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Moduli spaces and moduli stacks of affine bundles

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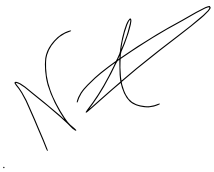
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Liste des publications et participation aux conférences

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Résumé

Nous étudions la classification des fibrés affines holomorphes sur une variété complexe compacte.

La thèse a deux objectifs : dans un premier temps, nous démontrons des théorèmes de classification pour les fibrés affines sur les surfaces de Riemann, notamment pour les fibrés affines sur la droite projective. Nous étudions en détail l'espace des modules des fibrés affines décorés ("framed") et non dégénérés de rang 2 sur $\mathbb{P}^1_{\mathbb{C}}$, dont la linéarisation, vue comme faisceau localement libre, est isomorphe à $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_2)$, avec $n_1 > n_2$. Nous montrons que cet espace des modules peut être identifié avec le "conoyau topologique" d'un morphisme d'espaces linéaires (au sens de la géométrie complexe) au-dessus de l'espace projectif $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ des formes binaires de degré $l \coloneqq -2 - n_2$. La fibre de ce conoyau topologique au-dessus de chaque point de cet espace projectif est donc un espace vectoriel complexe de dimension finie. Nous montrons que la stratification de $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ définie par les dimensions des fibres est déterminée explicitement par $d \coloneqq n_1 - n_2$ et la stratification définie par les ensemble de niveau de l'application "cactus rank" $\operatorname{crank} : \mathbb{P}(\mathbb{C}[X_0, X_1]_l) \to \mathbb{N}$. On met donc en évidence une relation inattendue entre le problème de classification des fibrés affines de rang 2 sur $\mathbb{P}^1_{\mathbb{C}}$ et la théorie des formes binaires.

Le second objectif a un caractère plus général : il concerne la classification des fibrés affines – sur une variété complexe compacte X – dont la linéarisation a un type d'isomorphisme fixé. Plus précisément, soit E un fibré vectoriel holomorphe fixé sur X et $\mathcal{E}$ le faisceau localement libre associé. Nous supposons qu'il existe un diviseur effectif D sur X tel que $H^1(X, \mathcal{E}(D)) = 0$. Cette condition est toujours satisfaite si X est une variété complexe projective.

Nous définissons de manière naturelle le champ ("the stack") des fibrés affines sur X de linéarisation isomorphe à E et nous montrons que ce champ est isomorphe au champ quotient

$$\left[H^0(X, \mathcal{E}(D)_D)/H^0(X, \mathcal{E}(D)) \ltimes \operatorname{Aut}(E)\right]$$

de l'espace vectoriel complexe $H^0(X, \mathcal{E}(D)_D)$ par le produit semi-direct du groupe des automorphismes $\operatorname{Aut}(E)$ (qui est toujours un groupe algébrique affine) avec l'espace vectoriel complexe $H^0(X, \mathcal{E}(D))$. Cette équivalence est obtenue par la construction *explicite* de foncteurs de comparaison $\mathscr{A}$, $\mathscr{T}$ entre les deux champs, ainsi que d'isomorphismes naturels $\mathscr{A} \circ \mathscr{T} \xrightarrow{\simeq} \mathrm{id}$, $\mathrm{id} \xrightarrow{\simeq} \mathscr{T} \circ \mathscr{A}$.

2020 *Mathematics Subject Classification* : 32L05, 32L10, 32G13, 14D20.

Abstract

We study the classification of affine holomorphic bundles over a compact complex manifold X.

The thesis has two goals: first we prove classification theorems for affine bundles over Riemann surfaces, notably for affine bundles over the projective line. In particular we study the moduli space of framed, non-degenerate rank 2 affine bundles over $\mathbb{P}^1_{\mathbb{C}}$ whose linearisation, viewed as locally free sheaf, is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_2)$ where $n_1 > n_2$. We show that this moduli space can be identified with the "topological cokernel" of a morphism of linear spaces over the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ of binary forms of degree $l \coloneqq -2 - n_2$, in particular it fibres over this projective space with vector spaces as fibres. We show that the stratification of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ defined by the level sets of the fibre dimension map is determined explicitly by $d \coloneqq n_1 - n_2$ and the cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$.

The second goal has a broader scope: the classification of affine bundles with fixed linearisation type over a compact complex manifold X. More precisely, let E be a fixed holomorphic vector bundle over X and $\mathcal{E}$ the associated locally free sheaf. We will assume that there exists an effective divisor D on X such that $H^1(X, \mathcal{E}(D)) = 0$. This condition is always satisfied if X is a projective complex manifold.

We define in a natural way the moduli stack of affine bundles on X whose linearisation is isomorphic to E, and we prove that this moduli stack is isomorphic to the quotient stack $\left[H^0(X, \mathcal{E}(D)_D)/H^0(X, \mathcal{E}(D)) \ltimes \mathrm{Aut}(E)\right]$ of the complex vector space $H^0(X, \mathcal{E}(D)_D)$ by the semi-direct product of the automorphism group $\mathrm{Aut}(E)$ (which is always an affine algebraic group) by the complex vector space $H^0(X, \mathcal{E}(D))$. Our isomorphism is obtained by constructing *explicitly* comparison functors $\mathscr{A}$, $\mathscr{T}$ between the two stacks and natural isomorphisms $\mathscr{A} \circ \mathscr{T} \xrightarrow{\simeq} \mathrm{id}$, $\mathrm{id} \xrightarrow{\simeq} \mathscr{T} \circ \mathscr{A}$.

2020 *Mathematics Subject Classification*: 32L05, 32L10, 32G13, 14D20.

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Contents

Liste des publications et participation aux conférences 4
Résumé 5
Abstract 6
Remerciements 7

Chapter 1. Introduction 11
1.1. Previous work on holomorphic affine bundles. Our goals 11
1.2. Our results 13

Chapter 2. Preliminary notions and results 17
2.1. Affine bundles 17
2.2. The classification theorem 19
2.3. Affine bundles over $\mathbb{P}^1_{\mathbb{C}}$ 21
2.4. The cokernel problem for linear spaces 22
2.5. Apolar ideals, the Sylvester theorem and the cactus rank 23
2.5.1. General Theory 23
2.5.2. The cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ for small l 25

Chapter 3. Moduli spaces of rank 2 affine bundles on $\mathbb{P}^1$ and apolar ideals 27
3.1. Affine bundles of rank 2 over Riemann surfaces 27
3.2. The case $r = 2$ and $X = \mathbb{P}^1_{\mathbb{C}}$ 28
3.2.1. General classification results 28
3.2.2. Framed non-degenerate rank 2 affine bundles over $\mathbb{P}^1_{\mathbb{C}}$ 29
3.2.3. The duality theorem, the fibres of $\mathrm{Coker}(H)$ over $\mathcal{P}$ and apolar ideals 30

Chapter 4. Moduli stacks of affine bundles with fixed linearisation type 35
4.1. Preliminaries 35
4.1.1. Gluing sheaves 35
4.1.2. Enlarged extensions 37
4.1.3. Principal bundles and torsors 38
4.1.4. The representability of the functor $S \mapsto \mathrm{Aut}_{S\times X}(S \times E)$ 39
4.2. Fibrewise constant families of vector bundles 40
4.3. An equivalence of groupoids 42
4.3.1. The functor $\mathfrak{T} : \mathcal{A}\mathrm{ff}_E \to \mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}$ 43
4.3.2. A family of elementary transformations 44
4.3.3. The functor $\mathfrak{A} : \mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D} \to \mathcal{A}\mathrm{ff}_E$ and its properties 46
4.3.4. The natural isomorphism $\mathfrak{A} \circ \mathfrak{T} \xrightarrow{\simeq} \mathrm{id}_{\mathcal{A}\mathrm{ff}_E}$ 48
4.3.5. The natural isomorphism $\mathrm{id}_{\mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}} \xrightarrow{\simeq} \mathfrak{T} \circ \mathfrak{A}$ 50
4.4. An equivalence of stacks 51
4.4.1. The functor ${}_S\mathfrak{T} : {}_S\mathcal{A}\mathrm{ff}_E \to {}_S\mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}$ 51
4.4.2. The functor ${}_S\mathfrak{A} : {}_S\mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D} \to {}_S\mathcal{A}\mathrm{ff}_E$ and its properties 52
4.4.3. The natural isomorphisms ${}_S\mathfrak{A} \circ {}_S\mathfrak{T} \to \mathrm{id}$, $\mathrm{id} \to {}_S\mathfrak{T} \circ {}_S\mathfrak{A}$ 55

Bibliography 57

CHAPTER 1

Introduction

1.1. Previous work on holomorphic affine bundles. Our goals

In his famous work [**We**], Weil introduces and studies affine line bundles over an algebraic curve C. Using the terminology of this thesis (see section 2) the objects studied by Weil are algebraic affine line bundles over C whose linearisation is isomorphic to the line bundle associated with a divisor D on C. His classification result [**We**, section 4] can be recovered as a special case of (the algebraic-geometric analogue of) our Theorem 2.2.4 in this thesis, taking into account Serre duality.

Affine line bundles in the holomorphic category have been considered in Enoki's article about class VII surfaces with curves [**En**]. In this article the author proves that any minimal class VII surface with $b_2 > 0$ which contains a non-zero homologically trivial divisor can be obtained as a compactification of the total space of an affine line bundle with negative degree over an elliptic curve. The compactifying divisor is a homologically trivial cycle of rational curves.

This thesis is dedicated to the classification problem of affine bundles in the complex geometric category. The problem has been previously studied by Gómez-Presas and Plechinger. We summarise their main results below:

In [**GoPr**] the authors state the equivalence between the groupoid of affine bundles of rank r and the groupoid of augmented bundles of rank $r+1$ (i.e. of epimorphisms $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X$, where $\tilde{\mathcal{E}}$ is a locally free sheaf of rank $r+1$), see also Proposition 2.1.8 in this thesis. The authors use this result to construct a bijection between the set of conjugacy classes of semi-simple representations of the fundamental group $\pi_1(X, x_0)$ of a compact Kähler manifold into the affine group $\mathrm{Aff}(r, \mathbb{C})$, and the set of isomorphism classes of "Higgs extensions" with polystable Higgs kernel (see [**GoPr**, Definition 3.4, Theorem 3.5]).

Plechinger's main results [**Pl1**], [**Pl2**] are dedicated to the classification of affine line bundles on a compact complex manifold:

- He proved (with different methods) a generalisation of Weil theorem for arbitrary complex manifolds.
- He obtained classification results for framed affine line bundles on compact complex manifolds. In Plechniger's formalism, an x_0-framed affine line bundle on X is a pair (A, ε), where A is an affine on X and ε is a linear isomorphism between $\mathbb{C}$ and the fibre $L(A)_{x_0}$ of the linearisation $L(A)$ of A at a fixed point $x_0 \in A$. He gives a sufficient and necessary condition for the existence of a moduli space in the sense of classical complex geometry which classifies x_0-framed affine line bundles A on X with fixed Chern class $c_1(L(A)) = c \in H^2(X, \mathbb{Z})$, up to isomorphism. If this existence condition is satisfied, the obtained moduli space is a linear space over the component $\mathrm{Pic}^c(X)$ of the Picard group $\mathrm{Pic}(X)$. The role of the framing is clear: the moduli space of non-framed affine line bundles with $c_1(L(A)) = c$ can be recovered as the (highly non-Hausdorff) $\mathbb{C}^*$-quotient of the moduli space of framed line bundles.
- For the general case, Plechinger constructed and studied the holomorphic stack which classifies framed and non-framed holomorphic affine line bundles which replaces (in the framed case) the classical moduli space when the existence condition is not satisfied.

Our main results (see section 2 in this chapter for details) concern the classification of higher rank holomorphic affine bundles over compact complex manifolds. We will start with a general classification theorem for affine bundles whose linearisation belongs to a fixed isomorphism class (Theorem 2.2.4). We will illustrate this general theorem in the special case $X = \mathbb{P}^1_{\mathbb{C}}$ (see Proposition 2.3.1), which shows already that, even on the simplest Riemann surface, the classification of affine bundles is much more complicated than the classification of vector bundles.

In the chapter 3 we will focus on affine rank 2-bundles on a Riemann surface C, more precisely on such affine bundles on C with *unstable* (i.e. not semi-stable) linearisation. We have two reasons for this choice:

- We have a natural way to frame such affine bundles by adding a framing parameter which varies, as in Plechinger's formalism, in a $\mathbb{C}^*$-torsor: we will consider pairs (A, ε) consisting of a rank 2-affine bundle on C and an isomorphism $\varepsilon : \mathbb{C} \xrightarrow{\simeq} D_{L(A),x_0}$, where $D_{L(A)}$ is the maximal destabilising line bundle of the linearisation $L(A)$, i.e. the first term of its Harder-Narasimhan filtration (see [**Sch**, Theorem 2.5.2.3] and [**HuLe**, I.1.3] for the algebraic framework).
- In the last section of chapter 3 we will consider affine rank 2-bundles on $\mathbb{P}^1_{\mathbb{C}}$, which admits no stable rank 2 vector bundles.

Note that for such affine bundles we have a natural non-degeneracy condition: an affine rank 2 bundle with unstable linearisation on C will be called *non-degenerate* if the image of the invariant $h_A \in H^1(X, \mathcal{L}(A))$ of A in the 1-cohomology of the minimal destabilising quotient $\mathcal{Q}_{L(A)} \coloneqq \mathcal{L}(A)/\mathcal{D}_{L(A)}$ of the locally free sheaf $\mathcal{L}(A)$ does not vanish. We will study in detail the moduli space – regarded as a topological space – of framed non-degenerate holomorphic affine rank 2-bundles on $\mathbb{P}^1_{\mathbb{C}}$ whose linearisation, viewed as locally free sheaf, is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_2)$ where $n_1 > n_2$. We will see that this moduli space can be identified with the topological cokernel (see section 2.4) of a morphism of linear spaces over the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ of binary forms of degree $l \coloneqq -2 - n_2$. In particular (although non-Hausdorff), this moduli space fibres naturally over this projective space with vector spaces as fibres. We will give an explicit formula for the dimension of the fibre over a point $[P] \in \mathbb{P}(\mathbb{C}[X_0, X_1]_{-2-n_2})$ in terms of $d \coloneqq n_1 - n_2$ and the *cactus rank* $\mathrm{crank}(P)$ of P, which is a fundamental invariant in the theory of binary forms (see section 2.5).

Our result highlights an unexpected relationship between the classification problem for affine rank 2-bundles over $\mathbb{P}^1_{\mathbb{C}}$ and the theory of apolar ideals (see section 2.5), and also gives a strong motivation for studying and describing geometrically the cactus rank stratification of the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ of binary forms of degree l (see section 2.5.2).

In chapter 4 we will use the formalism of Stack Theory. We refer to [**He**], [**Wa**] for the general Moduli Stack Theory for principal bundles in the algebraic framework.

Our goal is different: we will focus on the classification of affine bundles with fixed linearisation type on a compact complex manifold in full generality.

More precisely let X be a compact complex manifold, E be holomorphic vector bundle on X and $\mathcal{E}$ the associated locally free sheaf.

We define a holomorphic stack (see [**Mee**]) denoted $\mathscr{A}\mathrm{ff}_E$ on the category of complex spaces in the following way: we associate to a complex space S the groupoid ${}_S\mathcal{A}\mathrm{ff}_E$ whose objects are affine A bundles on $S \times X$, whose linearisation $L(A)$ is "fibrewise isomorphic to E" in the following sense: for any point $s \in S$ there exists an open neighbourhood U of s in S and a bundle isomorphism $p_X^{U\times X *}(E) \xrightarrow{\simeq} L(A)|_{U\times X}$ (see section 4.2 for details). This condition commutes obviously with pull-backs.

If S is reduced this condition reduces to

$$\forall s \in S,\ L(A)|_{\{s\}\times X} \simeq E$$

(via the obvious identification $\{s\} \times X \xrightarrow{\simeq} X$). Using Proposition 2.1.8 we will identify ${}_S\mathcal{A}\mathrm{ff}_E$ with the groupoid of augmented bundles $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X}$ over $S \times X$ with $\mathcal{E}_\varphi \coloneqq \ker(\varphi)$ fibrewise isomorphic to $\mathcal{E}$.

We will assume that there exists an effective divisor on X such that $H^1(X, \mathcal{E}(D)) = 0$, we fix such a divisor and put $V_D \coloneqq H^0(X, \mathcal{E}(D)_D)$, $W_D \coloneqq H^0(X, \mathcal{E}(D))$. The automorphism group $G \coloneqq \mathrm{Aut}(E)$ is always an affine algebraic group (see Proposition 2.2.6) and acts on the vector spaces V_D, W_D in a natural way. The semidirect product $G_D \coloneqq W_D \ltimes G$ comes with a natural action α_D on V_D induced by the mentioned above G-action on V_D and the action of W_D on V_D by translations via the natural linear map $W_D \to V_D$.

We will prove that the stack $\mathcal{A}\mathrm{ff}_E$ id equivalent to the quotient stack $[V_D/G_D]$. The fiber over S of this quotient stack is the groupoid ${}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$ whose objects are pairs (Θ, ρ) consisting of an $\mathcal{O}_S(G_D)$-torsor on S and a $\mathcal{O}_S(G_D)$-equivariant sheaf morphism $\rho : \Theta \to \mathcal{O}_S(V_D)$.

Our proof will be obtained by constructing explicitly functors

$$\mathscr{A} : [V_D/G_D] \to \mathscr{A}\mathrm{ff}_E,\ \mathscr{T} : \mathscr{A}\mathrm{ff}_E \to [V_D/G_D]$$

and explicit natural isomorphisms $\mathscr{A}\circ\mathscr{T}\xrightarrow{\simeq}\mathrm{id}_{\mathscr{A}\mathrm{ff}_E}$, $\mathscr{T}\circ\mathscr{A}\xrightarrow{\simeq}\mathrm{id}_{[V_D/G_D]}$. The functors $\mathscr{A}$, $\mathscr{T}$ will be defined by constructing, for any individual complex space S, functors ${}_S\mathfrak{A} : {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}\to {}_S\mathcal{A}\mathrm{ff}_E$, ${}_S\mathfrak{T} : {}_S\mathcal{A}\mathrm{ff}_E\to {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$ with clear geometric interpretations. The construction of these functors will be explained in the next section. Using the terminology of [**Gom**, p. 11], our result shows that the stack $\mathscr{A}\mathrm{ff}_E$ is isomorphic to the quotient stack $[V_D/G_D]$.

1.2. Our results

Chapter 2 will be dedicated to preliminaries: we will start with the definition and first properties of affine bundles and we will continue with classification results. In section 2.2 we will prove the following classification theorem (see [**Pl2**, Theorem 9] for the case $r = 1$):

THEOREM 2.2.4. Let X be a complex manifold, E a holomorphic vector bundle of rank r on X, $\mathcal{E}$ the associated locally free sheaf and $\mathrm{Aut}(E) = \mathrm{Aut}(\mathcal{E})$ its automorphism group. The map $A\mapsto h_A\in H^1(X,\mathcal{L}(A))$ induces a bijection between the set of isomorphism classes of holomorphic affine rank r-bundles A on X with $L(A)\simeq E$ and the quotient $H^1(X,\mathcal{E})/\mathrm{Aut}(E)$.

This statement is an easy consequence of the classification theorem [**GoPr**, Lemma 2.2] (see also Proposition 2.1.8 in this thesis). Taking into account Theorem 2.2.4, throughout chapters 2, 3 by the "moduli space" of affine bundles on X with fixed linearisation type $[E]$ we will mean the topological quotient space $H^1(X,\mathcal{E})/\mathrm{Aut}(E)$. This theorem shows that the classification of holomorphic affine bundles is much more interesting and complex than the classification of holomorphic vector bundles. We will illustrate this contrast in the simplest case $X=\mathbb{P}^1_{\mathbb{C}}$; in this case the classification of vector bundles is completely solved by Grothendieck's theorem and the answer is very simple. The classification of affine bundles on $\mathbb{P}^1_{\mathbb{C}}$ is much more complex, see Proposition 2.3.1. The last two sections of Chapter 2 are dedicated to the topological cokernel of a morphism of linear spaces and the theory of apolar ideals; they will play a fundamental role in chapter 3.

Our next results, proved in chapter 3, concern the classification of (framed) non-degenerate holomorphic affine bundles (A,ε) with $L(A)\simeq E$ over a Riemann surface C, where E is a fixed unstable rank 2 holomorphic vector bundle on C, and ε is a linear isomorphism $\mathbb{C}\xrightarrow{\simeq}D_{L(A),x_0}$ onto the fibre of the maximal destabilising line bundle of E at a fixed point $x_0\in C$. Let D_E be the maximal destabilising line bundle of E, $\mathrm{Aut}(E)_0$ be the closed subgroup of $\mathrm{Aut}(E)$ consisting of automorphisms which induce the identity on D_E, $\mathcal{E}$ the locally free sheaf associated with E and $H^1(C,\mathcal{E})_0\subset H^1(C,\mathcal{E})$ the open subspace of cohomology classes which are mapped non-trivially to $H^1(C,\mathcal{E}/\mathcal{D}_E)$. Using the terminology introduced in section 3.1, we will prove:

THEOREM 3.1.3. Let C be a Riemann surface, E be a fixed rank 2 unstable vector bundle on C, $\mathcal{E}$ the associated locally free sheaf, $x_0\in C$ and $\varepsilon_0:\mathbb{C}\to D_{E,x_0}$ be a fixed isomorphism. Then:

(1) The map $A\mapsto h_A$ induces a bijection between the set of isomorphism classes of non-degenerate affine holomorphic rank r-bundles A on X with $L(A)\simeq E$ and the quotient $H^1(C,\mathcal{E})_0/\mathrm{Aut}(E)$.
(2) For an x_0-framed affine bundle (A,ε) with $L(A)\simeq E$ choose an isomorphism $f:L(A)\to E$ such that $f\circ\varepsilon=\varepsilon_0$.

The map $(A,\varepsilon)\mapsto H^1(f)(h_A)$ induces a bijection between the set of isomorphism classes of x_0-framed (non-degenerate) affine holomorphic rank r-bundles A on C with $L(A)\simeq E$ and the quotient $H^1(C,\mathcal{E})/\mathrm{Aut}(E)_0$ (respectively $H^1(C,\mathcal{E})_0/\mathrm{Aut}(E)_0$).

Therefore, by the moduli space of framed (non-degenerate) affine holomorphic bundles with fixed linearisation type on C we will mean the topological space obtained by endowing the corresponding moduli set with the topology induced by the identifications given by Theorem 3.1.3.

Note that in general the groups $\mathrm{Aut}(E)$, $\mathrm{Aut}(E)_0$ are affine algebraic groups (see Proposition 2.2.6), but not necessarily reductive. Therefore the classical GIT methods do not apply.

Section 3.2 is dedicated to the classification of framed, non-degenerate affine rank 2 bundles over $\mathbb{P}^1_{\mathbb{C}}$ whose linearisation, viewed as locally free sheaf, is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_1)\oplus\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_2)$ where $-2\geqslant n_1>n_2$.

Making use of Theorem 3.1.3, we will show that the moduli space of isomorphism classes of such bundles, regarded as a topological space, can be identified with the topological cokernel (see section 2.4) $\mathrm{Coker}(H)$ of a morphism

$$H : L\to L'$$

of linear spaces over the projective space $\mathbb{P}(\mathbb{C}[X_0,X_1]^*_{-2-n_2})$. $\mathrm{Coker}(H)$ can be viewed as the underlying topological space of the relative quotient stack (over $\mathbb{P}(\mathbb{C}[X_0,X_1]^*_{-2-n_2})$) of the linear space L' by L, where the latter acts on L' via H. In a future article we will compare the moduli

stack of the framed, non-degenerate affine bundles we consider with the relative quotient stack of L' by L.

Although in general non-Hausdorff, our moduli space $\mathrm{Coker}(H)$ fibres over the projective space $\mathbb{P}(\mathbb{C}[X_0,X_1]^*_{-2-n_2})$ with vector spaces as fibres. Our next goal will be understanding geometrically the stratification of $\mathbb{P}(\mathbb{C}[X_0,X_1]^*_{-2-n_2})$ defined by the level sets of the fibre dimension map $\xi \mapsto \dim(\mathrm{Coker}(H)_\xi)$.

Writing $\xi = [\Phi]$ for a non-trivial linear form $\Phi \in \mathbb{C}[X_0,X_1]^*_{-2-n_2}$, we will see that the fibre $\mathrm{Coker}(H)_\xi$ is naturally identified with the cokernel of the linear map

$$\mathbb{C}[X_0,X_1]_{n_1-n_2} \ni Q \mapsto {}^t m_Q(\Phi) \in \mathbb{C}[X_0,X_1]^*_{-2-n_1}, \tag{1}$$

where, for a polynomial $Q \in (\mathbb{C}[X_0,X_1]_{n_1-n_2}$, we have denoted by

$$m_Q : \mathbb{C}[X_0,X_1]_{-2-n_1} \to \mathbb{C}[X_0,X_1]_{-2-n_2}$$

the multiplication by Q, and by ${}^t m_Q$ its transpose (algebraic adjoint).

The main idea for computing the dimensions of the cokernels of the family of linear maps (1): for $l \in \mathbb{N}$ identify the dual space $\mathbb{C}[X_0,X_1]_l^*$ with $\mathbb{C}[X_0,X_1]_l$ via the isomorphism $D_l : \mathbb{C}[X_0,X_1]_l \xrightarrow{\simeq} \mathbb{C}[X_0,X_1]_l^*$ which maps a polynomial $P \in \mathbb{C}[X_0,X_1]_l$ to the differential operator $P(\delta_0,\delta_1)$, where $\delta_i \coloneqq \frac{\partial}{\partial X_i}$.

Using these identifications we will prove (see section 3.2.3) the following duality theorem, which makes the connection between our classification problem for rank 2 affine bundles over $\mathbb{P}^1_{\mathbb{C}}$ and the theory of apolar ideals:

Theorem 3.2.6. Via the isomorphisms D_l, D_{l-d} we have

$${}^t m_Q(P(\delta_0,\delta_1)) = Q\left(\frac{\partial}{\partial\delta_0},\frac{\partial}{\partial\delta_1}\right)(P(\delta_0,\delta_1)).$$

for any $P(\delta_0,\delta_1) \in \mathbb{C}[\delta_0,\delta_1]_l$, $Q \in \mathbb{C}[X_0,X_1]_d$.

In other words, putting $\Phi \coloneqq D_l(P(\delta_0,\delta_1)) \in \mathbb{C}[X_0,X_1]_l^*$, the differential operator associated with ${}^t m_Q(\Phi)$ via D_{l-d}^{-1} is obtained by applying the differential operator $Q\left(\frac{\partial}{\partial\delta_0},\frac{\partial}{\partial\delta_1}\right)$ to $P(\delta_0,\delta_1)$, *regarded as polynomial in the abstract variables* δ_0, δ_1.

Put $l \coloneqq -2-n_2$, $d \coloneqq n_1-n-2$ and let $P \in \mathbb{C}[X_0,X_1]_l$ be the polynomial associated with Φ via D_l^{-1}. The duality theorem shows that the vector space $\mathrm{coker}(Q \mapsto {}^t m_Q(\Phi))$, in which we are interested, can be identified with the cokernel

$$C_P^d \coloneqq \mathrm{coker}\big(\mathbb{C}[X_0,X_1]_d \ni Q \mapsto Q(\frac{\partial}{\partial X_0},\frac{\partial}{\partial X_1})(P)\big).$$

At this stage we will make use essentially of the classical theory of binary forms and of the fundamental concepts of this theory: apolar ideals and cactus rank. The needed definitions and results are explained in section 2.5 in chapter 2 which is dedicated to preliminaries.

The kernel $\ker\big(\mathbb{C}[X_0,X_1]_d \ni Q \mapsto Q(\frac{\partial}{\partial X_0},\frac{\partial}{\partial X_1})(P)\big)$ is just the degree d homogeneous component $\mathrm{Ann}(P)_d$ of the apolar ideal $\mathrm{Ann}(P)$ of P. Its dimension depends on d and the cactus rank $\mathrm{crank}(P)$ of P (see Definition 2.5.2, Proposition 2.5.9, Remark 2.5.4). We will prove (see Proposition 3.2.8 in section 3.2.3):

Proposition 1.2.1. *The dimension $s := \dim(C_P^d)$ depends only on d and the cactus rank $r = \mathrm{crank}(P)$:*

(1) For $0 \leqslant d \leqslant [\frac{l}{2}]$, the function $s = s(r)$ is given by:

$$s(r) = \begin{cases} l-d-(r-1) & \text{if} \quad 1 \leqslant r \leqslant d+1, \\ l-2d & \text{if} \quad d+1 \leqslant r \leqslant [\frac{l}{2}]+1. \end{cases}$$

(2) For $[\frac{l}{2}] \leqslant d \leqslant l$, the function $s = s(r)$ is given by:

$$s(r) = \begin{cases} l-d-(r-1) & \text{if} \quad 1 \leqslant r \leqslant l-d+1, \\ 0 & \text{if} \quad l-d+1 \leqslant r \leqslant [\frac{l}{2}]+1. \end{cases}$$

It follows that the stratification of $\mathbb{P}(\mathbb{C}[X_0,X_1]_l^*) \simeq \mathbb{P}(\mathbb{C}[X_0,X_1]_l)$ defined by the level sets of the fibre dimension map $\xi \mapsto \dim(\mathrm{Coker}(H)_\xi)$ of our family of vector spaces is determined by d and the cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0,X_1]_l)$. Corollary 3.2.10 gives explicit formulae for the level sets of the fibre dimension map in terms of d and the level sets of the cactus rank.

In section 2.5.2 we describe explicitly the cactus rank stratification of the projective $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ for $l \leqslant 4$. The case $l = 4$ shows already that, in general, $\dim(C_P^d)$ is not determined only by d and the multiplicities of the roots of P, but depends effectively of the GIT invariants of P.

The main result of Chapter 4 is:

THEOREM 4.4.5. Let X be a compact complex manifold, and E be a fixed holomorphic vector bundle on X and $\mathcal{E}$ the associated locally free sheaf. Let D be an effective divisor on X such that $H^1(X, \mathcal{E}(D)) = 0$. The holomorphic stack $\mathscr{A}\mathrm{ff}_E$ is isomorphic to the quotient stack $[V_D/G_D]$.

The two stacks will be compared using a collection of functors ${}_S\mathfrak{A} : {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D} \to {}_S\mathcal{A}\mathrm{ff}_E$, ${}_S\mathfrak{T} : {}_S\mathcal{A}\mathrm{ff}_E \to {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$ associated with complex spaces S. We explain briefly below the construction of these functors in the special case when $S = \{*\}$.

In other words, let $\mathcal{A}\mathrm{ff}_E$ be the groupoid whose objects are augmented bundles $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X$ on X with $\mathcal{E}_\varphi \coloneqq \ker(\varphi) \simeq \mathcal{E}$ and let $\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$ be the groupoid whose objects are pairs (Θ, ρ) consisting of a Lie G_D-torsor Θ (see section 4.3) and an α_D-equivariant holomorphic map $\rho : \Theta \to V_D$, where $\alpha_D : G_D \times V_D \to V_D$ denotes the left action given by

$$\alpha_D((w, g), \beta) = g(D)_D(\beta) - w_D.$$

In section 4.3 we will construct explicitly functors $\mathfrak{T} : \mathcal{A}\mathrm{ff}_E \to \mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$, $\mathfrak{A} : \mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D} \to \mathcal{A}\mathrm{ff}_E$ and natural isomorphisms $\mathfrak{A} \circ \mathfrak{T} \xrightarrow{\simeq} \mathrm{id}_{\mathcal{A}\mathrm{ff}_E}$, $\mathrm{id}_{\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}} \xrightarrow{\simeq} \mathfrak{T} \circ \mathfrak{A}$.

- The functor $\mathfrak{T}$: An object $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X$ of the groupoid $\mathcal{A}\mathrm{ff}_E$ defines an extension

$$0 \to \mathcal{E}_\varphi \coloneqq \ker(\varphi) \hookrightarrow \tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X \to 0. \tag{e_φ}$$

  Consider D-enlarged extension (see sections 4.1.2, 4.3.1)

$$0 \to \mathcal{E}_\varphi(D) \xrightarrow{\iota_\varphi^D} \tilde{\mathcal{E}}_\varphi^D \xrightarrow{\varphi^D} \mathcal{O}_X \to 0 \tag{e_φ^D}$$

  of (e_φ). An important point: the passage from (e_φ) to (e_φ^D) is canonical. It is not just a correspondence between isomorphism classes of extensions, but a functor on the category of extensions. Our condition $H^1(X, \mathcal{E}(D)) = 0$ implies that (e_φ^D) is split. In section 4.3.1 we will show that:
  - ◇ The set $S_D^\varphi \coloneqq \{\sigma : \mathcal{O}_X \to \tilde{\mathcal{E}}_\varphi^D \mid \sigma \circ \varphi_D = \mathrm{id}\}$ of right splittings of (e_φ^D) is an affine subspace of $H^0(X, \tilde{\mathcal{E}}_\varphi^D)$ with model space $H^0(X, \mathcal{E}_\varphi(D))$, so it can be regarded as right Lie $H^0(X, \mathcal{E}_\varphi(D))$-torsor.
  - ◇ The set $\mathcal{Q}^\varphi$ of isomorphisms $f : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}_\varphi$ is naturally a right Lie G-torsor.
  - ◇ The product $\Theta_\varphi^D \coloneqq S_D^\varphi \times \mathcal{Q}^\varphi$ has a natural structural of a right G_D-torsor.

  We also define $\rho_\varphi^D : \Theta_\varphi^D \to V_D$ by

$$\rho_\varphi^D(\sigma, f) \coloneqq f(D)_D^{-1}(q_D^\varphi \circ \sigma) \in \mathrm{Hom}(\mathcal{O}_X, \mathcal{E}(D)_D) = H^0(X, \mathcal{E}(D)_D) = V_D,$$

  where $f(D)_D$ stands for the automorphism of $\mathcal{E}(D)_D$ induced by f and $q_D^\varphi : \tilde{\mathcal{E}}_\varphi^D \to \mathcal{E}_\varphi(D)_D$ stands for the canonical epimorphism on the central term of the D-enlarged extension (see section 4.1.2). Our functor $\mathfrak{T}$ is defined by

$$\mathfrak{T}\big(\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X\big) \coloneqq (\Theta_\varphi^D, \rho_\varphi^D).$$

- The functor $\mathfrak{A}$: Let (Θ, ρ) be an object of the groupoid $\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$.

  For a point $\theta \in \Theta$ we obtain an element $\rho(\theta) \in H^0(X, \mathcal{E}(D)_D)$ which gives a sheaf epimorphism $u_{\rho(\theta)} : \mathcal{E}(D) \oplus \mathcal{O}_X \to \mathcal{E}(D)_D$ given by

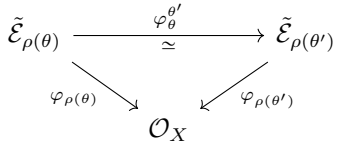
$$u_{\rho(\theta)}(s, \zeta) = s_D + \zeta\, \rho(\theta),$$

  where s_D stands for the restriction of s to D. The elementary transformation associated with this sheaf epimorphism is $\tilde{\mathcal{E}}_{\rho(\beta)} \coloneqq \ker(u_{\rho(\theta)})$. By the general properties of elementary transformations, $\tilde{\mathcal{E}}_{\rho(\beta)}$ is locally free of rank $\mathrm{rk}(\mathcal{E}) + 1$. The restriction $\varphi_{\rho(\theta)} : \tilde{\mathcal{E}}_{\rho(\theta)} \to \mathcal{O}_X$ of the second projection on $\mathcal{E}(D) \oplus \mathcal{O}_X$ is an augmented bundle whose kernel $\mathcal{E}_{\rho(\theta)} \coloneqq \varphi_{\rho(\theta)}$ is canonically isomorphic to $\mathcal{E}$.

  For two points $\theta, \theta' \in \Theta$ consider the unique $\gamma \in G_D$ be defined by $\theta = \theta'$. Writing $\gamma = (v, g)$ with $v \in H^0(X, \mathcal{E}(D))$ and $g \in G = \mathrm{Aut}(\mathcal{E})$ we obtain an automorphism

$$R_\gamma = \begin{pmatrix} g(D) & v \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathcal{E}(D) \oplus \mathcal{O}_X)$$

which induces an isomorphism



of augmented bundles on X. The pair $\big((\tilde{\mathcal{E}}_{\rho(\theta)})_{\theta\in\Theta}, (\varphi_\theta^{\theta'})\big)$ is an $\mathcal{O}_X$-module gluing data subordinate to the cover $(U_\theta)_{\theta\in\Theta}$ of X defined by $U_\theta = X$ (see section 4.1.1), so, applying the gluing functor given by Proposition 4.1.2, we obtain an augmented bundle

$$\mathfrak{A}(\Theta,\varphi) \coloneqq \tilde{\mathcal{E}}_{\Theta,\rho} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_X$$

which is canonically associated with (Θ,ρ).

We have an alternative way to interpret the functor $\mathfrak{A}$ using descent theory. Putting $D_\Theta \coloneqq \Theta \times D$ (regarded as effective divisor on the product $\Theta \times X$), we see that the V_D-valued map $\rho \in \mathcal{O}(\Theta, V_D)$ gives an element $\tilde{\rho} \in H^0(\Theta \times X, p_X^*(\mathcal{E})(D_\Theta)_{D_\Theta})$. The formula

$$U_{\tilde{\rho}}(s,\zeta) = s_{D_\Theta} + \zeta\,\tilde{\rho}$$

defines a sheaf epimorphism $p_X^*(\mathcal{E})(D_\Theta) \oplus \mathcal{O}_{\Theta\times X} \to p_X^*(\mathcal{E})(D_\Theta)_{D_\Theta}$.

The corresponding elementary transformation $\tilde{\mathcal{E}}(\Theta,\rho) \coloneqq \ker(U_{\tilde{\rho}}(s,\zeta))$ comes with a epimorphism $\tilde{\mathcal{E}}(\Theta,\rho) \xrightarrow{\varphi(\Theta,\rho)} \mathcal{O}_{\Theta\times X}$ induced by the second projection on $p_X^*(\mathcal{E})(D_\Theta) \oplus \mathcal{O}_{\Theta\times X}$. Using the automorphisms $(R_\gamma)_{\gamma\in G_D}$ defined above, we obtain the structure of a G_D-equivariant $\mathcal{O}_X$-module (see [**Sta**, section 2.39.12]) on $p_X^*(\mathcal{E})(D_\Theta) \oplus \mathcal{O}_{\Theta\times X}$. We regard $\Theta \times X$ as a principal G_D-bundle on X. The augmented bundle $\tilde{\mathcal{E}}(\Theta,\rho) \xrightarrow{\varphi(\Theta,\rho)} \mathcal{O}_{\Theta\times X}$ is G_D-invariant, so it descends to an augmented bundle $\tilde{\mathcal{E}}_{\Theta,\rho} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_X$ on X.

We preferred using gluing theory for sheaves instead of descent theory for equivariant sheaves for the following reason: whereas descent theory is usually developed only within Algebraic Geometry, gluing theory is available in the literature in the very general framework of modules on ringed spaces.

The general functors ${}_S\mathfrak{T}$, ${}_S\mathfrak{A}$ associated with a complex space S are just versions "with parameter" of the functors $\mathfrak{T}$, $\mathfrak{A}$. They will be introduced and studied in detail in sections 4.4.1, 4.4.2.

CHAPTER 2

Preliminary notions and results

2.1. Affine bundles

Let $\mathrm{Aff}(r,\mathbb{C})$ be the group of affine transformations of $\mathbb{C}^r$. Let A and X be complex manifolds, and $\pi : A \longrightarrow X$ a holomorphic map. A local trivialisation with standard fibre $\mathbb{C}^r$ of π is a biholomorphism

$$\tau = (\tau_1, \tau_2) : \pi^{-1}(U_\tau) \longrightarrow U_\tau \times \mathbb{C}^r$$

such that $\tau_1 = \pi|_{\pi^{-1}(U)}$. This implies that the restriction $\tau_x \coloneqq \tau_2|_{A_x} : A_x \to \mathbb{C}^r$ is a biholomorphism. Two such local trivialisations τ, θ are said to be $\mathrm{Aff}(r,\mathbb{C})$-compatible, if for every $x \in U_\tau \cap U_\theta$ the map

$$\tau_x \circ \theta_x^{-1} : \mathbb{C}^r \longrightarrow \mathbb{C}^r$$

is an affine transformation, i.e. it belongs to $\mathrm{Aff}(r,\mathbb{C})$. If this is the case, we obtain a comparison map

$$U_\tau \cap U_\theta \to \mathrm{Aff}(r,\mathbb{C}),$$

which is holomorphic.

Following the general formalism present in [**Sch**, section 1.2.1] we define: A bundle atlas with structure group $\mathrm{Aff}(r,\mathbb{C})$ on π is a set $\mathcal{T}$ of local trivialisations with standard fibre $\mathbb{C}^r$ of π satisfying the properties:

(1) $\bigcup_{\tau\in\mathcal{T}} U_\tau = X$.
(2) Any two elements of $\mathcal{T}$ are $\mathrm{Aff}(r,\mathbb{C})$-compatible.

Definition 2.1.1. *A holomorphic affine bundle of rank r on X is a couple*

$$(\pi : A \to X, \mathcal{T})$$

consisting of a holomorphic map π and a bundle atlas $\mathcal{T}$ with structure group $\mathrm{Aff}(r,\mathbb{C})$ on π which is maximal with respect to inclusion.

The classical theory of principal bundles in the differential geometric framework (see for instance [**KN**], [**Te**]) extends naturally in the holomorphic category. With any affine bundle $(\pi : A \to X, \mathcal{T})$ one can associate an $\mathrm{Aff}(r,\mathbb{C})$-principal bundle as follows: for any $x \in X$ we define

$$P_{A,x} \coloneqq \{f : \mathbb{C}^r \to A_x |\ \text{ for any } \tau \in \mathcal{T} \text{ with } x \in U_\tau, \text{ we have } \tau_x \circ f \in \mathrm{Aff}(r,\mathbb{C})\}.$$

The disjoint union $P_A \coloneqq \coprod_{x\in X} P_{A,x}$ has a natural complex manifold structure and comes with an obvious $\mathrm{Aff}(r,\mathbb{C})$-right action such that the natural map $P_\pi : P_A \to X$ becomes a holomorphic $\mathrm{Aff}(r,\mathbb{C})$-principal bundle.

Conversely, for any holomorphic principal $\mathrm{Aff}(r,\mathbb{C})$-bundle P on X we define the affine bundle

$$A^P \coloneqq P \times_{\mathrm{Aff}(r,\mathbb{C})} \mathbb{C}^r$$

to be the bundle associated with P and the standard action of its structure group $\mathrm{Aff}(r,\mathbb{C})$ on $\mathbb{C}^r$. Similarly to the case of vector bundles, we have

Remark 2.1.2. *The functors $A \mapsto P_A$, $P \mapsto A^P$ define an equivalence between the groupoid of affine rank r-bundles on X and the groupoid of principal $\mathrm{Aff}(r,\mathbb{C})$-bundles on X.*

We recall that the set of local sections of a principal bundle can always be identified with the set of its local trivialisations.

Remark 2.1.3. *Let $(A,\mathcal{T})$ be a holomorphic affine rank r bundle on X, and $\mathcal{S}$ be the set of local sections of P_A. The map $\mathcal{T} \to \mathcal{S}$ defined by*

$$\mathcal{T} \ni \tau \mapsto (U_\tau \ni x \mapsto \tau_x^{-1} \in P_{A,x}) \in \mathcal{S}$$

is a bijection.

Therefore $\mathcal{T}$ can be identified with the set of local trivialisations of P_A.

Let $(\pi : A \to X, \mathcal{T})$ be an affine bundle on X and $x \in X$. A biholomorphic automorphism u of A_x will be called *translation* if there exists (or, equivalently, if for any) $\tau \in \mathcal{T}$ with $x \in U_\tau$ the composition $\tau_x \circ u \circ \tau_x^{-1}$ is a translation of $\mathbb{C}^r$. The set $L(A)_x$ of translations of A_x has the natural structure of an r-dimensional complex vector space, and the natural action of $L(A)_x$ on A_x is free and transitive. In other words A_x has the natural structure of a complex affine space with associated (model) vector space $L(A)_x$.

The disjoint union $L(A) \coloneqq \coprod_{x\in X} L(A)_x$ has the natural structure of a holomorphic vector bundle, which can be identified with the associated vector bundle $P_A \times_\rho \mathbb{C}^r$, where $\rho : \mathrm{Aff}(r, \mathbb{C}) \to \mathrm{GL}(r, \mathbb{C})$ is the standard linear representation which assigns to each affine transformation its linear part.

Definition 2.1.4. *The holomorphic vector bundle $L(A)$ will be called the linearisation of A, and its associated locally free sheaf will be denoted by $\mathcal{L}(A)$.*

We will also need the representation

$$\tilde{\rho} : \mathrm{Aff}(r, \mathbb{C}) \to \mathrm{GL}(r + 1, \mathbb{C})$$

of $\mathrm{Aff}(r, \mathbb{C})$ defined by

$$\tilde{\rho}(z \mapsto Mz + \zeta) \coloneqq \begin{pmatrix} 1 & 0 \\ \zeta & M \end{pmatrix}.$$

The image $\tilde{\mathrm{Aff}}(r, \mathbb{C})$ of this group monomorphism is a closed affine subgroup of $\mathrm{GL}(r + 1, \mathbb{C})$. The associated holomorphic vector bundle

$$\tilde{L}(A) \coloneqq P_A \times_{\tilde{\rho}} \mathbb{C}^{r+1}$$

is a holomorphic rank $(r+1)$-vector bundle on X which comes with a canonical $\tilde{\mathrm{Aff}}(r, \mathbb{C})$-structure (reduction of structure group from $\mathrm{GL}(r+1, \mathbb{C})$ to $\tilde{\mathrm{Aff}}(r, \mathbb{C})$), see for instance [**Te**, Définition 2.3.1]. We will denote by $\tilde{\mathcal{L}}(A)$ the locally free sheaf associated with $\tilde{L}(A)$.

We recall that

Remark 2.1.5. *(1) The subgroup $\tilde{\mathrm{Aff}}(r, \mathbb{C}) \subset \mathrm{GL}(r+1, \mathbb{C})$ leaves invariant the projection*

$$p_0 : \mathbb{C}^{r+1} \to \mathbb{C},\ p_0 \begin{pmatrix} z_0 \\ \vdots \\ z_r \end{pmatrix} \coloneqq z_0.$$

(2) Let E be a holomorphic complex vector bundle of rank $r + 1$ on X. The data of an $\tilde{\mathrm{Aff}}(r, \mathbb{C})$-structure on E is equivalent to the data of a bundle epimorphism $\varphi : E \to X \times \mathbb{C}$.

Concerning (2): for a point $x \in X$, the linear epimorphism $\varphi_x : E_x \to \mathbb{C}$ is $p_0 \circ \tau_x$, where $\tau : E_{U_\tau} \to U_\tau \times \mathbb{C}^{r+1}$ is a local vector bundle trivialisation of E which is compatible with the fixed $\tilde{\mathrm{Aff}}(r, \mathbb{C})$-structure. By (1) it follows that $p_0 \circ \tau_x$ is independent of the choice of τ.

Applying this remark to $\tilde{L}(A)$ we obtain a canonical bundle epimorphism

$$\varphi_A : \tilde{L}(A) \to X \times \mathbb{C}$$

induced by the canonical $\tilde{\mathrm{Aff}}(r, \mathbb{C})$-structure of $\tilde{L}(A)$. We will denote by the same symbol the associated sheaf epimorphism $\tilde{\mathcal{L}}(A) \to \mathcal{O}_X$.

Definition 2.1.6. *An augmented rank $r + 1$ holomorphic vector bundle on X is a pair (E, φ) consisting of a holomorphic rank $r+1$-vector bundle E on X and a holomorphic bundle epimorphism $\varphi : E \to X \times \mathbb{C}$ onto the trivial line bundle $X \times \mathbb{C}$ on X.*

Following [**Pl1**], [**Pl2**] we will denote by E_X^r the category of augmented rank $r+1$ holomorphic vector bundles on X.

Remark 2.1.7. *Let $\mathfrak{a} = (E, \varphi)$ be an augmented rank $r + 1$ holomorphic vector bundle on X. Then the natural map $A_\mathfrak{a} \coloneqq \varphi^{-1}(X \times \{1\}) \to X$ comes with a natural structure $\mathcal{T}_\mathfrak{a}$ of a holomorphic affine bundle of rank r on X whose associated linearisation is canonically isomorphic to* $\ker(\varphi)$.

Proof. Note first that the fibre $A_{\mathfrak{a},x}$ of $A_\mathfrak{a}$ over a point $x \in X$ is an affine hyperplane in E_x, so it comes with a natural affine space space structure with model space $\ker(\varphi)_x$.

We define $\mathcal{T}_{\mathfrak{a}}$ to be the set of holomorphic local trivialisations τ of $A_{\mathfrak{a}}$ with the property that for any $x \in U$ the map $\tau_x : A_{\mathfrak{a},x} \to \mathbb{C}^r$ is affine, i.e. it satisfies the identity

$$\tau_x(\lambda u + (1-\lambda)v) = \lambda\tau_x(u) + (1-\lambda)\tau_x(v)$$

for u, $v \in A_{\mathfrak{a},x}$, $\lambda \in \mathbb{C}$. It is easy to check that $\mathcal{T}_{\mathfrak{a}}$ is a maximal bundle atlas with structure group $\mathrm{Aff}(r, \mathbb{C})$ on $A_{\mathfrak{a}}$. □

Moreover one has (see [**GoPr**, Lemma 2.2], [**Pl1**], [**Pl2**]):

PROPOSITION 2.1.8. *The functors $A \mapsto (\tilde{L}(A), \varphi_A)$, $\mathfrak{a} \mapsto (A_{\mathfrak{a}}, \mathcal{T}_{\mathfrak{a}})$ define an equivalence of groupoids between the groupoid of holomorphic affine rank r-bundles on X and the groupoid of augmented holomorphic rank $(r+1)$-vector bundles on X.*

Via the first functor the linearisation $L(A)$ of an affine bundle A is canonically isomorphic with $\ker(\varphi_A)$.

We will denote by (e_A) the obtained extension of locally free sheaves

$$0 \to \ker(\varphi_A) = \mathcal{L}(A) \hookrightarrow \tilde{\mathcal{L}}(A) \xrightarrow{\varphi_A} \mathcal{O}_X \to 0 \qquad (e_A)$$

which will play an importent role in this thesis.

2.2. The classification theorem

Recall (see for instance [**At**, Proposition 2 p. 184], [**Ha**, Proposition 6.3 p. 234]) that, in general, for any coherent sheaf $\mathcal{F}$ on X, isomorphism classes of extensions

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F}' \to \mathcal{O}_X \to 0 \qquad (e)$$

of $\mathcal{O}_X$ by $\mathcal{F}$ correspond bijectively to elements of $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{F}) = H^1(X, \mathcal{F})$ via the extension class (invariant) map

$$(e) \mapsto h_e \coloneqq \delta(1),$$

where $\delta : H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{F})$ is the connecting morphism associated with the exact sequence (e).

DEFINITION 2.2.1. *Let A be an affine bundle over a complex manifold X and (e_A) the associated short exact sequence (see Section 2.1). We define the cohomology invariant of A by $h_A \coloneqq h_{e_A} \in H^1(X, \mathcal{L}(A))$.*

Using the functoriality of the connecting morphism with respect to morphisms of short exact sequences, it follows easily:

REMARK 2.2.2. *Consider an exact sequence of the form (e), a sheaf isomorphism $f : \mathcal{G} \to \mathcal{F}$ and the exact sequence*

$$0 \to \mathcal{G} \xrightarrow{j \circ f} \mathcal{F}' \to \mathcal{O}_X \to 0 \qquad (f^*(e))$$

Then

$$h_{f^*(e)} = H^1(f)^{-1}(h_e).$$

We will apply the general results explained above to the extensions associated with affine bundles (see section 2).

LEMMA 2.2.3. *Let A, B be affine bundles on X and $g : L(A) \to L(B)$ be a vector bundle isomorphism such that $H^1(g)(h_A) = h_B$. There exists an affine bundle isomorphism $\gamma : A \to B$ such that $L(\gamma) = g$.*

PROOF. By Remark 2.2.2 we know that the extension invariant of

$$0 \to \mathcal{L}(A) \xrightarrow{\iota_B \circ g} \tilde{\mathcal{L}}(B) \xrightarrow{\varphi_B} \mathcal{O}_X \to 0 \qquad (g^*(e_B))$$

is $H^1(g^{-1})(h_B) = h_A$. Therefore this extension is isomorphic to (e_A), so there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}(A) & \xrightarrow{\iota_A} & \tilde{\mathcal{L}}(A) & \xrightarrow{\varphi_A} & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \wr\downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathcal{L}(A) & \xrightarrow{\iota_B \circ g} & \tilde{\mathcal{L}}(B) & \xrightarrow{\varphi_B} & \mathcal{O}_X & \longrightarrow & 0.
\end{array}$$

Using Remark 2.1.7 note now that $\tilde{\gamma}$ induces an isomorphism of affine bundles

$$\gamma : A = \varphi_A^{-1}(1) \xrightarrow{\simeq} \varphi_B^{-1}(1) = B.$$

The same diagram shows that the isomorphism $\ker(\varphi_A) = \mathcal{L}(A) \to \ker(\varphi_B) = \mathcal{L}(B)$ induced by $\tilde{\gamma}$ coincides with g.

□

We can prove now the following

THEOREM 2.2.4. *Let X be a complex manifold, E a holomorphic vector bundle of rank r on X, $\mathcal{E}$ the associated locally free sheaf and $\mathrm{Aut}(E) = \mathrm{Aut}(\mathcal{E})$ its automorphism group. The map $A \mapsto h_A \in H^1(X, \mathcal{L}(A))$ induces a bijection between the set of isomorphism classes of holomorphic affine rank r-bundles A on X with $L(A) \simeq E$ and the quotient $H^1(X, \mathcal{E})/\mathrm{Aut}(E)$.*

PROOF. For the surjectivity, let $\beta \in H^1(X, \mathcal{E})$. By Atiyah's classification theorem mentioned above, there exists an exact sequence

$$0 \to \mathcal{E} \xrightarrow{j} \mathcal{E}' \xrightarrow{\varphi} \mathcal{O}_X \to 0 \qquad (e)$$

with $h_e = \beta$. Let E' be the vector bundle associated with $\mathcal{E}'$ and use the same symbol for the bundle epimorphism $E' \to X \times \mathbb{C}$ associated with φ. Put $\mathfrak{a} \coloneqq (E', \varphi)$. By Remark 2.1.7 it follows that $A_{\mathfrak{a}} = \varphi^{-1}(X \times \{1\})$ is an affine bundle whose linearization $L(A_{\mathfrak{a}})$ is identified with $\ker(\varphi)$. But j induces an isomorphism $E \xrightarrow{\simeq} \ker(\varphi)$.

For the injectivity, let A, B be two affine bundles of rank r on X with $L(A) \simeq E$, $L(B) \simeq E$ and fix vector bundle isomorphisms $f : E \xrightarrow{\simeq} L(A)$, $g : E \xrightarrow{\simeq} L(B)$. We have to prove that, assuming that $H^1(f^{-1})(h_A)$, $H^1(g^{-1})(h_B)$ belong to the same $\mathrm{Aut}(E)$-orbit in $H^1(X, \mathcal{E})$, then $A \simeq B$.

Choose a vector bundle automorphism $\psi \in \mathrm{Aut}(E)$ such that

$$H^1(\psi^{-1})(H^1(f^{-1})(h_A)) = H^1(g^{-1})(h_B).$$

By Lemma 2.2.3, it follows that there exists an affine bundle isomorphism $\gamma : A \to B$ such that $L(\gamma) = g \circ \psi^{-1} \circ f^{-1}$. Then A is isomorphic to B. □

By Cartan's finiteness theorem we have:

REMARK 2.2.5. *If X is a compact complex manifold, then the space $H^1(X, \mathcal{E})$ intervening in Theorem 2.2.4 is a complex space of finite dimension.*

Moreover one has

PROPOSITION 2.2.6. *Let X be a compact connected complex manifold, E a holomorphic vector bundle on X and $\mathcal{E}$ the associated locally free sheaf. Then $\mathrm{Aut}(E) = \mathrm{Aut}(\mathcal{E})$ is an affine algebraic group.*

PROOF. We will prove that $\mathrm{Aut}(\mathcal{E})$ is an affine Zariski open (in the algebraic sense) subset of the finitely dimensional complex vector space $H^0(X, \mathcal{E}nd(E))$, and that the composition and the inversion maps are regular.

Let $\mathcal{E}nd(E)$ be the sheaf of (local) holomorphic sections of the bundle $\mathrm{End}(E)$. For any $f \in H^0(X, \mathcal{E}nd(E))$, the determinant map $\det(f) \in \mathcal{O}(X)$ is constant, because X is compact and connected. Fix $x_0 \in X$. A section $f \in H^0(X, \mathcal{E}nd(E))$ is an automorphism of E if and only if $\det(f)(x_0) \neq 0$. Therefore

$$\mathrm{Aut}(E) = H^0(X, \mathcal{E}nd(E)) \backslash \delta_{x_0}^{-1}(0),$$

where $\delta_{x_0} : H^0(X, \mathcal{E}nd(E)) \to \mathbb{C}$ is defined by

$$\delta_{x_0}(f) = \det(f)(x_0).$$

In the commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{E}nd(E)) & \xrightarrow{\mathrm{ev}_{x_0}} & \mathrm{End}(E_{x_0}) \\ & {\scriptstyle \delta_{x_0}} \searrow \quad \swarrow {\scriptstyle \det} & \\ & \mathbb{C} & \end{array}$$

the evaluation map ev_{x_0} at x_0 is linear, which proves that $\delta_{x_0}(f)$ is a (homogeneous) polynomial map of degree r, so (by [**Ha**, Exercice 2.1 p. 79]) the group $\mathrm{Aut}(\mathcal{E})$ is an open affine subvariety of the vector space $H^0(X, \mathcal{E}nd(E))$, as claimed.

The composition map $\mathrm{Aut}(E)\times\mathrm{Aut}(E)\to\mathrm{Aut}(E)$ is the restriction to $\mathrm{Aut}(E)$ of the bilinear map $H^0(X,\mathcal{E}nd(E))\times H^0(X,\mathcal{E}nd(E))\to H^0(X,\mathcal{E}nd(E))$ given by composition of endmorphisms, so it is regular.

For the inversion map: Via the canonical isomorphism $\mathcal{E}\xrightarrow{\simeq}(\wedge^{r-1}\mathcal{E})^\vee\otimes\det(\mathcal{E})$ (see [**Ha**, Exercice 5.16, section II.5]) the inversion map $f\mapsto f^{-1}$ is given by $f\mapsto {}^{\mathrm{t}}(\wedge^{r-1}(f))\det(f)^{-1}$. It suffices to note that $f\mapsto {}^{\mathrm{t}}(\wedge^{r-1}(f))$ is the restriction to $\mathrm{Aut}(\mathcal{E})$ of a homogenous polynomial map of degree $r-1$ on the vector space $H^0(X,\mathcal{E}nd(E))$ and, again by [**Ha**, Exercice 2.1 p. 79], $f\mapsto\det(f)^{-1}$ is a regular function on the affine subvariety $\mathrm{Aut}(\mathcal{E})$.

□

2.3. Affine bundles over $\mathbb{P}^1_{\mathbb{C}}$

Let $\mathcal{E}$ be a locally free sheaf of rank r on $\mathbb{P}^1_{\mathbb{C}}$. By Grothendieck's classification theorem [**Gro**] we may assume that $\mathcal{E}=\bigoplus_{i=1}^m\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_i)^{\oplus s_i}$ where $n_i\in\mathbb{Z}$ satisfy

$$n_1>\cdots>n_m,\ r=\sum_{i=1}^m s_i,$$

which gives an identification

$$\mathcal{E}^*\otimes\mathcal{E}=\bigoplus_{i,j=1}^m\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_i-n_j)^{\oplus s_is_j}$$

An element of the space $\mathrm{End}(\mathcal{E})=H^0(\mathbb{P}^1_{\mathbb{C}},\mathcal{E}^*\otimes\mathcal{E})$ of global endomorphisms of $\mathcal{E}$ can be identified with an upper triangular matrix

$$A=(A_{ij})_{1\leqslant i,j\leqslant m},$$

where

$$A_{ij}\in M_{s_is_j}(H^0(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_i-n_j))=M_{s_is_j}(\mathbb{C}[X_0,X_1]_{n_i-n_j}).$$

Here we use the notation $\mathbb{C}[X_0,X_1]_d$ for the complex vector space of homogeneous polynomials of degree d in the variables X_0, X_1. We will use the convention $\mathbb{C}[X_0,X_1]_d=0$ for $d<0$.

Note that the i-th diagonal element A_{ii} of A belongs to $M_{s_is_i}(\mathbb{C})$ and an element $A\in\mathrm{End}(\mathcal{E})$ is invertible if and only if $\det(A_{ii})\neq 0$ for $1\leqslant i\leqslant m$. Therefore

$$\begin{aligned}\mathrm{Aut}(\mathcal{E})=\big\{(A_{ij})_{1\leqslant i,j\leqslant m}\big|\ A_{ij}\in M_{s_is_j}(\mathbb{C}[X_0,X_1]_{n_i-n_j}),\\ A_{ii}\in\mathrm{GL}(s_i,\mathbb{C})\text{ for }1\leqslant i\leqslant m\big\}.\end{aligned}\tag{2}$$

This is obviously an affine algebraic group, which confirms the general Proposition 2.2.6 proved above.

Taking into account the isomorphism $\mathcal{K}_{\mathbb{P}^1_{\mathbb{C}}}\simeq\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$, and using the Serre Duality theorem we obtain the following:

$$\begin{aligned}H^1(X,\mathcal{E})=&\bigoplus_{i=1}^m H^1(\mathbb{P}^1_{\mathbb{C}},\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_i))^{\oplus s_i}\simeq\bigoplus_{i=1}^m H^0(\mathbb{P}^1_{\mathbb{C}},\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2-n_i))^{*\oplus s_i}\\ \simeq&\bigoplus_{i=1}^m(\mathbb{C}[X_0,X_1]^*_{l_i})^{\oplus s_i},\end{aligned}\tag{3}$$

where $l_i:=-2-n_i$.

Taking into account Theorem 2.2.4, we obtain the following general result which solves the classification problem of affine bundles of arbitrary rank on $\mathbb{P}^1_{\mathbb{C}}$:

Proposition 2.3.1. *The moduli space of affine bundles on $\mathbb{P}^1_{\mathbb{C}}$ with fixed linearisation type $[\bigoplus_{i=1}^m\mathcal{O}_{\mathbb{P}^1}(n_i)^{\oplus s_i}]$ is naturally identified with the quotient*

$$\bigoplus_{i=1}^m(\mathbb{C}[X_0,X_1]^*_{-2-n_i})^{\oplus s_i}/G$$

where $G=\mathrm{Aut}(\bigoplus_{i=1}^m\mathcal{O}_{\mathbb{P}^1}(n_i)^{\oplus s_i})$ is the affine algebraic group

$$G=\big\{(A_{ij})_{1\leqslant i,j\leqslant m}\big|\ A_{ij}\in M_{s_is_j}(\mathbb{C}[X_0,X_1]_{n_i-n_j}),A_{ii}\in\mathrm{GL}(s_i,\mathbb{C})\text{ for }1\leqslant i\leqslant m\big\}$$

acting on $\bigoplus_{i=1}^m(\mathbb{C}[X_0,X_1]^*_{-2-n_i})^{\oplus s_i}$ as follows: for any $A=(A_{ij})_{1\leqslant i,j\leqslant m}\in G$ and $\Phi:=(\Phi_i)_{1\leqslant i\leqslant m}\in H^1(X,\mathcal{E})$,

$$A\bullet\Phi=(\sum_{j=1}^m{}^{\mathrm{t}}m_{A_{ij}}(\Phi_j))_{1\leqslant i\leqslant m}.$$

Here $m_{A_{ij}}$ stands for the multiplication map $\mathbb{C}[X_0,X_1]^{\oplus s_i}_{-2-n_i} \to \mathbb{C}[X_0,X_1]^{\oplus s_j}_{-2-n_j}$ by the matrix A_{ij} from the right, and ${}^{\mathrm{t}}m_{A_{ij}}$ stands for its transpose.

This result shows already how difficult is the classification of affine bundles compared to the classification of vector bundles. Note that this quotient is in general highly non-Hausdorff and difficult to describe geometrically in full generality.

2.4. The cokernel problem for linear spaces

This subsection is not directly related to the theory of affine bundles but will play crucial role in section 3.2.2. For a morphism $\varphi : L \to L'$ of linear spaces over a complex manifoild X we will introduce two cokernels: the categorical cokernel $\mathrm{coker}(\varphi)$, which is again a linear space over X (but, in general, is not the union of the fibrewise cokernels) and the topological cokernel $\mathrm{Coker}(\varphi)$, which, in general, is not a linear space. $\mathrm{Coker}(\varphi)$ coincides with the union $\bigcup_{x\in X}\mathrm{coker}(\varphi_x)$ endowed with a natural topology.

Let X be a complex manifold. Recall [**Fi**] that category of linear spaces over X is anti-equivalent to the category of coherent sheaves over X under the functor which associates to a linear space L on X the coherent sheaf $\mathscr{L}(L)$ of linear forms on L.

Using this equivalence one can define

Definition 2.4.1. *Let $\varphi : L \to L'$ be a morphism of linear spaces over X and let $\varphi^\vee : \mathscr{L}(L') \to \mathscr{L}(L)$ be the associated morphism of coherent sheaves. The categorical cokernel of φ is the linear space* $\mathrm{coker}(\varphi)$ *associated with the coherent sheaf* $\ker(\varphi^\vee)$.

Note that $\mathrm{coker}(\varphi)$ comes with a natural epimorphism of linear spaces

$$c : L' \to \mathrm{coker}(\varphi)$$

with $c\circ\varphi = 0$ satisfying the following universal property: for any linear space morphism $\gamma : L' \to L''$ with $\gamma\circ\varphi = 0$, there exists an unique $\tilde\gamma : \mathrm{coker}(\varphi) \to L''$ such that $\gamma = \tilde\gamma \circ c$.

Remark 2.4.2. *In general*

1. *An epimorphism of linear spaces is not necessarily fibrewise surjective.*
2. *The underlying total space of the cokernel* $\mathrm{coker}(\varphi)$ *of a morphism of linear spaces over X does not coincide with the union* $\bigcup_{x\in X}\mathrm{coker}(\varphi_x)$ *of fibrewise cokernels of φ.*

Exemple 2.1. Let L be a line bundle over X with $H^0(X,\mathcal{L}) \neq 0$ and let $f : \mathcal{O}_X \to \mathcal{L}$ be a non-trivial section of L (or, equivalently, of the associated invertible sheaf $\mathcal{L}$). Note that φ defines a morphism of (locally trivial) linear spaces

$$\varphi : X \times \mathbb{C} \to L$$

over X. The morphism of coherent sheaves which corresponds to φ via the anti-equivalence $\mathscr{L}$ is ${}^{\mathrm{t}}f : \mathcal{L}^* \to \mathcal{O}_X^* = \mathcal{O}_X$.

Denoting by D the zero devisor of φ, we can identify this morphism with the canonical embedding $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$, and, via this identification, ${}^{\mathrm{t}}f$ coincides with the canonical embedding $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$. Therefore $\ker({}^{\mathrm{t}}f) = 0$, so $\mathrm{coker}(\varphi) = 0$. This shows that φ is always an epimorphism of linear spaces. But, when $D \neq \varnothing$, φ is not fibrewise surjective.

This justifies the following

Definition 2.4.3. *Let $p : L \to X$, $p' : L' \to X$ linear spaces over X and $\varphi : L \to L'$ be a morphism of linear spaces over X. The topological cokernel of φ is the topological quotient of L' by the equivalence relation R_φ given by*

$$y' \; R_\varphi \; z' \text{ if } p'(y') = p'(z) \text{ and } \exists y \in L \text{ such that } z' - y' = \varphi(y).$$

This quotient will be denote by $\mathrm{Coker}(\varphi)$.

In other words, $\mathrm{Coker}(\varphi)$ is the underlying space of the relative quotient stack L'/L over X, where L acts on L' via φ fibrewise over X. This relative quotient stack is called the *stacky cokernel* of φ in [**Pl2**, section I.3].

Exemple 2.2. Consider the line bundles $L = L' = \mathbb{C}\times\mathbb{C}$ over $\mathbb{C}$ and the morphism $\varphi : L \to L'$ defined by $\varphi(z,\zeta) = z\zeta$. The topological cokernel $\mathrm{Coker}(\varphi)$ is a "line with an infinite family of origins". The fibre $\mathrm{Coker}(\varphi)_z$ over a point $z \in \mathbb{C}^*$ is trivial, whereas $\mathrm{Coker}(\varphi)_0$ is a complex line. Any two points in $\mathrm{Coker}(\varphi)_0$ are non-separable by disjoint neighbourhoods, so $\mathrm{Coker}(\varphi)$ is non-Hausdorff.

EXEMPLE 2.3. Suppose that L, L' are holomorphic vector bundles over X, let $\mathcal{L}$, $\mathcal{L}'$ be the associated locally free shaves of locally defined sections and let $f : \mathcal{L} \to \mathcal{L}'$ be the sheaf morphism associated with φ. The underlying set of $\mathrm{Coker}(\varphi)$ can be identified with the disjoint union

$$\coprod_{x\in X} \mathrm{coker}(f)(x)$$

of the fibres $\mathrm{coker}(f)(x) = \mathrm{coker}(f)_x/\mathfrak{m}_x\mathrm{coker}(f)_x$ of the coherent sheaf $\mathrm{coker}(f)$. Taking into account a classical semi-continuity theorem (see [**GrRe**, section 4.4]) it follows that the map

$$X \ni x \mapsto \dim(\mathrm{coker}(f)(x)) = \dim(\mathrm{Coker}(\varphi)_x)$$

is upper semicontinuous with respect to the Zariski topology.

2.5. Apolar ideals, the Sylvester theorem and the cactus rank

2.5.1. General Theory. Consider the apolar action

$$\bullet : \mathbb{C}[\xi_0, \xi_1] \times \mathbb{C}[X_0, X_1] \to \mathbb{C}[X_0, X_1],\ (Q, P) \mapsto Q \bullet P \coloneqq Q(\frac{\partial}{\partial X_0}, \frac{\partial}{\partial X_1})(P(X_0, X_1)).$$

For a fixed homogeneous polynomial $P \in \mathbb{C}[X_0, X_1]$ denote by $\mathrm{Ann}(P)$ the *apolar ideal* of P defined as below:

$$\mathrm{Ann}(P) \coloneqq \{Q \in \mathbb{C}[\xi_0, \xi_1] |\ Q \bullet P = 0\ \}.$$

EXEMPLE 2.4. We have

$$\mathrm{Ann}(X_0^m X_1^m (X_0 + X_1)) = (\xi_0^{m+1} - \sum_{k=1}^{m} (-1)^k \xi_0^{m-k} \xi_1^k, \xi_1^{m+2}).$$

This formula is obtained by proving that the remainder of the Euclidian division of a polynomial $g \in \mathrm{Ann}(X_0^m X_1^m (X_0 + X_1))$ by ξ_1^{m+2} is divisible by $\xi_0^{m+1} - \sum_{k=1}^{m} (-1)^k \xi_0^{m-k} \xi_1^k$.

We denote by $\sim$ the equivalence relation on $\mathbb{C}[X_0, X_1]_l$ induced by the right $\mathrm{GL}(2, \mathbb{C})$-action induced by composition with linear transformations. One has the well-known result (see [**DS2**, Lemma 2.2]):

LEMMA 2.5.1. *(1)* $\mathrm{Ann}(P)_0 \neq 0$ *if and only if* $P = 0$.
(2) $\mathrm{Ann}(P)_0 = 0$ *and* $\mathrm{Ann}(P)_1 \neq 0$ *if and only if* $P \sim X_0^l$.
(3) Suppose $l \geqslant 3$. *We have simultaneously* $\mathrm{Ann}(P)_1 = 0$ *and* $\mathrm{Ann}(P)_2 = \mathbb{C}\lambda^2$ *(for a linear form* λ*) if and only if* $P \sim X_0^{l-1} X_1$.

Recall that the *Waring rank* $\mathrm{rk}(P)$ of a binary form $P \in \mathbb{C}[X_0, X_1]_l \backslash \{0\}$ is the smallest $r \in \mathbb{N}$ such that there exists linear forms $\lambda_1, \ldots, \lambda_r$ with

$$P = \lambda_1^l + \cdots + \lambda_r^l.$$

Such a decomposition always exists because $\mathbb{C}[X_0, X_1]_l$ is generated by the powers λ^l of linear forms on $\mathbb{C}^2$, see [**Re**, Theorem 4.2], which also establishes the inequality $\mathrm{rk}(P) \leqslant l$.

DEFINITION 2.5.2. *Let* P *be a non-trivial binary form. The cactus rank* $\mathrm{crank}(P)$ *of* P *is the minimal length* $l(Z)$ *of a* 0*-dimensional subscheme* Z *in* $\mathbb{P}^1_{\mathbb{C}}$ *defined by a homogeneous saturated ideal* I_Z *contained in* $\mathrm{Ann}(P)$.

Note that the Waring and the cactus ranks are obviously invariant under the natural $\mathrm{GL}(2, \mathbb{C})$-action on $\mathbb{C}[X_0, X_1]_l$.

We have the following important *structure theorem* due to Sylvester, see [**IK**, Theorem 1.44, Claim p. 31], [**DS2**, Theorem 2.3]:

THEOREM 2.5.3. *Let* $P \in \mathbb{C}[X_0, X_1]_l \backslash \{0\}$. *The ideal* $\mathrm{Ann}(P)$ *is generated by two homogeneous polynomials* G_1, G_2 *with no common zeros satisfying* $\deg(G_1) + \deg(G_2) = l + 2$ *and* $\deg(G_1) \leqslant \deg(G_2)$.

Moreover the Waring rank $\mathrm{rk}(P)$ *is equal to* $\deg(G_1)$ *if* G_1 *has no multiple roots, and it is equal to* $\deg(G_2)$ *otherwise.*

We will use the notation $d_i(P) \coloneqq \deg(G_i)$ for $1 \leqslant i \leqslant 2$.

The dimensions of the homogeneous summands $\mathrm{Ann}(P)_d$ of the apolar ideal $\mathrm{Ann}(P)$ in the interesting case $d \leqslant \deg(P)$ can be easily computed in terms of the degrees $d_i(P)$:

REMARK 2.5.4. *Let P be a non-trivial binary form. Then*

$$\dim(\mathrm{Ann}(P)_d) = \begin{cases} 2d - \deg(P) & \text{if } d_2(P) \leqslant d \leqslant \deg(P), \\ d + 1 - d_1(P) & \text{if } d_1(P) \leqslant d < d_2(P), \\ 0 & \text{if } d < d_1(P). \end{cases}$$

PROOF. Any polynomial $\Pi \in \mathrm{Ann}(P)_d$ decomposes as

$$\Pi = \Pi_1 G_1 + \Pi_2 G_2$$

where $\Pi_i \in \mathbb{C}[X_0, X_1]_{d-d_i(P)}$ (with the convention $\mathbb{C}[X_0, X_1]_s = 0$ for $s < 0$). This shows that the linear map

$$\mathbb{C}[X_0, X_1]_{d-d_1(P)} \times \mathbb{C}[X_0, X_1]_{d-d_2(P)} \ni (\Pi_1, \Pi_2) \mapsto \Pi_1 G_1 + \Pi_2 G_2$$

is surjective. The injectivity of this map is clear when $d < d_2(P)$, because in this case at least one of the two spaces $\mathbb{C}[X_0, X_1]_{d-d_i(P)}$ vanishes. For $d_2(P) \leqslant d \leqslant \deg(P)$ the injectivity follows taking into account that two homogeneous polynomials G_1, G_2 have no commun zero and that $d_1(P) + d_2(P) = \deg(P) + 2$. □

Taking into account that $\deg(G_1) + \deg(G_2) = \deg(P) + 2$, $\deg(G_1) \leqslant \deg(G_2)$ and that one of the two degrees $\deg(G_1)$, $\deg(G_2)$ coincides with the $\mathrm{rk}(P)$, it follows that ([**DS2**, Corollary 2.5]):

REMARK 2.5.5. *For any non-trivial binary form P of degree l we have*

$$d_1(P) = \begin{cases} \mathrm{rk}(P) & \text{if} \quad \mathrm{rk}(P) \leqslant \frac{l+2}{2} \\ l + 2 - \mathrm{rk}(P) & \text{if} \quad \mathrm{rk}(P) \geqslant \frac{l+2}{2}. \end{cases}$$

REMARK 2.5.6. *We have*

$$d_1(P) = \min\{\deg(H) |\ H \text{ homogeneous, } H \in \mathrm{Ann}(P) \backslash \{0\}\}.$$

In other words

$$d_1(P) = \min\{s \in \mathbb{N}^* |\ \mathbb{C}[\xi_0, \xi_1]_s \ni Q \mapsto Q \bullet P \in \mathbb{C}[X_0, X_1]_{l-s} \text{ is not injective}\}.$$

COROLLARY 2.5.7. *The map $\mathbb{C}[X_0, X_1]_l \backslash \{0\} \ni P \mapsto d_1(P) \in \mathbb{N}^*$ is lower semicontinuous.*

Note that the Waring rank is in general *not* semicontinuous on $\mathbb{C}[X_0, X_1]_l \backslash \{0\}$.

EXEMPLE 2.5.
(1) Let m, $n \in \mathbb{N}^*$. We have $\mathrm{Ann}(X_0^m X_1^n) = (\xi_0^{m+1}, \xi_1^{n+1})$, so, by Theorem 2.5.3, $\mathrm{rk}(X_0^m X_1^n) = \max(n, m) + 1$.
(2) Let $P(X_0, X_1) = X_0^3 X_1 (X_0 + X_1)$. Then $\mathrm{Ann}(P) = (X_1^3, G_2)$ where G_2 is a homogeneous polynomial of degree 4. Then $\mathrm{rk}(P) = 4$.
(3) Coming back to the Example 2.4 we obtain $\mathrm{rk}(X_0^m X_1^m (X_0 + X_1)) = m + 1$.

The following theorem (see [**DS2**, Theorem 2.4]) concerns binary forms of $\mathrm{rk}(P) \geqslant 2$.

THEOREM 2.5.8. *Let P be a binary form of degree l and Waring rank $\mathrm{rk}(P) \geqslant 2$. The known inequality $\mathrm{rk}(P) \leqslant \deg(P)$ becomes an equality if and only if $P \sim X_0^{l-1} X_1$. Moreover, for a generic binary form P of degree l we have*

$$\mathrm{rk}(P) = \left[\frac{l+2}{2} \right].$$

We are grateful to Alexandru Dimca for pointing me out the following result, which plays an important role in this chapter:

PROPOSITION 2.5.9. *Let $P \in \mathbb{C}[X_0, X_1]_l \backslash \{0\}$. Then $d_1(P) = \mathrm{crank}(P)$.*

PROOF. For a homogeneous ideal $I \subset \mathbb{C}[X_0, X_1]$ let $V(I)$ stand for subscheme of $\mathbb{P}^1_{\mathbb{C}}$ associated with I [**Ha**, section II.2]. Note that the assignment $[G] \mapsto V(G)$ defines bijection between $\mathbb{P}^1(\mathbb{C}[X_0, X_1]_d)$ and the set of 0-dimensional subschemes $V \subset \mathbb{P}^1_{\mathbb{C}}$ of length d. It suffices to note that the ideal (G) generated by a polynomial $G \in \mathbb{C}[X_0, X_1]_d \backslash \{0\}$ is saturated and to take into account Remark 2.5.6. □

Taking into account Proposition 2.5.9 and Remark 2.5.5 it follows:

REMARK 2.5.10. *The map* $\mathrm{crank} : \mathbb{C}[X_0, X_1]_l \backslash \{0\} \to \mathbb{N}^*$ *given by the cactus rank is lower semicontinuous. The cactus rank and the waring rank are related by the formula*

$$\mathrm{crank}(P) = \begin{cases} \mathrm{rk}(P) & \text{if} \quad \mathrm{rk}(P) \leqslant \frac{l+2}{2} \\ l + 2 - \mathrm{rk}(P) & \text{if} \quad \mathrm{rk}(P) \geqslant \frac{l+2}{2}. \end{cases}$$

This implies:

REMARK 2.5.11. *Let P be a non-trivial binary form of degree $l \geqslant 2$. Then $P \sim X_0^l$ if and only if* $\operatorname{crank}(P) = 1$.

Taking into account Theorem 2.5.8, we also infer:

REMARK 2.5.12. *Put $P_r^l \coloneqq \{[P] \in \mathbb{P}(\mathbb{C}[X_0, X_1]_l) |\ \operatorname{crank}(P) = r\}$.*
Then $(P_l^r)_{1 \leqslant r \leqslant [\frac{l+2}{2}]}$ is a partition of the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ in locally closed subsets. For $1 \leqslant s \leqslant [\frac{l+2}{2}]$ the union $\bigcup_{1 \leqslant r \leqslant s} P_r^l$ is Zariski closed and the union $\bigcup_{s \leqslant r \leqslant [\frac{l+2}{2}]} P_r^l$ is Zariski open. The highest index stratum $P^l_{[\frac{l+2}{2}]}$ is Zariski open and non-empty.

Note that

$$[P] \in P^l_{[\frac{l+2}{2}]} \Leftrightarrow \begin{cases} \operatorname{rk}(P) = \frac{l+2}{2} & \text{if} \quad l \in 2\mathbb{N}^*, \\ \operatorname{rk}(P) \in \{\frac{l+1}{2}, \frac{l+3}{2}\} & \text{if} \quad l \in 2\mathbb{N}+1. \end{cases}$$

Remark 2.5.12 shows that $(P_l^r)_{1 \leqslant r \leqslant [\frac{l+2}{2}]}$ is a stratification of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ (in the sense of [**Sta**, Definition 5.28.3, part 1]) with locally closed strata. This stratification will be called *the cactus rank stratification* of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$.

There is a lower bound for the Waring rank of binary forms based in terms of their factorization to a product of linear forms (See [**Ne**, Theorem 2.1]). Let $P \in \mathbb{C}[X_0, X_1]_l \backslash \{0\}$ be a nonzero binary form of degree l which factorizes as

$$P(X_0, X_1) \coloneqq \prod_{i=0}^{s} \lambda_i(X_0, X_1)^{n_i},$$

where $s \geqslant 1$, λ_i's are pairwise linearly independent linear forms and $n_i \geqslant 1$. Then:

$$\operatorname{rk}(P) \geqslant \max(n_0, \ldots, n_s) + 1. \tag{4}$$

For example, let $P = \lambda_1^{d-2}\lambda_2\lambda_3$, where $d \geqslant 3$ and the λ_i's are pairwise linearly independent linear forms. Combining the inequality (4) with [**DS2**, Theorem 2.4], we obtain

$$\operatorname{rk}(\lambda_1^{d-2}\lambda_2\lambda_3) = d - 1. \tag{5}$$

2.5.2. The cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ for small l. In this section we will give explicit geometric descriptions of the cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$ defined in the previous section.

We recall that the discriminant [**Mu**, p. 23] of binary forms defines a homogeneous map $\Delta : \mathbb{C}[X_0, X_1]_l \to \mathbb{C}$ of degree $2(l-1)$. A binary form P has multiple roots if and only if $\Delta(P) = 0$. For $l \in \mathbb{N}^*$ we will denote by

$$v_l : \mathbb{P}(\mathbb{C}[X_0, X_1]_1) \simeq \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}(\mathbb{C}[X_0, X_1]_l)$$

the Veronese map $[\lambda] \mapsto [\lambda^l]$.

The case $l = 2$. The level sets of the cactus rank on the plane $\mathbb{P}(\mathbb{C}[X_0, X_1]_2) \simeq \mathbb{P}^2$ are:

$$\operatorname{crank}^{-1}(1) = Z(\Delta) = \operatorname{im}(v_2),$$

$$\operatorname{crank}^{-1}(2) = \mathbb{P}(\mathbb{C}[X_0, X_1]_2) \backslash Z(\Delta).$$

Note that $\Delta(a_0X_0^2 + 2a_1X_0X_1 + a_2X_1^2) = a_0a_1 - a_1^2$, so the closed stratum

$$\operatorname{crank}^{-1}(1) = Z(\Delta)$$

is a conic.

The case $l = 3$. By Remark 2.5.11 the level sets of the cactus rank when on the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1]_3)$ of binary cubic forms are:

$$\operatorname{crank}^{-1}(1) = v_3(\mathbb{P}^1_{\mathbb{C}}),\ \operatorname{crank}^{-1}(2) = \mathbb{P}(\mathbb{C}[X_0, X_1]_3) \backslash \operatorname{im}(v_3).$$

The case $l = 4$. Recall that we have two semi-invariant polynomials g_2 and g_3 on the space $\mathbb{C}[X_0, X_1]_4$ with respect to the $\mathrm{GL}(2, \mathbb{C})$ action on the space of binary quartic forms defined by the formula

$$\begin{vmatrix} a_0 & a_1 & a_2 + 2t \\ a_1 & a_2 - t & a_3 \\ a_2 + 2t & a_3 & a_4 \end{vmatrix} = 4t^3 - g_2(Q)t + g_3(Q),$$

where

$$Q(X_0, X_1) = a_0X_0^4 + 4a_1X_0^3X_1 + 6a_2X_0^2X_1^2 + 4a_3X_0X_1^3 + a_4X_1^4.$$

(see [**Mu**, Proposition 1.25, p.27]). Their weights are 8 and 12 respectively. The explicit formulae for the g_i are

$$g_2(Q) = a_0a_4 - 4a_1a_3 + 3a_2^2,$$
$$g_3(Q) = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4.$$

Moreover, by [**Mu**, Remark 1.26], we have the following identity

$$\Delta = g_2^3 - 27g_3^2. \tag{6}$$

The J invariant is defined as

$$J(Q) \coloneqq \frac{g_2(Q)^3}{\Delta(Q)}$$

which is a regular function on $\mathbb{P}(\mathbb{C}[X_0, X_1]_4)\backslash Z(\Delta)$. The fibres of J are precisely the orbits of the $\mathrm{GL}(2, \mathbb{C})$-action on this Zariski open set.

We will use the following table which computes the Waring, the cactus rank and the value of g_3 for the canonical forms of binary quartics:

Q	$\mathrm{rk}(Q)$	$\mathrm{crank}(Q)$	$g_3(Q)$
X_0^4	1	1	0
$X_0^3X_1$	4	2	0
$X_0^2X_1^2$	3	3	$-\frac{1}{6^3}$
$X_0^2X_1(X_0 + X_1)$	3	3	$-\frac{1}{6^3}$
$X_0X_1(X_0 + X_1)(X_0 + tX_1),\ t \in \{-1, \frac{1}{2}, 2\}$	2	2	0
$X_0X_1(X_0 + X_1)(X_0 + tX_1),\ t \notin \{-1, \frac{1}{2}, 2\}$	3	3	$\frac{(t-2)(t+1)(2t-1)}{144}$

For the Waring rank in the first three rows we have used Example 2.5 and formula (5). For the Waring rank in the last two rows we have used [**DS1**, Theorem 3.1] which stated that the Waring rank of a quartic form without multiple roots is either 3, when $J(Q) \neq 1$, or 2, when $J(Q) = 1$. By formula (6) we have $J(Q) = 1$ if and only if $g_3(Q) = 0$. The cactus rank column has been filled in using Remark 2.5.10; the values in the third column have been obtained by direct computation.

We can now prove

REMARK 2.5.13. *The strata of the cactus rank stratification of the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1]_4)$ of quartic forms are given by*

$$\mathrm{crank}^{-1}(1) = \mathrm{im}(v_4),$$
$$\mathrm{crank}^{-1}(2) = Z(g_3)\backslash\mathrm{im}(v_4),$$
$$\mathrm{crank}^{-1}(3) = \mathbb{P}^3(\mathbb{C})\backslash Z(g_3).$$

PROOF. The first equality follows again by Remark 2.5.11. The second and the third equalities follow by comparing the second and the third columns of the above table and taking into account the $\mathrm{GL}(2, \mathbb{C})$-equivariance of g_3 and the $\mathrm{GL}(2, \mathbb{C})$-invariance of the cactus rank. □

Therefore

REMARK 2.5.14. *The filtration of $\mathbb{P}(\mathbb{C}[X_0, X_1]_4)$ with Zariski closed subsets associated with its cactus rank stratification is*

$$\mathbb{P}(\mathbb{C}[X_0, X_1]_4) \supset Z(g_3) \supset \mathrm{im}(v_4).$$

CHAPTER 3

Moduli spaces of rank 2 affine bundles on $\mathbb{P}^1$ and apolar ideals

3.1. Affine bundles of rank 2 over Riemann surfaces

Let C be a Riemann surface. We have two classes of affine bundles of rank r on C:

(1) The affine bundles A with $L(A)$ semi-stable,
(2) The affine bundles A with $L(A)$ unstable.

Since we are especially interested in the case $C = \mathbb{P}^1_{\mathbb{C}}$, we will focus on the second class. For an unstable bundle E of rank 2 we will denote by $D_E \subset E$ the maximal destabilising line subbundle of E and by

$$p_E : E \to Q_E \coloneqq E/D_E$$

the canonical epimorphism onto the corresponding quotient.

REMARK 3.1.1. *Let $f : F \to E$ be an isomorphism of unstable vector bundles. Then we have $f(D_F) = D_E$, so f induces an isomorphism $\tilde{f} : Q_F \to Q_E$ with the property*

$$\tilde{f} \circ p_F = p_E \circ f.$$

We define:

DEFINITION 3.1.2. *Let A be an affine bundle of rank 2 on C with $E \coloneqq L(A)$ unstable, and let $x_0 \in C$.*

(1) A will be called non-degenerate if $H^1(p_E)(h_A)$ is non-zero in $H^1(C, \mathcal{Q}_E)$, where $h_A \in H^1(C, \mathcal{E})$ stands for the cohomological invariant of A (see Definition 2.2.1).
(2) An x_0-framing of A is an isomorphism $\varepsilon : \mathbb{C} \xrightarrow{\simeq} D_{L(A),x_0}$.

An x_0-framed affine bundle of rank 2 is an affine bundle as above endowed with an x_0-framing. An isomorphism $(A, \varepsilon) \to (A', \varepsilon')$ of x_0-framed affine bundles is an isomorphism $\psi : A \to A'$ such that $L(\psi) \circ \varepsilon = \varepsilon'$.

Let now E be a fixed rank 2 unstable vector bundle on C. Our goal is the classification of isomorphism classes of affine bundle A on C with $L(A) \simeq E$. We will handle this problem in several steps:

(S1) Classify all x_0-framed non-degenerate affine bundles A with $L(A) \simeq E$.
(S2) Classify all non-degenerate affine bundles A with $L(A) \simeq E$.
(S3) Classify all affine bundles A with $L(A) \simeq E$

Note that, by Remark 3.1.1, any automorphism of E leaves invariants its maximal destabilising line bundle. Let $\mathrm{Aut}(E)_0$ be the closed subgroup of $\mathrm{Aut}(E)$ consisting of automorphism which induce the identity on D_E. Let also $H^1(C, \mathcal{E})_0 \subset H^1(C, \mathcal{E})$ be the Zariski open subspace defined by

$$H^1(C, \mathcal{E})_0 \coloneqq \{h \in H^1(C, \mathcal{E}) |\ H^1(p_E)(h) \neq 0\}. \tag{7}$$

Our next result concerns the classification of (framed) non-degenerate holomorphic affine bundles (A, ε) with $L(A) \simeq E$ over a Riemann surface C, where E is a fixed unstable rank 2 holomorphic vector bundle on C (see Definition 3.1.2). Taking into account Theorem 2.2.4, we will prove:

THEOREM 3.1.3. *Let C be a Riemann surface, E be a fixed rank 2 unstable vector bundle on C, $\mathcal{E}$ the associated locally free sheaf, $x_0 \in C$ and $\varepsilon_0 : \mathbb{C} \to D_{E,x_0}$ be a fixed isomorphism. Then:*

(1) The map $A \mapsto h_A$ induces a bijection between the set of isomorphism classes of non-degenerate affine holomorphic rank 2-bundles A on X with $L(A) \simeq E$ and the quotient $H^1(C, \mathcal{E})_0/\mathrm{Aut}(E)$.
(2) For an x_0-framed affine bundle (A, ε) with $L(A) \simeq E$ choose an isomorphism $f : L(A) \to E$ such that $f \circ \varepsilon = \varepsilon_0$.

The map $(A, \varepsilon) \mapsto H^1(f)(h_{e_A})$ induces a bijection between the set of isomorphism classes of x_0-framed (non-degenerate) affine holomorphic rank r-bundles A on C with $L(A) \simeq E$ and the quotient $H^1(C, \mathcal{E})/\mathrm{Aut}(E)_0$ (respectively $H^1(C, \mathcal{E})_0/\mathrm{Aut}(E)_0$).

PROOF. (1) The first statement is obvious.

(2) Note first that, under the assumption $L(A) \simeq E$, such an isomorphism f always exists.

The map

$$[(A,\varepsilon)] \mapsto [H^1(f)(h_A)]_{\mathrm{Aut}(E)_0}$$

between the set of isomorphism classes of x_0-framed non-degenerate affine holomorphic rank r-bundles A on C with $L(A) \simeq E$ and $H^1(C,\mathcal{E})_0/\mathrm{Aut}(E)_0$ is obviously well defined. It is also clearly surjective by Theorem 2.2.4.

For the injectivity, let now (A,ε), (B,σ) be two x_0-framed non-degenerate affine bundles, and $f : L(A) \xrightarrow{\simeq} E$, $g : L(B) \xrightarrow{\simeq} E$ vector bundle isomorphisms satisfying

$$f \circ \varepsilon = g \circ \sigma = \varepsilon_0,$$

such that $H^1(f)(h_B)$, $H^1(g)(h_B)$ belong to the same $\mathrm{Aut}(E)_0$-orbit.

Therefore, there exists an isomorphism $\psi \in \mathrm{Aut}(E)_0$ such that

$$H^1(\psi)(H^1(f)(h_A)) = H^1(g)(h_B).$$

Putting $b := g^{-1} \circ \psi \circ f$ note that $H^1(b)(h_A) = h_B$, so, by Lemma 2.2.3, there exists an affine bundle isomorphism $\beta : A \to B$ such that $L(\beta) = b$. It suffices to note that $L(\beta) \circ \varepsilon = b \circ \varepsilon = \sigma$, so β is an isomorphism of framed affine bundles $(A,\varepsilon) \to (B,\sigma)$.

□

In this thesis, by the moduli space of framed (non-degenerate) affine holomorphic bundles with fixed linearisation type on C we will mean the topological space obtained by endowing the corresponding moduli set with the topology induced by the identifications given by Theorem 3.1.3.

3.2. The case $r = 2$ and $X = \mathbb{P}^1_{\mathbb{C}}$

3.2.1. General classification results. In this section we will specify the action of the group $\mathrm{Aut}(\mathcal{E})$ on $H^1(\mathbb{P}^1_{\mathbb{C}},\mathcal{E})$ through the identifications given in section 2.3 and we will give explicit descriptions for the quotients intervening in Theorem 3.1.3 (in the case $X = \mathbb{P}^1_{\mathbb{C}}$).

To simplify the formalism, we will study in detail the case $r = 2$ and $s_1 = s_2 = 1$. We can assume $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_2)$ with $n_1 > n_2$. In other words we suppose that $\mathcal{E}$ is unstable (non-semistable). The maximal destabilising subsheaf of $\mathcal{E}$ is $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_1)$.

We also assume that $-2 > n_1$, so that the two cohomology spaces $H^1(\mathbb{P}^1_{\mathbb{C}},\mathcal{O}(n_i))$ are both non-trivial. Putting $l_i := -n_i - 2 > 0$, we obtain canonical identifications

$$\begin{gathered} H^1(\mathbb{P}^1_{\mathbb{C}},\mathcal{E}) = \mathbb{C}[X_0,X_1]^*_{l_1} \oplus \mathbb{C}[X_0,X_1]^*_{l_2}, \\ \mathrm{Aut}(\mathcal{E}) \simeq G := \left\{ \begin{pmatrix} u_1 & Q \\ 0 & u_2 \end{pmatrix} \middle|\ u_1,\ u_2 \in \mathbb{C}^*,\ Q \in \mathbb{C}[X_0,X_1]_d \right\} \text{ with } d := l_2 - l_1, \end{gathered} \tag{8}$$

and, via these identifications, $\mathrm{Aut}(E)$ acts on $H^1(\mathbb{P}^1_{\mathbb{C}},\mathcal{E})$ by

$$\begin{pmatrix} u_1 & Q \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} u_1\Phi_1 + {}^t m_Q(\Phi_2) \\ u_2\Phi_2 \end{pmatrix} \tag{9}$$

where $m_Q : \mathbb{C}[X_0,X_1]_{l_1} \to \mathbb{C}[X_0,X_1]_{l_2}$ denotes the linear map defined by multiplication with Q, and ${}^t m_Q$ stands for its transpose.

Note that that the group $\mathrm{Aut}(\mathcal{E})$ fits in the short exact sequence

$$\{1\} \to N := \left\{ \begin{pmatrix} 1 & Q \\ 0 & 1 \end{pmatrix} \middle|\ Q \in \mathbb{C}[X_0,X_1]_d \right\} \hookrightarrow \mathrm{Aut}(\mathcal{E}) \to \mathbb{C}^* \times \mathbb{C}^* \to \{1\} \tag{10}$$

whose kernel N is isomorphic to $\mathbb{C}[X_0,X_1]_d$ via the obvious map. Note that via the identification (8), $\mathrm{Aut}(\mathcal{E})_0$ corresponds to the subgroup $G_0 \subset G$ defined by the condition $u_1 = 1$. By the classification Theorems 2.2.4, 3.1.3 we obtain

PROPOSITION 3.2.1. *Let E be the underlying vector bundle of $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n_2)$ with $-2 > n_1 > n_2$. Put $l_i := -2 - n_i$, $d := l_2 - l_1$. Fix $x_0 \in \mathbb{P}^1_{\mathbb{C}}$. Let the groups*

$$\begin{aligned} G &:= \left\{ \begin{pmatrix} u_1 & Q \\ 0 & u_2 \end{pmatrix} \middle|\ u_1,\ u_2 \in \mathbb{C}^*,\ Q \in \mathbb{C}[X_0,X_1]_d \right\}, \\ G_0 &:= \left\{ \begin{pmatrix} 1 & Q \\ 0 & u_2 \end{pmatrix} \middle|\ u_2 \in \mathbb{C}^*,\ Q \in \mathbb{C}[X_0,X_1]_d \right\} \end{aligned} \tag{11}$$

act on $\mathbb{C}[X_0,X_1]^*_{l_1} \oplus \mathbb{C}[X_0,X_1]^*_{l_2}$ via formula (9).

(1) The moduli space of isomorphism classes of (non-degenerate) affine bundles A on $\mathbb{P}^1_{\mathbb{C}}$ with $L(A) \simeq E$ is homeomorphically identified with the quotient

$$\begin{aligned} \mathcal{Q} &\coloneqq \mathbb{C}[X_0, X_1]^*_{l_1} \oplus \mathbb{C}[X_0, X_1]^*_{l_2}/_G, \text{ respectively} \\ \mathcal{Q}_0 &\coloneqq \mathbb{C}[X_0, X_1]^*_{l_1} \times \left(\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}\right)/_G. \end{aligned} \tag{12}$$

(2) The moduli space of isomorphism classes of x_0-framed (non-degenerate) affine (A, ε) on $\mathbb{P}^1_{\mathbb{C}}$ with $L(A) \simeq E$ is homeomorphically identified with the quotient

$$\begin{aligned} \mathfrak{Q} &\coloneqq \mathbb{C}[X_0, X_1]^*_{l_1} \oplus \mathbb{C}[X_0, X_1]^*_{l_2}/_{G_0}, \text{ respectively} \\ \mathfrak{Q}_0 &\coloneqq \mathbb{C}[X_0, X_1]^*_{l_1} \times \left(\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}\right)/_{G_0}. \end{aligned} \tag{13}$$

The quotients $\mathcal{Q}$, $\mathcal{Q}_0$, $\mathfrak{Q}$, $\mathfrak{Q}_0$ will be endowed with the quotient topologies.

3.2.2. Framed non-degenerate rank 2 affine bundles over $\mathbb{P}^1_{\mathbb{C}}$. By Proposition 3.2.1 the set of isomorphism classes of x_0-framed non-degenerate affine bundles A on $\mathbb{P}^1_{\mathbb{C}}$ with $L(A) \simeq E$ is identified with the quotient $\mathfrak{Q}_0$. The goal of this section is to describe explicitly this quotient endowed with its natural quotient topology.

We have the following diagram

$$\begin{array}{ccc} (\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}) \times \mathbb{C}[X_0, X_1]_d & \xrightarrow{\quad \eta \quad} & (\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}) \times \mathbb{C}[X_0, X_1]^*_{l_1} \\ & \searrow \quad \swarrow & \\ & \mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\} & \end{array} \tag{14}$$

of trivial linear spaces over $\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}$, where η is the fibrewise linear map

$$\eta(\Phi_2, Q) = (\Phi_2, {}^{\mathrm{t}}m_Q(\Phi_2))$$

Factorising all the spaces in the above diagram by the subgroup

$$G_0^2 \coloneqq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix} \middle| \; u_2 \in \mathbb{C}^* \right\} \simeq \mathbb{C}^*$$

of G_0, we obtain the following diagram of linear spaces

$$\begin{array}{ccc} \Theta_{\mathcal{P}} \otimes_{\mathbb{C}} \mathbb{C}[X_0, X_1]_d & \xrightarrow{\quad H \quad} & \mathcal{P} \times \mathbb{C}[X_0, X_1]^*_{l_1}, \\ & \searrow \quad \swarrow & \\ & \mathcal{P} & \end{array} \tag{15}$$

over the projective space $\mathcal{P} = \mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}/\mathbb{C}^* = \mathbb{P}(\mathbb{C}[X_0, X_1]^*_{l_2})$, where $\Theta_{\mathcal{P}} = |\mathcal{O}_{\mathcal{P}}(-1)|$ is the tautological line bundle of $\mathcal{P}$. Note that we have an obvious set-theoretical identification

$$\mathfrak{Q}_0 = \coprod_{[\Phi] \in \mathcal{P}} \operatorname{coker}(Q \mapsto {}^{\mathrm{t}}m_Q(\Phi)).$$

For the topology of $\mathfrak{Q}_0$ we have the following result:

Proposition 3.2.2. *We have a natural homeomorphism $\mathfrak{Q}_0 \xrightarrow{\simeq} \operatorname{Coker}(H)$, where $\operatorname{Coker}(H)$ stands for the topological cokernel of H (see Definition 2.4.3 in section 2.4).*

Proof. Note first that the right hand space $\mathcal{P} \times \mathbb{C}[X_0, X_1]^*_{l_1}$ in (15) is precisely the quotient (in the topological sense) of $\mathbb{C}[X_0, X_1]^*_{l_1} \times \left(\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}\right)$ by the equivalence relation, say R, induced by the G_0^2-action on this space.

Let S be the equivalence relation on $\mathcal{P} \times \mathbb{C}[X_0, X_1]^*_{l_1}$ defined by

$$S \coloneqq \left\{ \left(([\Phi], \phi), ([\Phi], \phi + {}^{\mathrm{t}}m_Q(\Phi))\right) |\; [\Phi] \in \mathcal{P},\; Q \in \mathbb{C}[X_0, X_1]_d \right\}.$$

Note that the quotient space $\mathcal{P} \times \mathbb{C}[X_0, X_1]^*_{l_1}/S$ is precisely the topological cokernel of H, by the definition of this cokernel.

Using the elementary Proposition 3.2.3 stated below, it suffices to prove that the equivalence relation R_S on $\mathbb{C}[X_0, X_1]^*_{l_1} \times \left(\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}\right)$ is precisely the equivalence relation associated with the G_0-action on this space. This follows easily noting that the equivalence class of a pair $(\Phi, \phi) \in \mathbb{C}[X_0, X_1]^*_{l_1} \times \left(\mathbb{C}[X_0, X_1]^*_{l_2} \backslash \{0\}\right)$ with respect to R_S coincides with its G_0-orbit.

□

Note that Proposition 3.2.2 cannot be obtained using Corollary 3.2.4 stated below, because the subgroup G_0^2 is not normal in G_0. Note however that making use of this corollary we obtain an alternative description of $\mathfrak{Q}_0$ as the $\mathbb{C}^*$-quotient of a topological cokernel on $\mathbb{C}[X_0, X_1]_{l_2}^* \backslash \{0\}$.

PROPOSITION 3.2.3. *Let X be a topological space, $R \subset X \times X$ an equivalence relation on X, $p_R : X \to Q \coloneqq X/R$ the canonical surjection, $S \subset Q \times Q$ an equivalence relation on Q, and $p_S : Q \to Q/S$ the canonical surjection.*

(1) The subset

$$R_S \coloneqq \{(x, y) \in X \times X | (p_R(x), p_R(y)) \in S\} \subset X \times X$$

is an equivalence relation on X.

(2) The composition $p_S \circ p_R$ is compatible with R_S and the induced map

$$h : X/R_S \to Q/S$$

is a homeomorphism with respect to the quotient topologies.

COROLLARY 3.2.4. *Let X be a topological space, G an arbitrary group and*

$$\alpha : G \times X \to X$$

be an action by homeomorphisms. Let $N \subset G$ be a normal subgroup and $H \coloneqq G/N$. The quotient X/N comes with a H-action induced by α and the natural bijection

$$h : X/G \to (X/N)/H$$

is a homeomorphism.

3.2.3. The duality theorem, the fibres of $\mathrm{Coker}(H)$ over $\mathcal{P}$ and apolar ideals. The goal of this section is to describe explicitly:

(1) The projective space $\mathcal{P}$, the base of the topological cokernel $\mathrm{Coker}(H)$.
(2) The fibres of $\mathrm{Coker}(H)$ over $\mathcal{P}$, and notably their dimensions.

We will see that the computation of the dimensions of these fibres is related to the classical theory of apolar ideals.

The main idea of this section: we identify the dual space $\mathbb{C}[X_0, X_1]_l^*$ with the space of constant-coefficient homogeneous differential operators of order l. More precisely we put

$$\mathbb{C}[\delta_0, \delta_1]_l := \big\{P(\delta_0, \delta_1)|\ P \in \mathbb{C}[X_0, X_1]_l\big\},$$

where $\delta_i := \frac{\partial}{\partial X_i}$ and $\delta_0^k \delta_1^{l-k} = \frac{\partial^d}{\partial^k X_0 \partial^{d-k} X_1}$, and we define the linear isomorphism $D_l : \mathbb{C}[\delta_0, \delta_1]_l \to \mathbb{C}[X_0, X_1]_l^*$ by assigning to $\Delta \in \mathbb{C}[\delta_0, \delta_1]_l$ the linear form

$$D_l(\Delta) : \mathbb{C}[X_0, X_1]_l \to \mathbb{C}$$

which maps a polynomial $P \in \mathbb{C}[X_0, X_1]_l$ to $\Delta(P)$; here Δ is regarded as a differential operator.

REMARK 3.2.5. *Note that*

$$D_l(\delta_0^k \delta_1^{l-k})(X_0^s X_1^{l-s}) = \delta_0^k \delta_1^{l-k}(X_0^s X_1^{l-s}) = \begin{cases} s!(l-s)! & \text{if} \quad k = s, \\ 0 & \text{if} \quad k \neq s. \end{cases}$$

This formula compares, via D_l, the canonical basis of $\mathbb{C}[\delta_0, \delta_1]_l$ to the dual of the canonical basis of $\mathbb{C}[X_0, X_1]_l$. The inverse of D_l is given by the formula

$$D_l^{-1}(\Phi) = \sum_{s=0}^{l} \frac{\Phi(X_0^s X_1^{l-s})}{s!(l-s)!} \delta_0^s \delta_1^{l-s} = \frac{1}{l!} \sum_{s=0}^{l} \binom{l}{s} \Phi(X_0^s X_1^{l-s}) \delta_0^s \delta_1^{l-s}.$$

From now on we will put $l \coloneqq l_2$ to save on notations and will thus use the isomorphism D_l to identify the space $\mathcal{P} \coloneqq \mathbb{P}(\mathbb{C}[X_0, X_1]_l^*)$ to the projective space $\mathbb{P}(\mathbb{C}[\delta_0, \delta_1]_l)$, which can be obviously identified with the symmetric power $S^l(\mathbb{P}^1_{\mathbb{C}})$ via the map

$$P(\delta_0, \delta_1) \mapsto Z(P)$$

which assigns to a differential operator $P(\delta_0, \delta_1)$, the zero locus of P. Therefore, from now on we will identify $\mathcal{P}$ to $S^l(\mathbb{P}^1_{\mathbb{C}})$ via this map.

Our next goal: describe explicitly the fibres of the map $\mathrm{Coker}(H) \to \mathcal{P} \simeq S^l(\mathbb{P}^1_{\mathbb{C}})$. We will make use of the duality Theorem 3.2.6 stated in the introduction, which gives, for a fixed linear form $\Phi \in \mathbb{C}[X_0, X_1]_l^*$, a precise interpretation of the associated linear map

$$\mathbb{C}[X_0, X_1]_d \ni Q \mapsto {}^t m_Q(\Phi) \in \mathbb{C}[X_0, X_1]_{l-d}^*$$

whose cokernel $C_\Phi^d \coloneqq \operatorname{coker}(Q \mapsto {}^{\mathrm{t}}m_Q(\Phi))$ is the space we are interested in. We give below the proof of this theorem:

THEOREM 3.2.6. *(Duality Theorem) Via the isomorphisms D_l, D_{l-d} we have*

$$ {}^{\mathrm{t}}m_Q(P(\delta_0,\delta_1)) = Q\left(\frac{\partial}{\partial\delta_0},\frac{\partial}{\partial\delta_1}\right)(P(\delta_0,\delta_1)).$$

for any $P(\delta_0,\delta_1)\in\mathbb{C}[\delta_0,\delta_1]_l$, $Q\in\mathbb{C}[X_0,X_1]_d$.

In other words, putting $\Phi \coloneqq D_l(P(\delta_0,\delta_1))\in\mathbb{C}[X_0,X_1]_l^*$, the differential operator associated with ${}^{\mathrm{t}}m_Q(\Phi)$ via D_{l-d}^{-1} is obtained by applying the differential operator $Q\left(\frac{\partial}{\partial\delta_0},\frac{\partial}{\partial\delta_1}\right)$ to $P(\delta_0,\delta_1)$, regarded as polynomial in the abstract variables δ_0, δ_1.

PROOF. We have to prove that for any $P(\delta_0,\delta_1)\in\mathbb{C}[\delta_0,\delta_1]_l$ we have equality

$$ {}^{\mathrm{t}}m_Q\big(D_l(P(\delta_0,\delta_1))\big) = D_{l-d}\left(Q\left(\frac{\partial}{\partial\delta_0},\frac{\partial}{\partial\delta_1}\right)(P(\delta_0,\delta_1))\right) \tag{16}$$

in $\mathbb{C}[X_0,X_1]_{l-d}^*$. It suffices to check the equality for monomials. Choose $P(\delta_0,\delta_1)=\delta_0^{s_0}\delta_1^{s_1}$, $Q(X_0,X_1)=X_0^{k_0}X_1^{k_1}$ with $s_0+s_1=l$, $k_0+k_1=d$. For any monomial $X_0^{a_0}X_1^{a_1}\in\mathbb{C}[X_0,X_1]_{l-d}$ we have

$$\begin{aligned}\big\langle {}^{\mathrm{t}}m_Q\big(D_l(P(\delta_0,&\delta_1))\big), X_0^{a_0}X_1^{a_1}\big\rangle = \big\langle D_l(P(\delta_0,\delta_1)), X_0^{a_0+k_0}X_1^{a_1+k_1}\big\rangle = \\ &= \begin{cases} (a_0+k_0)!(a_1+k_1)! & \text{if} \quad (a_0+k_0,a_1+k_1)=(s_0,s_1)\\ 0 & \text{if} \quad (a_0+k_0,a_1+k_1)\neq(s_0,s_1),\end{cases}\end{aligned} \tag{17}$$

and

$$\begin{aligned}\left\langle D_{l-d}\left(Q\left(\frac{\partial}{\partial\delta_0},\frac{\partial}{\partial\delta_1}\right)(P(\delta_0,\delta_1))\right), X_0^{a_0}X_1^{a_1}\right\rangle &= \left\langle D_{l-d}\left(\frac{\partial^d(\delta_0^{s_0}\delta_1^{s_1})}{\partial\delta_0^{k_0}\partial\delta_1^{k_1}}\right), X_0^{a_0}X_1^{a_1}\right\rangle \\ &= \frac{s_0!}{(s_0-k_0)!}\frac{s_1!}{(s_1-k_1)!}\left\langle D_{l-d}(\delta_0^{s_0-k_0}\delta_1^{s_1-k_1}), X_0^{a_0}X_1^{a_1}\right\rangle \\ &= \begin{cases} s_0!s_1! & \text{if} \quad (s_0-k_0,s_1-k_1)=(a_0,a_1)\\ 0 & \text{if} \quad (s_0-k_0,s_1-k_1)\neq(a_0,a_1).\end{cases}\end{aligned} \tag{18}$$

The claim follows after comparing (17) with (18). □

We will make use of the duality theorem Theorem 3.2.6 is the following way: we will replace the variables δ_0, δ_1 in $P(\delta_0,\delta_1)$ by the standard variables X_0, X_1 used in the theory of polynomials and we will replace the differential operators $\frac{\partial}{\partial\delta_0}$, $\frac{\partial}{\partial\delta_1}$ intervening in $Q\left(\frac{\partial}{\partial\delta_0},\frac{\partial}{\partial\delta_1}\right)$ by the partial differentials $\frac{\partial}{\partial X_0}$, $\frac{\partial}{\partial X_1}$. In this way we will be able to make use of the theory of apolar ideals explained in section 2.5.

Let $\Phi\in\mathbb{C}[X_0,X_1]_l^*\backslash\{0\}$ be a non-trivial linear form and let $P\in\mathbb{C}[X_0,X_1]_l\backslash\{0\}$ be the polynomial associated with Φ using the identifications explained above. The cokernel C_Φ^d in which are interested is obviously identified with

$$C_P^d = \operatorname{coker}\big(\mathbb{C}[\xi_0,\xi_1]_d \ni Q\mapsto Q\bullet P\in\mathbb{C}[X_0,X_1]_{l-d}\big).$$

As explained above, the goal of this section is the computation of the dimensions $\dim(C_P^d)$ as $[P]$ varies in the projective space $\mathbb{P}(\mathbb{C}[X_0,X_1]_l)$. We will answer this problem in two ways:

- We will give an explicit formula for the value of $\dim(C_P^d)$ on each stratum P_r^l of the cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0,X_1]_l)$.
- We will express the level sets $P_{d,s}^l$ $(0\leqslant s\leqslant l-d+1)$ of the map

  $$\mathbb{P}(\mathbb{C}[X_0,X_1]_l)\ni[P]\mapsto\dim(C_P^d)\in\mathbb{N}$$

  in terms of the strata of the cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0,X_1]_l)$ (see section 2.5).

The definition of C_P^d gives:

REMARK 3.2.7. *Under the assumptions and with the notations above we have*

$$\dim(C_P^d) = l-2d+\dim(\operatorname{Ann}(P)_d)$$

for any $0\leqslant d\leqslant l$. In particular $\dim(C_P^d)\geqslant l-2d$.

By Theorem 2.5.3 explained in section 2.5.1, the apolar ideal $\mathrm{Ann}(P)_d$ is generated by two relative prime homogeneous polynomials G_1, G_2 satisfying $\deg(G_1) \leqslant \deg(G_2)$, $\deg(G_1) + \deg(G_2) = l + 2$.

The dimension $\dim(\mathrm{Ann}(P)_d)$ can be easily computed in terms of $d_i(P) \coloneqq \deg(G_i)$, see Remark 2.5.4 proved in section 2.5.1. We obtain:

PROPOSITION 3.2.8. *Let $P \in \mathbb{C}[X_0, X_1]_l \backslash \{0\}$ and $0 \leqslant d \leqslant l$.*

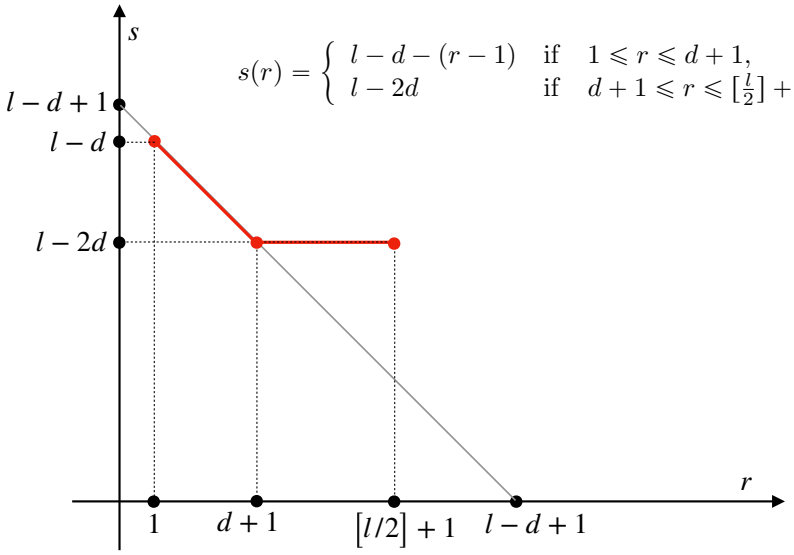
(1) We have

$$\dim(C_P^d) = \begin{cases} 0 & \text{if } l + 2 - \mathrm{crank}(P) \leqslant d \leqslant l, \\ l - d + 1 - \mathrm{crank}(P) & \text{if } \mathrm{crank}(P) \leqslant d < l + 2 - \mathrm{crank}(P), \\ l - 2d & \text{if } d < \mathrm{crank}(P). \end{cases} \tag{19}$$

(2) The dimension $s := \dim(C_P^d)$ depends only on d and the cactus rank $r = \mathrm{crank}(P)$:

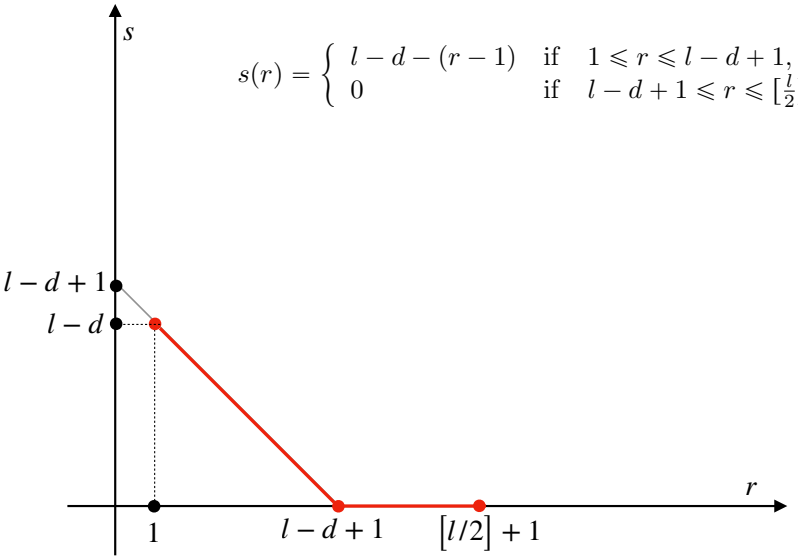
(a) For $0 \leqslant d \leqslant [\frac{l}{2}]$, the function $s = s(r)$ is given by:

$$s(r) = \begin{cases} l - d - (r - 1) & \text{if} \quad 1 \leqslant r \leqslant d + 1, \\ l - 2d & \text{if} \quad d + 1 \leqslant r \leqslant [\frac{l}{2}] + 1. \end{cases}$$



(b) For $[\frac{l}{2}] \leqslant d \leqslant l$, the function $s = s(r)$ is given by:

$$s(r) = \begin{cases} l - d - (r - 1) & \text{if} \quad 1 \leqslant r \leqslant l - d + 1, \\ 0 & \text{if} \quad l - d + 1 \leqslant r \leqslant [\frac{l}{2}] + 1. \end{cases}$$



We will now give explicit formulae for the level sets of the map $[P] \mapsto \dim(C_P^d)$ in terms of the strata of the cactus rank stratification of $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$:

For $1 \leqslant r \leqslant \left[\frac{l+2}{2}\right]$ consider the locally closed partition $(P_r^l))_{1 \leqslant r \leqslant [\frac{l+2}{2}]}$ of the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1]_l)$, where

$$P_r^l \coloneqq \{[P] \in \mathbb{P}(\mathbb{C}[X_0, X_1]_l) |\ \mathrm{crank}(P) = r\},$$

(see Remark 2.5.12 in section 2.5.1). For $0 \leqslant d \leqslant l$ and $0 \leqslant s \leqslant l - d + 1$ define

$$P_{d,s}^l \coloneqq \{[P] \in \mathbb{P}(\mathbb{C}[X_0, X_1]_l) |\ \dim(C_P^d) = s\}$$

Using Proposition 3.2.8, we obtain

PROPOSITION 3.2.9. *Let d, $s \in \mathbb{N}$ with $0 \leqslant d \leqslant l$ and $0 \leqslant s \leqslant l-d+1$. We have:*

$$P^l_{d,s} = \begin{cases} \bigcup_{r>d} P^l_r & \text{if} \quad s = l-2d, \\ P^l_{l-d+1-s} & \text{if} \quad s > l-2d \text{ and } s > 0, \\ \bigcup_{r \geqslant l-d+1} P^l_r & \text{if} \quad s > l-2d \text{ and } s = 0. \end{cases} \tag{20}$$

PROOF. 1) The case $s = l-2d$. The inclusion $P^l_{d,l-2d} \supset \bigcup_{r>d} P^l_r$ is obvious by the third formula in (19). For the opposite inclusion, let $[P] \in P^l_{d,s}$ with $s = l-2d$; suppose that $r = \operatorname{crank}(P) \leqslant d$. It follows $l+2-\operatorname{crank}(P)-d = d+s+2-\operatorname{crank}(P) > 0$, so the two inequalities defining the second case in (19), so $l-2d = l-d+1-\operatorname{crank}(P)$, so $\operatorname{crank}(P) = d+1$, which contradicts the assumption $\operatorname{crank}(P) \leqslant d$.

2) The case $s > l-2d$ and $s > 0$.

In this case we use the second formula in (19) and we see that we must have

$$s = l-d+1-\operatorname{crank}(P) \text{ with } \operatorname{crank}(P) \leqslant d < l+2-\operatorname{crank}(P).$$

We obtain $\operatorname{crank}(P) = l-d+1-s$ and the condition

$$l-d+1-s \leqslant d < l+2-(l-d+1-s) = d+1+s$$

is obviously satisfied.

3) The case $s > l-2d$ and $s = 0$.

In this case $l-2d < 0$, so the third formula (19) cannot yield 0. Taking into account the first two formulae in (19) we obtain the result $s = 0$ when

(a) $l+2-\operatorname{crank}(P) \leqslant d \leqslant l$, or

(b) $l-d+1-\operatorname{crank}(P) = 0$ and $\operatorname{crank}(P) \leqslant d < l+2-\operatorname{crank}(P)$

The case (a) gives $\operatorname{crank}(P) \geqslant l+2-d$ and the condition $d \leqslant l$ is satisfied. The case (b) gives $\operatorname{crank}(P) = l-d+1$ and the condition $\operatorname{crank}(P) \leqslant d < l+2-\operatorname{crank}(P)$ is satisfied. Therefore cases (a), (b) give the solutions $\operatorname{crank}(P) \geqslant l-d+1$. □

COROLLARY 3.2.10. *We have*

(1) For $0 \leqslant d \leqslant [\frac{l}{2}]$, we have

$$P^l_{d,s} = \begin{cases} P^l_{l-d+1-s} & \text{if} \quad l-2d < s \leqslant l-d, \\ \bigcup_{d+1 \leqslant r \leqslant [\frac{l}{2}]+1} P^l_r & \text{if} \quad s = l-2d, \\ \varnothing & \text{if} \quad s < l-2d \text{ or } s > l-d. \end{cases}$$

(2) For $[\frac{l}{2}] \leqslant d \leqslant l$, we have

$$P^l_{d,s} = \begin{cases} P^l_{l-d+1-s} & \text{if} \quad 0 < s \leqslant l-d, \\ \bigcup_{l-d+1 \leqslant r \leqslant [\frac{l}{2}]+1} P^l_r & \text{if} \quad s = 0, \\ \varnothing & \text{if} \quad s > l-d. \end{cases}$$

CHAPTER 4

Moduli stacks of affine bundles with fixed linearisation type

4.1. Preliminaries

4.1.1. Gluing sheaves. We refer to [**Go**, sections II.1.2, II.1.6] for the correspondence between sheaves of sets and étale spaces and the general formalism of sheaf homomorphisms.

We also refer to [**Ha**, Exercise 1.22, p. 69], [**Sta**, Section 1.6.33] for the following gluing principle for sheaves. In the following definition and proposition by sheaf we mean sheaf of sets or sheaf of abelian groups.

Let X be a topological space and $\mathcal{U} = (U_i)_{i\in I}$ an open cover of X.

Definition 4.1.1. *A sheaf gluing data subordinate to $\mathcal{U}$ is a pair $\big((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\big)$ consisting of*

(1) a family $(\mathcal{F}_i)_{i\in I}$ of sheaves, $\mathcal{F}_i$ being a sheaf on U_i for any $i \in I$.

(2) a family $(\varphi_i^j)_{(i,j)\in I^2}$ of sheaf isomorphisms, $\varphi_i^j : \mathcal{F}_i|_{U_i\cap U_j} \xrightarrow{\simeq} \mathcal{F}_j|_{U_i\cap U_j}$ satisfying the cocycle conditions:

(a) $\varphi_i^i = \mathrm{id}$ for any $i \in I$.

(b) For each $(i, j, k) \in I^3$ we have $\varphi_i^k = \varphi_j^k \circ \varphi_i^j$ on $U_i \cap U_j \cap U_k$.

A morphism

$$\big((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\big) \to \big((\mathcal{F}'_i)_{i\in I}, (\varphi'^j_i)_{(i,j)\in I^2}\big)$$

of gluing data subordinate to $\mathcal{U}$ is a family of morphisms $(u_i : \mathcal{F}_i \to \mathcal{F}'_i)_{i\in I}$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}_i|_{U_i\cap U_j} & \xrightarrow{u_i|_{U_i\cap U_j}} & \mathcal{F}'_i|_{U_i\cap U_j} \\ \Big\downarrow{\scriptstyle \varphi_i^j} & & \Big\downarrow{\scriptstyle \varphi'^j_i} \\ \mathcal{F}_j|_{U_i\cap U_j} & \xrightarrow{u_j|_{U_i\cap U_j}} & \mathcal{F}'_j|_{U_i\cap U_j}, \end{array}$$

is commutative for any $(i, j) \in I^2$.

We refer to [**Sta**, Lemma 6.33.2] for the following result which will play a crucial role in our results:

Proposition 4.1.2. *To any sheaf gluing data $\big((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\big)$ subordinate to $\mathcal{U}$ there is a canonically associated pair $(\mathcal{F}, (\psi_i : \mathcal{F}_i \xrightarrow{\simeq} \mathcal{F}|_{U_i})_{i\in I})$ where $\mathcal{F}$ is a sheaf on X and for any $i \in I$, $\psi_i : \mathcal{F}_i \xrightarrow{\simeq} \mathcal{F}|_{U_i}$ is a sheaf isomorphism on U_i which satisfies:*

(1) The compatibility condition:

$$\forall (i, j) \in I^2,\ \psi_i = \psi_j \circ \varphi_i^j \text{ on } U_i \cap U_j. \tag{Co}$$

(2) The following universal property:

For any pair $(\mathcal{G}, (g_i : \mathcal{F}_i \to \mathcal{G}|_{U_i})_{i\in I})$ consisting of a sheaf $\mathcal{G}$ on X and a family of sheaf homomorphisms satisfying the compatibility condition

$$\forall (i, j) \in I^2,\ g_i = g_j \circ \varphi_i^j \text{ on } U_i \cap U_j, \tag{Co$'$}$$

there exists a unique sheaf homomomorphism $g : \mathcal{F} \to \mathcal{G}$ such that

$$\forall i \in I,\ g|_{U_i} \circ \psi_i = g_i. \tag{21}$$

Note that the compatibility conditions (Co), (Co′) are equivalent to the commutativity of the diagrams

$$\begin{array}{ccccc} & & \mathcal{F}|_{U_i\cap U_j} & & \\ & \nearrow^{\psi_i|_{U_i\cap U_j}} & & \nwarrow_{\psi_j|_{U_i\cap U_j}} & \\ \mathcal{F}_i|_{U_i\cap U_j} & & \xrightarrow{\varphi_i^j} & & \mathcal{F}_j|_{U_i\cap U_j}, \end{array} \qquad \begin{array}{ccccc} & & \mathcal{G}|_{U_i\cap U_j} & & \\ & \nearrow^{g_i|_{U_i\cap U_j}} & & \nwarrow_{g_j|_{U_i\cap U_j}} & \\ \mathcal{F}_i|_{U_i\cap U_j} & & \xrightarrow{\varphi_i^j} & & \mathcal{F}_j|_{U_i\cap U_j}, \end{array}$$

whereas property (21) is equivalent to the commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{F}|_{U_i} & \xrightarrow{g|_{U_i}} & \mathcal{G}|_{U_i} \\ & {\scriptstyle\psi_i}\nwarrow \quad \nearrow{\scriptstyle g_i} & \\ & \mathcal{F}_i & \end{array} .$$

Proof. We give only the construction of $\mathcal{F}$. For each open set $U \subset X$ we put

$$\mathcal{F}(U) \coloneqq \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap U) |\ \forall (i,j) \in I^2,\ \varphi_i^j(s_i|_{U\cap U_i\cap U_j}) = s_j|_{U\cap U_i\cap U_j} \right\}.$$

The stalk $\mathcal{F}_x$ at a point $x \in X$ is the quotient

$$\mathcal{F}_x = \left(\coprod_{\substack{i\in I \\ x\in U_i}} \mathcal{F}_{i,x} \right) \Big/ \sim ,$$

where $\sim$ is the equivalence relation generated by the isomorphisms $(\varphi_i^j)_x : \mathcal{F}_{i,x} \xrightarrow{\simeq} \mathcal{F}_{j,x}$. □

Definition 4.1.3. *The (essentially unique) sheaf given by Proposition 4.1.2 will be called to sheaf associated to the sheaf gluing data* $\left((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\right)$ *and will be denoted by*

$$\coprod_{i\in I} \mathcal{F}_i \Big/_{(\varphi_i^j)}.$$

Remark 4.1.4. *For any morphism*

$$(u_i : \mathcal{F}_i \to \mathcal{F}'_i)_{i\in I} : \left((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\right) \to \left((\mathcal{F}'_i)_{i\in I}, (\varphi'^j_i)_{(i,j)\in I^2}\right)$$

of gluing data, there exists a unique morphism

$$\coprod_{i\in I} \mathcal{F}_i \Big/_{(\varphi_i^j)} =: \mathcal{F} \xrightarrow{\coprod_{i\in I} u_i} \mathcal{F}' \coloneqq \coprod_{i\in I} \mathcal{F}'_i \Big/_{(\varphi_i^j)}$$

such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_i & \xrightarrow{u_i} & \mathcal{F}'_i \\ {\scriptstyle\psi_i}\downarrow & & \downarrow{\scriptstyle\psi'_i} \\ \mathcal{F}\big|_{U_i} & \xrightarrow{\coprod_{i\in I} u_i|_{U_i}} & \mathcal{F}'\big|_{U_i} \end{array} ,$$

where $(\mathcal{F}, (\psi_i : \mathcal{F}_i \xrightarrow{\simeq} \mathcal{F}|_{U_i})_{i\in I})$, $(\mathcal{F}', (\psi'_i : \mathcal{F}'_i \xrightarrow{\simeq} \mathcal{F}'|_{U_i})_{i\in I})$ *are the pairs associated with the gluing data* $\left((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\right)$, $\left((\mathcal{F}'_i)_{i\in I}, (\varphi'^j_i)_{(i,j)\in I^2}\right)$.

Remark 4.1.5. *Let* $(X, \mathcal{R})$ *be a ringed space (complex space). Proposition 4.1.2 remains true in the category of (respectively coherent)* $\mathcal{R}$*-modules.*

We will need the following obvious properties of the sheaves and morphisms associated with gluing data (see [**Sta**, Lemma 6.33.4])

Remark 4.1.6. *Let* $(X, \mathcal{R})$ *be a ringed space and* $\mathcal{U} = (U_i)_{i\in I}$ *an open cover of* X*. The assignment*

$$\left((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\right) \mapsto \coprod_{i\in I} \mathcal{F}_i \Big/_{(\varphi_i^j)}$$

has the following properties:

(1) Is an exact functor on the category of $\mathcal{R}$*-module gluing data in the following sense: let*

$$\left((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\right),\ \left((\mathcal{F}'_i)_{i\in I}, (\varphi'^j_i)_{(i,j)\in I^2}\right),\ \left((\mathcal{F}_i)_{i\in I}, (\varphi''^j_i)_{(i,j)\in I^2}\right)$$

$\mathcal{R}$-module gluing data and let $(u_i : \mathcal{F}'_i \to \mathcal{F}_i)_{i\in I}$ $(v_i : \mathcal{F}_i \to \mathcal{F}''_i)_{i\in I}$ *be morphisms of gluing data. Then the sequence*

$$\coprod_{i\in I} \mathcal{F}'_i \Big/_{(\varphi'^j_i)} \xrightarrow{\coprod_{i\in I} u_i} \coprod_{i\in I} \mathcal{F}_i \Big/_{(\varphi_i^j)} \xrightarrow{\coprod_{i\in I} v_i} \coprod_{i\in I} \mathcal{F}''_i \Big/_{(\varphi''^j_i)}$$

is exact.

(2) *Commutes with tensor products in the following sense: let* $\big((\mathcal{F}_i)_{i\in I}, (\varphi_i^j)_{(i,j)\in I^2}\big)$ *be an* $\mathcal{R}$*-module gluing data,* $(\mathcal{F} = \coprod_{i\in I} \mathcal{F}_i/_{(\varphi_i^j)}, (\psi_i : \mathcal{F}_i \xrightarrow{\simeq} \mathcal{F}|_{U_i})_{i\in I})$ *the associated pair and* $\mathcal{A}$ *an* $\mathcal{R}$*-module on* X*. Then the family of* $\mathcal{R}$*-module isomorphisms*

$$\big(\psi_i \otimes \mathrm{id}_{\mathcal{A}} : \mathcal{F}_i \otimes \mathcal{A}|_{U_i} \to (\mathcal{F}\otimes\mathcal{A})|_{U_i}\big)_{i\in I}$$

induces a canonical isomorphism $\coprod_{i\in I}(\mathcal{F}_i \otimes \mathcal{A}|_{U_i})/_{(\varphi_i^j \otimes \mathrm{id}_{\mathcal{A}})} \xrightarrow{\simeq} \big(\coprod_{i\in I} \mathcal{F}_i/_{(\varphi_i^j)}\big) \otimes \mathcal{A}$.

4.1.2. Enlarged extensions. Let X be a topological space, $\iota : \mathcal{A} \to \mathcal{A}'$ a monomorphism of sheaves of abelian groups on X, and

$$0 \to \mathcal{A} \xrightarrow{u} \mathcal{B} \xrightarrow{v} \mathcal{C} \to 0 \tag{e}$$

an exact sequence of sheaves of abelian groups on X, regarded as an extension of $\mathcal{C}$ by $\mathcal{A}$.

Definition 4.1.7. *The enlarged extension associated with* (e) *and* ι *is*

$$0 \to \mathcal{A}' \xrightarrow{u'} \mathcal{B}' \xrightarrow{v'} \mathcal{C} \to 0 \tag{e'}$$

where

(1) $\mathcal{B}' \coloneqq \mathrm{coker}\big((\iota, -u) : \mathcal{A} \to \mathcal{A}' \oplus \mathcal{B}\big)$,
(2) u' *is the composition*

$$\mathcal{A}' \xrightarrow{i_1} \mathcal{A}' \oplus \mathcal{B} \to \mathcal{A}' \oplus \mathcal{B}/_{\mathrm{im}\big((\iota,-u)\big)} =: \mathcal{B}',$$

(3) v' *is induced by the composition*

$$\mathcal{A}' \oplus \mathcal{B} \xrightarrow{p_2} \mathcal{B} \xrightarrow{v} \mathcal{C}$$

whose kernel obviously contains $\mathrm{im}(\iota, -u)$.

Note that the kernel of the composition

$$\mathcal{A}' \oplus \mathcal{B} \xrightarrow{p_1} \mathcal{A} \longrightarrow \mathcal{A}'/_{\iota(\mathcal{A})} = \mathrm{coker}(\iota)$$

contains $\mathrm{im}(\iota, -u)$, so this composition induces an epimorphism $q_e^\iota : \mathcal{B}' \to \mathrm{coker}(\iota)$ whose kernel is $\mathrm{im}(b')$ and will be called *the canonical epimorphism* on the central term $\mathcal{B}'$ of the enlarged extension.

We have:

Remark 4.1.8. *The enlarged extension* (e') *defined above has the following properties:*

(1) *Fits in the commutative diagram*

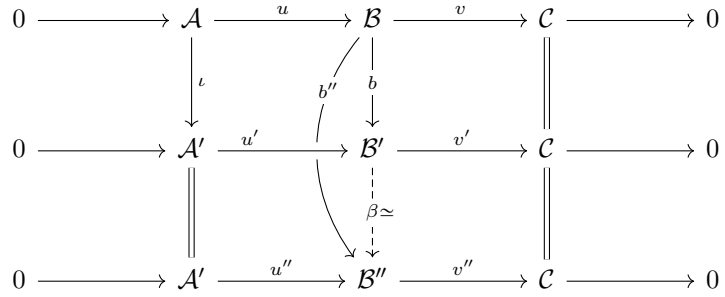
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A} & \xrightarrow{u} & \mathcal{B} & \xrightarrow{v} & \mathcal{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle b'} & & \| & & \\
0 & \longrightarrow & \mathcal{A}' & \xrightarrow{u'} & \mathcal{B}' & \xrightarrow{v'} & \mathcal{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q_e^\iota} & & & & \\
 & & \mathrm{coker}(\iota) & = & \mathrm{coker}(\iota) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns. The vertical arrow q_e^ι *is the unique morphism* $\mathcal{B}' \to \mathrm{coker}(\iota)$ *which makes the diagram commutative and the second column exact.*

(2) *Satisfies the following universal property: for any commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A} & \xrightarrow{u} & \mathcal{B} & \xrightarrow{v} & \mathcal{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle b''} & & \| & & \\
0 & \longrightarrow & \mathcal{A}' & \xrightarrow{u''} & \mathcal{B}'' & \xrightarrow{v''} & \mathcal{C} & \longrightarrow & 0
\end{array}$$

with exact rows there a unique isomorphism $\beta : \mathcal{B}' \xrightarrow{\simeq} \mathcal{B}''$ making commutative the following diagram



PROOF. (1) Direct checking.

(2) The unique isomorphism $\beta : \mathcal{B}' \to \mathcal{B}''$ which makes the above diagram commutative is the isomorphism induced by $(u'', b'') : \mathcal{A}' \oplus \mathcal{B} \to \mathcal{B}''$. □

REMARK 4.1.9. *Let $(X, \mathcal{R})$ be a ringed space (complex space). The definition of the enlarged extension and its properties stated in Remark 4.1.8 remain true in the category of sheaves of (respectively coherent) $\mathcal{R}$-modules.*

4.1.3. Principal bundles and torsors. Let S be a complex space and G a complex Lie group. Recall that

DEFINITION 4.1.10. *A (left) holomorphic G-action on S is a morphism $\alpha : G \times S \to S$ of complex spaces such that the diagrams*

$$\begin{array}{ccc} G \times G \times S & \xrightarrow{\mathrm{id}_G \times \alpha} & G \times S \\ {\scriptstyle m \times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times S & \xrightarrow[\alpha]{} & S \end{array} \quad , \qquad \begin{array}{ccc} G \times S & \xrightarrow{\alpha} & S \\ {\scriptstyle e_G \times \mathrm{id}_X}\uparrow & \nearrow_{\mathrm{id}_X} & \\ S & & \end{array}$$

are commutative in the category of complex spaces.

Similarly one defines holomorphic right actions. In the special case when S is a complex manifold we recover the definition of a holomorphic action in the well known sense.

DEFINITION 4.1.11. *A principal G-bundle over S is a smooth complex space morphism (see* [**Del**, Défintion 0.4 p. 2]*) $P \xrightarrow{\pi} S$ where P is endowed with a right holomorphic G-action $P \times G \xrightarrow{\alpha} P$ satisfying:*

(1) The diagram

$$\begin{array}{ccc} P \times G & \xrightarrow{\alpha} & P \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow[\pi]{} & S \end{array}$$

is commutative.

(2) The induced map $P \times G \to P \times_S P$ is an isomorphism of complex spaces.

We will denote by $\mathcal{O}_S(G)$ the sheaf $U \mapsto \mathcal{O}(U, G)$ of holomorphic G-valued morphisms defined on open subspaces of S. $\mathcal{O}_S(G)$ is obviously a sheaf of groups.

For a principal G-bundle $P \xrightarrow{\pi} S$ over S we will denote by $\mathcal{O}_S(P)$ the sheaf of sections defined on open subspaces of S which are complex space morphisms.

REMARK 4.1.12. *Let $P \xrightarrow{\pi} S$ be a principal G-bundle over S.*

(1) For any $s \in S$ there exists a section $\tau : U_\tau \to P$ defined on an open neighbourhood U_τ of s in S.

(2) The natural right action of $\mathcal{O}_S(G)$ on $\mathcal{O}_S(P)$ makes the sheaf $\mathcal{O}_S(P)$ an $\mathcal{O}_S(G)$-torsor. In other words, for any $s \in S$ the stalk $\mathcal{O}_S(P)_s$ is a right $\mathcal{O}_S(G)_s$-torsor under the natural right action of this group.

PROOF. (1) follows from the smoothness condition in Definition 4.1.11.

(2) The statement (1) shows that $\mathcal{O}_S(P)_s$ is non-empty. The isomorphism $P \times G \to P \times_S P$ gives (taking germes of sections at s) a bijection $\mathcal{O}_S(P)_s \times \mathcal{O}_S(G)_s \to \mathcal{O}_S(P)_s \times \mathcal{O}_S(P)_s$, given explicitly by

$$(t, \gamma) \mapsto (t, t\gamma)$$

which proves that the right action of $\mathcal{O}_S(G)_s$ on $\mathcal{O}_S(P)_s$ is free and transitive.

□

Conversely, let $\mathcal{F}$ be an $\mathcal{O}_S(G)$-torsor on S. Let $\tau \in \mathcal{F}(U_a)$, $\theta \in \mathcal{F}(U_b)$ be sections of $\mathcal{F}$ on open subsets U_τ, respectively $U_\theta \subset S$. We define $g_{\tau\theta} \in \mathcal{O}(U_\tau \cap U_\theta, G)$ by the formula

$$\theta|_{U_\tau \cap U_\theta} = \tau|_{U_\tau \cap U_\theta}\ g_{\tau\theta}.$$

We put

$$P^{\mathcal{F}} \coloneqq \left(\coprod_{\tau \text{ local section of } \mathcal{F}} U_\tau \times G \right) \Big/ \sim$$

where $\sim$ is the equivalence relation defined by the biholomorphisms

$$\begin{array}{ccc} (s, g_{ab}(s)\gamma) & \longleftarrow\!\shortmid & (s,\gamma) \\ \cap & & \cap \\ (U_\tau \cap U_\theta) \times G & \xleftarrow{\Phi_{\tau\theta}} & (U_\tau \cap U_\theta) \times G \\ \downarrow & & \downarrow \\ U_\tau \times G & & U_\theta \times G \end{array}$$

associated with pairs (τ, θ) of local sections pf P.

The following proposition, which is well known to experts, provides an equivalent way to define principal bundles:

Proposition 4.1.13. *The functors $P \mapsto \mathcal{O}_S(P)$ and $\mathcal{F} \mapsto P^{\mathcal{F}}$ yield and equivalence between the groupoid of principal G-bundles on S and the groupoid of $\mathcal{O}_S(G)$-torsors.*

A fundamental construction in differential geometry is the (locally trivial) bundle associated with a principal G-bundle and a G-manifold F (called standard fibre). Using the torsor of local sections associated with P, we can extend this construction to the framework of complex spaces:

Definition 4.1.14. *Let $P \to S$ be a principal G-bundle and $G \times F \to F$ be a left holomorphic G-action on a complex space F. We define the associated bundle*

$$P \times_\rho F \coloneqq \left(\coprod_{\tau \text{ local section of } P} U_\tau \times F \right) \Big/ \sim,$$

where this time $\sim$ is generated by the biholomorphisms

$$\begin{array}{ccc} (s, g_{ab}(s)y) & \longleftarrow\!\shortmid & (s,y) \\ \cap & & \cap \\ (U_\tau \cap U_\theta) \times F & \xleftarrow{\Phi_{\tau\theta}} & (U_\tau \cap U_\theta) \times F \\ \downarrow & & \downarrow \\ U_\tau \times F & & U_\theta \times F \end{array}$$

associated with pairs (τ, θ) of local sections of P.

4.1.4. The representability of the functor $S \mapsto \mathrm{Aut}_{S\times X}(S \times E)$.

Let $f : X \to S$ be a morphism of complex spaces and $\mathcal{F}$ be a coherent sheaf on X. For $s \in S$ we will denote by X_s the fibre of f over s and by $\mathcal{F}_s$ the restriction of $\mathcal{F}$ to X_s.

Now let

$$\begin{array}{ccc} X' & \xrightarrow{G} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

be a commutative diagram of complex spaces, and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

Recall (see for instance in [**BS**, p. 93]) that for any $q \in \mathbb{N}$ we have a canonical morphism of sheaves:

$$g^*(R^q f_*(\mathcal{F})) \to R^q(f')_*(G^*(\mathcal{F})).$$

If $f : X \to S$ is proper, $X' \coloneqq S' \times_S X$, $\mathcal{F}$ is flat over S and cohomologically flat in dimension q, this canonical morphism is in fact an isomorphism (see [**BS**, Theorem 4.10 (d), p. 133]).

On the other hand, by Grauert's Theorem (see [**BS**, Theorem 4.12 (ii), p. 134]), we know that if moreover

(1) S is reduced, and,
(2) the map $S \ni s \mapsto h^q(X_s, \mathcal{F}_s)$ is locally constant,

then $\mathcal{F}$ is cohomologically flat in dimension q. We now apply this result to the diagram

$$\begin{array}{ccc} S \times X & \xrightarrow{p_X} & X \\ {\scriptstyle p_S}\downarrow & & \downarrow \\ S & \longrightarrow & \{\star\} \end{array}$$

where $\{\star\}$ is a reduced singleton and $q = 0$. Since the conditions (1) and (2) above are obviously satisfied, we obtain the following:

REMARK 4.1.15. *Let X be a compact complex space and $\mathcal{F}$ a coherent sheaf on X. For any complex space S, the canonical morphism*

$$\mathcal{O}_S \otimes_{\mathbb{C}} H^0(X, \mathcal{F}) \to (p_S)_*(p_X^*(\mathcal{F}))$$

is an isomorphism. Applying the functor $H^0(S, -)$ to this sheaf isomorphism, we obtain a canonical isomorphism

$$\mathcal{O}(S, H^0(X, \mathcal{F})) \to H^0(S, (p_S)_*((p_X^*)(\mathcal{F}))) = H^0(S \times X, (p_X^*)(\mathcal{F})). \tag{22}$$

Let now X be a compact complex manifold, E a holomorphic vector bundle on X and $\mathcal{E}$ its associated locally free sheaf. Recall (see Proposition 2.2.6) that the automorphism group $\mathrm{Aut}(E) = \mathrm{Aut}(\mathcal{E})$ is an affine algebraic group. From now on we will denote by G the group $\mathrm{Aut}(E)$ endowed with its natural affine algebraic group structure given by this proposition.

We have the following

PROPOSITION 4.1.16. *The functor $S \mapsto \mathrm{Aut}_{S\times X}((p_X^{S\times X})^*(E))$ is representable by G.*

PROOF. Using the isomorphism (22) when $\mathcal{F} = \mathcal{E}nd(\mathcal{E}) = \mathcal{E}^\vee \otimes \mathcal{E}$, we obtain identifications

$$\mathcal{O}_S(\mathrm{End}(\mathcal{E}))(S) \xrightarrow{\simeq} H^0(S \times X, p_X^*(\mathcal{E}^\vee \otimes \mathcal{E})) = H^0(S \times X, p_X^*(\mathcal{E})^\vee \otimes p_X^*(\mathcal{E})) = \mathrm{End}(p_X^*(\mathcal{E})), \tag{23}$$

which are functorial with respect to S and compatible with compositions. Here $p_X = p_X^{S\times X}$ stands for the projection on the first factor. Now note that, since $G = \mathrm{Aut}(E)$ is open in $\mathrm{End}(E)$, the set $\mathcal{O}_S(G)(S)$ of complex space morphisms $S \to G$ coincides with the subset of $\mathcal{O}_S(\mathrm{End}(E))(S)$ consisting of complex space morphisms $S \to \mathrm{End}(E)$ whose underlying map takes values in G. On the other hand the set of invertible elements in $\mathcal{O}_S(\mathrm{End}(E))(S)$ is obviously $\mathcal{O}_S(G)(S)$. Taking the set of invertible elements in (23), we obtain a group isomorphism

$$\mathcal{O}_S(G)(S) \xrightarrow{\simeq} \mathrm{Aut}_{S\times X}(S \times E) = \mathrm{Aut}_{S\times X}(p_X^*(\mathcal{E}))$$

which is functorial with respect to S.

□

REMARK 4.1.17. *The analogue of the representability theorem given by Proposition 4.1.16 in the algebraic geometric category is known for principal G-bundles over proper algebraic varieties, where G is an arbitrary algebraic group (see* [**Br**, Theorem 4.2 p. 16, comments on p. 17]*).*

4.2. Fibrewise constant families of vector bundles

Let X be a compact complex manifold and S a complex space. For a holomorphic vector bundle F (locally free sheaf $\mathcal{F}$) on $S \times X$ we will denote by F_s (respectively $\mathcal{F}_s$) the restriction $F|_{\{s\}\times X}$ (respectively $\mathcal{F}|_{\{s\}\times X}$) regarded as a vector bundle (respectively locally free sheaf) on X. Let E be a holomorphic vector bundle of rank r on X and $\mathcal{E}$ its associated locally free sheaf.

DEFINITION 4.2.1. *A vector bundle F over $S \times X$ is said to be fibrewise isomorphic to E if for any $s \in S$ there exists an open neighbourhood U and an isomorphism*

$$(p_X^{U\times X})^*(E) \xrightarrow{\simeq} F|_{U\times X},$$

of holomorphic vector bundles over $U \times X$, where $p_X^{U\times X} : U \times X \to X$ stands for the projection on the second factor.

Similarly, a locally free sheaf $\mathcal{F}$ over $S \times X$ is said to be fibrewise isomorphic to $\mathcal{E}$ if for any $s \in S$ there exists an open neighbourhood U and an isomorphism $(p_X^{U\times X})^*(\mathcal{E}) \xrightarrow{\simeq} \mathcal{F}|_{U\times X}$ of $\mathcal{O}_{U\times X}$-modules.

REMARK 4.2.2. *Let $F \to S \times X$ be a vector bundle fibrewise isomorphic to E. For each $s \in S$, the vector bundle F_s is isomorphic to E.*

The converse of this implication holds when S is reduced, so for reduced space of parameters S the condition "F is fibrewise isomorphic to E" agrees with the intuitive interpretation of this condition. More precisely

PROPOSITION 4.2.3. *Let S be a **reduced** complex space and F an holomorphic bundle on $S \times X$ such that for any $s \in S$ the vector bundle F_s is isomorphic to E. Then F is fibrewise isomorphic to E in the sense of Definition 4.2.1.*

PROOF. Let $\mathcal{H}$ be the locally free sheaf $p_X^*(\mathcal{E})^\vee \otimes \mathcal{F}$ over $S \times X$. Since $p_S : S \times X \longrightarrow S$ is flat, $\mathcal{H}$ is flat over S. Since X is compact, the projection p_S is proper. We have:

$$\mathcal{H}_s \simeq \mathcal{E}^\vee \otimes \mathcal{F}_s \simeq \mathcal{E}^\vee \otimes \mathcal{E}.$$

Therefore the map $S \ni s \mapsto h^0(X_s, \mathcal{H}_s)$ is constant and equals $d \coloneqq \dim_{\mathbb{C}}(\mathrm{End}(\mathcal{E}))$. Since S is reduced, it follows by Grauert's Theorem [**BS**, Theorem 4.12 (ii)], that $\mathcal{H}$ is cohomologically flat in dimension 0 over S [**BS**, p. 133], and in particular:

(1) the sheaf $p_{S_*}(\mathcal{H})$ is locally free of rank d, and
(2) for each $s \in S$ the natural linear map

$$(p_S)_*(\mathcal{H})_s \big/ \mathfrak{m}_s (p_S)_*(\mathcal{H})_s \coloneqq (p_S)_*(\mathcal{H})(s) \to H^0(X_s, \mathcal{E}^\vee \otimes \mathcal{F}_s) \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{F}_s)$$

is an isomorphism (see [**BS**, Theorem 4.10 (c)]).

Let $s \in S$. Choose an isomorphism $\alpha \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}_s)$. Such an isomorphism always exists, because by assumption $\mathcal{F}_s \simeq \mathcal{E}$. Let $\tilde{\alpha} \in (p_S)_*(\mathcal{H})_s$ be an element which is mapped to α via the composition

$$(p_S)_*(\mathcal{H})_s \to (p_S)_*(\mathcal{H})_s \big/ \mathfrak{m}_s (p_S)_*(\mathcal{H})_s \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{F}_s),$$

and let U be an open neighbourhood of s such that $\tilde{\alpha}$ extends to an element of $(p_S)_*(\mathcal{H})(U) = H^0(U \times X, \mathcal{H})$. Taking U sufficiently small, we obtain an isomorphism $p_X^*(\mathcal{E})|_{U\times X} \xrightarrow{\simeq} \mathcal{F}|_{U\times X}$. □

Using similar arguments, one obtains the following result which gives an equivalent definition for the condition "fibrewise isomorphic to E":

PROPOSITION 4.2.4. *Let S be a complex space and F be vector bundle on $S \times X$. The following conditions are equivalent:*

(1) F is fibrewise isomorphic to E.
(2) For any $s \in S$, F_s is isomorphic to E and the locally free sheaf $p_X^*(\mathcal{E})^\vee \otimes \mathcal{F}$ on $S \times X$ is cohomogically flat in dimension 0.

PROPOSITION 4.2.5. *Let F be a holomorphic bundle on $S \times X$ fibrewise isomorphic to E. The sheaf of sets $\mathcal{T}^F$ on S defined by*

$$\mathcal{T}^F(U) \coloneqq \left\{ (p_X^{U\times X})^*(E) \xrightarrow{h\simeq} F|_{U\times X} \mid h \text{ isomorphism of holomorphic vector bundles over } U \times X \right\}$$

is naturally an $\mathcal{O}_S(G)$-torsor.

PROOF. Note first for every open set U of S, we have a natural right $\mathcal{O}_S(G)(U)$-action

$$\mathcal{T}^F(U) \times \mathcal{O}_S(G)(U) \to \mathcal{T}^F(U).$$

This action is free and transitive. Indeed, for any pair $(h, k) \in \mathcal{T}^F(U) \times \mathcal{T}^F(U)$ there exist a unique $g \in \mathcal{O}_S(G)(U)$ such that $k = h \circ g$, namely the element corresponding to $h^{-1} \circ k \in \mathrm{Aut}_{U\times X}((p_X^{U\times X})^*(E))$ under the isomorphism given by the representability Proposition 4.1.16.

On the other hand for any $s \in S$ the stalk $\mathcal{T}_s^F$ is non-empty because F is fibrewise isomorphic to E. Therefore $\mathcal{T}_s^F$ is a $\mathcal{O}_S(G)_s$-torsor, which completes the proof. □

Conversely, for any $\mathcal{O}_S(G)$-torsor $\mathcal{T}$, we can associate functorially an holomorphic vector bundle $F^{\mathcal{T}} \to S \times X$ which is fibrewise isomorphic to E which is given explicitly by:

$$F^{\mathcal{T}} \coloneqq \left(\coprod_{\tau \text{ local section of } \mathcal{T}} U_\tau \times E \right) \Big/ \sim,$$

where $\sim$ stands for the equivalence relation generated by the holomorphic vector bundle isomorphisms

$$\begin{array}{ccc} (U_\tau \cap U_\theta) \times E & \xleftarrow{g_{\tau\theta}} & (U_\tau \cap U_\theta) \times E \\ \downarrow & & \downarrow \\ U_\tau \times E & & U_\theta \times E \end{array}$$

where $g_{\tau\theta} \in \mathcal{O}_S(G)(U_\tau \cap U_\theta)$ is defined by the condition

$$\theta|_{U_\tau \cap U_\theta} = \tau|_{U_\tau \cap U_\theta}\ g_{\tau\theta}$$

(see section 4.1).

REMARK 4.2.6. *Let $F \to S \times X$ be a holomorphic vector bundle fibrewise isomorphic to E and $\mathcal{T}$ a $\mathcal{O}_S(G)$-torsor on S. We have a natural isomorphisms*

$$F \simeq F^{\mathcal{T}^F},\ \mathcal{T} \simeq \mathcal{T}^{F^{\mathcal{T}}}.$$

In particular, the functors $F \to \mathcal{T}^F$ and $\mathcal{T} \to F^{\mathcal{T}}$ define an equivalence between the groupoid of holomorphic vector bundles on $S \times X$ which are fibrewise isomorphic to E and the groupoid of $\mathcal{O}_S(G)$-torsors on S.

In particular, taking into account the equivalence between principal G-bundles and $\mathcal{O}_S(G)$-torsors (see Proposition 4.1.13), we obtain an equivalence between the groupoid of holomorphic vector bundles on $S \times X$ which are fibrewise isomorphic to E and the groupoid of principal G-bundles on S.

Using the formalism of associated principal bundles (see definition 4.1.14), we have

REMARK 4.2.7. *Let $P \to S$ be a principal G-bundle over S. The holomorphic vector bundle on $S \times X$ fibrewise isomorphic to E associated to P (via the equivalence given by Remark 4.2.6) is precisely the associated bundle $P \times_\rho E$, regarded as a bundle over $S \times X$, where $\rho : G \times E \to E$ is the natural left action.*

4.3. An equivalence of groupoids

Let G be a complex Lie group. By a right Lie G-torsor we will mean a non-empty complex manifold Θ endowed with a holomorphic free transitive right action $\Theta \times G \to \Theta$. Therefore, in the presence of such an object, the choice of a point $\theta \in \Theta$ yields a biholomorphism $G \xrightarrow{\simeq} \Theta$. Note that a right Lie G-torsor can be interpreted as a principal G-bundle over a singleton. We denote by $\mathcal{T}\mathrm{ors}_G$ the groupoid of right Lie G-torsors.

Now let $\alpha : G \times V \to V$ be a holomorphic G-action on a complex manifold V. We will denote by $\mathcal{T}\mathrm{ors}_G^\alpha$ the groupoid of pairs (Θ, ρ), where Θ is right Lie G-torsor and $\rho : \Theta \to V$ is an equivariant holomorphic map.

Let X be a compact complex manifold, E be a holomorphic vector bundle of rank r on X, $\mathcal{E}$ the associated locally free sheaf. We will assume that the pair $(X, \mathcal{E})$ satisfies:

ASSUMPTION. (Main Assumption) X admits an effective divisor D such that

$$H^1(X, \mathcal{E}(D)) = 0.$$

Note that this condition is automatically satisfied when X is a projective complex manifold. In this case one can consider an ample effective divisor $H \subset X$ and, making use of Cartan-Serre-Grothedieck's vanishing theorem, set $D = nH$ for sufficiently large n.

From now on

- We suppose that $(X, \mathcal{E})$ satisfies the main assumption,
- We fix an effective divisor D on X such that $H^1(X, \mathcal{E}(D)) = 0$.
- We put $G \coloneqq \mathrm{Aut}(\mathcal{E})$.

For $g \in G$ we will denote by $g(D)$, $g(D)_D$ the automorphisms induced by g on the sheaves $\mathcal{E}(D)$, respectively $\mathcal{E}(D)_D$ and also on their spaces of global sections.

We define

$$G \coloneqq \mathrm{Aut}(\mathcal{E}),\ \ V_D = H^0(X, \mathcal{E}(D)_D),\ \ W_D \coloneqq H^0(X, \mathcal{E}(D)),\ G_D \coloneqq W_D \rtimes G,$$

where the semidirect product giving G_D is associated with the group morphism

$$G \ni g \longmapsto H^0(g(D)) \in \mathrm{GL}(H^0(X, \mathcal{E}(D))) = \mathrm{GL}(W_D).$$

Taking into account the definition of semidirect product, the group law of G_D is given by

$$(v, g)(w, h) = (v + g(D)(w), g \circ h).$$

We also define a natural left G_D-action $\alpha_D : G_D \times V_D \to V_D$ on V_D by

$$\alpha_D((w, g), \beta) = g(D)_D(\beta) - w_D.$$

where w_D is the image of w under the obvious restriction morphism

$$H^0(X, \mathcal{E}(D)) = W_D \to V_D = H^0(X, \mathcal{E}(D)_D).$$

The aim of this section is to construct an equivalence of groupoids between

(1) the groupoid $\mathcal{A}\mathrm{ff}_E$ of holomorphic affine bundles over X with linearisation type E, and,
(2) the groupoid $\mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}$ of pairs (Θ, ρ) consisting of a Lie G_D-torsor Θ together with a G_D-equivariant holomorphic map $\rho : \Theta \to V_D$.

Taking into account Proposition 2.1.8 we will identify the the groupoid $\mathcal{A}\mathrm{ff}_E$ with the groupoid of augmented bundles (locally free sheaves) $\varphi : \tilde{\mathcal{E}} \to \mathcal{O}_X$ with $\ker(\varphi) \simeq \mathcal{E}$.

4.3.1. The functor $\mathfrak{T} : \mathcal{A}\mathrm{ff}_E \to \mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}$. Let $\varphi : \tilde{\mathcal{E}} \to \mathcal{O}_X$ be a surjective morphism of locally free sheaves such that $\mathcal{E}_\varphi \coloneqq \ker(\varphi) \simeq \mathcal{E}$, where $\mathcal{E}$ is the locally free sheaf associated with the vector bundle E. We have an exact sequence

$$0 \to \mathcal{E}_\varphi \hookrightarrow \tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X \to 0, \tag{e_φ}$$

which will be regarded as an extension of $\mathcal{O}_X$ by $\mathcal{E}_\varphi$. Since $\mathcal{E}_\varphi$ is locally free, we also have the exact sequence

$$0 \to \mathcal{E}_\varphi \xrightarrow{\iota^\varphi_D} \mathcal{E}_\varphi(D) \xrightarrow{\pi^\varphi_D} \mathcal{E}_\varphi(D)_D \to 0 \tag{c^D_φ}.$$

obtained by tensorizing by $\mathcal{E}_\varphi$ the canonical exact sequence $0 \to \mathcal{O}_X \hookrightarrow \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$ associated with D.

Definition 4.3.1. *We define the D-enlarged extension associated with (e_φ) to be the enlarged extension*

$$0 \to \mathcal{E}_\varphi(D) \xrightarrow{\iota^D_\varphi} \tilde{\mathcal{E}}^D_\varphi \xrightarrow{\varphi^D} \mathcal{O}_X \to 0. \tag{e^D_φ}$$

associated to (e_φ) and the canonical sheaf monomorphism $\mathcal{E}_\varphi \hookrightarrow \mathcal{E}_\varphi(D)$ in the sense of Definition 4.1.7.

In other words $\tilde{\mathcal{E}}^D_\varphi$ is the cokernel of the morphism $(\iota^\varphi_D, -j) : \mathcal{E}_\varphi \to \mathcal{E}_\varphi(D) \oplus \tilde{\mathcal{E}}$, and $\tilde{\mathcal{E}}^D_\varphi \xrightarrow{\varphi^D} \mathcal{O}_X$ is the epimorphism induced by φ whose kernel is obviously identified with $\mathcal{E}_\varphi(D)$.

The canonical monomorphism into the central term of $\tilde{\mathcal{E}}^D_\varphi$ of (e^D_φ) is the the sheaf monomorphism given by the composition

$$\tilde{\mathcal{E}} \xhookrightarrow{\quad} \mathcal{E}_\varphi(D) \oplus \tilde{\mathcal{E}} \xrightarrow{\quad} \tilde{\mathcal{E}}^D_\varphi.$$
$$\qquad\qquad\qquad

The canonical epimorphism $q_D^\varphi : \tilde{\mathcal{E}}_\varphi^D \to \mathcal{E}_\varphi(D)_D$ is the unique morphism $\tilde{\mathcal{E}}_\varphi^D \to \mathcal{E}_\varphi(D)_D$ which makes the second column exact and the diagram commutative.

Note that the extension invariant of (e_φ^D) is precisely the image of the extension invariant $h_\varphi \in H^1(X, \mathcal{E}_\varphi)$ of (e_φ) in $H^1(X, \mathcal{E}_\varphi(D))$ via the morphism $H^1(\iota_D^\varphi)$. Since we assumed in the Main Assumption that $H^1(X, \mathcal{E}(D)) = 0$, it follows that the extension (e_φ^D) is split.

The set

$$S_D^\varphi \coloneqq \left\{\sigma : \mathcal{O}_X \to \tilde{\mathcal{E}}_\varphi^D \middle|\ \sigma \circ \varphi_D = \mathrm{id}\right\}$$

of right splittings of (e_φ^D) is an affine subspace of $H^0(X, \tilde{\mathcal{E}}_\varphi^D)$ with model space $H^0(X, \mathcal{E}_\varphi(D))$, hence is naturally a right Lie $H^0(X, \mathcal{E}_\varphi(D))$-torsor.

On the other hand the set $\mathcal{Q}^\varphi$ of sheaf isomorphisms $f : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}_\varphi$ is naturally a right Lie G-torsor. We endow the product

$$\Theta_\varphi^D \coloneqq S_D^\varphi \times \mathcal{Q}^\varphi$$

with the structure of a right G_D-torsor by putting

$$(\sigma, f) \circ (v, g) = (\sigma + (\iota_\varphi^D \circ f(D))(v)\,,\ f \circ g), \tag{25}$$

where $f(D)$ stands for the linear isomorphism $W_D = H^0(X, \mathcal{E}(D)) \to H^0(X, \mathcal{E}_\varphi(D))$ induced by f.

We now define $\rho_\varphi^D : \Theta_\varphi^D \to V_D$ by

$$\rho_\varphi^D(\sigma, f) \coloneqq f(D)_D^{-1}(q_D^\varphi \circ \sigma),$$

where $f(D)_D$ stands for the linear isomorphism $H^0(X, \mathcal{E}(D)_D) \to H^0(X, \mathcal{E}_\varphi(D)_D)$ induced by f. The map ρ_φ^D is obviously holomorphic and one can easily check that it is G_D-equivariant.

Proof. □

We conclude this section with the

Remark 4.3.3. *The assignment*

$$(\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X) \mapsto (\Theta_\varphi^D, \rho_\varphi^D).$$

defines a functor $\mathfrak{T} : \mathcal{A}\mathrm{ff}_E \to \mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$.

4.3.2. A family of elementary transformations. We start this section by noting that any element $\beta \in V_D \coloneqq H^0(X, \mathcal{E}(D)_D)$ defines *in a canonical way* an augmented rank $(r+1)$-bundle

$$\varphi_\beta : \tilde{\mathcal{E}}_\beta \to \mathcal{O}_X$$

with $\ker(\varphi_\beta) \simeq \mathcal{E}$. This augmented bundle will be obtained using a well known construction method in the theory of holomorphic bundles: elementary transformation (see [**Ma**, 1.2 p. 244]).

We start with the canonical exact sequence

$$0 \to \mathcal{E} \xrightarrow{\iota_D} \mathcal{E}(D) \xrightarrow{\pi_D} \mathcal{E}(D)_D \to 0 \tag{c^D}$$

associated with the locally free sheaf $\mathcal{E}$ and the effective divisor D.

For $\beta \in H^0(X, \mathcal{E}(D)_D)$ we denote by $u_\beta : \mathcal{E}(D) \oplus \mathcal{O}_X \to \mathcal{E}(D)_D$ the sheaf morphism defined by

$$u_\beta(s, \zeta) = s_D + \zeta\beta, \tag{26}$$

(for local sections s, ζ of $\mathcal{E}(D)$, respectively $\mathcal{O}_X$), where s_D stands for the image of s under the canonical epimorphism $\pi_D : \mathcal{E}(D) \to \mathcal{E}(D)_D$.

It is easy to check that the morphism u_β is an epimorphism.

Definition 4.3.4. *The associated elementary transformation* $\ker(u_\beta)$ *(which is locally free by Maruyama's result quoted above) will be denoted by $\tilde{\mathcal{E}}_\beta$.*

Remark 4.3.5. *With the notations above we have:*

1. *The restriction $\varphi_\beta : \tilde{\mathcal{E}}_\beta \to \mathcal{O}_X$ of the second projection $p_2 : \mathcal{E}(D) \oplus \mathcal{O}_X \to \mathcal{O}_X$ is still an epimorphism.*
2. *The image of $\mathcal{E}$ in the direct sum $\mathcal{E}(D) \oplus \mathcal{O}_X$ under the composition*

$$\mathcal{E} \xrightarrow{\iota_D} \mathcal{E}(D) \hookrightarrow \mathcal{E}(D) \oplus \mathcal{O}_X$$

is contained in $\tilde{\mathcal{E}}_\beta$ and coincides with $\ker(\varphi_\beta)$*. In particular $\mathcal{E}_{\varphi_\beta} \coloneqq \ker(\varphi_\beta)$ is canonically isomorphic with $\mathcal{E}$, so $\tilde{\mathcal{E}}_\beta$ fits in the short exact sequence*

$$0 \to \mathcal{E} \longrightarrow \tilde{\mathcal{E}}_\beta \xrightarrow{\varphi_\beta} \mathcal{O}_X \to 0. \tag{e_{φ_β}}$$

(3) The D-enlarged extension associated with (e_{φ_β}) (in the sense of Definition 4.3.1) can be canonically identified with the split extension

$$0 \to \mathcal{E}(D) \to \mathcal{E}(D) \oplus \mathcal{O}_X \to \mathcal{O}_X \to 0.$$

(4) The extension invariant $h_{\varphi_\beta} \in H^1(X, \mathcal{E})$ of (e_{φ_β}) is precisely the image of $-\beta$ under the connecting morphism $H^0(X, \mathcal{E}(D)_D) \to H^1(X, \mathcal{E})$ associated with the short exact sequence (c^D).

Proof. The first two statements are clear. For the third statement, see [**Pl2**, Remark 2.1, p. 31]. For the fourth statement, see [**Pl2**, Proposition 2.2, p. 31]. □

The first two statements in the Remark 4.3.5 yield the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{E} & \hookrightarrow & \tilde{\mathcal{E}}_\beta & \xrightarrow{\varphi_\beta} & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{E}(D) & \hookrightarrow & \mathcal{E}(D) \oplus \mathcal{O}_X & \xrightarrow{p_2} & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle u_\beta} & & & & \\
 & & \mathcal{E}(D)_D & = & \mathcal{E}(D)_D & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (D_\beta)$$

in which the second row is canonically identified with the enlarged extension of the first.

Remark 4.3.6. *Identifying the second row in diagram (D_β) with the D-enlarged extension of (e_{φ_β}), we see by Remark 4.3.2, that the canonical epimorphism $q_{\varphi_\beta} : \mathcal{E}(D) \oplus \mathcal{O}_X \to \mathcal{E}(D)_D$ coincides with u_β.*

It is important to point out a natural equivariance property of the assignment $\beta \mapsto u_\beta$. More precisely note that we have a natural complex Lie group morphism

$$R : G_D \to \mathrm{Aut}(\mathcal{E}(D) \oplus \mathcal{O}_X),\ G_D \ni (v, g) \mapsto \begin{pmatrix} g(D) & v \\ 0 & 1 \end{pmatrix}$$

Proposition 4.3.7. *For any $\gamma = (v, g) \in G_D$, $\beta \in V_D$ we have*

$$u_{\gamma\beta} \circ R_\gamma = g(D)_D \circ u_\beta.$$

In other words, putting $\gamma = (v, g)$, $\gamma_0 = g$, $R^0_\gamma = g(D)_D \in \mathrm{Aut}(\mathcal{E}(D)_D)$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{E}(D) \oplus \mathcal{O}_X & \xrightarrow[\simeq]{R_\gamma} & \mathcal{E}(D) \oplus \mathcal{O}_X \\
{\scriptstyle u_\beta}\downarrow & & \downarrow{\scriptstyle u_{\gamma\beta}} \\
\mathcal{E}(D)_D & \xrightarrow[R^0_\gamma]{\simeq} & \mathcal{E}(D)_D
\end{array} \qquad (27)$$

with isomorphic rows.

This implies

Remark 4.3.8. *The automorphism $R_\gamma \in \mathrm{Aut}(\mathcal{E}(D) \oplus \mathcal{O}_X)$ maps $\tilde{\mathcal{E}}_\beta := \ker(u_\beta)$ isomorphically onto $\tilde{\mathcal{E}}_{\gamma\beta} := \ker(u_{\gamma\beta})$, so it induces an isomorphism*

$$R^\beta_\gamma : \tilde{\mathcal{E}}_\beta \to \tilde{\mathcal{E}}_{\gamma\beta}.$$

For any γ, $\gamma' \in G_D$ and $\beta \in V_D$ we have

$$R^{\gamma\beta}_{\gamma'} \circ R^\beta_\gamma = R^\beta_{\gamma'\gamma}.$$

4.3.3. The functor $\mathfrak{A} : \mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D} \to \mathcal{A}\mathrm{ff}_E$ and its properties. Using the constructions of section 4.3.2, we now define a functor $\mathfrak{A} : \mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D} \to \mathcal{A}\mathrm{ff}_E$ in the following way:

Let (Θ, ρ) be an object of the groupoid $\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$, so Θ is a Lie G_D-torsor and $\rho : \Theta \to V_D$ is a G_D equivariant holomorphic map.

Consider the open cover $\mathcal{U}_\Theta = (U_\theta)_{\theta\in\Theta}$ of X defined by $U_\theta = X$ for any $\theta \in \Theta$. We will apply the gluing principle given by Proposition 4.1.2 to the family of sheaves $(\tilde{\mathcal{E}}_{\rho(\theta)})_{\theta\in\Theta}$ and the family of sheaf isomorphisms

$$\varphi_\theta^{\theta'} \coloneqq R_{\gamma_\theta^{\theta'}}^{\rho(\theta)} : \tilde{\mathcal{E}}_{\rho(\theta)} \to \tilde{\mathcal{E}}_{\rho(\theta')}$$

where $\gamma_\theta^{\theta'} \in G_D$ is defined by the condition $\theta = \theta'\gamma_\theta^{\theta'}$. The compatibility condition required in Proposition 4.1.2 can be obtained by using Remark 4.3.8. Indeed, by the equivariance property of ρ, we have

$$\varphi_{\theta'}^{\theta''} \circ \varphi_\theta^{\theta'} = \varphi_\theta^{\theta''}.$$

We now put

$$\tilde{\mathcal{E}}_{\Theta,\rho} \coloneqq \coprod_{\theta\in\Theta} \tilde{\mathcal{E}}_{\rho(\theta)}\big/_{(\varphi_\theta^{\theta'})},$$

the sheaf obtained by gluing the sheaves $\tilde{\mathcal{E}}_{\rho(\theta)}$ via the sheaf isomorphisms $\varphi_\theta^{\theta'}$ in the sense of Proposition 4.1.2.

By the universal property given by Proposition 4.1.2, we see that the family of sheaf epimorphisms

$$(\varphi_{\rho(\theta)} : \mathcal{E}_{\rho(\theta)} \to \mathcal{O}_X)_{\theta\in\Theta}$$

yields a well defined sheaf epimorphism

$$\varphi_{\Theta,\rho} : \tilde{\mathcal{E}}_{\Theta,\rho} \to \mathcal{O}_X.$$

Remark 4.3.9. *The assignment*

$$(\Theta, \rho) \mapsto (\varphi_{\Theta,\rho} : \tilde{\mathcal{E}}_{\Theta,\rho} \to \mathcal{O}_X)$$

defines a functor $\mathfrak{A} : \mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D} \to \mathcal{A}\mathrm{ff}_E$.

In section 4.3.4 we will need an explicit description of

(1) The kernel $\mathcal{E}_{\varphi_{\Theta,\rho}}$ of the augmented bundle $\varphi_{\Theta,\rho} : \tilde{\mathcal{E}}_{\Theta,\rho} \to \mathcal{O}_X$.
(2) The D-enlarged extension

$$0 \to \mathcal{E}_{\varphi_{\Theta,\rho}}(D) \xrightarrow{\iota^D_{\varphi_{\Theta,\rho}}} \tilde{\mathcal{E}}^D_{\varphi_{\Theta,\rho}} \to \mathcal{O}_X \to 0$$

of the extension $0 \to \mathcal{E}_{\varphi_{\Theta,\rho}} \hookrightarrow \tilde{\mathcal{E}}_{\Theta,\rho} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_X \to 0$.
(3) The canonical epimorphism

$$q_D^{\varphi_{\Theta,\rho}} : \tilde{\mathcal{E}}^D_{\varphi_{\Theta,\rho}} \to \mathcal{E}_{\varphi_{\Theta,\rho}}(D)_D$$

on the central term of this enlarged extension.

We will see below that all these three objects are also associated with gluing data parameterised by points $\theta \in \Theta$.

Note first that for any $\theta \in \Theta$ we have the commutative diagram $D_\theta \coloneqq D_{\rho(\theta)}$ obtained by taking $\beta = \rho(\theta)$ in the diagram (D_β). An element $\gamma = (v, g) \in G_D$ gives an isomorphism

$$\mathfrak{R}_\gamma : D_\theta \to D_{\theta\gamma^{-1}}$$

of commutative diagrams (in the obvious sense) defined in the following way:

Definition 4.3.10. *(1) On the sheaves $\mathcal{E}$, $\mathcal{E}(D)$, $\mathcal{E}(D)_D$ the isomorphism $\mathfrak{R}_\gamma$ acts by g, $g(D)$, respectively $g(D)_D$.*
(2) On the sheaves $\mathcal{E}(D) \oplus \mathcal{O}_X$, $\tilde{\mathcal{E}}_{\rho(\theta)}$ the isomorphism $\mathfrak{R}_\gamma$ acts by R_γ, respectively by its restriction to $\tilde{\mathcal{E}}_{\rho(\theta)}$.
(3) On $\mathcal{O}_X$ the isomorphism $\mathfrak{R}_\gamma$ acts by identity.

Using Remark 4.1.6 we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \coprod_{\theta\in\Theta}\mathcal{E}/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) & \longrightarrow & \tilde{\mathcal{E}}_{\Theta,\rho}\coloneqq\coprod_{\theta\in\Theta}\tilde{\mathcal{E}}_{\rho(\theta)}/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) & \xrightarrow{\varphi_{\Theta,\rho}} & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \coprod_{\theta\in\Theta}\mathcal{E}(D)/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) & \xrightarrow{\mathfrak{j}} & \coprod_{\theta\in\Theta}(\mathcal{E}(D)\oplus\mathcal{O}_X)/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) & \xrightarrow{\mathfrak{p}} & \mathcal{O}_X & \longrightarrow & 0 \qquad (D_{\Theta,\rho}) \\
 & & \downarrow & & \downarrow{\scriptstyle \coprod_{\theta\in\Theta}u_{\rho(\theta)}} & & & & \\
 & & \coprod_{\theta\in\Theta}\mathcal{E}(D)_D/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) & = & \coprod_{\theta\in\Theta}\mathcal{E}(D)_D/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

of sheaves associated to gluing data, with canonical isomorphisms

$$\coprod_{\theta\in\Theta}\mathcal{E}/(\mathfrak{R}_{\gamma_\theta^{\theta'}})=\ker(\varphi_{\Theta,\rho})=:\mathcal{E}_{\varphi_{\Theta,\rho}},\ \coprod_{\theta\in\Theta}\mathcal{E}(D)/(\mathfrak{R}_{\gamma_\theta^{\theta'}})=\big(\coprod_{\theta\in\Theta}\mathcal{E}/(\mathfrak{R}_{\gamma_\theta^{\theta'}})\big)(D)=\mathcal{E}_{\varphi_{\Theta,\rho}}(D).$$

Note that via the above identifications, the vertical arrow on the left is the canonical monomorphism $\mathcal{E}_{\varphi_{\Theta,\rho}}\hookrightarrow\mathcal{E}_{\varphi_{\Theta,\rho}}(D)$.

Put $\tilde{\mathcal{E}}\coloneqq\tilde{\mathcal{E}}_{\Theta,\rho}$, $\varphi\coloneqq\varphi_{\Theta,\rho}$ to save on notations. Using the universal property of the enlarged extension (see Remark 4.1.8) and the characterisation of the canonical epimorphism (see Remark 4.3.2), we obtain:

Proposition 4.3.11. *With the notations and under assumptions above, we have:*

(1) The kernel $\mathcal{E}_\varphi$ of the augmented bundle $\tilde{\mathcal{E}}\xrightarrow{\varphi}\mathcal{O}_X$ defined above is canonically identified with the sheaf $\coprod_{\theta\in\Theta}\mathcal{E}/(\mathfrak{R}_{\gamma_\theta^{\theta'}})$ associated to the gluing data of $\mathcal{O}_X$-modules

$$\big((\theta\mapsto\mathcal{E})_{\theta\in\Theta}\,,\,(\mathfrak{R}_{\gamma_\theta^{\theta'}})\big).$$

(2) The D-enlarged extension

$$0\to\mathcal{E}_\varphi(D)\xrightarrow{\iota_\varphi^D}\tilde{\mathcal{E}}_\varphi^D\xrightarrow{\varphi^D}\mathcal{O}_X\to 0$$

of the extension

$$0\to\mathcal{E}_\varphi\hookrightarrow\tilde{\mathcal{E}}\xrightarrow{\varphi}\mathcal{O}_X\to 0$$

is canonically identified with the extension

$$0\longrightarrow\coprod_{\theta\in\Theta}\mathcal{E}(D)/(\mathfrak{R}_{\gamma_\theta^{\theta'}})\xrightarrow{\mathfrak{j}}\coprod_{\theta\in\Theta}(\mathcal{E}(D)\oplus\mathcal{O}_X)/(\mathfrak{R}_{\gamma_\theta^{\theta'}})\xrightarrow{\mathfrak{p}}\mathcal{O}_X\longrightarrow 0,$$

of sheaves associated with gluing data, where the monomorphism $\mathfrak{j}$ is is obtained by gluing (in the sense of Remark 4.1.4) the constant family of inclusions

$$\Theta\ni\theta\mapsto(j_1:\mathcal{E}(D)\hookrightarrow\mathcal{E}(D)\oplus\mathcal{O}_X),$$

and the epimorphism $\mathfrak{p}$ is obtained by gluing (in the sense of Remark 4.1.4) the constant family of projections

$$\Theta\ni\theta\mapsto(p_2:\mathcal{E}(D)\oplus\mathcal{O}_X\to\mathcal{O}_X).$$

(3) Via the identification $\tilde{\mathcal{E}}=\coprod_{\theta\in\Theta}(\mathcal{E}(D)\oplus\mathcal{O}_X)/(\mathfrak{R}_{\gamma_\theta^{\theta'}})$, the canonical epimorphism

$$q_D^\varphi:\tilde{\mathcal{E}}_\varphi^D\to\mathcal{E}_\varphi(D)_D$$

coincides with the morphism $\coprod_{\theta\in\Theta}u_{\rho(\theta)}$ obtained by gluing (in the sense of Remark 4.1.4) the family of morphisms $(\mathcal{E}(D)\oplus\mathcal{O}_X\xrightarrow{u_{\rho(\theta)}}\mathcal{E}(D)_D)_{\theta\in\Theta}$.

Using the properties of the sheaf associated with gluing data and the morphisms associated with gluing data morphisms (Proposition 4.1.2, Remarks 4.1.4, 4.1.5, 4.1.6), we obtain:

(1) a family $(\psi_\theta : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}_\varphi)_{\theta\in\Theta}$ of canonical sheaf isomorphisms satisfying the compatibility condition

$$\psi_{\theta\gamma} = \psi_\theta \circ \mathfrak{R}_\gamma = \psi_\theta \circ g \text{ for any } \theta \in \Theta,\ \gamma = (v, g) \in G_D. \tag{28}$$

(2) a family $(\Psi_\theta : \mathcal{E}(D) \oplus \mathcal{O}_X \xrightarrow{\simeq} \tilde{\mathcal{E}}^D_\varphi)_{\theta\in\Theta}$ of canonical sheaf isomorphisms making commutative the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}(D) & \xrightarrow{j_1} & \mathcal{E}(D)\oplus\mathcal{O}_X & \xrightarrow{p_2} & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_\theta(D)} & & \downarrow{\scriptstyle \Psi_\theta} & & \| & & \\ 0 & \longrightarrow & \mathcal{E}_\varphi(D) & \xrightarrow{\iota^D_\varphi} & \tilde{\mathcal{E}}^D_\varphi & \xrightarrow{\varphi^D} & \mathcal{O}_X & \longrightarrow & 0 \end{array}, \tag{29}$$

$$\begin{array}{ccc} \mathcal{E}(D)\oplus\mathcal{O}_X & \xrightarrow{\Psi_\theta} & \tilde{\mathcal{E}}^D_\varphi \\ {\scriptstyle u_{\rho(\theta)}}\downarrow & & \downarrow{\scriptstyle q^\varphi_D} \\ \mathcal{E}(D)_D & \xrightarrow{\psi_\theta(D)_D} & \mathcal{E}_\varphi(D)_D \end{array} \tag{30}$$

and satisfying the compatibility condition

$$\Psi_{\theta\gamma} = \Psi_\theta \circ R_\gamma \text{ for any } \theta \in \Theta,\ \gamma \in G_D. \tag{31}$$

4.3.4. The natural isomorphism $\mathfrak{A}\circ\mathfrak{T} \xrightarrow{\simeq} \mathrm{id}_{\mathcal{A}\mathrm{ff}_E}$. For any object $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X$ of $\mathcal{A}\mathrm{ff}_E$ we will construct a *canonical* isomorphism

$$\mathfrak{A}(\mathfrak{T}(\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X)) \xrightarrow{\simeq} (\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X)$$

of augmented bundles.

Recall that $\mathfrak{T}(\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X) = (\Theta^D_\varphi, \rho^D_\varphi)$ defined in section 4.3.1. Put $\Theta \coloneqq \Theta^D_\varphi$ and $\rho \coloneqq \rho^D_\varphi$ to save on notations. Then $\mathfrak{A}(\Theta, \rho)$ is defined using the gluing principle for sheaves (see Proposition 4.1.2) by

$$\mathfrak{A}(\Theta, \rho) = \Big(\coprod_{\theta\in\Theta} \tilde{\mathcal{E}}_{\rho(\theta)}/_{(\varphi^{\theta'}_\theta)} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_X \Big).$$

By the universal property given by Proposition 4.1.2 it is sufficient to construct, for any $\theta \in \Theta$, an isomorphism of augmented bundles

$$\begin{array}{ccc} \tilde{\mathcal{E}}_{\rho(\theta)} & \xrightarrow{e_\theta} & \tilde{\mathcal{E}} \\ & {\scriptstyle \varphi_{\rho(\theta)}}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & \mathcal{O}_X & \end{array} \tag{32}$$

such that the following compatibility condition is satisfied

$$e_{\theta'} \circ \varphi^{\theta'}_\theta = e_\theta \text{ for } \theta,\ \theta' \in \Theta. \tag{33}$$

The idea of the construction of e_θ is very natural: A point $\theta \in \Theta = \Theta^D_\varphi$ is a pair (σ, f) consisting of a splitting σ of the exact sequence

$$0 \to \mathcal{E}_\varphi(D) \xrightarrow{\iota^D_\varphi} \tilde{\mathcal{E}}^D_\varphi \xrightarrow{\varphi^D} \mathcal{O}_X \to 0. \tag{e^D_φ}$$

and an isomorphism of sheaves $f : \mathcal{E} \to \mathcal{E}_\varphi$. We will use σ to identify $\tilde{\mathcal{E}}^D_\varphi$ with $\mathcal{E}_\varphi(D) \oplus \mathcal{O}_X$ and f to further identify this direct sum to $\mathcal{E}(D) \oplus \mathcal{O}_X$. This will allow us to identify the subsheaf $\tilde{\mathcal{E}}$ of $\tilde{\mathcal{E}}^D_\varphi$ to precisely the elementary transformation $\tilde{\mathcal{E}}_{\rho(\theta)} \hookrightarrow \mathcal{E}(D) \oplus \mathcal{O}_X$.

More precisely, let $\theta \coloneqq (\sigma, f)$ and $\beta \coloneqq \rho(\theta) = f(D)^{-1}_D(q^\varphi_D \circ \sigma)$ (see section 4.3.1). The diagram

$$\begin{array}{ccc} \mathcal{E}(D)\oplus\mathcal{O}_X & \xrightarrow[\simeq]{(\iota^D_\varphi\circ f(D)\,,\,\sigma)} & \tilde{\mathcal{E}}^D_\varphi \\ {\scriptstyle u_\beta}\downarrow & & \downarrow{\scriptstyle q^D_\varphi} \\ \mathcal{E}(D)_D & \xrightarrow[\simeq]{f(D)_D} & \mathcal{E}_\varphi(D)_D \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{34}$$

is commutative. Indeed, we have

$$f(D)_D \circ u_\beta(s, \zeta) = f(D)_D(s_D + \zeta\beta) = (f(D)(s))_D + \zeta q^\varphi_D \circ \sigma,$$

and, using the commutativity of (24), we also obtain

$$q_D^{\varphi}\circ(\iota_{\varphi}^D\circ f(D),\sigma)(s,\zeta)=q_D^{\varphi}(\iota_{\varphi}^D\circ f(D)(s)+\zeta\sigma)=q_D^{\varphi}(\iota_{\varphi}^D\circ f(D)(s))+\zeta q_D^{\varphi}\circ\sigma=(f(D)(s))_D+\zeta q_D^{\varphi}\circ\sigma.$$

It follows that the sheaf isomorphism

$$I_\theta\coloneqq(\iota_{\varphi}^D\circ f(D)\,,\sigma):\mathcal{E}(D)\oplus\mathcal{O}_X\xrightarrow{\simeq}\tilde{\mathcal{E}}_{\varphi}^D$$

maps isomorphically $\tilde{\mathcal{E}}_\beta\coloneqq\ker(u_\beta)$ onto $\ker(q_{\varphi}^D)=\tilde{\mathcal{E}}$, so it induces an isomorphism

$$e_\theta:\tilde{\mathcal{E}}_{\rho(\theta)}\xrightarrow{\simeq}\tilde{\mathcal{E}}.$$

In other words, the commutative diagram (34) can be extended to the commutative diagram

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\tilde{\mathcal{E}}_\beta & \xrightarrow[\simeq]{e_\theta} & \tilde{\mathcal{E}}\\
\downarrow & & \downarrow{\scriptstyle j_{\varphi}^D}\\
\mathcal{E}(D)\oplus\mathcal{O}_X & \xrightarrow[\simeq]{I_\theta} & \tilde{\mathcal{E}}_{\varphi}^D\\
\downarrow{\scriptstyle u_\beta} & & \downarrow{\scriptstyle q_{\varphi}^D}\\
\mathcal{E}(D)_D & \xrightarrow[\simeq]{f(D)_D} & \mathcal{E}_{\varphi}(D)_D\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}\tag{35}$$

Now note that the diagram

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\tilde{\mathcal{E}}_\beta & \xrightarrow[\simeq]{e_\theta} & \tilde{\mathcal{E}}\\
\downarrow & \searrow^{\varphi_\beta}\quad\swarrow_{\varphi} & \downarrow{\scriptstyle j_{\varphi}^D}\\
 & \mathcal{O}_X & \\
 & \nearrow_{p_2}\quad\nwarrow_{\varphi^D} & \\
\mathcal{E}(D)\oplus\mathcal{O}_X & \xrightarrow[\simeq]{I_\theta} & \tilde{\mathcal{E}}_{\varphi}^D
\end{array}\tag{36}$$

is commutative. Indeed, the commutativity of the lower triangle follows taking into account that $\varphi^D\circ\iota_{\varphi}^D=0$ and, since σ is a right splitting of (e_{φ}^D), $\varphi^D\circ\sigma=\mathrm{id}_{\mathcal{O}_X}$. Therefore, the commutativity of the upper triangle follows from the commutativity of the lateral triangles and the commutativity of the rectangle. The latter is obvious since e_θ is induced by I_θ. The commutativity of the lateral triangles follow using the commutative diagrams (24), (D_β).

The commutativity of the upper triangle yields $\varphi\circ e_\theta=\varphi_{\rho(\theta)}$, so e_θ is an isomorphism of augmented bundles as claimed in (32).

It remains to prove the compatibility condition (33). Let θ, $\theta'\in\Theta$ and let again $\gamma_\theta^{\theta'}\in G_D$ be defined by the condition $\theta=\theta'\gamma_\theta^{\theta'}$. Now note that in the following diagram

$$\begin{array}{ccc}
\mathcal{E}(D)\oplus\mathcal{O}_X & \longleftarrow & \tilde{\mathcal{E}}_{\rho(\theta)}\\
\downarrow{\scriptstyle R_{\gamma_\theta^{\theta'}}} & \nearrow_{I_\theta}\quad\nwarrow_{e_\theta} & \downarrow{\scriptstyle \varphi_\theta^{\theta'}}\\
 & \tilde{\mathcal{E}}_\varphi & \\
 & \nwarrow_{I_{\theta'}}\quad\nearrow_{e_{\theta'}} & \\
\mathcal{E}(D)\oplus\mathcal{O}_X & \longleftarrow & \tilde{\mathcal{E}}_{\rho(\theta')}
\end{array}\tag{37}$$

the rectangle is commutative by the definition of $\varphi_\theta^{\theta'}$, the upper and lower triangles are commutative by the definitions of e_θ, $e_{\theta'}$. Therefore the commutativity of the right hand triangle follows from the commutativity of the left hand triangle. But, putting $\theta=(\sigma,f)$, $\theta'=(\sigma',f')$, $\gamma_\theta^{\theta'}=(v,g)$, we have

$$(\sigma,f)=(\sigma',f')(v,g)=(\sigma'+(\iota_{\varphi}^D\circ f'(D))(v)\,,\,f'\circ g),$$

so

$$I_{\theta'} \circ R_{\gamma_\theta^{\theta'}} = (\iota_\varphi^D \circ f'(D)\,,\, \sigma') \begin{pmatrix} g(D) & v \\ 0 & 1 \end{pmatrix} = (\iota_\varphi^D \circ f(D)\ \sigma) = I_\theta,$$

which proves that the left hand triangle is commutative, as claimed. Therefore in conclusion we have:

PROPOSITION 4.3.12. *The family of augmented bundle isomorphisms*

$$\begin{array}{ccc} \tilde{\mathcal{E}}_{\rho(\theta)} & \xrightarrow{\quad e_\theta \quad} & \tilde{\mathcal{E}} \\ & {\scriptstyle \varphi_{\rho(\theta)}} \searrow \quad \swarrow {\scriptstyle \varphi} & \\ & \mathcal{O}_X & \end{array}$$

defined above verify the compatibility condition (33), so, by the universal property given by Proposition 4.1.2, defines a canonical isomorphism

$$\mathfrak{A}(\mathfrak{T}(\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X)) \coloneqq \left(\coprod_{\theta\in\Theta} \tilde{\mathcal{E}}_{\rho(\theta)}/_{(\varphi_\theta^{\theta'})} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_X \right) \xrightarrow{\simeq} (\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_X).$$

4.3.5. The natural isomorphism $\mathrm{id}_{\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}} \xrightarrow{\simeq} \mathfrak{T} \circ \mathfrak{A}$**.** For an object (Θ, ρ) of the groupoid $\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$, we will construct a *canonical* isomorphism

$$(\Theta, \rho) \xrightarrow{\simeq \eta} (\mathfrak{T} \circ \mathfrak{A})(\Theta, \rho).$$

In other words, putting $\tilde{\mathcal{E}} \coloneqq \tilde{\mathcal{E}}_{\Theta,\rho}$, $\varphi \coloneqq \varphi_{\Theta,\rho}$, we will construct a canonical right Lie G_D-torsor isomorphism $\Theta \xrightarrow{\simeq\eta} \Theta_\varphi^D$ making commutative the triangle

$$\begin{array}{ccc} \Theta & \xrightarrow{\quad \eta\simeq \quad} & \Theta_\varphi^D \\ & {\scriptstyle \rho} \searrow \quad \swarrow {\scriptstyle \rho_\varphi^D} & \\ & V_D & \end{array} \quad . \tag{38}$$

Using the constructions explained at the end of section 4.3.3, and denoting by

$$j_2 : \mathcal{O}_X \to \mathcal{E}(D) \oplus \mathcal{O}_X$$

the obvious monomorphism, we define $\eta : \Theta \to \Theta_\varphi^D$ as follows:

$$\eta(\theta) \coloneqq (\Psi_\theta \circ j_2, \psi_\theta).$$

REMARK 4.3.13. *The map η defined above is an isomorphism of right Lie G_D-torsors.*

PROOF. For $\theta \in \Theta$, $\gamma = (v, g) \in G_D$ we have by the compatibility formulae (28), (31) the following:

$$\eta(\theta\gamma) = (\Psi_{\theta\gamma} \circ j_2, \psi_{\theta\gamma}) = (\Psi_\theta \circ R_\gamma \circ j_2, \psi_\theta \circ g).$$

We also have

$$R_\gamma \circ j_2(\zeta) = \begin{pmatrix} g(D) & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \zeta \end{pmatrix} = \begin{pmatrix} \zeta v \\ \zeta \end{pmatrix} = j_2(\zeta) + j_1(\zeta v).$$

Therefore, by the commutative diagram (29),

$$\Psi_\theta \circ R_\gamma \circ j_2(\zeta) = \Psi_\theta \circ j_2(\zeta) + \Psi_\theta \circ j_1(\zeta v) = \Psi_\theta \circ j_2(\zeta) + \iota_\varphi^D \circ \psi_\theta(D)(\zeta v).$$

Therefore, via the identification $\mathrm{Hom}(\mathcal{O}_X, \mathcal{E}_\varphi(D)) = H^0(X, \mathcal{E}_\varphi(D))$, we have

$$\Psi_{\theta\gamma} \circ j_2 = \Psi_\theta \circ j_2 + \iota_\varphi^D \circ \psi_\theta(D)(v).$$

On the other hand

$$\eta(\theta) \circ \gamma = (\Psi_\theta \circ j_2, \psi_\theta) \circ (v, g) = \big(\Psi_\theta \circ j_2 + \iota_\varphi^D \circ \psi_\theta(D)(v), \psi_\theta \circ g),$$

which proves the claim. □

The commutativity of the triangle (38) follows from:

REMARK 4.3.14. *We have $\rho_\varphi^D \circ \eta = \rho$.*

PROOF. For $\theta \in \Theta$ we have

$$\rho_\varphi^D \circ \eta(\theta) = \rho_\varphi^D(\Psi_\theta \circ j_2, \psi_\theta) = \psi_\theta(D)_D^{-1}(q_D^\varphi \circ \Psi_\theta \circ j_2) = (\psi_\theta(D)_D^{-1} \circ q_D^\varphi \circ \Psi_\theta) \circ j_2.$$

Using the commutativity of the diagram (30) we have $\psi_\theta(D)_D^{-1} \circ q_D^\varphi \circ \Psi_\theta = u_{\rho(\theta)}$. Therefore

$$\rho_\varphi^D \circ \eta(\theta) = u_{\rho(\theta)} \circ j_2,$$

which, by formula (26), obviously coincides with the morphism $\mathcal{O}_X \to \mathcal{E}(D)_D$ associated with the global section $\rho(\theta) \in H^0(X, \mathcal{E}(D)_D) = V_D$. □

4.4. An equivalence of stacks

In this section we will show that the stack $\mathscr{A}\mathrm{ff}_E$ defined in the introduction is isomorphic to the quotient stack $[V_D/G_D]$.

Let S be a complex space. As explained in the introduction, we denote by ${}_S\mathcal{A}\mathrm{ff}_E$ the fibre of $\mathscr{A}\mathrm{ff}_E$ over S, i.e. the groupoid whose objects are augmented bundles $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X}$ on $S \times X$ such that $\mathcal{E}_\varphi \coloneqq \ker(\varphi)$ is fibrewise isomorphic to $\mathcal{E}$ in the sense of Definition 4.2.1. For such augmented bundle $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X}$ over $S \times X$ and for an open subspace $U \subset S$, we will denote by $\tilde{\mathcal{E}}_U \xrightarrow{\varphi_U} \mathcal{O}_{U\times X}$ its restriction to $U \times X$.

By Proposition 4.2.5, the sheaf of sets $\mathcal{T}_\varphi$ on S defined by

$$\mathcal{T}_\varphi(U) \coloneqq \left\{(p_X^{U\times X})^*(\mathcal{E}) \xrightarrow{f\simeq} \mathcal{E}_{\varphi_U} \mid h \text{ is an isomorphism of } \mathcal{O}_{U\times X}\text{-modules}\right\}$$

(for open sets $U \subset S$) is naturally an $\mathcal{O}_S(G)$-torsor.

We will also denote by ${}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$ the groupoid whose objects are pairs (Θ, ρ) consisting of an $\mathcal{O}_S(G_D)$-torsor on S and a $\mathcal{O}_S(G_D)$-equivariant sheaf morphism $\rho : \Theta \to \mathcal{O}_S(V_D)$.

Note that ${}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$ is precisely the fibre over S of the quotient stack $[V_D/G_D]$ regarded as category fibred in groupoids.

We will define an equivalence of categories using the functors

$${}_S\mathfrak{T} : {}_S\mathcal{A}\mathrm{ff}_E \to {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D},\ {}_S\mathfrak{A} : {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D} \to {}_S\mathcal{A}\mathrm{ff}_E$$

which will be defined below by generalising the constructions developed in sections 4.3.1, 4.3.3.

4.4.1. The functor ${}_S\mathfrak{T} : {}_S\mathcal{A}\mathrm{ff}_E \to {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$.

Put $D_S \coloneqq S \times D$ regarded as a divisor on $S \times X$.

LEMMA 4.4.1. *We have* $R^1 p_{S*}(\mathcal{E}_\varphi(D_S)) = 0$.

PROOF. Indeed, since $\mathcal{E}_\varphi$ is locally (with respect to S) isomorphic to $p_X^*(\mathcal{E})$, it follows that $\mathcal{E}_\varphi(D_S)$ is locally (with respect to S) isomorphic to $p_X^*(\mathcal{E}(D))$. The claim follows using the condition $H^1(X, \mathcal{E}(D)) = 0$ (our Main Assumption) and the base change theorem applied to the morphism $X \to \{*\}$. □

Let $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X}$ be an object of the groupoid ${}_S\mathcal{A}\mathrm{ff}_E$. Consider the extension

$$0 \to \mathcal{E}_\varphi \hookrightarrow \tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X} \to 0$$

and its D_S-enlarged extension

$$0 \to \mathcal{E}_\varphi(D_S) \xrightarrow{\iota_\varphi^D} \tilde{\mathcal{E}}_\varphi^D \xrightarrow{\varphi^D} \mathcal{O}_{S\times X} \to 0 \tag{39}$$

(see Definitions 4.1.7, 4.3.1). Note that, in order to save on notations, we will use the superscript D instead of D_S in the notations $\mathcal{E}_\varphi^D$, φ^D, ι_φ^D.

Taking into account Lemma 4.4.1, we obtain the short exact sequence of sheaves on S:

$$0 \to p_{S*}(\mathcal{E}_\varphi(D_S)) \xrightarrow{\iota_\varphi^D} p_{S*}(\tilde{\mathcal{E}}_\varphi^D) \xrightarrow{\varphi^D} \mathcal{O}_S \to 0. \tag{40}$$

A local right inverse $\sigma : \mathcal{O}_U \to p_{S*}(\tilde{\mathcal{E}}_\varphi^D)_U$ of the epimorphism φ^D in (40) can be regarded as a fibrewise right splitting of (39) over $U \times X$ via the identifications

$$\mathrm{Hom}(\mathcal{O}_U, p_{U*}(\tilde{\mathcal{E}}_{\varphi_U}^D)) = H^0(U, p_{U*}(\tilde{\mathcal{E}}_{\varphi_U}^D)) = H^0(U \times X, \tilde{\mathcal{E}}_{\varphi_U}^D) = \mathrm{Hom}(\mathcal{O}_{U\times X}, \tilde{\mathcal{E}}_{\varphi_U}^D).$$

Let $\mathcal{S}_\varphi^D$ be the sheaf of local right inverses of the sheaf epimorphism φ^D in (40). The short exact sequence (40) shows that this sheaf is naturally a $p_{S*}(\mathcal{E}_\varphi(D_S))$-torsor. The sheaf $p_{S*}(\mathcal{E}_\varphi(D_S))$ which acts on this torsor is locally isomorphic to $\mathcal{O}_S(W_D)$. More precisely, any isomorphism

$$\left(p_X^{U\times X *}(\mathcal{E}) \xrightarrow{f\simeq} \mathcal{E}_{\varphi_U}\right) \in \mathcal{T}_\varphi(U)$$

defines isomorphisms

$$f(D) : p_{U*}(p_X^*(\mathcal{E}(D)) = \mathcal{O}(U, W_D) \to p_{U*}(\mathcal{E}_\varphi(D_S)) = p_{S*}(\mathcal{E}_\varphi(D_S))(U),$$

$$f(D)_D : p_{U*}(p_X^*(\mathcal{E}(D)_D) = \mathcal{O}(U, V_D) \to p_{U*}(\mathcal{E}_\varphi(D_S)_{D_S}) = p_{S*}(\mathcal{E}_\varphi(D_S)_{D_S})(U).$$

For the first equalities in the above formulae we have used Remark 4.1.15.

We put

$$\Theta_\varphi^D \coloneqq \mathcal{S}_\varphi^D \times \mathcal{T}_\varphi,$$

i.e. for any open subspace $U \subset S$ we set $\Theta_\varphi^D(U) \coloneqq \mathcal{S}_\varphi^D(U) \times \mathcal{T}_\varphi(U)$ and we endow this set with a right $\mathcal{O}(U, G_D)$-action by putting

$$(\sigma, f) \circ (v, g) \coloneqq (\sigma + (\iota_\varphi^D \circ f(D))(v)\,,\ f \circ g)$$

for any $(v, g) \in \mathcal{O}(U, W_D) \rtimes \mathcal{O}(U, G) = \mathcal{O}(U, W_D \rtimes G) = \mathcal{O}(U, G_D)$. Formally this is formula (25) used is section 4.3.1.

We now define an $\mathcal{O}_S(G_D)$-equivariant sheaf morphism $\rho_\varphi^D : \Theta_\varphi^D \to \mathcal{O}_S(V_D)$ as follows:

For any open subspace $U \subset X$ the short exact sequence

$$0 \to \mathcal{E}_{\varphi_U}(D_U) \xrightarrow{\iota_{\varphi_U}^D} \tilde{\mathcal{E}}_{\varphi_U}^D \xrightarrow{\varphi_U^D} \mathcal{O}_{U\times X} \to 0$$

is the D_U-enlarged extension of $0 \to \mathcal{E}_{\varphi_U} \hookrightarrow \tilde{\mathcal{E}}_U \xrightarrow{\varphi_U} \mathcal{O}_{U\times X} \to 0$, so it fits in the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{E}_{\varphi_U} & \hookrightarrow & \tilde{\mathcal{E}}_U & \xrightarrow{\varphi_U} & \mathcal{O}_{U\times X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle j_{\varphi_U}^D} & \overset{\sigma}{\swarrow} & \| & & \\
0 & \longrightarrow & \mathcal{E}_{\varphi_U}(D_U) & \xrightarrow{\iota_{\varphi_U}^D} & \tilde{\mathcal{E}}_{\varphi_U}^D & \xrightarrow[\varphi_U^D]{} & \mathcal{O}_{U\times X} & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow{\scriptstyle q_D^{\varphi_U}} & & & & \\
 & & \mathcal{E}_{\varphi_U}(D_U)_{D_U} & = & \mathcal{E}_{\varphi_U}(D_U)_{D_U} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{41}$$

For any $(\sigma, f) \in \mathcal{S}_\varphi^D(U) \times \mathcal{T}_\varphi(U)$ we put:

$$\begin{aligned}\rho_\varphi^D(\sigma, f) \coloneqq & f(D)_D^{-1}(q_D^{\varphi_U} \circ \sigma) \in H^0(U \times X, p_X^*(\mathcal{E})(D_U)_{D_U}) = H^0(U \times X, p_X^{U\times X*}(\mathcal{E}(D)_D)) = \\ & = \mathcal{O}(U, H^0(X, \mathcal{E}(D)_D)) = \mathcal{O}(U, V_D).\end{aligned}$$

The $\mathcal{O}_S(G_D)$-equivariance of ρ_φ^D is verified in the same way as the G_D-equivariance of ρ_φ^D in section 4.3.1.

The functor ${}_S\mathfrak{T} : {}_S\mathcal{A}\mathrm{ff}_E \to {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$ is defined by

$${}_S\mathfrak{T}(\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X}) \coloneqq (\Theta_\varphi^D, \rho_\varphi^D).$$

4.4.2. The functor ${}_S\mathfrak{A} : {}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D} \to {}_S\mathcal{A}\mathrm{ff}_E$ and its properties.

Let $U \subset S$ be an open subspace and let

$$\beta \in H^0(U \times X, p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U}) = H^0(U, p_{U*}^{U\times X}(p_X^{U\times X*}(\mathcal{E}(D)_D)) = \mathcal{O}(U, V_D).$$

(see Remark 4.1.15). We define a sheaf epimorphism

$$u_\beta : p_X^{U\times X*}(\mathcal{E})(D_U) \oplus \mathcal{O}_{U\times X} \to p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U}$$

by

$$u_\beta(\eta, \zeta) \coloneqq \eta_{D_U} + \zeta\beta,$$

we let $\tilde{\mathcal{E}}_\beta \coloneqq \ker(u_\beta)$ be the elementary transformation associated with u_β and we denote by $\varphi_\beta : \tilde{\mathcal{E}}_\beta \to \mathcal{O}_{U\times X}$ the epimorphism induced by the projection of $p_X^{U\times X*}(\mathcal{E})(D_U) \oplus \mathcal{O}_{U\times X}$ on the second factor.

As in section 4.3.2, for any $\beta \in \mathcal{O}(U, V_D)$ we obtain the commutative diagram (D_β) with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & p_X^{U\times X*}(\mathcal{E}) & \longrightarrow & \tilde{\mathcal{E}}_\beta & \xrightarrow{\varphi_\beta} & \mathcal{O}_{U\times X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & p_X^{U\times X*}(\mathcal{E})(D_U) & \hookrightarrow & p_X^{U\times X*}(\mathcal{E})(D_U)\oplus\mathcal{O}_{U\times X} & \xrightarrow{p_2} & \mathcal{O}_{U\times X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle u_\beta} & & & & \\
 & & p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U} & = & p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (D_\beta)$$

in which the second row is canonically identified with the enlarged extension of the first, and, via this identification, u_β coincides with the canonical epimorphism on $p_X^{U\times X*}(\mathcal{E})(D_U)\oplus\mathcal{O}_{U\times X}$.

We will need the equivariance property of the correspondences $\beta \mapsto u_\beta$, $\beta \mapsto \tilde{\mathcal{E}}_\beta$ with respect to the natural action of the group $\mathcal{O}(U, G_D)$. Let $\gamma = (v, g) \in \mathcal{O}(U, G_D)$. We define an automorphism

$$R_\gamma \in \mathrm{Aut}(p_X^{U\times X*}(\mathcal{E})(D_U)\oplus\mathcal{O}_{U\times X})$$

as follows: By the representability Theorem 4.1.16, g can be viewed as an automorphism of $p_X^{U\times X*}(\mathcal{E})$. We will use the same symbol g for this automorphism, and we will denote by $g(D_U)$, $g(D_U)_{D_U}$ the induced automorphisms of the sheaves $p_X^{U\times X*}(\mathcal{E})(D_U)$, $p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U}$. Similarly, by Remark 4.1.15, $v \in \mathcal{O}(U, W_D)$ can be regarded as an element of

$$H^0\big(U\times X, p_X^{U\times X*}(\mathcal{E}(D))\big) = H^0\big(U\times X, p_X^{U\times X*}(\mathcal{E})(D_U)\big) = \mathrm{Hom}\big(\mathcal{O}_{U\times X}, p_X^{U\times X*}(\mathcal{E})(D_U)\big).$$

Therefore, the matrix

$$R_\gamma = \begin{pmatrix} g(D_U) & v \\ 0 & 1 \end{pmatrix}$$

can be interpreted as an automorphism R_γ of $p_X^{U\times X*}(\mathcal{E})(D_U)\oplus\mathcal{O}_{U\times X}$. We have the following generalisation of Proposition 4.3.7:

Proposition 4.4.2. *For any $\gamma = (v, g) \in \mathcal{O}(U, G_D)$, $\beta \in \mathcal{O}(U, V_D)$ we have*

$$u_{\gamma\beta} \circ R_\gamma = g(D_U)_{D_U} \circ u_\beta.$$

In other words, putting $\gamma = (v, g)$, $\gamma_0 = g$, $R_\gamma^0 = g(D_U)_{D_U} \in \mathrm{Aut}(p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U})$, we have the following commutative diagram:

$$\begin{array}{ccc}
p_X^{U\times X*}(\mathcal{E})(D_U)\oplus\mathcal{O}_{U\times X} & \xrightarrow[\simeq]{R_\gamma} & p_X^{U\times X*}(\mathcal{E})(D_U)\oplus\mathcal{O}_{U\times X} \\
{\scriptstyle u_\beta}\downarrow & & \downarrow{\scriptstyle u_{\gamma\beta}} \\
p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U} & \xrightarrow[R_\gamma^0]{\simeq} & p_X^{U\times X*}(\mathcal{E})(D_U)_{D_U}
\end{array} \qquad (42)$$

with isomorphic rows.

In particular R_γ maps isomorphically $\tilde{\mathcal{E}}_\beta := \ker(u_\beta)$ onto $\tilde{\mathcal{E}}_{\gamma\beta} := \ker(u_{\gamma\beta})$.

Let now (Θ, ρ) be a object of the groupoid ${}_S\mathcal{T}\mathrm{ors}_{G_D}^{\alpha_D}$. For a local section $\theta \in \Theta(U_\theta)$ we obtain the locally free sheaf $\tilde{\mathcal{E}}_{\rho(\theta)}$ on $U_\theta \times X$.

For two local sections $\theta \in \Theta(U_\theta)$, $\theta' \in \Theta(U_{\theta'})$ of Θ we define $\gamma_\theta^{\theta'} \in \mathcal{O}(U_\theta \cap U_{\theta'}, G_D)$ by the formula

$$\theta|_{U_\theta\cap U_{\theta'}} = \theta'|_{U_\theta\cap U_{\theta'}}\, \gamma_\theta^{\theta'}.$$

This implies

$$\rho(\theta')|_{U_\theta\cap U_{\theta'}} = \gamma_\theta^{\theta'}\rho(\theta)|_{U_\theta\cap U_{\theta'}},$$

so, by Proposition 4.4.2, it follows that $R_{\gamma_\theta^{\theta'}}$ maps isomorphically $\tilde{\mathcal{E}}_{\rho(\theta)}|_{(U_\theta\cap U_{\theta'})\times X}$ onto $\tilde{\mathcal{E}}_{\rho(\theta')}|_{(U_\theta\cap U_{\theta'})\times X}$, so it induces an isomorphism

$$\varphi_\theta^{\theta'} : \tilde{\mathcal{E}}_{\rho(\theta)}|_{(U_\theta\cap U_{\theta'})\times X} \xrightarrow{\simeq} \tilde{\mathcal{E}}_{\rho(\theta')}|_{(U_\theta\cap U_{\theta'})\times X}.$$

Applying the gluing functor given by Proposition 4.1.2 to the open cover

$$(U_\theta \times X)_{\theta \underset{\text{of } \Theta}{\text{ loc. sect.}}}$$

of $S \times X$ and the pair $\big((\tilde{\mathcal{E}}_{\rho(\theta)}), (\varphi_\theta^{\theta'})\big)$, we obtain a locally free sheaf $\tilde{\mathcal{E}}_{\Theta,\rho}$ on $S \times X$. In the same way as in section 4.3.3, we see that $\tilde{\mathcal{E}}_{\Theta,\rho}$ comes with a canonical epimorphism $\varphi_{\Theta,\rho}$ induced by the family of sheaf epimorphisms

$$\tilde{\mathcal{E}}_{\rho(\theta)} \hookrightarrow p_X^*(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta\times X} \twoheadrightarrow \mathcal{O}_{U_\theta\times X}.$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \searrow \quad \nearrow$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varphi_{\rho(\theta)}$$

Definition 4.4.3. *Our functor ${}_S\mathfrak{A}$ is defined by*

$${}_S\mathfrak{A}(\Theta,\rho) \coloneqq \big(\tilde{\mathcal{E}}_{\Theta,\rho} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_{S\times X}\big).$$

For the construction of a natural isomorphism $\mathrm{id} \xrightarrow{\simeq} {}_S\mathfrak{T} \circ {}_S\mathfrak{A}$ we will need, as in section 4.3.3, an explicit description of the following objects:

(1) The kernel $\mathcal{E}_{\varphi_{\Theta,\rho}}$ of the augmented bundle $\varphi_{\Theta,\rho} : \tilde{\mathcal{E}}_{\Theta,\rho} \to \mathcal{O}_{S\times X}$.
(2) The D_S-enlarged extension

$$0 \to \mathcal{E}_{\varphi_{\Theta,\rho}}(D_S) \xrightarrow{\iota^D_{\varphi_{\Theta,\rho}}} \tilde{\mathcal{E}}^D_{\varphi_{\Theta,\rho}} \to \mathcal{O}_{S\times X} \to 0$$

of the extension $0 \to \mathcal{E}_{\varphi_{\Theta,\rho}} \hookrightarrow \tilde{\mathcal{E}}_{\Theta,\rho} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_{S\times X} \to 0$.
(3) The canonical epimorphism

$$q_D^{\varphi_{\Theta,\rho}} : \tilde{\mathcal{E}}^D_{\varphi_{\Theta,\rho}} \to \mathcal{E}_{\varphi_{\Theta,\rho}}(D_S)_{D_S}$$

on the central term of this enlarged extension.

Following the same idea as in section 4.3.3, one can see that these three objects are also associated with gluing data, but this time parameterised by local sections θ of Θ.

Let θ be a local section of Θ. Taking $\beta = \rho(\theta)$ in the diagram (D_β) of this section, we obtain the commutative diagram $D_\theta \coloneqq D_{\rho(\theta)}$. For two local sections θ, θ' of Θ we obtain an isomorphism of commutative diagrams

$$\mathfrak{R}_{\gamma_\theta^{\theta'}} : D_\theta|_{(U_\theta\cap U_{\theta'})\times X} \to D_{\theta'}|_{(U_\theta\cap U_{\theta'})\times X}$$

defined in a similar way as the isomorphism denoted by the same symbol considered in section 4.3.3 (see Definition 4.3.10).

Using the same method as in section 4.3.3, we obtain the following generalisation of Proposition 4.3.11 which is needed for the construction of the natural isomorphism $\mathrm{id}_{{}_S\mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}} \xrightarrow{\simeq} {}_S\mathfrak{T} \circ {}_S\mathfrak{A}$:

Proposition 4.4.4. *Let (Θ,ρ) be an object of the groupoid ${}_S\mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}$. Put $\tilde{\mathcal{E}} \coloneqq \tilde{\mathcal{E}}_{\Theta,\rho}$, $\varphi \coloneqq \varphi_{\Theta,\rho}$ to save on notations. With the notations and under assumptions above, we have:*

(1) The kernel $\mathcal{E}_\varphi$ of the augmented bundle $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X}$ defined above is canonically identified with the sheaf $\coprod_{\theta \underset{\text{of } \Theta}{\text{ loc. sect.}}} p_X^{U_\theta\times X *}(\mathcal{E})/(\mathfrak{R}_{\gamma_\theta^{\theta'}})$ associated to the $\mathcal{O}_{S\times X}$-module gluing data

$$\big((p_X^{U_\theta\times X *}(\mathcal{E}))_{\theta \underset{\text{of } \Theta}{\text{ loc. sect.}}}, (\mathfrak{R}_{\gamma_\theta^{\theta'}})\big).$$

(2) The D_S-enlarged extension

$$0 \to \mathcal{E}_\varphi(D_S) \xrightarrow{\iota^D_\varphi} \tilde{\mathcal{E}}^D_\varphi \to \mathcal{O}_{S\times X} \to 0$$

of the extension

$$0 \to \mathcal{E}_\varphi \hookrightarrow \tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S\times X} \to 0$$

is canonically identified with the extension

$$0 \longrightarrow \coprod_{\theta \underset{\text{of } \Theta}{\text{ loc. sect.}}} p_X^*(\mathcal{E})(D_{U_\theta})/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) \xrightarrow{\mathfrak{j}} \coprod_{\theta \underset{\text{of } \Theta}{\text{ loc. sect.}}} (p_X^*(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta\times X})/(\mathfrak{R}_{\gamma_\theta^{\theta'}}) \xrightarrow{\mathfrak{p}} \mathcal{O}_{U_\theta\times X} \longrightarrow 0,$$

of sheaves associated with gluing data, where the monomorphism $\mathfrak{j}$ *is obtained by gluing (in the sense of Remark 4.1.4) the family of inclusions*

$$\underset{\substack{\text{loc. sect.}\\ \text{of } \Theta}}{\theta} \mapsto (j_1 : p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \hookrightarrow p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X}),$$

and the epimorphism $\mathfrak{p}$ *is obtained by gluing (in the sense of Remark 4.1.4) the family of projections*

$$\underset{\substack{\text{loc. sect.}\\ \text{of } \Theta}}{\theta} \mapsto (p_2 : p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X} \to \mathcal{O}_{U_\theta \times X}).$$

(3) Via the identification $\tilde{\mathcal{E}} = \coprod_{\theta \text{ loc. sect. of } \Theta} (p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X}) / (\mathfrak{R}_{\gamma_\theta^{\theta'}})$*, the canonical epimorphism* $q_D^\varphi : \tilde{\mathcal{E}}_\varphi^D \to \mathcal{E}_\varphi(D_S)_{D_S}$ *coincides with the morphism* $\coprod_{\theta \text{ loc. sect. of } \Theta} u_{\rho(\theta)}$ *obtained by gluing the family of morphisms*

$$\left(p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X} \xrightarrow{u_{\rho(\theta)}} p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta})_{D_{U_\theta}} \right)_{\substack{\theta \text{ loc. sect.}\\ \text{of } \Theta}}$$

Using again the properties of the sheaf associated with gluing data stated and the morphisms associated with gluing data morphisms (Proposition 4.1.2, Remarks 4.1.4, 4.1.5, 4.1.6), we obtain:

(1) A family

$$(\psi_\theta : p_X^{U_\theta \times X *}(\mathcal{E}) \xrightarrow{\simeq} \mathcal{E}_{\varphi_{U_\theta}} = \mathcal{E}_\varphi|_{U_\theta \times X})_{\theta \text{ loc. sect. of } \Theta}$$

of canonical sheaf isomorphisms satisfying the compatibility condition

$$\psi_{\theta'}|_{(U_\theta \cap U_{\theta'}) \times X} = \psi_\theta|_{(U_\theta \cap U_{\theta'}) \times X} \circ \mathfrak{R}_{\gamma_\theta^{\theta'}} \text{ for any local sections } \theta,\ \theta' \text{ of } \Theta. \tag{43}$$

(2) A family

$$(\Psi_\theta : p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X} \xrightarrow{\simeq} \tilde{\mathcal{E}}_{\varphi_{U_\theta}}^D = \tilde{\mathcal{E}}_\varphi^D|_{U_\theta \times X})_{\theta \text{ loc. sect. of } \Theta}$$

of canonical sheaf isomorphisms making commutative the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) & \xrightarrow{j_1} & p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X} & \xrightarrow{p_2} & \mathcal{O}_{U_\theta \times X} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_\theta(D)} & & \downarrow{\scriptstyle \Psi_\theta} & & \| & & \\ 0 & \longrightarrow & \mathcal{E}_{\varphi_{U_\theta}}(D_{U_\theta}) & \xrightarrow{\iota_{\varphi_{U\theta}}^D} & \tilde{\mathcal{E}}_{\varphi_{U_\theta}}^D & \xrightarrow{\varphi_{U\theta}} & \mathcal{O}_{U_\theta \times X} & \longrightarrow & 0 \end{array}, \tag{44}$$

$$\begin{array}{ccc} p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X} & \xrightarrow{\Psi_\theta} & \tilde{\mathcal{E}}_{\varphi_{U_\theta}}^D \\ \downarrow{\scriptstyle u_{\rho(\theta)}} & & \downarrow{\scriptstyle q_D^\varphi} \\ p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta})_{D_{U_\theta}} & \xrightarrow{\psi_\theta(D_{U_\theta})_{D_{U_\theta}}} & \mathcal{E}_{\varphi_{U_\theta}}(D_{U_\theta})_{D_{U_\theta}} \end{array} \tag{45}$$

and satisfying the compatibility condition

$$\Psi_\theta|_{(U_\theta \cap U_{\theta'}) \times X} = \Psi_{\theta'}|_{(U_\theta \cap U_{\theta'}) \times X} \circ R_{\gamma_\theta^{\theta'}} \text{ for any local sections } \theta,\ \theta' \text{ of } \Theta. \tag{46}$$

4.4.3. The natural isomorphisms ${}_S\mathfrak{A} \circ {}_S\mathfrak{T} \to \mathrm{id}$, $\mathrm{id} \to {}_S\mathfrak{T} \circ {}_S\mathfrak{A}$.

Let $\tilde{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{S \times X}$ be an object of the groupoid ${}_S\mathcal{A}\mathrm{ff}_E$. Put $\Theta \coloneqq \Theta_\varphi^D$, $\rho \coloneqq \rho_\varphi^D$ to save on notations.

Then augmented bundle ${}_S\mathfrak{A}(\Theta, \rho)$ on $S \times X$ is defined using the gluing principle for sheaves by

$${}_S\mathfrak{A}(\Theta, \rho) = \left(\coprod_{\theta \text{ loc. sect. of } \Theta} \tilde{\mathcal{E}}_{\rho(\theta)} / {}_{(\varphi_\theta^{\theta'})} \xrightarrow{\varphi_{\Theta,\rho}} \mathcal{O}_X \right).$$

We will use again, as in section 4.3.4, the universal property given by Proposition 4.1.2. It suffices to construct, for any local section θ of Θ, an isomorphism of augmented bundles

$$\begin{array}{ccc} \tilde{\mathcal{E}}_{\rho(\theta)} & \xrightarrow{e_\theta} & \tilde{\mathcal{E}}_{U_\theta} \\ & {\scriptstyle \varphi_{\rho(\theta)}} \searrow \quad \swarrow {\scriptstyle \varphi_{U_\theta}} & \\ & \mathcal{O}_{U_\theta \times X} & \end{array} \tag{47}$$

such that the following compatibility condition is satisfied

$$e_{\theta'}|_{(U_\theta \cap U_{\theta'})\times X} \circ \varphi_\theta^{\theta'} = e_\theta|_{(U_\theta \cap U_{\theta'})\times X} \text{ for local sections } \theta,\ \theta' \text{ of } \Theta. \tag{48}$$

We define e_θ using the same idea as in section 4.3.4: A local section θ of $\Theta = \Theta_\varphi^D$ is a pair (σ, f) consisting of a splitting σ of the exact sequence

$$0 \to \mathcal{E}_{\varphi_{U_\theta}}(D_{U_\theta}) \xrightarrow{\iota^D_{\varphi_{U_\theta}}} \tilde{\mathcal{E}}^D_{\varphi_{U_\theta}} \xrightarrow{\varphi^D_{U_\theta}} \mathcal{O}_{U_\theta \times X} \to 0$$

and a sheaf isomorphism $f : p_X^{U_\theta \times X *}(\mathcal{E}) \to \mathcal{E}_{\varphi_{U_\theta}}$.

The right splitting σ identifies $\tilde{\mathcal{E}}^D_{\varphi_{U_\theta}}$ with $\mathcal{E}_{\varphi_{U_\theta}}(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X}$ and $f \oplus \mathrm{id}$ further identifies this direct sum with $p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X}$.

Via this identification, the subsheaf $\tilde{\mathcal{E}}_{U_\theta}$ of $\tilde{\mathcal{E}}^D_{\varphi_{U_\theta}}$ corresponds precisely to the elementary transformation $\tilde{\mathcal{E}}_{\rho(\theta)} \hookrightarrow p_X^{U_\theta \times X *}(\mathcal{E})(D_{U_\theta}) \oplus \mathcal{O}_{U_\theta \times X}$. This follows using the analogue (on the product $U_\theta \times X$) of the commutative diagram (34).

Similarly, one can use the analogue of diagram (36) to prove that the obtained isomorphism is also an isomorphism of augmented bundles on $U_\theta \times X$, and the analogue of diagram (37) – which will be a diagram of sheaves on $(U_\theta \cap U_{\theta'}) \times X$ – to prove the compatibility condition (48).

For the construction of a natural isomorphism $\mathrm{id} \to {}_S\mathfrak{T} \circ {}_S\mathfrak{A}$ we will use the same method as in section 4.3.5: Let (Θ, ρ) be an object of the groupoid ${}_S\mathcal{T}\mathrm{ors}^{\alpha_D}_{G_D}$. We construct a *canonical* isomorphism

$$(\Theta, \rho) \xrightarrow{\simeq \eta} {}_S\mathfrak{T} \circ {}_S\mathfrak{A}(\Theta, \rho).$$

In other words, putting $\tilde{\mathcal{E}} \coloneqq \tilde{\mathcal{E}}_{(\Theta,\rho)}$, $\varphi \coloneqq \varphi_{(\Theta,\rho)}$, we will construct a *canonical* isomorphism of $\mathcal{O}(G_D)$-torsors $\eta : \Theta \to \Theta_\varphi^D$ making commutative the triangle

$$\begin{array}{ccc} \Theta & \xrightarrow{\eta \simeq} & \Theta_\varphi^D \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\rho^D_\varphi} & \\ & \mathcal{O}_S(V_D) & \end{array} .$$

For any open subspace $U \subset S$ and $\theta \in \Theta(U)$, denote by $j_2^\theta : \mathcal{O}_{U\times X} \to p_X^{U\times X *}(\mathcal{E})(D_U) \oplus \mathcal{O}_{U\times X}$ the obvious monomorphism. Using the isomorphisms introduced at the end of section 4.4.2, we put

$$\eta(\theta) \coloneqq (\Psi_\theta \circ j_2^\theta, \psi_\theta).$$

This formula defines an isomorphism of $\mathcal{O}(G_D)$-torsors. This follows as in the Remark 4.3.13, but this time using the compatibility formulae (43), (46) and the commutative diagram (44).

The equality $\rho^D_\varphi \circ \eta = \rho$ follows as in Remark 4.3.14 but this time using the commutative diagram (45).

The collections of functors ${}_S\mathfrak{A}$, ${}_S\mathfrak{T}$ associated with complex spaces S define functors

$$\mathscr{A} : [V_D/G_D] \to \mathscr{A}\mathrm{ff}_E,\ \mathscr{T} : \mathscr{A}\mathrm{ff}_E \to [V_D/G_D]$$

over the category $\mathscr{A}\mathrm{nn}$ of complex spaces. These functors commute with pull-backs and define an equivalence of categories. In conclusion, using the terminology of [**Gom**, p. 11], we have:

THEOREM 4.4.5. *Let X be a compact complex manifold, and E be a fixed holomorphic vector bundle on X and $\mathcal{E}$ the associated locally free sheaf. Let D be an effective divisor on X such that $H^1(X, \mathcal{E}(D)) = 0$. The holomorphic stack $\mathscr{A}\mathrm{ff}_E$ is isomorphic to the quotient stack $[V_D/G_D]$.*

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