta = 0, & \text{sur } \Gamma_2 \times \mathbb{R}^*_+. \end{cases} \tag{1.34}$$

Pour des solutions lisses u, η, w et ζ du système (1.31)-(1.34), nous définissons l'énergie associée comme suit:

$$E(t) = \frac{1}{2}\left\{\int_{\Omega_1}\left(|\nabla u|^2 + |u_t|^2\right)dx + \int_{\Gamma_1}|\eta|^2 d\Gamma\right\} + \frac{1}{2}\left\{a(w,\overline{w}) + \int_{\Omega_2}|w_t|^2 dx + \int_{\Gamma_2}|\zeta|^2 d\Gamma\right\},$$

où la forme sesquilinéaire $a : H^2(\Omega_2) \times H^2(\Omega_2) \longmapsto \mathbb{C}$ est définie comme décrit précédemment. Un calcul direct montre que l'énergie $E(t)$ satisfait l'estimation suivante:

$$E'(t) = -\int_{\Gamma_1}|\eta|^2 d\Gamma - \int_{\Gamma_2}|\zeta|^2 d\Gamma \leq 0,$$

indiquant que le système (1.31)-(1.34) est dissipatif au sens où l'énergie $E(t)$ est décroissante par rapport à la variable temporelle t.

Pour étudier le caractère bien posé du système (1.31)-(1.34), nous introduisons les espaces $H^1_*(\Omega_1)$ et $H^2_*(\Omega_2)$ comme définis précédemment, ainsi que l'espace d'énergie $\mathcal{H}$ donné par:

$$\mathcal{H} = \left\{(u,v,\eta,w,z,\zeta) \in H^1_*(\Omega_1)\times L^2(\Omega_1)\times L^2(\Gamma_1)\times H^2_*(\Omega_2)\times L^2(\Omega_2)\times L^2(\Gamma_2) \mid \ u = w \ \text{ sur } \ I\right\},$$

qui est muni du produit scalaire habituel suivant

$$\left(U,\tilde{U}\right)_{\mathcal{H}} = \int_{\Omega_1}\left(\nabla u\cdot\nabla\overline{\tilde{u}} + v\overline{\tilde{v}}\right)dx + \int_{\Gamma_1}\eta\overline{\tilde{\eta}}d\Gamma + a(w,\overline{\tilde{w}}) + \int_{\Omega_2} z\overline{\tilde{z}}dx + \int_{\Gamma_2}\zeta\overline{\tilde{\zeta}}d\Gamma,$$

où $U = (u,v,\eta,w,z,\zeta)$, $\tilde{U} = (\tilde{u},\tilde{v},\tilde{\eta},\tilde{w},\tilde{z},\tilde{\zeta}) \in \mathcal{H}$. Nous définissons ensuite l'opérateur linéaire non borné $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ par

$$D(\mathcal{A}) = \left\{\begin{array}{l} U = (u,v,\eta,w,z,\zeta) \in \mathcal{H} \mid \Delta u \in L^2(\Omega_1), \quad v \in H^1_*(\Omega_1), \quad \Delta^2 w \in L^2(\Omega_2), \quad z \in H^2_*(\Omega_2), \\ \partial_{\nu_1} u + \eta = 0 \ \text{ sur } \ \Gamma_1, \quad \mathcal{B}_1 w = 0 \ \text{ sur } \ \Gamma_2, \quad \mathcal{B}_2 w - \zeta = 0 \ \text{ sur } \ \Gamma_2, \\ v = z, \quad \mathcal{B}_1 w = 0 \ \text{ et } \ \mathcal{B}_2 w = \partial_{\nu_1} u \ \text{ sur } \ I \end{array}\right\}$$

et

$$\mathcal{A}U = \left(v, \Delta u, \gamma_1(v) - \eta, z, -\Delta^2 w, \gamma_{2,1}(z) - \zeta\right), \qquad \forall U = (u,v,\eta,w,z,\zeta) \in D(\mathcal{A}),$$

où $\gamma_1 : H^1(\Omega_1) \longmapsto L^2(\Gamma_1)$ et $\gamma_2 : H^2(\Omega_2) \longmapsto L^2(\Gamma_2)\times L^2(\Gamma_2)$ sont les opérateurs de trace, tels que $\gamma_1(\varphi) = \varphi|_{\Gamma_1}$ et $\gamma_2(\psi) = (\gamma_{2,1}(\psi),\gamma_{2,2}(\psi)) = (\psi|_{\Gamma_2}, \partial_{\nu_2}\psi|_{\Gamma_2})$. En posant $U = (u, u_t, \eta, w, w_t, \zeta)$ comme l'état du système (1.31)-(1.34), nous réécrivons le problème sous forme d'une équation d'évolution du premier ordre:

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{1.35}$$

où la condition initiale $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \zeta_0) \in \mathcal{H}$ est donnée par les équations (1.32) et (1.33).

Nous prouvons que l'opérateur linéaire non borné $\mathcal{A}$ est m-dissipatif dans l'espace d'énergie $\mathcal{H}$. Ensuite, selon le théorème de Lumer-Phillips (voir [44]), nous en déduisons que l'opérateur $\mathcal{A}$ génère un C_0-semi-groupe de contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ dans $\mathcal{H}$. Ce résultat garantit le caractère bien posé du système (1.35), conduisant au théorème suivant:

Théorème 1.2.4. *Pour tout $U_0 \in \mathcal{H}$, le système (1.35) admet une solution faible unique*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

De plus, si $U_0 \in D(\mathcal{A})$, alors le système (1.35) admet une solution forte unique

$$U(t) \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

Ensuite, nous imposons des conditions géométriques suffisantes sur les domaines $\Omega_i, i = 1, 2$ (voir les Hypothèses (A1) et (A2) ci-dessus) afin d'assurer la régularité de la solution de ce système. Pour établir la stabilité forte du système (1.31)-(1.34) et garantir la décroissance stricte de l'énergie, c'est-à-dire que $E(t)$ tend vers zéro lorsque t tend vers l'infini, nous introduisons l'hypothèse géométrique suivante sur la plaque, qui est due aux exigences d'intégration dans Ω_2:

Hypothèse (A4) Supposons qu'il existe un point fixe $x_0 \in \mathbb{R}^2$ tel que, en posant $m(x) = x - x_0$, nous ayons

$$(m \cdot \nu_2) = 0, \quad \forall x \in I,$$

et

$$(m \cdot \nu_2) \geq 0, \quad \forall x \in \Gamma_2,$$

où $(\cdot)$ désigne le produit scalaire dans $\mathbb{R}^2$.

Notre résultat principal est résumé dans le théorème suivant:

Théorème 1.2.5. *Supposons que Ω_1 et Ω_2 soient des domaines polygonaux tels que décrits précédemment. Supposons également que les hypothèses (A1), (A2) et (A4) soient satisfaites. Alors, le semi-groupe de contractions $e^{t\mathcal{A}}$ est fortement stable dans l'espace d'énergie $\mathcal{H}$, au sens où*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

Il est évident que la décroissance exponentielle n'est pas atteignable lorsque trois contrôles dynamiques sont appliqués. Par conséquent, l'absence de stabilité exponentielle persiste dans le système (1.31)-(1.34) même lorsque seuls deux contrôles dynamiques sont appliqués. Il est donc naturel d'espérer une stabilité polynomiale du système (1.31)-(1.34) pour des solutions régulières. Pour établir notre résultat de décroissance polynomiale de l'énergie, nous devons imposer une hypothèse géométrique supplémentaire sur l'onde, due aux exigences d'intégration dans Ω_1:

Hypothèse (A5) Supposons qu'il existe un point fixe $x_0 \in \mathbb{R}^2$ tel que, en posant $m(x) = x - x_0$, nous ayons

$$(m \cdot \nu_1) \geq \delta, \quad \forall x \in \Gamma_1,$$

où δ est un nombre réel positif et $(\cdot)$ désigne le produit scalaire dans $\mathbb{R}^2$.

Nous pouvons maintenant présenter notre résultat sur le taux de décroissance de l'énergie:

Théorème 1.2.6. *Supposons que Ω_1 et Ω_2 soient des domaines polygonaux comme décrit précédemment. Supposons également que les hypothèses (A1), (A2), (A4) et (A5) soient satisfaites. Alors, il existe une constante $\mathcal{C} > 0$ telle que, pour toutes les données initiales $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \zeta_0) \in D(\mathcal{A})$, l'énergie du système (1.31)-(1.34) vérifie l'estimation suivante:*

$$E(t) \leq \frac{\mathcal{C}}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

Les résultats présentés dans ce chapitre ont été publiés dans l'article de conférence *Advances in Partial Differential Equations and Control*, dans le chapitre intitulé "Energy Decay Estimate for a Wave-Plate Interface Transmission Problem with Only Two Dynamical Boundary Controls" (voir [2]).

Dans le Chapitre 5, nous nous concentrons sur l'étude de la solution numérique d'un problème de transmission entre une équation des ondes et une poutre d'Euler-Bernoulli avec des contrôles aux limites dynamiques en une dimension spatiale. Pour ce faire, nous utilisons la méthode des volumes finis (FVM) pour la discrétisation spatiale. Les deux équations sont couplées par des connexions aux frontières entre les intervalles où évoluent l'onde et la poutre d'Euler-Bernoulli. Le système décrivant ce modèle est le suivant:

$$\begin{cases} u_{tt} - au_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ w_{tt} + bw_{xxxx} = 0, & (x,t) \in (-L,0) \times (0,\infty), \\ u(0,t) = w(0,t), \quad w_x(0,t) = 0, & t \in (0,\infty), \\ au_x(0,t) + bw_{xxx}(0,t) = 0, & t \in (0,\infty), \\ u_x(L,t) + \eta(t) = 0, & t \in (0,\infty), \\ w_{xx}(-L,t) - \xi(t) = 0, & t \in (0,\infty), \\ w_{xxx}(-L,t) + \zeta(t) = 0, & t \in (0,\infty), \end{cases} \tag{1.36}$$

avec des conditions initiales de la forme:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,L), \quad \eta(0) = \alpha, \tag{1.37}$$

$$w(x,0) = \phi(x), \quad w_t(x,0) = \theta(x), \quad x \in (-L,0), \quad \xi(0) = \beta, \quad \zeta(0) = \gamma, \tag{1.38}$$

où η, ξ et ζ désignent les contrôles dynamiques aux frontières. Ici, les coefficients a et b sont des constantes strictement positives représentant respectivement la vitesse de propagation de l'équation des ondes et de la poutre d'Euler-Bernoulli. L'amortissement du système est réalisé via un mécanisme d'amortissement indirect impliquant les équations différentielles du premier ordre suivantes:

$$\begin{cases} \eta_t(t) - u_t(L,t) + \eta(t) = 0, & t \in (0,\infty), \\ \xi_t(t) - w_{xt}(-L,t) + \xi(t) = 0, & t \in (0,\infty), \\ \zeta_t(t) - w_t(-L,t) + \zeta(t) = 0, & t \in (0,\infty). \end{cases} \tag{1.39}$$

L'énergie du système décrit par (1.36)-(1.39) est définie par

$$E(t) = \frac{1}{2}\left\{\int_0^L \left(|u_t(x,t)|^2 + a|u_x(x,t)|^2\right) dx + a|\eta(t)|^2\right\} + \frac{1}{2}\left\{\int_{-L}^0 \left(|w_t(x,t)|^2 + b|w_{xx}(x,t)|^2\right) dx + b|\xi(t)|^2 + b|\zeta(t)|^2\right\}.$$

En prenant la dérivée de $E(t)$ le long de la solution de (1.36)-(1.39), on obtient

$$\frac{dE(t)}{dt} = -\left(a|\eta(t)|^2 + b|\xi(t)|^2 + b|\zeta(t)|^2\right) \leq 0,$$

ce qui implique que l'énergie du système diminue au cours du temps. Notre objectif est d'étudier numériquement ce système.

Nous commençons par définir la solution faible du problème (1.36)-(1.39). Tout au long de ce texte, nous utiliserons la notation suivante pour les espaces de fonctions:

Définition des espaces fonctionnels. *Introduisons les espaces suivants:*

$$H^1_*(\ell_1,\ell_2)=\left\{u\in H^1(\ell_1,\ell_2)\ \ |\ \ \int_{\ell_1}^{\ell_2}udx=0\right\},$$

$$H^2_*(\ell_1,\ell_2)=\left\{w\in H^2(\ell_1,\ell_2)\ \ |\ \ \int_{\ell_1}^{\ell_2}wdx=w_x(\ell_2)=0\right\},$$

$$H^2_{**}(\ell_1,\ell_2)=\left\{w\in H^2(\ell_1,\ell_2)\ \ |\ \ w_x(\ell_2)=0\right\},$$

$$\mathbb{H}=\left\{(u,w)\in H^1_*(0,L)\times H^2_*(-L,0)\ \ |\ \ u(0)=w(0)\right\},$$

$$\mathbb{H}_*=\left\{(u,w)\in H^1(0,L)\times H^2_{**}(-L,0)\ \ |\ \ u(0)=w(0)\right\},$$

et

$$\mathbb{L}^2=L^2(0,L)\times L^2(-L,0).$$

La formulation faible du problème (1.36)-(1.39) est définie comme suit:

Définition 1.2.1. *On dit que la fonction (u,w,η,ξ,ζ) est une solution faible de (1.36)-(1.39) si, pour tout $T>0$, les conditions suivantes sont satisfaites:*

1. $(u,w)\in L^2([0,T];\mathbb{H})$ *avec* $(u_t,w_t)\in L^2([0,T];\mathbb{L}^2)$ *et* $(\eta,\xi,\zeta)\in L^2([0,T];\mathbb{R}^3)$,

2. $(u(\cdot,0),w(\cdot,0))=(\varphi(\cdot),\phi(\cdot))$,

3. (u,w,η,ξ,ζ) *est tel que*

$$\begin{aligned}&-\int_0^T\int_0^L u_tv_tdxdt+a\int_0^T\int_0^L u_xv_xdxdt-\int_0^L\psi(x)v(x,0)dx\\&-\int_0^T\int_{-L}^0 w_tz_tdxdt+b\int_0^T\int_{-L}^0 w_{xx}z_{xx}dxdt-\int_{-L}^0\theta(x)z(x,0)dx\\&+a\int_0^T\eta(t)v(L,t)dt+b\int_0^T\xi(t)z_x(-L,t)dt+b\int_0^T\zeta(t)z(-L,t)dt=0,\end{aligned}\tag{1.40}$$

   pour tout $(v,z)\in L^2([0,T];\mathbb{H}_*)$ *avec* $(v_t,z_t)\in L^2([0,T];\mathbb{L}^2)$, *tel que* $v(T,x)=z(T,x)=0$.

4. (u,w,η,ξ,ζ) *est tel que*

$$-\int_0^T\eta p_tdt-\alpha p(0)+\int_0^T u(L,t)p_t(t)dt+u(L,0)p(0)+\int_0^T\eta pdt=0,\tag{1.41}$$

$$-\int_0^T\xi q_tdt-\beta q(0)+\int_0^T w_x(-L,t)q_t(t)dt+w_x(-L,0)q(0)+\int_0^T\xi qdt=0,\tag{1.42}$$

   et

$$-\int_0^T\xi q_tdt-\gamma q(0)+\int_0^T w_x(-L,t)q_t(t)dt+w_x(-L,0)q(0)+\int_0^T\xi qdt=0,\tag{1.43}$$

   pour tout $(p,q,r)\in L^2([0,T];\mathbb{R}^3)$ *avec* $(p_t,q_t,r_t)\in L^2([0,T];\mathbb{R}^3)$, *tel que* $p(T)=q(T)=r(T)=0$.

Nous mentionnons ici que, selon le théorème de Lumer-Philips (voir [44]), nous pouvons prouver l'existence et l'unicité d'une solution forte $(u, w) \in L^2([0,T];\mathbb{H})$ avec $(u_t, w_t) \in L^2([0,T];\mathbb{L}^2)$ et $(\eta, \xi, \zeta) \in L^2([0,T];\mathbb{R}^3)$ pour le système (1.36)-(1.39). Ce résultat confirme également l'existence et l'unicité de la solution faible du problème (1.36)-(1.39).

Pour effectuer une discrétisation en volume fini de l'espace du problème, un maillage $\mathcal{T}$ de $(-L,0)\cup(0,L)$, sur lequel (1.36)-(1.39) sera étudié, est introduit comme suit:

Définition 1.2.2 (Maillage admissible unidimensionnel). *Un maillage admissible $\mathcal{T}$ du domaine spatial $(-L,0)\cup(0,L)$ est donné par une séquence croissante de valeurs réelles $\left(x_{i+\frac{1}{2}}\right)_{i=0,\dots,N_{\max}}$ telle que $[-L,L] = \bigcup_{i=1}^{N_{\max}} \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. Le maillage $\mathcal{T}$ est l'ensemble $\{K_i : \ i \in \{1,\dots,N_{\max}\}; \quad N_{\max} \in \mathbb{N}^*\}$ des volumes de contrôle définis par $K_i = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$ pour tout $i \in \{1,\dots,N_{\max}\}$. La famille $(x_i)_{i=1,\dots,N_{\max}}$ est supposée être le centre de $(K_i)_{i=1,\dots,N_{\max}}$ telle que*

$$-L = x_0 = x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < \cdots < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < \cdots < x_{N_{\max}-\frac{1}{2}} < x_{N_{\max}} < x_{N_{\max}+\frac{1}{2}} = x_{N_{\max}+1} = L.$$

Nous discrétisons les intervalles $[-L,0]$ et $[0,L]$ en N_1 et N_2 points internes respectivement, tels que N_1 et $N_2 \in \mathbb{N}^*$. La relation est donnée par $N_{\max} = N_1 + N_2$. Pour plus de clarté, notons que les tailles de maille spatiale sont $h_1 = L/N_1$ et $h_2 = L/N_2$, et procédons à la discrétisation comme suit:

Nous discrétisons l'intervalle $[-L,0]$ de la manière suivante:

- *Pour $i = 0,\dots,N_1$, $x_{i+\frac{1}{2}} = -L + ih_1$, ce qui donne $x_{\frac{1}{2}} = -L$ et $x_{N_1+\frac{1}{2}} = 0$.*
- *Pour $i = 1,\dots,N_1$, $x_i = -L + \left(i - \frac{1}{2}\right)h_1$, ce qui donne $x_1 = -L + \dfrac{h_1}{2}$ et $x_{N_1} = -\dfrac{h_1}{2}$.*

Nous discrétisons l'intervalle $[0,L]$ de la manière suivante:

- *Pour $i = N_1+1,\dots,N_{\max}$, $x_{i+\frac{1}{2}} = (i-N_1)h_2$, ce qui donne $x_{N_1+\frac{3}{2}} = h_2$ et $x_{N_{\max}+\frac{1}{2}} = L$.*
- *Pour $i = N_1+1,\dots,N_{\max}$, $x_i = \left(i - N_1 - \frac{1}{2}\right)h_2$, ce qui donne $x_{N_1+1} = \dfrac{h_2}{2}$ et $x_{N_{\max}} = L - \dfrac{h_2}{2}$.*

La longueur de chaque volume de contrôle K_i est notée:

$$h_i = m(K_i) = \begin{cases} h_1, & i = 1,\dots,N_1, \\ h_2, & i = N_1+1,\dots,N_{\max}. \end{cases}$$

Nous définissons également:

$$h_{i+\frac{1}{2}} = \begin{cases} x_{i+1} - x_i, & i = 0,\dots N_1-1,\ N_1+1,\dots,N_{\max}, \\ x_{N_1+1} - x_{N_1+\frac{1}{2}}, & i = N_1, \end{cases}$$

ce qui implique que

$$h_{i+\frac{1}{2}} = \begin{cases} \dfrac{h_1}{2}, & i = 0, \\ h_1, & i = 1, \ldots, N_1 - 1, \\ h_2, & i = N_1 + 1, \ldots, N_{\max} - 1, \\ \dfrac{h_2}{2}, & i = N_1,\ N_{\max}. \end{cases}$$

De plus, nous définissons les inconnues discrètes pour la discrétisation spatiale (en volumes finis) comme $\{u_i(t);\ i = N_1 + 1, \ldots, N_{\max},\ t \in [0, T]\}$ et $\{w_i(t);\ i = 1, \ldots, N_1,\ t \in [0, T]\}$, qui représentent respectivement l'approximation des valeurs moyennes de u et w sur les volumes de contrôle K_i. Cependant, ces données sont utilisées pour construire les approximations de u et w à l'intérieur des mailles. En effet, nous avons:

$$\begin{aligned} u_i(t) &= \frac{1}{h_i} \int_{K_i} u(x,t)dx, \quad \text{pour } i = N_1 + 1, \ldots, N_{\max}, \\ w_i(t) &= \frac{1}{h_i} \int_{K_i} w(x,t)dx, \quad \text{pour } i = 1, \ldots, N_1. \end{aligned}$$

Nous notons également

$$U_i = \begin{cases} u_i, & i = N_1 + 1, \ldots, N_{\max}, \\ w_i, & i = 1, \ldots, N_1, \end{cases}$$

et posons

$$\Xi = (\eta, \xi, \zeta).$$

Pour décrire les schémas de la manière la plus générale possible, nous introduisons quelques inconnues auxiliaires. Plus précisément, nous notons les valeurs de u et w en $x_{N_1+\frac{1}{2}}$ par $u_{N_1+\frac{1}{2}}$ et $w_{N_1+\frac{1}{2}}$, respectivement. De plus, nous définissons $u_{N_{\max}+1}$ comme la valeur de u en $x_{N_{\max}+1}$ et w_0 comme la valeur de w en x_0. En utilisant ces définitions, nous pouvons établir les opérateurs aux différences nécessaires pour discrétiser les dérivées spatiales. Ces opérateurs sont exprimés en fonction des inconnues discrètes $(u_i)_{i=N_1+1,\ldots,N_{\max}}$, $(w_i)_{i=1,\ldots,N_1}$ et d'autres données.

À partir de ces définitions, nous construisons un schéma numérique semi-discret pour le système (1.36)-(1.39), puis nous estimons l'énergie discrète en utilisant une discrétisation en volumes finis dans l'espace. Pour commencer, nous abordons la discrétisation des conditions initiales. Par exemple, nous initialisons la solution avec $\Phi_i = (\varphi_i, \phi_i)$ tel que

$$\begin{aligned} u_i(0) = \varphi_i \quad \text{où} \quad \varphi_i &= \frac{1}{h_i} \int_{K_i} \varphi(x)dx, \qquad \text{pour} \quad i = N_1 + 1, \ldots, N_{\max}, \\ w_i(0) = \phi_i \quad \text{où} \quad \phi_i &= \frac{1}{h_i} \int_{K_i} \phi(x)dx, \qquad \text{pour} \quad i = 1, \cdots, N_1. \end{aligned}$$

En conséquence, nous avons

$$U_i(0) = \Phi_i, \quad \text{pour} \quad i = 1, \ldots, N_{\max}.$$

De plus, définissons $\Psi_i = (\psi_i, \theta_i)$ en posant

$$\psi_i = \frac{1}{h_i}\int_{K_i} \psi(x)dx, \qquad \text{pour} \quad i = N_1+1, \ldots, N_{\max},$$

$$\theta_i = \frac{1}{h_i}\int_{K_i} \theta(x)dx, \qquad \text{pour} \quad i = 1, \ldots, N_1.$$

Ainsi, en utilisant les notations ci-dessus, nous approchons les conditions initiales de u_t et w_t de la manière suivante

$$u_{i,t}(0) = \psi_i, \qquad \text{pour} \quad i = N_1+1, \ldots, N_{\max},$$

$$w_{i,t}(0) = \theta_i, \qquad \text{pour} \quad i = 1, \ldots, N_1.$$

Par conséquent, nous avons

$$U_{i,t}(0) = \Psi_i, \quad \text{pour} \quad i = 1, \cdots, N_{\max},$$

et nous posons

$$\Xi(0) = \Theta.$$

À partir de ces définitions, nous dérivons le schéma numérique semi-discrétisé suivant pour le système (1.36)-(1.39):

$$\begin{cases} h_i U_{i,tt} + F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} = 0, & i = 1, \ldots, N_{\max}, \\ u_{N_1+\frac{1}{2}} = w_{N_1+\frac{1}{2}}, \quad \delta_x w_{N_1+\frac{1}{2}} = 0, \quad \delta_x^2 w_0 - \xi = 0, \end{cases} \tag{1.44}$$

où le flux numérique $F_{i+\frac{1}{2}}$ est défini comme suit:

$$F_{i+\frac{1}{2}} = \begin{cases} b\delta_x^3 w_{i+\frac{1}{2}}, & i = 0, \ldots, N_1 - 1, \\ -a\delta_x u_{i+\frac{1}{2}}, & i = N_1, \ldots, N_{\max}, \end{cases} \tag{1.45}$$

et les conditions aux limites sont données par

$$\delta_x u_{N_{\max}+\frac{1}{2}} + \eta = 0, \qquad \delta_x^3 w_{\frac{1}{2}} + \zeta = 0, \tag{1.46}$$

ainsi que les équations différentielles semi-discrétisées:

$$\begin{cases} \eta_t - u_{N_{\max}+1,t} + \eta = 0, \\ \xi_t - \delta_x w_{\frac{1}{2},t} + \xi = 0, \\ \zeta_t - w_{0,t} + \zeta = 0. \end{cases} \tag{1.47}$$

Pour ce système, nous définissons:

- L'énergie cinétique discrète pour $U_{\mathcal{T}}$ comme:

$$\mathcal{E}_k(U_{\mathcal{T}}) = \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i \left(U_{i,t}\right)^2.$$

- L'énergie potentielle discrète pour $U_{\mathcal{T}}$ comme:

$$\mathcal{E}_p(U_{\mathcal{T}}) = \frac{1}{2}\left(b\sum_{i=1}^{N_1} h_i(\delta_x^2 w_i)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}(\delta_x u_{i+\frac{1}{2}})^2 \right).$$

- L'énergie aux frontières dynamiques pour Ξ comme:

$$\mathcal{E}_b(\Xi) = \frac{1}{2}\left(a\eta^2 + b\xi^2 + b\zeta^2\right).$$

L'énergie totale discrète $\mathcal{E}(t)$ du système semi-discrétisé (1.44)-(1.47) est donnée par

$$\mathcal{E}(t) = \mathcal{E}_k(U_{\mathcal{T}}) + \mathcal{E}_p(U_{\mathcal{T}}) + \mathcal{E}_b(\Xi), \qquad \forall t \in [0,T].$$

La dérivée temporelle de cette énergie discrète, $\mathcal{E}(t)$, satisfait la relation:

$$\frac{d\mathcal{E}(t)}{dt} = -\left(a|\eta|^2 + b|\xi|^2 + b|\zeta|^2\right).$$

Cela implique que l'énergie discrétisée $\mathcal{E}(t)$ du système (1.44)-(1.47) est conservée en l'absence de contrôles dynamiques. Cependant, l'énergie est non croissante au cours du temps tant qu'au moins un contrôle dynamique aux frontières est actif, soit sur l'onde, soit sur la poutre d'Euler-Bernoulli.

Ensuite, nous établissons des estimations de stabilité pour la solution approchée et nous prouvons sa convergence vers la solution faible du problème (1.40)-(1.43) dans un sens approprié, lorsque le pas de discrétisation tend vers zéro. Afin d'établir un résultat de stabilité et de prouver la convergence de ce schéma, nous posons $\Phi = (\varphi, \phi)$, $\Psi = (\psi, \theta)$, $\Theta = (\alpha, \beta, \gamma)$ et nous étudions notre problème sous une certaine hypothèse, que nous appelons **Hypothesis (H1)**:

$$\Phi \in \mathbb{H}, \quad \Psi \in \mathbb{L}^2 \quad \text{et} \quad \Theta \in \mathbb{R}^3. \qquad \textbf{Hypothèse (H1)}$$

Pour tout $U_{\mathcal{T}}$ arbitraire, nous définissons la norme $\mathbb{L}^2$ suivante:

$$\|U_{\mathcal{T}}\| = \left(\sum_{i=1}^{N_{\max}} h_i (U_i)^2\right)^{\frac{1}{2}}.$$

Nous définissons également la norme discrète $\mathbb{H}$ de la fonction $U_{\mathcal{T}} = (u_{\mathcal{T}}, w_{\mathcal{T}})$ comme

$$\|U_{\mathcal{T}}\|_{\mathbb{H},\mathcal{T}} = \left(\|u_{\mathcal{T}}\|_{1,\mathcal{T}}^2 + \|w_{\mathcal{T}}\|_{2,\mathcal{T}}^2\right)^{\frac{1}{2}},$$

où

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}(\delta_x u_{i+\frac{1}{2}})^2$$

et

$$\|w_{\mathcal{T}}\|_{2,\mathcal{T}}^2 = b\sum_{i=1}^{N_1} h_i(\delta_x^2 w_i)^2$$

représentent les normes discrètes dans $H_*^1(0,L)$ et $H_*^2(-L,0)$ pour les fonctions $u_{\mathcal{T}}$ et $w_{\mathcal{T}}$, respectivement. Pour la fonction $\Xi = (\eta, \xi, \zeta)$, nous définissons la norme $\mathbb{R}^3$ comme

$$\|\Xi\|_{\mathbb{R}^3} = \left(a\eta^2 + b\xi^2 + b\zeta^2\right)^{\frac{1}{2}}.$$

Théorème 1.2.7. *Soit l'hypothèse* ***Hypothèse (H1)*** *vérifiée. Alors, la solution numérique $U_{\mathcal{T}}$ du schéma numérique (1.44)-(1.47) est uniformément bornée indépendamment des paramètres de discrétisation dans $L^\infty([0,T];\mathbb{L}^2)\cap L^\infty([0,T];\mathbb{H})$. De plus, la dérivée première en temps est bornée dans $L^\infty([0,T];\mathbb{L}^2)$ et la fonction Ξ est également bornée dans $L^\infty([0,T];\mathbb{R}^3)$. Par conséquent, l'estimation suivante est satisfaite:*

$$\|U_{\mathcal{T},t}\|^2+\|U_{\mathcal{T}}\|^2+\|U_{\mathcal{T}}\|^2_{\mathbb{H},\mathcal{T}}+\|\Xi\|^2_{\mathbb{R}^3}\leq C,$$

où C est une constante strictement positive indépendante du paramètre de discrétisation spatiale.

Nous démontrons ensuite que, lorsque le pas de maillage tend vers zéro, la solution numérique obtenue par le schéma numérique (1.44)-(1.47) converge vers la solution faible. La stratégie repose sur les estimations de stabilité obtenues et, en passant à la limite, nous obtenons la convergence vers la solution faible énoncée dans le théorème suivant:

Théorème 1.2.8. *Supposons que* ***Hypothèse (H1)*** *soit vérifiée. Pour $m\in\mathbb{N}$, soit $\mathcal{T}_m$ un maillage admissible décrit au sens de la Définition 1.2.2. Soit $(u_{\mathcal{T}_m},w_{\mathcal{T}_m},\eta,\xi,\zeta)$ la solution discrète de (1.44)-(1.47). Supposons que taille$(\mathcal{T}_m)\longrightarrow 0$ lorsque $m\longrightarrow\infty$. Alors, il existe une sous-suite de la suite des solutions approchées, encore notée $(u_{\mathcal{T}_m},w_{\mathcal{T}_m},\eta,\xi,\zeta)$, qui converge vers la solution faible (u,w,η,ξ,ζ) dans $L^\infty([0,T];L^2(0,L))\times L^\infty([0,T];L^2(-L,0))\times L^\infty([0,T];\mathbb{R}^3)$ du problème (1.40)-(1.43), lorsque m tend vers ∞, pour la topologie faible-$\star$.*

Pour la discrétisation en temps, soit $\mathcal{N}\in\mathbb{N}^*$ et définissons un pas de temps discret uniforme $\tau=T/\mathcal{N}$ où nous notons $t_n=n\tau$, pour $n=0,1,\ldots,\mathcal{N}$. Les inconnues discrètes sont notées par $\{u_i^n;\ \ i=N_1+1,\ldots,N_{\max},\ \ n=0,1,\ldots\mathcal{N}\}$ et $\{w_i^n;\ \ i=1,\ldots,N_1,\ \ n=0,1,\ldots\mathcal{N}\}$, représentant des approximations des valeurs moyennes de u et w au temps t_n sur chaque volume de contrôle K_i.

Nous imposons les conditions initiales comme suit. En partant de $\Phi_i=(\varphi_i,\phi_i)$, nous définissons

$$u_i^0=\varphi_i\quad\text{où}\quad\varphi_i=\frac{1}{h_i}\int_{K_i}\varphi(x)dx,\qquad\text{pour}\quad i=N_1+1,\ldots,N_{\max},$$

$$w_i^0=\phi_i\quad\text{où}\quad\phi_i=\frac{1}{h_i}\int_{K_i}\phi(x)dx,\qquad\text{pour}\quad i=1,\cdots,N_1.$$

Ainsi,

$$U_i^0=\Phi_i,\quad\text{pour}\quad i=1,\ldots,N_{\max}.$$

Pour le premier pas de temps, nous utilisons un niveau de temps artificiel -1 en désignant les valeurs de nos fonctions d'approximation à ce pas de temps par

$$u_i^{-1}\qquad\text{pour}\quad i=N_1+1,\ldots,N_{\max},$$

$$w_i^{-1}\qquad\text{pour}\quad i=1,\ldots,N_1.$$

Nous introduisons $\Psi_i=(\psi_i,\theta_i)$ tel que

$$\psi_i=\frac{1}{h_i}\int_{K_i}\psi(x)dx,\qquad\text{pour}\quad i=N_1+1,\ldots,N_{\max},$$

$$\theta_i=\frac{1}{h_i}\int_{K_i}\theta(x)dx,\qquad\text{pour}\quad i=1,\ldots,N_1.$$

En utilisant les notations ci-dessus, les conditions initiales pour u_t et w_t peuvent alors être approximées en utilisant les différences centrales:

$$\frac{u_i^1 - u_i^{-1}}{2\tau} = \psi_i, \qquad \text{pour} \quad i = N_1 + 1, \ldots, N_{\max},$$
$$\frac{w_i^1 - w_i^{-1}}{2\tau} = \theta_i, \qquad \text{pour} \quad i = 1, \ldots, N_1.$$

En réarrangeant ces expressions, nous obtenons:

$$u_i^{-1} = u_i^1 - 2\tau\psi_i, \qquad \text{pour} \quad i = N_1 + 1, \ldots, N_{\max},$$
$$w_i^{-1} = w_i^1 - 2\tau\theta_i, \qquad \text{pour} \quad i = 1, \ldots, N_1.$$

Ainsi, on peut écrire:

$$U_i^{-1} = U_i^1 - 2\tau\Psi_i, \quad \text{pour} \quad i = 1, \cdots, N_{\max}.$$

Enfin, nous définissons $\Theta = (\alpha, \beta, \gamma)$ de sorte que

$$\Xi^0 = \Theta.$$

Nous construisons le schéma numérique implicite pour le système (1.36)-(1.39) en utilisant la méthode des volumes finis pour la discrétisation en espace et une formulation implicite alternative basée sur une approximation centrée en temps du terme dérivé. Les dérivées temporelles sont discrétisées à l'aide de la méthode des différences finies. Le schéma implicite pour l'inconnue U_i^{n+1} s'écrit comme suit:

$$\begin{cases} h_i \dfrac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\tau^2} + \dfrac{1}{2}\left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1}\right) + \dfrac{1}{2}\left(F_{i+\frac{1}{2}}^{n-1} - F_{i-\frac{1}{2}}^{n-1}\right) = 0, \quad i = 1, \ldots, N_{\max}, \ n = 1, \ldots, \mathcal{N} - 1, \\ u_{N_1+\frac{1}{2}}^n = w_{N_1+\frac{1}{2}}^n, \quad \delta_x w_{N_1+\frac{1}{2}}^n = 0, \quad \delta_x^2 w_0^n - \xi^n = 0, \qquad\qquad n = 0, 1, \ldots \mathcal{N}, \end{cases} \tag{1.48}$$

où le flux numérique au point $x_{i+\frac{1}{2}}$ et au temps t_n est défini par

$$F_{i+\frac{1}{2}}^n = \begin{cases} b\delta_x^3 w_{i+\frac{1}{2}}^n, & i = 0, \ldots, N_1 - 1, \\ -a\delta_x u_{i+\frac{1}{2}}^n, & i = N_1, \ldots, N_{\max}, \end{cases} \tag{1.49}$$

avec les conditions aux limites

$$\delta_x u_{N_{\max}+\frac{1}{2}}^n + \eta^n = 0, \qquad \text{pour} \quad n = 0, 1, \ldots \mathcal{N},$$

et

$$\delta_x^3 w_{\frac{1}{2}}^n + \zeta^n = 0, \qquad \text{pour} \quad n = 0, 1, \ldots \mathcal{N}.$$

La forme discrétisée des équations de (1.39) est alors donnée par:

$$\begin{cases} \eta^{n+1} = \eta^{n-1} + u_{N_{\max}+1}^{n+1} - u_{N_{\max}+1}^{n-1} - \tau\left(\eta^{n+1} + \eta^{n-1}\right), & n = 1, \ldots, \mathcal{N} - 1, \\ \xi^{n+1} = \xi^{n-1} + \delta_x w_{\frac{1}{2}}^{n+1} - \delta_x w_{\frac{1}{2}}^{n-1} - \tau\left(\xi^{n+1} + \xi^{n-1}\right), & n = 1, \ldots, \mathcal{N} - 1, \\ \zeta^{n+1} = \zeta^{n-1} + w_0^{n+1} - w_0^{n-1} - \tau\left(\zeta^{n+1} + \zeta^{n-1}\right), & n = 1, \ldots, \mathcal{N} - 1. \end{cases} \tag{1.50}$$

Ensuite, nous présentons une estimation de l'énergie discrète. À cette fin, nous définissons:

- L'énergie potentielle discrète pour $U_{\mathcal{T}}^n$ comme:

$$\mathcal{E}_k\left(U_{\mathcal{T}}^n\right) = \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^{n+1} - U_i^n}{\tau}\right)^2.$$

- L'énergie potentielle discrète pour $U_{\mathcal{T}}^n$ est définie comme:

$$\mathcal{E}_p(U_{\mathcal{T}}^n) = \frac{1}{4}\left(b\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n+1}\right)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}\right)^2\right) + \frac{1}{4}\left(b\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n}\right)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n}\right)^2\right).$$

- L'énergie de contrôle aux frontières discrètes pour Ξ^n est donnée par :

$$\mathcal{E}_b(\Xi^n) = \frac{1}{4}\left(a\left(\eta^{n+1}\right)^2 + b\left(\xi^{n+1}\right)^2 + b\left(\zeta^{n+1}\right)^2\right) + \frac{1}{4}\left(a\left(\eta^{n}\right)^2 + b\left(\xi^{n}\right)^2 + b\left(\zeta^{n}\right)^2\right).$$

L'énergie discrète totale $\mathcal{E}^n$ du schéma numérique implicite (1.48)-(1.50) s'exprime comme suit:

$$\mathcal{E}^n = \mathcal{E}_k(U_{\mathcal{T}}^n) + \mathcal{E}_p(U_{\mathcal{T}}^n) + \mathcal{E}_b(\Xi^n).$$

Cette énergie satisfait l'estimation suivante:

$$\mathcal{E}^n - \mathcal{E}^{n-1} = -\frac{\tau}{4}\left(a|\eta^{n+1} + \eta^{n-1}|^2 + b|\xi^{n+1} + \xi^{n-1}|^2 + b|\zeta^{n+1} + \zeta^{n-1}|^2\right).$$

Cela implique que l'énergie discrète totale $\mathcal{E}^n$ est conservée en l'absence de contrôles dynamiques aux frontières et qu'elle est décroissante au cours du temps dès qu'au moins un contrôle dynamique est appliqué soit sur l'onde, soit sur la poutre d'Euler-Bernoulli.

Les estimations de stabilité de ce schéma sont établies sous certaines conditions sur les données initiales et un choix approprié du pas de temps. Pour des raisons de stabilité, nous définissons $\Phi = (\varphi, \phi)$, $\Psi = (\psi, \theta)$, $\Theta = (\alpha, \beta, \gamma)$ et nous étudions notre problème sous une hypothèse spécifique, que nous appelons **Hypothèse (H2)**, de la forme suivante:

$$\Phi \in \mathbb{H}, \quad \Psi \in H^2(0,L) \times H^4(-L,0) \quad \text{et} \quad \Theta \in \mathbb{R}^3. \qquad \textbf{Hypothèse (H2)}$$

Nous supposons également la condition suivante sur le pas de temps τ:

$$\text{Il existe un réel } \tau_0 \in [0,T] \text{ tel que } \tau \leq \tau_0. \qquad \textbf{Hypothèse (TA)}$$

Sous ces conditions, nous établissons les estimations de stabilité nécessaires pour le schéma numérique implicite en utilisant des normes discrètes et l'inégalité de Poincaré discrète. Ces estimations conduisent à la convergence du schéma. Ces résultats sont formalisés dans le théorème suivant:

Théorème 1.2.9. *Soit **Hypothèse (H2)** et **Hypothèse (TA)** vérifiées. Alors, la solution numérique $U_{\mathcal{T},\tau}$ du schéma numérique implicite (1.48)-(1.50) est uniformément bornée par rapport aux paramètres de discrétisation dans $L^\infty\left([0,T];\mathbb{L}^2\right) \cap L^\infty\left([0,T];\mathbb{H}\right)$. De plus, l'approximation de la dérivée temporelle première est bornée dans $L^\infty\left([0,T];\mathbb{L}^2\right)$ et la fonction Ξ_τ est également bornée dans $L^\infty\left([0,T];\mathbb{R}^3\right)$. Par conséquent, l'estimation suivante est satisfaite:*

$$\|\delta_t U_{\mathcal{T}}^m\|^2 + \|U_{\mathcal{T}}^m\|^2 + \frac{1}{2}\|U_{\mathcal{T}}^m\|_{\mathbb{H},\mathcal{T}}^2 + \frac{1}{2}\|\Xi^m\|_{\mathbb{R}^3}^2 \leq C, \qquad \textit{pour} \qquad m = 1,\ldots,\mathcal{N},$$

où C est une constante strictement positive indépendante des paramètres de discrétisation en espace et en temps.

Une preuve détaillée de la convergence de la solution approximative vers la solution faible est développée à mesure que la taille de la maille et le pas de temps tendent vers zéro. Cela est résumé dans le théorème de convergence suivant:

Théorème 1.2.10. *Supposons que* ***Hypothèse (H2)*** *soit vérifié. Pour $m \in \mathbb{N}$, soit $\mathcal{T}_m$ une maille admissible décrite au sens de la Définition 1.2.2. Soit τ_m un pas de temps satisfaisant* ***Hypothèse (TA)****. Soit $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m}, \eta_{\tau_m}, \xi_{\tau_m}, \zeta_{\tau_m})$ la solution du système décrit par (1.48)-(1.50). Supposons que la taille de $\mathcal{T}_m$ tende vers zéro et que τ_m tende vers zéro lorsque $m \to \infty$. Alors, il existe une sous-suite de la suite des solutions approximatives, toujours notée $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m}, \eta_{\tau_m}, \xi_{\tau_m}, \zeta_{\tau_m})$, qui converge, dans la topologie faible-$\star$, vers la solution faible (u, w, η, ξ, ζ) dans $L^\infty([0,T]; L^2(0,L)) \times L^\infty([0,T]; L^2(-L,0)) \times (L^\infty([0,T]))^3$ du problème (1.40)-(1.43), lorsque m tend vers ∞.*

En outre, nous étudions le comportement asymptotique de l'énergie du système (1.11)–(1.14) avec un seul contrôle dynamique au bord η, en obtenant une décroissance polynomiale optimale de l'énergie de type $1/\sqrt{t}$ pour des données initiales régulières. Toutefois, la démonstration complète n'est pas incluse dans ce chapitre, car sa nature technique et sa longueur détourneraient l'attention de l'objectif principal. Des exemples numériques confirment ce taux de décroissance et valident les résultats théoriques à travers une représentation graphique.

Enfin, dans le chapitre 6, nous résumons nos conclusions et fournissons des perspectives pour les développements futurs.

Chapter 2

Preliminaries

This chapter reviews basic concepts and results in semigroup theory and numerical analysis that are essential for the proofs in the subsequent chapters. For ease of reading, we collect theorems and definitions that will be used throughout this work. In the context of semigroup theory, we present well-known results related to strong, exponential, and polynomial stability of a C_0-semigroup. For numerical analysis, we include results concerning the Finite Volume Method that will be used to demonstrate our main findings. Most of the theorems are stated without proof, but relevant references are provided. Readers may skip this chapter initially and refer to it as needed. For more details and proofs, we refer to [7, 8, 12, 14, 17, 28, 29, 35, 38, 39, 44, 45].

2.1 Bounded and Unbounded Linear Operators

In this section, we list some important results about bounded and unbounded operators (see [14, 29]).

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two Banach spaces over $\mathbb{C}$ and H be a Hilbert space endowed with the scalar product $(\cdot,\cdot)_H$ and the induced norm $\|\cdot\|_H$.

A linear operator $T : E \longmapsto F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall\ u,\ v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 2.1.1. *A linear operator $T : E \to F$ is said to be bounded if there exists $C \geq 0$ such that*

$$\|Tu\|_F \leq C\|u\|_E, \quad \forall\ u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$.

Definition 2.1.2. *A bounded operator $T \in \mathcal{L}(E, F)$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}} \in E$ such that $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in F.*

The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity, one writes $\mathcal{K}(E) = \mathcal{K}(E, E)$.

Definition 2.1.3. *Let $T \in \mathcal{L}(E, F)$.*

- *We define the range of T by*

$$\mathcal{R}(T) = \{Tu :\ u \in E\} \subset F.$$

- *We define the Kernel of* T *by*

$$\ker(T) = \{u \in E : \ Tu = 0\} \subset E.$$

Theorem 2.1.1. *(Fredholm alternative [14, Theorem 6.6]) If* $T \in \mathcal{K}(E)$*, then*

- $\dim(\ker(I-T)) < \infty$*, where* dim *is the dimension of the space and* I *is the identity operator on* E*.*
- $\mathcal{R}(I-T)$ *is closed.*
- $\ker(I-T) = 0 \Leftrightarrow \mathcal{R}(I-T) = E$*.*

Definition 2.1.4. *An unbounded linear operator* T *from* E *into* F *is a pair* $(T, D(T))$ *presented by the following linear transformation, with* $D(T) \subset E$ *being a subspace called in particular the domain of* T

$$T : D(T) \subset E \longmapsto F.$$

Definition 2.1.5. *Let* $T : D(T) \subset E \longmapsto F$ *be an unbounded linear operator.*

- *We define the range of* T *by*

$$\mathcal{R}(T) = \{Tu : \ u \in D(T)\} \subset F.$$

- *We define the kernel of* T *by*

$$\ker(T) = \{u \in D(T) : \ Tu = 0\} \subset E.$$

- *We define the graph of* T *by*

$$\mathrm{G}(T) = \{(u, Tu) : \ u \in D(T)\} \subset E \times F.$$

Definition 2.1.6. *A map* T *is said to be closed if* $\mathrm{G}(T)$ *is closed in* $E \times F$*. The closedness of an unbounded linear operator* T *can be characterized as follows*

$$\text{if } u_n \in D(T) \text{ such that } u_n \to u \text{ in } E \text{ and } Tu_n \to v \text{ in } F, \text{ then } u \in D(T) \text{ and } Tu = v.$$

Definition 2.1.7. *Let* $T : D(T) \subset E \longmapsto F$ *be a closed unbounded linear operator.*

- *The resolvent set of* T *is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \ \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

- *The resolvent of* T *is defined by*

$$R(\lambda, T) = \left\{(\lambda I - T)^{-1} : \ \lambda \in \rho(T)\right\}.$$

- *The spectrum set of* T *is the complement of the resolvent set in* $\mathbb{C}$*, denoted by*

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

Definition 2.1.8. *Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator. We can split the spectrum $\sigma(T)$ of T into three disjoint sets: the ponctuel spectrum denoted by $\sigma_p(T)$, the continuous spectrum denoted by $\sigma_c(T)$ and the residual spectrum denoted by $\sigma_r(T)$, where these sets are defined as follows:*

- $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ \ker(\lambda I - T) \neq \{0\}\}$; *in this case λ is called an eigenvalue of T.*
- $\sigma_c(T) = \left\{\lambda \in \mathbb{C} : \ \ker(\lambda I - T) = 0, \ \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded}\right\}$.
- $\sigma_r(T) = \{\lambda \in \mathbb{C} : \ \ker(\lambda I - T) = 0 \text{ and } \mathcal{R}(\lambda I - T) \text{ is not dense in } F\}$.

Definition 2.1.9. *Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator and let λ be an eigenvalue of T. A non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0.$$

Moreover, if $n = 1$, then e is called an eigenvector.

Definition 2.1.10. *Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exists $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 2.1.2. *[29, Theorem 6.7] Let $(T, D(T))$ be a closed unbounded linear operator on H, then $\rho(T)$ is an open set of $\mathbb{C}$.*

Definition 2.1.11. *An unbounded operator $(T, D(T))$ on H is*

- *Positive if $(Tu, u)_H \geq 0, \quad \forall\ u \in D(T)$.*
- *Coercive if $\exists\ C > 0$, such that $(Tu, u)_H \geq C\|u\|_H^2, \quad \forall\ u \in D(T)$.*

2.2 Semigroups, Existence and Uniqueness of Solution

In this section, we present a few fundamental concepts concerning the semigroups. The vast majority of evolution equations can be reduced to the form

$$\begin{cases} U_t(x,t) &= AU(x,t), \quad t > 0, \\ U(x,0) &= U_0(x), \end{cases} \tag{2.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Let's start by basic definitions and theorems (see [14, 39, 44]).

Let $(X, \|\cdot\|_X)$ be a Banach space and H be a Hilbert space equipped with the inner product $(\cdot,\cdot)_H$ and the induced norm $\|\cdot\|_H$.

Definition 2.2.1. *A family $(S(t))_{t\geq 0}$ of bounded linear operators in X is called a strongly continuous semigroup (C_0-semigroup in short) if*

- $S(0) = I$ *(I is the identity operator on X).*
- $S(t+s) = S(t)S(s), \quad \forall\ t, s \geq 0.$
- *For each $u \in X$, $S(t)u$ is continuous in t on $[0, +\infty[$.*

We can denote $S(t)$ by e^{tA}.

Definition 2.2.2. *For a semigroup $(S(t))_{t\geq 0}$, we define a linear operator A with domain $D(A)$ consisting of point u such that the limit*

$$Au := \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad u \in D(A)$$

exists. Then, A is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

Proposition 2.2.1 (Theorem 2.2 [44])**.** *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup in X. Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

If $\omega = 0$, then the corresponding semigroup is uniformly bounded. Moreover, if $M = 1$, then $(S(t))_{t\geq 0}$ is said to be a C_0-semigroup of contractions.

Definition 2.2.3. *An unbounded linear operator $(A, D(A))$ on X is said to be dissipative if*

$$\|(\lambda I - A)x\|_X \geq \lambda \|x\|_X, \quad \forall\, x \in D(A) \text{ and } \forall\, \lambda > 0.$$

Proposition 2.2.2. *Let $(A, D(A))$ be an unbounded linear operator on H, then*

$$A \text{ is dissipative if and only if } \Re(Ax, x)_H \leq 0, \quad \forall\, x \in D(A).$$

Definition 2.2.4. *An unbounded linear operator $(A, D(A))$ on X is said to be maximal dissipative (m-dissipative) if*

- *A is a dissipative operator.*
- *$\exists\ \lambda_0 > 0$ such that $\mathcal{R}(\lambda_0 I - A) = X$.*

Theorem 2.2.1. *[44, Theorem 4.5] Let A be an m-dissipative operator, then*

- *$\mathcal{R}(\lambda I - A) = X, \quad \forall\ \lambda > 0.$*
- *$]0, \infty[\subseteq \rho(A)$.*

Theorem 2.2.2 (Hille-Yosida Theorem 3.1 [44])**.** *An unbounded linear operator $(A, D(A))$ on X is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ if and only if*

- *A is closed and $\overline{D(A)} = X$.*
- *The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{R}^+$ and for all $\lambda > 0$,*

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \lambda^{-1}.$$

Theorem 2.2.3 (Lummer-Phillips Theorem 4.3 [44])**.** *Let $(A, D(A))$ be an unbounded linear operator on X with dense domain $D(A)$ in X. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is an m-dissipative operator.*

Theorem 2.2.4 (Theorem 4.6 [44])**.** *Let $(A, D(A))$ be an unbounded linear operator on X. If A is dissipative with $\mathcal{R}(I - A) = X$ and X is reflexive, then $\overline{D(A)} = X$.*

Corollary 2.2.1. *Let $(A, D(A))$ be an unbounded linear operator on H. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is m-dissipative operator.*

Theorem 2.2.5 (Theorem 1.2.4 [39]). *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.*

Consequently, A is maximal dissipative operator on a Hilbert space H if and only if it generates a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on H. Then the existence of solution is justified by the following corollary which follows from Hille-Yosida Theorem.

Corollary 2.2.2 (Hille-Yosida Theorem 7.4 [14]). *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$.*

1. *For $U_0 \in D(A)$, problem (2.1) admits a unique strong solution*

$$U(t) = S(t)U_0 \in C^0(\mathbb{R}_+; D(A)) \cap C^1(\mathbb{R}_+; H).$$

2. *For $U_0 \in H$, problem (2.1) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}_+; H).$$

2.3 Stability of a C_0-Semigroup

In this section, we list some definitions and theorems about strong, exponential and polynomial stability of a C_0-semigroup. For more details, see [7, 8, 12, 28, 35, 38, 39, 45].

Let $(X, \|\cdot\|_X)$ be a Banach space and H be a Hilbert space equipped with the inner product $(\cdot,\cdot)_H$ and the induced norm $\|\cdot\|_H$.

Definition 2.3.1. *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is:*

- *Strongly stable if*
$$\lim_{t\to+\infty} \|S(t)u\|_X = 0, \quad \forall\ u \in X.$$
- *Uniformly stable if*
$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$
- *Exponentially stable if there exist two positive constants M and ϵ such that*
$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X, \quad \forall\ t > 0,\ \forall\ u \in X.$$
- *Polynomially stable if there exist two positive constants C and α such that*
$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_{D(A)}, \quad \forall\ t > 0,\ \forall\ u \in D(A).$$

Proposition 2.3.1. *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. The following statements are equivalent*

- *$(S(t))_{t\geq 0}$ is uniformly stable.*

- $(S(t))_{t\geq 0}$ *is exponentially stable.*

Now, we look for the necessary conditions of strong stability of a C_0-semigroup. To this end, we will recall two methods. The first result was obtained by Arendt and Batty in [7].

Theorem 2.3.1 (Arendt and Batty in [7]). *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If*

- *A has no pure imaginary eigenvalues*
- *$\sigma(A)\cap i\mathbb{R}$ is countable*

then, $S(t)$ is strongly stable.

Remark 2.3.1. *If the resolvent $(I-A)^{-1}$ of A is compact, then $\sigma(A)=\sigma_p(A)$; i.e., the spectrum of A becomes exactly the set of eigenvalues of A. Thus, the state of Theorem 2.3.1 reduces to $\sigma_p(\mathcal{A})\cap i\mathbb{R}=\emptyset$.*

The second one is a classical method based on Arendt and Batty theorem and the contradiction argument ([39, page 25]).

Lemma 2.3.1 (Liu and Zheng page 25 [39]). *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a Hilbert space H. Furthermore, assume that $0\in\rho(A)$. If $i\mathbb{R}\not\subseteq\rho(A)$, then there exists $\{(\lambda_n,U_n)\}_{n\geq 1}\subset\mathbb{R}\times D(A)$, with $\lambda_n\to\omega$ as $n\to\infty$, $|\lambda_n|<|\omega|$ and $\|U_n\|_H=1$, such that*

$$(i\lambda_n I-A)U_n\to 0 \text{ in } H, \qquad \text{as } n\to\infty.$$

We will check condition $i\mathbb{R}\subset\rho(A)$ by finding a contradiction with $\|U_n\|_H=1$ such as $\|U_n\|_H=o(1)$.

Next, after showing that the C_0-semigroup is strongly stable, we aim to look for the necessary and sufficient conditions of exponential stability of a C_0-semigroup. In fact, the results of exponential stability are obtained by using several methods like: multiplier method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them. In this thesis, we will recall only the frequency domain approach method which was obtained by Huang-Prüss in [28, 45].

Theorem 2.3.2 (Huang-Prüss in [28, 45]). *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if*

- $i\mathbb{R}\subset\rho(A)$.
- $\sup_{\beta\in\mathbb{R}}\|(i\beta I-A)^{-1}\|_{\mathcal{L}(H)}<+\infty$.

In the case that the C_0-semigroup is not exponentially stable, we look for polynomial stability. In general, polynomial stability results are also obtained using different methods like: multiplier method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them. In this thesis, we will only mention the frequency domain approach method that was obtained by Batty and Duyckaerts in [8], Borichev and Tomilov in [12], Liu and Rao in [38].

Theorem 2.3.3 (Batty-Duyckaerts in [8], Borichev-Tomilov in [12], Liu-Rao in [38]). *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R}\subset\rho(A)$, then for a fixed $\ell>0$, the following conditions are equivalent*

1. $\limsup_{|\lambda|\to\infty}\frac{1}{|\lambda|^{\ell}}\left\|(i\lambda I-A)^{-1}\right\|_{\mathcal{L}(H)}<+\infty$.
2. $\|S(t)U_0\|_H\leq\frac{C}{t^{\frac{1}{\ell}}}\|U_0\|_{D(A)}\quad\forall\, t>0,\ U_0\in D(A),$ *for some* $C>0$.

2.4 Spatial Discretization and Results for the Finite Volume Method

This section focuses on spatial discretization for the finite volume method and presents results that will be used in the numerical analysis throughout this thesis (see [14, 17]).

Consider the interval $(0,1)$ and define a mesh $\mathcal{T}$ for this interval that satisfies the following assumptions:

Definition 2.4.1. *For $N \in \mathbb{N}^*$ and $h = \frac{1}{N}$, an admissible mesh of $(0,1)$, denoted by $\mathcal{T}$, is defined by a family of subintervals $(K_i)_{i=1,\dots,N}$ such that $K_i = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$ and a family $(x_i)_{i=0,\dots,N+1}$ such that:*

$$0 = x_0 = x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < \cdots < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < \cdots < x_{N-\frac{1}{2}} < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1,$$

where $x_{i+\frac{1}{2}} = ih$ for $i = 0, \dots, N$ and $x_i = \left(i - \frac{1}{2}\right) h$ for $i = 1, \dots, N$.

We set $h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = h$ for $i = 1, \dots, N$ and $h_{i+\frac{1}{2}} = x_{i+1} - x_i$ for $i = 0, \dots N$, which implies that

$$h_{i+\frac{1}{2}} = \begin{cases} \dfrac{h}{2}, & i = 0,\ N, \\ h, & i = 1, \dots, N-1. \end{cases}$$

Suppose u is a smooth function and $(u_i)_{i=1,\dots,N}$ represents the approximation of the mean value of u over K_i given by

$$u_i = \frac{1}{h_i} \int_{K_i} u(x)dx, \quad \text{for } i = 1, \dots, N.$$

Moreover, we denote the value of u at 0 as u_0 and the value of u at x_{N+1} as u_{N+1}. We proceed to define the difference operators $\delta_x u_{i+\frac{1}{2}}$, $\delta_x^2 u_i$ and $\delta_x^3 u_{i+\frac{1}{2}}$ as follows

$$\begin{aligned}
\delta_x u_{i+\frac{1}{2}} &= \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}, \quad \text{for } i = 0, \dots, N, \\
\delta_x^2 u_0 &= \frac{4}{3}\left(\frac{u_2 - 3u_1 + 2u_0}{h^2}\right), \\
\delta_x^2 u_i &= \frac{\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}}{h_i}, \quad \text{for } i = 1, \dots, N, \\
\delta_x^3 u_{\frac{1}{2}} &= \frac{2}{5}\left(\frac{3u_3 - 10u_2 + 15u_1 - 8u_0}{h^3}\right), \\
\delta_x^3 u_{i+\frac{1}{2}} &= \frac{\delta_x^2 u_{i+1} - \delta_x^2 u_i}{h_{i+\frac{1}{2}}}, \quad \text{for } i = 1, \dots, N-1.
\end{aligned}$$

According to Taylor expansion, it is easy to see that the above difference operators are of first order at the geometric center of stencils, i.e.,

$$\begin{aligned}
&\delta_x u_{i+\frac{1}{2}} = u_x(x_{i+\frac{1}{2}}) + \mathcal{O}(h), \qquad \delta_x u_{i+\frac{1}{2}} = u_x(x_{i+\frac{1}{2}}) + \mathcal{O}(h), \\
&\delta_x^2 u_i = u_{xx}(x_i) + \mathcal{O}(h), \qquad \delta_x^3 u_{i+\frac{1}{2}} = u_{xxx}(x_{i+\frac{1}{2}}) + \mathcal{O}(h).
\end{aligned}$$

We then define the set $\mathcal{X}(\mathcal{T})$ as follows

$$\mathcal{X}(\mathcal{T}) = \{u_{\mathcal{T}} : (0,1) \to \mathbb{R} \;\mid\; u_{\mathcal{T}}(x) = u_i, \;\text{ if }\; x \in K_i,\; i = 1, \dots, N\}\,.$$

For any $u_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$, we define the $L^2(0,1)$ norm as

$$\|u_{\mathcal{T}}\| = \left(\sum_{i=1}^{N} h_i (u_i)^2\right)^{\frac{1}{2}}.$$

Additionally, we introduce the respective discrete norms in $H^1_*(0,1)$ and $H^2_*(0,1)$ as follows

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}} = \left(\sum_{i=0}^{N} h_{i+\frac{1}{2}} (\delta_x u_{i+\frac{1}{2}})^2\right)^{\frac{1}{2}}$$

and

$$\|u_{\mathcal{T}}\|_{2,\mathcal{T}} = \left(\sum_{i=1}^{N} h_i (\delta_x^2 u_i)^2\right)^{\frac{1}{2}}.$$

Within the notations introduced above, we have the following lemmas, which are crucial for the proof of stability estimates and the convergence of the numerical scheme.

Lemma 2.4.1 (Discrete Norms Inequality)**.** *Let $u_{\mathcal{T}}$ be the discrete representation of u on the mesh $\mathcal{T}$. The following inequalities hold:*

1. *If $u \in L^2(0,1)$, then*
$$\|u_{\mathcal{T}}\| \le \|u\|_{L^2(0,1)}\,. \tag{2.2}$$
2. *If $u \in H^1(0,1)$, there exists a constant $C > 0$, independent of the space discretization parameter, such that*
$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}} \le C\,\|u_x\|_{L^2(0,1)}\,. \tag{2.3}$$
3. *If $u \in H^2(0,1)$, there exists a constant $C > 0$, independent of the space discretization parameter, such that*
$$\|u_{\mathcal{T}}\|_{2,\mathcal{T}} \le C\,\|u_{xx}\|_{L^2(0,1)}\,. \tag{2.4}$$

Proof.

1. Using the Cauchy-Schwarz inequality, we obtain
$$\|u_{\mathcal{T}}\|^2 = \sum_{i=1}^{N} \frac{1}{h_i}\left(\int_{K_i} u dx\right)^2 \le \sum_{i=1}^{N} \frac{1}{h_i}\left(\int_{K_i} 1^2_{K_i} dx\right)\left(\int_{K_i} u^2 dx\right) = \int_0^1 u^2 dx = \|u\|^2_{L^2(0,1)}\,.$$

2. For $u \in H^1(0,1)$, the norm can be expressed as
$$\begin{aligned}\|u_{\mathcal{T}}\|^2_{1,\mathcal{T}} &= \sum_{i=0}^{N} h_{i+\frac{1}{2}} (\delta_x u_{i+\frac{1}{2}})^2 \\ &= \sum_{i=0}^{N} \frac{1}{h_{i+\frac{1}{2}}} (u_{i+1} - u_i)^2 \\ &= \frac{2}{h}(u_1 - u_0)^2 + \frac{1}{h}\sum_{i=1}^{N-1} (u_{i+1} - u_i)^2 + \frac{2}{h}(u_{N+1} - u_N)^2.\end{aligned} \tag{2.5}$$

For $i = 1, \dots, N-1$, applying Cauchy-Schwarz inequality to the following term gives

$$\begin{aligned} |u_{i+1} - u_i|^2 &= \frac{1}{h^2} \left| \int_{K_i} u(x+h) - u(x) dx \right|^2 \\ &= \frac{1}{h^2} \left(\int_{K_i} \int_x^{x+h} |u_s(s)| \, ds dx \right)^2 \\ &\leq \int_{K_i} \int_x^{x+h} (u_s(s))^2 \, ds dx, \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{h} \sum_{i=1}^{N-1} (u_{i+1} - u_i)^2 &= \frac{1}{h} \sum_{i=1}^{N-1} \int_{K_i} \int_x^{x+h} (u_s(s))^2 \, ds dx \\ &\leq \frac{1}{h} \int_0^1 \left(\int_x^{x+h} (u_s(s))^2 \, ds \right) dx \\ &= \frac{1}{h} \int_0^1 \int_0^1 1_{[x,x+h[}(s) \, (u_s(s))^2 \, ds dx \\ &= \frac{1}{h} \int_0^1 \int_0^1 1_{[s-h,s[}(x) \, (u_s(s))^2 \, ds dx \\ &= \frac{1}{h} \int_0^1 \left(\int_0^1 1_{[s-h,s[}(x) dx \right) (u_s(s))^2 \, ds = \int_0^1 (u_s(s))^2 \, ds. \end{aligned} \tag{2.6}$$

For the first term, applying Cauchy-Schwarz inequality gives

$$|u_1 - u_0|^2 = \frac{1}{h^2} \left| \int_{K_1} u(x) - u(x_0) dx \right|^2 = \frac{1}{h^2} \left(\int_{K_1} \int_{x_0}^x |u_s(s)| \, ds dx \right)^2 \leq \int_{K_1} \int_{x_0}^x (u_s(s))^2 \, ds dx.$$

Since $(u_s(s))^2$ is non-negative and $(x_0, x) \subset (0,1)$, we deduce

$$\int_{x_0}^x (u_s(s))^2 \, ds \leq \int_0^1 (u_s(s))^2 \, ds,$$

for all $x \in K_1$, and therefore

$$\frac{2}{h}(u_1 - u_0)^2 \leq 2 \int_0^1 (u_s(s))^2 \, ds. \tag{2.7}$$

By similar computations, the third term on the right hand side of (2.5) satisfies

$$\frac{2}{h}(u_{N+1} - u_N)^2 \leq 2 \int_0^1 (u_s(s))^2 \, ds. \tag{2.8}$$

Substituting (2.6), (2.7) and (2.8) into (2.5), we obtain

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq 5 \|u_s\|_{L^2(0,1)}^2,$$

which, in turn, yields (2.3).

3. For $u \in H^2(0,1)$, the norm can be expressed as

$$\begin{aligned}\|u_{\mathcal{T}}\|_{2,\mathcal{T}}^2 &= \sum_{i=1}^{N} h_i(\delta_x^2 u_i)^2 \\ &= \frac{1}{h^3}(u_2 - 3u_1 + 2u_0)^2 + \frac{1}{h^3}\sum_{i=2}^{N-1}(u_{i+1} - 2u_i + u_{i-1})^2 + \frac{1}{h^3}(2u_{N+1} - 3u_N + u_{N-1})^2.\end{aligned} \tag{2.9}$$

For $i = 2, \ldots, N-1$, applying Cauchy-Schwarz inequality to the following term gives

$$\begin{aligned}|u_{i+1} - 2u_i + u_{i-1}|^2 &= \frac{1}{h^2}\left|\int_{K_i} u(x+h) - 2u(x) + u(x-h)dx\right|^2 \\ &= \frac{1}{h^2}\left|\int_{K_i}\left(\int_x^{x+h} u_s(s) - u_s(s-h)ds\right)dx\right|^2 \\ &= \frac{1}{h^2}\left(\int_{K_i}\int_x^{x+h}\int_{s-h}^{s} |u_{tt}(t)|\,dtdsdx\right)^2 \\ &\leq h\int_{K_i}\int_x^{x+h}\int_{s-h}^{s}(u_{tt}(t))^2\,dtdsdx,\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{h^3}\sum_{i=2}^{N-1}(u_{i+1} - 2u_i + u_{i-1})^2 &\leq \frac{1}{h^2}\sum_{i=2}^{N-1}\int_{K_i}\int_x^{x+h}\int_{s-h}^{s}(u_{tt}(t))^2\,dtdsdx \\ &\leq \frac{1}{h^2}\int_0^1\left(\int_x^{x+h}\left(\int_{s-h}^{s}(u_{tt}(t))^2\,dt\right)ds\right)dx \\ &= \frac{1}{h^2}\int_0^1\left(\int_0^1 1_{[x,x+h[}(s)\left(\int_0^1 1_{[s-h,s[}(t)\,(u_{tt}(t))^2\,dt\right)ds\right)dx \\ &= \frac{1}{h^2}\int_0^1\left(\int_0^1 1_{[x,x+h[}(s)\left(\int_0^1 1_{[t,t+h[}(s)\,(u_{tt}(t))^2\,dt\right)ds\right)dx \\ &= \frac{1}{h^2}\int_0^1\int_0^1\int_0^1 1_{[x,x+h[}(s)1_{[t,t+h[}(s)ds\,(u_{tt}(t))^2\,dtdx \\ &= \int_0^1 (u_{tt}(t))^2\,dt.\end{aligned} \tag{2.10}$$

On the other hand, we have

$$\begin{aligned}|u_2 - 3u_1 + 2u_0|^2 &= \frac{1}{h^2}\left|\int_{K_1} u(x+h) - 3u(x) + 2u(x_0)dx\right|^2 \\ &= \frac{1}{h^2}\left|\int_{K_1}\left(\int_x^{x+h} u_s(s)ds - 2\int_{x_0}^{x} u_s(s)ds\right)dx\right|^2.\end{aligned}$$

Extending u by $u(0)$ for $s \in (-\infty, x_0)$ and noting $u_s(s) = 0$ for $s \in (-\infty, x_0)$, we write

$$\begin{aligned}
|u_2 - 3u_1 + 2u_0|^2 &= \frac{1}{h^2}\left|\int_{K_1}\left(\int_x^{x+h} u_s(s)ds - \int_{x-h}^{x} u_s(s)ds - \int_{x_0}^{x} u_s(s)ds + \int_{x_0-h}^{x-h} u_s(s)ds\right)dx\right|^2 \\
&= \frac{1}{h^2}\left|\int_{K_1}\left(\int_x^{x+h}\int_{s-h}^{s} u_{tt}(t)dtds - \int_{x_0}^{x}\int_{s-h}^{s} u_{tt}(t)dtds\right)dx\right|^2 \\
&\leq \frac{2}{h^2}\left[\left(\int_{K_1}\int_x^{x+h}\int_{s-h}^{s}|u_{tt}(t)|\,dtdsdx\right)^2 + \left(\int_{K_1}\int_{x_0}^{x}\int_{s-h}^{s}|u_{tt}(t)|\,dtdsdx\right)^2\right].
\end{aligned}$$

Applying Cauchy-Schwarz inequality to the right-hand side gives

$$|u_2 - 3u_1 + 2u_0|^2 \leq 2h\left(\int_{K_1}\int_x^{x+h}\int_{s-h}^{s}(u_{tt}(t))^2\,dtdsdx + \int_{K_1}\int_{x_0}^{x}\int_{s-h}^{s}(u_{tt}(t))^2\,dtdsdx\right).$$

Now, since

$$\int_{K_1}\int_x^{x+h}\int_{s-h}^{s}(u_{tt}(t))^2\,dtdsdx \leq h^2\int_0^1(u_{tt}(t))^2\,dt,$$

and

$$\int_{K_1}\int_{x_0}^{x}\int_{x-h}^{x}(u_{tt}(t))^2\,dtdsdx \leq h^2\int_0^1(u_{tt}(t))^2\,dt,$$

we can conclude that

$$\frac{1}{h^3}(u_2 - 3u_1 + 2u_0)^2 \leq 4\int_0^1(u_{tt}(t))^2\,dt. \tag{2.11}$$

By a similar calculation, the third term on the right-hand side of (2.9) satisfies

$$\frac{1}{h^3}(2u_{N+1} - 3u_N + u_{N-1})^2 \leq 4\int_0^1(u_{tt}(t))^2\,dt. \tag{2.12}$$

Substituting (2.10), (2.11) and (2.12) into (2.9), we obtain

$$\|u_{\mathcal{T}}\|_{2,\mathcal{T}}^2 \leq 9\,\|u_{tt}\|_{L^2(0,1)}^2,$$

which, in turn, yields (2.4).

■

Remark 2.4.1. *The same result holds for non-uniform meshes and the proof can be adapted accordingly.*

Lemma 2.4.2 (Discrete Trace Inequality)**.** *Let u be a function defined on $(0,1)$. The following discrete trace inequalities hold:*

1. *If $u \in H^1(0,1)$, there exists a constant $C > 0$ such that*

$$|u_0| \leq C\,\|u\|_{H^1(0,1)}. \tag{2.13}$$

2. *If $u \in H^2(0,1)$, there exists a constant $C > 0$, independent of the space discretization parameter, such that*

$$\left|\delta_x u_{\frac{1}{2}}\right| \leq C\,\|u\|_{H^2(0,1)}. \tag{2.14}$$

Proof.

1. A similar result is proved in Lemma 9.9 of [14].
2. For $u \in H^2(0,1)$, we have

$$\left|\delta_x u_{\frac{1}{2}}\right| = \frac{2}{h}\left|u_1 - u_0\right| = \frac{2}{h^2}\left|\int_{K_1}\left(u(x) - u(x_0)\right)dx\right| \leq \frac{2}{h^2}\left(\int_{K_1}\int_{x_0}^{x}\left|u_s(s)\right|dsdx\right) \leq 2\left\|u_s\right\|_{L^\infty(0,1)}.$$

Moreover, since the injection $H^1(0,1) \subset C([0,1])$ is continuous (see [14]), there exists a constant $C > 0$ such that

$$\left\|u_s\right\|_{L^\infty(0,1)} \leq C\left\|u_s\right\|_{H^1(0,1)} \leq C\left\|u\right\|_{H^2(0,1)}.$$

By combining these inequalities, we obtain (2.14), which completes the proof.

■

Remark 2.4.2. *The same results hold when taking the trace of u on the right boundary of the interval. Specifically, there exists a constant $C > 0$ such that*

$$\left|u_{N+1}\right| \leq C\left\|u\right\|_{H^1(0,1)}, \quad \forall\ u \in H^1(0,1).$$

Additionally, there exists $C > 0$, independent of the space discretization parameter, such that

$$\left|\delta_x u_{N+\frac{1}{2}}\right| \leq C\left\|u\right\|_{H^2(0,1)}, \quad \forall\ u \in H^2(0,1).$$

Remark 2.4.3. *Using similar arguments, one can also state that: There exists a constant $C > 0$, independent of the space discretization parameter, such that*

$$\left|\delta_x^2 u_0\right| \leq C\left\|u\right\|_{H^3(0,1)}, \quad \forall\ u \in H^3(0,1).$$

Moreover, there exists a constant $C > 0$, independent of the space discretization parameter, such that

$$\left|\delta_x^3 u_{\frac{1}{2}}\right| \leq C\left\|u\right\|_{H^4(0,1)}, \quad \forall\ u \in H^4(0,1).$$

One of the tools used also is the Discrete Poincare' inequality (see Lemma 10.2 in [17]), which is stated as follows:

Lemma 2.4.3 (Discrete Poincare' inequality, Lemma 10.2 in [17])**.** *For any admissible mesh $\mathcal{T}$ described in the sense of Definition 2.4.1, and for all $u_\mathcal{T} \in \mathcal{X}(\mathcal{T})$, there exist strictly positive constants c_1, $c_2 \in \mathbb{R}_+$ such that the following inequalities hold:*

$$\left\|u_\mathcal{T}\right\|^2 \leq c_1\left\|u_\mathcal{T}\right\|_{1,\mathcal{T}}^2, \tag{2.15}$$

and

$$\left\|u_\mathcal{T}\right\|^2 \leq c_2\left\|u_\mathcal{T}\right\|_{2,\mathcal{T}}^2, \tag{2.16}$$

where $\|\cdot\|_{1,\mathcal{T}}$ denotes the discrete $H_*^1(0,1)$ norm and $\|\cdot\|_{2,\mathcal{T}}$ denotes the discrete $H_*^2(0,1)$ norm, as defined above.

Chapter 3

Stability for an Interface Transmission Problem of Wave-Plate Equations with Dynamical Boundary Controls

This chapter investigates a two-dimensional transmission model consisting of a wave equation and a Kirchhoff plate equation with dynamical boundary controls under geometric conditions. The two equations are coupled through transmission conditions along a steady interface between the domains in which the wave and plate equations evolve, respectively. Our primary concern is the stability analysis of the system, which has not appeared in the literature. For this aim, using a unique continuation theorem, the strong stability of the system is proved without any geometric condition and in the absence of compactness of the resolvent. Then, we show that our system lacks exponential (uniform) stability. However, we establish a polynomial energy decay estimate of type $1/t$ for smooth initial data using the frequency domain approach from semigroup theory, which combines a contradiction argument with the multiplier technique. This method leads to certain geometrical conditions concerning the wave's and the plate's domains.

3.1 Introduction

3.1.1 Presentation of the System

In this chapter, we investigate a wave-plate system coupled through transmission conditions along a steady interface between the domains in which the wave and plate equations evolve respectively with dynamical boundary controls. More precisely, let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary $\partial\Omega$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where Ω_i is a bounded set with Lipschitz boundary $\partial\Omega_i$, $i = 1, 2$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. We denote by I the interior of $\overline{\Omega}_1 \cap \overline{\Omega}_2$, which is called the interface between Ω_1 and Ω_2, and $\Gamma_i = \partial\Omega_i \backslash \overline{I}$ represents the exterior boundary of Ω_i with positive measure, $i = 1, 2$. We consider the following interface transmission wave-plate model:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega_1 \times \mathbb{R}_+^*, \\ w_{tt} + \Delta^2 w = 0, & \text{in } \Omega_2 \times \mathbb{R}_+^*, \\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I \times \mathbb{R}_+^*, \\ \partial_{\nu_1} u + \eta = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ \mathcal{B}_1 w + \xi = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*, \\ \mathcal{B}_2 w - \zeta = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*, \end{cases} \tag{3.1}$$

with the initial conditions:

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega_1,\quad \eta(x,0)=\eta_0(x),\quad x\in\Gamma_1, \tag{3.2}$$

$$w(x,0)=w_0(x),\quad w_t(x,0)=w_1(x),\quad x\in\Omega_2,\quad \xi(x,0)=\xi_0(x),\quad \zeta(x,0)=\zeta_0(x),\quad x\in\Gamma_2, \tag{3.3}$$

where η, ξ and ζ denote the dynamical boundary controls, $\nu_i=(\nu_{i1},\nu_{i2})$ is the unit outward normal vector along $\partial\Omega_i$, and $\tau_i=(-\nu_{i2},\nu_{i1})$ is the unit tangent vector along $\partial\Omega_i$, $i=1,2$. Now, let's proceed with defining the gradient of a function f as follows:

$$\nabla f=(f_{x_1},f_{x_2}),$$

where f_{x_1} and f_{x_2} represent the partial derivatives of f with respect to x_1 and x_2, respectively. Next, the Laplacian operator is defined as the divergence of the gradient of f, given by

$$\Delta f=\operatorname{div}(\nabla f)=f_{x_1x_1}+f_{x_2x_2},$$

where $f_{x_1x_1}$ and $f_{x_2x_2}$ are the second partial derivatives of f with respect to x_1 and x_2. Additionally, in this problem's context, we introduce the constant $\mu\in\left(0,\frac{1}{2}\right)$ as the Poisson coefficient. Moreover, we define the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ on $\partial\Omega_2$ as follows:

$$\mathcal{B}_1 f=\Delta f+(1-\mu)\mathcal{C}_1 f$$

and

$$\mathcal{B}_2 f=\partial_{\nu_2}\Delta f+(1-\mu)\partial_{\tau_2}\mathcal{C}_2 f,$$

where

$$\mathcal{C}_1 f=2\nu_{21}\nu_{22}f_{x_1x_2}-\nu_{21}^2 f_{x_2x_2}-\nu_{22}^2 f_{x_1x_1}\quad\text{and}\quad \mathcal{C}_2 f=(\nu_{21}^2-\nu_{22}^2)f_{x_1x_2}-\nu_{21}\nu_{22}\left(f_{x_1x_1}-f_{x_2x_2}\right).$$

The damping of the system is made via the indirect damping mechanism that involves the following first-order differential equations:

$$\begin{cases}\eta_t-u_t+\eta=0, & \text{on}\ \ \Gamma_1\times\mathbb{R}_+^*,\\ \xi_t-\partial_{\nu_2}w_t+\xi=0, & \text{on}\ \ \Gamma_2\times\mathbb{R}_+^*,\\ \zeta_t-w_t+\zeta=0, & \text{on}\ \ \Gamma_2\times\mathbb{R}_+^*.\end{cases} \tag{3.4}$$

Moreover, by trivial computation, we know that

$$\mathcal{C}_1 f=-\partial_{\tau_2}^2 f-\partial_{\tau_2}\nu_{22}f_{x_1}+\partial_{\tau_2}\nu_{21}f_{x_2}\quad\text{and}\quad \mathcal{C}_2 f=\partial_{\nu_2\tau_2}f-\partial_{\tau_2}\nu_{21}f_{x_1}-\partial_{\tau_2}\nu_{22}f_{x_2}. \tag{3.5}$$

Let u,η,w,ξ and ζ be smooth solutions of system (3.1)-(3.4). We define their associated energy by

$$E(t)=\frac{1}{2}\left\{\int_{\Omega_1}\left(|\nabla u|^2+|u_t|^2\right)dx+\int_{\Gamma_1}|\eta|^2 d\Gamma\right\}+\frac{1}{2}\left\{a(w,\overline{w})+\int_{\Omega_2}|w_t|^2dx+\int_{\Gamma_2}\left(|\xi|^2+|\zeta|^2\right)d\Gamma\right\},$$

where the sesquilinear form $a:H^2(\Omega_2)\times H^2(\Omega_2)\longmapsto\mathbb{C}$ is defined by

$$a(f,\overline{g})=\int_{\Omega_2}\left[f_{x_1x_1}\overline{g}_{x_1x_1}+f_{x_2x_2}\overline{g}_{x_2x_2}+\mu\left(f_{x_1x_1}\overline{g}_{x_2x_2}+f_{x_2x_2}\overline{g}_{x_1x_1}\right)+2(1-\mu)f_{x_1x_2}\overline{g}_{x_1x_2}\right]dx. \tag{3.6}$$

For further analysis, the following Green's formula (see [32]) is used in the study:

$$a(f,\overline{g})=\int_{\Omega_2}\Delta^2 f\overline{g}dx+\int_{\partial\Omega_2}\left(\mathcal{B}_1 f\partial_{\nu_2}\overline{g}-\mathcal{B}_2 f\overline{g}\right)d\Gamma,\quad\forall f\in H^4(\Omega_2),\ g\in H^2(\Omega_2). \tag{3.7}$$

Lemma 3.1.1. *Let $U = (u, u_t, \eta, w, w_t, \xi, \zeta)$ be a regular solution of system (3.1)-(3.4). Then, the energy $E(t)$ satisfies the following estimation*

$$E'(t) = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} \left(|\xi|^2 + |\zeta|^2\right) d\Gamma \leq 0.$$

Proof. Firstly, multiplying the first and second equations of (3.1) by $\overline{u}_t$ and $\overline{w}_t$ respectively, then using Green's formula in Ω_1 and Ω_2 and taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}\left\{\int_{\Omega_1}\left(|\nabla u|^2 + |u_t|^2\right)dx + a(w,\overline{w}) + \int_{\Omega_2}|w_t|^2 dx\right\} + \Re\left\{\int_{\Gamma_1}\eta\overline{u}_t d\Gamma + \int_{\Gamma_2}\left(\xi\partial_{\nu_2}\overline{w}_t + \zeta\overline{w}_t\right)d\Gamma\right\} = 0, \quad (3.8)$$

where $\Re$ stands for the real part of a complex number. Secondly, by inserting the equations (3.4) into the second term of (3.8), we get

$$\begin{aligned}\Re\left\{\int_{\Gamma_1}\eta\overline{u}_t d\Gamma + \int_{\Gamma_2}\left(\xi\partial_{\nu_2}\overline{w}_t + \zeta\overline{w}_t\right)d\Gamma\right\} = \frac{1}{2}\frac{d}{dt}\left\{\int_{\Gamma_1}|\eta|^2 d\Gamma + \int_{\Gamma_2}\left(|\xi|^2 + |\zeta|^2\right)d\Gamma\right\} \\ + \int_{\Gamma_1}|\eta|^2 d\Gamma + \int_{\Gamma_2}\left(|\xi|^2 + |\zeta|^2\right)d\Gamma.\end{aligned} \quad (3.9)$$

Combining equations (3.8) and (3.9), we obtain

$$E'(t) = -\int_{\Gamma_1}|\eta|^2 d\Gamma - \int_{\Gamma_2}\left(|\xi|^2 + |\zeta|^2\right)d\Gamma.$$

■

Thus, from Lemma 3.1.1, we deduce that the system (3.1)-(3.4) is dissipative in the sense that the energy $E(t)$ is non-increasing with respect to time variable t.

Remark 3.1.1. *Note that the dissipative mechanism of the system (3.1)-(3.4) remains constructed by a single dynamical boundary control of the wave or the plate.*

3.1.2 Motivation and Aims

Dynamical boundary controls refer to approaches in real-life applications of mathematical physics and engineering. Since Ö. Morgül proposed such a damping mechanism on the boundary of elastic beams (see, for instance, [41], [42] and [43]), many authors have been interested in studying similar problems in the context of plates (see [23], [47] and [49]). Recently in [48], B. Rao et al. considered the boundary stabilization of a wave equation by means of singular dynamical boundary control. In particular, they considered the following system on a given open bounded set Ω of $\mathbb{R}^N$ with boundary Γ of class C^2 divided into two disjoint parts Γ_0 and Γ_1 :

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times [0,T], \\ u = 0, & \text{on } \Gamma_0 \times [0,T], \\ \dfrac{\partial u}{\partial \nu} + \eta = 0, & \text{on } \Gamma_1 \times [0,T], \\ \eta_t - u_t = -\eta, & \text{on } \Gamma_1 \times [0,T] \end{cases} \quad (3.10)$$

with the initial condition:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1, \quad (3.11)$$

where η denotes the dynamical boundary control. The authors proved that the energy of the system (3.10)-(3.11) does not decay uniformly (exponentially) to zero. However, using a multiplier method, they showed the polynomial stability of the system with an energy decay rate of $1/t$ under certain geometrical conditions for all smooth initial data.

On the other hand, stimulated by many practical applications in the modeling and control of engineering, transmission problems have attracted considerable attention over the past several years (see [5], [21], [31], [58], [57], [27], [26], [62] and the references therein). For instance, K. Ammari and S. Nicaise in [6] considered the stabilization of a system coupling the wave equation with a Kirchhoff plate and damped through frictional boundary dissipation laws. Mainly, they deal with the following system:

$$\begin{cases} \partial_t^2 u_1(x,t) - \Delta u_1(x,t) = 0, & \text{in } \Omega_1 \times (0,+\infty), \\ \partial_t^2 u_2(x,t) + \Delta^2 u_2(x,t) = 0, & \text{in } \Omega_2 \times (0,+\infty), \\ u_i(x,0) = u_i^0(x), \quad \partial_t u_i(x,0) = u_i^1(x), & \text{in } \Omega_i, \quad i = 1,2, \\ u_1 = u_2, \quad \mathcal{B}_1 u_2 = 0, \quad \mathcal{B}_2 u_2 = \partial_{\nu_1} u_1, & \text{on } I \times (0,+\infty), \\ \partial_{\nu_1} u_1 = -\alpha_1 u_1 - \partial_t u_1, & \text{on } \Gamma_1 \times (0,+\infty), \\ \mathcal{B}_1 u_2 = -\beta \partial_{\nu_2} u_2 - \partial_{\nu_2} \partial_t u_2, & \text{on } \Gamma_2 \times (0,+\infty), \\ \mathcal{B}_2 u_2 = \alpha_2 u_2 + \partial_t u_2, & \text{on } \Gamma_2 \times (0,+\infty), \end{cases} \tag{3.12}$$

where Ω refers to a bounded domain in $\mathbb{R}^2$, as described in Subsection 3.1.1. The unit normal vector of $\partial\Omega_i$ for $i = 1, 2$ and the unit tangent vector along $\partial\Omega_i$ are as described in Subsection 3.1.1. The boundary operator $\mathcal{B}_j$ with $j = 1, 2$, is also defined on $\partial\Omega_2$ as mentioned above. Additionally, α_1, α_2, and β are three fixed positive constants. The authors proved, under certain geometric assumptions on the polygonal domains Ω_i, $i = 1, 2$, that the semi-group of the problem (3.12) is exponentially stable in the energy space when the boundary damping is of frictional type.

To the best of our knowledge, the stabilization problem of a transmission system of wave and Kirchhoff plate equations with dynamical boundary controls remains unexplored in the literature and remains an open problem. It is essential to acknowledge that much of the motivation for studying the stabilization of this system arises from its significance in engineering and its strong physical foundation. This approach is expected to solve various control challenges in flexible structures composed of two physically distinct materials, where dynamical control plays a key role. Consequently, we focus on the transmission wave-plate model (3.1)-(3.4) and the primary objective of this chapter is to estimate the decay rate of the energy for this system under specific geometric conditions.

Before going on, let us situate our work in comparison with previously published results in [6]. Significantly, our work differs from [6] in terms of the damping choices applied to the boundary. While [6] utilized frictional boundary damping, we employ dynamical boundary controls, introducing novel mathematical challenges and leading to distinct results. One of the most intriguing issues in the mathematical theory of (3.1)-(3.4) is the regularity of solutions, which we emphasize in this study. We have found that the proposed dynamical boundary conditions reduce the regularity of the solution compared to [6]. This reduction stems primarily from the lack of regularity in the boundary control $\partial_{\nu_1} u$ near the boundary Γ_1, posing significant challenges. The mathematical theory of polygonal domains is known to be challenging, and it becomes even more complex when dealing with solutions of poor regularity. Additionally, the non-compactness of the resolvent of the associated operator introduces further difficulties in establishing the stability of (3.1)-(3.4). Furthermore, the uniform decay of the natural energy observed in [6] does not hold in our case due to the implementation of

dynamical boundary controls, leading to a different decay behavior.

In part to answer these objections, we first prove the well-posedness of the system (3.1)-(3.4). Then, we impose sufficient geometric restrictions on the domains $\Omega_i, i = 1, 2$ (see Assumptions (A1) and (A2) below) to ensure the regularity of the solution to this system and guarantee the strict decay of the energy, i.e., $E(t)$ tends to zero as t goes to infinity. The key question we are interested in, under Assumptions (A1) and (A2), is the energy decay rate of the solution of (3.1)-(3.4). For this purpose, we prove the lack of exponential decay of the system and establish a polynomial energy decay estimate of type $1/t$ for smooth initial data, provided that the exterior boundaries Γ_1, Γ_2 and the interface I satisfy an additional geometrical assumption described below (see Assumption (A3)). The frequency domain approach is the main tool used for the proof of polynomial stability. More precisely, we combine a contradiction argument with a new multiplier technique to carry out a special analysis of the resolvent.

3.1.3 Literature

Dynamical controls are part of the indirect damping mechanisms proposed by Russell in [52], which do not arise from the insertion of damping terms into the original equations describing the mechanical motion, but by coupling those equations to further equations describing other processes in the structure. A significant number of papers dealing with the mathematical theory of dynamical boundary control in elastic structures have appeared. The emphasis is usually placed upon elastic beams modeled by the Euler-Bernoulli beam or Timoshenko beam, including [41], [42] and [43] in our list of references. Let us start now by recalling some previous studies related to Kirchhoff plates and wave equations with dynamical boundary controls, which are relevant to this study. In 2005, A. Wehbe and B. Rao considered a plate equation in an open bounded domain $\Omega \subset \mathbb{R}^2$ with dynamical controls on the boundary Γ with partition $\overline{\Gamma} = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where the dynamical controls are applied on Γ_1 (see [49]). They proved that the energy decays polynomially with the rate $1/t$ for all smooth initial data, assuming that the boundary Γ satisfies the Multiplier Geometric Control Condition (MGC). This condition requires the existence of $\delta > 0$ such that $m(x) \cdot \nu \geq \delta^{-1}$ for all $x \in \Gamma_1$ and $m(x) \cdot \nu \leq 0$ for all $x \in \Gamma_0$, where $m(x) = x - x_0$ for x_0 fixed in $\mathbb{R}^2$. In 2015, B. Rao et al. considered the stabilization of a wave equation by means of dynamical boundary controls (see [48]). They proved that a singular dynamical control applied to a portion of the boundary is sufficient to polynomially stabilize the wave equation with a decay rate $1/t$ under the MGC geometrical condition. Finally, in [4], M. Akil et al. studied the stabilization of a Kirchhoff plate equation with time delay added to the dynamical boundary controls. They found that the energy of the system decays polynomially with a decay rate of type $1/t$, provided that an appropriate condition on the delay terms is satisfied.

On the other hand, transmission systems are mathematical models that arise most naturally in the description of structures that are partly composed of interactive or interconnected materials. Transmission systems can be found in many practical applications, such as spacecraft [10], satellite antennas [40], road traffic [34], and many other interactive physical processes. Such problems have also received attention and there have been fruitful results concerning the control design and stability analysis of the solutions to different types of problems (see, for instance, [65], [61], [5], [6], [37] and [24]).

3.1.4 Organization of the Chapter

This chapter is organized as follows: In Section 3.2, we formulate system (3.1)-(3.4) into a first-order evolution equation and then deduce the well-posedness property by using semigroup theory. Some

regularity results needed in the following sections are rigorously investigated in Section 3.3. In Section 3.4, a general criteria of Arendt-Batty theorem is used to prove the strong stability of the system in the absence of compactness of the resolvent. We show the lack of exponential stability of (3.1)-(3.4) in Section 3.5. Finally, Section 3.6 is devoted to study the energy decay rate under certain geometric conditions given by Theorem 3.6.1. We prove that the energy of our system has a polynomial decay rate of type $1/t$.

Let us finish this introduction by introducing some notations that will be used throughout this chapter: The notation $a \lesssim b$ signifies that there exists a constant $\mathcal{C} > 0$, independent of a, b, and the natural parameter n, such that $a \leq \mathcal{C}b$.

3.2 Well-Posedness of the Problem

This section is devoted to study the well-posedness property of system (3.1)-(3.4) using the semigroup approach. We first introduce the following spaces:

$$H^1_*(\Omega_1) = \left\{ u \in H^1(\Omega_1) \mid \int_{\Omega_1} u dx = 0 \right\},$$

$$H^2_*(\Omega_2) = \left\{ w \in H^2(\Omega_2) \mid \int_{\Omega_2} w dx = \int_{\Omega_2} \nabla w dx = 0 \right\}$$

and the energy space $\mathcal{H}$ by

$$\mathcal{H} = \left\{ (u, v, \eta, w, z, \xi, \zeta) \in H^1_*(\Omega_1) \times L^2(\Omega_1) \times L^2(\Gamma_1) \times H^2_*(\Omega_2) \times L^2(\Omega_2) \times L^2(\Gamma_2) \times L^2(\Gamma_2) \mid \ u = w \ \text{ on } \ I \right\},$$

which is endowed with the following usual inner product

$$\left(U, \tilde{U}\right)_{\mathcal{H}} = \int_{\Omega_1} \left(\nabla u \cdot \nabla \overline{\tilde{u}} + v\overline{\tilde{v}}\right) dx + \int_{\Gamma_1} \eta\overline{\tilde{\eta}} d\Gamma + a(w, \overline{\tilde{w}}) + \int_{\Omega_2} z\overline{\tilde{z}} dx + \int_{\Gamma_2} \left(\xi\overline{\tilde{\xi}} + \zeta\overline{\tilde{\zeta}}\right) d\Gamma, \tag{3.13}$$

where $U = (u, v, \eta, w, z, \xi, \zeta)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{w}, \tilde{z}, \tilde{\xi}, \tilde{\zeta}) \in \mathcal{H}$.

Remark 3.2.1. *The condition*

$$\int_{\Omega_2} \nabla w \, dx = 0,$$

signifies that both

$$\int_{\Omega_2} w_{x_1} \, dx = 0 \quad \text{and} \quad \int_{\Omega_2} w_{x_2} \, dx = 0.$$

Remark 3.2.2. *By considering the conditions of the function spaces $H^1_*(\Omega_1)$ and $H^2_*(\Omega_2)$, it follows that $(\cdot,\cdot)_{\mathcal{H}}$ is a scalar product in the space $\mathcal{H}$. Consequently, this observation confirms that $\mathcal{H}$ is a Hilbert space, a fundamental prerequisite for the application of Theorem 2.3.3.*

We next define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, \eta, w, z, \xi, \zeta) \in \mathcal{H} \mid \Delta u \in L^2(\Omega_1), \quad v \in H^1_*(\Omega_1), \quad \Delta^2 w \in L^2(\Omega_2), \quad z \in H^2_*(\Omega_2), \\ \partial_{\nu_1} u + \eta = 0 \ \text{ on } \ \Gamma_1, \quad \mathcal{B}_1 w + \xi = 0 \ \text{ on } \ \Gamma_2, \quad \mathcal{B}_2 w - \zeta = 0 \ \text{ on } \ \Gamma_2, \\ v = z, \quad \mathcal{B}_1 w = 0 \ \text{ and } \ \mathcal{B}_2 w = \partial_{\nu_1} u \ \text{ on } \ I \end{array} \right\} \tag{3.14}$$

and

$$\mathcal{A}U = \big(v, \Delta u, \gamma_1(v) - \eta, z, -\Delta^2 w, \gamma_{2,2}(z) - \xi, \gamma_{2,1}(z) - \zeta\big), \quad \forall U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A}), \tag{3.15}$$

where $\gamma_1 : H^1(\Omega_1) \longmapsto L^2(\Gamma_1)$ and $\gamma_2 : H^2(\Omega_2) \longmapsto L^2(\Gamma_2) \times L^2(\Gamma_2)$ are the trace operators such that $\gamma_1(\varphi) = \varphi|_{\Gamma_1}$ and $\gamma_2(\psi) = (\gamma_{2,1}(\psi), \gamma_{2,2}(\psi)) = (\psi|_{\Gamma_2}, \partial_{\nu_2}\psi|_{\Gamma_2})$. Now, setting $U = (u, u_t, \eta, w, w_t, \xi, \zeta)$ as the state of system (3.1)-(3.4), we rewrite the problem into a first-order evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{3.16}$$

where the initial value $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \xi_0, \zeta_0) \in \mathcal{H}$ is given by the equations (3.2) and (3.3).

Remark 3.2.3. *From (3.6) and the fact that* $2\Re\left(w_{x_1x_1}\overline{w}_{x_2x_2}\right) = |w_{x_1x_1} + w_{x_2x_2}|^2 - |w_{x_1x_1}|^2 - |w_{x_2x_2}|^2$, *we remark that*

$$\begin{aligned} a(w, \overline{w}) &= \int_{\Omega_2} \left[|w_{x_1x_1}|^2 + |w_{x_2x_2}|^2 + \mu\left(w_{x_1x_1}\overline{w}_{x_2x_2} + w_{x_2x_2}\overline{w}_{x_1x_1}\right) + 2(1-\mu)|w_{x_1x_2}|^2\right] dx \\ &= \int_{\Omega_2} \left[|w_{x_1x_1}|^2 + |w_{x_2x_2}|^2 + 2\mu\Re\left(w_{x_1x_1}\overline{w}_{x_2x_2}\right) + 2(1-\mu)|w_{x_1x_2}|^2\right] dx \\ &= \int_{\Omega_2} \left[(1-\mu)|w_{x_1x_1}|^2 + (1-\mu)|w_{x_2x_2}|^2 + \mu|w_{x_1x_1} + w_{x_2x_2}|^2 + 2(1-\mu)|w_{x_1x_2}|^2\right] dx \geq 0. \end{aligned}$$

For further purposes, set the Hilbert space H by

$$\mathsf{H} = \big\{(f, g) \in H^1_*(\Omega_1) \times H^2_*(\Omega_2) \mid f = g \quad \text{on} \quad I\big\},$$

equipped with the norm

$$\|(f, g)\|^2_{\mathsf{H}} = \|\nabla f\|^2_{L^2(\Omega_1)} + a(g, \overline{g}). \tag{3.17}$$

Note that using the compactness injection from $H^1(\Omega_1)$ into $L^2(\Omega_1)$ and from $H^2(\Omega_2)$ into $H^1(\Omega_2)$, we can easily show that $\|\nabla f\|^2_{L^2(\Omega_1)}$ is equivalent to the usual norm of $H^1(\Omega_1)$ on $H^1_*(\Omega_1)$ and $a(g, \overline{g})$ is equivalent to the usual norm of $H^2(\Omega_2)$ on $H^2_*(\Omega_2)$.

Proposition 3.2.1. *The unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof. First, let $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$. Using (3.13) and (3.15), we have

$$\begin{aligned} \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = \Re\Bigg\{ &\int_{\Omega_1} \nabla v \cdot \nabla \overline{u} dx + \int_{\Omega_1} \Delta u \overline{v} dx + \int_{\Gamma_1} (v - \eta)\overline{\eta} d\Gamma + a(z, \overline{w}) - \int_{\Omega_2} \Delta^2 w \overline{z} dx \\ &+ \int_{\Gamma_2} (\partial_{\nu_2} z - \xi)\overline{\xi} d\Gamma + \int_{\Gamma_2} (z - \zeta)\overline{\zeta} d\Gamma \Bigg\}. \end{aligned}$$

We can proceed by integrating by parts and utilizing Green's formula (3.7). Given that $U \in D(\mathcal{A})$, we have

$$\Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} |\xi|^2 d\Gamma - \int_{\Gamma_2} |\zeta|^2 d\Gamma \leq 0, \tag{3.18}$$

which implies that $\mathcal{A}$ is dissipative. Now, let us prove that $\mathcal{A}$ is maximal. For this aim, let $F = (f^1, g^1, h^1, f^2, g^2, h^2, h^3) \in \mathcal{H}$, we look for $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$ unique solution of

$$U - \mathcal{A}U = F. \tag{3.19}$$

Equivalently, we have the following system

$$\begin{aligned} u - v &= f^1, \quad \text{in } \Omega_1, && (3.20)\\ v - \Delta u &= g^1, \quad \text{in } \Omega_1, && (3.21)\\ \eta - v + \eta &= h^1, \quad \text{on } \Gamma_1, && (3.22)\\ w - z &= f^2, \quad \text{in } \Omega_2, && (3.23)\\ z + \Delta^2 w &= g^2, \quad \text{in } \Omega_2, && (3.24)\\ \xi - \partial_{\nu_2} z + \xi &= h^2, \quad \text{on } \Gamma_2, && (3.25)\\ \zeta - z + \zeta &= h^3, \quad \text{on } \Gamma_2, && (3.26) \end{aligned}$$

with the following transmission and boundary conditions

$$\begin{cases} u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I,\\ \partial_{\nu_1} u + \eta = 0, & \text{on } \Gamma_1,\\ \mathcal{B}_1 w + \xi = 0, & \text{on } \Gamma_2,\\ \mathcal{B}_2 w - \zeta = 0, & \text{on } \Gamma_2. \end{cases} \tag{3.27}$$

It follows from (3.20), (3.22), (3.23), (3.25) and (3.26) that

$$\begin{gathered} v = u - f^1 \quad \text{in } \Omega_1, \quad 2\eta = v + h^1 \quad \text{on } \Gamma_1,\\ z = w - f^2 \quad \text{in } \Omega_2, \quad 2\xi = \partial_{\nu_2} z + h^2 \quad \text{on } \Gamma_2, \quad \text{and} \quad 2\zeta = z + h^3 \quad \text{on } \Gamma_2. \end{gathered} \tag{3.28}$$

By elimination of v, z, η, ξ and ζ in (3.21), (3.24) and (3.27), we find that u and w satisfy the following system:

$$\begin{cases} u - \Delta u = f^1 + g^1, & \text{in } \Omega_1,\\ w + \Delta^2 w = f^2 + g^2, & \text{in } \Omega_2,\\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I,\\ 2\partial_{\nu_1} u + u = f^1 - h^1, & \text{on } \Gamma_1,\\ 2\mathcal{B}_1 w + \partial_{\nu_2} w = \partial_{\nu_2} f^2 - h^2, & \text{on } \Gamma_2,\\ 2\mathcal{B}_2 w - w = h^3 - f^2, & \text{on } \Gamma_2. \end{cases} \tag{3.29}$$

Let $\phi = (\varphi, \psi) \in \mathsf{H}$. Multiplying the first equation of (3.29) by $\overline{\varphi}$ and integrating over Ω_1, multiplying the second equation of (3.29) by $\overline{\psi}$ and integrating over Ω_2, then using Green's formula, we get

$$\int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + \frac{1}{2}\int_{\Gamma_1} u\overline{\varphi} d\Gamma - \int_I \partial_{\nu_1} u \overline{\varphi} d\Gamma = \int_{\Omega_1} F^1 \overline{\varphi} dx + \frac{1}{2}\int_{\Gamma_1} F^2 \overline{\varphi} d\Gamma, \tag{3.30}$$

$$\begin{aligned} &\int_{\Omega_2} w\overline{\psi} dx + a(w, \overline{\psi}) + \frac{1}{2}\int_{\Gamma_2} \left(\partial_{\nu_2} w \partial_{\nu_2} \overline{\psi} + w\overline{\psi}\right) d\Gamma - \int_I \left(\mathcal{B}_1 w \partial_{\nu_2} \overline{\psi} - \mathcal{B}_2 w \overline{\psi}\right) d\Gamma\\ &\quad = \int_{\Omega_2} F^3 \overline{\psi} dx + \frac{1}{2}\int_{\Gamma_2} F^4 \partial_{\nu_2} \overline{\psi} d\Gamma + \frac{1}{2}\int_{\Gamma_2} F^5 \overline{\psi} d\Gamma, \end{aligned} \tag{3.31}$$

where $F^1 = f^1 + g^1$, $F^2 = f^1 - h^1$, $F^3 = f^2 + g^2$, $F^4 = \partial_{\nu_2} f^2 - h^2$ and $F^5 = f^2 - h^3$. Adding the resulting equations and using the transmission conditions, we obtain a variational formulation of (3.29): Find $(u, w) \in \mathsf{H}$ such that

$$b((u, w), (\varphi, \psi)) = l(\varphi, \psi), \qquad \forall (\varphi, \psi) \in \mathsf{H}, \tag{3.32}$$

where

$$b((u,w),(\varphi,\psi)) = \int_{\Omega_1} u\overline{\varphi}dx + \int_{\Omega_1} \nabla u\cdot\nabla\overline{\varphi}dx + \frac{1}{2}\int_{\Gamma_1} u\overline{\varphi}d\Gamma + \int_{\Omega_2} w\overline{\psi}dx + a(w,\overline{\psi}) + \frac{1}{2}\int_{\Gamma_2}\left(\partial_{\nu_2}w\partial_{\nu_2}\overline{\psi} + w\overline{\psi}\right)d\Gamma$$

and

$$l(\varphi,\psi) = \int_{\Omega_1} F^1\overline{\varphi}dx + \frac{1}{2}\int_{\Gamma_1} F^2\overline{\varphi}d\Gamma + \int_{\Omega_2} F^3\overline{\psi}dx + \frac{1}{2}\int_{\Gamma_2} F^4\partial_{\nu_2}\overline{\psi}d\Gamma + \frac{1}{2}\int_{\Gamma_2} F^5\overline{\psi}d\Gamma.$$

It is easy to see that b is a sesquilinear, continuous, and coercive form on the space $\mathsf{H}\times\mathsf{H}$ and l is an antilinear and continuous form on H. Then, it follows by Lax-Milgram's theorem that (3.32) admits a unique solution $(u,w)\in\mathsf{H}$. By choosing $\varphi\in C_0^\infty(\Omega_1)$, $\psi=0$ in (3.32) and applying Green's formula, we have

$$\int_{\Omega_1}(u-\Delta u)\,\overline{\varphi}dx = \int_{\Omega_1}\left(f^1+g^1\right)\overline{\varphi}dx, \qquad \forall\varphi\in C_0^\infty(\Omega_1),$$

which implies that the first equation of (3.29) holds in the sense of distributions in Ω_1 and hence is satisfied in $L^2(\Omega_1)$. As $u-f^1-g^1$ belongs to $L^2(\Omega_1)$, the same holds for Δu, i.e., $\Delta u\in L^2(\Omega_1)$. In the same way, choosing $\varphi=0$ and $\psi\in C_0^\infty(\Omega_2)$ in (3.32), we see that the second equation of (3.29) holds as equality in $L^2(\Omega_2)$ and therefore $\Delta^2 w\in L^2(\Omega_2)$. Now, let us define the spaces

$$H^1_{*,I}(\Omega_1) = \left\{f\in H^1_*(\Omega_1)\mid\ f=0\ \text{ on }\ I\right\},$$

and

$$H^2_{*,I}(\Omega_2) = \left\{f\in H^2_*(\Omega_2)\mid\ f=\partial_{\nu_2}f=0\ \text{ on }\ I\right\}.$$

By taking $\varphi\in H^1_{*,I}(\Omega_1)$, $\psi=0$ and applying Green's formula in (3.32), we calculate

$$\int_{\Gamma_1}(2\partial_{\nu_1}u+u-f^1+h^1)\overline{\varphi}d\Gamma = 0, \qquad \forall\varphi\in H^{\frac{1}{2}}_{*,I}(\Gamma_1),$$

where $H^{\frac{1}{2}}_{*,I}(\Gamma_1)$ is the corresponding trace space of $H^1_{*,I}(\Omega_1)$ through the operator γ_1. This implies that $2\partial_{\nu_1}u+u-f^1+h^1\in(H^{\frac{1}{2}}_{*,I}(\Gamma_1))^\perp$ in $L^2(\Gamma_1)$, and since $H^{\frac{1}{2}}_{*,I}(\Gamma_1)$ is dense in $L^2(\Gamma_1)$, we deduce that u satisfies

$$2\partial_{\nu_1}u+u = f^1-h^1, \qquad \text{on }\ \Gamma_1.$$

Similarly, by taking $\varphi=0$, $\psi\in H^2_{*,I}(\Omega_2)$ and applying Green's formula in (3.32), we deduce that w satisfies

$$2\mathcal{B}_1w+\partial_{\nu_2}w = \partial_{\nu_2}f^2-h^2, \qquad \text{on }\ \Gamma_2,$$

as well as

$$2\mathcal{B}_2w-w = h^3-f^2, \qquad \text{on }\ \Gamma_2.$$

Coming back to (3.32) and again applying Green's formula, we calculate

$$\int_I \mathcal{B}_1w\partial_{\nu_2}\overline{\psi}d\Gamma - \int_I\left(\mathcal{B}_2w-\partial_{\nu_1}u\right)\overline{\psi}d\Gamma = 0, \qquad \forall\psi\in H^{\frac{3}{2}}_*(I),$$

where $H^{\frac{3}{2}}_*(I)$ is the corresponding trace space of $H^2_*(\Omega_2)$ through the operator $\psi\longmapsto\psi|_I$. Due to the density of $H^{\frac{3}{2}}_*(I)$ into $L^2(I)$, we can easily check that u and w satisfy the transmission conditions

of (3.29). Therefore, we deduce that system (3.29) has a unique solution $(u, w) \in \mathsf{H}$ such that $\Delta u \in L^2(\Omega_1)$ and $\Delta^2 w \in L^2(\Omega_2)$. Finally, by setting

$$v := u - f^1,\ \eta := -\partial_{\nu_1} u,\ z := w - f^2,\ \xi := -\mathcal{B}_1 w \ \text{ and } \ \zeta := \mathcal{B}_2 w,$$

we conclude that there exists a unique $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$ solution of equation (3.19) and thus the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$. The proof is thus complete. ■

According to Lumer-Philips theorem (see [44]), Proposition 3.2.1 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ in $\mathcal{H}$ which gives the well-posedness of (3.16). Then, we have the following result:

Theorem 3.2.1. *For all $U_0 \in \mathcal{H}$, system (3.16) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the system (3.16) admits a unique strong solution

$$U(t) \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

Remark 3.2.4. *Note that Proposition 3.2.1 remains true and system (3.16) admits a unique solution even if the domains Ω_1 and Ω_2 have no dynamical controls at their exterior boundaries Γ_1 and Γ_2.*

3.3 Some Regularity Results

In the following sections, our goal is to establish energy estimates of the system (3.16). Our stability results are based on a frequency domain approach combined with a multiplier technique and hence require some Green's formulas in Ω_1 and Ω_2. In order to justify these formulas, we need some regularity results proved in [11] and [16]. Hence, from now on, we assume that Ω_1 and Ω_2 have a polygonal boundary in the sense that their boundary is piecewise smooth. For $i = 1, 2$, we denote by $\omega_{i,j}$, $j = 1, \ldots, N_i$ the interior angles at the corners of Ω_i enclosed between two consecutive curves. These angles may vary within the range $0 < \omega_{i,j} \leq 2\pi$. To obtain the needed regularity, we require the following assumptions concerning the angles at the corners of Ω_1 and Ω_2:

Assumption (A1). *The inner angles $\omega_{1,j} < \pi$, for all $j = 1, \ldots, N_1$.*

Assumption (A2). *There exists a minimal angle ω_0 where ω_0 depends on the Poisson coefficient μ (for instance, $\omega_0 \simeq 77.753311\ldots^\circ$ when $\mu = 0.3$) such that $\omega_{2,j} < \omega_0$, for all $j = 1, \ldots, N_2$.*

The regularity results are summarized by the following proposition:

Proposition 3.3.1. *Let $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$. Assume that Ω_1 and Ω_2 are polygonal domains as described above, satisfying Assumptions (A1) and (A2), respectively. Then, there exists $\varepsilon \in (0, \frac{1}{2})$ such that u belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ and there exists a sequence $\{w_k\}_{k\geq 0} \subset H^4(\Omega_2)$ such that w_k converges to w in $H^2(\Omega_2)$ and $\Delta^2 w_k$ converges to $\Delta^2 w$ in $L^2(\Omega_2)$.*

Now, let us introduce some lemmas that are sufficient to prove our regularity result.

Lemma 3.3.1. *Assume that Ω_1 is a polygonal domain as described above, satisfying Assumption (A1). Then, there exists $\varepsilon \in (0, \frac{1}{2})$ such that the solution $y \in H^1(\Omega_1)$ of*

$$\begin{cases} \Delta y \in L^2(\Omega_1), & \\ \partial_{\nu_1} y = v_1 \in L^2(\Gamma_1), & \text{on } \ \Gamma_1, \\ y = v_2 \in H^{\frac{3}{2}}(I), & \text{on } \ I, \end{cases} \tag{3.33}$$

belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega_1)$.

Proof. We define the operator:

$$\begin{aligned} P : H^{1+s}(\Omega_1) \to \Xi^s(\Omega_1) &:= H^{s-1}(\Omega_1) \times H^{s+\frac{1}{2}}(I) \times H^{s-\frac{1}{2}}(\Gamma_1) \\ y \mapsto P(y) &:= (\Delta y, y|_I, \partial_{\nu_1} y|_{\Gamma_1}) \end{aligned}$$

for all $0 < s < 1$ such that $s \neq \frac{1}{2}$. From the elliptic regularity theory (see Theorem 23.3 of [16]), we deduce that there exists $\varepsilon \in (0, \frac{1}{2})$ such that for any $s = \frac{1}{2} - \varepsilon$, the operator P is $[0, s]-$regular in the sense that if $y \in H^1(\Omega_1)$ and $P(y) \in \Xi^s(\Omega_1)$, then $y \in H^{1+s}(\Omega_1)$. This implies that the solution $y \in H^1(\Omega_1)$ of system (3.33) belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ for some $\varepsilon \in (0, \frac{1}{2})$. ■

Lemma 3.3.2. *Assume that Ω_2 is a polygonal domain as described above, satisfying Assumption (A2). Then, for any $\varepsilon \in (0, \frac{1}{2})$ and $y \in H^2(\Omega_2)$ solution of*

$$\begin{cases} \Delta^2 y \in L^2(\Omega_2), & \\ \mathcal{B}_1 y = 0, & \text{on } I, \\ \mathcal{B}_2 y = v_3 \in H^{-\varepsilon}(I), & \text{on } I, \\ \mathcal{B}_1 y = v_4 \in L^2(\Gamma_2), & \text{on } \Gamma_2, \\ \mathcal{B}_2 y = v_5 \in L^2(\Gamma_2), & \text{on } \Gamma_2, \end{cases} \tag{3.34}$$

there exists a sequence $\{y_k\}_{k\geq 0} \subset H^4(\Omega_2)$ such that y_k converges to y in $H^2(\Omega_2)$ and $\Delta^2 y_k$ converges to $\Delta^2 y$ in $L^2(\Omega_2)$.

Proof. Given $v_4, v_5 \in L^2(\Gamma_2)$ and $v_3 \in H^{-\varepsilon}(I)$, by density, there exist three sequences $\{v_{4,k}\}_{k\geq 0}$, $\{v_{5,k}\}_{k\geq 0} \subset C_0^\infty(\Gamma_2)$ and $\{v_{3,k}\}_{k\geq 0} \subset C_0^\infty(I)$ such that

$$v_{4,k} \underset{k\to\infty}{\longrightarrow} v_4, \quad v_{5,k} \underset{k\to\infty}{\longrightarrow} v_5 \text{ in } L^2(\Gamma_2) \quad \text{and} \quad v_{3,k} \underset{k\to\infty}{\longrightarrow} v_3 \text{ in } H^{-\varepsilon}(I). \tag{3.35}$$

Then we consider the unique solution y_k of

$$\begin{cases} y_k + \Delta^2 y_k = y + \Delta^2 y \in L^2(\Omega_2), & \text{in } \Omega_2, \\ \mathcal{B}_1 y_k = 0, & \text{on } I, \\ \mathcal{B}_2 y_k = v_{3,k}, & \text{on } I, \\ \mathcal{B}_1 y_k = v_{4,k}, & \text{on } \Gamma_2, \\ \mathcal{B}_2 y_k = v_{5,k}, & \text{on } \Gamma_2. \end{cases} \tag{3.36}$$

As the system $(\Delta^2, \mathcal{B}_1, \mathcal{B}_2)$ is a strongly elliptic system, we can use the elliptic regularity theory (see Theorem 2 of [11]). This allows us to deduce that $y_k \in H^4(\Omega_2)$ and that there exists a positive constant $C > 0$ such that

$$\|y - y_k\|_{H^2(\Omega_2)} \leq C \left(\|v_4 - v_{4,k}\|_{L^2(\Gamma_2)} + \|v_5 - v_{5,k}\|_{L^2(\Gamma_2)} + \|v_3 - v_{3,k}\|_{H^{-\varepsilon}(I)} \right). \tag{3.37}$$

By (3.35) and (3.37), we deduce that y_k converges to y in $H^2(\Omega_2)$. Thanks to (3.36), we have $\Delta^2 y_k = y + \Delta^2 y - y_k$, which converges to $\Delta^2 y$ in $L^2(\Omega_2)$. ■

Proof of Proposition 3.3.1. Indeed, u may be seen as the unique solution $u \in H^1(\Omega_1)$ of

$$\begin{cases} \Delta u \in L^2(\Omega_1), & \\ \partial_{\nu_1} u = -\eta \in L^2(\Gamma_1), & \text{on } \Gamma_1, \\ u = w \in H^{\frac{3}{2}}(I) \subset H^1(I), & \text{on } I, \end{cases}$$

and using Lemma 3.3.1, we conclude that $u \in H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ for some $\varepsilon \in (0, \frac{1}{2})$. By the definition of $D(\mathcal{A})$, we notice that w is a solution of system (3.34) with $v_3 = \partial_{\nu_1} u$, $v_4 = -\xi$ and $v_5 = \zeta$. We know that v_3 belongs to $H^{-\varepsilon}(I)$ for some $\varepsilon > 0$, while the $L^2(\Gamma_2)$ regularity of v_4 and v_5 follows from the regularity ξ and $\zeta \in L^2(\Gamma_2)$. We then conclude using Lemma 3.3.2.

Remark 3.3.1. *The situation is much more complicated for the regularity of w due to the lack of regularity of the boundary conditions $\mathcal{B}_1 w = -\xi$ and $\mathcal{B}_2 w = \zeta$ on Γ_2, as ξ and ζ belong at most to $L^2(\Gamma_2)$, which is not sufficient to achieve the required regularity of w. For this reason, we use arguments inspired by Lemma 3.1 of [46] and Lemma 3.2 of [6].*

3.4 Strong Stability with Non-Compact Resolvent

In this section, we prove the strong stability of the system (3.1)-(3.4) in the sense that its energy $E(t)$ converges to zero as t goes to infinity for all initial data in $\mathcal{H}$. As the resolvent of $\mathcal{A}$ is not compact, classical methods such as Lasalle's invariance principle [55] or the spectrum decomposition theory of Benchimol [9] are not applicable in this case. Instead, we prove the strong stability using a more general criteria of Arendt-Batty [7] which states that in a reflexive Banach space, a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable if $\mathcal{A}$ has no eigenvalues on the imaginary axis, and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$. Our main result in this section is summarized by the following theorem:

Theorem 3.4.1. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Assume also that Assumptions (A1) and (A2) hold. Then, the semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable in the energy space $\mathcal{H}$ in the sense that*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

In the proof of Theorem 3.4.1, we shall use the following lemmas:

Lemma 3.4.1. *Assuming that Assumptions (A1) and (A2) hold, the operator $\mathcal{A}$ has no pure imaginary eigenvalues.*

Proof. Let $\lambda \in \mathbb{R}$ and let $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U. \tag{3.38}$$

Equivalently, we have the following system of equations

$$\begin{aligned}
v &= i\lambda u, \quad \text{in } \Omega_1, && (3.39)\\
\Delta u &= i\lambda v, \quad \text{in } \Omega_1, && (3.40)\\
v - \eta &= i\lambda \eta, \quad \text{on } \Gamma_1, && (3.41)\\
z &= i\lambda w, \quad \text{in } \Omega_2, && (3.42)\\
-\Delta^2 w &= i\lambda z, \quad \text{in } \Omega_2, && (3.43)\\
\partial_{\nu_2} z - \xi &= i\lambda \xi, \quad \text{on } \Gamma_2, && (3.44)\\
z - \zeta &= i\lambda \zeta, \quad \text{on } \Gamma_2. && (3.45)
\end{aligned}$$

By using equations (3.38) and (3.18), a direct computation leads to the following:

$$0 = \Re\left\{i\lambda \|U\|_{\mathcal{H}}^2\right\} = \Re\{(\mathcal{A}U, U)_{\mathcal{H}}\} = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} |\xi|^2 d\Gamma - \int_{\Gamma_2} |\zeta|^2 d\Gamma,$$

which implies that

$$\eta = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \xi = \zeta = 0 \text{ on } \Gamma_2. \tag{3.46}$$

It follows from (3.14) that

$$\partial_{\nu_1} u = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \mathcal{B}_1 w = \mathcal{B}_2 w = 0 \text{ on } \Gamma_2, \tag{3.47}$$

and from (3.41), (3.44) and (3.45) that

$$v = 0 \text{ on } \Gamma_1 \quad \text{and} \quad z = \partial_{\nu_2} z = 0 \text{ on } \Gamma_2. \tag{3.48}$$

Now, we need to consider two distinct cases:

Case 1. If $\lambda = 0$, then $v = 0$ in Ω_1 and $z = 0$ in Ω_2, and we obtain the following system

$$\begin{aligned} -\Delta u &= 0, \quad \text{in } \Omega_1, && (3.49)\\ \Delta^2 w &= 0, \quad \text{in } \Omega_2, && (3.50)\\ \partial_{\nu_1} u &= 0, \quad \text{on } \Gamma_1, && (3.51)\\ \mathcal{B}_1 w = \mathcal{B}_2 w &= 0, \quad \text{on } \Gamma_2, && (3.52) \end{aligned}$$

with the following transmission conditions

$$u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u \text{ on } I. \tag{3.53}$$

Multiplying equations (3.49) and (3.50) by $\overline{u}$ and $\overline{w}$ respectively, integrating over Ω_1 and Ω_2, then using Green's formula and equations (3.51) and (3.52), we get

$$\int_{\Omega_1} |\nabla u|^2 dx - \int_I \partial_{\nu_1} u \overline{u} d\Gamma = 0, \tag{3.54}$$

$$a(w, \overline{w}) - \int_I (\mathcal{B}_1 w \partial_{\nu_2} \overline{w} - \mathcal{B}_2 w \overline{w})\, d\Gamma = 0. \tag{3.55}$$

Adding the resulting equations and taking (3.53) into consideration, we obtain

$$\int_{\Omega_1} |\nabla u|^2 dx + a(w, \overline{w}) = 0, \tag{3.56}$$

which leads to

$$u = 0 \text{ in } \Omega_1 \quad \text{and} \quad w = 0 \text{ in } \Omega_2. \tag{3.57}$$

Hence, we get

$$U = 0.$$

Case 2. If $\lambda \neq 0$, then using equations (3.39), (3.42) and (3.48), we get

$$u = 0 \text{ on } \Gamma_1 \quad \text{and} \quad w = \partial_{\nu_2} w = 0 \text{ on } \Gamma_2. \tag{3.58}$$

Inserting (3.39) in (3.40), we obtain

$$\begin{cases} \lambda^2 u + \Delta u = 0, & \text{in } \Omega_1, \\ u = \partial_{\nu_1} u = 0, & \text{on } \Gamma_1. \end{cases} \tag{3.59}$$

Thus, from the above system and by using a unique continuation theorem (see [35]), we obtain

$$u = 0 \quad \text{in} \quad \Omega_1. \tag{3.60}$$

Now, from (3.47), we get

$$\mathcal{B}_1 w = \Delta w + (1-\mu)\mathcal{C}_1 w = 0 \quad \text{on} \quad \Gamma_2 \quad \text{and} \quad \mathcal{B}_2 w = \partial_{\nu_2}\Delta w + (1-\mu)\partial_{\tau_2}\mathcal{C}_2 w = 0 \quad \text{on} \quad \Gamma_2. \tag{3.61}$$

Using (3.58) and the fact that $\nabla w = \partial_{\tau_2} w \tau_2 + \partial_{\nu_2} w \nu_2$ on Γ_2, we obtain

$$\nabla w = (w_{x_1}, w_{x_2}) = (0,0) \quad \text{on} \quad \Gamma_2 \times \Gamma_2 \quad \text{and consequently} \quad w_{x_1} = w_{x_2} = 0 \quad \text{on} \quad \Gamma_2. \tag{3.62}$$

From equations (3.5), (3.58) and (3.62), we get

$$\mathcal{C}_1 w = \mathcal{C}_2 w = 0 \quad \text{on} \quad \Gamma_2, \tag{3.63}$$

consequently, from (3.61), we get

$$\Delta w = \partial_{\nu_2}\Delta w = 0 \quad \text{on} \quad \Gamma_2. \tag{3.64}$$

Inserting (3.42) into (3.43), we obtain

$$\begin{cases} \lambda^2 w - \Delta^2 w = 0, & \text{in} \quad \Omega_2, \\ w = \partial_{\nu_2} w = \Delta w = \partial_{\nu_2}\Delta w = 0, & \text{on} \quad \Gamma_2. \end{cases} \tag{3.65}$$

Again, from the above system and by using a unique continuation theorem (see [35]), we obtain

$$w = 0 \quad \text{in} \quad \Omega_2. \tag{3.66}$$

Consequently, from equations (3.39), (3.42), (3.46), (3.60) and (3.66), we get

$$U = 0.$$

The proof is thus complete. ■

Remark 3.4.1. *The preceding result is true with some slight modifications, where only two dynamical controls ξ and ζ are applied. First, one can show that $w = 0$ in Ω_2. Then, due to some useful information provided by the interface, we obtain that $u = 0$ in Ω_1. The transmission conditions play an essential role in this case. However, this may happen under certain geometric conditions in the case where only one dynamical control η is applied. In such cases, we get $u = 0$ in Ω_1, but the transmission conditions introduced are not enough to conclude that $w = 0$ in Ω_2.*

Lemma 3.4.2. *Assume that Assumptions (A1) and (A2) hold. Then, the operator $i\lambda\mathcal{I} - \mathcal{A}$ is surjective for all real number $\lambda \in \mathbb{R}$.*

Proof. For any given $F = (f^1, g^1, h^1, f^2, g^2, h^2, h^3) \in \mathcal{H}$, we are looking for $U = (u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$ that solves the following system:

$$(i\lambda\mathcal{I} - \mathcal{A})U = F, \tag{3.67}$$

which is equivalent to the following equations:

$$
\begin{aligned}
i\lambda u - v &= f^1, \quad \text{in} \quad \Omega_1, && (3.68)\\
i\lambda v - \Delta u &= g^1, \quad \text{in} \quad \Omega_1, && (3.69)\\
i\lambda \eta - v + \eta &= h^1, \quad \text{on} \quad \Gamma_1, && (3.70)\\
i\lambda w - z &= f^2, \quad \text{in} \quad \Omega_2, && (3.71)\\
i\lambda z + \Delta^2 w &= g^2, \quad \text{in} \quad \Omega_2, && (3.72)\\
i\lambda \xi - \partial_{\nu_2} z + \xi &= h^2, \quad \text{on} \quad \Gamma_2, && (3.73)\\
i\lambda \zeta - z + \zeta &= h^3, \quad \text{on} \quad \Gamma_2. && (3.74)
\end{aligned}
$$

By eliminating v, z, η, ξ and ζ from the above equations and using (3.14), we obtain the following system:

$$
\begin{cases}
\lambda^2 u + \Delta u = -(i\lambda f^1 + g^1), & \text{in} \quad \Omega_1,\\
\lambda^2 w - \Delta^2 w = -(i\lambda f^2 + g^2), & \text{in} \quad \Omega_2,\\
u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on} \quad I,\\
\partial_{\nu_1} u + \dfrac{i\lambda}{i\lambda+1} u = \dfrac{1}{i\lambda+1}(f^1 - h^1), & \text{on} \quad \Gamma_1,\\
\mathcal{B}_1 w + \dfrac{i\lambda}{i\lambda+1}\partial_{\nu_2} w = \dfrac{1}{i\lambda+1}(\partial_{\nu_2} f^2 - h^2), & \text{on} \quad \Gamma_2,\\
\mathcal{B}_2 w - \dfrac{i\lambda}{i\lambda+1} w = \dfrac{1}{i\lambda+1}(h^3 - f^2), & \text{on} \quad \Gamma_2.
\end{cases} \tag{3.75}
$$

Let $\phi = (\varphi, \psi) \in \mathsf{H}$. Multiplying the first equation of (3.75) by $\overline{\varphi}$ and integrating over Ω_1, and multiplying the second equation of (3.75) by $\overline{\psi}$ and integrating over Ω_2, we use Green's formula to get the following equations:

$$
-\lambda^2 \int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + \frac{i\lambda}{i\lambda+1} \int_{\Gamma_1} u\overline{\varphi} d\Gamma - \int_I \partial_{\nu_1} u \overline{\varphi} d\Gamma = \int_{\Omega_1} F^1 \overline{\varphi} dx + \frac{1}{i\lambda+1} \int_{\Gamma_1} F^2 \overline{\varphi} d\Gamma, \tag{3.76}
$$

and

$$
\begin{aligned}
&-\lambda^2 \int_{\Omega_2} w\overline{\psi} dx + a(w, \overline{\psi}) + \frac{i\lambda}{i\lambda+1} \int_{\Gamma_2} \left(\partial_{\nu_2} w \partial_{\nu_2} \overline{\psi} + w\overline{\psi}\right) d\Gamma - \int_I \left(\mathcal{B}_1 w \partial_{\nu_2} \overline{\psi} - \mathcal{B}_2 w \overline{\psi}\right) d\Gamma\\
&\quad = \int_{\Omega_2} F^3 \overline{\psi} dx + \frac{1}{i\lambda+1} \int_{\Gamma_2} F^4 \partial_{\nu_2} \overline{\psi} d\Gamma + \frac{1}{i\lambda+1} \int_{\Gamma_2} F^5 \overline{\psi} d\Gamma,
\end{aligned} \tag{3.77}
$$

where

$$
F^1 = i\lambda f^1 + g^1, \quad F^2 = f^1 - h^1, \quad F^3 = i\lambda f^2 + g^2, \quad F^4 = \partial_{\nu_2} f^2 - h^2 \quad \text{and} \quad F^5 = f^2 - h^3.
$$

Adding equations (3.76) and (3.77) and using the transmission conditions, we obtain

$$
\begin{aligned}
&-\lambda^2 \left(\int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_2} w\overline{\psi} dx \right) + \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + i\lambda\Lambda \int_{\Gamma_1} u\overline{\varphi} d\Gamma\\
&\qquad + a(w, \overline{\psi}) + i\lambda\Lambda \int_{\Gamma_2} \left(\partial_{\nu_2} w \partial_{\nu_2} \overline{\psi} + w\overline{\psi}\right) d\Gamma\\
&= \int_{\Omega_1} F^1 \overline{\varphi} dx + \Lambda \int_{\Gamma_1} F^2 \overline{\varphi} d\Gamma + \int_{\Omega_2} F^3 \overline{\psi} dx + \Lambda \int_{\Gamma_2} F^4 \partial_{\nu_2} \overline{\psi} d\Gamma + \Lambda \int_{\Gamma_2} F^5 \overline{\psi} d\Gamma,
\end{aligned} \tag{3.78}
$$

where

$$\Lambda = \frac{1 - i\lambda}{\lambda^2 + 1}. \tag{3.79}$$

Here we note that Lax-Milgram's theorem cannot be applied because coercivity is not available. Therefore, we use a compact perturbation argument. For that purpose, let us introduce the sesquilinear form

$$\begin{aligned} b_\lambda((u,w),(\varphi,\psi)) &= \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + i\lambda\Lambda \int_{\Gamma_1} u\overline{\varphi} d\Gamma + a(w,\overline{\psi}) \\ &+ i\lambda\Lambda \int_{\Gamma_2} \left(\partial_{\nu_2} w \partial_{\nu_2}\overline{\psi} + w\overline{\psi}\right) d\Gamma, \quad \forall\, (\varphi,\psi) \in \mathsf{H}. \end{aligned} \tag{3.80}$$

This sesquilinear form b_λ is continuous and coercive on $\mathsf{H} \times \mathsf{H}$. Then, by Lax-Milgram's theorem, the operator

$$B_\lambda : \mathsf{H} \to \mathsf{H}' : (\boldsymbol{u}, \boldsymbol{w}) \to B_\lambda(\boldsymbol{u}, \boldsymbol{w}),$$

with $B_\lambda(\boldsymbol{u}, \boldsymbol{w})((\boldsymbol{\varphi}, \boldsymbol{\psi})) := b_\lambda((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{\varphi}, \boldsymbol{\psi})) \quad \forall (\varphi, \psi) \in \mathsf{H}$, is an isomorphism, where H' is the dual space of H. Now, let us set

$$R_\lambda : \mathsf{H} \to \mathsf{H}' : (\boldsymbol{u}, \boldsymbol{w}) \to R_\lambda(\boldsymbol{u}, \boldsymbol{w}),$$

with

$$R_\lambda(\boldsymbol{u}, \boldsymbol{w})((\boldsymbol{\varphi}, \boldsymbol{\psi})) = -\lambda^2 \left(\int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_2} w\overline{\psi} dx \right), \quad \forall\, (\varphi, \psi) \in \mathsf{H}.$$

As R_λ is a compact operator, we deduce that $B_\lambda + R_\lambda$ is a Fredholm operator of index zero from H to H'. Now by setting

$$L_\lambda(\varphi, \psi) = \int_{\Omega_1} F^1 \overline{\varphi} dx + \Lambda \int_{\Gamma_1} F^2 \overline{\varphi} d\Gamma + \int_{\Omega_2} F^3 \overline{\psi} dx + \Lambda \int_{\Gamma_2} F^4 \partial_{\nu_2} \overline{\psi} d\Gamma + \Lambda \int_{\Gamma_2} F^5 \overline{\psi} d\Gamma,$$

we notice that (3.78) is equivalent to

$$(B_\lambda + R_\lambda)(\boldsymbol{u}, \boldsymbol{w}) = L_\lambda \quad \text{in } \mathsf{H}'. \tag{3.81}$$

Hence, problem (3.78) admits a unique solution $(\boldsymbol{u}, \boldsymbol{w}) \in \mathsf{H}$ if and only if $B_\lambda + R_\lambda$ is invertible. Since $B_\lambda + R_\lambda$ is a Fredholm operator, it is enough to prove that $B_\lambda + R_\lambda$ is injective, i.e.,

$$\ker(B_\lambda + R_\lambda) = \{0\}.$$

Let us now fix $(\boldsymbol{u}, \boldsymbol{w}) \in \ker(B_\lambda + R_\lambda)$, which means it satisfies

$$\begin{aligned} &-\lambda^2 \left(\int_{\Omega_1} \boldsymbol{u}\overline{\boldsymbol{\varphi}} dx + \int_{\Omega_2} \boldsymbol{w}\overline{\boldsymbol{\psi}} dx \right) + \int_{\Omega_1} \nabla \boldsymbol{u} \cdot \nabla \overline{\boldsymbol{\varphi}} dx + i\lambda\Lambda \int_{\Gamma_1} \boldsymbol{u}\overline{\boldsymbol{\varphi}} d\Gamma \\ &+ a(\boldsymbol{w}, \overline{\boldsymbol{\psi}}) + i\lambda\Lambda \int_{\Gamma_2} \left(\partial_{\nu_2} \boldsymbol{w} \partial_{\nu_2} \overline{\boldsymbol{\psi}} + \boldsymbol{w}\overline{\boldsymbol{\psi}}\right) d\Gamma = 0, \quad \forall\, (\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \mathsf{H}. \end{aligned} \tag{3.82}$$

Thus, if we set

$$\boldsymbol{v} = i\lambda\boldsymbol{u}, \quad \boldsymbol{\eta} = i\lambda\Lambda\boldsymbol{u}, \quad \boldsymbol{z} = i\lambda\boldsymbol{w}, \quad \boldsymbol{\xi} = i\lambda\Lambda\partial_{\nu_2}\boldsymbol{w} \quad \text{and} \quad \boldsymbol{\zeta} = i\lambda\Lambda\boldsymbol{w},$$

we can conclude that $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{z}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in D(\mathcal{A})$ is a solution of

$$(i\lambda - \mathcal{A})\boldsymbol{U} = 0.$$

Using Lemma 3.4.1, we deduce that $\boldsymbol{U} = 0$. This shows that $B_\lambda + R_\lambda$ is invertible, and therefore a unique solution $(u, w) \in \mathsf{H}$ of (3.81) exists. At this stage, by setting

$$v = i\lambda u - f^1, \quad \eta = i\lambda\Lambda u + \Lambda(h^1 - f^1), \quad z = i\lambda w - f^2,$$

$$\xi = i\lambda\Lambda\partial_{\nu_2} w + \Lambda(h^2 - \partial_{\nu_2} f^2) \quad \text{and} \quad \zeta = i\lambda\Lambda w + \Lambda(h^3 - f^2),$$

we conclude that $(u, v, \eta, w, z, \xi, \zeta) \in D(\mathcal{A})$ is a solution of (3.67) and the proof is thus complete. ■

Proof of Theorem 3.4.1. Using Lemma 3.4.1, we conclude that the operator $\mathcal{A}$ has no pure imaginary eigenvalues. Additionally, by Lemma 3.4.2, we have $Im(i\lambda\mathcal{I} - \mathcal{A}) = \mathcal{H}$ for all real numbers $\lambda \in \mathbb{R}$. Therefore, the closed graph theorem of Banach implies that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Following Arendt-Batty (see [7]), the C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable and the proof is complete.

3.5 Lack of Exponential Stability

In this section, we prove that the system (3.1)-(3.4) is not exponentially stable. Concerning the characterization of exponential stability of a C_0-semigroup of contractions, we use Theorem 2.3.2 due to Huang [28] and Prüss [45]. The main result of this section is stated in the following theorem:

Theorem 3.5.1. *The C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is not uniformly stable in the energy space $\mathcal{H}$.*

According to Theorem 2.3.2 due to Huang [28] and Prüss [45], it is sufficient to prove that the resolvent of the operator $\mathcal{A}$ is not uniformly bounded on the imaginary axis. For this aim, we begin with the following technical lemma:

Lemma 3.5.1. *Define the linear unbounded operator $\mathcal{O}_{\Delta,R} : D(\mathcal{O}_{\Delta,R}) \longmapsto L^2(\Omega_1) \times L^2(\Omega_2)$ by*

$$D(\mathcal{O}_{\Delta,R}) = \left\{ \begin{array}{l} (f,g) \in \mathsf{H} \mid \left(\Delta f, \Delta^2 g\right) \in L^2(\Omega_1) \times L^2(\Omega_2), \quad \partial_{\nu_1} f + f = 0 \quad on \quad \Gamma_1, \\ \mathcal{B}_1 g + \partial_{\nu_2} g = 0 \quad on \quad \Gamma_2, \quad \mathcal{B}_2 g - g = 0 \quad on \quad \Gamma_2, \quad \mathcal{B}_1 g = 0 \quad and \quad \mathcal{B}_2 g = \partial_{\nu_1} f \quad on \quad I \end{array} \right\}, \tag{3.83}$$

and

$$\mathcal{O}_{\Delta,R}(f,g) = \left(-\Delta f, \Delta^2 g\right), \quad \forall\, (f,g) \in D(\mathcal{O}_{\Delta,R}). \tag{3.84}$$

Then, $\mathcal{O}_{\Delta,R}$ is a positive self-adjoint operator with a compact resolvent.

Proof. To prove that $\mathcal{O}_{\Delta,R}$ is a positive self-adjoint operator, we will show that it is a symmetric m-accretive operator. For this purpose, we will divide the proof into steps:

Step 1. ($\mathcal{O}_{\Delta,R}$ is symmetric.) Indeed, for all $(f,g),\ (h,k) \in D(\mathcal{O}_{\Delta,R})$, we have

$$\begin{aligned} (\mathcal{O}_{\Delta,R}(f,g),(h,k))_{L^2(\Omega_1)\times L^2(\Omega_2)} &= -\int_{\Omega_1} (\Delta f)\overline{h}dx + \int_{\Omega_2} (\Delta^2 g)\overline{k}dx \\ &= \int_{\Omega_1} \nabla f \cdot \nabla \overline{h} dx + \int_{\Gamma_1} f\overline{h}d\Gamma + a(g,\overline{k}) + \int_{\Gamma_2} \left(\partial_{\nu_2} g \partial_{\nu_2}\overline{k} + g\overline{k}\right) d\Gamma \\ &= ((f,g), \mathcal{O}_{\Delta,R}(h,k))_{L^2(\Omega_1)\times L^2(\Omega_2)}. \end{aligned} \tag{3.85}$$

Thus, $\mathcal{O}_{\Delta,R}$ is symmetric.

Step 2. ($\mathcal{O}_{\Delta,R}$ is $m-$accretive.) Indeed, for all $(f,g) \in D(\mathcal{O}_{\Delta,R})$, we have

$$\begin{aligned}\Re\left(\mathcal{O}_{\Delta,R}(f,g),(f,g)\right)_{L^2(\Omega_1)\times L^2(\Omega_2)} &= \Re\left\{-\int_{\Omega_1}(\Delta f)\overline{f}dx+\int_{\Omega_2}(\Delta^2 g)\overline{g}dx\right\}\\ &= \int_{\Omega_1}|\nabla f|^2dx+\int_{\Gamma_1}|f|^2d\Gamma+a(g,\overline{g})+\int_{\Gamma_2}\left(|\partial_{\nu_2}g|^2+|g|^2\right)d\Gamma\geq 0.\end{aligned} \tag{3.86}$$

Thus, $\mathcal{O}_{\Delta,R}$ is an accretive operator. Now, let $(F,G)\in L^2(\Omega_1)\times L^2(\Omega_2)$ and $\lambda>0$, looking for $(f,g)\in D\left(\mathcal{O}_{\Delta,R}\right)$ solution of

$$\left(\lambda\mathcal{I}+\mathcal{O}_{\Delta,R}\right)(f,g)=(F,G). \tag{3.87}$$

Equivalently, we have the following system:

$$\lambda f-\Delta f=F, \tag{3.88}$$

$$\lambda g+\Delta^2 g=G. \tag{3.89}$$

Taking $(\varphi,\psi)\in\mathsf{H}$, then integrating after multiplying (3.88) by $\overline{\varphi}$ and (3.89) by $\overline{\psi}$ yields the two equations added as follows

$$\begin{aligned}\int_{\Omega_1}\nabla f\cdot\nabla\overline{\varphi}dx+a(g,\overline{\psi})+\int_{\Gamma_1}f\overline{\varphi}d\Gamma+\int_{\Gamma_2}\left(\partial_{\nu_2}g\partial_{\nu_2}\overline{\psi}+g\overline{\psi}\right)d\Gamma+\lambda\int_{\Omega_1}f\overline{\varphi}dx+\lambda\int_{\Omega_2}g\overline{\psi}dx\\ =\int_{\Omega_1}F\overline{\varphi}dx+\int_{\Omega_2}G\overline{\psi}dx.\end{aligned} \tag{3.90}$$

Let

$$\begin{aligned}S\left((f,g),(\varphi,\psi)\right)=\int_{\Omega_1}\nabla f\cdot\nabla\overline{\varphi}dx+a(g,\overline{\psi})+\int_{\Gamma_1}f\overline{\varphi}d\Gamma+\int_{\Gamma_2}\left(\partial_{\nu_2}g\partial_{\nu_2}\overline{\psi}+g\overline{\psi}\right)d\Gamma\\ +\lambda\int_{\Omega_1}f\overline{\varphi}dx+\lambda\int_{\Omega_2}g\overline{\psi}dx,\end{aligned} \tag{3.91}$$

and

$$L(\varphi,\psi)=\int_{\Omega_1}F\overline{\varphi}dx+\int_{\Omega_2}G\overline{\psi}dx. \tag{3.92}$$

It is easy to see that S is a sesquilinear, continuous, and coercive form on the space $\mathsf{H}\times\mathsf{H}$, and L is an antilinear and continuous form on H. Then, it follows by Lax-Milgram's theorem that $S((f,g),(\varphi,\psi))=L(\varphi,\psi)$ admits a unique solution $(f,g)\in\mathsf{H}$. By classical elliptic regularity, we deduce that $(f,g)\in D(\mathcal{O}_{\Delta,R})$ solution of system (3.88)-(3.89). Thus, $\mathcal{O}_{\Delta,R}$ is m-accretive.

Step 3. ($\mathcal{O}_{\Delta,R}$ has a compact resolvent.) Let

$$R_\lambda(\mathcal{O}_{\Delta,R})=\left(\lambda\mathcal{I}+\mathcal{O}_{\Delta,R}\right)^{-1}.$$

Due to Sobolev embeddings, $R_0(\mathcal{O}_{\Delta,R})$ is compact. Then, using the following resolvent identity

$$R_\lambda-R_\mu=(\mu-\lambda)R_\mu R_\lambda,$$

we deduce that the resolvent $(\lambda\mathcal{I}+\mathcal{O}_{\Delta,R})^{-1}$ of the operator $\mathcal{O}_{\Delta,R}$ is compact, and the proof is thus complete. ■

Proof of Theorem 3.5.1. According to Theorem 2.3.2 due to Huang [28] and Prüss [45], it is sufficient to show that the resolvent of $\mathcal{A}$ is not uniformly bounded on the imaginary axis. In other words, it is enough to show the existence of a positive real number M and some sequences $\lambda_n \in i\mathbb{R}$, $U_n = (u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n)^\top \in D(\mathcal{A})$, and $F_n = (f_n^1, g_n^1, h_n^1, f_n^2, g_n^2, h_n^2, h_n^3)^\top \in \mathcal{H}$, where $n \in \mathbb{N}$, such that

$$(\lambda_n \mathcal{I} - \mathcal{A})U_n = F_n, \quad \forall n \in \mathbb{N}, \tag{3.93}$$

$$\|U_n\|_{\mathcal{H}} \geq M, \quad \forall n \in \mathbb{N}, \tag{3.94}$$

$$\lim_{n\to\infty} \|F_n\|_{\mathcal{H}} = 0. \tag{3.95}$$

From Lemma 3.5.1, we can consider the sequence of eigenfunctions $(\varphi_n, \psi_n)_{n\in\mathbb{N}}$ (that form an orthonormal basis of $L^2(\Omega_1) \times L^2(\Omega_2)$) of the operator $\mathcal{O}_{\Delta,R}$, corresponding to the eigenvalues $(\mu_n^2)_{n\in\mathbb{N}}$, such that μ_n^2 tends to infinity as n goes to infinity. Consequently, for all $n \in \mathbb{N}$, they satisfy the following system

$$\begin{cases} -\Delta\varphi_n = \mu_n^2\varphi_n, & \text{in} \quad \Omega_1, \\ \Delta^2\psi_n = \mu_n^2\psi_n, & \text{in} \quad \Omega_2, \\ \varphi_n - \psi_n = 0, \quad \mathcal{B}_1\psi_n = 0, \quad \mathcal{B}_2\psi_n = \partial_{\nu_1}\varphi_n & \text{on} \quad I, \\ \partial_{\nu_1}\varphi_n + \varphi_n = 0, & \text{on} \quad \Gamma_1, \\ \mathcal{B}_1\psi_n + \partial_{\nu_2}\psi_n = 0, & \text{on} \quad \Gamma_2, \\ \mathcal{B}_2\psi_n - \psi_n = 0, & \text{on} \quad \Gamma_2, \end{cases} \tag{3.96}$$

with

$$\|(\varphi_n, \psi_n)\|^2_{L^2(\Omega_1)\times L^2(\Omega_2)} = \int_{\Omega_1} |\varphi_n|^2 dx + \int_{\Omega_2} |\psi_n|^2 dx = 1. \tag{3.97}$$

Now, let us choose

$$\begin{aligned} u_n = \frac{\varphi_n}{i\mu_n}, \quad v_n = \varphi_n, \quad \eta_n = \frac{1}{i\mu_n}\gamma_1(\varphi_n), \\ w_n = \frac{\psi_n}{i\mu_n}, \quad z_n = \psi_n, \quad \xi_n = \frac{1}{i\mu_n}\gamma_{2,2}(\psi_n), \quad \text{and} \quad \zeta_n = \frac{1}{i\mu_n}\gamma_{2,1}(\psi_n). \end{aligned} \tag{3.98}$$

So, by setting

$$F_n = \left(0, 0, \frac{1}{i\mu_n}\gamma_1(\varphi_n), 0, 0, \frac{1}{i\mu_n}\gamma_{2,2}(\psi_n), \frac{1}{i\mu_n}\gamma_{2,1}(\psi_n)\right), \tag{3.99}$$

we deduce that

$$U_n = (u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n), \tag{3.100}$$

is the solution in $D(\mathcal{A})$ of the following equation

$$(i\mu_n\mathcal{I} - \mathcal{A})U_n = F_n. \tag{3.101}$$

Now, multiplying equation $(3.96)_1$ and $(3.96)_2$ by $\overline{\varphi_n}$ and $\overline{\psi_n}$ respectively, integrating by parts, we get

$$\begin{aligned} \int_{\Omega_1} |\nabla\varphi_n|^2 dx + \int_{\Gamma_1} |\varphi_n|^2 d\Gamma + a(\psi_n, \overline{\psi_n}) + \int_{\Gamma_2} \left(|\partial_{\nu_2}\psi_n|^2 + |\psi_n|^2\right) d\Gamma \\ = \mu_n^2 \int_{\Omega_1} |\varphi_n|^2 dx + \mu_n^2 \int_{\Omega_2} |\psi_n|^2 dx = \mu_n^2. \end{aligned} \tag{3.102}$$

Since the norm defined on the left-hand side of the above equation is equivalent to the usual norm of $H^1(\Omega_1)\times H^2(\Omega_2)$ on $H^1(\Omega_1)\times H^2(\Omega_2)$, we get

$$\|(\varphi_n,\psi_n)\|^2_{H^1(\Omega_1)\times H^2(\Omega_2)}\lesssim \mu_n^2. \tag{3.103}$$

On the other hand, we have

$$\begin{aligned}\|U_n\|^2_{\mathcal{H}} = \int_{\Omega_1}|\varphi_n|^2dx+\int_{\Omega_2}|\psi_n|^2dx+\mu_n^{-2}\int_{\Omega_1}|\nabla\varphi_n|^2dx+\mu_n^{-2}a(\psi_n,\overline{\psi_n})\\ +\mu_n^{-2}\int_{\Gamma_1}|\varphi_n|^2d\Gamma)+\mu_n^{-2}\int_{\Gamma_2}\left(|\partial_{\nu_2}\psi_n|^2+|\psi_n|^2\right)d\Gamma\geq 1,\end{aligned} \tag{3.104}$$

which implies that (3.94) holds for $M=1$. By using the trace theorem of interpolation type (see Theorem 1.4.4 in [39] and Theorem 1.5.1.10 in [22]), we obtain

$$\begin{aligned}\|F_n\|^2_{\mathcal{H}} &= \mu_n^{-2}\|\varphi_n\|^2_{L^2(\Gamma_1)}+\mu_n^{-2}\|\partial_{\nu_2}\psi_n\|^2_{L^2(\Gamma_2)}+\mu_n^{-2}\|\psi_n\|^2_{L^2(\Gamma_2)}\\ &\lesssim \mu_n^{-2}\|\varphi_n\|_{H^1(\Omega_1)}\|\varphi_n\|_{L^2(\Omega_1)}+\mu_n^{-2}\|\psi_n\|_{H^2(\Omega_2)}\|\psi_n\|_{H^1(\Omega_2)}+\mu_n^{-2}\|\psi_n\|_{H^1(\Omega_2)}\|\psi_n\|_{L^2(\Omega_2)}.\end{aligned} \tag{3.105}$$

Further, by using Theorem 4.17 in [3], equation (3.97) and equation (3.103), we obtain from the above inequality that

$$\begin{aligned}\|F_n\|^2_{\mathcal{H}} &\lesssim \mu_n^{-2}\|\varphi_n\|_{H^1(\Omega_1)}\|\varphi_n\|_{L^2(\Omega_1)}+\mu_n^{-2}\|\psi_n\|^{\frac{3}{2}}_{H^2(\Omega_2)}\|\psi_n\|^{\frac{1}{2}}_{L^2(\Omega_2)}+\mu_n^{-2}\|\psi_n\|^{\frac{1}{2}}_{H^2(\Omega_2)}\|\psi_n\|^{\frac{3}{2}}_{L^2(\Omega_2)}\\ &\lesssim \mu_n^{-1}+\mu_n^{-\frac{1}{2}}+\mu_n^{-\frac{3}{2}}\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{3.106}$$

Then, the resolvent of the operator $\mathcal{A}$ is not uniformly bounded on the imaginary axis, and consequently, our system is not uniformly (exponentially) stable. The proof is thus complete.

3.6 Polynomial Stability

In this section, we study the polynomial decay of the system (3.1)-(3.4) for smooth solutions using the multiplier method, as the system is not uniformly stable. One of the main ingredients is the frequency domain approach; specifically, the proof of polynomial stability relies on Theorem 2.4 in [12] (see also [8] and [38]). To proceed with our polynomial energy decay result, we introduce an additional geometric assumption on the wave and the plate, which is due to the requirements of integration in Ω_1 and Ω_2:

Assumption (A3). *Assume that there exists a fixed point $x_0\in\mathbb{R}^2$ such that, putting $m(x)=x-x_0$, we have*

$$(m\cdot\nu_1)=0, \quad \forall x\in I, \tag{3.107}$$

and

$$(m\cdot\nu)\geq\delta, \quad \forall x\in\partial\Omega, \tag{3.108}$$

where δ is a positive real number and $(\cdot)$ designates the scalar product in $\mathbb{R}^2$.

Remark 3.6.1. *We remark that if condition (3.107) is satisfied, then the interface I is straight. Hence, from now on, we assume that the interface is straight and the interior angles $\omega_{i,1}$ and ω_{i,N_i}, for $i=1,2$, are the angles at the extremities of I.*

The following figures provide some examples of geometries that satisfy the previously mentioned assumptions:

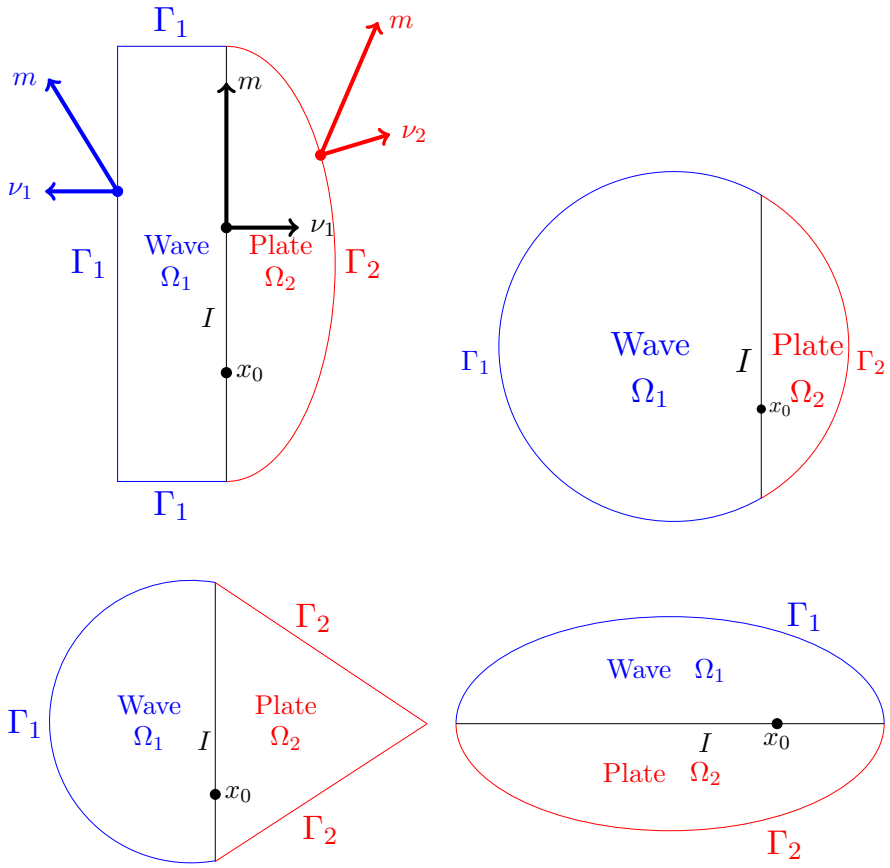

Figure 3.1: Transmission wave-plate models satisfying Assumptions (A1), (A2) and (A3)

Now, we can state our energy decay rate result.

Theorem 3.6.1. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Also, assume that Assumptions (A1), (A2) and (A3) hold. Then, there exists a constant $\mathcal{C} > 0$ such that for all initial data $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \xi_0, \zeta_0) \in D(\mathcal{A})$, the energy of system (3.1)-(3.4) satisfies the following estimate:*

$$E(t) \leq \frac{\mathcal{C}}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{3.109}$$

Before stating the proof of the above theorem, we shall preliminarily present the following two technical lemmas to be used in the proof of our polynomial-type decay estimate:

Lemma 3.6.1. *Assume that Ω_1 is a polygonal domain as described above. Assume also that Assumptions (A1) and (A3) hold. Then, there exists $\varepsilon \in (0, \frac{1}{2})$ such that the solution $y \in H^1(\Omega_1)$ of system (3.33) satisfies the following inequality*

$$-\Re\left\{\int_{\Omega_1} \Delta y(m \cdot \nabla \overline{y})dx\right\} \geq -\frac{R_1^2}{\delta}\int_{\Gamma_1} |\partial_{\nu_1} y|^2 d\Gamma - \Re\left\{\langle \partial_{\nu_1} y, m \cdot \nabla \overline{y}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}, \tag{3.110}$$

where $R_1 = \max\limits_{x\in\Gamma_1} |m(x)|$.

Proof. Before starting the proof, we assume that $y \in H^2(\Omega_1)$. Using Green's formula, we have the following Rellich's identity (see [30], also see identity 3.17 of [48]):

$$-\int_{\Omega_1} \Delta y(m\cdot\nabla\overline{y})dx = \frac{1}{2}\int_{\partial\Omega_1}(m\cdot\nu_1)|\nabla y|^2 d\Gamma - \int_{\partial\Omega_1}\partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma. \tag{3.111}$$

According to the geometric condition (3.107), we have from (3.111) that

$$-\int_{\Omega_1} \Delta y(m\cdot\nabla\overline{y})dx = \frac{1}{2}\int_{\Gamma_1}(m\cdot\nu_1)|\nabla y|^2 d\Gamma - \int_{\Gamma_1}\partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma - \int_{I}\partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma. \tag{3.112}$$

Next, for any $\varepsilon_1 > 0$, using Young's inequality, we have

$$\Re\left\{\int_{\Gamma_1}\partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma\right\} \le \frac{1}{2\varepsilon_1}\int_{\Gamma_1}|\partial_{\nu_1} y|^2 d\Gamma + \frac{R_1^2\varepsilon_1}{2}\int_{\Gamma_1}|\nabla y|^2 d\Gamma. \tag{3.113}$$

Using the geometric condition (3.108), we deduce from (3.113) that

$$\frac{1}{2}\int_{\Gamma_1}(m\cdot\nu_1)|\nabla y|^2 d\Gamma - \Re\left\{\int_{\Gamma_1}\partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma\right\} \ge \left(\frac{\delta}{2} - \frac{R_1^2\varepsilon_1}{2}\right)\int_{\Gamma_1}|\nabla y|^2 d\Gamma - \frac{1}{2\varepsilon_1}\int_{\Gamma_1}|\partial_{\nu_1} y|^2 d\Gamma. \tag{3.114}$$

Choosing $\varepsilon_1 = \frac{\delta}{2R_1^2} > 0$ in (3.114), we get

$$\begin{aligned}\frac{1}{2}\int_{\Gamma_1}(m\cdot\nu_1)|\nabla y|^2 d\Gamma - \Re\left\{\int_{\Gamma_1}\partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma\right\} &\ge \frac{\delta}{4}\int_{\Gamma_1}|\nabla y|^2 d\Gamma - \frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1} y|^2 d\Gamma \\ &\ge -\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1} y|^2 d\Gamma.\end{aligned} \tag{3.115}$$

Taking the real part of (3.112) and then inserting (3.115), we obtain

$$-\Re\left\{\int_{\Omega_1}\Delta y(m\cdot\nabla\overline{y})dx\right\} \ge -\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1} y|^2 d\Gamma - \Re\left\{\int_{I}\partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma\right\}. \tag{3.116}$$

Now, for $y \in H^1(\Omega_1)$, the solution of (3.33), we deduce from Lemma 3.3.1 and Assumption (A1) that $y \in H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ for some $\varepsilon \in (0, \frac{1}{2})$. By density, there exists a sequence $\{y_k\}_{k\ge 0} \subset H^2(\Omega_1)$ such that

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{in} \quad H^{\frac{3}{2}-\varepsilon}(\Omega_1).$$

Applying (3.116) to y_k and then passing to the limit in k, we obtain

$$-\Re\left\{\langle \Delta y, m\cdot\nabla\overline{y}\rangle_{H^{-\frac{1}{2}-\varepsilon}(\Omega_1)\times H^{\frac{1}{2}+\varepsilon}(\Omega_1)}\right\} \ge -\frac{R_1^2}{\delta}\|\partial_{\nu_1} y\|^2_{H^{-\varepsilon}(\Gamma_1)} - \Re\left\{\langle \partial_{\nu_1} y, m\cdot\nabla\overline{y}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}.$$

Since y satisfies (3.33), we see that the duality pairings become integrals in the first two terms of the above identity, which gives (3.110), as desired. ■

Lemma 3.6.2. *Assume that Ω_2 is a polygonal domain as described above. Assume also that Assumptions (A2) and (A3) hold. Then, there exists $\varepsilon \in (0, \frac{1}{2})$ such that the solution $y \in H^2(\Omega_2)$ of system (3.34) satisfies the following inequality*

$$\begin{aligned}\Re\left\{\int_{\Omega_2}\Delta^2 y(m\cdot\nabla\overline{y})dx\right\} \ge \frac{1}{2}a(y,\overline{y}) - \frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2} y|^2 d\Gamma - \frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1 y|^2 d\Gamma \\ -\frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 y|^2 d\Gamma + \Re\left\{\langle \mathcal{B}_2 y, m\cdot\nabla\overline{y}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\},\end{aligned} \tag{3.117}$$

where $R_2 = \max\limits_{x\in\Gamma_2}|m(x)|$ and M is a positive constant.

Proof. Before starting, we assume that $\mathcal{B}_1 y$ and $\mathcal{B}_2 y$ are smooth enough so that $y \in H^4(\Omega_2)$ (see [22]). Using Green's formula and identity 3.4 of [49] (see also [32] and [46]), we have

$$\begin{aligned}\int_{\Omega_2} \Delta^2 y(m\cdot\nabla\overline{y})\,dx = a(y,\overline{y}) - \int_{\partial\Omega_2} [\mathcal{B}_1 y\partial_{\nu_2}(m\cdot\nabla\overline{y}) - \mathcal{B}_2 y(m\cdot\nabla\overline{y})]\,d\Gamma \\ +\frac{1}{2}\int_{\partial\Omega_2}(m\cdot\nu_2)b(y)d\Gamma,\end{aligned} \tag{3.118}$$

where

$$b(y) = \left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\mu\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2} + 2(1-\mu)\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2.$$

It follows from the geometric condition (3.107) and the boundary condition $\mathcal{B}_1 y = 0$ on I that

$$\begin{aligned}\int_{\Omega_2} \Delta^2 y(m\cdot\nabla\overline{y})dx = a(y,\overline{y}) - \int_{\Gamma_2}\mathcal{B}_1 y\partial_{\nu_2}(m\cdot\nabla\overline{y})d\Gamma + \int_{\Gamma_2}\mathcal{B}_2 y(m\cdot\nabla\overline{y})d\Gamma \\ +\frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)b(y)d\Gamma + \int_I \mathcal{B}_2 y(m\cdot\nabla\overline{y})d\Gamma.\end{aligned} \tag{3.119}$$

A straightforward computation shows that

$$\begin{aligned}b(y) &= \left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\mu\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2} + 2(1-\mu)\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2 \\ &= \mu\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2}\right\} + (1-\mu)\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\} \\ &= \mu\,(\Delta y)^2 + (1-\mu)\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\},\end{aligned} \tag{3.120}$$

and

$$|\partial_{\nu_2}(m\cdot\nabla y)| \le |\partial_{\nu_2} y| + R_2\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\}^{\frac{1}{2}}. \tag{3.121}$$

Indeed, applying Young's inequality, we obtain from (3.121) that

$$\begin{aligned}\Re\left\{\int_{\Gamma_2}\mathcal{B}_1 y\partial_{\nu_2}(m\cdot\nabla\overline{y})d\Gamma\right\} &\le \frac{1}{2\varepsilon_2}\int_{\Gamma_2}|\mathcal{B}_1 y|^2 d\Gamma + \frac{\varepsilon_2}{2}\int_{\Gamma_2}|\partial_{\nu_2}(m\cdot\nabla\overline{y})|^2\,d\Gamma \\ &\le \frac{1}{2\varepsilon_2}\int_{\Gamma_2}|\mathcal{B}_1 y|^2 d\Gamma + \varepsilon_2\int_{\Gamma_2}|\partial_{\nu_2} y|^2\,d\Gamma \\ &\quad + R_2^2\varepsilon_2\int_{\Gamma_2}\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\}d\Gamma,\end{aligned} \tag{3.122}$$

for any $\varepsilon_2 > 0$. Taking the geometric condition (3.108) into consideration, it follows from (3.120) that

$$\begin{aligned}\frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)b(y)d\Gamma &\ge \frac{\delta}{2}\int_{\Gamma_2} b(y)d\Gamma \\ &\ge \frac{\delta\mu}{2}\int_{\Gamma_2}(\Delta y)^2\,d\Gamma + \frac{\delta(1-\mu)}{2}\int_{\Gamma_2}\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\}d\Gamma \\ &\ge \frac{\delta(1-\mu)}{2}\int_{\Gamma_2}\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\}d\Gamma.\end{aligned} \tag{3.123}$$

Combining (3.122) and (3.123), we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)b(y)d\Gamma-\Re\left\{\int_{\Gamma_2}\mathcal{B}_1y\partial_{\nu_2}(m\cdot\nabla\overline{y})d\Gamma\right\}\geq-\frac{1}{2\varepsilon_2}\int_{\Gamma_2}|\mathcal{B}_1y|^2d\Gamma-\varepsilon_2\int_{\Gamma_2}|\partial_{\nu_2}y|^2\,d\Gamma\\+\left(\frac{\delta(1-\mu)}{2}-R_2^2\varepsilon_2\right)\int_{\Gamma_2}\left\{\left(\frac{\partial^2y}{\partial x_1^2}\right)^2+\left(\frac{\partial^2y}{\partial x_2^2}\right)^2+2\left(\frac{\partial^2y}{\partial x_1\partial x_2}\right)^2\right\}d\Gamma.\end{aligned}\tag{3.124}$$

By taking $\varepsilon_2=\frac{\delta(1-\mu)}{4R_2^2}>0$ in (3.124), we deduce

$$\begin{aligned}\frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)b(y)d\Gamma-\Re\left\{\int_{\Gamma_2}\mathcal{B}_1y\partial_{\nu_2}(m\cdot\nabla\overline{y})d\Gamma\right\}\geq-\frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1y|^2d\Gamma-\frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2}y|^2\,d\Gamma\\+\frac{\delta(1-\mu)}{4}\int_{\Gamma_2}\left\{\left(\frac{\partial^2y}{\partial x_1^2}\right)^2+\left(\frac{\partial^2y}{\partial x_2^2}\right)^2+2\left(\frac{\partial^2y}{\partial x_1\partial x_2}\right)^2\right\}d\Gamma,\end{aligned}$$

from which we conclude

$$\begin{aligned}\frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)b(y)d\Gamma-\Re\left\{\int_{\Gamma_2}\mathcal{B}_1y\partial_{\nu_2}(m\cdot\nabla\overline{y})d\Gamma\right\}\geq-\frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1y|^2d\Gamma\\-\frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2}y|^2\,d\Gamma.\end{aligned}\tag{3.125}$$

On the other hand, applying Young's inequality once more gives

$$\Re\left\{\int_{\Gamma_2}\mathcal{B}_2y(m\cdot\nabla\overline{y})d\Gamma\right\}\geq-\frac{1}{2\varepsilon_3}\int_{\Gamma_2}|\mathcal{B}_2y|^2d\Gamma-\frac{R_2^2\varepsilon_3}{2}\int_{\Gamma_2}|\nabla y|^2\,d\Gamma,\tag{3.126}$$

for any $\varepsilon_3>0$. Thanks to the trace theorem (see Theorem 1.6.6 of [13]), we know that there exists a constant $C_1>0$ such that

$$\|\nabla y\|^2_{L^2(\Gamma_2)}\leq C_1\|\nabla y\|_{L^2(\Omega_2)}\|\nabla y\|_{H^1(\Omega_2)}\leq C_1\|y\|^2_{H^2(\Omega_2)}.\tag{3.127}$$

Moreover, according to the fact that $a(y,\overline{y})$ is equivalent to the usual norm of $H^2(\Omega_2)$ on $H^2_*(\Omega_2)$, we know that there exists a constant $C_2>0$ such that

$$\|y\|^2_{H^2(\Omega_2)}\leq C_2a(y,\overline{y}).\tag{3.128}$$

The combination of (3.127) and (3.128) yields

$$\|\nabla y\|^2_{L^2(\Gamma_2)}\leq Ma(y,\overline{y}),\tag{3.129}$$

where the constant $M=C_1C_2>0$. This, along with (3.126), implies that

$$\Re\left\{\int_{\Gamma_2}\mathcal{B}_2y(m\cdot\nabla\overline{y})d\Gamma\right\}\geq-\frac{1}{2\varepsilon_3}\int_{\Gamma_2}|\mathcal{B}_2y|^2d\Gamma-\frac{R_2^2M\varepsilon_3}{2}a(y,\overline{y}).\tag{3.130}$$

By taking the real part of (3.119) and inserting (3.125) and (3.130), we obtain

$$\begin{aligned}\Re\left\{\int_{\Omega_2}\Delta^2y(m\cdot\nabla\overline{y})dx\right\}\geq\left(1-\frac{R_2^2M\varepsilon_3}{2}\right)a(y,\overline{y})-\frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1y|^2d\Gamma\\-\frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2}y|^2\,d\Gamma-\frac{1}{2\varepsilon_3}\int_{\Gamma_2}|\mathcal{B}_2y|^2d\Gamma+\Re\left\{\int_I\mathcal{B}_2y(m\cdot\nabla\overline{y})d\Gamma\right\}.\end{aligned}\tag{3.131}$$

By letting $\varepsilon_3 = \frac{1}{R_2^2 M} > 0$ in (3.131), we conclude that

$$\begin{aligned}\Re\left\{\int_{\Omega_2}\Delta^2 y(m\cdot\nabla\overline{y})dx\right\} \geq \frac{1}{2}a(y,\overline{y}) - \frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1 y|^2 d\Gamma - \frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2} y|^2\, d\Gamma \\ -\frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 y|^2 d\Gamma + \Re\left\{\int_I \mathcal{B}_2 y(m\cdot\nabla\overline{y})d\Gamma\right\}.\end{aligned} \tag{3.132}$$

Now, for $y \in H^2(\Omega_2)$, which is a solution of (3.34), we deduce, based on Lemma 3.3.2 and Assumption (A2) that there exists a sequence $\{y_k\}_{k\geq 0} \subset H^4(\Omega_2)$ such that

$$y_k \underset{k\to\infty}{\longrightarrow} y \ \text{ in } \ H^2(\Omega_2) \quad \text{and} \quad \Delta^2 y_k \underset{k\to\infty}{\longrightarrow} \Delta^2 y \ \text{ in } \ L^2(\Omega_2).$$

By applying (3.132) to y_k and passing the limit in k, we obtain the desired estimation (3.117). ■

We are now ready to present the proof of Theorem 3.6.1.

Proof. In accordance to Theorem 2.3.3, a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ on a Hilbert space $\mathcal{H}$ verifies (3.109) if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}) \qquad (P1)$$

and

$$\limsup_{|\lambda|\to\infty} \frac{1}{\lambda^2}\left\|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty \qquad (P2)$$

are satisfied. Since the resolvent of the operator $\mathcal{A}$ is not compact in the energy space $\mathcal{H}$ and $0 \in \rho(\mathcal{A})$, proving $i\mathbb{R} \subset \rho(\mathcal{A})$ is equivalent to showing that $i\lambda\mathcal{I} - \mathcal{A}$ is bijective in the energy space $\mathcal{H}$ for all $\lambda \in \mathbb{R}^*$. This is established in Section 3.4 using a unique continuation theorem and Fredholm's alternative. Then, we still need to prove condition $(P2)$. This is checked by a contradiction argument. Indeed, suppose that there exists a sequence $\{\lambda_n\}_{n\geq 1} \subset \mathbb{R}^*$ and a sequence $\{U_n := (u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n)\}_{n\geq 1} \subset D(\mathcal{A})$ such that

$$|\lambda_n| \longrightarrow +\infty, \qquad \|U_n\|_{\mathcal{H}} = 1, \tag{3.133}$$

and

$$\lambda_n^2(i\lambda_n\mathcal{I}-\mathcal{A})U_n = (f_n^1, g_n^1, h_n^1, f_n^2, g_n^2, h_n^2, h_n^3) \longrightarrow 0 \quad \text{in } \mathcal{H}. \tag{3.134}$$

Our aim is to show that $\|(u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n)\|_{\mathcal{H}} \longrightarrow 0$. This condition provides a contradiction with (3.133). By detailing equation (3.134), we get the following system

$$\lambda_n^2(i\lambda_n u_n - v_n) = f_n^1 \to 0 \quad \text{in} \quad H_*^1(\Omega_1), \tag{3.135}$$
$$\lambda_n^2(i\lambda_n v_n - \Delta u_n) = g_n^1 \to 0 \quad \text{in} \quad L^2(\Omega_1), \tag{3.136}$$
$$\lambda_n^2(i\lambda_n \eta_n - v_n + \eta_n) = h_n^1 \to 0 \quad \text{in} \quad L^2(\Gamma_1), \tag{3.137}$$
$$\lambda_n^2(i\lambda_n w_n - z_n) = f_n^2 \to 0 \quad \text{in} \quad H_*^2(\Omega_2), \tag{3.138}$$
$$\lambda_n^2(i\lambda_n z_n + \Delta^2 w_n) = g_n^2 \to 0 \quad \text{in} \quad L^2(\Omega_2), \tag{3.139}$$
$$\lambda_n^2(i\lambda_n \xi_n - \partial_{\nu_2} z_n + \xi_n) = h_n^2 \to 0 \quad \text{in} \quad L^2(\Gamma_2), \tag{3.140}$$
$$\lambda_n^2(i\lambda_n \zeta_n - z_n + \zeta_n) = h_n^3 \to 0 \quad \text{in} \quad L^2(\Gamma_2). \tag{3.141}$$

We notice from (3.133) that v_n and z_n are uniformly bounded in $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively. It follows from Equations (3.135) and (3.138) that

$$\|u_n\|_{L^2(\Omega_1)} = \frac{O(1)}{\lambda_n} \qquad \text{and} \qquad \|w_n\|_{L^2(\Omega_2)} = \frac{O(1)}{\lambda_n}. \tag{3.142}$$

On the other hand, inserting Equation (3.135) (resp. (3.138)) into Equation (3.136) (resp. (3.139)), we obtain the following system

$$-\lambda_n^2 u_n - \Delta u_n = \frac{i f_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2}, \tag{3.143}$$

$$-\lambda_n^2 w_n + \Delta^2 w_n = \frac{i f_n^2}{\lambda_n} + \frac{g_n^2}{\lambda_n^2}. \tag{3.144}$$

For clarity, the proof is divided into several lemmas:

Lemma 3.6.3. *The solution $(u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n) \in D(\mathcal{A})$ of system (3.135)-(3.141) satisfies the following asymptotic behavior estimation*

$$\int_{\Gamma_1} |\eta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2}, \quad \int_{\Gamma_2} |\xi_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2} \quad \text{and} \quad \int_{\Gamma_2} |\zeta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2}. \tag{3.145}$$

Proof. Taking the inner product of (3.134) with $U_n = (u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n)$ in $\mathcal{H}$, then using the fact that U_n is uniformly bounded in $\mathcal{H}$, we get

$$\int_{\Gamma_1} |\eta_n|^2 d\Gamma + \int_{\Gamma_2} |\xi_n|^2 d\Gamma + \int_{\Gamma_2} |\zeta_n|^2 d\Gamma = \Re\{((i\lambda_n \mathcal{I} - \mathcal{A})U_n, U_n)_{\mathcal{H}}\} = \frac{o(1)}{\lambda_n^2},$$

which implies that

$$\int_{\Gamma_1} |\eta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2}, \qquad \int_{\Gamma_2} |\xi_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2} \qquad \text{and} \qquad \int_{\Gamma_2} |\zeta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2}.$$

■

Lemma 3.6.4. *The solution $(u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n) \in D(\mathcal{A})$ of system (3.135)-(3.141) satisfies the following asymptotic behavior estimation*

$$\|u_n\|_{L^2(\Gamma_1)} = \frac{o(1)}{\lambda_n}, \tag{3.146}$$

$$\|\partial_{\nu_2} w_n\|_{L^2(\Gamma_2)} = \frac{o(1)}{\lambda_n}, \tag{3.147}$$

$$\|w_n\|_{L^2(\Gamma_2)} = \frac{o(1)}{\lambda_n}. \tag{3.148}$$

Proof. First, using Equation (3.137) and the first estimation of (3.145), we get

$$\|v_n\|_{L^2(\Gamma_1)} = o(1). \tag{3.149}$$

From Equation (3.135) and the fact that the trace operator $\varphi \longmapsto \varphi|_{\Gamma_1}$ is a linear and continuous mapping from $H^1_*(\Omega_1)$ into $L^2(\Gamma_1)$, we have

$$i\lambda_n u_n - v_n = \frac{f_n^1}{\lambda_n^2} \to 0 \quad \text{in} \quad L^2(\Gamma_1).$$

Multiplying the above equation by $-i\lambda_n\overline{u_n}$ and integrating over Γ_1, we obtain

$$\int_{\Gamma_1}|\lambda_n u_n|^2d\Gamma+\int_{\Gamma_1}i\lambda_n\overline{u_n}v_nd\Gamma=-\int_{\Gamma_1}\frac{if_n^1\overline{u_n}}{\lambda_n}d\Gamma.$$

Using estimations (3.142), (3.149) and the fact that f_n^1 converges to zero in $L^2(\Gamma_1)$, we deduce from the above equation that

$$\int_{\Gamma_1}|\lambda_n u_n|^2d\Gamma=o(1),$$

which gives the desired estimation (3.146). Next, using Equations (3.140), (3.141) and the second two estimations of (3.145), we obtain

$$\|\partial_{\nu_2}z_n\|_{L^2(\Gamma_2)}=o(1)\quad\text{and}\quad\|z_n\|_{L^2(\Gamma_2)}=o(1).\tag{3.150}$$

Similar computation performed on Equation (3.138), using estimation (3.150) and the fact that f_n^2 converges to zero in $L^2(\Gamma_2)$, yield, as well,

$$\int_{\Gamma_2}|\lambda_n\partial_{\nu_2}w_n|^2d\Gamma=o(1)\quad\text{and}\quad\int_{\Gamma_2}|\lambda_n w_n|^2d\Gamma=o(1),$$

which gives the desired estimations (3.147) and (3.148). ■

Lemma 3.6.5. *The solution $(u_n,v_n,\eta_n,w_n,z_n,\xi_n,\zeta_n)\in D(\mathcal{A})$ of system (3.135)-(3.141) satisfies the following asymptotic behavior estimation*

$$\|\partial_{\nu_1}u_n\|_{L^2(\Gamma_1)}=\frac{o(1)}{\lambda_n},\tag{3.151}$$

$$\|\mathcal{B}_1w_n\|_{L^2(\Gamma_2)}=\frac{o(1)}{\lambda_n},\tag{3.152}$$

$$\|\mathcal{B}_2w_n\|_{L^2(\Gamma_2)}=\frac{o(1)}{\lambda_n}.\tag{3.153}$$

Proof. From the boundary conditions of (3.14), we have $\partial_{\nu_1}u_n=-\eta_n$ on Γ_1, $\mathcal{B}_1w_n=-\xi_n$ and $\mathcal{B}_2w_n=\zeta_n$ on Γ_2. Therefore, using estimation (3.145), the desired asymptotic estimations (3.151), (3.152) and (3.153) follow. ■

Lemma 3.6.6. *Assume that Assumptions (A1), (A2) and (A3) hold. Then, the solution $(u_n,v_n,\eta_n,w_n,z_n,\xi_n,\zeta_n)\in D(\mathcal{A})$ of system (3.135)-(3.141) satisfies the following estimation*

$$\begin{aligned}\Re\left\{\int_{\Omega_2}\Delta^2w_n(m\cdot\nabla\overline{w_n})dx-\int_{\Omega_1}\Delta u_n(m\cdot\nabla\overline{u_n})dx\right\}&\geq\frac{1}{2}a(w_n,\overline{w_n})-\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}u_n|^2d\Gamma\\&-\frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2}w_n|^2d\Gamma-\frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1w_n|^2d\Gamma-\frac{R_2^2M}{2}\int_{\Gamma_2}|\mathcal{B}_2w_n|^2d\Gamma,\end{aligned}\tag{3.154}$$

where $R_1=\max\limits_{x\in\Gamma_1}|m(x)|$, $R_2=\max\limits_{x\in\Gamma_2}|m(x)|$ and M is a positive constant independent of n.

Proof. By the definition of $D(\mathcal{A})$, u_n may be seen as the unique solution $u_n\in H^1(\Omega_1)$ of system (3.33) with $v_1=-\eta_n$ and $v_2=w_n$. As Assumptions (A1) and (A3) hold, it suffices to apply Lemma 3.6.1 for u_n to obtain

$$-\Re\left\{\int_{\Omega_1}\Delta u_n(m\cdot\nabla\overline{u_n})dx\right\}\geq-\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}u_n|^2d\Gamma-\Re\left\{\langle\partial_{\nu_1}u_n,m\cdot\nabla\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\},\tag{3.155}$$

for some $\varepsilon \in (0, \frac{1}{2})$. Again, since $U_n \in D(\mathcal{A})$, we see that $w_n \in H^2(\Omega_2)$ is a solution of system (3.34) with $v_3 = \partial_{\nu_1} u_n$, $v_4 = -\xi_n$ and $v_5 = \zeta_n$. It follows from Lemma 3.6.2, Assumptions (A2) and (A3) that

$$\begin{aligned}\Re\left\{\int_{\Omega_2} \Delta^2 w_n (m\cdot\nabla\overline{w_n})dx\right\} \geq \frac{1}{2}a(w_n,\overline{w_n}) - \frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2}w_n|^2 d\Gamma \\ - \frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1 w_n|^2 d\Gamma - \frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 w_n|^2 d\Gamma + \Re\left\{\langle \mathcal{B}_2 w_n, m\cdot\nabla\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}.\end{aligned} \tag{3.156}$$

Now, by adding (3.155) and (3.156), we obtain

$$\begin{aligned}\Re\left\{\int_{\Omega_2} \Delta^2 w_n (m\cdot\nabla\overline{w_n})dx\right\} - \Re\left\{\int_{\Omega_1}\Delta u_n (m\cdot\nabla\overline{u_n})dx\right\} \geq \frac{1}{2}a(w_n,\overline{w_n}) - \frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}u_n|^2 d\Gamma \\ -\frac{\delta(1-\mu)}{4R_2^2}\int_{\Gamma_2}|\partial_{\nu_2}w_n|^2 d\Gamma - \frac{2R_2^2}{\delta(1-\mu)}\int_{\Gamma_2}|\mathcal{B}_1 w_n|^2 d\Gamma - \frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 w_n|^2 d\Gamma \\ +\Re\left\{\langle \mathcal{B}_2 w_n, m\cdot\nabla\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} - \langle \partial_{\nu_1}u_n, m\cdot\nabla\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}.\end{aligned} \tag{3.157}$$

The boundary conditions $u_n = w_n$ and $\mathcal{B}_2 w_n = \partial_{\nu_1} u_n$ on the interface I, along with the geometric condition (3.107), lead to

$$\begin{aligned}&\langle \mathcal{B}_2 w_n, m\cdot\nabla\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} - \langle \partial_{\nu_1}u_n, m\cdot\nabla\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} \\ &= \langle \mathcal{B}_2 w_n, m\cdot(\nu_2\partial_{\nu_2}\overline{w_n} + \tau_2\partial_{\tau_2}\overline{w_n})\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} - \langle \partial_{\nu_1}u_n, m\cdot(\nu_1\partial_{\nu_1}\overline{u_n} + \tau_1\partial_{\tau_1}\overline{u_n})\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} \\ &= \langle \mathcal{B}_2 w_n, m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} - \langle \partial_{\nu_1}u_n, m\cdot\tau_1\partial_{\tau_1}\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} \\ &= \langle \mathcal{B}_2 w_n, m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} - \langle \partial_{\nu_1}u_n, m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} \\ &= \langle \mathcal{B}_2 w_n - \partial_{\nu_1}u_n, m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)} = 0.\end{aligned}$$

This, together with (3.157), proves (3.154), as desired. ■

Lemma 3.6.7. *Assume that Assumptions (A1) and (A2) hold. Then, the solution $(u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n) \in D(\mathcal{A})$ of system (3.135)-(3.141) satisfies the following asymptotic behavior estimation*

$$\int_{\Omega_1}|\lambda_n u_n|^2 dx + \int_{\Omega_2}|\lambda_n w_n|^2 dx = \int_{\Omega_1}|\nabla u_n|^2 dx + a(w_n,\overline{w_n}) + \frac{o(1)}{\lambda_n^2}. \tag{3.158}$$

Proof. Before starting the proof, we assume that $u_n \in H^2(\Omega_1)$ and $w_n \in H^4(\Omega_2)$. Multiplying Equation (3.143) (resp. (3.144)) by $-\overline{u_n}$ (resp. $-\overline{w_n}$), integrating over Ω_1 (resp. Ω_2) and applying Green's formula, we obtain

$$\int_{\Omega_1}|\lambda_n u_n|^2 dx - \int_{\Omega_1}|\nabla u_n|^2 dx + \int_{\partial\Omega_1}\partial_{\nu_1}u_n\overline{u_n}d\Gamma = -\int_{\Omega_1}\left(\frac{if_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2}\right)\overline{u_n}dx, \tag{3.159}$$

and

$$\int_{\Omega_2}|\lambda_n w_n|^2 dx - a(w_n,\overline{w_n}) + \int_{\partial\Omega_2}(\mathcal{B}_1 w_n\partial_{\nu_2}\overline{w_n} - \mathcal{B}_2 w_n\overline{w_n})d\Gamma = -\int_{\Omega_2}\left(\frac{if_n^2}{\lambda_n} + \frac{g_n^2}{\lambda_n^2}\right)\overline{w_n}dx. \tag{3.160}$$

Taking into consideration that f_n^1, g_n^1 converge to zero in $L^2(\Omega_1)$, f_n^2, g_n^2 converge to zero in $L^2(\Omega_2)$ and using (3.142), we get from (3.159) and (3.160) that

$$\int_{\Omega_1} |\lambda_n u_n|^2 \, dx - \int_{\Omega_1} |\nabla u_n|^2 \, dx + \int_{\partial\Omega_1} \partial_{\nu_1} u_n \overline{u_n} d\Gamma = \frac{o(1)}{\lambda_n^2},$$

and

$$\int_{\Omega_2} |\lambda_n w_n|^2 \, dx - a(w_n, \overline{w_n}) + \int_{\partial\Omega_2} (\mathcal{B}_1 w_n \partial_{\nu_2} \overline{w_n} - \mathcal{B}_2 w_n \overline{w_n}) \, d\Gamma = \frac{o(1)}{\lambda_n^2}.$$

Adding both equations and taking into account the boundary conditions on the interface, as well as the estimations (3.146), (3.147), (3.148), (3.151), (3.152) and (3.153), we obtain

$$\int_{\Omega_1} |\lambda_n u_n|^2 \, dx - \int_{\Omega_1} |\nabla u_n|^2 \, dx + \int_{\Omega_2} |\lambda_n w_n|^2 \, dx - a(w_n, \overline{w_n}) = \frac{o(1)}{\lambda_n^2}.$$

By following a similar argument to the end of the proof of Lemma 3.6.1 and Lemma 3.6.2, we obtain the desired estimation (3.158). ■

Lemma 3.6.8. *Assume that Assumptions (A1), (A2) and (A3) hold. Then, the solution* $(u_n, v_n, \eta_n, w_n, z_n, \xi_n, \zeta_n) \in D(\mathcal{A})$ *of system* (3.135)-(3.141) *satisfies the following asymptotic behavior estimation*

$$\int_{\Omega_1} |\lambda_n u_n|^2 \, dx = o(1), \quad \int_{\Omega_2} |\lambda_n w_n|^2 \, dx = o(1) \quad \text{and} \quad a(w_n, \overline{w_n}) = o(1). \tag{3.161}$$

Proof. Multiplying Equation (3.143) by $m \cdot \nabla \overline{u_n}$ and integrating over Ω_1, and similarly for Equation (3.144) over Ω_2, we obtain

$$-\int_{\Omega_1} \lambda_n^2 u_n \, (m \cdot \nabla \overline{u_n}) \, dx - \int_{\Omega_1} \Delta u_n \, (m \cdot \nabla \overline{u_n}) \, dx = \int_{\Omega_1} \left(\frac{i f_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2} \right) (m \cdot \nabla \overline{u_n}) \, dx, \tag{3.162}$$

and

$$-\int_{\Omega_2} \lambda_n^2 w_n \, (m \cdot \nabla \overline{w_n}) \, dx + \int_{\Omega_2} \Delta^2 w_n \, (m \cdot \nabla \overline{w_n}) \, dx = \int_{\Omega_2} \left(\frac{i f_n^2}{\lambda_n} + \frac{g_n^2}{\lambda_n^2} \right) (m \cdot \nabla \overline{w_n}) \, dx. \tag{3.163}$$

Using Cauchy-Schwarz inequality and the fact that $a(w_n, \overline{w_n})$ is equivalent to the usual norm of $H^2(\Omega_2)$ on $H^2_*(\Omega_2)$, we can observe that ∇u_n is uniformly bounded in $L^2(\Omega_1)$ and $a(w_n, \overline{w_n})$ is uniformly bounded. Additionally, f_n^1 and g_n^1 converge to zero in $L^2(\Omega_1)$, while f_n^2 and g_n^2 converge to zero in $L^2(\Omega_2)$. Hence, we obtain

$$\left| \int_{\Omega_1} \left(\frac{i f_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2} \right) (m \cdot \nabla \overline{u_n}) \, dx \right| \leq \|m\|_\infty \left(\frac{\|f_n^1\|_{L^2(\Omega_1)}}{\lambda_n} + \frac{\|g_n^1\|_{L^2(\Omega_1)}}{\lambda_n^2} \right) \|\nabla u_n\|_{L^2(\Omega_1)} = \frac{o(1)}{\lambda_n}, \tag{3.164}$$

and

$$\left| \int_{\Omega_2} \left(\frac{i f_n^2}{\lambda_n} + \frac{g_n^2}{\lambda_n^2} \right) (m \cdot \nabla \overline{w_n}) \, dx \right| \leq \|m\|_\infty \left(\frac{\|f_n^2\|_{L^2(\Omega_2)}}{\lambda_n} + \frac{\|g_n^2\|_{L^2(\Omega_2)}}{\lambda_n^2} \right) \|\nabla w_n\|_{L^2(\Omega_2)} = \frac{o(1)}{\lambda_n}. \tag{3.165}$$

On the other hand, applying Green's formula to the first terms of (3.162) and (3.163) yields

$$\int_{\Omega_1} \lambda_n^2 u_n \, (m \cdot \nabla \overline{u_n}) \, dx = \frac{1}{2} \int_{\partial\Omega_1} (m \cdot \nu_1) \, |\lambda_n u_n|^2 \, d\Gamma - \int_{\Omega_1} |\lambda_n u_n|^2 \, dx, \tag{3.166}$$

and

$$\int_{\Omega_2} \lambda_n^2 w_n \left(m \cdot \nabla \overline{w_n}\right) dx = \frac{1}{2} \int_{\partial\Omega_2} (m \cdot \nu_2) \left|\lambda_n w_n\right|^2 d\Gamma - \int_{\Omega_2} \left|\lambda_n w_n\right|^2 dx. \tag{3.167}$$

Then, substituting (3.166) (resp. (3.167)) into (3.162) (resp. (3.163)) and considering the estimates (3.164) and (3.165), we obtain

$$\int_{\Omega_1} \left|\lambda_n u_n\right|^2 dx - \frac{1}{2} \int_{\partial\Omega_1} (m \cdot \nu_1) \left|\lambda_n u_n\right|^2 d\Gamma - \int_{\Omega_1} \Delta u_n \left(m \cdot \nabla \overline{u_n}\right) dx = \frac{o(1)}{\lambda_n},$$

and

$$\int_{\Omega_2} \left|\lambda_n w_n\right|^2 dx - \frac{1}{2} \int_{\partial\Omega_2} (m \cdot \nu_2) \left|\lambda_n w_n\right|^2 d\Gamma + \int_{\Omega_2} \Delta^2 w_n \left(m \cdot \nabla \overline{w_n}\right) dx = \frac{o(1)}{\lambda_n}.$$

Adding both equations and using the boundary condition $u_n = w_n$ on the interface I and the estimates (3.146) and (3.148), then taking the real part, we obtain

$$\int_{\Omega_1} \left|\lambda_n u_n\right|^2 dx + \int_{\Omega_2} \left|\lambda_n w_n\right|^2 dx + \Re \left\{ \int_{\Omega_2} \Delta^2 w_n (m \cdot \nabla \overline{w_n}) dx - \int_{\Omega_1} \Delta u_n (m \cdot \nabla \overline{u_n}) dx \right\} = o(1). \tag{3.168}$$

As Assumptions (A1), (A2) and (A3) hold, it suffices to apply Lemma 3.6.6. Hence, inserting (3.154) into (3.168), we get

$$\begin{aligned} &\int_{\Omega_1} \left|\lambda_n u_n\right|^2 dx + \int_{\Omega_2} \left|\lambda_n w_n\right|^2 dx + \frac{1}{2} a(w_n, \overline{w_n}) - \frac{R_1^2}{\delta} \int_{\Gamma_1} |\partial_{\nu_1} u_n|^2 d\Gamma \\ &- \frac{\delta(1-\mu)}{4R_2^2} \int_{\Gamma_2} |\partial_{\nu_2} w_n|^2 d\Gamma - \frac{2R_2^2}{\delta(1-\mu)} \int_{\Gamma_2} |\mathcal{B}_1 w_n|^2 d\Gamma - \frac{R_2^2 M}{2} \int_{\Gamma_2} |\mathcal{B}_2 w_n|^2 d\Gamma = o(1). \end{aligned} \tag{3.169}$$

Finally, using the estimations (3.147), (3.151), (3.152) and (3.153), it follows from (3.169) that

$$\int_{\Omega_1} \left|\lambda_n u_n\right|^2 dx + \int_{\Omega_2} \left|\lambda_n w_n\right|^2 dx + \frac{1}{2} a(w_n, \overline{w_n}) = o(1),$$

which proves the estimation (3.161), as desired. ■

Proof of Theorem 3.6.1. Using (3.158) and (3.161), we deduce that

$$\int_{\Omega_1} |\nabla u_n|^2 dx = o(1). \tag{3.170}$$

Therefore, combining the estimates (3.145), (3.161) and (3.170), we obtain $\|U_n\|_{\mathcal{H}} \longrightarrow 0$, which leads to the desired contradiction with (3.133). Consequently, condition $(P2)$ holds and this permits us to conclude that the energy of system (3.1)-(3.4) decays polynomially to zero as t goes to infinity. The proof is thus complete.

3.7 Conclusion and Open Problems

In this chapter, we present a study of the stabilization of a transmission wave-plate model coupled through the interface with dynamical boundary controls. By employing a general criterion proposed by Arendt-Batty, we have successfully demonstrated the strong stability of the system. Additionally, we have proven the lack of exponential stability. Notably, under certain geometric assumptions on

the boundary, particularly when the interface between the wave and the plate is straight, we establish a polynomial decay rate of the energy of type $1/t$.

An intriguing question arising from the problem studied in this chapter is the optimality of the polynomial decay rate through spectral analysis. Furthermore, it would be interesting to extend the obtained results to the case of a wave-plate model subject to only one or two dynamical boundary feedbacks at the exterior boundaries of the wave's and the plate's domains. Such generalizations could lead to further insights and applications in related fields.

Chapter 4

Energy Decay Estimate for a Wave-Plate Interface Transmission Problem with Only Two Dynamical Boundary Controls

In this chapter, we consider the stabilization of a transmission model consisting of a wave equation coupled with a Kirchhoff plate equation, where only two dynamical boundary controls are applied instead of the conventional three. Specifically, we estimate the energy decay rate under the condition in which the control ξ is absent, leaving the system influenced by the remaining two controls, η and ζ. To this end, we establish the strong stability of the system based on specific geometrical conditions concerning the plate's domain. However, a polynomial decay rate of type $1/t$ is proved using a frequency domain approach. This method also imposes particular geometric conditions on the wave's domain.

4.1 Introduction

In this chapter, we extend the results obtained in the previous chapter to a wave-plate model subject to only two dynamical boundary feedbacks at the exterior boundaries of the wave's and the plate's domains. Specifically, let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary $\partial\Omega$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where Ω_i is a bounded set with Lipschitz boundary $\partial\Omega_i$, $i = 1, 2$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. We denote by I the interior of $\overline{\Omega}_1 \cap \overline{\Omega}_2$, which is called the interface between Ω_1 and Ω_2, and $\Gamma_i = \partial\Omega_i \backslash \overline{I}$ represents the exterior boundary of Ω_i with positive measure, $i = 1, 2$. We then consider the following interface transmission wave-plate model:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega_1 \times \mathbb{R}_+^*, \\ w_{tt} + \Delta^2 w = 0, & \text{in } \Omega_2 \times \mathbb{R}_+^*, \\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I \times \mathbb{R}_+^*, \\ \partial_{\nu_1} u + \eta = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ \mathcal{B}_1 w = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*, \\ \mathcal{B}_2 w - \zeta = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*, \end{cases} \tag{4.1}$$

with the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega_1, \quad \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1, \tag{4.2}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in \Omega_2, \quad \zeta(x,0) = \zeta_0(x), \quad x \in \Gamma_2, \tag{4.3}$$

where η and ζ denote the dynamical boundary controls, $\nu_i = (\nu_{i1}, \nu_{i2})$ is the unit outward normal vector along $\partial\Omega_i$, and $\tau_i = (-\nu_{i2}, \nu_{i1})$ is the unit tangent vector along $\partial\Omega_i$, $i = 1, 2$. In this context, we also introduce the constant $\mu \in \left(0, \frac{1}{2}\right)$ as the Poisson coefficient. Moreover, we define the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ on $\partial\Omega_2$ as follows:

$$\mathcal{B}_1 f = \Delta f + (1-\mu)\mathcal{C}_1 f$$

and

$$\mathcal{B}_2 f = \partial_{\nu_2}\Delta f + (1-\mu)\partial_{\tau_2}\mathcal{C}_2 f,$$

where

$$\mathcal{C}_1 f = 2\nu_{21}\nu_{22} f_{x_1x_2} - \nu_{21}^2 f_{x_2x_2} - \nu_{22}^2 f_{x_1x_1} \quad \text{and} \quad \mathcal{C}_2 f = (\nu_{21}^2 - \nu_{22}^2) f_{x_1x_2} - \nu_{21}\nu_{22}\left(f_{x_1x_1} - f_{x_2x_2}\right).$$

The damping of the system is made via the indirect damping mechanism that involves the following first-order differential equations:

$$\begin{cases} \eta_t - u_t + \eta = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ \zeta_t - w_t + \zeta = 0, & \text{on } \Gamma_2 \times \mathbb{R}_+^*. \end{cases} \tag{4.4}$$

Let u, η, w and ζ be smooth solutions of system (4.1)-(4.4). We define their associated energy by

$$E(t) = \frac{1}{2}\left\{\int_{\Omega_1}\left(|\nabla u|^2 + |u_t|^2\right)dx + \int_{\Gamma_1}|\eta|^2 d\Gamma\right\} + \frac{1}{2}\left\{a(w,\overline{w}) + \int_{\Omega_2}|w_t|^2 dx + \int_{\Gamma_2}|\zeta|^2 d\Gamma\right\},$$

where the sesquilinear form $a : H^2(\Omega_2) \times H^2(\Omega_2) \longmapsto \mathbb{C}$ is defined by

$$a(f,\overline{g}) = \int_{\Omega_2}\left[f_{x_1x_1}\overline{g}_{x_1x_1} + f_{x_2x_2}\overline{g}_{x_2x_2} + \mu\left(f_{x_1x_1}\overline{g}_{x_2x_2} + f_{x_2x_2}\overline{g}_{x_1x_1}\right) + 2(1-\mu)f_{x_1x_2}\overline{g}_{x_1x_2}\right]dx. \tag{4.5}$$

For further analysis, the following Green's formula (see [32]) is used in the study:

$$a(f,\overline{g}) = \int_{\Omega_2}\Delta^2 f\overline{g}dx + \int_{\partial\Omega_2}\left(\mathcal{B}_1 f\partial_{\nu_2}\overline{g} - \mathcal{B}_2 f\overline{g}\right)d\Gamma, \quad \forall f \in H^4(\Omega_2),\ g \in H^2(\Omega_2). \tag{4.6}$$

Lemma 4.1.1. *Let $U = (u, u_t, \eta, w, w_t, \zeta)$ be a regular solution of system (4.1)-(4.4). Then, the energy $E(t)$ satisfies the following estimation*

$$E'(t) = -\int_{\Gamma_1}|\eta|^2 d\Gamma - \int_{\Gamma_2}|\zeta|^2 d\Gamma \leq 0.$$

Proof. Firstly, multiplying the first and second equations of (4.1) by $\overline{u}_t$ and $\overline{w}_t$ respectively, then using Green's formula in Ω_1 and Ω_2 and taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}\left\{\int_{\Omega_1}\left(|\nabla u|^2 + |u_t|^2\right)dx + a(w,\overline{w}) + \int_{\Omega_2}|w_t|^2 dx\right\} + \Re\left\{\int_{\Gamma_1}\eta\overline{u}_t d\Gamma + \int_{\Gamma_2}\zeta\overline{w}_t d\Gamma\right\} = 0, \tag{4.7}$$

where $\Re$ stands for the real part of a complex number. Secondly, by inserting the equations (4.4) into the second term of (4.7), we get

$$\Re\left\{\int_{\Gamma_1}\eta\overline{u}_t d\Gamma + \int_{\Gamma_2}\zeta\overline{w}_t d\Gamma\right\} = \frac{1}{2}\frac{d}{dt}\left\{\int_{\Gamma_1}|\eta|^2 d\Gamma + \int_{\Gamma_2}|\zeta|^2 d\Gamma\right\} + \int_{\Gamma_1}|\eta|^2 d\Gamma + \int_{\Gamma_2}|\zeta|^2 d\Gamma. \tag{4.8}$$

Combining equations (4.7) and (4.8), we obtain

$$E'(t) = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} |\zeta|^2 d\Gamma.$$

■

Thus, from Lemma 4.1.1, we deduce that the system (4.1)-(4.4) is dissipative in the sense that the energy $E(t)$ is non-increasing with respect to time variable t.

In Chapter 3, we investigated the stabilization of a transmission model involving a wave equation and a Kirchhoff plate equation, employing a set of three dynamical boundary controls denoted as η, ξ, and ζ. We established the strong stability of the system without any geometric conditions. Additionally, under certain geometric assumptions, we showed the polynomial stability of the system with an energy decay rate $1/t$. We remark that the preceding results remain constructed with slight modifications when reducing the number of dynamical controls to two, specifically ξ and ζ. This is due to the useful information provided by the transmission conditions at the interface. However, an intriguing issue that has remained unexplored is the stabilization problem when one of the controls, either ξ or ζ, is removed. This chapter's primary objective is to estimate the energy decay rate when the control ξ is omitted, thereby subjecting the system to the influence of two dynamical controls, namely η and ζ. Under certain geometric assumptions, we establish the strong stability of the system, confirming that removing the control ξ does not impact the system's polynomial decay rate.

The rest of this chapter is organized as follows: In Section 4.2, we reformulate system (4.1)-(4.4) as an evolution equation and establish its well-posedness property using semigroup theory. Section 4.3 presents regularity results needed for the analyses in subsequent sections. In Section 4.4, we apply a general criteria of Arendt-Batty theorem to prove strong stability under specific geometric conditions. Finally, Section 4.5 investigates the energy decay rate, proving that the system exhibits a polynomial decay rate of type $1/t$ under the geometric conditions specified in Theorem 4.5.1.

4.2 Well-Posedness of the Problem

This section is devoted to study the well-posedness property of system (4.1)-(4.4) using the semigroup approach. We first introduce the following spaces:

$$H^1_*(\Omega_1) = \left\{ u \in H^1(\Omega_1) \mid \int_{\Omega_1} u dx = 0 \right\},$$

$$H^2_*(\Omega_2) = \left\{ w \in H^2(\Omega_2) \mid \int_{\Omega_2} w dx = \int_{\Omega_2} \nabla w dx = 0 \right\}$$

and the energy space $\mathcal{H}$ by

$$\mathcal{H} = \left\{ (u, v, \eta, w, z, \zeta) \in H^1_*(\Omega_1) \times L^2(\Omega_1) \times L^2(\Gamma_1) \times H^2_*(\Omega_2) \times L^2(\Omega_2) \times L^2(\Gamma_2) \mid u = w \text{ on } I \right\},$$

which is endowed with the following usual inner product

$$\left(U, \tilde{U}\right)_{\mathcal{H}} = \int_{\Omega_1} \left(\nabla u \cdot \nabla \overline{\tilde{u}} + v\overline{\tilde{v}}\right) dx + \int_{\Gamma_1} \eta \overline{\tilde{\eta}} d\Gamma + a(w, \overline{\tilde{w}}) + \int_{\Omega_2} z\overline{\tilde{z}} dx + \int_{\Gamma_2} \zeta \overline{\tilde{\zeta}} d\Gamma, \tag{4.9}$$

where $U = (u, v, \eta, w, z, \zeta)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{w}, \tilde{z}, \tilde{\zeta}) \in \mathcal{H}$. We next define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, \eta, w, z, \zeta) \in \mathcal{H} \mid \Delta u \in L^2(\Omega_1), \quad v \in H^1_*(\Omega_1), \quad \Delta^2 w \in L^2(\Omega_2), \quad z \in H^2_*(\Omega_2), \\ \partial_{\nu_1} u + \eta = 0 \ \text{ on } \ \Gamma_1, \quad \mathcal{B}_1 w = 0 \ \text{ on } \ \Gamma_2, \quad \mathcal{B}_2 w - \zeta = 0 \ \text{ on } \ \Gamma_2, \\ v = z, \quad \mathcal{B}_1 w = 0 \ \text{ and } \ \mathcal{B}_2 w = \partial_{\nu_1} u \ \text{ on } \ I \end{array} \right\} \tag{4.10}$$

and

$$\mathcal{A}U = \big(v, \Delta u, \gamma_1(v) - \eta, z, -\Delta^2 w, \gamma_{2,1}(z) - \zeta\big), \qquad \forall U = (u, v, \eta, w, z, \zeta) \in D(\mathcal{A}), \tag{4.11}$$

where $\gamma_1 : H^1(\Omega_1) \longmapsto L^2(\Gamma_1)$ and $\gamma_2 : H^2(\Omega_2) \longmapsto L^2(\Gamma_2) \times L^2(\Gamma_2)$ are the trace operators such that $\gamma_1(\varphi) = \varphi|_{\Gamma_1}$ and $\gamma_2(\psi) = (\gamma_{2,1}(\psi), \gamma_{2,2}(\psi)) = (\psi|_{\Gamma_2}, \partial_{\nu_2}\psi|_{\Gamma_2})$. Now, setting $U = (u, u_t, \eta, w, w_t, \zeta)$ as the state of system (4.1)-(4.4), we rewrite the problem into a first-order evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{4.12}$$

where the initial value $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \zeta_0) \in \mathcal{H}$ is given by the equations (4.2) and (4.3). For further purposes, set the Hilbert space H by

$$\mathsf{H} = \big\{(f, g) \in H^1_*(\Omega_1) \times H^2_*(\Omega_2) \mid f = g \quad \text{on} \quad I\big\},$$

equipped with the norm

$$\|(f, g)\|^2_{\mathsf{H}} = \|\nabla f\|^2_{L^2(\Omega_1)} + a(g, \overline{g}).$$

Note that using the compactness injection from $H^1(\Omega_1)$ into $L^2(\Omega_1)$ and from $H^2(\Omega_2)$ into $H^1(\Omega_2)$, we can easily show that $\|\nabla f\|^2_{L^2(\Omega_1)}$ is equivalent to the usual norm of $H^1(\Omega_1)$ on $H^1_*(\Omega_1)$ and $a(g, \overline{g})$ is equivalent to the usual norm of $H^2(\Omega_2)$ on $H^2_*(\Omega_2)$.

Proposition 4.2.1. *The unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof. First, let $U = (u, v, \eta, w, z, \zeta) \in D(\mathcal{A})$. Using (4.9) and (4.11), we have

$$\Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = \Re\Bigg\{ \int_{\Omega_1} \nabla v \cdot \nabla \overline{u} dx + \int_{\Omega_1} \Delta u \overline{v} dx + \int_{\Gamma_1} (v - \eta)\overline{\eta} d\Gamma + a(z, \overline{w}) \\ - \int_{\Omega_2} \Delta^2 w \overline{z} dx + \int_{\Gamma_2} (z - \zeta)\overline{\zeta} d\Gamma \Bigg\}.$$

We can proceed by integrating by parts and utilizing Green's formula (4.6). Given that $U \in D(\mathcal{A})$, we have

$$\Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} |\zeta|^2 d\Gamma \leq 0, \tag{4.13}$$

which implies that $\mathcal{A}$ is dissipative. Now, let us prove that $\mathcal{A}$ is maximal. For this aim, let $F = (f^1, g^1, h^1, f^2, g^2, h^2) \in \mathcal{H}$, we look for $U = (u, v, \eta, w, z, \zeta) \in D(\mathcal{A})$ unique solution of

$$U - \mathcal{A}U = F. \tag{4.14}$$

Equivalently, we have the following system

$$\begin{aligned} u - v &= f^1, \quad \text{in } \Omega_1, && (4.15)\\ v - \Delta u &= g^1, \quad \text{in } \Omega_1, && (4.16)\\ \eta - v + \eta &= h^1, \quad \text{on } \Gamma_1, && (4.17)\\ w - z &= f^2, \quad \text{in } \Omega_2, && (4.18)\\ z + \Delta^2 w &= g^2, \quad \text{in } \Omega_2, && (4.19)\\ \zeta - z + \zeta &= h^2, \quad \text{on } \Gamma_2, && (4.20) \end{aligned}$$

with the following transmission and boundary conditions

$$\begin{cases} u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I,\\ \partial_{\nu_1} u + \eta = 0, & \text{on } \Gamma_1,\\ \mathcal{B}_1 w = 0, & \text{on } \Gamma_2,\\ \mathcal{B}_2 w - \zeta = 0, & \text{on } \Gamma_2. \end{cases} \tag{4.21}$$

It follows from (4.15), (4.17), (4.18) and (4.20) that

$$v = u - f^1 \text{ in } \Omega_1, \quad 2\eta = v + h^1 \text{ on } \Gamma_1, \quad z = w - f^2 \text{ in } \Omega_2 \text{ and } 2\zeta = z + h^2 \text{ on } \Gamma_2. \tag{4.22}$$

By elimination of v, z, η and ζ in (4.16), (4.19) and (4.21), we find that u and w satisfy the following system:

$$\begin{cases} u - \Delta u = f^1 + g^1, & \text{in } \Omega_1,\\ w + \Delta^2 w = f^2 + g^2, & \text{in } \Omega_2,\\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I,\\ 2\partial_{\nu_1} u + u = f^1 - h^1, & \text{on } \Gamma_1,\\ \mathcal{B}_1 w = 0 & \text{on } \Gamma_2,\\ 2\mathcal{B}_2 w - w = h^2 - f^2, & \text{on } \Gamma_2. \end{cases} \tag{4.23}$$

Let $\phi = (\varphi, \psi) \in \mathsf{H}$. Multiplying the first equation of (4.23) by $\overline{\varphi}$ and integrating over Ω_1, multiplying the second equation of (4.23) by $\overline{\psi}$ and integrating over Ω_2, then using Green's formula, we get

$$\int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + \frac{1}{2}\int_{\Gamma_1} u\overline{\varphi} d\Gamma - \int_I \partial_{\nu_1} u \overline{\varphi} d\Gamma = \int_{\Omega_1} F^1 \overline{\varphi} dx + \frac{1}{2}\int_{\Gamma_1} F^2 \overline{\varphi} d\Gamma, \tag{4.24}$$

$$\int_{\Omega_2} w\overline{\psi} dx + a(w, \overline{\psi}) + \frac{1}{2}\int_{\Gamma_2} w\overline{\psi} d\Gamma - \int_I \left(\mathcal{B}_1 w \partial_{\nu_2} \overline{\psi} - \mathcal{B}_2 w \overline{\psi}\right) d\Gamma = \int_{\Omega_2} F^3 \overline{\psi} dx + \frac{1}{2}\int_{\Gamma_2} F^4 \overline{\psi} d\Gamma, \tag{4.25}$$

where $F^1 = f^1 + g^1$, $F^2 = f^1 - h^1$, $F^3 = f^2 + g^2$ and $F^4 = f^2 - h^2$. Adding the resulting equations and using the transmission conditions, we obtain a variational formulation of (4.23): Find $(u, w) \in \mathsf{H}$ such that

$$b((u, w), (\varphi, \psi)) = l(\varphi, \psi), \qquad \forall (\varphi, \psi) \in \mathsf{H}, \tag{4.26}$$

where

$$b((u, w), (\varphi, \psi)) = \int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + \frac{1}{2}\int_{\Gamma_1} u\overline{\varphi} d\Gamma + \int_{\Omega_2} w\overline{\psi} dx + a(w, \overline{\psi}) + \frac{1}{2}\int_{\Gamma_2} w\overline{\psi} d\Gamma$$

and

$$l(\varphi,\psi)=\int_{\Omega_1}F^1\overline{\varphi}dx+\frac{1}{2}\int_{\Gamma_1}F^2\overline{\varphi}d\Gamma+\int_{\Omega_2}F^3\overline{\psi}dx+\frac{1}{2}\int_{\Gamma_2}F^4\overline{\psi}d\Gamma.$$

It is easy to see that b is a sesquilinear, continuous, and coercive form on the space $\mathsf{H}\times\mathsf{H}$ and l is an antilinear and continuous form on H. Then, it follows by Lax-Milgram's theorem that (4.26) admits a unique solution $(u,w)\in\mathsf{H}$. By choosing $\varphi\in C_0^\infty(\Omega_1)$, $\psi=0$ in (4.26) and applying Green's formula, we have

$$\int_{\Omega_1}(u-\Delta u)\,\overline{\varphi}dx=\int_{\Omega_1}\left(f^1+g^1\right)\overline{\varphi}dx,\qquad\forall\varphi\in C_0^\infty(\Omega_1),$$

which implies that the first equation of (4.23) holds in the sense of distributions in Ω_1 and hence is satisfied in $L^2(\Omega_1)$. As $u-f^1-g^1$ belongs to $L^2(\Omega_1)$, the same holds for Δu, i.e., $\Delta u\in L^2(\Omega_1)$. In the same way, choosing $\varphi=0$ and $\psi\in C_0^\infty(\Omega_2)$ in (4.26), we see that the second equation of (4.23) holds as equality in $L^2(\Omega_2)$ and therefore $\Delta^2 w\in L^2(\Omega_2)$. Now, let us define the spaces

$$H^1_{*,I}(\Omega_1)=\left\{f\in H^1_*(\Omega_1)\mid\ f=0\ \text{ on }\ I\right\},$$

and

$$H^2_{*,I}(\Omega_2)=\left\{f\in H^2_*(\Omega_2)\mid\ f=\partial_{\nu_2}f=0\ \text{ on }\ I\right\}.$$

By taking $\varphi\in H^1_{*,I}(\Omega_1)$, $\psi=0$ and applying Green's formula in (4.26), we calculate

$$\int_{\Gamma_1}(2\partial_{\nu_1}u+u-f^1+h^1)\overline{\varphi}d\Gamma=0,\qquad\forall\varphi\in H^{\frac{1}{2}}_{*,I}(\Gamma_1),$$

where $H^{\frac{1}{2}}_{*,I}(\Gamma_1)$ is the corresponding trace space of $H^1_{*,I}(\Omega_1)$ through the operator γ_1. This implies that $2\partial_{\nu_1}u+u-f^1+h^1\in(H^{\frac{1}{2}}_{*,I}(\Gamma_1))^\perp$ in $L^2(\Gamma_1)$, and since $H^{\frac{1}{2}}_{*,I}(\Gamma_1)$ is dense in $L^2(\Gamma_1)$, we deduce that u satisfies

$$2\partial_{\nu_1}u+u=f^1-h^1,\qquad\text{on }\ \Gamma_1.$$

Similarly, by taking $\varphi=0$, $\psi\in H^2_{*,I}(\Omega_2)$ and applying Green's formula in (4.26), we deduce that w satisfies

$$\mathcal{B}_1w=0,\qquad\text{on }\ \Gamma_2,$$

as well as

$$2\mathcal{B}_2w-w=h^2-f^2,\qquad\text{on }\ \Gamma_2.$$

Coming back to (4.26) and again applying Green's formula, we calculate

$$\int_I\mathcal{B}_1w\partial_{\nu_2}\overline{\psi}d\Gamma-\int_I(\mathcal{B}_2w-\partial_{\nu_1}u)\,\overline{\psi}d\Gamma=0,\qquad\forall\psi\in H^{\frac{3}{2}}_*(I),$$

where $H^{\frac{3}{2}}_*(I)$ is the corresponding trace space of $H^2_*(\Omega_2)$ through the operator $\psi\longmapsto\psi|_I$. Due to the density of $H^{\frac{3}{2}}_*(I)$ into $L^2(I)$, we can easily check that u and w satisfy the transmission conditions of (4.23). Therefore, we deduce that system (4.23) has a unique solution $(u,w)\in\mathsf{H}$ such that $\Delta u\in L^2(\Omega_1)$ and $\Delta^2w\in L^2(\Omega_2)$. Finally, by setting

$$v:=u-f^1,\ \eta:=-\partial_{\nu_1}u,\ z:=w-f^2\ \text{ and }\ \zeta:=\mathcal{B}_2w,$$

we conclude that there exists a unique $U=(u,v,\eta,w,z,\zeta)\in D(\mathcal{A})$ solution of equation (4.14) and thus the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$. The proof is thus complete. ■

According to Lumer-Philips theorem (see [44]), Proposition 4.2.1 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq0}$ in $\mathcal{H}$ which gives the well-posedness of (4.12). Then, we have the following result:

Theorem 4.2.1. *For all $U_0 \in \mathcal{H}$, system (4.12) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the system (4.12) admits a unique strong solution

$$U(t) \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

4.3 Some Regularity Results

In the following sections, we aim to establish energy estimates for the system (4.12). Our stability results rely on a frequency domain approach combined with a multiplier technique, which in turn requires the application of certain Green's formulas in Ω_1 and Ω_2. To justify these formulas, we need some regularity results proved in [11] and [16]. Therefore, from now on, we assume that Ω_1 and Ω_2 have polygonal boundaries, meaning that their boundaries are piecewise smooth. For $i = 1, 2$, let $\omega_{i,j}$, $j = 1, \ldots, N_i$ represent the interior angles at the corners of Ω_i enclosed between two consecutive curves. To achieve the required regularity, we rely on assumptions (A1) and (A2) from Chapter 3, which pertain to the corner angles of Ω_1 and Ω_2. The regularity result is summarized in the following proposition:

Proposition 4.3.1. *Let $U = (u, v, \eta, w, z, \zeta) \in D(\mathcal{A})$. Assume that Ω_1 and Ω_2 are polygonal domains as described above, satisfying Assumptions (A1) and (A2), respectively. Then, there exists $\varepsilon \in (0, \frac{1}{2})$ such that u belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ and there exists a sequence $\{w_k\}_{k\geq 0} \subset H^4(\Omega_2)$ such that w_k converges to w in $H^2(\Omega_2)$ and $\Delta^2 w_k$ converges to $\Delta^2 w$ in $L^2(\Omega_2)$.*

Proof. We apply arguments inspired by Lemma 3.3.1 and Lemma 3.3.2 from Chapter 3. ■

4.4 Strong Stability with Non-Compact Resolvent

In this section, we establish the strong stability of the system (4.1)-(4.4) in the sense that its energy $E(t)$ converges to zero as t goes to infinity for all initial data in $\mathcal{H}$. Since the resolvent of $\mathcal{A}$ is not compact, classical methods such as Lasalle's invariance principle [55] or the spectrum decomposition theory of Benchimol [9] are not applicable here. Instead, we rely on the general criteria provided by Arendt-Batty [7], which states that in a reflexive Banach space, a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable if $\mathcal{A}$ has no eigenvalues on the imaginary axis, and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$. To proceed, we impose the following geometric condition on the plate, required for integration in Ω_2:

Assumption (A4). *Assume that there exists a fixed point $x_0 \in \mathbb{R}^2$ such that, putting $m(x) = x - x_0$, we have*

$$(m \cdot \nu_2) = 0, \quad \forall x \in I, \tag{4.27}$$

and

$$(m \cdot \nu_2) \geq 0, \quad \forall x \in \Gamma_2, \tag{4.28}$$

where $(\cdot)$ designates the scalar product in $\mathbb{R}^2$.

Remark 4.4.1. *We remark that if condition (4.27) is satisfied, then the interface I is straight. Hence, from now on, we assume that the interface is straight and the interior angles $\omega_{i,1}$ and ω_{i,N_i}, for $i = 1, 2$, are the angles at the extremities of I.*

The main result of this section is summarized in the following theorem:

Theorem 4.4.1. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Assume also that Assumptions (A1), (A2) and (A4) hold. Then, the semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable in the energy space $\mathcal{H}$ in the sense that*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

In the proof of Theorem 4.4.1, we shall use the following lemmas:

Lemma 4.4.1. *Assume that Assumptions (A1), (A2) and (A4) hold. Then, the operator $\mathcal{A}$ has no pure imaginary eigenvalues.*

Proof. Let $\lambda \in \mathbb{R}$ and let $U = (u, v, \eta, w, z, \zeta) \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U. \tag{4.29}$$

Equivalently, we have the following system of equations

$$\begin{aligned}
v &= i\lambda u, \quad \text{in } \Omega_1, && (4.30)\\
\Delta u &= i\lambda v, \quad \text{in } \Omega_1, && (4.31)\\
v - \eta &= i\lambda \eta, \quad \text{on } \Gamma_1, && (4.32)\\
z &= i\lambda w, \quad \text{in } \Omega_2, && (4.33)\\
-\Delta^2 w &= i\lambda z, \quad \text{in } \Omega_2, && (4.34)\\
z - \zeta &= i\lambda \zeta, \quad \text{on } \Gamma_2. && (4.35)
\end{aligned}$$

By using equations (4.29) and (4.13), a direct computation leads to the following:

$$0 = \Re\left\{i\lambda \|U\|^2_{\mathcal{H}}\right\} = \Re\{(\mathcal{A}U, U)_{\mathcal{H}}\} = -\int_{\Gamma_1} |\eta|^2 d\Gamma - \int_{\Gamma_2} |\zeta|^2 d\Gamma,$$

which implies that

$$\eta = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \zeta = 0 \text{ on } \Gamma_2. \tag{4.36}$$

It follows from (4.10) that

$$\partial_{\nu_1} u = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \mathcal{B}_2 w = 0 \text{ on } \Gamma_2, \tag{4.37}$$

and from (4.32) and (4.35) that

$$v = 0 \text{ on } \Gamma_1 \quad \text{and} \quad z = 0 \text{ on } \Gamma_2. \tag{4.38}$$

Now, we need to consider two distinct cases:

Case 1. If $\lambda = 0$, then $v = 0$ in Ω_1 and $z = 0$ in Ω_2, and we obtain the following system

$$\begin{aligned}
-\Delta u &= 0, \quad \text{in } \Omega_1, && (4.39)\\
\Delta^2 w &= 0, \quad \text{in } \Omega_2, && (4.40)\\
\partial_{\nu_1} u &= 0, \quad \text{on } \Gamma_1, && (4.41)\\
\mathcal{B}_1 w = \mathcal{B}_2 w &= 0, \quad \text{on } \Gamma_2, && (4.42)
\end{aligned}$$

with the following transmission conditions

$$u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u \quad \text{on} \quad I. \tag{4.43}$$

Multiplying equations (4.39) and (4.40) by $\overline{u}$ and $\overline{w}$ respectively, integrating over Ω_1 and Ω_2, then using Green's formula and equations (4.41) and (4.42), we get

$$\int_{\Omega_1} |\nabla u|^2 dx - \int_I \partial_{\nu_1} u \overline{u} d\Gamma = 0, \tag{4.44}$$

$$a(w, \overline{w}) - \int_I (\mathcal{B}_1 w \partial_{\nu_2} \overline{w} - \mathcal{B}_2 w \overline{w}) \, d\Gamma = 0. \tag{4.45}$$

Adding the resulting equations and taking (4.43) into consideration, we obtain

$$\int_{\Omega_1} |\nabla u|^2 dx + a(w, \overline{w}) = 0, \tag{4.46}$$

which leads to

$$u = 0 \quad \text{in} \quad \Omega_1 \quad \text{and} \quad w = 0 \quad \text{in} \quad \Omega_2. \tag{4.47}$$

Hence, we get

$$U = 0.$$

Case 2. If $\lambda \neq 0$, then using equations (4.30), (4.33) and (4.38), we get

$$u = 0 \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad w = 0 \quad \text{on} \quad \Gamma_2. \tag{4.48}$$

Inserting (4.30) in (4.31), we obtain

$$\begin{cases} \lambda^2 u + \Delta u = 0, & \text{in} \quad \Omega_1, \\ u = \partial_{\nu_1} u = 0, & \text{on} \quad \Gamma_1. \end{cases} \tag{4.49}$$

Thus, from the above system and by using a unique continuation theorem (see [35]), we obtain

$$u = 0 \quad \text{in} \quad \Omega_1. \tag{4.50}$$

Inserting (4.33) into (4.34) and using the definition of $D(\mathcal{A})$, we notice that w satisfies

$$\begin{cases} \lambda^2 w - \Delta^2 w = 0, & \text{in} \quad \Omega_2, \\ w = \mathcal{B}_1 w = \mathcal{B}_2 w = 0, & \text{on} \quad \partial\Omega_2. \end{cases} \tag{4.51}$$

Multiplying the first equation of (4.51) by $-m \cdot \nabla \overline{w}$ and integrating over Ω_2, we obtain

$$-\int_{\Omega_2} \lambda^2 w \, (m \cdot \nabla \overline{w}) \, dx + \int_{\Omega_2} \Delta^2 w \, (m \cdot \nabla \overline{w}) \, dx = 0. \tag{4.52}$$

Applying Green's formula and identity 3.4 of [49] (see also [32] and [46]), we have

$$\int_{\Omega_2} \Delta^2 w (m \cdot \nabla \overline{w}) dx = a(w, \overline{w}) - \int_{\partial\Omega_2} [\mathcal{B}_1 w \partial_{\nu_2} (m \cdot \nabla \overline{w}) - \mathcal{B}_2 w (m \cdot \nabla \overline{w})] \, d\Gamma + \frac{1}{2} \int_{\partial\Omega_2} (m \cdot \nu_2) b(w) d\Gamma,$$

where

$$b(w) = \left(\frac{\partial^2 w}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 w}{\partial x_2^2} \right)^2 + 2\mu \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 w}{\partial x_2^2} + 2(1-\mu) \left(\frac{\partial^2 w}{\partial x_1 \partial x_2} \right)^2.$$

It follows from the geometric condition (4.27) and the boundary condition $\mathcal{B}_1 w = \mathcal{B}_2 w = 0$ on $\partial\Omega_2$ that

$$\int_{\Omega_2} \Delta^2 w(m \cdot \nabla \overline{w}) dx = a(w, \overline{w}) + \frac{1}{2} \int_{\Gamma_2} (m \cdot \nu_2) b(w) d\Gamma. \tag{4.53}$$

A straightforward computation shows that

$$\begin{aligned} b(w) &= \left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2\mu \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 w}{\partial x_2^2} + 2(1-\mu) \left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 \\ &= \mu \left\{ \left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2 \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 w}{\partial x_2^2} \right\} + (1-\mu) \left\{ \left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2 \left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 \right\} \\ &= \mu \left(\Delta w\right)^2 + (1-\mu) \left\{ \left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2 \left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 \right\} \geq 0. \end{aligned} \tag{4.54}$$

Taking the geometric condition (4.28) into consideration, it follows from (4.53) and (4.54) that

$$\int_{\Omega_2} \Delta^2 w(m \cdot \nabla \overline{w}) dx \geq a(w, \overline{w}). \tag{4.55}$$

On the other hand, applying Green's formula to the first term of (4.52) yields

$$-\int_{\Omega_2} \lambda^2 w \left(m \cdot \nabla \overline{w}\right) dx = -\frac{1}{2} \int_{\partial\Omega_2} (m \cdot \nu_2) \left|\lambda w\right|^2 d\Gamma + \int_{\Omega_2} \left|\lambda w\right|^2 dx. \tag{4.56}$$

Inserting (4.55) and (4.56) into (4.52) and using the boundary condition $w = 0$ on $\partial\Omega_2$, we obtain

$$a(w, \overline{w}) + \int_{\Omega_2} \left|\lambda w\right|^2 dx \leq 0,$$

from which one concludes that

$$w = 0 \quad \text{in} \quad \Omega_2. \tag{4.57}$$

Consequently, from equations (4.30), (4.33), (4.36), (4.50) and (4.57), we get

$$U = 0.$$

The proof is thus complete. ■

Lemma 4.4.2. *Assume that Assumptions (A1), (A2) and (A4) hold. Then, the operator $i\lambda\mathcal{I} - \mathcal{A}$ is surjective for all real number $\lambda \in \mathbb{R}$.*

Proof. For any given $F = (f^1, g^1, h^1, f^2, g^2, h^2) \in \mathcal{H}$, we are looking for $U = (u, v, \eta, w, z, \zeta) \in D(\mathcal{A})$ that solves the following system:

$$(i\lambda\mathcal{I} - \mathcal{A})U = F, \tag{4.58}$$

which is equivalent to the following equations:

$$\begin{aligned} i\lambda u - v &= f^1, \quad \text{in} \quad \Omega_1, && (4.59)\\ i\lambda v - \Delta u &= g^1, \quad \text{in} \quad \Omega_1, && (4.60)\\ i\lambda \eta - v + \eta &= h^1, \quad \text{on} \quad \Gamma_1, && (4.61)\\ i\lambda w - z &= f^2, \quad \text{in} \quad \Omega_2, && (4.62)\\ i\lambda z + \Delta^2 w &= g^2, \quad \text{in} \quad \Omega_2, && (4.63)\\ i\lambda \zeta - z + \zeta &= h^2, \quad \text{on} \quad \Gamma_2. && (4.64) \end{aligned}$$

By eliminating v, z, η, and ζ from the above equations and using (4.10), we obtain the following system:

$$\begin{cases} \lambda^2 u + \Delta u = -(i\lambda f^1 + g^1), & \text{in } \Omega_1, \\ \lambda^2 w - \Delta^2 w = -(i\lambda f^2 + g^2), & \text{in } \Omega_2, \\ u = w, \quad \mathcal{B}_1 w = 0, \quad \mathcal{B}_2 w = \partial_{\nu_1} u, & \text{on } I, \\ \partial_{\nu_1} u + \dfrac{i\lambda}{i\lambda+1} u = \dfrac{1}{i\lambda+1}(f^1 - h^1), & \text{on } \Gamma_1, \\ \mathcal{B}_1 w = 0, & \text{on } \Gamma_2, \\ \mathcal{B}_2 w - \dfrac{i\lambda}{i\lambda+1} w = \dfrac{1}{i\lambda+1}(h^2 - f^2), & \text{on } \Gamma_2. \end{cases} \tag{4.65}$$

Let $\phi = (\varphi, \psi) \in \mathsf{H}$. Multiplying the first equation of (4.65) by $\overline{\varphi}$ and integrating over Ω_1, and multiplying the second equation of (4.65) by $\overline{\psi}$ and integrating over Ω_2, we use Green's formula to get the following equations:

$$\begin{aligned} -\lambda^2 \int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + \frac{i\lambda}{i\lambda+1} \int_{\Gamma_1} u\overline{\varphi} d\Gamma - \int_I \partial_{\nu_1} u \overline{\varphi} d\Gamma \\ = \int_{\Omega_1} F^1 \overline{\varphi} dx + \frac{1}{i\lambda+1} \int_{\Gamma_1} F^2 \overline{\varphi} d\Gamma, \end{aligned} \tag{4.66}$$

and

$$\begin{aligned} -\lambda^2 \int_{\Omega_2} w\overline{\psi} dx + a(w, \overline{\psi}) + \frac{i\lambda}{i\lambda+1} \int_{\Gamma_2} w\overline{\psi} d\Gamma - \int_I \left(\mathcal{B}_1 w \partial_{\nu_2} \overline{\psi} - \mathcal{B}_2 w \overline{\psi} \right) d\Gamma \\ = \int_{\Omega_2} F^3 \overline{\psi} dx + \frac{1}{i\lambda+1} \int_{\Gamma_2} F^4 \overline{\psi} d\Gamma, \end{aligned} \tag{4.67}$$

where

$$F^1 = i\lambda f^1 + g^1, \quad F^2 = f^1 - h^1, \quad F^3 = i\lambda f^2 + g^2 \quad \text{and} \quad F^4 = f^2 - h^2.$$

Adding equations (4.66) and (4.67) and using the transmission conditions, we obtain

$$\begin{aligned} -\lambda^2 \left(\int_{\Omega_1} u\overline{\varphi} dx + \int_{\Omega_2} w\overline{\psi} dx \right) + \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + i\lambda\Lambda \int_{\Gamma_1} u\overline{\varphi} d\Gamma + a(w, \overline{\psi}) + i\lambda\Lambda \int_{\Gamma_2} w\overline{\psi} d\Gamma \\ = \int_{\Omega_1} F^1 \overline{\varphi} dx + \Lambda \int_{\Gamma_1} F^2 \overline{\varphi} d\Gamma + \int_{\Omega_2} F^3 \overline{\psi} dx + \Lambda \int_{\Gamma_2} F^4 \overline{\psi} d\Gamma, \end{aligned} \tag{4.68}$$

where

$$\Lambda = \frac{1 - i\lambda}{\lambda^2 + 1}. \tag{4.69}$$

Here we note that Lax-Milgram's theorem cannot be applied because coercivity is not available. Therefore, we use a compact perturbation argument. For that purpose, let us introduce the sesquilinear form

$$b_\lambda((u,w),(\varphi,\psi)) = \int_{\Omega_1} \nabla u \cdot \nabla \overline{\varphi} dx + i\lambda\Lambda \int_{\Gamma_1} u\overline{\varphi} d\Gamma + a(w, \overline{\psi}) + i\lambda\Lambda \int_{\Gamma_2} w\overline{\psi} d\Gamma, \quad \forall\, (\varphi, \psi) \in \mathsf{H}. \tag{4.70}$$

This sesquilinear form b_λ is continuous and coercive on $\mathsf{H} \times \mathsf{H}$. Then, by Lax-Milgram's theorem, the operator

$$B_\lambda : \mathsf{H} \to \mathsf{H}' : (\boldsymbol{u}, \boldsymbol{w}) \to B_\lambda(\boldsymbol{u}, \boldsymbol{w}),$$

with $B_\lambda(\boldsymbol{u},\boldsymbol{w})((\boldsymbol{\varphi},\boldsymbol{\psi})) := b_\lambda((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{\varphi},\boldsymbol{\psi}))\ \ \forall(\varphi,\psi)\in \mathsf{H}$, is an isomorphism, where H' is the dual space of H. Now, let us set

$$R_\lambda : \mathsf{H}\to\mathsf{H}' : (\boldsymbol{u},\boldsymbol{w})\to R_\lambda(\boldsymbol{u},\boldsymbol{w}),$$

with

$$R_\lambda(\boldsymbol{u},\boldsymbol{w})((\boldsymbol{\varphi},\boldsymbol{\psi})) = -\lambda^2\left(\int_{\Omega_1} u\overline{\varphi}dx+\int_{\Omega_2} w\overline{\psi}dx\right),\quad \forall\,(\varphi,\psi)\in\mathsf{H}.$$

As R_λ is a compact operator, we deduce that $B_\lambda+R_\lambda$ is a Fredholm operator of index zero from H to H'. Now by setting

$$L_\lambda(\varphi,\psi)=\int_{\Omega_1}F^1\overline{\varphi}dx+\Lambda\int_{\Gamma_1}F^2\overline{\varphi}d\Gamma+\int_{\Omega_2}F^3\overline{\psi}dx+\Lambda\int_{\Gamma_2}F^4\overline{\psi}d\Gamma,$$

we notice that (4.68) is equivalent to

$$(B_\lambda+R_\lambda)(\boldsymbol{u},\boldsymbol{w})=L_\lambda\quad\text{in }\mathsf{H}'. \tag{4.71}$$

Hence, problem (4.68) admits a unique solution $(\boldsymbol{u},\boldsymbol{w})\in\mathsf{H}$ if and only if $B_\lambda+R_\lambda$ is invertible. Since $B_\lambda+R_\lambda$ is a Fredholm operator, it is enough to prove that $B_\lambda+R_\lambda$ is injective, i.e.,

$$\ker(B_\lambda+R_\lambda)=\{0\}.$$

Let us now fix $(\boldsymbol{u},\boldsymbol{w})\in\ker(B_\lambda+R_\lambda)$, which means it satisfies

$$\begin{aligned}-\lambda^2\left(\int_{\Omega_1}\boldsymbol{u}\overline{\boldsymbol{\varphi}}dx+\int_{\Omega_2}\boldsymbol{w}\overline{\boldsymbol{\psi}}dx\right)+\int_{\Omega_1}\nabla\boldsymbol{u}\cdot\nabla\overline{\boldsymbol{\varphi}}dx+i\lambda\Lambda\int_{\Gamma_1}\boldsymbol{u}\overline{\boldsymbol{\varphi}}d\Gamma\\ +a(\boldsymbol{w},\overline{\boldsymbol{\psi}})+i\lambda\Lambda\int_{\Gamma_2}\boldsymbol{w}\overline{\boldsymbol{\psi}}d\Gamma=0,\quad\forall\,(\boldsymbol{\varphi},\boldsymbol{\psi})\in\mathsf{H}.\end{aligned} \tag{4.72}$$

Thus, if we set

$$\boldsymbol{v}=i\lambda\boldsymbol{u},\quad \boldsymbol{\eta}=i\lambda\Lambda\boldsymbol{u},\quad \boldsymbol{z}=i\lambda\boldsymbol{w},\quad\text{and}\quad \boldsymbol{\zeta}=i\lambda\Lambda\boldsymbol{w},$$

we can conclude that $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{w},\boldsymbol{z},\boldsymbol{\zeta})\in D(\mathcal{A})$ is a solution of

$$(i\lambda-\mathcal{A})\boldsymbol{U}=0.$$

Using Lemma 4.4.1, we deduce that $\boldsymbol{U}=0$. This shows that $B_\lambda+R_\lambda$ is invertible, and therefore a unique solution $(u,w)\in\mathsf{H}$ of (4.71) exists. At this stage, by setting

$$v=i\lambda u-f^1,\quad \eta=i\lambda\Lambda u+\Lambda(h^1-f^1),\quad z=i\lambda w-f^2\quad\text{and}\quad \zeta=i\lambda\Lambda w+\Lambda(h^2-f^2),$$

we conclude that $(u,v,\eta,w,z,\zeta)\in D(\mathcal{A})$ is a solution of (4.58) and the proof is thus complete. ■

Proof of Theorem 4.4.1. Using Lemma 4.4.1, we conclude that the operator $\mathcal{A}$ has no pure imaginary eigenvalues. Additionally, by Lemma 4.4.2, we have $Im(i\lambda\mathcal{I}-\mathcal{A})=\mathcal{H}$ for all real numbers $\lambda\in\mathbb{R}$. Therefore, the closed graph theorem of Banach implies that $\sigma(\mathcal{A})\cap i\mathbb{R}=\emptyset$. Following Arendt-Batty (see [7]), the C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq0}$ is strongly stable and the proof is complete.

4.5 Polynomial Stability

Following Theorem 3.5.1 in Chapter 3, exponential decay does not occur when three dynamical controls are employed. Consequently, exponential stability is also absent for the system (4.1)-(4.4) with only two dynamical controls. Given this, it is natural to hope for polynomial stability for smooth solutions of the system (4.1)-(4.4) using a multiplier method. A key component of the analysis is the frequency domain approach, specifically the key ingredient is Theorem 2.4 of [12] (see also [8] and [38]). To proceed with our polynomial energy decay result, we impose an additional geometric condition on the wave domain, due to the integration requirements in Ω_1:

Assumption (A5). *Assume that there exists a fixed point $x_0 \in \mathbb{R}^2$ such that, putting $m(x) = x - x_0$, we have*

$$(m \cdot \nu_1) \geq \delta, \quad \forall x \in \Gamma_1, \tag{4.73}$$

where δ is a positive real number and $(\cdot)$ designates the scalar product in $\mathbb{R}^2$.

We are now ready to present our result on the energy decay rate.

Theorem 4.5.1. *Assume that Ω_1 and Ω_2 are polygonal domains as described above. Also, assume that Assumptions (A1), (A2), (A4) and (A5) hold. Then, there exists a constant $\mathcal{C} > 0$ such that for all initial data $U_0 = (u_0, u_1, \eta_0, w_0, w_1, \zeta_0) \in D(\mathcal{A})$, the energy of system (4.1)-(4.4) satisfies the following estimate:*

$$E(t) \leq \frac{\mathcal{C}}{t} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{4.74}$$

Before starting the proof of the above theorem, we first introduce the following two technical lemmas, which will be used in establishing our polynomial-type decay estimate.

Lemma 4.5.1. *Assume that Ω_1 is a polygonal domain as described above. Assume also that Assumptions (A1), (A4) and (A5) hold. Then, there exists $\varepsilon \in (0, \frac{1}{2})$ such that the solution $y \in H^1(\Omega_1)$ of system*

$$\begin{cases} \Delta y \in L^2(\Omega_1), \\ \partial_{\nu_1} y = v_1 \in L^2(\Gamma_1), \quad on \ \ \Gamma_1, \\ y = v_2 \in H^{\frac{3}{2}}(I), \quad\quad on \ \ I, \end{cases} \tag{4.75}$$

satisfies the following inequality

$$-\Re\left\{\int_{\Omega_1} \Delta y (m \cdot \nabla \overline{y}) dx\right\} \geq -\frac{R_1^2}{\delta} \int_{\Gamma_1} |\partial_{\nu_1} y|^2 d\Gamma - \Re\left\{\langle \partial_{\nu_1} y, m \cdot \nabla \overline{y} \rangle_{H^{-\varepsilon}(I) \times H^{\varepsilon}(I)}\right\}, \tag{4.76}$$

where $R_1 = \max\limits_{x \in \Gamma_1} |m(x)|$.

Proof. Before starting the proof, we assume that $y \in H^2(\Omega_1)$. Using Green's formula, we have the following Rellich's identity (see [30], also see identity 3.17 of [48]):

$$-\int_{\Omega_1} \Delta y (m \cdot \nabla \overline{y}) dx = \frac{1}{2} \int_{\partial\Omega_1} (m \cdot \nu_1) |\nabla y|^2 d\Gamma - \int_{\partial\Omega_1} \partial_{\nu_1} y (m \cdot \nabla \overline{y}) d\Gamma. \tag{4.77}$$

According to the geometric condition (4.27), we have from (4.77) that

$$-\int_{\Omega_1} \Delta y (m \cdot \nabla \overline{y}) dx = \frac{1}{2} \int_{\Gamma_1} (m \cdot \nu_1) |\nabla y|^2 d\Gamma - \int_{\Gamma_1} \partial_{\nu_1} y (m \cdot \nabla \overline{y}) d\Gamma - \int_I \partial_{\nu_1} y (m \cdot \nabla \overline{y}) d\Gamma. \tag{4.78}$$

Next, for any $\varepsilon_1 > 0$, using Young's inequality, we have

$$\Re\left\{\int_{\Gamma_1} \partial_{\nu_1} y(m\cdot\nabla\overline{y})d\Gamma\right\} \leq \frac{1}{2\varepsilon_1}\int_{\Gamma_1}|\partial_{\nu_1}y|^2 d\Gamma + \frac{R_1^2\varepsilon_1}{2}\int_{\Gamma_1}|\nabla y|^2 d\Gamma. \tag{4.79}$$

Using the geometric condition (4.73), we deduce from (4.79) that

$$\begin{aligned}\frac{1}{2}\int_{\Gamma_1}(m\cdot\nu_1)|\nabla y|^2 d\Gamma - \Re\left\{\int_{\Gamma_1}\partial_{\nu_1}y(m\cdot\nabla\overline{y})d\Gamma\right\} \geq \left(\frac{\delta}{2}-\frac{R_1^2\varepsilon_1}{2}\right)\int_{\Gamma_1}|\nabla y|^2 d\Gamma \\ -\frac{1}{2\varepsilon_1}\int_{\Gamma_1}|\partial_{\nu_1}y|^2 d\Gamma.\end{aligned} \tag{4.80}$$

Choosing $\varepsilon_1 = \frac{\delta}{2R_1^2} > 0$ in (4.80), we get

$$\begin{aligned}\frac{1}{2}\int_{\Gamma_1}(m\cdot\nu_1)|\nabla y|^2 d\Gamma - \Re\left\{\int_{\Gamma_1}\partial_{\nu_1}y(m\cdot\nabla\overline{y})d\Gamma\right\} &\geq \frac{\delta}{4}\int_{\Gamma_1}|\nabla y|^2 d\Gamma - \frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}y|^2 d\Gamma \\ &\geq -\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}y|^2 d\Gamma.\end{aligned} \tag{4.81}$$

Taking the real part of (4.78) and then inserting (4.81), we obtain

$$-\Re\left\{\int_{\Omega_1}\Delta y(m\cdot\nabla\overline{y})dx\right\} \geq -\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}y|^2 d\Gamma - \Re\left\{\int_I \partial_{\nu_1}y(m\cdot\nabla\overline{y})d\Gamma\right\}. \tag{4.82}$$

Now, for $y\in H^1(\Omega_1)$, the solution of (4.75), we deduce from Lemma 3.3.1 and Assumption (A1) that $y\in H^{\frac{3}{2}-\varepsilon}(\Omega_1)$ for some $\varepsilon\in(0,\frac{1}{2})$. By density, there exists a sequence $\{y_k\}_{k\geq 0}\subset H^2(\Omega_1)$ such that

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{in} \quad H^{\frac{3}{2}-\varepsilon}(\Omega_1).$$

Applying (4.82) to y_k and then passing to the limit in k, we obtain

$$-\Re\left\{\langle\Delta y, m\cdot\nabla\overline{y}\rangle_{H^{-\frac{1}{2}-\varepsilon}(\Omega_1)\times H^{\frac{1}{2}+\varepsilon}(\Omega_1)}\right\} \geq -\frac{R_1^2}{\delta}\|\partial_{\nu_1}y\|^2_{H^{-\varepsilon}(\Gamma_1)} - \Re\left\{\langle\partial_{\nu_1}y, m\cdot\nabla\overline{y}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}.$$

Since y satisfies (4.75), we see that the duality pairings become integrals in the first two terms of the above identity, which gives (4.76), as desired. ■

Lemma 4.5.2. *Assume that Ω_2 is a polygonal domain as described above. Assume also that Assumptions (A2) and (A4) hold. Then, there exists $\varepsilon\in(0,\frac{1}{2})$ such that the solution $y\in H^2(\Omega_2)$ of system*

$$\begin{cases}\Delta^2 y\in L^2(\Omega_2),\\ \mathcal{B}_1 y = 0, & \text{on } I,\\ \mathcal{B}_2 y = v_3\in H^{-\varepsilon}(I), & \text{on } I,\\ \mathcal{B}_1 y = v_4\in L^2(\Gamma_2), & \text{on } \Gamma_2,\\ \mathcal{B}_2 y = v_5\in L^2(\Gamma_2), & \text{on } \Gamma_2,\end{cases} \tag{4.83}$$

satisfies the following inequality

$$\Re\left\{\int_{\Omega_2}\Delta^2 y(m\cdot\nabla\overline{y})dx\right\} \geq \frac{1}{2}a(y,\overline{y}) - \frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 y|^2 d\Gamma + \Re\left\{\langle\mathcal{B}_2 y, m\cdot\nabla\overline{y}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}, \tag{4.84}$$

where $R_2 = \max\limits_{x\in\Gamma_2}|m(x)|$ and M is a positive constant.

Proof. Before starting, we assume that $\mathcal{B}_2 y$ is smooth enough so that $y \in H^4(\Omega_2)$ (see [22]). Using Green's formula and identity 3.4 of [49] (see also [32] and [46]), we have

$$\int_{\Omega_2} \Delta^2 y(m \cdot \nabla \overline{y}) dx = a(y, \overline{y}) - \int_{\partial\Omega_2} \left[\mathcal{B}_1 y \partial_{\nu_2}(m \cdot \nabla \overline{y}) - \mathcal{B}_2 y(m \cdot \nabla \overline{y})\right] d\Gamma + \frac{1}{2} \int_{\partial\Omega_2} (m \cdot \nu_2) b(y) d\Gamma, \tag{4.85}$$

where

$$b(y) = \left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\mu \frac{\partial^2 y}{\partial x_1^2} \frac{\partial^2 y}{\partial x_2^2} + 2(1-\mu) \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^2.$$

It follows from the geometric condition (4.27) and the boundary conditions $\mathcal{B}_1 y = 0$ on I and $\mathcal{B}_1 y = 0$ on Γ_2 that

$$\int_{\Omega_2} \Delta^2 y(m \cdot \nabla \overline{y}) dx = a(y, \overline{y}) + \int_{\Gamma_2} \mathcal{B}_2 y(m \cdot \nabla \overline{y}) d\Gamma + \frac{1}{2} \int_{\Gamma_2} (m \cdot \nu_2) b(y) d\Gamma + \int_I \mathcal{B}_2 y(m \cdot \nabla \overline{y}) d\Gamma. \tag{4.86}$$

Applying Young's inequality to the second term on the right-hand side of (4.86) gives

$$\Re \left\{ \int_{\Gamma_2} \mathcal{B}_2 y(m \cdot \nabla \overline{y}) d\Gamma \right\} \geq -\frac{1}{2\varepsilon_3} \int_{\Gamma_2} |\mathcal{B}_2 y|^2 d\Gamma - \frac{R_2^2 \varepsilon_3}{2} \int_{\Gamma_2} |\nabla y|^2 d\Gamma, \tag{4.87}$$

for any $\varepsilon_3 > 0$. Thanks to the trace theorem (see Theorem 1.6.6 of [13]), we know that there exists a constant $C_1 > 0$ such that

$$\|\nabla y\|^2_{L^2(\Gamma_2)} \leq C_1 \|\nabla y\|_{L^2(\Omega_2)} \|\nabla y\|_{H^1(\Omega_2)} \leq C_1 \|y\|^2_{H^2(\Omega_2)}. \tag{4.88}$$

Moreover, according to the fact that $a(y, \overline{y})$ is equivalent to the usual norm of $H^2(\Omega_2)$ on $H^2_*(\Omega_2)$, we know that there exists a constant $C_2 > 0$ such that

$$\|y\|^2_{H^2(\Omega_2)} \leq C_2 a(y, \overline{y}). \tag{4.89}$$

The combination of (4.88) and (4.89) yields

$$\|\nabla y\|^2_{L^2(\Gamma_2)} \leq M a(y, \overline{y}), \tag{4.90}$$

where the constant $M = C_1 C_2 > 0$. This, along with (4.87), implies that

$$\Re \left\{ \int_{\Gamma_2} \mathcal{B}_2 y(m \cdot \nabla \overline{y}) d\Gamma \right\} \geq -\frac{1}{2\varepsilon_3} \int_{\Gamma_2} |\mathcal{B}_2 y|^2 d\Gamma - \frac{R_2^2 M \varepsilon_3}{2} a(y, \overline{y}). \tag{4.91}$$

By taking the real part of (4.86) and inserting (4.91), we obtain

$$\Re \left\{ \int_{\Omega_2} \Delta^2 y(m \cdot \nabla \overline{y}) dx \right\} \geq \left(1 - \frac{R_2^2 M \varepsilon_3}{2}\right) a(y, \overline{y}) - \frac{1}{2\varepsilon_3} \int_{\Gamma_2} |\mathcal{B}_2 y|^2 d\Gamma + \frac{1}{2} \int_{\Gamma_2} (m \cdot \nu_2) b(y) d\Gamma + \Re \left\{ \int_I \mathcal{B}_2 y(m \cdot \nabla \overline{y}) d\Gamma \right\}. \tag{4.92}$$

By letting $\varepsilon_3 = \frac{1}{R_2^2 M} > 0$ in (4.92), we conclude that

$$\Re\left\{\int_{\Omega_2}\Delta^2 y(m\cdot\nabla\overline{y})dx\right\} \geq \frac{1}{2}a(y,\overline{y}) - \frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 y|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)b(y)d\Gamma + \Re\left\{\int_I \mathcal{B}_2 y(m\cdot\nabla\overline{y})d\Gamma\right\}. \tag{4.93}$$

A straightforward computation shows that

$$\begin{aligned} b(y) &= \left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\mu\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2} + 2(1-\mu)\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2 \\ &= \mu\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2}\right\} + (1-\mu)\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\} \\ &= \mu\left(\Delta y\right)^2 + (1-\mu)\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\}. \end{aligned} \tag{4.94}$$

Taking the geometric condition (4.28) into consideration, it follows from (4.94) that

$$\begin{aligned} \Re\left\{\int_{\Omega_2}\Delta^2 y(m\cdot\nabla\overline{y})dx\right\} &\geq \frac{1}{2}a(y,\overline{y}) - \frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 y|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)\left(\Delta y\right)^2 d\Gamma \\ &+ \frac{1}{2}\int_{\Gamma_2}(m\cdot\nu_2)\left\{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 y}{\partial x_1\partial x_2}\right)^2\right\}d\Gamma + \Re\left\{\int_I \mathcal{B}_2 y(m\cdot\nabla\overline{y})d\Gamma\right\}, \end{aligned} \tag{4.95}$$

from which we conclude

$$\Re\left\{\int_{\Omega_2}\Delta^2 y(m\cdot\nabla\overline{y})dx\right\} \geq \frac{1}{2}a(y,\overline{y}) - \frac{R_2^2 M}{2}\int_{\Gamma_2}|\mathcal{B}_2 y|^2 d\Gamma + \Re\left\{\int_I \mathcal{B}_2 y(m\cdot\nabla\overline{y})d\Gamma\right\}. \tag{4.96}$$

Now, for $y \in H^2(\Omega_2)$, which is a solution of (4.83), we deduce, based on Lemma 3.3.2 and Assumption (A2) that there exists a sequence $\{y_k\}_{k\geq 0} \subset H^4(\Omega_2)$ such that

$$y_k \underset{k\to\infty}{\longrightarrow} y \ \text{ in } \ H^2(\Omega_2) \quad \text{and} \quad \Delta^2 y_k \underset{k\to\infty}{\longrightarrow} \Delta^2 y \ \text{ in } \ L^2(\Omega_2).$$

By applying (4.96) to y_k and passing the limit in k, we obtain the desired estimation (4.84). ■

We are now ready to present the proof of Theorem 4.5.1.

Proof. In accordance to Theorem 2.3.3, a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ on a Hilbert space $\mathcal{H}$ verifies (4.74) if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}) \qquad (P1)$$

and

$$\limsup_{|\lambda|\to\infty}\frac{1}{\lambda^2}\left\|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty \qquad (P2)$$

are satisfied. Since the resolvent of the operator $\mathcal{A}$ is not compact in the energy space $\mathcal{H}$ and $0 \in \rho(\mathcal{A})$, proving $i\mathbb{R} \subset \rho(\mathcal{A})$ is equivalent to showing that $i\lambda\mathcal{I} - \mathcal{A}$ is bijective in the energy

space $\mathcal{H}$ for all $\lambda \in \mathbb{R}^*$. This is established in Section 4.4 using a unique continuation theorem and Fredholm's alternative. Then, we still need to prove condition $(P2)$. This is checked by a contradiction argument. Indeed, suppose that there exists a sequence $\{\lambda_n\}_{n\geq 1} \subset \mathbb{R}^*$ and a sequence $\{U_n := (u_n, v_n, \eta_n, w_n, z_n, \zeta_n)\}_{n\geq 1} \subset D(\mathcal{A})$ such that

$$|\lambda_n| \longrightarrow +\infty, \qquad \|U_n\|_{\mathcal{H}} = 1, \tag{4.97}$$

and

$$\lambda_n^2(i\lambda_n\mathcal{I} - \mathcal{A})U_n = (f_n^1, g_n^1, h_n^1, f_n^2, g_n^2, h_n^2) \longrightarrow 0 \quad \text{in } \mathcal{H}. \tag{4.98}$$

Our aim is to show that $\|(u_n, v_n, \eta_n, w_n, z_n, \zeta_n)\|_{\mathcal{H}} \longrightarrow 0$. This condition provides a contradiction with (4.97). By detailing equation (4.98), we get the following system

$$\begin{aligned}
\lambda_n^2(i\lambda_n u_n - v_n) &= f_n^1 \to 0 \quad \text{in} \quad H_*^1(\Omega_1), && (4.99)\\
\lambda_n^2(i\lambda_n v_n - \Delta u_n) &= g_n^1 \to 0 \quad \text{in} \quad L^2(\Omega_1), && (4.100)\\
\lambda_n^2(i\lambda_n \eta_n - v_n + \eta_n) &= h_n^1 \to 0 \quad \text{in} \quad L^2(\Gamma_1), && (4.101)\\
\lambda_n^2(i\lambda_n w_n - z_n) &= f_n^2 \to 0 \quad \text{in} \quad H_*^2(\Omega_2), && (4.102)\\
\lambda_n^2(i\lambda_n z_n + \Delta^2 w_n) &= g_n^2 \to 0 \quad \text{in} \quad L^2(\Omega_2), && (4.103)\\
\lambda_n^2(i\lambda_n \zeta_n - z_n + \zeta_n) &= h_n^2 \to 0 \quad \text{in} \quad L^2(\Gamma_2). && (4.104)
\end{aligned}$$

We notice from (4.97) that v_n and z_n are uniformly bounded in $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively. It follows from Equations (4.99) and (4.102) that

$$\|u_n\|_{L^2(\Omega_1)} = \frac{O(1)}{\lambda_n} \qquad \text{and} \qquad \|w_n\|_{L^2(\Omega_2)} = \frac{O(1)}{\lambda_n}. \tag{4.105}$$

On the other hand, inserting Equation (4.99) (resp. (4.102)) into Equation (4.100) (resp. (4.103)), we obtain the following system

$$\begin{aligned}
-\lambda_n^2 u_n - \Delta u_n &= \frac{if_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2}, && (4.106)\\
-\lambda_n^2 w_n + \Delta^2 w_n &= \frac{if_n^2}{\lambda_n} + \frac{g_n^2}{\lambda_n^2}. && (4.107)
\end{aligned}$$

For clarity, the proof is divided into several lemmas:

Lemma 4.5.3. *The solution $(u_n, v_n, \eta_n, w_n, z_n, \zeta_n) \in D(\mathcal{A})$ of system (4.99)-(4.104) satisfies the following asymptotic behavior estimation*

$$\int_{\Gamma_1} |\eta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2} \quad \textit{and} \quad \int_{\Gamma_2} |\zeta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2}. \tag{4.108}$$

Proof. Taking the inner product of (4.98) with $U_n = (u_n, v_n, \eta_n, w_n, z_n, \zeta_n)$ in $\mathcal{H}$, then using the fact that U_n is uniformly bounded in $\mathcal{H}$, we get

$$\int_{\Gamma_1} |\eta_n|^2 d\Gamma + \int_{\Gamma_2} |\zeta_n|^2 d\Gamma = \Re\{((i\lambda_n\mathcal{I} - \mathcal{A})U_n, U_n)_{\mathcal{H}}\} = \frac{o(1)}{\lambda_n^2},$$

which implies that

$$\int_{\Gamma_1} |\eta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2} \qquad \text{and} \qquad \int_{\Gamma_2} |\zeta_n|^2 d\Gamma = \frac{o(1)}{\lambda_n^2}.$$

■

Lemma 4.5.4. *The solution* $(u_n, v_n, \eta_n, w_n, z_n, \zeta_n) \in D(\mathcal{A})$ *of system* (4.99)-(4.104) *satisfies the following asymptotic behavior estimation*

$$\|u_n\|_{L^2(\Gamma_1)} = \frac{o(1)}{\lambda_n}, \tag{4.109}$$

$$\|w_n\|_{L^2(\Gamma_2)} = \frac{o(1)}{\lambda_n}. \tag{4.110}$$

Proof. First, using Equation (4.101) and the first estimation of (4.108), we get

$$\|v_n\|_{L^2(\Gamma_1)} = o(1). \tag{4.111}$$

From Equation (4.99) and the fact that the trace operator $\varphi \longmapsto \varphi|_{\Gamma_1}$ is a linear and continuous mapping from $H^1_*(\Omega_1)$ into $L^2(\Gamma_1)$, we have

$$i\lambda_n u_n - v_n = \frac{f^1_n}{\lambda^2_n} \to 0 \quad \text{in} \quad L^2(\Gamma_1).$$

Multiplying the above equation by $-i\lambda_n \overline{u_n}$ and integrating over Γ_1, we obtain

$$\int_{\Gamma_1} |\lambda_n u_n|^2 d\Gamma + \int_{\Gamma_1} i\lambda_n \overline{u_n} v_n d\Gamma = -\int_{\Gamma_1} \frac{i f^1_n \overline{u_n}}{\lambda_n} d\Gamma.$$

Using estimations (4.105), (4.111) and the fact that f^1_n converges to zero in $L^2(\Gamma_1)$, we deduce from the above equation that

$$\int_{\Gamma_1} |\lambda_n u_n|^2 d\Gamma = o(1),$$

which gives the desired estimation (4.109). Next, using Equation (4.104) and the second estimation of (4.108), we obtain

$$\|z_n\|_{L^2(\Gamma_2)} = o(1). \tag{4.112}$$

Similar computation performed on Equation (4.102), using estimation (4.112) and the fact that f^2_n converges to zero in $L^2(\Gamma_2)$, yield, as well,

$$\int_{\Gamma_2} |\lambda_n w_n|^2 d\Gamma = o(1),$$

which gives the desired estimation (4.110). ■

Lemma 4.5.5. *The solution* $(u_n, v_n, \eta_n, w_n, z_n, \zeta_n) \in D(\mathcal{A})$ *of system* (4.99)-(4.104) *satisfies the following asymptotic behavior estimation*

$$\|\partial_{\nu_1} u_n\|_{L^2(\Gamma_1)} = \frac{o(1)}{\lambda_n}, \tag{4.113}$$

$$\|\mathcal{B}_2 w_n\|_{L^2(\Gamma_2)} = \frac{o(1)}{\lambda_n}. \tag{4.114}$$

Proof. From the boundary conditions of (4.10), we have $\partial_{\nu_1} u_n = -\eta_n$ on Γ_1 and $\mathcal{B}_2 w_n = \zeta_n$ on Γ_2. Therefore, using estimation (4.108), the desired estimations (4.113) and (4.114) follow. ■

Lemma 4.5.6. *Assume that Assumptions (A1), (A2), (A4) and (A5) hold. Then, the solution* $(u_n, v_n, \eta_n, w_n, z_n, \zeta_n) \in D(\mathcal{A})$ *of system* (4.99)-(4.104) *satisfies the following estimation*

$$\Re\left\{\int_{\Omega_2}\Delta^2 w_n(m\cdot\nabla\overline{w_n})dx-\int_{\Omega_1}\Delta u_n(m\cdot\nabla\overline{u_n})dx\right\}\geq\frac{1}{2}a(w_n,\overline{w_n})-\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}u_n|^2d\Gamma \\ -\frac{R_2^2M}{2}\int_{\Gamma_2}|\mathcal{B}_2w_n|^2d\Gamma, \tag{4.115}$$

where $R_1=\max\limits_{x\in\Gamma_1}|m(x)|$, $R_2=\max\limits_{x\in\Gamma_2}|m(x)|$ *and* M *is a positive constant independent of* n.

Proof. By the definition of $D(\mathcal{A})$, u_n may be seen as the unique solution $u_n\in H^1(\Omega_1)$ of system (4.75) with $v_1=-\eta_n$ and $v_2=w_n$. As Assumptions (A1), (A4) and (A5) hold, it suffices to apply Lemma 4.5.1 for u_n to obtain

$$-\Re\left\{\int_{\Omega_1}\Delta u_n(m\cdot\nabla\overline{u_n})dx\right\}\geq-\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}u_n|^2d\Gamma-\Re\left\{\langle\partial_{\nu_1}u_n,m\cdot\nabla\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}, \tag{4.116}$$

for some $\varepsilon\in(0,\frac{1}{2})$. Again, since $U_n\in D(\mathcal{A})$, we see that $w_n\in H^2(\Omega_2)$ is a solution of system (4.83) with $v_3=\partial_{\nu_1}u_n$ and $v_4=\zeta_n$. It follows from Lemma 4.5.2, Assumptions (A2) and (A4) that

$$\Re\left\{\int_{\Omega_2}\Delta^2 w_n(m\cdot\nabla\overline{w_n})dx\right\}\geq\frac{1}{2}a(w_n,\overline{w_n})-\frac{R_2^2M}{2}\int_{\Gamma_2}|\mathcal{B}_2w_n|^2d\Gamma+\Re\left\{\langle\mathcal{B}_2w_n,m\cdot\nabla\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}. \tag{4.117}$$

Now, by adding (4.116) and (4.117), we obtain

$$\Re\left\{\int_{\Omega_2}\Delta^2 w_n(m\cdot\nabla\overline{w_n})dx\right\}-\Re\left\{\int_{\Omega_1}\Delta u_n(m\cdot\nabla\overline{u_n})dx\right\}\geq\frac{1}{2}a(w_n,\overline{w_n})-\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}u_n|^2d\Gamma \\ -\frac{R_2^2M}{2}\int_{\Gamma_2}|\mathcal{B}_2w_n|^2d\Gamma+\Re\left\{\langle\mathcal{B}_2w_n,m\cdot\nabla\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}-\langle\partial_{\nu_1}u_n,m\cdot\nabla\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\right\}. \tag{4.118}$$

The boundary conditions $u_n=w_n$ and $\mathcal{B}_2w_n=\partial_{\nu_1}u_n$ on the interface I, along with the geometric condition (4.27), lead to

$$\begin{aligned}
&\langle\mathcal{B}_2w_n,m\cdot\nabla\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}-\langle\partial_{\nu_1}u_n,m\cdot\nabla\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\\
&=\langle\mathcal{B}_2w_n,m\cdot(\nu_2\partial_{\nu_2}\overline{w_n}+\tau_2\partial_{\tau_2}\overline{w_n})\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}-\langle\partial_{\nu_1}u_n,m\cdot(\nu_1\partial_{\nu_1}\overline{u_n}+\tau_1\partial_{\tau_1}\overline{u_n})\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\\
&=\langle\mathcal{B}_2w_n,m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}-\langle\partial_{\nu_1}u_n,m\cdot\tau_1\partial_{\tau_1}\overline{u_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\\
&=\langle\mathcal{B}_2w_n,m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}-\langle\partial_{\nu_1}u_n,m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}\\
&=\langle\mathcal{B}_2w_n-\partial_{\nu_1}u_n,m\cdot\tau_2\partial_{\tau_2}\overline{w_n}\rangle_{H^{-\varepsilon}(I)\times H^{\varepsilon}(I)}=0.
\end{aligned}$$

This, together with (4.118), proves (4.115), as desired. ■

Lemma 4.5.7. *Assume that Assumptions (A1) and (A2) hold. Then, the solution* $(u_n, v_n, \eta_n, w_n, z_n, \zeta_n) \in D(\mathcal{A})$ *of system* (4.99)-(4.104) *satisfies the following asymptotic behavior estimation*

$$\int_{\Omega_1}|\lambda_nu_n|^2\,dx+\int_{\Omega_2}|\lambda_nw_n|^2\,dx=\int_{\Omega_1}|\nabla u_n|^2\,dx+a(w_n,\overline{w_n})+\frac{o(1)}{\lambda_n^2}. \tag{4.119}$$

Proof. Before starting the proof, we assume that $u_n \in H^2(\Omega_1)$ and $w_n \in H^4(\Omega_2)$. Multiplying Equation (4.106) (resp. (4.107)) by $-\overline{u_n}$ (resp. $-\overline{w_n}$), integrating over Ω_1 (resp. Ω_2) and applying Green's formula, we obtain

$$\int_{\Omega_1} |\lambda_n u_n|^2 \, dx - \int_{\Omega_1} |\nabla u_n|^2 \, dx + \int_{\partial\Omega_1} \partial_{\nu_1} u_n \overline{u_n} d\Gamma = -\int_{\Omega_1} \left(\frac{i f_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2} \right) \overline{u_n} dx, \tag{4.120}$$

and

$$\int_{\Omega_2} |\lambda_n w_n|^2 \, dx - a(w_n, \overline{w_n}) + \int_{\partial\Omega_2} (\mathcal{B}_1 w_n \partial_{\nu_2} \overline{w_n} - \mathcal{B}_2 w_n \overline{w_n}) \, d\Gamma = -\int_{\Omega_2} \left(\frac{i f_n^2}{\lambda_n} + \frac{g_n^2}{\lambda_n^2} \right) \overline{w_n} dx. \tag{4.121}$$

Taking into consideration that f_n^1, g_n^1 converge to zero in $L^2(\Omega_1)$, f_n^2, g_n^2 converge to zero in $L^2(\Omega_2)$ and using (4.105), we get from (4.120) and (4.121) that

$$\int_{\Omega_1} |\lambda_n u_n|^2 \, dx - \int_{\Omega_1} |\nabla u_n|^2 \, dx + \int_{\partial\Omega_1} \partial_{\nu_1} u_n \overline{u_n} d\Gamma = \frac{o(1)}{\lambda_n^2},$$

and

$$\int_{\Omega_2} |\lambda_n w_n|^2 \, dx - a(w_n, \overline{w_n}) + \int_{\partial\Omega_2} (\mathcal{B}_1 w_n \partial_{\nu_2} \overline{w_n} - \mathcal{B}_2 w_n \overline{w_n}) \, d\Gamma = \frac{o(1)}{\lambda_n^2}.$$

Adding both equations and taking into account the boundary conditions on the interface and Γ_2, as well as the estimations (4.109), (4.110), (4.113) and (4.114), we obtain

$$\int_{\Omega_1} |\lambda_n u_n|^2 \, dx - \int_{\Omega_1} |\nabla u_n|^2 \, dx + \int_{\Omega_2} |\lambda_n w_n|^2 \, dx - a(w_n, \overline{w_n}) = \frac{o(1)}{\lambda_n^2}.$$

By following a similar argument to the end of the proof of Lemma 4.5.1 and Lemma 4.5.2, we obtain the desired estimation (4.119). ■

Lemma 4.5.8. *Assume that Assumptions (A1), (A2), (A4) and (A5) hold. Then, the solution* $(u_n, v_n, \eta_n, w_n, z_n, \zeta_n) \in D(\mathcal{A})$ *of system* (4.99)-(4.104) *satisfies the following asymptotic behavior estimation*

$$\int_{\Omega_1} |\lambda_n u_n|^2 \, dx = o(1), \quad \int_{\Omega_2} |\lambda_n w_n|^2 \, dx = o(1) \quad \text{and} \quad a(w_n, \overline{w_n}) = o(1). \tag{4.122}$$

Proof. Multiplying Equation (4.106) by $m \cdot \nabla \overline{u_n}$ and integrating over Ω_1, and similarly for Equation (4.107) over Ω_2, we obtain

$$-\int_{\Omega_1} \lambda_n^2 u_n \, (m \cdot \nabla \overline{u_n}) \, dx - \int_{\Omega_1} \Delta u_n \, (m \cdot \nabla \overline{u_n}) \, dx = \int_{\Omega_1} \left(\frac{i f_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2} \right) (m \cdot \nabla \overline{u_n}) \, dx, \tag{4.123}$$

and

$$-\int_{\Omega_2} \lambda_n^2 w_n \, (m \cdot \nabla \overline{w_n}) \, dx + \int_{\Omega_2} \Delta^2 w_n \, (m \cdot \nabla \overline{w_n}) \, dx = \int_{\Omega_2} \left(\frac{i f_n^2}{\lambda_n} + \frac{g_n^2}{\lambda_n^2} \right) (m \cdot \nabla \overline{w_n}) \, dx. \tag{4.124}$$

Using Cauchy-Schwarz inequality and the fact that $a(w_n, \overline{w_n})$ is equivalent to the usual norm of $H^2(\Omega_2)$ on $H^2_*(\Omega_2)$, we can observe that ∇u_n is uniformly bounded in $L^2(\Omega_1)$ and $a(w_n, \overline{w_n})$ is uniformly bounded. Additionally, f_n^1 and g_n^1 converge to zero in $L^2(\Omega_1)$, while f_n^2 and g_n^2 converge to zero in $L^2(\Omega_2)$. Hence, we obtain

$$\left| \int_{\Omega_1} \left(\frac{i f_n^1}{\lambda_n} + \frac{g_n^1}{\lambda_n^2} \right) (m \cdot \nabla \overline{u_n}) \, dx \right| \leq \|m\|_\infty \left(\frac{\|f_n^1\|_{L^2(\Omega_1)}}{\lambda_n} + \frac{\|g_n^1\|_{L^2(\Omega_1)}}{\lambda_n^2} \right) \|\nabla u_n\|_{L^2(\Omega_1)} = \frac{o(1)}{\lambda_n}, \tag{4.125}$$

and

$$\left|\int_{\Omega_2}\left(\frac{if_n^2}{\lambda_n}+\frac{g_n^2}{\lambda_n^2}\right)(m\cdot\nabla\overline{w_n})\,dx\right|\leq\|m\|_\infty\left(\frac{\|f_n^2\|_{L^2(\Omega_2)}}{\lambda_n}+\frac{\|g_n^2\|_{L^2(\Omega_2)}}{\lambda_n^2}\right)\|\nabla w_n\|_{L^2(\Omega_2)}=\frac{o(1)}{\lambda_n}. \quad (4.126)$$

On the other hand, applying Green's formula to the first terms of (4.123) and (4.124) yields

$$\int_{\Omega_1}\lambda_n^2u_n\,(m\cdot\nabla\overline{u_n})\,dx=\frac{1}{2}\int_{\partial\Omega_1}(m\cdot\nu_1)\,|\lambda_nu_n|^2\,d\Gamma-\int_{\Omega_1}|\lambda_nu_n|^2\,dx, \quad (4.127)$$

and

$$\int_{\Omega_2}\lambda_n^2w_n\,(m\cdot\nabla\overline{w_n})\,dx=\frac{1}{2}\int_{\partial\Omega_2}(m\cdot\nu_2)\,|\lambda_nw_n|^2\,d\Gamma-\int_{\Omega_2}|\lambda_nw_n|^2\,dx. \quad (4.128)$$

Then, substituting (4.127) (resp. (4.128)) into (4.123) (resp. (4.124)) and considering the estimates (4.125) and (4.126), we obtain

$$\int_{\Omega_1}|\lambda_nu_n|^2\,dx-\frac{1}{2}\int_{\partial\Omega_1}(m\cdot\nu_1)\,|\lambda_nu_n|^2\,d\Gamma-\int_{\Omega_1}\Delta u_n\,(m\cdot\nabla\overline{u_n})\,dx=\frac{o(1)}{\lambda_n},$$

and

$$\int_{\Omega_2}|\lambda_nw_n|^2\,dx-\frac{1}{2}\int_{\partial\Omega_2}(m\cdot\nu_2)\,|\lambda_nw_n|^2\,d\Gamma+\int_{\Omega_2}\Delta^2w_n\,(m\cdot\nabla\overline{w_n})\,dx=\frac{o(1)}{\lambda_n}.$$

Adding both equations and using the boundary condition $u_n=w_n$ on the interface I and the estimates (4.109) and (4.110), then taking the real part, we obtain

$$\int_{\Omega_1}|\lambda_nu_n|^2\,dx+\int_{\Omega_2}|\lambda_nw_n|^2\,dx+\Re\left\{\int_{\Omega_2}\Delta^2w_n(m\cdot\nabla\overline{w_n})dx-\int_{\Omega_1}\Delta u_n(m\cdot\nabla\overline{u_n})dx\right\}=o(1). \quad (4.129)$$

As Assumptions (A1), (A2), (A4) and (A5) hold, it suffices to apply Lemma 4.5.6. Hence, inserting (4.115) into (4.129), we get

$$\int_{\Omega_1}|\lambda_nu_n|^2\,dx+\int_{\Omega_2}|\lambda_nw_n|^2\,dx+\frac{1}{2}a(w_n,\overline{w_n})-\frac{R_1^2}{\delta}\int_{\Gamma_1}|\partial_{\nu_1}u_n|^2d\Gamma-\frac{R_2^2M}{2}\int_{\Gamma_2}|\mathcal{B}_2w_n|^2d\Gamma=o(1). \quad (4.130)$$

Finally, using the estimations (4.113) and (4.114), it follows from (4.130) that

$$\int_{\Omega_1}|\lambda_nu_n|^2\,dx+\int_{\Omega_2}|\lambda_nw_n|^2\,dx+\frac{1}{2}a(w_n,\overline{w_n})=o(1),$$

which proves the estimation (4.122), as desired. ■

Proof of Theorem 4.5.1. Using (4.119) and (4.122), we deduce that

$$\int_{\Omega_1}|\nabla u_n|^2dx=o(1). \quad (4.131)$$

Therefore, combining the estimates (4.108), (4.122) and (4.131), we obtain $\|U_n\|_{\mathcal{H}}\longrightarrow 0$, which leads to the desired contradiction with (4.97). Consequently, condition $(P2)$ holds and this permits us to conclude that the energy of system (4.1)-(4.4) decays polynomially to zero as t goes to infinity. The proof is thus complete.

4.6 Conclusion and Open Problems

This chapter investigates the stabilization of a transmission wave-plate model using only two dynamical boundary controls. By applying a general criterion of Arendt-Batty, we have established the system's strong stability under specific geometric conditions. Our analysis reveals a polynomial decay rate for the system's energy, characterized by $1/t$. Notably, while the dynamical boundary control ξ plays a pivotal role in establishing strong stability in the absence of geometric conditions, it does not affect the energy decay rate of the system.

An interesting question arising from this study is the stabilization when the wave-plate system is damped by two dynamical boundary controls η and ξ. We conjecture that removing the control ζ will not affect the system's polynomial stability. Further, we aim to develop a finite volume method for the problem and conduct numerical experiments to validate our findings.

Chapter 5

Finite Volume Method for a 1-D Euler-Bernoulli Beam\Wave Transmission Problem with Dynamical Boundary Controls

In this chapter, we investigate the numerical solution of a transmission problem consisting of a wave equation coupled with an Euler-Bernoulli beam with dynamical boundary controls in one spatial dimension. The coupling occurs via boundary connections where the wave and beam equations are evolved across adjacent intervals. Using the Finite Volume Method for spatial discretization, we establish stability estimates and prove the convergence of the numerical solution to the weak solution of the original continuous system. Additionally, we present a numerical experiment to validate the theoretical study, particularly regarding the decay rate of the solution under the influence of a single dynamical boundary control.

5.1 Introduction

Numerical simulation is a computational approach that implements mathematical models to represent physical systems. It is required to study the behavior of systems representing complicated mathematical models to provide good approximations to the analytical solutions. Numerical modeling, particularly useful in engineering, uses mathematical frameworks to describe necessary physical conditions. However, many of the equations involved, especially partial differential equations, are often too complex to solve analytically. Through numerical methods such as the Finite Difference Method (FDM), Finite Element Method (FEM), or Finite Volume Method (FVM), approximate solutions can be achieved, with each method having its own advantages and limitations. Numerical experiments play an integral role in the engineering process, helping interpret results from these models. For mathematicians, these studies not only validate theoretical results but also offer insights into potential solutions for unresolved conjectures. Numerous studies in the literature explore FDM, FEM, and FVM. For instance, Larsson et al. in [33] applied FEM to a strongly damped wave equation. In [56], Zuazua et al. modeled damped vibrations of a string with fixed ends, applying FDM with artificial viscosity to demonstrate exponential decay of discrete energy and convergence of the scheme. Error estimates were later derived by Rincon et al. using spatial FEM (see [51]). Additionally, Gao et al., in [19], proposed an unconditionally stable finite difference scheme for solving a one-dimensional linear hyperbolic equation with global damping, while Wang et al., in [59], employed high-order FDM to solve a second-order wave equation in two spatial dimensions. Xu et al., in [60], studied the Euler-Bernoulli beam equation with local Kelvin-Voigt damping and used FEM followed by the control parameterization method to design a control input distributed over a subinterval, aiming to minimize the total energy of the beam and control within a specified time period.

From the industrial point of view, the FVM is known as a robust and cheap method for the discretization of particularly difficult equations, such as nonlinear systems of hyperbolic equations, and it can easily be extended to more realistic and physical contexts. In FVM, discrete values are approximations of cell averages based on an integral formulation. For instance, the application of FVM to hyperbolic equations is detailed in [36]. Wave equations, which are key hyperbolic equations in acoustics, have drawn significant attention in numerical studies. These studies focus on wave-based modeling of room acoustics using both FDM and FVM, as discussed in [25]. Moreover, explicit numerical simulations for time-dependent viscoelastic flow problems have been developed by combining FVM and FEM (see [53]). Zhang et al., in [63], introduced a new spectral FVM for 2D and 3D elastic wave equations with external sources on unstructured meshes and later presented an efficient FVM for 3D elastic wave simulations on unstructured tetrahedral meshes in [64]. Additionally, Riečanová et al., in [50], obtained stability estimates and convergence for a wave equation with Dirichlet boundary conditions on a rectangular domain. Their method employed FVM for spatial discretization, combined with the average of $n+1$ and $n-1$ time step diffusion.

The FVM is distinct from, yet sometimes related to, the FDM and the FEM. Roughly speaking, the principle of the FDM is as follows: Given a set of discretization points, which may be defined by a mesh, one discrete unknown is assigned per discretization point, and one equation is written for each of these points. At each discretization point, the derivatives of the unknown are replaced by finite differences, derived through the use of Taylor expansions. The FDM becomes difficult to use when the equation involves discontinuous coefficients, such as in heterogeneous media. With the FVM, discontinuities of the coefficients will not be any problem if the mesh is chosen such that the discontinuities of the coefficients occur on the boundaries of the control volumes. For instance, in [20], the authors investigated the numerical solution of a transmission problem for a wave equation with localized Kelvin-Voigt damping, where the damping acts away from the boundary through a non-smooth coefficient, resulting in a discontinuity at the interface. Their approach, based on FVM, successfully addressed this complexity.

To the best of our knowledge, there are currently no results in the literature regarding the convergence and stability estimates for the transmission problem of a one-dimensional Euler-Bernoulli beam \ wave equation with dynamical boundary controls. This chapter aims to address this gap. Specifically, we focus on studying the numerical solution for a transmission problem of a wave equation and an Euler-Bernoulli beam with dynamical boundary controls, utilizing the FVM. The two equations are coupled via boundary connections between the intervals in which the wave and the Euler-Bernoulli beam evolve respectively. The system that describes this model is as follows:

$$\begin{cases} u_{tt} - a u_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ w_{tt} + b w_{xxxx} = 0, & (x,t) \in (-L,0) \times (0,\infty), \\ u(0,t) = w(0,t), \quad w_x(0,t) = 0, & t \in (0,\infty), \\ a u_x(0,t) + b w_{xxx}(0,t) = 0, & t \in (0,\infty), \\ u_x(L,t) + \eta(t) = 0, & t \in (0,\infty), \\ w_{xx}(-L,t) - \xi(t) = 0, & t \in (0,\infty), \\ w_{xxx}(-L,t) + \zeta(t) = 0, & t \in (0,\infty), \end{cases} \tag{5.1}$$

with the initial conditions of the type:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,L), \quad \eta(0) = \alpha, \tag{5.2}$$

$$w(x,0)=\phi(x),\quad w_t(x,0)=\theta(x),\quad x\in(-L,0),\quad \xi(0)=\beta,\quad \zeta(0)=\gamma, \tag{5.3}$$

where η, ξ and ζ denote the dynamical boundary controls. Here, the coefficients a and b are strictly positive constant numbers representing the speed of propagation of the wave equation and the Euler-Bernoulli beam respectively.

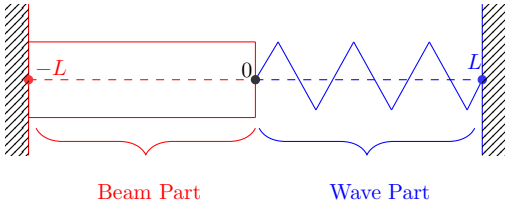

Figure 5.1: Euler-Bernoulli beam \ wave transmission model

The damping of the system is made via the indirect damping mechanism that involves the following first-order differential equations:

$$\begin{cases} \eta_t(t)-u_t(L,t)+\eta(t)=0, & t\in(0,\infty),\\ \xi_t(t)-w_{xt}(-L,t)+\xi(t)=0, & t\in(0,\infty),\\ \zeta_t(t)-w_t(-L,t)+\zeta(t)=0, & t\in(0,\infty). \end{cases} \tag{5.4}$$

The energy of the system (5.1)-(5.4) is given by

$$\begin{aligned} E(t)=\frac{1}{2}\left\{\int_0^L \left(|u_t(x,t)|^2+a|u_x(x,t)|^2\right)dx+a|\eta(t)|^2\right\}\\ +\frac{1}{2}\left\{\int_{-L}^0 \left(|w_t(x,t)|^2+b|w_{xx}(x,t)|^2\right)dx+b|\xi(t)|^2+b|\zeta(t)|^2\right\}. \end{aligned}$$

Finding the derivative of $E(t)$ along the solution of (5.1)-(5.4) gives

$$E'(t)=-\left(a|\eta(t)|^2+b|\xi(t)|^2+b|\zeta(t)|^2\right)\le 0, \tag{5.5}$$

which implies that the energy of the system decreases as time goes on. In this chapter, our goal is to study this system numerically.

The rest of this chapter is organized as follows. In Section 5.2, we present the definition of the weak solution for the problem (5.1)-(5.4). Section 5.3 introduces the concept of an admissible mesh for the finite volume discretization of the system. For a pedagogical construction of the numerical schemes, we first construct the semi-discrete numerical scheme in Subsection 5.4.1, followed by the explicit numerical scheme in Subsection 5.5.1, and the implicit numerical scheme in Subsection 5.5.2, which incorporates an average of $n+1$ and $n-1$ time step diffusion. In Subsections 5.4.2, 5.5.1 and 5.5.2, we design a discrete energy that dissipates when control is applied and is conserved in its absence. Under suitable assumptions, satisfied by the initial data, Subsections 5.4.3 and 5.5.2 derive stability estimates for the approximate solution, which are essential for establishing the convergence of the numerical schemes. Subsections 5.4.4 and 5.5.2 provide proofs of convergence of the discrete solution to the continuous one. Additionally, we investigate the asymptotic behavior of the energy in the continuous case of the system (5.1)–(5.4), with a single dynamical boundary control η. For smooth initial data, we establish an optimal polynomial energy decay rate of $1/\sqrt{t}$. A central component

of the analysis is the frequency domain approach, which combines a contradiction argument with the multiplier method. A key tool in demonstrating polynomial stability is Theorem 2.4 from [12]. However, the full proof is not included in this chapter due to its technical nature and length, which would divert from the main focus of the chapter. Section 5.6 provides a numerical example that illustrates this theoretical result. Finally, the chapter concludes with a summary in Section 5.7.

5.2 Weak Formulation of the Problem

We denote by $I := [0, T]$ for $T > 0$. In the whole chapter, we use the notation for function spaces as presented in the following definition:

Definition of the Functional Spaces. *Let us introduce the following spaces:*

$$H^1_*(\ell_1, \ell_2) = \left\{ u \in H^1(\ell_1, \ell_2) \;\; | \;\; \int_{\ell_1}^{\ell_2} u dx = 0 \right\},$$

$$H^2_*(\ell_1, \ell_2) = \left\{ w \in H^2(\ell_1, \ell_2) \;\; | \;\; \int_{\ell_1}^{\ell_2} w dx = w_x(\ell_2) = 0 \right\},$$

$$H^2_{**}(\ell_1, \ell_2) = \left\{ w \in H^2(\ell_1, \ell_2) \;\; | \;\; w_x(\ell_2) = 0 \right\},$$

$$\mathbb{H} = \left\{ (u, w) \in H^1_*(0, L) \times H^2_*(-L, 0) \;\; | \;\; u(0) = w(0) \right\},$$

$$\mathbb{H}_* = \left\{ (u, w) \in H^1(0, L) \times H^2_{**}(-L, 0) \;\; | \;\; u(0) = w(0) \right\},$$

and

$$\mathbb{L}^2 = L^2(0, L) \times L^2(-L, 0).$$

The weak formulation of the problem (5.1)-(5.4) is presented in the following definition:

Definition 5.2.1. *We say that the function (u, w, η, ξ, ζ) is a weak solution of (5.1)-(5.4) if, for all $T > 0$, the following conditions hold:*

1. $(u, w) \in L^2([0, T]; \mathbb{H})$ *with* $(u_t, w_t) \in L^2([0, T]; \mathbb{L}^2)$ *and* $(\eta, \xi, \zeta) \in L^2([0, T]; \mathbb{R}^3)$,
2. $(u(\cdot, 0), w(\cdot, 0)) = (\varphi(\cdot), \phi(\cdot))$,
3. (u, w, η, ξ, ζ) *is such that*

$$\begin{aligned} &- \int_0^T \int_0^L u_t v_t dx dt + a \int_0^T \int_0^L u_x v_x dx dt - \int_0^L \psi(x) v(x, 0) dx \\ &- \int_0^T \int_{-L}^0 w_t z_t dx dt + b \int_0^T \int_{-L}^0 w_{xx} z_{xx} dx dt - \int_{-L}^0 \theta(x) z(x, 0) dx \\ &+ a \int_0^T \eta(t) v(L, t) dt + b \int_0^T \xi(t) z_x(-L, t) dt + b \int_0^T \zeta(t) z(-L, t) dt = 0, \end{aligned} \tag{5.6}$$

   for all $(v, z) \in L^2([0, T]; \mathbb{H}_*)$ with $(v_t, z_t) \in L^2([0, T]; \mathbb{L}^2)$, such that $v(T, x) = z(T, x) = 0$.

4. (u, w, η, ξ, ζ) *is such that*

$$- \int_0^T \eta p_t dt - \alpha p(0) + \int_0^T u(L, t) p_t(t) dt + u(L, 0) p(0) + \int_0^T \eta p dt = 0, \tag{5.7}$$

$$- \int_0^T \xi q_t dt - \beta q(0) + \int_0^T w_x(-L, t) q_t(t) dt + w_x(-L, 0) q(0) + \int_0^T \xi q dt = 0, \tag{5.8}$$

and

$$-\int_0^T \zeta r_t dt - \gamma r(0) + \int_0^T w(-L,t) r_t(t) dt + w(-L,0) r(0) + \int_0^T \zeta r dt = 0, \tag{5.9}$$

for all $(p,q,r) \in L^2([0,T];\mathbb{R}^3)$ *with* $(p_t,q_t,r_t) \in L^2([0,T];\mathbb{R}^3)$, *such that* $p(T) = q(T) = r(T) = 0$.

We mention here that, according to Lumer-Philips theorem (see [44]), we can prove the existence and uniqueness of a strong solution $(u,w) \in L^2([0,T];\mathbb{H})$ with $(u_t,w_t) \in L^2([0,T];\mathbb{L}^2)$ and $(\eta,\xi,\zeta) \in L^2([0,T];\mathbb{R}^3)$ to the system (5.1)-(5.4). This result also confirms the existence and uniqueness of the weak solution to problem (5.1)-(5.4).

5.3 Finite Volume Discretization of the Spatial Domain

In order to perform a space finite volume discretization of the problem, a mesh $\mathcal{T}$ of $(-L,0)\cup(0,L)$, over which (5.1)-(5.4) to be studied, is introduced as follows:

Definition 5.3.1 (**One-dimensional admissible mesh**). *An admissible mesh* $\mathcal{T}$ *of the spatial domain* $(-L,0)\cup(0,L)$, *is given by an increasing sequence of real values* $\left(x_{i+\frac{1}{2}}\right)_{i=0,\dots,N_{\max}}$ *such that* $[-L,L] = \bigcup_{i=1}^{N_{\max}} \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. *The mesh* $\mathcal{T}$ *is the set* $\{K_i : i \in \{1,\dots,N_{\max}\};\quad N_{\max} \in \mathbb{N}^*\}$ *of control volumes defined by* $K_i = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$ *for all* $i \in \{1,\dots,N_{\max}\}$. *The family* $(x_i)_{i=1,\dots,N_{\max}}$ *is assumed to be the center of* $(K_i)_{i=1,\dots,N_{\max}}$ *such that*

$$-L = x_0 = x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < \cdots < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < \cdots < x_{N_{\max}-\frac{1}{2}} < x_{N_{\max}} < x_{N_{\max}+\frac{1}{2}} = x_{N_{\max}+1} = L.$$

We discretize the intervals $[-L,0]$ *and* $[0,L]$ *into* N_1 *and* N_2 *internal points respectively such that* N_1 *and* $N_2 \in \mathbb{N}^*$. *The relation is* $N_{\max} = N_1 + N_2$. *To be clear, let the spatial mesh size* $h_1 = L/N_1$ *and* $h_2 = L/N_2$ *and discretize as the following:*

We discretize the interval $[-L,0]$ *such that:*

- *For* $i = 0,\dots,N_1$, $x_{i+\frac{1}{2}} = -L + ih_1$, *which yields,* $x_{\frac{1}{2}} = -L$ *and* $x_{N_1+\frac{1}{2}} = 0$.
- *For* $i = 1,\dots,N_1$, $x_i = -L + \left(i - \frac{1}{2}\right)h_1$, *which yields,* $x_1 = -L + \dfrac{h_1}{2}$ *and* $x_{N_1} = -\dfrac{h_1}{2}$.

We discretize the interval $[0,L]$ *such that:*

- *For* $i = N_1+1,\dots,N_{\max}$, $x_{i+\frac{1}{2}} = (i-N_1)h_2$, *which yields,* $x_{N_1+\frac{3}{2}} = h_2$ *and* $x_{N_{\max}+\frac{1}{2}} = L$.
- *For* $i = N_1+1,\dots,N_{\max}$, $x_i = \left(i - N_1 - \frac{1}{2}\right)h_2$, *which yields,* $x_{N_1+1} = \dfrac{h_2}{2}$ *and* $x_{N_{\max}} = L - \dfrac{h_2}{2}$.

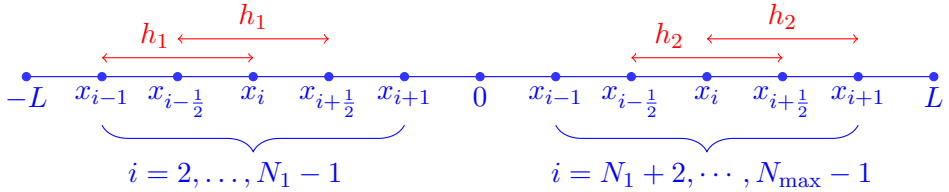

Figure 5.2: A model representing the admissible one-dimensional finite volume mesh

One sets the length of the control volume K_i by

$$h_i = m(K_i) = \begin{cases} h_1, & i = 1, \ldots, N_1, \\ h_2, & i = N_1 + 1, \ldots, N_{\max}. \end{cases}$$

Also, we set

$$h_{i+\frac{1}{2}} = \begin{cases} x_{i+1} - x_i, & i = 0, \ldots N_1 - 1,\ N_1 + 1, \ldots, N_{\max}, \\ x_{N_1+1} - x_{N_1+\frac{1}{2}}, & i = N_1, \end{cases}$$

which implies that

$$h_{i+\frac{1}{2}} = \begin{cases} \dfrac{h_1}{2}, & i = 0, \\ h_1, & i = 1, \ldots, N_1 - 1, \\ h_2, & i = N_1 + 1, \ldots, N_{\max} - 1, \\ \dfrac{h_2}{2}, & i = N_1,\ N_{\max}. \end{cases}$$

The mesh size is then defined by: $\text{size}(\mathcal{T}) = h = \max\{h_1,\ h_2\}$. Further, we need to define the discrete unknowns for the (finite volume) space discretization as $\{u_i(t);\quad i = N_1 + 1, \ldots, N_{\max},\quad t \in [0, T]\}$ and $\{w_i(t);\quad i = 1, \ldots, N_1,\quad t \in [0, T]\}$ which stand for the approximation of the mean values of u and w over the control volumes K_i, respectively. However, this data is used to construct the approximations of u and w within the cells. In fact, we have:

$$\begin{aligned} u_i(t) &= \frac{1}{h_i} \int_{K_i} u(x,t)dx, \quad \text{for } i = N_1 + 1, \ldots, N_{\max}, \\ w_i(t) &= \frac{1}{h_i} \int_{K_i} w(x,t)dx, \quad \text{for } i = 1, \ldots, N_1. \end{aligned}$$

Remark 5.3.1. ***(Reformulated Uniqueness Condition)*** *To ensure the uniqueness of the exact weak solution (u, w) for problem (5.1)-(5.4), it is necessary that (u, w) satisfies the condition:*

$$\int_0^L u(x,t)dx = \int_{-L}^0 w(x,t)dx = 0,$$

for all $t \in [0, T]$. In terms of the discrete values u_i and w_i defined by the problem, this condition can be expressed as:

$$\sum_{i=N_1+1}^{N_{\max}} h_i u_i(t) = \sum_{i=1}^{N_1} h_i w_i(t) = 0, \qquad \forall t \in [0, T]. \tag{5.10}$$

Let us define

$$U_i = \begin{cases} u_i, & i = N_1+1, \dots, N_{\max}, \\ w_i, & i = 1, \dots, N_1, \end{cases}$$

and

$$\Xi = (\eta, \xi, \zeta).$$

In the following, we introduce auxiliary unknowns to describe the scheme in the most general form. Specifically, the values of u and w at $x_{N_1+\frac{1}{2}}$ are denoted by $u_{N_1+\frac{1}{2}}$ and $w_{N_1+\frac{1}{2}}$ respectively. Let $u_{N_{\max}+1}$ represent the value of u at $x_{N_{\max}+1}$ and w_0 represent the value of w at x_0.

Before proceeding, we define the following difference operators for the discretization of the space derivatives. These operators are expressed in terms of the discrete unknowns $(u_i)_{i=N_1+1,\dots,N_{\max}}$, $(w_i)_{i=1,\dots,N_1}$, and the given data.

For the Euler-Bernoulli beam: A reasonable choice for the approximation of $w_{xxx}(x_{i+\frac{1}{2}})$ for $i = 0, \dots, N_1 - 1$ seems to be the following discrete difference operators:

- For $i = 0$,

$$\delta_x w_{\frac{1}{2}} = 2\frac{w_1 - w_0}{h_1}, \quad \delta_x w_{\frac{3}{2}} = \frac{w_2 - w_1}{h_1},$$

$$\delta_x^2 w_0 = \frac{4}{3}\left(\frac{w_2 - 3w_1 + 2w_0}{h_1^2}\right), \quad \delta_x^2 w_1 = \frac{\delta_x w_{\frac{3}{2}} - \delta_x w_{\frac{1}{2}}}{h_1} = \frac{w_2 - 3w_1 + 2w_0}{h_1^2},$$

$$\delta_x^3 w_{\frac{1}{2}} = \frac{2}{5}\left(\frac{3w_3 - 10w_2 + 15w_1 - 8w_0}{h_1^3}\right).$$

- For $i = 1$,

$$\delta_x w_{\frac{1}{2}} = 2\frac{w_1 - w_0}{h_1}, \quad \delta_x w_{\frac{3}{2}} = \frac{w_2 - w_1}{h_1}, \quad \delta_x w_{\frac{5}{2}} = \frac{w_3 - w_2}{h_1},$$

$$\delta_x^2 w_1 = \frac{\delta_x w_{\frac{3}{2}} - \delta_x w_{\frac{1}{2}}}{h_1} = \frac{w_2 - 3w_1 + 2w_0}{h_1^2}, \quad \delta_x^2 w_2 = \frac{\delta_x w_{\frac{5}{2}} - \delta_x w_{\frac{3}{2}}}{h_1} = \frac{w_3 - 2w_2 + w_1}{h_1^2},$$

$$\delta_x^3 w_{\frac{3}{2}} = \frac{\delta_x^2 w_2 - \delta_x^2 w_1}{h_1} = \frac{w_3 - 3w_2 + 4w_1 - 2w_0}{h_1^3}.$$

- For $i = 2, \dots N_1 - 2$,

$$\delta_x w_{i+\frac{1}{2}} = \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} = \frac{w_{i+1} - w_i}{h_1},$$

$$\delta_x^2 w_i = \frac{\delta_x w_{i+\frac{1}{2}} - \delta_x w_{i-\frac{1}{2}}}{h_i} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h_1^2},$$

$$\delta_x^3 w_{i+\frac{1}{2}} = \frac{\delta_x^2 w_{i+1} - \delta_x^2 w_i}{h_{i+\frac{1}{2}}} = \frac{w_{i+2} - 3w_{i+1} + 3w_i - w_{i-1}}{h_1^3}.$$

- For $i = N_1 - 1$,

$$\delta_x w_{N_1-\frac{3}{2}} = \frac{w_{N_1-1} - w_{N_1-2}}{h_1}, \quad \delta_x w_{N_1-\frac{1}{2}} = \frac{w_{N_1} - w_{N_1-1}}{h_1}, \quad \delta_x w_{N_1+\frac{1}{2}} = 2\frac{w_{N_1+\frac{1}{2}} - w_{N_1}}{h_1},$$

$$\delta_x^2 w_{N_1-1} = \frac{\delta_x w_{N_1-\frac{1}{2}} - \delta_x w_{N_1-\frac{3}{2}}}{h_1} = \frac{w_{N_1} - 2w_{N_1-1} + w_{N_1-2}}{h_1^2}, \delta_x^2 w_{N_1} = \frac{\delta_x w_{N_1+\frac{1}{2}} - \delta_x w_{N_1-\frac{1}{2}}}{h_1} = \frac{2w_{N_1+\frac{1}{2}} - 3w_{N_1} + w_{N_1-1}}{h_1^2},$$

$$\delta_x^3 w_{N_1-\frac{1}{2}} = \frac{\delta_x^2 w_{N_1} - \delta_x^2 w_{N_1-1}}{h_1} = \frac{2w_{N_1+\frac{1}{2}} - 4w_{N_1} + 3w_{N_1-1} - w_{N_1-2}}{h_1^3}.$$

For the Wave equation: A reasonable choice for the approximation of $u_x(x_{i+\frac{1}{2}})$ for $i = N_1, \ldots, N_{\max}$ seems to be the following discrete difference operators:

- For $i = N_1$,

$$\delta_x u_{N_1+\frac{1}{2}} = 2\frac{u_{N_1+1} - u_{N_1+\frac{1}{2}}}{h_2}.$$

- For $i = N_1 + 1, \ldots, N_{\max}$,

$$\delta_x u_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}.$$

5.4 Semi-Discretization of the Problem

In this section, we construct a semi-discrete numerical scheme for System (5.1)-(5.4) and estimate the discrete energy using space finite volume discretization. Then, we derive the stability estimates on the approximate solution and prove its convergence towards the weak solution of the problem (5.6)-(5.9) in an adequate sense, as the mesh size tends to zero. In order to give a stability result and prove the convergence for this scheme, we set $\Phi = (\varphi, \phi)$, $\Psi = (\psi, \theta)$, $\Theta = (\alpha, \beta, \gamma)$ and study our problem under a certain hypothesis, which we call **Hypothesis (H1)** of the following form:

$$\Phi \in \mathbb{H}, \quad \Psi \in \mathbb{L}^2 \quad \text{and} \quad \Theta \in \mathbb{R}^3. \qquad \textbf{Hypothesis (H1)}$$

The discretization of the initial conditions is first taken into account. For example, it is possible to start the solution with $\Phi_i = (\varphi_i, \phi_i)$ regarding that

$$u_i(0) = \varphi_i \quad \text{where} \quad \varphi_i = \frac{1}{h_i}\int_{K_i} \varphi(x)dx, \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$w_i(0) = \phi_i \quad \text{where} \quad \phi_i = \frac{1}{h_i}\int_{K_i} \phi(x)dx, \qquad \text{for} \quad i = 1, \cdots, N_1.$$

Consequently, we have

$$U_i(0) = \Phi_i, \quad \text{for} \quad i = 1, \ldots, N_{\max} \tag{5.11}$$

Additionally, we define $\Psi_i = (\psi_i, \theta_i)$ such that

$$\psi_i = \frac{1}{h_i}\int_{K_i} \psi(x)dx, \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$\theta_i = \frac{1}{h_i}\int_{K_i} \theta(x)dx, \qquad \text{for} \quad i = 1, \ldots, N_1.$$

Using the above notations, we can approximate the initial conditions of u_t and w_t as follows

$$u_{i,t}(0) = \psi_i, \qquad \text{for} \quad i = N_1 + 1, \dots, N_{\max},$$
$$w_{i,t}(0) = \theta_i, \qquad \text{for} \quad i = 1, \dots, N_1.$$

Thus, we have

$$U_{i,t}(0) = \Psi_i, \quad \text{for} \quad i = 1, \cdots, N_{\max}. \tag{5.12}$$

Further, we set

$$\Xi(0) = \Theta.$$

Finally, we define $\Phi_{\mathcal{T}}(x) = \Phi_i$, $x \in K_i$ and $\Psi_{\mathcal{T}}(x) = \Psi_i$, $x \in K_i$ for $i = 1, \dots, N_{\max}$.

5.4.1 Formulation of the Semi-Discrete Numerical Scheme

In this subsection, we construct a semi-discrete numerical scheme of System (5.1)-(5.4) using the finite volume method in space.

Discretization of the Euler-Bernoulli beam: For $i = 1, \dots, N_1$, integrate the second Equation of (5.1)-(5.4) on an arbitrary finite volume K_i to obtain

$$\int_{K_i} w_{tt}(x,t)dx + b \int_{K_i} w_{xxxx}(x,t)dx = 0, \qquad \forall t \in [0,T]. \tag{5.13}$$

For the first term of Equation (5.13), the Leibniz integral rule for differentiation under integral sign reads

$$\int_{K_i} w_{tt}(x,t)dx = h_i \left(\frac{1}{h_i} \int_{K_i} w(x,t)dx \right)_{tt} = h_i \left(w_i(t) \right)_{tt}, \qquad \forall t \in [0,T].$$

Integrating by parts for the second term of Equation (5.13), we get

$$\int_{K_i} w_{xxxx}(x,t)dx = w_{xxx}\left(x_{i+\frac{1}{2}}, t \right) - w_{xxx}\left(x_{i-\frac{1}{2}}, t \right), \qquad \forall t \in [0,T].$$

Thus, (5.13) can be reformulated, for $i = 1, \dots, N_1$, into

$$h_i \left(w_i(t) \right)_{tt} + b \left(w_{xxx}\left(x_{i+\frac{1}{2}}, t \right) - w_{xxx}\left(x_{i-\frac{1}{2}}, t \right) \right) = 0, \qquad \forall t \in [0,T].$$

☐ For $i = 1$, we have

$$h_1 \left(w_1(t) \right)_{tt} + b \left(w_{xxx}\left(x_{\frac{3}{2}}, t \right) - w_{xxx}\left(x_{\frac{1}{2}}, t \right) \right) = 0, \qquad \forall t \in [0,T].$$

Combining the above equation with the dynamical boundary condition $w_{xxx}(-L,t) + \zeta(t) = 0$ as the position $x_{\frac{1}{2}}$ represents the point $-L$ and replacing the second term by its corresponding difference operator, we obtain

$$h_1 \left(w_1(t) \right)_{tt} + b \left(\delta_x^3 w_{\frac{3}{2}}(t) + \zeta(t) \right) = 0, \qquad \forall t \in [0,T].$$

☐ For $i = 2, \dots, N_1 - 1$, we have

$$h_i \left(w_i(t) \right)_{tt} + b \left(w_{xxx}\left(x_{i+\frac{1}{2}}, t \right) - w_{xxx}\left(x_{i-\frac{1}{2}}, t \right) \right) = 0, \qquad \forall t \in [0,T].$$

By replacing the differential operators for the second and third terms of the above equation with the corresponding difference operators and dropping the infinitesimal of higher orders, we get

$$h_i \left(w_i(t)\right)_{tt} + b\left(\delta_x^3 w_{i+\frac{1}{2}}(t) - \delta_x^3 w_{i-\frac{1}{2}}(t)\right) = 0, \qquad \forall t \in [0,T].$$

□ For $i = N_1$, we have

$$h_{N_1} \left(w_{N_1}(t)\right)_{tt} + b\left(w_{xxx}\left(x_{N_1+\frac{1}{2}}, t\right) - w_{xxx}\left(x_{N_1-\frac{1}{2}}, t\right)\right) = 0, \qquad \forall t \in [0,T].$$

Using the transmission condition $au_x(0,t) + bw_{xxx}(0,t) = 0$ as the position $x_{N_1} + \frac{1}{2}$ is exactly the point zero, and by replacing the second and third terms of the above equation by the corresponding difference operators dropping the infinitesimal of higher orders, we get

$$h_{N_1} \left(w_{N_1}(t)\right)_{tt} - \left(a\delta_x u_{N_1+\frac{1}{2}}(t) + b\delta_x^3 w_{N_1-\frac{1}{2}}(t)\right) = 0, \qquad \forall t \in [0,T].$$

Discretization of the Wave equation: Secondly, for $i = N_1+1, \ldots, N_{\max}$, integrate the first Equation of (5.1)-(5.4) on an arbitrary finite volume K_i to obtain

$$\int_{K_i} u_{tt}(x,t)dx - a\int_{K_i} u_{xx}(x,t)dx = 0, \qquad \forall t \in [0,T]. \tag{5.14}$$

By Leibniz integral rule for differentiation under the integral sign, the first term of Equation (5.14) has

$$\int_{K_i} u_{tt}(x,t)dx = h_i\left(\frac{1}{h_i}\int_{K_i} u(x,t)dx\right)_{tt} = h_i\left(u_i(t)\right)_{tt}, \qquad \forall t \in [0,T].$$

For the second term of Equation (5.14), the Integration by Parts formula claims

$$\int_{K_i} u_{xx}(x,t)dx = u_x\left(x_{i+\frac{1}{2}}, t\right) - u_x\left(x_{i-\frac{1}{2}}, t\right), \qquad \forall t \in [0,T].$$

Thus, (5.14) can be reformulated, for $i = N_1 + 1, \ldots, N_{\max}$, into

$$h_i \left(u_i(t)\right)_{tt} - a\left(u_x\left(x_{i+\frac{1}{2}}, t\right) - u_x\left(x_{i-\frac{1}{2}}, t\right)\right) = 0, \qquad \forall t \in [0,T].$$

□ For $i = N_1 + 1, \ldots, N_{\max} - 1$, we have

$$h_i \left(u_i(t)\right)_{tt} - a\left(u_x\left(x_{i+\frac{1}{2}}, t\right) - u_x\left(x_{i-\frac{1}{2}}, t\right)\right) = 0, \qquad \forall t \in [0,T].$$

By replacing the differential operators for the second and third terms of the above equation by the corresponding difference operators and dropping the infinitesimal of higher orders, we get

$$h_i \left(u_i(t)\right)_{tt} - a\left(\delta_x u_{i+\frac{1}{2}}(t) - \delta_x u_{i-\frac{1}{2}}(t)\right) = 0, \qquad \forall t \in [0,T].$$

□ For $i = N_{\max}$, we have

$$h_{N_{\max}} \left(u_{N_{\max}}(t)\right)_{tt} - a\left(u_x\left(x_{N_{\max}+\frac{1}{2}}, t\right) - u_x\left(x_{N_{\max}-\frac{1}{2}}, t\right)\right) = 0, \qquad \forall t \in [0,T].$$

Combining the above equation with the dynamical boundary condition $u_x(L,t) + \eta(t) = 0$ as the position $x_{N_{\max}+\frac{1}{2}}$ represents the point L and replacing the third term by its corresponding difference operator, we obtain

$$h_{N_{\max}} \left(u_{N_{\max}}(t)\right)_{tt} + a\left(\eta(t) + \delta_x u_{N_{\max}-\frac{1}{2}}(t)\right) = 0, \qquad \forall t \in [0,T].$$

Remark 5.4.1. *Here, the transmission condition* $au_x(0,t)+bw_{xxx}(0,t)=0$ *and the dynamical boundary conditions* $u_x(L,t)+\eta(t)=0$ *and* $w_{xxx}(-L,t)+\zeta(t)=0$ *are incorporated exactly into the numerical scheme without discretized errors.*

Discretization of the Continuity Conditions: The continuity conditions $u(0,t)=w(0,t)$ and $w_x(0,t)=0$ are discretized as

$$u_{N_1+\frac{1}{2}}(t)=w_{N_1+\frac{1}{2}}(t), \qquad \forall t\in[0,T]$$

and

$$\delta_x w_{N_1+\frac{1}{2}}(t)=0, \qquad \forall t\in[0,T].$$

Discretization of the Dynamical Boundary Condition: The dynamical boundary condition $w_{xx}(-L,t)-\xi(t)=0$ is also approximated by

$$\delta_x^2 w_0(t)-\xi(t)=0, \qquad \forall t\in[0,T].$$

Discretization of the First-Order Differential Equations: The remaining step in order to define our scheme is therefore the approximation of the first-order differential equations (5.4). The first Equation of (5.4) can be written as:

$$\eta_t(t)-u_{N_{\max}+1,t}(t)+\eta(t)=0, \qquad \forall t\in[0,T].$$

Using forward difference discretization in space for the second term of the second Equation of (5.4) yields the following

$$\xi_t(t)-\delta_x w_{\frac{1}{2},t}(t)+\xi(t)=0, \qquad \forall t\in[0,T],$$

and the third Equation of (5.4) can be also written as:

$$\zeta_t(t)-w_{0,t}(t)+\zeta(t)=0, \qquad \forall t\in[0,T].$$

Finally, by omitting the dependence $U_i(t)$ in t and simply denoting U_i as $U_i(t)$ unless it needs special specifications, we derive the semi-discretized numerical scheme for System (5.1)-(5.4)

$$\begin{cases} h_i U_{i,tt}+F_{i+\frac{1}{2}}-F_{i-\frac{1}{2}}=0, & i=1,\dots,N_{\max},\\ u_{N_1+\frac{1}{2}}=w_{N_1+\frac{1}{2}}, \quad \delta_x w_{N_1+\frac{1}{2}}=0, \quad \delta_x^2 w_0-\xi=0, \end{cases} \tag{5.15}$$

where the numerical flux $F_{i+\frac{1}{2}}$ is given by

$$F_{i+\frac{1}{2}}=\begin{cases} b\delta_x^3 w_{i+\frac{1}{2}}, & i=0,\dots,N_1-1,\\ -a\delta_x u_{i+\frac{1}{2}}, & i=N_1,\dots,N_{\max}, \end{cases} \tag{5.16}$$

along with the boundary conditions:

$$\delta_x u_{N_{\max}+\frac{1}{2}}+\eta=0, \qquad \delta_x^3 w_{\frac{1}{2}}+\zeta=0, \tag{5.17}$$

and the corresponding semi-discretized differential equations

$$\begin{cases} \eta_t-u_{N_{\max}+1,t}+\eta=0,\\ \xi_t-\delta_x w_{\frac{1}{2},t}+\xi=0,\\ \zeta_t-w_{0,t}+\zeta=0. \end{cases} \tag{5.18}$$

The scheme (5.15)-(5.18) is then used to build an approximate solution $U_{\mathcal{T}}$ defined by

$$U_{\mathcal{T}}(x,t)=U_i(t),$$

for $x\in K_i$, $i=1,\dots,N_{\max}$ and $t\in[0,T]$.

5.4.2 Estimation of the Discrete Energy for the Semi-Discrete Form

In this subsection, we plan to design the discrete energy of the semi-discretized numerical scheme (5.15)-(5.18) that conserves when no dynamical boundary controls are applied and dissipates when at least one dynamical boundary control is applied to the wave or the Euler-Bernoulli beam. For this aim, we define:

- The discrete kinetic energy for $U_{\mathcal{T}}$ as:

$$\mathcal{E}_k(U_{\mathcal{T}}) = \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i \left(U_{i,t}\right)^2.$$

- The discrete potential energy for $U_{\mathcal{T}}$ as:

$$\mathcal{E}_p(U_{\mathcal{T}}) = \frac{1}{2}\left(b\sum_{i=1}^{N_1} h_i(\delta_x^2 w_i)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}(\delta_x u_{i+\frac{1}{2}})^2\right).$$

- The dynamical boundary energy for Ξ as:

$$\mathcal{E}_b(\Xi) = \frac{1}{2}\left(a\eta^2 + b\xi^2 + b\zeta^2\right).$$

The total discrete energy $\mathcal{E}(t)$ of the semi-discretized system (5.15)-(5.18) is defined as

$$\mathcal{E}(t) = \mathcal{E}_k(U_{\mathcal{T}}) + \mathcal{E}_p(U_{\mathcal{T}}) + \mathcal{E}_b(\Xi), \qquad \forall t \in [0,T].$$

We want to prove that the above-stated goals for the discrete energy $\mathcal{E}(t)$ are fulfilled. For this purpose, we aim to prove the following theorem:

Theorem 5.4.1. *The derivative of the total discrete energy $\mathcal{E}(t)$ of system (5.15)-(5.18) satisfies the following estimation:*

$$\frac{d\mathcal{E}(t)}{dt} = -\left(a|\eta|^2 + b|\xi|^2 + b|\zeta|^2\right).$$

Proof. Multiplying both sides of the first Equation of the discrete problem (5.15)-(5.18) by $U_{i,t}$ and summing up i from 1 to $N_{\max}$, gives

$$\sum_{i=1}^{N_{\max}} h_i U_{i,tt} U_{i,t} + \sum_{i=1}^{N_{\max}} \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) U_{i,t} = 0,$$

so that

$$\sum_{i=1}^{N_{\max}} h_i U_{i,tt} U_{i,t} + \sum_{i=1}^{N_{\max}} F_{i+\frac{1}{2}} U_{i,t} - \sum_{i=1}^{N_{\max}} F_{i-\frac{1}{2}} U_{i,t} = 0.$$

By (5.16), the numerical flux $F_{i+\frac{1}{2}}$ yields

$$\begin{aligned}\sum_{i=1}^{N_{\max}} h_i U_{i,tt} U_{i,t} + b\sum_{i=1}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}} w_{i,t} - a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}} u_{i,t} \\ - b\sum_{i=1}^{N_1} \delta_x^3 w_{i-\frac{1}{2}} w_{i,t} + a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}} u_{i,t} - a\delta_x u_{N_1+\frac{1}{2}} w_{N_1,t} = 0.\end{aligned} \tag{5.19}$$

By the translation of index i for the second term of the following equation, we obtain

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}u_{i,t}+a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}u_{i,t}-a\delta_x u_{N_1+\frac{1}{2}}w_{N_1,t}\\
&=-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}u_{i,t}+a\sum_{i=N_1}^{N_{\max}-1}\delta_x u_{i+\frac{1}{2}}u_{i+1,t}-a\delta_x u_{N_1+\frac{1}{2}}w_{N_1,t}\\
&=a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}\left(u_{i+1,t}-u_{i,t}\right)+a\delta_x u_{N_1+\frac{1}{2}}\left(u_{N_1+1,t}-w_{N_1,t}\right)-a\delta_x u_{N_{\max}+\frac{1}{2}}u_{N_{\max}+1,t}\\
&=a\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2},t}+a\delta_x u_{N_1+\frac{1}{2}}\left(u_{N_1+1,t}-w_{N_1,t}\right)-a\delta_x u_{N_{\max}+\frac{1}{2}}u_{N_{\max}+1,t}.
\end{aligned}\tag{5.20}$$

Since $\delta_x w_{N_1+\frac{1}{2}}=0$, we have

$$w_{N_1+\frac{1}{2}}=w_{N_1}.\tag{5.21}$$

Thanks to (5.21) and the fact that $u_{N_1+\frac{1}{2}}=w_{N_1+\frac{1}{2}}$ due to the continuity of the functions at the interface, one has

$$\delta_x u_{N_1+\frac{1}{2}}=\frac{u_{N_1+1}-w_{N_1}}{h_{N_1+\frac{1}{2}}}.\tag{5.22}$$

Therefore, (5.20) together with (5.22) implies that

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}u_{i,t}+a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}u_{i,t}-a\delta_x u_{N_1+\frac{1}{2}}w_{N_1,t}\\
&=a\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2},t}+ah_{N_1+\frac{1}{2}}\delta_x u_{N_1+\frac{1}{2}}\delta_x u_{N_1+\frac{1}{2},t}-a\delta_x u_{N_{\max}+\frac{1}{2}}u_{N_{\max}+1,t}\\
&=a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2},t}-a\delta_x u_{N_{\max}+\frac{1}{2}}u_{N_{\max}+1,t}.
\end{aligned}$$

Noticing that $\delta_x u_{N_{\max}+\frac{1}{2}}=-\eta$, one has from the last equality that

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}u_{i,t}+a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}u_{i,t}-a\delta_x u_{N_1+\frac{1}{2}}w_{N_1,t}\\
&=a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2},t}+a\eta u_{N_{\max}+1,t}.
\end{aligned}\tag{5.23}$$

Multiplying both sides of the first Equation of (5.18) by η yields

$$\eta\eta_t-\eta u_{N_{\max}+1,t}+|\eta|^2=0.\tag{5.24}$$

Hence, (5.23) together with (5.24) implies that

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}u_{i,t}+a\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}u_{i,t}-a\delta_x u_{N_1+\frac{1}{2}}w_{N_1,t}\\
&=a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2},t}+a\eta\eta_t+a|\eta|^2.
\end{aligned}\tag{5.25}$$

Now, by the translation of index i for the second term of the following equation, we obtain

$$\begin{aligned} b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}w_{i,t} - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}w_{i,t} &= b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}w_{i,t} - b\sum_{i=0}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}w_{i+1,t} \\ &= b\sum_{i=0}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}\left(w_{i,t}-w_{i+1,t}\right) - b\delta_x^3 w_{\frac{1}{2}}w_{0,t} \\ &= -b\sum_{i=0}^{N_1-1} h_{i+\frac{1}{2}}\delta_x^3 w_{i+\frac{1}{2}}\delta_x w_{i+\frac{1}{2},t} - b\delta_x^3 w_{\frac{1}{2}}w_{0,t} \\ &= b\sum_{i=0}^{N_1-1}\left(\delta_x^2 w_i - \delta_x^2 w_{i+1}\right)\delta_x w_{i+\frac{1}{2},t} - b\delta_x^3 w_{\frac{1}{2}}w_{0,t} \\ &= b\sum_{i=0}^{N_1-1}\delta_x^2 w_i\delta_x w_{i+\frac{1}{2},t} - b\sum_{i=0}^{N_1-1}\delta_x^2 w_{i+1}\delta_x w_{i+\frac{1}{2},t} - b\delta_x^3 w_{\frac{1}{2}}w_{0,t}. \end{aligned}$$

Once again, the translation of index i for the second term of the right-hand side of the above equation with the help of the fact that $\delta_x w_{N_1+\frac{1}{2}} = 0$ gives

$$\begin{aligned} b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}w_{i,t} - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}w_{i,t} &= b\sum_{i=0}^{N_1-1}\delta_x^2 w_i\delta_x w_{i+\frac{1}{2},t} - b\sum_{i=1}^{N_1}\delta_x^2 w_i\delta_x w_{i-\frac{1}{2},t} - b\delta_x^3 w_{\frac{1}{2}}w_{0,t} \\ &= b\sum_{i=1}^{N_1}\delta_x^2 w_i\left(\delta_x w_{i+\frac{1}{2},t} - \delta_x w_{i-\frac{1}{2},t}\right) + b\delta_x^2 w_0\delta_x w_{\frac{1}{2},t} - b\delta_x^3 w_{\frac{1}{2}}w_{0,t} \\ &= b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i\delta_x^2 w_{i,t} + b\delta_x^2 w_0\delta_x w_{\frac{1}{2},t} - b\delta_x^3 w_{\frac{1}{2}}w_{0,t}. \end{aligned}$$

Noticing that $\delta_x^2 w_0 = \xi$ and $\delta_x^3 w_{\frac{1}{2}} = -\zeta$, this last equality yields

$$b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}w_{i,t} - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}w_{i,t} = b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i\delta_x^2 w_{i,t} + b\xi\delta_x w_{\frac{1}{2},t} + b\zeta w_{0,t}. \tag{5.26}$$

Multiplying both sides of the second and third Equations of (5.18) by ξ and ζ respectively, we obtain

$$\xi\xi_t - \xi\delta_x w_{\frac{1}{2},t} + |\xi|^2 = 0, \tag{5.27}$$

and

$$\zeta\zeta_t - \zeta w_{0,t} + |\zeta|^2 = 0. \tag{5.28}$$

Therefore, (5.26), together with (5.27) and (5.28), implies

$$b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}w_{i,t} - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}w_{i,t} = b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i\delta_x^2 w_{i,t} + b\xi\xi_t + b\zeta\zeta_t + b|\xi|^2 + b|\zeta|^2. \tag{5.29}$$

In order to complete the proof, plug (5.25) and (5.29) into (5.19) giving

$$\begin{aligned} \sum_{i=1}^{N_{\max}} h_i U_{i,tt}U_{i,t} + b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i\delta_x^2 w_{i,t} + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2},t} \\ +a\eta\eta_t + b\xi\xi_t + b\zeta\zeta_t + a|\eta|^2 + b|\xi|^2 + b|\zeta|^2 = 0, \end{aligned}$$

which implies that

$$\frac{d\mathcal{E}(t)}{dt} = -\left(a|\eta|^2 + b|\xi|^2 + b|\zeta|^2\right).$$

This completes the proof of the theorem. ■

According to Theorem 5.4.1, the discretized energy $\mathcal{E}(t)$ of system (5.15)-(5.18) is conserved when no dynamical controls are applied to the system, and it is non-increasing in time as long as at least one dynamical boundary control is applied to the wave or the Euler-Bernoulli beam.

5.4.3 Stability Estimates on the Approximate Solution

In this subsection, necessary estimates that ensure the stability of the Scheme (5.15)-(5.18) are derived. First, we express the set of all functions from $(-L, 0) \cup (0, L)$ to $\mathbb{R}$ which are constant over each finite volume of the mesh by $\mathcal{X}(\mathcal{T})$. Then, for arbitrary $U_\mathcal{T} \in \mathcal{X}(\mathcal{T})$, let the following $\mathbb{L}^2$ norm be defined by

$$\|U_\mathcal{T}\| = \left(\sum_{i=1}^{N_{\max}} h_i (U_i)^2\right)^{\frac{1}{2}}.$$

Further, we use the discrete $\mathbb{H}$ norm of the function $U_\mathcal{T} = (u_\mathcal{T}, w_\mathcal{T})$ in the form

$$\|U_\mathcal{T}\|_{\mathbb{H},\mathcal{T}} = \left(\|u_\mathcal{T}\|_{1,\mathcal{T}}^2 + \|w_\mathcal{T}\|_{2,\mathcal{T}}^2\right)^{\frac{1}{2}},$$

where

$$\|u_\mathcal{T}\|_{1,\mathcal{T}}^2 = a \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} (\delta_x u_{i+\frac{1}{2}})^2$$

and

$$\|w_\mathcal{T}\|_{2,\mathcal{T}}^2 = b \sum_{i=1}^{N_1} h_i (\delta_x^2 w_i)^2$$

denote the discrete norms in $H_*^1(0, L)$ and $H_*^2(-L, 0)$ of the functions $u_\mathcal{T}$ and $w_\mathcal{T}$ respectively. The $\mathbb{R}^3$ norm of the function $\Xi = (\eta, \xi, \zeta)$ is also defined by

$$\|\Xi\|_{\mathbb{R}^3} = \left(a\eta^2 + b\xi^2 + b\zeta^2\right)^{\frac{1}{2}}.$$

The discrete norms are used to obtain stability properties which are given by some estimates on the approximate solution of (5.15)-(5.18) under **Hypothesis (H1)**.

Theorem 5.4.2. *The numerical solution $U_\mathcal{T}$ of the numerical scheme (5.15)-(5.18) satisfies the following inequality:*

$$\|U_{\mathcal{T},t}\|^2 + \|U_\mathcal{T}\|_{\mathbb{H},\mathcal{T}}^2 + \|\Xi\|_{\mathbb{R}^3}^2 \leq \|\Psi_\mathcal{T}\|^2 + \|\Phi_\mathcal{T}\|_{\mathbb{H},\mathcal{T}}^2 + \|\Theta\|_{\mathbb{R}^3}^2. \tag{5.30}$$

Proof. From Theorem 5.4.1 and using the fact that $-\left(a|\eta|^2 + b|\xi|^2 + b|\zeta|^2\right) \leq 0$, we deduce that the discrete energy $\mathcal{E}(t)$ is decreasing over $[0, T]$. This implies that

$$\mathcal{E}(t) \leq \mathcal{E}(0), \qquad \forall t \in [0, T],$$

which gives,

$$\sum_{i=1}^{N_{\max}} h_i \left(U_{i,t}\right)^2 + b\sum_{i=1}^{N_1} h_i(\delta_x^2 w_i)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}(\delta_x u_{i+\frac{1}{2}})^2 + a\eta^2 + b\xi^2 + b\zeta^2$$
$$\leq \sum_{i=1}^{N_{\max}} h_i \left(\Psi_i\right)^2 + b\sum_{i=1}^{N_1} h_i(\delta_x^2 \phi_i)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}(\delta_x \varphi_{i+\frac{1}{2}})^2 + a\alpha^2 + b\beta^2 + b\gamma^2.$$

Writing the above terms as norms (since $\mathcal{E}(t) \geq 0,\ \forall t \in [0,T]$), we obtain the desired estimation (5.30) which concludes the proof. ■

Theorem 5.4.3. *Let the **Hypothesis (H1)** hold. Then, the numerical solution $U_\mathcal{T}$ of the numerical scheme (5.15)-(5.18) is bounded independently on discretization parameters in $L^\infty([0,T];\mathbb{L}^2) \cap L^\infty([0,T];\mathbb{H})$. Moreover, the first time derivation is bounded in $L^\infty([0,T];\mathbb{L}^2)$ and the function Ξ is bounded in $L^\infty([0,T];\mathbb{R}^3)$ as well. Therefore, the following estimation holds*

$$\|U_{\mathcal{T},t}\|^2 + \|U_\mathcal{T}\|^2 + \|U_\mathcal{T}\|^2_{\mathbb{H},\mathcal{T}} + \|\Xi\|^2_{\mathbb{R}^3} \leq C, \tag{5.31}$$

where C is a strictly positive constant independent of the space discretization parameter.

Proof. Using the properties of the functions Φ, Ψ and Θ, i.e., using **Hypothesis (H1)** and the results obtained in Lemma 2.4.1 and Theorem 5.4.2, we can conclude the estimate

$$\|U_{\mathcal{T},t}\|^2 + \|U_\mathcal{T}\|^2_{\mathbb{H},\mathcal{T}} + \|\Xi\|^2_{\mathbb{R}^3} \leq \|\Psi_\mathcal{T}\|^2 + \|\Phi_\mathcal{T}\|^2_{\mathbb{H},\mathcal{T}} + \|\Theta\|^2_{\mathbb{R}^3} \leq C, \tag{5.32}$$

with C being a strictly positive constant independent of the discretization parameter and depending only on the initial data of the problem. Then, using the Discrete Poincare's inequality, i.e., using Lemma 2.4.3, we immediately get

$$\begin{aligned}
\|U_\mathcal{T}\|^2 &= \|u_\mathcal{T}\|^2 + \|w_\mathcal{T}\|^2 \\
&\leq c_1 \|u_\mathcal{T}\|^2_{1,\mathcal{T}} + c_2 \|w_\mathcal{T}\|^2_{2,\mathcal{T}} \\
&\leq \max\{c_1, c_2\} \left(\|u_\mathcal{T}\|^2_{1,\mathcal{T}} + \|w_\mathcal{T}\|^2_{2,\mathcal{T}}\right) \\
&= \max\{c_1, c_2\} \|U_\mathcal{T}\|^2_{\mathbb{H},\mathcal{T}},
\end{aligned}$$

with $c_1,\ c_2$ are strictly positive constants. We can deduce from (5.32) that

$$\|U_\mathcal{T}\|^2 \leq C, \tag{5.33}$$

which concludes the proof. ■

5.4.4 Convergence of the Semi-Discrete Scheme

In this subsection, we prove the convergence of the numerical solution obtained by the numerical scheme (5.15)-(5.18) when the size of the mesh tends to zero. The strategy is based on the stability estimates obtained in Subsection 5.4.3 and passing through the limit, we obtain the convergence to the weak solution. To this end, we will write the continuous weak formulation problem (5.6)-(5.9) in its discrete form. First, we introduce the following space:

$$C^\infty_*([-L,0]) = \{z \in C^\infty([-L,0]) \quad | \quad z_x(0) = 0\}.$$

As we will use density arguments, we set the function $V = (v, z) \in C^{\infty}([0,T]; C_*)$ where

$$C_* = \{(v,z) \in C^{\infty}([0,L]) \times C_*^{\infty}([-L,0]) \quad | \quad v(0) = z(0)\},$$

such that $V(x,T) = 0$ and let size$(\mathcal{T})$ be small enough. Define

$$\begin{aligned} v_i(t) &= \frac{1}{h_i}\int_{K_i} v(x,t)dx, \quad \text{for } i = N_1+1,\dots,N_{\max}, \\ z_i(t) &= \frac{1}{h_i}\int_{K_i} z(x,t)dx, \quad \text{for } i = 1,\dots,N_1. \end{aligned}$$

Let us define also

$$V_i = \begin{cases} v_i, & i = N_1+1,\dots,N_{\max}, \\ z_i, & i = 1,\dots,N_1. \end{cases}$$

Now, we want to get the discrete form of (5.6)-(5.9) passing through several steps:

Multiplying both sides of the first Equation of the numerical scheme (5.15)-(5.18) by V_i, summing up i from 1 to $N_{\max}$ and integrating over $[0,T]$, we obtain

$$\int_0^T \sum_{i=1}^{N_{\max}} h_i U_{i,tt} V_i dt + \int_0^T \sum_{i=1}^{N_{\max}} \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) V_i dt = 0,$$

so that

$$\int_0^T \sum_{i=1}^{N_{\max}} h_i U_{i,tt} V_i dt + \int_0^T \sum_{i=1}^{N_{\max}} F_{i+\frac{1}{2}} V_i dt - \int_0^T \sum_{i=1}^{N_{\max}} F_{i-\frac{1}{2}} V_i dt = 0.$$

By (5.16), the numerical flux $F_{i+\frac{1}{2}}$ yields

$$\begin{aligned} &\int_0^T \sum_{i=1}^{N_{\max}} h_i U_{i,tt} V_i dt + b\int_0^T \sum_{i=1}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}} z_i dt - a\int_0^T \sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}} v_i dt \\ &\quad - b\int_0^T \sum_{i=1}^{N_1} \delta_x^3 w_{i-\frac{1}{2}} z_i dt + a\int_0^T \sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}} v_i dt - a\int_0^T \delta_x u_{N_1+\frac{1}{2}} z_{N_1} dt = 0. \end{aligned} \tag{5.34}$$

The translation of index i for the second term of the following equation gives

$$\begin{aligned} &-a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}} v_i + a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}} v_i - a\delta_x u_{N_1+\frac{1}{2}} z_{N_1} \\ &= -a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}} v_i + a\sum_{i=N_1}^{N_{\max}-1} \delta_x u_{i+\frac{1}{2}} v_{i+1} - a\delta_x u_{N_1+\frac{1}{2}} z_{N_1} \\ &= a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}} (v_{i+1} - v_i) + a\delta_x u_{N_1+\frac{1}{2}} (v_{N_1+1} - z_{N_1}) - a\delta_x u_{N_{\max}+\frac{1}{2}} v_{N_{\max}+1} \\ &= a\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}} + a\delta_x u_{N_1+\frac{1}{2}} (v_{N_1+1} - z_{N_1}) - a\delta_x u_{N_{\max}+\frac{1}{2}} v_{N_{\max}+1}. \end{aligned} \tag{5.35}$$

Since $\delta_x z_{N_1+\frac{1}{2}} = 0$, we have

$$z_{N_1+\frac{1}{2}} = z_{N_1}. \tag{5.36}$$

Thanks to (5.36) and the fact that $v_{N_1+\frac{1}{2}} = z_{N_1+\frac{1}{2}}$, one has

$$\delta_x v_{N_1+\frac{1}{2}} = \frac{v_{N_1+1} - z_{N_1}}{h_{N_1+\frac{1}{2}}}. \tag{5.37}$$

Therefore, (5.35) together with (5.37) implies that

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}} v_i + a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}} v_i - a\delta_x u_{N_1+\frac{1}{2}} z_{N_1} \\
&= a\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}} + a h_{N_1+\frac{1}{2}} \delta_x u_{N_1+\frac{1}{2}} \delta_x v_{N_1+\frac{1}{2}} - a\delta_x u_{N_{\max}+\frac{1}{2}} v_{N_{\max}+1} \\
&= a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}} - a\delta_x u_{N_{\max}+\frac{1}{2}} v_{N_{\max}+1}.
\end{aligned}$$

Noticing that $\delta_x u_{N_{\max}+\frac{1}{2}} = -\eta$, one has from the last equality that

$$-a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}} v_i + a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}} v_i - a\delta_x u_{N_1+\frac{1}{2}} z_{N_1} = a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}} + a\eta v_{N_{\max}+1}. \tag{5.38}$$

Now, by the translation of index i for the second term of the following equation, a direct calculation gives

$$\begin{aligned}
b\sum_{i=1}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}} z_i - b\sum_{i=1}^{N_1} \delta_x^3 w_{i-\frac{1}{2}} z_i &= b\sum_{i=1}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}} z_i - b\sum_{i=0}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}} z_{i+1} \\
&= b\sum_{i=0}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}} (z_i - z_{i+1}) - b\delta_x^3 w_{\frac{1}{2}} z_0 \\
&= -b\sum_{i=0}^{N_1-1} h_{i+\frac{1}{2}} \delta_x^3 w_{i+\frac{1}{2}} \delta_x z_{i+\frac{1}{2}} - b\delta_x^3 w_{\frac{1}{2}} z_0 \\
&= b\sum_{i=0}^{N_1-1} \left(\delta_x^2 w_i - \delta_x^2 w_{i+1}\right) \delta_x z_{i+\frac{1}{2}} - b\delta_x^3 w_{\frac{1}{2}} z_0 \\
&= b\sum_{i=0}^{N_1-1} \delta_x^2 w_i \delta_x z_{i+\frac{1}{2}} - b\sum_{i=0}^{N_1-1} \delta_x^2 w_{i+1} \delta_x z_{i+\frac{1}{2}} - b\delta_x^3 w_{\frac{1}{2}} z_0.
\end{aligned}$$

Once again, the translation of index i for the second term of the right-hand side of the above equation

with the help of the fact that $\delta_x z_{N_1+\frac{1}{2}} = 0$ gives

$$\begin{aligned} b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}} z_i - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}} z_i &= b\sum_{i=0}^{N_1-1}\delta_x^2 w_i \delta_x z_{i+\frac{1}{2}} - b\sum_{i=1}^{N_1}\delta_x^2 w_i \delta_x z_{i-\frac{1}{2}} - b\delta_x^3 w_{\frac{1}{2}} z_0 \\ &= b\sum_{i=1}^{N_1}\delta_x^2 w_i \left(\delta_x z_{i+\frac{1}{2}} - \delta_x z_{i-\frac{1}{2}}\right) + b\delta_x^2 w_0 \delta_x z_{\frac{1}{2}} - b\delta_x^3 w_{\frac{1}{2}} z_0 \\ &= b\sum_{i=1}^{N_1} h_i \delta_x^2 w_i \delta_x^2 z_i + b\delta_x^2 w_0 \delta_x z_{\frac{1}{2}} - b\delta_x^3 w_{\frac{1}{2}} z_0. \end{aligned}$$

Noticing that $\delta_x^2 w_0 = \xi$ and $\delta_x^3 w_{\frac{1}{2}} = -\zeta$, this last equality yields

$$b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}} z_i - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}} z_i = b\sum_{i=1}^{N_1} h_i \delta_x^2 w_i \delta_x^2 z_i + b\xi\delta_x z_{\frac{1}{2}} + b\zeta z_0. \tag{5.39}$$

After these operations, plug (5.38) and (5.39) into (5.34) to obtain

$$\begin{aligned} &\int_0^T \sum_{i=1}^{N_{\max}} h_i U_{i,tt} V_i dt + a\int_0^T \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}} dt + b\int_0^T \sum_{i=1}^{N_1} h_i \delta_x^2 w_i \delta_x^2 z_i dt \\ &\qquad + a\int_0^T \eta v_{N_{\max}+1} dt + b\int_0^T \xi\delta_x z_{\frac{1}{2}} dt + b\int_0^T \zeta z_0 dt = 0. \end{aligned} \tag{5.40}$$

The Integration by Parts formula for the first term of (5.40) and the fact that $V_i(T) = 0$ and $U_{i,t}(0) = \Psi_i$ achieves,

$$\int_0^T \sum_{i=1}^{N_{\max}} h_i U_{i,tt} V_i dt = -\int_0^T \sum_{i=1}^{N_{\max}} h_i U_{i,t} V_{i,t} dt - \sum_{i=1}^{N_{\max}} h_i \Psi_i V_i(0). \tag{5.41}$$

Substituting (5.41) into (5.40), we get

$$\begin{aligned} &-\int_0^T \sum_{i=1}^{N_{\max}} h_i U_{i,t} V_{i,t} dt + a\int_0^T \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}} dt + b\int_0^T \sum_{i=1}^{N_1} h_i \delta_x^2 w_i \delta_x^2 z_i dt \\ &\qquad - \sum_{i=1}^{N_{\max}} h_i \Psi_i V_i(0) + a\int_0^T \eta v_{N_{\max}+1} dt + b\int_0^T \xi\delta_x z_{\frac{1}{2}} dt + b\int_0^T \zeta z_0 dt = 0. \end{aligned} \tag{5.42}$$

Now, using the notations introduced previously, we obtain from (5.42) that

$$\begin{aligned} &-\int_0^T \int_{-L}^{L} U_{\mathcal{T},t} V_{\mathcal{T},t} dx dt + a\int_0^T \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}} dt + b\int_0^T \sum_{i=1}^{N_1} h_i \delta_x^2 w_i \delta_x^2 z_i dt \\ &\qquad - \int_{-L}^{L} \Psi_{\mathcal{T}} V_{\mathcal{T}}(0) dx + a\int_0^T \eta v_{N_{\max}+1} dt + b\int_0^T \xi\delta_x z_{\frac{1}{2}} dt + b\int_0^T \zeta z_0 dt = 0. \end{aligned}$$

The above equation can be reformulated as the following

$$\begin{aligned}-\int_0^T\int_0^L u_{\mathcal{T},t}v_{\mathcal{T},t}dxdt + a\int_0^T\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x v_{i+\frac{1}{2}}dt - \int_0^L \psi_{\mathcal{T}}(x)v_{\mathcal{T}}(x,0)dx\\ -\int_0^T\int_{-L}^0 w_{\mathcal{T},t}z_{\mathcal{T},t}dxdt + b\int_0^T\sum_{i=1}^{N_1} h_i\delta_x^2 w_i\delta_x^2 z_i dt - \int_{-L}^0 \theta_{\mathcal{T}}(x)z_{\mathcal{T}}(x,0)dx\\ +a\int_0^T \eta v_{N_{\max}+1}dt + b\int_0^T \xi\delta_x z_{\frac{1}{2}}dt + b\int_0^T \zeta z_0 dt = 0.\end{aligned} \tag{5.43}$$

Now, let us define the function $(p,q,r) \in C^\infty\left([0,T];\mathbb{R}^3\right)$ such that $p(T) = q(T) = r(T) = 0$. Multiplying the Equations in (5.18) by p, q and r respectively and integrating over $[0,T]$, then using the Integration by Parts formula with the fact that $\eta(0) = \alpha$, $\xi(0) = \beta$ and $\zeta(0) = \gamma$, we get

$$-\int_0^T \eta p_t dt - \alpha p(0) + \int_0^T u_{N_{\max}+1}p_t dt + u_{N_{\max}+1}(0)p(0) + \int_0^T \eta p dt = 0, \tag{5.44}$$

$$-\int_0^T \xi q_t dt - \beta q(0) + \int_0^T \delta_x w_{\frac{1}{2}} q_t dt + \delta_x w_{\frac{1}{2}}(0)q(0) + \int_0^T \xi q dt = 0, \tag{5.45}$$

and

$$-\int_0^T \zeta r_t dt - \gamma r(0) + \int_0^T w_0 r_t dt + w_0(0)r(0) + \int_0^T \zeta r dt = 0. \tag{5.46}$$

We are now able to state the convergence result in the following theorem:

Theorem 5.4.4. *Assume that **Hypothesis (H1)** holds. For $m \in \mathbb{N}$, let $\mathcal{T}_m$ be an admissible mesh described in the sense of Definition 5.3.1. Let $(u_{\mathcal{T}_m}, w_{\mathcal{T}_m}, \eta, \xi, \zeta)$ be the discrete solution of (5.15)-(5.18). Assume that size$(\mathcal{T}_m) \longrightarrow 0$ as $m \longrightarrow \infty$. Then, there exists a subsequence of the sequence of approximate solutions, still denoted by $(u_{\mathcal{T}_m}, w_{\mathcal{T}_m}, \eta, \xi, \zeta)$, which converges to the weak solution (u, w, η, ξ, ζ) in $L^\infty([0,T];L^2(0,L))\times L^\infty([0,T];L^2(-L,0))\times L^\infty([0,T];\mathbb{R}^3)$ of the problem (5.6)-(5.9), as m tends to ∞, for the weak-$\star$ topology.*

Proof. Thanks to the Theorem 5.4.3, we have that the sequence $U_{\mathcal{T}_m} = (u_{\mathcal{T}_m}, w_{\mathcal{T}_m})$ is bounded in $L^\infty([0,T];L^2(0,L)) \times L^\infty([0,T];L^2(-L,0))$. Then, there exists a subsequence, still denoted by $U_{\mathcal{T}_m} = (u_{\mathcal{T}_m}, w_{\mathcal{T}_m})$, such that $(u_{\mathcal{T}_m}, w_{\mathcal{T}_m})$ converges to the function (u,w) in $L^\infty([0,T];L^2(0,L)) \times L^\infty([0,T];L^2(-L,0))$ as $m \longrightarrow \infty$ for the weak-$\star$ topology. Further, following [18], there exists $(u,w) \in C^0([0,T];\mathbb{L}^2)$ with $(u(\cdot,0), w(\cdot,0)) = (\varphi(\cdot), \phi(\cdot))$ and a subsequence of approximation solutions denoted again by $(u_{\mathcal{T}_m}, w_{\mathcal{T}_m})$ converging to (u,w) in $L^\infty([0,T];\mathbb{L}^2)$. Moreover, due to the stability assumption in the discrete $\mathbb{H}$-norm, we have $(u,w) \in L^\infty([0,T],\mathbb{H})$. Furthermore, from the stability estimates, we know that both $u_{\mathcal{T}_m}$ and $\delta_x u_{\mathcal{T}_m}$ are bounded in $L^\infty([0,T];L^2(0,L))$. To establish compactness in time, we apply Simon's compactness criterion (see Theorem 1, page 71 of [54]), which characterizes relatively compact subsets in $L^\infty([0,T];L^2(0,L))$. It follows that the sequences $\{u_{\mathcal{T}_m}\}$ and $\{\delta_x u_{\mathcal{T}_m}\}$ are relatively compact in $C([0,T];L^2(0,L))$. Specifically, the boundedness of $u_{\mathcal{T}_m}$ in $L^\infty([0,T];L^2(0,L))$ implies that the integrals $\int_{t_1}^{t_2} u_{\mathcal{T}_m}(t)\,dt$ are bounded in $L^2(0,L)$, hence the family $\left\{\int_{t_1}^{t_2} u_{\mathcal{T}_m}(t)\,dt\right\}$ is relatively compact in $L^2(0,L)$ for all $0 < t_1 < t_2 < T$. Additionally, using Lemma 9.3 from [17], we show that the sequence $\{u_{\mathcal{T}_m}\}$ is uniformly equicontinuous in time; that is, for every $\epsilon > 0$, , there exists $\delta > 0$ such that $\|u_{\mathcal{T}_m}(t+h) - u_{\mathcal{T}_m}(t)\|_{L^2(0,L)} \leq \epsilon$ for $|h| < \delta$. As a result, there exists a subsequence (still denoted by $u_{\mathcal{T}_m}$) that converges to u in $L^\infty([0,T];L^2(0,L))$ as

$m \longrightarrow \infty$. Similarly, the sequence $\delta_x u_{\mathcal{T}_m}$ admits a subsequence converging to u_x in $L^\infty([0,T];L^2(0,L))$ as $m \longrightarrow \infty$. Analogously, from the stability estimates, we deduce that $w_{\mathcal{T}_m}$, $\delta_x w_{\mathcal{T}_m}$, and $\delta_x^2 w_{\mathcal{T}_m}$ are bounded in $L^\infty([0,T];L^2(-L,0))$. Thus, there exists a subsequence (still denoted by $w_{\mathcal{T}_m}$) that converges to w in $L^\infty([0,T];L^2(-L,0))$ as $m \to \infty$. Moreover, there exist subsequences of $\delta_x w_{\mathcal{T}_m}$ and $\delta_x^2 w_{\mathcal{T}_m}$ (denoted similarly) that converge respectively to w_x and w_{xx} in $L^\infty([0,T];L^2(-L,0))$ as $m \to \infty$. We now move to show the convergence of the numerical solution to the weak solution of our problem, i.e., it only remains to prove that (u,w,η,ξ,ζ) is the weak solution of the problem (5.6)-(5.9). The proof of Theorem 5.4.4 is divided into several steps:

Step 1. The regularity properties of the function V gives, for $m \longrightarrow \infty$, the strong convergence of $v_{\mathcal{T}_m,t}$ to v_t and $z_{\mathcal{T}_m,t}$ to z_t in $L^2([0,T]\times(0,L))$ and $L^2([0,T]\times(-L,0))$ respectively. The stability estimate results show that $U_{\mathcal{T}_m,t}$ is bounded in $L^\infty([0,T];\mathbb{L}^2)$ which implies that $u_{\mathcal{T}_m,t}$ and $w_{\mathcal{T}_m,t}$ are bounded in $L^\infty([0,T];L^2(0,L))$ and $L^\infty([0,T];L^2(-L,0))$ respectively, and thus $u_{\mathcal{T}_m,t} \rightharpoonup u_t$ weakly and $w_{\mathcal{T}_m,t} \rightharpoonup w_t$ weakly in $L^2([0,T]\times(0,L))$ and $L^2([0,T]\times(-L,0))$ respectively (as $L^2([0,T]\times(0,L))$ and $L^2([0,T]\times(-L,0))$ are weakly compact). We then obtain

$$-\int_0^T\int_0^L u_{\mathcal{T}_m,t}v_{\mathcal{T}_m,t}dxdt \longrightarrow -\int_0^T\int_0^L u_t v_t dxdt, \qquad \text{as } m \longrightarrow \infty,$$

and

$$-\int_0^T\int_{-L}^0 w_{\mathcal{T}_m,t}z_{\mathcal{T}_m,t}dxdt \longrightarrow -\int_0^T\int_{-L}^0 w_t z_t dxdt, \qquad \text{as } m \longrightarrow \infty.$$

Also, using the fact that for all $\psi \in L^2(0,L)$, $\psi_{\mathcal{T}_m} \rightharpoonup \psi$ in $L^2(0,L)$, we deduce that

$$-\int_0^L \psi_{\mathcal{T}_m}(x)v_{\mathcal{T}_m}(x,0)dx \longrightarrow -\int_0^L \psi(x)v(x,0)dx, \qquad \text{as } m \longrightarrow \infty,$$

for function v regular enough. Using the previous arguments, a similar result is achieved for

$$-\int_{-L}^0 \theta_{\mathcal{T}_m}(x)z_{\mathcal{T}_m}(x,0)dx \longrightarrow -\int_{-L}^0 \theta(x)z(x,0)dx, \qquad \text{as } m \longrightarrow \infty.$$

Step 2. We denote the second and fifth terms of (5.43) in the following form:

$$\begin{aligned} T_1 &= a\int_0^T \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x v_{i+\frac{1}{2}}dt = a\int_0^T \frac{h_2}{2}\delta_x u_{N_1+\frac{1}{2}}\delta_x v_{N_1+\frac{1}{2}}dt \\ &\quad + a\int_0^T \sum_{i=N_1+1}^{N_{\max}-1} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x v_{i+\frac{1}{2}}dt + a\int_0^T \frac{h_2}{2}\delta_x u_{N_{\max}+\frac{1}{2}}\delta_x v_{N_{\max}+\frac{1}{2}}dt, \\ T_2 &= b\int_0^T \sum_{i=1}^{N_1} h_i\delta_x^2 w_i\delta_x^2 z_i dt. \end{aligned}$$

Let us turn to the study of T_1 and T_2. For the term T_1, we consider the following auxiliary expression:

$$\tilde{T}_1 = -a\int_0^T\int_0^L u_{\mathcal{T}_m}\ v_{xx}(x,t)dxdt + a\int_0^T u_{N_{\max}+1}\ v_x(L)dt - a\int_0^T u_{N_1+\frac{1}{2}}\ v_x(0)dt.$$

It is important to note that $x_{N_1+\frac{1}{2}} = 0$ and $x_{N_{\max}+1} = L$. Since $u_{\mathcal{T}_m}$ converges to u in $L^\infty([0,T];L^2(0,L))$ and $\delta_x u_{\mathcal{T}_m}$ converges to u_x in $L^\infty([0,T];L^2(0,L))$ as $m \longrightarrow \infty$, it follows that the $u_{\mathcal{T}_m}(0)$ and $u_{\mathcal{T}_m}(L)$ converge to $u(0)$ and $u(L)$, respectively, in $L^\infty([0,T])$. Consequently, we obtain

$$\begin{aligned} u_{N_1+\frac{1}{2}} &\longrightarrow u(0), \\ u_{N_{\max}+1} &\longrightarrow u(L), \end{aligned}$$

in $L^\infty([0,T])$ as $m \longrightarrow \infty$. Furthermore, since $u_{\mathcal{T}_m} \longrightarrow u$ in $L^\infty([0,T];L^2(0,L))$, we immediately obtain

$$\tilde{T}_1 \longrightarrow -a\int_0^T\int_0^L u\ v_{xx}dxdt + a\int_0^T u(L)\ v_x(L)dt - a\int_0^T u(0)\ v_x(0)dt = a\int_0^T\int_0^L u_x\ v_x dxdt.$$

Now, we reformulate $\tilde{T}_1$ as follows:

$$\begin{aligned} \tilde{T}_1 &= -a\int_0^T \sum_{i=N_1+1}^{N_{\max}} u_i \int_{K_i} v_{xx}(x,t)dxdt + a\int_0^T u_{N_{\max}+1}\ v_x(L)dt - a\int_0^T u_{N_1+\frac{1}{2}}\ v_x(0)dt \\ &= \tilde{T}_{11} + \tilde{T}_{12}. \end{aligned}$$

We can write $\tilde{T}_{11}$ in the following form using integral formulation:

$$\tilde{T}_{11} = -a\int_0^T \sum_{i=N_1+1}^{N_{\max}} u_i \int_{K_i} v_{xx}(x,t)dxdt = -a\int_0^T \sum_{i=N_1+1}^{N_{\max}} u_i\ \delta_x v_{i+\frac{1}{2}}dt + a\int_0^T \sum_{i=N_1+1}^{N_{\max}} u_i\ \delta_x v_{i-\frac{1}{2}}dt.$$

Using the translation of index i for the second term of the right-hand side of the above equation, we obtain

$$\tilde{T}_{11} = a\int_0^T \sum_{i=N_1+1}^{N_{\max}-1} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}\delta_x v_{i+\frac{1}{2}}dt - a\int_0^T u_{N_{\max}}\delta_x v_{N_{\max}+\frac{1}{2}}dt + a\int_0^T u_{N_1+1}\delta_x v_{N_1+\frac{1}{2}}dt.$$

We can also write

$$\begin{aligned} \tilde{T}_{12} &= a\int_0^T u_{N_{\max}+1}\ v_x(L)dt - a\int_0^T u_{N_1+\frac{1}{2}}\ v_x(0)dt \\ &= a\int_0^T u_{N_{\max}+1}\ \delta_x v_{N_{\max}+\frac{1}{2}}dt - a\int_0^T u_{N_1+\frac{1}{2}}\ \delta_x v_{N_1+\frac{1}{2}}dt. \end{aligned}$$

For the term T_2, we consider the following auxiliary expression:

$$\begin{aligned} \tilde{T}_2 = b\int_0^T\int_{-L}^0 w_{\mathcal{T}_m}\ z_{xxxx}(x,t)dxdt - b\int_0^T w_{N_1+\frac{1}{2}}\ z_{xxx}(0)dt + b\int_0^T w_0\ z_{xxx}(-L)dt \\ -b\int_0^T \delta_x w_{\frac{1}{2}}\ z_{xx}(-L)dt. \end{aligned}$$

We recall that $x_0 = x_{\frac{1}{2}} = -L$ and $x_{N_1+\frac{1}{2}} = 0$. Since $w_{\mathcal{T}_m}$ converges to w in $L^\infty([0,T];L^2(-L,0))$ as $m \longrightarrow \infty$, and similarly, $\delta_x w_{\mathcal{T}_m} \to w_x$ and $\delta_x^2 w_{\mathcal{T}_m} \to w_{xx}$ in $L^\infty([0,T];L^2(-L,0))$, we deduce the

following convergence in $L^\infty([0,T])$: $w_{\mathcal{T}_m}(-L) \longrightarrow w(-L)$, $w_{\mathcal{T}_m}(0) \longrightarrow w(0)$ and $\delta_x w_{\mathcal{T}_m}(-L) \longrightarrow w_x(-L)$. Consequently, we obtain

$$\begin{aligned} w_0 &\longrightarrow w(-L), \\ w_{N_1+\frac{1}{2}} &\longrightarrow w(0), \\ \delta_x w_{\frac{1}{2}} &\longrightarrow w_x(-L), \end{aligned}$$

in $L^\infty([0,T])$ as $m \longrightarrow \infty$. Furthermore, since $w_{\mathcal{T}_m} \longrightarrow w$ in $L^\infty([0,T];L^2(-L,0))$, we immediately obtain

$$\begin{aligned} \tilde{T}_2 \longrightarrow b\int_0^T\int_{-L}^0 w\ z_{xxxx}dxdt - b\int_0^T w(0)\ z_{xxx}(0)dt + b\int_0^T w(-L)\ z_{xxx}(-L)dt \\ -b\int_0^T w_x(-L)\ z_{xx}(-L)dt = b\int_0^T\int_{-L}^0 w_{xx}\ z_{xx}dxdt. \end{aligned}$$

Now, we reformulate $\tilde{T}_2$ as follows:

$$\begin{aligned} \tilde{T}_2 &= b\int_0^T \sum_{i=1}^{N_1} w_i \int_{K_i} z_{xxxx}(x,t)dxdt - b\int_0^T w_{N_1+\frac{1}{2}}\ z_{xxx}(0)dt + b\int_0^T w_0\ z_{xxx}(-L)dt \\ &\quad - b\int_0^T \delta_x w_{\frac{1}{2}}\ z_{xx}(-L)dt \\ &= \tilde{T}_{21} + \tilde{T}_{22}. \end{aligned}$$

We can write $\tilde{T}_{21}$ in the following form using integral formulation:

$$\tilde{T}_{21} = b\int_0^T \sum_{i=1}^{N_1} w_i \int_{K_i} z_{xxxx}(x,t)dxdt = b\int_0^T \sum_{i=1}^{N_1} w_i\ \delta_x^3 z_{i+\frac{1}{2}}dt - b\int_0^T \sum_{i=1}^{N_1} w_i\ \delta_x^3 z_{i-\frac{1}{2}}dt.$$

Using the translation of index i for the second term of the right-hand side of the above equation, we obtain

$$\begin{aligned} \tilde{T}_{21} &= -b\int_0^T \sum_{i=0}^{N_1-1} h_{i+\frac{1}{2}}\ \delta_x w_{i+\frac{1}{2}}\ \delta_x^3 z_{i+\frac{1}{2}}dt - b\int_0^T w_0\ \delta_x^3 z_{\frac{1}{2}}dt + b\int_0^T w_{N_1}\ \delta_x^3 z_{N_1+\frac{1}{2}}dt \\ &= b\int_0^T \sum_{i=0}^{N_1-1} \delta_x w_{i+\frac{1}{2}}\ \delta_x^2 z_i dt - b\int_0^T \sum_{i=0}^{N_1-1} \delta_x w_{i+\frac{1}{2}}\ \delta_x^2 z_{i+1}dt - b\int_0^T w_0\ \delta_x^3 z_{\frac{1}{2}}dt + b\int_0^T w_{N_1}\ \delta_x^3 z_{N_1+\frac{1}{2}}dt. \end{aligned}$$

Again, using the translation of index i for the second term of the right-hand side of the above equation and using the fact that $\delta_x w_{N_1+\frac{1}{2}} = 0$, we obtain

$$\tilde{T}_{21} = b\int_0^T \sum_{i=1}^{N_1} h_i\ \delta_x^2 w_i\ \delta_x^2 z_i dt + b\int_0^T \delta_x w_{\frac{1}{2}}\ \delta_x^2 z_0 dt - b\int_0^T w_0\ \delta_x^3 z_{\frac{1}{2}}dt + b\int_0^T w_{N_1}\ \delta_x^3 z_{N_1+\frac{1}{2}}dt.$$

We can also write

$$\begin{aligned} \tilde{T}_{22} &= -b\int_0^T w_{N_1+\frac{1}{2}}\ z_{xxx}(0)dt + b\int_0^T w_0\ z_{xxx}(-L)dt - b\int_0^T \delta_x w_{\frac{1}{2}}\ z_{xx}(-L)dt \\ &= -b\int_0^T w_{N_1+\frac{1}{2}}\ \delta_x^3 z_{N_1+\frac{1}{2}}dt + b\int_0^T w_0\ \delta_x^3 z_{\frac{1}{2}}dt - b\int_0^T \delta_x w_{\frac{1}{2}}\ \delta_x^2 z_0 dt. \end{aligned}$$

Summing all the results together and using the fact that $\delta_x w_{N_1+\frac{1}{2}} = 0$, we derive

$$\tilde{T}_1 = T_1 \qquad \text{and} \qquad \tilde{T}_2 = T_2,$$

and therefore,

$$T_1 \longrightarrow a \int_0^T \int_0^L u_x v_x dx dt, \qquad \text{as } m \longrightarrow \infty,$$

and

$$T_2 \longrightarrow b \int_0^T \int_{-L}^0 w_{xx} z_{xx} dx dt, \qquad \text{as } m \longrightarrow \infty.$$

Step 3. Due to the regularity of the functions v and z, we have that the sequences $v_{N_{\max}+1}$, $\delta_x z_{\frac{1}{2}}$ and z_0 converge pointwisely in $[0,T]$ to $v(L)$, $z_x(-L)$ and $z(-L)$ respectively, as $m \longrightarrow \infty$. Moreover, since η, ξ and ζ are bounded in $L^\infty([0,T])$, we obtain

$$a \int_0^T \eta v_{N_{\max}+1} dt \longrightarrow a \int_0^T \eta(t) v(L,t) dt, \qquad \text{as } m \longrightarrow \infty,$$

$$b \int_0^T \xi \delta_x z_{\frac{1}{2}} dt \longrightarrow b \int_0^T \xi(t) z_x(-L,t) dt, \qquad \text{as } m \longrightarrow \infty,$$

and

$$b \int_0^T \zeta z_0 dt \longrightarrow b \int_0^T \zeta(t) z(-L,t) dt, \qquad \text{as } m \longrightarrow \infty.$$

Step 4. Finally, using the convergence $u_{N_{\max}+1} \longrightarrow u(L)$, $\delta_x w_{\frac{1}{2}} \longrightarrow w_x(-L)$ and $w_0 \longrightarrow w(-L)$ in $L^\infty([0,T])$, we obtain

$$\int_0^T u_{N_{\max}+1} p_t dt \longrightarrow \int_0^T u(L,t) p_t(t) dt, \qquad \text{as } m \longrightarrow \infty,$$

$$\int_0^T \delta_x w_{\frac{1}{2}} q_t dt \longrightarrow \int_0^T w_x(-L,t) q_t(t) dt, \qquad \text{as } m \longrightarrow \infty,$$

and

$$\int_0^T w_0 r_t dt \longrightarrow \int_0^T w(-L,t) r_t(t) dt, \qquad \text{as } m \longrightarrow \infty.$$

Remark 5.4.2. *Given the definition of $U_{\mathcal{T}_m} = (u_{\mathcal{T}_m}, w_{\mathcal{T}_m})$, it is worth noting that the following holds for all $t \in [0,T]$:*

$$\int_0^L u_{\mathcal{T}_m}(x,t) dx = \int_{-L}^0 w_{\mathcal{T}_m}(x,t) dx = 0.$$

As $m \longrightarrow \infty$, if $u_{\mathcal{T}m}(t)$ converges weakly to $u(t)$ in $L^2(0,L)$ and $w_{\mathcal{T}_m}(t)$ converges weakly to $w(t)$ in $L^2(-L,0)$ for each $t \in [0,T]$, then we can conclude that:

$$\int_0^L u(x,t) dx = \int_{-L}^0 w(x,t) dx = 0,$$

for each $t \in [0,T]$.

Consequently, we have established the validity of equation (5.6) for the limit function (u, w, η, ξ, ζ), holding true for an arbitrary function $V = (v, z) \in C^\infty([0,T]; C_*)$ satisfying $V(T,x) = 0$, alongside fulfilling equations (5.7)-(5.9) for any function $(p, q, r) \in C^\infty\left([0,T]; \mathbb{R}^3\right)$ with the constraint $p(T) = q(T) = r(T) = 0$. Thus, due to the density argument, Problem (5.6) holds also for $V = (v, z) \in L^2([0,T]; \mathbb{H}_*)$ where $V_t = (v_t, z_t) \in L^2([0,T]; \mathbb{L}^2)$ satisfying $v(T,x) = z(T,x) = 0$, and equations (5.7)-(5.9) hold for $(p, q, r) \in L^2([0,T]; \mathbb{R}^3)$ with $(p_t, q_t, r_t) \in L^2([0,T]; \mathbb{R}^3)$ subject to the condition $p(T) = q(T) = r(T) = 0$. This concludes the proof of Theorem 5.4.4. ■

5.5 Fully-Discretization of the Problem

For the discretization in time, let $\mathcal{N} \in \mathbb{N}^*$ and consider the uniform discrete time step $\tau = T/\mathcal{N}$ where we denote $t_n = n\tau$, for $n = 0, 1, \ldots \mathcal{N}$. Further, we denote the discrete unknowns as $\{u_i^n;\ \ i = N_1 + 1, \ldots, N_{\max},\ \ n = 0, 1, \ldots \mathcal{N}\}$ and $\{w_i^n;\ \ i = 1, \ldots, N_1,\ \ n = 0, 1, \ldots \mathcal{N}\}$ which stand for the approximation of the mean values of u and w at time t_n over the control volume K_i. This data, however, is used to construct the approximations of u and w within the cells. In fact, we have:

$$\begin{aligned} u_i(t) &= \frac{1}{h_i}\int_{K_i} u(x,t)dx, \quad \text{for } i = N_1 + 1, \ldots, N_{\max}, \\ w_i(t) &= \frac{1}{h_i}\int_{K_i} w(x,t)dx, \quad \text{for } i = 1, \ldots, N_1. \end{aligned}$$

Also, we denote by η^n, ξ^n and ζ^n an expected approximation of the values of η, ξ and ζ at time t_n, respectively. Let us define

$$U_i = \begin{cases} u_i, & i = N_1 + 1, \ldots, N_{\max}, \\ w_i, & i = 1, \ldots, N_1, \end{cases}$$

and

$$\Xi = (\eta, \xi, \zeta).$$

Further, we denote the first time derivative at time t_n using a backward difference formula and a centered time discretization by the differential quotients:

$$\delta_t v^n = \frac{v^n - v^{n-1}}{\tau} \quad \text{and} \quad \delta_t^* v^n = \frac{v^{n+1} - v^{n-1}}{2\tau},$$

respectively. A second-order central finite difference is also used to approximate the second-time derivative. This approximation at time t_n is denoted by:

$$\delta_t^2 v^n = \frac{v^{n+1} - 2v^n + v^{n-1}}{\tau^2}.$$

The initial conditions are imposed in the following way: First, we start the solution with $\Phi_i = (\varphi_i, \phi_i)$ regarding that

$$u_i^0 = \varphi_i \quad \text{where} \quad \varphi_i = \frac{1}{h_i}\int_{K_i} \varphi(x)dx, \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$w_i^0 = \phi_i \quad \text{where} \quad \phi_i = \frac{1}{h_i}\int_{K_i} \phi(x)dx, \qquad \text{for} \quad i = 1, \cdots, N_1.$$

Consequently,

$$U_i^0 = \Phi_i, \quad \text{for} \quad i = 1, \ldots, N_{\max} \tag{5.47}$$

For the solution in the first time step, we use the artificial time level -1 and we symbolize the values of our approximation functions at this time step by:

$$u_i^{-1} \qquad \text{for} \quad i = N_1 + 1, \ldots, N_{\max},$$

$$w_i^{-1} \qquad \text{for} \quad i = 1, \ldots, N_1.$$

Then, we set $\Psi_i = (\psi_i, \theta_i)$ such that

$$\psi_i = \frac{1}{h_i}\int_{K_i} \psi(x)dx, \qquad \text{for} \quad i = N_1+1, \dots, N_{\max},$$

$$\theta_i = \frac{1}{h_i}\int_{K_i} \theta(x)dx, \qquad \text{for} \quad i = 1, \dots, N_1.$$

Using the above notations, the initial conditions for u_t and w_t can be approximated using central differences, as follows:

$$\frac{u_i^1 - u_i^{-1}}{2\tau} = \psi_i, \qquad \text{for} \quad i = N_1+1, \dots, N_{\max},$$

$$\frac{w_i^1 - w_i^{-1}}{2\tau} = \theta_i, \qquad \text{for} \quad i = 1, \dots, N_1,$$

and therefore,

$$u_i^{-1} = u_i^1 - 2\tau\psi_i, \qquad \text{for} \quad i = N_1+1, \dots, N_{\max},$$

$$w_i^{-1} = w_i^1 - 2\tau\theta_i, \qquad \text{for} \quad i = 1, \dots, N_1.$$

Consequently,

$$U_i^{-1} = U_i^1 - 2\tau\Psi_i, \quad \text{for} \quad i = 1, \cdots, N_{\max}. \tag{5.48}$$

Finally, we set $\Theta = (\alpha, \beta, \gamma)$ regarding that

$$\Xi^0 = \Theta. \tag{5.49}$$

5.5.1 Explicit Discretization of the Problem

This section is concerned with the construction of an explicit numerical scheme for System (5.1)-(5.4) using the finite volume mesh $\mathcal{T}$ for space discretization and a time finite difference discretization. An estimate of the discrete energy for the explicit form is then established.

Formulation of the Explicit Numerical Scheme

In this section, the concept of the finite volume method is used to discretize the explicit numerical scheme for System (5.1)-(5.4). The finite difference method is used to approximate the time derivatives.

Discretization of the Euler-Bernoulli beam: The second Equation of (5.1)-(5.4) is integrated (in space) formally over an arbitrary cell K_i of $\mathcal{T}$, $i = 1, \dots, N_1$, and yields the following

$$\int_{K_i} w_{tt}(x,t)dx + b\left(w_{xxx}\left(x_{i+\frac{1}{2}}, t\right) - w_{xxx}\left(x_{i-\frac{1}{2}}, t\right)\right) = 0, \qquad \forall t \in [0,T]. \tag{5.50}$$

For the first term of Equation (5.50), the Leibniz integral rule for differentiation under integral sign reads

$$\int_{K_i} w_{tt}(x,t)dx = h_i\left(\frac{1}{h_i}\int_{K_i} w(x,t)dx\right)_{tt} = h_i\left(w_i(t)\right)_{tt}, \qquad \forall t \in [0,T].$$

Thus, (5.50) can be reformulated, for $i = 1, \dots, N_1$, at time $t = t_n$ into

$$h_i w_{i,tt}(t_n) + b\left(w_{xxx}\left(x_{i+\frac{1}{2}}, t_n\right) - w_{xxx}\left(x_{i-\frac{1}{2}}, t_n\right)\right) = 0. \tag{5.51}$$

There remains to write the approximation of the various terms in Equation (5.51) with respect to the discrete unknowns (and the data).

☐ For $i = 1$, we have

$$h_1 w_{1,tt}(t_n) + b\left(w_{xxx}\left(x_{\frac{3}{2}}, t_n\right) - w_{xxx}\left(x_{\frac{1}{2}}, t_n\right)\right) = 0.$$

Combining the above equation with the dynamical boundary condition $w_{xxx}(-L, t_n) + \zeta(t_n) = 0$ as the position $x_{\frac{1}{2}}$ represents the point $-L$ yields

$$h_1 w_{1,tt}(t_n) + b\left(w_{xxx}\left(x_{\frac{3}{2}}, t_n\right) + \zeta(t_n)\right) = 0.$$

For the first term of the above equation, we apply the second-order central time discretization. However, we replace the space differential operator with the corresponding difference operator to obtain

$$h_1 \frac{w_1^{n+1} - 2w_1^n + w_1^{n-1}}{\tau^2} + b\left(\delta_x^3 w_{\frac{3}{2}}^n + \zeta^n\right) = 0, \qquad \forall n \in \{1, \dots, \mathcal{N} - 1\}.$$

☐ For $i = 2, \dots, N_1 - 1$, we have

$$h_i w_{i,tt}(t_n) + b\left(w_{xxx}\left(x_{i+\frac{1}{2}}, t_n\right) - w_{xxx}\left(x_{i-\frac{1}{2}}, t_n\right)\right) = 0.$$

The time derivative is approximated by a second-order central difference. However, we replace the differential operators for the second and third terms of the above equation with the corresponding difference operators to obtain

$$h_i \frac{w_i^{n+1} - 2w_i^n + w_i^{n-1}}{\tau^2} + b\left(\delta_x^3 w_{i+\frac{1}{2}}^n - \delta_x^3 w_{i-\frac{1}{2}}^n\right) = 0, \qquad \forall n \in \{1, \dots, \mathcal{N} - 1\}.$$

☐ For $i = N_1$, we have

$$h_{N_1} w_{N_1,tt}(t_n) + b\left(w_{xxx}\left(x_{N_1+\frac{1}{2}}, t_n\right) - w_{xxx}\left(x_{N_1-\frac{1}{2}}, t_n\right)\right) = 0.$$

Using the transmission condition $a u_x(0, t_n) + b w_{xxx}(0, t_n) = 0$ as the position $x_{N_1+\frac{1}{2}}$ is exactly the point zero, yields

$$h_{N_1} w_{N_1,tt}(t_n) - \left(a u_x\left(x_{N_1+\frac{1}{2}}, t_n\right) + b w_{xxx}\left(x_{N_1-\frac{1}{2}}, t_n\right)\right) = 0.$$

We apply the second-order central difference approximation in time for the first term of the above equation. However, we replace the second and third terms with the corresponding difference operators to obtain

$$h_{N_1} \frac{w_{N_1}^{n+1} - 2w_{N_1}^n + w_{N_1}^{n-1}}{\tau^2} - \left(a\delta_x u_{N_1+\frac{1}{2}}^n + b\delta_x^3 w_{N_1-\frac{1}{2}}^n\right) = 0, \qquad \forall n \in \{1, \dots, \mathcal{N} - 1\}.$$

Discretization of the Wave equation: The first Equation of (5.1)-(5.4) is integrated (in space) formally over an arbitrary cell K_i of $\mathcal{T}$, $i = N_1 + 1, \dots, N_{\max}$, and yields the following

$$\int_{K_i} u_{tt}(x, t)dx - a\left(u_x\left(x_{i+\frac{1}{2}}, t\right) - u_x\left(x_{i-\frac{1}{2}}, t\right)\right) = 0, \qquad \forall t \in [0, T]. \tag{5.52}$$

By Leibniz integral rule for differentiation under the integral sign, the first term of Equation (5.52) has

$$\int_{K_i} u_{tt}(x,t)dx = h_i \left(\frac{1}{h_i} \int_{K_i} u(x,t)dx \right)_{tt} = h_i \left(u_i(t) \right)_{tt}, \qquad \forall t \in [0,T].$$

Thus, (5.52) can be reformulated, for $i = N_1 + 1, \ldots, N_{\max}$, at time $t = t_n$ into

$$h_i u_{i,tt}(t_n) - a \left(u_x \left(x_{i+\frac{1}{2}}, t_n \right) - u_x \left(x_{i-\frac{1}{2}}, t_n \right) \right) = 0. \tag{5.53}$$

There now remains to replace in Equation (5.53) each term by its approximation with respect to the discrete unknowns (and the data).

☐ For $i = N_1 + 1, \ldots, N_{\max} - 1$, we have

$$h_i u_{i,tt}(t_n) - a \left(u_x \left(x_{i+\frac{1}{2}}, t_n \right) - u_x \left(x_{i-\frac{1}{2}}, t_n \right) \right) = 0.$$

Using the second-order central time discretization and replacing the space differential operators for the second and third terms of the above equation with the corresponding difference operators, we obtain

$$h_i \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2} - a \left(\delta_x u_{i+\frac{1}{2}}^n - \delta_x u_{i-\frac{1}{2}}^n \right) = 0, \qquad \forall n \in \{1, \ldots, \mathcal{N} - 1\}.$$

☐ For $i = N_{\max}$, we have

$$h_{N_{\max}} u_{N_{\max},tt}(t_n) - a \left(u_x \left(x_{N_{\max}+\frac{1}{2}}, t_n \right) - u_x \left(x_{N_{\max}-\frac{1}{2}}, t_n \right) \right) = 0.$$

Combining the above equation with the dynamical boundary condition $u_x(L,t_n) + \eta(t_n) = 0$ as the position $x_{N_{\max}+\frac{1}{2}}$ represents the point L yields

$$h_{N_{\max}} u_{N_{\max},tt}(t_n) + a \left(\eta(t_n) + u_x \left(x_{N_{\max}-\frac{1}{2}}, t_n \right) \right) = 0.$$

Applying the second-order central difference in time and replacing the space differential operator with the corresponding difference operator, we obtain

$$h_{N_{\max}} \frac{u_{N_{\max}}^{n+1} - 2u_{N_{\max}}^n + u_{N_{\max}}^{n-1}}{\tau^2} + a \left(\eta^n + \delta_x u_{N_{\max}-\frac{1}{2}}^n \right) = 0, \qquad \forall n \in \{1, \ldots, \mathcal{N} - 1\}.$$

Discretization of the Continuity Conditions: The continuity conditions are discretized at time $t = t_n$, $n = 0, 1, \ldots \mathcal{N}$ as

$$u_{N_1+\frac{1}{2}}^n = w_{N_1+\frac{1}{2}}^n,$$

and

$$\delta_x w_{N_1+\frac{1}{2}}^n = 0.$$

Discretization of the Dynamical Boundary Condition: The dynamical boundary condition $w_{xx}(-L,t) - \xi(t) = 0$ is also approximated at time $t = t_n$, $n = 0, 1, \ldots \mathcal{N}$ by

$$\delta_x^2 w_0^n - \xi^n = 0.$$

Discretization of the First-Order Differential Equations: There remains to approximate the first-order differential equations of (5.4). Writing the first Equation of (5.4) at time $t = t_n$ and

discretizing the time partial derivatives using the centered difference formula yields the following equation:

$$\eta^{n+1} = \eta^{n-1} + u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1} - 2\tau\eta^n, \qquad \forall n \in \{1, \dots, \mathcal{N}-1\}.$$

Similarly, we write the second Equation of (5.4) at time $t = t_n$ and use a centered time discretization for the first term. For the second term, however, a forward difference in space and a centered difference in time is applied to obtain the following equation:

$$\xi^{n+1} = \xi^{n-1} + \delta_x w^{n+1}_{\frac{1}{2}} - \delta_x w^{n-1}_{\frac{1}{2}} - 2\tau\xi^n, \qquad \forall n \in \{1, \dots, \mathcal{N}-1\}.$$

Writing the third Equation of (5.4) at time $t = t_n$ and using a centered time discretization for the first two terms yields the following equation:

$$\zeta^{n+1} = \zeta^{n-1} + w^{n+1}_0 - w^{n-1}_0 - 2\tau\zeta^n, \qquad \forall n \in \{1, \dots, \mathcal{N}-1\}.$$

Therefore, an explicit version of the numerical scheme for the approximation of System (5.1)-(5.4) can then be described in the following way:

$$\begin{cases} h_i \dfrac{U^{n+1}_i - 2U^n_i + U^{n-1}_i}{\tau^2} + F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} = 0, & i = 1, \dots, N_{\max},\ n = 1, \dots, \mathcal{N}-1, \\ u^n_{N_1+\frac{1}{2}} = w^n_{N_1+\frac{1}{2}}, \quad \delta_x w^n_{N_1+\frac{1}{2}} = 0, \quad \delta^2_x w^n_0 - \xi^n = 0, & n = 0, 1, \dots \mathcal{N}, \end{cases} \tag{5.54}$$

where the numerical flux at point $x_{i+\frac{1}{2}}$ and time t_n is defined as

$$F^n_{i+\frac{1}{2}} = \begin{cases} b\delta^3_x w^n_{i+\frac{1}{2}}, & i = 0, \dots, N_1 - 1, \\ -a\delta_x u^n_{i+\frac{1}{2}}, & i = N_1, \dots, N_{\max}, \end{cases} \tag{5.55}$$

with the following boundary conditions:

$$\delta_x u^n_{N_{\max}+\frac{1}{2}} + \eta^n = 0, \qquad \text{for} \quad n = 0, 1, \dots \mathcal{N},$$

$$\delta^3_x w^n_{\frac{1}{2}} + \zeta^n = 0, \qquad \text{for} \quad n = 0, 1, \dots \mathcal{N}.$$

The equations of (5.4) are discretized as follows:

$$\begin{cases} \eta^{n+1} = \eta^{n-1} + u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1} - 2\tau\eta^n, & n = 1, \dots, \mathcal{N}-1, \\ \xi^{n+1} = \xi^{n-1} + \delta_x w^{n+1}_{\frac{1}{2}} - \delta_x w^{n-1}_{\frac{1}{2}} - 2\tau\xi^n, & n = 1, \dots, \mathcal{N}-1, \\ \zeta^{n+1} = \zeta^{n-1} + w^{n+1}_0 - w^{n-1}_0 - 2\tau\zeta^n, & n = 1, \dots, \mathcal{N}-1. \end{cases} \tag{5.56}$$

On the other hand, we discretize the equations in (5.4) at time t_1 using the backward difference formula without centering the differences. This leads to the following system

$$\begin{cases} \eta^1 = \eta^0 + u^1_{N_{\max}+1} - u^0_{N_{\max}+1} - \tau\eta^1, \\ \xi^1 = \xi^0 + \delta_x w^1_{\frac{1}{2}} - \delta_x w^0_{\frac{1}{2}} - \tau\xi^1, \\ \zeta^1 = \zeta^0 + w^1_0 - w^0_0 - \tau\zeta^1. \end{cases} \tag{5.57}$$

The scheme (5.54)-(5.56) is then used to build an approximate solution $U_{\mathcal{T},\tau}$ defined by

$$U_{\mathcal{T},\tau}(x,t) = U_i^n, \quad \forall x \in K_i,\ \forall t \in [t_{n-1}, t_n[,\ \forall i \in \{1,\dots,N_{\max}\},\ \forall n \in \{1,\dots\mathcal{N}\}.$$

We also define

$$\Xi_\tau(t) = (\eta^n, \xi^n, \zeta^n), \quad \forall t \in [t_{n-1}, t_n[,\ \forall n \in \{1,\dots\mathcal{N}\}.$$

Now, we require a so-called CFL condition necessary for the stability of the explicit numerical scheme (5.54)-(5.56). Firstly, the CFL condition leads to a constraint on the Courant number corresponding to the wave equation, and this constraint happens to be:

$$C_{Wave}^2 = \frac{a\tau^2}{h_2^2} \leq 1,$$

which is equivalent to

$$\tau \leq \frac{h_2}{\sqrt{a}}. \tag{5.58}$$

However, the following condition controls the Courant number corresponding to the Euler-Bernoulli beam:

$$C_{EBB}^2 = \frac{b\tau^2}{h_1^4} \leq 1,$$

so that

$$\tau \leq \frac{h_1^2}{\sqrt{b}}. \tag{5.59}$$

As a result of (5.58) and (5.59), the time step τ is limited by the following condition:

$$\tau \leq \min\left\{\frac{h_2}{\sqrt{a}}, \frac{h_1^2}{\sqrt{b}}\right\}. \tag{5.60}$$

The condition (5.60), which imposes an upper bound on the time step that can be used, is known as the Courant-Friedrichs-Lewy condition or CFL condition.

Estimation of the Discrete Energy for the Explicit Form

In this subsection, we plan to design the discrete energy of the explicit numerical scheme (5.54)-(5.56) that conserves when no dynamical boundary controls are applied and dissipates when at least one dynamical boundary control is applied to the system. For this purpose, we define:

- The discrete kinetic energy for $U_{\mathcal{T}}^n$ as:

$$\mathcal{E}_k\left(U_{\mathcal{T}}^n\right) = \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^{n+1} - U_i^n}{\tau}\right)^2.$$

- The discrete potential energy for $U_{\mathcal{T}}^n$ as:

$$\mathcal{E}_p(U_{\mathcal{T}}^n) = \frac{1}{2}\left(b\sum_{i=1}^{N_1} h_i \delta_x^2 w_i^{n+1}\delta_x^2 w_i^n + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\delta_x u_{i+\frac{1}{2}}^n\right).$$

- The discrete boundary control energy for Ξ^n as:

$$\mathcal{E}_b(\Xi^n) = \frac{1}{2}\left(a\eta^{n+1}\eta^n + b\xi^{n+1}\xi^n + b\zeta^{n+1}\zeta^n\right).$$

The total discrete energy $\mathcal{E}^n$ of the explicit numerical scheme (5.54)-(5.56) is defined by

$$\mathcal{E}^n = \mathcal{E}_k(U_{\mathcal{T}}^n) + \mathcal{E}_p(U_{\mathcal{T}}^n) + \mathcal{E}_b(\Xi^n).$$

Now, we want to prove that the goals above for the discrete energy $\mathcal{E}^n$ are fulfilled. For this purpose, we aim to prove the following theorem:

Theorem 5.5.1. *The total discrete energy $\mathcal{E}^n$ of the numerical scheme (5.54)-(5.56) satisfies the following estimation:*

$$\mathcal{E}^n - \mathcal{E}^{n-1} = -\tau\left(a|\eta^n|^2 + b|\xi^n|^2 + b|\zeta^n|^2\right). \tag{5.61}$$

Proof. Multiplying both sides of the first Equation of (5.54)-(5.56) by $U_i^{n+1} - U_i^{n-1}$ and summing up i from 1 to $N_{\max}$, gives

$$\sum_{i=1}^{N_{\max}} h_i \frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\tau^2}\left(U_i^{n+1} - U_i^{n-1}\right) + \sum_{i=1}^{N_{\max}} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right)\left(U_i^{n+1} - U_i^{n-1}\right) = 0,$$

so that

$$\sum_{i=1}^{N_{\max}} h_i \frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\tau^2}\left(U_i^{n+1} - U_i^{n-1}\right) + \sum_{i=1}^{N_{\max}} F_{i+\frac{1}{2}}^n\left(U_i^{n+1} - U_i^{n-1}\right) - \sum_{i=1}^{N_{\max}} F_{i-\frac{1}{2}}^n\left(U_i^{n+1} - U_i^{n-1}\right) = 0.$$

Using (5.55), we can write

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\tau^2}\left(U_i^{n+1} - U_i^{n-1}\right) + b\sum_{i=1}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}}^n\left(w_i^{n+1} - w_i^{n-1}\right) \\
&\qquad -a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}}^n\left(u_i^{n+1} - u_i^{n-1}\right) - b\sum_{i=1}^{N_1} \delta_x^3 w_{i-\frac{1}{2}}^n\left(w_i^{n+1} - w_i^{n-1}\right) \\
&\qquad +a\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}}^n\left(u_i^{n+1} - u_i^{n-1}\right) - a\delta_x u_{N_1+\frac{1}{2}}^n\left(w_{N_1}^{n+1} - w_{N_1}^{n-1}\right) = 0.
\end{aligned} \tag{5.62}$$

A direct calculation applied to the first term of (5.62) gives

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\tau^2}\left(U_i^{n+1} - U_i^{n-1}\right) \\
&\quad = \sum_{i=1}^{N_{\max}} h_i \frac{\left(U_i^{n+1} - U_i^n\right) - \left(U_i^n - U_i^{n-1}\right)}{\tau^2}\left(\left(U_i^{n+1} - U_i^n\right) + \left(U_i^n - U_i^{n-1}\right)\right) \\
&\quad = \sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^{n+1} - U_i^n}{\tau}\right)^2 - \sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^n - U_i^{n-1}}{\tau}\right)^2.
\end{aligned} \tag{5.63}$$

By the translation of index i for the second term of the following equation, we obtain

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i+\frac{1}{2}}\left(u_i^{n+1}-u_i^{n-1}\right)+a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i-\frac{1}{2}}\left(u_i^{n+1}-u_i^{n-1}\right)-a\delta_x u^n_{N_1+\frac{1}{2}}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)\\
&=-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i+\frac{1}{2}}\left(u_i^{n+1}-u_i^{n-1}\right)+a\sum_{i=N_1}^{N_{\max}-1}\delta_x u^n_{i+\frac{1}{2}}\left(u_{i+1}^{n+1}-u_{i+1}^{n-1}\right)-a\delta_x u^n_{N_1+\frac{1}{2}}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)\\
&=a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i+\frac{1}{2}}\left(u_{i+1}^{n+1}-u_{i+1}^{n-1}-u_i^{n+1}+u_i^{n-1}\right)+a\delta_x u^n_{N_1+\frac{1}{2}}\left(u_{N_1+1}^{n+1}-u_{N_1+1}^{n-1}-w_{N_1}^{n+1}+w_{N_1}^{n-1}\right)\\
&\quad-a\delta_x u^n_{N_{\max}+\frac{1}{2}}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right)\\
&=a\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}-\delta_x u_{i+\frac{1}{2}}^{n-1}\right)+a\delta_x u^n_{N_1+\frac{1}{2}}\left(u_{N_1+1}^{n+1}-u_{N_1+1}^{n-1}-w_{N_1}^{n+1}+w_{N_1}^{n-1}\right)\\
&\quad-a\delta_x u^n_{N_{\max}+\frac{1}{2}}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right).
\end{aligned}\tag{5.64}$$

Since $\delta_x w^n_{N_1+\frac{1}{2}}=0$, one can deduce that

$$w^n_{N_1+\frac{1}{2}}=w^n_{N_1}.\tag{5.65}$$

Thanks to (5.65) and using the fact that $u^n_{N_1+\frac{1}{2}}=w^n_{N_1+\frac{1}{2}}$, one may show that

$$\delta_x u^n_{N_1+\frac{1}{2}}=\frac{u^n_{N_1+1}-w^n_{N_1}}{h_{N_1+\frac{1}{2}}}.\tag{5.66}$$

Therefore, (5.64) together with (5.91) implies that

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i+\frac{1}{2}}\left(u_i^{n+1}-u_i^{n-1}\right)+a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i-\frac{1}{2}}\left(u_i^{n+1}-u_i^{n-1}\right)-a\delta_x u^n_{N_1+\frac{1}{2}}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)\\
&=a\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}-\delta_x u_{i+\frac{1}{2}}^{n-1}\right)+ah_{N_1+\frac{1}{2}}\delta_x u^n_{N_1+\frac{1}{2}}\left(\delta_x u_{N_1+\frac{1}{2}}^{n+1}-\delta_x u_{N_1+\frac{1}{2}}^{n-1}\right)\\
&\quad-a\delta_x u^n_{N_{\max}+\frac{1}{2}}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right)\\
&=a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}-\delta_x u_{i+\frac{1}{2}}^{n-1}\right)-a\delta_x u^n_{N_{\max}+\frac{1}{2}}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right)\\
&=a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^{n+1}_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}-a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\delta_x u^{n-1}_{i+\frac{1}{2}}-a\delta_x u^n_{N_{\max}+\frac{1}{2}}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right).
\end{aligned}$$

Noticing that $\delta_x u^n_{N_{\max}+\frac{1}{2}}=-\eta^n$, one has from the last equality that

$$\begin{aligned}
&-a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i+\frac{1}{2}}\left(u_i^{n+1}-u_i^{n-1}\right)+a\sum_{i=N_1+1}^{N_{\max}}\delta_x u^n_{i-\frac{1}{2}}\left(u_i^{n+1}-u_i^{n-1}\right)-a\delta_x u^n_{N_1+\frac{1}{2}}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)\\
&=a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^{n+1}_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}-a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\delta_x u^{n-1}_{i+\frac{1}{2}}+a\eta^n\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right).
\end{aligned}\tag{5.67}$$

Multiplying both sides of the first Equation of (5.56) by η^n yields

$$\eta^{n+1}\eta^n = \eta^n\eta^{n-1} + \eta^n\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right) - 2\tau|\eta^n|^2. \tag{5.68}$$

Hence, (5.67) together with (5.68) implies that

$$\begin{aligned} &- a\sum_{i=N_1+1}^{N_{\max}} \delta_x u^n_{i+\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) + a\sum_{i=N_1+1}^{N_{\max}} \delta_x u^n_{i-\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) - a\delta_x u^n_{N_1+\frac{1}{2}}\left(w^{n+1}_{N_1} - w^{n-1}_{N_1}\right) \\ &= a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^{n+1}_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}} - a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\delta_x u^{n-1}_{i+\frac{1}{2}} + a\eta^{n+1}\eta^n - a\eta^n\eta^{n-1} + 2\tau a|\eta^n|^2. \end{aligned} \tag{5.69}$$

Now, by the translation of index i for the second term of the following equation, we obtain

$$\begin{aligned} &b\sum_{i=1}^{N_1-1} \delta_x^3 w^n_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) - b\sum_{i=1}^{N_1} \delta_x^3 w^n_{i-\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) \\ &= b\sum_{i=1}^{N_1-1} \delta_x^3 w^n_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) - b\sum_{i=0}^{N_1-1} \delta_x^3 w^n_{i+\frac{1}{2}}\left(w^{n+1}_{i+1} - w^{n-1}_{i+1}\right) \\ &= b\sum_{i=0}^{N_1-1} \delta_x^3 w^n_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i - w^{n+1}_{i+1} + w^{n-1}_{i+1}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) \\ &= -b\sum_{i=0}^{N_1-1} h_{i+\frac{1}{2}}\delta_x^3 w^n_{i+\frac{1}{2}}\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) \\ &= b\sum_{i=0}^{N_1-1} \left(\delta_x^2 w^n_i - \delta_x^2 w^n_{i+1}\right)\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) \\ &= b\sum_{i=0}^{N_1-1} \delta_x^2 w^n_i\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - b\sum_{i=0}^{N_1-1} \delta_x^2 w^n_{i+1}\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right). \end{aligned}$$

Once again, the translation of index i for the second term of the right-hand side of the above equation gives, under the condition $\delta_x w^n_{N_1+\frac{1}{2}} = 0$

$$\begin{aligned} &b\sum_{i=1}^{N_1-1} \delta_x^3 w^n_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) - b\sum_{i=1}^{N_1} \delta_x^3 w^n_{i-\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) \\ &= b\sum_{i=0}^{N_1-1} \delta_x^2 w^n_i\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - b\sum_{i=1}^{N_1} \delta_x^2 w^n_i\left(\delta_x w^{n+1}_{i-\frac{1}{2}} - \delta_x w^{n-1}_{i-\frac{1}{2}}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) \\ &= b\sum_{i=1}^{N_1} \delta_x^2 w^n_i\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}} - \delta_x w^{n+1}_{i-\frac{1}{2}} + \delta_x w^{n-1}_{i-\frac{1}{2}}\right) + b\delta_x^2 w^n_0\left(\delta_x w^{n+1}_{\frac{1}{2}} - \delta_x w^{n-1}_{\frac{1}{2}}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) \\ &= b\sum_{i=1}^{N_1} h_i\delta_x^2 w^n_i\left(\delta_x^2 w^{n+1}_i - \delta_x^2 w^{n-1}_i\right) + b\delta_x^2 w^n_0\left(\delta_x w^{n+1}_{\frac{1}{2}} - \delta_x w^{n-1}_{\frac{1}{2}}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) \\ &= b\sum_{i=1}^{N_1} h_i\delta_x^2 w^{n+1}_i\delta_x^2 w^n_i - b\sum_{i=1}^{N_1} h_i\delta_x^2 w^n_i\delta_x^2 w^{n-1}_i + b\delta_x^2 w^n_0\left(\delta_x w^{n+1}_{\frac{1}{2}} - \delta_x w^{n-1}_{\frac{1}{2}}\right) - b\delta_x^3 w^n_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right). \end{aligned}$$

Noticing that $\delta_x^2 w_0^n = \xi^n$ and $\delta_x^3 w_{\frac{1}{2}}^n = -\zeta^n$, this last equality yields

$$\begin{aligned} b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^n\left(w_i^{n+1}-w_i^{n-1}\right) - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^n\left(w_i^{n+1}-w_i^{n-1}\right) = b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n+1}\delta_x^2 w_i^n \\ - b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^n\delta_x^2 w_i^{n-1} + b\xi^n\left(\delta_x w_{\frac{1}{2}}^{n+1} - \delta_x w_{\frac{1}{2}}^{n-1}\right) + b\zeta^n\left(w_0^{n+1}-w_0^{n-1}\right). \end{aligned} \tag{5.70}$$

Multiplying both sides of the second Equation of (5.56) by ξ^n, we obtain

$$\xi^{n+1}\xi^n = \xi^n\xi^{n-1} + \xi^n\left(\delta_x w_{\frac{1}{2}}^{n+1} - \delta_x w_{\frac{1}{2}}^{n-1}\right) - 2\tau|\xi^n|^2. \tag{5.71}$$

Similarly, multiplying both sides of the third Equation of (5.56) by ζ^n, yields

$$\zeta^{n+1}\zeta^n = \zeta^n\zeta^{n-1} + \zeta^n\left(w_0^{n+1}-w_0^{n-1}\right) - 2\tau|\zeta^n|^2. \tag{5.72}$$

Therefore, (5.70) together with (5.71) and (5.72), implies

$$\begin{aligned} b\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^n\left(w_i^{n+1}-w_i^{n-1}\right) - b\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^n\left(w_i^{n+1}-w_i^{n-1}\right) = b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n+1}\delta_x^2 w_i^n \\ - b\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^n\delta_x^2 w_i^{n-1} + b\xi^{n+1}\xi^n - b\xi^n\xi^{n-1} + b\zeta^{n+1}\zeta^n - b\zeta^n\zeta^{n-1} + 2\tau|\xi^n|^2 + 2\tau|\zeta^n|^2. \end{aligned} \tag{5.73}$$

In order to complete the proof, plug (5.63), (5.69) and (5.73) into (5.62) giving

$$\mathcal{E}_k\left(U_{\mathcal{T}}^n\right) - \mathcal{E}_k\left(U_{\mathcal{T}}^{n-1}\right) + \mathcal{E}_p\left(U_{\mathcal{T}}^n\right) - \mathcal{E}_p\left(U_{\mathcal{T}}^{n-1}\right) + \mathcal{E}_b(\Xi^n) - \mathcal{E}_b(\Xi^{n-1}) = -\tau\left(a|\eta^n|^2 + b|\xi^n|^2 + b|\zeta^n|^2\right),$$

from which one deduces that

$$\mathcal{E}^n - \mathcal{E}^{n-1} = -\tau\left(a|\eta^n|^2 + b|\xi^n|^2 + b|\zeta^n|^2\right).$$

This completes the proof of the theorem. ■

By Theorem 5.5.1, we know that the total discrete energy $\mathcal{E}^n$ of the system (5.54)-(5.56) is conserved when no dynamical boundary controls are used and it is non-increasing over time as long as at least one dynamical boundary control is applied on the wave or the Euler-Bernoulli beam.

5.5.2 Implicit Discretization of the Problem

We present an implicit numerical scheme of System (5.1)-(5.4) using a space finite volume discretization combined with an average of $n+1$ and $n-1$ time step diffusion and time finite difference schemes in this section. Following that, we present an estimate of the discrete energy. The stability estimates are established under adequate assumptions on the initial data of the problem and the time step. Finally, a detailed proof of convergence of the approximate solution is developed as the mesh size and time step tend to zero. For stability reasons, we set $\Phi = (\varphi, \phi)$, $\Psi = (\psi, \theta)$, $\Theta = (\alpha, \beta, \gamma)$ and study our problem under a certain hypothesis, which we call **Hypothesis (H2)** of the following form:

$$\Phi \in \mathbb{H}, \quad \Psi \in H^2(0,L)\times H^4(-L,0) \quad \text{and} \quad \Theta \in \mathbb{R}^3. \tag{Hypothesis (H2)}$$

For the time step τ, we also set the following assumption

$$\text{There exists a real number } \tau_0 \in [0,T] \text{ such that } \tau \le \tau_0. \tag{Assumption (TA)}$$

Formulation of the Implicit Numerical Scheme

In this subsection, we construct the implicit numerical scheme of System (5.1)-(5.4) using the finite volume method in space and an alternative implicit formulation that follows from time-centering the derivative term. The time derivatives are approximated using the finite difference method. The discretization of our numerical scheme is inspired by the schemes in [15] and [50], where the implicit scheme to compute the $n+1$ time step uses the average of diffusion terms in $n+1$ and $n-1$ time steps in diffusion term.

Discretization of the Euler-Bernoulli beam: Integrating the second Equation of (5.1)-(5.4) on each control volume K_i, $i = 1, \dots, N_1$, yields the following

$$\int_{K_i} w_{tt}(x,t)dx + b\left(w_{xxx}\left(x_{i+\frac{1}{2}},t\right) - w_{xxx}\left(x_{i-\frac{1}{2}},t\right)\right) = 0, \qquad \forall t \in [0,T]. \tag{5.74}$$

For the first term of Equation (5.74), the Leibniz integral rule for differentiation under integral sign reads

$$\int_{K_i} w_{tt}(x,t)dx = h_i\left(\frac{1}{h_i}\int_{K_i} w(x,t)dx\right)_{tt} = h_i\left(w_i(t)\right)_{tt}, \qquad \forall t \in [0,T].$$

Thus, (5.74) can be reformulated, for $i = 1, \dots, N_1$, at time t_n into

$$h_i w_{i,tt}(t_n) + b\left(w_{xxx}\left(x_{i+\frac{1}{2}},t_n\right) - w_{xxx}\left(x_{i-\frac{1}{2}},t_n\right)\right) = 0. \tag{5.75}$$

There now remains to write the approximation of the various terms in Equation (5.75) with respect to the discrete unknowns (and the data).

☐ For $i = 1$, we have

$$h_1 w_{1,tt}(t_n) + b\left(w_{xxx}\left(x_{\frac{3}{2}},t_n\right) - w_{xxx}\left(x_{\frac{1}{2}},t_n\right)\right) = 0.$$

Combining the above equation with the dynamical boundary condition $w_{xxx}(-L,t_n) + \zeta(t_n) = 0$ as the position $x_{\frac{1}{2}}$ represents the point $-L$ yields

$$h_1 w_{1,tt}(t_n) + b\left(w_{xxx}\left(x_{\frac{3}{2}},t_n\right) + \zeta(t_n)\right) = 0.$$

For the first term of the above equation, we apply the second-order central time discretization. However, we replace the space differential operator with the corresponding difference operator with an average of diffusion terms in $n+1$ and $n-1$ time steps to obtain

$$h_1\frac{w_1^{n+1} - 2w_1^n + w_1^{n-1}}{\tau^2} + \frac{b}{2}\left(\delta_x^3 w_{\frac{3}{2}}^{n+1} + \zeta^{n+1}\right) + \frac{b}{2}\left(\delta_x^3 w_{\frac{3}{2}}^{n-1} + \zeta^{n-1}\right) = 0, \quad \forall n \in \{1, \dots, \mathcal{N}-1\}.$$

☐ For $i = 2, \dots, N_1 - 1$, we have

$$h_i w_{i,tt}(t_n) + b\left(w_{xxx}\left(x_{i+\frac{1}{2}},t_n\right) - w_{xxx}\left(x_{i-\frac{1}{2}},t_n\right)\right) = 0.$$

The time derivative is approximated by a second-order central difference. However, we replace the differential operators for the second and third terms of the above equation with the corresponding difference operators together with an average of $n+1$ and $n-1$ time steps diffusion to obtain

$$h_i\frac{w_i^{n+1} - 2w_i^n + w_i^{n-1}}{\tau^2} + \frac{b}{2}\left(\delta_x^3 w_{i+\frac{1}{2}}^{n+1} - \delta_x^3 w_{i-\frac{1}{2}}^{n+1}\right) + \frac{b}{2}\left(\delta_x^3 w_{i+\frac{1}{2}}^{n-1} - \delta_x^3 w_{i-\frac{1}{2}}^{n-1}\right) = 0, \quad \forall n \in \{1, \dots, \mathcal{N}-1\}.$$

☐ For $i = N_1$, we have

$$h_{N_1} w_{N_1,tt}(t_n) + b\left(w_{xxx}\left(x_{N_1+\frac{1}{2}}, t_n\right) - w_{xxx}\left(x_{N_1-\frac{1}{2}}, t_n\right)\right) = 0.$$

Using the transmission condition $au_x(0,t_n) + bw_{xxx}(0,t_n) = 0$ as the position $x_{N_1} + \frac{1}{2}$ is exactly the point zero, yields

$$h_{N_1} w_{N_1,tt}(t_n) - \left(au_x\left(x_{N_1+\frac{1}{2}}, t_n\right) + bw_{xxx}\left(x_{N_1-\frac{1}{2}}, t_n\right)\right) = 0.$$

We apply the second-order central difference approximation in time for the first term of the above equation. However, we replace the second and third terms with the corresponding difference operators combined with an average of diffusion terms in $n+1$ and $n-1$ time steps to obtain

$$h_{N_1}\frac{w_{N_1}^{n+1} - 2w_{N_1}^{n} + w_{N_1}^{n-1}}{\tau^2} - \frac{1}{2}\left(a\delta_x u_{N_1+\frac{1}{2}}^{n+1} + b\delta_x^3 w_{N_1-\frac{1}{2}}^{n+1}\right) - \frac{1}{2}\left(a\delta_x u_{N_1+\frac{1}{2}}^{n-1} + b\delta_x^3 w_{N_1-\frac{1}{2}}^{n-1}\right) = 0, \quad \forall n \in \{1,\dots,\mathcal{N}-1\}.$$

Discretization of the Wave equation: Secondly, integrating the first Equation of (5.1)-(5.4) over each control volume K_i, $i = N_1+1,\dots,N_{\max}$, yields the following

$$\int_{K_i} u_{tt}(x,t)dx - a\left(u_x\left(x_{i+\frac{1}{2}}, t\right) - u_x\left(x_{i-\frac{1}{2}}, t\right)\right) = 0, \qquad \forall t \in [0,T]. \tag{5.76}$$

By Leibniz integral rule for differentiation under the integral sign, the first term of Equation (5.76) has

$$\int_{K_i} u_{tt}(x,t)dx = h_i\left(\frac{1}{h_i}\int_{K_i} u(x,t)dx\right)_{tt} = h_i\left(u_i(t)\right)_{tt}, \qquad \forall t \in [0,T].$$

Thus, (5.76) can be reformulated, for $i = N_1+1,\dots,N_{\max}$, at time t_n into

$$h_i u_{i,tt}(t_n) - a\left(u_x\left(x_{i+\frac{1}{2}}, t_n\right) - u_x\left(x_{i-\frac{1}{2}}, t_n\right)\right) = 0. \tag{5.77}$$

There now remains to replace in Equation (5.77) each term by its approximation with respect to the discrete unknowns (and the data).

☐ For $i = N_1+1,\dots,N_{\max}-1$, we have

$$h_i u_{i,tt}(t_n) - a\left(u_x\left(x_{i+\frac{1}{2}}, t_n\right) - u_x\left(x_{i-\frac{1}{2}}, t_n\right)\right) = 0.$$

Using the second-order central time discretization and replacing the space differential operators for the second and third terms of the above equation with the corresponding difference operators, together with an average of diffusion terms in $n+1$ and $n-1$ time steps, we obtain

$$h_i\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2} - \frac{1}{2}a\left(\delta_x u_{i+\frac{1}{2}}^{n+1} - \delta_x u_{i-\frac{1}{2}}^{n+1}\right) - \frac{1}{2}a\left(\delta_x u_{i+\frac{1}{2}}^{n-1} - \delta_x u_{i-\frac{1}{2}}^{n-1}\right) = 0, \quad \forall n \in \{1,\dots,\mathcal{N}-1\}.$$

☐ For $i = N_{\max}$, we have

$$h_{N_{\max}} u_{N_{\max},tt}(t_n) - a\left(u_x\left(x_{N_{\max}+\frac{1}{2}}, t_n\right) - u_x\left(x_{N_{\max}-\frac{1}{2}}, t_n\right)\right) = 0.$$

Combining the above equation with the dynamical boundary condition $u_x(L,t_n) + \eta(t_n) = 0$ as the position $x_{N_{\max}+\frac{1}{2}}$ represents the point L yields

$$h_{N_{\max}} u_{N_{\max},tt}(t_n) + a\left(\eta(t_n) + u_x\left(x_{N_{\max}-\frac{1}{2}}, t_n\right)\right) = 0.$$

Applying the second-order central difference in time and replacing the space differential operator with the corresponding difference operator, together with an average of $n+1$ and $n-1$ time steps diffusion, we obtain

$$h_{N_{\max}}\frac{u_{N_{\max}}^{n+1}-2u_{N_{\max}}^{n}+u_{N_{\max}}^{n-1}}{\tau^2}+\frac{a}{2}\left(\eta^{n+1}+\delta_x u_{N_{\max}-\frac{1}{2}}^{n+1}\right)+\frac{a}{2}\left(\eta^{n-1}+\delta_x u_{N_{\max}-\frac{1}{2}}^{n-1}\right)=0,\quad \forall n\in\{1,\dots,\mathcal{N}-1\}.$$

Discretization of the Continuity Conditions: The continuity conditions are discretized at time t_n, $n=0,1,\dots\mathcal{N}$ as

$$u_{N_1+\frac{1}{2}}^{n}=w_{N_1+\frac{1}{2}}^{n},$$

and

$$\delta_x w_{N_1+\frac{1}{2}}^{n}=0.$$

Discretization of the Dynamical Boundary Condition: The dynamical boundary condition $w_{xx}(-L,t)-\xi(t)=0$ is also approximated at time t_n, $n=0,1,\dots\mathcal{N}$ by

$$\delta_x^2 w_0^n-\xi^n=0.$$

Discretization of the First-Order Differential Equations: There now remains to approximate the first-order differential equations of (5.4). Writing the first Equation of (5.4) at time t_n, discretizing the time partial derivatives using the centered difference formula, then using an average of $n+1$ and $n-1$ time steps diffusion yields the following equation:

$$\eta^{n+1}=\eta^{n-1}+u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}-\tau\left(\eta^{n+1}+\eta^{n-1}\right),\qquad \forall n\in\{1,\dots,\mathcal{N}-1\}.$$

Similarly, we write the second Equation of (5.4) at time t_n and use a centered time discretization for the first term. For the second term, however, a forward difference in space and a centered difference in time are applied. Additionally, an average of $n+1$ and $n-1$ time steps diffusion is used to obtain the following equation:

$$\xi^{n+1}=\xi^{n-1}+\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}-\tau\left(\xi^{n+1}+\xi^{n-1}\right),\qquad \forall n\in\{1,\dots,\mathcal{N}-1\}.$$

Writing the third Equation of (5.4) at time t_n and using a centered time discretization for the first two terms with an average of $n+1$ and $n-1$ time steps diffusion for the third term yields the following equation:

$$\zeta^{n+1}=\zeta^{n-1}+w_0^{n+1}-w_0^{n-1}-\tau\left(\zeta^{n+1}+\zeta^{n-1}\right),\qquad \forall n\in\{1,\dots,\mathcal{N}-1\}.$$

Finally, we have derived the implicit numerical scheme for the unknown value U_i^{n+1}, which can be expressed as follows

$$\begin{cases} h_i\dfrac{U_i^{n+1}-2U_i^{n}+U_i^{n-1}}{\tau^2}+\dfrac{1}{2}\left(F_{i+\frac{1}{2}}^{n+1}-F_{i-\frac{1}{2}}^{n+1}\right)+\dfrac{1}{2}\left(F_{i+\frac{1}{2}}^{n-1}-F_{i-\frac{1}{2}}^{n-1}\right)=0, & i=1,\dots,N_{\max},\ n=1,\dots,\mathcal{N}-1,\\ u_{N_1+\frac{1}{2}}^{n}=w_{N_1+\frac{1}{2}}^{n},\quad \delta_x w_{N_1+\frac{1}{2}}^{n}=0,\quad \delta_x^2 w_0^n-\xi^n=0, & n=0,1,\dots\mathcal{N}, \end{cases} \tag{5.78}$$

where the numerical flux at point $x_{i+\frac{1}{2}}$ and time t_n is defined as

$$F_{i+\frac{1}{2}}^{n}=\begin{cases} b\delta_x^3 w_{i+\frac{1}{2}}^{n}, & i=0,\dots,N_1-1,\\ -a\delta_x u_{i+\frac{1}{2}}^{n}, & i=N_1,\dots,N_{\max}, \end{cases} \tag{5.79}$$

with the boundary conditions

$$\delta_x u^n_{N_{\max}+\frac{1}{2}} + \eta^n = 0, \qquad \text{for} \quad n = 0, 1, \ldots \mathcal{N},$$

and

$$\delta_x^3 w^n_{\frac{1}{2}} + \zeta^n = 0, \qquad \text{for} \quad n = 0, 1, \ldots \mathcal{N}.$$

The discretized form of the equations in (5.4) is given by:

$$\begin{cases} \eta^{n+1} = \eta^{n-1} + u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1} - \tau\left(\eta^{n+1} + \eta^{n-1}\right), & n = 1, \ldots, \mathcal{N}-1, \\ \xi^{n+1} = \xi^{n-1} + \delta_x w^{n+1}_{\frac{1}{2}} - \delta_x w^{n-1}_{\frac{1}{2}} - \tau\left(\xi^{n+1} + \xi^{n-1}\right), & n = 1, \ldots, \mathcal{N}-1, \\ \zeta^{n+1} = \zeta^{n-1} + w^{n+1}_0 - w^{n-1}_0 - \tau\left(\zeta^{n+1} + \zeta^{n-1}\right), & n = 1, \ldots, \mathcal{N}-1. \end{cases} \tag{5.80}$$

In order to formulate the first equation of (5.78)-(5.80) for $n = 0$, let us introduce

$$G_{i+\frac{1}{2}} = \begin{cases} b\delta_x^3 \theta_{i+\frac{1}{2}}, & i = 0, \ldots, N_1 - 1, \\ -a\delta_x \psi_{i+\frac{1}{2}}, & i = N_1, \ldots, N_{\max}. \end{cases} \tag{5.81}$$

After straightforward rearranging, we have

$$h_i \frac{U_i^1 - U_i^0}{\tau^2} + \frac{1}{2}\left(F^1_{i+\frac{1}{2}} - F^1_{i-\frac{1}{2}}\right) = h_i \frac{U_i^0 - U_i^{-1}}{\tau^2} - \frac{1}{2}\left(F^{-1}_{i+\frac{1}{2}} - F^{-1}_{i-\frac{1}{2}}\right), \quad \text{for} \quad i = 1, \ldots, N_{\max}. \tag{5.82}$$

Substituting (5.48) into (5.82), we obtain

$$\begin{aligned} h_i \frac{U_i^1 - U_i^0}{\tau^2} + \frac{1}{2}\left(F^1_{i+\frac{1}{2}} - F^1_{i-\frac{1}{2}}\right) = h_i \frac{U_i^0 - U_i^1}{\tau^2} + 2h_i \frac{\Psi_i}{\tau} - \frac{1}{2}\left(F^1_{i+\frac{1}{2}} - F^1_{i-\frac{1}{2}}\right) \\ + \tau\left(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}}\right), \quad \text{for} \quad i = 1, \ldots, N_{\max}, \end{aligned} \tag{5.83}$$

and finally,

$$h_i \frac{U_i^1 - U_i^0}{\tau^2} + \frac{1}{2}\left(F^1_{i+\frac{1}{2}} - F^1_{i-\frac{1}{2}}\right) = h_i \frac{\Psi_i}{\tau} + \frac{\tau}{2}\left(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}}\right), \quad \text{for} \quad i = 1, \ldots, N_{\max}. \tag{5.84}$$

On the other hand, we discretize the equations of (5.4) at time t_1 using the backward difference formula without centering the differences. This yields the following

$$\begin{cases} \eta^1 = \eta^0 + u^1_{N_{\max}+1} - u^0_{N_{\max}+1} - \tau\eta^1, \\ \xi^1 = \xi^0 + \delta_x w^1_{\frac{1}{2}} - \delta_x w^0_{\frac{1}{2}} - \tau\xi^1, \\ \zeta^1 = \zeta^0 + w^1_0 - w^0_0 - \tau\zeta^1. \end{cases} \tag{5.85}$$

The numerical solution can be defined as a piecewise constant function on each finite volume K_i and each time interval $[t_{n-1}, t_n[$. We denote it by

$$U_{\mathcal{T},\tau}(x,t) = U_i^n,$$

where $x \in K_i$, $i = 1, \ldots, N_{\max}$ and $t \in [t_{n-1}, t_n[$, $n = 1, \ldots \mathcal{N}$. We also define

$$\Xi_\tau(t) = \Xi^n,$$

for $t \in [t_{n-1}, t_n[$, $n = 1, \ldots \mathcal{N}$.

Estimation of the Discrete Energy for the Implicit Form

In this subsection, we aim to design the discrete energy of the implicit numerical scheme that is preserved when no dynamical boundary controls are applied. When applying at least one dynamical boundary control to the system, however, we aim to have a dissipation of the discrete energy. To this end, we define:

- The discrete kinetic energy for $U_{\mathcal{T}}^n$ as:

$$\mathcal{E}_k\left(U_{\mathcal{T}}^n\right)=\frac{1}{2}\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{n+1}-U_i^n}{\tau}\right)^2.$$

- The discrete potential energy for $U_{\mathcal{T}}^n$ as:

$$\mathcal{E}_p(U_{\mathcal{T}}^n)=\frac{1}{4}\left(b\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{n+1}\right)^2+a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}\right)^2\right)+\frac{1}{4}\left(b\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{n}\right)^2+a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n}\right)^2\right).$$

- The discrete boundary control energy for Ξ^n as:

$$\mathcal{E}_b(\Xi^n)=\frac{1}{4}\left(a\left(\eta^{n+1}\right)^2+b\left(\xi^{n+1}\right)^2+b\left(\zeta^{n+1}\right)^2\right)+\frac{1}{4}\left(a\left(\eta^{n}\right)^2+b\left(\xi^{n}\right)^2+b\left(\zeta^{n}\right)^2\right).$$

The total discrete energy $\mathcal{E}^n$ of the implicit numerical scheme (5.78)-(5.80) is defined by

$$\mathcal{E}^n=\mathcal{E}_k(U_{\mathcal{T}}^n)+\mathcal{E}_p(U_{\mathcal{T}}^n)+\mathcal{E}_b(\Xi^n).$$

Now, we want to prove that the properties mentioned above for the discrete energy $\mathcal{E}^n$ are fulfilled. To this end, we prove the following theorem:

Theorem 5.5.2. *The total discrete energy $\mathcal{E}^n$ of the numerical scheme (5.78)-(5.80) satisfies the following estimation:*

$$\mathcal{E}^n-\mathcal{E}^{n-1}=-\frac{\tau}{4}\left(a|\eta^{n+1}+\eta^{n-1}|^2+b|\xi^{n+1}+\xi^{n-1}|^2+b|\zeta^{n+1}+\zeta^{n-1}|^2\right). \tag{5.86}$$

Proof. Multiplying both sides of the first Equation of (5.78)-(5.80) by $U_i^{n+1}-U_i^{n-1}$ and summing up i from 1 to $N_{\max}$, gives

$$\sum_{i=1}^{N_{\max}}h_i\frac{U_i^{n+1}-2U_i^n+U_i^{n-1}}{\tau^2}\left(U_i^{n+1}-U_i^{n-1}\right)+\frac{1}{2}\sum_{i=1}^{N_{\max}}\left(F_{i+\frac{1}{2}}^{n+1}-F_{i-\frac{1}{2}}^{n+1}\right)\left(U_i^{n+1}-U_i^{n-1}\right)$$
$$+\frac{1}{2}\sum_{i=1}^{N_{\max}}\left(F_{i+\frac{1}{2}}^{n-1}-F_{i-\frac{1}{2}}^{n-1}\right)\left(U_i^{n+1}-U_i^{n-1}\right)=0,$$

so that

$$\sum_{i=1}^{N_{\max}}h_i\frac{U_i^{n+1}-2U_i^n+U_i^{n-1}}{\tau^2}\left(U_i^{n+1}-U_i^{n-1}\right)+\frac{1}{2}\sum_{i=1}^{N_{\max}}F_{i+\frac{1}{2}}^{n+1}\left(U_i^{n+1}-U_i^{n-1}\right)-\frac{1}{2}\sum_{i=1}^{N_{\max}}F_{i-\frac{1}{2}}^{n+1}\left(U_i^{n+1}-U_i^{n-1}\right)$$
$$+\frac{1}{2}\sum_{i=1}^{N_{\max}}F_{i+\frac{1}{2}}^{n-1}\left(U_i^{n+1}-U_i^{n-1}\right)-\frac{1}{2}\sum_{i=1}^{N_{\max}}F_{i-\frac{1}{2}}^{n-1}\left(U_i^{n+1}-U_i^{n-1}\right)=0.$$

By (5.79), the numerical flux $F^n_{i+\frac{1}{2}}$ yields

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \frac{U_i^{n+1}-2U_i^n+U_i^{n-1}}{\tau^2}\left(U_i^{n+1}-U_i^{n-1}\right)+\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}\left(w_i^{n+1}-w_i^{n-1}\right)-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(u_i^{n+1}-u_i^{n-1}\right)\\
&-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n+1}\left(w_i^{n+1}-w_i^{n-1}\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}^{n+1}\left(u_i^{n+1}-u_i^{n-1}\right)+\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}\left(w_i^{n+1}-w_i^{n-1}\right)\\
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(u_i^{n+1}-u_i^{n-1}\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n-1}\left(w_i^{n+1}-w_i^{n-1}\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}^{n-1}\left(u_i^{n+1}-u_i^{n-1}\right)\\
&-\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n+1}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)-\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n-1}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)=0.
\end{aligned}\tag{5.87}$$

A direct calculation applied to the first term of (5.87) gives

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \frac{U_i^{n+1}-2U_i^n+U_i^{n-1}}{\tau^2}\left(U_i^{n+1}-U_i^{n-1}\right)\\
&=\sum_{i=1}^{N_{\max}} h_i \frac{\left(U_i^{n+1}-U_i^n\right)-\left(U_i^n-U_i^{n-1}\right)}{\tau^2}\left(\left(U_i^{n+1}-U_i^n\right)+\left(U_i^n-U_i^{n-1}\right)\right)\\
&=\sum_{i=1}^{N_{\max}} h_i\left(\frac{U_i^{n+1}-U_i^n}{\tau}\right)^2-\sum_{i=1}^{N_{\max}} h_i\left(\frac{U_i^n-U_i^{n-1}}{\tau}\right)^2.
\end{aligned}\tag{5.88}$$

By the translation of index i for the second and fourth terms of the following equation, we obtain

$$\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(u_i^{n+1}-u_i^{n-1}\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}^{n+1}\left(u_i^{n+1}-u_i^{n-1}\right)-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(u_i^{n+1}-u_i^{n-1}\right)\\
&\quad+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i-\frac{1}{2}}^{n-1}\left(u_i^{n+1}-u_i^{n-1}\right)-\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n+1}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)-\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n-1}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)\\
&=-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(u_i^{n+1}-u_i^{n-1}\right)+\frac{a}{2}\sum_{i=N_1}^{N_{\max}-1}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(u_{i+1}^{n+1}-u_{i+1}^{n-1}\right)-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(u_i^{n+1}-u_i^{n-1}\right)\\
&\quad+\frac{a}{2}\sum_{i=N_1}^{N_{\max}-1}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(u_{i+1}^{n+1}-u_{i+1}^{n-1}\right)-\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n+1}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)-\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n-1}\left(w_{N_1}^{n+1}-w_{N_1}^{n-1}\right)\\
&=\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(u_{i+1}^{n+1}-u_{i+1}^{n-1}-u_i^{n+1}+u_i^{n-1}\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(u_{i+1}^{n+1}-u_{i+1}^{n-1}-u_i^{n+1}+u_i^{n-1}\right)\\
&\quad+\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n+1}\left(u_{N_1+1}^{n+1}-u_{N_1+1}^{n-1}-w_{N_1}^{n+1}+w_{N_1}^{n-1}\right)+\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n-1}\left(u_{N_1+1}^{n+1}-u_{N_1+1}^{n-1}-w_{N_1}^{n+1}+w_{N_1}^{n-1}\right)\\
&\quad-\frac{a}{2}\delta_x u_{N_{\max}+\frac{1}{2}}^{n+1}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right)-\frac{a}{2}\delta_x u_{N_{\max}+\frac{1}{2}}^{n-1}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right)\\
&=\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}-\delta_x u_{i+\frac{1}{2}}^{n-1}\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}-\delta_x u_{i+\frac{1}{2}}^{n-1}\right)\\
&\quad+\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n+1}\left(u_{N_1+1}^{n+1}-u_{N_1+1}^{n-1}-w_{N_1}^{n+1}+w_{N_1}^{n-1}\right)+\frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^{n-1}\left(u_{N_1+1}^{n+1}-u_{N_1+1}^{n-1}-w_{N_1}^{n+1}+w_{N_1}^{n-1}\right)\\
&\quad-\frac{a}{2}\delta_x u_{N_{\max}+\frac{1}{2}}^{n+1}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right)-\frac{a}{2}\delta_x u_{N_{\max}+\frac{1}{2}}^{n-1}\left(u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}\right).
\end{aligned}\tag{5.89}$$

Since $\delta_x w^n_{N_1+\frac{1}{2}} = 0$, one can deduce that

$$w^n_{N_1+\frac{1}{2}} = w^n_{N_1}. \tag{5.90}$$

Thanks to (5.90) and the fact that $u^n_{N_1+\frac{1}{2}} = w^n_{N_1+\frac{1}{2}}$, one has

$$\delta_x u^n_{N_1+\frac{1}{2}} = \frac{u^n_{N_1+1} - w^n_{N_1}}{h_{N_1+\frac{1}{2}}}. \tag{5.91}$$

Therefore, (5.89) together with (5.91) imply that

$$\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i+\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i-\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) - \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i+\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right)\\
&\quad + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i-\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) - \frac{a}{2}\delta_x u^{n+1}_{N_1+\frac{1}{2}}\left(w^{n+1}_{N_1} - w^{n-1}_{N_1}\right) - \frac{a}{2}\delta_x u^{n-1}_{N_1+\frac{1}{2}}\left(w^{n+1}_{N_1} - w^{n-1}_{N_1}\right)\\
&= \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^{n+1}_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}} - \delta_x u^{n-1}_{i+\frac{1}{2}}\right) + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^{n-1}_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}} - \delta_x u^{n-1}_{i+\frac{1}{2}}\right)\\
&\quad + \frac{a}{2}h_{N_1+\frac{1}{2}}\delta_x u^{n+1}_{N_1+\frac{1}{2}}\left(\delta_x u^{n+1}_{N_1+\frac{1}{2}} - \delta_x u^{n-1}_{N_1+\frac{1}{2}}\right) + \frac{a}{2}h_{N_1+\frac{1}{2}}\delta_x u^{n-1}_{N_1+\frac{1}{2}}\left(\delta_x u^{n+1}_{N_1+\frac{1}{2}} - \delta_x u^{n-1}_{N_1+\frac{1}{2}}\right)\\
&\quad - \frac{a}{2}\delta_x u^{n+1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right) - \frac{a}{2}\delta_x u^{n-1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right)\\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^{n+1}_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}} - \delta_x u^{n-1}_{i+\frac{1}{2}}\right) + \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^{n-1}_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}} - \delta_x u^{n-1}_{i+\frac{1}{2}}\right)\\
&\quad - \frac{a}{2}\delta_x u^{n+1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right) - \frac{a}{2}\delta_x u^{n-1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right)\\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}} + \delta_x u^{n-1}_{i+\frac{1}{2}}\right)\left(\delta_x u^{n+1}_{i+\frac{1}{2}} - \delta_x u^{n-1}_{i+\frac{1}{2}}\right) - \frac{a}{2}\delta_x u^{n+1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right)\\
&\quad - \frac{a}{2}\delta_x u^{n-1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right)\\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}}\right)^2 - \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^{n-1}_{i+\frac{1}{2}}\right)^2 - \frac{a}{2}\delta_x u^{n+1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right)\\
&\quad - \frac{a}{2}\delta_x u^{n-1}_{N_{\max}+\frac{1}{2}}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right).
\end{aligned}$$

Noticing that $\delta_x u^n_{N_{\max}+\frac{1}{2}} = -\eta^n$, one has from the last equality that

$$\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i+\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i-\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) - \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i+\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right)\\
&\quad + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i-\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) - \frac{a}{2}\delta_x u^{n+1}_{N_1+\frac{1}{2}}\left(w^{n+1}_{N_1} - w^{n-1}_{N_1}\right) - \frac{a}{2}\delta_x u^{n-1}_{N_1+\frac{1}{2}}\left(w^{n+1}_{N_1} - w^{n-1}_{N_1}\right)\\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}}\right)^2 - \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^{n-1}_{i+\frac{1}{2}}\right)^2 + \frac{a}{2}\eta^{n+1}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right) + \frac{a}{2}\eta^{n-1}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right).
\end{aligned} \tag{5.92}$$

Multiplying both sides of the first Equation of (5.80) by η^{n+1} and η^{n-1} yields

$$\left(\eta^{n+1}\right)^2 = \eta^{n+1}\eta^{n-1} + \eta^{n+1}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right) - \tau\eta^{n+1}\left(\eta^{n+1} + \eta^{n-1}\right), \tag{5.93}$$

and

$$\eta^{n-1}\eta^{n+1} = \left(\eta^{n-1}\right)^2 + \eta^{n-1}\left(u^{n+1}_{N_{\max}+1} - u^{n-1}_{N_{\max}+1}\right) - \tau\eta^{n-1}\left(\eta^{n+1} + \eta^{n-1}\right), \tag{5.94}$$

respectively. Hence, (5.92) together with (5.93) and (5.94) implies that

$$\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^{n+1}_{i+\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^{n+1}_{i-\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) - \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^{n-1}_{i+\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right)\\
&\quad + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^{n-1}_{i-\frac{1}{2}}\left(u^{n+1}_i - u^{n-1}_i\right) - \frac{a}{2}\delta_x u^{n+1}_{N_1+\frac{1}{2}}\left(w^{n+1}_{N_1} - w^{n-1}_{N_1}\right) - \frac{a}{2}\delta_x u^{n-1}_{N_1+\frac{1}{2}}\left(w^{n+1}_{N_1} - w^{n-1}_{N_1}\right)\\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^{n+1}_{i+\frac{1}{2}}\right)^2 - \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^{n-1}_{i+\frac{1}{2}}\right)^2 + \frac{a}{2}\left(\eta^{n+1}\right)^2 - \frac{a}{2}\left(\eta^{n-1}\right)^2 + \frac{\tau a}{2}\left|\eta^{n+1} + \eta^{n-1}\right|^2.
\end{aligned} \tag{5.95}$$

Now, by the translation of index i for the second and fourth terms of the following equation, we obtain

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^{n+1}_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) - \frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w^{n+1}_{i-\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) + \frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^{n-1}_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right)\\
&\quad - \frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w^{n-1}_{i-\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right)\\
&= \frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^{n+1}_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right) - \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w^{n+1}_{i+\frac{1}{2}}\left(w^{n+1}_{i+1} - w^{n-1}_{i+1}\right) + \frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^{n-1}_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i\right)\\
&\quad - \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w^{n-1}_{i+\frac{1}{2}}\left(w^{n+1}_{i+1} - w^{n-1}_{i+1}\right)\\
&= \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w^{n+1}_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i - w^{n+1}_{i+1} + w^{n-1}_{i+1}\right) + \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w^{n-1}_{i+\frac{1}{2}}\left(w^{n+1}_i - w^{n-1}_i - w^{n+1}_{i+1} + w^{n-1}_{i+1}\right)\\
&\quad - \frac{b}{2}\delta_x^3 w^{n+1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) - \frac{b}{2}\delta_x^3 w^{n-1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right)\\
&= -\frac{b}{2}\sum_{i=0}^{N_1-1} h_{i+\frac{1}{2}}\delta_x^3 w^{n+1}_{i+\frac{1}{2}}\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - \frac{b}{2}\sum_{i=0}^{N_1-1} h_{i+\frac{1}{2}}\delta_x^3 w^{n-1}_{i+\frac{1}{2}}\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right)\\
&\quad - \frac{b}{2}\delta_x^3 w^{n+1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) - \frac{b}{2}\delta_x^3 w^{n-1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right)\\
&= \frac{b}{2}\sum_{i=0}^{N_1-1}\left(\delta_x^2 w^{n+1}_i - \delta_x^2 w^{n+1}_{i+1}\right)\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) + \frac{b}{2}\sum_{i=0}^{N_1-1}\left(\delta_x^2 w^{n-1}_i - \delta_x^2 w^{n-1}_{i+1}\right)\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right)\\
&\quad - \frac{b}{2}\delta_x^3 w^{n+1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) - \frac{b}{2}\delta_x^3 w^{n-1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right)\\
&= \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w^{n+1}_i\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w^{n+1}_{i+1}\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) + \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w^{n-1}_i\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right)\\
&\quad - \frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w^{n-1}_{i+1}\left(\delta_x w^{n+1}_{i+\frac{1}{2}} - \delta_x w^{n-1}_{i+\frac{1}{2}}\right) - \frac{b}{2}\delta_x^3 w^{n+1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right) - \frac{b}{2}\delta_x^3 w^{n-1}_{\frac{1}{2}}\left(w^{n+1}_0 - w^{n-1}_0\right).
\end{aligned}$$

Once again, the translation of index i for the second and fourth terms of the right-hand side of the above equation with the help of the fact that $\delta_x w^n_{N_1+\frac{1}{2}} = 0$ gives

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}\left(w_i^{n+1}-w_i^{n-1}\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n+1}\left(w_i^{n+1}-w_i^{n-1}\right)+\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}\left(w_i^{n+1}-w_i^{n-1}\right)\\
&\quad-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n-1}\left(w_i^{n+1}-w_i^{n-1}\right)\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_i^{n+1}\left(\delta_x w_{i+\frac{1}{2}}^{n+1}-\delta_x w_{i+\frac{1}{2}}^{n-1}\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n+1}\left(\delta_x w_{i-\frac{1}{2}}^{n+1}-\delta_x w_{i-\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_i^{n-1}\left(\delta_x w_{i+\frac{1}{2}}^{n+1}-\delta_x w_{i+\frac{1}{2}}^{n-1}\right)\\
&\quad-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n-1}\left(\delta_x w_{i-\frac{1}{2}}^{n+1}-\delta_x w_{i-\frac{1}{2}}^{n-1}\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}\left(w_0^{n+1}-w_0^{n-1}\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}\left(w_0^{n+1}-w_0^{n-1}\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n+1}\left(\delta_x w_{i+\frac{1}{2}}^{n+1}-\delta_x w_{i+\frac{1}{2}}^{n-1}-\delta_x w_{i-\frac{1}{2}}^{n+1}+\delta_x w_{i-\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n-1}\left(\delta_x w_{i+\frac{1}{2}}^{n+1}-\delta_x w_{i+\frac{1}{2}}^{n-1}-\delta_x w_{i-\frac{1}{2}}^{n+1}+\delta_x w_{i-\frac{1}{2}}^{n-1}\right)\\
&\quad+\frac{b}{2}\delta_x^2 w_0^{n+1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\delta_x^2 w_0^{n-1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}\left(w_0^{n+1}-w_0^{n-1}\right)\\
&\quad-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}\left(w_0^{n+1}-w_0^{n-1}\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 w_i^{n+1}-\delta_x^2 w_i^{n-1}\right)+\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 w_i^{n+1}-\delta_x^2 w_i^{n-1}\right)+\frac{b}{2}\delta_x^2 w_0^{n+1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)\\
&\quad+\frac{b}{2}\delta_x^2 w_0^{n-1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}\left(w_0^{n+1}-w_0^{n-1}\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}\left(w_0^{n+1}-w_0^{n-1}\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^{n+1}+\delta_x^2 w_i^{n-1}\right)\left(\delta_x^2 w_i^{n+1}-\delta_x^2 w_i^{n-1}\right)+\frac{b}{2}\delta_x^2 w_0^{n+1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\delta_x^2 w_0^{n-1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)\\
&\quad-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}\left(w_0^{n+1}-w_0^{n-1}\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}\left(w_0^{n+1}-w_0^{n-1}\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^{n+1}\right)^2-\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^{n-1}\right)^2+\frac{b}{2}\delta_x^2 w_0^{n+1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\delta_x^2 w_0^{n-1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)\\
&\quad-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}\left(w_0^{n+1}-w_0^{n-1}\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}\left(w_0^{n+1}-w_0^{n-1}\right).
\end{aligned}$$

Noticing that $\delta_x^2 w_0^n = \xi^n$ and $\delta_x^3 w_{\frac{1}{2}}^n = -\zeta^n$, this last equality yields

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}\left(w_i^{n+1}-w_i^{n-1}\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n+1}\left(w_i^{n+1}-w_i^{n-1}\right)+\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}\left(w_i^{n+1}-w_i^{n-1}\right)\\
&\quad-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n-1}\left(w_i^{n+1}-w_i^{n-1}\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^{n+1}\right)^2-\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^{n-1}\right)^2+\frac{b}{2}\xi^{n+1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\xi^{n-1}\left(\delta_x w_{\frac{1}{2}}^{n+1}-\delta_x w_{\frac{1}{2}}^{n-1}\right)\\
&\quad+\frac{b}{2}\zeta^{n+1}\left(w_0^{n+1}-w_0^{n-1}\right)+\frac{b}{2}\zeta^{n-1}\left(w_0^{n+1}-w_0^{n-1}\right).
\end{aligned}\tag{5.96}$$

Multiplying both sides of the second Equation of (5.80) by ξ^{n+1} and ξ^{n-1}, we obtain

$$\left(\xi^{n+1}\right)^2 = \xi^{n+1}\xi^{n-1} + \xi^{n+1}\left(\delta_x w_{\frac{1}{2}}^{n+1} - \delta_x w_{\frac{1}{2}}^{n-1}\right) - \tau\xi^{n+1}\left(\xi^{n+1} + \xi^{n-1}\right), \tag{5.97}$$

and

$$\xi^{n-1}\xi^{n+1} = \left(\xi^{n-1}\right)^2 + \xi^{n-1}\left(\delta_x w_{\frac{1}{2}}^{n+1} - \delta_x w_{\frac{1}{2}}^{n-1}\right) - \tau\xi^{n-1}\left(\xi^{n+1} + \xi^{n-1}\right), \tag{5.98}$$

respectively. Similarly, multiplying both sides of the third Equation of (5.80) by ζ^{n+1} and ζ^{n-1}, yields

$$\left(\zeta^{n+1}\right)^2 = \zeta^{n+1}\zeta^{n-1} + \zeta^{n+1}\left(w_0^{n+1} - w_0^{n-1}\right) - \tau\zeta^{n+1}\left(\zeta^{n+1} + \zeta^{n-1}\right), \tag{5.99}$$

and

$$\zeta^{n-1}\zeta^{n+1} = \left(\zeta^{n-1}\right)^2 + \zeta^{n-1}\left(w_0^{n+1} - w_0^{n-1}\right) - \tau\zeta^{n-1}\left(\zeta^{n+1} + \zeta^{n-1}\right), \tag{5.100}$$

respectively. Therefore, (5.96), together with (5.97), (5.98), (5.99) and (5.100), implies

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}\left(w_i^{n+1} - w_i^{n-1}\right) - \frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n+1}\left(w_i^{n+1} - w_i^{n-1}\right) + \frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}\left(w_i^{n+1} - w_i^{n-1}\right) \\
&\quad - \frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n-1}\left(w_i^{n+1} - w_i^{n-1}\right) \\
&= \frac{b}{2}\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n+1}\right)^2 - \frac{b}{2}\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n-1}\right)^2 + \frac{b}{2}\left(\xi^{n+1}\right)^2 - \frac{b}{2}\left(\xi^{n-1}\right)^2 + \frac{b}{2}\left(\zeta^{n+1}\right)^2 - \frac{b}{2}\left(\zeta^{n-1}\right)^2 \\
&\quad + \frac{b\tau}{2}\left|\xi^{n+1} + \xi^{n-1}\right|^2 + \frac{\tau b}{2}\left|\zeta^{n+1} + \zeta^{n-1}\right|^2.
\end{aligned} \tag{5.101}$$

In order to complete the proof, plug (5.88), (5.95) and (5.101) into (5.87) giving

$$\begin{aligned}
&\mathcal{E}_k\left(U_{\mathcal{T}}^n\right) - \mathcal{E}_k\left(U_{\mathcal{T}}^{n-1}\right) + \mathcal{E}_p\left(U_{\mathcal{T}}^n\right) - \mathcal{E}_p\left(U_{\mathcal{T}}^{n-1}\right) + \mathcal{E}_b(\Xi^n) - \mathcal{E}_b(\Xi^{n-1}) \\
&= -\frac{\tau}{4}\left(a|\eta^{n+1} + \eta^{n-1}|^2 + b|\xi^{n+1} + \xi^{n-1}|^2 + b|\zeta^{n+1} + \zeta^{n-1}|^2\right),
\end{aligned}$$

which implies that

$$\mathcal{E}^n - \mathcal{E}^{n-1} = -\frac{\tau}{4}\left(a|\eta^{n+1} + \eta^{n-1}|^2 + b|\xi^{n+1} + \xi^{n-1}|^2 + b|\zeta^{n+1} + \zeta^{n-1}|^2\right).$$

This completes the proof of our theorem. ■

By Theorem 5.5.2, we know that the total discrete energy $\mathcal{E}^n$ of system (5.78)-(5.80) is preserved when no dynamical boundary controls are used, and it is non-increasing over time as long as at least one dynamical boundary control is applied on the wave or the Euler-Bernoulli beam.

Stability Results for the Implicit Scheme

In this subsection, necessary stability estimates for the numerical solution of the implicit scheme are derived. These estimates, which lead to the convergence of the scheme, are obtained using the discrete norms and the "Discrete Poincare' inequality" given in subsection 2.4.

Theorem 5.5.3. *Let the **Hypothesis (H2)** hold. Let the time step fulfill **Assumption (TA)**. Then, the numerical solution on the first time step $U_{\mathcal{T}}^1$ of the numerical scheme (5.84)-(5.85) is bounded independently on discretization parameters in $\mathbb{L}^2$. Moreover, its discrete $\mathbb{H}$ norm is bounded, the approximation of the first time derivation is bounded in $\mathbb{L}^2$ and the $\mathbb{R}^3$ norm of Ξ^1 is bounded as well. Therefore, the following estimation holds:*

$$\left\|U_{\mathcal{T}}^1\right\| + \left\|\delta_t U_{\mathcal{T}}^1\right\| + \left\|U_{\mathcal{T}}^1\right\|_{\mathbb{H},\mathcal{T}} + \left\|\Xi^1\right\|_{\mathbb{R}^3} \leq C, \tag{5.102}$$

where C is a strictly positive constant independent of space and time discretization parameters.

Proof. First, multiplying both sides of Equation (5.84) by $2\tau^2 U_i^1$ and summing up i from 1 to $N_{\max}$, gives

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \left(U_i^1\right)^2 + \sum_{i=1}^{N_{\max}} h_i \left(U_i^1 - U_i^0\right)^2 + \tau^2 \sum_{i=1}^{N_{\max}} \left(F_{i+\frac{1}{2}}^1 - F_{i-\frac{1}{2}}^1\right) U_i^1 \\
&\qquad = \sum_{i=1}^{N_{\max}} h_i \left(U_i^0\right)^2 + 2\tau \sum_{i=1}^{N_{\max}} h_i \Psi_i U_i^1 + \tau^3 \sum_{i=1}^{N_{\max}} \left(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}}\right) U_i^1,
\end{aligned}$$

that is

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \left(U_i^1\right)^2 + \sum_{i=1}^{N_{\max}} h_i \left(U_i^1 - U_i^0\right)^2 + \tau^2 \sum_{i=1}^{N_{\max}} F_{i+\frac{1}{2}}^1 U_i^1 - \tau^2 \sum_{i=1}^{N_{\max}} F_{i-\frac{1}{2}}^1 U_i^1 \\
&\qquad = \sum_{i=1}^{N_{\max}} h_i \left(U_i^0\right)^2 + 2\tau \sum_{i=1}^{N_{\max}} h_i \Psi_i U_i^1 + \tau^3 \sum_{i=1}^{N_{\max}} G_{i+\frac{1}{2}} U_i^1 - \tau^3 \sum_{i=1}^{N_{\max}} G_{i-\frac{1}{2}} U_i^1.
\end{aligned}$$

From (5.79) and (5.81), one deduces that

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \left(U_i^1\right)^2 + \sum_{i=1}^{N_{\max}} h_i \left(U_i^1 - U_i^0\right)^2 + b\tau^2 \sum_{i=1}^{N_1 - 1} \delta_x^3 w_{i+\frac{1}{2}}^1 w_i^1 \\
&\quad - a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}}^1 u_i^1 - b\tau^2 \sum_{i=1}^{N_1} \delta_x^3 w_{i-\frac{1}{2}}^1 w_i^1 + a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}}^1 u_i^1 - a\tau^2 \delta_x u_{N_1+\frac{1}{2}}^1 w_{N_1}^1 \\
&= \sum_{i=1}^{N_{\max}} h_i \left(U_i^0\right)^2 + 2\tau \sum_{i=1}^{N_{\max}} h_i \Psi_i U_i^1 + b\tau^3 \sum_{i=1}^{N_1 - 1} \delta_x^3 \theta_{i+\frac{1}{2}} w_i^1 \\
&\quad - a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} u_i^1 - b\tau^3 \sum_{i=1}^{N_1} \delta_x^3 \theta_{i-\frac{1}{2}} w_i^1 + a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i-\frac{1}{2}} u_i^1 - a\tau^3 \delta_x \psi_{N_1+\frac{1}{2}} w_{N_1}^1.
\end{aligned} \tag{5.103}$$

By the translation of index i for the second and fourth terms of the following equation, we obtain

$$\begin{aligned}
&- a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u^1_{i+\frac{1}{2}} u^1_i + a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u^1_{i-\frac{1}{2}} u^1_i + a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} u^1_i \\
&\quad - a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i-\frac{1}{2}} u^1_i - a\tau^2 \delta_x u^1_{N_1+\frac{1}{2}} w^1_{N_1} + a\tau^3 \delta_x \psi_{N_1+\frac{1}{2}} w^1_{N_1} \\
&= -a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u^1_{i+\frac{1}{2}} u^1_i + a\tau^2 \sum_{i=N_1}^{N_{\max}-1} \delta_x u^1_{i+\frac{1}{2}} u^1_{i+1} + a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} u^1_i \\
&\quad - a\tau^3 \sum_{i=N_1}^{N_{\max}-1} \delta_x \psi_{i+\frac{1}{2}} u^1_{i+1} - a\tau^2 \delta_x u^1_{N_1+\frac{1}{2}} w^1_{N_1} + a\tau^3 \delta_x \psi_{N_1+\frac{1}{2}} w^1_{N_1} \\
&= a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u^1_{i+\frac{1}{2}} \left(u^1_{i+1} - u^1_i\right) - a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} \left(u^1_{i+1} - u^1_i\right) + a\tau^2 \delta_x u^1_{N_1+\frac{1}{2}} \left(u^1_{N_1+1} - w^1_{N_1}\right) \\
&\quad - a\tau^3 \delta_x \psi_{N_1+\frac{1}{2}} \left(u^1_{N_1+1} - w^1_{N_1}\right) - a\tau^2 \delta_x u^1_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1} + a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1} \\
&= a\tau^2 \sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u^1_{i+\frac{1}{2}}\right)^2 - a\tau^3 \sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u^1_{i+\frac{1}{2}} \delta_x \psi_{i+\frac{1}{2}} + a\tau^2 \delta_x u^1_{N_1+\frac{1}{2}} \left(u^1_{N_1+1} - w^1_{N_1}\right) \\
&\quad - a\tau^3 \delta_x \psi_{N_1+\frac{1}{2}} \left(u^1_{N_1+1} - w^1_{N_1}\right) - a\tau^2 \delta_x u^1_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1} + a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1}.
\end{aligned} \tag{5.104}$$

Since $\delta_x w^1_{N_1+\frac{1}{2}} = 0$, one can deduce that

$$w^1_{N_1+\frac{1}{2}} = w^1_{N_1}. \tag{5.105}$$

Thanks to (5.105) and the fact that $u^1_{N_1+\frac{1}{2}} = w^1_{N_1+\frac{1}{2}}$, one has

$$\delta_x u^1_{N_1+\frac{1}{2}} = \frac{u^1_{N_1+1} - w^1_{N_1}}{h_{N_1+\frac{1}{2}}}. \tag{5.106}$$

Therefore, (5.104) together with (5.106) implies that

$$\begin{aligned}
&- a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u^1_{i+\frac{1}{2}} u^1_i + a\tau^2 \sum_{i=N_1+1}^{N_{\max}} \delta_x u^1_{i-\frac{1}{2}} u^1_i + a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} u^1_i \\
&\quad - a\tau^3 \sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i-\frac{1}{2}} u^1_i - a\tau^2 \delta_x u^1_{N_1+\frac{1}{2}} w^1_{N_1} + a\tau^3 \delta_x \psi_{N_1+\frac{1}{2}} w^1_{N_1} \\
&= a\tau^2 \sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u^1_{i+\frac{1}{2}}\right)^2 - a\tau^3 \sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u^1_{i+\frac{1}{2}} \delta_x \psi_{i+\frac{1}{2}} + a\tau^2 h_{N_1+\frac{1}{2}} \left(\delta_x u^1_{N_1+\frac{1}{2}}\right)^2 \\
&\quad - a\tau^3 h_{N_1+\frac{1}{2}} \delta_x u^1_{N_1+\frac{1}{2}} \delta_x \psi_{N_1+\frac{1}{2}} - a\tau^2 \delta_x u^1_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1} + a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1} \\
&= a\tau^2 \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u^1_{i+\frac{1}{2}}\right)^2 - a\tau^3 \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u^1_{i+\frac{1}{2}} \delta_x \psi_{i+\frac{1}{2}} - a\tau^2 \delta_x u^1_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1} + a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} u^1_{N_{\max}+1}.
\end{aligned}$$

Noticing that $\delta_x u^1_{N_{\max}+\frac{1}{2}} = -\eta^1$, one has from the last equality that

$$\begin{aligned}
&-a\tau^2\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i+\frac{1}{2}}u^1_i + a\tau^2\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i-\frac{1}{2}}u^1_i + a\tau^3\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i+\frac{1}{2}}u^1_i\\
&\quad - a\tau^3\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i-\frac{1}{2}}u^1_i - a\tau^2\delta_x u^1_{N_1+\frac{1}{2}}w^1_{N_1} + a\tau^3\delta_x\psi_{N_1+\frac{1}{2}}w^1_{N_1} \qquad (5.107)\\
&= a\tau^2\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u^1_{i+\frac{1}{2}}\right)^2 - a\tau^3\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\delta_x\psi_{i+\frac{1}{2}} + a\tau^2\eta^1u^1_{N_{\max}+1} + a\tau^3\delta_x\psi_{N_{\max}+\frac{1}{2}}u^1_{N_{\max}+1}.
\end{aligned}$$

Multiplying both sides of the first Equation of (5.85) by η^1 and $\delta_x\psi_{N_{\max}+\frac{1}{2}}$ yields

$$\left(\eta^1\right)^2 = \eta^1\eta^0 + \eta^1\left(u^1_{N_{\max}+1} - u^0_{N_{\max}+1}\right) - \tau\left(\eta^1\right)^2, \qquad (5.108)$$

and

$$\eta^1\delta_x\psi_{N_{\max}+\frac{1}{2}} = \delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^0 + \delta_x\psi_{N_{\max}+\frac{1}{2}}\left(u^1_{N_{\max}+1} - u^0_{N_{\max}+1}\right) - \tau\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^1, \qquad (5.109)$$

respectively. Hence, (5.107) together with (5.108) and (5.109) implies that

$$\begin{aligned}
&-a\tau^2\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i+\frac{1}{2}}u^1_i + a\tau^2\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i-\frac{1}{2}}u^1_i + a\tau^3\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i+\frac{1}{2}}u^1_i\\
&\quad - a\tau^3\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i-\frac{1}{2}}u^1_i - a\tau^2\delta_x u^1_{N_1+\frac{1}{2}}w^1_{N_1} + a\tau^3\delta_x\psi_{N_1+\frac{1}{2}}w^1_{N_1} \qquad (5.110)\\
&= a\tau^2\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u^1_{i+\frac{1}{2}}\right)^2 - a\tau^3\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\delta_x\psi_{i+\frac{1}{2}} + a\tau^2\left(\eta^1\right)^2 - a\tau^2\eta^1\eta^0 + a\tau^2\eta^1u^0_{N_{\max}+1}\\
&\quad + a\tau^3\left(\eta^1\right)^2 + a\tau^3\eta^1\delta_x\psi_{N_{\max}+\frac{1}{2}} - a\tau^3\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^0 + a\tau^3\delta_x\psi_{N_{\max}+\frac{1}{2}}u^0_{N_{\max}+1} + a\tau^4\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^1.
\end{aligned}$$

Now, by the translation of index i for the second and fourth terms of the following equation, we obtain

$$\begin{aligned}
&b\tau^2\sum_{i=1}^{N_1-1}\delta_x^3w^1_{i+\frac{1}{2}}w^1_i - b\tau^2\sum_{i=1}^{N_1}\delta_x^3w^1_{i-\frac{1}{2}}w^1_i - b\tau^3\sum_{i=1}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}w^1_i + b\tau^3\sum_{i=1}^{N_1}\delta_x^3\theta_{i-\frac{1}{2}}w^1_i\\
&= b\tau^2\sum_{i=1}^{N_1-1}\delta_x^3w^1_{i+\frac{1}{2}}w^1_i - b\tau^2\sum_{i=0}^{N_1-1}\delta_x^3w^1_{i+\frac{1}{2}}w^1_{i+1} - b\tau^3\sum_{i=1}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}w^1_i + b\tau^3\sum_{i=0}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}w^1_{i+1}\\
&= b\tau^2\sum_{i=0}^{N_1-1}\delta_x^3w^1_{i+\frac{1}{2}}\left(w^1_i - w^1_{i+1}\right) - b\tau^3\sum_{i=0}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w^1_i - w^1_{i+1}\right) - b\tau^2\delta_x^3w^1_{\frac{1}{2}}w^1_0 + b\tau^3\delta_x^3\theta_{\frac{1}{2}}w^1_0\\
&= -b\tau^2\sum_{i=0}^{N_1-1}h_{i+\frac{1}{2}}\delta_x^3w^1_{i+\frac{1}{2}}\delta_x w^1_{i+\frac{1}{2}} + b\tau^3\sum_{i=0}^{N_1-1}h_{i+\frac{1}{2}}\delta_x^3\theta_{i+\frac{1}{2}}\delta_x w^1_{i+\frac{1}{2}} - b\tau^2\delta_x^3w^1_{\frac{1}{2}}w^1_0 + b\tau^3\delta_x^3\theta_{\frac{1}{2}}w^1_0\\
&= b\tau^2\sum_{i=0}^{N_1-1}\left(\delta_x^2w^1_i - \delta_x^2w^1_{i+1}\right)\delta_x w^1_{i+\frac{1}{2}} - b\tau^3\sum_{i=0}^{N_1-1}\left(\delta_x^2\theta_i - \delta_x^2\theta_{i+1}\right)\delta_x w^1_{i+\frac{1}{2}} - b\tau^2\delta_x^3w^1_{\frac{1}{2}}w^1_0 + b\tau^3\delta_x^3\theta_{\frac{1}{2}}w^1_0\\
&= b\tau^2\sum_{i=0}^{N_1-1}\delta_x^2w^1_i\delta_x w^1_{i+\frac{1}{2}} - b\tau^2\sum_{i=0}^{N_1-1}\delta_x^2w^1_{i+1}\delta_x w^1_{i+\frac{1}{2}} - b\tau^3\sum_{i=0}^{N_1-1}\delta_x^2\theta_i\delta_x w^1_{i+\frac{1}{2}} + b\tau^3\sum_{i=0}^{N_1-1}\delta_x^2\theta_{i+1}\delta_x w^1_{i+\frac{1}{2}}\\
&\quad - b\tau^2\delta_x^3w^1_{\frac{1}{2}}w^1_0 + b\tau^3\delta_x^3\theta_{\frac{1}{2}}w^1_0.
\end{aligned}$$

Once again, the translation of index i for the second and fourth terms of the right-hand side of the above equation with the help of the fact that $\delta_x w^1_{N_1+\frac{1}{2}} = 0$ gives

$$
\begin{aligned}
&b\tau^2 \sum_{i=1}^{N_1-1} \delta_x^3 w^1_{i+\frac{1}{2}} w_i^1 - b\tau^2 \sum_{i=1}^{N_1} \delta_x^3 w^1_{i-\frac{1}{2}} w_i^1 - b\tau^3 \sum_{i=1}^{N_1-1} \delta_x^3 \theta_{i+\frac{1}{2}} w_i^1 + b\tau^3 \sum_{i=1}^{N_1} \delta_x^3 \theta_{i-\frac{1}{2}} w_i^1 \\
&= b\tau^2 \sum_{i=0}^{N_1-1} \delta_x^2 w_i^1 \delta_x w^1_{i+\frac{1}{2}} - b\tau^2 \sum_{i=1}^{N_1} \delta_x^2 w_i^1 \delta_x w^1_{i-\frac{1}{2}} - b\tau^3 \sum_{i=0}^{N_1-1} \delta_x^2 \theta_i \delta_x w^1_{i+\frac{1}{2}} + b\tau^3 \sum_{i=1}^{N_1} \delta_x^2 \theta_i \delta_x w^1_{i-\frac{1}{2}} \\
&\quad - b\tau^2 \delta_x^3 w^1_{\frac{1}{2}} w_0^1 + b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} w_0^1 \\
&= b\tau^2 \sum_{i=1}^{N_1} \delta_x^2 w_i^1 \left(\delta_x w^1_{i+\frac{1}{2}} - \delta_x w^1_{i-\frac{1}{2}} \right) - b\tau^3 \sum_{i=1}^{N_1} \delta_x^2 \theta_i \left(\delta_x w^1_{i+\frac{1}{2}} - \delta_x w^1_{i-\frac{1}{2}} \right) + b\tau^2 \delta_x^2 w_0^1 \delta_x w^1_{\frac{1}{2}} - b\tau^3 \delta_x^2 \theta_0 \delta_x w^1_{\frac{1}{2}} \\
&\quad - b\tau^2 \delta_x^3 w^1_{\frac{1}{2}} w_0^1 + b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} w_0^1 \\
&= b\tau^2 \sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1 \right)^2 - b\tau^3 \sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 \theta_i + b\tau^2 \delta_x^2 w_0^1 \delta_x w^1_{\frac{1}{2}} - b\tau^3 \delta_x^2 \theta_0 \delta_x w^1_{\frac{1}{2}} - b\tau^2 \delta_x^3 w^1_{\frac{1}{2}} w_0^1 + b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} w_0^1.
\end{aligned}
$$

Noticing that $\delta_x^2 w_0^1 = \xi^1$ and $\delta_x^3 w^1_{\frac{1}{2}} = -\zeta^1$, this last equality yields

$$
\begin{aligned}
&b\tau^2 \sum_{i=1}^{N_1-1} \delta_x^3 w^1_{i+\frac{1}{2}} w_i^1 - b\tau^2 \sum_{i=1}^{N_1} \delta_x^3 w^1_{i-\frac{1}{2}} w_i^1 - b\tau^3 \sum_{i=1}^{N_1-1} \delta_x^3 \theta_{i+\frac{1}{2}} w_i^1 + b\tau^3 \sum_{i=1}^{N_1} \delta_x^3 \theta_{i-\frac{1}{2}} w_i^1 \\
&= b\tau^2 \sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1 \right)^2 - b\tau^3 \sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 \theta_i + b\tau^2 \xi^1 \delta_x w^1_{\frac{1}{2}} - b\tau^3 \delta_x^2 \theta_0 \delta_x w^1_{\frac{1}{2}} + b\tau^2 \zeta^1 w_0^1 + b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} w_0^1.
\end{aligned} \tag{5.111}
$$

Multiplying both sides of the second Equation of (5.85) by ξ^1 and $\delta_x^2 \theta_0$, we obtain

$$
\left(\xi^1 \right)^2 = \xi^1 \xi^0 + \xi^1 \left(\delta_x w^1_{\frac{1}{2}} - \delta_x w^0_{\frac{1}{2}} \right) - \tau \left(\xi^1 \right)^2, \tag{5.112}
$$

and

$$
\xi^1 \delta_x^2 \theta_0 = \delta_x^2 \theta_0 \xi^0 + \delta_x^2 \theta_0 \left(\delta_x w^1_{\frac{1}{2}} - \delta_x w^0_{\frac{1}{2}} \right) - \tau \delta_x^2 \theta_0 \xi^1, \tag{5.113}
$$

respectively. Similarly, multiplying both sides of the third Equation of (5.85) by ζ^1 and $\delta_x^3 \theta_{\frac{1}{2}}$, yields

$$
\left(\zeta^1 \right)^2 = \zeta^1 \zeta^0 + \zeta^1 \left(w_0^1 - w_0^0 \right) - \tau \left(\zeta^1 \right)^2, \tag{5.114}
$$

and

$$
\zeta^1 \delta_x^3 \theta_{\frac{1}{2}} = \delta_x^3 \theta_{\frac{1}{2}} \zeta^0 + \delta_x^3 \theta_{\frac{1}{2}} \left(w_0^1 - w_0^0 \right) - \tau \delta_x^3 \theta_{\frac{1}{2}} \zeta^1, \tag{5.115}
$$

respectively. Therefore, (5.111), together with (5.112), (5.113), (5.114) and (5.115), implies

$$
\begin{aligned}
&b\tau^2 \sum_{i=1}^{N_1-1} \delta_x^3 w^1_{i+\frac{1}{2}} w_i^1 - b\tau^2 \sum_{i=1}^{N_1} \delta_x^3 w^1_{i-\frac{1}{2}} w_i^1 - b\tau^3 \sum_{i=1}^{N_1-1} \delta_x^3 \theta_{i+\frac{1}{2}} w_i^1 + b\tau^3 \sum_{i=1}^{N_1} \delta_x^3 \theta_{i-\frac{1}{2}} w_i^1 \\
&= b\tau^2 \sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1 \right)^2 - b\tau^3 \sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 \theta_i + b\tau^2 \left(\xi^1 \right)^2 - b\tau^2 \xi^1 \xi^0 + b\tau^2 \xi^1 \delta_x w^0_{\frac{1}{2}} + b\tau^3 \left(\xi^1 \right)^2 \\
&\quad + b\tau^2 \left(\zeta^1 \right)^2 - b\tau^2 \zeta^1 \zeta^0 + b\tau^2 \zeta^1 w_0^0 + b\tau^3 \left(\zeta^1 \right)^2 - b\tau^3 \xi^1 \delta_x^2 \theta_0 + b\tau^3 \delta_x^2 \theta_0 \xi^0 - b\tau^3 \delta_x^2 \theta_0 \delta_x w^0_{\frac{1}{2}} \\
&\quad - b\tau^4 \delta_x^2 \theta_0 \xi^1 + b\tau^3 \zeta^1 \delta_x^3 \theta_{\frac{1}{2}} - b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} \zeta^0 + b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} w_0^0 + b\tau^4 \delta_x^3 \theta_{\frac{1}{2}} \zeta^1.
\end{aligned} \tag{5.116}
$$

Inserting (5.110) and (5.116) into (5.103) gives

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \left(U_i^1\right)^2 + \sum_{i=1}^{N_{\max}} h_i \left(U_i^1 - U_i^0\right)^2 + b\tau^2 \sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1\right)^2 + a\tau^2 \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^1\right)^2 \\
&\quad + a\tau^2 \left(\eta^1\right)^2 + b\tau^2 \left(\xi^1\right)^2 + b\tau^2 \left(\zeta^1\right)^2 + a\tau^3 \left(\eta^1\right)^2 + b\tau^3 \left(\xi^1\right)^2 + b\tau^3 \left(\zeta^1\right)^2 \\
&= \sum_{i=1}^{N_{\max}} h_i \left(U_i^0\right)^2 + 2\tau \sum_{i=1}^{N_{\max}} h_i \Psi_i U_i^1 + b\tau^3 \sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 \theta_i + a\tau^3 \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^1 \delta_x \psi_{i+\frac{1}{2}} \\
&\quad + a\tau^2 \eta^1 \eta^0 - a\tau^2 \eta^1 u_{N_{\max}+1}^0 + b\tau^2 \xi^1 \xi^0 - b\tau^2 \xi^1 \delta_x w_{\frac{1}{2}}^0 + b\tau^2 \zeta^1 \zeta^0 - b\tau^2 \zeta^1 w_0^0 - a\tau^3 \eta^1 \delta_x \psi_{N_{\max}+\frac{1}{2}} \\
&\quad + a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} \eta^0 - a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} u_{N_{\max}+1}^0 - a\tau^4 \delta_x \psi_{N_{\max}+\frac{1}{2}} \eta^1 + b\tau^3 \xi^1 \delta_x^2 \theta_0 - b\tau^3 \delta_x^2 \theta_0 \xi^0 \\
&\quad + b\tau^3 \delta_x^2 \theta_0 \delta_x w_{\frac{1}{2}}^0 + b\tau^4 \delta_x^2 \theta_0 \xi^1 - b\tau^3 \zeta^1 \delta_x^3 \theta_{\frac{1}{2}} + b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} \zeta^0 - b\tau^3 \delta_x^3 \theta_{\frac{1}{2}} w_0^0 - b\tau^4 \delta_x^3 \theta_{\frac{1}{2}} \zeta^1.
\end{aligned} \tag{5.117}$$

Applying Young's inequality, the terms on the right-hand side of (5.117) may be estimated as follows:

$$2\tau \sum_{i=1}^{N_{\max}} h_i \Psi_i U_i^1 \leq \frac{1}{2} \sum_{i=1}^{N_{\max}} h_i \left(U_i^1\right)^2 + 2\tau^2 \sum_{i=1}^{N_{\max}} h_i \left(\Psi_i\right)^2,$$

with

$$a\tau^3 \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^1 \delta_x \psi_{i+\frac{1}{2}} \leq \frac{a\tau^2}{2} \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^1\right)^2 + \frac{a\tau^4}{2} \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x \psi_{i+\frac{1}{2}}\right)^2,$$

and

$$b\tau^3 \sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 \theta_i \leq \frac{b\tau^2}{2} \sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1\right)^2 + \frac{b\tau^4}{2} \sum_{i=1}^{N_1} h_i \left(\delta_x^2 \theta_i\right)^2.$$

Similarly, Young's inequality applied to the following terms gives

$$a\tau^2 \eta^1 \eta^0 \leq \frac{a\tau^2}{2} \left(\eta^1\right)^2 + \frac{a\tau^2}{2} \left(\eta^0\right)^2,$$

$$a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} \eta^0 \leq \frac{a\tau^4}{2} \left(\delta_x \psi_{N_{\max}+\frac{1}{2}}\right)^2 + \frac{a\tau^2}{2} \left(\eta^0\right)^2,$$

and

$$a\tau^3 \delta_x \psi_{N_{\max}+\frac{1}{2}} u_{N_{\max}+1}^0 \leq \frac{a\tau^4}{2} \left(\delta_x \psi_{N_{\max}+\frac{1}{2}}\right)^2 + \frac{a\tau^2}{2} \left(u_{N_{\max}+1}^0\right)^2.$$

Next, for any $\varepsilon > 0$, applying Young's inequality once more to the given terms results in

$$a\tau^2 \eta^1 u_{N_{\max}+1}^0 \leq \frac{a\tau^2}{2\varepsilon} \left(\eta^1\right)^2 + \frac{a\tau^2 \varepsilon}{2} \left(u_{N_{\max}+1}^0\right)^2,$$

with

$$a\tau^3 \eta^1 \delta_x \psi_{N_{\max}+\frac{1}{2}} \leq \frac{a\tau^2}{2\varepsilon} \left(\eta^1\right)^2 + \frac{a\tau^4 \varepsilon}{2} \left(\delta_x \psi_{N_{\max}+\frac{1}{2}}\right)^2,$$

and finally,

$$a\tau^4\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^1 \leq \frac{a\tau^2}{2\varepsilon}\left(\eta^1\right)^2 + \frac{a\tau^6\varepsilon}{2}\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2.$$

The remaining terms on the right-hand side of (5.117) can be estimated in a similar manner. As a result, applying these estimates to (5.117) leads to the following inequality:

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N_{\max}} h_i\left(U_i^1\right)^2 + \sum_{i=1}^{N_{\max}} h_i\left(U_i^1-U_i^0\right)^2 + \frac{b\tau^2}{2}\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^1\right)^2 + \frac{a\tau^2}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^1\right)^2 \\
&\quad + \frac{a\tau^2}{2}\left(1-\frac{3}{\varepsilon}\right)\left(\eta^1\right)^2 + \frac{b\tau^2}{2}\left(1-\frac{3}{\varepsilon}\right)\left(\xi^1\right)^2 + \frac{b\tau^2}{2}\left(1-\frac{3}{\varepsilon}\right)\left(\zeta^1\right)^2 + a\tau^3\left(\eta^1\right)^2 + b\tau^3\left(\xi^1\right)^2 + b\tau^3\left(\zeta^1\right)^2 \\
&\leq \sum_{i=1}^{N_{\max}} h_i\left(U_i^0\right)^2 + 2\tau^2\sum_{i=1}^{N_{\max}} h_i\left(\Psi_i\right)^2 + \frac{b\tau^4}{2}\sum_{i=1}^{N_1} h_i\left(\delta_x^2\theta_i\right)^2 + \frac{a\tau^4}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x\psi_{i+\frac{1}{2}}\right)^2 \\
&\quad + a\tau^2\left(\eta^0\right)^2 + a\tau^2\left(\frac{\varepsilon+1}{2}\right)\left(u_{N_{\max}+1}^0\right)^2 + b\tau^2\left(\xi^0\right)^2 + b\tau^2\left(\frac{\varepsilon+1}{2}\right)\left(\delta_x w_{\frac{1}{2}}^0\right)^2 + b\tau^2\left(\zeta^0\right)^2 \\
&\quad + b\tau^2\left(\frac{\varepsilon+1}{2}\right)\left(w_0^0\right)^2 + a\tau^4\left(\frac{\varepsilon}{2}+1\right)\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2 + \frac{a\tau^6\varepsilon}{2}\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2 + b\tau^4\left(\frac{\varepsilon}{2}+1\right)\left(\delta_x^2\theta_0\right)^2 \\
&\quad + \frac{b\tau^6\varepsilon}{2}\left(\delta_x^2\theta_0\right)^2 + b\tau^4\left(\frac{\varepsilon}{2}+1\right)\left(\delta_x^3\theta_{\frac{1}{2}}\right)^2 + \frac{b\tau^6\varepsilon}{2}\left(\delta_x^3\theta_{\frac{1}{2}}\right)^2.
\end{aligned} \tag{5.118}$$

Now, let us proceed to define the vectors

$$\vartheta_1 = \left(u_{N_{\max}+1}^0, \delta_x w_{\frac{1}{2}}^0, w_0^0\right) \qquad \text{and} \qquad \vartheta_2 = \left(\delta_x\psi_{N_{\max}+\frac{1}{2}}, \delta_x^2\theta_0, \delta_x^3\theta_{\frac{1}{2}}\right).$$

By setting $\varepsilon = 4$ and using the properties of the functions Φ, Ψ and Θ along with the condition on the time step τ, we can deduce

$$\begin{aligned}
\left\|U_{\mathcal{T}}^1\right\|^2 = \sum_{i=1}^{N_{\max}} h_i\left(U_i^1\right)^2 \leq 2\left\|\Phi_{\mathcal{T}}\right\|^2 + 4\tau^2\left\|\Psi_{\mathcal{T}}\right\|^2 + \tau^4\left\|\Psi_{\mathcal{T}}\right\|_{\mathbb{H},\mathcal{T}}^2 + 2\tau^2\left\|\Theta\right\|_{\mathbb{R}^3}^2 \\
+5\tau^2\left\|\vartheta_1\right\|_{\mathbb{R}^3}^2 + 6\tau^4\left\|\vartheta_2\right\|_{\mathbb{R}^3}^2 + 4\tau^6\left\|\vartheta_2\right\|_{\mathbb{R}^3}^2 \leq C,
\end{aligned} \tag{5.119}$$

where the constant C is independent of discretization parameters and only depends on the initial data of the problem and the constant τ_0. Consequently, we establish that $U_{\mathcal{T}}^1$ belongs to $\mathbb{L}^2$.

Now, multiplying both sides of Equation (5.84) by $U_i^1 - U_i^0$ and summing up i from 1 to $N_{\max}$, gives

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i\left(\frac{U_i^1-U_i^0}{\tau}\right)^2 + \frac{1}{2}\sum_{i=1}^{N_{\max}}\left(F_{i+\frac{1}{2}}^1 - F_{i-\frac{1}{2}}^1\right)\left(U_i^1-U_i^0\right) \\
&\qquad = \frac{1}{\tau}\sum_{i=1}^{N_{\max}} h_i\Psi_i\left(U_i^1-U_i^0\right) + \frac{\tau}{2}\sum_{i=1}^{N_{\max}}\left(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}}\right)\left(U_i^1-U_i^0\right),
\end{aligned}$$

that is

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i\left(\frac{U_i^1-U_i^0}{\tau}\right)^2 + \frac{1}{2}\sum_{i=1}^{N_{\max}} F_{i+\frac{1}{2}}^1\left(U_i^1-U_i^0\right) - \frac{1}{2}\sum_{i=1}^{N_{\max}} F_{i-\frac{1}{2}}^1\left(U_i^1-U_i^0\right) \\
&\qquad = \frac{1}{\tau}\sum_{i=1}^{N_{\max}} h_i\Psi_i\left(U_i^1-U_i^0\right) + \frac{\tau}{2}\sum_{i=1}^{N_{\max}} G_{i+\frac{1}{2}}\left(U_i^1-U_i^0\right) - \frac{\tau}{2}\sum_{i=1}^{N_{\max}} G_{i-\frac{1}{2}}\left(U_i^1-U_i^0\right).
\end{aligned}$$

From (5.79) and (5.81), one deduces that

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^1 - U_i^0}{\tau}\right)^2 + \frac{b}{2}\sum_{i=1}^{N_1-1} \delta_x^3 w_{i+\frac{1}{2}}^1 \left(w_i^1 - w_i^0\right) - \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}}^1 \left(u_i^1 - u_i^0\right) \\
&\quad - \frac{b}{2}\sum_{i=1}^{N_1} \delta_x^3 w_{i-\frac{1}{2}}^1 \left(w_i^1 - w_i^0\right) + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}}^1 \left(u_i^1 - u_i^0\right) - \frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^1 \left(w_{N_1}^1 - w_{N_1}^0\right) \\
&= \frac{1}{\tau}\sum_{i=1}^{N_{\max}} h_i \Psi_i \left(U_i^1 - U_i^0\right) + \frac{b\tau}{2}\sum_{i=1}^{N_1-1} \delta_x^3 \theta_{i+\frac{1}{2}} \left(w_i^1 - w_i^0\right) - \frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} \left(u_i^1 - u_i^0\right) \\
&\quad - \frac{b\tau}{2}\sum_{i=1}^{N_1} \delta_x^3 \theta_{i-\frac{1}{2}} \left(w_i^1 - w_i^0\right) + \frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i-\frac{1}{2}} \left(u_i^1 - u_i^0\right) - \frac{a\tau}{2}\delta_x \psi_{N_1+\frac{1}{2}} \left(w_{N_1}^1 - w_{N_1}^0\right).
\end{aligned} \tag{5.120}$$

By the translation of index i for the second and fourth terms of the following equation, we obtain

$$\begin{aligned}
&- \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}}^1 \left(u_i^1 - u_i^0\right) + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i-\frac{1}{2}}^1 \left(u_i^1 - u_i^0\right) + \frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} \left(u_i^1 - u_i^0\right) \\
&\quad - \frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i-\frac{1}{2}} \left(u_i^1 - u_i^0\right) - \frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^1 \left(w_{N_1}^1 - w_{N_1}^0\right) + \frac{a\tau}{2}\delta_x \psi_{N_1+\frac{1}{2}} \left(w_{N_1}^1 - w_{N_1}^0\right) \\
&= -\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}}^1 \left(u_i^1 - u_i^0\right) + \frac{a}{2}\sum_{i=N_1}^{N_{\max}-1} \delta_x u_{i+\frac{1}{2}}^1 \left(u_{i+1}^1 - u_{i+1}^0\right) + \frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} \left(u_i^1 - u_i^0\right) \\
&\quad - \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}-1} \delta_x \psi_{i+\frac{1}{2}} \left(u_{i+1}^1 - u_{i+1}^0\right) - \frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^1 \left(w_{N_1}^1 - w_{N_1}^0\right) + \frac{a\tau}{2}\delta_x \psi_{N_1+\frac{1}{2}} \left(w_{N_1}^1 - w_{N_1}^0\right) \\
&= \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u_{i+\frac{1}{2}}^1 \left(u_{i+1}^1 - u_{i+1}^0 - u_i^1 + u_i^0\right) - \frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x \psi_{i+\frac{1}{2}} \left(u_{i+1}^1 - u_{i+1}^0 - u_i^1 + u_i^0\right) \\
&\quad + \frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^1 \left(u_{N_1+1}^1 - u_{N_1+1}^0 - w_{N_1}^1 + w_{N_1}^0\right) - \frac{a\tau}{2}\delta_x \psi_{N_1+\frac{1}{2}} \left(u_{N_1+1}^1 - u_{N_1+1}^0 - w_{N_1}^1 + w_{N_1}^0\right) \\
&\quad - \frac{a}{2}\delta_x u_{N_{\max}+\frac{1}{2}}^1 \left(u_{N_{\max}+1}^1 - u_{N_{\max}+1}^0\right) + \frac{a\tau}{2}\delta_x \psi_{N_{\max}+\frac{1}{2}} \left(u_{N_{\max}+1}^1 - u_{N_{\max}+1}^0\right) \\
&= \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^1 \left(\delta_x u_{i+\frac{1}{2}}^1 - \delta_x u_{i+\frac{1}{2}}^0\right) - \frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x \psi_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^1 - \delta_x u_{i+\frac{1}{2}}^0\right) \\
&\quad + \frac{a}{2}\delta_x u_{N_1+\frac{1}{2}}^1 \left(u_{N_1+1}^1 - u_{N_1+1}^0 - w_{N_1}^1 + w_{N_1}^0\right) - \frac{a\tau}{2}\delta_x \psi_{N_1+\frac{1}{2}} \left(u_{N_1+1}^1 - u_{N_1+1}^0 - w_{N_1}^1 + w_{N_1}^0\right) \\
&\quad - \frac{a}{2}\delta_x u_{N_{\max}+\frac{1}{2}}^1 \left(u_{N_{\max}+1}^1 - u_{N_{\max}+1}^0\right) + \frac{a\tau}{2}\delta_x \psi_{N_{\max}+\frac{1}{2}} \left(u_{N_{\max}+1}^1 - u_{N_{\max}+1}^0\right).
\end{aligned} \tag{5.121}$$

Since $\delta_x w_{N_1+\frac{1}{2}}^n = 0$, one can deduce that

$$w_{N_1+\frac{1}{2}}^n = w_{N_1}^n. \tag{5.122}$$

Thanks to (5.122) and the fact that $u_{N_1+\frac{1}{2}}^n = w_{N_1+\frac{1}{2}}^n$, one has

$$\delta_x u_{N_1+\frac{1}{2}}^n = \frac{u_{N_1+1}^n - w_{N_1}^n}{h_{N_1+\frac{1}{2}}}. \tag{5.123}$$

Therefore, (5.121) together with (5.123) implies that

$$
\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i+\frac12}\left(u^1_i-u^0_i\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i-\frac12}\left(u^1_i-u^0_i\right)+\frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i+\frac12}\left(u^1_i-u^0_i\right)\\
&\quad-\frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i-\frac12}\left(u^1_i-u^0_i\right)-\frac{a}{2}\delta_x u^1_{N_1+\frac12}\left(w^1_{N_1}-w^0_{N_1}\right)+\frac{a\tau}{2}\delta_x\psi_{N_1+\frac12}\left(w^1_{N_1}-w^0_{N_1}\right)\\
&=\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac12}\delta_x u^1_{i+\frac12}\left(\delta_x u^1_{i+\frac12}-\delta_x u^0_{i+\frac12}\right)-\frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac12}\delta_x\psi_{i+\frac12}\left(\delta_x u^1_{i+\frac12}-\delta_x u^0_{i+\frac12}\right)\\
&\quad+\frac{a}{2}h_{N_1+\frac12}\delta_x u^1_{N_1+\frac12}\left(\delta_x u^1_{N_1+\frac12}-\delta_x u^0_{N_1+\frac12}\right)-\frac{a\tau}{2}h_{N_1+\frac12}\delta_x\psi_{N_1+\frac12}\left(\delta_x u^1_{N_1+\frac12}-\delta_x u^0_{N_1+\frac12}\right)\\
&\quad-\frac{a}{2}\delta_x u^1_{N_{\max}+\frac12}\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right)+\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac12}\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right)\\
&=\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x u^1_{i+\frac12}\left(\delta_x u^1_{i+\frac12}-\delta_x u^0_{i+\frac12}\right)-\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x\psi_{i+\frac12}\left(\delta_x u^1_{i+\frac12}-\delta_x u^0_{i+\frac12}\right)\\
&\quad-\frac{a}{2}\delta_x u^1_{N_{\max}+\frac12}\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right)+\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac12}\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right)\\
&=\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\left(\delta_x u^1_{i+\frac12}\right)^2-\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x u^1_{i+\frac12}\delta_x u^0_{i+\frac12}-\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x u^1_{i+\frac12}\delta_x\psi_{i+\frac12}\\
&+\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x\psi_{i+\frac12}\delta_x u^0_{i+\frac12}-\frac{a}{2}\delta_x u^1_{N_{\max}+\frac12}\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right)+\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac12}\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right).
\end{aligned}
$$

Noticing that $\delta_x u^1_{N_{\max}+\frac12}=-\eta^1$, one has from the last equality that

$$
\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i+\frac12}\left(u^1_i-u^0_i\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i-\frac12}\left(u^1_i-u^0_i\right)+\frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i+\frac12}\left(u^1_i-u^0_i\right)\\
&\quad-\frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i-\frac12}\left(u^1_i-u^0_i\right)-\frac{a}{2}\delta_x u^1_{N_1+\frac12}\left(w^1_{N_1}-w^0_{N_1}\right)+\frac{a\tau}{2}\delta_x\psi_{N_1+\frac12}\left(w^1_{N_1}-w^0_{N_1}\right)\\
&=\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\left(\delta_x u^1_{i+\frac12}\right)^2-\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x u^1_{i+\frac12}\delta_x u^0_{i+\frac12}-\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x u^1_{i+\frac12}\delta_x\psi_{i+\frac12}\\
&\quad+\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac12}\delta_x\psi_{i+\frac12}\delta_x u^0_{i+\frac12}+\frac{a}{2}\eta^1\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right)+\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac12}\left(u^1_{N_{\max}+1}-u^0_{N_{\max}+1}\right).
\end{aligned}
\tag{5.124}
$$

Hence, (5.124) together with (5.108) and (5.109) implies that

$$
\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i+\frac{1}{2}}\left(u^1_i-u^0_i\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x u^1_{i-\frac{1}{2}}\left(u^1_i-u^0_i\right)+\frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i+\frac{1}{2}}\left(u^1_i-u^0_i\right)\\
&\quad-\frac{a\tau}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_x\psi_{i-\frac{1}{2}}\left(u^1_i-u^0_i\right)-\frac{a}{2}\delta_x u^1_{N_1+\frac{1}{2}}\left(w^1_{N_1}-w^0_{N_1}\right)+\frac{a\tau}{2}\delta_x\psi_{N_1+\frac{1}{2}}\left(w^1_{N_1}-w^0_{N_1}\right)\\
&=\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u^1_{i+\frac{1}{2}}\right)^2-\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\delta_x u^0_{i+\frac{1}{2}}-\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\delta_x\psi_{i+\frac{1}{2}}+\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x\psi_{i+\frac{1}{2}}\delta_x u^0_{i+\frac{1}{2}}\\
&\quad+\frac{a}{2}\left(\eta^1\right)^2-\frac{a}{2}\eta^1\eta^0+\frac{a\tau}{2}\left(\eta^1\right)^2+\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^1-\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^0+\frac{a\tau^2}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^1.
\end{aligned}
\tag{5.125}
$$

Now, by the translation of index i for the second and fourth terms of the following equation, we obtain

$$
\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^1_{i+\frac{1}{2}}\left(w^1_i-w^0_i\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w^1_{i-\frac{1}{2}}\left(w^1_i-w^0_i\right)-\frac{b\tau}{2}\sum_{i=1}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w^1_i-w^0_i\right)+\frac{b\tau}{2}\sum_{i=1}^{N_1}\delta_x^3\theta_{i-\frac{1}{2}}\left(w^1_i-w^0_i\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^1_{i+\frac{1}{2}}\left(w^1_i-w^0_i\right)-\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w^1_{i+\frac{1}{2}}\left(w^1_{i+1}-w^0_{i+1}\right)-\frac{b\tau}{2}\sum_{i=1}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w^1_i-w^0_i\right)+\frac{b\tau}{2}\sum_{i=0}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w^1_{i+1}-w^0_{i+1}\right)\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w^1_{i+\frac{1}{2}}\left(w^1_i-w^0_i-w^1_{i+1}+w^0_{i+1}\right)-\frac{b\tau}{2}\sum_{i=0}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w^1_i-w^0_i-w^1_{i+1}+w^0_{i+1}\right)\\
&\quad-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w^1_0-w^0_0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w^1_0-w^0_0\right)\\
&=-\frac{b}{2}\sum_{i=0}^{N_1-1}h_{i+\frac{1}{2}}\delta_x^3 w^1_{i+\frac{1}{2}}\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)+\frac{b\tau}{2}\sum_{i=0}^{N_1-1}h_{i+\frac{1}{2}}\delta_x^3\theta_{i+\frac{1}{2}}\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)\\
&\quad-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w^1_0-w^0_0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w^1_0-w^0_0\right)\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\left(\delta_x^2 w^1_i-\delta_x^2 w^1_{i+1}\right)\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)-\frac{b\tau}{2}\sum_{i=0}^{N_1-1}\left(\delta_x^2\theta_i-\delta_x^2\theta_{i+1}\right)\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)\\
&\quad-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w^1_0-w^0_0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w^1_0-w^0_0\right)\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w^1_i\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)-\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w^1_{i+1}\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)-\frac{b\tau}{2}\sum_{i=0}^{N_1-1}\delta_x^2\theta_i\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)\\
&\quad+\frac{b\tau}{2}\sum_{i=0}^{N_1-1}\delta_x^2\theta_{i+1}\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w^1_0-w^0_0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w^1_0-w^0_0\right).
\end{aligned}
$$

Once again, the translation of index i for the second and fourth terms of the right-hand side of the

above equation with the help of the fact that $\delta_x w^n_{N_1+\frac{1}{2}} = 0$ gives

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^1_{i+\frac{1}{2}}\left(w_i^1-w_i^0\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w^1_{i-\frac{1}{2}}\left(w_i^1-w_i^0\right)-\frac{b\tau}{2}\sum_{i=1}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w_i^1-w_i^0\right)+\frac{b\tau}{2}\sum_{i=1}^{N_1}\delta_x^3\theta_{i-\frac{1}{2}}\left(w_i^1-w_i^0\right)\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_i^1\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^1\left(\delta_x w^1_{i-\frac{1}{2}}-\delta_x w^0_{i-\frac{1}{2}}\right)-\frac{b\tau}{2}\sum_{i=0}^{N_1-1}\delta_x^2\theta_i\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}\right)\\
&\quad+\frac{b\tau}{2}\sum_{i=1}^{N_1}\delta_x^2\theta_i\left(\delta_x w^1_{i-\frac{1}{2}}-\delta_x w^0_{i-\frac{1}{2}}\right)-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w_0^1-w_0^0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w_0^1-w_0^0\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^1\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}-\delta_x w^1_{i-\frac{1}{2}}+\delta_x w^0_{i-\frac{1}{2}}\right)-\frac{b\tau}{2}\sum_{i=1}^{N_1}\delta_x^2\theta_i\left(\delta_x w^1_{i+\frac{1}{2}}-\delta_x w^0_{i+\frac{1}{2}}-\delta_x w^1_{i-\frac{1}{2}}+\delta_x w^0_{i-\frac{1}{2}}\right)\\
&\quad+\frac{b}{2}\delta_x^2 w_0^1\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)-\frac{b\tau}{2}\delta_x^2\theta_0\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w_0^1-w_0^0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w_0^1-w_0^0\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\left(\delta_x^2 w_i^1-\delta_x^2 w_i^0\right)-\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2\theta_i\left(\delta_x^2 w_i^1-\delta_x^2 w_i^0\right)+\frac{b}{2}\delta_x^2 w_0^1\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)\\
&\quad-\frac{b\tau}{2}\delta_x^2\theta_0\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w_0^1-w_0^0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w_0^1-w_0^0\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^1\right)^2-\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\delta_x^2 w_i^0-\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\delta_x^2\theta_i+\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2\theta_i\delta_x^2 w_i^0\\
&\quad+\frac{b}{2}\delta_x^2 w_0^1\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)-\frac{b\tau}{2}\delta_x^2\theta_0\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)-\frac{b}{2}\delta_x^3 w^1_{\frac{1}{2}}\left(w_0^1-w_0^0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w_0^1-w_0^0\right).
\end{aligned}$$

Noticing that $\delta_x^2 w_0^1=\xi^1$ and $\delta_x^3 w^1_{\frac{1}{2}}=-\zeta^1$, this last equality yields

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^1_{i+\frac{1}{2}}\left(w_i^1-w_i^0\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w^1_{i-\frac{1}{2}}\left(w_i^1-w_i^0\right)-\frac{b\tau}{2}\sum_{i=1}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w_i^1-w_i^0\right)+\frac{b\tau}{2}\sum_{i=1}^{N_1}\delta_x^3\theta_{i-\frac{1}{2}}\left(w_i^1-w_i^0\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^1\right)^2-\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\delta_x^2 w_i^0-\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\delta_x^2\theta_i+\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2\theta_i\delta_x^2 w_i^0\\
&\quad+\frac{b}{2}\xi^1\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)-\frac{b\tau}{2}\delta_x^2\theta_0\left(\delta_x w^1_{\frac{1}{2}}-\delta_x w^0_{\frac{1}{2}}\right)+\frac{b}{2}\zeta^1\left(w_0^1-w_0^0\right)+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\left(w_0^1-w_0^0\right).
\end{aligned}\tag{5.126}$$

Therefore, (5.126), together with (5.112), (5.113), (5.114) and (5.115), implies

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w^1_{i+\frac{1}{2}}\left(w_i^1-w_i^0\right)-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w^1_{i-\frac{1}{2}}\left(w_i^1-w_i^0\right)-\frac{b\tau}{2}\sum_{i=1}^{N_1-1}\delta_x^3\theta_{i+\frac{1}{2}}\left(w_i^1-w_i^0\right)+\frac{b\tau}{2}\sum_{i=1}^{N_1}\delta_x^3\theta_{i-\frac{1}{2}}\left(w_i^1-w_i^0\right)\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^1\right)^2-\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\delta_x^2 w_i^0-\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\delta_x^2\theta_i+\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2\theta_i\delta_x^2 w_i^0+\frac{b}{2}\left(\xi^1\right)^2-\frac{b}{2}\xi^1\xi^0+\frac{b\tau}{2}\left(\xi^1\right)^2\\
&\quad+\frac{b}{2}\left(\zeta^1\right)^2-\frac{b}{2}\zeta^1\zeta^0+\frac{b\tau}{2}\left(\zeta^1\right)^2-\frac{b\tau}{2}\delta_x^2\theta_0\xi^1+\frac{b\tau}{2}\delta_x^2\theta_0\xi^0-\frac{b\tau^2}{2}\delta_x^2\theta_0\xi^1+\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\zeta^1-\frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\zeta^0+\frac{b\tau^2}{2}\delta_x^3\theta_{\frac{1}{2}}\zeta^1.
\end{aligned}\tag{5.127}$$

Plugging (5.125) and (5.127) into (5.120) gives finally

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^1 - U_i^0}{\tau}\right)^2 + \frac{b}{2}\sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1\right)^2 + \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^1\right)^2 + \frac{a}{2}\left(\eta^1\right)^2 + \frac{b}{2}\left(\xi^1\right)^2 + \frac{b}{2}\left(\zeta^1\right)^2 \\
&\quad + \frac{a\tau}{2}\left(\eta^1\right)^2 + \frac{b\tau}{2}\left(\xi^1\right)^2 + \frac{b\tau}{2}\left(\zeta^1\right)^2 \\
&= \frac{b}{2}\sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 w_i^0 + \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^1 \delta_x u_{i+\frac{1}{2}}^0 + \frac{1}{\tau}\sum_{i=1}^{N_{\max}} h_i \Psi_i \left(U_i^1 - U_i^0\right) + \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 \theta_i \\
&\quad - \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i \delta_x^2 \theta_i \delta_x^2 w_i^0 + \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^1 \delta_x \psi_{i+\frac{1}{2}} - \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x \psi_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^0 \\
&\quad + \frac{a}{2}\eta^1\eta^0 + \frac{b}{2}\xi^1\xi^0 + \frac{b}{2}\zeta^1\zeta^0 - \frac{a\tau}{2}\delta_x \psi_{N_{\max}+\frac{1}{2}}\eta^1 + \frac{a\tau}{2}\delta_x \psi_{N_{\max}+\frac{1}{2}}\eta^0 - \frac{a\tau^2}{2}\delta_x \psi_{N_{\max}+\frac{1}{2}}\eta^1 \\
&\quad + \frac{b\tau}{2}\delta_x^2\theta_0\xi^1 - \frac{b\tau}{2}\delta_x^2\theta_0\xi^0 + \frac{b\tau^2}{2}\delta_x^2\theta_0\xi^1 - \frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\zeta^1 + \frac{b\tau}{2}\delta_x^3\theta_{\frac{1}{2}}\zeta^0 - \frac{b\tau^2}{2}\delta_x^3\theta_{\frac{1}{2}}\zeta^1.
\end{aligned} \tag{5.128}$$

On the right-hand side, we can use Young's inequality to estimate the first three terms of (5.128) in the following manner:

$$\frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^1 \delta_x u_{i+\frac{1}{2}}^0 \leq \frac{a}{8}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^1\right)^2 + \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^0\right)^2,$$

and

$$\frac{b}{2}\sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 w_i^0 \leq \frac{b}{8}\sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1\right)^2 + \frac{b}{2}\sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^0\right)^2,$$

with

$$\frac{1}{\tau}\sum_{i=1}^{N_{\max}} h_i \Psi_i \left(U_i^1 - U_i^0\right) \leq \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i \left(\frac{U_i^1 - U_i^0}{\tau}\right)^2 + \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i \left(\Psi_i\right)^2.$$

Similarly, applying Young's inequality once more to the next four terms on the right-hand side of (5.128) results in

$$\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^1 \delta_x \psi_{i+\frac{1}{2}} \leq \frac{a}{8}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^1\right)^2 + \frac{a\tau^2}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x \psi_{i+\frac{1}{2}}\right)^2,$$

with

$$\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x \psi_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^0 \leq \frac{a\tau^2}{8}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x \psi_{i+\frac{1}{2}}\right)^2 + \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u_{i+\frac{1}{2}}^0\right)^2,$$

$$\frac{b\tau}{2}\sum_{i=1}^{N_1} h_i \delta_x^2 w_i^1 \delta_x^2 \theta_i \leq \frac{b}{8}\sum_{i=1}^{N_1} h_i \left(\delta_x^2 w_i^1\right)^2 + \frac{b\tau^2}{2}\sum_{i=1}^{N_1} h_i \left(\delta_x^2 \theta_i\right)^2,$$

and

$$\frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\delta_x^2\theta_i\delta_x^2 w_i^0 \le \frac{b\tau^2}{8}\sum_{i=1}^{N_1} h_i\left(\delta_x^2\theta_i\right)^2 + \frac{b}{2}\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^0\right)^2.$$

Once more, applying Young's inequality to the following terms on the right-hand side of (5.128) leads to

$$\frac{a}{2}\eta^1\eta^0 \le \frac{a}{8}\left(\eta^1\right)^2 + \frac{a}{2}\left(\eta^0\right)^2,$$

with

$$\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^1 \le \frac{a}{8}\left(\eta^1\right)^2 + \frac{a\tau^2}{2}\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2,$$

$$\frac{a\tau}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^0 \le \frac{a\tau^2}{8}\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2 + \frac{a}{2}\left(\eta^0\right)^2,$$

and finally,

$$\frac{a\tau^2}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}}\eta^1 \le \frac{a\tau}{2}\left(\eta^1\right)^2 + \frac{a\tau^3}{8}\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2.$$

The remaining terms on the right-hand side of (5.128) can be estimated analogously. Replacing these estimates into (5.128) yields the following inequality:

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N_{\max}} h_i\left(\frac{U_i^1-U_i^0}{\tau}\right)^2 + \frac{b}{4}\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^1\right)^2 + \frac{a}{4}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^1\right)^2 + \frac{a}{4}\left(\eta^1\right)^2 + \frac{b}{4}\left(\xi^1\right)^2 + \frac{b}{4}\left(\zeta^1\right)^2 \\
&\le b\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^0\right)^2 + a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^0\right)^2 + \frac{5b\tau^2}{8}\sum_{i=1}^{N_1} h_i\left(\delta_x^2\theta_i\right)^2 + \frac{5a\tau^2}{8}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x\psi_{i+\frac{1}{2}}\right)^2 \\
&\quad + \frac{1}{2}\sum_{i=1}^{N_{\max}} h_i\left(\Psi_i\right)^2 + a\left(\eta^0\right)^2 + b\left(\xi^0\right)^2 + b\left(\zeta^0\right)^2 + \frac{5a\tau^2}{8}\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2 + \frac{a\tau^3}{8}\left(\delta_x\psi_{N_{\max}+\frac{1}{2}}\right)^2 \\
&\quad + \frac{5b\tau^2}{8}\left(\delta_x^2\theta_0\right)^2 + \frac{b\tau^3}{8}\left(\delta_x^2\theta_0\right)^2 + \frac{5b\tau^2}{8}\left(\delta_x^3\theta_{\frac{1}{2}}\right)^2 + \frac{b\tau^3}{8}\left(\delta_x^3\theta_{\frac{1}{2}}\right)^2.
\end{aligned} \tag{5.129}$$

Under the assumptions of the theorem, we immediately obtain

$$\begin{aligned}
\left\|\delta_t U_{\mathcal{T}}^1\right\|^2 + \frac{1}{2}\left\|U_{\mathcal{T}}^1\right\|_{\mathbb{H},\mathcal{T}}^2 + \frac{1}{2}\left\|\Xi^1\right\|_{\mathbb{R}^3}^2 \le 2\left\|\Phi_{\mathcal{T}}\right\|_{\mathbb{H},\mathcal{T}}^2 + \frac{5\tau^2}{4}\left\|\Psi_{\mathcal{T}}\right\|_{\mathbb{H},\mathcal{T}}^2 + \left\|\Psi_{\mathcal{T}}\right\|^2 + 2\left\|\Theta\right\|_{\mathbb{R}^3}^2 \\
+\frac{5\tau^2}{4}\left\|\vartheta_2\right\|_{\mathbb{R}^3}^2 + \frac{\tau^3}{4}\left\|\vartheta_2\right\|_{\mathbb{R}^3}^2 \le C,
\end{aligned} \tag{5.130}$$

where the constant C depends only on the initial data of the problem and on the constant τ_0 and does not depend on the discretization parameters. This concludes the proof of the theorem. ■

Now, let's go back to prove stability estimates for the solution in other time steps that fulfill (5.78)-(5.80). For this aim, we prove the following theorem:

Theorem 5.5.4. *Let **Hypothesis (H2)** and **Assumption (TA)** hold. Then, the numerical solution $U_{\mathcal{T},\tau}$ of the numerical scheme (5.78)-(5.80) is bounded independently on discretization parameters in $L^\infty\left([0,T];\mathbb{L}^2\right)\cap L^\infty\left([0,T];\mathbb{H}\right)$. Moreover, the first time derivative approximation is bounded in $L^\infty\left([0,T];\mathbb{L}^2\right)$ and the function Ξ_τ is bounded in $L^\infty\left([0,T];\mathbb{R}^3\right)$ as well. Therefore, the following estimation holds:*

$$\|\delta_t U_{\mathcal{T}}^m\|^2+\|U_{\mathcal{T}}^m\|^2+\frac{1}{2}\|U_{\mathcal{T}}^m\|_{\mathbb{H},\mathcal{T}}^2+\frac{1}{2}\|\Xi^m\|_{\mathbb{R}^3}^2\leq C,\qquad \text{for}\qquad m=1,\dots,\mathcal{N},\tag{5.131}$$

where C is a strictly positive constant independent of space and time discretization parameters.

Proof. From Theorem 5.5.2, we can have

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{n+1}-U_i^n}{\tau}\right)^2+\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{n+1}\right)^2+\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}\right)^2+\frac{a}{2}\left(\eta^{n+1}\right)^2+\frac{b}{2}\left(\xi^{n+1}\right)^2+\frac{b}{2}\left(\zeta^{n+1}\right)^2\\
&=\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{n}-U_i^{n-1}}{\tau}\right)^2+\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{n-1}\right)^2+\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n-1}\right)^2+\frac{a}{2}\left(\eta^{n-1}\right)^2+\frac{b}{2}\left(\xi^{n-1}\right)^2+\frac{b}{2}\left(\zeta^{n-1}\right)^2\\
&\quad-\frac{\tau}{2}\left(a|\eta^{n+1}+\eta^{n-1}|^2+b|\xi^{n+1}+\xi^{n-1}|^2+b|\zeta^{n+1}+\zeta^{n-1}|^2\right),
\end{aligned}$$

and therefore, using the fact that

$$-\frac{\tau}{2}\left(a|\eta^{n+1}+\eta^{n-1}|^2+b|\xi^{n+1}+\xi^{n-1}|^2+b|\zeta^{n+1}+\zeta^{n-1}|^2\right)\leq 0,$$

we obtain directly

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{n+1}-U_i^n}{\tau}\right)^2+\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{n+1}\right)^2+\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}\right)^2+\frac{a}{2}\left(\eta^{n+1}\right)^2+\frac{b}{2}\left(\xi^{n+1}\right)^2+\frac{b}{2}\left(\zeta^{n+1}\right)^2\\
&\leq\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{n}-U_i^{n-1}}{\tau}\right)^2+\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{n-1}\right)^2+\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n-1}\right)^2+\frac{a}{2}\left(\eta^{n-1}\right)^2+\frac{b}{2}\left(\xi^{n-1}\right)^2+\frac{b}{2}\left(\zeta^{n-1}\right)^2.
\end{aligned}$$

After summing for $n=1,\dots,m-1$, we get

$$\begin{aligned}
&\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{m}-U_i^{m-1}}{\tau}\right)^2+\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{m}\right)^2+\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{m}\right)^2+\frac{a}{2}\left(\eta^{m}\right)^2+\frac{b}{2}\left(\xi^{m}\right)^2+\frac{b}{2}\left(\zeta^{m}\right)^2\\
&\leq\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{m}-U_i^{m-1}}{\tau}\right)^2+\frac{1}{2}\sum_{\ell=m-1}^{m}\left(b\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{\ell}\right)^2+a\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{\ell}\right)^2+a\left(\eta^{\ell}\right)^2+b\left(\xi^{\ell}\right)^2+b\left(\zeta^{\ell}\right)^2\right)\\
&\leq\sum_{i=1}^{N_{\max}}h_i\left(\frac{U_i^{1}-U_i^{0}}{\tau}\right)^2+\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{0}\right)^2+\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{0}\right)^2+\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2w_i^{1}\right)^2+\frac{a}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{1}\right)^2\\
&\quad+\frac{a}{2}\left(\eta^{0}\right)^2+\frac{b}{2}\left(\xi^{0}\right)^2+\frac{b}{2}\left(\zeta^{0}\right)^2+\frac{a}{2}\left(\eta^{1}\right)^2+\frac{b}{2}\left(\xi^{1}\right)^2+\frac{b}{2}\left(\zeta^{1}\right)^2.
\end{aligned}$$

We can conclude the following estimate:

$$\|\delta_t U_{\mathcal{T}}^m\|^2+\frac{1}{2}\|U_{\mathcal{T}}^m\|_{\mathbb{H},\mathcal{T}}^2+\frac{1}{2}\|\Xi^m\|_{\mathbb{R}^3}^2\leq\left\|\delta_t U_{\mathcal{T}}^1\right\|^2+\frac{1}{2}\|\Phi_{\mathcal{T}}\|_{\mathbb{H},\mathcal{T}}^2+\frac{1}{2}\left\|U_{\mathcal{T}}^1\right\|_{\mathbb{H},\mathcal{T}}^2+\frac{1}{2}\|\Theta\|_{\mathbb{R}^3}^2+\frac{1}{2}\left\|\Xi^1\right\|_{\mathbb{R}^3}^2\leq C,$$

with C being a strictly positive constant independent of the discretization parameters and depending only on the initial data of the problem and the constant τ_0 due to the properties of the functions Φ,

Ψ and Θ and the results obtained in Theorem 5.5.3. Finally, using the Discrete Poincare's inequality, i.e., using Lemma 2.4.3, we immediately get

$$\begin{aligned}\|U_{\mathcal{T}}^m\|^2 &= \|u_{\mathcal{T}}^m\|^2 + \|w_{\mathcal{T}}^m\|^2 \\ &\leq c_1 \|u_{\mathcal{T}}^m\|_{1,\mathcal{T}}^2 + c_2 \|w_{\mathcal{T}}^m\|_{2,\mathcal{T}}^2 \\ &\leq \max\{c_1, c_2\} \left(\|u_{\mathcal{T}}^m\|_{1,\mathcal{T}}^2 + \|w_{\mathcal{T}}^m\|_{2,\mathcal{T}}^2\right) \\ &= \max\{c_1, c_2\} \|U_{\mathcal{T}}^m\|_{\mathbb{H},\mathcal{T}}^2,\end{aligned}$$

with $c_1,\ c_2$ are strictly positive constants, which, in turn, gives

$$\|U_{\mathcal{T}}^m\|^2 \leq C,$$

from the previous result and thus completes the proof of the theorem. ■

Convergence towards the Weak Solution

To prove the convergence of the numerical solution, we use the stability estimations obtained in the previous section, and passing through the limit, we can obtain the convergence of the approximate solution towards the weak solution. As an alternative to the integral identity (5.6)-(5.9), we may now reformulate our discrete scheme by writing the continuous weak formulation problem in its discrete form. First, we introduce the following space:

$$C_*^\infty([-L,0]) = \{z \in C^\infty([-L,0]) \quad | \quad z_x(0) = 0\}.$$

As we will use density arguments, let the function $V = (v,z) \in C^\infty([0,T]; C_*)$ where

$$C_* = \{(v,z) \in C^\infty([0,L]) \times C_*^\infty([-L,0]) \quad | \quad v(0) = z(0)\},$$

such that $V(x,T) = 0$ and let size$(\mathcal{T})$ be small enough. We denote by

$$\begin{aligned} v_i(t) &= \frac{1}{h_i}\int_{K_i} v(x,t)dx, \quad \text{for } i = N_1+1,\dots,N_{\max}, \\ z_i(t) &= \frac{1}{h_i}\int_{K_i} z(x,t)dx, \quad \text{for } i = 1,\dots,N_1. \end{aligned}$$

Let us also define

$$V_i = \begin{cases} v_i, & i = N_1+1,\dots,N_{\max}, \\ z_i, & i = 1,\dots,N_1. \end{cases}$$

Now, we want to get the discrete form of (5.6)-(5.9) passing through several steps:

First, multiplying both sides of the first Equation of the numerical scheme (5.78)-(5.80) by τV_i^n and summing up i from 1 to $N_{\max}$ and n from 1 to $\mathcal{N}-1$, we obtain

$$\begin{aligned}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}} h_i \frac{\left(U_i^{n+1}-U_i^n\right)-\left(U_i^n-U_i^{n-1}\right)}{\tau^2} V_i^n + \frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}\left(F_{i+\frac{1}{2}}^{n+1}-F_{i-\frac{1}{2}}^{n+1}\right)V_i^n \\ +\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}\left(F_{i+\frac{1}{2}}^{n-1}-F_{i-\frac{1}{2}}^{n-1}\right)V_i^n = 0,\end{aligned}$$

so that

$$\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}h_i\frac{\left(U_i^{n+1}-U_i^n\right)-\left(U_i^n-U_i^{n-1}\right)}{\tau^2}V_i^n+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}F_{i+\frac{1}{2}}^{n+1}V_i^n-\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}F_{i-\frac{1}{2}}^{n+1}V_i^n$$
$$+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}F_{i+\frac{1}{2}}^{n-1}V_i^n-\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}F_{i-\frac{1}{2}}^{n-1}V_i^n=0.$$

By (5.79), the numerical flux $F_{i+\frac{1}{2}}^n$ yields

$$\begin{aligned}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}}h_i\frac{\left(U_i^{n+1}-U_i^n\right)-\left(U_i^n-U_i^{n-1}\right)}{\tau^2}V_i^n&+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1-1}\delta_x^3w_{i+\frac{1}{2}}^{n+1}z_i^n-\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n+1}v_i^n\\
-\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1}\delta_x^3w_{i-\frac{1}{2}}^{n+1}z_i^n&+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i-\frac{1}{2}}^{n+1}v_i^n+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1-1}\delta_x^3w_{i+\frac{1}{2}}^{n-1}z_i^n\\
-\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n-1}v_i^n&-\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1}\delta_x^3w_{i-\frac{1}{2}}^{n-1}z_i^n+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i-\frac{1}{2}}^{n-1}v_i^n\\
-\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\delta_xu_{N_1+\frac{1}{2}}^{n+1}z_{N_1}^n&-\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\delta_xu_{N_1+\frac{1}{2}}^{n-1}z_{N_1}^n=0.\end{aligned}\tag{5.132}$$

The translation of index i for the second and fourth terms of the following equation gives

$$\begin{aligned}&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n+1}v_i^n+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i-\frac{1}{2}}^{n+1}v_i^n-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n-1}v_i^n+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i-\frac{1}{2}}^{n-1}v_i^n\\
&\quad-\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n+1}z_{N_1}^n-\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n-1}z_{N_1}^n\\
&=-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n+1}v_i^n+\frac{a}{2}\sum_{i=N_1}^{N_{\max}-1}\delta_xu_{i+\frac{1}{2}}^{n+1}v_{i+1}^n-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n-1}v_i^n+\frac{a}{2}\sum_{i=N_1}^{N_{\max}-1}\delta_xu_{i+\frac{1}{2}}^{n-1}v_{i+1}^n\\
&\quad-\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n+1}z_{N_1}^n-\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n-1}z_{N_1}^n\\
&=\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n+1}\left(v_{i+1}^n-v_i^n\right)+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}\delta_xu_{i+\frac{1}{2}}^{n-1}\left(v_{i+1}^n-v_i^n\right)+\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n+1}\left(v_{N_1+1}^n-z_{N_1}^n\right)\\
&\quad+\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n-1}\left(v_{N_1+1}^n-z_{N_1}^n\right)-\frac{a}{2}\delta_xu_{N_{\max}+\frac{1}{2}}^{n+1}v_{N_{\max}+1}^n-\frac{a}{2}\delta_xu_{N_{\max}+\frac{1}{2}}^{n-1}v_{N_{\max}+1}^n\\
&=\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_xu_{i+\frac{1}{2}}^{n+1}\delta_xv_{i+\frac{1}{2}}^n+\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_xu_{i+\frac{1}{2}}^{n-1}\delta_xv_{i+\frac{1}{2}}^n+\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n+1}\left(v_{N_1+1}^n-z_{N_1}^n\right)\\
&\quad+\frac{a}{2}\delta_xu_{N_1+\frac{1}{2}}^{n-1}\left(v_{N_1+1}^n-z_{N_1}^n\right)-\frac{a}{2}\delta_xu_{N_{\max}+\frac{1}{2}}^{n+1}v_{N_{\max}+1}^n-\frac{a}{2}\delta_xu_{N_{\max}+\frac{1}{2}}^{n-1}v_{N_{\max}+1}^n.\end{aligned}\tag{5.133}$$

Since $\delta_xz_{N_1+\frac{1}{2}}^n=0$, we have

$$z_{N_1+\frac{1}{2}}^n=z_{N_1}^n.\tag{5.134}$$

Thanks to (5.134) and the fact that $v^n_{N_1+\frac{1}{2}} = z^n_{N_1+\frac{1}{2}}$, one has

$$\delta_x v^n_{N_1+\frac{1}{2}} = \frac{v^n_{N_1+1} - z^n_{N_1}}{h_{N_1+\frac{1}{2}}}. \tag{5.135}$$

Therefore, (5.133) together with (5.135) implies that

$$\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i+\frac{1}{2}} v^n_i + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i-\frac{1}{2}} v^n_i - \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i+\frac{1}{2}} v^n_i + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i-\frac{1}{2}} v^n_i \\
&\quad - \frac{a}{2}\delta_x u^{n+1}_{N_1+\frac{1}{2}} z^n_{N_1} - \frac{a}{2}\delta_x u^{n-1}_{N_1+\frac{1}{2}} z^n_{N_1} \\
&= \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u^{n+1}_{i+\frac{1}{2}} \delta_x v^n_{i+\frac{1}{2}} + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u^{n-1}_{i+\frac{1}{2}} \delta_x v^n_{i+\frac{1}{2}} + \frac{a}{2} h_{N_1+\frac{1}{2}} \delta_x u^{n+1}_{N_1+\frac{1}{2}} \delta_x v^n_{N_1+\frac{1}{2}} \\
&\quad + \frac{a}{2} h_{N_1+\frac{1}{2}} \delta_x u^{n-1}_{N_1+\frac{1}{2}} \delta_x v^n_{N_1+\frac{1}{2}} - \frac{a}{2}\delta_x u^{n+1}_{N_{\max}+\frac{1}{2}} v^n_{N_{\max}+1} - \frac{a}{2}\delta_x u^{n-1}_{N_{\max}+\frac{1}{2}} v^n_{N_{\max}+1} \\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u^{n+1}_{i+\frac{1}{2}} \delta_x v^n_{i+\frac{1}{2}} + \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u^{n-1}_{i+\frac{1}{2}} \delta_x v^n_{i+\frac{1}{2}} - \frac{a}{2}\delta_x u^{n+1}_{N_{\max}+\frac{1}{2}} v^n_{N_{\max}+1} - \frac{a}{2}\delta_x u^{n-1}_{N_{\max}+\frac{1}{2}} v^n_{N_{\max}+1} \\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u^{n+1}_{i+\frac{1}{2}} + \delta_x u^{n-1}_{i+\frac{1}{2}}\right) \delta_x v^n_{i+\frac{1}{2}} - \frac{a}{2}\left(\delta_x u^{n+1}_{N_{\max}+\frac{1}{2}} + \delta_x u^{n-1}_{N_{\max}+\frac{1}{2}}\right) v^n_{N_{\max}+1}.
\end{aligned}$$

Noticing that $\delta_x u^n_{N_{\max}+\frac{1}{2}} = -\eta^n$, one has from the last equality that

$$\begin{aligned}
&-\frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i+\frac{1}{2}} v^n_i + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n+1}_{i-\frac{1}{2}} v^n_i - \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i+\frac{1}{2}} v^n_i + \frac{a}{2}\sum_{i=N_1+1}^{N_{\max}} \delta_x u^{n-1}_{i-\frac{1}{2}} v^n_i \\
&\quad - \frac{a}{2}\delta_x u^{n+1}_{N_1+\frac{1}{2}} z^n_{N_1} - \frac{a}{2}\delta_x u^{n-1}_{N_1+\frac{1}{2}} z^n_{N_1} \\
&= \frac{a}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \left(\delta_x u^{n+1}_{i+\frac{1}{2}} + \delta_x u^{n-1}_{i+\frac{1}{2}}\right) \delta_x v^n_{i+\frac{1}{2}} + \frac{a}{2}\left(\eta^{n+1} + \eta^{n-1}\right) v^n_{N_{\max}+1}.
\end{aligned} \tag{5.136}$$

Now, by the translation of index i for the second and fourth terms of the following equation, a direct

calculation gives

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}z_i^n-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n+1}z_i^n+\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}z_i^n-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n-1}z_i^n\\
&=\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}z_i^n-\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}z_{i+1}^n+\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}z_i^n-\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}z_{i+1}^n\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}\left(z_i^n-z_{i+1}^n\right)+\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}\left(z_i^n-z_{i+1}^n\right)-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}z_0^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}z_0^n\\
&=-\frac{b}{2}\sum_{i=0}^{N_1-1}h_{i+\frac{1}{2}}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}\delta_x z_{i+\frac{1}{2}}^n-\frac{b}{2}\sum_{i=0}^{N_1-1}h_{i+\frac{1}{2}}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}\delta_x z_{i+\frac{1}{2}}^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}z_0^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}z_0^n\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\left(\delta_x^2 w_i^{n+1}-\delta_x^2 w_{i+1}^{n+1}\right)\delta_x z_{i+\frac{1}{2}}^n+\frac{b}{2}\sum_{i=0}^{N_1-1}\left(\delta_x^2 w_i^{n-1}-\delta_x^2 w_{i+1}^{n-1}\right)\delta_x z_{i+\frac{1}{2}}^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}z_0^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}z_0^n\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_i^{n+1}\delta_x z_{i+\frac{1}{2}}^n-\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_{i+1}^{n+1}\delta_x z_{i+\frac{1}{2}}^n+\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_i^{n-1}\delta_x z_{i+\frac{1}{2}}^n-\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_{i+1}^{n-1}\delta_x z_{i+\frac{1}{2}}^n\\
&\quad-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}z_0^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}z_0^n.
\end{aligned}$$

Once again, the translation of index i for the second and fourth terms of the right-hand side of the above equation with the help of the fact that $\delta_x z_{N_1+\frac{1}{2}}^n=0$ gives

$$\begin{aligned}
&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1}z_i^n-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n+1}z_i^n+\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1}z_i^n-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n-1}z_i^n\\
&=\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_i^{n+1}\delta_x z_{i+\frac{1}{2}}^n-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n+1}\delta_x z_{i-\frac{1}{2}}^n+\frac{b}{2}\sum_{i=0}^{N_1-1}\delta_x^2 w_i^{n-1}\delta_x z_{i+\frac{1}{2}}^n-\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n-1}\delta_x z_{i-\frac{1}{2}}^n\\
&\quad-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}z_0^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}z_0^n\\
&=\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n+1}\left(\delta_x z_{i+\frac{1}{2}}^n-\delta_x z_{i-\frac{1}{2}}^n\right)+\frac{b}{2}\sum_{i=1}^{N_1}\delta_x^2 w_i^{n-1}\left(\delta_x z_{i+\frac{1}{2}}^n-\delta_x z_{i-\frac{1}{2}}^n\right)+\frac{b}{2}\delta_x^2 w_0^{n+1}\delta_x z_{\frac{1}{2}}^n+\frac{b}{2}\delta_x^2 w_0^{n-1}\delta_x z_{\frac{1}{2}}^n\\
&\quad-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}z_0^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}z_0^n\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n+1}\delta_x^2 z_i^n+\frac{b}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n-1}\delta_x^2 z_i^n+\frac{b}{2}\delta_x^2 w_0^{n+1}\delta_x z_{\frac{1}{2}}^n+\frac{b}{2}\delta_x^2 w_0^{n-1}\delta_x z_{\frac{1}{2}}^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n+1}z_0^n-\frac{b}{2}\delta_x^3 w_{\frac{1}{2}}^{n-1}z_0^n\\
&=\frac{b}{2}\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^{n+1}+\delta_x^2 w_i^{n-1}\right)\delta_x^2 z_i^n+\frac{b}{2}\left(\delta_x^2 w_0^{n+1}+\delta_x^2 w_0^{n-1}\right)\delta_x z_{\frac{1}{2}}^n-\frac{b}{2}\left(\delta_x^3 w_{\frac{1}{2}}^{n+1}+\delta_x^3 w_{\frac{1}{2}}^{n-1}\right)z_0^n.
\end{aligned}$$

Noticing that $\delta_x^2 w_0^n = \xi^n$ and $\delta_x^3 w_{\frac{1}{2}}^n = -\zeta^n$, this last equality gives

$$\begin{aligned}&\frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n+1} z_i^n - \frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n+1} z_i^n + \frac{b}{2}\sum_{i=1}^{N_1-1}\delta_x^3 w_{i+\frac{1}{2}}^{n-1} z_i^n - \frac{b}{2}\sum_{i=1}^{N_1}\delta_x^3 w_{i-\frac{1}{2}}^{n-1} z_i^n \\ &= \frac{b}{2}\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n+1} + \delta_x^2 w_i^{n-1}\right)\delta_x^2 z_i^n + \frac{b}{2}\left(\xi^{n+1}+\xi^{n-1}\right)\delta_x z_{\frac{1}{2}}^n + \frac{b}{2}\left(\zeta^{n+1}+\zeta^{n-1}\right) z_0^n.\end{aligned} \tag{5.137}$$

After these operations, plug (5.136) and (5.137) into (5.132) to obtain

$$\begin{aligned}&\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}} h_i \frac{\left(U_i^{n+1}-U_i^n\right)-\left(U_i^n-U_i^{n-1}\right)}{\tau^2} V_i^n + \frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1} + \delta_x u_{i+\frac{1}{2}}^{n-1}\right)\delta_x v_{i+\frac{1}{2}}^n \\ &+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n+1}+\delta_x^2 w_i^{n-1}\right)\delta_x^2 z_i^n + \frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\eta^{n+1}+\eta^{n-1}\right) v_{N_{\max}+1}^n \\ &+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\xi^{n+1}+\xi^{n-1}\right)\delta_x z_{\frac{1}{2}}^n + \frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\zeta^{n+1}+\zeta^{n-1}\right) z_0^n = 0.\end{aligned} \tag{5.138}$$

Now, we rearrange all the terms on the left-hand side of (5.138). The translation of index n for the first term of (5.138) and the fact that $V_i^{\mathcal{N}} = 0$ achieves

$$\begin{aligned}&\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}} h_i \frac{\left(U_i^{n+1}-U_i^n\right)-\left(U_i^n-U_i^{n-1}\right)}{\tau^2} V_i^n \\ &= \sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}} h_i \frac{U_i^{n+1}-U_i^n}{\tau^2} V_i^n - \sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}} h_i \frac{U_i^n-U_i^{n-1}}{\tau^2} V_i^n \\ &= \sum_{n=2}^{\mathcal{N}}\tau\sum_{i=1}^{N_{\max}} h_i \frac{U_i^n-U_i^{n-1}}{\tau^2} V_i^{n-1} - \sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_{\max}} h_i \frac{U_i^n-U_i^{n-1}}{\tau^2} V_i^n \\ &= -\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=1}^{N_{\max}} h_i \frac{\left(U_i^n-U_i^{n-1}\right)\left(V_i^n-V_i^{n-1}\right)}{\tau^2} - \tau\sum_{i=1}^{N_{\max}} h_i \frac{\left(U_i^1-U_i^0\right)V_i^1}{\tau^2}.\end{aligned}$$

For the second and third terms of (5.138), we apply the following modifications to get

$$
\begin{aligned}
&\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}+\delta_x u_{i+\frac{1}{2}}^{n-1}\right)\delta_x v_{i+\frac{1}{2}}^{n}+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n+1}+\delta_x^2 w_i^{n-1}\right)\delta_x^2 z_i^n\\
&=\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\delta_x v_{i+\frac{1}{2}}^{n}+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\,\delta_x u_{i+\frac{1}{2}}^{n-1}\delta_x v_{i+\frac{1}{2}}^{n}+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n+1}\delta_x^2 z_i^n\\
&\quad+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n-1}\delta_x^2 z_i^n\\
&=\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(\delta_x v_{i+\frac{1}{2}}^{n+1}+\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n+1}\right)+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\,\delta_x u_{i+\frac{1}{2}}^{n-1}\left(\delta_x v_{i+\frac{1}{2}}^{n-1}+\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n-1}\right)\\
&\quad+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 z_i^{n+1}+\delta_x^2 z_i^{n}-\delta_x^2 z_i^{n+1}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 z_i^{n-1}+\delta_x^2 z_i^{n}-\delta_x^2 z_i^{n-1}\right)\\
&=\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\delta_x v_{i+\frac{1}{2}}^{n+1}+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n-1}\delta_x v_{i+\frac{1}{2}}^{n-1}+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n+1}\delta_x^2 z_i^{n+1}\\
&\quad+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n-1}\delta_x^2 z_i^{n-1}+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n+1}\right)\\
&\quad+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 z_i^{n}-\delta_x^2 z_i^{n+1}\right)\\
&\quad+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 z_i^{n}-\delta_x^2 z_i^{n-1}\right).
\end{aligned}
$$

By the translation of index n for the first, second, third, and fourth terms of the right-hand side of

the above equation, we obtain

$$
\begin{aligned}
&\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}+\delta_x u_{i+\frac{1}{2}}^{n-1}\right)\delta_x v_{i+\frac{1}{2}}^{n}+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1}h_i\left(\delta_x^2 w_i^{n+1}+\delta_x^2 w_i^{n-1}\right)\delta_x^2 z_i^n\\
&=\frac{a}{2}\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n}\delta_x v_{i+\frac{1}{2}}^{n}+\frac{a}{2}\sum_{n=0}^{\mathcal{N}-2}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n}\delta_x v_{i+\frac{1}{2}}^{n}+\frac{b}{2}\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^n\delta_x^2 z_i^n\\
&\quad+\frac{b}{2}\sum_{n=0}^{\mathcal{N}-2}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^n\delta_x^2 z_i^n+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n+1}\right)\\
&\quad+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 z_i^n-\delta_x^2 z_i^{n+1}\right)\\
&\quad+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 z_i^n-\delta_x^2 z_i^{n-1}\right)\\
&=a\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n}\delta_x v_{i+\frac{1}{2}}^{n}+\frac{a}{2}\sum_{n=0}^{1}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n}\delta_x v_{i+\frac{1}{2}}^{n}+b\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^n\delta_x^2 z_i^n\\
&\quad+\frac{b}{2}\sum_{n=0}^{1}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^n\delta_x^2 z_i^n+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n+1}\right)\\
&\quad+\frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(\delta_x v_{i+\frac{1}{2}}^{n}-\delta_x v_{i+\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 z_i^n-\delta_x^2 z_i^{n+1}\right)\\
&\quad+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 z_i^n-\delta_x^2 z_i^{n-1}\right).
\end{aligned}
$$

By using analogous modifications, the remaining terms of (5.138) may also be expressed in a similar manner

$$
\begin{aligned}
&\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\eta^{n+1}+\eta^{n-1}\right)v_{N_{\max}+1}^{n}+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\xi^{n+1}+\xi^{n-1}\right)\delta_x z_{\frac{1}{2}}^{n}+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\zeta^{n+1}+\zeta^{n-1}\right)z_0^n\\
&=a\sum_{n=2}^{\mathcal{N}}\tau\eta^n v_{N_{\max}+1}^{n}+\frac{a}{2}\sum_{n=0}^{1}\tau\eta^n v_{N_{\max}+1}^{n}+b\sum_{n=2}^{\mathcal{N}}\tau\xi^n\delta_x z_{\frac{1}{2}}^{n}+\frac{b}{2}\sum_{n=0}^{1}\tau\xi^n\delta_x z_{\frac{1}{2}}^{n}+b\sum_{n=2}^{\mathcal{N}}\tau\zeta^n z_0^n+\frac{b}{2}\sum_{n=0}^{1}\tau\zeta^n z_0^n\\
&\quad+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}\left(v_{N_{\max}+1}^{n}-v_{N_{\max}+1}^{n+1}\right)+\frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau\eta^{n-1}\left(v_{N_{\max}+1}^{n}-v_{N_{\max}+1}^{n-1}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\xi^{n+1}\left(\delta_x z_{\frac{1}{2}}^{n}-\delta_x z_{\frac{1}{2}}^{n+1}\right)\\
&\quad+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\xi^{n-1}\left(\delta_x z_{\frac{1}{2}}^{n}-\delta_x z_{\frac{1}{2}}^{n-1}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\zeta^{n+1}\left(z_0^n-z_0^{n+1}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\zeta^{n-1}\left(z_0^n-z_0^{n-1}\right).
\end{aligned}
$$

Now, after these operations, we obtain from (5.138)

$$
\begin{aligned}
-\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=1}^{N_{\max}}h_i\frac{\left(U_i^n-U_i^{n-1}\right)\left(V_i^n-V_i^{n-1}\right)}{\tau^2}-\tau\sum_{i=1}^{N_{\max}}h_i\frac{\left(U_i^1-U_i^0\right)V_i^1}{\tau^2}+a\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^n\delta_x v_{i+\frac{1}{2}}^n\\
+\frac{a}{2}\sum_{n=0}^{1}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^n\delta_x v_{i+\frac{1}{2}}^n+b\sum_{n=2}^{\mathcal{N}}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^n\delta_x^2 z_i^n+\frac{b}{2}\sum_{n=0}^{1}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^n\delta_x^2 z_i^n\\
+a\sum_{n=2}^{\mathcal{N}}\tau\eta^n v_{N_{\max}+1}^n+\frac{a}{2}\sum_{n=0}^{1}\tau\eta^n v_{N_{\max}+1}^n+b\sum_{n=2}^{\mathcal{N}}\tau\xi^n\delta_x z_{\frac{1}{2}}^n+\frac{b}{2}\sum_{n=0}^{1}\tau\xi^n\delta_x z_{\frac{1}{2}}^n\\
+b\sum_{n=2}^{\mathcal{N}}\tau\zeta^n z_0^n+\frac{b}{2}\sum_{n=0}^{1}\tau\zeta^n z_0^n=R_1,
\end{aligned}
\tag{5.139}
$$

where the other rest terms R_1 are in the form

$$
\begin{aligned}
R_1=&\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(\delta_x v_{i+\frac{1}{2}}^{n+1}-\delta_x v_{i+\frac{1}{2}}^n\right)+\frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(\delta_x v_{i+\frac{1}{2}}^{n-1}-\delta_x v_{i+\frac{1}{2}}^n\right)\\
&+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 z_i^{n+1}-\delta_x^2 z_i^n\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 z_i^{n-1}-\delta_x^2 z_i^n\right)\\
&+\frac{a}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}\left(v_{N_{\max}+1}^{n+1}-v_{N_{\max}+1}^n\right)+\frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau\eta^{n-1}\left(v_{N_{\max}+1}^{n+1}-v_{N_{\max}+1}^n\right)\\
&+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\xi^{n+1}\left(\delta_x z_{\frac{1}{2}}^{n+1}-\delta_x z_{\frac{1}{2}}^n\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\xi^{n-1}\left(\delta_x z_{\frac{1}{2}}^{n+1}-\delta_x z_{\frac{1}{2}}^n\right)\\
&+\frac{b}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\zeta^{n+1}\left(z_0^{n+1}-z_0^n\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\zeta^{n-1}\left(z_0^{n+1}-z_0^n\right).
\end{aligned}
$$

Now, we multiply both sides of the Equation (5.84) by τV_i^0 and sum up i from 1 to $N_{\max}$. By using the same method of translating the index i, a direct computation yields the following result

$$
\begin{aligned}
&\tau\sum_{i=1}^{N_{\max}}h_i\frac{\left(U_i^1-U_i^0\right)V_i^0}{\tau^2}-\sum_{i=1}^{N_{\max}}h_i\Psi_i V_i^0+\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^1\delta_x v_{i+\frac{1}{2}}^0\\
&-\frac{a\tau^2}{2}\sum_{i=N_1}^{N_{\max}}h_{i+\frac{1}{2}}\delta_x\psi_{i+\frac{1}{2}}\delta_x v_{i+\frac{1}{2}}^0+\frac{b\tau}{2}\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^1\delta_x^2 z_i^0-\frac{b\tau^2}{2}\sum_{i=1}^{N_1}h_i\delta_x^2\theta_i\delta_x^2 z_i^0\\
&+\frac{a\tau}{2}\eta^1 v_{N_{\max}+1}^0+\frac{a\tau^2}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}}v_{N_{\max}+1}^0+\frac{b\tau}{2}\xi^1\delta_x z_{\frac{1}{2}}^0-\frac{b\tau^2}{2}\delta_x^2\theta_0\delta_x z_{\frac{1}{2}}^0+\frac{b\tau}{2}\zeta^1 z_0^0+\frac{b\tau^2}{2}\delta_x^3\theta_{\frac{1}{2}}z_0^0=0.
\end{aligned}
$$

Further, we modify the following terms as follows:

$$\begin{aligned}
&\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\delta_x v^0_{i+\frac{1}{2}} + \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^1\delta_x^2 z_i^0\\
&= \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\left(\delta_x v^1_{i+\frac{1}{2}} + \delta_x v^0_{i+\frac{1}{2}} - \delta_x v^1_{i+\frac{1}{2}}\right) + \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^1\left(\delta_x^2 z_i^1 + \delta_x^2 z_i^0 - \delta_x^2 z_i^1\right)\\
&= \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\delta_x v^1_{i+\frac{1}{2}} + \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^1\delta_x^2 z_i^1 + \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\left(\delta_x v^0_{i+\frac{1}{2}} - \delta_x v^1_{i+\frac{1}{2}}\right)\\
&\quad + \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^1\left(\delta_x^2 z_i^0 - \delta_x^2 z_i^1\right).
\end{aligned}$$

Similarly, we can express the following term as

$$\begin{aligned}
\frac{a\tau}{2}\eta^1 v^0_{N_{\max}+1} + \frac{b\tau}{2}\xi^1\delta_x z^0_{\frac{1}{2}} + \frac{b\tau}{2}\zeta^1 z_0^0 &= \frac{a\tau}{2}\eta^1 v^1_{N_{\max}+1} + \frac{b\tau}{2}\xi^1\delta_x z^1_{\frac{1}{2}} + \frac{b\tau}{2}\zeta^1 z_0^1\\
&\quad + \frac{a\tau}{2}\eta^1\left(v^0_{N_{\max}+1} - v^1_{N_{\max}+1}\right) + \frac{b\tau}{2}\xi^1\left(\delta_x z^0_{\frac{1}{2}} - \delta_x z^1_{\frac{1}{2}}\right) + \frac{b\tau}{2}\zeta^1\left(z_0^0 - z_0^1\right).
\end{aligned}$$

These modifications lead us to the following rearranged equation

$$\begin{aligned}
&\tau\sum_{i=1}^{N_{\max}} h_i\frac{\left(U_i^1 - U_i^0\right)V_i^0}{\tau^2} - \sum_{i=1}^{N_{\max}} h_i\Psi_i V_i^0 + \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\delta_x v^1_{i+\frac{1}{2}}\\
&- \frac{a\tau^2}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x\psi_{i+\frac{1}{2}}\delta_x v^0_{i+\frac{1}{2}} + \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^1\delta_x^2 z_i^1 - \frac{b\tau^2}{2}\sum_{i=1}^{N_1} h_i\delta_x^2\theta_i\delta_x^2 z_i^0\\
&+ \frac{a\tau}{2}\eta^1 v^1_{N_{\max}+1} + \frac{a\tau^2}{2}\delta_x\psi_{N_{\max}+\frac{1}{2}} v^0_{N_{\max}+1} + \frac{b\tau}{2}\xi^1\delta_x z^1_{\frac{1}{2}} - \frac{b\tau^2}{2}\delta_x^2\theta_0\delta_x z^0_{\frac{1}{2}} + \frac{b\tau}{2}\zeta^1 z_0^1 + \frac{b\tau^2}{2}\delta_x^3\theta_{\frac{1}{2}} z_0^0 = R_2,
\end{aligned} \tag{5.140}$$

where R_2 is the rest in the form

$$\begin{aligned}
R_2 =& \frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^1_{i+\frac{1}{2}}\left(\delta_x v^1_{i+\frac{1}{2}} - \delta_x v^0_{i+\frac{1}{2}}\right) + \frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^1\left(\delta_x^2 z_i^1 - \delta_x^2 z_i^0\right)\\
&+ \frac{a\tau}{2}\eta^1\left(v^1_{N_{\max}+1} - v^0_{N_{\max}+1}\right) + \frac{b\tau}{2}\xi^1\left(\delta_x z^1_{\frac{1}{2}} - \delta_x z^0_{\frac{1}{2}}\right) + \frac{b\tau}{2}\zeta^1\left(z_0^1 - z_0^0\right).
\end{aligned}$$

Summing (5.139) and (5.140), we obtain

$$\begin{aligned}
&-\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=1}^{N_{\max}} h_i\frac{\left(U_i^n - U_i^{n-1}\right)\left(V_i^n - V_i^{n-1}\right)}{\tau^2} - \sum_{i=1}^{N_{\max}} h_i\Psi_i V_i^0 + a\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\delta_x v^n_{i+\frac{1}{2}}\\
&+ b\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^n\delta_x^2 z_i^n + a\sum_{n=1}^{\mathcal{N}}\tau\eta^n v^n_{N_{\max}+1} + b\sum_{n=1}^{\mathcal{N}}\tau\xi^n\delta_x z^n_{\frac{1}{2}} + b\sum_{n=1}^{\mathcal{N}}\tau\zeta^n z_0^n = R,
\end{aligned} \tag{5.141}$$

where

$$
\begin{aligned}
R =& \frac{a}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac12}\delta_x u_{i+\frac12}^{n+1}\left(\delta_x v_{i+\frac12}^{n+1}-\delta_x v_{i+\frac12}^{n}\right)+\frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=N_1}^{N_{\max}} h_{i+\frac12}\delta_x u_{i+\frac12}^{n-1}\left(\delta_x v_{i+\frac12}^{n-1}-\delta_x v_{i+\frac12}^{n}\right)\\
&+\frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 z_i^{n+1}-\delta_x^2 z_i^{n}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\sum_{i=1}^{N_1} h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 z_i^{n-1}-\delta_x^2 z_i^{n}\right)\\
&+\frac{a}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\eta^{n+1}\left(v_{N_{\max}+1}^{n+1}-v_{N_{\max}+1}^{n}\right)+\frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau\eta^{n-1}\left(v_{N_{\max}+1}^{n-1}-v_{N_{\max}+1}^{n}\right)+\frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\xi^{n+1}\left(\delta_x z_{\frac12}^{n+1}-\delta_x z_{\frac12}^{n}\right)\\
&+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\xi^{n-1}\left(\delta_x z_{\frac12}^{n-1}-\delta_x z_{\frac12}^{n}\right)+\frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\zeta^{n+1}\left(z_0^{n+1}-z_0^{n}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau\zeta^{n-1}\left(z_0^{n-1}-z_0^{n}\right)\\
&+\frac{a\tau}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac12}\left(\tau\delta_x\psi_{i+\frac12}-\delta_x u_{i+\frac12}^{0}\right)\delta_x v_{i+\frac12}^{0}+\frac{b\tau}{2}\sum_{i=1}^{N_1} h_i\left(\tau\delta_x^2\theta_i-\delta_x^2 w_i^{0}\right)\delta_x^2 z_i^{0}\\
&-\frac{a\tau}{2}\left(\tau\delta_x\psi_{N_{\max}+\frac12}+\eta^{0}\right)v_{N_{\max}+1}^{0}+\frac{b\tau}{2}\left(\tau\delta_x^2\theta_0-\xi^{0}\right)\delta_x z_{\frac12}^{0}-\frac{b\tau}{2}\left(\tau\delta_x^3\theta_{\frac12}+\zeta^{0}\right)z_0^{0}.
\end{aligned}
$$

Now, using the notations introduced in the previous sections, we have from (5.141) that

$$
\begin{aligned}
&-\int_0^T\int_{-L}^{L}\delta_t U_{\mathcal{T},\tau}\delta_t V_{\mathcal{T},\tau}dxdt+a\int_0^T\sum_{i=N_1}^{N_{\max}} h_{i+\frac12}\delta_x u_{i+\frac12,\tau}\delta_x v_{i+\frac12,\tau}dt+b\int_0^T\sum_{i=1}^{N_1} h_i\delta_x^2 w_{i,\tau}\delta_x^2 z_{i,\tau}dt\\
&\qquad-\int_{-L}^{L}\Psi_{\mathcal{T}}V_{\mathcal{T}}^{0}dx+a\int_0^T\eta_\tau v_{N_{\max}+1,\tau}dt+b\int_0^T\xi_\tau\delta_x z_{\frac12,\tau}dt+b\int_0^T\zeta_\tau z_{0,\tau}dt=R.
\end{aligned}
$$

The above equation can be reformulated as the following

$$
\begin{aligned}
&-\int_0^T\int_0^{L}\delta_t u_{\mathcal{T},\tau}\delta_t v_{\mathcal{T},\tau}dxdt+a\int_0^T\sum_{i=N_1}^{N_{\max}} h_{i+\frac12}\delta_x u_{i+\frac12,\tau}\delta_x v_{i+\frac12,\tau}dt-\int_0^{L}\psi_{\mathcal{T}}v_{\mathcal{T}}^{0}dx\\
&\qquad-\int_0^T\int_{-L}^{0}\delta_t w_{\mathcal{T},\tau}\delta_t z_{\mathcal{T},\tau}dxdt+b\int_0^T\sum_{i=1}^{N_1} h_i\delta_x^2 w_{i,\tau}\delta_x^2 z_{i,\tau}dt-\int_{-L}^{0}\theta_{\mathcal{T}}z_{\mathcal{T}}^{0}dx\\
&\qquad+a\int_0^T\eta_\tau v_{N_{\max}+1,\tau}dt+b\int_0^T\xi_\tau\delta_x z_{\frac12,\tau}dt+b\int_0^T\zeta_\tau z_{0,\tau}dt=R.
\end{aligned}
\tag{5.142}
$$

Next, we consider the function $(p,q,r)\in C^\infty\left([0,T];\mathbb{R}^3\right)$ with $p(T)=q(T)=r(T)=0$. Our objective is to derive the discrete representation of equations (5.7)-(5.9) by passing through several steps.

Remark 5.5.1. *In the discrete formulation of (5.7)-(5.9), we employ an index translation of n. To ensure continuity, we extend the functions p, q, and r by 0 for $t>T$.*

Multiplying both sides of the first equation in (5.80) by τp^n and summing up n from 1 to $\mathcal{N}-1$, we obtain

$$
\sum_{n=1}^{\mathcal{N}-1}\tau\frac{\eta^{n+1}-\eta^{n-1}}{2\tau}p^n-\sum_{n=1}^{\mathcal{N}-1}\tau\frac{u_{N_{\max}+1}^{n+1}-u_{N_{\max}+1}^{n-1}}{2\tau}p^n+\frac12\sum_{n=1}^{\mathcal{N}-1}\tau\left(\eta^{n+1}+\eta^{n-1}\right)p^n=0.
\tag{5.143}
$$

The index translation for the first two terms in (5.143), along with the conditions $p^{\mathcal{N}+1}=p^{\mathcal{N}}=0$,

leads to

$$\begin{aligned}
&\sum_{n=1}^{\mathcal{N}-1}\tau\frac{\eta^{n+1}-\eta^{n-1}}{2\tau}p^n-\sum_{n=1}^{\mathcal{N}-1}\tau\frac{u^{n+1}_{N_{\max}+1}-u^{n-1}_{N_{\max}+1}}{2\tau}p^n\\
&\quad=\sum_{n=2}^{\mathcal{N}}\tau\frac{\eta^n}{2\tau}p^{n-1}-\sum_{n=0}^{\mathcal{N}-2}\tau\frac{\eta^n}{2\tau}p^{n+1}-\sum_{n=2}^{\mathcal{N}}\tau\frac{u^n_{N_{\max}+1}}{2\tau}p^{n-1}+\sum_{n=0}^{\mathcal{N}-2}\tau\frac{u^n_{N_{\max}+1}}{2\tau}p^{n+1}\\
&\quad=-\sum_{n=2}^{\mathcal{N}}\tau\eta^n\left(\frac{p^{n+1}-p^{n-1}}{2\tau}\right)-\sum_{n=0}^{1}\tau\frac{\eta^n}{2\tau}p^{n+1}+\sum_{n=2}^{\mathcal{N}}\tau u^n_{N_{\max}+1}\left(\frac{p^{n+1}-p^{n-1}}{2\tau}\right)+\sum_{n=0}^{1}\tau\frac{u^n_{N_{\max}+1}}{2\tau}p^{n+1}.
\end{aligned}$$

Regarding the third term in (5.143), we apply the following modifications

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\eta^{n+1}+\eta^{n-1}\right)p^n=\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}p^n+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n-1}p^n\\
&\quad=\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}\left(p^{n+1}+p^n-p^{n+1}\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n-1}\left(p^{n-1}+p^n-p^{n-1}\right)\\
&\quad=\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}p^{n+1}+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n-1}p^{n-1}+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}\left(p^n-p^{n+1}\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n-1}\left(p^n-p^{n-1}\right).
\end{aligned}$$

By the translation of index n for the first two terms of the right-hand side of the above equation, we derive

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\left(\eta^{n+1}+\eta^{n-1}\right)p^n&=\frac{1}{2}\sum_{n=2}^{\mathcal{N}}\tau\eta^np^n+\frac{1}{2}\sum_{n=0}^{\mathcal{N}-2}\tau\eta^np^n+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}\left(p^n-p^{n+1}\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n-1}\left(p^n-p^{n-1}\right)\\
&=\sum_{n=2}^{\mathcal{N}}\tau\eta^np^n+\frac{1}{2}\sum_{n=0}^{1}\tau\eta^np^n+\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}\left(p^n-p^{n+1}\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}}\tau\eta^{n-1}\left(p^n-p^{n-1}\right).
\end{aligned}$$

Following these operations on (5.143), we derive

$$\begin{aligned}
-\sum_{n=2}^{\mathcal{N}}\tau\eta^n\left(\frac{p^{n+1}-p^{n-1}}{2\tau}\right)-\sum_{n=0}^{1}\tau\frac{\eta^n}{2\tau}p^{n+1}+\sum_{n=2}^{\mathcal{N}}\tau u^n_{N_{\max}+1}\left(\frac{p^{n+1}-p^{n-1}}{2\tau}\right)+\sum_{n=0}^{1}\tau\frac{u^n_{N_{\max}+1}}{2\tau}p^{n+1}\\
+\sum_{n=2}^{\mathcal{N}}\tau\eta^np^n+\frac{1}{2}\sum_{n=0}^{1}\tau\eta^np^n=R_3,
\end{aligned}\tag{5.144}$$

where the remaining terms R_3 take the form

$$R_3=\frac{1}{2}\sum_{n=1}^{\mathcal{N}-1}\tau\eta^{n+1}\left(p^{n+1}-p^n\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}}\tau\eta^{n-1}\left(p^{n-1}-p^n\right).$$

Now, multiplying both sides of the first equation of (5.85) by τp^0, we obtain

$$\tau\frac{\eta^1}{2\tau}p^0-\tau\frac{\eta^0}{2\tau}p^0-\tau\frac{u^1_{N_{\max}+1}}{2\tau}p^0+\tau\frac{u^0_{N_{\max}+1}}{2\tau}p^0+\frac{1}{2}\tau\eta^1p^0=0.$$

Further modification of the following term yields

$$\frac{1}{2}\tau\eta^1p^0=\frac{1}{2}\tau\eta^1\left(p^1+p^0-p^1\right)=\frac{1}{2}\tau\eta^1p^1+\frac{1}{2}\tau\eta^1\left(p^0-p^1\right).$$

These modifications lead to the rearranged equation

$$\tau\frac{\eta^1}{2\tau}p^0-\eta^0p^0-\tau\frac{u^1_{N_{\max}+1}}{2\tau}p^0+u^0_{N_{\max}+1}p^0+\frac{1}{2}\tau\eta^1p^1=R_4, \tag{5.145}$$

where R_4 represents the remaining terms in the form

$$R_4=\frac{1}{2}\tau\eta^1\left(p^1-p^0\right)-\tau\frac{\eta^0}{2\tau}p^0+\tau\frac{u^0_{N_{\max}+1}}{2\tau}p^0.$$

Summing equations (5.144) and (5.145), we obtain

$$-\sum_{n=1}^{\mathcal{N}}\tau\eta^n\left(\frac{p^{n+1}-p^{n-1}}{2\tau}\right)-\eta^0p^0+\sum_{n=1}^{\mathcal{N}}\tau u^n_{N_{\max}+1}\left(\frac{p^{n+1}-p^{n-1}}{2\tau}\right)+u^0_{N_{\max}+1}p^0+\sum_{n=1}^{\mathcal{N}}\tau\eta^np^n=R', \tag{5.146}$$

where

$$R'=\frac{1}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\eta^{n+1}\left(p^{n+1}-p^n\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}}\tau\eta^{n-1}\left(p^{n-1}-p^n\right)+\frac{1}{2}\eta^0\left(p^1-p^0\right)+\frac{1}{2}u^0_{N_{\max}+1}\left(p^0-p^1\right)-\frac{1}{2}\tau\eta^0p^0.$$

Using the notations introduced earlier, we can deduce from (5.146) that

$$-\int_0^T\eta_\tau\delta_t^*p_\tau dt-\eta^0p^0+\int_0^T u_{N_{\max}+1,\tau}\delta_t^*p_\tau dt+u^0_{N_{\max}+1}p^0+\int_0^T\eta_\tau p_\tau dt=R'. \tag{5.147}$$

By employing similar computations for the second and third equations of (5.80), we derive

$$-\int_0^T\xi_\tau\delta_t^*q_\tau dt-\xi^0q^0+\int_0^T\delta_xw_{\frac{1}{2},\tau}\delta_t^*q_\tau dt+\delta_xw^0_{\frac{1}{2}}q^0+\int_0^T\xi_\tau q_\tau dt=R'', \tag{5.148}$$

$$-\int_0^T\zeta_\tau\delta_t^*r_\tau dt-\zeta^0r^0+\int_0^T w_{0,\tau}\delta_t^*r_\tau dt+w_0^0r^0+\int_0^T\zeta_\tau r_\tau dt=R''', \tag{5.149}$$

where

$$R''=\frac{1}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\xi^{n+1}\left(q^{n+1}-q^n\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}}\tau\xi^{n-1}\left(q^{n-1}-q^n\right)+\frac{1}{2}\xi^0\left(q^1-q^0\right)+\frac{1}{2}\delta_xw^0_{\frac{1}{2}}\left(q^0-q^1\right)-\frac{1}{2}\tau\xi^0q^0,$$

and

$$R'''=\frac{1}{2}\sum_{n=0}^{\mathcal{N}-1}\tau\zeta^{n+1}\left(r^{n+1}-r^n\right)+\frac{1}{2}\sum_{n=1}^{\mathcal{N}}\tau\zeta^{n-1}\left(r^{n-1}-r^n\right)+\frac{1}{2}\zeta^0\left(r^1-r^0\right)+\frac{1}{2}w_0^0\left(r^0-r^1\right)-\frac{1}{2}\tau\zeta^0r^0.$$

The convergence result is given in the following theorem:

Theorem 5.5.5. *Assume that **Hypothesis (H2)** holds. For $m\in\mathbb{N}$, let $\mathcal{T}_m$ be an admissible mesh described in the sense of Definition 5.3.1. Let τ_m be a time step satisfying **Assumption (TA)**. Let $(u_{\mathcal{T}_m,\tau_m},w_{\mathcal{T}_m,\tau_m},\eta_{\tau_m},\xi_{\tau_m},\zeta_{\tau_m})$ be the solution of the system described by (5.84)-(5.85) and (5.78)-(5.80). Assume that size$(\mathcal{T}_m)\longrightarrow 0$ and $\tau_m\longrightarrow 0$ as $m\longrightarrow\infty$. Then, there exists a subsequence of the sequence of approximate solutions, still denoted by $(u_{\mathcal{T}_m,\tau_m},w_{\mathcal{T}_m,\tau_m},\eta_{\tau_m},\xi_{\tau_m},\zeta_{\tau_m})$, which converges, in the weak-$\star$ topology, to the weak solution (u,w,η,ξ,ζ) in $L^\infty([0,T];L^2(0,L))\times L^\infty([0,T];L^2(-L,0))\times\left(L^\infty([0,T])\right)^3$ of the problem (5.6)-(5.9), as m tends to ∞.*

Proof. Thanks to the Theorems 5.5.3 and 5.5.4, we have that the sequence $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m}, \eta_{\tau_m}, \xi_{\tau_m}, \zeta_{\tau_m})$ is bounded in $L^\infty([0,T];L^2(0,L))\times L^\infty([0,T];L^2(-L,0))\times L^\infty\left([0,T];\mathbb{R}^3\right)$. Then, there exists a subsequence, denoted again as $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m}, \eta_{\tau_m}, \xi_{\tau_m}, \zeta_{\tau_m})$, such that $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m}, \eta_{\tau_m}, \xi_{\tau_m}, \zeta_{\tau_m})$ converges to the function (u, w, η, ξ, ζ) in $L^\infty([0,T];L^2(0,L)) \times L^\infty([0,T];L^2(-L,0)) \times L^\infty\left([0,T];\mathbb{R}^3\right)$ as $m \longrightarrow \infty$ for the weak-$\star$ topology. Additionally, following Theorem 6.1 of [18], there exists $(u,w) \in C^0([0,T];\mathbb{L}^2)$ with $(u(\cdot,0), w(\cdot,0)) = (\varphi(\cdot), \phi(\cdot))$ and a subsequence of approximation solutions, again represented as $(u_{\mathcal{T}_m,\tau_m}, w_{\mathcal{T}_m,\tau_m})$, which converges to (u,w) in $L^\infty([0,T];\mathbb{L}^2)$. Furthermore, due to the stability assumption in the discrete $\mathbb{H}$-norm, it follows that $(u,w) \in L^\infty([0,T],\mathbb{H})$. In addition, the stability results show that $u_{\mathcal{T}_m,\tau_m}$ and $\delta_x u_{\mathcal{T}_m,\tau_m}$ are bounded in $L^\infty([0,T];L^2(0,L))$. To establish time compactness, we apply Simon's compactness criterion (see Theorem 1, page 71 of [54]), which characterizes compactness in $L^\infty([0,T];L^2(0,L))$. It follows that the sequences $\{u_{\mathcal{T}_m,\tau_m}\}$ and $\{\delta_x u_{\mathcal{T}_m,\tau_m}\}$ are relatively compact in $C([0,T];L^2(0,L))$. Therefore, there exist subsequences (still denoted by the same notation) such that

$$u_{\mathcal{T}_m,\tau_m} \to u \quad \text{and} \quad \delta_x u_{\mathcal{T}_m,\tau_m} \to u_x \quad \text{in} \quad L^\infty([0,T];L^2(0,L)), \quad \text{as} \quad m \to \infty.$$

Analogously, from the stability estimates, the functions $w_{\mathcal{T}_m,\tau_m}$, $\delta_x w_{\mathcal{T}_m,\tau_m}$ and $\delta_x^2 w_{\mathcal{T}_m,\tau_m}$ are bounded in $L^\infty([0,T];L^2(-L,0))$. Consequently, there exist subsequences (still denoted the same) such that

$$w_{\mathcal{T}_m,\tau_m} \to w, \quad \delta_x w_{\mathcal{T}_m,\tau_m} \to w_x, \quad \delta_x^2 w_{\mathcal{T}_m,\tau_m} \to w_{xx} \quad \text{in} \quad L^\infty([0,T];L^2(-L,0)), \quad \text{as} \quad m \to \infty.$$

Now, we proceed to demonstrate the convergence of the numerical solution to the weak solution of our problem. Our objective is to prove that (u,w,η,ξ,ζ) is the weak solution of problem (5.6)-(5.9). First, using the notation for subsequence, we denote the terms of (5.142) and (5.147)-(5.149) in the following form:

$$
\begin{aligned}
T_1 &= -\int_0^T \int_0^L \delta_t u_{\mathcal{T}_m,\tau_m} \delta_t v_{\mathcal{T}_m,\tau_m} dx dt, \\
T_2 &= -\int_0^T \int_{-L}^0 \delta_t w_{\mathcal{T}_m,\tau_m} \delta_t z_{\mathcal{T}_m,\tau_m} dx dt, \\
T_3 &= a\int_0^T \sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2},\tau_m} \delta_x v_{i+\frac{1}{2},\tau_m} dt, \\
T_4 &= b\int_0^T \sum_{i=1}^{N_1} h_i \delta_x^2 w_{i,\tau_m} \delta_x^2 z_{i,\tau_m} dt, \\
T_5 &= -\int_0^L \psi_{\mathcal{T}_m} v^0_{\mathcal{T}_m} dx, \\
T_6 &= -\int_{-L}^0 \theta_{\mathcal{T}_m} z^0_{\mathcal{T}_m} dx, \\
T_7 &= a\int_0^T \eta_{\tau_m} v_{N_{\max}+1,\tau_m} dt, \\
T_8 &= b\int_0^T \xi_{\tau_m} \delta_x z_{\frac{1}{2},\tau_m} dt, \\
T_9 &= b\int_0^T \zeta_{\tau_m} z_{0,\tau_m} dt, \\
T_{10} &= -\int_0^T \eta_{\tau_m} \delta_t^* p_{\tau_m} dt, \\
T_{11} &= \int_0^T u_{N_{\max}+1,\tau_m} \delta_t^* p_{\tau_m} dt, \\
T_{12} &= \int_0^T \eta_{\tau_m} p_{\tau_m} dt, \\
T_{13} &= -\int_0^T \xi_{\tau_m} \delta_t^* q_{\tau_m} dt, \\
T_{14} &= \int_0^T \delta_x w_{\frac{1}{2},\tau_m} \delta_t^* q_{\tau_m} dt, \\
T_{15} &= \int_0^T \xi_{\tau_m} q_{\tau_m} dt, \\
T_{16} &= -\int_0^T \zeta_{\tau_m} \delta_t^* r_{\tau_m} dt, \\
T_{17} &= \int_0^T w_{0,\tau_m} \delta_t^* r_{\tau_m} dt, \\
T_{18} &= \int_0^T \zeta_{\tau_m} r_{\tau_m} dt.
\end{aligned}
$$

The proof of Theorem 5.5.5 is divided into several steps:

Step 1. First, remark that as $m \longrightarrow \infty$, the regularity properties of the functions v and z allows us to observe the strong convergence of $\delta_t v_{\mathcal{T}_m,\tau_m}$ to v_t and $\delta_t z_{\mathcal{T}_m,\tau_m}$ to z_t in $L^2([0,T]\times(0,L))$ and $L^2([0,T]\times(-L,0))$ respectively. However, since $\delta_t U_{\mathcal{T}_m,\tau_m}$ is bounded in $L^\infty([0,T];\mathbb{L}^2)$, we can deduce that $\delta_t u_{\mathcal{T}_m,\tau_m} \rightharpoonup u_t$ and $\delta_t w_{\mathcal{T}_m,\tau_m} \rightharpoonup w_t$ in $L^2([0,T]\times(0,L))$ and $L^2([0,T]\times(-L,0))$ respectively as $m \longrightarrow \infty$. This implies the following

$$T_1 \longrightarrow -\int_0^T\int_0^L u_t v_t dxdt, \qquad \text{as } m \longrightarrow \infty,$$

$$T_2 \longrightarrow -\int_0^T\int_{-L}^0 w_t z_t dxdt, \qquad \text{as } m \longrightarrow \infty.$$

Similar results are also achieved for the terms T_5 and T_6 by using the previous arguments for v and z (indeed, v and z are regular enough) along with the fact that $(\psi,\theta)\in H^2(0,L)\times H^4(-L,0)$, which implies that $\psi_{\mathcal{T}_m}\longrightarrow\psi$ in $L^2(0,L)$ and $\theta_{\mathcal{T}_m}\longrightarrow\theta$ in $L^2(-L,0)$. We deduce that

$$T_5 \longrightarrow -\int_0^L \psi(x)v(x,0)dx, \qquad \text{as } m \longrightarrow \infty,$$

and

$$T_6 \longrightarrow -\int_{-L}^0 \theta(x)z(x,0)dx, \qquad \text{as } m \longrightarrow \infty.$$

Step 2. Let us now turn to the study of T_3 and T_4. For the term T_3, we introduce an auxiliary expression:

$$\tilde{T_3} = -a\int_0^T\int_0^L u_{\mathcal{T}_m,\tau_m}\ v_{xx}dxdt + a\int_0^T u_{N_{\max}+1,\tau_m}\ v_x(L)dt - a\int_0^T u_{N_1+\frac{1}{2},\tau_m}\ v_x(0)dt.$$

It is important to note that $x_{N_1+\frac{1}{2}} = 0$ and $x_{N_{\max}+1} = L$. Since $u_{\mathcal{T}_m,\tau_m}\to u$ in $L^\infty([0,T];L^2(0,L))$, and $\delta_x u_{\mathcal{T}_m,\tau_m}\to u_x$ in $L^\infty([0,T];L^2(0,L))$ as $m\to\infty$, it follows that $u_{\mathcal{T}_m,\tau_m}(0)$ and $u_{\mathcal{T}_m,\tau_m}(L)$ converge to $u(0)$ and $u(L)$, respectively, in $L^\infty([0,T])$. Hence, we have

$$\begin{aligned} u_{N_1+\frac{1}{2},\tau_m} &\longrightarrow u(0),\\ u_{N_{\max}+1,\tau_m} &\longrightarrow u(L),\end{aligned}$$

in $L^\infty([0,T])$ as $m\longrightarrow\infty$. Moreover, since $u_{\mathcal{T}_m,\tau_m}\to u$ in $L^\infty([0,T];L^2(0,L))$, we immediately obtain

$$\tilde{T_3} \longrightarrow -a\int_0^T\int_0^L u\ v_{xx}dxdt + a\int_0^T u(L)\ v_x(L)dt - a\int_0^T u(0)\ v_x(0)dt = a\int_0^T\int_0^L u_x\ v_x dxdt.$$

Now, let's reformulate $\tilde{T_3}$ as the following:

$$\begin{aligned}\tilde{T_3} = &-a\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=N_1+1}^{N_{\max}} u_i^n\int_{K_i} v_{xx}(x,t_n)dx + a\sum_{n=1}^{\mathcal{N}}\tau_m u^n_{N_{\max}+1}\ v_x(L,t_n) - a\sum_{n=1}^{\mathcal{N}}\tau_m u^n_{N_1+\frac{1}{2}}\ v_x(0,t_n)\\ &- a\int_0^T\int_0^L u_{\mathcal{T}_m,\tau_m}\ \left(v_{xx}(x,t)-v_{xx}(x,t_n)\right)dxdt + a\int_0^T u_{N_{\max}+1,\tau_m}\ \left(v_x(L,t)-v_x(L,t_n)\right)dt\\ &- a\int_0^T u_{N_1+\frac{1}{2},\tau_m}\ \left(v_x(0,t)-v_x(0,t_n)\right)dt = \tilde{T}_{31}+\tilde{T}_{32}+\tilde{T}_{33}.\end{aligned}$$

The functions $u_{N_1+\frac{1}{2},\tau_m}$ and $u_{N_{\max}+1,\tau_m}$ are bounded in $L^\infty([0,T])$. Moreover, since $u_{\mathcal{T}_m,\tau_m}$ is bounded in $L^\infty([0,T],L^2(0,L))$ and the function v is sufficiently smooth, we can immediately obtain

$$\tilde{T}_{33} \longrightarrow 0, \qquad \text{as } m \longrightarrow \infty.$$

We now express $\tilde{T}_{31}$ in the following form using integral formulation:

$$\tilde{T}_{31} = -a\sum_{n=1}^{\mathcal{N}}\tau_m \sum_{i=N_1+1}^{N_{\max}} u_i^n \int_{K_i} v_{xx}(x,t_n)dx = -a\sum_{n=1}^{\mathcal{N}}\tau_m \sum_{i=N_1+1}^{N_{\max}} u_i^n\ \delta_x v^n_{i+\frac{1}{2}} + a\sum_{n=1}^{\mathcal{N}}\tau_m \sum_{i=N_1+1}^{N_{\max}} u_i^n\ \delta_x v^n_{i-\frac{1}{2}}.$$

By translating the index i for the second term of the right-hand side, we can rewrite this as

$$\tilde{T}_{31} = a\sum_{n=1}^{\mathcal{N}}\tau_m \sum_{i=N_1+1}^{N_{\max}-1} h_{i+\frac{1}{2}}\delta_x u^n_{i+\frac{1}{2}}\delta_x v^n_{i+\frac{1}{2}} - a\sum_{n=1}^{\mathcal{N}}\tau_m u_{N_{\max}}\delta_x v^n_{N_{\max}+\frac{1}{2}} + a\sum_{n=1}^{\mathcal{N}}\tau_m u^n_{N_1+1}\delta_x v^n_{N_1+\frac{1}{2}}.$$

Moreover, for $\tilde{T}_{32}$

$$\begin{aligned}\tilde{T}_{32} &= a\sum_{n=1}^{\mathcal{N}}\tau_m u^n_{N_{\max}+1}\ v_x(L,t_n) - a\sum_{n=1}^{\mathcal{N}}\tau_m u^n_{N_1+\frac{1}{2}}\ v_x(0,t_n)\\ &= a\sum_{n=1}^{\mathcal{N}}\tau_m u^n_{N_{\max}+1}\ \delta_x v^n_{N_{\max}+\frac{1}{2}} - a\sum_{n=1}^{\mathcal{N}}\tau_m u^n_{N_1+\frac{1}{2}}\ \delta_x v^n_{N_1+\frac{1}{2}}.\end{aligned}$$

As a result, we can deduce

$$\tilde{T}_3 = T_3 + \tilde{T}_{33}.$$

Summing up all the results, we obtain

$$T_3 \longrightarrow a\int_0^T\int_0^L u_x v_x dxdt, \qquad \text{as } m \longrightarrow \infty.$$

For the term T_4, we employ analogous arguments, yielding the following auxiliary expression:

$$\begin{aligned}\tilde{T}_4 = b\int_0^T\int_{-L}^0 w_{\mathcal{T}_m,\tau_m}\ z_{xxxx}dxdt - b\int_0^T w_{N_1+\frac{1}{2},\tau_m}\ z_{xxx}(0)dt + b\int_0^T w_{0,\tau_m}\ z_{xxx}(-L)dt\\ -b\int_0^T \delta_x w_{\frac{1}{2},\tau_m}\ z_{xx}(-L)dt.\end{aligned}$$

We recall that $x_0 = x_{\frac{1}{2}} = -L$ and $x_{N_1+\frac{1}{2}} = 0$. Since $w_{\mathcal{T}_m,\tau_m} \to w$ in $L^\infty([0,T];L^2(-L,0))$ as $m \to \infty$, and similarly, $\delta_x w_{\mathcal{T}_m,\tau_m} \to w_x$ and $\delta_x^2 w_{\mathcal{T}_m,\tau_m} \to w_{xx}$ in the same space, we deduce the following convergence in $L^\infty([0,T])$: $w_{\mathcal{T}_m,\tau_m}(-L) \longrightarrow w(-L)$, $w_{\mathcal{T}_m,\tau_m}(0) \longrightarrow w(0)$, and $\delta_x w_{\mathcal{T}_m,\tau_m}(-L) \longrightarrow w_x(-L)$. Consequently, we obtain

$$\begin{aligned} w_{0,\tau_m} &\longrightarrow w(-L),\\ w_{N_1+\frac{1}{2},\tau_m} &\longrightarrow w(0),\\ \delta_x w_{\frac{1}{2},\tau_m} &\longrightarrow w_x(-L),\end{aligned}$$

in $L^\infty([0,T])$ as $m \longrightarrow \infty$. Furthermore, since $w_{\mathcal{T}_m,\tau_m} \longrightarrow w$ in $L^\infty([0,T];L^2(-L,0))$, we can immediately deduce

$$\tilde{T}_4 \longrightarrow b\int_0^T\int_{-L}^0 w\ z_{xxxx}dxdt - b\int_0^T w(0)\ z_{xxx}(0)dt + b\int_0^T w(-L)\ z_{xxx}(-L)dt$$
$$-b\int_0^T w_x(-L)\ z_{xx}(-L)dt = b\int_0^T\int_{-L}^0 w_{xx}\ z_{xx}dxdt.$$

Once again, we re-express $\tilde{T}_4$ as follows:

$$\tilde{T}_4 = b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=1}^{N_1} w_i^n\int_{K_i} z_{xxxx}(x,t_n)dx - b\sum_{n=1}^{\mathcal{N}}\tau_m w^n_{N_1+\frac{1}{2}}\ z_{xxx}(0,t_n) + b\sum_{n=1}^{\mathcal{N}}\tau_m w_0^n\ z_{xxx}(-L,t_n)$$
$$- b\sum_{n=1}^{\mathcal{N}}\tau_m\delta_x w^n_{\frac{1}{2}}\ z_{xx}(-L,t_n) + b\int_0^T\int_{-L}^0 w_{\mathcal{T}_m,\tau_m}\ \left(z_{xxxx}(x,t) - z_{xxxx}(x,t_n)\right)dxdt$$
$$- b\int_0^T w_{N_1+\frac{1}{2},\tau_m}\ \left(z_{xxx}(0,t) - z_{xxx}(0,t_n)\right)dt + b\int_0^T w_{0,\tau_m}\ \left(z_{xxx}(-L,t) - z_{xxx}(-L,t_n)\right)dt$$
$$- b\int_0^T \delta_x w_{\frac{1}{2},\tau_m}\ \left(z_{xx}(-L,t) - z_{xx}(-L,t_n)\right)dt = \tilde{T}_{41} + \tilde{T}_{42} + \tilde{T}_{43}.$$

The functions w_{0,τ_m}, $w_{N_1+\frac{1}{2},\tau_m}$ and $\delta_x w_{\frac{1}{2},\tau_m}$ are bounded in $L^\infty([0,T])$. Furthermore, given the boundedness of $w_{\mathcal{T}_m,\tau_m}$ in $L^\infty([0,T],L^2(-L,0))$ and the smoothness of the function z (which is sufficiently regular), we directly obtain

$$\tilde{T}_{43} \longrightarrow 0, \qquad \text{as } m \longrightarrow \infty.$$

We reformulate $\tilde{T}_{41}$ as follows using integral formulation:

$$\tilde{T}_{41} = b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=1}^{N_1} w_i^n\int_{K_i} z_{xxxx}(x,t_n)dx = b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=1}^{N_1} w_i^n\ \delta_x^3 z^n_{i+\frac{1}{2}} - b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=1}^{N_1} w_i^n\ \delta_x^3 z^n_{i-\frac{1}{2}}.$$

Using the translation of index i for the second term of the right-hand side of the above equation, we obtain

$$\tilde{T}_{41} = -b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=0}^{N_1-1} h_{i+\frac{1}{2}}\ \delta_x w^n_{i+\frac{1}{2}}\ \delta_x^3 z^n_{i+\frac{1}{2}} - b\sum_{n=1}^{\mathcal{N}}\tau_m w_0^n\ \delta_x^3 z^n_{\frac{1}{2}} + b\sum_{n=1}^{\mathcal{N}}\tau_m w^n_{N_1}\ \delta_x^3 z^n_{N_1+\frac{1}{2}}$$
$$= b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=0}^{N_1-1}\ \delta_x w^n_{i+\frac{1}{2}}\ \delta_x^2 z_i^n - b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=0}^{N_1-1}\ \delta_x w^n_{i+\frac{1}{2}}\ \delta_x^2 z^n_{i+1} - b\sum_{n=1}^{\mathcal{N}}\tau_m w_0^n\ \delta_x^3 z^n_{\frac{1}{2}} + b\sum_{n=1}^{\mathcal{N}}\tau_m w^n_{N_1}\ \delta_x^3 z^n_{N_1+\frac{1}{2}}.$$

Again, using the translation of index i for the second term of the right-hand side of the above equation and applying the fact that $\delta_x w_{N_1+\frac{1}{2}} = 0$, we obtain

$$\tilde{T}_{41} = b\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=1}^{N_1} h_i\ \delta_x^2 w_i^n\ \delta_x^2 z_i^n + b\sum_{n=1}^{\mathcal{N}}\tau_m\delta_x w^n_{\frac{1}{2}}\ \delta_x^2 z_0^n - b\sum_{n=1}^{\mathcal{N}}\tau_m w_0^n\ \delta_x^3 z^n_{\frac{1}{2}} + b\sum_{n=1}^{\mathcal{N}}\tau_m w^n_{N_1}\ \delta_x^3 z^n_{N_1+\frac{1}{2}}.$$

We can also express $\tilde{T}_{42}$ as follows

$$\tilde{T}_{42} = -b\sum_{n=1}^{\mathcal{N}}\tau_m w^n_{N_1+\frac{1}{2}}\ z_{xxx}(0,t_n) + b\sum_{n=1}^{\mathcal{N}}\tau_m w_0^n\ z_{xxx}(-L,t_n) - b\sum_{n=1}^{\mathcal{N}}\tau_m\delta_x w^n_{\frac{1}{2}}\ z_{xx}(-L,t_n)$$
$$= -b\sum_{n=1}^{\mathcal{N}}\tau_m w^n_{N_1+\frac{1}{2}}\ \delta_x^3 z^n_{N_1+\frac{1}{2}} + b\sum_{n=1}^{\mathcal{N}}\tau_m w_0^n\ \delta_x^3 z^n_{\frac{1}{2}} - b\sum_{n=1}^{\mathcal{N}}\tau_m\delta_x w^n_{\frac{1}{2}}\ \delta_x^2 z_0^n.$$

Thus, by using the fact that $\delta_x w_{N_1+\frac{1}{2}} = 0$, we conclude

$$\tilde{T}_4 = T_4 + \tilde{T}_{43}.$$

In summary, when summing up all the previous results, we can derive

$$T_4 \longrightarrow b\int_0^T \int_{-L}^0 w_{xx} z_{xx} dx dt, \qquad \text{as } m \longrightarrow \infty.$$

Step 3. Due to the regularity of the functions v and z, we observe that $v_{N_{\max}+1,\tau_m}$ converges to $v(L)$, $\delta_x z_{\frac{1}{2},\tau_m}$ converges to $z_x(-L)$ and z_{0,τ_m} converges to $z(-L)$ in $L^1([0,T])$ as $m \longrightarrow \infty$. This can be established through Lebesgue's dominated convergence theorem, given that $v_{N_{\max}+1,\tau_m}$, $\delta_x z_{\frac{1}{2},\tau_m}$, and z_{0,τ_m} are bounded real sequences, thereby converging pointwise to $v(L)$, $z_x(-L)$, and $z(-L)$, respectively, in the interval $[0,T]$ as m tends to infinity. Additionally, considering the convergence $\eta_{\tau_m} \longrightarrow \eta$, $\xi_{\tau_m} \longrightarrow \xi$ and $\zeta_{\tau_m} \longrightarrow \zeta$ in $L^\infty([0,T])$ under the weak-$\star$ topology, we obtain the following results:

$$\begin{aligned}
T_7 &\longrightarrow a\int_0^T \eta(t) v(L,t) dt, \qquad \text{as } m \longrightarrow \infty,\\
T_8 &\longrightarrow b\int_0^T \xi(t) z_x(-L,t) dt, \qquad \text{as } m \longrightarrow \infty,\\
T_9 &\longrightarrow b\int_0^T \zeta(t) z(-L,t) dt, \qquad \text{as } m \longrightarrow \infty.
\end{aligned}$$

Step 4. The regularity properties of functions p, q and r, along with the convergence of $\eta_{\tau_m} \longrightarrow \eta$, $\xi_{\tau_m} \longrightarrow \xi$ and $\zeta_{\tau_m} \longrightarrow \zeta$ in $L^\infty([0,T])$ under the weak-$\star$ topology, allows us to observe that

$$\begin{aligned}
T_{10} &\longrightarrow -\int_0^T \eta p_t dt, \qquad \text{as } m \longrightarrow \infty,\\
T_{12} &\longrightarrow \int_0^T \eta p dt, \qquad \text{as } m \longrightarrow \infty,\\
T_{13} &\longrightarrow -\int_0^T \xi q_t dt, \qquad \text{as } m \longrightarrow \infty,\\
T_{15} &\longrightarrow \int_0^T \xi q dt, \qquad \text{as } m \longrightarrow \infty,\\
T_{16} &\longrightarrow -\int_0^T \zeta r_t dt, \qquad \text{as } m \longrightarrow \infty,\\
T_{18} &\longrightarrow \int_0^T \zeta r dt, \qquad \text{as } m \longrightarrow \infty.
\end{aligned}$$

Moreover, with $u_{N_{\max}+1,\tau_m} \longrightarrow u(L)$, $\delta_x w_{\frac{1}{2},\tau_m} \longrightarrow w_x(-L)$ and $w_{0,\tau_m} \longrightarrow w(-L)$ in $L^\infty([0,T])$, we

deduce

$$\begin{aligned} T_{11} &\longrightarrow \int_0^T u(L,t)p_t(t)dt, \qquad \text{as } m \longrightarrow \infty, \\ T_{14} &\longrightarrow \int_0^T w_x(-L,t)q_t(t)dt, \qquad \text{as } m \longrightarrow \infty, \\ T_{17} &\longrightarrow \int_0^T w(-L,t)r_t(t)dt, \qquad \text{as } m \longrightarrow \infty. \end{aligned}$$

Step 5. Finally, we need to show that

$$\begin{aligned} R_m &\longrightarrow 0, \qquad \text{as } m \longrightarrow \infty, \\ R_m^{'} &\longrightarrow 0, \qquad \text{as } m \longrightarrow \infty, \\ R_m^{''} &\longrightarrow 0, \qquad \text{as } m \longrightarrow \infty, \\ R_m^{'''} &\longrightarrow 0, \qquad \text{as } m \longrightarrow \infty. \end{aligned}$$

We denote the terms of R_m as follows:

$$A_1 = \frac{a}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n+1}\left(\delta_x v_{i+\frac{1}{2}}^{n+1}-\delta_x v_{i+\frac{1}{2}}^{n}\right)+\frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n+1}\left(\delta_x^2 z_i^{n+1}-\delta_x^2 z_i^n\right),$$

$$A_2 = \frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\delta_x u_{i+\frac{1}{2}}^{n-1}\left(\delta_x v_{i+\frac{1}{2}}^{n-1}-\delta_x v_{i+\frac{1}{2}}^{n}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau_m\sum_{i=1}^{N_1}h_i\delta_x^2 w_i^{n-1}\left(\delta_x^2 z_i^{n-1}-\delta_x^2 z_i^n\right),$$

$$A_3 = \frac{a}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\eta^{n+1}\left(v_{N_{\max}+1}^{n+1}-v_{N_{\max}+1}^{n}\right)+\frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\xi^{n+1}\left(\delta_x z_{\frac{1}{2}}^{n+1}-\delta_x z_{\frac{1}{2}}^{n}\right)+\frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\zeta^{n+1}\left(z_0^{n+1}-z_0^n\right),$$

$$A_4 = \frac{a}{2}\sum_{n=1}^{\mathcal{N}}\tau_m\eta^{n-1}\left(v_{N_{\max}+1}^{n-1}-v_{N_{\max}+1}^{n}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau_m\xi^{n-1}\left(\delta_x z_{\frac{1}{2}}^{n-1}-\delta_x z_{\frac{1}{2}}^{n}\right)+\frac{b}{2}\sum_{n=1}^{\mathcal{N}}\tau_m\zeta^{n-1}\left(z_0^{n-1}-z_0^n\right),$$

$$A_5 = \frac{a\tau_m}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\tau_m\delta_x\psi_{i+\frac{1}{2}}-\delta_x u_{i+\frac{1}{2}}^0\right)\delta_x v_{i+\frac{1}{2}}^0+\frac{b\tau_m}{2}\sum_{i=1}^{N_1}h_i\left(\tau_m\delta_x^2\theta_i-\delta_x^2 w_i^0\right)\delta_x^2 z_i^0,$$

$$A_6 = -\frac{a\tau_m}{2}\left(\tau_m\delta_x\psi_{N_{\max}+\frac{1}{2}}+\eta^0\right)v_{N_{\max}+1}^0+\frac{b\tau_m}{2}\left(\tau_m\delta_x^2\theta_0-\xi^0\right)\delta_x z_{\frac{1}{2}}^0-\frac{b\tau_m}{2}\left(\tau_m\delta_x^3\theta_{\frac{1}{2}}+\zeta^0\right)z_0^0.$$

The two terms A_1 and A_2 can be estimated similarly, so we do it for the first term A_1. We can bound

the first term as follows using the Cauchy-Schwarz inequality:

$$
\begin{aligned}
|A_1| &\leq \frac{a}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left|\delta_x u_{i+\frac{1}{2}}^{n+1}\right|\left|\delta_x v_{i+\frac{1}{2}}^{n+1}-\delta_x v_{i+\frac{1}{2}}^{n}\right| + \frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\sum_{i=1}^{N_1} h_i\left|\delta_x^2 w_i^{n+1}\right|\left|\delta_x^2 z_i^{n+1}-\delta_x^2 z_i^{n}\right| \\
&\leq \frac{1}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left(a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u_{i+\frac{1}{2}}^{n+1}\right)^2\right)^{\frac{1}{2}}\left(a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_t\delta_x v_{i+\frac{1}{2}}^{n+1}\right)^2\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left(b\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^{n+1}\right)^2\right)^{\frac{1}{2}}\left(b\sum_{i=1}^{N_1} h_i\left(\delta_t\delta_x^2 z_i^{n+1}\right)^2\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left\|u_{\mathcal{T}_m}^{n+1}\right\|_{1,\mathcal{T}_m}\left\|\delta_t v_{\mathcal{T}_m}^{n+1}\right\|_{1,\mathcal{T}_m} + \frac{1}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left\|w_{\mathcal{T}_m}^{n+1}\right\|_{2,\mathcal{T}_m}\left\|\delta_t z_{\mathcal{T}_m}^{n+1}\right\|_{2,\mathcal{T}_m} \\
&\leq \sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left\|U_{\mathcal{T}_m}^{n+1}\right\|_{\mathbb{H},\mathcal{T}_m}\left\|\delta_t V_{\mathcal{T}_m}^{n+1}\right\|_{\mathbb{H},\mathcal{T}_m} \\
&= \tau_m\int_0^T \left\|U_{\mathcal{T}_m,\tau_m}\right\|_{\mathbb{H},\mathcal{T}_m}\left\|\delta_t V_{\mathcal{T}_m,\tau_m}\right\|_{\mathbb{H},\mathcal{T}_m} dt.
\end{aligned}
$$

Considering the regularity properties of the function V and the stability estimates, we find that $A_1 \longrightarrow 0$ as $m \longrightarrow \infty$ (with $\tau_m \longrightarrow 0$). Next, A_3 and A_4 can be estimated analogously, so let us begin by examining the term A_3. We can bound A_3 as follows:

$$
\begin{aligned}
|A_3| &\leq \frac{a}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\left|\eta^{n+1}\right|\left|v_{N_{\max}+1}^{n+1}-v_{N_{\max}+1}^{n}\right| + \frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\left|\xi^{n+1}\right|\left|\delta_x z_{\frac{1}{2}}^{n+1}-\delta_x z_{\frac{1}{2}}^{n}\right| + \frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m\left|\zeta^{n+1}\right|\left|z_0^{n+1}-z_0^{n}\right| \\
&= \frac{a}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left|\eta^{n+1}\right|\left|\delta_t v_{N_{\max}+1}^{n+1}\right| + \frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left|\xi^{n+1}\right|\left|\delta_t\delta_x z_{\frac{1}{2}}^{n+1}\right| + \frac{b}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left|\zeta^{n+1}\right|\left|\delta_t z_0^{n+1}\right| \\
&\leq \frac{3}{2}\sum_{n=0}^{\mathcal{N}-1}\tau_m^2\left\|\Xi^{n+1}\right\|_{\mathbb{R}^3}\left\|\delta_t\vartheta^{n+1}\right\|_{\mathbb{R}^3} \\
&= \frac{3\tau_m}{2}\int_0^T \left\|\Xi_{\tau_m}\right\|_{\mathbb{R}^3}\left\|\delta_t\vartheta_{\tau_m}\right\|_{\mathbb{R}^3} dt,
\end{aligned}
$$

where

$$
\vartheta_{\tau_m} = \left(v_{N_{\max}+1,\tau_m}, \delta_x z_{\frac{1}{2},\tau_m}, z_{0,\tau_m}\right).
$$

Thanks to the regularity of v and z and the stability estimates, we conclude that $A_3 \longrightarrow 0$ as $m \longrightarrow \infty$.

Now, let us consider the term A_5 and apply Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|A_5| &\leq \frac{a\tau_m}{2}\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left|\tau_m\delta_x\psi_{i+\frac{1}{2}}-\delta_x u^0_{i+\frac{1}{2}}\right|\left|\delta_x v^0_{i+\frac{1}{2}}\right| + \frac{b\tau_m}{2}\sum_{i=1}^{N_1} h_i\left|\tau_m\delta_x^2\theta_i-\delta_x^2 w_i^0\right|\left|\delta_x^2 z_i^0\right| \\
&\leq \frac{\tau_m}{2}\left[\tau_m\left(a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x\psi_{i+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}+\left(a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x u^0_{i+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\right]\left(a\sum_{i=N_1}^{N_{\max}} h_{i+\frac{1}{2}}\left(\delta_x v^0_{i+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\
&\quad+\frac{\tau_m}{2}\left[\tau_m\left(b\sum_{i=1}^{N_1} h_i\left(\delta_x^2\theta_i\right)^2\right)^{\frac{1}{2}}+\left(b\sum_{i=1}^{N_1} h_i\left(\delta_x^2 w_i^0\right)^2\right)^{\frac{1}{2}}\right]\left(b\sum_{i=1}^{N_1} h_i\left(\delta_x^2 z_i^0\right)^2\right)^{\frac{1}{2}} \\
&= \frac{\tau_m}{2}\left(\tau_m\left\|\psi_{\mathcal{T}_m}\right\|_{1,\mathcal{T}_m}+\left\|\varphi_{\mathcal{T}_m}\right\|_{1,\mathcal{T}_m}\right)\left\|v^0_{\mathcal{T}_m}\right\|_{1,\mathcal{T}_m}+\frac{\tau_m}{2}\left(\tau_m\left\|\theta_{\mathcal{T}_m}\right\|_{2,\mathcal{T}_m}\left\|\phi_{\mathcal{T}_m}\right\|_{2,\mathcal{T}_m}\right)\left\|z^0_{\mathcal{T}_m}\right\|_{2,\mathcal{T}_m} \\
&\leq \tau_m\left(\tau_m\left\|\Psi_{\mathcal{T}_m}\right\|_{\mathbb{H},\mathcal{T}_m}+\left\|\Phi_{\mathcal{T}_m}\right\|_{\mathbb{H},\mathcal{T}_m}\right)\left\|V^0_{\mathcal{T}_m}\right\|_{\mathbb{H},\mathcal{T}_m},
\end{aligned}$$

from which one deduces, due to the regularity of V and the properties of Φ and Ψ, that $A_5 \longrightarrow 0$ as $m \longrightarrow \infty$. We continue to estimate the term A_6

$$\begin{aligned}
|A_6| &\leq \frac{a\tau_m}{2}\left|\tau_m\delta_x\psi_{N_{\max}+\frac{1}{2}}+\eta^0\right|\left|v^0_{N_{\max}+1}\right| + \frac{b\tau_m}{2}\left|\tau_m\delta_x^2\theta_0-\xi^0\right|\left|\delta_x z^0_{\frac{1}{2}}\right| + \frac{b\tau_m}{2}\left|\tau_m\delta_x^3\theta_{\frac{1}{2}}+\zeta^0\right|\left|z_0^0\right| \\
&\leq \frac{a\tau_m}{2}\left(\tau_m\left|\delta_x\psi_{N_{\max}+\frac{1}{2}}\right|+\left|\eta^0\right|\right)\left|v^0_{N_{\max}+1}\right| + \frac{b\tau_m}{2}\left(\tau_m\left|\delta_x^2\theta_0\right|+\left|\xi^0\right|\right)\left|\delta_x z^0_{\frac{1}{2}}\right| + \frac{b\tau_m}{2}\left(\tau_m\left|\delta_x^3\theta_{\frac{1}{2}}\right|+\left|\zeta^0\right|\right)\left|z_0^0\right| \\
&\leq \frac{3\tau_m}{2}\left(\tau_m\left\|\chi_1\right\|_{\mathbb{R}^3}+\|\Theta\|_{\mathbb{R}^3}\right)\left\|\chi_2\right\|_{\mathbb{R}^3},
\end{aligned}$$

where

$$\chi_1 = \left(\delta_x\psi_{N_{\max}+\frac{1}{2}}, \delta_x^2\theta_0, \delta_x^3\theta_{\frac{1}{2}}\right) \qquad \text{and} \qquad \chi_2 = \left(v^0_{N_{\max}+1}, \delta_x z^0_{\frac{1}{2}}, z_0^0\right).$$

With the regularity of the functions v and z and the properties imposed on Ψ and Θ, we conclude that $A_6 \longrightarrow 0$ as $m \longrightarrow \infty$. Finally, we obtain the convergence of $R_m \longrightarrow 0$ as $m \longrightarrow \infty$. Likewise, R_m', R_m'' and R_m''' also tend to zero as m tends to infinity.

Remark 5.5.2. *Given the definition of $U_{\mathcal{T}_m} = (u_{\mathcal{T}_m}, w_{\mathcal{T}_m})$, it is worth noting that the following holds for all $t \in [0,T]$:*

$$\int_0^L u_{\mathcal{T}_m}(x,t)dx = \int_{-L}^0 w_{\mathcal{T}_m}(x,t)dx = 0.$$

As $m \longrightarrow \infty$, if $u_{\mathcal{T}m}(t)$ converges weakly to $u(t)$ in $L^2(0,L)$ and $w_{\mathcal{T}_m}(t)$ converges weakly to $w(t)$ in $L^2(-L,0)$ for each $t \in [0,T]$, then we can conclude that:

$$\int_0^L u(x,t)dx = \int_{-L}^0 w(x,t)dx = 0,$$

for each $t \in [0,T]$.

Consequently, as we let $m \longrightarrow \infty$ in equations (5.142) and (5.147)-(5.149), the limit function (u,w,η,ξ,ζ) satisfies equation (5.6) for an arbitrary function $V = (v,z) \in C^\infty([0,T];C_*)$ subject to the constraint $V(x,T) = 0$. Moreover, it satisfies equations (5.7)-(5.9) for any function $(p,q,r) \in C^\infty\left([0,T];\mathbb{R}^3\right)$ with $p(T) = q(T) = r(T) = 0$. Hence, due to the density argument, equation (5.6) also holds for $(v,z) \in L^2([0,T];\mathbb{H})$ with $(v_t,z_t) \in L^2([0,T];\mathbb{L}^2)$ under the condition $v(T,x) = z(T,x) = 0$. Additionally, equations (5.7)-(5.9) hold for $(p,q,r) \in L^2([0,T];\mathbb{R}^3)$ with $(p_t,q_t,r_t) \in L^2([0,T];\mathbb{R}^3)$ provided $p(T) = q(T) = r(T) = 0$. This conclusion completes the proof. ■

5.6 Numerical Example: Validation of the Theoretical Result

In this section, we consider the transmission problem of a wave equation coupled with an Euler-Bernoulli beam, subjected to a single dynamical boundary control η. The system is given by:

$$\begin{cases} u_{tt} - au_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ w_{tt} + bw_{xxxx} = 0, & (x,t) \in (-L,0) \times (0,\infty), \\ u(0,t) = w(0,t), \quad w_x(0,t) = 0, & t \in (0,\infty), \\ au_x(0,t) + bw_{xxx}(0,t) = 0, & t \in (0,\infty), \\ u_x(L,t) + \eta(t) = 0, & t \in (0,\infty), \\ w_{xx}(-L,t) = 0, & t \in (0,\infty), \\ w_{xxx}(-L,t) = 0, & t \in (0,\infty), \\ \eta_t(t) - u_t(L,t) + \eta(t) = 0, & t \in (0,\infty), \end{cases} \tag{5.150}$$

with initial conditions:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,L), \quad \eta(0) = \alpha, \tag{5.151}$$

$$w(x,0) = \phi(x), \quad w_t(x,0) = \theta(x), \quad x \in (-L,0). \tag{5.152}$$

We have studied the effect of the dynamical boundary control η on the stability of the system (5.150)-(5.152) and proved a polynomial energy decay of type $1/\sqrt{t}$ for smooth initial data. A central component of the analysis is the frequency domain approach, which combines a contradiction argument with the multiplier method. A key tool in demonstrating polynomial stability is Theorem 2.4 from [12]. Due to the technical nature and length of the full proof, it is not included in this chapter. Instead, we provide a numerical example to illustrate and confirm the theoretical decay rate. When implementing the explicit numerical scheme (5.54)–(5.56), we observe a decrease in energy; however, it does not preserve positivity, leading to negative values. To overcome this limitation, we tested the proposed implicit numerical scheme described by (5.84)–(5.85) and (5.78)–(5.80). The numerical illustration is carried out for specific values of the parameters: $a = 2$ in the wave equation and $b = 4$ in the Euler–Bernoulli beam. To proceed, we assume $L = 1$ in (5.150)-(5.152) and the computation is carried out up to a final time $T = 1000$. The discretization is performed for $N_1 = 40$ and $N_2 = 40$, yielding step sizes of $h_1 = L/N_1 = 0.025$ and $h_2 = L/N_2 = 0.025$. Additionally, we set $\mathcal{N} = 3200000$ ensuring that the time step τ satisfies the relation $\tau = T/\mathcal{N} = \min\left\{\frac{h_2}{\sqrt{a}}, \frac{h_1^2}{\sqrt{b}}\right\} = 3.1250 \times 10^{-4}$. We now study the asymptotic behavior for the following initial conditions in $D(\mathcal{A})$ of the form:

$$\varphi(x) = \frac{1}{L}\left(\frac{2b}{aL} - \frac{\alpha}{2}\right)x^2 - \frac{4b}{aL}x + 1, \ \psi(x) = -\left(\frac{1}{L}\left(\frac{2b}{aL} - \frac{\alpha}{2}\right)x^2 - \frac{4b}{aL}x + 1\right), \ x \in (0,L), \ \alpha = 1,$$

$$\phi(x) = \frac{1}{6L^2}x^4 + \frac{2}{3L}x^3 + x^2 + 1, \quad \theta(x) = -\left(\frac{1}{6L^2}x^4 + \frac{2}{3L}x^3 + x^2 + 1\right), \quad x \in (-L,0).$$

Refer to Figure 5.3 for a visualization of the initial conditions. The numerical results are shown in the graphs below.

- **No Damping (Conservation of the total energy):** Figure 5.4 shows that, in the absence of dynamical boundary control, the total energy remains conserved over time. This implies that the numerical scheme described by equations (5.84)-(5.85) and (5.78)-(5.80) does not produce any numerical dissipation, meaning the observed numerical behavior is only due to the inherent dynamics of the considered model.

- **Damped with η (Polynomial stability):** When the system is damped using a single dynamical boundary control η, Figure 5.5 shows that the total energy decreases over time and eventually goes to zero as $t \to \infty$, indicating dissipation. With this, we expect the rate of energy decay in the discrete case to be an accurate reflection of the behavior in the continuous case. Therefore, we intend to study the decay rate numerically. To further examine this rate of decay, Figure 5.6 provides a zoom of the same graph plotted on a log-log scale. This zoom shows an asymptotic behavior of the energy, approximating a straight line, which in the log-log scale corresponds to polynomial decay. Using simple linear regression (least squares), the rate of decay was numerically determined. The result indicates an effective polynomial decaying of the energy with rate $1/t^{\ell}$, where $\ell = 0.49$, confirming the optimal result obtained by the theory. This result aligns with the theoretical expectations and is clearly illustrated in the figure.

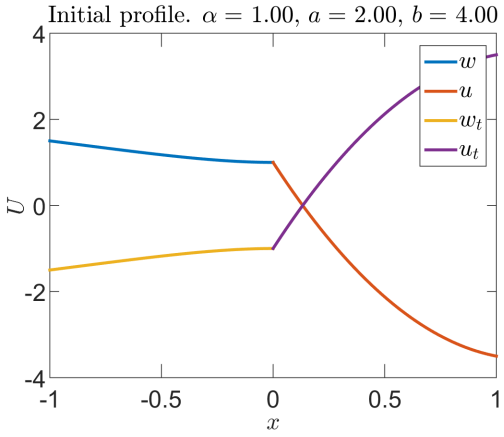

Figure 5.3: Initial profile.

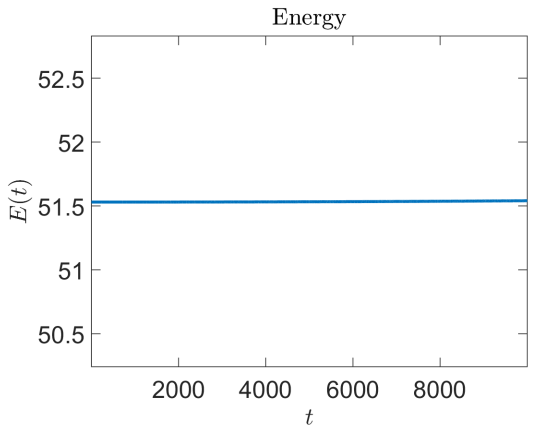

Figure 5.4: Conservation of the energy.

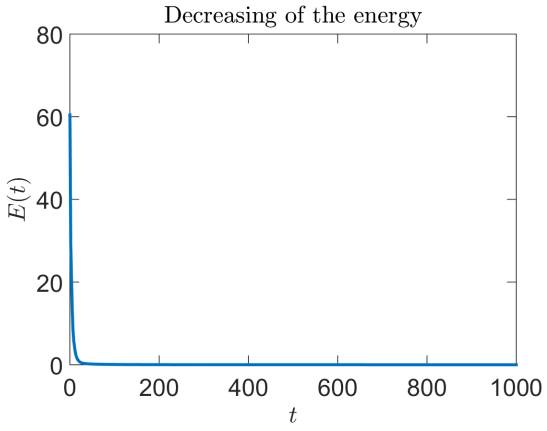

Figure 5.5: Decreasing of the energy.

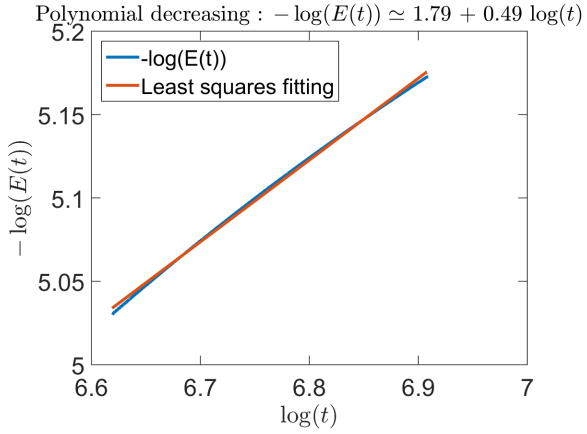

Figure 5.6: Polynomial decay.

5.7 Conclusion

In conclusion, we have developed a finite volume space discretization method to obtain a numerical solution for the transmission problem of a one-dimensional Euler-Bernoulli beam \ wave equation

with dynamical boundary controls. We have demonstrated the stability of the method and proved that the discretized solution converges to the original continuous weak solution. Additionally, our numerical experiment confirms that the decay rate of the discrete solution towards the null solution, under the influence of a single dynamical boundary control, aligns with the theoretical findings. For future work, the investigation of the spatial discretization order could be explored. This could be verified through detailed numerical analysis and further experimentation.

Chapter 6

Conclusion and Open Problems

6.1 Conclusion and Open Problems in English

The goal of this thesis has been to address the stabilization of interface transmission problems with dynamical boundary controls. By employing semigroup theory in conjunction with numerical analysis, we have made significant progress toward achieving this objective.

As a brief summary of this thesis, in Chapter 3, we began with the stabilization of a wave-plate model employing three dynamical boundary controls. Under specific geometric conditions, we established a polynomial energy decay rate of type $1/t$. An intriguing question that arise from this study is the optimality of the decay rate through spectral analysis, where we conjecture that $1/t$ is the optimal rate. In Chapter 4, we extended this discussion to a wave-plate model with only two dynamical boundary controls instead of the conventional three. Our analysis reveals that this reduction did not affect the decay rate obtained. Future research could explore models with one or two dynamical boundary controls, which may yield further insights and applications. In Chapter 5, we used the finite volume method to obtain a numerical solution of the transmission problem for a one-dimensional Euler-Bernoulli beam/wave equation with dynamical boundary controls, with the goal of demonstrating the stability of the method and proving the convergence of the discretized solution to the continuous one. Additionally, a numerical experiment is presented confirming the decay rate of the solution. Future work could extend these findings to N-dimensional cases ($N \geq 2$) on unstructured meshes. Further exploration of spatial discretization order, supported by detailed numerical analysis and experimentation, would also be valuable.

Overall, the insights presented in this thesis offer a robust foundation for further advancements in the stabilization of interface transmission problems. These results represent a step forward in mathematical control theory and its applications to engineering, physics and related fields.

6.2 Conclusion et Problèmes Ouverts en Français

L'objectif de cette thèse a été d'aborder la stabilisation des problèmes de transmission d'interface avec des contrôles de frontière dynamiques. En utilisant la théorie des semi-groupes en combinaison avec l'analyse numérique, nous avons réalisé des progrès significatifs vers l'atteinte de cet objectif.

En résumé, dans le chapitre 3, nous avons commencé par la stabilisation d'un modèle de plaque-onde en utilisant trois contrôles de frontière dynamiques. Sous des conditions géométriques spécifiques, nous avons établi un taux de décroissance de l'énergie polynomiale de type $1/t$. Une question intéressante qui découle de cette étude est l'optimalité du taux de décroissance par analyse spectrale, où nous

conjecturons que $1/t$ est le taux optimal. Dans le chapitre 4, nous avons étendu cette discussion à un modèle de plaque-onde avec seulement deux contrôles de frontière dynamiques au lieu des trois conventionnels. Notre analyse montre que cette réduction n'a pas affecté le taux de décroissance obtenu. Les recherches futures pourraient explorer des modèles avec un ou deux contrôles de frontière dynamiques, ce qui pourrait offrir de nouvelles perspectives et applications. Dans le chapitre 5, nous avons utilisé la méthode des volumes finis pour obtenir une solution numérique du problème de transmission pour une équation de type Euler-Bernoulli en une dimension avec des contrôles de frontière dynamiques, dans le but de démontrer la stabilité de la méthode et de prouver la convergence de la solution discrétisée vers la solution continue. De plus, une expérience numérique est présentée, confirmant le taux de décroissance de la solution. Les travaux futurs pourraient étendre ces résultats à des cas en N dimensions ($N \geq 2$) sur des maillages non structurés. Une exploration plus approfondie de l'ordre de discrétisation spatiale, soutenue par une analyse numérique détaillée et des expérimentations, serait également précieuse.

Dans l'ensemble, les résultats présentés dans cette thèse offrent une base solide pour de futures avancées dans la stabilisation des problèmes de transmission d'interface. Ces résultats représentent un pas en avant dans la théorie du contrôle mathématique et ses applications en ingénierie, physique et domaines connexes.

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