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On the stable norm of flat surfaces

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Sous la direction de Daniel Massart et Ivan Babenko

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- Bilbon Sacquet

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Résumé

La norme stable d'une variété riemannienne singulière est une norme définie sur le premier groupe d'homologie de la variété, et qui étend aux classes d'homologie la notion de longueur des courbes. Il est connu qu'une grande quantité d'information géométrique sur la variété est encodée par sa norme stable: mais pour accéder à cette information, il faut d'abord être capable de calculer la norme stable elle-même. C'est un problème difficile pour lequel très peu d'exemples sont connus.

L'objet de cette thèse est le calcul de nouveaux exemples de surfaces sur lesquelles la norme stable peut être calculée. Dans une première partie, nous calculons la norme stable des tores plats fendus. Dans une seconde partie, nous recollons des tores fendus à des cylindres plats pour construire des surfaces de demi-translation sur lesquelles la norme stable peut être calculée. Dans une troisième partie, nous montrons que, pour un certain problème de comptage lié à la norme stable, les surfaces construites dans la partie précédente présentent un comportent qualitativement différent des exemples connus jusqu'alors. Enfin, dans une quatrième partie, nous présentons des résultats mineurs et expériences numériques sur la norme stable d'une surface de translation particulière.

Abstract

The stable norm of a (possibly singular) Riemannian manifold is a norm defined on the first homology group of the manifold, that extends to homology classes the notion of lengths of curves. It is known that a lot of geometric information on the manifold is encoded by the stable norm. But to access this information, one must first be able to compute the stable norm itself. This is a difficult problem for which very few examples are known.

The object of this thesis is the computation of the stable norm on new examples of surfaces. In a first part, we compute the stable norm of flat slit tori. In a second part, we glue these slit tori to flat cylinders in order to construct half-translation surfaces on which the stable norm can be computed. In a third part we prove that, regarding a counting problem related to the stable norm, the surfaces we constructed in the previous part exhibit qualitatively different behaviour than what was previously known. Finally, in a fourth part we give some partial results and numerical experiments related to the stable norm of a specific translation surface.

Introduction

Le sujet de cette thèse est le calcul explicite de la norme stable sur de nouveaux exemples de surfaces plates, plus précisément sur des surfaces de translation et de demi-translation.

La norme stable d'une variété différentielle compacte M munie d'une métrique riemannienne (éventuellement singulière) est une norme définie sur le premier groupe d'homologie $H_1(M,\mathbb{R})$ qui étend la notion de longueur des courbes fermées aux classes d'homologie. Plus précisément, la norme stable $\|h\|_M$ d'une classe d'homologie entière $h \in H_1(M,\mathbb{Z})$ est définie par (voir [GLP81]):

$$\|h\|_M := \lim_{n\longrightarrow\infty} \frac{\inf\{l(\gamma), \gamma \text{ multicourbe fermée qui représente } n.h\}}{n}$$

où une multicourbe est une union de courbes fermées et où $l(\gamma)$ est la longueur de la multicourbe γ relativement à la métrique de M. Cette définition s'étend à l'homologie à coefficients rationnels $H_1(M,\mathbb{Q})$ par homogénéité, puis à l'homologie réelle $H_1(M,\mathbb{R})$ par continuité uniforme, et on vérifie que cette extension est une norme. Dans le cas des surfaces fermées, on peut donner une définition plus simple de la norme stable ([Mas96]) qui s'affranchit du passage à la limite : si S est une surface fermée munie d'une métrique riemannienne (possiblement singulière) et $h \in H_1(S,\mathbb{Z})$ est une classe d'homologie entière sur S, on a

$$\|h\|_S = \inf\left\{\sum_i |p_i| l(\gamma_i) \text{ avec } \gamma_i \text{ multicourbe et } h = \sum p_i[\gamma_i], p_i \in \mathbb{R}\right\}$$

et cette définition est à nouveau étendue à tout $H_1(S,\mathbb{R})$.

La donnée de la norme stable d'une variété différentielle contient une grande quantité d'information géométrique, menant notamment à des résultats de rigidité qui contraignent la géométrie globale de la variété ([BIK97],[Osu10]). Le prix à payer pour cette information est que la norme stable est très difficile à calculer explicitement en pratique, si bien que les exemples sont rares et qu'on ne sait pas à quoi ressemble la norme stable en général. Le but premier de cette thèse est donc d'exhiber de nouveaux exemples non-triviaux de variétés dont la norme stable a pu être calculée.

Les quelques exemples connus de variétés pour lesquelles la norme stable a été calculée suggèrent de s'intéresser au cas des surfaces. En effet, la dimension 1 n'offre pas une grande diversité de comportements possibles de la norme stable ([Bal05]),

tandis que la dimension $n \geq 3$ semble au contraire autoriser trop de comportements exotiques ([BB06]) pour espérer pouvoir y trouver des exemples pertinents, c'est à dire des exemples pouvant prétendre à un certain degré d'universalité. À mi-chemin entre ces deux extrêmes se trouve la dimension 2, où des exemples qualitativement très différents les uns des autres sont connus ([Bab88],[MR95],[KS17],[BM08]) mais où des contraintes générales peuvent tout de même être énoncées ([Mas96]). La dimension 2 semble alors être un cadre prometteur pour la recherche d'exemples concernant la norme stable, et c'est donc dans ce cadre que s'inscrit cette thèse.

Calculer la norme stable sur une surface requiert implicitement d'être capable de calculer les longueurs de nombreuses courbes, et plus précisément, de calculer les longueurs des géodésiques de la surface. Cette tâche peut être grandement simplifiée par un choix judicieux de la métrique dont la surface est munie : idéalement on voudrait une métrique pour laquelle les géodésiques sont faciles à calculer. Le choix fait dans cette thèse est de considérer des métriques plates, et plus précisément des métriques dites de translation (et de demi-translation) : ainsi, les géodésiques sont des segments de droite et le calcul de leur longueurs est trivial. Les surfaces de translation et de demi-translation sont des cas particuliers de surfaces plates à singularités coniques dont l'étude occupe une place prépondérante au sein de la recherche mathématique de ces dernières décennies. Originellement étudiées pour des questions de dynamique de part leur lien avec les billards polygonaux, ces surfaces ont rapidement été l'objet de nombreux questionnements provenant d'horizons mathématiques très divers, et se trouvent aujourd'hui à l'intersection de la dynamique, de la géométrie différentielle, de la géométrie algébrique, de la théorie des nombres, de la théorie des espaces de Teichmüller et de la théorie géométrique des groupes. Cette grande richesse d'outils disponibles pour leur étude, couplée à la grande simplicité de leur définition et à l'aisance à les manipuler qui en découle, font des surfaces de translation et de demi-translation un cadre privilégié pour la recherche de nouveaux exemples pour la norme stable.

Question 1: Peut-on calculer explicitement la norme stable sur un exemple de surface de translation ou de demi-translation?

Un problème associé au calcul d'une norme est la description de sa boule unité. La boule unité d'une norme est un convexe symétrique par rapport à l'origine, mais deux tels convexes peuvent avoir des aspects qualitativement très différents l'un de l'autre: par exemple, une sphère euclidienne est lisse et chacun de ses points est extrémal, tandis qu'un cube possède peu de points extrémaux, à savoir ses sommets qui sont également des points de non-lissité.

Question 2: Si l'on arrive à calculer la norme stable d'une surface plate, que dire de la géométrie de sa boule unité? Est-elle lisse, y a-t-il des sommets? Possède-t-elle des faces plates?

Calculer la norme stable d'une surface de translation donnée est une question difficile. La stratégie employée dans cette thèse est de découper les surfaces qui nous intéressent en plus petits morceaux sur lesquels la norme stable stable peut être calculée pour

ensuite les recoller, en espérant que trop d'information n'a pas été perdue dans l'opération et ainsi pouvoir calculer la norme stable de la surface toute entière. Plus précisément, nous découpons les surface en tores fendus : un tore fendu est une surface homéomorphe à un tore privé d'un disque ouvert, muni d'une métrique plate pour laquelle la composante de bord peut être vue comme deux segments parallèles de même longueur.

Dans une première partie, nous calculons la norme stable des tores fendus et nous décrivons la géométrie de sa boule unité. Surprenamment, le résultat fait intervenir la suite de Farey, une suite d'ensembles de nombres rationnels célèbre pour ses excellentes propriétés combinatoires et d'approximation diophantienne. La suite de Farey d'ordre $L \in \mathbb{N}^*$ est définie comme la suite ordonnée des fractions irréductibles dont le dénominateur ne dépasse pas L:

$$\mathscr{F}_L := \left\{ \frac{p}{q}, p \text{ et } q \text{ premiers entre eux}, 0 < q \leq L \right\}.$$

Les relations que satisfont des termes consécutifs dans la suite de Farey permettent de définir une généalogie des nombres et notamment la notion de parents de Farey d'un nombre. Le problème du calcul de la norme stable des tores fendus se ramène en fait à des considérations arithmétiques et combinatoires, encodées par la suite de Farey. Dans le cas du tore carré avec une fente verticale, on montre le résultat suivant.

Théorème A. *Soit X un tore carré de côté 1 avec une fente verticale de longueur $0 < \rho < 1$ et soit $L = \lfloor 1/\rho \rfloor$, où $\lfloor . \rfloor$ désigne la partie entière. Soit $h = (m,n) \in H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^2$ une classe d'homologie entière primitive sur X, c'est à dire que m et n sont premiers entre eux, et soient $a/b < c/d$ les parents de Farey du rationnel n/m. La boule unité $\mathscr{B}$ de la norme stable de X a un sommet dans la direction (m,n) si, et seulement si, ou bien*

- $\dfrac{n}{m} \in \mathscr{F}_L$, *et alors*
$$\|(m,n)\|_X = \sqrt{m^2+n^2}.$$
ou
- $\dfrac{n}{m} \notin \mathscr{F}_L$ *mais* $\dfrac{a}{b} \in \mathscr{F}_L$ *ou* $\dfrac{c}{d} \in \mathscr{F}_L$, *et alors*
$$\|(m,n)\|_X = \sqrt{b^2+(a+\rho)^2} + \sqrt{d^2+(c-\rho)^2}.$$

De plus, $\mathscr{B}$ est strictement convexe dans les directions $\pm(0,1)$, et $\| \pm (0,1)\|_X = 1$. Le point de $\partial\mathscr{B}$ dans n'importe quelle autre direction se trouve dans l'intérieur d'un segment contenu dans $\partial\mathscr{B}$. De plus dans ce cas, si $h = (m,n)$ est entière avec les notations précédentes, on a $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$.

On généralise ensuite le Théorème A à n'importe quel tore plat fendu (voir Théorème 4.2.5) par déformation du tore carré avec une fente verticale. Si le théorème que l'on obtient est qualitativement similaire au Théorème A, son énoncé est plus laborieux car il nécessite l'introduction de notations supplémentaires, d'où le choix de plutôt présenter le cas particulier dans cette introduction.

Dans une seconde partie nous recollons des tores fendus pour construire des surfaces sur lesquelles l'on peut calculer la norme stable. Cette approche ne permet pas de calculer la norme stable de la surface recollée en toute généralité, mais pour tout $g \geq 2$ nous pouvons construire une surface de demi-translation Σ_g de genre g sur laquelle la norme stable peut être complètement calculée. La surface Σ_g est obtenue en recollant g tores fendus $X_1, ..., X_g$ identiques et des longs cylindres plats, où "long" signifie plus long qu'une certaine constante explicite. Pour être exact, si ρ est la longueur de la fente des X_i on suppose que les cylindres composant Σ_g sont plus longs que ρ. Rappelons que $H_1(\Sigma_g, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$. On montre alors:

Théorème B. *La boule unité de la norme stable de Σ_g est l'enveloppe convexe de*

$$\bigcup_{i=1}^{g} \left\{ \frac{(0,0,...,a_i,b_i,...,0,0)}{\sqrt{a_i^2+b_i^2}}, (a_i,b_i) \in V(X_i) \right\}$$

où $V(X_i)$ est l'ensemble des directions strictement convexes de la norme stable du tore fendu X_i.

On montre ensuite que dans le cas du genre 2 le Théorème B reste vrai pour toute surface obtenue par une déformation suffisamment petite de la surface Σ_2 au sein de la strate $\mathcal{Q}(1,1,1,1)$ de l'espace des modules des différentielles quadratiques.

La norme stable permet d'étendre aux classes d'homologie la notion de longueur des courbes fermées ; ainsi certains problèmes classiques concernant les courbes fermées ont une version en homologie qui fait intervenir la norme stable. La question classique dont nous nous inspirons concerne le comptage des courbes fermées sur une surface fixée, où la question est de compter les géodésiques fermées dans une certaine orbite de l'action du groupe modulaire et dont la longueur est inférieure ou égale à un nombre $x > 0$ donné. Cette question a été résolue par Mirzakhani [Mir16] dans le cas des surfaces hyperboliques d'aire finie par des méthodes ergodiques (entre autres), et a connu depuis de nombreuses variantes (voir par exemple [ES23]).
La version naturelle en homologie de ce type de questions est de compter le nombre de classes d'homologie entières, puisqu'elles sont la généralisation naturelle des courbes fermées, de norme stable inférieure ou égale à un nombre $x > 0$. Cette question n'est toutefois pas la plus pertinente à poser, car elle implique de nombreuses redondances dans le comptage. En effet, si la norme stable d'une classe d'homologie h est réalisée par la longueur d'une courbe fermée γ, alors γ peut apparaître à nouveau comme une composante connexe d'une multicourbe dont la longueur réalise la norme stable d'une autre classe d'homologie h' : en clair, la contribution de γ est comptée plusieurs fois. Pour éviter cet écueil, une solution est de ne compter que certaines classes d'homologie bien particulières. On dit qu'une classe d'homologie entière est *simple* si sa norme stable est réalisée par la longueur d'une courbe fermée sans auto-intersection qui la représente.

Question 3: Combien y a-t-il de classes d'homologie simples de norme stable inférieure ou égale à $x > 0$ sur une surface donnée?

Les exemples présents dans la littérature donnent tous une croissance quadratique en x du nombre de classes d'homologie simples de norme stable inférieure ou égale à $x > 0$. Pour les surfaces de translation, un argument simple reposant sur les travaux d'Eskin et Masur [EM01], dont les méthodes ne sont pas sans rappeler les travaux de Mirzakhani, donne à nouveau une croissance (au moins) quadratique en x. Cet argument échoue pour les surfaces de demi-translation pour lesquelles la question reste ouverte. Dans une troisième partie nous calculons précisément une estimation asymptotique sous-quadratique du nombre de classes d'homologie simples sur la surface de demi-translation Σ_g construite dans la partie précédente.

Théorème C. *Soit ρ la longueur des fentes des tores fendus qui composent Σ_g. Soit $p_{\Sigma_g}(x) = \#\{h \in H_1(\Sigma_g, \mathbb{Z})$ simple, $\|h\|_{\Sigma_g} \leq x\}$. Pour x assez grand on a*

$$p_{\Sigma_g}(x) = 4g \left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b} \right) x \ln x + O(x)$$

où φ est la fonction indicatrice d'Euler.

La preuve du Théorème C est qualitativement différente des résultats antérieurs concernant le comptage des classes d'homologie simples. Ici, il n'est pas question de méthodes ergodiques mais plutôt de techniques issues de la théorie analytique des nombres, via les séries de Dirichlet, une comparaison avec la fonction ζ de Riemann et une application de la formule de Perron. De plus, la structure de la suite de Farey joue de nouveau un rôle central dans ce calcul.

Enfin, dans une quatrième et dernière partie, nous présentons un travail en cours sur la norme stable d'une surface de translation particulière, la surface $L(2,2)$. Cette partie contient peu de résultats démontrés et présente plutôt une première approche empirique du problème via un code informatique réalisé sous `sagemath`. Ce code permet notamment de réaliser des images de la boule unité de la norme stable de $L(2,2)$ dont la structure possède une grande valeur esthétique.

Organisation du manuscrit. Le chapitre 1 introduit la norme stable et résume l'état de la littérature à son sujet, notamment en ce qui concerne les exemples connus. Le chapitre 2 introduit les connaissances de base concernant les surfaces de translation et de demi-translation, ainsi que les résultats plus spécifiques qui seront requis dans la suite du manuscrit. Le chapitre 3 introduit la suite de Farey et les quelques propriétés clés qui seront utilisées dans les chapitres suivants, puis tente par quelques exemples historiques et de culture générale de communiquer la richesse des connexions de cette suite avec le reste des mathématiques. Les chapitres 4, 5 et 6 présentent respectivement les preuves des Théorèmes A,B et C ainsi que des résultats associés. Enfin, le chapitre 7 présente quelques résultats partiels concernant la norme stable de la surface $L(2,2)$ et contient une galerie d'images de sa boule unité.

English Introduction

The topic of this thesis is the explicit computation of the stable norm on new examples of flat surfaces, more precisely on translation surfaces and on half-translation surfaces.

The stable norm of a compact differential manifold M equipped with a (possibly singular) Riemannian metric is a norm defined on the first homology group $H_1(M,\mathbb{R})$ that extends to homology classes the notion of the length of closed curves. More precisely, the stable norm $\|h\|_M$ of an integral homology class $h \in H_1(M,\mathbb{Z})$ is defined by (see [GLP81]):

$$\|h\|_M := \lim_{n\longrightarrow\infty} \frac{\inf\{l(\gamma), \gamma \text{ closed multicurve that represents } n.h\}}{n}$$

where a multicurve is a union of closed curves and where $l(\gamma)$ is the length of the multicurve γ with respect to the metric on M. This definition is extended by homogeneity to the homology with rational coefficients $H_1(M,\mathbb{Q})$ and then to the real homology $H_1(M,\mathbb{R})$ by uniform continuity, and the extension is a norm. For closed surfaces, one can give a simpler definition of the stable norm ([Mas96]) that gets rid of the limit: if S is a closed surface equipped with a (possibly singular) Riemannian metric and $h \in H_1(S,\mathbb{Z})$ is an integral homology class on S, we have

$$\|h\|_S = \inf\left\{\sum_i |p_i| l(\gamma_i) \text{ where } \gamma_i \text{ are multicurves and } h = \sum p_i[\gamma_i], p_i \in \mathbb{R}\right\}$$

and this definition can again be extended to the whole $H_1(S,\mathbb{R})$.

The data of the stable norm of a differential manifold contains a large amount of geometric information, which leads to rigidity results that constrain the global geometry of the manifold ([BIK97],[Osu10]). The cost for this information is that the stable norm is very difficult to compute explicitly in practice, so much so that examples are rare and that it is unclear what the stable norm looks like in general. Thus the main goal of this thesis is to find new non-trivial examples of manifolds whose stable norm can be computed.

The few previously known examples of manifolds on which the stable norm has been computed suggest to look at the case of surfaces. Indeed, dimension 1 does not offer a sufficient diversity of possible behaviours for the stable norm ([Bal05]), whereas on the contrary dimension $n \geq 3$ seem to allow too many exotic behaviours ([BB06]) to hope finding relevant examples, that is, examples with some degree of universality. Between those two opposite cases is dimension 2, where examples qualitatively very

different from one another have been found ([Bab88],[MR95],[KS17],[BM08]) but where general results can still be obtained ([Mas96]). Dimension 2 appears to be a promising ground for the search of new examples of the stable norm, so this is the set-up chosen for the work presented in this thesis.

Computing the stable norm on a given surface implicitly requires to be able to compute the lengths of many curves; more precisely, to compute the lengths of the geodesics of the surface. This task can be greatly simplified by a judicious choice of the metric the surface is endowed with; ideally we would want a metric for which the geodesics are easy to find. The choice made in this thesis is to consider flat metrics, and more precisely the so-called translation and half-translation metrics: this way, the geodesics are straight line segments and computing their lengths is trivial. Translation and half-translation surfaces are special cases of flat surfaces with conical singularities that hold a preponderant place within the mathematical research of the last few decades. Studied originally for dynamical considerations because of their link with polygonal billiards, these surfaces quickly became the focus of many questions originating from various mathematical fields, so much so that now they can be found at the intersection of dynamics, differential geometry, algebraic geometry, number theory, Teichmüller theory and geometric group theory. Thus, because of the many sophisticated tools available, and because of the great simplicity of their definition that makes them very easy to handle, translation and half-translation surfaces are great candidates for new examples for the stable norm.

Question 1: Can we compute explicitly the stable norm of a given translation or half-translation surface?

A problem associated to the computation of any norm is to describe its unit ball. The unit ball of a norm is convex set that is symmetric with respect to the origin. However, two such convex sets can look qualitatively very different from one another. For example, a Euclidean sphere is smooth and each of its point it extremal, whereas a cube has few extremal points: namely these are its vertices, that are also non-smooth points.

Question 2: If we can compute the stable norm of a flat surface, what can we say of the geometry of its unit ball? Is it smooth, are there any vertices? Are there any flat faces?

To compute the stable norm of a given translation surface is a difficult question. The strategy used in this thesis is to cut the surfaces we are interested in into smaller pieces on which the stable norm can be computed, and then to glue them back together and hope that not too much information was lost in the process so that we can compute the stable norm of the whole surface. More precisely, we cut the surfaces into slit tori: a slit torus is a surface homeomorphic to a torus minus an open disk, equipped with a flat metric so that the boundary component can be seen as two parallel segments with the same length.

In a first part, we compute the stable norm of slit tori and we describe the geometry

of its unit ball. Surprisingly, the result involves the Farey sequence, a sequence of sets of rational numbers famous for its excellent combinatorial properties and its link to diophantine approximations. The Farey sequence of order $L \in \mathbb{N}^*$ is defined as the ordered sequence of irreducible fractions whose denominators do not exceed L:

$$\mathscr{F}_L := \left\{ \frac{p}{q}, p \text{ and } q \text{ coprime }, 0 < q \leq L \right\}.$$

Consecutive terms inside the Farey sequence satisfy relations that allow to define a genealogy of numbers, and in particular the notion of the Farey parents of a rational number. It turns out that the computation of the stable norm of slit tori can be reduced to arithmetical and combinatorial considerations, encoded by the Farey sequence. In the case of a square torus with a vertical slit, we get the following result.

Theorem A. *Let X be a square torus of area 1 with a vertical slit of length $0 < \rho < 1$ and let $L = \lfloor 1/\rho \rfloor$, where $\lfloor . \rfloor$ denotes the integral part. Let $h = (m,n) \in H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^2$ be a primitive integral homology class on X, i.e m and n are coprime, and let $a/b < c/d$ be the Farey parents of the rational n/m. The unit ball $\mathscr{B}$ of the stable norm of X has a vertex in the direction (m,n) if and only if, either*

- $\dfrac{n}{m} \in \mathscr{F}_L$, *and then*
$$\|(m,n)\|_X = \sqrt{m^2+n^2}.$$
or
- $\dfrac{n}{m} \notin \mathscr{F}_L$ *but* $\dfrac{a}{b} \in \mathscr{F}_L$ *or* $\dfrac{c}{d} \in \mathscr{F}_L$, *and then*
$$\|(m,n)\|_X = \sqrt{b^2+(a+\rho)^2} + \sqrt{d^2+(c-\rho)^2}.$$

Moreover, $\mathscr{B}$ is strictly convex in the directions $\pm(0,1)$ and $\|\pm(0,1)\|_X = 1$. The point of $\partial\mathscr{B}$ in any other direction lies in the interior of a segment contained in $\partial\mathscr{B}$. In particular, if $h = (m,n)$ is an integral class, using the above notations we have $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$.

We then generalize Theorem A to any flat slit torus (see Theorem 4.2.5) by deforming the square torus with a vertical slit. Even though the result we get is qualitatively similar to Theorem A, its statement is more tedious as it requires extra notations, hence the choice of presenting the less general case in this introduction.

In a second part we glue together several slit tori to construct surfaces on which we can compute the stable norm. This approach does not allow us to compute the stable norm of the glued surface in full generality, but for every $g \geq 2$ we are able to construct a half-translation surface Σ_g of genus g on which the stable norm can be completely computed. The surface Σ_g is obtained by glueing together g identical slit tori $X_1, ..., X_g$ to long flat cylinders, where "long" means longer than some explicit constant. To be precise, if ρ denotes the length of the slits of the X_i's we need the cylinders to be longer than ρ. Recall that $H_1(\Sigma_g,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. We show:

Theorem B. *The unit ball of the stable norm of Σ_g is the convex hull of*

$$\bigcup_{i=1}^{g} \left\{ \frac{(0,0,...,a_i,b_i,...,0,0)}{\sqrt{a_i^2+b_i^2}}, (a_i,b_i) \in V(X_i) \right\}$$

where $V(X_i)$ denotes the set of strictly convex directions of the stable norm of the slit torus X_i.

We then show that in the case of genus 2, Theorem B remains true for any surface obtained by a sufficiently small deformation of Σ_2 inside the stratum $\mathcal{Q}(1,1,1,1)$ of the moduli space of quadratic differentials.

The stable norm extends the notion of length of closed curves to homology classes; thus some classical problems about closed curves can be generalized to homology classes thanks to the stable norm. The classical question we draw inspiration from is a counting one: the question is to count the closed geodesics inside a fixed orbit of the action of the mapping class group and whose length is less than some positive number x. This problem was solved by Mirzakhani [Mir16] in the case of hyperbolic surfaces with finite area using (among other things) ergodic methods, and since then many other variants of the question have been studied (see for instance [ES23]).
The natural homological counterpart of this type of questions is to find the number of integral homology classes -that are the natural generalization of closed curves- whose stable norm is less than $x > 0$. However, this question is not the most interesting one to ask, as the counting involves a lot of redundancies. Indeed, if the stable norm of a homology class h is realized by the length of a closed curve γ, then γ appears again as a connected component of a closed multicurve whose length realizes the stable norm of another homology class h': in other words, the contribution of γ is counted multiple times. To avoid this issue, a solution is to count solely specific homology classes. We say that an integral homology class is *simple* if its stable norm is realized by the length of a closed curve representative of this class with no self-intersection.

Question 3: How many simple homology classes are there with stable norm less than $x > 0$ on a given surface?

The examples that can be found in the literature all exhibit a quadratic growth in x for the number of simple homology classes whose stable norm is less than x. On translation surfaces, a simple argument that relies on the work of Eskin and Masur [EM01], which Mirzakhani's work is reminiscent of, also provides a quadratic (at least) growth in x. This argument does not work on half-translation surfaces, on which the question remains open. In a third part, we compute a precise subquadratic asymptotic estimate of the number of simple homology classes on the half-translation surface Σ_g constructed previously.

Theorem C. *Let ρ be the length of the slits of the slit tori that compose Σ_g. Let $p_{\Sigma_g}(x) = \#\{h \in H_1(\Sigma_g, \mathbb{Z})$ simple $\|h\|_{\Sigma_g} \leq x\}$. For x large enough we have*

$$p_{\Sigma_g}(x) = 4g \left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b} \right) x \ln x + O(x)$$

where φ is Euler's totient function.

The proof of Theorem C is qualitatively different from the proofs of prior results about counting simple homology classes. Here, instead of ergodic methods, we rely on techniques imported from analytic number theory, via Dirichlet series, comparison with the Riemann ζ function and an application of Perron formula. Moreover, the structure of the Farey sequence plays once again a major part in this computation.

Finally, in a fourth and last part, we present a work in progress about the stable norm of a specific translation surface, namely the surface $L(2,2)$. This part contains few proven results and shows instead a first empirical approach to the problem using a computer program made using `sagemath`. In particular this program allows us to create pictures of the unit ball of the stable norm of $L(2,2)$, whose structure possess great aesthetic qualities.

Organization of the manuscript. In chapter 1 we introduce the stable norm and summarize the state of the art, especially the know examples. In chapter 2 we introduce the basic knowledge regarding translation and half-translation surfaces, along with more specific results that are needed in the rest of this manuscript. In chapter 3 we introduce the Farey sequence and the few key properties we will need in the following chapters, then through historical and general knowledge facts we attempt to convey the depth of the link between this sequence and the rest of mathematics. The chapters 4, 5 and 6 are devoted respectively to proving Theorems A,B and C, along with some associated results. Finally, chapter 7 contains some partial results about the stable norm of $L(2,2)$ and a gallery of pictures of its unit ball.

1 The stable norm

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In this first chapter we introduce the main object we will be interested in throughout this thesis: the stable norm. The stable norm of a manifold is a norm on its first homology group that measures the shortest possible length of a curve with prescribed homology class. Many qualitative results are known regarding the stable norm, but quantitative results are rare since computations are difficult to conduct. The goal of this thesis is to present new quantitative statements concerning the stable norm of surfaces. Thus in this chapter we recall the basic definitions and the main known results on the stable norm of manifolds.

1.1 Definition

1.1.1 General case

In this section we follow [Mas96]. Let (M, g) be a closed orientable Riemannian manifold. Recall that a multicurve on M is a finite union of closed curves; its homology class is the sum of the homology classes of its connected components.

We denote $f : H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R}_+$ the function that maps a homology class to the smallest possible length of a rectifiable multicurve that represents it, i.e for $h \in H_1(M, \mathbb{Z})$

we have

$$f(h) = \inf\{l_g(\gamma) \text{ where } \gamma \text{ is a rectifiable multicurve such that } [\gamma] = h\}$$

where l_g denotes the length of a curve with respect to the metric g.

The function f measures the shortest multicurve in a given homology class, but this is not quite the stable norm yet. Given a homology class h, an obvious way to get a representative of the class $2h$ is to follow a representative γ of h twice; this way, we get a representative of $2h$ of length exactly $2l_g(\gamma)$. However, it is not true in general that $f(n.h) = n.f(h)$. Indeed, it may happen that for some very large multiple of a homology class h -say $10^6 h$- some "new" curve γ' representing it appears, in the sense that it does not arise as a representative of h ran multiple times. When this happens, it is possible that this curve is relatively short, i.e $l_g(\gamma') < 10^6 f(h)$, so $10^{-6} f(10^6 h) < f(h)$.

Definition 1.1.1. The stable norm of a integral homology class $h \in H_1(M,\mathbb{Z})$, denoted $\|h\|_M$, is defined as the limit

$$\|h\|_M = \lim_{N\to\infty} \frac{f(Nh)}{N}$$

Taking this limit is called a stabilization process, hence the name *stable* norm.

The stable norm of integral classes is well-defined as one can show the limit always exists. This definition extends by homogeneity to $H_1(M,\mathbb{Q})$ the space of rational homology classes: for $h \in H_1(M,\mathbb{Z})$ and $p/q \in \mathbb{Q}$ we simply define

$$\left\| \frac{p}{q} h \right\|_M := \left| \frac{p}{q} \right| \|h\|_M$$

Then, since $H_1(M,\mathbb{Q})$ is dense in $H_1(M,\mathbb{R})$, by uniform continuity one can extend the above definition to the whole $H_1(M,\mathbb{R})$.

Proposition 1.1.2. *The extension is a norm on $H_1(M,\mathbb{R})$.*

Hence the name stable *norm*. Although the stable norm is defined on homology, a topological invariant of the manifold M, it is clear from the definition that it depends of the metric g of M. Hence the stable norm is a metric object that only really requires a notion of length to be defined: indeed, one can extend the definition to spaces with much wilder metrics.

1.1.2 Special case of surfaces

In this thesis we will be interested in the stable norm in the case of dimension 2. For surfaces the stable norm is easier to handle, as the following theorem holds:

Theorem 1.1.3 ([Mas96])**.** *Let S be a closed smooth surface equipped with a Riemannian metric m. Let $h \in H_1(S,\mathbb{Z})$ be an integral homology class. Then*

$$\|h\|_S = \inf\left\{ \sum_i |p_i| l_m(\gamma_i),\ \textit{with } p_i \in \mathbb{Z} \textit{ and } \gamma_i \textit{ closed curves such that } \sum_i p_i[\gamma_i] = h \right\}.$$

The above formula can then be taken as a definition of the stable norm in the case of surfaces.

Proposition 1.1.4. *The infimum in the previous theorem is actually a minimum.*

This comes from the compactness of the surface via a classical Arzelà-Ascoli argument.

Definition 1.1.5. A weighted multicurve $\gamma = \bigcup_i p_i \gamma_i$ on a surface S that realizes the minimum total length in its homology class h, i.e such that $\sum_i |p_i| l_m(\gamma_i) = \|h\|_S$, is called *minimizing* for h, or minimizing for the sake of brevity.

Note that minimizing multicurves in a homology class might not be unique. In general a minimizing multicurve is not connected: one very interesting question is to characterize which curves arise as connected components of minimizing multicurves, and what are their associated homology classes. This question will be discussed later in this thesis, along with counting problems related to it. For now let us simply point out the following important fact:

Proposition 1.1.6. *The connected components of a minimizing multicurve do not self-intersect.*

Proof. Let γ be a connected component of a minimizing multicurve $\tilde{\gamma}$, and assume γ has a self-intersection point p. Then $\gamma - \{p\}$ is made of two arcs; closing each arc with a short path we obtain two closed curves α and β such that $[\alpha] + [\beta] = [\gamma]$. Then by slightly contracting those curves we obtain a multicurve γ' in the same homology class as γ, but shorter. Thus replacing γ by γ' in $\tilde{\gamma}$ we obtain a shorter multicurve, contradicting the minimality of $\tilde{\gamma}$.

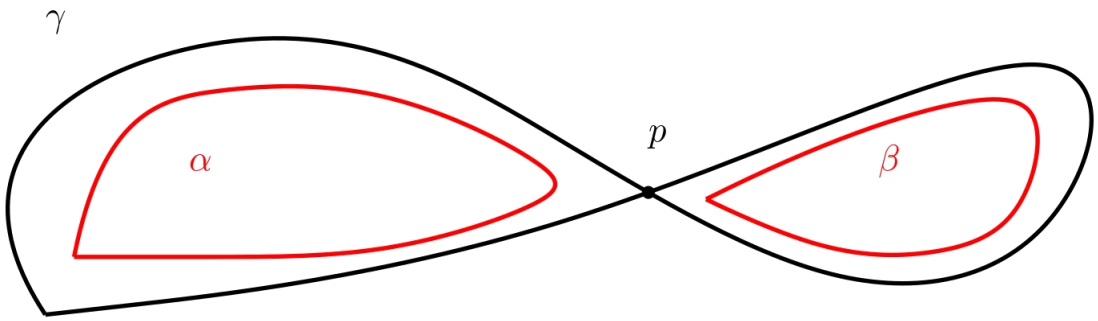

Figure 1.1: Removing intersections saves length

□

1.2 A quick survey on the stable norm

Here is a short state of the art on the stable norm of Riemannian manifolds.

1.2.1 Structure of the unit ball

When studying the stable norm, a good question to ask is the following: what is the shape of the unit ball of the norm? Indeed, understanding a norm is equivalent to understanding the combinatorial structure of its unit ball: we know it is a convex set of some Euclidean space, containing zero and symmetric around the origin. But what about its boundary? We want to know whether or not the boundary of the unit ball of the stable norm is smooth, if its contains faces like a polytope or rather smooth arcs like a Euclidean sphere, if some extremal points are exposed...

The unit ball of the stable norm of Riemannian manifolds takes qualitatively different shapes depending on the dimension of the manifold. In this section we present some prior results to illustrate these differences.

Dimension 1. Following [Bal05], let (Γ, w) be a weighted graph with first Betti number b. Let $\mathbb{E}$ be the set of edges of Γ, where each edge is oriented arbitrarily. The stable norm of a class $h \in H_1(\Gamma, \mathbb{R})$ is

$$\|h\|_\Gamma = \inf\left\{ \sum_i |\alpha_i| w(\sigma_i) \text{ where } h = \sum_i \alpha_i [\sigma_i] \right\}$$

Theorem 1.2.1 (Balacheff)**.** *Let $\{C_i\}_i$ be the set of oriented closed circuits in Γ. Then the unit ball of the stable norm of (Γ, w) is the convex hull of the set $\{C_i / \|C_i\|_\Gamma\}$.*

We can then describe what the unit ball of the stable looks like.

Corollary 1.2.1.1. *The unit ball of the stable norm of (Γ, w) is a b-dimensional polytope with at most $2(2^b - 1)$ vertices.*

Dimension $n \geq 3$. In dimension 1 there was essentially not enough room for the stable norm to be wild: to construct a curve representing a homology class one has very few available possibilities, so there ends up being very few candidates to being minimizing curves. In dimension 3 and above the situation is the complete opposite: there is so much room available for curves that the unit ball of the stable norm can be almost anything *a priori*.

But in particular, the stable norm can sometimes be nice. Hedlund constructed in [Hed32] a metric on the 3-torus for which the unit ball of the stable norm is a octahedron. The idea is to pick some disjoint curves that generate the space $H_1(\mathbb{T}^3, \mathbb{R})$ and construct a metric that is very small in a narrow tubular neighborhood of those curves, and very large elsewhere. This ensures that any curve that ventures out of these tubular regions cannot be minimizing in its homology class; thus the unit ball of the stable norm is the convex hull of the homology classes associated to these artificially short curves. Note that this type of construction is not possible in dimension 2 as one cannot pick *disjoint* curves that generate the entire homology of the surface.

Thus polytopes can arise as the unit ball of the stable norm of manifolds. But there is more:

Theorem 1.2.2 ([BB06])**.** *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$ with first Betti number $b \geq 1$. Let P be a convex polytope of $H_1(M,\mathbb{R})$ which is centrally symmetric, with non-empty interior and such that its vertices all have rational direction, i.e for each vertex there exists a point of $H_1(M,\mathbb{Z})$ on the line from the origin passing through this vertex.*
Then there exists a metric g' on M conformal to g such that P is the unit ball of the stable norm of (M,g').

Dimension 2. In dimension 2 the situation is somewhat unique: intuitively, there is now enough room for the stable norm to be complicated (contrary to dimension 1) but still not enough for the stable norm not to be heavily constrained. Thus dimension 2 is the right place to look for interesting phenomena involving the stable norm.

Let us be more precise. Even if we are currently interested in the case of surfaces, we state the following definitions (following [BM08]) in general dimension.

Definition 1.2.3. Let (M,g) be a closed Riemannian manifold of dimension $n \geq 1$.

- A supporting subspace to the unit ball of the stable norm of (M,g) is an affine subspace of $H_1(M,\mathbb{R})$ that intersects the unit sphere but not the open unit ball.
- A flat of the unit ball is the intersection of the unit sphere of the stable norm with a supporting subspace.
- The dimension of a flat is the dimension of the affine subspace it generates in $H_1(M,\mathbb{R})$.

Massart showed (see [Mas96]) there exists a partial ordering on flats: if two flats F_1, F_2 contain a common point that is interior to at least one of them, then there exists a flat F containing $F_1 \cup F_2$. Maximal flats for the inclusion are called *faces* of the unit ball of the stable norm.

In dimension 2, it turns out that flats cannot be too big:

Theorem 1.2.4 ([Mas96])**.** *Let S be a closed smooth orientable surface of genus g equipped with a Riemannian manifold. Then the dimension of the flats of the unit ball of the stable norm is bounded above by $g-1$.*

Thus the unit ball of the stable norm of a surface of genus 2 or greater cannot be a polytope: this is a striking difference with the other dimensions. So the unit ball of the stable norm of surfaces cannot be "flat"; on the other side of the spectrum, can it be strictly convex (such as a Euclidean sphere)? In other words, are there any flats at all?

Theorem 1.2.5 ([Mas96])**.** *Let S be a genus g closed Riemannian surface. Then every homology class $h \in H_1(S,\mathbb{R})$ of rational direction is contained in a flat of dimension $g-1$. Moreover at $h' := h/\|h\|_S$ the stable norm is differentiable only in the directions tangent to the maximal flat containing h' in its interior.*

So the unit ball of the stable norm of surfaces is not strictly convex nor smooth. A point of the unit sphere of the stable norm is said to be *extremal* if the maximal flat containing it in its interior is reduced to a point; in particular, vertices of polytopes are extremal. Massart showed that the unit sphere of the stable norm of surfaces has infinitely many extremal points. Thus the unit ball of surfaces is extremely difficult to visualize in general.

1.2.2 Known examples in dimension 2

There are extremely few explicit examples of orientable surfaces on which the stable norm has been computed explicitly; in fact, there are only three families of examples.

The first explicit example is the flat 2-torus $\mathbf{T}$, studied by Babenko in [Bab88]. The stable norm of flat tori turns out to be Euclidean, i.e it comes from a scalar product in $\mathbb{R}^2$. Thus the unit ball $\mathbf{B_T}$ of the stable norm of a flat torus is simply an ellipse. It is smooth everywhere, all the points in its boundary are exposed and it has no non-trivial flats, as predicted by Theorem 1.2.4.

The second example is the rotational torus in $\mathbb{R}^3$, studied by Klempnauer and Schröder in [KS17]. They provide an explicit parametrization of the unit sphere of the stable norm, which is made of two smooth curves glued together. This time the unit ball is not smooth everywhere, but rather it is piecewise smooth.

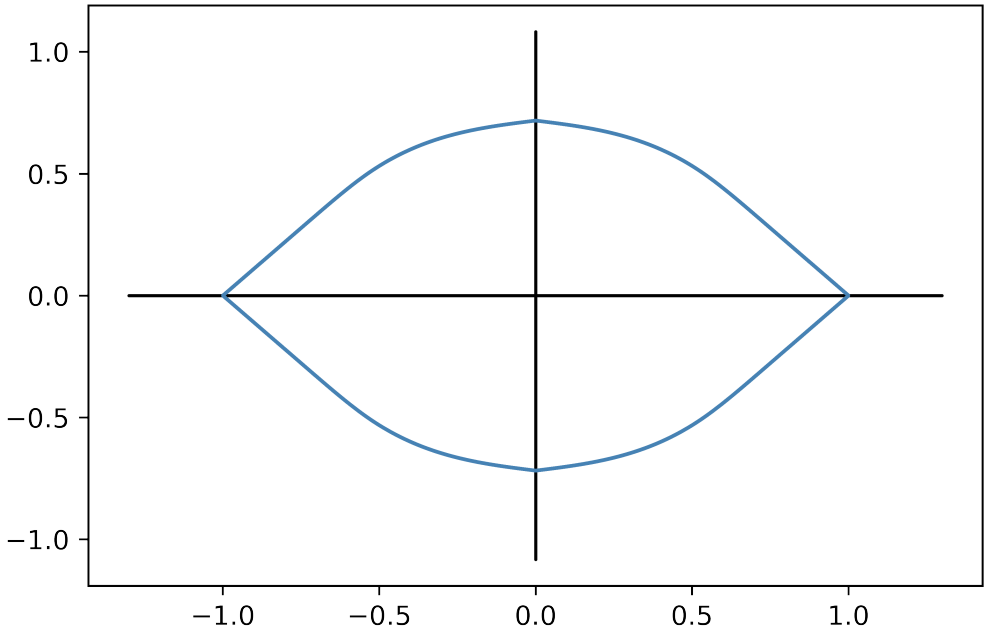

Figure 1.2: Unit ball of the stable norm of a rotational torus

The third and last explicit example in this short list is the case of hyperbolic punctured tori by McShane and Rivin in [MR95]. Although the surfaces considered are not compact, the definition of the stable norm still makes sense in this specific setting. They provide in their paper a computer-generated picture of the unit ball of the modular torus T_M: at first glance it may seem that the unit ball of the stable norm of T_M is polygonal, but this is far from the truth. Indeed, the triangle inequality of the stable norm of T_M happens to be *strict* for every non colinear vectors $(m,n),(p,q)$ in $\mathbb{Z}^2 \simeq H_1(T_M,\mathbb{Z})$, i.e $\|(m+p,n+q)\|_{T_M} < \|(m,n)\|_{T_M} + \|(p,q)\|_{T_M}$. Thus the unit ball of the stable norm of T_M cannot have flats with non-empty interior, as it must be strictly convex in *every* rational direction.

1.2.3 Miscellaneous results related to the stable norm

We mention here a few results involving the stable norm that connect it to the rest of mathematics.

Rigidity. The stable norm measures the minimal length for curves in every homology class, hence constraining the length of every topological type of curves. Thus one should expect that prescribing the stable norm of a manifold puts serious constraints on its metric.

However, prescribing the stable norm is not enough to uniquely determine the metric in general, as we can easily construct counter-examples. Indeed, take a hyperbolic punctured torus and remove a thin neighbourhood of its cusp: if this neighbourhood is thin enough then no simple geodesic enters it, so the curves minimizing in their homology class are the same on this surface as on the hyperbolic punctured torus. Now glue a Euclidean disk to fill the hole we made on the surface. The resulting surface has the same stable norm as the punctured torus we started with, and they are obviously not isometric.

Still in some settings one can obtain rigidity results with the stable norm. Perhaps one of the weakest criterion related to the stable norm is the volume of its unit ball; for Riemannian tori it turns out to be enough to constrain the metric.

Theorem 1.2.6 ([BIK97])**.** *Let T^n be a Riemannian n-torus: we have $H_1(T^n,\mathbb{R}) \simeq \mathbb{R}^n$. Let $V_{T^n,r}$ be the volume of the ball in $\mathbb{R}^n$ of radius r with respect to the stable norm of T^n, and let V_n be the volume of the standard Euclidean ball in $\mathbb{R}^n$. Then*

$$\liminf_{r\to\infty} \frac{V_{T^n,r}}{V_n r^n} \geq 1$$

with equality if and only if the metric on T^n is flat.

Thus prescribing the stable norm can force the metric to be flat. We defined the stable norm on the first homology group of a manifold M of dimension n, but the definition can naturally be extended to the homology group of every dimension: again, define the stable norm of a homology class of higher order to be the infimum of the volumes of its representatives. Now, prescribing the stable norm on the first and the $(n-1)$-th homology groups, one can obtain actual rigidity results on the metric.

Theorem 1.2.7 ([Osu10]). *Let T^n be a Riemannian n-torus, and let g_0 and g be metrics on T^n. If g_0 is flat and the stable norms of g_0 and g coincide on $H_1(T^n,\mathbb{R})$ and $H_{n-1}(T^n,\mathbb{R})$ then g_0 and g are isometric.*

Note that for $n = 2$ this means the stable norm on the first homology group is enough to uniquely determine the flat metric on a torus.

Counting closed geodesics. A classical problem in hyperbolic geometry is the following: given a hyperbolic surface S of genus g with r punctures, how many simple closed geodesics are there on S of length less than a positive number L ? Before Mirzakhani's work ([Mir16]) it was already known that this quantity is bounded above and below by polynomials of degree $6g-6+2r$, but there was no sharp result yet. The first precise estimate comes from McShane's and Rivin's work on the stable norm of hyperbolic punctured tori:

Theorem 1.2.8 ([MR95]). *Let T be a hyperbolic punctured torus. Then the number of simple closed geodesics on T of length less than $L > 0$ is asymptotic to $c.L^2$ for some constant $c > 0$.*

They showed that in this setting the simple closed geodesics identify to the primitive integral homology classes. Thus the stable norm extends continuously the hyperbolic length on T to $H_1(T,\mathbb{R})$, and counting simple closed geodesics on T comes down to counting lattice points in $\mathbb{R}^2$ with (stable) norm less than L. Putting a Dirac measure on each of these points and scaling everything down by a factor $1/L$ we get a counting measure on $\mathbb{R}^2$. Then we take the limit of those measures when L goes to infinity, which is none other than the Lebesgue measure on $\mathbb{R}^2$. Thus the final estimate is $\mathrm{Vol}(B_T).L^{\dim\mathbb{R}^2} = \mathrm{Vol}(B_T).L^2$ where B_T is the unit ball of the stable norm of T.

Lower bound on the total volume of a manifold. Let (M,g) be a m-dimensional Riemannian manifold. The homological systole of M is the length of the shortest geodesic on M whose homology class in not trivial: this positive number has been shown to contain valuable information on the geometry of M, in particular under some topological assumption it provides a lower bound on the total volume of (M,g). The stable norm provides information of the same nature, but more precise: indeed, the stable norm gives the length of the shortest closed geodesic inside *every* homology class, not just the shortest overall.
Thus extracting this extra information one can expect to obtain stronger versions of the results involving the homological systole by replacing it by the stable norm of M. Here is an example of such a result:

Theorem 1.2.9 ([BM17]). *Let M be an orientable m-dimensional differentiable manifold with first Betti number b_1. For any Riemannian metric g on M we have*

$$\left(\frac{v_{b_1}V[M]}{Vol(B(g))}\right)^m \leq Vol(M,g)^{b_1}$$

where $B(g)$ is the unit ball of the stable norm associated to the metric g on M, v_{b_1} is a universal constant that depends only on b_1 and $V[M]$ is a topological invariant of M, called the algebraic volume.

1.3 Continuity of the stable norm

On a Riemannian manifold, one might expect the stable norm to stay roughly the same under a small deformation of the metric. The aim of this section is to give a few results going into that direction. This will become useful later on, as we will approximate surfaces with sequences of surfaces on which the stable norm is already known. The correct topology to work with in that case is the Hausdorff-Lipschitz topology on the set of metric spaces.

The Hausdorff-Lipschitz topology. The Hausdorff-Lipschitz distance is the combination of the Hausdorff distance and of the Lipschitz distance between metric spaces. We will not use those distances in this thesis but we give their definitions for the record. We follow [GLP81].

The Lipschitz distance measures the dilatation between two homeomorphic metric spaces.

Definition 1.3.1. Let X, Y be metric spaces. The Lipschitz distance between X and Y is defined as

$$d_L(X,Y) := \inf_f \left|\log \mathrm{dil}(f)\right| + \left|\log \mathrm{dil}(f^{-1})\right|$$

where the infimum is taken over all bi-Lipschitz homeomorphisms $f : X \longrightarrow Y$ and

$$\mathrm{dil}(f) = \sup_{x_1,x_2 \in X} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1,x_2))}$$

The Hausdorff distance measures how far two metric spaces are from each other when embedded into a bigger common metric space.

Definition 1.3.2. Let X, Y be metric spaces. The Hausdorff distance between X and Y is defined as

$$d_H(X,Y) := \inf_{f,g,Z} d_H^Z(f(X), g(Y))$$

where the infimum is taken over all metric spaces Z and embeddings $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, and where for $A, B \subset Z$ we define

$$d_H^Z(A,B) := \inf\{\epsilon > 0, B \subset U_\epsilon(A) \text{ and } A \subset U_\epsilon(B)\}$$

with $U_\epsilon(A) = \{z \in Z, d(z,A) \leq \epsilon\}$.

Both these distances provide interesting geometric information, but they remain unsatisfying. Indeed, the Hausdorff distance is not finite in general for unbounded spaces, and the Lipschitz distance can be rendered infinite by small singularities, making it too restrictive. These limitations are the motivation that lead to the introduction of the Hausdorff-Lipschitz distance.

Definition 1.3.3. The Hausdorff-Lipschitz distance $d_{HL}(X,Y)$ between the metric spaces X and Y is

$$\inf_{X_1,Y_1} d_H(X,X_1) + d_L(X_1,Y_1) + d_H(Y_1,Y)$$

where the infimum is taken on X_1 and Y_1 arbitrary metric spaces.

The topology induced by this distance on the set of metric spaces is called the Hausdorff-Lipschitz topology.

Stable norm of nearby surfaces. The continuity of the stable norm of Riemannian surfaces with respect to the Hausdorff-Lipschitz topology has been studied by Gutkin and Massart. In particular, they proved the following result:

Proposition 1.3.4. *[Lemma 4 in [GM02]] Let M be a Riemannian surface. There exists a number c_M such that for any $\epsilon > 0$, there exists $\alpha > 0$ such that for any surface N homeomorphic to M with $d_{HL}(M,N) \leq \alpha$, for any integral homology class h,*

$$|\|h\|_M - \|h\|_N| \leq \epsilon.c_M.\|h\|_M$$

Corollary 1.3.4.1 ([GM02])**.** *The volume of the unit ball of the stable norm is uniformly continuous with respect to the Hausdorff-Lipschitz topology.*

2 Translation surfaces

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Translation and half-translation surfaces are two very special classes of surfaces that enjoy many great geometrical, dynamical or even algebraic properties. Thus, these surfaces provide a snug setting to start from when trying to understand new difficult geometrical objects, such as the stable norm: this is the strategy of this thesis. There is a very rich and well documented theory regarding translation surfaces that could easily fill several books; but since we will only use this theory as a tool rather than study it for itself we will not go into much details here. See [Zor06] and [Wri14] for surveys on the subject. In this preliminary chapter we will instead restrict ourselves to the basic knowledge about translation surfaces that we will need later on.

2.1 Riemann surfaces

As its name suggests, a Riemann surface is a surface, that is to say a 2-dimensional real manifold, equipped with some additional complex structure. These surfaces are the underlying objects throughout this thesis; thus this section aims to recall some basic facts about Riemann surfaces.

Definition 2.1.1. A Riemann surface is a complex manifold of dimension 1, i.e it is a Hausdorff topological space X with countable basis equipped with an atlas

$$\mathscr{A} = \{(U,\varphi) \text{ where } U \text{ is an open set and } \varphi : U \longrightarrow \mathbb{C} \text{ is a homeomorphism onto its image}\}$$

such that:

1. $\bigcup_{(U,\varphi)\in\mathscr{A}} U = X$
2. The transition maps are holomorphic, i.e if $(U,\varphi),(V,\psi)\in\mathscr{A}$ are two overlapping charts then $\psi\circ\varphi^{-1}:\varphi(U\cap V)\longrightarrow\psi(U\cap V)$ is holomorphic.

A map $f:X\longrightarrow Y$ between two Riemann surfaces X and Y is holomorphic if its restriction to any chart is holomorphic, as a map from $\mathbb{C}$ to itself.

Definition 2.1.2. An Abelian differential, or holomorphic 1-form, on a Riemann surface X is a collection of holomorphic 1-forms $f(z)dz$ on open sets of $\mathbb{C}$ corresponding to the charts of $\mathscr{A}$ that satisfy the following compatibility condition. If $f(z)dz$ and $g(z)dz$ are two holomorphic 1-forms associated to the charts (U,φ) and (V,ψ) of X, then we require $g(z)=f(\psi\circ\varphi^{-1}(z))(\psi\circ\varphi^{-1})'(z)$. The set of Abelian differentials on X is denoted $\Omega^1(X)$.

Note that there does not exist any non-zero Abelian differential on the Riemann sphere $\hat{\mathbb{C}}$. Indeed, if ω is a holomorphic 1-form on $\hat{\mathbb{C}}$ that reads as $f(z)dz$ in the chart z and as $g(w)dw$ in the chart $w=1/z$, then the overlapping condition yields, for $z\in\mathbb{C}-\{0\}$:

$$f(z)=\frac{-g(1/z)}{z^2}$$

Hence $|f(z)|\longrightarrow 0$ as $|z|$ goes to infinity, so f is bounded on $\mathbb{C}$. By Liouville's theorem it must be constant and equal to 0. Thus $f\equiv g\equiv 0$ and $\Omega^1(\hat{\mathbb{C}})=\{0\}$.

The space of Abelian differentials will be especially important in the rest of this chapter.

Theorem 2.1.3 (Riemann-Roch)**.** *Let X be a compact Riemann surface of genus g. Then $\Omega^1(X)$ is a complex vector space of dimension g.*

Theorem 2.1.4 (Consequence of the Riemann-Hurwitz formula)**.** *Let X be a compact Riemann surface of genus g, and let $\omega\in\Omega^1(X)$. Then ω has exactly $2g-2$ zeroes, counted with multiplicity.*

Classifying Riemann surfaces. A classical problem is to parameterize the set of all Riemann surfaces up to biholomorphism. The first result in this direction is the following theorem.

Theorem 2.1.5 (Riemann's Uniformization)**.** *A Riemann surface that is simply connected is biholomorphic to one of the following Riemann surfaces: the complex plane $\mathbb{C}$, the hyperbolic half-plane $\mathbb{H}$ or the sphere $\hat{\mathbb{C}}$. Thus, any closed Riemann surface is a quotient of one of these three surfaces by the free action of some discrete subgroup of their isometry group.*

Since biholomorphisms preserve the topological data of the surface, it only makes sense to compare Riemann surfaces that have the same genus, as they cannot be biholomorphic to one another otherwise. This leads to the following definition.

Definition 2.1.6. The moduli space of Riemann surfaces of genus g, denoted $\mathscr{M}_g$, is the set of equivalence classes of closed Riemann surfaces of genus g, up to biholomorphism.

Since any quotient of $\mathbb{C}$ or $\mathbb{H}$ by an isometry group has genus greater than 1, the only Riemann surfaces of genus 0 come from taking a quotient of the sphere. But any quotient of $\hat{\mathbb{C}}$ by an isometry group can be seen to be biholomorphic to $\hat{\mathbb{C}}$, thus the moduli space $\mathcal{M}_0$ of Riemann surfaces of genus 0 contains a single point. In genus 1, one can show that $\mathcal{M}_1 = \{\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \text{ with } \Im(\tau) > 0\}$, that is to say $\mathcal{M}_1 = \mathbb{H}$. In general the space $\mathcal{M}_g$ carries a lot more interesting structure.

Proposition 2.1.7. *For $g \geq 2$ the moduli space of compact Riemann surfaces of genus g is a complex orbifold of (complex) dimension $3g-3$.*

Unfortunately, we can see that $\mathcal{M}_g$ is not a manifold in general. One way around this problem is to see the moduli space as the quotient of some bigger, and hopefully nicer, space, namely the Teichmüller space. Let us fix a closed Riemann surface X of genus g. A marked Riemann surface is a pair (X_1, f_1) where X_1 is a Riemann surface of genus g and $f_1 : X \longrightarrow X$ is an orientation-preserving diffeomorphism. Two marked Riemann surfaces (X_1, f_1) and (X_2, f_2) are said to be equivalent if there exists a biholomorphism $h : X_1 \longrightarrow X_2$ such that $f_2 \circ f_1^{-1}$ is isotopic to h. We define the Teichmüller space of X denoted $\mathcal{T}(X)$ as the set of equivalence classes of marked Riemann surfaces (with marking relative to X).

Definition 2.1.8. The Teichmüller space $\mathcal{T}(X)$ of a Riemann surface X of genus g is independant of the choice of X: in fact it depends only on g. Thus we can define *the* Teichmüller space $\mathcal{T}_g$ of Riemann surfaces of genus g as the Teichmüller space of any Riemann surface of genus g.

The space $\mathcal{T}_g$ enjoys many great geometrical properties, starting with the following proposition.

Proposition 2.1.9. *The Teichmüller space $\mathcal{T}_g$ is a complex manifold of dimension $3g-3$.*

Definition 2.1.10. Let S be a compact smooth surface of genus g. The Mapping Class Group of genus g is the quotient group

$$MCG(g) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S)$$

where $\mathrm{Diff}^+(S)$ is the group of orientation-preserving diffeomorphisms of S and $\mathrm{Diff}_0^+(S)$ is the subgroup of $\mathrm{Diff}^+(S)$ consisting of diffeomorphisms isotopic to the identity. Since all smooth surfaces of genus g are diffeomorphic to each other, the definition of $MCG(g)$ is independent of the choice of S.

Theorem 2.1.11. *The group $MCG(g)$ acts properly discontinuously on $\mathcal{T}_g$ and*

$$\mathcal{M}_g = \mathcal{T}_g / MCG(g).$$

2.2 The three definitions of translation surfaces

The surfaces we will be interested in throughout this thesis are translation and half-translation surfaces. These surfaces are closed Riemann surfaces with some additional structure, namely a flat metric with a finite number of conical singularities whose angles are all integral multiples of π. There are three main ways to define translation surfaces, that all have some merit. We will now give these definitions, starting with the most intuitive one and progressively getting more abstract.

By glueing polygons. The simplest way of defining translation surfaces is by glueing polygons side by side.

Definition 2.2.1 (First definition of a translation surface)**.** Let $P_1, ..., P_n$ be polygons of the complex plane such that for every edge e of a polygon, there exists another edge e' (possibly in the same polygon) that is the image of e under a non-trivial translation. We identify the pair (e, e'). The quotient of $P_1 \cup ... \cup P_n$ with pairs of edges identified by translation is called a translation surface.

Note that there might be several edges of the polygons that are all image of the same original edge under various translations: the different choices of pairs to identify then give different translation surfaces. For instance, in the following picture the two surfaces made by glueing the sides of the same polygon in two different ways do not even have the same genus! Indeed, the left hand side surface has genus 2 while the right hand side surface has genus 1, as one can see by computing their Euler's characteristic through a triangulation.

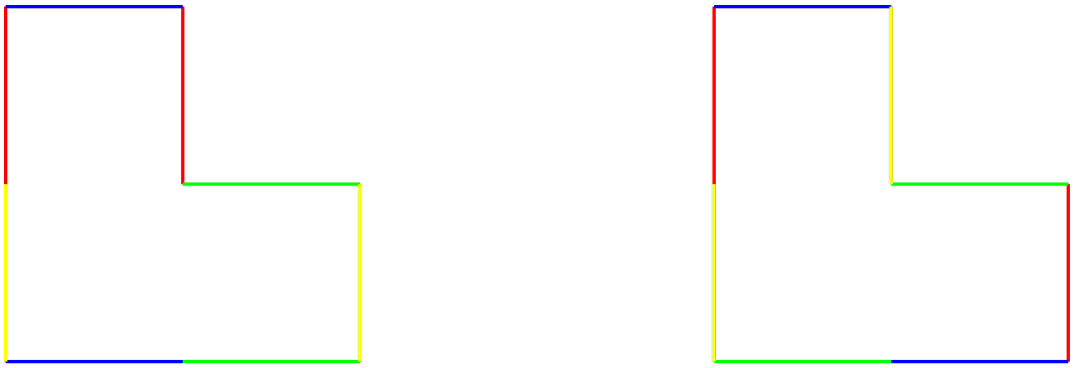

Figure 2.1: Two different identifications of a polygon give two different surfaces

A translation surface is an orientable closed topological surface: indeed, one can construct an explicit atlas on the surface. We will not provide formulas, but it is important to understand what the charts of this atlas are. There are three types of points to consider: the image of points that lie in the interior of a polygon, of points that lie in the interior of an edge and of the vertices of the polygons.

- For the image of a point in the interior of a polygon, the quotient map is essentially the identity on a small disk neighborhood of that point.
- For the image of a point that lies inside an edge of one the polygons, the lift to $\mathbb{C}$ of a small disk neighborhood of that point is made of two half-disks. On one of those half-disks the chart is the identity, and the other half-disk the chart is a translation putting the two half-disks together.
- Finally, the images of the vertices are a bit more tricky. When glueing together many edges that share a common vertex, it can happen that the total angle at this vertex is more than 2π; in fact one can check that it must be $2k\pi$ for some $k \in \mathbb{N}$. For example see Figure 2.2: the red points are all identified and the red disks are glued together, forming a total angle of measure 6π. Thus the chart around the vertex is modelled on $z \mapsto z^{1/k}$.

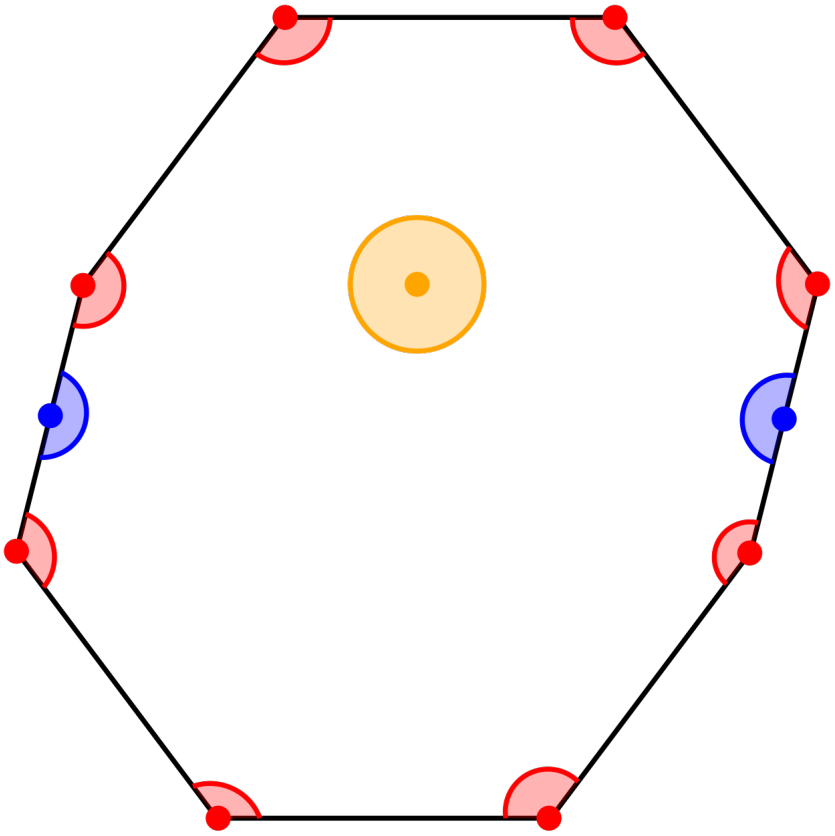

Figure 2.2: The three different types of points

Definition 2.2.2. Let X be a translation surface. The set of points that are the images of the vertices of the polygons with total angle greater than 2π are called the singularities of X. If the total angle around a singularity is $2(k+1)\pi$, that is with excess angle $2k\pi$, we say that the singularity has order k. The set of singularities of X is denoted $\Sigma(X)$, or Σ for short when there is no ambiguity on the surface.

Through a translation atlas. In the previous paragraph we have seen that a translation surface X obtained by glueing polygons together comes with a natural atlas. One can check that the transition maps are holomorphic, thus giving X the structure of a Riemann surface. In particular, removing the tricky points corresponding to Σ, we obtain an atlas on $X - \Sigma$ whose transition maps are special: they are all translations. This atlas is called a translation atlas. We can use this as a definition:

Definition 2.2.3 (Second definition of translation surfaces)**.** A translation surface is the data of a closed Riemann surface X, a finite subset Σ of X and an atlas whose restriction to $X - \Sigma$ is a translation atlas, and whose charts around the points of Σ are of the form $z \mapsto z^{1/k}$.

This second definition of translation surfaces is equivalent to the first one. We already know that glueing provide the required atlas. Conversely, starting with the second definition, the translation atlas naturally endows $X - \Sigma$ with a flat metric: this is simply the Euclidean metric on $\mathbb{C}$, which is preserved by translation. Draw geodesics on X connecting the points of Σ so that these geodesics cut X into simply connected

pieces. Each of these pieces is simply connected and bounded by straight line segments: they are polygons of the complex plane. Note that a geodesic of the flat metric on X is given in charts as *parallel* straight line segments in $\mathbb{C}$.

With Abelian differentials. Since translations are extremely nice maps of $\mathbb{C}$, they preserve almost any structure we might be interested into. In particular, a translation surface X comes naturally with an Abelian differential, which on $X - \Sigma$ is none other than the form dz on $\mathbb{C}$ which is preserved by translations. Again, we can use this as a definition:

Definition 2.2.4 (Third definition of translation surfaces)**.** A translation surface is a pair (X, ω) where X is a closed Riemann surface equipped with a non-zero Abelian differential ω.

One can check that this definition is equivalent to the previous one: indeed, one can construct an atlas on X whose transition maps outside of some finite subset of X are translations. This finite subset is exactly the set Σ previously defined, and one can see that ω has zeroes of order k at the singularities of X or order k. Thus Σ can also be defined as the zero locus of ω. The flat metric can also be defined with the Abelian differential:

Proposition 2.2.5. *Let (X, ω) be a translation surface. Then X is equipped with a flat metric $|\omega|$ with a finite number conical singularities whose cone angles are* even *multiples of π.*

2.3 Moduli space of translation surfaces

Just as in the case of Riemann surfaces, an interesting problem is to classify the translation surfaces up to some equivalence relation. Since the main feature of translation surfaces is their flat metric, it makes sense to classify them up to isometry.

Definition 2.3.1. Two translation surfaces (X_1, ω_1) and (X_2, ω_2) are said to be equivalent if there exists a biholomorphism $h : X_1 - \Sigma(X_1) \longrightarrow X_2 - \Sigma(X_2)$ such that $h^*\omega_2 = \omega_1$. We define the moduli space of translation surfaces $\Omega\mathcal{M}_g$, also called the moduli space of Abelian differentials, as the set of equivalence classes of translation surfaces.

Note that the space $\Omega\mathcal{M}_g$ is the Hodge bundle of $\mathcal{M}_g$. Replacing the underlying Riemann surfaces X_i by *marked* Riemann surfaces (X_i, f_i) in the previous definition, one can define in the same way the Teichmüller space $\Omega\mathcal{T}_g$ of translation surfaces of genus g. Just as in the case of Riemann surfaces the moduli space $\Omega\mathcal{M}_g$ can be obtained as the quotient $\Omega\mathcal{T}_g / MCG(g)$. Although we will not use it in the present thesis, we want to mention for the record an interesting dynamical point of view on the space $\Omega\mathcal{T}_g$. Given a Riemann surface X, the set of quadratic differentials on X are a subset of the cotangent bundle of $\mathcal{T}_g$ at $[X]$. Thus, given a metric on the Teichmüller space space $\mathcal{T}_g$ of Riemann surfaces, a translation surface can thus be seen as a point of $\mathcal{T}_g$ plus the additional data of a tangent vector: that is, the data allowing to define a geodesic of $\mathcal{T}_g$ with respect to a choice of a metric.

Proposition 2.3.2. *The moduli space $\Omega\mathcal{M}_g$ is a complex orbifold of dimension $4g-3$.*

Proof. The result follows directly from the fact that $\mathcal{M}_g$ is a complex orbifold or dimension g and that for a Riemann surface X the space $\Omega^1(X)$ has dimension g. □

There is a natural way to stratify the space $\Omega\mathcal{M}_g$. Let (X,ω) be a translation surface of genus g. By the Gauss-Bonnet formula, we know the Abelian differential ω must have exactly $2g-2$ zeroes counted with multiplicity; note that these zeroes, along with their respective multiplicity, are preserved by biholomorphism. We can then distinguish Abelian differentials based on the combinatorics of their zero set.

Proposition 2.3.3. *The moduli space $\Omega\mathcal{M}_g$ of Abelian differentials of genus g is stratified in the following way:*

$$\Omega\mathcal{M}_g = \bigcup_{d_1+...+d_n=2g-2} \mathcal{H}(d_1,...,d_n)$$

where $\mathcal{H}(d_1,...,d_n)$ is the set of (equivalence classes of) translation surfaces (X,ω) where ω has exactly n zeroes with multiplicity $d_1,...,d_n$ respectively.

The sets $\mathcal{H}(d_1,...,d_n)$ are called the *strata* of $\Omega\mathcal{M}_g$. The stratum $\mathcal{H}(2g-2)$ is called the minimal stratum; the stratum $\mathcal{H}(1,...,1)$ is called the maximal stratum.

Proposition 2.3.4. *The stratum $\mathcal{H}(d_1,...,d_n)$ is a complex orbifold of dimension $2g+n-1$.*

Here is a sketch of the proof. The local coordinates are obtained by integrating the Abelian differentials on some common homology basis. More precisely, let $(X,\omega) \in \mathcal{H}(d_1,...,d_n)$ be a translation surface with $\Sigma \subset X$ the zero locus of ω on X. Let S be the underlying topological surface and let $(\gamma_1,...,\gamma_{2g+n-1})$ be a basis of the relative homology group $H_1(S,\Sigma,\mathbb{C})$. Let $\mathrm{hol}_\omega : H_1(S,\Sigma,\mathbb{C}) \longrightarrow \mathbb{C}$ the *holonomy map* defined by $\mathrm{hol}_\omega([\gamma]) = \int_\gamma \omega$ for $[\gamma]$ a (relative) homology class on S. Then the *periods map*

$$\Phi : (X',\omega') \mapsto \big((\gamma_1,...,\gamma_{2g+n-1}) \mapsto (\mathrm{hol}_{\omega'}(\gamma_1),...,\mathrm{hol}_{\omega'}(\gamma_{2g+n-1})\big)$$

is a local homeomorphism around (X,ω), providing local coordinates from the stratum $\mathcal{H}(d_1,...,d_n)$ to $H^1(S,\Sigma,\mathbb{C}) \simeq \mathbb{C}^{2g+n-1}$. A nice feature of these charts is that, outside of the singular points of the stratum, the coordinates changes are affine maps that preserve the Lebesgue measure in the charts. This allows us to endow the stratum with a "Lebesgue measure". A powerful result from Masur and Veech states that this measure has *finite* total volume on any stratum (after some normalization, namely considering only translation surfaces of area 1). This provides a probability measure on the stratum and gives meaning to the notion of generic translation surfaces.

The local coordinates on surfaces can also be seen as the complex length of the sides of the polygon defining the surface. Let $(X,\omega) \in \mathcal{H}(2)$ be the translation surface defined by the following polygon:

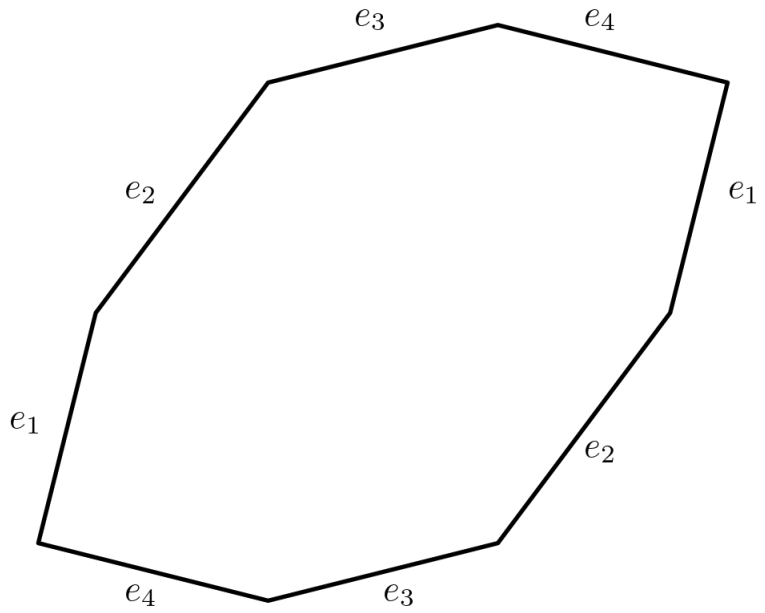

Figure 2.3: The sides parametrize the surface

The edges $e_1, ..., e_4$ of the polygon project onto closed curves that form a basis $\gamma_1, ..., \gamma_2$ of $H_1(X, \Sigma, \mathbb{C})$. The complex length of the e_i is $\int_{e_i} dz$. Since the form dz projects onto X as ω, this is equal to $\int_{\gamma_i} \omega$, that is, $\mathrm{hol}_\omega(\gamma_i)$ the local coordinates at (X, ω).

Theorem 2.3.5 (Konsevich-Zorich)**.** *In genus* 2 *every stratum is connected. This is not true anymore if* $g \geq 3$.

2.4 Action of $SL(2,\mathbb{R})$ and the Veech group

There is a natural action of the special linear group $SL(2,\mathbb{R})$ on translation surfaces, that also preserves the strata. The most intuitive way to picture it is the following. Let (X, ω) be a translation surface obtained by glueing the edges of a polygon P of the complex plane. Let $M \in SL(2,\mathbb{R})$ be a determinant 1 real matrix: M acts linearly on $\mathbb{C}$ and thus on P, creating a new polygon $P' = M.P$. Since the linear action of M on P preserves parallelism and the relative lengths, each edge of P' is part of a pair of parallel edges of the same length.

Definition 2.4.1. With the notations above, glueing the pairs of parallel egdes of same length of P' we obtain a new translation surface $(X', \omega') =: M.(X, \omega)$.

In terms of translation atlases the matrix M acts by post-composition of the charts, providing a new translation atlas on the underlying topological surface, thus a new translation surface altogether. With this point to view it is easy to see that the action of M does not change the number of zeroes of the Abelian differential nor their multiplicity: thus (X, ω) and $M.(X, \omega)$ lie in the same stratum of $\Omega\mathcal{M}_g$.

Definition 2.4.2. Let (X_1,ω_1) and (X_2,ω_2) be two translation surfaces. A map $f : X_1 \longrightarrow X_2$ is *affine* if it is locally affine: that is, in every chart there exists a matrix A such that f reads as $z \mapsto Az + c$ in those coordinates, with $c \in \mathbb{C}$.

Definition 2.4.3. Let (X,ω) be a translation surface. A map $f : X \longrightarrow X$ is an affine diffeomorphism of X if it is a diffeomorphism of $X - \Sigma$, affine, and preserves the set Σ. We denote $\mathrm{aff}(X)$ the group of affine diffeomorphisms of X, and $\mathrm{aff}^+(X)$ the group of affine diffeomorphisms of X that preserve the orientation.

Let $f \in \mathrm{aff}^+(X)$ be an affine diffeomorphism. Since it is affine, f reads in some chart of $X - \Sigma$ as $z \mapsto Az + c$ with $A \in SL(2,\mathbb{R})$ and $c \in \mathbb{C}$. But since the transition maps on the translation atlas are translations, in any overlapping chart f reads as $w \mapsto Aw + c'$ with $c' \in \mathbb{C}$. In other words, the matrix A representing the linear part of f is the same in every chart; this matrix is called the *derivative* of f. With the previous notations, we define a homomorphism $\mathrm{der} : \mathrm{aff}^+(X) \longrightarrow SL(2,\mathbb{R})$ by letting $\mathrm{der}(f) = A$.

Definition 2.4.4. The Veech group of a translation surface (X,ω), denoted $\Gamma(X,\omega)$ or $\Gamma(\omega)$ for short, is the image of the homomorphism der:

$$\Gamma(\omega) = \mathrm{der}(\mathrm{aff}^+(X)) \subset SL(2,\mathbb{R})$$

An equivalent definition of $\Gamma(\omega)$ is that it is the stabilizer of (X,ω) under the action of $SL(2,\mathbb{R})$.

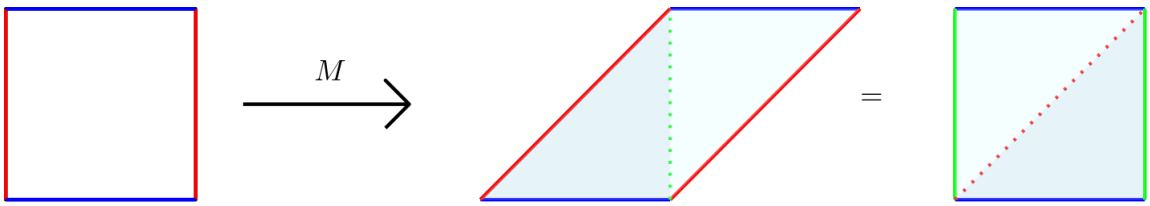

Figure 2.4: The square torus is stabilized by the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

For example, the Veech group of flat tori is the full $SL(2,\mathbb{Z})$, while the Veech group of the L-shaped surface made of three identical squares is generated by the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In general, the Veech group of a translation surface is discrete and non-cocompact: it is a Fuschian group. Note that having an interesting Veech group is somewhat rare: indeed, for almost every translation surface the Veech group is trivial. But there are some especially interesting Veech groups:

Definition 2.4.5. A translation surface whose Veech group is a lattice of $SL(2,\mathbb{R})$ is called a Veech surface.

Veech surfaces enjoy many good geometrical and dynamical properties, such as the following result about their geodesic flow.

Theorem 2.4.6 (Veech Dichotomy)**.** *Let (X,ω) be a Veech surface. Then for each direction θ the geodesic flow on X with direction θ is either periodic or uniquely ergodic.*

One can also be interested in the action of matrices of $SL(2,\mathbb{R})$ that do not stabilize a surface, and try to understand what its orbit may look like. Given a translation surface (X,ω), its orbit under the $SL(2,\mathbb{R})$-action is called its *Teichmüller disk*. While studying the whole $SL(2,\mathbb{R})$-action on $\Omega\mathcal{M}_g$ is certainly interesting, there are some subsets whose action is worth looking at. The Teichmüller flow is the action of the one parameter subgroup $(g_t)_t$ of $SL(2,\mathbb{R})$ defined by

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

where the $1/2$ factor appears for normalization purpose. This flow is also called the geodesic flow on the Teichmüller space, as one can define a metric on it (called the Teichmüller metric) for which the orbits under the action of (g_t) happen to follow the geodesics of the metric.

Theorem 2.4.7 (Masur-Veech)**.** *The Teichmüller flow is ergodic on each connected component of a stratum.*

2.5 Quadratic differentials

There is another class of interesting flat surfaces that are closely related to translation surfaces: the half-translation surfaces. As the basic theory about these surfaces is very similar to the case of translation surfaces, in this section we will briefly cover the main features of half-translation surfaces.

The definition of half-translation surfaces is similar to translation surfaces, with one small difference: when glueing polygons by identifying pair of parallel edges with the same length, this time we allow the glueing maps to be not only translations but also translations with a twist, i.e maps of the form $z \mapsto \pm z + c$ with some $c \in \mathbb{C}$. The same applies to the definition using atlases: this time the transition maps can be translation with a twist, also called half-translations. The following definition is the half-translation surface equivalent of the definition of translation surfaces using Abelian differentials:

Definition 2.5.1. A half-translation surface is a pair (X, q) where X is a closed Riemann surface and q is a quadratic differential on X, that is to say a section of the symmetric square of the holomorphic cotangent bundle of X.

To put things in a less wordy fashion, just as an Abelian differential locally looks like dz, a quadratic differential locally looks like dz^2. Note that every square of an Abelian differential is a quadratic differential, but the converse is not true; thus half-translation surfaces constitute a more general class of surfaces than translation surfaces. If a quadratic differential is the global square of some Abelian differential then it gives the surface a translation surface structure: in what follows we will then assume that the quadratic differentials we consider are *not* squares of Abelian differentials.

Proposition 2.5.2. *Let (X, q) be a half-translation surface. It is naturally equipped with a flat metric $|q|$ with a finite number of conical singularities, with cone angle integral multiples of π. The excess angle (divided by π) is called the multiplicity of the singularity.*

A key difference with translation surfaces is the following. On a translation surface (X,ω), the real (resp. imaginary) part of ω define a horizontal (resp. vertical) foliation of the surface; moreover this foliation is orientable. On a half-translation surface one can define in the same fashion a horizontal (resp. vertical) foliation, but it is not orientable anymore, as identifying sides of polygons with a half-translation introduces a twist. Thus on a half-translation surface, while the notion of verticality still makes sense, there is no concept of "up" and "down", contrary to the case of translation surfaces.

Just as we have done in the previous section, we can define the moduli space of quadratic differentials, denoted $\mathcal{Q}_g$. By the Gauss-Bonnet formula, the sum of multiplicities of a quadratic differential on a genus g surface is equal to $4g-4$. This time again, the moduli space of quadratic differential is stratified by sets of half-translation surfaces with prescribed type of singularities, called the strata:

$$\mathcal{Q}_g = \bigcup_{k_1+...+k_n=4g-4} \mathcal{Q}(k_1,...,k_n)$$

Proposition 2.5.3. *A stratum $\mathcal{Q}(k_1,...,k_n)$ is a complex orbifold of dimension $2g+n-1$.*

As in the case of translation surfaces, matrices act linearly on polygons defining half-translation surfaces. This time, since the vertical foliation on the surface is not orientable, the linear action of a matrix on the polygons is defined only up to multiplication by $\pm Id$. Thus the right matrix group to consider is $PSL(2,\mathbb{R})$; its action preserves the strata of $\mathcal{Q}_g$. Again, there is a natural finite volume measure on each stratum, and with respect to that measure the action of the matrices $\pm\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ is ergodic on each connected component of the stratum.

Definition 2.5.4. A connected component C of a stratum $\mathcal{Q}(k_1,...,k_n)$ is said to be hyperelliptic if for every half-translation surface $(X,q) \in C$ the underlying Riemann surface X is hyperelliptic, and the hyperelliptic involution τ fixes the quadratic differential, i.e $\tau^* q = q$.

The hyperelliptic components of the strata were actually classified by Lanneau. In general, not all half-translation surfaces are hyperelliptic. However, there is a miracle happening is genus 2:

Theorem 2.5.5 ([Lan04])**.** *The strata of $\mathcal{Q}_g$ having a hyperelliptic component are not connected, except when $g=2$ where the whole $\mathcal{Q}(1,1,1,1), \mathcal{Q}(1,1,2)$ and $\mathcal{Q}(2,2)$ are hyperelliptic and connected.*

2.6 Saddle connections and cylinders of closed geodesics

In classical differential geometry of surfaces, a closed geodesic is a somewhat rare object; in general, slightly deforming a closed geodesic does not result in another geodesic of the surface. For example, a classical result of hyperbolic geometry states that on hyperbolic surfaces there is exactly one closed geodesic inside of a free homotopy class. The situation is radically different on translation (and half-translation) surfaces: closed

geodesics come in families of infinitely many parallel, freely homotopic geodesics of the surface. We discuss some of the properties of these families of closed geodesics in this section.

Definition 2.6.1. A cylinder on a translation surface is a maximal family of closed geodesics of same length and direction that are freely homotopic to one another. The direction of the cylinder is defined as the direction of the closed geodesics it contains.

The geodesics of a cylinder fill an open connected domain of the surface, that is, unless the surface is a torus, homeomorphic to an annulus. Since a cylinder is defined as a *maximal* family of parallel geodesics, a cylinder on a translation surface always contains singularities in its boundary (unless the surface is a torus): otherwise we would be able to find a larger cylinder that contains it, contradicting its maximality.

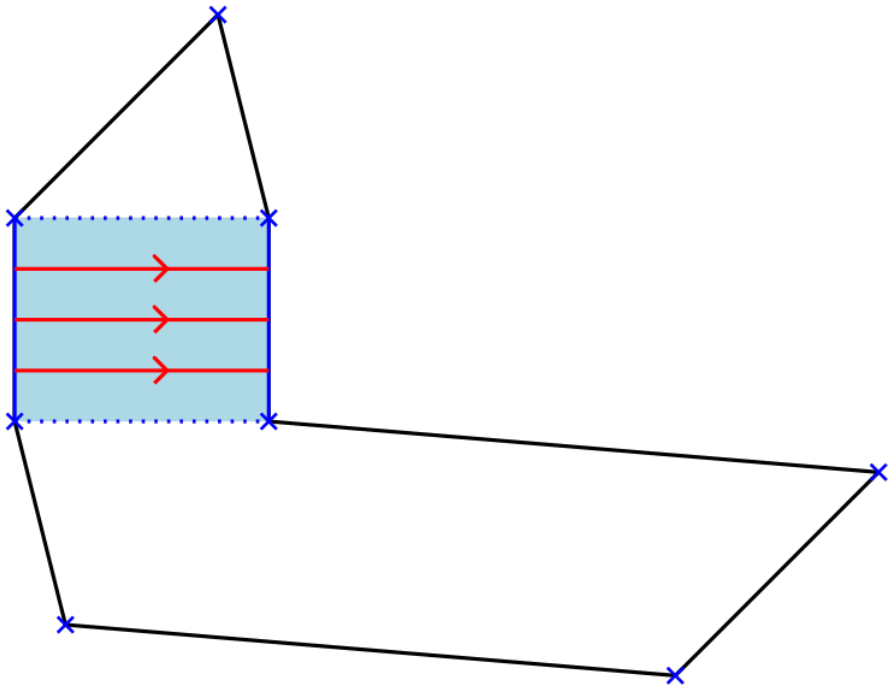

Figure 2.5: A horizontal cylinder on a translation surface

By definition, on a cylinder of a given direction, the geodesic flow of the surface in that direction is periodic. While in general the geodesic flow in a given direction is not globally periodic, meaning we cannot find a single period, it can happen that every geodesic in that direction is periodic: the surface can then be seen as the union of the periodic components of the geodesic flow. Geometrically, each of these components is a cylinder parallel to the flow: we say that the surface decomposes into cylinders of said direction, and that this direction is completely periodic. We have already encountered cylinder decompositions: Veech Dichotomy Theorem 2.4.6 says that on a Veech surface, on a given direction either the geodesic flow is uniquely ergodic or the surface decomposes into cylinders. Note that there may exist several directions in which a surface can be decomposed into cylinders, and the number of cylinders involved may change.

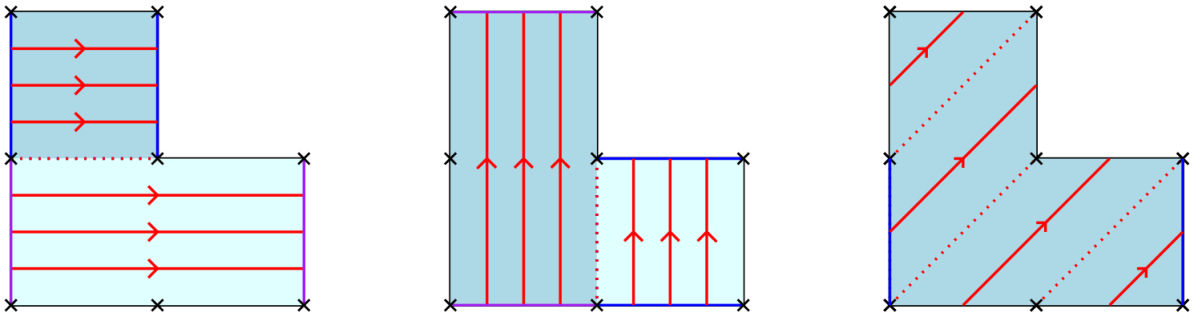

Figure 2.6: Multiple cylinder decompositions of the same surface, in one and two cylinder(s)

Definition 2.6.2. A saddle connection on a translation surface is a geodesic segment joining two singularities (possibly the same one) of the surface and having no singularity in its interior.

Thus the boundary of a cylinder is made of a union of parallel saddle connections, which make cylinders and saddle connections closely related objects on translation surfaces.

An interesting question concerning cylinders and saddle connections is a counting one. On a translation surface (X,ω) in a stratum $\mathscr{H}$, denote $N_c(\omega,L)$ (resp. $N_{sc}(\omega,L)$) the number of cylinders (resp. saddle connections) on X of length at most L. Then, what are the asymptotic behaviours of $N_c(\omega,L)$ and $N_{sc}(\omega,L)$ as L goes to infinity? It turns out these functions have similar asymptotic behaviour, hence we refer to both of them as $N(\omega,L)$ when the distinction is not useful.

Theorem 2.6.3 (Masur [Mas88])**.** *For* any *translation surface* $(X,\omega) \in \mathscr{H}$ *the counting function* $N(\omega,L)$ *grows quadratically in* L*. More precisely, there exist positive constants* $c_1(\omega) < c_2(\omega)$ *such that*

$$c_1(\omega) \leq \frac{N(\omega,L)}{L^2} \leq c_2(\omega)$$

for L *sufficiently large.*

The optimal values of the constants c_1 and c_2 are not known. Moreover, while the growth of $N(\omega,L)$ is bounded by quadratic functions, the theorem does not say that it is actually equivalent to a quadratic polynomial, i.e we do not know whether or not there exists a constant c such that $N(\omega,L)/L^2 \longrightarrow c$ as L goes to infinity. There have been progress on this question shortly after Masur's work, as Veech proved in [Vee89] a L^1-quadratic asymptotic result: there exists a constant $c = c(\mathscr{H})$ that depends only on the stratum such that

$$\lim_{L\longrightarrow\infty} \int_{\mathscr{H}} \left| \frac{N(\omega,L)}{L^2} - c \right| d\mu(\omega) = 0$$

where μ is the natural Lebesgue probability measure on $\mathscr{H}$. The constant $c(\mathscr{H})$ is called the Siegel-Veech constant of the stratum $\mathscr{H}$. This result has since been further improved to an almost everywhere pointwise result:

Theorem 2.6.4 (Eskin-Masur [EM01])**.** *Let $c = c(\mathscr{H})$ be the Siegel-Veech constant of the stratum $\mathscr{H}$. Then for μ-almost every $(X,\omega) \in \mathscr{H}$,*

$$\lim_{L\longrightarrow\infty} \frac{N(\omega,L)}{L^2} = c$$

3 The Farey sequence

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At first glance this chapter may seem out of place and largely off topic, as we discuss basic number theory facts; however the notions introduced here are actually essential to the main results developed in this thesis. The Farey sequence is a sequence of (sets of) numbers whose consecutive terms satisfy deep arithmetical and combinatorial relations. As a result, the Farey sequence is involved in numerous different topics in mathematics; among those topics is the present thesis. The aim of this chapter is to introduce the basic notions we will need, along with some general knowledge and history about the Farey sequence.

3.1 Definition and basic properties

Sir John Farey (1766-1826) was a British geologist whose interests spanned across various fields of science, including mathematics. He wrote a letter to the *Philosophical Magazine* in 1816 in which he conjectured, without proof, some of the properties of the Farey sequence. Not long after Cauchy proved the conjectures, rediscovering results of Haros from 1802 that were not known to Farey nor Cauchy at the time. The name "Farey sequence" arose as an inaccuracy from this confusion and remains the name in use today.

The Farey sequence should in fact be called the Farey *sequences*, with plural. Indeed, strictly speaking it is a sequence of ordered sets: a common abuse of terminology makes it singular.

Definition 3.1.1. The Farey sequence of order $k \in \mathbb{N}^*$, denoted $\mathscr{F}_k$, is the ordered sequence of irreducible fractions in $\mathbb{Q}$ whose denominator is less than k.

$$\mathscr{F}_k = \left\{ \frac{p}{q}, \text{ with } p \text{ and } q \text{ coprime}, q \leq k \right\}$$

For example

$$\mathscr{F}_1 = \left\{ \ldots, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots \right\} = \mathbb{Z}$$

$$\mathscr{F}_2 = \left\{ \ldots, \frac{-1}{1}, \frac{-1}{2}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \ldots \right\}$$

and

$$\mathscr{F}_3 = \left\{ \ldots, \frac{-1}{1}, \frac{-2}{3}, \frac{-1}{2}, \frac{-1}{3}, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}, \ldots \right\}$$

Note that the historical definition of the Farey sequence only involved the rationals between 0 and 1, with some authors referring to the above definition as the *extended Farey sequence*. This does not make a big difference anyway: in this thesis we will need to consider all rational numbers, so we will use the not-historically-accurate Definition 3.1.1.

Proposition 3.1.2 ([HW08])**.** *Let $a/b < c/d$ be two irreducible fractions that are consecutive in some Farey sequence $\mathscr{F}_k$ of order k. These fractions are called a Farey pair and are said to be Farey neighbours. They satisfy the following properties:*

1. *bc-ad=1. Note that this property is actually equivalent to being Farey neighbours.*
2. *If p/q is an irreducible fraction such that a/b, p/q and c/d are consecutive in some Farey sequence, then*
$$\frac{p}{q} = \frac{a+c}{b+d}$$
We denote $p/q = a/b \oplus c/d$ and we say that p/q is the mediant of a/b and c/d.

Say that a/b and c/d are Farey neighbours, meaning they are consecutive in some Farey sequence of order k_0. As the order k of the sequence increases, more and more fractions appear, as $\mathscr{F}_{k_0} \subset \mathscr{F}_{k_0+1} \subset \ldots \subset \mathscr{F}_k$. According to the previous proposition, the first fraction p/q that appears between a/b and c/d as the order of the sequence increases is their mediant $a/b \oplus c/d$. The fractions a/b,c/d and p/q satisfy very strong relations that will be at the core of the next chapter.

Definition 3.1.3. Let a/b and c/d be Farey neighbours. The first fraction $p/q = a/b \oplus c/d$ that appears between them as the order of the sequence increases is called the *Farey child* of a/b and c/d, and a/b and c/d are called the *Farey parents* of p/q.

Extending the definition, one can define a whole genealogy on the rational numbers. Note that according to this definition, the integers are the Farey ancestors of all the rationals. In other words, starting with integers and taking their successive mediants one can obtain any given rational number within a finite amount of operations. According to this definition, the integers do not have any Farey parent.

Proposition 3.1.4. *Let $p/q = a/b \oplus c/d$ be an irreducible fraction, with $a/b < c/d$ its Farey parents. The Farey parents are the best rational approximations of the number p/q with denominator less than q. More precisely, $a/b < p/q < c/d$ and there are no other rational number than p/q with denominator less than q in the interval $[a/b, c/d]$. In other words, the last two convergents in the continued fraction expansion of p/q are its Farey parents.*

A convenient way to picture the genealogical relations between the rationals is through the Farey graph. The vertices of the graph are the rationals arranged in rows, with the topmost vertices corresponding to the integers, and with a downward edge from a fraction to each of its Farey children.

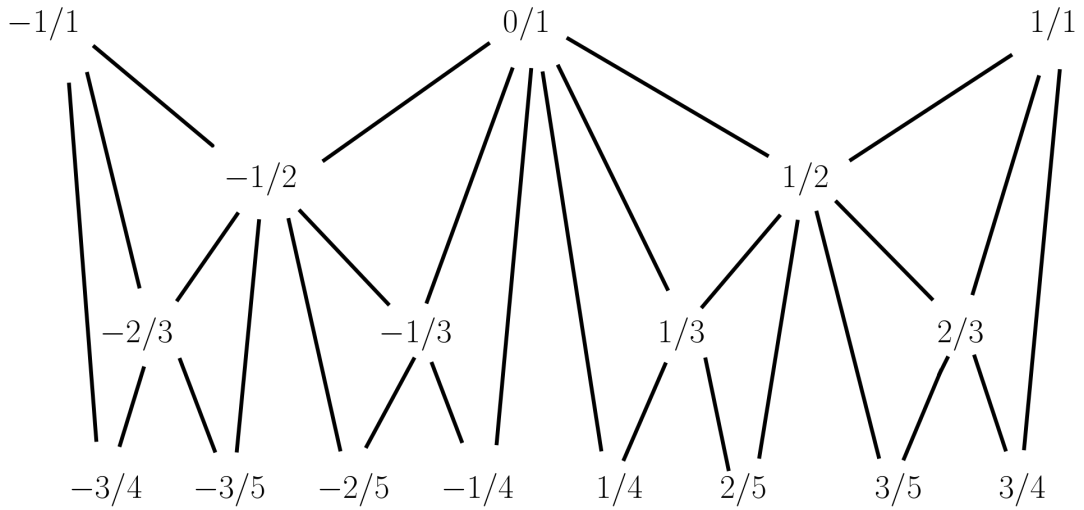

Figure 3.1: A small part of the Farey graph

This graph captures all the parenting relations within $\mathbb{Q}$, but at the cost of being difficult to read as every vertex of the graph has infinite valency. Indeed, each rational has infinitely many Farey children: if a/b and c/d are the Farey parents of p/q, then the mediant $a/b \oplus p/q$ is also a Farey child of a/b. One simple solution is to only draw the edges between a rational a/b and its *first generation* Farey children, i.e those obtained by taking the mediant of a/b and another rational that is not already a Farey child of a/b. The resulting graph is a bivalent tree, called the Stern-Brocot tree.

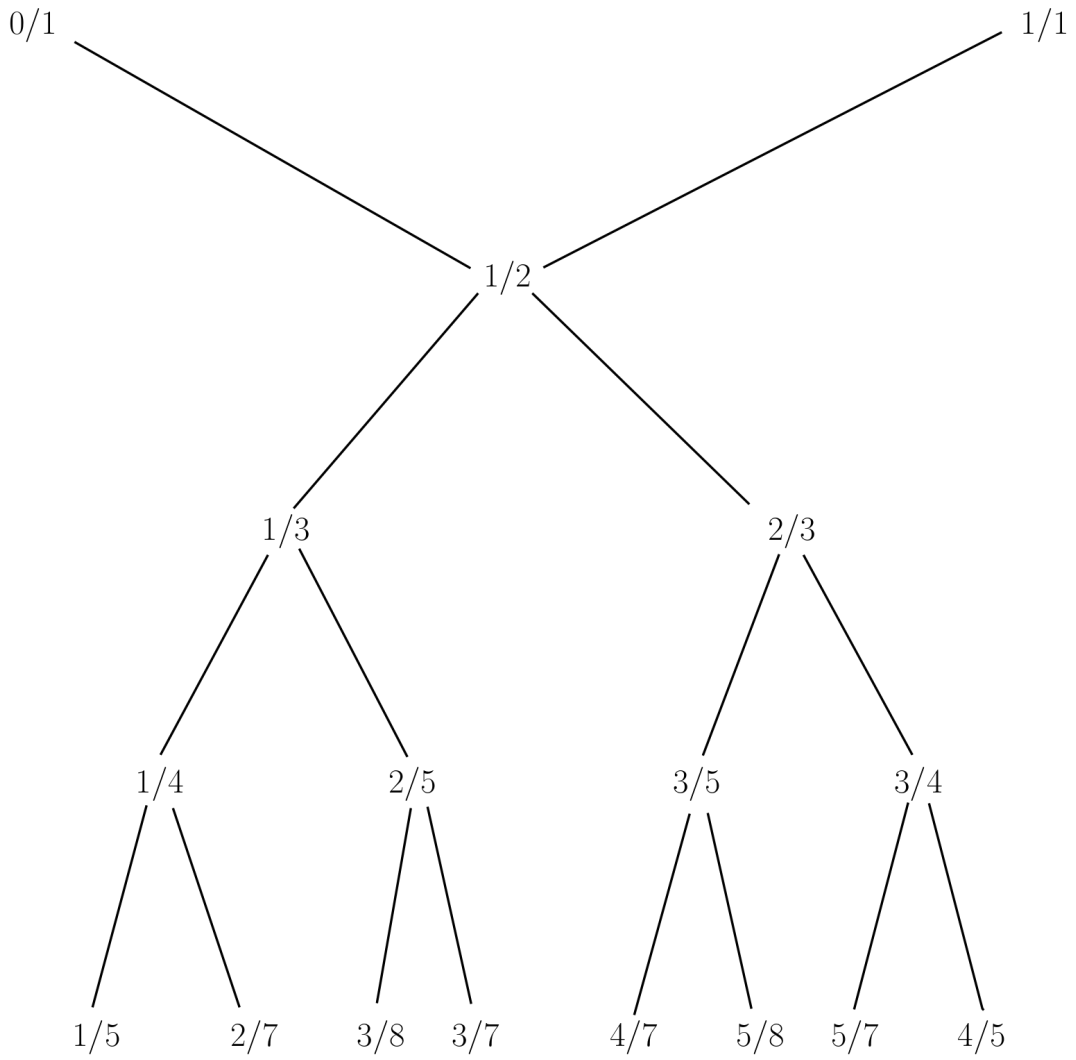

Figure 3.2: The Stern-Brocot tree

3.2 General knowledge about the Farey sequence

The Farey sequence finds applications to countless topics in mathematics, physics, engineering... We mention a few of those in this section, selected according to historical relevancy and personal preference.

3.2.1 Clockmaking

The Stern-Brocot tree was introduced independently by the German mathematician Mauritz Stern (1807-1894) in his work on number theory, and by the French clockmaker Achille Brocot (1817-1878) who used the tree as a tool for his craft. The latter is the first historical application of the Farey sequence outside of mathematics, and this short section aims to explain how it was done. We follow [Ain+12].

In a clock, some energy source (battery, spring, suspended weight...) rotates a shaft at a constant speed. This shaft transfers its movement to the hands of the clock through

gears that rotate at various speeds, depending on the hand they are associated to, to slow down the motion transferred to the hands from the shaft. These different gear speeds must then be precise multiples of one another.

For example, since a minute is divided into 60 seconds, on a clock the minute hand must rotate exactly 60 times slower than the second hand. Thus the gear responsible for the movement of the second hand must transfer 1/60-th of its speed to the gear moving the minute hand. One revolution of the minute gear corresponds to 60 revolutions of the second gear: we say that the system has a *gear ratio* of 1/60. How does one realize such a gear ratio?

The first solution is straightforward: make it so that the minute gear has 60 times more teeth than the second gear. For instance if the second gear has 10 teeth then the minute gear must have 600 teeth in order to obtain the desired gear ratio. There is an obvious problem with this approach: a 600 teeth gear is huge and does not fit inside a clock! The following picture (made using Gear Generator 2 Beta) gives an idea of the scale difference of those gears.

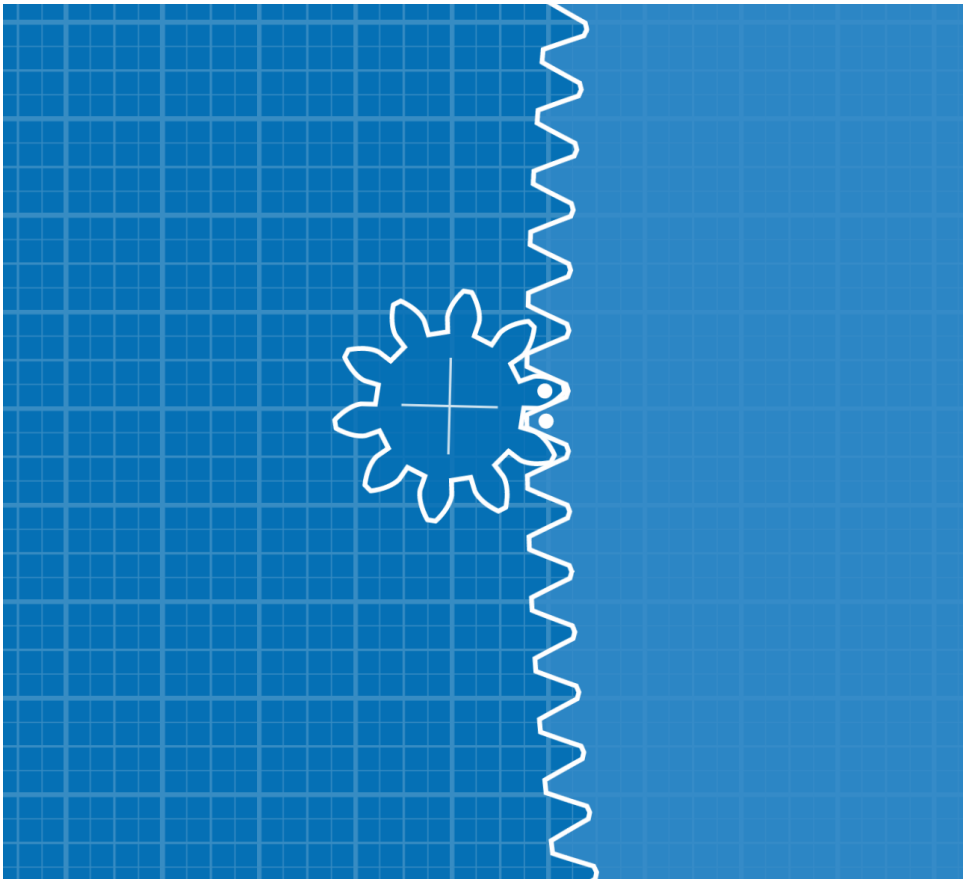

Figure 3.3: A 10 teeth gear and a 600 teeth gear for a gear ration of $\frac{1}{60}$

The second solution is to use gear trains. The idea is to use a chain of gears, with small gear ratio from one gear to another, so that all the gear ratios multiply to produce the desired gear ratio in the end. In order to prevent the gears to get too big, one can use multiple gears mounted on the same axle: then they must all rotate at the same

speed, and the connection to the rest of the gear chain is made *via* the extra gear that can be taken to be small.
Concretely, if the starting gear has 10 teeth again, we use a 60 teeth gear to realize a $1/6$ gear ratio. Then it transfers its speed to a smaller gear with 10 teeth mounted *on the same axle*, that in turn moves a 100 teeth gear for a $1/10$ gear ratio. This way, the final gear ratio is $1/6 \times 1/10 = 1/60$ as required, and the gears used are all relatively small. Of course, one can further reduce the size of the system using some more gear trains to realize the gear ratios $1/6$ and $1/10$ with even smaller gears.

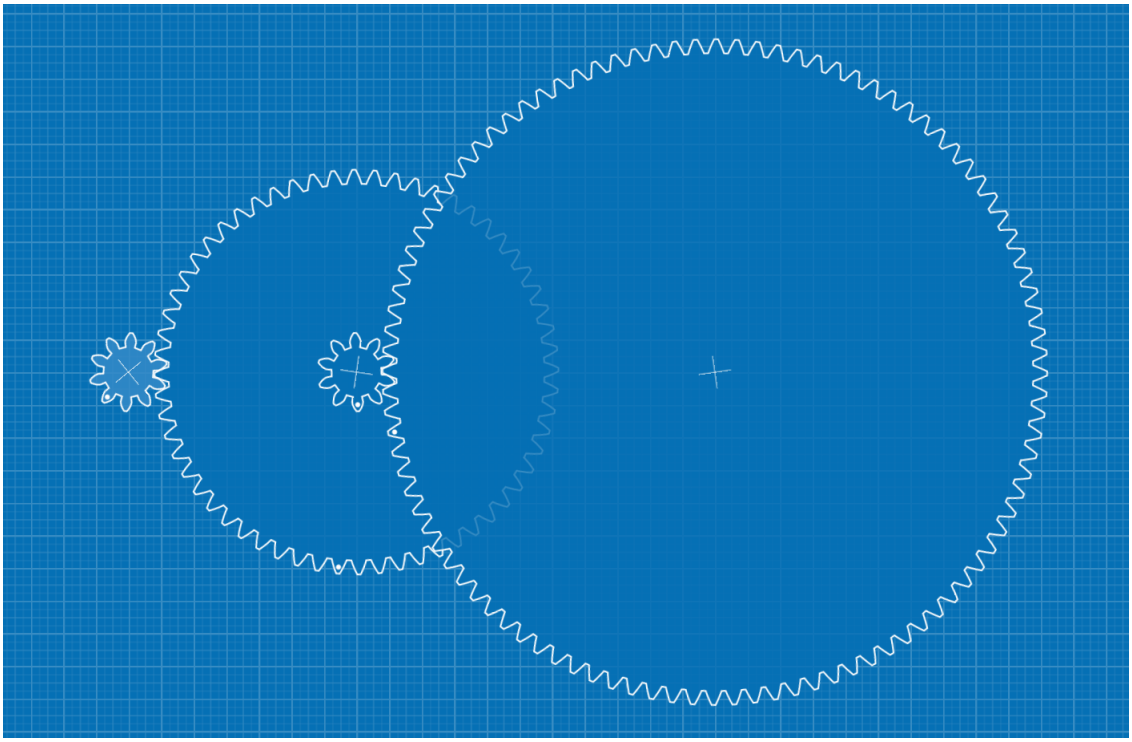

Figure 3.4: A gear train realizing the desired gear ratio $\frac{1}{60}$

Using this method, given a target gear ratio all one has to do is to factorize it and realize the smaller gear ratios of each of its factors. Gear trains work very well for the daily life clocks that mostly involve the numbers $60, 24$ and 12, all of those numbers having plenty of divisors (which is precisely the reason why we measure time in base 60 to begin with). However, Brocot was asked to make special clocks showcasing some astronomical indications, requiring to craft some unusual gear ratios such as $21/191$. This reveals one major flaw of the gear train approach: what happens if we want to realize a gear ratio p/q where p or q is *prime*?

This is where the Farey sequence becomes useful: there is no simple solution to realize the desired gear ratio, but we can approximate it. Brocot figured out in his paper [Bro61] he could use the Farey sequence to find excellent rational approximations of its target gear ratio that will not have prime factors, meaning they can be obtained using gear trains. The method Brocot used to approximate a gear ratio p/q is the following:

1. Locate p/q in the Stern-Brocot tree, which Brocot invented for this specific problem.

2. Simply read the Farey parents of p/q on the tree.

3. If one of the Farey parents of p/q does not have a prime numerator and denominator, use a gear train to realize this gear ratio, provided it satisfies some technical requirements such as manufacture constraints.

4. If both the Farey parents have a prime numerator or denominator, go up in the tree to the next Farey ancestors of p/q to see if they are suitable, etc.

While the idea of approximating the ratios was not new, the main mathematical tool clockmakers had at the time was the continued fraction expansion, which requires a substantial amount of computations and was impractical (of course at the time all computations had to be done by hand). Moreover, most clockmakers lacked proper mathematical education, so using advanced arithmetic was unrealistic for them. The point of Brocot's method is that the only pre-requisite is to have the Stern-Brocot tree already written somewhere: Brocot produced tables that all clockmakers could easily use. Even without the table, constructing the tree is extremely easy as the only operations involved are sums, as taking the mediant of two fractions simply comes down to summing their numerators (resp. denominators) together. Thus, thanks to this new method, approximating fractions became considerably faster for the 19th century clockmakers.

How good is the approximation? Another upside of Brocot's method is that the approximation error is very easy to compute. Brocot remarked that if $a/b < p/q < c/d$ (not necessarily consecutive in a Farey sequence of any order) then

$$\frac{a+c}{b+d} - \frac{p}{q} = \frac{(aq-pb)+(cq-pd)}{qb+qd}$$

If other words, the error when using $a/b \oplus c/d$, the mediant of a/b and c/d, to approximate p/q is itself the mediant of the errors made by approximating p/q by a/b and by c/d respectively. Thanks to this formula, it becomes very easy to compute the error recursively by going up in the Stern-Brocot tree.

A concrete example of the strength of this method is the following, which was given as an exercise for clockmakers students after Brocot's method was published. The students were asked to design a system, bound to the hour gear, which completes one revolution in a mean year and whose gear ratio was $720/525949$. This is a problem using traditional gear trains as 525949 is prime, so it is necessary to approximate using Brocot's method. The Farey parents of this ratio are not helpful as they both have prime denominators aswell, but one the Farey grandparents can be factorized nicely: indeed, the number $196/143175$ is equal to $2/3 \times 2/25 \times 7/23 \times 7/83$. Using Brocot's error formula, the error is

$$E = \frac{143175 \times 720 - 196 \times 525949}{196}$$

that is $\frac{4}{196}$-th of an hour every time the system completes a full rotation. In the end, the whole system is too fast by slightly more than *a second per year*. Not bad.

3.2.2 The Ford circle packing

The Farey sequence can be found in a special case of circle packing problems. Such a problem is generally stated in the following way: given n mutually tangent circles of radii $r_1, ..., r_n$, what is the biggest possible radius r of a circle that fits into the gap between those circles?

One of the simplest possible versions of circle packing problems is the case $n = 3$, with two $r_1 = r_2$ and $r_3 = \infty$ so that the first two circles are identical and the third circle is pictured as the real line. Say the first (resp. second) circle has center $(0, 1/2)$ (resp. $(1, 1/2)$) and radius $r_1 = 1/2$ so that is is tangent to the real line at 0 (resp. at 1). Then the biggest circle that fits between these two circles is tangent to the horizontal at $x = 1/2$, which is the mediant of $0 = 0/1$ and $1 = 1/1$.

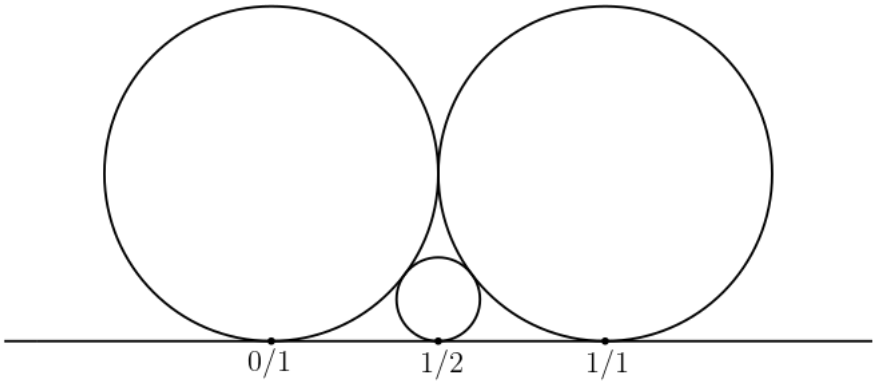

Figure 3.5: The biggest circle that fits in the gap

Now, one may ask what is the biggest circle that fits between one of the big circles and the new smaller one. We obtain two new circles, with which we can then ask the same question again. Repeating this process, we finally obtain a packing of infinitely many circles called the Ford packing.

One can show that each circle of the packing is tangent to the real line at a rational point. Conversely, for every rational p/q with p and q coprime there exists a unique circle of the Ford packing that is tangent to the real line at p/q; we denote $C(p/q)$ this circle. Direct computations show that the center of $C(p/q)$ is the point $(p/q, 1/2q^2)$ and its radius is equal to $1/2q^2$.

Proposition 3.2.1 (Tangency of circles)**.** *The circles $C(p/q)$ and $C(p'/q')$ are tangent if and only if p/q and p'/q' are a Farey pair, i.e they are neighbour in some Farey sequence.*

Thus this circle packing encodes the Farey sequence; in fact it can be seen as dual to the Farey graph. As a nice feature of the Ford packing, if two circles of different size are tangent, then the rational associated to the bigger one is the Farey parent of the rational associated to the smaller one.

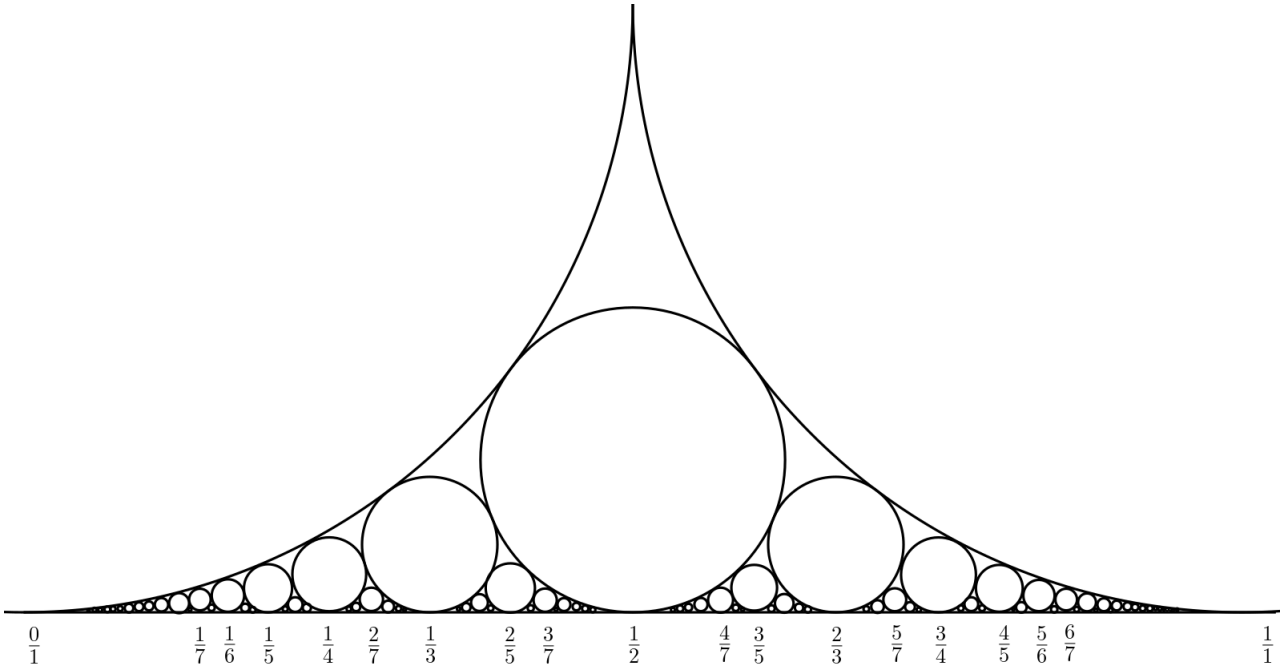

Figure 3.6: The Ford circle packing

Proposition 3.2.2 (Mediant property of the circles)**.** *The only circle in the packing that is tangent to both $C(p/q)$ and $C(p'/q')$ is the circle $C((p+p')/(q+q'))$.*

The Ford circle packing features many algebraic and number theoretic properties that we will not state here. We will see that it is linked to hyperbolic geometry in the next section.

3.2.3 Tessellation of the hyperbolic plane

The Farey sequence also appears in hyperbolic geometry, mainly through a tessellation of the hyperbolic plane called the Farey tessellation.
Denote by $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$ the Poincaré half-plane equipped with the hyperbolic metric $ds^2 = \dfrac{|dz|^2}{\Im(z)^2}$. The geodesics of $\mathbb{H}$ are the vertical lines and the half-circles orthogonal to $\mathbb{R}$: we denote a geodesic with a pair $(x,y) \in \mathbb{R} \cup \{\infty\}$ representing its endpoints. The isometries of $\mathbb{H}$ are identified with the group $SL(2,\mathbb{R})$, with the action

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.z = \frac{az+c}{bz+d}$$

We claim that the $SL(2,\mathbb{Z})$-orbit of the vertical geodesic $(0,\infty)$ triangulates $\mathbb{H}$. Recall that $SL(2,\mathbb{Z})$ is generated by the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Notice that the geodesic $(0,\infty)$ along with the geodesics

$$T.(0,\infty) = (1,\infty)$$

and

$$S^3 T^{-1} S.(0,\infty) = (0,1)$$

form an ideal hyperbolic triangle of area π.

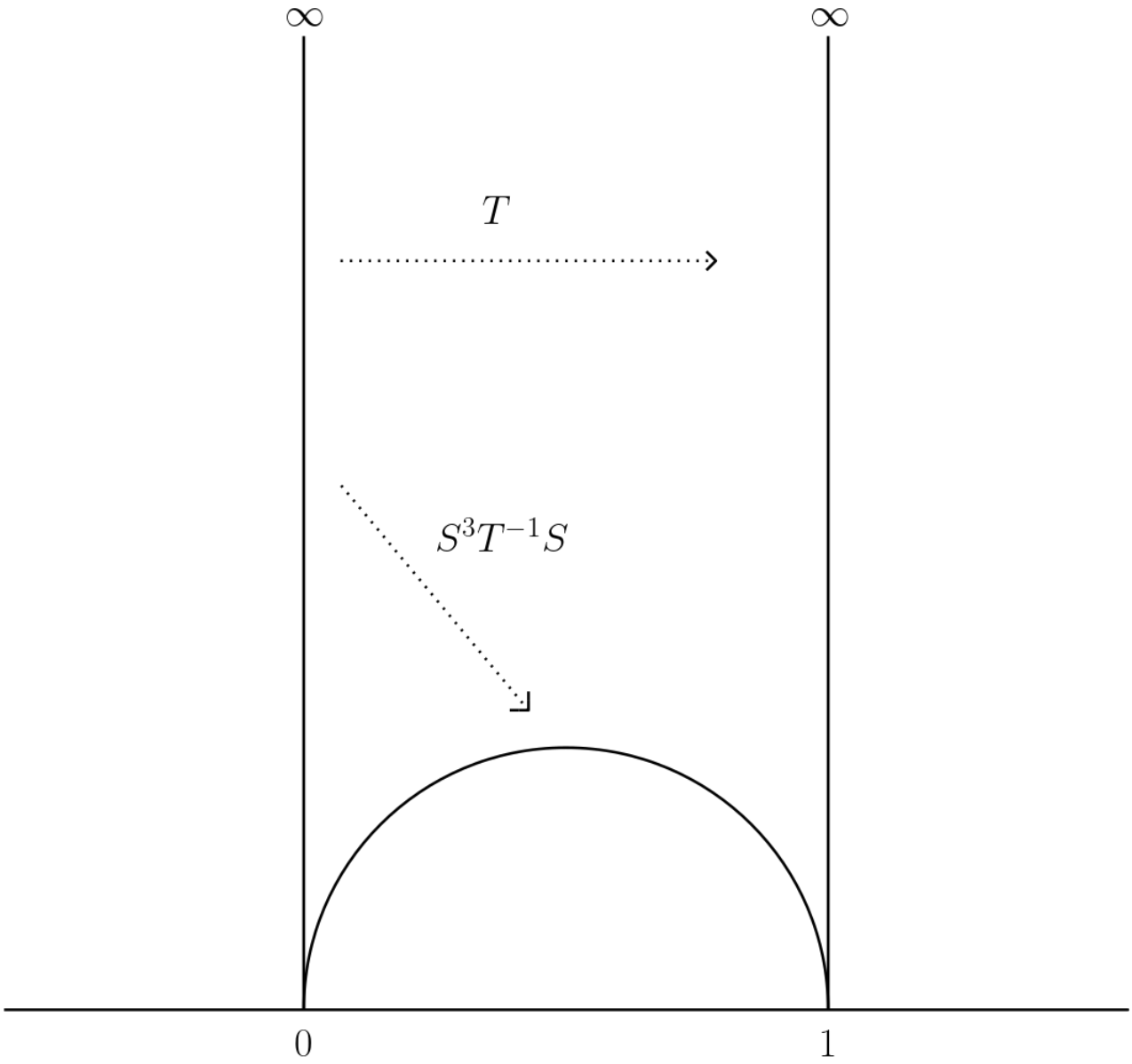

Figure 3.7: Hyperbolic triangle of area π

The $SL(2,\mathbb{Z})$-orbit of this triangle tessellates the hyperbolic plane. To find out what the edges of this tessellation are, notice that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.\infty = \frac{a}{b}$$

and

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.0 = \frac{c}{d}$$

Thus the geodesic $(a/b, c/d)$ is an edge of the Farey tessellation if and only if $ad-bc = 1$, i.e if and only if a/b and c/d are Farey neighbours. This observation allows us to draw pictures of this tessellation.

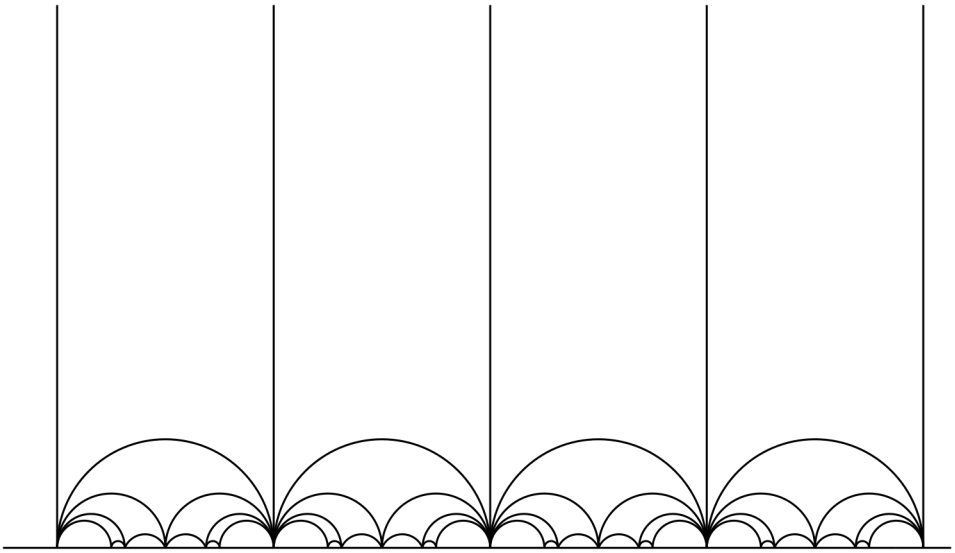

Figure 3.8: The Farey tessellation of the hyperbolic plane

All the ideal triangles of the tessellation have the same hyperbolic area π.

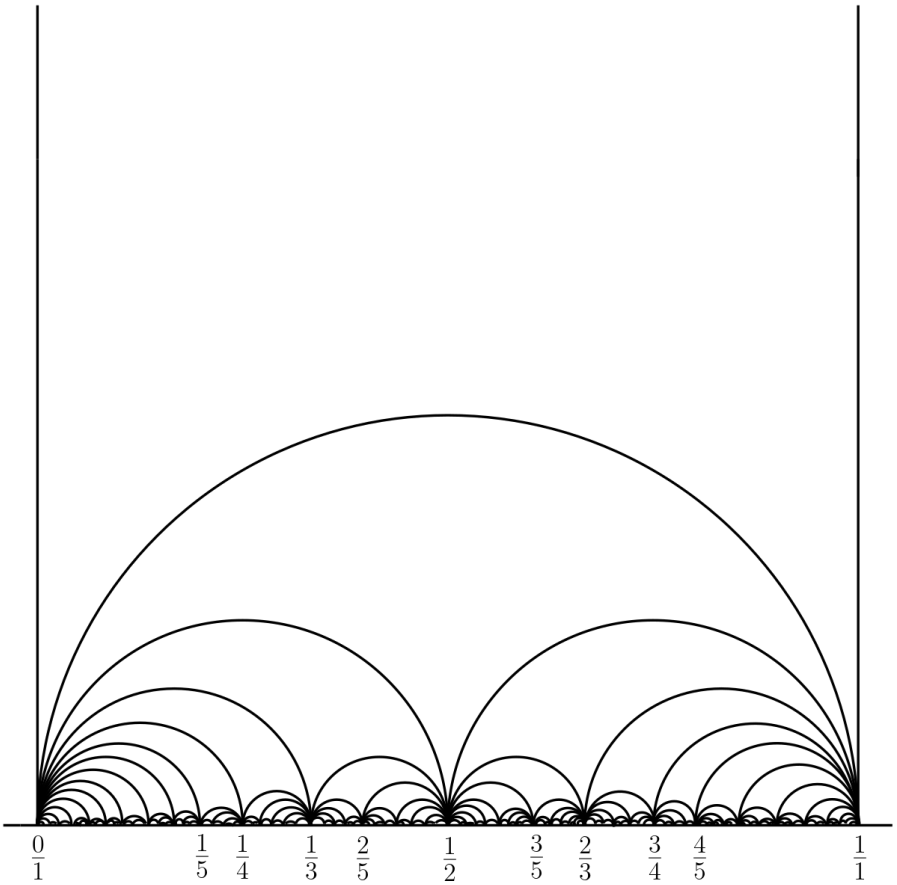

Figure 3.9: Close-up view of the Farey tessellation

The edges of the Farey tessellation clearly are an embedding of the Farey graph in the hyperbolic plane. This picture can in turn help better understand the Farey sequence. Indeed, it suggests to see ∞ as part of the Farey sequence, which is neighbour

to all integers as all the geodesics (k,∞) with $k \in \mathbb{Z}$ belong to the tessellation. It is tempting to see ∞ as $1/0$: thus, since $1/0$, $k/1$ with $k \in \mathbb{Z}$ and $(k+1)/1$ are neighbours and

$$\frac{1}{0} \oplus \frac{k}{1} = \frac{k+1}{1}$$

we can define Farey parents for integers, the Farey parents of $(k+1)/1$ being $k/1$ and $1/0$. This point of view has some merit as it nicely extends to the integers some of the combinatorial relations related to Farey parents; moreover the Farey graph/tree now has a single root (the point at $\infty = 1/0$) instead of infinitely many (all the integers). However this definition is not canonical as one could instead define ∞ as $-1/0$. This time, the natural Farey parents of $k/1$ are $-1/0$ and $(k+1)/1$, reversing the order of the previous parenting relation.

There is a third possibility when dealing with ∞: seeing it as *both* $1/0$ and $-1/0$, as we will see that it makes more sense geometrically. We can see an irreducible fraction q/p as a primitive $\mathbb{Z}^2$-vector (p,q) of the plane: the slope of the line directed by this vector is q/p. Note that by convention an irreducible fraction always have a positive denominator. Now, given another primitive vector $(m,n) \in \mathbb{Z}^2$, the sum $(p,q)+(m,n) = (p+m, n+q)$ has slope $(n+q)/(m+p)$ which is precisely the mediant of the slopes q/p and n/m. To put it in an informal way, the sum of two directions is their mediant; this observation will be very important in the next chapter. We then define the Farey parents of the vector (m,n) to be the vectors (b,a) and (d,c) where a/b and c/d are the Farey parents of n/m.
According to this definition, every vector (m,n) in with $m,n \geq 0$ can be obtained by taking successive sums of the vectors $(1,0)$ and $(0,1)$, implying that $0/1$ and $1/0$ are the Farey ancestors of all positive rationals. Thus when dealing with positive rationals one must see ∞ as being $1/0$. Conversely, every vector (m,n) with $m \geq 0$ and $n \leq 0$ can be obtained by summing $(1,0)$ and $(0,-1)$, so $0/1$ and $-1/0$ are the Farey ancestors of all negative rationals and we are right to see ∞ as $-1/0$ when dealing with negative rationals. Thus $0 = 0/1$ and $\infty = 1/0 = -1/0$ are the Farey ancestors of all rationals; with this point of view, the Farey graph/tree has two roots.

Another argument in favor of this point of view involves some more hyperbolic geometry. A *horocycle* is a curve all of whose normal curves converge to the same point at infinity; in $\mathbb{H}$ the horocycles are circles tangent to $\mathbb{R}$. Given a tiling of $\mathbb{H}$ obtained by acting on a given polygon by an isometry group G one can define a unique associated collection of horocycles. Each vertex τ of the tiling is associated to a horocycle $C(\tau)$, with the additional condition that if there exists an isometry $\phi \in G$ that sends a vertex τ to another vertex τ' then $\phi(C(\tau)) = C(\tau')$. The Farey tessellation is obtained by taking the $SL(2,\mathbb{Z})$-orbit of a triangle; the associated horocycle collection provides a familiar picture.

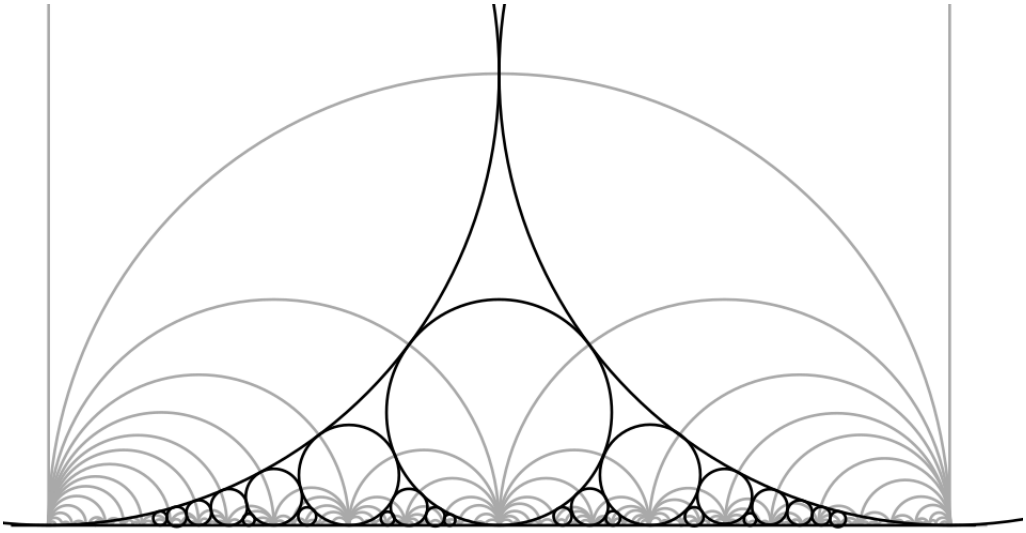

Figure 3.10: The Ford packing and the Farey tessellation

Sending this horocycle collection isometrically into the hyperbolic disk we get:

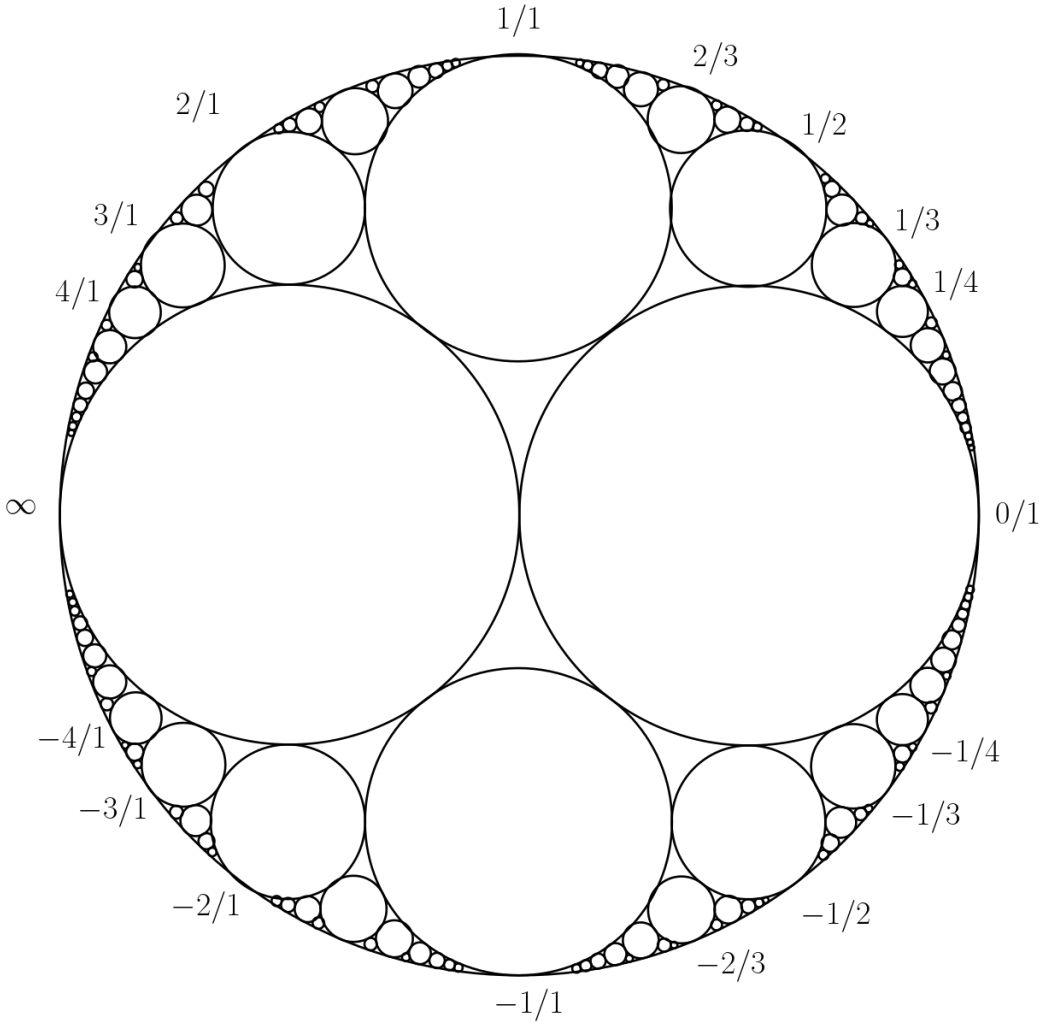

Figure 3.11: The Ford packing in the hyperbolic disk

The fractions on the above picture indicate which point of the boundary of $\mathbb{H}$ were sent to this point of the circle. Now, remember that in the Ford circle packing, if a circle is tangent to a bigger circle then the rational associated to the big circle is the Farey parent of the rational associated to the small circle. Here in our picture the two biggest circles are associated to $0/1$ and ∞, which reinforces the idea that they are the Farey ancestors of all other rationals. Moreover, the circles associated to ∞ and $0/1$ have exactly the same size, so neither one is the Farey parent of the other. Finally, by the Mediant Property of the Ford packing (Proposition 3.2.2), the only circle that is tangent to both $C(p/q)$ (the circle associated to the rational p/q) and $C(p'/q')$ is the circle $C(p/q \oplus p'/q')$. Since the circles associated to positive integers are tangent to $C(\infty)$, by the mediant rule we deduce that ∞ is represented by $1/0$ in that case. Conversely for negative integers, we find that ∞ is this time viewed as $-1/0$.

For all these reasons we conclude this chapter by expressing the opinion that Figure 3.11 is the most effective, most natural and last but not least the most aesthetic way to picture the Farey sequence.

4 The stable norm of flat slit tori

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The problem that initially motivated the present thesis was to explicitly compute the stable norm on some translation surfaces. Since the dimension of the first real homology group of a surface is twice its genus g, considering surfaces with a small genus is a natural way to reduce the complexity of the problem. The question has already been answered in the case $g = 1$ of flat tori; thus, the next simplest genus to look at is $g = 2$. Even in this case, computing the stable norm of a given genus 2 translation surface is very difficult; too difficult, in fact, to be approached in a direct way. Thus, we need to further simplify the problem.

The world of translation surfaces can be thought of as two distinct main topics: surfaces of genus 2 and surfaces of genus $g \geq 3$ (the case of genus 1 surfaces is trivial). Indeed, numerous powerful theorem that hold for genus 2 surfaces completely fail for surfaces of genus $g \geq 3$. We have already seen an example of such a result in Theorem 2.5.5; the one we are now interested in is the following, about cutting surfaces into slit tori. A *slit torus* is a flat torus on which a slit has been made, so that its boundary is made of two parallel line segments.

Theorem 4.0.1 (Proposition 1.16 [Wri14]). *Every translation surface in $\mathcal{H}(1,1)$ can be obtained by glueing two slit tori together along their slits. Every translation surface in $\mathcal{H}(2)$ can be obtained by glueing a flat cylinder along the slit of a slit torus.*

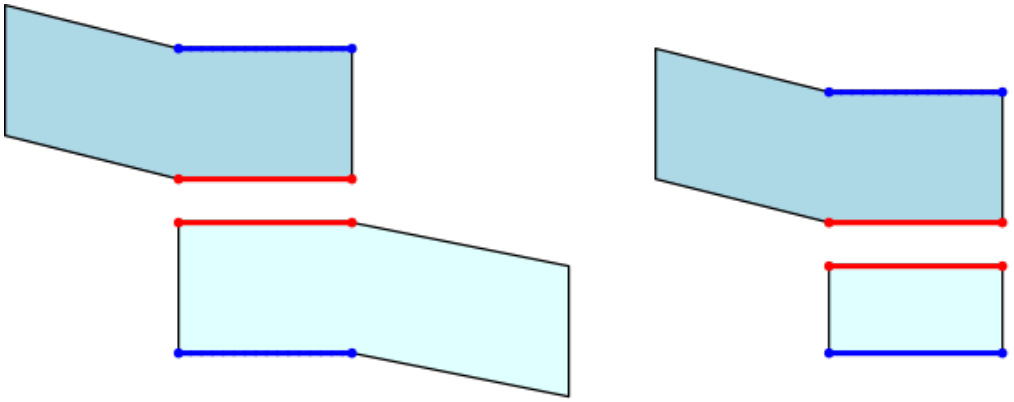

Figure 4.1: A surface of $\mathcal{H}(1,1)$ (left) and a surface of $\mathcal{H}(2)$ (right)

The slit tori appear to be the elementary building blocks of the genus 2 translation surfaces. Thus, a natural first step towards computing the stable norm of translation surfaces is to compute the stable norm of slit tori: this is what is done in this chapter, and it constitutes the main body of this thesis.

4.1 Stable norm of the square slit torus with a vertical slit

The aim of this chapter is to provide explicit computations of the stable norm of general flat slit tori, but the general case can be derived from the special case of flat tori whose metric come from a square, and with the additional assumption that the slit is vertical. Thus we will first compute the stable norm in this setting, as it involves simpler notations and computations.

4.1.1 Preliminaries

In this section we introduce the notations we will need, along with reformulating the problem in a way that makes it easier to solve.

The slit torus. A flat slit torus is a compact surface that is homeomorphic to a torus with one boundary component, equipped with a flat Riemannian metric relative to which the boundary of the surface is made of two parallel straight line segments of same length ρ. Formally, the boundary of the surface can be written as a union $S_1 \cup S_2$, where S_1 and S_2 are homeomorphic to a segment and $S_1 \cap S_2$ is equal to two distinct points $\{P, P'\}$, such that there exists two local charts φ_1 and φ_2 that map isometrically S_1 and S_2 to the *same* segment $[(0,0),(0,\rho)] \subset \mathbb{R}^2$ of length ρ. More precisely, we have $\varphi_1(S_1) = \varphi_2(S_2) = [(0,0),(0,\rho)]$, with $\varphi_1(P) = \varphi_2(P) = (0,0)$ and $\varphi_1(P') = \varphi_2(P') = (0,\rho)$. The boundary of the surface is called the slit, the two segments S_1, S_2 that compose the slit are called the sides of the slit and their common endpoints P, P' are the endpoints

of the slit.

Throughout this section we will be interested in *the* flat slit torus, denoted X, whose metric comes from a *square* of area 1 (i.e $X\backslash\{S_1, S_2\}$ is isometric to an open subset of a flat square torus) and whose slit of length $0 < \rho < 1$ is *vertical*, as these assumptions make the computations much easier. We fix an orientation on X, so that the notions of left and right side of the slit and the notions of upper and lower endpoint of the slit are no longer ambiguous. Informally, the surface X can be visualized with the following polygonal model. Take a square of area 1 and cut it open along a vertical open interval of length ρ, then glue the opposites edges of the square by translation. Finally glue one segment to each side of the cut part, so that the final surface is a compact genus 1 surface with boundary homeomorphic to a circle. The metric on the surface is then simply coming from the canonical Euclidean metric on the square.

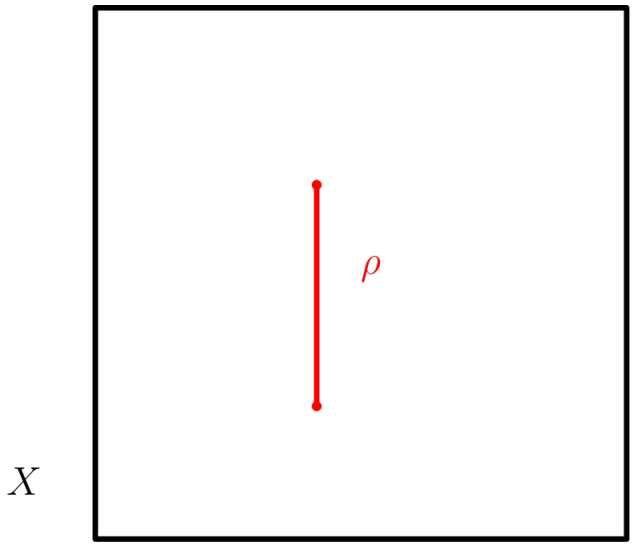

Figure 4.2: The square torus with a vertical slit

The slit torus is homotopy equivalent to the wedge sum of two circle, so for $x \in X$ we have

$$\pi_1(X, x) \simeq \mathbb{F}_2$$

where $\mathbb{F}_2$ is the free group on two elements. Moreover, we have

$$H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^2$$

so from now on we will identify integral homology classes $h \in H_1(X, \mathbb{Z})$ on X with pairs of integers $(m, n) \in \mathbb{Z}^2$.

The Abelian covering space of the slit torus. Let $p : \tilde{X} \longrightarrow X$ be the Abelian covering space of the slit torus: this is the covering space of X whose transformation group is isomorphic to $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^2$. We fix a flat metric on the space $\tilde{X}$ so that the covering map p is a local isometry.

One convenient way to picture $\tilde{X}$ is the Euclidean plane from which cut along all $\mathbb{Z}^2$-translate of some vertical segment (i.e we make a slit along those segments). The

choice of this segment does not matter, so in order to simplify future notations we choose the vertical segment $[(0,0),(0,\rho)]$. This segment and all its $\mathbb{Z}^2$-translate are the copies of the slit, that we will also call the slits. The choice we made ensures the lower endpoint of the slits are the integer points of the plane.

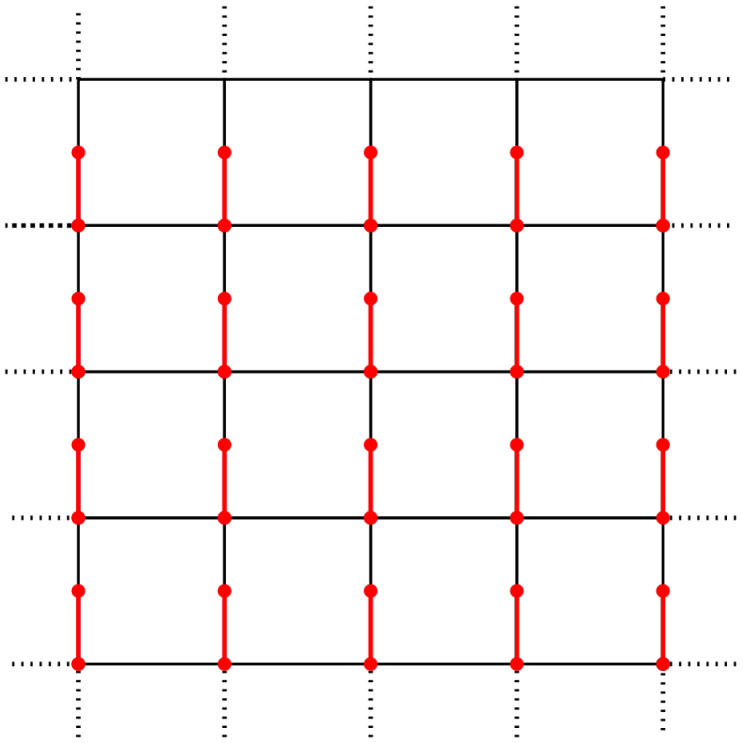

Figure 4.3: The Abelian covering space $\tilde{X}$

A more practical way to compute the stable norm of X. Since the slit torus has non-empty boundary, we cannot apply Theorem 1.1.3 to express its stable norm. Instead we have to rely on the general definition (see 1.1.1) and go through the stabilization process: if $h \in H_1(X,\mathbb{Z})$ is an integral homology class on X, the stable norm $\|h\|_X$ is the limit of the sequence $f(Nh)/N$ as N goes to infinity, where

$$f(h) = \inf_{\gamma}\{l(\gamma), \text{ where } \gamma \text{ is a closed multicurve such that } [\gamma] = h\}$$

However, in our setting a miracle occurs:

Proposition 4.1.1. *Let $h \in H_1(X,\mathbb{Z})$ be an integral homology class of the slit torus X. Then the sequence $\big(f(Nh)/N\big)_{N\in\mathbb{N}^*}$ is constant.*

First we make an observation: given an integral homology class $h = (m,n)$, when computing $f(Nh)$ we can only look for closed curves on X passing through the lower endpoint of the slit. Indeed, if γ is a closed (multi)curve on X that represents the homology class Nh that do not already passes through the lower endpoint of the slit, then we can translate (each connected component of) γ upwards until it encounters the lower end of the slit. Since translations preserve the metric on X and act trivially on $H_1(X,\mathbb{Z})$, we obtain a new curve γ' that has the same length and represents the same homology class as γ. The curve γ' can be lifted isometrically to the Abelian cover $\tilde{X}$ in a

unique way as a path from the origin $(0,0)$ to the point (Nm,Nn). Thus, $f(Nh)$ is equal to the infimum of the lengths of paths from the origin in $\tilde{X}$ to (Nm,Nn).

Proof. Let $h=(m,n)\in H_1(X,\mathbb{Z})$ be an integral homology class of X and let $N\in\mathbb{N}^*$ be a positive integer.

We reproduce an old argument from Hedlund [Hed32], originally stated on the torus. Let γ be a rectifiable path on $\tilde{X}$ from $(0,0)$ to (Nm,Nn). We will show by induction on N that γ projects onto X as the union of N closed curves representing h, thus $f(Nh)\geq Nf(h)$, and since the converse inequality is trivial we will finally get $f(Nh)=Nf(h)$. To do this, we apply the following algorithm.

Let Δ be the line from the origin to (m,n) in $\tilde{X}$; note that it might encounter the slit. Let Δ_l (resp. Δ_r) the leftmost (resp. rightmost) parallel to Δ which meets γ. If $\Delta_l=\Delta=\Delta_r$ then $\gamma\subset\Delta$ and there is nothing left to prove, so let us assume that $\Delta_l\neq\Delta_r$. Let $U\in\Delta_l\cap\gamma$, and let $U'=T(m,n).U$, where $T(m,n)$ is the translation by the (m,n) vector.

If $U'\in\gamma$ then we are done: since U and U' project onto the same point in X, the arc of γ from U to U' projects onto a closed curve in X representing $h=(m,n)$. We then remove this arc from γ to obtain two subpaths of γ. By concatenation of the subpath of γ from $(0,0)$ to U with the subpath from U' to (Nm,Nn) translated by $T(-m,-n)$ we obtain a new (and shorter) path from $(0,0)$ to $((N-1)m,(N-1)n)$, and we conclude by induction.

If U' does not belong to γ, then U' lies on the boundary of some connected compact region δ, bounded by Δ_l, γ and the line orthogonal to Δ_l passing through (Nm,Nn). Let $V\in\Delta_r\cap\gamma$ and assume that $V'=T(m,n).V$ does not belong to γ (just like before, if $V'\in\gamma$ then we are done). See figure below. By construction, V' lies in the exterior of δ: the only way it could belong to δ would be if he lied on γ, which we assumed is not the case. Consider the path $\gamma'=T(m,n)\gamma$: it is well-defined because γ does not encounter the slit as the copies of the slit are all related by integral translations. By construction, the path γ' passes through U' and V', one in the interior of δ and the other on its exterior: since γ' is continuous, it must cross transversally the boundary of δ at some point P'. Since γ' cannot intersect transversally the line Δ_l, the point P' must lie on γ instead. Thus, the arc of γ from $P:=T(-m,-n)P'$ to P' projects onto a closed curve on C representing h. We then refer to the previous case: as before, we remove this arc from γ to get by concatenation a shorter path from $(0,0)$ to $((N-1)m,(N-1)n)$ and we conclude by induction.

This process is repeated N times until we have decomposed γ as N arcs, each projecting onto representatives of h in X.

□

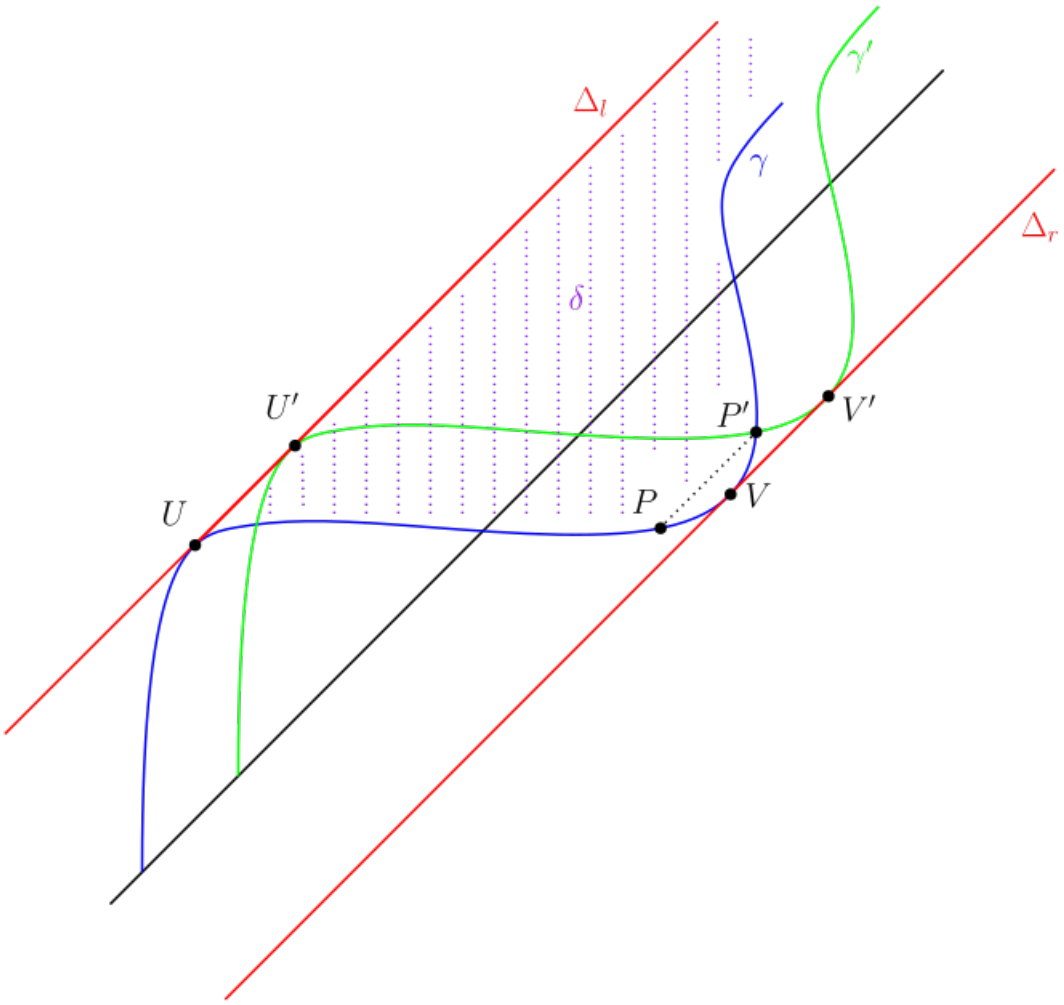

Figure 4.4: The geometric construction from the proof

Since the slit torus is compact Proposition 1.1.4 still holds in our setting, so the stable norm of a homology class is realized by a *minimizing* curve.

Corollary 4.1.1.1. *Let $h \in H_1(X,\mathbb{Z})$. Then $\|h\|_X = \min\{l(\gamma),\ \gamma$ closed curve on X that represents $h\}$.*

Since a simple closed curve on X and its lift to $\tilde{X}$ have the same length, computing $\|(m,n)\|_X$ comes down to finding the shortest path in $\tilde{X}$ from the origin to (m,n) that avoids the slits.

Reformulation of the problem. Our goal is to compute the stable norm of any homology class on the slit torus X, but we do not need to do it for every class as most of them provide redundant information.

It is sufficient to compute the stable norm of *integral* homology classes. Indeed, if $h \in H_1(X,\mathbb{Z})$ is an integral homology class then by homogeneity of the norm for any rational number $p/q \in \mathbb{Q}$ we have

$$\left\| \frac{p}{q} h \right\|_X = \left| \frac{p}{q} \right| \|h\|_X$$

so we can compute the stable norm on $H_1(X,\mathbb{Q})$ provided that we already know how to compute it on $H_1(X,\mathbb{Z})$. Then, by density of $H_1(X,\mathbb{Q})$ inside $H_1(X,\mathbb{R})$, the stable norm is completely determined by its values on rational classes. In general, establishing a tricky formula on the rational homology classes does not guarantee that we can provide explicit formulas for the stable norm of irrational homology classes. In our case however we will be able to compute explicitly the stable norm of *any* homology class, although it requires tedious computations (that can be automated).

In fact we can further reduce the set of homology classes on which we need to compute the stable norm, as we only need to consider *primitive* integral homology classes, that is integral classes (m,n) such that m and n are coprime. This is obvious since by homogeneity of the norm, for $k \in \mathbb{Z}$ we have $\|(km,kn)\|_X = |k|\|(m,n)\|_X$.

Finally, we can also assume without loss of generality that $m,n \geq 0$. Indeed, by homogeneity $\|(m,n)\|_X = \|(-m,-n)\|_X$ so the stable norm is centrally symmetric with respect to the origin in $H_1(X,\mathbb{Z})$. Moreover, since the metric on the slit torus X (and on its Abelian covering space $\tilde{X}$) is symmetric with respect to vertical reflections we have $\|(m,n)\|_X = \|(-m,n)\|_X$, so the stable norm is symmetric with respect to the vertical axis in $H_1(X,\mathbb{Z})$. Composing those two symmetries we finally obtain $\|(\pm m,\pm n)\|_X = \|(m,n)\|_X$, so up to replacing m (resp. n) by $-m$ (resp. $-n$) we can always assume $m,n \geq 0$.

Thus in order to compute the stable norm of the slit torus X we need to solve the following problem:

Given a primitive integral homology class (m,n) *with* $m,n \geq 0$*, compute* $\|(m,n)\|_X$.

According to the previous part, this comes down to finding the length of the shortest closed curve on X passing through the lower endpoint of the slits that represents the class (m,n). This in turn is equivalent to finding the shortest path from $(0,0)$ to (m,n) in the space $\tilde{X}$.

4.1.2 Visible homology classes

Let $h = (m,n) \in H_1(X,\mathbb{Z})$ be an integral homology class on X and assume that h is primitive, i.e $gcd(m,n) = 1$. Moreover assume $m,n \geq 0$. We want to compute its stable norm $\|h\|_X$: this comes down to finding the length of the shortest path from $(0,0)$ to (m,n) in the Abelian cover $\tilde{X}$. The obvious candidate for being such a path is the straight line segment $[(0,0),(m,n)]$: if it does not intersect the slit we are done. Hence we need to check whether or not it intersects the slit.

If $(m,n) = (0,1)$ then this is trivial. Indeed, since the slit is vertical the geodesics representing $(0,1)$ are parallel to the slit, so they never intersect it. Thus any vertical geodesic on X is a minimizing representative of $(0,1)$, so $\|(0,1)\|_X = 1$. With this special case out of the way, from now on we will assume $m > 0$.

The segment $[(0,0),(m,n)]$ satisfies the equation $y = \frac{n}{m}x$, with $x \in [0,m]$. It crosses the vertical of the slits at points $\left(k, k\frac{n}{m}\right)$, $k = 1,...,m-1$: those are the points where an intersection with the slit may occur. Recall that the slit has length ρ. Since the slits are the open segments $\{(p,q+s) \text{ with } s \in]0,\rho[\}$ with $(p,q) \in \mathbb{Z}^2$, we have to check how the fractional part $\left\{k\frac{n}{m}\right\}$ of $k\frac{n}{m}$ compares with ρ for all $k = 1,...,m-1$. More precisely, there is an intersection with the slit at the vertical $x = k$ if and only if

$$\left\{k\frac{n}{m}\right\} < \rho$$

Remark that $\left\{k\frac{n}{m}\right\} = \frac{r}{m}$, where r is the remainder in the Euclidean division of kn by m, i.e $kn = r$ in $\mathbb{Z}/m\mathbb{Z}$. Since m and n are coprime n is invertible in $\mathbb{Z}/m\mathbb{Z}$ and multiplication by n is an automorphism of $\mathbb{Z}/m\mathbb{Z}$, so r takes all the possible values $1,2,...,m-1$ when k varies from 1 to $m-1$. Hence, the smallest value $\left\{k\frac{n}{m}\right\}$ takes is $\frac{1}{m}$. Now,

$$\frac{1}{m} < \rho \iff m\rho > 1$$

So if $m\rho \leq 1$, the segment $[(0,0),(m,n)]$ does not encounter the slit and projects onto a minimizing representative of (m,n) on X of length $\sqrt{m^2+n^2}$. We assumed at the beginning that m was positive: if instead we had assumed $m < 0$ the condition would become $-m\rho \leq 1$. Thus, we have shown:

Proposition 4.1.2. *Let $h = (m,n) \in H_1(X,\mathbb{Z})$ be a primitive integral homology class. Let ρ be the length of the slit. The stable norm coincides with the Euclidean norm in the direction (m,n) if and only if $|m|\rho \leq 1$. Moreover, if $|m|\rho > 1$ then $\|(m,n)\|_X > \|(m,n)\|_2$.*

Geometrically, the previous proposition simply says that the segment from the origin to $(1,n)$ can never intersect any of the slits, as all the slits are contained in the vertical lines $x = k$ with $k \in \mathbb{Z}$ and the segment never crosses such a line.

By the previous proposition, if $|m|\rho \leq 1$ then a light ray emitted from $(0,0)$ and travelling in a straight line would illuminate the point (m,n) on $\tilde{X}$. Hence the following terminology, borrowed from [HW08]: we say that primitive integral homology class (m,n) is *visible* if $|m|\rho \leq 1$, or equivalently if its stable norm coincides with the Euclidean norm. A visible direction in $H_1(X,\mathbb{R})$ is the direction of a visible homology class. Note that a primitive homology class (m,n) with $m \neq 0$ is visible if and only if n/m belongs to $\mathscr{F}_L$ the Farey sequence of order L, where $L = \lfloor 1/\rho \rfloor$ is the integral part of $1/\rho$. Note that the class $(0,1)$ is visible.

Corollary 4.1.2.1. *Let $\mathscr{B}$ be the unit ball of the stable norm of X and let $\mathbb{D}$ be the unit Euclidean disk in $\mathbb{R}^2$, with boundary $\mathbb{S}^1$. Then $\mathscr{B} \subset \mathbb{D}$ and $\partial\mathscr{B} \cap \mathbb{S}^1$ is non empty, with intersections occurring exactly in the visible directions.*

Proof. The stable norm coincides with the Euclidean norm exactly in the visible directions, so they unit sphere intersect in those directions. In other directions the stable norm is strictly greater than the Euclidean norm, so the point of $\partial\mathscr{B}$ in this direction lies in the interior of the unit disk $\mathbb{D}$. □

Proposition 4.1.3. *There are infinitely many visible directions.*

Proof. The visible directions are the directions of the classes (m,n) with $n/m \in \mathscr{F}_L$. Since $\rho < 1$ the classes of the form $(1,n)$ are always visible, whatever value ρ may take. □

4.1.3 Short non-simple curves

Let $h = (m,n) \in H_1(X,\mathbb{Z})$ be a primitive homology class on X. Assume h is not visible, i.e $|m|\rho > 1$ where ρ is the length of the slit. By Proposition 4.1.2 we know a minimizing representative of h cannot be the projection of the straight line segment from $(0,0)$ to (m,n). To find exactly what a minimizing representative of h looks like, we will search separately for the shortest simple and non-simple closed curves, as those two cases need to be dealt with differently. We will start with the latter, as it is the easier case. Note that when we say simple curves we actually mean *injective* curves: there are no self-intersection points, even if the intersections are not transverse.

Intuition of the link with the Farey sequence. Here is an informal idea of why the Farey sequence appears in the computation of the stable norm of the slit torus. Given a non-visible homology class (m,n), we want to find the shortest path from the point $(0,0)$ to the point (m,n) in $\tilde{X}$, but since the class is not visible the straight line path is forbidden as it eventually intersects the slit. Hence we have to deviate from this direction at some point. Since the straight line segment would be the shortest path to the point (m,n) if it did not intersect the slit, we want to deviate from it as little as possible in order to minimize the total length of the path.

We have to deviate from the straight line path as it intersects the slit at some point. One natural idea would be to deviate as little as possible so that our path goes under the copy of the slit where the intersection was occurring, as depicted below with $(p,q) \in \mathbb{Z}^2$:

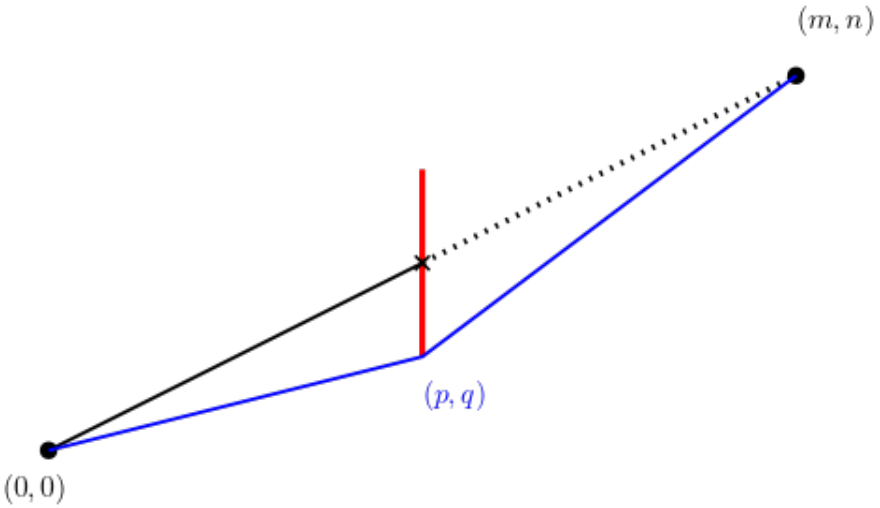

Figure 4.5: Deviating to pass under the slit

This is a simplification as it is far from clear that it is possible to go in a straight line from the origin to (p,q), or from (p,q) to (m,n): in fact, in general it is not possible

as there are again some slits in the way. However, there are cases where it is actually possible, and those cases lead to a key observation, so let us assume for now that the above picture is correct for the sake of communicating an idea.

Now, the segment from $(0,0)$ to (p,q) has slope q/p and the segment from (p,q) to (m,n) has slope $(n-q)/(m-p)$. Here we notice that

$$\frac{n}{m} = \frac{q}{p} \oplus \frac{n-q}{m-p}$$

the mediant of the slopes of each segment. Thus in some sense "the sum of two directions is their mediant". Since we want our path from the origin to (m,n) to deviate as little as possible from the straight line, we need to chose (p,q) so that the slopes of the segments are as close as possible to the slope of the straight line path, that is to n/m. In other words this suggests to consider the best rational approximations of n/m, that is to say its Farey parents.

Note that the points $(0,0),(p,q)$ and (m,n) all project to the same point on X, thus the projection to X of the blue path is a closed *non-simple* curve as it has a self-intersection point, hence the title of this section. If instead we try to deviate from the straight line path by going above the slit, the situation is different as we may get simple curves this time: these curves will be the object of the next section.

Shortest non-simple curves and Farey parents. We now add rigor to the above vague intuition. Note that the Farey parents of a rational n/m are not defined if $m = 1$. This is no concern to us as we have pointed out that homology classes of the form $(1,n)$ are always visible, so we already know what their stable norm is. Thus we are now only interested in classes (m,n) with $m \geq 2$.

Proposition 4.1.4. *Let $h = (m,n)$ be a primitive integral homology class with $m \geq 2$. Let a/b and c/d be the Farey parents of the rational n/m. Then the shortest non-simple closed curve representing h has length $\|(b,a)\|_X + \|(d,c)\|_X$.*

Proof. Let γ be a non-simple closed curve representing $h = (m,n)$. Since γ has a self-intersection point we can see γ as the union of two closed curves γ_1 and γ_2 representing two non-trivial integral homology classes (p,q) and $(m-p,n-q)$ with $p,q \in \mathbb{Z}$. Recall that γ lifts to a path $\tilde{\gamma}$ from $(0,0)$ to (m,n) in the Abelian covering $\tilde{X}$ of the slit torus which is the concatenation of a path from $(0,0)$ to (p,q) and of another path from (p,q) to (m,n).

Moreover, since we are looking for *short* curves representing $h = (m,n)$, we can assume $0 \leq p \leq m$, $0 \leq q \leq n$. Indeed, if it was not the case, then at least one of the classes (p,q) and $(m-p,n-q)$ would have a negative coordinate. This means $\tilde{\gamma}$ would eventually escape the $m \times n$ box having the segment $[(0,0),(m,n)]$ as one of its diagonals. Then, removing the part of $\tilde{\gamma}$ that goes outside of that box and replacing it by the straight line segment between its exit and re-entry points in that box one would produce a new representative of h that would be *shorter* than $\tilde{\gamma}$.

Now, if the homology classes of γ_1 and γ_2 are fixed, i.e if (p,q) is fixed, the length of γ is minimal if both the lengths of these two curves are minimal in their homology class, so we get the following inequality:

$$l(\gamma) \geq \|(p,q)\|_X + \|(m-p,n-q)\|_X$$

with equality if and only if γ is the union of two minimizing curves for (p,q) and $(m-p,n-q)$ respectively (each of those possibly translated upwards so it starts at the bottom of the slit). Hence, the minimal length of a non-simple closed curve representing h is

$$\min_{(p,q)} \|(p,q)\|_X + \|(m-p,n-q)\|_X$$

with $0 \leq p \leq m$ and $0 \leq q \leq n$ and (p,q) different from $(0,0)$ or (m,n).

Let C be the convex hull of

$$\left\{ \frac{(p,q)}{\|(p,q)\|_X} | (p,q) \in [\![0,m]\!] \times [\![0,n]\!] \backslash \{(m,n),(0,0)\} \right\} \cup (0,0)$$

Since $\mathscr{B}$ is convex, it contains the convex hull of any of its subsets, so $C \subset \mathscr{B}$. Thus, since $(m,n)/\|(m,n)\|_X$ belongs to $\partial\mathscr{B}$, either $(m,n)/\|(m,n)\|_X$ belongs to ∂C or it lies outside of C. For $(p,q) \in \mathbb{Z}^2$ with $p \leq m$ and $q \leq n$ with $(p,q) \neq (0,0),(m,n)$, denote $z_{p,q}$ the intersection point of the segment of endpoints $(p,q)/\|(p,q)\|_X$ and $(m-p,n-q)/\|(m-p,n-q)\|_X$ and of the line Δ from the origin passing through (m,n), that is the line of slope n/m. Direct computation yields

$$z_{p,q} = \frac{(m,n)}{\|(p,q)\|_X + \|(m-p,n-q)\|_X}$$

Denote $|z_{p,q}|$ the Euclidean length of the segment from the origin to $z_{p,q}$. By construction,

$$\left(|z_{p,q}| \leq |z_{p',q'}|\right) \iff \left(\|(p',q')\|_X + \|(m-p',n-q')\|_X \leq \|(p,q)\|_X + \|(m-p,n-q)\|_X\right)$$

Thus finding (p,q) such that $\|(p,q)\|_X + \|(m-p,n-q)\|_X$ is minimal is equivalent to finding he rightmost point of the set $\{z_{a,b}\}_{a,b}$ on Δ, where Δ is oriented positively from the origin to (m,n). Up to exchanging (p,q) and $(m-p,n-q)$ we can assume $q/p < n/m$ (thus $(n-q)/(m-p) > n/m$) and we finally have:

$$|z_{p,q}| \leq |z_{p',q'}| \iff \frac{q}{p} \leq \frac{q'}{p'}$$

This means that we have to find the greatest rational $q/p < n/m$ with $p < m$. This is precisely the characterization of a Farey parent of n/m, hence $(p,q) = (b,a)$ and $(m-p,n-q) = (d,c)$. We have shown:

$$\min_{(p,q)} \|(p,q)\|_X + \|(m-p,n-q)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$$

□

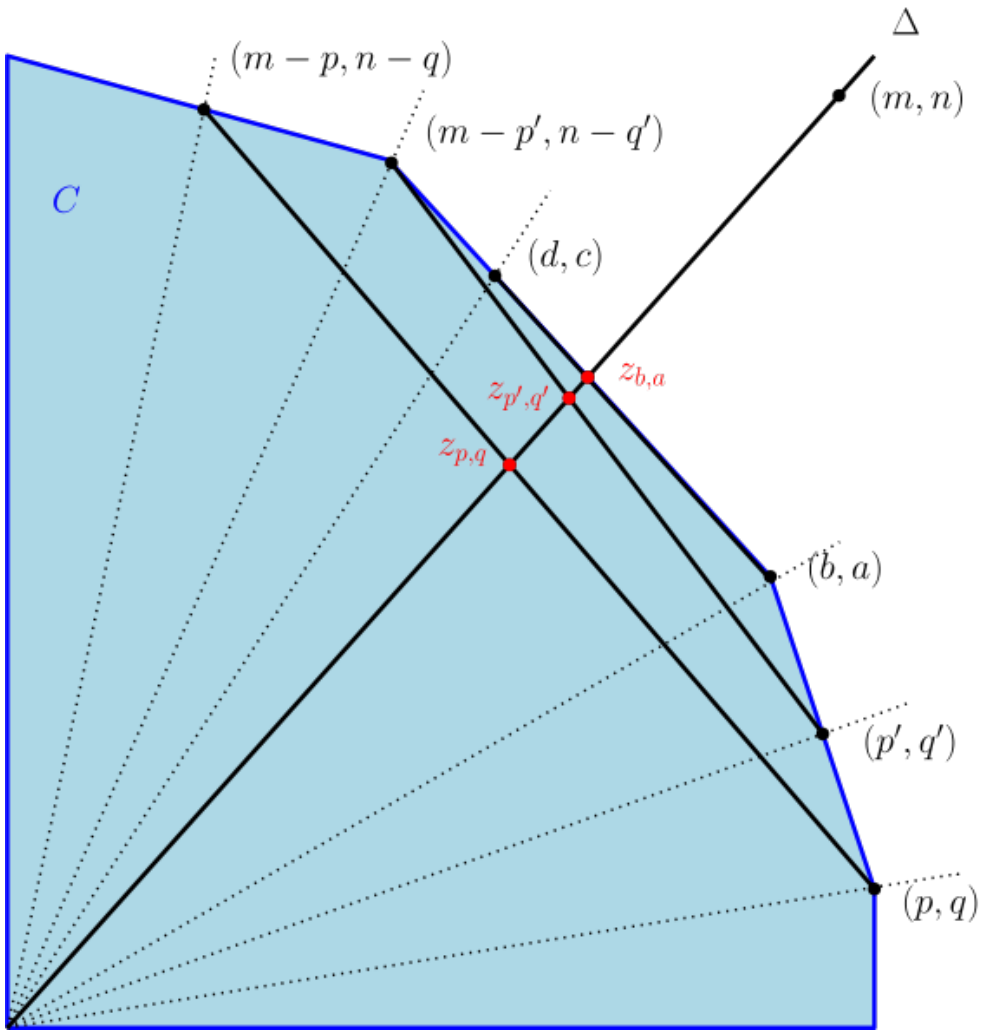

Figure 4.6: The construction from the proof

4.1.4 Short simple curves

In the discussion at the beginning of the previous section we looked at a simplified version of the problem to develop some intuition: see Figure 4.1.3. We tried to avoid the slit by going underneath it: so now, what happens if we try to go over it? Unfortunately, while there is still a little bit of intuition to obtain in that case, this simplified approach has reached its limits and does not provide a good enough starting point to write a proof.

So for now, simply note that going over the slit is qualitatively different from going under it, as it is not obvious that the path we obtain by going over the slit passes through points of $\tilde{X}$ that project to the same point in X. Hence this path might project onto a *simple* curve on X, contrary to what happens when we consider a path going under the slit. Remark that it could very well have self-intersection points, so that the curve it projects onto is non-simple: but this is redundant as we already treated the case of non-simple curves in the previous section.

Thus, since we already fully understand short non-simple curves, the goal of this section is to compute the length of the shortest simple curve that represents a non-visible

primitive homology class $h \in H_1(X,\mathbb{Z})$. As we will see, the result also depends on the Farey parents of n/m, although not exactly for the same reasons as before.

Simple closed curves up to free homotopy. We will need an algebraic result due to Osborne and Zieschang. Recall that $\mathbb{F}_2$ denotes the free group on two elements, and that an element $w \in \mathbb{F}_2$ is primitive if there exists $w' \in \mathbb{F}_2$ such that $\langle w, w' \rangle = \mathbb{F}_2$. Equivalently, given a system of generators (s,t) of $\mathbb{F}_2$, an element w is primitive if and only if there exists $\phi \in \mathrm{Aut}(\mathbb{F}_2)$ such that $\phi(s) = w$. Recall that (m,n) is called a primitive element of $\mathbb{Z}^2$ if $\gcd(m,n) = 1$; note that even if they are named using the same word, primitive elements of a free group and primitive elements of a lattice have different definitions.

Osborne and Zieschang perform the following geometric construction. Let (s,t) be a basis of $\mathbb{F}_2$. On a square grid, draw the open segment from $(0,0)$ to (m,n). Following this segment from the origin to (m,n), write s (resp. t) whenever it crosses a vertical (resp. horizontal) line of the grid. We obtain a word $V'_{m,n} \in \mathbb{F}_2$ in the two letters s and t. Note that by taking an *open* segment we do not take into account the intersections with the lines of the grid at the points $(0,0)$ and (m,n) when writing this word.

Theorem 4.1.5 ([OZ81, Corollary 3.2, Proposition 2.3])**.** *Let (m,n) be a primitive element of $\mathbb{Z}^2$ and let $\psi : \mathbb{F}_2 \longrightarrow \mathbb{Z}^2$ be the canonical abelianizing homomorphism. The word $V_{m,n}$ satisfies the following assertions.*

1. *$V_{m,n}$ is primitive in $\mathbb{F}_2$.*
2. *$\psi(V_{m,n}) = (m,n)$.*
3. *If $w \in \mathbb{F}_2$ is primitive and $\psi(w) = (m,n)$ then w and $V_{m,n}$ are conjugate in $\mathbb{F}_2$.*

Thus on the slit torus, where for all $x \in X, \pi_1(X,x) \cong \mathbb{F}_2$ and $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^2$, up to conjugacy there is a one-to-one correspondance between primitive homotopy classes and primitive integral homology classes. In other words, since the conjugacy classes of the fundamental group can be identified with free homotopy classes, there is a one-to-one correspondance between primitive free homotopy classes on X and primitive elements of $H_1(X,\mathbb{Z})$.

Corollary 4.1.5.1. *If $h \in H_1(X,\mathbb{Z})$ is a primitive integral homology class on X then there exist simple closed curves that represent h, and these curves are all freely homotopic to one another.*

Proof. It remains to prove that a free homotopy class is primitive if and only if it contains simple closed curves. Denote by (s,t) a basis of $\mathbb{F}_2 \cong \pi_1(X)$ such that $\psi(s) = (1,0)$ and $\psi(t) = (0,1)$, and let a be a simple closed curve representing s.
If γ is a simple closed curve on X then by the theorem of classification of (non-separating) simple closed curves on surfaces (see [FM12, Paragraph 1.3.1]) there exists a homeomorphism Φ of X such that $\Phi(a) = \gamma$. This homeomorphism induces an automorphism φ of $\pi_1(X)$ such that $\varphi(s) = \bar{\gamma}$, hence the free homotopy class $\bar{\gamma}$ is primitive.
Conversely, if $\bar{\gamma} \in \pi_1(X)$ is primitive there exists $\varphi \in \mathrm{Aut}(\pi_1(X))$ such that $\varphi(s) = \bar{\gamma}$. Since X is a $K(\mathbb{F}_2,1)$, by [Hat02, Proposition 1B.9] φ is induced by a homeomorphism Φ of X. Then, since a is simple and Φ is a homeomorphism, $\Phi(a)$ is a simple closed curve that represents the free homotopy class $\bar{\gamma}$. □

Constraints on simple representatives of a homology class. Osborne and Zieschang also proved a remarkable property of this word $V_{m,n}$ that we rephrase here using the Farey sequence:

Theorem 4.1.6 (Lemma 2.2 in [OZ81])**.** *Let a/b and c/d be the Farey parents of n/m, with $a/b < c/d$. Then*

$$V_{m,n} = V_{d,c}V_{b,a} = V_{b,a}tsV'_{d,c}$$

This theorem contains a lot of information that can be extracted to provide geometric constraints on the simple representatives of a primitive homology class (m,n) on the slit torus. The Farey parents of n/m appear once again, but, at first glance, for different reasons than before. Indeed, the Farey parents of n/m were relevant in the previous section because they are the best rational approximations of n/m, whereas this time they seem to come from an algebraic background. However, one possible proof of Theorem 4.1.6 relies on... best rational approximations, so seeing the Farey parents appear once again in our problem is far from being a coincidence.

The first equality $V_{m,n} = V_{d,c}V_{b,a}$ tells us that in order to avoid the slits the lift to $\tilde{X}$ of a simple curve representing (m,n) on X must cross the vertical line $x = d$ between the points $(d, c-1+\rho)$ and (d,c), as illustrated by the following picture:

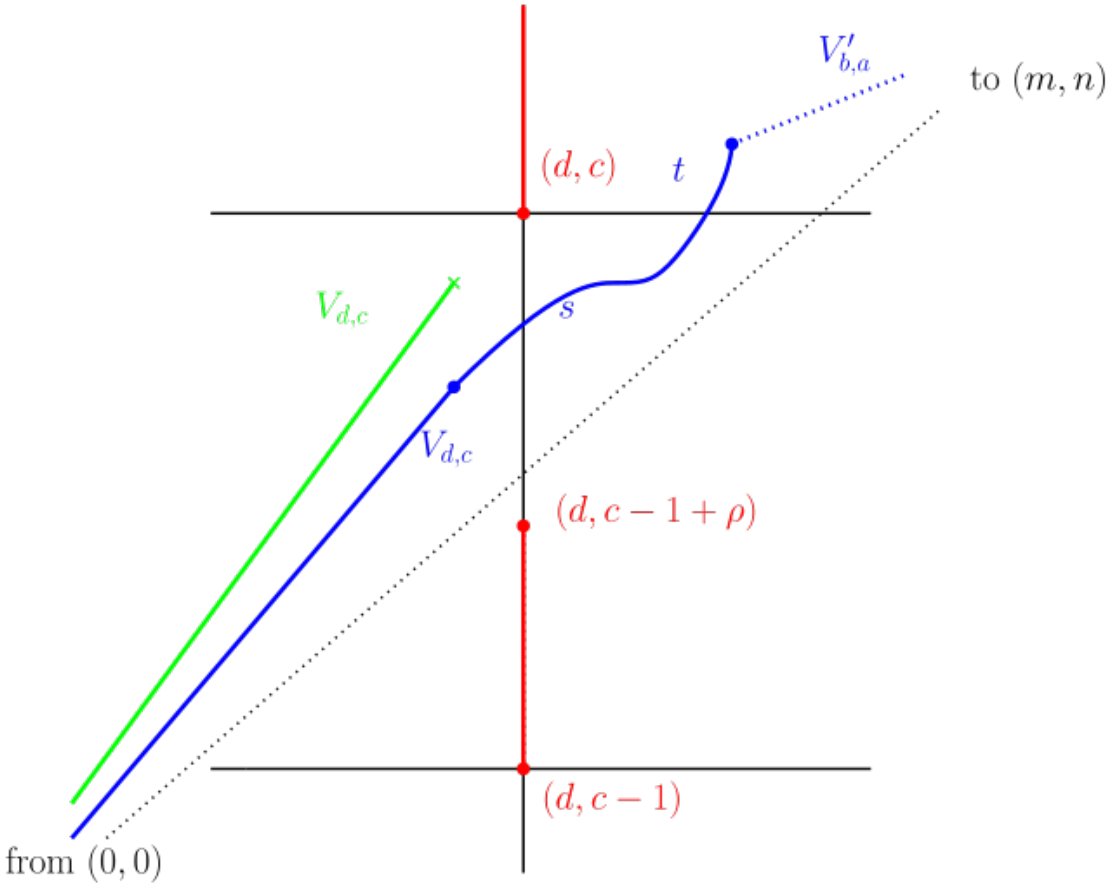

Figure 4.7: First constraint for a curve in $V_{m,n}$

Similarly, the second equality $V_{m,n} = V_{b,a}tsV'_{d,c}$ tells us that the lift of a simple curve representing (m,n) on X must cross the vertical line $x = b$ between the points $(b, a+\rho)$ and $(b, a+1)$:

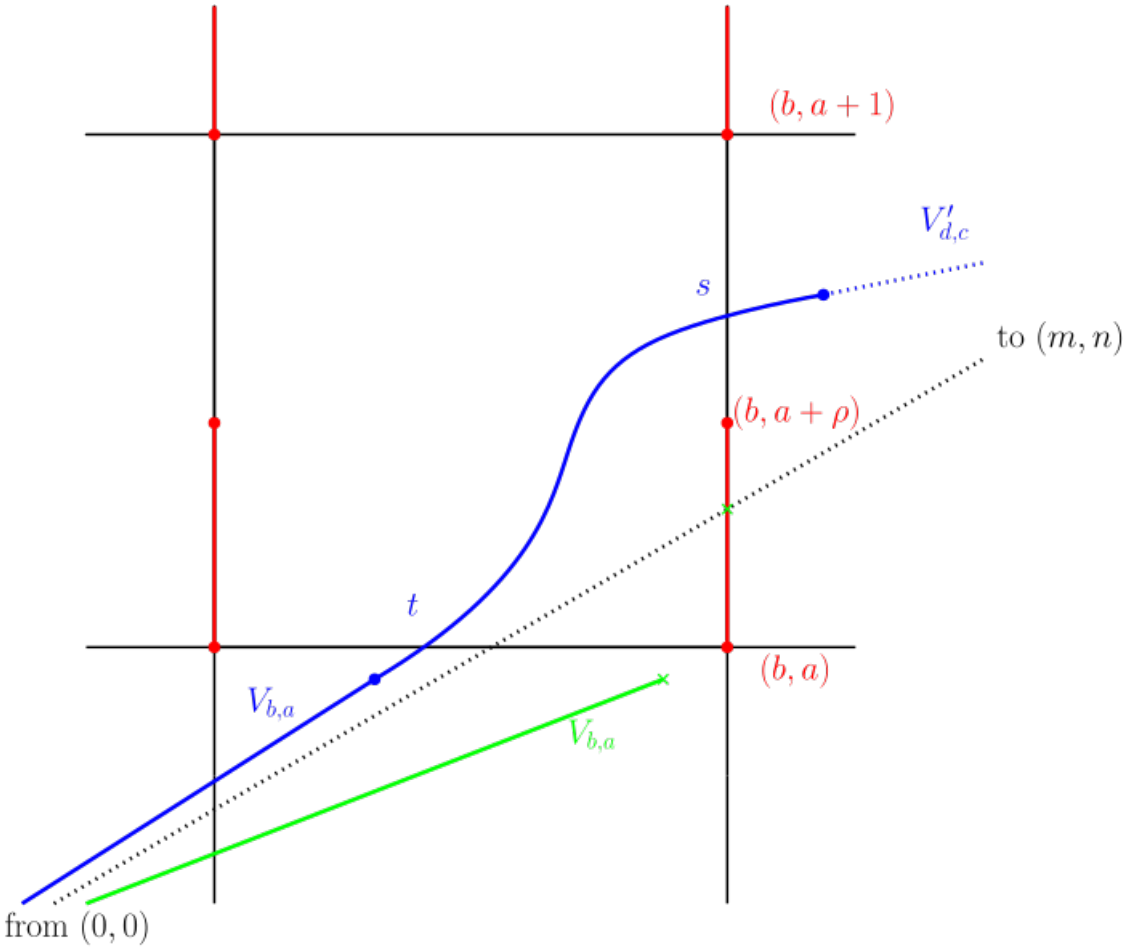

Figure 4.8: Second constraint for a curve in $V_{m,n}$

We will start by considering this second constraint on simple curves, illustrated by Figure 4.8; as we will see, it already provides plenty of information. Satisfying this constraint is a necessary condition for a curve representing a non-visible class (m,n) to be simple. Amongst all the curves (simple or not) satisfying this constraint, the obvious candidate to be the shortest is the path γ made by concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$. Thus, provided that γ projects onto a closed curve on X, that is to say provided that it does not encounter the slit, it will be the shortest curve satisfying the constraint illustrated by Figure 4.8. Its length will then provide a lower bound on the length of any curve satisfying the constraint of Figure 4.8, and in particular on the length of simple curves representing (m,n).

Moreover, if γ projects onto a simple curve this lower bound would be a minimum and we would have found the shortest simple curve representing h. Hence we have to check if this path γ satisfies the required properties, that is not to intersect the slit and to project onto a simple curve on X. As we will see, the answers to these questions depend on the position of n/m in the Farey sequence. There are two distinct cases to consider, corresponding to whether or not the Farey parents of n/m are associated to visible homology classes.

If one of the Farey parents is visible. Let $h=(m,n)\in H_1(X,\mathbb{Z})$ be a primitive homology class on X that is not visible. Let $a/b<c/d$ be the Farey parents of the rational n/m, and assume that at least one of the homology classes (b,a), (d,c) is visible. In that sense we say that one of the Farey parents of n/m is visible.

Lemma 4.1.7. *Let γ be the path in $\tilde{X}$ from $(0,0)$ to (m,n), obtained by the concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$. If at least one of the Farey parents of n/m is visible then the path γ does not transversally intersect the slit.*

Proof. Assume that at least one of the classes (b,a) and (d,c) is visible. We need to fix the visible class in order to be able to conduct computations, so let us assume that (d,c) is visible: the computations are similar if (b,a) is visible instead. We check that each segment composing γ avoids the slits.

• The segment $[(0,0),(b,a+\rho)]$ crosses the vertical of the slit at points $\big(k,k(a+\rho)/b\big)$ with $k=1,...,b-1$, with intersection with the slit if and only if $\big\{k(a+\rho)/b\big\}<\rho$ where $\{.\}$ denotes the fractional part. Since a/b and c/d are a Farey pair, we have

$$\frac{a}{b}=\frac{c}{d}-\frac{1}{bd}$$

so

$$\left\{k\frac{a+\rho}{b}\right\}=\left\{k\left(\frac{c}{d}-\frac{1}{bd}+\frac{\rho}{b}\right)\right\}=\left\{\frac{r}{d}+k\frac{\rho-1/d}{b}\right\}$$

where r is the remainder in the Euclidean division of kc by d. Remark that since (d,c) is visible we have $d\rho\leq 1$ so $\rho-1/d\leq 0$. On one hand

$$\frac{r}{d}+k\frac{\rho-1/d}{b}\leq\frac{r}{d}\leq\frac{d-1}{d}<1$$

and on the other hand

$$\frac{r}{d}+k\frac{\rho-1/d}{b}\geq\frac{1}{d}+\frac{k(\rho-1/d)}{b}\geq\frac{1}{d}+\frac{(b-1)(\rho-1/d)}{b}$$

so

$$\frac{r}{d}+k\frac{\rho-1/d}{b}\geq\rho+\frac{1/d-\rho}{b}\geq\rho.$$

Thus $\left\{\frac{r}{d}+k\frac{\rho-1/d}{b}\right\}=\frac{r}{d}+k\frac{\rho-1/d}{b}\geq\rho$ and the segment $[(0,0),(b,a+\rho)]$ does not intersect the slit.

• For the case of the segment from $(b,a+\rho)$ to (m,n), the computations become easier when we translate the problem by $(-b,-a-\rho)$. Since $m=b+d$ and $n=a+c$, the problem is equivalent to determining whether or not the segment $[(0,0),(d,c-\rho)]$ intersects the slit, with the copies of the slit being this time the sets $\big\{(p,q-s),s\in[0,\rho]\big\}$. The segment $[(0,0),(d,c-\rho)]$ has slope $(c-\rho)/d$ and crosses the vertical of the slit at the points $(k,k(c-\rho)/d)$, $k=1,...,d-1$, with intersection with the slit if and only if

$$\left\{k\frac{c-\rho}{d}\right\}>1-\rho.$$

Let r be the remainder in the Euclidean division of kc by d. We have $\{k(c-\rho)/=d\} = \{(r-k\rho)/d\}$. Moreover,

$$\frac{r-k\rho}{d} > \frac{1-k\rho}{d} \geq \frac{1-(d-1)\rho}{d} > 0$$

because $(d-1)\rho \leq 1-\rho < 1$, and since $1/d > \rho$ we have

$$\frac{r-k\rho}{d} < \frac{d-1-k\rho}{d} = 1 - \frac{1+k\rho}{d} < 1 - \frac{1}{d} \leq 1-\rho.$$

Hence the segment $[(b, a+\rho), (m,n)]$ does not intersect the slit.

□

Thus the path γ, defined as the concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$, projects onto a closed curve on X that represents h that has the same length as γ. Is this curve simple? We start with the easier special case where *both* the classes associated to the Farey parents of n/m are visible.

Lemma 4.1.8. *With the previous notations, if both the homology classes (b,a) and (d,c) are visible, then γ projects onto a simple closed curve on X. Moreover, this curve is minimizing in the homology class (m,n).*

Proof. Remind that $m, n \geq 0$. Assume that both the classes (b,a) and (d,c) are visible, where $a/b < c/d$ are the Farey parents of n/m. By Proposition 4.1.4 we know that the shortest non-simple curve representing h has length $\|(b,a)\|_X + \|(d,c)\|_X$, and since we assumed that both (b,a) and (d,c) are visible by Proposition 4.1.2 we know that $\|(b,a)\|_X = \sqrt{b^2+a^2}$ and $\|(d,c)\|_X = \sqrt{d^2+c^2}$. Thus the shortest non-simple curve representing h on X has length

$$\|(b,a)\|_X + \|(d,c)\|_X = \sqrt{b^2+a^2} + \sqrt{d^2+c^2}$$

and we only need to check that γ is shorter. Indeed, if so the projected curve is automatically simple as it represents the homology class (m,n).

Since a/b and c/d are Farey neighbours they satisfy the relation $bc-ad=1$, so

$$\frac{n}{m} - \frac{a+\rho}{b} = \frac{a+c}{b+d} - \frac{a}{b} - \frac{\rho}{b} = \frac{1}{b}\left(\frac{1}{b+d} - \rho\right) < 0$$

because $b+d = m > 1/\rho$. Since $d\rho \leq 1$ we have $\rho \leq 1/d$ and

$$\frac{a+\rho}{b} = \frac{a}{b} + \frac{\rho}{b} \leq \frac{a}{b} + \frac{1}{bd} = \frac{c}{d}$$

hence we have

$$\frac{a}{b} \leq \frac{c-\rho}{d} < \frac{n}{m} < \frac{a+\rho}{b} \leq \frac{c}{d}.$$

Thus the triangle formed by the points $(0,0)$, (d,c) and (m,n) contains the triangle formed by the points $(0,0),(b,a+\rho)$ and (m,n) (see Figure 4.9). We deduce that the perimeter of the first triangle is greater than the perimeter of the second. But their perimeters are respectively

$$\|(b,a)\|_X + \|(d,c)\|_X + \sqrt{m^2+n^2}$$

and

$$\sqrt{b^2+(a+\rho)^2}+\sqrt{d^2+(c-\rho)^2}+\sqrt{m^2+n^2}=l(\gamma)+\sqrt{m^2+n^2}$$

so we finally have $l(\gamma) < \|(b,a)\|_X + \|(d,c)\|_X$, hence γ projects onto a curve that is shorter than the shortest non-simple curve that represents h. Thus γ projects onto a simple curve on X.

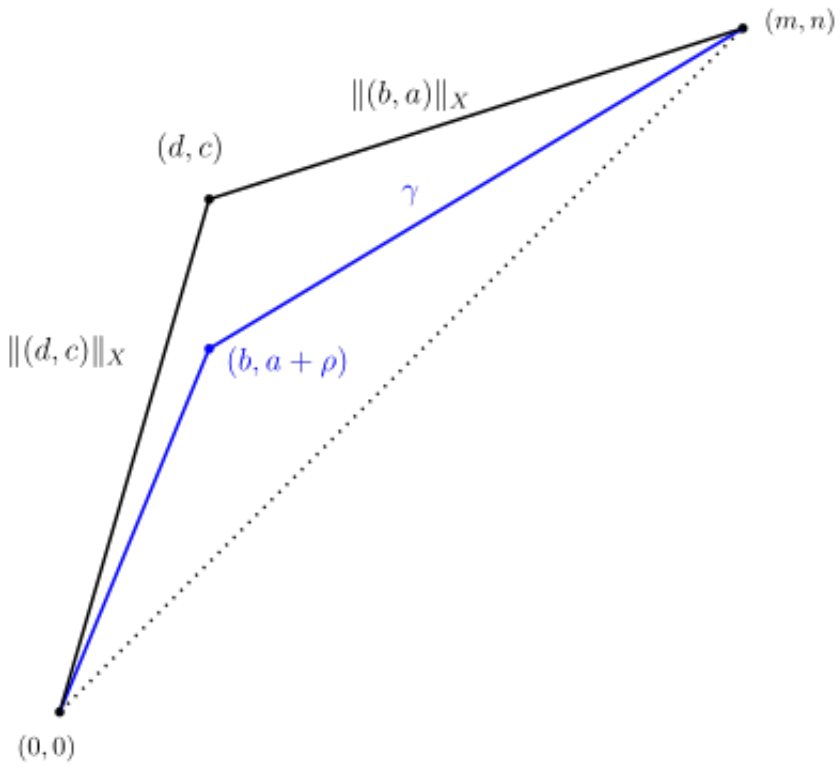

Figure 4.9: The path γ is shorter

By construction γ is the shortest path satisfying the constraint illustrated by Figure 4.8. Since all (lifts to $\tilde{X}$ of) simple curves representing the class (m,n) on X must satisfy this constraint, the path γ projects onto is the shortest simple closed curve that represents (m,n). Since we have already shown it is shorter than the shortest non-simple closed curve representing (m,n) on X, we conclude that it is minimizing in the homology class (m,n).

□

We can now compute the stable norm of the class (m,n).

Proposition 4.1.9. *Let $h=(m,n)$ be a primitive non-visible integral homology class on X. Let $a/b < c/d$ be the Farey parents of n/m. If at least one of the classes (b,a) ad (d,c) is visible then*

$$\|(m,n)\|_X=\sqrt{b^2+(a+\rho)^2}+\sqrt{d^2+(c-\rho)^2}.$$

We will need the following *ad hoc* definition of the distance to a couple of Farey ancestors of a rational number to conduct the proof. Recall that starting from any pair of Farey neighbours $p_1/q_1 < p_2/q_2$ and by taking mediants with p_1/q_1 and p_2/q_2, one

can obtain every rational number in the interval $]p_1/q_1, p_2/q_2[$. In other words, for any rational $r/s \in]p_1/q_1, p_2/q_2[$ there exist $k_1, k_2 \in \mathbb{N}^*$ such that

$$\frac{r}{s} = \frac{k_1 p_1 + k_2 p_2}{k_1 q_1 + k_2 q_2}.$$

We say that the distance from r/s to the couple $(p_1/q_1, p_2/q_2)$ of its Farey ancestors is $k_1 + k_2 - 1$, as it is the number of steps one has to take to obtain r/s from p_1/q_1 and p_2/q_2 with the operations "taking the mediant with p_1/q_1" and "taking the mediant with p_2/q_2". This distance is simply the distance between r/s and $p_1/q_1 \oplus p_2/q_2$ in the Stern-Brocot tree.

Proof. Let γ be the path in $\tilde{X}$ obtained by concatenation of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$. We follow the same strategy as in Lemma 4.1.8: we will show that γ projects on X onto a curve that is shorter than the shortest non-simple closed curve on X that represents the homology class (m,n). More precisely, we will show that

$$l(\gamma) < \|(b,a)\|_X + \|(d,c)\|_X.$$

Then the curve γ projects onto will automatically be simple and, because it is the projection of the shortest path satisfying the constraint illustrated in Figure 4.8, minimizing in the homology class (m,n) so $\|(m,n)\|_X = l(\gamma)$. We proceed by induction on $\kappa \geq 1$, where κ is the distance from n/m to its latest couple of visible Farey ancestors, that is, the smallest possible distance from n/m to a couple of its Farey ancestors that are both visible.

Let us describe the Farey ancestors of n/m more precisely. By assumption, one of the classes $(b,a),(d,c)$ is visible. To fix notations assume that (b,a) is visible, as (b,a) and (d,c) play similar roles in this proof. Since (b,a) is visible the number a/b, which is a Farey parent of n/m, is part of the latest couple of visible Farey ancestors of n/m. Let α/β be the other one; possibly $\alpha/\beta = c/d$ if both the Farey parents of n/m are visible. By definition, there exist $k_1, k_2 \in \mathbb{N}^*$ such that $n/m = (k_1 a + k_2 \alpha)/(k_1 b + k_2 \beta)$ and $\kappa = k_1 + k_2 - 1$. Since a/b and n/m (resp. a/b and α/β) are Farey neighbours we have $bn - am = 1$ (resp. $b\alpha - a\beta = 1$). Then

$$bn - am = b(k_1 a + k_2 \alpha) - a(b + k_2 \beta) = k_2(b\alpha - a\beta) = k_2.$$

Hence $k_2 = 1$ and

$$\frac{n}{m} = \frac{\kappa a + \alpha}{\kappa b + \beta} = \left(\left(\left(\frac{a}{b} \oplus \frac{\alpha}{\beta}\right) \oplus \frac{a}{b}\right) \oplus \ldots \oplus \frac{a}{b}\right)$$

i.e n/m is obtained from its latest visible Farey ancestors by taking κ times the mediant with a/b. Thus, the Farey ancestors of n/m up to α/β in the Farey genealogy are of the form $(ka+\alpha)/(kb+\beta)$ with k running from $\kappa - 1$ down to 1. In particular the second Farey parent of n/m is $c/d = ((\kappa-1)a+\alpha)/((\kappa-1)b+\beta)$.

If $\kappa = 1$ then $\alpha/\beta = c/d$, meaning both the classes (b,a) and (d,c) are visible. By Lemma 4.1.8 we know γ projects onto a minimizing simple closed curve for the class (m,n) on X, so

$$\|(m,n)\|_X = l(\gamma) = \sqrt{b^2 + (a+\rho)^2} + \sqrt{d^2 + (c-\rho)^2}$$

and we are done.

Assume the proposition is proven up to $\kappa-1$ with $\kappa \geq 2$. We want to compare $l(\gamma)$ to $\|(b,a)\|_X + \|(d,c)\|_X$. Since (b,a) is visible, we have $\|(b,a)\|_X = \sqrt{b^2+a^2}$. Since $\kappa \neq 1$ we have $c/d \neq \alpha/\beta$, so (d,c) is not visible. However, the distance from c/d to its latest couple of Farey ancestors $(a/b, \alpha/\beta)$ is $\kappa-1$ so by the induction hypothesis we know how to express the stable norm of (d,c) using only its Farey parents. More precisely, the Farey parents of $c/d = ((\kappa-1)a+\alpha)/((\kappa-1)b+\beta)$ are a/b and $((\kappa-2)a+\alpha)/((\kappa-2)b+\beta)$, at least one of whose being visible, so according to the induction hypothesis at rank $\kappa-1$ we have

$$\|(d,c)\|_X = \sqrt{b^2+(a+\rho)^2} + \sqrt{((\kappa-2)b+\beta)^2 + ((\kappa-2)a+\alpha-\rho)^2}.$$

We then have

$$\begin{aligned}\|(b,a)\|_X + \|(d,c)\|_X - l(\gamma) &= \sqrt{b^2+a^2} + \sqrt{((\kappa-2)b+\beta)^2 + ((\kappa-2)a+\alpha-\rho)^2} - \sqrt{d^2+(c-\rho)^2} \\ &= \|(b,a)\|_2 + \|((\kappa-2)b+\beta, (\kappa-2)a+\alpha-\rho)\|_2 - \|(d,c-\rho)\|_2\end{aligned}$$

and since $(d,c-\rho) = (b,a) + ((\kappa-2)b+\beta, (\kappa-2)a+\alpha-\rho)$, by the triangle inequality of the Euclidean norm we finally have $\|(b,a)\|_X + \|(d,c)\|_X - l(\gamma) > 0$.

Thus γ projects onto a simple closed curve on X that represents the homology class (m,n). Since any simple curve representing (m,n) on X must lift to a path in $\tilde{X}$ that satisfies the constraint illustrated by Figure 4.8, and since among the paths that satisfy this constraint γ is the shortest, we deduce that the curve γ projects onto is the shortest simple closed curve on X that represent (m,n). We have seen this curve is shorter than the shortest non-simple closed curve representing (m,n), so it must be minimizing inside its homology class. Hence we finally have

$$\|(m,n)\|_X = l(\gamma) = \sqrt{b^2+(a+\rho)^2} + \sqrt{d^2+(c-\rho)^2}.$$

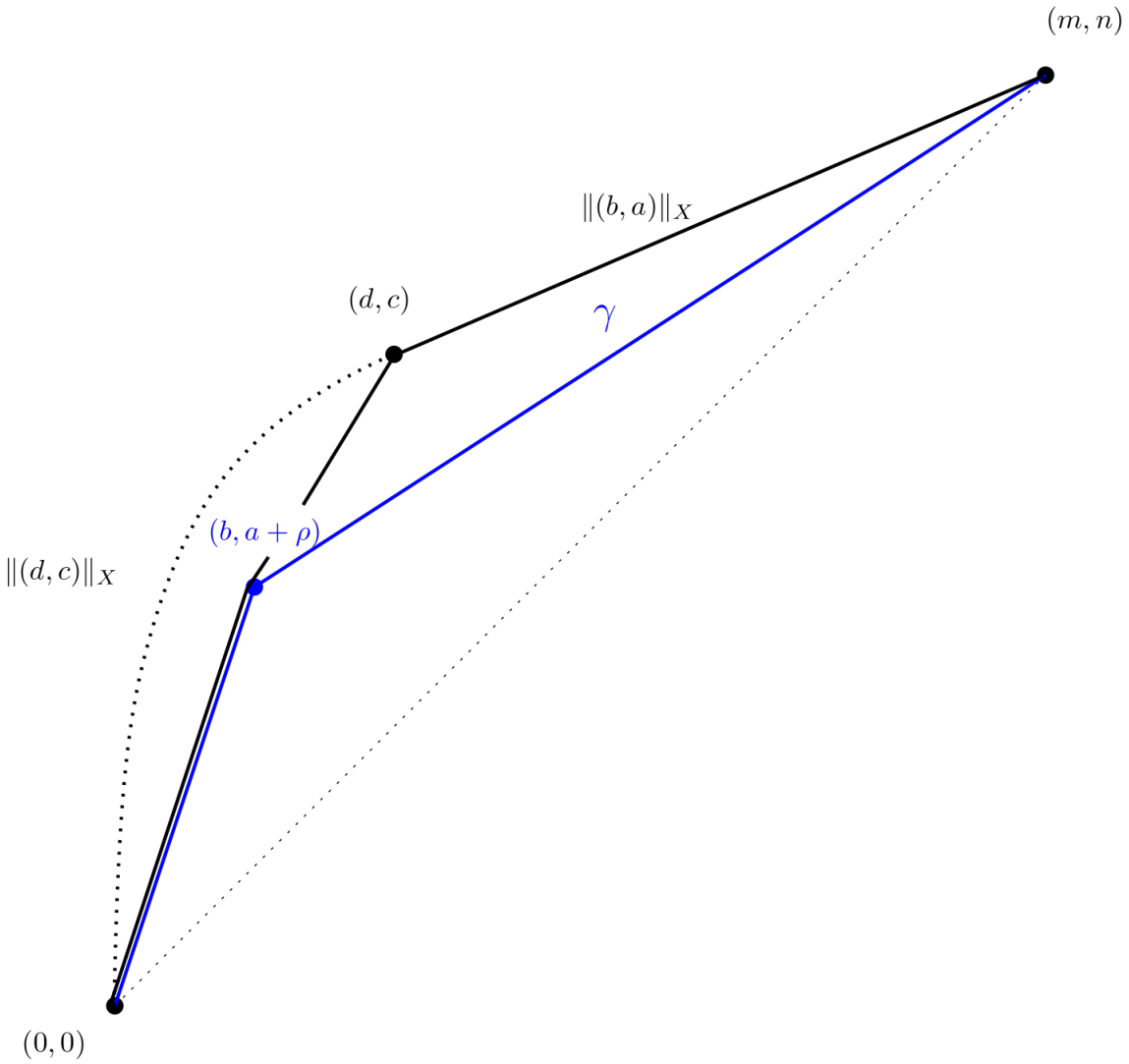

Figure 4.10: The path γ is shorter

□

If none of the Farey parents is visible. Finally we investigate what happens when none of the classes associated to the Farey parents a/b and c/d of n/m is visible. This time, the path which is the union of the segments $[(0,0),(b,a+\rho)]$ and $[(b,a+\rho),(m,n)]$ does not project onto a simple closed curve on the slit torus. Indeed it intersects the slit, for instance at $x = d$ as one can check with a direct computation that we will not detail. Instead we will prove a stronger statement:

Proposition 4.1.10. *If none of the Farey parents of n/m is visible, there is no simple geodesic in the free homotopy class $V_{m,n}$, where $V_{m,n}$ is the unique free homotopy class of simple curves that represent the homology class (m,n).*

First we need the following lemma:

Lemma 4.1.11. *Let γ be the lift to $\tilde{X}$ of a geodesic in the free homotopy class $V_{m,n}$ starting at the origin. Then γ changes direction at least twice.*

Proof. Assume that $m, n \geq 0$ to simplify notations. Let $a/b < c/d$ be the Farey parents of n/m, by assumption (b, a) and (d, c) are not visible so $b, d > 1/\rho$.

- If $d < b$, since $d > 1/\rho$ and $bc - ad = 1$, straightforward computations yield

$$\frac{c-1+\rho}{d} < \frac{a}{b} < \frac{n}{m} < \frac{c}{d} < \frac{a+\rho}{b}$$

so by putting together the constraints illustrated by Figure 4.7 and Figure 4.8, we obtain the following picture, on which it is clear that the lift of a curve in $V_{m,n}$ must change direction at least twice.

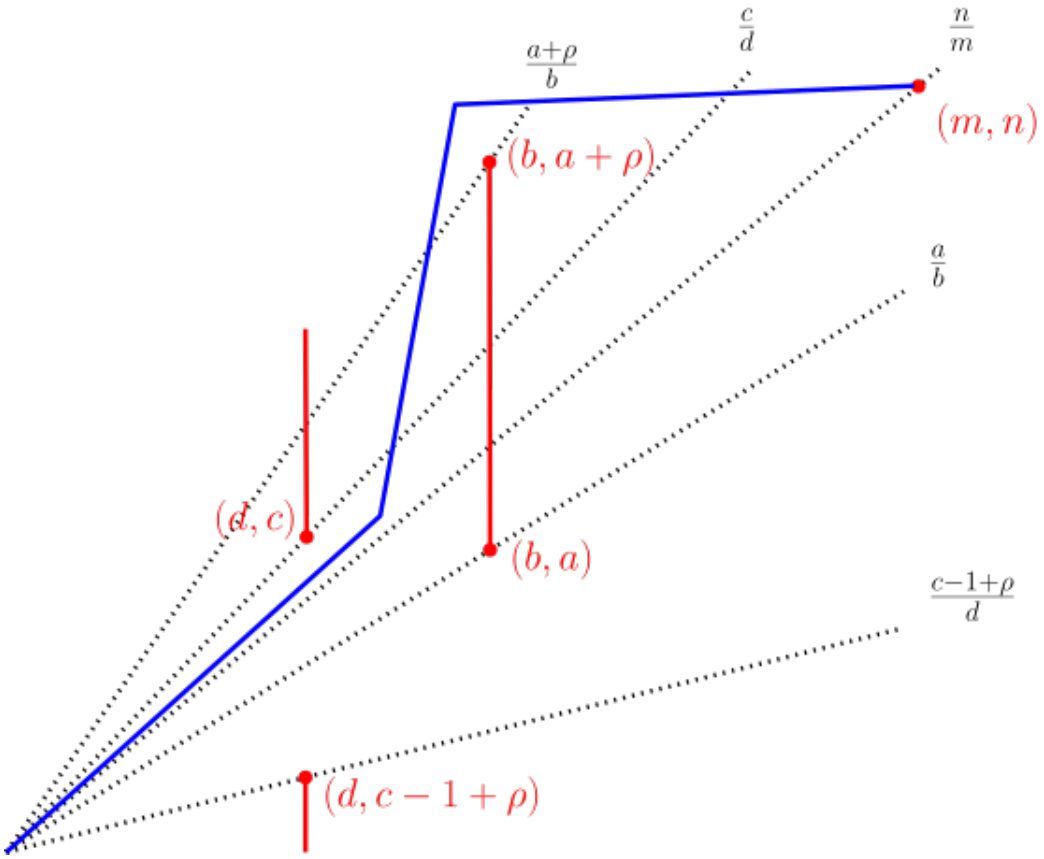

Figure 4.11: The blue curve in $V_{m,n}$ must change direction twice

- If $b < d$, similarly we get the following picture:

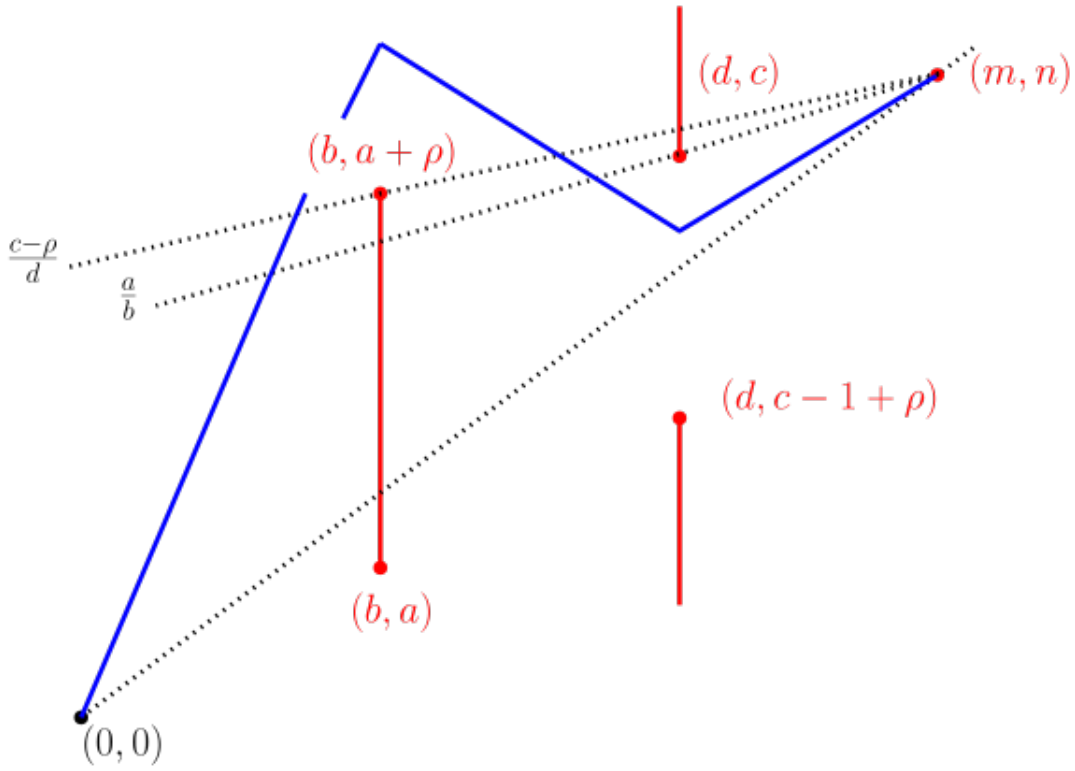

Figure 4.12: Again, the blue curve in $V_{m,n}$ must change direction twice

and since $b > 1/\rho$ we have $\dfrac{c-\rho}{d} < \dfrac{a}{b}$, so the lift to $\tilde{X}$ of a curve in $V_{m,n}$ has to change direction in order to pass under (d,c).

□

We can now prove Proposition 4.1.10.

Proof. Since the metric is flat, a geodesic on X must be a broken line with angles located at the slit's endpoints. Indeed, if a geodesic changes direction outside of an endpoint of the slit, then it has a corner and the geodesic is not locally length-minimizing, which it should be, hence a contradiction. Since we know that every lift of a geodesic in $V_{m,n}$ must change direction at least twice between $(0,0)$ and (m,n), that means that any geodesic in $V_{m,n}$ passes at least three times through an endpoint of the slit: the two times when changing direction and the third time corresponding to start and finish, i.e the projection of the points $(0,0)$ and (m,n). The slit has only two endpoints, hence any geodesic in $V_{m,n}$ meets a single endpoint of the slit at least twice, so it is not simple. □

Obviously, a minimizing curve for a homology class h has to be a geodesic. Thus, if there are no simple geodesics representing h this means that h is minimized by a non-simple curve, and we know its length according to Proposition 4.1.4. We have shown:

Proposition 4.1.12. *Let $h = (m,n)$ be a primitive integral homology class, and let $a/b, c/d$ be the Farey parents of n/m. Assume that none of the Farey parents of n/m is visible. Then $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$.*

4.1.5 Structure of the unit ball

Putting together Proposition 4.1.2, 4.1.4, 4.1.9 and 4.1.12 we have completely computed the stable norm of the slit torus X: given a primitive homology class (m,n) we have explicit formulas for its stable norm. To sum things up, we have shown there are three different types of classes:

Proposition 4.1.13. *Let $(m,n) \in H_1(X,\mathbb{Z})$ be a primitive integral homology class on X.*

1. *If (m,n) is visible then it is minimized by a simple curve and*
$$\|(m,n)\|_X = \sqrt{m^2+n^2} = \|(m,n)\|_2.$$
2. *If (m,n) is not visible but (b,a) or (d,c) is visible, where $a/b < c/d$ are the Farey parents of n/m then (m,n) is minimized by a simple curve and*
$$\|(m,n)\|_X = \sqrt{b^2+(a+\rho)^2} + \sqrt{d^2+(c-\rho)^2} > \|(m,n)\|_2$$
where ρ denotes the length of the slit.
3. *If $(m,n),(b,a)$ and (d,c) are not visible then (m,n) is minimized by a non-simple curve and*
$$\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X > \|(m,n)\|_2.$$

Note that for classes of type 1 and 2 in Proposition 4.1.13 we have found a simple minimizing curve that is *strictly shorter* than any non-simple curve representing (m,n), so *every* minimizing curve of such a class must be simple.

The unit ball of the stable norm $\mathscr{B}$ is a convex set of the plane, symmetric with respect to the origin. We now want to describe its combinatorial structure. More precisely, given a homology class $h \in H_1(X,\mathbb{R})$, we would like to know what does $\mathscr{B}$ look like in the direction h: does it have an extremal point (such as a vertex), or does the point $h/\|h\|_X$ lie in the interior of a segment contained in $\partial\mathscr{B}$?

We say that $\mathscr{B}$ has *a flat* in a direction h if the unit vector $h/\|h\|_X$ lies in the interior of a segment contained in the boundary $\partial\mathscr{B}$ of the unit ball. If $\mathscr{B}$ does not have a flat in the direction (m,n) we say that $\mathscr{B}$ is strictly convex in the direction (m,n), and the corresponding point on the boundary of $\mathscr{B}$ is called an extreme point. Equivalently, this means that the triangle inequality for h is *strict*: if $h_1+h_2=h$ then
$$\|h\|_X < \|h_1\|_X + \|h_2\|_X$$
unless h_1 and h_2 are multiples of h.

Non-simple minimizing curves and flats of the unit ball. First we prove the following lemma, linking homology classes minimized by non-simple closed curves to flats of the unit ball of the stable norm.

Lemma 4.1.14. *Let $h \in H_1(X,\mathbb{Z})$. The unit ball of the stable norm $\mathscr{B}$ has a flat in the direction h if and only if there exists a connected minimizing curve for h with at least one self-intersection. Equivalently, the unit ball $\mathscr{B}$ is strictly convex in the direction h if and only if every minimizing curve for h is connected and simple.*

Proof. Let $h = (m,n)$ be a primitive integral homology class, and assume that the unit ball of the stable norm $\mathscr{B}$ has a flat in the direction h. By Proposition 4.1.14, the classes $h_1 = (b,a)$ and $h_2 = (d,c)$, where $a/b, c/d$ are the Farey parents of n/m, belong to the same flat of $\mathscr{B}$ as h. Indeed, since $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$ we have

$$\frac{(m,n)}{\|(m,n)\|_X} = \lambda \frac{(b,a)}{\|(b,a)\|_X} + (1-\lambda)\frac{(d,c)}{\|(d,c)\|_X}$$

where $\lambda = \|(b,a)\|_X / \|(m,n)\|_X$. Since $h = h_1 + h_2$, one can get a multicurve that is minimizing for h by taking the union of two multicurves, representing (and minimizing for) h_1 and h_2 respectively. Let γ be such a minimizing multicurve. By translating each of its component upwards until they encounter the lower endpoint of the slit, we get another representative of h that is also minimizing, because applying a translation does not change the homology class nor the length of each curve. This representative is connected and has at least one self-intersection, located at the lower endpoint of the slit.

Conversely, if an integral class h is minimized by a closed curve γ with at least one self-intersection point $p \in X$, one can see γ as the union of two closed curves γ_1 and γ_2 based at p. These curves necessarily have non-trivial homology classes h_1 and h_2: otherwise, by simply deleting one of them from γ we would get another closed curve representing h that is strictly shorter than γ, which contradicts its minimality. Moreover, these curves must be minimizing in their homology class, simply because if one of them is not we could replace it by a minimizing curve of its homology class to get another representative of h shorter than γ, again contradicting the minimality of γ. Hence $h = h_1 + h_2$ and $\|h\|_X = \|h_1\|_X + \|h_2\|_X$, so $h/\|h\|_X$ lies in the interior of the segment $\left[h_1/\|h_1\|_X, h_2/\|h_2\|_X\right]$ and the unit ball of the stable norm has a flat in the direction h. □

According to Proposition 4.1.13, the only integral homology classes that are minimized by a non-simple curve are type 3 classes. We have shown:

Proposition 4.1.15. *The unit ball of the stable norm of X has a flat in the direction $(m,n) \in H_1(X,\mathbb{Z})$ if and only if (m,n) is not visible* and *none of the classes $(b,a),(d,c)$ is visible, where a/b and c/d are the Farey parents of n/m.*

Note that necessarily, irrational homology classes must lie in a flat of the unit ball. Indeed, by Proposition 4.1.13, all but possibly a discrete subset of the rational classes lie in a flat of the unit ball, thus by a density argument so do the irrational classes.

Strictly convex directions of the stable norm. According to Lemma 4.1.14, the unit ball of the stable norm is strictly convex in the direction (m,n) if and only if every minimizing curve is simple. Recall that for classes of type 1 and 2 in Proposition 4.1.13 we have found a simple minimizing curve that is *strictly shorter* than any non-simple

curve representing (m,n), so *every* minimizing curve of such a class must be simple. Thus the strictly convex directions of the stable norm of X are exactly the directions of classes of type 1 and 2.

Let $\mathscr{F}_k$ be the Farey sequence of order $k \in \mathbb{N}^*$; it is a subset of the vertices of the Stern-Brocot tree (see Figure 3.1). Note that $\mathscr{F}_k$ is *not* the k-th floor of this tree. We denote $\mathscr{F}_k^{\oplus 1}$ the set of vertices of the Stern-Brocot tree whose distance from $\mathscr{F}_k$ is at most 1.

If $L = \lfloor 1/\rho \rfloor$, where ρ is the length of the slit, another way to state Proposition 4.1.13 is by saying that $(m,n) \in H_1(X,\mathbb{Z})$ is a strictly convex direction of the stable norm if and only if $n/m \in \mathscr{F}_L^{\oplus 1} \cup \{1/0\}$.

A very important remark is that the strictly convex directions accumulate in the direction of visible homology classes: indeed, the accumulation points $(\mathscr{F}_L^{\oplus 1})'$ of $\mathscr{F}_L^{\oplus 1}$ for the induced topology from $\mathbb{R}$ are

$$(\mathscr{F}_L^{\oplus 1})' = \mathscr{F}_L$$

as the sequence of successive Farey children of a visible class n/m converges to n/m.

Vertices of the unit ball. We now know what the strictly convex directions of $\mathscr{B}$ are, but we can ask a more precise question: we would like to know what is the geometry of $\mathscr{B}$ at those points. We will see that $\mathscr{B}$ has vertices in those directions, except for the vertical directions $\pm(0,1)$.

Let $p \in \partial\mathscr{B}$ be a point of the unit sphere. A *supporting line* for $\mathscr{B}$ at p is a line whose intersection with $\mathscr{B}$ is reduced to $\{p\}$. We say that p is a *vertex* of $\mathscr{B}$ if there are infinitely many supporting lines for $\mathscr{B}$ at p.

Proposition 4.1.16. *Let $L = \lfloor 1/\rho \rfloor$. Let $(m,n) \in H_1(X,\mathbb{Z})$ be an integral homology class. Then the unit ball of the stable norm of X has a vertex in the direction (m,n) if and only if $n/m \in \mathscr{F}_L^{\oplus 1}$.*

Proof. First let us consider of classes of type 2 (labeled as in Proposition 4.1.13) as it is the easier case. Let $(m,n) \in H_1(X,\mathbb{Z})$ be a type 2 homology class, i.e (m,n) is *not* a visible class, but at least one of the homology classes associated to the Farey parents of n/m is visible. In other words, we have $n/m \in \mathscr{F}_L^{\oplus 1} - \mathscr{F}_L$. We have already seen that the set of accumulation points of $\mathscr{F}_L^{\oplus 1}$ is $(\mathscr{F}_L^{\oplus 1})' = \mathscr{F}_L$: hence every point in $\mathscr{F}_L^{\oplus 1} - \mathscr{F}_L$ is *isolated*. Thus, the point of $\partial\mathscr{B}$ in the direction (m,n) lies on the edge of two flats of $\mathscr{B}$: since the stable norm is strictly convex in the direction (m,n) the only possibility is that $\mathscr{B}$ has a vertex in the direction (m,n).

Now we move on to the case of visible classes. Let (m,n) be a visible homology class and assume $m \geq 1$, so that $(m,n) \neq \pm(0,1)$ as we will deal with these classes later; we have $n/m \in \mathscr{F}_L$.

According to the previous point, since $(\mathscr{F}_L^{\oplus 1})' = \mathscr{F}_L$ we know there are infinitely many vertices of $\mathscr{B}$ that accumulate to $p := (m,n)/\|(m,n)\|_X$: it might happen that there is only one supporting line at p, namely the line tangent to the circle at p. Those

vertices accumulating on p correspond to the Farey children of n/m, so they are of the form

$$u_k := \frac{(\beta + km, \alpha + kn)}{\|(\beta + km, \alpha + kn)\|_X}$$

on the left side of p and

$$v_k := \frac{(\delta + km, \gamma + kn)}{\|(\delta + km, \gamma + kn)\|_X}$$

on the right side of p, where $k \in \mathbb{N}^*$ and (β, α) and (δ, γ) are two visible classes such that $\alpha/\beta, n/m$ and γ/δ are consecutive in $\mathscr{F}_L$. Indeed, the k-th Farey child of n/m with α/β is obtained by taking the k successive mediants with n/m, i.e it is

$$\frac{n}{m} \oplus \ldots \oplus \frac{n}{m} \oplus \frac{\alpha}{\beta}.$$

Let $\Delta_k = p - u_k$. Since (m, n) is visible, we have $\|(m, n)\|_X = \sqrt{m^2 + n^2}$, and since u_k is the direction of a type 2 class, by Proposition 4.1.13 we have

$$\|(\beta + km, \alpha + kn)\|_X = \sqrt{m^2 + (n \pm \rho)^2} + \sqrt{(\beta + (k-1)m)^2 + (\alpha + (k-1)n \mp \rho)^2}.$$

Injecting this formula back in Δ_k and taking the Taylor series expansion of each of its coordinates then yields

$$\Delta_k = \left(\frac{c_1}{k} + O(1/k^2), \frac{c_2}{k} + O(1/k^2)\right)$$

where c_1 and c_2 are non-zero constants depending only on m, n, α, β and ρ. Thus the slopes of the vectors Δ_k goes to the non-zero constant c_2/c_1 as k goes to infinity: in particular those slopes are bounded away from zero. Hence a line passing through p with a slope in between 0 and the slopes of the Δ_k's does not intersect $\mathscr{B}$ outside of p: it is a supporting line of $\mathscr{B}$ at p, thus p is a vertex of $\mathscr{B}$.

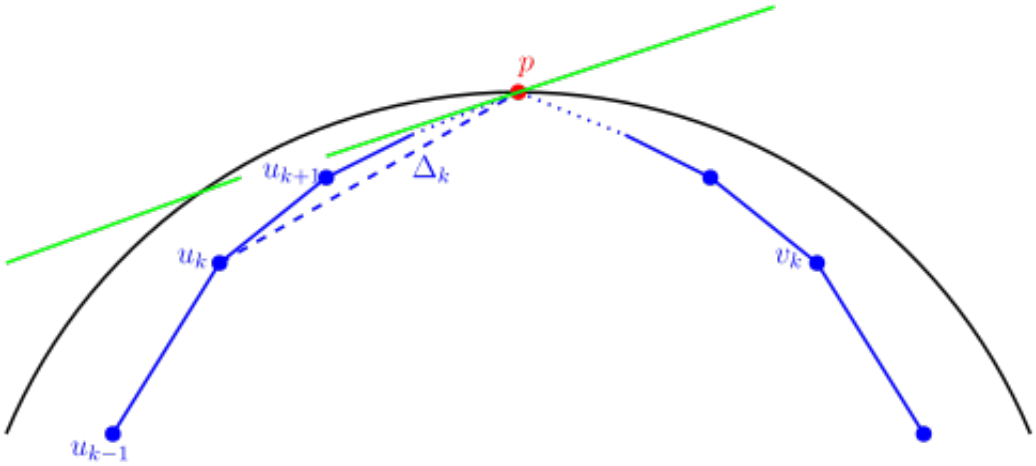

Figure 4.13: The green line is a supporting line of $\mathscr{B}$ at p

Finally, the classes $\pm(0, 1)$ are visible but are not the direction of vertices of $\mathscr{B}$. The other visible classes accumulate in the vertical direction. But the points of $\partial\mathscr{B}$ in the direction of the visible classes lie on the unit circle $\mathbb{S}^1$, so any supporting line of $\mathscr{B}$ at

$\pm(0,1)$ must not intersect the unit circle elsewhere; otherwise its intersection with $\mathscr{B}$ would not be reduced to a point. Thus such a line must be tangent to $\mathbb{S}^1$ at $\pm(0,1)$, meaning there is a *unique* supporting line at $\mathscr{B}$ at $\pm(0,1)$, so this point is not a vertex of $\mathscr{B}$.

□

4.1.6 Statement of the theorem and pictures of the unit ball

Putting together all the previous results, we have completely computed the stable norm of the slit torus X and described its unit ball.

Theorem 4.1.17. *Let X be the square flat torus with a vertical slit of length ρ. Let $L = \lfloor 1/\rho \rfloor$ be the integral part of $1/\rho$. Let $h = (m,n) \in H_1(X,\mathbb{Z})$ be a primitive integral homology class, and let a/b and c/d be the Farey parents of n/m, with $a/b < c/d$. The unit ball $\mathscr{B}$ of the stable norm of X has a vertex in the direction h if and only if either*

- $\dfrac{n}{m} \in \mathscr{F}_L$, *that is to say* $1 \leq |m| \leq L$. *In that case* $\|(m,n)\|_X = \sqrt{m^2+n^2}$.

  or

- $\dfrac{n}{m}$ *does not belong to* $\mathscr{F}_L$, *but* $\dfrac{a}{b} \in \mathscr{F}_L$ *or* $\dfrac{c}{d} \in \mathscr{F}_L$. *In that case we have*

$$\|(m,n)\|_X = \sqrt{b^2+(a+\rho)^2} + \sqrt{d^2+(c-\rho)^2}.$$

Moreover, the stable norm is strictly convex in the vertical directions $\pm(0,1)$, and $\|(0,1)\|_X = 1$. Finally, the point of the unit sphere $\partial\mathscr{B}$ of the stable norm in every other direction of $H_1(X,\mathbb{R})$ lies in the interior of a segment. More precisely, for an integral homology class with the previous notations, if $|m|,|b|,|d| > L$ then $\|(m,n)\|_X = \|(b,a)\|_X + \|(d,c)\|_X$.

It is difficult to produce a good picture of the unit ball of the stable norm of the slit torus. Indeed, it is so close to the unit circle that they are barely distinguishable from one another by eyesight; moreover, the angles at the vertices are nearly flat so they appear almost invisible, making $\mathscr{B}$ look like a polygon. The best pictures are obtained when ρ is large, so that the stable norm differs as much as possible from the Euclidean norm, but not too close to 1, so that at least some of the angles at the vertices are visible by eyesight.

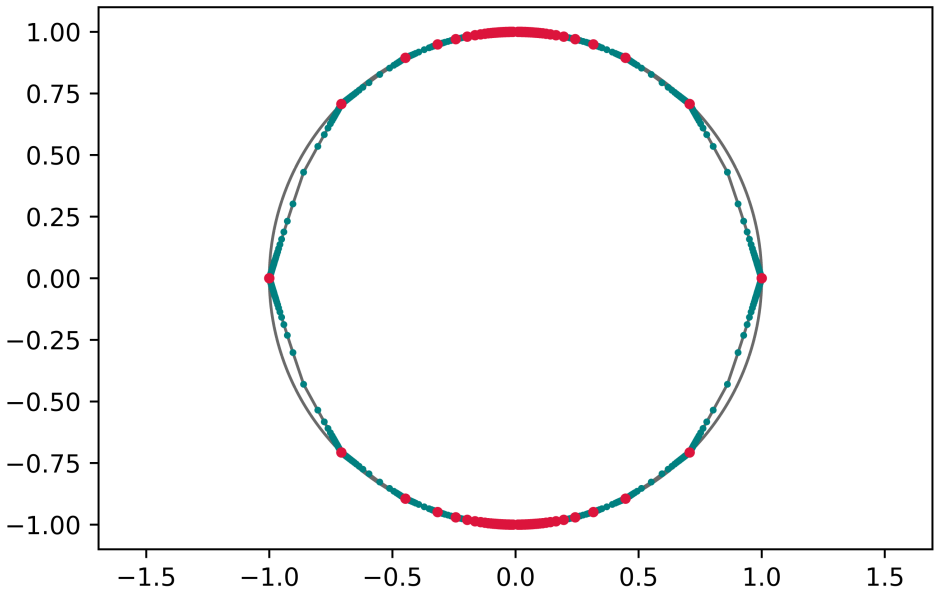

Figure 4.14: The unit ball $\mathscr{B}$ inside $\mathbb{S}^1$ (grey) for $\rho = 0.85$.

The red dots are the vertices of $\mathscr{B}$ corresponding to visible classes, and the blue dots are the vertices corresponding to the Farey children of visible classes. Here is a zoomed image of the part of $\mathscr{B}$ for directions of slope $n/m < 2.2$ (the axis have been stretched by a linear map to further improve the readability of the picture):

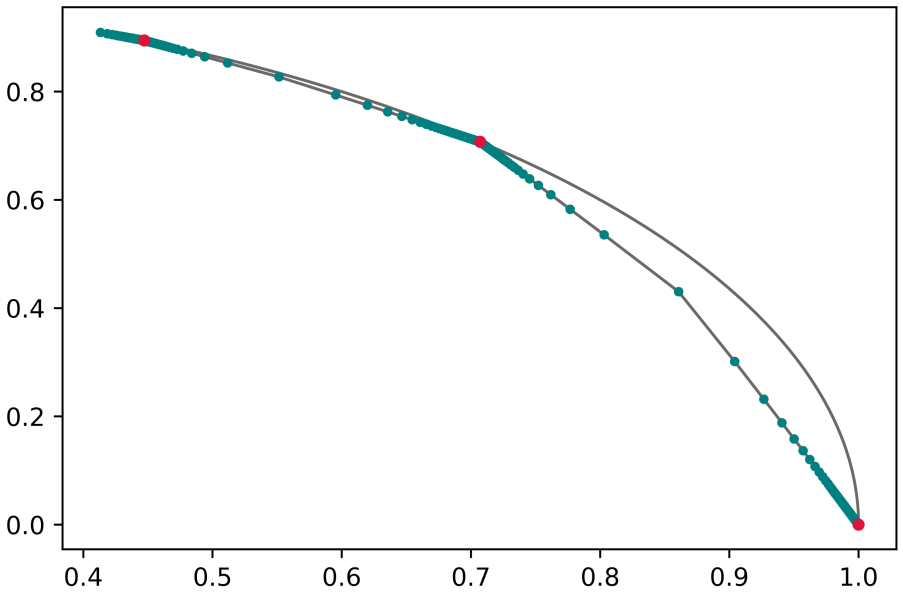

Figure 4.15: Arc of the unit ball for $\rho = 0.85$

Even after zooming and stretching the axis, the resulting picture is not really satisfying. In order to grasp the structure of $\mathscr{B}$, we flatten the arc of the unit circle between angles 0 and $\pi/4$ onto the segment $[0,1]$, and to $x \in [0,1]$ we associate the distance between the unit circle $\mathbb{S}^1$ and the unit ball of the stable norm $\mathscr{B}$ in the direction of slope x. The points of non-differentiability of the resulting curve correspond to the vertices of $\mathscr{B}$. On the left figure, with $1/3 < \rho < 1/2$, we can clearly see the three visible directions where $\mathscr{B}$ intersects $\mathbb{S}^1$, namely $(1,0), (2,1)$ and $(1,1)$. The right figure, with $1/5 < \rho < 1/4$, highlights the accumulation of vertices on the visible direction $(1,0)$.

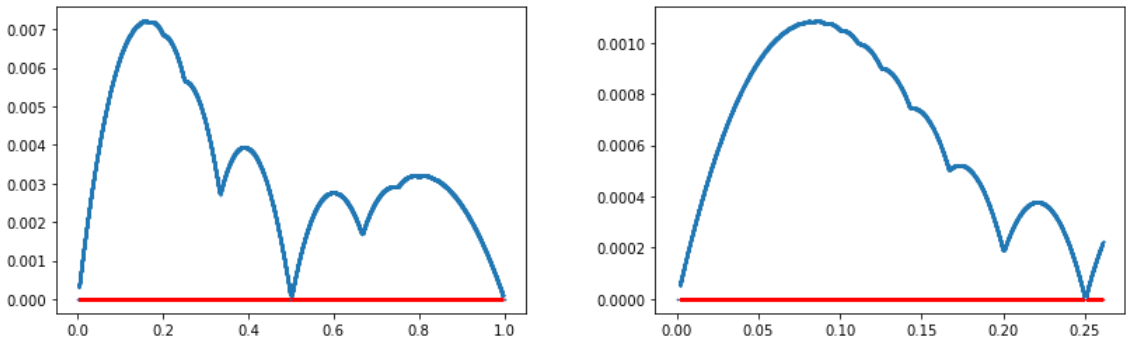

Figure 4.16: Unit ball of the stable norm (blue) near an arc of the unit circle (red)

Computing the stable norm of a given homology class. So far we have provided explicit formulas for the stable norm of integral homology classes in strictly convex directions of $\mathscr{B}$, and said the other directions were flats of the stable norm. This data is sufficient to compute the stable norm of any other homology class $(x,y) \in H_1(X,\mathbb{R})$, even irrational ones. Note that x and y are non-zero, as we already know the stable norm of the classes $(1,0)$ and $(0,1)$. Here is how to proceed.
First, find which flat of $\partial\mathscr{B}$ contains the point $(x,y)/\|(x,y)\|_X$. More precisely, find the directions (q,p) and (q',p') with $p/q < y/x < p'/q'$ of the endpoints of this flat, which are type 2 homology classes such that p/q and p'/q' are Farey neighbours. While rather easy since we know the set of type 2 homology classes corresponds to the slopes $\mathscr{F}_L^{\oplus 1} - \mathscr{F}_L$, this step has to be done numerically (note that for integral classes we simply have to look at their Farey ancestors). Then, there exists $\lambda \in [0,1]$ such that

$$\frac{(x,y)}{\|(x,y)\|_X} = \lambda \frac{(q,p)}{\|(q,p)\|_X} + (1-\lambda)\frac{(q',p')}{\|(q',p')\|_X}$$

so we can write the system

$$\begin{cases} \dfrac{x}{\|(x,y)\|_X} = \dfrac{\lambda q}{\|(q,p)\|_X} + \dfrac{(1-\lambda)q'}{\|(q',p')\|_X} \\ \dfrac{y}{\|(x,y)\|_X} = \dfrac{\lambda p}{\|(q,p)\|_X} + \dfrac{(1-\lambda)p'}{\|(q',p')\|_X} \end{cases}$$

where the unknown quantities are λ and $\|(x,y)\|_X$. Solving this system, which is always possible since x and y are non-zero, we then obtain a formula for the stable norm of the class (x,y).

For example, let us compute the stable norm of the irrational homology class $(\pi, \sqrt{2})$ on X with a slit of length $\rho = 2/5$. The visible homology classes are the classes (m,n) with $|m| \leq 2$ and since

$$\frac{\sqrt{2}}{\pi} \simeq 0.450158...$$

we know the flat of $\mathscr{B}$ the class $(\pi, \sqrt{2})$ belongs to has its endpoints associated to Farey children of $1/2$. Going through the numerical values of the successive Farey children of $1/2$ we find $4/9 \approx 0.44444$ and $5/11 \approx 0.454545$ so $4/9 < \sqrt{2}/\pi < 5/11$.

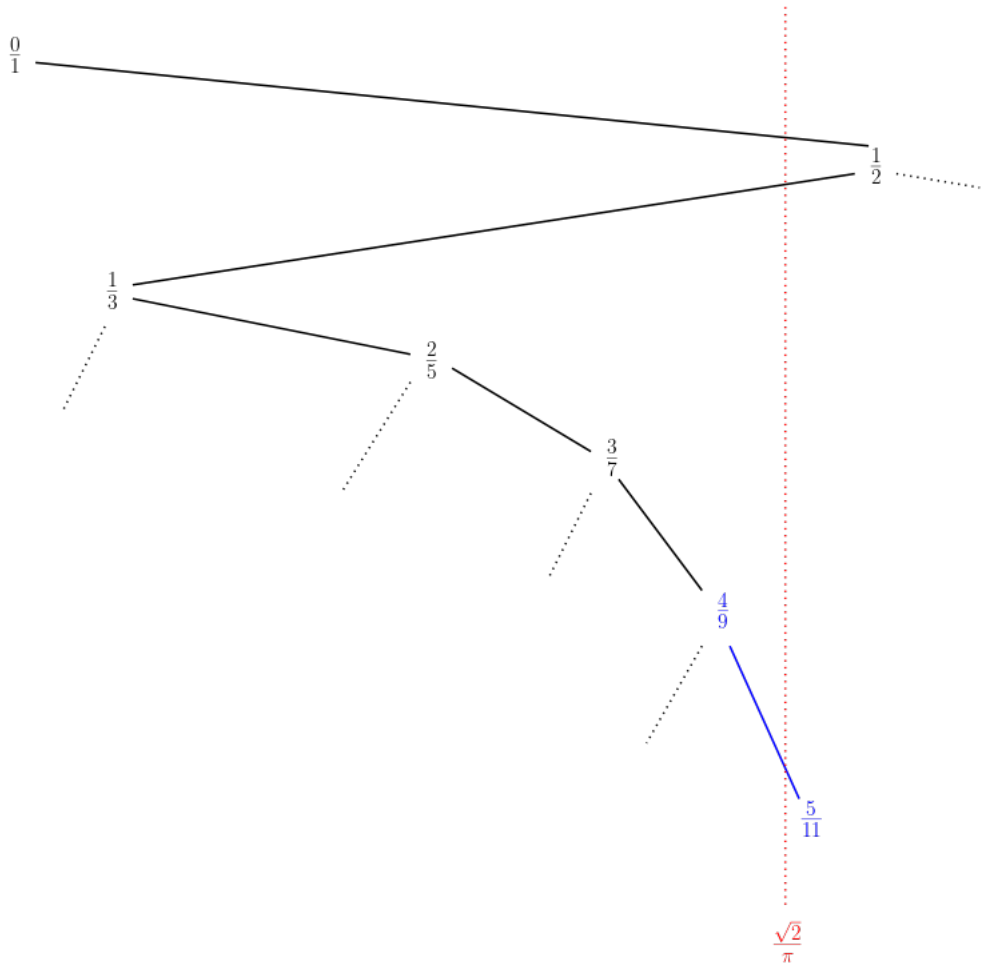

Figure 4.17: Approximating $\sqrt{2}/\pi$ by slopes of vertices of $\mathscr{B}$

We have

$$\begin{cases} \dfrac{\pi}{\|(\pi,\sqrt{2})\|_X} = \lambda \dfrac{9}{\|(9,4)\|_X} + (1-\lambda)\dfrac{11}{\|(11,5)\|_X} \\ \dfrac{\sqrt{2}}{\|(\pi,\sqrt{2})\|_X} = \lambda \dfrac{4}{\|(9,4)\|_X} + (1-\lambda)\dfrac{5}{\|(11,5)\|_X} \end{cases}$$

By Theorem 4.1.17 we have $\|(9,4)\|_X = \frac{1}{5}(\sqrt{1514}+\sqrt{109})$ and $\|(11,5)\|_X = \frac{1}{5}(\sqrt{2509}+\sqrt{109})$ so solving the system we finally get

$$\frac{1}{\|(\pi,\sqrt{2})\|_X} = \frac{\frac{45}{\pi}\left(\frac{5\sqrt{2}\pi}{\sqrt{2509}+\sqrt{109}} - \frac{18}{\sqrt{1514}+\sqrt{109}}\right)}{\left(\sqrt{2}\pi\left(\frac{5}{\sqrt{2509}+\sqrt{109}} - \frac{4}{\sqrt{1514}+\sqrt{109}}\right) + \frac{22}{\sqrt{2509}+\sqrt{109}} - \frac{18}{\sqrt{1514}+\sqrt{109}}\right)\left(\sqrt{1514}+\sqrt{109}\right)} - \frac{\frac{55}{\pi}\left(\frac{\frac{5\sqrt{2}\pi}{\sqrt{2509}+\sqrt{109}} - \frac{18}{\sqrt{1514}+\sqrt{109}}}{\sqrt{2}\pi\left(\frac{5}{\sqrt{2509}+\sqrt{109}} - \frac{4}{\sqrt{1514}+\sqrt{109}}\right) + \frac{22}{\sqrt{2509}+\sqrt{109}} - \frac{18}{\sqrt{1514}+\sqrt{109}}} - 1\right)}{\sqrt{2509}+\sqrt{109}}$$

which is terrifying, but a closed formula nonetheless. We displayed $1/\|(\pi,\sqrt{2})\|_X$ instead of $\|(\pi,\sqrt{2})\|_X$ itself in order for the formula to fit inside the page.

4.2 Stable norm of general flat slit tori

We now fully understand the stable norm of the *square* flat torus with a *vertical* slit. Next we want to understand the stable norm of *any* slit torus, that is to say a slit torus whose metric comes from a generic parallelogram and with a slit of general direction. We will proceed by taking the problem back to something that we already understand, that is the square torus with a vertical slit.

Here is the general strategy of this section.

Let us consider a general slit torus, that is a flat torus whose metric comes from a generic parallelogram and with a slit of general direction. We deform this surface by the linear action of a $SL(2,\mathbb{R})$ matrix so that the parallelogram defining the metric is sent to a square. Providing we have a way to keep track of the stable norm under $SL(2,\mathbb{R})$-deformation of the surface (which will be discussed in the first paragraph), the problem of computing the stable norm of a general slit torus comes down to computing the stable norm for *square* slit tori, with a slit of any direction.

So let X' be a square torus with a slit of general direction. If the slope of the slit is rational we say that X' is a *rational* slit torus, and if it is not we say X' is an *irrational* slit torus.
If X' is rational we will see it is the image of a square torus X with a *vertical* slit under the action of a $SL(2,\mathbb{Z})$ matrix, so the problem comes down to computing the stable norm of X, which we already solved in the previous section.
If X' is irrational then it can no longer be seen as the image of a square torus with a vertical slit by a linear deformation. However, it is the limit of a sequence of rational slit tori in the Hausdorff-Lipschitz topology; in this topology, the stable norm is known to be a continuous function of the surface. Thus, taking the limit of the stable norms we computed in the rational case we will finally be able to compute the stable norm of any slit torus.

4.2.1 The $SL(2,\mathbb{R})$-action on slit tori

In this short section we make the somewhat trivial but useful observation that the results we obtained in the case of the square torus with a vertical slit still hold under linear deformation of the surface.

Invertible matrices act linearly on $\mathbb{R}^2$ and transform the polygonal model of the flat square torus X with a vertical slit (Figure 4.1.1) into a parallelogram with a slit parallel to a side of the parallelogram. Glueing the opposites sides of the parallelogram together by translation, we obtain a new flat slit torus that we denote MX. In general, the flat metric of MX does not come from a square. More formally, an invertible matrix $M \in GL(2,\mathbb{R})$ induces a diffeomorphism $M : X \longrightarrow MX$ between the slit tori X and MX.

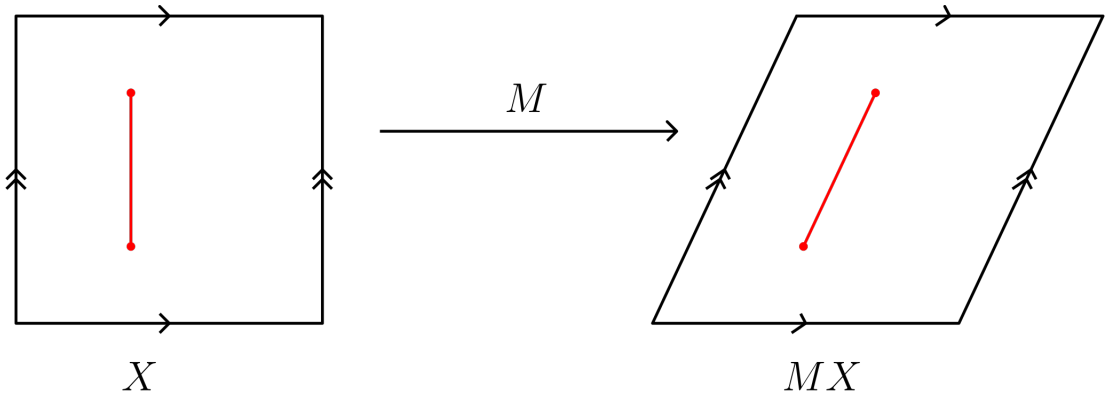

Figure 4.18: Diffeomorphism between slit tori induced by M

We want to compute the stable norm of MX. Since the stable norm transforms homogeneously with respect to the scaling of the surface, we can assume that MX also has area 1, i.e we can only consider determinant 1 matrices of $SL(2,\mathbb{R})$. The sides of the parallelogram modelling MX project onto closed curves that form a natural basis for $H_1(MX,\mathbb{Z})$. As the diffeomorphism M maps the sides of the square to the sides of the parallelogram, with this choice of basis the application between homology groups $M_* : H_1(X,\mathbb{Z}) \longrightarrow H_1(MX,\mathbb{Z})$ induced by the diffeomorphism M is the identity. Repeating the same arguments as in the previous section in this new setting, we get a slightly more general version of Theorem 4.1.17:

Proposition 4.2.1. *Let X be a square torus with a vertical slit of length $0 < \rho < 1$, and let $L = \lfloor 1/\rho \rfloor$. Let $M \in SL(2,\mathbb{R})$ be a determinant 1 matrix. The unit ball $\mathscr{B}$ of the stable norm of the slit torus MX has a vertex in the direction $h = (m,n) \in H_1(MX,\mathbb{Z})$ if and only if either*

- *$n/m \in \mathscr{F}_L$, and in that case $\|(m,n)\|_{MX} = \left\| M.\begin{pmatrix} m \\ n \end{pmatrix} \right\|_2$*

  or

- *$n/m \notin \mathscr{F}_L$ and $a/b \in \mathscr{F}_L$ or $c/d \in \mathscr{F}_L$ where $a/b < c/d$ are the Farey parents of n/m, and in that case*

$$\|(m,n)\|_{MX} = \left\| M.\left[\begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \rho \end{pmatrix}\right] \right\|_2 + \left\| M.\left[\begin{pmatrix} d \\ c \end{pmatrix} - \begin{pmatrix} 0 \\ \rho \end{pmatrix}\right] \right\|_2$$

The unit ball $\mathscr{B}$ is strictly convex in the vertical direction $\pm(0,1)n$ and

$$\|(0,1]\|_{MX} = \left\| M.\begin{pmatrix}0\\1\end{pmatrix}\right\|_2$$

Finally, the unit ball $\mathscr{B}$ has a flat in every other direction of $H_1(MX,\mathbb{R})$. In particular, for an integral class with the previous notations, if $|m|,|b|,|d| > L$ then $\|(m,n)\|_{MX} = \|(b,a)\|_{MX} + \|(d,c)\|_{MX}$.

Proof. The proof is exactly the same as in the case of a square slit torus with a vertical slit, up to shearing all the pictures under the linear action of M. In fact, M maps minimizing curves on X to minimizing curves on MX. More precisely, going through the proof step by step to make sure all the arguments still hold in this new setting:

- As the linear action of M obviously preserves straight line paths, $(m,n) \in H_1(MX,\mathbb{Z})$ is visible on MX if and only if $(m,n) \in H_1(X,\mathbb{Z})$ is visible on X, that is if and only if $n/m \in \mathscr{F}_L$. Hence, in that case we have $\|(m,n)\|_{MX} = \left\| M.\begin{pmatrix}m\\n\end{pmatrix}\right\|_2$.

If (m,n) is not visible on MX, with $a/b < c/d$ the Farey parents of n/m:

- As the application on homology groups $M_* : H_1(X,\mathbb{R}) \longrightarrow H_1(MX,\mathbb{R})$ induced by the diffeomorphism M is the identity, the proof of Proposition 4.1.4 still holds in this new setting. Hence the shortest non-simple curve representing (m,n) on MX has length $\|(b,a)\|_{MX} + \|(d,c)\|_{MX}$.

- Since M is a diffeomorphism, it maps bijectively simple curves on X to simple curves on MX: in particular we know what the free homotopy classes of simple closed geodesics on MX are, and geometrical constraints on the curves can be derived from them. Thus if (b,a) or (d,c) is visible we obtain the shortest simple curve γ representing (m,n) in the exact same way as in Proposition 4.1.9. It is the image by M of the minimizing curve of (m,n) on X. The action of M shears Figure 4.9 and Figure 4.10 but does not change how lengths compare to one another, so γ is minimizing for (m,n). Hence

$$\|(m,n)\|_{MX} = l(\gamma) = \left\| M.\left[\begin{pmatrix}b\\a\end{pmatrix} + \begin{pmatrix}0\\\rho\end{pmatrix}\right]\right\|_2 + \left\| M.\left[\begin{pmatrix}d\\c\end{pmatrix} - \begin{pmatrix}0\\\rho\end{pmatrix}\right]\right\|_2$$

- Again, since M maps simple curves to simple curves if none of (b,a) and (d,c) are visible, since there are no simple geodesics representing (m,n) on X, there are no simple geodesics representing (m,n) on MX. So the unit ball of the stable norm of MX has a flat in the direction (m,n).

□

Remark that although the strictly convex directions of the stable norm are the same for X and MX (with the natural but arbitrary choice of homology basis we have made) the lengths do not all transform linearly, so the unit ball of the stable norm of MX is *not* the image of the unit ball of X under the linear action of M on $H_1(X,\mathbb{R}) \simeq \mathbb{R}^2$.

4.2.2 Rational slit tori

In this technical section we focus on a special case of $SL(2,\mathbb{R})$ deformations of square tori with a vertical slit. The goal is to reformulate Proposition 4.2.1 in a more convenient setting in order to simplify the arguments given in the next section.

Motivation. Let X be a square torus with a vertical slit. An interesting special case of the $SL(2,\mathbb{R})$ linear deformation of X is the case of $SL(2,\mathbb{Z})$ matrices. The linear action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2$ preserves the lattice $\mathbb{Z}^2$: thus by applying a $SL(2,\mathbb{Z})$ matrix to X we get a parallelogram that can be rearranged into a square by cut and paste operations. These operations on the polygonal model do not change the metric of the underlying surface. In other words, the image of X under a diffeomorphism induced by an $SL(2,\mathbb{Z})$ matrix is again a square slit torus, with a slit that is *a priori* not vertical but has rational slope.

More precisely, if X is the square torus with a vertical slit and $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2,\mathbb{Z})$, the slit torus MX comes from a parallelogram that can isometrically be rearranged by cut and paste operations into a square slit torus X' with a slit of slope $c/d \in \mathbb{Q}$. We say that X' is a *rational* slit torus. Conversely, a square torus with a slit of irrational slope is called an *irrational* slit torus: it is clear from the discussion above that no irrational slit torus can be obtained by acting on X by a $SL(2,\mathbb{R})$ matrix, as it is not possible to obtain simultaneously a metric coming from a square *and* a slit of irrational slope.

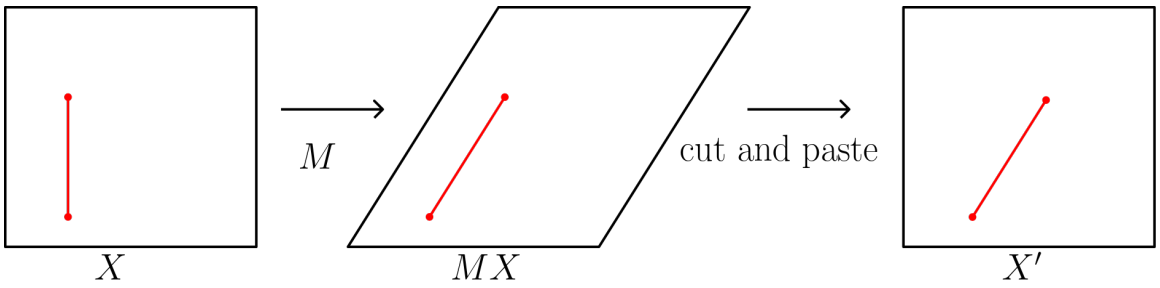

Figure 4.19: Operations on X leading to X'

Note that by Proposition 4.2.1 we already know how to compute the stable norm of rational slit tori, as they are merely a special case of the $SL(2,\mathbb{R})$-action on X. However, we would like to reformulate this proposition for rational tori in a way that will be helpful to deal with irrational tori afterwards.

The starting observation is that even if MX and X' are isometric surfaces, we do not want to completely identify them together as they are naturally equipped with two different coordinate systems on the torus. When we denote MX we want to think about the surface obtained from a parallelogram obtained by shearing the polygonal model of X: in particular in this model the slit is parallel to the sides of the parallelogram. When we denote X' we instead consider the surface that comes from a square, with a slit whose direction has no special link with the sides of the square apart from being

rational. Why do we bother? For both MX and X', the natural choice for a homology basis is given by the sides of the polygon we use as a model. On MX, the slit is parallel to a side of the parallelogram; if we were to slightly change the direction of the slit, we would need to change the parallelogram (hence the metric) along to still be able to compute the stable norm of the surface. This forces us to always think of MX as a surface obtained from X, as we need to keep track of where the surface is coming from. This is no longer the case on X', as the choice of the homology basis is independent of the direction of the slit. Far from anecdotal, this point of view will greatly facilitate dealing with sequences of rational tori, which we will do in the next section. Proposition 4.2.1 provides formulas for the stable norm in the coordinate system associated to MX; we now want to adapt this proposition to the coordinate system associated to X'.

The setting. We represent the slit of X' with the vector (β,α). More precisely, if $\tilde{X}'$ is the Abelian covering space of X' such that the lower endpoint of the slit is lifted to the point $(0,0)$, then the point (β,α) is the upper endpoint of this copy of the slit. By assumption the slit has rational direction, that is $\alpha/\beta \in \mathbb{Q}$. Note that α and β might be irrational themselves, the only requirement is for their quotient to be rational: thus, there exists an irreducible fraction $p/q \in \mathbb{Q}$ such that $\alpha/\beta = p/q$. Since X' is a rational slit torus, by definition there exists a square torus X with a vertical slit of length ρ and a matrix $M \in SL(2,\mathbb{Z})$ such that X' is isometric to MX through a cut and paste isometry $\varphi : MX \longrightarrow X'$ whose induced map in homology $\varphi_* : H_1(MX,\mathbb{Z}) \longrightarrow H_1(X',\mathbb{Z})$ acts as M.

By definition, the diffeomorphism $\varphi \circ M : X \longrightarrow X'$ sends the slit of X to the slit of X'. In homology, this means that

$$\varphi_* \circ M_* \left(\begin{pmatrix} 0 \\ \rho \end{pmatrix} \right) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

Since M_* is the identity and φ_* acts as M, we have $M \begin{pmatrix} 0 \\ \rho \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. Since M has integer entries and $p/q = \alpha/\beta$ with p and q coprime, necessarily the second column of M is the vector $\begin{pmatrix} q \\ p \end{pmatrix}$. The matrix M is of the form

$$M = \begin{pmatrix} v & q \\ u & p \end{pmatrix}$$

with $u, v \in \mathbb{Z}$ such that $vp - uq = 1$. We have

$$M^{-1} = \begin{pmatrix} p & -q \\ -u & v \end{pmatrix}$$

so

$$M^{-1} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} p\beta - q\alpha \\ v\alpha - u\beta \end{pmatrix} = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

so the length ρ of the vertical slit of X, the torus from which X' is obtained, is

$$\rho = v\alpha - u\beta$$

We now have all the tools to apply Proposition 4.2.1 to X'.

Visible classes. Let $(m,n) \in H_1(X',\mathbb{Z})$ be a primitive integral homology class on X'. Since $\varphi_* : H_1(MX,\mathbb{Z}) \longrightarrow H_1(X',\mathbb{Z})$ acts as M, the class (m,n) on X' reads as

$$M^{-1}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} mp-nq \\ vn-um \end{pmatrix}$$

on MX (we identify line and column vectors for the sake of simplicity). Thus

$$\|(m,n)\|_{X'} = \|(mp-nq, vn-um)\|_{MX}$$

As before, we say that a homology class is visible on X' if the lift to $\tilde{X}'$ of a minimizing curve for this class is a straight line segment. Thus the class (m,n) is visible on X' if and only if the class $(mp-nq, vn-um)$ is visible of MX. According to Proposition 4.2.1, this is in turn equivalent to

$$|mp-nq| \leq \left\lfloor \frac{1}{\rho} \right\rfloor \iff |(v\alpha-u\beta)(mp-nq)| \leq 1$$

Remembering that $p/q = \alpha/\beta$, straightforward computations yield

$$(m,n) \text{ visible on } X' \iff |m\alpha - n\beta| \leq 1$$

and in that case

$$\|(m,n)\|_{X'} = \|(m,n)\|_2$$

as it can be seen directly from the definition or from the formula in Proposition 4.2.1.

Non-visible classes. Let $(m,n) \in H_1(X',\mathbb{Z})$ be a non-visible homology class on X': this is equivalent to $|m\alpha - n\beta| > 1$. Back on MX, the class $(mp-nq, vn-um)$ is not visible either. Let $a/b < c/d$ be the Farey parents of the rational $(vn-um)/(mp-nq)$. By Proposition 4.2.1, we know that if (b,a) or (d,c) is visible on MX then $(mp-nq, vn-um)$ is a strictly convex direction of the stable norm of MX, and so must be (m,n) for the stable norm of X'. However, establishing *useful* formulas for the stable norm of (m,n) on X' is a bit tricky this time. By useful we mean *intrinsic* formulas, that is formulas that do not refer to X, as the whole point of looking at X' was to get rid of references to X. Let us see why it is complicated.

Assume at least one of the classes (b,a) and (d,c) is visible on MX. By Proposition 4.2.1, we then have

$$\|(m,n)\|_{X'} = \left\| M^{-1}\begin{pmatrix} m \\ n \end{pmatrix} \right\|_{MX} = \left\| M.\left[\begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \rho \end{pmatrix} \right] \right\|_2 + \left\| M.\left[\begin{pmatrix} d \\ c \end{pmatrix} - \begin{pmatrix} 0 \\ \rho \end{pmatrix} \right] \right\|_2$$

so

$$\|(m,n)\|_{X'} = \left\| M\begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \right\|_2 + \left\| M\begin{pmatrix} d \\ c \end{pmatrix} - \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \right\|_2$$

The problem is to express the classes $M\begin{pmatrix} b \\ a \end{pmatrix} =: \begin{pmatrix} m' \\ n' \end{pmatrix}$ and $M\begin{pmatrix} d \\ c \end{pmatrix} =: \begin{pmatrix} m'' \\ n'' \end{pmatrix}$ on X' in an intrinsic way, without referencing to X or M. The difficulty lies in the fact that the rationals n'/m' and n''/m'' are Farey neighbours to n/m but in general they are *not* the

Farey parents of n/m. Indeed, since a/b is a Farey parent of $(vn-um)/(mp-nq)$ and *a fortiori* one of its Farey neighbours, we have

$$\det\begin{pmatrix} b & mp-nq \\ a & vn-um \end{pmatrix} = 1$$

so since $\det M = 1$,

$$\det\begin{pmatrix} m' & m \\ n' & n \end{pmatrix} = \det\left[M.\begin{pmatrix} m'' & mp-nq \\ n'' & vn-um \end{pmatrix}\right] = 1$$

and n'/m' and n/m are Farey neighbours; similarly, n/m and n''/m'' are Farey neighbours too. However in general these are not the Farey parents of n/m: the $SL(2,\mathbb{Z})$ action on the rational numbers (seen as slopes of vectors of $\mathbb{Z}^2$ on which the matrices act linearly) preserve the property of being Farey neighbours, but not the Farey parent relation. For example, the class $\begin{pmatrix}1\\0\end{pmatrix}$ is associated to the rational $0/1$, which is the Farey parent of the rational $-1/2$ associated to the class $\begin{pmatrix}2\\-1\end{pmatrix}$. These homology classes are sent by the matrix $\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}$ respectively to $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$ of slopes $1/2$ and $0/1$. These two rationals still form a Farey pair, but the parenting relation has been reversed: indeed, the Farey parent has been sent to the Farey child and the Farey child to the Farey parent. Thus the rationals n'/m' and n''/m'' could be any pair of Farey neighbours of n/m among infinitely many others, with *a priori* no canonical way (such as the Farey parents) to distinguish them in order to get intrinsic formulas for the stable norm of (m,n). Still, there are things we can say.

If *both* the classes (b,a) and (d,c) are visible on MX then the classes (m',n') and (m'',n'') are visible on X'. This case is easy: since a homology class on MX has at most two visible neighbours (its Farey parents) then a class on X' also has at most two visible neighbours. Thus on X', ignoring what we know about the classes (m',n') and (m'',n''), if we simply look at the neighbours of the class (m,n) and find two visible classes we know these have to be the images of the Farey parents of $M^{-1}\begin{pmatrix}m\\n\end{pmatrix}$. Thus, injecting these classes in the formula for $\|(m,n)\|_{X'}$ we finally get

$$\|(m,n)\|_{X'} = \|(m',n') \pm (\beta,\alpha)\|_2 + \|(m'',n'') \mp (\beta,\alpha)\|_2$$

The last difficulty is to figure out which term contains the "+" sign and which term contains the "−" sign: according to Proposition 4.2.1 we know the term containing the "+" sign corresponds to the term involving the image of the smallest Farey parent of $M^{-1}\begin{pmatrix}m\\n\end{pmatrix}$. Since we do not allow going back to MX to figure out which one it is, the only solution we have left is to try both formulas and pick the one that provides the smaller numerical value, that is $\min(\|(m',n')+(\beta,\alpha)\|_2+\|(m'',n'')-(\beta,\alpha)\|_2, \|(m',n')-(\beta,\alpha)\|_2+\|(m'',n'')+(\beta,\alpha)\|_2)$.

If exactly one of the classes (b,a) and (d,c) is visible on MX then things are a bit trickier, as this time (m,n) only has one visible neighbour on X' that we denote (m',n'). With the same argument as in the previous case, we know (m',n') must be either $M\begin{pmatrix}b\\a\end{pmatrix}$ or $M\begin{pmatrix}d\\c\end{pmatrix}$. What about the other one? We know it has to be neighbour to both (m,n) and (m',n'); unfortunately, there are exactly two classes (m_1'',n_1'') and (m_2'',n_2'') that satisfy this condition: see the Ford circle packing (Figure 3.11) where there are always two circles that are tangent to a fixed pair of tangent circles. There is no qualitative way to favor one of these classes over the other, so we have to rely on numerical values again. The stable norm of (m,n) is then equal to

$$\|(m,n)\|_{X'} = \min_{i=1,2}(\|(m',n') \pm (\beta,\alpha)\|_2 + \|(m_i'',n_i'') \mp (\beta,\alpha)\|_2)$$

for a total of four numerical values to compare, as we also need to figure out where the "+" and "−" are placed.

Finally, if neither (b,a) nor (d,c) is visible on MX then the unit ball of the stable norm of MX has a flat in the direction of the class $M^{-1}\begin{pmatrix}m\\n\end{pmatrix}$, and so does the unit ball of the stable norm of X' in the direction (m,n). This time, we cannot produce a formula as there is no way to determine intrinsically what the images of (b,a) and (d,c) are on X'.

Among other things, we have shown:

Corollary 4.2.1.1 (Intrinsic reformulation of Proposition 4.2.1)**.** *Let X' be a rational square torus with a slit represented by the vector (β,α) with $\alpha/\beta \in \mathbb{Q}$. Take the sides of the square representing X' as a basis for $H_1(X',\mathbb{R})$. In this basis, the unit ball of the stable norm of X' has a vertex in the direction $(m,n) \in H_1(X',\mathbb{Z})$, with $n/m \neq \alpha/\beta$, if and only if either*

- *(m,n) is visible, that is to say $|m\alpha - n\beta| \leq 1$. In that case, $\|(m,n)\|_{X'} = \|(m,n)\|_2$*

  or

- *$|m\alpha - n\beta| > 1$ and there exists $(m',n') \in H_1(X',\mathbb{Z})$ visible such that n'/m' and n/m are Farey neighbours. In that case, $\|(m,n)\|_{X'} > \|(m,n)\|_2$.*

The unit ball of the stable norm of X' is strictly convex in the directions $\pm(\beta,\alpha)$, with $\|(\beta,\alpha)\|_{X'} = \|(\beta,\alpha)\|_2$, and it has a flat in every other directions of $H_1(X',\mathbb{R})$.

Note there is no mention to Farey parents in the statement of this corollary: as we have already mentioned, the notion of Farey parents is not compatible with the $SL(2,\mathbb{Z})$-action on the rationals, contrary to the notion of Farey neighbours. The notion of Farey neighbours appears to be more robust than Farey parents; thus the right criterion for a homology class to be a non-visible strictly convex direction of the stable norm of rational slit tori (case of Theorem 4.1.17 included!) is being neighbour to a visible

class. In the case of the square torus with a vertical slit, it just happened that the visible classes are the Farey ancestors of the other classes. This does not mean that we could easily get rid of any reference to the Farey parents, as many arguments used to prove Theorem 4.1.17 relied heavily on their remarkable arithmetic properties, but rather than Farey parents are no longer the most relevant concept when dealing with general slit tori; one should think of Farey neighbours instead.

Also note that the statement of the Corollary is intrinsic, as it does not mention X nor M at all. The formulas given are simply qualitative, as the actual formulas we proved were complicated and impractical. This is due to the fact that the notion of Farey parents is not compatible with the linear $SL(2,\mathbb{Z})$-action on $\mathbb{Q}$; instead, we have to rely on Farey neighbours which are far less precise. Thus, using our approach to this problem, it is not possible to give a statement about the stable norm of X' that is both fully *quantitative* and *intrinsic*.

4.2.3 Irrational slit tori

Let X' be an irrational slit torus, i.e a square slit torus whose slit is given by the vector (β,α) with $\alpha/\beta \in \mathbb{R}\backslash\mathbb{Q}$. Let $\rho = \sqrt{\beta^2+\alpha^2}$ be the length of the slit. This time, X' is *not* in the $SL(2,\mathbb{R})$-orbit of a square torus with a vertical slit. The idea is to approximate X' by a sequence of rational slit tori.

The rational tori X_k. We will approximate the irrational torus X' with the following sequence of rational slit tori. Let $(p_k/q_k)_k$ be a sequence of rational numbers written as irreducible fractions such that $p_k/q_k \longrightarrow \alpha/\beta$ as k goes to infinity. For now any such sequence of rationals numbers would work, but in order to simplify future computations we pick the best possible one: more precisely, we assume that p_k/q_k is the k-th convergent to the continued fraction expansion of α/β, i.e if α/β has (unique) infinite continued fraction expansion

$$\frac{\alpha}{\beta} = [a_0, a_1, ..., a_n, ...] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + ... + \cfrac{1}{a_n + ...}}}$$

then we define

$$\frac{p_k}{q_k} = [a_0, a_1, ..., a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{... + \cfrac{1}{a_k}}}}$$

Let X_k be the square torus with a slit of length $\rho = \sqrt{\beta^2+\alpha^2}$ and slope p_k/q_k. For k large enough, one would expect X_k to be close to X' in any reasonable topology; in particular, it is easy to see that this is true in the Hausdorff-Lipschitz topology. By Proposition 1.3.4, assuming we have fixed a common homology basis for the X_k's and X', for h an integral homology class we have

$$\|h\|_{X_k} \longrightarrow \|h\|_{X'}$$

Hopefully the purpose of the previous section is now clear: we expressed the stable norm of rational tori in a homology basis that is independent of the slit, i.e that depends

only on the polygon used to model the surface (in our case, a square). Hence we now have a natural common homology basis for the X_k and X' that allows us to apply the above formula.

Thus, in order to compute the stable norm of a homology class on X' we only have to take the limit of the stable norm of this class on the X_k's as k goes to infinity, which we already know how to compute. This, however, is not fully satisfying for two reasons. First, as we mentioned earlier, it is rather tedious to produce explicit formulas for the stable norm of X_k, let alone taking their limit as k goes to infinity. Second and most importantly, we do not get a combinatorial description of the unit ball of the stable norm of X'. Indeed, many things can happen when taking a limit of convex sets: for instance, the regular n-gon becomes a circle when n goes to infinity, but the combinatorial structure of the n-gon and of the circle are very different. In order to describe the structure of the unit ball $\mathscr{B}$ of the stable norm of X' we will proceed as in the case of rational slit tori: we find what the visible classes are, and among non-visible classes we determine which classes correspond to a strictly convex direction of $\mathscr{B}$ and which classes lie inside a flat of $\mathscr{B}$.

Visible classes. As in the case of rational tori, we say that an integral homology class on X' is *visible* if the lift of a minimizing curve to the Abelian covering space $\tilde{X}'$ is a straight line. Equivalently, a class is visible if its stable norm coincides with the Euclidean norm on $H_1(X',\mathbb{R})$. Notice that the *irrational* homology classes $\pm(\beta,\alpha)$ are visible on X'; this is a big difference with rational tori, where the visible classes were all rational.

Proposition 4.2.2. *Let $(m,n)\in H_1(X',\mathbb{Z})$ be a primitive integral homology class on X'. Recall that the slit is represented by the vector (β,α), with $\alpha/\beta\in\mathbb{R}\backslash\mathbb{Q}$. Then (m,n) is visible if and only if $|m\alpha-n\beta|\leq 1$.*

Proof. Note that this is the same condition as for rational tori (see Corollary 4.2.1.1), but this time we cannot pull things back to a torus with a vertical slit by a linear map, so we need a different argument. Note that the following proof is more general as it also covers the case of rational tori.

By definition, the class (m,n) is visible if it is possible to go in a straight line from $(0,0)$ to (m,n) in the Abelian covering space of X'. Such a path projects onto X' as parallel segments of slope n/m. In the square model of X', draw a line of slope n/m starting from the bottom left corner of the square: as in the case of the torus with a vertical slit, we make the arbitrary choice that this corner is the lower endpoint of the slit. Draw the parallel line passing through the upper endpoint of the slit; this line intersects the left side of the square at height $\alpha-\frac{n}{m}\beta$. These two lines bound a band containing the slit. Now, a line of slope n/m in the Abelian cover of X' intersects the slit if and only if its projection to X' enters this band at any point. Since the projection of any line of slope n/m is parallel to this band, if it enters the band at any point it means it is trapped inside the band forever; in particular, it intersects the left side of the square inside of the band, i.e at height *below* $\alpha-\frac{n}{m}\beta$.

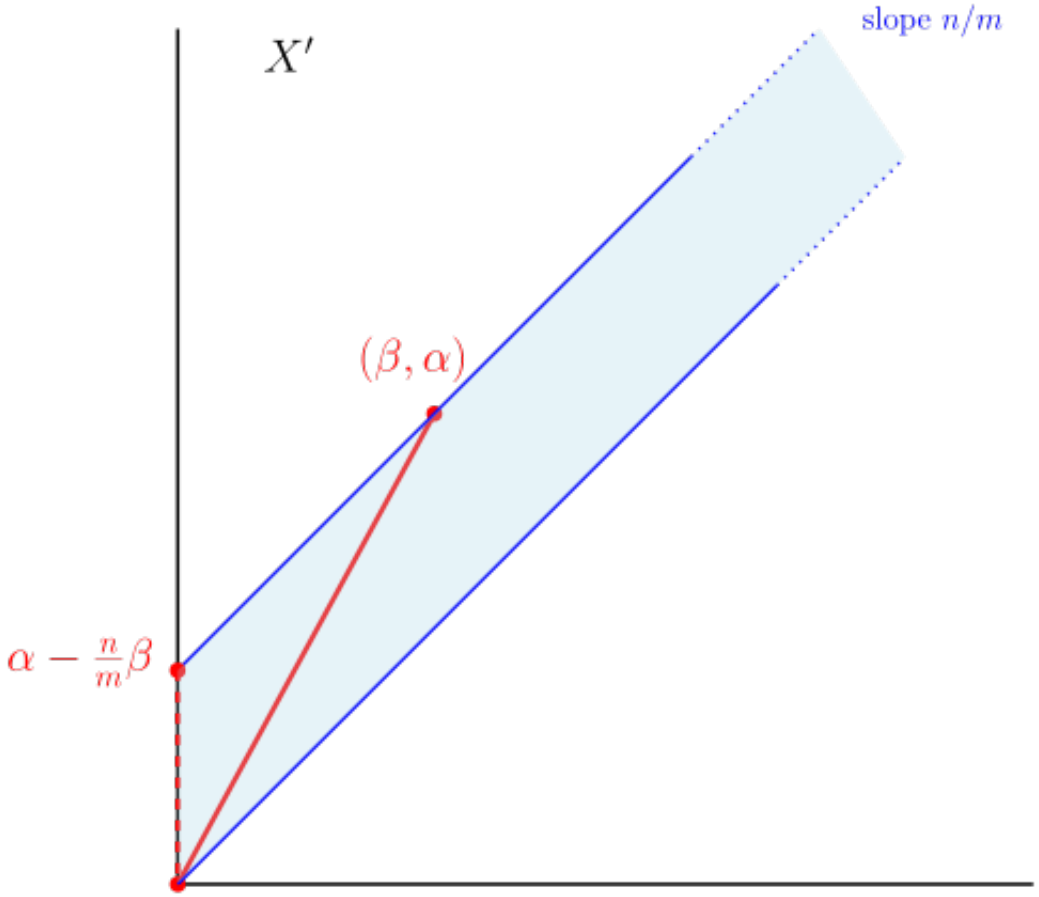

Figure 4.20: The slit (β, α) casts the same shadow as a vertical slit of length $\alpha - \frac{n}{m}\beta$

So the problem comes back to a vertical slit torus: more precisely, the class (m, n) is visible on X' if and only if it is visible on a square slit torus with a *vertical* slit of length $\alpha - \frac{n}{m}\beta$. By Proposition 4.1.2, we know this is equivalent to

$$|m.(\alpha - \frac{n}{m}\beta)| \leq 1$$

that is if and only if $|m\alpha - n\beta| \leq 1$. □

Remark that the quantity $|m\alpha - n\beta|$ is the area of the forbidden band, i.e the size of the shadow cast by the slit on X' when hit by parallel light rays of slope n/m. Since X' has area 1, the condition $|m\alpha - n\beta| \leq 1$ simply means that this shadow does not cover the entire torus, so there is still room for straight line curves representing (m, n) to exist.

Corollary 4.2.2.1. *Integral visible classes are isolated.*

Here by isolated we mean that if n/m is the slope of a visible direction on X', there exists $\delta > 0$ such that there is no other slope of a visible direction in the interval $]n/m - \delta, n/m + \delta[$.

Proof. Let (m, n) be an integral visible class, i.e $|m\alpha - n\beta| \leq 1$. This implies

$$\left|\frac{\alpha}{\beta} - \frac{n}{m}\right| \leq \frac{1}{\beta m}.$$

So for m fixed, there is only up to a finite number of visible classes. Moreover, since α/β is irrational, there exists a constant $c(m)$ that depends only on m such that

$$0 < c(m) \leq \left| \frac{\alpha}{\beta} - \frac{n}{m} \right| \leq \frac{1}{\beta m}.$$

But since $1/q\beta$ goes to 0 as q goes to infinity there are only finitely many integral visible classes (q,p) on X' such that $|\alpha/\beta - p/q| \geq c(m)$. This means there is only a finite number of visible classes (q,p) that *may* be close to (m,n). So (m,n) is isolated from other visible classes.

□

We want to characterize visible classes on X' through the approximation by the rational tori X_k's. So, are the visible classes on X' linked in some way to the visible classes of the X_k's?

Proposition 4.2.3. *An integral homology class (m,n) is visible on X' if and only if (m,n) is visible on infinitely many X_k's.*

Recall that since the common homology basis on X_k and on X' is given by the sides of the square it makes sense to talk about a same homology class (m,n) "on X_k" and "on X'" .

Proof. Let (m,n) be an integral homology class, and assume it is visible on X' : we have $|m\alpha - n\beta| \leq 1$. We deal separately with the cases $|m\alpha - n\beta| < 1$ and $|m\alpha - n\beta| = 1$.

First, assume $|m\alpha - n\beta| < 1$. The quantity $|m\alpha - n\beta|$ is the area of the parallelogram generated by the vector (m,n) and the slit of X', represented by the vector (β,α). On the rational torus X_k the slit has length $\rho = \sqrt{\beta^2+\alpha^2}$ and slope p_k/q_k, thus it is represented by the vector $\dfrac{\rho}{q_k^2+p_k^2}(q_k,p_k)$. Then by Corollary 4.2.1.1, the class (m,n) is visible on X_k if and only if

$$\frac{\rho}{q_k^2+p_k^2}|mp_k - nq_k| \leq 1$$

that it to say if and only if the parallelogram generated by the vector (m,n) and the slit of X_k has area less than 1. But by construction the slit of X_k goes to (β,α) when k goes to infinity, so this parallelogram goes to the parallelogram generated by (m,n) and (β,α) of area strictly less than 1. Thus for k big enough, the parallelogram generated by the vector (m,n) and the slit of X_k must have area less than 1, so (m,n) is visible on X_k for all k big enough.

Now assume $|m\alpha - n\beta| = 1$. Note that necessarily α and β must be linearly dependant over $\mathbb{Q}$, which is not the case in general. This time, there is a slight precaution to take: since this time the limit area of the parallelograms is not strictly less than 1, we cannot conclude that all parallelograms of the sequence have area less than 1 for k big enough. Instead, we claim that half of them have area less than 1; this is where the choice of p_k/q_k as being the k-th convergent to α/β becomes useful. Indeed, since the value of a continued fraction lies between any two of its consecutive convergents, for all k we have $p_k/q_k \leq \alpha/\beta \leq p_{k+1}/q_{k+1}$ or $p_{k+1}/q_{k+1} \leq \alpha/\beta \leq p_k/q_k$. Let $k_0 \in \mathbb{N}$ such that

p_{k_0}/q_{k_0} lies between n/m and α/β. Then the parallelogram generated by (m,n) and the slit of X_{k_0} has area *strictly* less than 1, so (m,n) is visible on X_{k_0}. Moreover, this is also true for any $k \geq k_0$ with the same parity as k_0 because $p_{k_0}/q_{k_0} < p_k/q_k < \alpha/\beta$ (or $p_{k_0}/q_{k_0} > p_k/q_k > \alpha/\beta$). Thus (m,n) is visible on infinitely many of the X_k's.

Conversely if (m,n) is visible on infinitely many X_k's, say on a subsequence $X_{\varphi(k)}$, then the parallelograms generated by (m,n) and the slit of $X_{\varphi(k)}$ have area less than 1 and so does their limit, i.e $|m\alpha - n\beta| \leq 1$ and (m,n) is visible on X'. □

We now understand what the visible classes on X' are; what about the non-visible ones? Obviously, by Proposition 4.2.3, if a class (m,n) is not visible on X' this is also the case on infinitely many X_k's, and conversely. As in the case of rational tori the notion of Farey parents is no longer relevant, but the notion of Farey neighbours still is.

Lemma 4.2.4. *Let (m,n) be an integral homology class on X', and assume that (m,n) is not visible. Then it has at most two visible neighbours. More precisely, there exists at most two rationals p/q and p'/q' neighbours to n/m in two Farey sequences (*a priori *of different order) such that the homology classes (q,p) and (q',p') are visible on X'.*

Proof. On a square torus with a vertical slit a non-visible class has up to two visible neighbours, which correspond to its Farey parents. Since rational tori are obtained by applying a $SL(2,\mathbb{Z})$ matrix transformation to a square torus with a vertical slit, this means that on any rational torus a non-visible class has at most two visible neighbours aswell. Since the X_k are rational, for any $k \in \mathbb{N}$ the homology class (m,n) has up to two visible neighbours. So, by contradiction, if (m,n) had three or more visible neighbours on X', by Proposition 4.2.3 they should also be visible on the X_k for k big enough, which according to the previous argument is not possible. □

We will treat separately the cases of classes with at least one visible neighbour and of classes with no visible neighbour. This is the same idea as in the case of the square torus with a vertical slit where we distinguished the homology classes according to whether or not their Farey parents were visible.

Classes with visible neighbours. Let (m,n) be a non-visible integral class on X' and assume that it has at least one visible neighbour, denoted (m',n'). By Proposition 4.2.3 for k large enough the situation is the same on X_k (up to taking a sub-sequence), that is to say the class (m,n) is not visible on X_k and the class (m',n') is visible. By Corollary 4.2.1.1, we already know that (m,n) is a strictly convex direction of the unit ball of the stable norm of X_k, and now we would like to find a formula for its stable norm. Taking the limit when k goes to infinity we will finally obtain a formula for the stable norm of (m,n) in X'.

Recall that for all k there exists a matrix $M_k \in SL(2,\mathbb{Z})$ and a square torus with a vertical slit T_k such that X_k is isometric to $M_k T_k$. Let $\begin{pmatrix}\mu_k\\ \nu_k\end{pmatrix} := M_k^{-1}\begin{pmatrix}m\\ n\end{pmatrix}$ and let $a_k/b_k < c_k/d_k$ be the Farey parents of ν_k/μ_k.

If (m,n) has a second visible neighbour (m'',n'') on X' (or equivalently on *all* X_k for k big enough) then we are done, as repeating the computations made to prove Corollary 4.2.1.1 we obtain, for all k big enough,

$$||(m,n)||_{X_k} = ||(m',n') \pm (\beta_k,\alpha_k)||_2 + ||(m'',n'') \mp (\beta_k,\alpha_k)||_2 \tag{4.1}$$

where (β_k,α_k) denotes the slit of X_k, and the position of the "+" sign corresponds to which of (m',n') and (m'',n'') is the image by M_k of the class (b_k,a_k). What is important here is that the *only* thing that depends on k in this formula is the vector (β_k,α_k): the homology classes (m',n') and (m'',n'') are fixed for k large enough, as shown in Proposition 4.2.3. More precisely, even if the homology classes $(\mu_k,\nu_k),(b_k,a_k)$ and (d_k,c_k) depend on k (as does the matrix M_k), their images by M_k *do not*. This is a very important observation that is made possible by the reformulating work we have done in the previous section for rational tori.

If (m,n) does not have another visible neighbour on X', then the above formula still holds, but this time it is a bit trickier to establish as it is not clear what the class (m'',n'') is. There are exactly two numbers $n_1''/m_1'', n_2''/m_2''$ in the Farey sequence that are neighbour to *both* n/m and n'/m': they correspond to their Farey child and their common Farey parent (possibly being $1/0$, using the extended notion of Farey parents discussed at the end of Chapter 3). Note that it is independent of k. In other words, there are exactly two classes (m_1'',n_1'') and (m_2'',n_2'') that are neighbour to *both* (m,n) and (m',n'), and only one of these is the class (m'',n'') we are looking for: we want to find the image by M_k of the second Farey parent of (μ_k,ν_k), see Corollary 4.2.1.1. To fix notations let us say it is (m_1'',n_1''). The problem is that we have no guarantee that this choice is independent of k: on X_{k+1} the image by M_{k+1} of the second Farey parent of (μ_k,ν_k) might be (m_2'',n_2'') (see Figure below for a representation using the Ford circle packing). However, since *independently of* k there are only two possibilities for the image of this second Farey parent of (μ_k,ν_k) to be, one of them has to be the right one for infinitely many values of k: again to fix notation, say it is (m_1'',n_1''). Thus, up to taking a sub-sequence we can assume that $(m_1'',n_1'') =: (m'',n'')$ is *always* the image of the second Farey parent of (μ_k,ν_k) under the action of M_k, that is, for all value of k. Hence in this case the formula 4.1 still holds for all k sufficiently large.

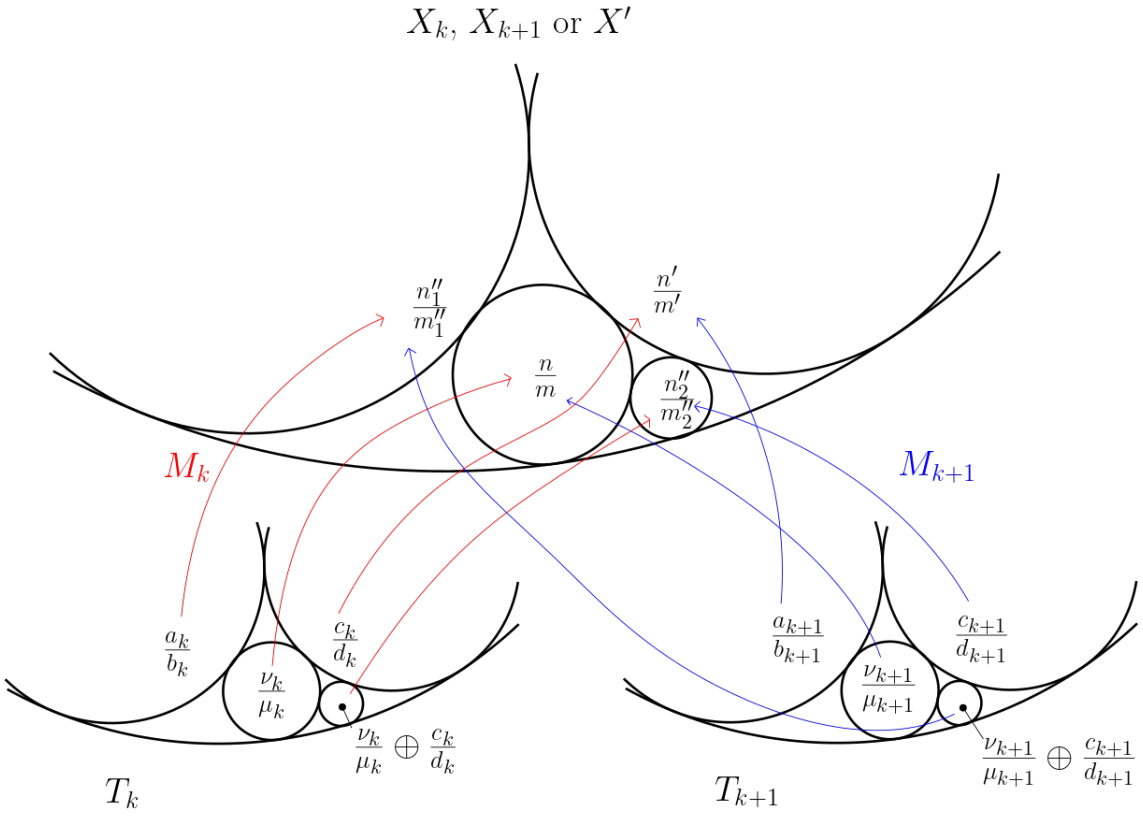

Figure 4.21: A possibility for the action of M_k and M_{k+1} on $\mathbb{Q}$, pictured with the Ford circle packing

Thus, since $(\beta_k, \alpha_k) \longrightarrow (\beta, \alpha)$ as $k \to \infty$, by taking the limit of the expression 4.1 as k goes to infinity we finally get

$$||(m,n)||_{X'} = ||(m',n') \pm (\beta,\alpha)||_2 + ||(m'',n'') \mp (\beta,\alpha)||_2$$

By the same argument as in Proposition 4.1.9, this is strictly less than the length of any non-simple curve representing (m,n) on X', so the unit ball of the stable norm of X' is strictly convex in the direction (m,n).

Classes with no visible neighbour. Let (m,n) be a non-visible integral class on X', and assume it has no visible neighbour. As in the case of rational tori, we will show that the unit ball of the stable norm of X' has a flat in the direction (m,n).

By Proposition 4.2.3, for all k sufficiently large the class (m,n) is not visible of X_k and has not any visible neighbour. By Corollary 4.2.1.1, this means that the unit ball of the stable norm of X_k has a flat in the direction (m,n). In other words, there exist two integral classes (q,p) and (q',p') such that the point $\dfrac{(m,n)}{||(m,n)||_{X_k}}$ lies in the interior of the segment $I = \left[\dfrac{(q,p)}{||(q,p)||_{X_k}}, \dfrac{(q',p')}{||(q',p')||_{X_k}}\right]$ contained in the unit sphere of the stable norm of X_k . Moreover, we can assume that no visible class appears inside of I as k goes to infinity. Indeed, if a class inside this segment would become visible as k increases we would just replace I by a shorter segment (we would need to change (q,p) and (q',p')

as well). In the worst case scenario, we would have to shrink I only a finite number of times, as if infinitely many visible classes would continue to appear as we take shorter and shorter segments this would give us a sequence of visible classes whose slopes converge to n/m. But the only accumulation point of slopes of visible classes is the *irrational* number α/β, so this is not possible.

For k large enough we have

$$||(m,n)||_{X_k} = ||(q,p)||_{X_k} + ||(q',p')||_{X_k}$$

By taking the limit as k goes to infinity, we get

$$||(m,n)||_{X'} = ||(q,p)||_{X'} + ||(q',p')||_{X'}$$

In particular the point $\dfrac{(m,n)}{||(m,n)||_{X'}}$ lies inside of the segment $\left[\dfrac{(q,p)}{||(q,p)||_{X'}}, \dfrac{(q',p')}{||(q',p')||_{X'}}\right]$, so the unit ball of the stable norm of X' has a flat in the direction (m,n).

4.2.4 Statement of the theorem and pictures of the unit ball

Putting all the previous results together we have shown

Theorem 4.2.5. *Let X' be a square torus with a slit (β,α) of any direction. The stable norm of X' is strictly convex in the direction (β,α) and $\|(\beta,\alpha)\|_{X'} = \|(\beta,\alpha)\|_2$. The unit ball of the stable norm of X' has a vertex in the direction $(m,n) \in H_1(X',\mathbb{Z})$, such that $n/m \neq \alpha/\beta$, if and only if either*

- *(m,n) is visible, i.e $|m\alpha - n\beta| \leq 1$. In that case, $\|(m,n)\|_{X'} = \|(m,n)\|_2$.*

  or

- *(m,n) is not visible but is neighbour to at least one visible class, and in that case $\|(m,n)\|_{X'} > \|(m,n)\|_2$.*

The point of the unit ball of the stable norm of X' in any other direction of $H_1(X',\mathbb{R})$ in contained in the interior of a flat.

The pictures we get are qualitatively similar to the case of the square torus with a vertical slit: the unit ball of the stable norm is a convex contained inside the unit disk of $\mathbb{R}^2$, with a few flats clearly identifiable, and the rest of the convex being so close to the unit circle that they are not distinguishable by eyesight. Here is a picture of the unit ball of the stable norm of the rational torus with a slit of length 1.5 and slope $4/7$.

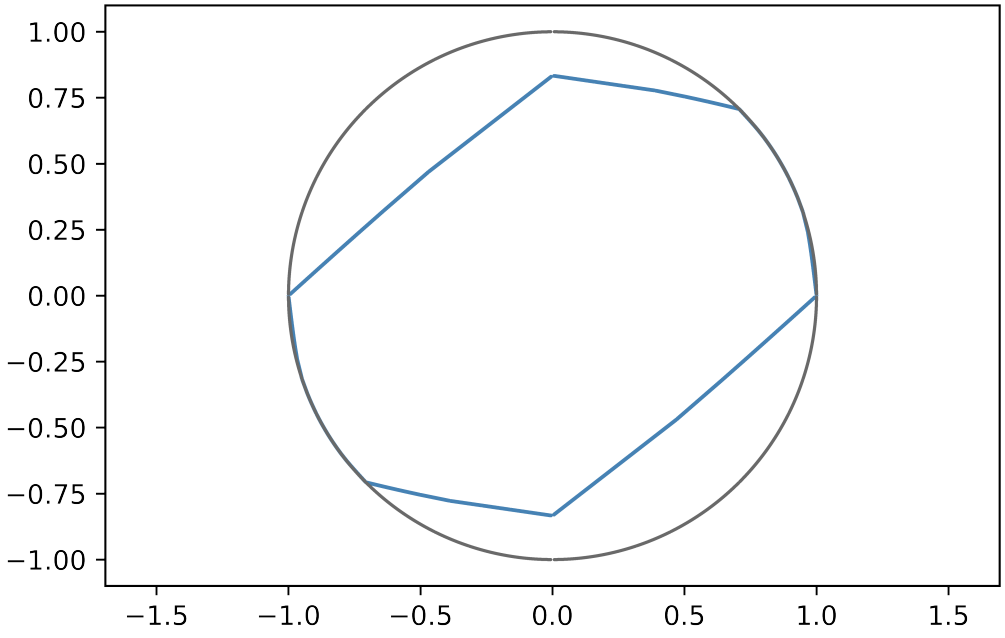

Figure 4.22: Unit ball of the stable norm of a rational torus

It is not possible to represent perfectly the unit ball of the stable norm of an irrational torus as we would need to take a limit, which requires an infinite amount of computations. However, we can represent the unit ball of a sequence of rational tori that converge to an irrational torus. As we will see, the pictures we get are close enough. The irrational torus we try to approximate is the square torus with a slit of length 1.2 and slope given by the continued fraction expansion $[u_0, u_1, ..., u_n, ...]$ where u_n is the n-th decimal of π (why not). The next figure displays the unit ball of the stable norm of the rational tori X_i with a slit of length 1.2 and slope $[u_0, u_1, ..., u_i]$ with i ranging from 1 to 6, increasing from left to right.

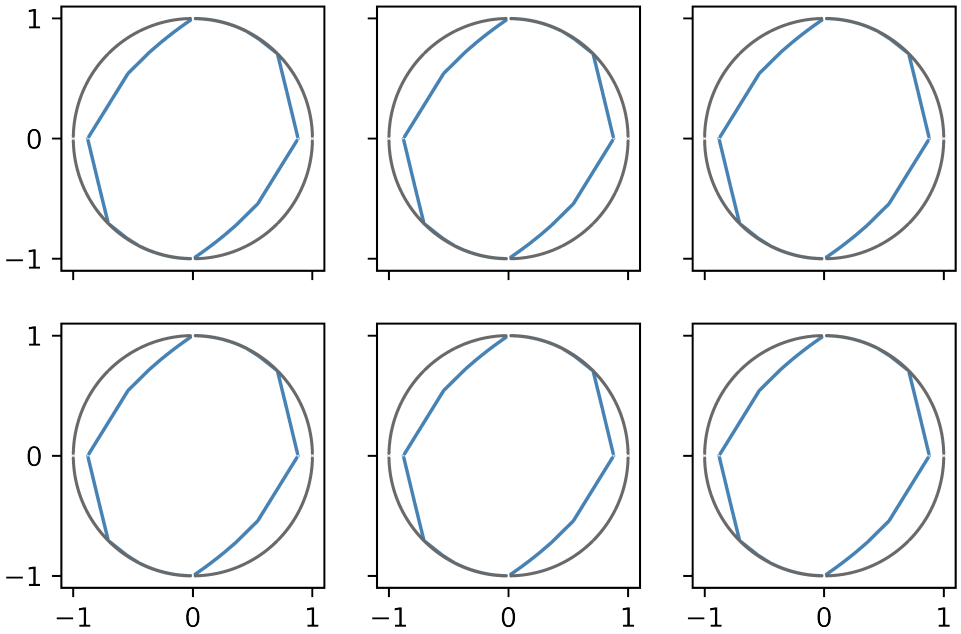

Figure 4.23: Unit ball of the stable norm of rational tori converging to an irrational torus

Although they look very similar, these six pictures are actually different, as it can be seen by numerically evaluating of the stable norm of a fixed homology class on X_i as i increases. Alternatively, we can repeat the idea of Figure 4.1.6: we flatten a small arc of the unit circle onto a horizontal line (the red segment on the picture) and we represent the unit ball of the stable norm of X_i seen from that line. More precisely, we draw the graphs of the functions $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f_i(n/m) = 1 - \frac{\|(m,n)\|_2}{\|(m,n)\|_{X_i}}$$

which is the distance between the unit ball of the stable norm of X_i and $\mathbb{S}^1$ in the direction (m,n) (of slope n/m). According to the previous theorem, as i increases there are more and more visible classes near the direction of the slit: thus the graph of the functions f_i should look qualitatively different near the slopes of the slits of the tori X_i. Since $[3,1,4,1,5,9,2,6,...] \approx 3.828656$, here are the graphs of f_1, f_2 and f_3 for $3.815 < x < 3.845$:

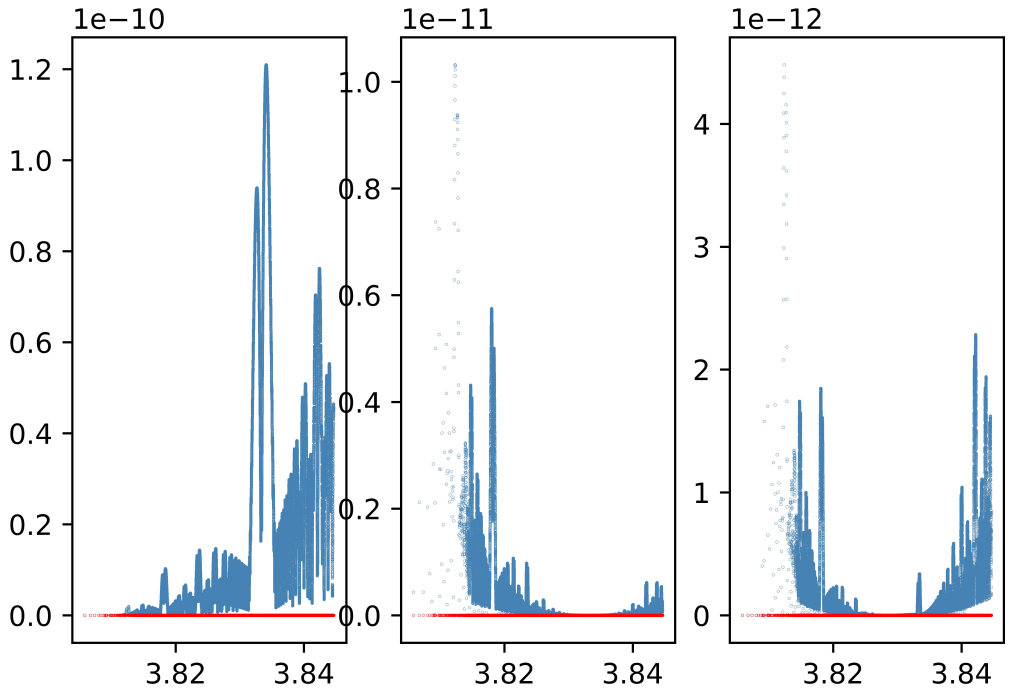

Figure 4.24: The unit ball (blue) of X_i seen from the circle (red), $i = 1,2,3$

As expected, these three graphs look different from one another. The unit ball of the stable norm of the X_i's look very similar away from the direction of the slit, i.e the visible classes and more generally the strictly convex directions of the unit ball are common, but differences appear as we get closer and closed to the direction of the slits. This is where the limitations of numerical methods are reached: distinguishing the graphs of f_4, f_5 and f_6 from the graph of f_3 would require further zooming and (a lot) more computation time. Also, the numerical values involved in the computations would get in range of rounding errors (using `python`'s default precision), which may cause additional problems.

5 Stable norm of half-translation surfaces

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We have seen in Chapter 4 that any translation surface of genus 2 can be obtained by glueing either a slit torus to a cylinder or two slit tori together. This was the initial motivation to study the stable norm of slit tori, as they are the building blocks of translation surfaces. Now that we fully understand the stable norm of slit tori, the natural thing to do is to glue them together to produce translation surfaces on which we will hopefully be able to compute the stable norm. Unfortunately, even if some minor results might be obtained with this approach, a complete computation of the stable norm of the glued surface seems out of reach. However the situation is better when looking at half-translation surfaces, as we will be able to explicitly compute the stable norm of some surfaces.

5.1 Glueing slit tori to long cylinders

The problem when trying to compute the stable norm of a surface made by glueing two slit tori along their slit is that, even if we now know a lot of things about the length of curves that are fully contained inside of one of the tori, we don't really know what happens for curves that "cross the slits" and are not contained inside only one of the tori. An easy fix to this problem would be to create a surface on which these curves that "cross the slit" have no chance to ever be minimizing in their respective homology classes, so that the problem is reduced to curves that we already understand very well. More precisely, the idea is to glue each slit torus at the extremity of a long cylinder so that a curve that would "cross the slit" would also have to cross the full length of the cylinder, thus being very long itself and very likely not minimizing inside of its homology class. However such a surface is not a translation surface, but a half-translation surface instead.

In this section we will glue two *identical* slit tori to a cylinder; moreover we will assume these are square tori with a *vertical* slit. A similar construction can be done with any number of slit tori instead of two; we will detail this construction at the end of the section.

The surface Σ. Let X denote the square torus of area 1 with a vertical slit of length $\rho \in]0,1[$. Let X_1, X_2 be two identical copies of X, and let C be a flat cylinder of width w, with boundary components of length 2ρ. Note that we do not assume yet that the cylinder is long. Let Σ be the closed genus 2 surface obtained by glueing C along the slits of X_1 and X_2. By construction, the surface Σ is endowed with a flat metric, obtained by glueing the flat metrics of the two slit tori and of the cylinder. By Riemann's Uniformization Theorem 2.1.5, we know that this metric cannot be smooth; indeed, a closer inspection shows the metric has four conical singularities of angle 3π located at each endpoint of the slits, and is smooth elsewhere. Thus Σ is not a translation surface, as the conical singularities of translation surfaces are *even* integral multiples of π. Drawing a polygonal model of Σ one can check that it is instead a half-translation surface.

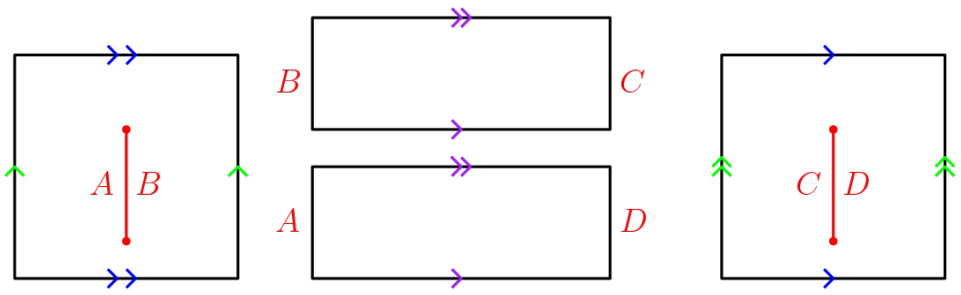

Figure 5.1: Polygonal model of the surface Σ

We have

$$H_1(\Sigma,\mathbb{Z}) = H_1(X_1,\mathbb{Z}) \oplus H_1(X_2,\mathbb{Z}) \simeq \mathbb{Z}^4$$

so if (e_i, f_i) is a basis of $H_1(X_i,\mathbb{Z})$, we have $H_1(\Sigma,\mathbb{Z}) = \langle e_1, f_1, e_2, f_2\rangle$.

Restriction to the slit tori's homology planes. We denote $||.||_{X_i}$ the stable norm of the slit tori X_i and $\|.\|_\Sigma$ the stable norm of Σ. The following Lemma is true independently of the width w of the cylinder.

Lemma 5.1.1. *The stable norm of* Σ *coincides with the stable norm of* X_i *on the plane* $Span(e_i, f_i)$.

Proof. Take for instance $i = 1$, and let $h = (m,n,0,0) \in H_1(\Sigma,\mathbb{Z}) \cap \mathrm{Span}(e_1, f_1)$ be an integral homology class and γ be a minimizing multicurve representing h. We will show that γ has to be fully contained in X_1. Thus its length must be $\|(m,n)\|_{X_1}$ as it must be minimizing for the class (m,n) in the slit torus X_1, so $\|(m,n,0,0)\|_\Sigma = \|(m,n)\|_{X_1}$.

By contradiction, suppose γ goes outside of X_1 and enters X_2. Let $\gamma_2 := \gamma \cap X_2$ be the part of γ inside of X_2; γ_2 may have several connected components, one of whose

being an arc. Close this arc with a path μ_2 contained in X_2 that goes around the slit by following its boundary to obtain a closed curve $\tilde{\gamma}_2$ in X_2. Note that μ_2 has length at most ρ. Let $\tilde{\gamma}$ be the closed multicurve obtained by closing $\gamma \backslash \gamma_2$ with another copy of the path μ_2.

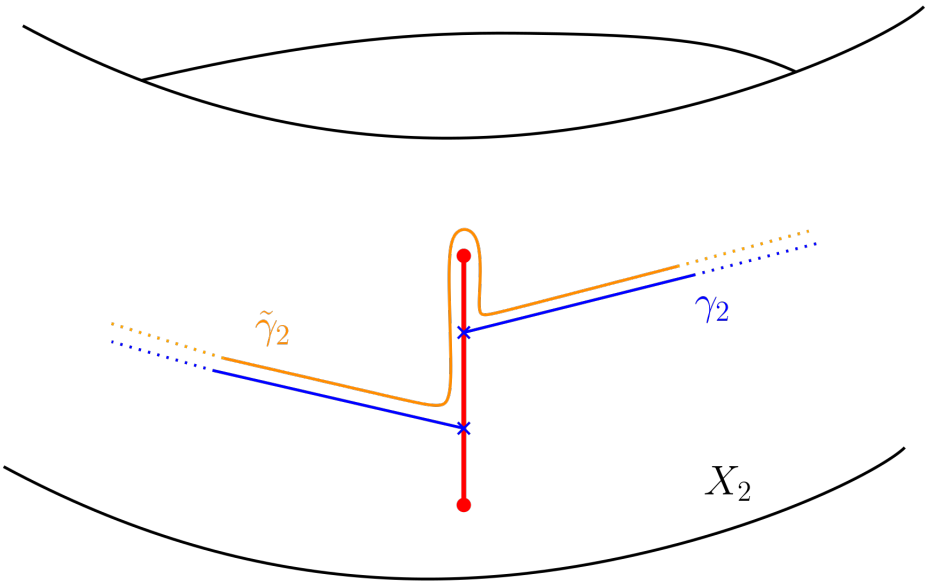

Figure 5.2: Closing γ_2 by going around the slit in X_2

By construction, $[\gamma] = [\tilde{\gamma}] + [\tilde{\gamma}_2] = (m, n, 0, 0)$. As $\tilde{\gamma}_2$ has intersection number zero with the canonical representatives of the homology classes $(1,0,0,0)$ and $(0,1,0,0)$, it represents a homology class whose first two coordinates are zero. Similarly, the multicurve $\tilde{\gamma}$ represents a homology class whose last two coordinates are zero. Since $[\tilde{\gamma}] + [\tilde{\gamma}_2] = (m, n, 0, 0)$ we get $[\tilde{\gamma}_2] = 0$, so $\tilde{\gamma}$ represents the same homology class as γ. We have

$$l(\tilde{\gamma}) = l(\gamma) + l(\mu_2) - l(\gamma_2)$$

We assumed that $\gamma_2 = \gamma \cap X_2$ was non-trivial, i.e $l(\gamma_2) > 0$. Necessarily the connected components of $\tilde{\gamma}_2$ all have non-trivial homotopy class in the slit torus X_2: otherwise we could shrink a connected component of $\tilde{\gamma}_2$ (and thus of γ_2 and γ) onto a point and reduce the length of γ, contradicting its minimality. Thus any connected component of $\tilde{\gamma}_2$

- either has to loop around the torus X_2 at least once, and since X_2 is obtained from a square of side of length 1 we have $l(\gamma_2) \geq 1$.
- either is homotopic to a circle going around the slit so $l(\tilde{\gamma}_2) \geq 2\rho$, and since $l(\tilde{\gamma}_2) = l(\gamma_2) + l(\mu_2)$ and $l(\mu_2) \leq \rho$ we have $l(\gamma_2) \geq \rho$.

In both cases we have $l(\gamma_2) \geq l(\mu_2)$. Hence $l(\tilde{\gamma}) = l(\gamma) + l(\mu_2) - l(\gamma_2) < l(\gamma)$, which contradicts the minimality of γ. Thus $l(\gamma_2) = 0$ and the minimizing curve γ does not enter X_2 at all.

We repeat the previous process to show that γ does not enter the cylinder C. Let $\gamma_1 := \gamma \cap X_1$ and $\gamma_C := \gamma \cap C$. We get a closed curve $\tilde{\gamma}_1$ by closing γ_1 with the path

$\mu_1 \subset X_1$ going around the slit in the shortest way. Again, $l(\mu_1) \leq \rho$. Set $\tilde{\gamma_C} = \gamma_C \cup \mu_1$. It is a closed curve contained in C with trivial homology class, hence $\tilde{\gamma}_1$ represents the same homology class as γ because $[\gamma] = [\tilde{\gamma}_1] + [\tilde{\gamma_C}]$. If $l(\gamma_C) > 0$ clearly we have $l(\gamma_C) > l(\mu_1)$ so

$$l(\gamma_1) + l(\mu_1) = l(\tilde{\gamma}_1) < l(\gamma) = l(\gamma_1) + l(\gamma_C)$$

which contradicts the minimality of γ as $\tilde{\gamma}_1$ would be a shorter curve representing the same homology class (m,n). Thus γ does not enter C and is fully contained in X_1.

□

Note that Lemma 5.1.1 is true even if there is no cylinder, i.e if $w = 0$.

Isolating the tori with a long cylinder. Outside of the homology planes associated to each torus composing Σ, computing the stable norm is complicated. However if the cylinder is long enough then the problem becomes much easier. The idea is the following: if a connected curve represents a homology class (m,n,p,q) then at some point it has to cross the cylinder, but if the cylinder is very long, so must be the curve. Hence it is much more length-efficient to have two (at least) connected components that avoid the cylinder by being fully contained inside the slit tori, representing the classes $(m,n,0,0)$ and $(0,0,p,q)$ respectively.

Proposition 5.1.2. *If the cylinder C is long enough, more precisely if $w > \rho$ with ρ the length of the slits of the X_i's, then for all $(m,n,p,q) \in H_1(\Sigma,\mathbb{Z})$ we have*

$$\|(m,n,p,q)\|_\Sigma = ||(m,n)||_{X_1} + ||(p,q)||_{X_2}$$

Proof. Let $h = (m,n,p,q) \in H_1(\Sigma,\mathbb{Z})$ be an integral class in Σ. If $(m,n) = (0,0)$ or $(p,q) = (0,0)$ then we apply Lemma 5.1.1 and there is nothing left to prove, so let us assume $(m,n) \neq (0,0)$ and $(p,q) \neq (0,0)$. Let γ be a representative of h. Note that γ is not necessarily connected. We consider two distinct cases, depending on whether or not γ crosses the cylinder.

First, consider the case where γ does not cross the cylinder: more precisely, let us assume that there does not exist an arc of γ whose endpoints lie in X_1 and X_2 respectively. By the same argument as in the proof of Lemma 5.1.1 it is not length-efficient for γ to venture into the cylinder without fully crossing it, so for γ to be as short as possible necessarily $\gamma \cap C = \emptyset$. Thus $\gamma = \gamma_1 \cup \gamma_2$, where $\gamma_1 = \gamma \cap X_1$ (resp. $\gamma_2 = \gamma \cap X_2$) is a multicurve contained in X_1 (resp. X_2) representing the homology class $(m,n,0,0)$ (resp. $(0,0,p,q)$). The length of γ is minimal if and only if both γ_1 and γ_2 are minimizing in their homology class, so by Lemma 5.1.1 the minimal length for γ is $||(m,n)||_{X_1} + ||(p,q)||_{X_2}$.

Now assume that γ crosses the cylinder at least once, i.e there exists an arc of γ whose endpoint lie in X_1 and X_2 respectively. As a matter of fact there exist at least two such arcs, going in opposite direction, as γ must go back eventually in order for all of its connected components to be closed curves. For simplicity assume that it crosses the cylinder exactly once; a similar argument still holds if γ crosses the cylinder more

than once. Denote $\gamma_C := \gamma \cap C$ the part of γ that lies in the cylinder, and $\gamma_i = \gamma \cap X_i$ the part of γ inside of X_i. We have

$$l(\gamma) = l(\gamma_C) + l(\gamma_1) + l(\gamma_2).$$

As we have cut some piece of γ away, γ_1 and γ_2 are composed of an arc and possibly some closed curves. Now close the arc in γ_1 by adding the shortest path between its endpoints (the points where γ exits and re-enters X_1) that follows the edges of the slit. Note that this path has length at most ρ. We obtain a union $\tilde{\gamma}_1$ of closed curves in X_1, with $l(\tilde{\gamma}_1) \leq l(\gamma_1) + \rho$. Moreover, $\tilde{\gamma}_1$ represents the homology class $(m,n,0,0)$, so $||(m,n)||_{X_1} \leq l(\tilde{\gamma}_1)$ and finally

$$l(\gamma_1) \geq ||(m,n)||_{X_1} - \rho.$$

The same construction in X_2 yields

$$l(\gamma_2) \geq ||(p,q)||_{X_2} - \rho.$$

Finally, since γ crosses the cylinder, there is an arc going from X_1 to X_2 through the cylinder; but this arc is part of a closed curve so γ has to go back to X_1 eventually. Hence γ_C in a union of *two* arcs whose endpoints lie in X_1 and X_2, so

$$l(\gamma_C) \geq 2w$$

where w is the width of the cylinder. Putting everything together we finally get that in this case

$$l(\gamma) \geq ||(m,n)||_{X_1} + ||(p,q)||_{X_2} + 2(w - \rho)$$

so if $w > \rho$, any representative of (m,n,p,q) that crosses the cylinder is strictly longer than the shortest representative of (m,n,p,q) that do not cross the cylinder. Hence if $w > \rho$ the shortest representative of (m,n,p,q) in Σ is a multicurve that does not encounter the cylinder, and it has length $||(m,n)||_{X_1} + ||(p,q)||_{X_2}$.

□

The stable norm of Σ. By the previous proposition, if $(m,n) \neq (0,0)$ and $(p,q) \neq (0,0)$, the point $\dfrac{(m,n,p,q)}{\|(m,n,p,q)\|_\Sigma}$ of the unit sphere of the stable norm of Σ in the direction (m,n,p,q) lies in the interior of the segment $\left[\dfrac{(m,n,0,0)}{\|(m,n,0,0)\|_\Sigma}, \dfrac{(0,0,p,q)}{\|(0,0,p,q)\|_\Sigma}\right]$.

Thus the strictly convex directions of the unit ball of the stable norm of Σ must lie in the homology planes associated to the slit tori X_1 and X_2. Repeating the argument given to prove Theorem 4.1.17, one can check that these strictly convex directions are exactly the directions of the vertices of the stable norm of the X_i's, and that the stable norm of Σ has vertices in those directions. More precisely, $(m,n,0,0)$ (resp. $(0,0,p,q)$) is the direction of a vertex of the stable norm of Σ if and only if (m,n) (resp. (p,q)) is the direction of a vertex of the stable norm of X_1 (resp. X_2). We have shown:

Theorem 5.1.3. *Let us assume the cylinder in Σ is long, more precisely assume $w > \rho$ where ρ is the length of the slits of the slit tori X_1, X_2. The unit ball of the stable norm of Σ is the convex hull of the set*

$$\left\{ \frac{(m,n,0,0)}{\|(m,n,0,0\|_\Sigma} \text{ with } (m,n) \in V(X_1) \right\} \cup \left\{ \frac{(0,0,p,q)}{\|(0,0,p,q)\|_\Sigma} \text{ with } (p,q) \in V(X_2) \right\}$$

where $V(X_i)$ is the set of directions of strictly convex directions of the stable norm of X_i.

As in the case of flat tori, the unit ball $\mathscr{B}_\Sigma$ of the stable norm of Σ is not a polytope but looks a lot like one: to picture $\mathscr{B}_\Sigma$, imagine two copies of Figure 4.1.6 in two planes intersecting only at the origin, then connect any pair of two points with a segment. Note that the unit ball of the stable norm of Σ has flats of dimension 3, which is maximal since $H_1(\Sigma,\mathbb{R})$ has dimension 4. This is a striking difference with the case of smooth surfaces, as according to Theorem 1.2.5 the flats of the unit ball of a genus 2 smooth Riemannian surface have dimension at most 1.

5.1.1 Genus g construction

A construction similar to the one of Σ provides half-translation surfaces of genus g, with $g \geq 2$ for which an analogue of Theorem 5.1.3 holds. Let $X_1, ..., X_g$ be g identical copies of the slit torus X. On X_i, label the left (resp. right) hand side of the slit A_i (resp. B_i). Take $2g$ identical $w \times \rho$ rectangles $r_1, ..., r_{2g}$, with $w > \rho$. Now glue as follows:

- The left side of r_{2i-1} (resp. r_{2i}) to A_i (resp. B_i).
- The upper (resp. lower) sides of r_{2i-1} and r_{2i} together.
- For $i \leq g-1$ the right side of r_{2i} to the right side of r_{2i+1}, and the right side of r_{2g} to the right side of r_1.

The resulting surface Σ_g is a closed genus g surface equipped with a flat metric with 3π angle conical singularities at each endpoint of the slits, and two $g\pi$ angle conical singularities at points where the top and bottom right corners of the rectangles were glued together.

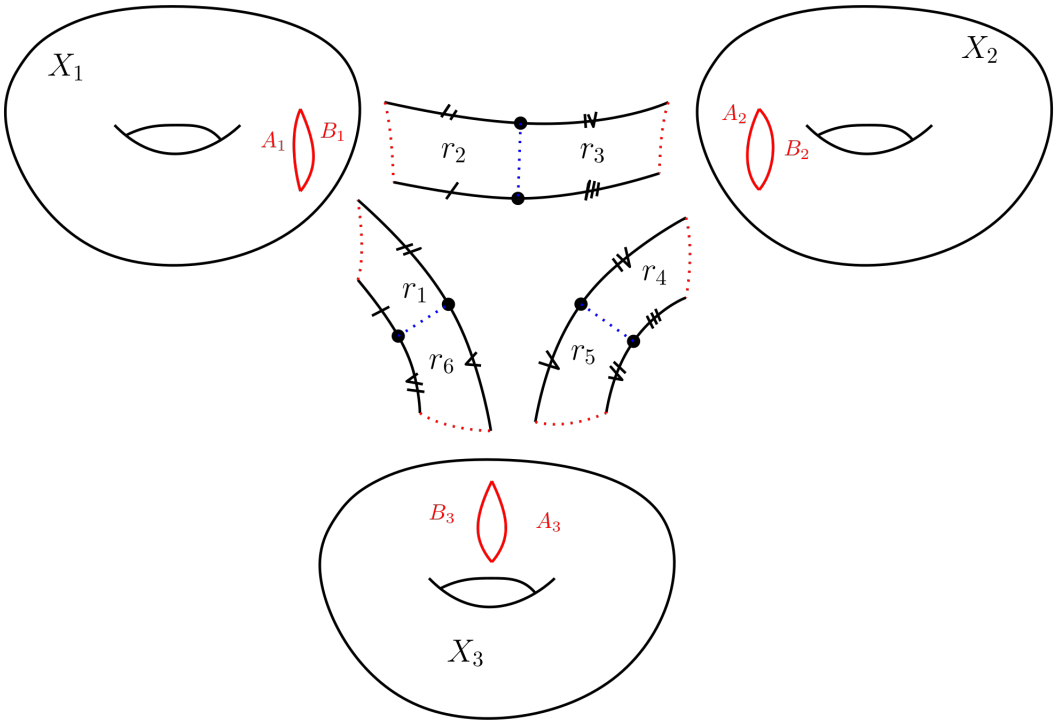

Figure 5.3: The construction for $g = 3$

The same arguments as is the case of Σ hold in this setting to prove an analogue of Theorem 5.1.3. More precisely, if $(a_1, b_1, ..., a_g, b_g)$ is an integral homology class on Σ_g then

$$\|(a_1, b_1, ..., a_g, b_g)\|_{\Sigma_g} = \sum_i \|(a_i, b_i)\|_{X_i}$$

and the unit ball of the stable norm of Σ_g is the convex hull of the set

$$\bigcup_i \left\{ \frac{(0, 0, ..., a_i, b_i, ...0, 0)}{\|(0, 0, ..., a_i, b_i, ...0, 0)\|_{\Sigma_g}}, (a_i, b_i) \in V(X_i) \right\}$$

where $V(X_i)$ is the set of strictly convex directions of X_i.

5.2 Deformation of quadratic differentials in genus 2

In this section Σ denotes the genus 2 half-translation surface constructed in the previous section by glueing two identical slit tori to a very long flat cylinder. Recall that Σ has four conical singularities of angle 3π, thus $\Sigma \in \mathcal{Q}(1,1,1,1)$. The aim of this short section is to extend Theorem 5.1.3 to an open neighborhood of the surface Σ in its stratum $\mathcal{Q}(1,1,1,1)$ of the moduli space of quadratic differentials $\mathcal{Q}_2$. We will show that on a sufficiently small neighbourhood of Σ the surfaces are also obtained by glueing (general) two slit tori and a cylinder, and that the proof of Theorem 5.1.3 can be adapted in that case.

Deforming a quadratic differential in $\mathcal{Q}_2$ can be seen geometrically by deforming its polygonal model. It might not be obvious that the deformed surface can be obtained

by glueing together two slit tori and a flat cylinder. Indeed, on a slit torus the boundary component (that is, the slit) is homeomorphic to a circle; this circle can be divided into two arcs of same length (that is, the two sides of the slit) that join the endpoints of the slit. When glueing slit tori with a flat cylinder to obtain a genus two half-translation surface, the endpoints of the slits become conical singularities of the flat metric. Thus, there are two singularities inside each slit torus composing Σ that correspond to the endpoints of the slit, and there are two saddle connections *of same length* joining these two singularities, whose union is homeomorphic to a circle. Moreover, cutting Σ along these saddle connections gives three connected components. Now, deforming the metric, one of these saddle connections could become shorter than the other: then it is not obvious anymore that the surface can be obtained by glueing slit tori with a cylinder.

Lemma 5.2.1. *Let Σ' be a genus 2 half-translation surface close to Σ in $\mathcal{Q}(1,1,1,1)$. Then Σ' can be obtained by glueing two slit tori to a long flat cylinder.*

Proof. Since Σ' is close to Σ, the general picture is the same: there is a long cylinder between two compact neighbourhood of the genus 1 components T_1, T_2 (that are both homeomorphic to a torus minus a disk), Σ' has four conical singularities, two in each genus 1 component, that we denote v_1, v_2, w_1, w_2. More importantly, there are two saddle connections γ_1, γ_2 (resp. μ_1, μ_2) joining v_1 to v_2 in T_1 (resp. w_1 to w_2 in T_2), and cutting Σ' along these saddle connections gives three connected component once again. We orient γ_1 (resp. μ_1) positively when going from v_1 to v_2 (resp. w_1 to w_2) and γ_2 (resp. μ_2) positively when going from v_2 to v_1 (resp. w_2 to w_1).

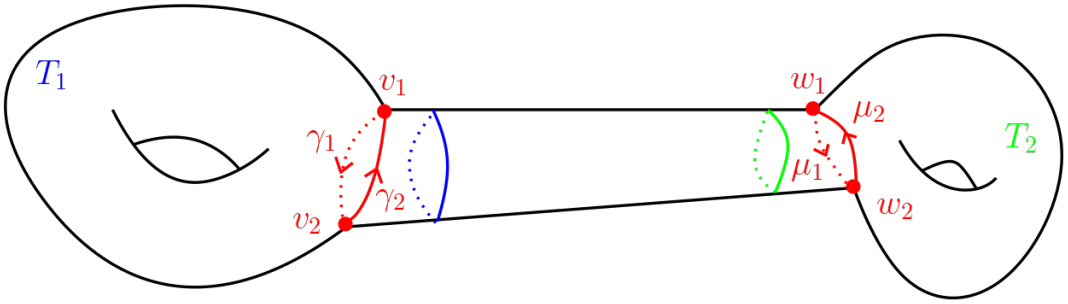

Figure 5.4: Sketch of the surface Σ'

By Theorem 2.5.5, the stratum component is hyperelliptic. By definition, there exists a hyperelliptic involution τ on Σ' that preserves the metric, i.e if q denotes the quadratic differential on Σ' generating the flat metric $|q|$, then $\tau^* q = q$. Since the metric is preserved by τ, so are its singularities: thus $\tau(\{v_1, v_2, w_1, w_2\}) = \{v_1, v_2, w_1, w_2\}$. It is a classical fact (see [FM12, Proposition 4.12]) that the hyperelliptic τ sends any homotopy class to its inverse, i.e it reverses the orientation of the curves. Thus $\tau(\{v_1, v_2\}) = \{v_1, v_2\}$ and $\tau(\{w_1, w_2\} = \{w_1, w_2\}$ or else we would get a contradiction, as we would be able to construct loops on Σ' whose homotopy class is not sent to its inverse by τ. Finally, since τ is an isometry of Σ' it preserves the length of curves. If $\tau(v_1) = v_1$ and $\tau(w_1) = w_1$ then the entire straight line segment joining v_1 and w_1 has to be fixed by τ, otherwise its length would not be preserved by τ. This is absurd because the hyperelliptic involution τ has exactly $2 \times \text{genus}(\Sigma') + 2 = 6$ fixed points. Hence

necessarily $\tau(v_1) = v_2, \tau(v_2) = v_1, \tau(w_1) = w_2$ and $\tau(w_2) = w_1$.

Since τ fixes the set $\{v_1, v_2\}$ and preserves the length of curves, we have $\tau(\gamma_1 \cup \gamma_2) = \gamma_1 \cup \gamma_2$. If $\tau(\gamma_1) = \gamma_1$ then the middle point of γ_1 is fixed by τ; similarly the middle points of γ_2, μ_1 and μ_2 are fixed by τ, along with the straight line segments joining the middle point of γ_1 (resp. γ_2) to the one of μ_1 (resp. μ_2) across the cylinder. Again, this leads to a contradiction as it gives τ too much fixed points. Hence $\tau(\gamma_1) = \gamma_2, \tau(\gamma_2) = \gamma_1, \tau(\mu_1) = \mu_2$ and $\tau(\mu_2) = \mu_1$.

Thus γ_1 and γ_2 (resp. μ_1 and μ_2) have same length, so they may form the opposite sides of a slit. Cutting Σ' along $\gamma_1, \gamma_2, \mu_1$ and μ_2 then gives three connected components, two of which being flat slit tori and the other one being a cylinder. □

Let Σ' be a surface in $\mathcal{Q}(1,1,1,1)$ close to Σ; by the previous Lemma, we know it can be obtained by glueing two slit tori X_1', X_2' along a long flat cylinder C'. Contrary to the case of Σ, the two slit tori composing Σ' are not isometric a priori, nor they are square tori with vertical slits.

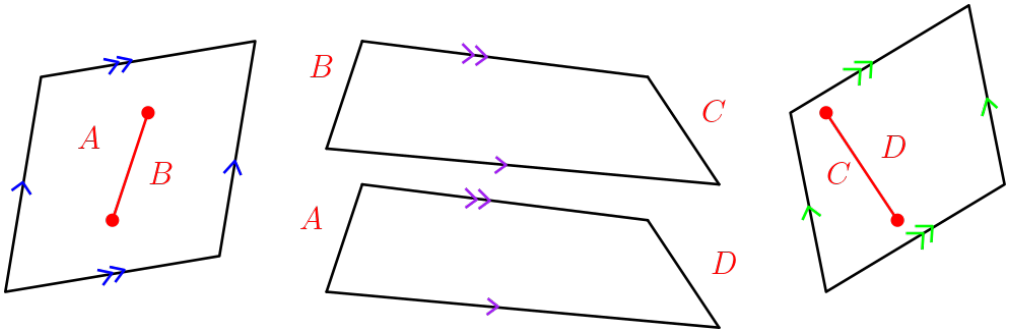

Figure 5.5: Polygonal model of the surface Σ'

In turns out the fact that Σ was made of *identical square* tori with *vertical* slits was a simplification to get much simpler notations, but these assumptions were unnecessary; a more general version of Theorem 5.1.3 that applies to surfaces such as Σ' can be proved in the same way as we did for Σ. Thus, we will now comment on the parts of the proof of Theorem 5.1.3 that need to be adjusted. Again, denote by (e_i, f_i) a basis of $H_1(X_i', \mathbb{Z})$ and identify these elements with their images in $H_1(\Sigma', \mathbb{Z})$ so that $H_1(\Sigma', \mathbb{Z}) = \langle e_1, f_1, e_2, f_2 \rangle$.

Lemma 5.1.1 is still valid in the setting of Σ', i.e the stable norm of Σ' coincides with the stable norm of X_i' on $\mathrm{Span}(e_i, f_i) = H_1(X_i', \mathbb{R})$. The only required adjustment is the following. At some point in the proof we had to show that some closed curve contained in one of the tori with a non-trivial homotopy class was longer than the slit. This was easy because either it looped around the slit so it was at least twice as long as the slit, either it looped around the torus so its length was at least 1, and since the slit had length $\rho < 1$ we were done. What mattered here is that the *systole* of the slit torus, i.e the shortest length of a closed curve that is not homotopic to a point, was greater than the length of the slit, as it was equal to $\min(1, 2\rho) > \rho$. If Σ' is sufficiently close to Σ in $\mathcal{Q}(1,1,1,1)$ then

$$\mathrm{sys}(\Sigma') = \min(\mathrm{sys}(X_1'), \mathrm{sys}(X_2')) > \max(\rho_1, \rho_2)$$

where ρ_i is the length of the slit of X_i', since for small enough deformations of the metric the systole of each of the slit tori remains longer than its slit. In other words, the systole of Σ' is still greater than the lengths of the slits of the X_i's, so we can repeat the arguments given in the proof of Lemma 5.1.1 on Σ'.

The proof of Proposition 5.1.2 can be repeated identically, apart from the fact that since now the slits of the X_i's may have different lengths, we need the cylinder to be longer than both the slits, i.e longer than $\max(\rho_1, \rho_2)$. Finally, the proof of Theorem 5.1.3 is repeated completely unchanged. We have shown:

Proposition 5.2.2. *Let $\Sigma' \in \mathcal{Q}(1,1,1,1)$ be a genus 2 half-translation surface. If Σ' is sufficiently close to Σ then the unit ball of the stable norm of Σ' is the convex hull of the points*

$$\left\{ \frac{(m,n,0,0)}{\|(m,n,0,0)\|_{\Sigma'}}, (m,n) \in V(X_1') \right\} \cup \left\{ \frac{(0,0,p,q)}{\|(0,0,p,q)\|_{\Sigma'}}, (p,q) \in V(X_2') \right\}$$

where Σ' can be decomposed as two slit tori X_1', X_2' and $V(X_i')$ is the set of strictly convex directions of the stable norm of X_i'.

Thus we can compute the stable norm on a open set of positive measure of the stratum $\mathcal{Q}(1,1,1,1)$. This is qualitatively very different than Theorem 5.1.3 that only gave us the stable norm on a measure zero set of the stratum.

6 Counting simple homology classes

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A classical counting problem in differential geometry is to count the number of geodesics of a fixed topological type on a given surface equipped with a (possibly singular) Riemannian metric whose length does not exceed a certain threshold $x > 0$. More precisely, since there are in general infinitely many curves of a given topological type, the problem is to find asymptotic estimates of this number when x goes to infinity (recall that two curves are said to be of the same topological type if they are in the same orbit under the action of the mapping class group of the surface).

Many results have already been obtained on this type of problem, the most notable of them arguably being due to Mirzakhani on hyperbolic surfaces, first proved for simple closed curves and a few years later for any topological type of closed curves:

Theorem 6.0.1 (Mirzakhani, [Mir16])**.** *Let S be an orientable hyperbolic surface of finite area with genus g and r punctures. Then the limit*

$$\lim_{x\longrightarrow\infty} \frac{\#\{\gamma \text{ closed geodesic of type } \gamma_0, l_S(\gamma) \leq x\}}{x^{6g-6+2r}}$$

exists and is positive for every closed geodesic γ_0.

This powerful theorem, along with its numerous later improvements, relies heavily on ergodic methods; this is worth mentioning as it will be a recurring observation in this chapter.

Taking a step back, the previous counting problem follows the general form: "How many are there closed curves satisfying a given topological constraint that are short?". Changing the imposed topological constraint on the curves leads to many variants of Theorem 6.0.1. One can also obtain new counting problems that are somehow

orthogonal to the classical ones by swapping the places of the group of words in the previous question. To put it informally as before, the question is: "How many are there topological constraints satisfied by at least one short curve?". Instead of counting all the short curves satisfying a topological requirement, we want to count all the families of curves, regrouped according to some topological property, that contain at least one short curve.

The topological constraint we will be interested is to prescribe the homology class. The question then becomes "How many are there homology classes that are representend by a short curve?". In other words, given a (possibly singular) Riemannian surface S of finite area and genus g, we want to count the number of homology classes on S whose stable norm is less than x, with $x > 0$. However, this is not a very interesting problem as by Minkowski's theorem this is easily seen to grow as $\mathrm{Vol}(\mathscr{B}_S)x^{2g}$, where $\mathscr{B}_S$ is the unit ball of the stable norm of S. It turns out there is a much more interesting question to be asked. If γ is a multicurve on S that is minimizing inside its homology class, then each of its connected components are simple closed curves that represent non trivial homology classes themselves: otherwise γ would not be minimizing. Moreover, examples show us that these simple closed curves also appear as connected components of minimizing multicurves representing different homology classes than γ. Thus, when counting all homology classes with stable norm less than x, we implicitly count these simple closed curves multiple times: a more interesting problem would then be to count the homology classes minimized by such simple curves.

Definition 6.0.2. An integral homology class h on S is *simple* if it admits a minimizing representative for the stable norm of S that is a simple closed curve.

These homology classes are especially interesting as their are linked to the geometry of the unit ball of the stable norm of S. Indeed, the unit ball of the stable norm is strictly convex in the direction of an integral class h if and only if every minimizing representative of h is a simple closed curve. Thus being simple is a weaker but necessary condition for a homology class to be a strictly convex direction of the stable norm. Throughout this chapter we will be interested in computing the quantity

$$p_S(x) = \#\{h \text{ simple homology class on } S \text{ such that } \|h\|_S \leq x\}$$

on new examples that exhibit qualitatively different behaviours than what was previously known.

6.1 Explicit examples

The first type of results on the simple homology classes counting problem comes from explicit examples of surfaces on which the stable norm is known, so that the counting function $p(x)$ can be computed.

Non-orientable surfaces. Although non-orientable surfaces are not one of the interests of the current thesis, we want to mention that Balacheff and Massart found in [BM08] non-orientable surfaces whose unit ball of the stable norm was a polytope. In particular,

there are only finitely many strictly convex directions of the stable norm, thus finitely many simple homology classes. This is never the case on orientable surfaces as, according to Theorem 1.2.4 the unit ball of the stable norm of orientable surfaces can never be a polytope, meaning there always are infinitely many simple homology classes on orientable surfaces.

Flat tori. The first example of orientable surfaces on which the stable norm is computed is the flat tori, where the stable norm is Euclidean, i.e it is associated to a scalar product. A minimizing curve for the primitive integral homology class (m,n) is then the obvious torus knot that winds around the torus m times in one direction and n times in the other. On the flat *square* torus $\mathbb{T}^2$ the stable norm is equal to the Euclidean norm $\|.\|_2$, and the set of simple homology classes is composed by all the primitive elements of $\mathbb{Z}^2$, that we denote $\mathbb{Z}^2_{\text{prim}}$. We have

$$p_{\mathbb{T}^2}(x) \sim \frac{6}{\pi}x^2$$

as x goes to infinity, where the symbol $\sim$ means the ratio of these two quantities converges to 1.

We give a brief overview of the proof of this asymptotic estimate, as most of the proofs of the other counting results we will see in this section share the same mathematical flavor of renormalization and ergodic theory; for a more detailed proof, see Arana-Herrera's survey [Ara22]. We want to count the number of points of $\mathbb{Z}^2_{\text{prim}}$ inside the disk of radius x in $\mathbb{R}^2$. Note that

$$\#\left\{v \in \mathbb{Z}^2_{\text{prim}}, \|v\|_2 \leq x\right\} = \#\left\{v \in \frac{1}{x}.\mathbb{Z}^2_{\text{prim}}, \|v\|_2 \leq 1\right\}$$

so it is natural to rescale the lattice by a factor $1/x$. Then define the counting measure

$$\nu_x = \frac{1}{x^2}\sum_{v\in\mathbb{Z}^2_{\text{prim}}} \delta_{\frac{1}{x}.v}$$

where δ_p is the Dirac measure at the point p. Then the problem of computing an asymptotic estimate for $p_{\mathbb{T}^2}(x)$ comes down to identifying the weak-$\star$ limit points, if any, of the measures ν_x. Since the set $\mathbb{Z}^2_{\text{prim}}$ is invariant under the linear action of $SL(2,\mathbb{Z})$, one can show that any weak-$\star$ limit point of $(\nu_x)_x$ is a $SL(2,\mathbb{Z})$ invariant measure on $\mathbb{R}^2$. Since the Lebesgue measure λ on $\mathbb{R}^2$ is ergodic with respect to the action of $SL(2,\mathbb{Z})$, with some additional measure theory tools (namely, showing that ν_x is absolutely continuous with respect to the Lebesgue measure) one can show that any weak-$\star$ limit point ν of the sequence $(\nu_x)_x$ is of the form $\nu = c.\lambda$, with $c \geq 0$. Last but not least, with additional analysis that involves, among other things, taking averages on the Teichmüller space of flat tori one can finally show that the sequence $(\nu_x)_x$ only has one weak-$\star$ limit point $(6/\pi^2).\lambda$. Since the unit disk of $\mathbb{R}^2$ has Lebesgue measure $\lambda(\mathbb{D}) = \pi$, the result follows.

Rotational tori. For a rotational torus $\mathbb{T}_{\text{rot}}$, again the simple homology classes are the set $\mathbb{Z}^2_{\text{prim}}$. Similar arguments as in the case of flat tori then yield

$$p_{\mathbb{T}_{\text{rot}}}(x) \sim c.x^2$$

for some positive constant c.

Hyperbolic punctured tori. For a hyperbolic punctured torus $\mathbb{T}_{\text{hyp}}$ again the same computations lead to

$$p_{\mathbb{T}_{\text{hyp}}}(x) \sim c'.x^2$$

for some constant $c' > 0$. This case is historically special as it was an intermediate step towards the first result of the type of Theorem 6.0.1, due to McShane and Rivin.

The 3-squares staircase $L(2,2)$. There is one final surface on which the asymptotic number of simple homology classes is known: the surface $L(2,2)$. It is a translation surface of genus 2 in $\mathscr{H}(2)$ made of three square assembled in a L-shape, with opposite edges identified together: see the left hand side surface in Figure 2.2. Although the stable norm of the surface $L(2,2)$ has not been completely computed yet, Cheboui, Kessi and Massart managed in a work-in-progress paper [CKMss] to compute an estimate of the number of simple homology classes. More precisely, they proved

$$p_{L(2,2)}(x) \sim \frac{137}{54}.\frac{6}{\pi}.x^2$$

which is again quadratic.

6.2 Quadratic growth on translation surfaces

Apart from the trivial case of tori, there are no translation surface on which the stable norm has been completely computed. However the asymptotic number of simple homology classes on translation surfaces can still be roughly estimated.

Let (S,ω) be a translation surface. Let γ be a closed curve on S: recall that the holonomy vector of γ, denoted by $\text{hol}(\gamma)$ or z_γ, is defined as

$$z_\gamma = \int_\gamma \omega \in \mathbb{C}.$$

Moreover, if γ is a geodesic of (S,ω) then

$$|z_\gamma| = \left|\int_\gamma \omega\right| = \int_\gamma |\omega| = l(\gamma)$$

The holonomy vector of a union of two closed curves is the sum of the holonomy vector of each curve. Thus, since ω is a differential form, the above integral only depends on the homology class of γ, hence the following definition:

Definition 6.2.1. Let $h \in H_1(S,\mathbb{Z})$ be an integral homology class on S. The holonomy vector z_h of h is defined as

$$z_h = \int_\gamma \omega$$

where γ is any multicurve representing h.

The holonomy vector of a homology class is simply the value of the Poincaré pairing with ω on this class. Let γ be a multicurve representing an integral homology class h, and denote $\gamma_1, ..., \gamma_k$ the connected components of γ. If all the γ_i are geodesics of S, since

$$z_h = \int_{\gamma_1} \omega + ... + \int_{\gamma_k} \omega = z_1 + ... + z_k$$

we have

$$|z_h| \leq |z_1| + ... + |z_k| = l(\gamma).$$

Thus, taking the infimum over all representatives of h, in general we have $|z_h| \leq \|h\|_S$.

Lemma 6.2.2. *Let (S,ω) be a translation surface. The homology classes of closed saddle connections and of cylinders are simple.*

Proof. Closed saddle connections (i.e closed geodesics that encounter exactly once a singularity of S) and cylinders are obviously closed curves: it remains to check that they are minimizing inside their homology class. Let γ be a closed saddle connection or a closed geodesic inside a cylinder on S representing the class $h \in H_1(S,\mathbb{Z})$. On one hand, as seen before we have $|z_\gamma| = |z_h| \leq \|h\|_L \leq l(\gamma)$. On the other hand, since γ is a simple closed geodesic we have $l(\gamma) = |z_\gamma|$. Hence $l(\gamma) = \|h\|_L$, so γ is minimizing inside its homology class. Thus h is simple. □

Proposition 6.2.3. *Let (S,ω) be a translation surface, and let $p_S(x)$ be the number of simple homology classes on S whose stable norm is less than $x > 0$. Then there exists $c(\omega) > 0$ such that $c(\omega)x^2 \leq p_S(x)$.*

Proof. By Theorem 2.6.3, the number $N(\omega, x)$ of saddle connections and cylinders of length less than x is bounded below by $c_1(\omega)x^2$, with $c_1(\omega) > 0$. By the previous lemma, the homology classes of cylinders and of closed saddle connections are simple. While several saddle connections or cylinders might represent the same homology class, the number of redundancies $r(\omega)$ is bounded above, independently on the homology class. Indeed, it is easy to check that two closed geodesics representing the same homology class must be parallel: since the number of cylinders of S in a given direction is uniformly bounded, $r(\omega)$ is necessarily less than the number of saddle connections in the direction in which S has the most cylinders. Thus

$$c(\omega)x^2 \leq p_S(x)$$

where $c(\omega) = c_1(\omega)/r(\omega)$. □

Corollary 6.2.3.1. *Let (S,ω) be a translation surface with a single singularity. Then there exists $c, c' > 0$ that depend on ω such that $cx^2 \leq p_s(x) \leq c'x^2$.*

Proof. Let h be a simple homology class on S, and let γ be a closed simple minimizing representative of h. If γ does not encounter the singularity of S, then it lies inside a cylinder of closed parallel geodesics (all of those being homologous to one another), so h is the homology class of a cylinder. If γ passes through exactly one singularity of S then it is a closed saddle connection and h is the homology class of a saddle connection. If γ encounters more than once the singularity of S then it cannot be simple. Thus on S the simple homology classes are exactly the classes of cylinders and saddle connections. By Theorem 2.6.3, the number of such classes is bounded above by $c'x^2$ for some $c' > 0$ depending on ω. □

6.3 Subquadratic growth on half-translation surfaces

It is known that on translation surfaces the number of simple homology classes with stable norm less than x grows (at least) quadratically in x. Since half-translation surfaces are very similar to translation surfaces, one would expect the number of simple homology classes on half-translation surfaces to be quadratic in x as well. But this is not true, and the aim of this section is to provide a counter-example. Note that the approach that is used is qualitatively very different from the proofs of the other counting results, as it relies on analytic number theory rather than ergodic theory.

Let X be the square torus with a vertical slit of length ρ. The surface we are interested in is the genus 2 half-translation surface Σ from the previous chapter, obtained by glueing two copies of X along a flat cylinder. With Proposition 5.1.2 we have seen there are no connected minimizing representative of a homology class $(m,n,p,q) \in H_1(\Sigma,\mathbb{Z})$, provided that $(m,n) \neq (0,0)$ and $(p,q) \neq (0,0)$, as any connected representative must cross the cylinder and therefore is too long to be minimizing. In particular, such a homology class cannot be simple: hence a necessary condition for a class (m,n,p,q) to be simple is $m = n = 0$ or $p = q = 0$. But by Lemma 5.1.1 the stable norm of Σ coincides with the stable norm of X on the homology planes $\mathrm{Span}((1,0,0,0),(0,1,0,0))$ and $\mathrm{Span}((0,0,1,0),(0,0,0,1))$. Thus counting the simple classes on Σ comes down to counting twice the simple classes on X.

Proposition 6.3.1. *Let $p(x) = \#\{h \in H_1(X,\mathbb{Z})$, with h simple and $\|h\|_X \leq x\}$. Then*

$$p(x) = 4\left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b}\right) x\ln x + O(x)$$

where φ is Euler's totient function and $\lfloor . \rfloor$ denotes the integral part.

Proof. We use classical counting methods from analytic number theory, see for instance [Apo76] or [Ten95].

Expressing $p(x)$ as an integral. Let $L := \{\|h\|_X$ with $h \in H_1(X,\mathbb{Z})$ simple$\}$ be the simple length spectrum of X, where lengths are repeated with multiplicity. This is a countable set of positive numbers, and we order it by increasing order: $L = \{l_1 \leq l_2 \leq \ldots \leq l_n \leq \ldots\}$. For any complex number s with $\Re(s) > 1$, let

$$F(s) := \sum_{n=1}^{\infty} \frac{1}{l_n^s}.$$

This is a generalized Dirichlet series of the form $F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ where for all n, $a_n = 1$ and $\lambda_n = \ln l_n$. Applying Perron formula, for any $\sigma > 1$ and $y \in]\lambda_n, \lambda_{n+1}[$ we get

$$\sum_{k=1}^{n} a_k = \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) \frac{e^{sy}}{s} ds.$$

The left hand term is the number of λ_k such that $\lambda_k \leq y$, or equivalently this is the number of l_k such that $l_k \leq e^y$. By setting $x := e^y$, we get

$$p(x) = \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) \frac{x^s}{s} ds$$

so all we have to do is to compute the right hand side integral. To do this, we need to find a more practical expression of $F(s)$.

Rearranging the sum $F(s)$. How are simple classes distributed in $H_1(X, \mathbb{Z})$? These classes correspond to the directions of the vertices of the unit ball of the stable norm of X, i.e they are either visible classes or the Farey child of some visible class. Each visible class has infinitely many Farey children, that are all simple classes. Thus we can rearrange the sum in $F(s)$:

$$F(s) = \sum_{(b,a) \text{ visible}} F_{b,a}(s) - E(s)$$

where, abusing the notations to say that (q,p) is the child of (b,a) if p/q is the Farey child of a/b,

$$F_{b,a}(s) = \frac{1}{\|(b,a)\|_X^s} + \sum_{(q,p) \text{ child of } (b,a)} \frac{1}{\|(q,p)\|_X^s}$$

and $E(s)$ is an excess correcting term, as we may count some classes multiple times.

A more precise expression for $F_{b,a}(s)$. The successive Farey children of (b,a) are of the form $(kb+\beta, ka+\alpha)$ and $(kb+\delta, ka+\gamma)$ where (β,α) and (δ,γ) are two other visible classes such that α/β, a/b and γ/δ are consecutive neighbours in some Farey sequence, and $k \in \mathbb{N}^*$.

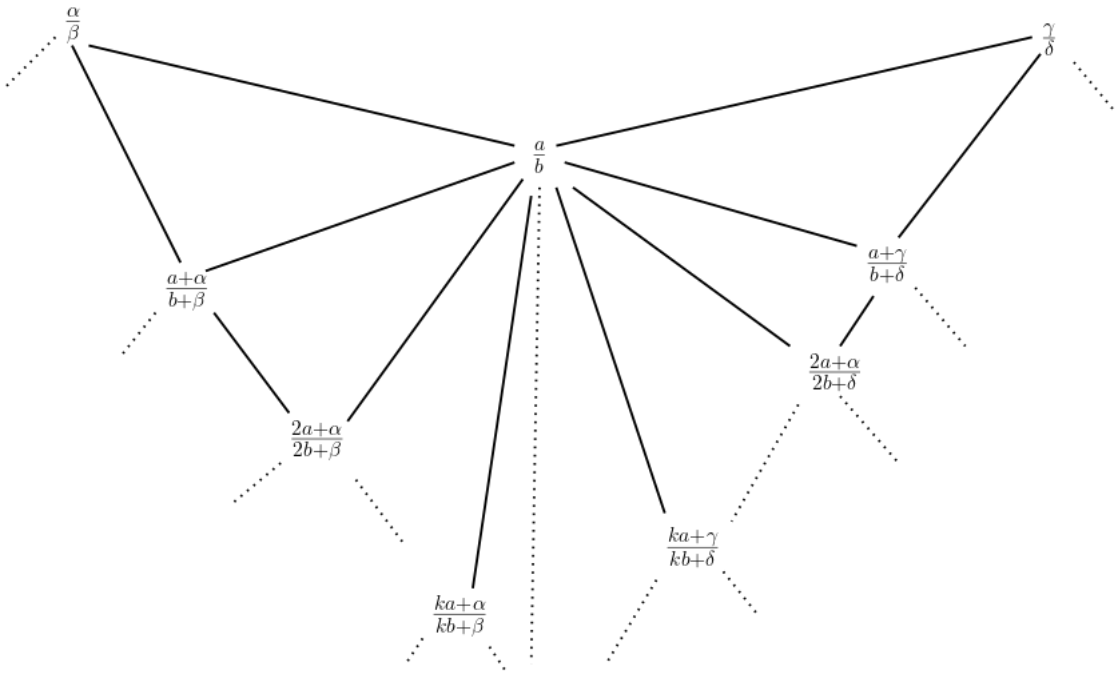

Figure 6.1: The Farey children of a/b

Since (b,a) is visible, we have

$$\|(b,a)\|_X = \|(b,a)\|_2.$$

For k big enough, $(kb+\beta, ka+\alpha)$ is not a visible class so by Theorem 4.1.17

$$\|(kb+\beta, ka+\alpha)\|_X = \sqrt{((k-1)b+\beta)^2 + ((k-1)a+\alpha \pm \rho)^2} + \sqrt{b^2 + (a \mp \rho)^2}.$$

Taking the Taylor series expansion of this expression, we get

$$\|(kb+\beta, ka+\alpha)\|_X = k\|(b,a)\|_2 + \frac{c}{k} + O\left(\frac{1}{k^2}\right)$$

where c is a constant independent of k. Thus,

$$\frac{1}{\|(kb+\beta, ka+\alpha)\|_X^s} = \frac{1}{k^s \|(b,a)\|_2^s} - \frac{sc}{k^s} + r_{1,k}(s)$$

where $|r_{1,k}(s)/k^{s+2}| \longrightarrow 0$ when k goes to infinity. Similarly, we have

$$\frac{1}{\|(kb+\delta, ka+\gamma)\|_X^s} = \frac{1}{k^s \|(b,a)\|_2^s} + \frac{sc}{k^s} + r_{2,k}(s)$$

where $|r_{2,k}(s)/k^{s+2}| \longrightarrow 0$ when k goes to infinity. Injecting this back into $F_{b,a}(s)$, we obtain

$$F_{b,a}(s) = \frac{1}{\|(b,a)\|_2^s} + 2\sum_{k=1}^{\infty} \frac{1}{k^s \|(b,a)\|_2^s} + \sum_{k=1}^{\infty} (r_{1,k}(s) + r_{2,k}(s))$$

so

$$F_{b,a}(s) = \frac{1}{\|(b,a)\|_2^s}(2\zeta(s)+1) + R_{b,a}(s)$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

is the Riemann zeta function, and $|R_{b,a}(s)/\zeta(s+2)|$ is bounded for $\Re(s) > 1$.

Summing the $F_{b,a}$'s. Summing over all visible classes, we have

$$\sum_{(b,a) \text{ visible}} F_{b,a}(s) = (2\zeta(s)+1) \sum_{(b,a) \text{ visible}} \frac{1}{\|(b,a)\|_2^s} + \sum_{(b,a) \text{ visible}} R_{b,a}(s).$$

Since the terms of the Farey sequence between two integers n and $n+1$ are simply the terms of the Farey sequence between 0 and 1 translated by n, we can rewrite the first right hand side sum as follows:

$$\sum_{(b,a) \text{ visible}} \frac{1}{\|(b,a)\|_2^s} = \sum_{\substack{(b,a) \text{ visible} \\ 0<a/b<1}} \sum_{k\in\mathbb{Z}} \frac{1}{\|(b,a+kb)\|_2^s} = 2 \sum_{\substack{(b,a) \text{ visible} \\ 0<a/b<1}} \sum_{k\geq 0} \frac{1}{\|(b,a+kb)\|_2^s}.$$

When k goes to infinity, we have

$$\frac{1}{\|(b,a+kb)\|_2^s} = \frac{1}{bk^s} + O\left(\frac{1}{k^{s+2}}\right)$$

so

$$\sum_{(b,a) \text{ visible}} \frac{1}{\|(b,a)\|_2^s} = 2\left(\sum_{\substack{(b,a) \text{ visible} \\ 0<a/b<1}} \frac{\zeta(s)}{b}\right) + R_1(s)$$

where $|R_1(s)/\zeta(s+2)|$ is bounded for $\Re(s) > 1$. Thus we have

$$\sum_{(b,a) \text{ visible}} F_{b,a}(s) = 4\left(\sum_{\substack{b\leq \lfloor 1/\rho \rfloor \\ \gcd(b,a)=1}} \frac{1}{b}\right)\zeta^2(s) + c'\zeta(s) + R_2(s)$$

where c' is a constant and $|R_2(s)/\zeta^2(s+2)|$ is bounded for $\Re(s) > 1$. Finally, we have

$$\sum_{(b,a) \text{ visible}} F_{b,a}(s) = 4\left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b}\right)\zeta^2(s) + c'\zeta(s) + R_2(s)$$

where φ is Euler's totient function.

Estimating the error term. Did we count classes multiple times? Yes we did: every visible class (m,n) has been counted as many times as it has Farey ancestors. Indeed, it has been counted as a Farey child of each of its ancestors, and since the first coordinate of the Farey parents are strictly less than the first coordinate of this class this means (m,n) has been counted at most m times. Now, the first coordinate of visible classes is bounded above by $\lfloor 1/\rho \rfloor$, so any visible class that we counted multiple times has been counted at most $\lfloor 1/\rho \rfloor$ times. Moreover, the first Farey child of two visible classes has been counted twice, once for each of its visible parents. Thus, even if we can't give a precise expression for $E(s)$, we deduce that $E(s)$ is dominated by

$$\lfloor 1/\rho \rfloor \sum_{(q,p) \text{ with both visible parents}} \frac{1}{\|(q,p)\|_X^s}.$$

This in turn is dominated by

$$c'' \sum_{(b,a)\text{visible}} \frac{1}{\|(b,a)\|_X^s}$$

where c'' is a positive constant. As we have already seen, this quantity is equal to $c''\zeta(s)$ plus a remainder that is small compared to $\zeta(s)$ when $\Re(s) \longrightarrow 1$.

An expression of $F(s)$. Putting all the formulas together, we finally get

$$F(s) = 4\left(\sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b}\right)\zeta^2(s) + C\zeta(s) + R(s)$$

where C is a non-zero constant and $|R(s)/\zeta(s)|$ bounded for $\Re(s) > 1$.

Computing the integral. Recall that by Perron formula for any $\sigma > 1$ we have

$$p(x) = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\frac{x^s}{s}ds.$$

Replacing $F(s)$ in the integral by its expression from the previous paragraph, we obtain three terms that we can estimate separately.

- By the Perron formula,

$$\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \zeta^2(s)\frac{x^s}{s}ds = \sum_{n\leq x}\tau(n)$$

  where $\tau(n) = \#\{d \text{ such that } d \text{ divides } n\}$. This is due to a classical property of Riemann zeta function: the Dirichlet series expansion of $\zeta^2(s)$ is known to be $\sum_{n\geq 1}\tau(n)n^{-s}$. Another classical number theory computation yields

$$\sum_{n\leq x}\tau(n) = x\ln x + O(x).$$

- Again, by the Perron formula, since $\zeta(s) = \sum_{n\geq 1} n^{-s}$ we have

$$\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \zeta(s)\frac{x^s}{s}ds = \sum_{n\leq x} 1 = O(x).$$

- We cannot compute explicitly the term containing the remainder $R(s)$. However, since $R(s)$ is small compared to $\zeta(s)$ when $\Re(s)$ goes to 1, and since the Perron formula holds for *any* $\sigma > 1$, by letting σ go to 1 we are assured that the integral term containing $R(s)$ is small compared to the other integral terms. In particular, this integral term is at most linear in x, so for σ close to 1

$$\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} R(s)\frac{x^s}{s}ds = O(x).$$

Summing these three terms with the right multiplicative constants we finally get

$$p(x) = 4\left(\sum_{b=1}^{\lfloor 1/\rho \rfloor}\frac{\varphi(b)}{b}\right)x\ln x + O(x).$$

This concludes the proof of Proposition 6.3.1. □

Since the surface Σ from the previous section is made by glueing two identical slit tori we have $p_\Sigma(x) = 2p(x)$, so multiplying the above expression by 2 we have computed the stable norm of Σ. In fact, we can glue more than two slit tori to construct a genus $g \geq 2$ half-translation surface Σ_g (see previous chapter) such that $p_{\Sigma_h}(x) = gp(x)$. We have shown

Theorem 6.3.2. *For every genus $g \geq 2$, there exists an explicit half-translation surface Σ_g of genus g on which the number of simple homology classes of stable norm less than x grows as*

$$p_{\Sigma_g}(x) = 4g\left(\sum_{b=1}^{\lfloor 1/\rho \rfloor}\frac{\varphi(b)}{b}\right)x\ln x + O(x)$$

where ρ is the length of the slits of the slit tori that compose Σ_g and φ is Euler's totient function.

Translation vs half-translation surfaces. To conclude we would like to comment on the difference between translation surfaces and half-translation surfaces with respect to the stable norm. We have seen in Lemma 6.2.2 that geodesics in a cylinder and saddle connections are minimizing in their homology class: thus, when counting simple homology classes one can obtain a (quadratic) lower bound by counting cylinders and saddle connections, and we get a quadratic estimate for simple classes on translation surfaces.
The situation is different on half-translation surfaces. This time if γ is a saddle connection or a closed geodesic in a cylinder it may happen that it represents the trivial homology class. Indeed, since the metric comes from a quadratic differential q that is not a differential form, the formula $l(\gamma) = \int_\gamma |q| > 0$ does not say anything about the pairing $\eta \in H^1(S,\mathbb{R}) \mapsto \langle[\gamma],\eta\rangle$ and the homology class $[\gamma]$. For instance, on our genus 2 surface Σ any closed geodesic looping around the central cylinder has trivial homology. Hence on half-translation surfaces there are fewer simple homology classes than there are cylinders of parallel geodesics. Thus, because of this difference between translation surfaces and half-translation surfaces, it makes sense to find a subquadratic estimate on the number of simple closed geodesics on a half-translation surface such as Σ.

7 Work in progress: the stable norm of $L(2,2)$

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We have already mentioned how difficult it is to explicitly compute the stable norm of a surface; translation surfaces are no exception to the rule. The stable norm of flat tori -that is, of translation surfaces of genus 1- is known but there are no other examples so far. Trying to compute the stable norm on translation surfaces makes a lot of sense. Indeed, to compute the stable norm on a surface one has first to compute the lengths of many geodesics: this task is far easier on a surface with a metric that is easy to describe, and translation metrics are arguably one of the best possible metrics from that point of view. The aim of this chapter is to provide an idea of what the stable norm of a specific translation surface looks like through partial results and numerical experiments.

The surface we will be interested in is the surface $L(2,2)$, denoted by L for short, that we already encountered several times in this thesis. It is a genus 2 square-tiled translation surface with a single singularity that belongs in the stratum $\mathcal{H}(2)$. The surface $L(2,2)$ can be made by glueing the opposite edges of three squares disposed in an L shape; the sides of the squares then provide a natural basis (e_1, e_2, f_1, f_2) for the space $H_1(L, \mathbb{Z})$.

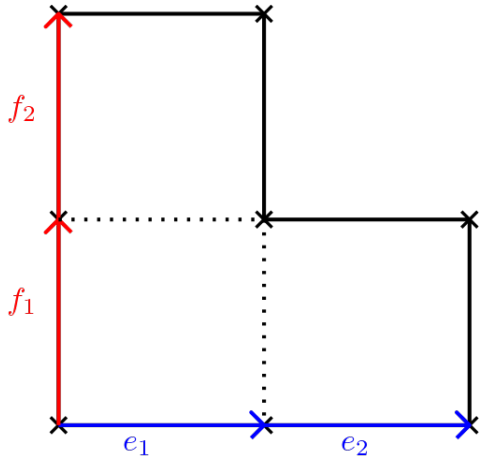

Figure 7.1: The surface $L(2,2)$

7.1 Thoughts on the stable norm of $L(2,2)$

We would like to compute the stable norm of any integral homology class on $L(2,2)$. While this is difficult in general, there is one easy case: homology classes of saddle connections. This is well defined on $L(2,2)$ because there is only one singularity, so any saddle connection is a closed curve and then represents an integral homology class.

Lemma 7.1.1. *If $h = (m,n,p,q) \in H_1(L,\mathbb{Z})$ is the homology class of a saddle connection γ, then*

$$\|h\|_L = l(\gamma) = \sqrt{(m+n)^2 + (p+q)^2}.$$

Proof. We have seen in the previous chapter that $\|h\|_L \geq |z_h|$, where z_h is the holonomy vector of h. On one hand, $\|h\|_L \leq l(\gamma)$ by definition of the stable norm, and on the other hand $|z_h| = |z_\gamma| = l(\gamma)$ since γ is a closed geodesic on L, hence $\|h\|_L = l(\gamma)$. Finally, since

$$h = me_1 + ne_2 + pf_1 + qf_2$$

we have

$$z_h = mz_{e_1} + nz_{e_2} + pz_{f_1} + qz_{f_2} = (m+n) + i(p+q)$$

as

$$z_{e_1} = z_{e_2} = 1 \text{ and } z_{f_1} = z_{f_2} = i.$$

Thus $l(\gamma) = |z_\gamma| = \sqrt{(m+n)^2 + (p+q)^2}$. □

Homology classes of saddle connections are essential to understand the stable norm of $L(2,2)$. Indeed, the stable norm of every other integral homology class depend on saddle connections. Let γ be a simple closed curve which is a connected component of a minimizing multicurve for an integral homology class on $L(2,2)$. Obviously γ must be minimizing in its own homology class, so it is a closed geodesic. If γ encounters the singularity of $L(2,2)$ it can only do it once, otherwise it would not be a simple curve; but then γ is a saddle connection itself. On the contrary, if γ does not encounter the singularity of $L(2,2)$ then it is contained inside a cylinder of closed geodesics. This

cylinder is bounded by saddle connections $\gamma_1, ..., \gamma_k$ parallel to γ, so the homology class of γ is the sum of the homology classes of these saddle connections. We have

$$[\gamma] = [\gamma_1] + ... + [\gamma_k]$$

and more importantly

$$\|\gamma\|_L = \|\gamma_1\|_L + ... + \|\gamma_k\|_L.$$

Hence the stable norm of any integral homology class which is not represented by a saddle connection is the sum of the stable norms of classes of saddle connections. However, there are many ways to write a single homology class as a sum of classes of saddles connections, so the challenge is to find which decomposition into classes of saddles connections yields the shortest total length.

Intuitively, most of the decompositions into classes of saddle connections cannot be minimizing, for example if the curve goes "backwards" at some point:

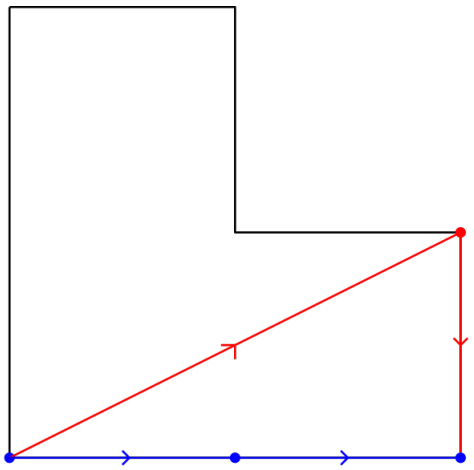

Figure 7.2: A minimizing representative (blue) and a non-minimizing representative (red) of $(1,1,0,0)$

On the above picture, it is clear that the red curve cannot be minimizing at it goes backwards: the first saddle connection goes up in the cylinder and has a positive third coordinate in the basis (e_1, e_2, f_1, f_2) whereas the second saddle connection goes down and represents the class $-f_1$. This is clearly length inefficient as such a curve makes unnecessary detours. Formally we have the following Lemma, proven by Cheboui, Kessi and Massart.

Lemma 7.1.2 ([CKMss]). *Let $(m,n,p,q) \in H_1(L,\mathbb{Z})$ be an integral homology class minimized by a multicurve γ with connected components $\gamma_1, ..., \gamma_k$, with $[\gamma_i] = (m_i, n_i, p_i, q_i)$. Then $m_1, ..., m_k$ (resp. $n_1, ..., n_k$, resp. $p_1, ..., p_k$, resp. $q_1, ..., q_k$) all have the same sign.*

Thus, the homology classes of saddle connections in the minimizing decomposition of a class h all have smaller coordinates than h. This is reminiscent of the stable norm of slit tori: recall that on the slit torus, in order to find short non-simple closed curves representing a given homology class (m,n) we decomposed the homology class into sums of classes with smaller coordinates for which the stable norm was known. The

best decomposition, meaning the one that provided the shortest total length, was the one such that the classes in the decomposition had slopes close to the slope of (m,n); this decomposition was then found using the Farey parents of n/m.
In the case of $L(2,2)$ the classes of saddle connections play the role of the visible classes on the slit torus, being classes whose stable norm is trivial to compute. Intuitively, given an integral homology class (m,n,p,q) on $L(2,2)$ that is not represented by a saddle connection, the decomposition into classes of saddle connections that yields the shortest total length should involve homology classes whose direction is close to the direction of (m,n,p,q). Thus, finding the minimizing decomposition would again come down to a combinatorial process, as was finding the Farey ancestors of a class on the slit torus.

The difficulty here is to find the right notion of direction of a homology class, and what it means for directions to be close to each other. On the slit torus we considered the slope of a class; on $L(2,2)$ the same approach cannot work as the homology is 4-dimensional this time. However in some cases we can reduce the problem to a 2-dimensional problem once again, using the holonomy vectors of the homology classes. The idea is that even if $H_1(L,\mathbb{R})$ has dimension 4, both the basis vector e_1, e_2 (resp. f_1, f_2) are horizontal (resp. vertical), so it makes sense to define a "slope" for a homology class (m,n,p,q) by grouping the "horizontal" and "vertical" coordinates together:

Definition 7.1.3. Let $a/b \in \mathbb{Q}$. The space of homology classes on $L(2,2)$ of slope, or direction, a/b is

$$E\left(\frac{a}{b}\right) := \left\{ me_1 + ne_2 + pf_1 + qf_2 \quad | \quad \frac{p+q}{m+n} = \frac{a}{b} \right\}.$$

In other words, $E(a/b)$ is the space of homology classes on $L(2,2)$ whose holonomy vector has slope a/b.

Obviously too many information on a class (m,n,p,q) is lost when looking at its slope $(p+q)/(m+n)$ to hope this data would be enough to compute the stable norm of (m,n,p,q) in general. However, there is one case where it *is* enough, and it involves (again!) the Farey sequence.

Proposition 7.1.4. *Let $h_1 \in E(a/b)$ and $h_2 \in E(c/d)$ be homology classes of saddle connections such that $a/b < c/d$ are Farey neighbours. If $h := h_1 + h_2 \in E(a/b \oplus c/d)$ is* not *the class of a saddle connection then $\|h\|_L = \|h_1\|_L + \|h_2\|_L$.*

Proof. By assumption h is not the class of a saddle connection, so a minimizing multicurve for h is obtained as a union of saddle connections. We already know one decomposition of h into classes of saddle connections, namely $h = h_1 + h_2$, and we need to check that this decomposition yields the shortest total length to conclude. We will compare the lengths of the various decompositions into saddle connections using their holonomy vectors.
Let z_h be the holonomy vector of h, and let z_i, $i = 1,2$, be the holonomy vector of h_i: necessarily we have $z_h = z_1 + z_2$. Let $\gamma_1, ..., \gamma_k$ be saddle connections on $L(2,2)$ such that $[\gamma_1] + ... + [\gamma_k] = h$. Denote by w_j the holonomy vector of γ_j: again necessarily we have $w_1 + ... + w_k = z_h$. Since they all are classes of saddle connections, by Lemma 7.1.1 we

have $\|h_i\|_L = |z_i|$ and $\|[\gamma_j]\|_L = |w_j|$. Thus, in order to show that the decomposition $h = h_1 + h_2$ is minimizing we only need to show that $|z_1| + |z_2| \leq |w_1| + ... + |w_k|$.

Since $h_1 \in E(a/b)$ (resp. $h_2 \in E(c/d)$) the slope of z_1 (resp. z_2) is a/b (resp. c/d). Since $h = h_1 + h_2$ and since a/b and c/d are Farey neighbours, the slope of z_h is equal to $(a+c)/(b+d)$ the Farey child of a/b and c/d. Let p_j/q_j be the slope of the holonomy vector w_j; up to renumbering them we can assume $p_1/q_1 > ... > p_k/q_k$. We can assume that for all $j = 1, ..., k$ we have $q_j < b+d$. Indeed if this is not the case then, in order for the sum $w_1 + ... + w_k$ to be equal to $z_h = (a+c) + i(b+d)$, all third or fourth coordinates of the classes $[\gamma_j]$ would not have the same sign. Then by Lemma 7.1.2 the decomposition $[\gamma_1] + ... + [\gamma_k]$ of h cannot possibly be minimizing, so we can ignore it. Thus, $p_j/q_j \in \mathscr{F}_{b+d}$ where $\mathscr{F}_{b+d}$ is the Farey sequence of order $b+d$. Since a/b and c/d are the Farey parents of $(a+c)/(b+d)$, no p_j/q_j lies in the interval $[a/b, c/d]$. Intuitively, no vector w_j is closer to z_h that z_1 and z_2. In the case $k = 4$ we have the following picture, on which it is clear that $|z_1| + |z_2| \leq |w_1| + ... + |w_k|$. The general case gives similar pictures.

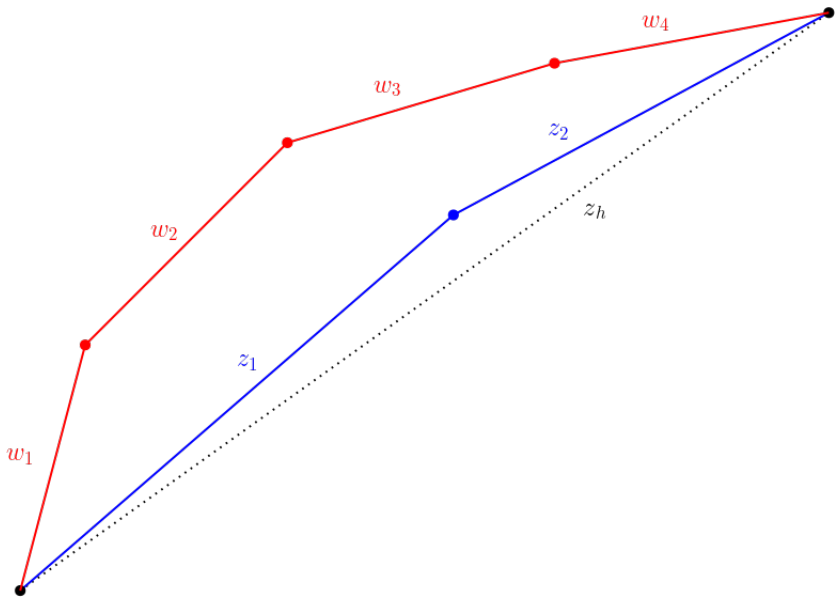

Figure 7.3: Geometric representation of $z_1 + z_2$ (blue) and $w_1 + ... + w_k$ (red)

Thus, for any decomposition $[\gamma_1] + ... + [\gamma_k]$ of the class h into classes of saddle connections, with holonomy vectors $w_1, ..., w_k$ we have

$$\|h_1\|_L + \|h_2\|_L = |z_1| + |z_2| \leq |w_1| + ... + |w_k| = \|[\gamma_1]\|_L + ... + \|[\gamma_k]\|_L$$

so $h_1 + h_2$ is a minimizing decomposition of h and finally $\|h\|_L = \|h_1\|_L + \|h_2\|_L$.

□

It is tempting to conjecture that if $h \in E(n/m)$ is an integral homology class that is not represented by a saddle connection, then there exist two integral classes $h_1 \in E(a/b)$ and $h_2 \in E(c/d)$ (possibly not classes of saddle connection themselves, meaning they must be split in order to compute their stable norm) where a/b and c/d are the Farey parents of n/m such that $\|h\|_L = \|h_1\|_L + \|h_2\|_L$.
However this is not true: it can be checked (for example using a computer program,

see next section) that the class $(4,4,5,0) \in E(5/8)$ is minimized by the decomposition $(0,1,1,0)+(4,3,4,0)$, with $(0,1,1,0) \in E(1/1)$ and $(4,3,4,0) \in E(4/7)$; but the Farey parents of $5/8$ are $2/3$ and $3/5$, not $1/1$ and $4/7$. Even worse, the classes $(0,1,1,0)$ and $(4,3,4,0)$ turn out to be classes of saddle connections, so the problem does not come from the fact that these classes must be further decomposed in order to compute their stable norm. This counter-example provides enough evidence to conclude that, even if some more small results such as Proposition 7.1.4 may still be obtainable this way, trying to compute the stable norm of $L(2,2)$ using slopes of holonomy vectors is not the correct approach.

7.2 An algorithm to compute the stable norm of $L(2,2)$

In the absence of formulas for the stable norm of $L(2,2)$, it is still possible to produce a naive recursive algorithm that computes the stable norm of any integral class through brute force. The computation time is a severe limiting factor for such an algorithm, but it remains reasonable for homology classes with relatively small coordinates. Thus we can look at examples to test our ideas and try to guess patterns taken by the values of the stable norm of $L(2,2)$.

7.2.1 General idea of the algorithm

We want to compute the stable norm of an integral homology class $h = (m,n,p,q) \in H_1(L,\mathbb{Z})$; the algorithm works recursively, so we assume that the stable norm of every integral class (m',n',p',q') with $|m'| \leq |m|, |n'| \leq |n|, |p'| \leq |p|$ and $|q'| \leq |q|$ is already known.

1. Check if $h = (m,n,p,q)$ is primitive, i.e if $\gcd(m,n,p,q) = 1$. If not, if instead $\gcd(m,n,p,q) = d$, return the value $d.\|(m/d, n/d, p/d, q/d)\|_L$.

2. If h is primitive, check if it is is the homology class of a saddle connection on $L(2,2)$. If so, by Lemma 7.1.1, return the value $\sqrt{(m+n)^2+(p+q)^2}$.

3. Optional step to save some computation time: if h is not the class of a saddle connection, check if it can be written as $h_1 + h_2$ with $h_1 \in E(a/b), h_2 \in E(c/d)$ classes of saddle connections and $a/b, c/d$ the Farey parents of $(p+q)/(m+n)$. If so, by Proposition 7.1.4 return $\|h_1\|_L + \|h_2\|_L$; these values are already known because by Lemma 7.1.2 the coordinates of h_1, h_2 are smaller than the coordinates of h.

4. If h is not the class of a saddle connection (nor the sum of two saddle connections whose directions are Farey neighbours) we know it is minimized by a union of saddle connections, i.e by a decomposition $h_1 + ... + h_k$ of homology classes. These k terms can be grouped to form only two terms si h is always minimized by a union of two multicurves representing homology classes h'_1, h'_2 such that $h'_1 + h'_2 = h$. So for all the possible decompositions $h = h'_1 + h'_2$ compute $\|h'_1\|_L + \|h'_2\|_L$ and return the minimum of these numbers. Again, we know recursively what the

stable norms of h'_1 and h'_2 are, as by Lemma 7.1.2 they have smaller coordinates than h.

Obviously the last step of this algorithm is extremely inefficient as the vast majority of the decompositions $h = h'_1 + h'_2$ are not minimizing and waste computation time for nothing, hence the wording *naive* algorithm. Unfortunately, to rule some of these decompositions out before even having to needlessly compute their stable norm, some more mathematics are required. Although we can develop some intuition on what an eliminating criterion might be, no significant result in this direction is known so far so we have to stick with this inefficient algorithm for now.

Homology classes of saddle connections. For the second step of the computation of the stable norm we need to be able to check whether or not a homology class can be represented by a saddle connection. Fortunately, this is an easy question to solve.

The set of saddle connections (and cylinders) of $L(2,2)$ is preserved under the action of the Veech group $\Gamma(L)$. More precisely, since $L(2,2)$ can be decomposed into either one or two cylinders (see Figure 2.6), there are exactly two distinct orbits of saddle connections under the action of $\Gamma(L)$, one orbit consisting of saddle connections in the direction of which $L(2,2)$ can be decomposed into one cylinder, and the other of saddle connections in the direction of which $L(2,2)$ can be decomposed into two cylinders.

Since saddle connections are preserved by the action of $\Gamma(L)$, so is the set of holonomy vectors of saddle connections under the linear action of the Veech group on $\mathbb{C} \simeq \mathbb{R}^2$. To test whether or not a given homology class $h \in E(q/p)$ is represented by a saddle connection, we compute all the homology classes of saddle connections inside $E(q/p)$ and then simply check if h is one of them. To do this, we first find the element $M \in \Gamma(L)$ of the Veech group that sends the holonomy vector $z_h = p + iq$ (that we identify with the vector (p,q)) to an "elementary direction", that is a vector in $\mathbb{R}^2$ whose coordinates are $-1, 0$ or 1. Note that it is always possible to do so because there are one-cylinder directions and two-cylinders directions of $L(2,2)$ among the elementary directions, i.e at least one element of each of the two possible orbits of saddle connections under the action of $\Gamma(L)$. It is then trivial to determine the homology classes of all saddle connections corresponding to an elementary direction (see tables below). Applying the map induced by M^{-1} on $H_1(L,\mathbb{Z})$ to these classes we finally obtain all the homology classes of saddle connections with the same direction as h.

On $L(2,2)$, there are three distinct homology classes of saddle connections in the directions in which the surface decomposes into a single cylinder, and only two distinct homology classes of saddle connections in the directions in which the surface decomposes into two cylinders.

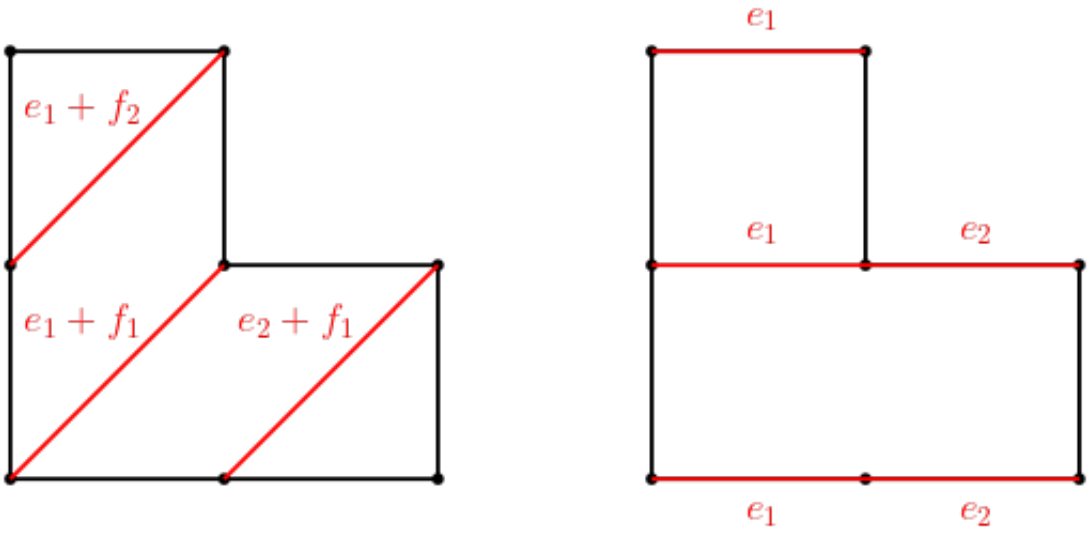

Figure 7.4: Homology classes of saddle connections in a one-cylinder direction (left) and in a two-cylinders direction (right)

Here is the list of homology classes of saddle connections in the elementary directions.

Elementary direction	Holonomy vector	Classes of saddle connections
$E(0/1)$	$(1,0)$	$(1,0,0,0),(0,1,0,0)$
$E(1/1)$	$(1,1)$	$(1,0,1,0),(0,1,1,0),(1,0,0,1)$
$E(1/0)$	$(0,1)$	$(0,0,1,0),(0,0,0,1)$
$E(1/-1)$	$(-1,1)$	$(-1,0,1,0),(0,-1,1,0),(-1,0,0,1)$
$E(0/-1)$	$(-1,0)$	$(-1,0,0,0),(0,-1,0,0)$
$E(-1/-1)$	$(-1,-1)$	$(-1,0,-1,0),(0,-1,-1,0),(-1,0,0,-1)$
$E(-1/0)$	$(0,-1)$	$(0,0,-1,0),(0,0,0,-1)$
$E(-1/1)$	$(1,-1)$	$(1,0,-1,0),(0,1,-1,0),(1,0,0,-1)$

The Veech group of $L(2,2)$ is $\Gamma(L) = \langle A, S\rangle \subset SL(2,\mathbb{Z})$ where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We also need the matrix $B = A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. The matrices A and B act on $H_1(L,\mathbb{Z})$ respectively as

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

7.2.2 Sagemath code

In this section we present the Sagemath code written to compute the stable norm of $L(2,2)$. My apologies to the expert reader for how badly the code may be written, as it is the work of a complete beginner in the matter.

Finding the homology classes of saddle connections. Here we introduce the function that, given a rational direction, returns all the homology classes of saddle connections in that direction.

First, given a rational direction q/p (given to the program under the form of a vector (p,q)), the following function returns an elementary vector v (named `vec`), the matrix $M \in \Gamma(L)$ such that $Mv = (p,q)^T$ and the matrix $\tilde{M}$ (named `matHom`) induced by the diffeomorphism M on $H_1(L,\mathbb{Z})$. The algorithm is similar to a classical algorithm computing continued fraction expansions of rational numbers, the elementary operations being this time multiplication by A and B. The goal is to transform (p,q) into an elementary vector by multiplying repeatedly by A, B, A^{-1} or B^{-1}. The idea is that for every vector (p,q), by multiplication by a suitable choice of one of those four matrices we can make the absolute value of the coordinates of the vector decrease. Thus, within a finite amount of steps, we are assured to transform (p,q) into an elementary vector.

```
def reduction(p,q):

    #The matrices A and B
    A = matrix(2,2,[1,2,0,1])
    invA = A.inverse()
    B = transpose(A)
    invB = B.inverse()

    #The matrices induced by A and B in homology
    gen1 = matrix(4,4,[[1,0,1,2],[0,1,1,0],[0,0,1,0],[0,0,0,1]])
    inv1 = gen1.inverse()
    gen2 = matrix(4,4,[[1,0,0,0],[0,1,0,0],[1,2,1,0],[1,0,0,1]])
    inv2 = gen2.inverse()

    if gcd(p,q) != 1:
        raise ValueError("input ({}, {}) are not coprime".format(p,q))
    else:

        #Target direction
        vec = vector([p,q])
        #Starting matrix, the identity
        M = matrix(2,2,[1,0,0,1])
        #Starting matrix induced in homology, the identity
        matHom = matrix(4,4,[[1,0,0,0],[0,1,0,0], [0,0,1,0],[0,0,0,1]])


        while (abs(vec[0])>1 or abs(vec[1])>1):
            if abs(vec[0]) > abs(vec[1]):
                if vec[0]*vec[1] >0:
                    M = M*A
                    matHom = matHom*gen1
                    vec = invA*vec
```

```
                elif vec[0]*vec[1] <0:
                    M = M*invA
                    matHom = matHom*inv1
                    vec = A*vec
            if abs(vec[0]) < abs(vec[1]):
                if vec[0]*vec[1] >0:
                    M = M*B
                    matHom = matHom*gen2
                    vec = invB*vec
                elif vec[0]*vec[1] <0:
                    M = M*invB
                    matHom = matHom*inv2
                    vec = B*vec

        return vec, M, matHom
```

The following function, given a direction (p,q), returns all the homology classes of saddle connections in $E(q/p)$. This is done by multiplying the homology classes of saddle connections in the suitable elementary direction by matrix induced on $H_1(L,\mathbb{Z})$ by the element of $\Gamma(L)$ that sends this vector to (p,q); see tables in the previous section.

```
def separatrices(p, q):
    e1 = vector([1,0,0,0])
    e2 = vector([0,1,0,0])
    f1 = vector([0,0,1,0])
    f2 = vector([0,0,0,1])

    (vec, M, matHom) =reduction(p,q)
    if list(vec)==[1,0]:
        sep1 = matHom*e1
        sep2 = matHom*e1
        sep3 = matHom*e2
    if list(vec)==[-1,0]:
        sep1 = matHom*(-e1)
        sep2 = matHom*(-e1)
        sep3 = matHom*(-e2)
    if list(vec)==[0,1]:
        sep1 = matHom*f1
        sep2 = matHom*f1
        sep3 = matHom*f2
    if  list(vec)==[0,-1]:
        sep1 = matHom*(-f1)
        sep2 = matHom*(-f1)
        sep3 = matHom*(-f2)
    if list(vec)==[1,1]:
        sep1 = matHom*(e1+f1)
```

```
        sep2 = matHom*(e2+f1)
        sep3 = matHom*(e1+f2)
    if  list(vec)==[-1,-1]:
        sep1 = -matHom*(e1+f1)
        sep2 = -matHom*(e2+f1)
        sep3 = -matHom*(e1+f2)
    if list(vec)==[1,-1] :
        sep1 = matHom*(e1-f1)
        sep2 = matHom*(e2-f1)
        sep3 = matHom*(e1-f2)
    if  list(vec)==[-1,1]:
        sep1 = matHom*(-e1+f1)
        sep2 = matHom*(-e2+f1)
        sep3 = matHom*(-e1+f2)
    return [sep1, sep2, sep3]
```

Finally, the following function tests if an integral class $(m,n,p,q) \in H_1(L,\mathbb{Z})$ is represented by a saddle connection. To do this, the function computes all the classes of saddle connections inside $E((p+q)/(m,n))$ and tests if (m,n,p,q) is one of them.

```
def is_sc(m,n,p,q):
    a=m+n
    b=p+q
    d=gcd(a,b)
    if d!=1:
        return False
    else:
        s=separatrices(a,b)
        for i in range(3):
            if m==s[i][0] and n==s[i][1] and p==s[i][2] and q==s[i][3]:
                return True
    return False
```

Decompositions of a homology class. According to the general idea of the algorithm explained in the previous section, if a homology class h is not represented by a saddle connection, we must look at all possible decompositions $h_1 + h_2 = h$ to find which one yields the shortest length. The intermediate functions combine to create a list that contains all of those decompositions.

A function that deletes all extra copies of an element inside of a list.

```
def doublons(liste):
    nliste=[]
    for part in liste:
        if [part[1],part[0]] not in nliste:
```

```
            nliste.append(part)
    return nliste
```

A function that returns all possible decompositions h_1+h_2 of the class $h=(m,n,p,q)$.

```
def split(m,n,p,q):
    resu=[]
    for i in range(abs(m)+1):
        for j in range(abs(n)+1):
            for k in range(abs(p)+1):
                for l in range(abs(q)+1):
                    I=sign(m)*i
                    J=sign(n)*j
                    K=sign(p)*k
                    L=sign(q)*l
                    if (i,j,k,l)!=(0,0,0,0) and (I,J,K,L)!=(m,n,p,q):
                        resu.append([(I,J,K,L),(m-I,n-J,p-K,q-L)])

    #If no decomposition is found,
    #adds the trivial decomposition to the list
    if resu==[]:
        resu=[[(0,0,0,0),(m,n,p,q)]]

    #Removes all the extra copies of a single decomposition
    #to account for permutations
    resu=doublons(resu)

    return resu
```

The stable norm of an integral class. We have all the tools to compute the stable norm of integral homology classes.

```
#Intermediate function, gcd of four integers
    def pgcd(m,n,p,q):
    if (m,n,p,q)==(0,0,0,0):
        return 1
    else:
        if m!=0 or n!=0:
            return gcd(gcd(gcd(m,n),p),q)
        else:
            if p!=0:
                m,p=p,m
                return gcd(gcd(gcd(m,n),p),q)
            else:
                m,q=q,m
                return gcd(gcd(gcd(m,n),p),q)
```

Here is the main function of this section, that, given a homology class (m,n,p,q), returns its stable norm.

```
@cached_function
def stable(m,n,p,q):

    div=pgcd(m,n,p,q)

    #Deals with the case of non-primitive classes
    if div!=1:
        return div*stable(m/div,n/div,p/div,q/div)

    #Case of primitive classes
    else:

        #Basic case: if (m,n,p,q) is the homology class
        #of a saddle connection
        if is_sc(m,n,p,q):
            return sqrt((m+n)**2 + (p+q)**2)

        #If not, computes recursively
        else:

        #The list of all possible decompositions of the class
            decomp=split(m,n,p,q)
            L=len(decomp)

        #The list that stores the length of each decomposition
            long=[0 for i in range(L)]

            for i in range(L):
                (a,b,c,d)=decomp[i][0]
                (A,B,C,D)=decomp[i][1]
                alpha=pgcd(a,b,c,d)
                beta=pgcd(A,B,C,D)
                a,b,c,d= a/alpha, b/alpha, c/alpha, d/alpha
                A,B,C,D= A/beta, B/beta, C/beta, D/beta
                long[i]= alpha*stable(a,b,c,d)+beta*stable(A,B,C,D)

        #The stable norm of (m,n,p,q) is the minimum
        #of the computed values
        res=min(long)

    return res
```

Other useful functions. Aside from the value of the stable norm of a class, we would like to know a minimizing decomposition of that class.

The function `sc_decomp` is an improved version of the function `split` that, given an integral homology class $h = (m,n,p,q)$, returns all possible decomposition of the class into sums of classes of saddle connections. The function returns a list of dictionaries, each dictionary corresponding to a decomposition of h into saddle connections: a key of the dictionary is the class of saddle connection and the associated value is its multiplicity in the decomposition. For example the decomposition $2(1,0,0,1)+(1,0,0,0)$ is represented by `{(1,0,0,1):2,(1,0,0,0):1}`.

```
#Intermediate function that fuses two dictionaries
def fusion(dict1,dict2):
    cles=dict2.keys()
    res=dict1.copy()
    for vec in cles:
        if vec not in dict1:
            val=dict2[vec]
            res[vec]=val
        else:
            res[vec]+=dict2[vec]
    return res



@cached_function
def sc_decomp(m,n,p,q):

    res=[]
    Res=[]

    #Trivial case
    if is_sc(m,n,p,q):
        Res=[{(m,n,p,q):1}]

    else:
        divi=pgcd(m,n,p,q)

        #Case of a primitive class
        if divi==1:
            decomp=split(m,n,p,q)
            L=len(decomp)
            for i in range(L):
                a,b,c,d=decomp[i][0]
                e,f,g,h=decomp[i][1]
                if is_sc(a,b,c,d) and is_sc(e,f,g,h):
```

```
                    dd={(a,b,c,d):1,(e,f,g,h):1}
                    res.append(dd)
                else:
                    l1=sc_decomp(a,b,c,d)
                    l2=sc_decomp(e,f,g,h)
                    for d1 in l1:
                        for d2 in l2:
                            dd=fusion(d1,d2)
                            res.append(dd)

        #Case of a non-primitive class
        else:
            ens=sc_decomp(m/divi,n/divi,p/divi,q/divi)
            for dd in ens:
                cles=dd.keys()
                DD={}
                for vec in cles:
                    val=dd[vec]
                    DD[vec]=val*divi
                res.append(DD)

    #Deletes extra copies of a decomposition from the result
    for dec in res:
        if dec not in Res:
            Res.append(dec)

    return Res
```

The following function returns a minimizing decomposition of a class (m,n,p,q).

```
def min_long(m,n,p,q):
    decomp=sc_decomp(m,n,p,q)
    Norm=stable(m,n,p,q)
    res=[]
    for dd in decomp:
        cles=dd.keys()
        long=0
        for vec in cles:
            a,b,c,d=vec
            long+= dd[vec]*stable(a,b,c,d)
        if long==Norm:
            res.append(dd)
    return res
```

7.3 Gallery: sections of the unit ball

Thanks to the algorithm presented in the previous section we can compute the stable norm of homology classes on $L(2,2)$, along with their minimizing decompositions. Aside from testing conjectures, this allows us to draw pictures of the unit ball $\mathscr{B}_L$ of the stable norm of $L(2,2)$. The ball $\mathscr{B}_L$ is a 4-dimensional convex set: obviously it is not possible to fully picture it, but we can represent some sections of it, 2-dimensional and 3-dimensional.

Note that the surface $L(2,2)$ can be rearranged by cut and paste equivalence into a cross, on which several symmetries of the surface are obvious.

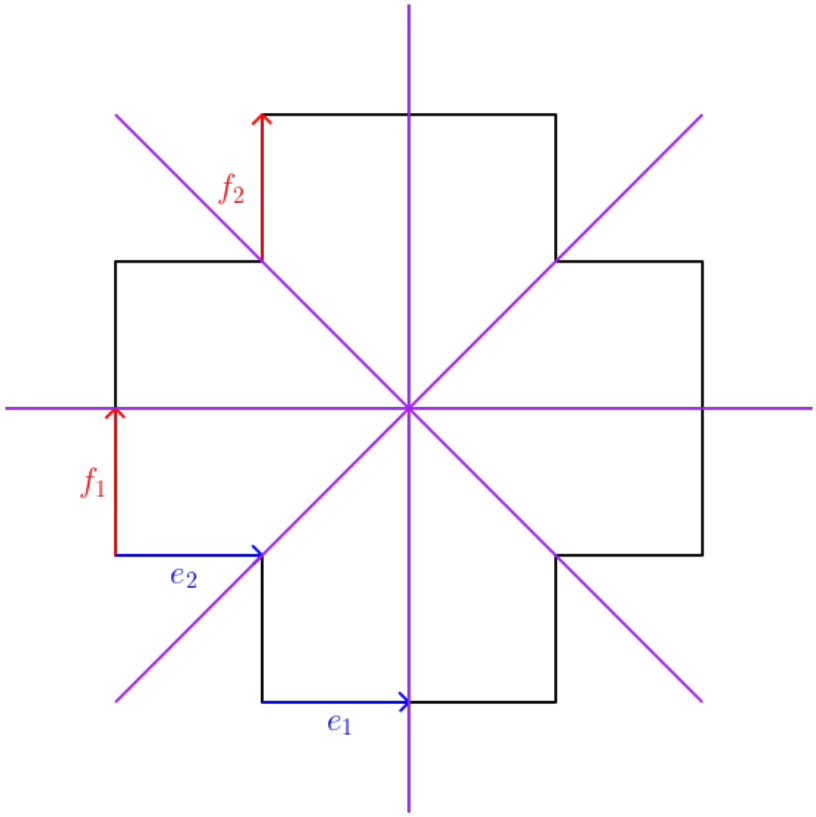

Figure 7.5: Symmetries of $L(2,2)$

It is then clear that there exist three isometries σ_1, σ_2 and σ_3 of $L(2,2)$ such that, for $i = 1,2$:

$$\sigma_1(e_i) = e_i \text{ and } \sigma_1(f_i) = -f_i$$

is the symmetry with respect to the horizontal axis,

$$\sigma_2(e_i) = -ei \text{ and } \sigma_2(f_i) = f_i$$

is the symmetry with respect to the vertical axis and

$$\sigma_3(e_i) = f_i \text{ and } \sigma_3(f_i) = e_i$$

is the symmetry with respect to the diagonal. As a consequence, each section of $\mathscr{B}_L$ by a subspace V of $H_1(L,\mathbb{Z})$ that we will draw is identical to the section of $\mathscr{B}_L$ by the subspace $\sigma_j(V)$. So in order to avoid repetitions, we will not draw those extra sections.

7.3.1 Two-dimensional sections of the unit ball of the stable norm

In this section we present pictures of sections of $\mathscr{B}_L$ by planes in $H_1(L,\mathbb{Z})$.

The tautological plane. There is one special homology plane that we would like to showcase. For a translation surface (S,ω), the *tautological* plane of S is the homology plane generated by the homology classes h_{ver}, h_{hor} of the vertical and horizontal foliations on S. Then (see [CKMss]) the stable norm of S, restricted to the tautological plane, is Euclidean: $\|ah_{ver} + bh_{hor}\|_S = vol(S)\sqrt{a^2+b^2}$. On $L(2,2)$, the tautological plane is generated by the classes $2e_1 + e_2$ and $2f_1 + f_2$ and $vol(L(2,2)) = 3$, so we expect the section of $\mathscr{B}_L$ by the tautological plane to be a circle of radius $1/3$. Our algorithm returns the following picture, as expected.

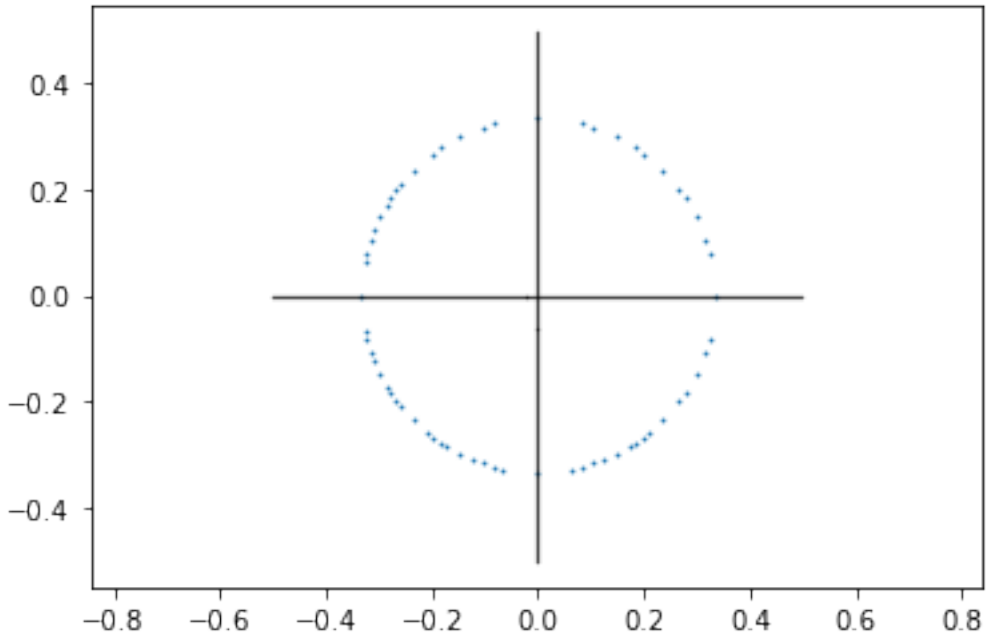

Figure 7.6: Section of $\mathscr{B}_L$ by the tautological plane

Only a few points of the circle are drawn, because the computation time of `min_long` increases very rapidly as the absolute value of the coordinates of the class grow. If anything, this picture confirms that the algorithm does what it is supposed to do.

Other planes. Since $\mathscr{B}_L$ is convex, if three points of its boundary are aligned then the entire segment that contains them lies in the boundary of $\mathscr{B}$. Thus, using the function `min_long` we can draw segments of $\partial\mathscr{B}_L$. For example, `min_long(1,1,2,2)` returns

```
[{(0, 1, 1, 0): 1, (1, 0, 1, 2): 1}].
```

This means that a minimizing decomposition of $(1,1,2,2)$ into classes of saddle connections is $(0,1,1,0) + (1,0,1,2)$, so

$$\|(1,1,2,2)\|_L = \|(0,1,1,0)\|_L + \|(1,0,1,2)\|_L$$

and the segment

$$\left[\frac{(0,1,1,0)}{\|(0,1,1,0)\|_L}, \frac{(1,0,1,2)}{\|(1,0,1,2)\|_L}\right]$$

is contained in $\partial\mathscr{B}_L$. We say that this segment is *certified*. Hopefully, by computing enough decompositions of classes inside a plane of $H_1(L,\mathbb{Z})$, we would eventually find enough certified segments to obtain a complete picture of the section of $\mathscr{B}_L$ by this plane.

We obtain various polygonal sections of $\mathscr{B}_L$. The red dots represent classes of saddle connections. Here is a parallelogram:

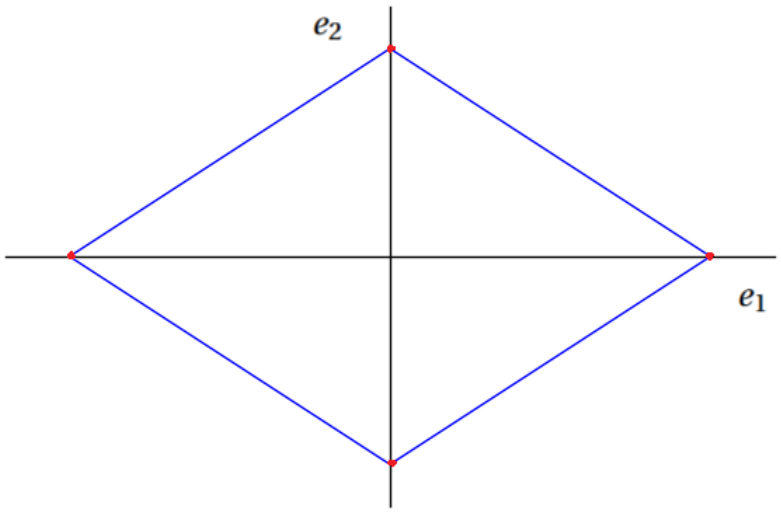

Figure 7.7: Section of $\mathscr{B}_L$ by $\mathrm{Span}(e_1, e_2)$

An octagon:

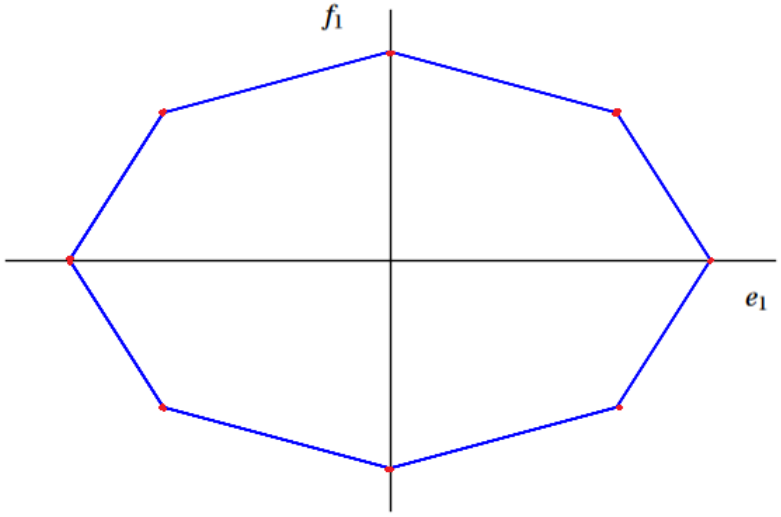

Figure 7.8: Section of $\mathscr{B}_L$ by $\mathrm{Span}(e_1, f_1)$

A different octagon:

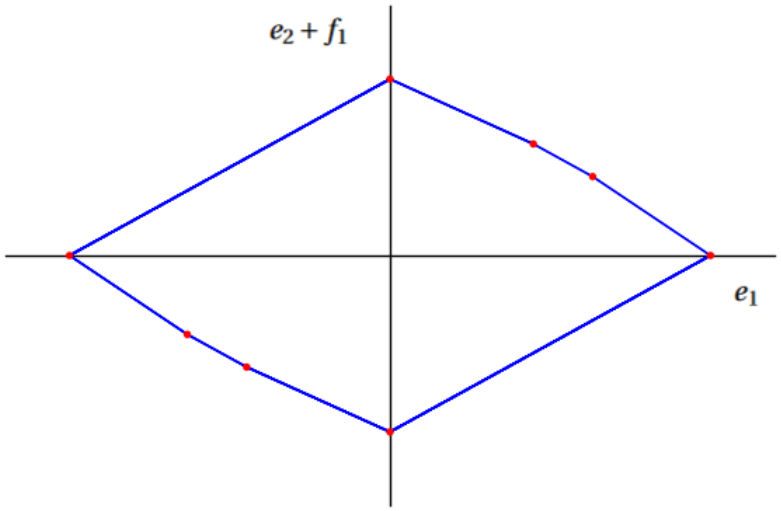

Figure 7.9: Section of $\mathscr{B}_L$ by Span$(e_1, e_2 + f_1)$

We also find more exotic shapes, such as

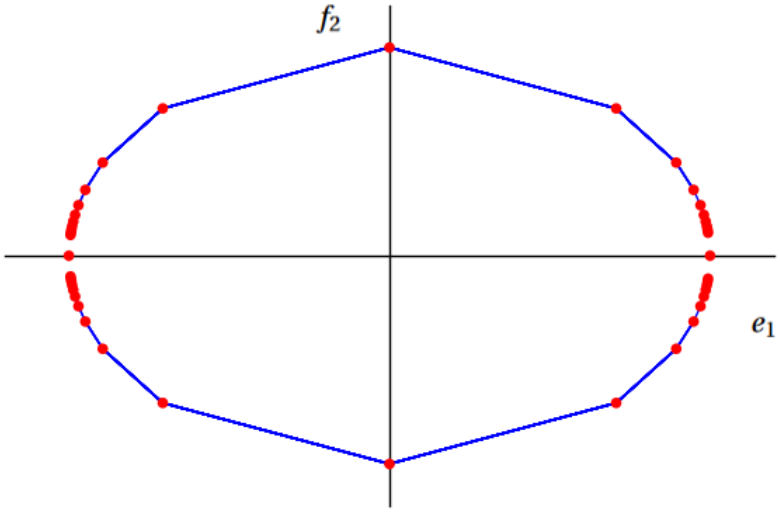

Figure 7.10: Section of $\mathscr{B}_L$ by Span(e_1, f_2)

or

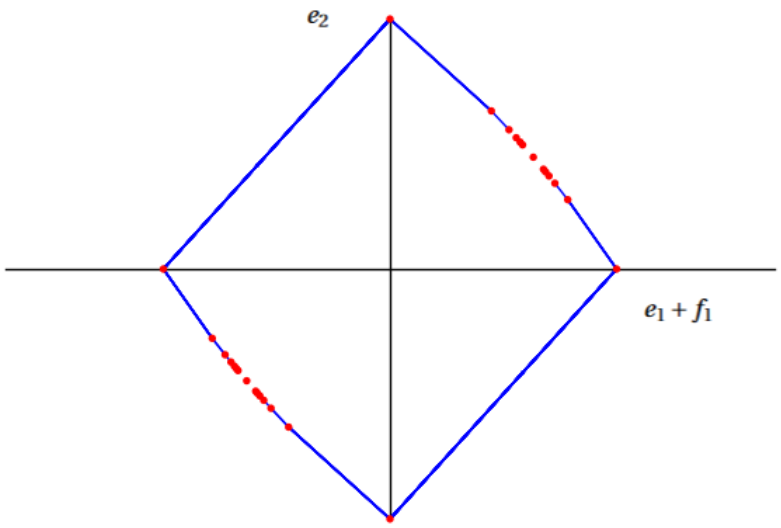

Figure 7.11: Section of $\mathscr{B}_L$ by Span$(e_1 + f_1, e_2)$

However in general, when picking a plane at random to cut $\mathscr{B}_L$ with, we do not get nice pictures. Indeed there are few, if any, segments of $\mathscr{B}_L$ contained in that plane (that are computable in a reasonable amount of time) so we can oly draw individual points and the pictures look like:

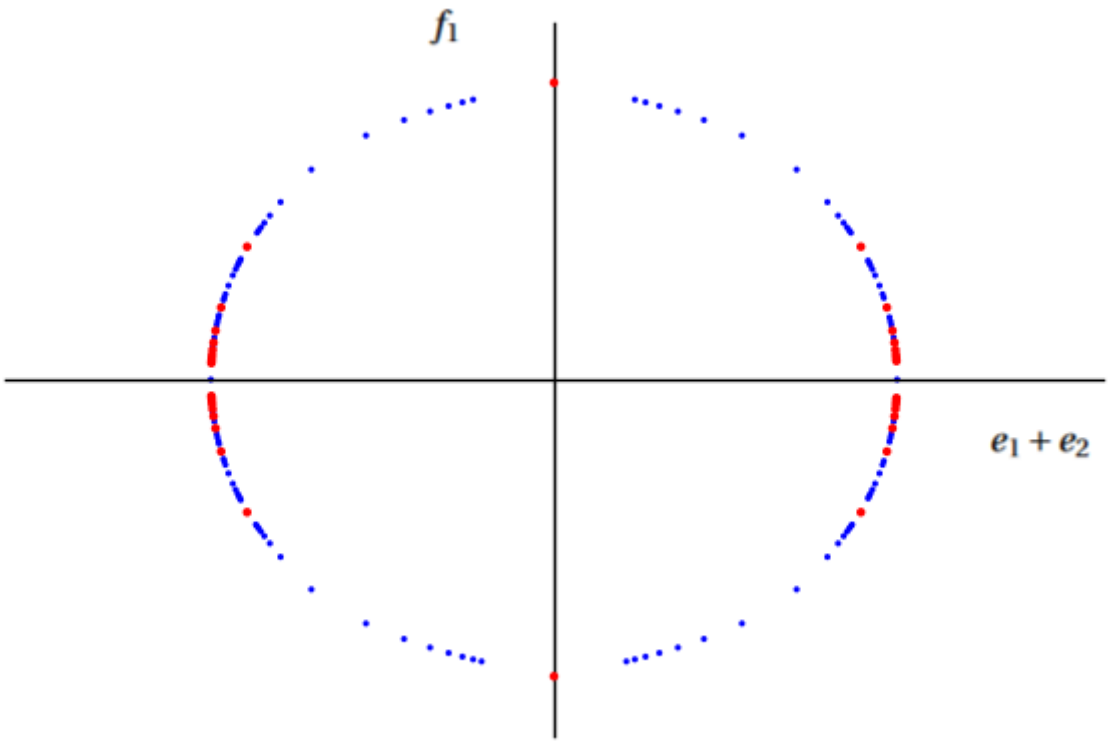

Figure 7.12: Section of $\mathscr{B}_L$ by Span$(e_1 + e_2, f_1)$

7.3.2 Three-dimensional sections of the unit ball of the stable norm

Similarly to the case of 2-dimensional sections, since $\mathscr{B}_L$ is convex we can find entire faces of its boundary. For example, `min_long(2,1,2,1)` returns

```
[{(0, 1, 1, 0): 1, (1, 0, 0, 1): 1, (1, 0, 1, 0): 1}]
```

so the entire triangle

$$\mathrm{Hull}\left(\frac{(0,1,1,0)}{\|(0,1,1,0)\|_L}, \frac{(1,0,0,1)}{\|(1,0,0,1)\|_L}, \frac{(1,0,1,0)}{\|(1,0,1,0)\|_L}\right)$$

is contained in $\partial\mathscr{B}_L$, and we can have our algorithm draw it. This time the pictures we get are incomplete, time being a limiting factor for our program: we have drawn every face and segment that the program has found within a reasonable amount of time, but there remains blank spaces; however in some cases the pattern is clear.

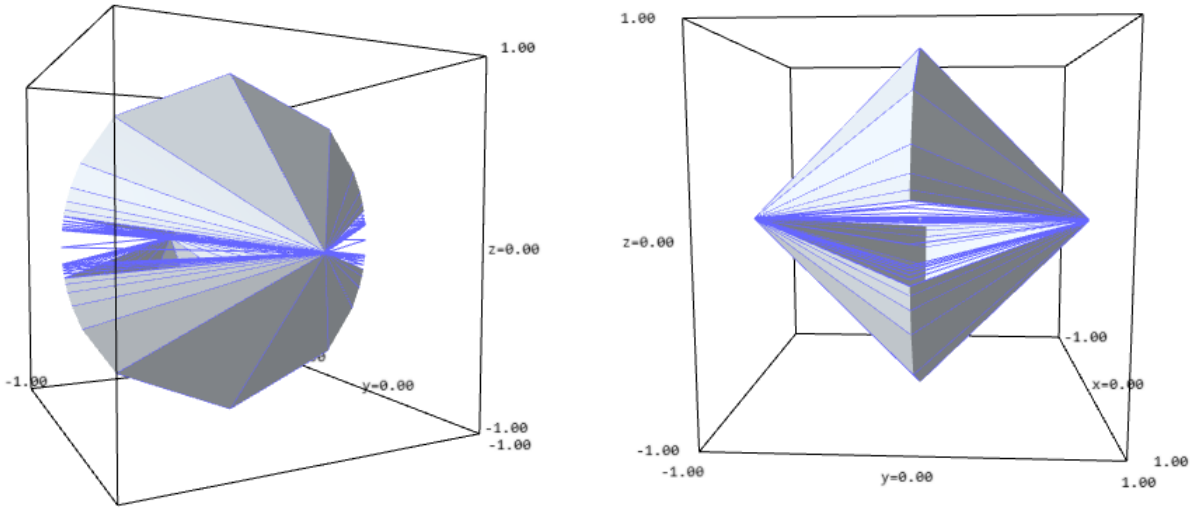

Figure 7.13: Section of $\mathscr{B}_L$ by $\mathrm{Span}(e_1, e_2, f_2)$

In this first picture all the faces seem to be triangles.

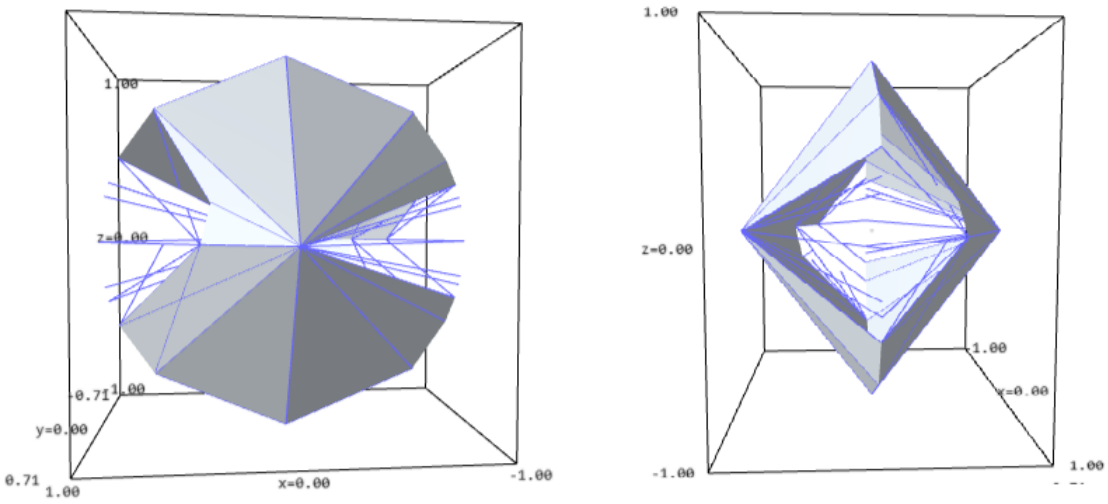

Figure 7.14: Section of $\mathscr{B}_L$ by $\mathrm{Span}(e_1, e_2 + f_1, f_2)$

Although very similar to the first picture, we can see that one of the faces seems to be a quadrilateral; after verification with the function `min_long`, it turns out that this face is indeed a quadrilateral.

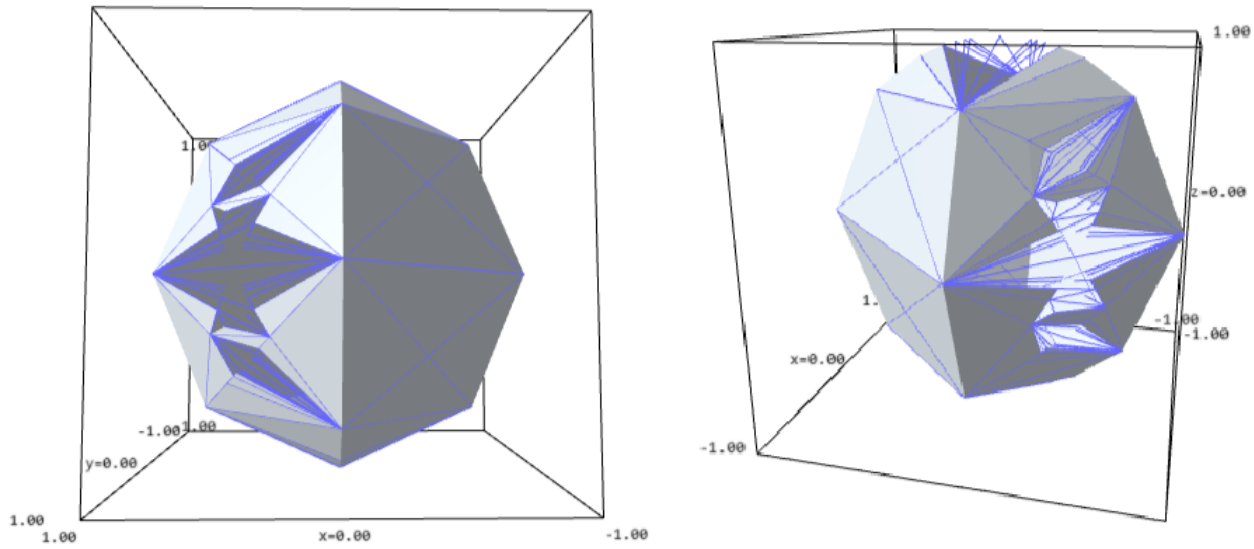

Figure 7.15: Section of $\mathscr{B}_L$ by $\mathrm{Span}(e_1, e_2, f_1)$

Again, we find quadrilateral faces, and a very complicated pattern elsewhere. We would like to represent this pattern in a more readable way, as it is confusing on a non-interactive 3-dimensional drawing. The idea is to project parts of the above picture onto simplices, in order to obtain a 2-dimensional sketch of a 3-dimensional object that will hopefully be more readable. More precisely, we flatten the picture by projecting radially the part of $\mathscr{B}_L \cap \mathrm{Span}(e_1, e_2, f_1)$ with *positive* coordinates onto the simplex

$$\Delta_{e_1,e_2,f_1} = \{(x, y, z, 0) \in H_1(L, \mathbb{R}), x + y + z = 1\}$$

keeping track of the vertices, segments and faces of $\mathscr{B}_L$ found by the algorithm. In the following picture the part of $\mathscr{B}_L$ (in blue and grey) is projected onto the simplex Δ_{e_1,e_2,f_1} (in red and gold):

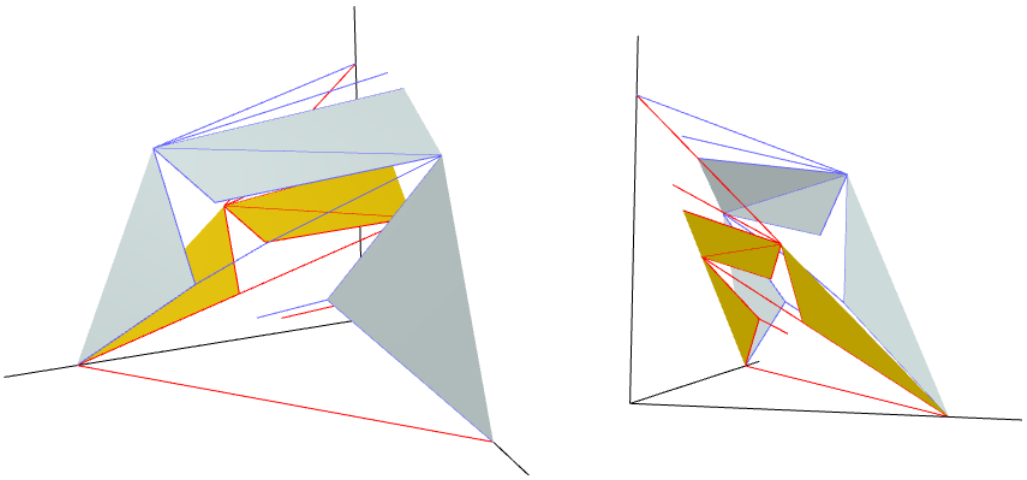

Figure 7.16: Projection of $\mathscr{B}_L \cap \mathrm{Span}(e_1, e_2, f_1)$ onto Δ_{e_1,e_2,f_1}

Pushing the computations for a (much) longer time the pattern of segments and faces becomes clearer. Coloring the adjacent faces of $\mathscr{B}_L$ in different colors to better distinguish them, we finally obtain the following picture. The blank spaces correspond to places where the program could not finish the computations in time.

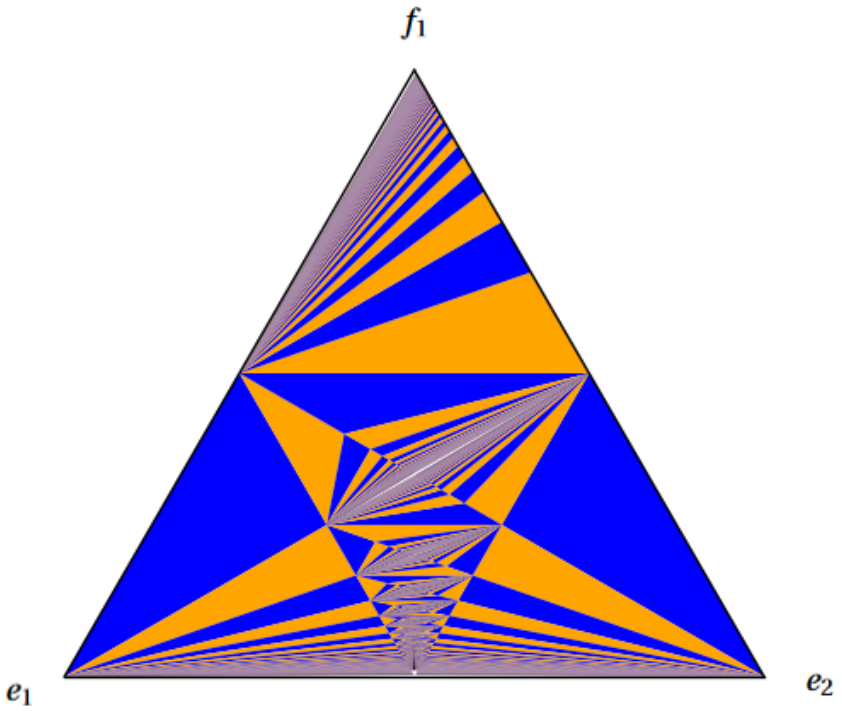

Figure 7.17: Projection of $\mathscr{B}_L$ onto Δ_{e_1,e_2,f_1}

Similarly, we get

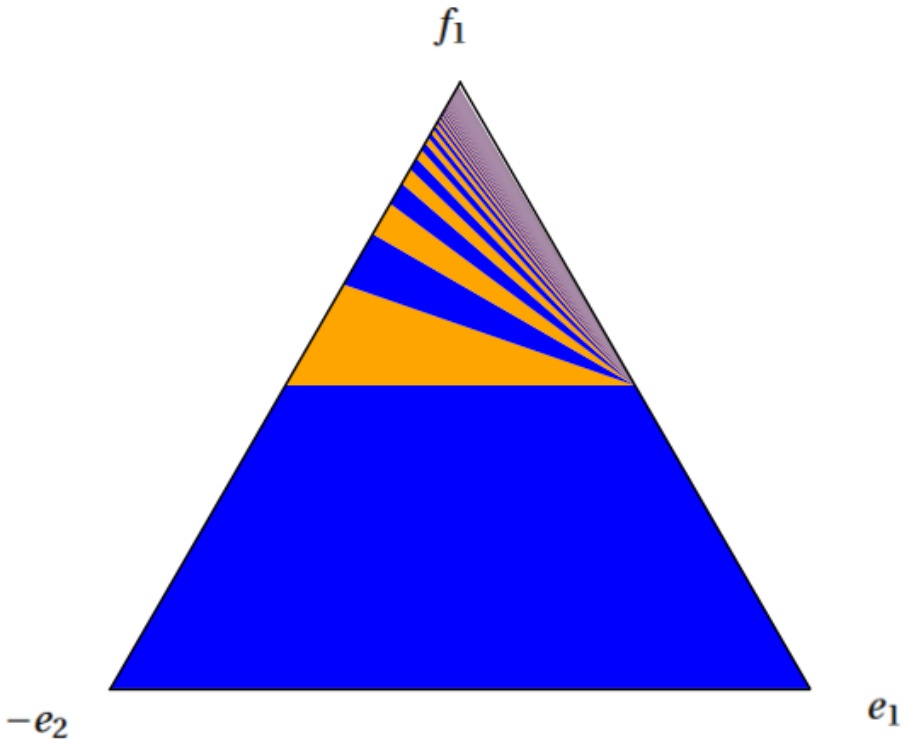

Figure 7.18: Projection of $\mathscr{B}_L$ onto $\Delta_{e_1,-e_2,f_1}$

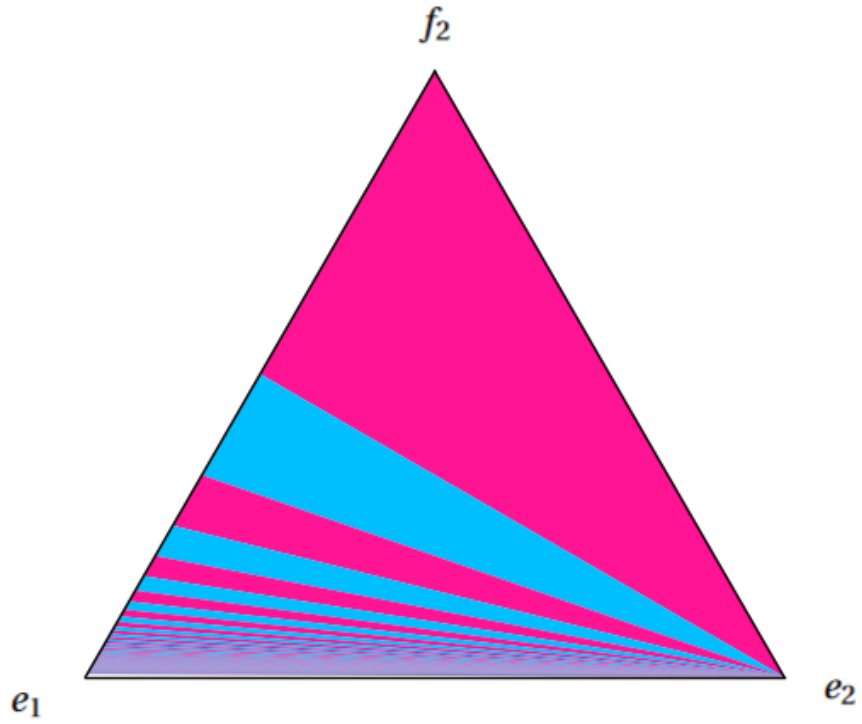

Figure 7.19: Projection of $\mathscr{B}_L$ onto Δ_{e_1,e_2,f_2}

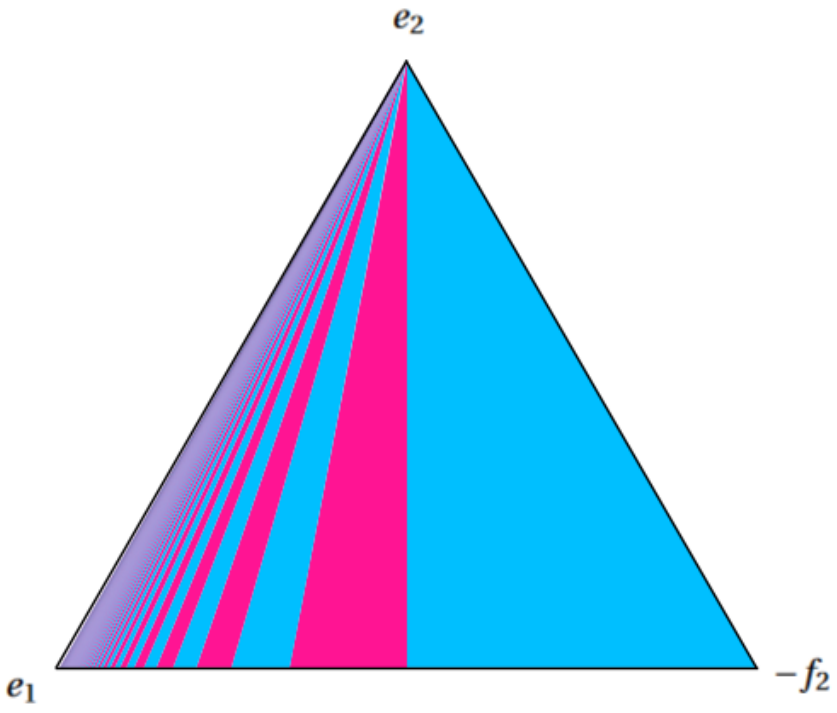

Figure 7.20: Projection of $\mathscr{B}_L$ onto $\Delta_{e_1,e_2,-f_2}$

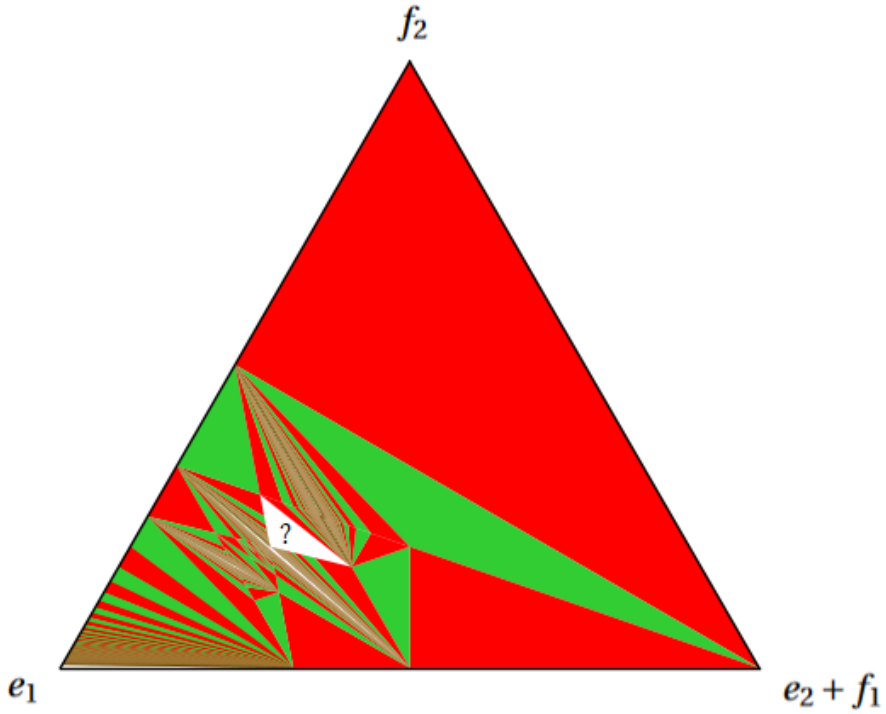

Figure 7.21: Projection of $\mathscr{B}_L$ onto Δ_{e_1,e_2+f_1,f_2}

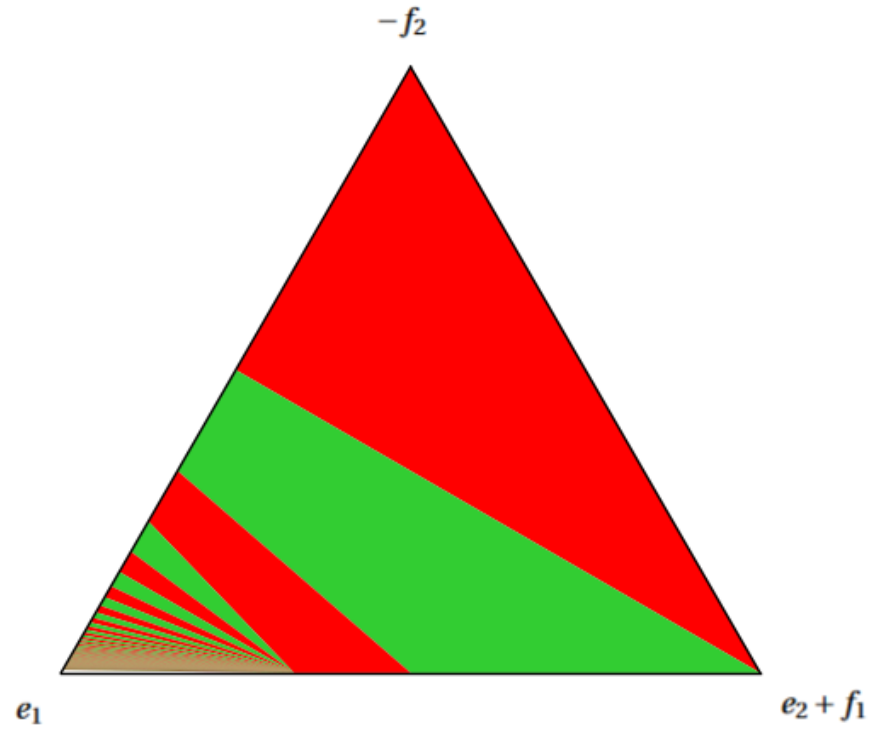

Figure 7.22: Projection of $\mathscr{B}_L$ onto $\Delta_{e_1,e_2+f_1,-f_2}$

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