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Couplages de processus stochastiques en géométrie sous-riemannienne

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Titre: Couplages de processus stochastiques en géométrie sous-riemannienne

Résumé :
Dans cette thèse on s'intéresse à l'étude des couplages des mouvements browniens sous-elliptiques sur plusieurs variétés sous-riemaniennes: les groupes de Carnot libres d'ordre 2, incluant le groupe d'Heisenberg, ainsi que les groupes de matrices $SU(2)$ et $SL(2,\mathbb{R})$. Ces derniers peuvent être vus, avec le groupe d'Heisenberg, comme des espaces modèles de courbure constante en géométrie sous-riemanienne. On utilise la décomposition du mouvement brownien sous la forme d'un mouvement brownien sur une variété riemannienne, que nous nommerons bruit directeur, ainsi que d'un processus pouvant s'interpréter comme l'aire balayée par ce bruit directeur. La méthode de couplage utilisée ici consiste alors à définir un couplage pour les bruits directeurs et à en étudier l'effet sur les aires balayées.

En s'inspirant de précédents travaux sur le groupe d'Heisenberg on s'intéresse d'abord aux couplages avec succès, c'est à dire les couplages pour lesquels le temps de rencontre entre les processus est fini presque surement. En utilisant la décomposition de Karhunen-Loève pour les ponts browniens, on construit des couplages non co-adaptés avec succès sur $SU(2)$ et sur les groupes de Carnot libres de profondeur 2 sur $\mathbb{R}^n$ avec $n \geq 3$. Lorsque les processus partent de la même fibre on obtient également un résultat sur $SL(2,\mathbb{R})$. En particulier ces constructions mènent à des estimées du taux de couplage dont on déduit des inégalités pour le semi-groupe de la chaleur et pour les fonctions harmoniques. On propose aussi la construction explicite d'un couplage co-adapté sur $SU(2)$.

Enfin, on développe un nouveau modèle de couplage "en un coup" basé sur résolution d'un problème inverse pour obtenir de bonnes estimées de la distance en variation totale entre les lois des processus. Ce travail est effectué sur les groupes de Carnot libres de profondeur 2 et utilise une décomposition du mouvement brownien via les polynômes de Legendre. Il permet aussi d'obtenir des relations similaires à la formule de Bismut-Elworthy-Li pour les gradients de semi-groupes, via l'étude d'un changement de probabilité sur l'espace des vecteurs Gaussiens.

Mots-clés : Géométrie sous-riemannienne, Processus stochastiques, Inégalités fonctionnelles, mouvement Brownien sous-elliptique

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Title: Coupling of stochastic processes in subRiemannian geometry

Abstract:
This thesis deals with the study of the couplings of subelliptic Brownian motions in several subRiemannian manifolds: the free, step 2 Carnot groups, including the Heisenberg group, as well as the groups of matrices $SU(2)$ et $SL(2,\mathbb{R})$. These last ones can be seen, together with the Heisenberg group, as some model spaces with constant curvature in subRiemannian geometry. We use the decomposition of the Brownian motion as a Brownian motion on a Riemannian manifold, that we will call driving noise, as well as a process that can be interpreted as the area swept by the driving noise. The coupling method used here consists in defining a coupling for the driving noises and then study its impact on the swept areas.

Taking inspiration from previous works on the Heisenberg group, we first focus on successful couplings, that is couplings where the processes meet at an almost surely finite time. Using the Karhunen-Loève expansion of the Brownian bridges, we construct successful non co-adapted couplings on $SU(2)$ and on the free step 2 Carnot groups with rank $n \geq 3$. When the processes start from the same fiber we also get a result on $SL(2,\mathbb{R})$. In particular we obtain estimates of the coupling rate, leading to gradient inequalities for the heat semi-group and for harmonic functions. We also describe the explicit construction of a co-adapted successful coupling on $SU(2)$.

Finally, we develop a new coupling model "in one sweep", which is based on the study of an inverse problem to obtain good estimates of the total variation distance between the laws of the processes. This work is done for any free, step 2 Carnot groups and uses a decomposition of the Brownian motion via the Legendre polynomials. This method also allows us to obtain relations similar to the Bismut-Elworthy-Li formula for the gradient of the semi-group by studying a change of probability on the Gaussian space.

Keywords: SubRiemannian geometry, Stochastic processes, Functional inequalities, Subelliptic Brownian motion

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Chapter 1

Résumé étendu en français

Ce court chapitre est la seule partie de cette thèse en français. Il a pour but de présenter le problème et de donner un aperçu des résultats obtenus.

1.1 Contexte d'étude

1.1.1 Notion de couplage

Nous donnons tout d'abord une définition de la notion de couplage. Soit $\mathbb{G}$ une variété (dans notre cas une variété sous-riemanienne) et $g, \tilde{g} \in \mathbb{G}$. On appelle couplage de deux mouvements browniens partant de $(g, \tilde{g})$ toute mesure de probabilité sur $\mathcal{C}([0, T[, G \times G)$ (pour $T \in [0, +\infty]$) pour laquelle la première marginale suit la loi d'un mouvement Brownien $(\mathbb{B}_t)_t$ partant de g et la seconde marginale suit la loi d'un mouvement brownien $(\tilde{\mathbb{B}}_t)_t$ partant de $\tilde{g}$. En d'autres termes, il s'agit d'étudier la loi jointe du processus $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$.

Voici quelques exemples simples (mais non moins utiles) de couplages partant de $(x, \tilde{x})$ sur $\mathbb{R}^2$:

1. **Le couplage synchrone:** On construit $\mathbb{B}_t$ et $\tilde{\mathbb{B}}_t$ égaux à une translation près:

$$\tilde{\mathbb{B}}_t - \tilde{x} = \mathbb{B}_t - x.$$

2. **Le couplage par réflexion**: On choisit une base orthonormée (e_1, e_2) de $\mathbb{R}^2$ telle que $e_1 := \frac{x-\tilde{x}}{\|x-\tilde{x}\|_2}$. Soit $(B_t^1, B_t^2)_t$ un mouvement brownien 2-dimensionel partant de 0. On construit alors:

$$\mathbb{B}_t = x + B_t^1 e_1 + B_t^2 e_2 \text{ et } \tilde{\mathbb{B}}_t = \tilde{x} - B_t^1 e_1 + B_t^2 e_2.$$

   Les deux mouvements browniens $(\mathbb{B}_t)_t$ et $(\tilde{\mathbb{B}}_t)_t$ sont en fait construits grâce à une symétrie axiale.

   Le couplage par réflection produit des mouvements browniens qui se rencontrent en un temps p.s. fini. Usuellement, et tout particulièrement dans la sous-section 1.1.3, une fois que les processus se sont rencontrés on choisit $\mathbb{B}_t = \tilde{\mathbb{B}}_t$.

Notons que ces deux couplages sont **co-adaptés**, c'est à dire que le futur des deux processus construits ne dépend que de leur passé commun. En fait $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ est même markovien. Voici un exemple de couplage non co-adapté sur $\mathbb{R}$:

3. **Un couplage avec connaissance du futur:** Soit $T > 0$, on peut décomposer le mouvement brownien dans une base de $L^2([0,T])$. Par exemple, en utilisant la décomposition de Karhunen Loève, on a:

$$\mathbb{B}_t = x + \sqrt{T}\sum_{j\geq 1} \xi_j \frac{\sqrt{2}}{j\pi}\sin\left(\frac{j\pi t}{T}\right) + \frac{t}{\sqrt{T}}\xi_0$$
$$\text{et } \tilde{\mathbb{B}}_t = \tilde{x} + \sqrt{T}\sum_{j\geq 0} \tilde{\xi}_j \frac{\sqrt{2}}{j\pi}\sin\left(\frac{j\pi t}{T}\right) + \frac{t}{\sqrt{T}}\tilde{\xi}_0$$

où $(\xi_j)_{j\geq 0}$ (resp. $(\tilde{\xi}_j)_{j\geq 0}$) sont identiquement et indépendemment distribuées et suivent une loi normale $\mathcal{N}(0,1)$. On peut alors construire le couplage $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ en choisissant un couplage $(\xi_j, \tilde{\xi}_j)$ pour tout $j \geq 0$. En prenant par exemple $\xi_1 \neq \tilde{\xi}_1$ et $\xi_0 = \tilde{\xi}_0$, on a $\tilde{\mathbb{B}}_T = \mathbb{B}_T - x + \tilde{x}$: le couplage n'est pas co-adapté puisqu'on a besoin de la connaissance du futur (la valeur $\mathbb{B}_T$) de l'un des processus pour construire le second.

Ces exemples se généralisent à $\mathbb{R}^n$ en toute dimension ainsi qu'à certaines variétés riemanniennes comme la sphère ou le plan hyperbolique. Notons que l'on peut aussi considérer des couplages statiques: il s'agit de définir un couplage ponctuellement et non sur toutes la trajectoire du processus. Ce type de couplage sera uniquement considéré dans le chapitre 7 et dans le chapitre 8).

1.1.2 Temps de couplage

Pour un couplage $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ donné, il peut être intéressant d'étudier le premier temps de couplage τ, c'est à dire le premier temps de rencontre des mouvements browniens. En effet, il est aisé de montrer l'inégalité de couplage suivante:

$$d_{TV}(\mathcal{L}(\mathbb{B}_s), \mathcal{L}(\tilde{\mathbb{B}}_s)) \leq \mathbb{P}(\tau > s)\ \forall s > 0$$

où d_{TV} est la distance en variation totale. En particulier, les couplages pour lesquels cette inégalité devient une égalité sont appelés **couplages maximaux**. Si $\mathbb{P}(\tau > t)$ et $d_{TV}(\mathcal{L}(\mathbb{B}_t), \mathcal{L}(\tilde{\mathbb{B}}_t))$ ont le même ordre de grandeur pour t grand, le couplage est dit **efficace**. Dans le cas de variétés riemanniennes qui ont de bonnes propriétés de symétrie, les couplages maximaux sont obtenus avec un couplage par réflexion. En étudiant le taux de couplage, c'est à dire la quantité $\mathbb{P}(\tau > t)$, on peut aussi obtenir des inégalités pour le gradient du semi groupe de la chaleur et des résultats sur les fonctions harmoniques.

1.1.3 Mouvements browniens en géométrie sous-riemannienne

Sur une variété sous-riemannienne $\mathbb{G}$, le mouvement brownien peut s'écrire sous la forme $(\mathbb{B}_t)_t = (X_t, z_t)_t$ avec $(X_t)_t$ un processus de diffusion elliptique et $z_t = f\left((X_s)_{s\leq t}\right)$ où f est une fonctionnelle. Grâce à cette structure, il suffit de construire un couplage de processus elliptiques (bien mieux connus) $(X_t, \tilde{X}_t)_t$ pour obtenir un couplage sur $\mathbb{G}$. Dans ce travail nous étudions les couplages de mouvements browniens sur les variétés sous-riemanniennes suivantes:

- Le groupe d'Heisenberg $\mathbb{H}$, pour lequel $(X_t)_t$ est un mouvement brownien sur $\mathbb{R}^2$ et z_t est l'aire signée (à valeurs dans $\mathbb{R}$) balayée par $(X_s)_{s\leq t}$ (appelée aire de Lévy).

- Le groupe spécial unitaire $SU(2)$, où $(X_t)_t$ est un mouvement brownien sur la sphère S^2 et où z_t est l'aire signée (modulo 4π) balayée par $(X_s)_{s\leq t}$.
- Le groupe spécial linéaire $SL(2,\mathbb{R})$, où $(X_t)_t$ est un mouvement brownien sur le plan hyperbolique et où z_t est l'aire signée (modulo 4π) balayée par $(X_s)_{s\leq t}$.

Ces trois espaces peuvent êtres vus comme des espaces modèles à courbure constante en géométrie sous-riemannienne.

Nous nous intéressons aussi aux groupe de Carnot libres de profondeur 2 sur $\mathbb{R}^n$ avec $n \geq 2$. Dans ce cas, $(X_t)_t$ est un mouvement brownien sur $\mathbb{R}^n$ et $z_t = (z_t^{i,j})_{1\leq i<j\leq n} \in \mathbb{R}^{\frac{n(n-1)}{2}}$ où $z_t^{i,j}$ est l'aire signée (à valeurs dans $\mathbb{R}$) balayée par le processus $(X_s^i, X_s^j)_{s\leq t}$, c'est à dire une aire de Lévy. Pour $n = 2$, il s'agit exactement du groupe d'Heisenberg.

1.1.4 Couplages *avec succès* sur le groupe d'Heisenberg

Dans le cas du groupe d'Heisenberg nous pouvons citer deux constructions notables (à quelques variantes près). La première est un couplage co-adapté, *avec succès* proposé par Kendall [37]. Cette construction utilise en alternance les couplages miroir et synchrone. Notons que ce couplage a été généralisé par Kendall [37] aux groupes de Carnot libres d'ordre 2. Le second est un couplage non co-adapté proposé par Banerjee, Gordina et Mariano [6]. Ce couplage est construit en se ramenant à deux mouvements browniens partant de la même fibre et en utilisant un couplage avec connaissance du futur. On choisit en particulier $\xi_k = \tilde{\xi}_k$ pour tout $k \neq 1$. Il a été prouvé que ce couplage non co-adapté est efficace. Il a aussi été montré qu'un couplage co-adapté (et donc en particulié celui de Kendall) ne peut être efficace. En notant d_{cc} la métrique sous-riemannienne (distance de Carnot-Carathéodory), on obtient:

$$\mathbb{P}(\tau > t) \leq C \left(\frac{d_{cc}(\mathbb{B}_0, \tilde{\mathbb{B}}_0)}{\sqrt{t}} \mathbb{1}_{\{X_0 \neq \tilde{X}_0\}} + \frac{d_{cc}(\mathbb{B}_0, \tilde{\mathbb{B}}_0)^2}{t} \mathbb{1}_{\{X_0 = \tilde{X}_0\}} \right). \tag{1.1}$$

Ce résultat permet d'obtenir des inégalités de gradients de deux types:

- La première concerne les gradients du semi groupe de la chaleur $(P_t f)_t$ associé au mouvement Brownien sur le groupe d'Heisenberg. Il existe une constante C telle que, pour toute fonction f bornée et $\mathcal{C}^\infty$:

$$||\nabla P_t f||_\infty \leq \frac{C}{\sqrt{t}} ||f||_\infty \ \forall t \geq 1. \tag{1.2}$$

- La seconde concerne les gradients de fonctions harmoniques. Il existe une constante C telle que, pour toute fonction f définie sur un domaine compact D, continue sur la fermeture de D, positive et harmonique, et pour tout $x \in D$:

$$\|\nabla_{\mathcal{H}} f(x)\|_{\mathcal{H}} < C \left(1 + \frac{1}{\delta_x} + \frac{1}{\delta_x^4} + \frac{(1+\delta_x)^3}{\delta_x^4} \right) f(x) \tag{1.3}$$

  avec $\delta_x := d_{cc}(x, D^c)$.

Ici $||\nabla \cdot||_\infty$ est la norme sup de la longueur de gradient et $\|\nabla_{\mathcal{H}} \cdot\|_{\mathcal{H}}$ est la norme du gradient horizontal associée à la structure sous-riemannienne. Les définitions et les relations entre ces objets sont donnés dans la sous-section 3.1.6.

1.2 Résultats

Dans cette thèse, nous nous intéressons à la construction explicite de couplages pour lesquels les estimées du taux de couplage sont suffisamment bonnes. Les travaux présentés concernent les publications suivantes:

[18] Magalie Bénéfice, Marc Arnaudon, and Michel Bonnefont. Couplings of Brownian motions on $SU(2, \mathbb{C})$. In *Geometric science of information. Part I*, volume 14071 of *Lecture Notes in Comput. Sci.*, pages 592–600. Springer, Cham, [2023] ©2023

[25] Magalie Bénéfice. Couplings of Brownian motions on $SU(2)$ and $SL(2, \mathbb{R})$. *Stochastic Processes and their Applications*, page 104434, 2024

[24] Magalie Bénéfice. Non co-adapted couplings of Brownian motions on subRiemannian manifolds. `https://arxiv.org/abs/2312.14512`, 2023

[26] Magalie Bénéfice. Non co-adapted successful couplings of Brownian motions on the free, step 2 carnot groups. `https://arxiv.org/abs/2407.06593`, 2024

[3] Marc Arnaudon, Magalie Bénéfice, Michel Bonnefont, and Delphine Féral. A coupling strategy for Brownian motions at fixed time on Carnot groups using Legendre expansion. `https://arxiv.org/abs/2407.04321`, 2024

1.2.1 Couplages *avec succès*

Nous avons pu généraliser les couplages *avec succès* présentés pour le groupe d'Heisenberg (le couplage co-adapté de Kendall [37] et le couplage non co-adapté de Banerjee, Gordina et Mariano [6]) au cas de $SU(2)$ dans [18, 25, 24]. Une des difficultés majeures de ce travail est la prise en compte de la courbure de la sphère dans laquelle vit $(X_t, \tilde{X}_t)_t$. Sur $SU(2)$, en notant d_{cc} la distance sous-riemannienne usuelle et pour t suffisamment grand, le couplage non co-adapté entraine l'inégalité suivante:

$$\mathbb{P}(\tau > t) \leq Ce^{-ct} d_{cc}\left((X_0, z_0), (X_0', z_0')\right) \tag{1.4}$$

où C et c sont des constantes indépendantes de la distance initiale. En considérant des mouvements browniens partant de la même fibre ($X_0 = \tilde{X}_0$), nous avons aussi généralisé ce couplage non co-adapté, *avec succès*, au cas de $SL(2, \mathbb{R})$. En particulier, pour $SL(2, \mathbb{R})$, (1.4) reste vraie pourvu que $X_0 = X_0'$. De l'inégalité (1.4), nous avons pu déduire une inégalité de gradient sur $SU(2)$, similaire à (1.2). Pour t suffisamment grand:

$$||\nabla_{\mathcal{H}} P_t f(g)||_{\mathcal{H}} \leq 2||f||_\infty Ce^{-ct} \text{ p.s..}$$

Sur $SL(2, \mathbb{R})$, nous obtenons un argument direct pour prouver que toute fonction harmonique bornée est constante sur chaque fibre.

Nous nous sommes aussi intéressés à la généralisation du couplage non co-adapté de Banerjee, Gordina et Mariano [6] et de toutes ses applications au cas général des groupes de Carnot libres de profondeur 2. La difficulté majeure est ici de devoir travailler sur une des aires balayées $(z_t^{i,j})_t$ tout en gardant un certain contrôle sur les autres. Toujours

en utilisant un couplage avec connaissance du futur comme décrit en Sous-section 1.1.1, notre stratégie consiste à travailler sur la matrice $(z_t^{i,j})_{1\leq i,j\leq n}$ ligne par ligne et à choisir $\xi_k = \tilde{\xi}_k$ uniquement pour $k \geq n$ et $k = 0$. En particulier, nous utilisons la théorie des matrices de Wishart et de leurs inverses. Nous obtenons un résultat similaire à (1.1), avec des constantes explicites, ainsi que des inégalités de gradients semblables à (1.2) et (1.3). En utilisant des relèvements sur les groupes de Carnot libres, cette stratégie offre aussi des résultats directs sur tous les groupes de Carnot homogènes de profondeur 2.

1.2.2 Couplage en un coup

Les derniers résultats présentés ici concernent un couplage sur les groupes de Carnot libres de profondeur 2 sur $\mathbb{R}^n$, que nous noterons $\mathbb{G}_n$, avec $n \geq 2$. Ce couplage n'est pas *avec succès* mais il permet une bonne estimation de la quantité $\mathbb{P}(\mathbb{B}_T \neq \tilde{\mathbb{B}}_T)$ et donc de la distance en variation totale $d_{TV}\left(\mathcal{L}(\mathbb{B}_T), \mathcal{L}(\tilde{\mathbb{B}}_T)\right)$ pour un temps préalablement fixé T:

$$d_{TV}\left(\mathcal{L}(\mathbb{B}_T), \mathcal{L}(\tilde{\mathbb{B}}_T)\right) \leq C_1(n)\frac{d_{cc}(\mathbb{B}_0, \tilde{\mathbb{B}}_0)}{\sqrt{T}} + C_2(n)\frac{d_{cc}(\mathbb{B}_0, \tilde{\mathbb{B}}_0)^2}{T}. \tag{1.5}$$

avec

$$C_2(n) := \frac{1}{\sqrt{\pi}}\left(6\sqrt{n} + \frac{4}{\sqrt{n}}\right) \quad \text{and} \quad C_1(n) := \frac{1}{\sqrt{2\pi}} + \sqrt{\frac{2(n-1)}{3}}C_2(n). \tag{1.6}$$

Notons que l'inégalité est similaire à celle obtenue avec le couplage *avec succès* mais que les constantes $C_1(n)$ et $C_2(n)$ obtenues avec cette méthode sont meilleures. Cette méthode permet à nouveau d'obtenir l'inégalité de gradient de gradient (1.2) avec $C = C_1(n)$ explicite. Ce travail est effectué en collaboration avec Marc Arnaudon, Michel Bonnefont et Delphine Féral.

Nous traitons les parties horizontales (X_t) et verticales (z_t) des processus en même temps. Nous utilisons une décomposition du mouvement brownien semblable à celle proposée pour le couplage non co-adapté de la sous-section 1.1.1 mais utilisant cette fois les polynômes de Legendre. Nous nous ramenons alors à l'étude d'un problème inverse, faisant intervenir à nouveau des matrices de Wishart, ainsi qu'à l'étude des couplages de vecteurs gaussiens.

En utilisant le même problème inverse et un changement de probabilité sur les vecteurs gaussiens, nous obtenons une inégalité de type Bismut-Elworthy-Li pour $d_g P_t f(\cdot)$. Nous pouvons en déduire des inégalités de gradient de type Poincarré-inverse: pour $T > 0$, $p \in (1, \infty]$, q son conjugué, pour tout $g, h \in \mathbb{G}_n$:

$$\|d_g P_T f(h)\| \leq \left(P_T |f|^p\right)^{1/p} M_q(T) \tag{1.7}$$

avec $M_q(T)$ une constante dépendant du temps T mais pas de g et h. Il est à noter que nous espérons pouvoir améliorer ces résultats dans le futur en ayant de meilleures estimées pour l'étude des moments des khi-deux pondérés.

Enfin, nous obtenons aussi une inégalité de type log Harnack. Pour toute fonction f positive sur $\mathbb{G}_n$, $T > 0$ et $g, \tilde{g} \in \mathbb{G}_n$:

$$P_T(\ln f)(\tilde{g}) \leq \ln(P_T f(g)) + C(n) d_{cc}(g, \tilde{g})^2 \tag{1.8}$$

où $C(n)$ est une constante explicite.

1.2.3 Couplage statique

Nous proposons également un couplage de type différent sur les groupes de Carnot. Ce couplage est statique dans le sens où l'on ne couple que les lois pour un temps fixé (et pas les trajectoires) et conduit à des estimées de Wasserstein de type L^1, voir Théorème 8.2.1. Il généralise les résultats de Bonnefont et Juillet [22].

Chapter 2

Presentation of the results

In this Chapter, we give an extensive presentation of the problem. We present the coupling method and the subRiemannian structure. We also recall the existing results on the Heisenberg group which gave the basis of this study. We then give a presentation of the results obtained in this thesis. Finally, we present the organisation for the remainder of the document.

2.1 Research context

2.1.1 Introduction to the couplings of Brownian motions

We first answer to a basic question: what is a coupling? A **coupling** of two probability measures μ and ν on a measurable space $(\Omega, \mathcal{F})$ is a probability measure π on $(\Omega^2, \mathcal{F}^2)$ such that μ is its first marginal distribution and ν its second one. In this thesis, we consider couplings of the paths of two continuous Markov processes $(X_t)_t$, $(\tilde{X}_t)_t$ starting at $x, \tilde{x}$, more precisely of subelliptic Brownian motions, on a manifold E. In this case the probability measure π lives on $\mathcal{C}([0,T[, E \times E)$ (with $T \in [0, +\infty]$), its first marginal is the law of a Brownian motion $(X_t)_t$ starting from x and its second marginal is the law of a Brownian motion $(\tilde{X})_t$ starting from $\tilde{x}$. In other words, we consider the joint law of $(X_t, \tilde{X}_t)_t$.

We first give two simple but useful examples in the case where $E = \mathbb{R}^n$ and where the processes are the usual (elliptic) n-dimensional Brownian motions. Note that these constructions can be extended to Riemannian manifolds (see [48]). This last point will also be discussed in Subsection 2.3.2.

- The reflection coupling (or mirror coupling):

  This coupling was introduced by Lindvall and Rogers in [43]. It can be described as follows. Supposing that $x \neq \tilde{x}$, we denote $e_1 := \frac{x-\tilde{x}}{\|x-\tilde{x}\|_2}$ and $(e_2, \dots, e_n)$ such that $(e_1, e_2, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$. Let $(B_t^1, B_t^2, \dots, B_t^n)_t$ be a n-dimensional Brownian motion starting from 0. The reflection coupling $(X_t, \tilde{X}_t)_t$ of two Brownian motions on $\mathbb{R}^n$ starting from $(x, \tilde{x})$ can then be constructed by taking:

$$X_t = x + B_t^1 e_1 + \sum_{i=2}^{n} B_t^i e_i \text{ and } \tilde{X}_t = \tilde{x} - B_t^1 e_1 + \sum_{i=2}^{n} B_t^i e_i. \tag{2.1}$$

By construction the two Brownian motions are symmetric with respect to the affine hyperplane $\frac{x+\tilde{x}}{2} + e_1^{\perp}$. Then the Brownian motions are equal if and only if one of them hits this hyperplane. The Euclidean distance between X_t and $\tilde{X}_t$ is given by $R_t = \|x - \tilde{x}\|_2 + 2B_t^1$ and acts like a Brownian motion. In general we consider this construction by reflection until $X_t = \tilde{X}_t$. After this time we shall take the Brownian motions to be equal.

- The synchronous coupling (or parallel transport coupling):

  Let $(e_1, \dots, e_n)$ be any orthonormal basis on $\mathbb{R}^n$ and $(B_t^1, \dots, B_t^n)$ be a n-dimensional Brownian motion starting from 0. We construct the synchronous coupling $(X_t, \tilde{X}_t)$ starting from $(x, \tilde{x})$ such that:

$$X_t = x + \sum_{i=1}^{n} B_t^i e_i \text{ and } \tilde{X}_t = \tilde{x} + \sum_{i=1}^{n} B_t^i e_i. \tag{2.2}$$

  In fact X_t and $\tilde{X}_t$ are equal, up to a translation. In particular the distance R_t is constant. Some reader could be quizzical about the "complexity" of Equation (2.2), as it simply means that $X_t - x = \tilde{X}_t - \tilde{x}$ a.s. However this construction using a basis will be useful to understand the description of the synchronous coupling in Riemannian manifolds made in Subsection 4.1.4 (Example 4.1.10).

Note that the two above couplings are **co-adapted**, i.e., the future of the processes only depends on their common past. In fact these two couplings are even Markovian. We now present a last coupling which is not co-adapted and uses Brownian bridges. Such a construction has first been introduced by Banerjee and Kendall to deal with the Kolmogorov diffusions [8]. A similar construction has also been used by Banerjee, Gordina and Mariano [6] to obtain results for the Brownian motion in the Heisenberg group. This last construction will be more detailed in Subsection 2.2.2.

- A *finite-look-ahead coupling*:

  Let $T > 0$, $(B_t^{br}, \tilde{B}_t^{br})_{t\in[0,T]}$ a coupling of Brownian bridges on the interval of time $[0, T]$, $\xi_0, \tilde{\xi}_0 \overset{\mathcal{L}}{\sim} \mathcal{N}(0, 1)$ and $x, \tilde{x} \in \mathbb{R}$. We construct:

$$X_t := x + B_t^{br} + \frac{t}{\sqrt{T}}\xi_0 \text{ and } \tilde{X}_t := \tilde{x} + \tilde{B}_t^{br} + \frac{t}{\sqrt{T}}\tilde{\xi}_0. \tag{2.3}$$

  This defines a coupling $(X_t, \tilde{X}_t)_{t\in[0,T]}$ of one dimensional Brownian motions starting from $(x, \tilde{x})$ on the time interval $[0, T]$. In particular, if we have $x = \tilde{x}$ and we choose $\xi_0 = \tilde{\xi}_0$, we get $X_T = \tilde{X}_T$. If $(B_t^{br})_{t\in[0,T]} \neq (\tilde{B}_t^{br})_{t\in[0,T]}$, this coupling is not co-adapted as we need to know the value of $(X_t)_t$ at time T to construct $(\tilde{X}_t)_{t\in[0,T]}$.

  We now explain how to construct an explicit coupling of Brownian bridges. The method applied in [6] is to use expansions of the Brownian bridges by using complete orthonormal systems of functions in the space $L^2([0, T])$. For example, using the Karhunen-Loève expansion we can write:

$$B_t^{br} = \sqrt{T}\sum_{j\geq 1} \xi_j \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi t}{T}\right) \text{ and } \tilde{B}_t^{br} = \sqrt{T}\sum_{j\geq 1} \tilde{\xi}_j \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi t}{T}\right) \tag{2.4}$$

  with $(\xi_j)_{j\geq 1}$ (resp. $(\tilde{\xi}_j)_{j\geq 1}$) a sequence of independent variables with the same distribution $\mathcal{N}(0, 1)$. Then the coupling is described if we construct a coupling $(\xi_j, \tilde{\xi}_j)_{j\geq 1}$ of the two Gaussian sequences.

2.1.2 Some usual applications

Aside from providing a better understanding of the geometry of the state space, the coupling method for Brownian motions is a great tool for many analysis results involving the harmonic functions and the heat semi-group like Harnack, Poincaré, Sobolev or Wasserstein inequalities (see [41, 56, 28, 27] for some examples). We now presents two types of coupling leading to some of these inequalities.

- For a coupling $(X_t, \tilde{X}_t)_t$, one can first define the first **coupling time** (or first meeting time) of two Brownian motions:

$$\tau := \inf\{t \geq 0 \mid X_t = \tilde{X}_t\}. \tag{2.5}$$

The first category of coupling, and the one we will study in the major part of this work, is about couplings such that the **coupling rate** $\mathbb{P}(\tau > t)$ can be estimated. This includes the **successful couplings**, i.e., the couplings such that τ is a.s. finite. In fact, one can get good estimates of the total variation between the laws of the two coupled processes by using the following Coupling inequality (or Aldous inequality, see [5], chapter VII) which is true for all Markov processes:

$$d_{TV}\left(\mathcal{L}(X_t), \mathcal{L}(\tilde{X}_t)\right) \leq \mathbb{P}(\tau > t) \text{ for all } t > 0. \tag{2.6}$$

As an example on $\mathbb{R}^n$, we invite the reader to look at the reflection coupling described above. Using the fact that the distance between the two Brownian motions is, up to a multiplicative constant, a one-dimensional Brownian motion, the first coupling time τ is the first hitting time of this one-dimensional Brownian motion whose distribution is well known. In particular, for t large enough, we obtain:

$$d_{TV}\left(\mathcal{L}(X_t), \mathcal{L}(\tilde{X}_t)\right) \leq \mathbb{P}(\tau > t) \leq \frac{1}{\sqrt{2\pi}} \frac{\|x - \tilde{x}\|_2}{\sqrt{t}}. \tag{2.7}$$

Moreover, the reflection coupling is **maximal** in the sense that the Coupling inequality (2.6) can be changed into an equality. If it has been proven that such couplings always exist in the case of càdlàg processes on Polish spaces [52], they can be very difficult to study (explicit construction, simulation, estimation of a coupling rate) as one will often need some knowledge of the future of one of the process. For a Riemannian manifold having a sort of "reflection structure" just as the Brownian motion on the plane or on the sphere (see [40, 33]), this can also be done by using a reflection coupling. In this specific case the coupling is co-adapted and even Markovian. More generally, the existence of Markovian maximal couplings of regular elliptic diffusions has been studied in [9], with the proof that this existence depends on the same rigidity properties of the Riemannian manifold as well as on strong conditions on the drift part of the the diffusion processes. In fact, Markovian or co-adapted maximal couplings are rare in Riemannian manifolds.

Another application of the successful couplings is to get gradient inequalities. In fact, if f is a $\mathcal{C}^1$ bounded function on the manifold E, then, denoting by P_t the heat semi group, we have for $x, \tilde{x} \in E$:

$$|P_t f(x) - P_t f(\tilde{x})| \leq \mathbb{E}[|f(X_t) - f(\tilde{X}_t)|] \leq 2\|f\|_\infty \mathbb{P}(\tau > t). \tag{2.8}$$

Using the estimate of the coupling rate $\mathbb{P}(\tau > t)$, in the case of the reflection coupling on $E = \mathbb{R}^n$, we directly obtain, for t large enough:

$$|\nabla P_t f(x)| \leq \frac{1}{\sqrt{2\pi}} \frac{2\|f\|_\infty}{\sqrt{t}}. \tag{2.9}$$

A last application that we will report here has been developed by Cranston on $\mathbb{R}^n$ in [28] and then for complete Riemannian manifolds with Ricci curvature admitting some lower bound in [27]. We consider the following well known result. Let $D \subset \mathbb{R}^n$ be an Euclidean domain, $x \in D$, $\delta_x = d(x, D^c)$. Let also f be a positive and harmonic function on D. Then:

$$\|\nabla f(x)\| < \frac{C}{\delta_x} f(x) \tag{2.10}$$

with C a constant that does not depend on f. For $x, \tilde{x} \in \mathbb{R}^n$ and any coupling $(X_t, \tilde{X}_t)$ of Brownian motions starting from $(x, \tilde{x})$, Cranston noticed that:

$$|f(x) - f(\tilde{x})| \leq \mathrm{osc}_D(f) \mathbb{P}\left(\tau > \tau_D(X) \wedge \tau_D(\tilde{X})\right) \tag{2.11}$$

with $\mathrm{osc}_D(f) := \sup_D(f) - \inf_D(f)$ and $\tau_D(Y)$ denoting the first exit time of the set D for a process $(Y_t)_t$. Using the reflection coupling on $\mathbb{R}^n$, Cranston obtained a good estimate for (2.11). To be more precise, for some specific cube $Q(x, \tilde{x}) \subset D$, depending on x and $\tilde{x}$, he obtained: $\mathbb{P}(\tau > \tau_Q(Y)) \leq \frac{|x-\tilde{x}|}{\delta_x}$ with $Y = (X_t)_t$ or $(\tilde{X}_t)_t$. With this inequality, he got (2.10).

- The second category of couplings, which will only be briefly discussed in Chapter 8 covers the couplings satisfying inequalities of the type :

$$d(X_t, \tilde{X}_t) \leq C(t) d(X_0, \tilde{X}_0). \tag{2.12}$$

This immediately gives an inequality for the L^∞ Wasserstein distance between the laws of the Brownian motions:

$$W_\infty(\mathcal{L}(X_t), \mathcal{L}(\tilde{X}_t)) \leq C(t) d(X_0, \tilde{X}_0). \tag{2.13}$$

This then induces an inequality of commutation of Bakry Emery type between the gradient and the semi group: $|\nabla P_t f| \leq C(t) P_t |\nabla f|$. Note that, under some conditions on the considered state space and the real p, Kuwada [41] proved that there is a duality between these semi-group inequalities and these type of control of the L^p Wassertein distance (see Section 8.1).

In $\mathbb{R}^n$, this inequality is trivially obtained for $C(t) = 1$ with the synchronous coupling. In the case of the Brownian motion in Riemannian manifold with Ricci curvature bounded below by k, (2.13) can be obtained with $C(t) = e^{-\frac{k}{2}t}$ by considering a synchronous coupling.

2.1.3 The subRiemannian structure

We now give a presentation of the subRiemannian structure of dimension N and of constant rank n. For a better understanding of these structures, the reader can see [1, 19]

Let $\mathbb{G}$ be a smooth connected Riemannian manifold of dimension N and $n \leq N$ an integer. For each $x \in \mathbb{G}$, we define a vector subspace $\mathcal{H}_x$ of dimension n of $T_x\mathbb{G}$, the tangent space in x. This way we define a subbundle of the tangent bundle $T\mathbb{G}$ denoted by $\mathcal{H}$ and called **horizontal bundle**. We can then define the **horizontal curves**, that is, the smooth curves $\gamma : I \subset \mathbb{R} \to G$ such that $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$: the horizontal curves are "moving" only with directions in $\mathcal{H}$. Defining, for all $x \in \mathbb{G}$, a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}_x}$ on $\mathcal{H}_x$, smooth in x, we obtain the length $L(\gamma)$ of the horizontal curve γ:

$$L(\gamma) := \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\mathcal{H}_{\gamma(t)}}} dt.$$

The **Carnot-Carathéodory distance** between x and $y \in \mathbb{G}$ is finally defined by:

$$d_{cc}(x, y) := \inf\{L(\gamma) \mid \gamma \text{ horizontal curve between } x \text{ and } y\}.$$

In what follows we consider subRiemannian manifolds whose horizontal bundle satisfies the **Hörmander condition**. This means that, provided that $(Y_1, ..., Y_n)$ is a local basis of vector fields in $\mathcal{H}$, the tangent bundle $T\mathbb{G}$ is spanned by the vectors $Y_1, ..., Y_n$ along with all their commutators obtained by operation with Lie brackets (the family $(Y_1, \ldots, Y_n)$ is then also called Lie bracket-generating). Then the Carnot-Carathéodory distance is finite and the subRiemannian structure is well defined.

In the case when $(\mathbb{G}, \star)$ is a stratified Lie group, we have an easy way to define a subRiemannian structure by choosing some globally defined and left-invariant basis of vector fields for $\mathcal{H}$. Indeed, if $\mathbb{G}$ is stratified, the associated Lie algebra $\mathfrak{g}$ (which we identify with the tangent space $T_0\mathbb{G}$) satisfies:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$$

with

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, [\mathfrak{g}_1, \mathfrak{g}_r] = 0 \text{ for all } 1 \leq i \leq r-1.$$

Let $X_1, \ldots, X_n$ be a linear basis of $\mathfrak{g}_1$. We define the family of linearly independent vector fields $\bar{X}_1, ..., \bar{X}_n$ on $T\mathbb{G}$ such that for all $g \in \mathbb{G}$:

$$\bar{X}_i f(g) = \frac{\partial}{\partial \epsilon}_{|\epsilon=0} f\left(g \star \exp(\epsilon X_i)\right) \text{ for } i \in \{1, \ldots n\}. \tag{2.14}$$

We obtain a basis of $\mathfrak{g}_1$ which is **left-invariant**, i.e., for all $g,\ a \in \mathbb{G}$ and for every smooth function f, $\bar{X}_i\left(f \circ trans_a\right)(g) = \bar{X}_i\left(f\right)\left(trans_a(g)\right)$ with $trans_a := g \in \mathbb{G} \mapsto a \star g \in \mathbb{G}$ the left translation. Moreover, if $X_1, \ldots, X_n$ satisfies the Hörmander condition, we obtain a well defined subRiemannian structure by taking $\mathcal{H} = \text{Span}\{\bar{X}_1, \ldots, \bar{X}_n\}$. Note that, with this last condition, $\mathcal{H}$ is called **r-step bracket generating**.

We can then introduce the **sub-Laplacian operator**:

$$L := \frac{1}{2} \sum_1^n {\bar{X}_i}^2. \tag{2.15}$$

Because Hörmander conditions is satisfied, this operator is hypoelliptic. The **subelliptic Brownian motion** on $\mathbb{G}$ is then defined as the diffusion process whose infinitesimal generator is L. Let denote by $(P_t)_t$ the **heat semi-group**, that is, the semi-group associated to L. For all $g \in \mathbb{G}$, let consider a Brownian motion $(\mathbb{B}_t)_t$ starting at g. Then for all bounded measurable function f on $\mathbb{G}$ we have:

$$P_t f(g) = \mathbb{E}[f(\mathbb{B}_t)]. \tag{2.16}$$

In this thesis we will consider some 2-steps bracket generating structures. These are:

- The free, step 2 homogeneous Carnot groups, including the Heisenberg group.
- The matrices groups $SU(2)$ and $SL(2,\mathbb{R})$ which may be seen, together with the Heisenberg group, as three model spaces of constant curvature for the left-invariant subRiemannian manifolds.

In the two following subsections we give a brief introduction to these spaces and their Brownian motions. A more complete description will be done in Chapter 3.

2.1.4 The free, step 2 homogeneous Carnot groups

For $n \geq 2$, we denote by $\mathbb{G}_n$ the free step 2 homogeneous Carnot group of rank n. Denoting by $\mathfrak{so}(n)$ the set of real skew-symmetric matrices of dimension $n \times n$, $\mathbb{G}_n$ can be defined as the Lie group $(\mathbb{R}^n \times \mathfrak{so}(n), \star)$ where:

$$(x,z) \star (x',z') = \left(x + x', z + z' + \frac{1}{2} x \odot x' \right) \text{ for all } (x,z), (x',z') \in \mathbb{R}^n \times \mathfrak{so}(n)$$

with $x \odot x' := x \cdot x'^t - x' \cdot x^t$ where $x^t \in M_{1,n}(\mathbb{R})$ denotes the transpose of x.

For all $1 \leq i \leq n$, we can use the fact that the associated Lie algebra is stratified to define $X_i := \partial_{x_i}$ and the associated left invariant vector fields $\bar{X}_1, \dots, \bar{X}_n$ as explained in Subsection 2.1.3. In particular we choose the horizontal subbundle $\mathcal{H} = \text{Span}\{\bar{X}_1, \dots, \bar{X}_n\}$ which is Lie bracket-generating. Then, the Hörmander condition is satisfied and we have a well defined structure of connected subRiemannian manifold for $\mathbb{G}_n$. The free, step 2 Carnot group $\mathbb{G}_n$ is **homogeneous**, i.e., the dilation $\text{dil}_\lambda := (x,z) \in \mathbb{G}_n \mapsto (\lambda x, \lambda^2 z) \in \mathbb{G}_n$ is a Lie group automorphism for all $\lambda > 0$. Moreover we can define a pseudo-norm $\| \cdot \|_{\mathbb{G}_n}$:

$$||(x,z)||_{\mathbb{G}_n} := \sqrt{||x||_2^2 + \|z\|_2} \text{ for all} (x,z) \in \mathbb{R}^n \times \mathfrak{so}(n) \tag{2.17}$$

where $\|z\|_2^2 := \sum_{1 \leq i < j \leq n} (z^{i,j})^2$. In fact, for all $z \in \mathfrak{so}(n)$, $\|z\|_2$ is, up to a multiplicative constant, the usual Hilbert-Schmidt norm of z.

This norm is homogeneous, that is, $\| \text{dil}_\lambda(x,z) \|_{\mathbb{G}_n} = \lambda ||(x,z)||_{\mathbb{G}_n}$ for all $\lambda > 0$. As every homogeneous norm induces a pseudo-distance on $\mathbb{G}_n$ which is equivalent to the Carnot-Carathéodory distance, there exist two constants c_1 and c_2 (depending on $\mathbb{G}_n$) such that, for any $g, g' \in \mathbb{G}_n$:

$$c_1 \|g^{-1} \star g'\|_{\mathbb{G}_n} \leq d_{cc}(g,g') \leq c_2 \|g^{-1} \star g'\|_{\mathbb{G}_n}. \tag{2.18}$$

In fact, as we will see later, some of the results presented in this thesis can be extended from the free, step 2 Carnot groups to every step 2 homogeneous Carnot groups. Indeed, for every step 2 homogeneous Lie group $\mathbb{G}$ of rank n, there exists a surjective Lie group morphism ϕ from $\mathbb{G}_n$ to $\mathbb{G}$ such that, for all $g, g' \in \mathbb{G}_n$, $d_{cc}(\phi(g), \phi(g')) \leq d_{cc}(g,g')$. Moreover, for all $a, a' \in \mathbb{G}$, there exist $g, g' \in \mathbb{G}_n$ such that:

$$\phi(g) = a, \ \phi(g') = a' \text{ and } d_{cc}(g,g') = d_{cc}(a,a'). \tag{2.19}$$

On $\mathbb{G}_n$, the subelliptic Brownian motion starting from $(x,z) \in \mathbb{G}_n$ can be written as $\mathbb{B}_t = (X_t, z_t)$ with $(X_t)_t$ a Brownian motion starting at x on $\mathbb{R}^n$ and $z_t := z + \frac{1}{2} \int_0^t X_s \odot dX_s$.

In fact, $z_t^{i,j} - z_0^{i,j}$ is the Lévy area between $(X_s^i)_{s\leq t}$ and $(X_s^j)_{s\leq t}$. That is, the quantity $\int_0^t X_s^i dX_s^j - \int_0^t X_s^j dX_s^j$. It is important to highlight, that, if we know the path of $(X_s)_{s\leq t}$, then we know the process $(\mathbb{B}_s)_{s\leq t}$. With this in mind, in what follows, we will often call $(X_t)_t$ the **leading noise**.

Remark 2.1.1. *For $n = 2$, $\mathbb{G}_2$ is the well known Heisenberg group which is isomorphic to $\mathbb{H} = (\mathbb{R}^3, \star)$ with $(x_1, x_2, z) \star (\tilde{x}_1, \tilde{x}_2, \tilde{z}) = (x_1 + \tilde{x}_1, x_2 + \tilde{x}_2, z + \tilde{z} + \frac{1}{2}(x_1\tilde{x}_2 - \tilde{x}_1 x_2))$.*

On $\mathbb{H}$, the Brownian motion can be written $\mathbb{B}_t = (X_t, z_t)$ with $(X_t)_t$ a Brownian motion on $\mathbb{R}^2$ and $z_t - z_0$ the signed area swept by $(X_s)_{s\leq t}$ with respect to the origin of the coordinate system, i.e., the Lévy area $\int_0^t X_s^1 dX_s^2 - \int_0^t X_s^2 dX_s^1$.

In what follows we will identify these two spaces and mostly use the notation $\mathbb{H}$ instead of $\mathbb{G}_2$ except to present general results on the free, step 2 Carnot groups. We will also denote by $\|\cdot\|_{\mathbb{H}}$ the homogeneous norm:

$$\|(x, z)\|_{\mathbb{H}} := \sqrt{\|x\|^2 + |z|}.$$

2.1.5 Model spaces for three-dimensional subRiemannian manifolds

We look at the three model spaces: the Heisenberg group $E_0 := \mathbb{H}$ which has be mentioned in Remark 2.1.1, the special unitary group $E_1 := SU(2)$ and the special linear group $E_{-1} := SL(2, \mathbb{R})$. For $k \in \{-1, 0, 1\}$, E_k is a stratified Lie group. In particular, we can choose X, Y and Z such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 = \text{Span}\{X, Y\}$, $\mathfrak{g}_2 = \text{Span}\{Z\}$:

$$[X, Y] = Z,\ [Y, Z] = kX,\ [Z, X] = kY. \tag{2.20}$$

In particular (X, Y) satisfies the Hörmander condition. Then, as explained in subsection 2.1.3, we can endow E_k with a natural left-invariant subRiemannian structure by taking $\mathcal{H} := \text{Span}\{\bar{X}, \bar{Y}\}$. In particular $\mathcal{H}$ is 2-step bracket generating.

Remark 2.1.2. *For every complete simply connected 3D contact subRiemannian manifold E, we have two functional invariant constants χ and κ that characterize the curvature. Then, if $\chi = 0$ and κ is constant we have:*

- *if $\kappa = 0$, E is isomorphic to E_0;*
- *if $\kappa = 1$, E is isomorphic to E_1;*
- *if $\kappa = -1$, E is isomorphic to $\widetilde{SL}(2, \mathbb{R})$ the universal cover or E_{-1}.*

This explains why E_0, E_1 and E_{-1} can be seen as the model spaces. Note that we will not develop any theory about the curvature in this thesis. The reader can be oriented to [1] for more information.

For every $k \in \{-1, 0, 1\}$, there exists a natural submersion Π_k from E_k to the simply connected Riemannian surface M_k endowed with the Riemannian metric of constant sectional curvature k , that is:

- if $k = 0$, the plane $\mathbb{R}^2$;

- if $k = 1$, the sphere S^2;
- if $k = -1$, the hyperbolic plane $\mathbf{H}^2$.

Note that this submersion is trivial on $E_0 = \mathbb{H}$ (this is the projection on the two first coordinates). As $E_1 = SU(2)$ is isomorphic to S^3, Π_1 can be obtained by using the Hopf fibration. We want to emphasize that every element of $E_1 = SU(2)$ and $E_{-1} = SL(2, \mathbb{R})$ can be written on the form

$$\exp(\varphi(\cos(\theta)X + \sin(\theta)Y)) \exp(zZ) \tag{2.21}$$

with:

- for $k = 1$: $\varphi \in [0, \pi[$, $\theta \in [0, 2\pi[$ and $z \in]-2\pi, 2\pi]$;
- for $k = -1$: $\varphi \in [0, +\infty[$, $\theta \in [0, 2\pi[$ and $z \in]-2\pi, 2\pi]$.

Note that $\varphi(\cos(\theta)X + \sin(\theta)Y)$ and zZ are unique. This provides some coordinates (φ, θ, z) called the **cylindrical coordinates**. They will be of particular importance in this manuscript. In particular we can chose Π_k such that it sends every point (φ, θ, z) of E_k to the point of M_k described by the spherical (polar coordinates for $k = -1$) (φ, θ) according to a fixed pole $N_0 \in M_k$ and a vector $e_0 \in T_{N_0}M_k$. Still using this cylindrical coordinates, it has been proven in [10, 20, 21] that there exist two constants c_1 and c_2 such that, for all elements of E_k (with $k = \pm 1$) written in the cylindrical coordinates (φ, θ, z):

$$c_1(\varphi^2 + |z|) \leq d_{cc}^2(0, (\varphi, \theta, z)) \leq c_2(\varphi^2 + |z|). \tag{2.22}$$

By using the left-invariance of the Carnot-Carathéodory distance, the inequalities (2.22) provides a convenient way to compare two elements in E_k.

As the Hörmander condition is satisfied, the sub-Laplacian operator L is hypoelliptic and we can define the Brownian motion $(\mathbb{B}_t)_t$. By expressing L in cylindrical coordinates, there exist φ_t, θ_t and z_t three real diffusion processes, such that $\mathbb{B}_t = \exp(\varphi_t(\cos(\theta_t)X + \sin(\theta_t)Y)) \exp(z_t Z)$ and:

$$\begin{cases} d\varphi_t = dB_t^1 + \frac{1}{2}\sqrt{k}\cot(\sqrt{k}\varphi_t)dt \\ d\theta_t = \frac{\sqrt{k}}{\sin(\sqrt{k}\varphi_t)} dB_t^2 \\ dz_t = \frac{\tan\left(\frac{\sqrt{k}\varphi_t}{2}\right)}{\sqrt{k}} dB_t^2 \end{cases} \tag{2.23}$$

with $(B_t^1)_t$ and $(B_t^2)_t$ two real independent Brownian motions.

In particular we have a nice geometrical interpretation of this Brownian motion obtained by Baudoin and Bonnefont in [10, 21, 20]:

- The process $(X_t := \Pi_k(\mathbb{B}_t))_t$ is a Brownian motion on M_k. Its spherical coordinates (polar coordinates for $k = -1$) are given by (φ_t, θ_t) with respect to (N_0, e_0).
- $z_t - z_0$ is the signed swept area (modulo 4π) of X_t with respect to the fixed pole N_0.

With this interpretation, as for the case of the free, step 2 Carnot groups, the Brownian motion $\mathbb{B}_t$ is all characterized by $(X_s)_{s \leq t}$.

Remark 2.1.3. *Let consider the Brownian motion $\mathbb{B}_t = (X_t, z_t)$ on the Heisenberg group $\mathbb{H}$. By identifying (φ_t, θ_t) with the polar coordinates of X_t, $(\varphi_t, \theta_t, z_t)_t$ satisfies the stochastic system (2.23) with $k = 0$ (by passing to the limit):*

$$\begin{cases} d\varphi_t = dB_t^1 + \frac{1}{2\varphi_t} dt \\ d\theta_t = \frac{1}{\varphi_t} dB_t^2 \\ dz_t = \frac{\varphi_t}{2} dB_t^2 \end{cases} \tag{2.24}$$

with $(B_t^1)_t$ and $(B_t^2)_t$ two real independent Brownian motions. In fact, we can also identify the Heisenberg group with a group of matrices. Then the cylindrical coordinates make sense and the expression of the Brownian motion can be obtained from (2.24). Note also that (2.22) is also true on the Heisenberg group with $(\varphi, \theta, z) \mapsto \sqrt{\varphi^2 + |z|}$ defining an homogeneous norm.

2.2 Couplings of Brownian motions on the Heisenberg group

As in the examples given in the previous sections, a lot of subelliptic, and more generally hypoelliptic diffusions, are written under the form $(X_t, z_t := f((X_s)_{s \leq t}))_t$, with $(X_t)_t$ an elliptic diffusion on a Riemannian manifold that we will call "the driving noise" and f a functional (see [13] for some examples). Then, a strategy for coupling such processes consists in choosing a coupling $(X_t, \tilde{X}_t)_t$ of the driving noises and study the induced process $\Big((X_t, z_t), (\tilde{X}_t, \tilde{z}_t)\Big)_t$. It gives numerous examples for couplings, successful or not [29, 16, 22, 17, 39, 38, 37, 6, 7, 25, 6, 8]. This is the strategy used in all this thesis.

As seen in Subsection 2.1.4 and Subsection 2.1.5, the Heisenberg group seems to be the simplest case (smallest dimension, curvature equal to 0) of all the structures we are studying. The coupling problem on $\mathbb{H}$ has been dealt numerous times: see [17, 16, 39, 38, 37, 6, 22, 7, 44]. The starting idea of this thesis was to extend some of these couplings to $SU(2)$ and to $SL(2, \mathbb{R})$. In particular, we have focused on two successful couplings:

- A successful co-adapted coupling proposed by Kendall in [37];
- A successful (and even efficient) non co-adapted coupling proposed by Banerjee, Gordina and Mariano in [6].

In the rest of this section we are going to give a presentation of these two couplings as it will give a better understanding of the new results we present in this work.

2.2.1 Kendall's co-adapted successful coupling

On the Heisenberg group we have a nice and direct estimate for the subRiemannian distance d_{cc} between two Brownian motions. Let $(\mathbb{B}_t)_t := (X_t, z_t)_t$ and $(\tilde{\mathbb{B}}_t)_t := (\tilde{X}_t, \tilde{z}_t)_t$ be two Brownian motions on $\mathbb{H}$ starting from (x, z) and $(\tilde{x}, \tilde{z})$ respectively. We have $\mathbb{B}_t^{-1} \star \tilde{\mathbb{B}}_t = (\tilde{X}_t - X_t, a + A_t)$ with A_t the signed swept area (starting from 0) between the

paths $(X_s)_{s\leq t}$ and $(\tilde{X}_s)_{s\leq t}$ and $a = \tilde{z} - z - \frac{1}{2}(x_1\tilde{x}_2 - x_2\tilde{x}_1)$. Then using the homogeneous norm defined in (2.17), we directly have:

$$c_1\left(R_t + \sqrt{|A_t|}\right) \leq d_{cc}\left((X_t, z_t), (\tilde{X}_t, \tilde{z}_t)\right) \leq c_2\left(R_t + \sqrt{|A_t|}\right) \tag{2.25}$$

with $R_t = \|X_t - \tilde{X}_t\|_2$. To construct his successful coupling, Kendall defines a coupling of the driving noises $(X_t, \tilde{X}_t)_t$. As explained before this induces a coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ on $\mathbb{H}$ and the distance between $\mathbb{B}_t$ and $\tilde{\mathbb{B}}_t$ can be controlled by looking simultaneously at the two processes R_t and A_t. The coupling strategy consists in switching between the two Markovian couplings introduced in Subsection 2.1.1: the synchronous coupling and the reflection coupling. We have already described the behaviour of $(R_t)_t$ during these two couplings. It is also easy to study the process $(A_t)_t$. In particular:

- During the synchronous coupling $(A_t)_t$ acts (up to a multiplicative constant) like a Brownian motion and $(R_t)_t$ stays constant.
- During the reflection coupling this is $\left(\frac{R_t}{2}\right)_t$ which acts (up to a multiplicative constant) like a Brownian motion.

By using these two couplings, we can assume for simplicity that $R_0 > 0$ and $a = 0$. The construction is the following (it is also pictured by Figure 2.1):

1. The reflection coupling is used until the process $\frac{|A_t|}{R_t^2}$ starting at 0 takes the value κ.
2. While the process $\frac{|A_t|}{R_t^2}$ starting at κ satisfies $\frac{|A_t|}{R_t^2} > \kappa - \epsilon$, the synchronous coupling is used.
   Note that, as R_t stays constant during this step, $\frac{|A_t|}{R_t^2}$ is a Brownian motion, up to a multiplicative constant, and will hit $\kappa - \epsilon$ in an almost surely finite time.
3. While the process $\frac{|A_t|}{R_t^2}$ starting at $\kappa - \epsilon$ satisfies $\frac{|A_t|}{R_t^2} < \kappa$, the reflection coupling is used.

Steps 2 and 3 are iterated until $R_t = 0$. Because of this construction, when it occurs, $A_t = 0$. As the distance stays constant during each fixed-distance coupling step, if we omit these times, R_t is moving as for a simple reflection coupling and so will hit 0 in an almost surely finite time.

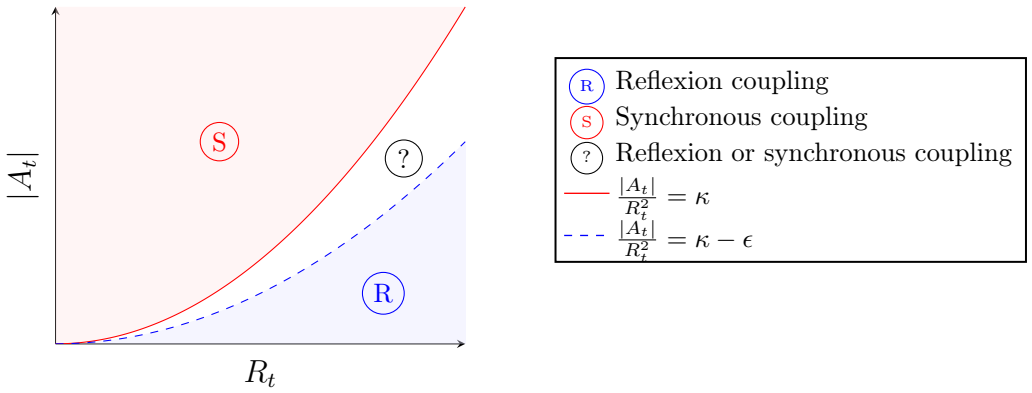

Figure 2.1: Kendall's strategy

To prove that this coupling is successful, it is necessary to show that there are not too many switches between the two coupling steps. As the different coupling steps are co-adapted, the final coupling is co-adapted too. However it is not Markovian because of the hysteresis phenomenon which occurs for $\kappa - \epsilon \leq \frac{|A_t|}{R_t^2} \leq \kappa$ (see the region labelled with the symbol ⓘ in Figure 2.1): to continue the construction after time t one will need the knowledge of all $\left(\frac{|A_s|}{R_s^2}\right)_{s\leq t}$ to decide if the strategy uses the reflection coupling or the synchronous coupling.

In [37], Kendall, also generalised this coupling to free, step 2 Carnot groups by adding rotations to the usual reflection coupling.

Remark 2.2.1. *Although the co-adapted successful coupling constructed by Kendall is interesting, it does not provide any upper bound for the coupling rate, as pointed in [6]. However, there is a lower bound estimate: for any starting points, there exists a constant c such that, for t large enough, the coupling rate satisfies:*

$$\mathbb{P}(\tau > t) \geq c \times \frac{1}{\sqrt{t}}. \tag{2.26}$$

In fact, (2.26) is satisfied by any co-adapted coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)$ of Brownian motions on $\mathbb{H}$. The main idea behind this inequality is that, unless we use a synchronous coupling in which case $\tau = +\infty$, there will exists $t_0 < \tau$ such that $X_{t_0} \neq \tilde{X}_{t_0}$. As the coupling rate after time t_0 cannot be better than the one obtained with a reflection coupling on $\mathbb{R}^2$, one get (2.26).

2.2.2 The non co-adapted successful coupling of Banerjee-Gordina-Mariano

The second strategy of coupling on $\mathbb{H}$, that we present here, is not co-adapted but successful and even efficient. It has been proposed by Banerjee, Gordina and Mariano in [6] and leads to the following theorem:

Theorem 2.2.2 (Banerjee, Gordina, Mariano [6])**.** *Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z}) \in \mathbb{H}$ and $\zeta \in \mathbb{R}$ such that $g^{-1} \star \tilde{g} = (\tilde{x} - x, \zeta)$. There exists a non co-adapted successful coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ on $\mathbb{H}$ starting at $(g, \tilde{g})$ and a constant C that does not depend on the starting points g and $\tilde{g}$ such that for all $t \geq max(2\pi||x - \tilde{x}||_2, 4\pi|\zeta|)$:*

$$\mathbb{P}(\tau > t) \leq C_1 \frac{||x - \tilde{x}||_2}{\sqrt{t}} + C_2 \frac{|\zeta|}{t}. \tag{2.27}$$

Moreover, this coupling is efficient.

Before explaining the strategy, we make some important remarks.

Remark 2.2.3. *For $|\zeta| < 1$, using the equivalence relation (2.18), we deduce from (2.27), that $\mathbb{P}(\tau > t) \leq \tilde{C} \frac{d_{cc}(g,\tilde{g})}{\sqrt{t}}$ with $\tilde{C}$ a constant that does not depend on g and $\tilde{g}$. Moreover, if $x = \tilde{x}$, then $\|g^{-1} \star \tilde{g}\|_{\mathbb{H}}^2 = |z - \tilde{z}|$ and we have $\mathbb{P}(\tau > t) \leq \tilde{C} \frac{d_{cc}(g,\tilde{g})}{t}$.*

Remark 2.2.4. *Using the results from Remarks 2.2.1 and 2.2.3, we can deduce that, if the starting points are in the same fiber, co-adapted coupling strategies cannot produce*

any efficient couplings. It is quite interesting to notice that Banerjee and Kendall also proposed in [8] a non co-adapted efficient coupling of Kolmogorov diffusions by using a strategy similar to the one used in Theorem 2.2.2. This suggests that using non co-adapted constructions should be better to obtain efficient couplings for hypoelliptic diffusions.

We now explain the strategy used by Banerjee, Gordina and Mariano to prove Theorem 2.2.2. The main point is to describe the strategy when the Brownian motions $(\mathbb{B}_t)_t$ and $(\tilde{\mathbb{B}}_t)_t$ on $\mathbb{H}$ start from the same fiber, in the sense that $X_0 = \tilde{X}_0$. Indeed, by using a reflection coupling on $\mathbb{R}^2$, one can obtain $X_t = \tilde{X}_t$ in an a.s. finite time. Assuming that $X_0 = \tilde{X}_0$, the main idea of this coupling is to divide the time in intervals of constant length and to define $\left(X_t, \tilde{X}_t\right)_t$ such that the two processes are equal at the end of each interval. Let us explain the method on the first interval $[0, T]$:

- One take $X_t^1 = \tilde{X}_t^1$.
- 0ne use the *finite-look-ahead* coupling with the Karhunen-Loève expansion to define a coupling of $(X_t^2, \tilde{X}_t^2)$, as presented in Subsection 2.1.1.

According to (2.3) and (2.4), one have:

$$X_t^2 - \tilde{X}_t^2 = \sqrt{T} \sum_{j \geq 1} \left(\xi_j - \tilde{\xi}_j\right) \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi t}{T}\right) + \frac{t}{\sqrt{T}}(\xi_0 - \tilde{\xi}_0). \tag{2.28}$$

By taking $\xi_k = \tilde{\xi}_k$ for $k = 0$ and $k \geq 2$, one obtain $X_t^2 - \tilde{X}_t^2 = \left(\xi_1 - \tilde{\xi}_1\right) \frac{\sqrt{2T}}{\pi} \sin\left(\frac{\pi t}{T}\right)$. In particular, at time $t = T$, one have $X_T = \tilde{X}_T$ and $z_T - \tilde{z}_T = z_0 - \tilde{z}_0 + \left(\frac{\xi_1 - \tilde{\xi}_1}{2}\right) K(T)$ with $K(T) := \frac{\sqrt{2T}}{\pi} \int_0^T \sin(\frac{\pi t}{T}) dX_t^1 \overset{\mathcal{L}}{\sim} \mathcal{N}\left(0, \frac{4T^2}{\pi^2}\right)$. Then the coupling is successful at time T if and only if

$$\frac{\xi_1 - \tilde{\xi}_1}{2} = \frac{(z_T - \tilde{z}_T) - (z_0 - \tilde{z}_0)}{K(T)}. \tag{2.29}$$

The coupling of the two Gaussian variables $(\xi_1, \tilde{\xi}_1)$ is then chosen such that (2.29) occurs with a probability large enough. Note that, here Banerjee Gordina and Mariano obtain this by constructing a mixed reflection/synchronous coupling of Brownian motions $(W_t, \tilde{W}_t)_t$ starting from $(0, 0)$ and by choosing $\xi_1 = W_1$ and $\tilde{\xi}_1 = \tilde{W}_1$. This particular strategy for coupling two Gaussian variables permits to compare the coupling time τ to the first hitting time of a time-changed Brownian motion (this change of time depends on $K(T)$). We note that the knowledge of the path of X^1 on all the interval $[0, T]$ is needed to construct the coupling $\left(X_t^2, \tilde{X}_t^2\right)_{t \in [0,T]}$. In particular, the obtained coupling is not co-adapted.

This coupling provides inequalities for the gradient of the heat semi group and for harmonic functions as in Subsection 2.1.2. The first result is obtained by using Inequality (2.8) together with the estimate of the coupling rate (2.27). For any bounded measurable function f on $\mathbb{H}$, there exists a positive constant C such that:

$$||\nabla_{\mathcal{H}} P_t f||_\infty \leq \frac{C}{\sqrt{t}} ||f||_\infty \ \forall t \geq 1 \tag{2.30}$$

with $|\nabla_{\mathcal{H}}(u)|(x)$ denoting the length of the horizontal gradient associated to $\mathbb{H}$ for any function $u : \mathbb{H} \to \mathbb{R}$ Lipschitz and $\|\nabla_{\mathcal{H}}(u)\|_\infty$ its sup-norm. In particular, if f is harmonic and bounded on $\mathbb{H}$ it is constant.

The second result is similar to Inequality (2.10) obtained by Cranston. Let denote by $\|\cdot\|_{\mathcal{H}}$ the norm induced by the subRiemannian metric on $\mathbb{H}$. We consider a domain D on $\mathbb{H}$, $g \in D$ and $\delta_g := d(g, D^c)$. Then there exists a constant C that do not depend on the domain D or g such that, for any $f \in \mathcal{C}(\bar{D})$ smooth, non-negative and harmonic on D:

$$\|\nabla_{\mathcal{H}} f(g)\|_{\mathcal{H}} \leq C \left(1 + \frac{1}{\delta_g} + \frac{1 + (1+\delta_g)^3}{\delta_g^4}\right) f(g). \tag{2.31}$$

This is obtained by using Inequality (2.11) and by obtaining an upper bound of

$$\mathbb{P}\left(\tau > \tau_D(\mathbb{B}) \wedge \tau_D(\tilde{\mathbb{B}})\right)$$

proportional to $d_{cc}(g, \tilde{g})$. This result is not direct and more difficult than for the Riemannian case. One of the main idea is to use the Burkholder-Davis-Gundy Inequalities. However, as the coupling is not co-adapted, the coupling time τ is not a stopping time for the natural filtration of each of the Brownian motions. Then it is necessary to use an enlarged filtration. We do not give more details as this result will be generalised in Chapter 6.

From this, we can also obtain the Cheng-Yau inequality: there exists a constant C such that, for any $g_0 \in \mathbb{H}$, $r > 0$ and for any positive harmonic function f on the open ball $B(g_0, 2r)$,

$$\sup_{g \in B(g_0, r)} \|\nabla_{\mathcal{H}} \ln f(g)\|_{\mathcal{H}} \leq \frac{C}{r}. \tag{2.32}$$

2.3 Coupling results on $SU(2)$ and $SL(2, \mathbb{R})$

As announced before, one of the main line of this thesis is to use the similarities between the Brownian motion on $\mathbb{H}$, $SU(2)$ and $SL(2, \mathbb{R})$ to extend the two couplings presented in Subsections 2.2.1 and 2.2.2 from $\mathbb{H}$ to $SU(2)$ and $SL(2, \mathbb{R})$. Indeed for these three model spaces, the Brownian motion can be written $(X_t, z_t)_t$ with $(X_t)_t$ a Brownian motion on $\mathbb{R}^2/S^2/\mathbf{H}^2$ and $(z_t)_t$ its associated swept area.

We have obtained two results on $SU(2)$: one successful co-adapted coupling and one successful non co-adapted coupling. In particular, for the non co-adapted coupling, the coupling rate is exponentially decreasing and proportional to the initial distance between the processes. For the special case when the Brownian motions start from the same fiber we have also obtained a non co-adapted successful coupling (with also exponential coupling rates) on $SL(2, \mathbb{R})$. Please note that, more recently another non co-adapted coupling strategy have been developed by Luo and Neel [44] for the subelliptic Brownian motions on $\mathbb{H}$, $SU(2)$, $SL(2, \mathbb{R})$ and also on the universal cover of $SL(2, \mathbb{R})$ and on non-isotropic Heisenberg groups of any dimension. Their strategy, based on the fact that the Riemannian manifolds M_k have good properties of symmetry, uses isometries defined conditional to the coupling time of the *area parts* of the processes. For Brownian motions starting from the same fiber, their strategy is maximal. For $SU(2)$ and $SL(2, \mathbb{R})$ they also obtain an exponential decreasing for the estimation of the coupling rate.

The results presented in this subsection have been the object of three publications:

[18] Magalie Bénéfice, Marc Arnaudon, and Michel Bonnefont. Couplings of Brownian motions on $SU(2,\mathbb{C})$. In *Geometric science of information. Part I*, volume 14071 of *Lecture Notes in Comput. Sci.*, pages 592–600. Springer, Cham, [2023] ©2023

[25] Magalie Bénéfice. Couplings of Brownian motions on $SU(2)$ and $SL(2,\mathbb{R})$. *Stochastic Processes and their Applications*, page 104434, 2024

[24] Magalie Bénéfice. Non co-adapted couplings of Brownian motions on subRiemannian manifolds. `https://arxiv.org/abs/2312.14512`, 2023.

As in subsection 2.1.5, we use the notations E_k, M_k and Π_k with $k \in \{\pm 1\}$. We also use the decomposition $\mathbb{B}_t = (X_t, z_t)$ of the Brownian motion described previously.

2.3.1 Comparison between two Brownian motions

As for the Heisenberg group we have a nice (but not immediate) estimate of the sub-Riemannian distance between two Brownian motions. Let $(\mathbb{B}_t)_t := (X_t, z_t)_t$ and $(\tilde{\mathbb{B}}_t)_t = (\tilde{X}_t, \tilde{z}_t)_t$ be two Brownian motions on E_k starting from $g = (x, z)$ and $\tilde{g} = (\tilde{x}, \tilde{z})$. Using cylindrical coordinates, we prove (see Subsection 3.2.6) that $\mathbb{B}_t^{-1}\tilde{\mathbb{B}}_t = (\rho_t, \Theta_t, \zeta_t)$ with:

- ρ_t the usual Riemannian distance on M_k between X_t and $\tilde{X}_t$;
- $\zeta_t \equiv A_t + \zeta_0 \mod (4\pi)$ where A_t is the signed swept area between the paths $(X_s)_{s\leq t}$ and $(\tilde{X}_s)_{s\leq t}$ (a more consistent definition of A_t will be given in Definition 3.2.1).

Then we obtain the following equivalence which is quite similar to (2.25):

$$c_1\left(\rho_t^2 + |\zeta_t|\right) \leq d_{cc}^2(\mathbb{B}_t, \tilde{\mathbb{B}}_t) \leq c_2\left(\rho_t^2 + |\zeta_t|\right)$$

with c_1 and c_2 two constants as in (2.22).

2.3.2 General construction of couplings

To construct general couplings of Brownian motions on E_k we use two different strategies. The first strategy will be used to define and study general co-adapted couplings whereas the second strategy will be used to construct non co-adapted couplings and only some specific co-adapted couplings.

First strategy To extend the co-adapted successful coupling result from Kendall described in Subsection 2.2.1 from $\mathbb{H}$ (the flat case) to $SU(2)$ (a curved manifold case), we need to define co-adapted couplings and the behaviour of the two processes $(\rho_t)_t$ and $(A_t)_t$ introduced above. Then it is convenient to define a coupling $(X_t, \tilde{X}_t)_t$ on M_k by using the Itô depiction of the Brownian motions $(X_t)_t$ and $(\tilde{X}_t)_t$ in some well chosen moving frames (in the sense given by Emery in [30]).

Supposing that $X_t \neq \tilde{X}_t$ and that $\tilde{X}_t$ is not in the (Riemannian) cut locus of X_t in M_k for all $0 \leq t \leq T$ (with T the stopping time associated to these hypothesis), we can define two Brownian motions $(X_t)_{t\in[0,T]}$ and $(\tilde{X}_t)_{t\in[0,T]}$ described by the equations:

$$d^\nabla X_t = dU_1(t)e_1^X(t) + dU_2(t)e_2^X(t) \text{ and } d^\nabla \tilde{X}_t = dV_1(t)e_1^{\tilde{X}}(t) + dV_2(t)e_2^{\tilde{X}}(t) \qquad (2.33)$$

where:

- $e_1^X(t) := \frac{\exp_{X_t}^{-1}(\tilde{X}_t)}{\rho_t}$, with $\exp_{X_t}^{-1}(\tilde{X}_t) = \dot{\gamma}(0)$, γ the unique geodesic such that $\gamma(0) = X_t$ and $\gamma(1) = \tilde{X}_t$;
- $e_2^X(t)$ such that $(e_1^X(t), e_2^X(t))$ is a direct orthonormal basis on $T_{X_t}S^2$;
- $(e_1^{\tilde{X}}(t), e_2^{\tilde{X}}(t))$ is the direct orthonormal basis on $T_{\tilde{X}_t}S^2$ obtained by parallel transport of $(e_1^X(t), e_2^X(t))$ along the geodesic γ joining X_t and $\tilde{X}_t$.

This defines a coupling $(X_t, \tilde{X}_t)_t$ which is characterized by its starting points and the joint law of $(U(t), V(t))_t$. By choosing a co-adapted coupling of $(U(t), V(t))_t$, this construction describes a wide range of co-adapted couplings $(X_t, \tilde{X}_t)_t$ on M_k. This also produces a co-adapted coupling on E_k. Recalling that $t < T$ (i.e. $X_t \neq \tilde{X}_t$ and $\tilde{X}_t$ is not in the cut locus of X_t), we prove that the behaviour of the processes ρ_t and A_t is then given by the following system of stochastic differential equations:

$$d\rho_t = dV_1(t) - dU_1(t) + \sqrt{k}\cot(\sqrt{k}\rho_t)dt - \frac{\sqrt{k}}{\sin(\sqrt{k}\rho_t)}dU_2(t)\cdot dV_2(t) \tag{2.34}$$

$$dA_t = \frac{\tan(\frac{\sqrt{k}\rho_t}{2})}{\sqrt{k}}(dU_2(t) + dV_2(t)) + \frac{1}{2\cos^2(\frac{\sqrt{k}\rho_t}{2})}(dU_2 \cdot dV_1(t) - dV_2(t)\cdot dU_1(t)) \tag{2.35}$$

$$d\rho_t \cdot dA_t = \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}\rho_t}{2}\right)(dV_1(t)\cdot dU_2(t) - dU_1(t)\cdot dV_2(t)) \tag{2.36}$$

with $dU_i(t)\cdot dV_j(t)$ denoting the derivative of the joint quadratic variation of U_i and V_j. These results will be stated later in Proposition 4.1.5.

Different examples of interesting choices of (U, V) will be presented in Subsection 4.1.4. In particular, to construct a successful co-adapted coupling on $SU(2)$, we shall need:

- a reflection coupling on M_k for which ρ_t is a Brownian motion (up to a multiplicative constant) with a drift having the same constant sign than k (see Example 4.1.11). This is the coupling obtained when $(U_t, V_t)_t$ is the reflection coupling on $\mathbb{R}^2$.
- a synchronous coupling on M_k for which $(R_t)_t$ is deterministic. Because of the curvature, $(R_t)_t$ is not constant (see Example 4.1.10). In particular, for $k = 1$, this $(R_t)_t$ is decreasing. This is the coupling obtained by taking $U_t = V_t$.
- a *"fixed distance coupling"* on M_k for which $(\rho_t)_t$ is constant and $(A_t)_t$ is a Brownian motion up to a multiplicative constant (see Example 4.1.13). This coupling is obtained by adding a noise to the the synchronous coupling.

Second strategy To describe the coupling $(\mathbb{B}_t = (X_t, z_t), \tilde{\mathbb{B}}_t = (\tilde{X}_t, \tilde{z}_t))_t$ on E_k, we can also directly use the cylindrical coordinates satisfying the system (2.23):

$$\begin{cases} d\varphi_t = dB_t^1 + \frac{\sqrt{k}}{2}\cot(\sqrt{k}\varphi_t)dt \\ d\theta_t = \frac{\sqrt{k}}{\sin(\sqrt{k}\varphi_t)}dB_t^2 \\ dz_t = \frac{\tan\left(\frac{\sqrt{k}\varphi_t}{2}\right)}{\sqrt{k}}dB_t^2 \end{cases} \quad \text{and} \quad \begin{cases} d\tilde{\varphi}_t = d\tilde{B}_t^1 + \frac{\sqrt{k}}{2}\cot(\sqrt{k}\tilde{\varphi}_t)dt \\ d\tilde{\theta}_t = \frac{\sqrt{k}}{\sin(\sqrt{k}\tilde{\varphi}_t)}d\tilde{B}_t^2 \\ d\tilde{z}_t = \frac{\tan\left(\frac{\sqrt{k}\tilde{\varphi}_t}{2}\right)}{\sqrt{k}}d\tilde{B}_t^2 \end{cases}.$$

It is then enough to construct a coupling $\left((\varphi_t,\theta_t),(\tilde{\varphi}_t,\tilde{\theta}_t)\right)_t$, that is, the coupling of the spherical/polar coordinates of the Brownian motions $(X_t)_t$ and $(\tilde{X}_t)_t$ on M_k relative to the reference pole N_0. As an example for $k=1$, when $(X_t)_t$ and $(\tilde{X}_t)_t$ start from a same meridian of the sphere, that is, $\tilde{\varphi}_0 = \pi - \varphi_0$, we easily obtain the reflection coupling, defining the Brownian motion by symmetry with respect to the equator. To do that we just have to take $\tilde{B}^1_t := -B^1_t$ and $\tilde{B}^2_t := B^2_t$. When the processes start from a same parallel, we also have an easy construction of the processes by symmetry with respect to a meridian. When the initial positions of X_0 and $\tilde{X}_0$ do not satisfy the two previous cases, a solution is to consider another system of spherical coordinates by choosing another couple $(N,e) \in TM_1$ of reference such that the new process $\left(\left(\varphi^{(N,e)}_t,\theta^{(N,e)}_t\right),\left(\tilde{\varphi}^{(N,e)}_t,\tilde{\theta}^{(N,e)}_t\right)\right)_t$ satisfies one of the two previous initial conditions.

This change of coordinates is convenient and can be done in general for $k \in \{-1,1\}$. However, one has to take extra care to reconstruct the full coupling $(\mathbb{B}_t,\tilde{\mathbb{B}}_t)_t$ on E_k. Indeed, it is natural to introduce the process $\left(I^{(N,e)}_t\right)_t$ starting at z_0 such that $\left(\varphi^{(N,e)}_t,\theta^{(N,e)}_t,I^{(N,e)}_t\right)_t$ satisfies the system (2.23). Then, $I^{(N,e)}_t$ is (up to a constant) the area swept by $(X_s)_{s\leq t}$ relative to the new pole N and in general it is will not be equal to z_t. In Section 3.2.7, we study the relation between $\left(\left(X_t,I^{(N,e)}_t\right),\left(\tilde{X}_t,\tilde{I}^{(N,e)}_t\right)\right)_t$ and $\left((X_t,z_t),(\tilde{X}_t,\tilde{z}_t)\right)_t$. More precisely we give an expression (modulo 4π) of the coordinate $(\zeta_t)_t$ of $\mathbb{B}_t^{-1}\tilde{\mathbb{B}}_t$ in function of $\left(X_t,I^{(N,e)}_t\right)_t$ and $\left(\tilde{X}_t,\tilde{I}^{(N,e)}_t\right)_t$.

Comparison of the two strategies If this first described strategy is well adapted to the construction of co-adapted couplings on M_k (and thus on E_k), it cannot be used to construct non co-adapted couplings. Indeed, (2.33) makes sense only if the moving frames are adapted to the same filtration which would not be the case with a non co-adapted strategy. With the second strategy, we are able to define non co-adapted couplings. As explained above it also permits the construction of co-adapted couplings, such as the reflection coupling for $k=1$. However, it is not obvious that we can define a fixed distance coupling on M_k by using spherical/polar coordinates whatever the couple (N,e) of reference.

2.3.3 Co-adapted successful coupling on $SU(2)$

We present here the construction of our first coupling result (found later in Subsection 4.2.1) :

Theorem 2.3.1. *For all $a \in \mathbb{R}$, there exists a co-adapted coupling $(X_t,\tilde{X}_t)$ of Brownian motions on S^2 starting from $(x,\tilde{x})$ such that $\tau := \inf\{t \geq 0 \mid \rho_t = 0 \text{ and } A_t = a\}$ is a.s finite. In particular, as $\zeta_t \equiv \zeta_0 + A_t \mod (4\pi)$, this induces a successful coupling on $SU(2)$.*

The strategy is similar with the one given in Subsection 2.2.1 but needs some adjustments due to the curvature. The first major difference is that we use the fixed distance coupling instead of the synchronous coupling. The second one is that if ρ_t is too close to π, it is possible to switch too fast from reflection coupling to fixed-distance coupling as the quadratic variation of the swept area will be very large. To avoid this possibility, an idea

is to introduce a constant $0 < \eta < \pi$ and the stopping time $\tau_\eta := \inf\{t \geq 0 \mid \rho_t \geq \pi - \eta\}$. If $\tau > \tau_\eta$, which does not occur a.s., we interrupt the coupling and we use the synchronous coupling to decrease ρ_t until it takes the initial value ρ_0. This strategy can then be iterated independently.

Note that, to obtain a coupling in $SU(2)$, we only need to have $A_t \equiv -\zeta_0(4\pi)$ instead of $A_t = -\zeta_0$. Thus we only need to stop reflection coupling if A_t enters intervals of the form:

$$[-\zeta_0 + 4n\pi + \kappa\rho_t^2, -\zeta_0 + 4(n+1)\pi - \kappa\rho_t^2] \ , \ n \in \mathbb{Z}.$$

For a good choice of κ, this doesn't occur for ρ_t close to π (see Figure 2.2). Then we don't need to introduce τ_η to obtain a successful coupling.

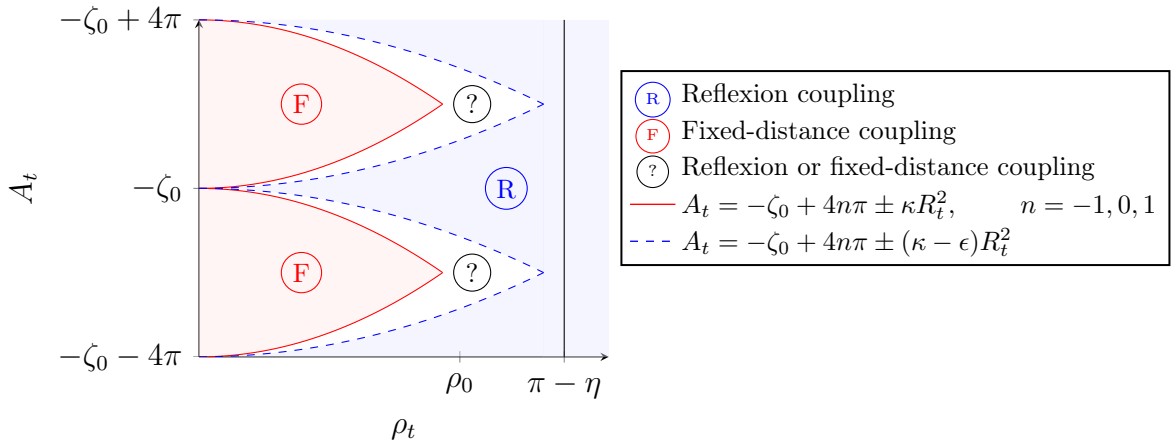

Figure 2.2: Co-adapted coupling on $SU(2)$

2.3.4 Non co-adapted successful coupling on $SU(2)$

The second result we present here is the explicit construction of a non co-adapted successful coupling on $SU(2)$ by using the *finite-look-ahead coupling*. It will be proven later in Chapter 5.

Theorem 2.3.2. *Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z}) \in SU(2)$. There exists a non co-adapted successful coupling of Brownian motions $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ on $SU(2)$ starting at $(g, \tilde{g})$. Moreover, denoting $\tau := \inf\{t \geq 0 |\ \mathbb{B}_t = \tilde{\mathbb{B}}_t\}$, there exist C and c some non negative constants that do not depend on the starting points of the process, such that, for all $t > 2$:*

$$\mathbb{P}(\tau > t) \leq \left(Ce^{-ct} d_{cc}(g, \tilde{g})\right) \wedge 1. \tag{2.37}$$

Moreover, if $x = \tilde{x}$, for all $0 < q < 1$, there exist C_q, $\tilde{c}$ some non negative constants that do not depend on the starting points g and $\tilde{g}$ such that for all $t > 1$:

$$\mathbb{P}(\tau > t) \leq \left(C_q e^{-\tilde{c}t} d_{cc}(g, \tilde{g})^{2q}\right) \wedge 1. \tag{2.38}$$

Please note that $\tilde{c}$ does not depend on the choice of q.

Remark 2.3.3. *Note that, in the announcement of this theorem we made the constraints $t > 1$ and $t > 2$. However, by changing the constants C and C_q, this can be replaced by $t > \frac{t_0}{2}$ and $t > t_0$ for any $t_0 > 0$. For t_0 small enough, the new constants $C(t_0)$ and $C_q(t_0)$ would be of order $\frac{1}{t_0}$. Note also that the coupling strategy then depends on the choice of t_0. More details on this will be found in Remark 5.2.6.*

As for the strategy used for the Heisenberg group in Subsection 2.2.2, the key point is to treat the case when two Brownian motion start from the same fiber, that is $X_0 = \tilde{X}_0$. Indeed, using first a reflection coupling on S^2, it is possible to reach this situation. Note that it is then important to control the value of the swept area at the end of this reflection coupling (this is also the case in $\mathbb{H}$).

As announced before, we construct a coupling of the spherical coordinates (φ_t, θ_t) and $\left(\tilde{\varphi}_t, \tilde{\theta}_t\right)$ satisfying the two first equations in (2.23). In particular, we notice that θ_t is a martingale, and thus can be written as a changed-time Brownian motion $(C_{\sigma(t)})_t$ with $\sigma(t)$ only depending on $(\varphi_s)_{s\leq t}$. We can do the same for $\tilde{\theta}_t = \tilde{C}_{\tilde{\sigma}(t)}$. On a deterministic interval of time $[0, T]$, we take:

1. $\varphi_t = \tilde{\varphi}_t$. In particular, $\sigma(t) = \tilde{\sigma}(t)$.

2. We make the similar decomposition as for the Heisenberg group, using Brownian bridges and the Karhunen-Loève formula, but applied on $(C_\sigma, \tilde{C}_\sigma)_\sigma$ on the interval of time $[0, \sigma(T)]$.

With this construction we obtain again $X_T = \tilde{X}_T$ and $z_T - \tilde{z}_T = z_0 - \tilde{z}_0 + \left(\frac{\xi_1 - \tilde{\xi}_1}{2}\right) K(T)$. However, contrary to the case of the Heisenberg group the random variable $K(T)$ is not Gaussian and, in fact, is quite difficult to study (we are not even sure that this variable admits a density) because of the curvature and especially because of the change of time $\sigma(t)$. The strategy then needs more adaptations. Using the fact that z_t takes its values in the compact $[-2\pi, 2\pi]$, we can obtain a positive lower bound for $\mathbb{P}(z_T = z'_T)$ that does not depend on the starting points. The iteration of the previous construction can then be compared to the iteration of identically and independently distributed experiments. Thus, we obtain a successful coupling with a coupling rate exponentially decreasing. Moreover, for any $t_0 > 0$ and $0 < q < 1$, we can obtain some constants C_q and c that do not depend on the starting points and such that:

$$\mathbb{P}(\tau > t) \leq C_q e^{-ct} |z - \tilde{z}|^q \text{ for all } t > t_0. \tag{2.39}$$

In particular, under the above condition $x = \tilde{x}$, we have:

$$c_1 \sqrt{|z - \tilde{z}|} \leq d_{cc}(g, \tilde{g}) \leq c_2 \sqrt{|z - \tilde{z}|}$$

with c_1 and c_2 two constant only depending on the structure of $SU(2)$ (see relation (2.22)). This leads to Inequality (2.38).

Remark 2.3.4. *Note that the above construction can also be done on the Heisenberg group. It corresponds to work in polar coordinates and not in Cartesian coordinates as it is done in [6]. However we cannot say if this new coupling is successful on $\mathbb{H}$. Indeed, as the swept area is not bounded in the Heisenberg group, to obtain a successful coupling it seems that we would need to iterate our construction for geometrically increasing intervals of time T. However, as we lack information on $K(T)$ for T too large, we cannot make any conclusion. Note that we recover Cartesian coordinates when the pole of the polar coordinates goes to infinity in $\mathbb{R}^2$.*

2.3.5 Successful couplings on $SL(2,\mathbb{R})$

If the Brownian motions do not start from the same fiber, there does not exist any (almost surely) successful couplings on $SL(2,\mathbb{R})$. If such a coupling had existed, there would have been an (almost surely) successful coupling $(X_t, \tilde{X}_t)_t$ on the hyperbolic plane $\mathbf{H}^2$. However, because of the negative curvature of $\mathbf{H}^2$, the reflection coupling have a repulsive effect on the Brownian motions. In fact, there do not exist any successful couplings on $\mathbf{H}^2$. As a proof, we can use Theorem 5.4 from Wang [55] (this result can be applied to every Riemannian manifold whose Ricci curvature is bounded below): as there exist some non constant bounded harmonic functions on the Riemannian manifold $\mathbf{H}^2$, there is no successful coupling of Brownian motions on it.

We thus look at the case where the Brownian motions start from the same fiber. Using exactly the same methods than for $SU(2)$, we obtain:

Theorem 2.3.5. *Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z}) \in SL(2,\mathbb{R})$. We suppose that $x = \tilde{x}$.*
There exists a non co-adapted successful coupling of Brownian motions $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ on $SL(2,\mathbb{R})$ starting at $(g, \tilde{g})$. Moreover, for all $0 < q < 1$, there exist C_q and c some non negative constants that do not depend on the starting points of the process, such that, (2.38) is still true for all $t > 1$.

This result will be later stated in Theorem 5.1.1.

Please note that, contrary to the results by Luo and Neel [44], we cannot extend the strategy above to the universal covering of $SL(2,\mathbb{R})$. Indeed, as for the Heisenberg group, for instance we are not able to know if the coupling strategy is successful when the swept area is not considered in a bounded set. In the future, we hope that we obtain a better understanding of $K(T)$ that would allow us to answer this problem. Note also that it is not possible to obtain a successful co-adapted coupling if the starting points are in the same fiber. Indeed, except by taking $\mathbb{B}_t = \tilde{\mathbb{B}}_t$ (which is clearly not a solution to our problem), any co-adapted coupling would get $(X_t)_t$ away from $(\tilde{X}_t)_t$ with a probability strictly less than 1 to make them meet again.

2.3.6 Gradient inequalities

As for the Heisenberg group, by using the results from Theorems 2.3.2 and 2.3.5 together with (2.8), we directly obtain:

Corollary 2.3.6. *There exists $C > 0$ such that, for every function $f \in \mathcal{C}(SU(2))$ and $t > 2$:*

$$||\nabla_{\mathcal{H}} P_t f(g)||_{\mathcal{H}} \leq 2||f||_\infty C e^{-ct} \text{ a.e..} \tag{2.40}$$

In particular, if f is harmonic on $SU(2)$, then it is constant.

We also get some results for $SL(2,\mathbb{R})$:

Corollary 2.3.7. *Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z}) \in SL(2,\mathbb{R})$. We suppose that $x = \tilde{x}$ and we consider a bounded measurable function f on $SL(2,\mathbb{R})$. For all $q \in]0, 1[$, there exist C_q, $\tilde{c}$ some constants that do not depend on g, $\tilde{g}$ such that for $t > 1$:*

$$|P_t f(g) - P_t f(\tilde{g})| \leq 2||f||_\infty C_q e^{-ct} d_{cc}(g, \tilde{g})^{2q}.$$

Moreover, if f is harmonic and bounded, it is constant on each fiber above x, that is, on the sets of the form $\{(x, z) \in SL(2,\mathbb{R}) \mid z \in [-2\pi, 2\pi]\}$.

These results will be proven in Section 5.4.

Note that in [4], Arnaudon and Thalmaier obtained some expressions for $\nabla_{\mathcal{H}} P_t f(g)$ on $SU(2)$ (Theorem 3.2 from [4]) but also on $SL(2,\mathbb{R})$ (Theorem 7.1 from [4]). These expressions are obtained in function of an adapted process $(\phi_t)_t$ leaving in the cotangent bundle at the point g. In particular, this leads to:

$$||\nabla_{\mathcal{H}} P_t f(g)||_{\mathcal{H}} \leq ||f||_\infty \times C(t) \text{ with } C(t) < \infty.$$

For now, it does not seem to exist any easy estimate for $(\phi_t)_t$ and thus for $C(t)$ contrary to our result in Corollary 2.3.6.

2.4 Coupling results on the free, step 2 Carnot groups

After this work on $SU(2)$ and $SL(2,\mathbb{R})$, we have looked at couplings on the free, step 2 Carnot groups where we have more than one Lévy area. The main idea is to consider a *finite-look-ahead coupling* (as in 2.1.1) and to chose more than one non trivial coupling in the coupling of the Gaussian sequences $(\xi_j, \tilde{\xi}_j)$. With this idea we have extended the coupling from [6], as well as its applications, to the general case of free, step 2 Carnot groups. In particular, we obtain explicit constants for the coupling rate. We have also obtained an explicit construction of a non co-adapted successful coupling for any homogeneous, step 2 Carnot groups.

The same idea also has permitted to develop one new strategy of coupling on the Heisenberg group and more generally on all free, step 2 Carnot groups. The goal of this last strategy is not to obtain a successful coupling but to obtain an efficient coupling for the Brownian motions at a fixed time $T > 0$ that we call here a *One sweep coupling*. It is based on a nice representation of the Lévy area obtained through a Legendre expansion of the standard Brownian motion. The obtained coupling rate $\mathbb{P}(\tau > T)$ has then the same order than the one obtained with the non co-adapted successful coupling strategy but the method is more simple. It also leads to explicit constants. We have just begun to look at the applications but we present here a result based on a change of probability that provides several gradient inequalities, including a Bismut-Elworthy-Li type representation of the semi-group.

The articles associated to these works are the following:

[26] Magalie Bénéfice. Non co-adapted successful couplings of Brownian motions on the free, step 2 carnot groups. `https://arxiv.org/abs/2407.06593`, 2024

[3] Marc Arnaudon, Magalie Bénéfice, Michel Bonnefont, and Delphine Féral. A coupling strategy for Brownian motions at fixed time on Carnot groups using Legendre expansion. `https://arxiv.org/abs/2407.04321`, 2024

We remind that we denote by $\mathbb{G}_n$ the free, step 2 Carnot group. In particular, we identify $\mathbb{G}_2$ with the Heisenberg group $\mathbb{H}$.

2.4.1 Non co-adapted successful coupling on the free, step 2 Carnot group

The first result presented here will be stated later as Theorem 6.1.1. It is the extension of Theorem 2.2.2 to $\mathbb{G}_n$:

Theorem 2.4.1. *Set $n \geq 2$. Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z})$ be two points in $\mathbb{G}_n$ and $\zeta \in \mathfrak{so}(n)$ such that $g^{-1} \star \tilde{g} = (\tilde{x} - x, \zeta)$. There exists a successful coupling of Brownian motions $\left(\mathbb{B}_t, \tilde{\mathbb{B}}_t\right)_t$ on $\mathbb{G}_n$ starting from $(g, \tilde{g})$. Moreover, for all $t \geq \beta_n \|x - \tilde{x}\|_2^2$ we have:*

$$\mathbb{P}(\tau > t) \leq \left(C_1(n) \frac{||x - \tilde{x}||_2}{\sqrt{t}} + C_2(n) \frac{||\zeta||_2}{t} \right) \tag{2.41}$$

with:

- $\beta_n = 2\pi(n-1)^2\sqrt{2(n-1)}$;
- $C_1(n) = 4\sqrt{\beta_n}(n-1) + \frac{1}{\sqrt{\pi}} = 4\sqrt{2\pi}(n-1)^2(2(n-1))^{\frac{1}{4}} + \frac{1}{\sqrt{\pi}}$;
- $C_2(n) = 2\beta_n = 4\pi(n-1)^2\sqrt{2(n-1)}$.

In particular Remark 2.2.3 is still true. Compared to the strategy in Subsection 2.2.2 that is used for $n = 2$, the main difficulty for $n \geq 3$, is that we have to deal with $\frac{n(n-1)}{2}$ Lévy areas instead of just one and that, from a quick look, there is no reason that moving one area will not impact another one. As in the previous sections we denote $\mathbb{B}_t = (X_t, z_t)$, $\tilde{\mathbb{B}}_t = (\tilde{X}_t, \tilde{z}_t)$ and $\mathbb{B}_t^{-1}\tilde{\mathbb{B}}_t = (\tilde{X}_t - X_t, \zeta_t)$.

By using the reflection coupling if needed, we can suppose that the processes start from the same fiber, that is, $X_0 = \tilde{X}_0$. The strategy is then to work on the area matrix ζ_t line by line and to use the fact that ζ_t is skew symmetric. We explain the strategy on the first line of the matrix. Let $T > 0$ and $t \in [0, T]$.

- We chose $X_t^j = \tilde{X}_t^j$ for all $2 \leq j \leq n$. In particular, this induces that for all $2 \leq i, j \leq n$, $\zeta_t^{i,j}$ stays constant;
- We couple $(X_t^1, \tilde{X}_t^1)$ by using the *finite-look-ahead coupling* given by Equations (2.4) and (2.28) with $\xi_i = \tilde{\xi}_i$ for $i = 0$ and $i \geq n$. Let denote $\xi := (\xi_i)_{1 \leq i \leq n-1}$ and $\tilde{\xi} := (\tilde{\xi}_i)_{1 \leq i \leq n-1}$ the two Gaussian vectors left to couple and v_t the $n-1$-dimensional vector induced by first line of ζ_t (by skew-symmetry, $\zeta_t^{1,1} = 0$ and can be forgotten). Then we have:
$$v_T = v_0 + A(T)\frac{\xi - \tilde{\xi}}{2} \tag{2.42}$$
with $A(T)$ a matrix whose entries are some independent Gaussian variables constructed from the paths of all the synchronous coupled coordinates $(X_t^j)_{t \in [0,T]}$, $2 \leq j \leq n$. The idea used in this work is to chose a coupling of $(\xi, \tilde{\xi})$ which is trivial in the directions induced by $(A(T)^{-1}v_0)^{\perp}$. This way, we get:
$$\frac{\xi - \tilde{\xi}}{2} = \frac{\alpha - \tilde{\alpha}}{2} \frac{A(T)^{-1}v_0}{\|A(T)^{-1}v_0\|_2} \tag{2.43}$$
with α and $\tilde{\alpha}$ two real Gaussians left to couple. In particular, with this choice, at time T, either v_T is 0 (and thus the first line and the first column of the matrix ζ_T are 0), either, it is collinear to v_0. The rest of the construction is then done according to the strategy used on $\mathbb{H}$. In particular, using the fact that the entries of the matrix $A(T)$ are some independent Gaussian variables, Inequality (2.41) is obtained from some properties of the Wishart distribution.

From this construction, we directly obtain an estimate of the total variation distance between the laws of the Brownian motion (see Corollary 6.1.11). We also obtain the estimate (2.30) of the horizontal gradient inequality for the heat semi group on $\mathbb{G}_n$ with an explicit constant (see Corollary 6.3.1). Please note that these results are also obtained with the strategy presented in the next subsection (Subsection 2.4.2) where we obtain better estimates.

As for the Heisenberg group we can obtain results for the gradient of harmonic functions similar to (2.31) and (2.32). The details of the estimates can be found in Subsection 6.3.2. The strategy is not so different from the one developed by Banerjee, Gordina and Mariano aside from the fact that we have to deal with the increase of the dimension. We also have to take extra care as the choice of the coupling $(\xi, \tilde{\xi})$ in (2.43) is constructed according to a direction depending on all the paths $(X_t^j)_{t\in[0,T]}$ with $2 \leq j \leq n$.

As a corollary of Theorem 2.4.1 we also obtain the existence of a successful coupling on all homogeneous Carnot groups. Indeed, considering any homogeneous Carnot group $\mathbb{G}$ of rank n, we have raised in subsection 2.1.4 the existence of a surjective morphism ϕ from $\mathbb{G}_n$ to $\mathbb{G}$. The point is that, if $(\mathbb{B}_t)_t$ is a Brownian motion in $\mathbb{G}_n$, $(\phi(\mathbb{B}_t))_t$ is a Brownian motion on $\mathbb{G}$. By choosing preimages that conserves the distance $d_{cc}(a, \tilde{a})$ as in Relation (2.19), we also obtain the gradient inequality (2.30) with explicit constants.

2.4.2 A new strategy of coupling on the free, step 2 Carnot groups (One Sweep coupling)

The last result of this thesis that we will present here is a new strategy of non co-adapted coupling on the free, step 2 Carnot groups. Its is based on the expansion of the Brownian motion with Legendre polynomials. This is a joint work with Marc Arnaudon, Michel Bonnefont and Delphine Féral [3]. For $T > 0$ and for two starting points $(g, \tilde{g})$, the idea is to construct a coupling of Brownian motions $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_{0\leq t\leq T}$ with an estimate of $\mathbb{P}(\mathbb{B}_T \neq \tilde{\mathbb{B}}_T)$ comparable to the one obtained for the quantity $\mathbb{P}(\tau > T)$ in Subsections 2.2.2 and 2.4.1.

Setting $n \geq 2$, we obtain the following theorem (it will be also stated later as Theorem 7.2.1):

Theorem 2.4.2. *Let $T > 0$, $(g, \tilde{g}) \in \mathbb{G}_n^2$ and $\zeta \in \mathfrak{so}(n)$ such that $g^{-1}\tilde{g} = (\tilde{x} - x, \zeta)$. There exists a coupling $(\mathbb{B}_T, \tilde{\mathbb{B}}_T)$ of Brownian motions at time T and starting at $(g, \tilde{g})$ such that:*

$$d_{TV}\left(\mathcal{L}(\mathbb{B}_T), \mathcal{L}(\tilde{\mathbb{B}}_T)\right) \leq \mathbb{P}(\mathbb{B}_T \neq \tilde{\mathbb{B}}_T) \leq C_1(n)\frac{\|\tilde{x} - x\|_2}{\sqrt{T}} + C_2(n)\frac{\sqrt{2}\|\zeta\|_2}{T} \tag{2.44}$$

with

$$C_2(n) := \frac{1}{\sqrt{\pi}}\left(6\sqrt{n} + \frac{4}{\sqrt{n}}\right) \quad \text{and} \quad C_1(n) := \frac{1}{\sqrt{2\pi}} + \sqrt{\frac{2(n-1)}{3}}C_2(n). \tag{2.45}$$

As said before, we obtain better constants than with the previous successful coupling and this coupling also induces estimates of the horizontal gradient of the heat-semi group (2.30) with these update constants.

For $g = (x, z)$ and $\tilde{g} = (\tilde{x}, \tilde{g})$, we explain briefly the strategy used to construct $(\mathbb{B}_T, \tilde{\mathbb{B}}_T) = \Big((X_T, z_T), (\tilde{X}_T, \tilde{z}_T)\Big)$. Note that, contrary to the previous non co-adapted constructions presented here, we do not need to make any assumption on x and $\tilde{x}$.

Using the expansion of the Brownian motion in Legendre polynomials as done by Kuznetsov in [42], we have, for X_T:

$$X_T = x + \sqrt{T}\xi_0, \qquad Z_T = z + \frac{1}{2}\sqrt{T}x \odot \xi_0 + T\sum_{k\geq 0}\alpha_k \xi_k \odot \xi_{k+1}, \tag{2.46}$$

$$\tilde{X}_T = \tilde{x} + \sqrt{T}\tilde{\xi}_0, \qquad \tilde{Z}_T = \tilde{z} + \frac{1}{2}\sqrt{T}\tilde{x} \odot \tilde{\xi}_0 + T\sum_{k\geq 0}\alpha_k \tilde{\xi}_k \odot \tilde{\xi}_{k+1}, \tag{2.47}$$

where $(\alpha_k)_{k\geq 0}$ are some deterministic constants that will be given in Chapter 7 and $(\xi_k)_k$ (resp. $(\xi_k)_k$) is a sequence of independent $\mathbb{R}^n$-valued Gaussian with the distribution $\mathcal{N}(0, I_n)$. Note that, here we choose an expansion in a basis built with the Legendre polynomials instead of the trigonometric polynomials used in the Karhunen Loève expansion. Indeed, when the starting points are not in the same fiber, this decomposition is more adapted to the computations of the Lévy areas as notice by Kuznetsov in [42].

Let $m \geq 1$. By choosing $\xi_k = \tilde{\xi}_k$ for any $k \notin \{3j\ , 0 \leq j \leq m\}$, we see that $\mathbb{B}_T = \tilde{\mathbb{B}}_T$ if and only if:

$$\begin{cases} \tilde{\xi}_0 - \xi_0 = \frac{x-\tilde{x}}{\sqrt{T}} \\ \sum_{k=1}^{m} T\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}\left(\tilde{\xi}_{3k} - \xi_{3k}\right) \odot V_k = W, \end{cases} \tag{2.48}$$

with:

$$V_k = \frac{\alpha_{3k}\xi_{3k+1} - \alpha_{3k-1}\xi_{3k-1}}{\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}}, \quad k = 1, \dots, m.$$

and

$$W = -\zeta + (x - \tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right).$$

If we can solve the equation $\sum_{k=1}^{m} U_k \odot V_k = W$ with unknown variables $U_1, \dots, U_m \in \mathbb{R}^n$ and if the obtained $\mathbb{E}[||U_k||_2]$ are finite for all $0 \leq k \leq m$, then we choose the coupling of $(\xi_{3k}, \tilde{\xi}_{3k})$ such that $\mathbb{P}\left(\xi_{3k} - \tilde{\xi}_{3k} \neq \frac{U_k}{T\sqrt{\alpha_{3k}^2+\alpha_{3k-1}^2}}\right)$ is minimal.

We get a positive result for $m \geq n+2$ by studying the moments of an inverted weight chi-two distribution with $k \geq m-n$ degrees of freedom. In fact, we can also get solutions for $m = n+1$ and for $m = n-1$. The case $m = n+1$ can be dealt similarly to the general case $m \geq n+2$, however the estimates are not optimal. The case $m = n-1$ is treated for the Heisenberg group in Theorem 7.1.1, that is, for $n = 2$. For $n \geq 3$ if the inverse problem for $m = n-1$ can be solved, it is much more difficult to deal with and the estimates are not so good. For $m = n$, the problem is of course invertible but the obtained quantities $\mathbb{E}[||U_k||_2]$ are not finite because we are brought back to estimate the moments of an inverted chi-two with only one degree of freedom.

For now, the better estimates for $C_1(n)$ and $C_2(n)$ are obtained for $m = 2n+1$. This could be improved by getting a better understanding of the inverted weight chi-two distribution.

Solving the same inverse problem, we can consider a coupling defined by a change of probability. In this case, the sequences $(\xi_k)_{k\geq 0}$ and $(\tilde{\xi}_k)_{k\geq 0}$ are defined such that:

- the sequence $(\xi_k)_{k\geq 0}$ is identically distributed will law $\mathcal{N}(0, I_n)$ under the usual probability $\mathbb{P}$;

- the sequence $(\tilde{\xi}_k)_{k\geq 0}$ is defined such that $(\xi_k - \tilde{\xi}_k)_{k\geq 0}$ satisfies (2.48) with $m \in \{k,\ k \geq n+2\} \cup \{+\infty\}$.

At time T, $\mathbb{B}_T^g = \mathbb{B}_T^{\tilde{g}}$ a.s.. However, the sequence $(\tilde{\xi}_k)_{k\geq 0}$ is not a standard Gaussian and $\mathbb{B}_T^{\tilde{g}}$ is not a Brownian motion under the usual probability $\mathbb{P}$. We thus define another probability $\mathbb{P}(\tilde{g})$ equivalent to $\mathbb{P}$ and such that $(\tilde{\xi}_k)_{k\geq 0}$ is independent and identically distributed with law $\mathcal{N}(0, I_n)$ under $\mathbb{P}(\tilde{g})$. In particular, under $\mathbb{P}(\tilde{g})$, $\mathbb{B}_T^{\tilde{g}}$ is distributed as a Brownian motion starting at $\tilde{g}$ at time T.

By studying $R(g) := \frac{d\mathbb{P}(\tilde{g})}{d\mathbb{P}}(\omega)$, we first obtain a log Harnack inequality (see also Proposition 7.3.7): for any positive function f in $\mathbb{G}_n$, $T > 0$ and $g, \tilde{g} \in \mathbb{G}_n$:

$$P_T(\ln f)(\tilde{g}) \leq \ln(P_T f(g)) + \frac{\|x - \tilde{x}\|_2^2}{2T} + \left(6\sqrt{n} + \frac{4}{\sqrt{n}}\right)^2 \left(\frac{2}{T^2}\|\zeta\|_2^2 + \frac{2(n-1)}{3T}\|x - \tilde{x}\|_2^2\right). \tag{2.49}$$

We also obtain a relation similar to the Bismut-Elworthy-Li formula by looking at:

$$d_g P_T f(\cdot) = \mathbb{E}[f(\mathbb{B}_T) d_g R(\cdot)].$$

In particular, we obtain an inverse Poincaré-type inequality. Let $T > 0$, $p \in (1, \infty]$ and q the conjugate number of p. For all $g, h \in \mathbb{G}_n$:

$$\|d_g P_T f(\cdot)\| \leq (P_T |f|^p)^{1/p} M_q(T) \tag{2.50}$$

with $M_q(T)$ a constant that depends of the time T but not on g. More details about these results can be found later in Proposition 7.3.8.

Note that, in a different hypoelliptic setting, this change of probability method was investigated at least by Guillin and Wang [31] and by Baudoin, Gordina and Mariano [16] to study some kinetic Fokker-Planck equation.

Another approach to obtain these gradient estimates is through the generalized curvature-dimension criterion developped by Baudoin and Garofalo [14]. Step 2 Carnot groups are examples of non-negative curved subRiemannian manifolds and thus a reverse log-Sobolev is known to hold, see *Proposition 3.1* in [11]. See also [12] where the reverse Poincaré inequality and its constant is studied on general Carnot groups by analytic methods. A general stochastic method which also provides local estimates can be found in [4] Theorem 7.1, but the constants are not explicit.

For the moment, we do not recover the full reverse log-Sobolev inequality. We still hope to obtain it with our method.

2.4.3 Static coupling and Wasserstein distance

We also obtain a slightly different result. We do not consider successful couplings but Wasserstein inequalities, (see the second category of couplings in Subsection 2.1.2). The main result, Theorem 8.2.1 is an extension of [22] to the case of homogeneous step 2 Carnot groups. The coupling is static, i.e., we only consider the coupling of the laws at time t. It leads to a L^1-Wasserstein inequality.

2.5 Organisation of the Thesis

Let us sum up the organisation of this research work. Note that, apart from Chapter 3, it follows the chronology of the research.

Chapter 3

In this chapter, we make a description of the model spaces, their basic properties and their associated Brownian motions. We study their basic properties and set the notation used in the remainder of the work.

In Section 3.1 we give some well known information about homogeneous step 2 Carnot groups and in particular about free, step 2 Carnot groups. In Section 3.2 we present the subRiemannian manifolds $SU(2)$ and $SL(2,\mathbb{R})$. By studying the cylindrical coordinates, we give three new results on $SU(2)$ and $SL(2,\mathbb{R})$:

- Proposition 3.2.9 and Corollary 3.2.10 that provides a nice estimate of the distance between two Brownian motions. It will be used later in Chapter 4
- Lemma 3.2.12, that study the behaviour of the area coordinate under changes of coordinates. It will be useful in Chapter 5.

We also present a computation of the sub-Laplacian operator using the Campbell Hausdorff formula and the Lie Bracket relations of the left invariant vector fields on $SU(2)$ and $SL(2,\mathbb{R})$.

Chapter 4

In this chapter, we study co-adapted couplings on $SU(2)$ and $SL(2,\mathbb{R})$. We use the Itô depiction of the Brownian motion in a moving frame and study the behaviour of $(R_t)_t$ and $(A_t)_t$ in Proposition 4.1.5 and give some examples of co-adapted couplings on these two spaces in Subsection 4.1.4. We then construct the announced co-adapted successful coupling on $SU(2)$ in Theorem 4.2.1 and Corollary 4.2.2.

Chapter 5

This chapter is devoted to the study of the non co-adapted coupling strategy on $SU(2)$ and $SL(2,\mathbb{R})$ using the looking-ahead coupling and the Karhunen-Loève expansion. In particular, we use the spherical/polar coordinates to define the coupling. The coupling results are stated in Theorem 5.1.1 and Theorem 5.1.2. We also prove some associated gradient inequality results: Corollary 5.4.1 and Corollary 5.4.2.

Chapter 6

This chapter deals with the non co-adapted successful coupling strategy on all the free, step 2 Carnot groups $\mathbb{G}_n$, $n \geq 2$ (Theorem 6.1.1). Direct results for the gradient of the heat semi-group on homogeneous Carnot groups of step 2 are also proven in Subsection

6.3.1. In a second part we prove the results associated to the comparison between the coupling time and the first exit time of a domain on $\mathbb{G}_n$ (Theorem 6.2.2). We deduce the gradient estimates for the harmonic functions on domains in Subsection 6.3.2.

Chapter 7

In this chapter, we describe the construction of the One-sweep coupling. We first deal on the case of the Heisenberg group $\mathbb{H}$ (Theorem 7.1.1) and then on the general case of the Carnot group G_n for $n \geq 2$ (Theorem 7.2.1). We shall make vary the number of non trivial couplings of the Gaussian modes $(\xi_j, \tilde{\xi}_j)$. In the last section of the chapter (section 7.3.2), we provide applications to gradient inequalities with a change of probability approach.

Chapter 8

In this short and independent chapter, we do not consider successful couplings but we discuss the couplings and their link with Wasserstein inequalities. In particular we construct a (static) L_1 Wasserstein coupling on all homogeneous step 2 Carnot groups; see Theorem 8.2.1.

Chapter 9

This last chapter is a conclusion for this thesis. We present some ideas to improve the results as well as some possible research tracks for the pursue of the study.

Chapter 3

Description of the model spaces and their Brownian motions

In this chapter we give some preliminaries about the studied model spaces, that is, the homogeneous step 2 Carnot groups and the Lie groups $SU(2)$ and $SL(2,\mathbb{R})$. We also give some important new results on $SU(2)$ and $SL(2,\mathbb{R})$ in Subsections 3.2.6 and 3.2.7. These two results will be fundamental to apprehend Chapter 4 and Chapter 5 respectively.

We recall to the reader that we gave some preliminaries about general subRiemannian structures in Subsection 2.1.3.

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3.1 Homogeneous step 2 Carnot groups

3.1.1 Homogeneous Carnot Groups

We first give a global definition for homogeneous Carnot groups. A global theory about these structures can be found in [19].

Definition 3.1.1. *We consider a Lie group $G = (\mathbb{R}^N, \star)$ with $\mathbb{R}^N = \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_r}$ and the dilation* $\begin{cases} \mathrm{dil}_\lambda : & \mathbb{R}^N & \to & \mathbb{R}^N \\ & \left(x^{(1)}, \ldots, x^{(r)}\right) & \mapsto & \left(\lambda x^{(1)}, \ldots, \lambda^r x^{(r)}\right) \end{cases}$ *with $\lambda > 0$. We also define $\left(\bar{X}_1, \ldots \bar{X}_{N_1}\right)$ the family of left invariant vector fields such that $\bar{X}_i(0) = \partial_{x_i}$ for $1 \leq i \leq N_1$. Then $\mathbb{G}$ is a homogeneous Carnot group with step r and N_1 generators if the two following conditions are satisfied:*

- dil_λ *is a Lie group automorphism for all $\lambda > 0$;*
- $\mathcal{H} := \mathrm{Span}\{\bar{X}_1, \ldots, \bar{X}_{N_1}\}$ *is Lie-generating, that is, $Lie\{\bar{X}_1, \ldots, \bar{X}_{N_1}\}$ is of rank N, or in other words, $(\bar{X}_1, \ldots, \bar{X}_{N_1})$ satisfies the Hörmander condition.*

In fact, one can prove that the group law $\star$ of all homogeneous Carnot groups has polynomial components.

On any homogeneous Carnot group $\mathbb{G}$, we can define an homogeneous norm:

Definition 3.1.2. *We call homogeneous norm on $\mathbb{G}$ any function $\|\cdot\| : \mathbb{G} \to [0, \infty]$ satisfying the two following properties:*

1. $\|\mathrm{dil}_\lambda(g)\| = \lambda \|g\|$ *for all $\lambda > 0$ and $g \in \mathbb{G}$;*
2. $\|g\| > 0$ *if and only if $g \neq 0$.*

Moreover, if $\|g^{-1}\| = \|g\|$ for all $g \in \mathbb{G}$, then the homogeneous norm is called symmetric.

For each homogeneous group, there exist multiple ways to define a homogeneous norm. As an example, we can consider:

$$||(x^{(1)}, \ldots, x^{(r)})|| := \left(\sum_{i=1}^{r} ||x^{(i)}||_2^{\frac{r}{i}} \right)^{\frac{1}{r}} \text{ or } ||(x^{(1)}, \ldots, x^{(r)})|| := \left(\sum_{i=1}^{r} ||x^{(i)}||_2^{\frac{2r!}{i}} \right)^{\frac{1}{2r!}}.$$

All the homogeneous norms are equivalent in the sense that, for any couple of homogeneous norms $\|\cdot\|$ and $\|\cdot\|_{bis}$ on $\mathbb{G}$, there exist c and C two positive constants such that for all $g \in \mathbb{G}$:

$$c\|g\| \leq \|g\|_{bis} \leq C\|g\|.$$

3.1.2 Properties of the homogeneous, step 2, Carnot Groups

In what follows we focus only on homogeneous, step 2, Carnot groups. In the remainder we simply denote N_1 by n, N_2 by m, $x^{(1)}$ by x and $x^{(2)}$ by z and we consider the homogeneous norm $\|\cdot\|_G$ given by:

$$||(x, z)||_G = \sqrt{||x||_2^2 + \|z\|_2} \tag{3.1}$$

Please note that, for the Hörmander condition to be satisfied, the family induced by $(X_i)_{1\leq i\leq n}$ and $([X_i,X_j])_{1\leq i<j\leq n}$ must be of rank $n+m$. In particular, we get that

$$m\leq \frac{n(n-1)}{2}.$$

By using polynomial components, we have the following characterisation of the group law :

Proposition 3.1.1. *Set $m\leq \frac{n(n-1)}{2}$ and let $\mathbb{G}=(\mathbb{R}^{n+m},\star)$ be an homogeneous Carnot group with step 2 and n generators. There exist $C^{(1)},\dots,C^{(m)}\in M_{n\times n}(\mathbb{R})$ such that, for $(x,z):=(x_1,\dots,x_n,z_1,\dots z_m)$, $(x',z'):=(x'_1,\dots,x'_n,z'_1,\dots z'_m)\in\mathbb{G}$:*

$$(x,z)\star(x',z')=\left(x+x',\left(z_k+z'_k+\frac{1}{2}\langle C^{(k)}x|x'\rangle\right)_{1\leq k\leq m}\right). \tag{3.2}$$

Moreover, the skew-symmetric parts $D^{(k)}=\frac{1}{2}\left(C^{(k)}-\left(C^{(k)}\right)^t\right)$ of $C^{(k)}$ are linearly independent.

Note that the linear independence of the matrices $D^{(k)}$ comes from the Hörmander condition.

The first example for these Carnot groups is the Heisenberg group $\mathbb{H}=(\mathbb{R}^3,\star)$ with:

$$(x_1,x_2,z)\star(\tilde{x}_1,\tilde{x}_2,\tilde{z})=(x_1+\tilde{x}_1,x_2+\tilde{x}_2,z+\tilde{z}+\frac{1}{2}(x_1\tilde{x}_2-\tilde{x}_1x_2))$$

In particular, this is an homogeneous Carnot group with step 2 and 2 generators. A generalisation to n generators is given by the free, step 2 Carnot groups:

Definition 3.1.3. *The free, step 2 Carnot group with n generators is the homogeneous Lie group $\mathbb{F}_n=\left(\mathbb{R}^{n+\frac{n(n-1)}{2}},\star\right)$ such that, for $(x,z)=\left((x_i)_{1\leq i\leq n},(z_{i,j})_{1\leq i<j\leq n}\right)$ and $(x',z')=\left((x'_i)_{1\leq i\leq n},(z'_{i,j})_{1\leq i<j\leq n}\right)$:*

$$(x,z)\star(x',z')=\left((x_i+x'_i)_{1\leq i\leq n},\left(z_{i,j}+z'_{i,j}+\frac{1}{2}(x_ix'_j-x'_ix_j)\right)_{i<j}\right).$$

Here for $1\leq i<j\leq n$, we take $C^{(i,j)}=\left(c^{(i,j)}_{l,r}\right)_{l,r}$ with $c^{(i,j)}_{l,r}=\begin{cases}-1 & \text{if } (l,r)=(i,j)\\ 1 & \text{if } (l,r)=(j,i)\\ 0 & \text{else}\end{cases}$.

We will give more details on this group in Subsection 3.1.5.

Let $n\geq 2$ and $1\leq m\leq\frac{n(n-1)}{2}$. We denote by $(e_1,\dots,e_{n+m})$ the canonical basis on $\mathbb{R}^{n+m}$ and $(\tilde{e}_1,\dots,\tilde{e}_n)$ the canonical basis on $\mathbb{R}^m$. Let $\mathbb{G}=(\mathbb{R}^{n+m},\star)$ be a homogeneous Carnot group with step 2 and n generator. Defining $C^{(1)},\dots,C^{(m)}$ as in Proposition 3.1.1, the "horizontal" left-invariant vector fields $(\bar{X}_1,\dots\bar{X}_n)$ can be explicitly computed:

$$\bar{X}_if(x,z)=\frac{d}{dt}_{|t=0}f((x,z)\star te_i) \tag{3.3}$$

$$=\partial_{x_i}f(x,z)+\frac{1}{2}\sum_{k=1}^m\sum_{j=1}^n c^{(k)}_{i,j}x_j\partial_{z_k}f(x,z) \text{ for } 1\leq i\leq n. \tag{3.4}$$

We can also look at the "vertical" left-invariant vector fields $(\bar{Z}_1, \dots \bar{Z}_m)$ such that $\bar{Z}_k(0) = \partial_{z_k}$. In fact, by commutation of the "vertical coordinate" z we get:

$$\begin{aligned}\bar{Z}_k f(x,z) &= \frac{d}{dt}_{|t=0} f((x,z) \star te_{k+n}) \\ &= \partial_{z_k} f(x,z) \text{ for } 1 \leq k \leq m.\end{aligned}$$

Note also that, for any smooth function f on $\mathbb{G}$ and $\lambda > 0$:

$$\begin{aligned}\bar{X}_i \left(f \circ \mathrm{dil}_\lambda\right)(x,z) &= \lambda \bar{X}_i(f)\left(\mathrm{dil}_\lambda\right) \text{ for } 1 \leq i \leq n \\ \bar{Z}_k \left(f \circ \mathrm{dil}_\lambda\right)(x,z) &= \lambda \bar{Z}_k(f)\left(\mathrm{dil}_\lambda\right) \text{ for } 1 \leq k \leq m.\end{aligned}$$

The sub-Laplacian operator is given by $L = \frac{1}{2}\sum_{i=1}^{n} \bar{X}_i^2$. This is a diffusion operator and, because the Hörmander condition is satisfied, this is a hypoelliptic operator. As before, it can be explicitly computed:

$$\begin{aligned}L = \frac{1}{2}\left(\sum_{i=1}^{n} \partial^2_{x_i,x_i} + \frac{1}{2}\sum_{k=1}^{m}\sum_{i,j=1}^{n} c_{i,j}^{(k)} x_j \partial^2_{x_i,z_k} + \frac{1}{4}\sum_{k,l=1}^{m}\sum_{i,j,r=1}^{n} c_{i,j}^{(k)} c_{i,r}^{(l)} x_j x_r \partial^2_{z_k,z_l}\right) \\ + \frac{1}{4}\sum_{k=1}^{m}\sum_{i=1}^{n} c_{i,i}^{(k)} \partial_{z_k}. \qquad (3.5)\end{aligned}$$

The Hörmander condition and the generalised Chow Theorem permit the definition of the Carnot-Carathéodory distance d_{cc}. Thus $(G, \mathcal{H}, d_{cc})$ is a connected metric space.

Remark 3.1.2. *As for the Heisenberg group,*

$$d_{cc}(0,(x,0)) = \|x\|_2. \qquad (3.6)$$

More generally $g = (x,z)$, $g' = (x',z') \in \mathbb{G}$, we have:

$$\|x - x'\|_2 \leq d_{cc}(g,g'). \qquad (3.7)$$

Indeed, if $t \mapsto \gamma(t) = (x(t), z(t))$ is an horizontal curve joining g and g', $t \mapsto \gamma_x(t)$ is a curve on $\mathbb{R}^n$ joining x and x'. As γ is horizontal, we can denote $\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{\alpha}_i(t)\bar{X}_i(\gamma(t))$ and $L(\gamma) = \int_0^1 \sqrt{\sum_{i=1}^{n} \dot{\alpha}_i(t)^2}$. By construction of $\bar{X}_i$, we then have $\dot{\gamma}_x(t) = \sum_{i=1}^{n} \dot{\alpha}_i(t)\partial_{x_i}(\gamma(t))$ and $L_{\mathbb{R}^n}(\gamma_x) = L(\gamma)$ (here we denote $L_{\mathbb{R}^n}$ the length of a curve on $\mathbb{R}^n$ associated to the Euclidean norm). In particular $\|x - x'\|_2 \leq L(\gamma)$ for any γ horizontal joining g and g'. This induces (3.7). To show (3.6), that is, when $g = 0$ and $g' = (x,0)$, it is now enough to verify that $\gamma(t) := (tx, 0)$ is an horizontal curve.

Note that the function $g \in \mathbb{G} \mapsto d_{cc}(0,g) \in \mathbb{R}^+$ defines a homogeneous norm. Thus, using the left-invariance of d_{cc} and the equivalence between homogeneous norms, there exist c and C two positive constant such that:

$$c\|g^{-1} \star g'\|_{\mathbb{G}} \leq d_{cc}(g,g') \leq C\|g^{-1} \star g'\|_{\mathbb{G}}. \qquad (3.8)$$

3.1.3 The Brownian motion on homogeneous, step two, Carnot groups

Let $\mathbb{G}$ be a homogeneous, step 2, Carnot group with n generators whose law is characterized by the matrices $(C^{(k)})_{1\leq k\leq m}$ as in Proposition 3.1.1. We define the Brownian motion as the diffusion process $(\mathbb{B}_t)_t$ which admits the sub-Laplacian L as infinitesimal generator. In particular it satisfies the stochastic differential equation:

$$d\mathbb{B}_t = \sum_{i=1}^{n} \bar{X}_i(\mathbb{B}_t) \circ dB_t^i \tag{3.9}$$

with $\circ d$ the Stratonovitch differential and $(B_t^1, \dots, B_t^n)$ a Brownian motion on $\mathbb{R}^n$. Using the explicit computation of L obtained in (3.5), the Brownian motion $(\mathbb{B}_t)_t$ starting from $g = (x, z)$ can be described as the continuous process $(\mathbb{B}_t)_t = \left(\left(X_t, \left(z_t^k\right)_{1\leq k\leq m}\right)\right)_t$ with:

- $(X_t)_t = (X_t^1, \dots, X_t^n)_t$ a $\mathbb{R}^n$-Brownian motion starting from x;
- For all $1 \leq k \leq m$, $dz_t^k = \frac{1}{2}\sum\limits_{i,j=1}^{n} c_{i,j}^{(k)} X_t^j dX_t^i + \frac{1}{4}\sum\limits_{i=1}^{n} c_{i,i}^{(k)} dt$. Denoting by $\mathcal{A}_t^{i,j} := \frac{1}{2}\left(\int_0^t X_s^i dX_s^j - \int_0^t X_s^j dX_s^i\right)$ the Lévy area associated to (X_t^i, X_t^j), we can also write:

$$z_t^k = z_0^k + \sum_{1\leq i<j\leq n} \frac{1}{2}(c_{j,i}^{(k)} - c_{i,j}^{(k)})\mathcal{A}_t^{i,j} + \frac{1}{4}\sum_{i,j=1}^{n} c_{i,j}^{(k)}\left(X_t^i X_t^j - X_0^i X_0^j\right)$$

In particular, the Brownian motion on $\mathbb{G}$ is a continuous process compatible with the dilation and with stationary independent increments.

Note that if the matrices $\left(C^{(k)}\right)_{1\leq k\leq m}$ are skew-symmetric, for all $1 \leq k \leq m$: $z_t^k = z_0^k + \sum\limits_{1\leq i<j\leq n} c_{j,i}^{(k)} \mathcal{A}_t^{i,j}$. In particular, on the free, step 2 Carnot group $\mathbb{F}_n$, for $1 \leq i < j \leq n$, we have $z_t^{i,j} = z_0^{i,j} + \mathcal{A}_t^{i,j}$.

3.1.4 Lie groups morphisms

In this subsection, we aim to describe some properties to simplify the study of homogeneous step 2 Carnot groups. The first one permit us to reduce the study to the cases where the matrices $C^{(k)}$ are skew-symmetric.

Proposition 3.1.3. *Let $\mathbb{G} = (\mathbb{R}^{n+m}, \star)$ be a homogeneous, step 2, Carnot group with n generator whose group law is characterized by the matrices $(C^{(k)})_{1\leq k\leq m}$ as in (3.2). Let $\tilde{\mathbb{G}} = (\mathbb{R}^{n+m}, \circ)$ denote the homogeneous, step 2, Carnot group satisfying:*

$$(x, z) \circ (x', z') = \left(x + x', \left(z_k + z_k' + \frac{1}{2}\langle D^{(k)} x | x'\rangle\right)_{1\leq k\leq m}\right) \text{ for all } (x, z),\ (x', z') \in \tilde{\mathbb{G}} \tag{3.10}$$

with $D^{(k)}$ the skew-symmetric parts of $C^{(k)}$. There exists a Lie group isomorphism $\varphi : \mathbb{G} \to \tilde{\mathbb{G}}$ such that:

- *For all $g,\ g' \in \mathbb{G}$, we have:*

$$d_{cc}(g,g') = d_{cc}(\varphi(g),\varphi(g')). \tag{3.11}$$

- *$(\mathbb{B}_t)_t$ is a Brownian motion on $\mathbb{G}$ if and only if $(\varphi(\mathbb{B}_t))_t$ is a Brownian motion on $\tilde{\mathbb{G}}$.*

Proof. We just define the map:

$$\left\{ \begin{array}{rccc} \varphi: & \mathbb{G} & \to & \tilde{\mathbb{G}} \\ & (x,z) & \mapsto & \left(x, \left(z_k - \frac{1}{4}\langle C^{(k)}x|x\rangle\right)_{1\leq k\leq m}\right) \end{array} \right. .$$

It is direct to check that this map is bijective, polynomial and respects the group structure, i.e., $\varphi((x,z)\star(x',z')) = \varphi(x,z)\circ\varphi(x',z')$.

Let $\left(\bar{X}_1,\dots,\bar{X}_n\right)$ be the left-invariant vector fields on $\mathbb{G}$ satisfying $\bar{X}_i(0) = \partial_{x_i}$ for all $1\leq i\leq n$. As φ is an isomorphism and preserves the first coordinate x, the natural left invariant family of vector fields on $\tilde{\mathbb{G}}$ computed in (3.3) can be obtained by lifting the family $\left(\bar{X}_1,\dots,\bar{X}_n\right)$. Indeed, using the bijectivity we can define on $\tilde{\mathbb{G}}$ the family of vector fields $\left(\bar{\tilde{X}}_1,\dots,\bar{\tilde{X}}_n\right)$ such that for $1\leq i\leq n$, $g\in\mathbb{G}$: $\bar{\tilde{X}}_i(\varphi(g)) = (d\varphi)_g(\bar{X}_i(g))$. These vector fields are left-invariant and, as φ preserves the coordinate x, we have $\bar{\tilde{X}}_i(0) = \partial_{x_i}$.

We now prove that φ defines an isometry. We consider an horizontal curve γ joining g and g' two points in $\mathbb{G}$ and we define $\tilde{\gamma} = \varphi\circ\gamma$ a smooth curve on $\tilde{\mathbb{G}}$ joining $\varphi(g)$ and $\varphi(g')$. Since γ is horizontal, there exist $\alpha_1,\dots,\alpha_n$ some smooth functions such that $\dot{\gamma}(t) = \sum_{i=1}^n \alpha_i(t)\bar{X}_i(\gamma(t))$ and we have:

$$\dot{\tilde{\gamma}}(t) = (d\varphi)_{\gamma(t)}\left(\sum_{i=1}^n \alpha_i(t)\bar{X}_i(\gamma(t))\right) = \sum_{i=1}^n \alpha_i(t)(d\varphi)_{\gamma(t)}\left(\bar{X}_i(\gamma(t))\right) = \sum_{i=1}^n \alpha_i(t)\bar{\tilde{X}}_i(\tilde{\gamma}(t)).$$

Thus, not only $\tilde{\gamma}$ is an horizontal curve on $\tilde{\mathbb{G}}$ joining $\varphi(a)$ and $\varphi(a')$, but the length of γ and $\tilde{\gamma}$ are equal. We obtain $d_{cc}(g,g')\geq d_{cc}(\phi(g),\phi(g'))$ for all $g,g'\in\mathbb{G}$. As φ is bijective, with the same method we get $d_{cc}(a,a')\geq d_{cc}(\phi^{-1}(a),\phi^{-1}(a'))$ for all $a,a'\in\tilde{\mathbb{G}}$. We obtain (3.11).
Finally, let $(\mathbb{B}_t)_t$ be a Brownian motion on $\mathbb{G}$. In particular, $(\mathbb{B}_t)_t$ satisfies the stochastic equation (3.9). As, the equation is given with the Stratonovitch integral and φ is smooth, we get:

$$d\varphi(\mathbb{B}_t) = \left(d\varphi\cdot\sum_{i=1}^n \bar{X}_i\right)(\varphi(\mathbb{B}_t))\circ dB_t^i = \sum_{i=1}^n d\bar{\tilde{X}}_i(\varphi(\mathbb{B}_t))\circ dB_t^i.$$

Then $\varphi(\mathbb{B}_t)$ is a Brownian motion on $\tilde{\mathbb{G}}$. By bijectivity of φ, we obtain the converse property. □

The following proposition permits to reduce the study to the case of free, step 2 Carnot groups.

Proposition 3.1.4. *Let $\mathbb{G} = (\mathbb{R}^{n+m},\circ)$ be a homogeneous Carnot group with step 2 and n generators. As previously, $\mathbb{F}_n$ denotes the free, step 2 Carnot group with n generators. There exists a surjective Lie group morphism $\phi:\mathbb{F}_n\to\mathbb{G}$ such that:*

1. *for all $g,g'\in\mathbb{F}_n$, $d_{cc}(g,g')\geq d_{cc}(\phi(g),\phi(g'))$;*

2. *for all $a, a' \in \mathbb{G}$, there exist $g, g' \in \mathbb{F}_n$ such that $\phi(g) = a$, $\phi(g') = a'$ and $d_{cc}(g, g') = d_{cc}(a, a')$;*

3. *if $(\mathbb{B}_t)_t$ is a Brownian motion on $\mathbb{F}_n$, then $(\phi(\mathbb{B}_t))_t$ is a Brownian motion on $\mathbb{G}$.*

Proof. Using the isomorphism from Proposition 3.1.3 if needed, we can suppose that the group law of $\mathbb{G}$ is characterized by the skew-symmetric matrices $\left(C^{(k)}\right)_{1\leq j\leq m}$ with $C^{(k)}$ skew-symmetric for all $1 \leq k \leq m$. We define:

$$\begin{cases} \phi : & \mathbb{F}_n & \to & \mathbb{G} \\ & \left(x, (z_{i,j})_{1\leq i<j\leq n}\right) & \mapsto & \left(x, \left(\sum\limits_{1\leq i<j\leq n} c_{j,i}^{(k)} z_{i,j}\right)_{1\leq k\leq m}\right) \end{cases}$$

It is immediate to verify that ϕ is a Lie group morphism. Moreover, as the matrices $\left(C^{(k)}\right)_{1\leq k\leq m}$ are linearly independent, ϕ is surjective.

As in the proof of Proposition 3.1.3, we define for $g \in \mathbb{F}_n$ $\bar{\tilde{X}}_i := \phi(g) \mapsto (d\phi)_g(\bar{X}_i(g))$ where $\bar{X}_1, \ldots, \bar{X}_n$ are the left invariant vector fields on $\mathbb{F}_n$ with $\bar{X}_i(0) = \partial_{x_i}$. We recall that for all $g = (x, z) \in \mathbb{F}_n$:

$$\bar{X}_i(g) = \partial_{x_i} + \frac{1}{2}\left(\sum_{1\leq l<i} x_l \partial_{z_{l,i}} - \sum_{i<l\leq n} x_l \partial_{z_{i,l}}\right).$$

As ϕ is linear, we have $d\phi = \phi$. Thus for all $g \in \mathbb{F}_n$, using the fact that the matrices $C^{(k)}$ are skew-symmetric for all $1 \leq k \leq n$, we get:

$$\begin{aligned} \bar{\tilde{X}}_i(\phi(g)) &= \partial_{x_i} + \frac{1}{2}\sum_{k=1}^{m}\left(\sum_{1\leq l<i} x_l c_{l,i}^{(k)} - \sum_{i<l\leq n} x_l c_{i,l}^{(k)}\right)\partial_{z_k} \\ &= \partial_{x_i} + \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n} x_l c_{l,i}^{(k)} \partial_{z_k} \end{aligned}$$

As ϕ preserves the coordinate x, we obtain the relation (3.3) for $\mathbb{G}$: $\bar{\tilde{X}}_i$ describe the left invariant vector field on $\mathbb{G}$ such that $\bar{\tilde{X}}_i(0) = \partial_{x_i}$. As in the proof of Proposition 3.1.3 this implies that if $(\mathbb{B}_t)_t$ is a Brownian motion on $\mathbb{F}_n$, then $(\phi(\mathbb{B}_t))_t$ is a Brownian motion on $\mathbb{G}$. We also get:

$$d_{cc}(g, g') \geq d_{cc}(\phi(g), \phi(g')). \tag{3.12}$$

Let now $a, a' \in \mathbb{G}$. By Chow's Theorem, there exists an horizontal curve $\tilde{\gamma} : [0, 1] \to G$ such that $\tilde{\gamma}(0) = a$, $\tilde{\gamma}(1) = a'$ and $L(\tilde{\gamma}) = d_{cc}(a, a')$.
By definition of an horizontal curve there exist $\alpha_1, \ldots, \alpha_n$ smooth functions such that $\dot{\tilde{\gamma}}(t) = \sum\limits_i \alpha_i(t)(\bar{X}_i)(\tilde{\gamma}(t))$.
As ϕ is surjective, we can chose $g \in \phi^{-1}(\{a\}) \neq \emptyset$. We then define the horizontal curve $\gamma : [0, 1] \to \mathbb{F}_n$ such that $\gamma(0) = g$ and $\dot{\gamma}(t) = \sum\limits_i \alpha_i(t)\bar{X}_i(\gamma(t))$. In particular, $\tilde{\gamma}(t) = \phi(\gamma(t))$ and the lengths of γ and $\tilde{\gamma}$ are equal. Thus $d_{cc}(a, a') \geq d_{cc}(\gamma(0), \gamma(1)) = d_{cc}(g, g')$ with $g' := \gamma(1)$. As $a = \phi(g)$ and $a' = \phi(g')$, using (3.12), we get $d_{cc}(g, g') = d_{cc}(a, a')$. □

3.1.5 Free, step 2 Carnot groups

In this subsection, we give more details on the free, step 2 Carnot groups $\mathbb{F}_n$ for all $n \geq 2$. In particular, as in Chapter 2, we introduce a representation of this group using skew-symmetric matrices. This is the representation we will use in all the remainder of this work.

Representation of the free, step 2 Carnot group by using skew-symmetric matrices Let first notice that $\mathbb{F}_n$ is isomorphic (Lie-group isomorphism) to the group $\mathbb{G}_n = (\mathbb{R}^n \times \mathfrak{so}(n), \star)$ with $\mathfrak{so}(n)$ the set of real skew-symmetric matrices of dimension $n \times n$. Indeed, we simply have to consider the skew-symmetric matrix $z = (z_{i,j})_{1 \leq i,j \leq n}$ induced by the $\frac{n(n-1)}{2}$ vector $(z_{i,j})_{1 \leq i < j \leq n}$. For $x, x' \in \mathbb{R}^n$, we define the symplectic form $x \odot x' := x \cdot x'^t - x' \cdot x^t$ where $x^t \in M_{1,n}(\mathbb{R})$ denotes the transpose of x. The group law then satisfies:

$$(x, z) \star (x', z') = \left(x + x', z + z' + \frac{1}{2} x \odot x' \right) \text{ for all } (x, z), (x', z') \in \mathbb{R}^n \times \mathfrak{so}(n).$$

In the remainder of this work, we will use $\mathbb{G}_n$ instead of $\mathbb{F}_n$ (except when dealing with the Heisenberg group). Note that Proposition 3.1.4 is obviously still true when considering $\mathbb{G}_n$ instead of $\mathbb{F}_n$.

Remark 3.1.5. *In fact, for $n = 3$, up to a reorganisation of the order and signs of the coefficients, we can replace the symplectic product $\odot$ used above by the traditional vector product $\wedge$ in $\mathbb{R}^3$. Indeed, let consider $g, g' \in \mathbb{F}_3 = \mathbb{R}^{3 \times 3}$, only by reorganising the "z" coordinate, we can denote $g = (x, z_{2,3}, -z_{1,3}, z_{1,2})$ and $g' = (x', z'_{2,3}, -z'_{1,3}, z'_{1,2})$. Then:*

$$g \star g' = \left(x + x', z + z' + \frac{1}{2} x \wedge x' \right).$$

Pseudo-distance For $z = (z_{i,j})_{1 \leq i < j \leq n} \in \mathfrak{so}(n)$, we define:

$$||z||_p := \left(\sum_{1 \leq i < j \leq n} |z_{i,j}|^p \right)^{\frac{1}{p}}.$$

In particular, denoting by $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm, we have $\|z\|_{HS} = \sqrt{2}\|z\|_2$ for all $z \in \mathfrak{so}(n)$. Note that, for any $x, x' \in \mathbb{R}^n$, we have $\|x \odot x'\|_2 \leq \|x\|_2 \|x'\|_2$.

We can then define the homogeneous norm $\|(x, z)\|_{\mathbb{G}_n} := \sqrt{\|x\|_2^2 + \|z\|_2}$. In particular, $\|(x, z)\|_{\mathbb{G}_n} = \|(x, (z_{i,j})_{1 \leq i < j \leq n}\|_{\mathbb{F}_n}$ with $\| \cdot \|_{\mathbb{F}_n}$ defined in Subsection 3.1.2. As we use quadratic norms, $\| \cdot \|_{\mathbb{G}_n}$ is invariant under any change of basis on $\mathbb{R}^n$ in the sense that for any $P \in \mathcal{O}(n)$, $\|(Px, PzP^t)\|_{\mathbb{G}_n} = \|(x, z)\|_{\mathbb{G}_n}$.

Let define $\begin{cases} \boldsymbol{\delta} : \mathbb{G}_n \times \mathbb{G}_n & \to & \mathbb{R} \\ (g, g') & \mapsto & \|g^{-1} g'\|_{\mathbb{G}_n} \end{cases}$. This application is a pseudo-distance on $\mathbb{G}_n$. In particular, it satisfies the pseudo triangular inequality for all (x, z), (x', z'), $(y, v) \in \mathbb{G}_n$:

$$\boldsymbol{\delta}\left((x, z), (x', z')\right) \leq \frac{3}{2} \big(\boldsymbol{\delta}\left((x, z), (y, v)\right) + \boldsymbol{\delta}\left((y, v), (x', z')\right) \big) \tag{3.13}$$

Indeed,

$$\|x-x'\|_2^2+\|z'-z-\frac{1}{2}x\odot x'\|_2\leq 2\|x-y\|_2^2+2\|y-x'\|_2^2+\|v-z-\frac{1}{2}x\odot y\|_2 \\ +\|z'-v-\frac{1}{2}y\odot x'\|_2+\frac{1}{2}\|-x\odot x'+x\odot y+y\odot x'\|_2$$

and

$$\|-x\odot x'+x\odot y+y\odot x'\|_2=\|(x-y)\odot(y-x')\|_2\leq\|x-y\|_2\|x'-y\|_2.$$

Because of the equivalence between the homogeneous norms, we also have a relation of equivalence between $\boldsymbol{\delta}$ and d_{cc} in the sense that, there exist two constants $m_1(n)$, $m_2(n)$ depending on n such that:

$$m_1(n)\boldsymbol{\delta}(g,g')\leq d_{cc}(g,g')\leq m_2(n)\boldsymbol{\delta}(g,g') \text{ for all } g,g'\in\mathbb{G}_n. \tag{3.14}$$

Let $\gamma:t\in[0,1]\mapsto(x(t),z(t))\in\mathbb{G}_n$ be an horizontal curve joining g and $\tilde{g}$. For any $1\leq i<j\leq n$, $\gamma^{i,j}(t):=\begin{pmatrix}x_i(t)\\x_j(t)\\z_{i,j}(t)\end{pmatrix}$ is an horizontal curve on the Heisenberg group $\mathbb{H}$ and $L(\gamma^{i,j})\leq L(\gamma)$. Thus, $d_{cc}\left((x_i,x_j,z_{i,j}),(\tilde{x}_i,\tilde{x}_j,\tilde{z}_{i,j})\right)\leq d_{cc}(g,\tilde{g})$. Using the equivalence relation (3.14) on $\mathbb{H}$, we get:

$$m_1(1)\sqrt{(n-1)\|\tilde{x}-x\|_2^2+\|\zeta\|_2}\leq\frac{n(n-1)}{2}d_{cc}(g,\tilde{g}) \tag{3.15}$$

where $\zeta\in\mathfrak{so}(n)$ such that $g^{-1}\tilde{g}=(\tilde{x}-x,\zeta)$. In particular, $m_1(1)\frac{2}{n(n-1)}\leq m_1(n)$. Note that, when $x=\tilde{x}$, this estimate can be more accurate by using *Proposition 13.18* from [2] and the left invariant property of the Carnot-Carathéodory distance: we have $d_{cc}(g,\tilde{g})=d_{cc}(0,(0,\zeta))^2=4\pi\sum_{i=1}^{r}i\lambda_i\geq 4\sqrt{2}\pi\|\zeta\|_2$ with $\lambda_1>\ldots>\lambda_r$ the absolute values of the non zero eigenvalues of ζ and $\zeta=\tilde{z}-z$.

With this same relation we get:

$$d_{cc}(0,(0,\zeta))^2\leq 4\pi n^{\frac{3}{2}}\sqrt{\mathrm{Tr}(\zeta^t\zeta)}\leq 4\sqrt{2}\pi n^{\frac{3}{2}}\|\zeta\|_2. \tag{3.16}$$

This provides an estimate for the constant $m_2(n)$ in (3.14):

$$d_{cc}(g,\tilde{g})\leq d_{cc}(0,(\tilde{x}-x,0))+d_{cc}(0,(0,\zeta))\leq\sqrt{2\pi}(2n)^{\frac{3}{4}}\boldsymbol{\delta}(g,\tilde{g}). \tag{3.17}$$

Brownian motion On $\mathbb{G}_n=(\mathbb{R}^n\times\mathfrak{so}(n))$, the Brownian motion $(\mathbb{B}_t)_t=\left(X_t,\left(z_t^{i,j}\right)_{i<j}\right)_t$ can be written $\mathbb{B}_t=(X_t,z_t)$ with X_t a Brownian motion on $\mathbb{R}^n$ and $z_t=z_0+\frac{1}{2}\int_0^t X_s\odot dX_s$.

In particular, we have a stability of the law of the Brownian motion under any change on the orthonormal basis on $\mathbb{R}^n$:

Lemma 3.1.6. *Let P be an orthogonal matrix, then $d(Pz_tP^t)=\frac{1}{2}(PX_t)\odot d(PX_t)$. In particular, if we define $\underline{\mathbb{B}}_t:=(\underline{X}_t,\underline{z}_t)$ with $\underline{X}_t:=PX_t$ and $\underline{z}_t=Pz_tP^t$, $(\underline{\mathbb{B}}_t)_t$ is a Brownian motion on $\mathbb{G}_n$.*

Proof. Let P be an orthogonal matrix, and $\underline{X} = PX$. We get:

$$\begin{aligned} z_t &= z_0 + \frac{1}{2}\int_0^t \left(P^t \underline{X}_s\right) \odot d\left(P^t \underline{X}_s\right) \\ &= z_0 + \frac{1}{2}\int_0^t \left(\left(P^t \underline{X}_s\right)\left(d\underline{X}_s^t P\right) - \left(P^t d\underline{X}_s\right)\left(\underline{X}_s^t P\right)\right) \\ &= P^t\left(P z_0 P^t + \frac{1}{2}\int_0^t \underline{X}_s \odot d\underline{X}_s\right) P. \end{aligned}$$

□

3.1.6 Norms of gradients

Let $\mathbb{G} = (\mathbb{R}^{n+m}, \star)$ be a homogeneous Carnot group with step 2 and rank n. For any f smooth enough, we define the horizontal gradient by

$$\nabla_{\mathcal{H}} f := \sum_{i=1}^{n} \bar{X}_i(f)\bar{X}_i.$$

Denoting by $\|\cdot\|_{\mathcal{H}}$ the Euclidean norm on $\mathcal{H}$, for any $g \in \mathbb{G}$, we have:

$$||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} := \sqrt{\sum_{i=1}^{n} (\bar{X}_i f)^2(g)}.$$

To obtain estimates of this norm, we use upper gradients. Let us first recall the definition of an upper gradient. We say that a function u on $\mathbb{G}$ is an upper gradient of f if, for every horizontal curve $\gamma : [0,T] \to \mathbb{G}$ parameterized with the arc-length, we have:

$$|f(\gamma(0)) - f(\gamma(t))| \leq \int_0^t u(\gamma(s))ds.$$

In particular, $||\nabla_{\mathcal{H}} \cdot ||_{\mathcal{H}}$ is an upper-gradient.

As $\mathbb{G}$ has a left-invariant structure, it is a regular subRiemannian manifold as described in [34], and, for all u upper gradient of f:

$$||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} \leq u(g) \text{ a.e. in } E_k. \tag{3.18}$$

See [32, 34] for some proofs and more details. In particular, for f Lipschitz, an upper gradient will be given by the gradient length associated to the metric space $(\mathbb{G}, d_{cc})$ (see [41, 32]):

$$|\nabla f|(g) := \lim_{r \downarrow 0} \sup_{\substack{g \neq g' \\ d_{cc}(g,g') < r}} \frac{|f(g) - f(g')|}{d_{cc(g,g')}}. \tag{3.19}$$

Thus, using (3.18), we have:

$$||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} \leq |\nabla f|(g) \text{ a.e. in } \mathbb{G}. \tag{3.20}$$

In the remainder of this work, when $\mathbb{G} = \mathbb{G}_n$, we will sometimes consider the gradient length associated to the pseudo-metric $\boldsymbol{\delta}$:

$$|\nabla f|_{\boldsymbol{\delta}}(g) := \lim_{r \downarrow 0} \sup_{\substack{g \neq g' \\ \boldsymbol{\delta}(g,g') < r}} \frac{|f(g) - f(g')|}{\boldsymbol{\delta}(g,g')} \text{ for any function } f \text{ Lipschitz on } (\mathbb{G}, d_{cc}).$$

By equivalence between the pseudo-metric $\boldsymbol{\delta}$ and the Carnot-Carathéodory metric d_{cc}, this notation makes sense. In particular, as $\boldsymbol{\delta}(g,\tilde{g}) \leq \frac{1}{m_1(n)} d_{cc}(g,\tilde{g})$ (see (3.14)), we have

$$|\nabla P_t f|_{\boldsymbol{\delta}}(g) \geq m_1(n)|\nabla P_t f|. \tag{3.21}$$

We can also define the vertical gradient:

$$\nabla_{\mathfrak{v}} f = \sum_{k=1}^{m} \bar{Z}_k(f)\bar{Z}_k.$$

Considering the Euclidean metric on each fiber $G_x := \{(x,z) \in \mathbb{G} \mid z \in \mathbb{R}^m\} \approx \mathbb{R}^m$, as for the horizontal gradient, we can evaluate $\|\nabla_{\mathfrak{v}} f(g)\| = \sqrt{\sum_{k=1}^{m} \bar{Z}_k(f)^2(g)}$. By using the gradient length associated to G_x endowed with the Euclidean norm, for $g=(x,z)$ and f Lipschitz, we have:

$$\|\nabla_{\mathfrak{v}} f(x,z)\| \leq \lim_{r \downarrow 0} \sup_{\substack{z \neq z' \\ \|z-z'\|_2 < r}} \frac{|f(x,z)-f(x,z')|}{\|z-z'\|_2} a.e.. \tag{3.22}$$

3.2 The Lie groups $SU(2)$ and $SL(2,\mathbb{R})$

3.2.1 Definition of $SU(2)$ and $SL(2,\mathbb{R})$ and cylindrical coordinates

We begin with the presentation of the two model spaces and the cylindrical coordinates.

- By $SU(2)$, we denote the group of the unitary two dimensional matrices with complex coefficients and with determinant 1. Considering the manifold structure induced of the usual topology on the matrices group, this is a Lie group. Note that we have:

$$\begin{aligned} SU(2) &= \left\{ \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix},\ z_1,\ z_2 \in \mathbb{C},\ |z_1|^2+|z_2|^2 = 1 \right\} \\ &= \left\{ \begin{pmatrix} \cos(\eta)e^{i\theta_1} & \sin(\eta)e^{i\theta_2} \\ -\sin(\eta)e^{-i\theta_2} & \cos(\eta)e^{-i\theta_1} \end{pmatrix},\ \eta \in \left[0,\frac{\pi}{2}\right],\ \theta_1,\ \theta_2 \in [0,2\pi] \right\}. \end{aligned}$$

  The associated Lie algebra is $\mathfrak{su}(2) = \{A \in M_{2,2}(\mathbb{C}),\ \exp(tA) \in SU(2)\ \forall t > 0\}$. It is constituted by the skew-adjoint two dimensional matrices with complex coefficients and trace 0. It is also the tangent space of $SU(2)$ at point I_2. A basis of this algebra can be formed by the Pauli matrices. We will use these Pauli matrices up to the multiplicative coefficient $\frac{1}{2}$. We denote:

$$X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \text{ and } Z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

  Then (X,Y,Z) is a basis of $\mathfrak{su}(2)$ and, because of the multiplicative coefficient, it also satisfies:

$$[X,Y] = Z\ ,\ [Y,Z] = X \text{ and } [Z,X] = Y. \tag{3.23}$$

It is important to notice that all the matrices in $SU(2)$ can be written on the form:

$$\exp(\varphi(\cos(\theta)X+\sin(\theta)Y))\exp(zZ)=\begin{pmatrix}\cos\left(\frac{\varphi}{2}\right)e^{i\frac{z}{2}} & e^{i(\theta-\frac{z}{2})}\sin\left(\frac{\varphi}{2}\right)\\ -e^{-i(\theta-\frac{z}{2})}\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right)e^{-i\frac{z}{2}}\end{pmatrix}$$

(this result is trivial taking $\varphi=2\eta$, $z=2\theta_1$ and $\theta\equiv\theta_2+\theta_1 \mod(2\pi)$). Thus, we have described a system of coordinates (φ,θ,z) for $SU(2)$ with $\varphi\in[0,\pi]$, $z\in\ ]-2\pi,2\pi]$ and $\theta\in[0,2\pi[$ called the cylindrical coordinates. We can also consider the coordinate system induced by $\exp(xX+yY)\exp(zZ)$ with $(x,y)\in\mathbb{R}^2$, $z\in]-2\pi,2\pi]$.

Let us remark that the cylindrical coordinates are a good way to observe the link between the sphere S^2 and $SU(2)$. Indeed, as there is a trivial diffeormophism between $SU(2)$ and S^3, using the Hopf fibration, we can define a submersion from $SU(2)$ to S^2. For example, we can define it by using the quaternions. For $\begin{pmatrix}z_1 & z_2\\ -\bar{z_2} & \bar{z_1}\end{pmatrix}\in SU(2)$, with $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, we define a unique quarternion $q=x_1+x_2i+y_2j+y_1k$. Denoting $N_0=k$ the north pole in S^2, we define:

$$\begin{array}{rccl}\Pi_1: SU(2) & \to & S^2 & \\ q & \mapsto & qkq^* & =2(x_1y_2+x_2y_1)i+2(y_1y_2-x_1x_2)j+(x_1^2-x_2^2-y_2^2+y_1^2)k\end{array}.$$

One can show (see [20]) that Π_1 define a submersion. Using the cylindrical coordinates (φ,θ,z) for q, we obtain:

$$\Pi_1(q)=\sin(\varphi)\sin(\theta)i-\sin(\varphi)\cos(\theta)j+\cos(\varphi)k.$$

Thus Π_1 sends every element of $SU(2)$ described by the cylindrical coordinates (φ,θ,z) onto the point of S^2 described by the spherical coordinates (φ,θ). Moreover, the fiber over (φ,θ) of this projection is described by $\{(\varphi,\theta,z),\ z\in]-2\pi,2\pi]\}$.

- We now deal with the $SL(2,\mathbb{R})$ group. It is the group of two dimensional matrices with real coefficients and with determinant 1. As for $SU(2)$ this is a Lie group with the topology of the matrices groups. The associated Lie algebra, denoted $\mathfrak{sl}(2)$, is constituted by the two dimensional matrices with real coefficients and trace 0. It is also the tangent space of $SL(2,\mathbb{R})$ at point I_2. The following matrices, using the same notation as for $SU(2)$, form a basis of $\mathfrak{sl}(2)$:

$$X=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ Y=\frac{1}{2}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\text{ and }Z=\frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

This time, the relation induced by the Lie brackets are:

$$[X,Y]=Z\ ,\ [Y,Z]=-X\ ,\ [Z,X]=-Y. \tag{3.24}$$

As before, we can prove that every element of $SL(2,\mathbb{R})$ can be written on the form:

$$\exp(\varphi(\cos(\theta)X+\sin(\theta)Y))\exp(zZ)=$$
$$\begin{pmatrix}\cosh\left(\frac{\varphi}{2}\right)\cos\left(\frac{z}{2}\right)+\sinh\left(\frac{\varphi}{2}\right)\cos\left(\theta+\frac{z}{2}\right) & -\cosh\left(\frac{\varphi}{2}\right)\sin\left(\frac{z}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\sin\left(\theta+\frac{z}{2}\right)\\ \cosh\left(\frac{\varphi}{2}\right)\sin\left(\frac{z}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\sin\left(\theta+\frac{z}{2}\right) & \cosh\left(\frac{\varphi}{2}\right)\cos\left(\frac{z}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\cos\left(\theta+\frac{z}{2}\right)\end{pmatrix}$$

with $z\in]-2\pi,2\pi]$, $\varphi>0$ and $\theta\in[0,2\pi[$ as seen in [23] for example. We can thus define as well a system of cylindrical coordinates on $SL(2,\mathbb{R})$.

Similarly to $SU(2)$, we can define a submersion Π_{-1} from $SL(2,\mathbb{R})$ to $\mathbf{H}^2$, the Poincaré upper half-plane:

$$\begin{aligned} \Pi_{-1} : SL(2) &\rightarrow \mathbf{H}^2 \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \frac{ai+b}{ci+d} \cdot \end{aligned}$$

Using the cylindrical coordinates (φ, θ, z) for A, we get:

$$\Pi_{-1}(A) = \frac{i - \sinh(\varphi)\sin(\theta)}{\cosh(\varphi) - \sinh(\varphi)\cos(\theta)}.$$

With the help of the Cartesian formula of the hyperbolic metric and trigonometric relations, we obtain that $\Pi_{-1}(\varphi, \theta, z)$ is described by the polar coordinates (φ, θ) relative to the pole $N_0 = i$. Thus the fiber over (φ, θ) of this projection is given by $\{(\varphi, \theta, z),\ z \in]-2\pi, 2\pi]\}$.

Remark 3.2.1. *In what follows we will often use the notations E_k and M_k with $k \in \{-1, 1\}$ with:*

- *$E_1 = SU(2)$ and $M_1 = S^2$;*
- *$E_{-1} = SL(2,\mathbb{R})$ and $M_{-1} = \mathbf{H}^2$.*

For the two cases we denote $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the left-invariant vector fields associated to X, Y, Z, that is, the vector fields induced by:

$$\frac{\partial}{\partial \epsilon}_{|\epsilon=0} \left(\exp\left(\varphi(\cos(\theta)X + \sin(\theta)Y)\right)\exp(zZ)\exp(\epsilon A)\right) \text{ for } A = X, Y, Z.$$

We can then provide a subRiemannian structure to $SU(2)$ (respectively $SL(2,\mathbb{R})$) by considering the horizontal bundle $\mathcal{H} = \text{Span}\{\bar{X}, \bar{Y}\}$ and the associated Carnot-Carathéodory distance d_{cc}. It has been proven by Baudoin and Bonnefont in [10, 21, 20], that there exist two constants c_1 and c_2 such that, for all elements of $SU(2)$ (resp. $SL(2,\mathbb{R})$) written in cylindrical coordinates (φ, θ, z):

$$c_1(\varphi^2 + |z|) \leq d_{cc}^2(0, (\varphi, \theta, z)) \leq c_2(\varphi^2 + |z|). \tag{3.25}$$

We now look at the sub-Laplacian operator. Here it is given by $L = \frac{1}{2}\left(\bar{X}^2 + \bar{Y}^2\right)$. Note that, as the considered vector fields are chosen left invariant, the subRiemannian structure is in fact only determined by the horizontal bundle at point I_2, $\mathcal{H}_{I_2} = \text{Span}\{X, Y\}$. In cylindrical coordinates, we have the following expression for the sub-Laplacian operator:

$$\text{for } SU(2): \qquad L = \frac{1}{2}\left(\partial_{\varphi,\varphi}^2 + \frac{1}{\sin^2(\varphi)}\partial_{\theta,\theta}^2 + \tan^2\left(\frac{\varphi}{2}\right)\partial_{z,z}^2 + \frac{1}{\cos^2\left(\frac{\varphi}{2}\right)}\partial_{\theta,z}^2 + \cot(\varphi)\partial_\varphi\right); \tag{3.26}$$

$$\text{for } SL(2,\mathbb{R}): \quad L = \frac{1}{2}\left(\partial_{\varphi,\varphi}^2 + \frac{1}{\sinh^2(\varphi)}\partial_{\theta,\theta}^2 + \tanh^2\left(\frac{\varphi}{2}\right)\partial_{z,z}^2 + \frac{1}{\cosh^2\left(\frac{\varphi}{2}\right)}\partial_{\theta,z}^2 + \coth(\varphi)\partial_\varphi\right). \tag{3.27}$$

In the literature, these results are obtained by using direct matrix computations as in [20] or (only for $SU(2)$) in [51][1] . We give a different way to prove this result in Subsections 3.2.2 and 3.2.3. Our proof only use the Lie bracket relations (3.23) and (3.24). Moreover, the proof for $SU(2)$ is easy to adapt to $SL(2,\mathbb{R})$.

[1]Here the author uses Euler angle parametrization which is quite similar from the one with cylindrical coordinates.

3.2.2 Computation of the sub-Laplacian on SU(2)

To obtain the expression (3.26) of the sub-Laplacian operator on $SU(2)$ in cylindrical coordinates, the first step is the computation of the left invariant vector fields $\bar{X}$ and $\bar{Y}$ in cylindrical coordinates. In order to do that, we need to express the following expressions in cylindrical coordinates:

$$\exp\left(\varphi(\cos(\theta)X + \sin(\theta)Y)\right)\exp(zZ)\exp(\epsilon X)$$
$$\text{and } \exp\left(\varphi(\cos(\theta)X + \sin(\theta)Y)\right)\exp(zZ)\exp(\epsilon Y).$$

We will then study the derivation of these coordinates in $\epsilon = 0$.

We recall that the triplet (X, Y, Z) is a basis of the Lie Algebra $\mathfrak{su}(2)$ satisfying (3.23). Thus it can be seen as a direct orthonormal basis for the action of rotation $[\cdot,\cdot]$. In order to simplify the computations, during this proof, we will denote this action of rotation $\cdot \wedge \cdot$ as the tensor operator. In the following lemmas we will deal with some basis (u, v, w) of the Lie algebra satisfying (3.23), that is, such that $u \wedge v = w$, $v \wedge w = u$ and $w \wedge u = v$. We will focus on expressions of the type

$$\exp\left(\alpha u + \beta v + \gamma w\right)\exp\left(\epsilon\left(\alpha' u + \beta' v + \gamma' w\right)\right)$$

for $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ real numbers and ϵ real near 0.

Lemma 3.2.2. *Let (u, v, w) be a basis of $\mathfrak{su}(2)$ satisfying (3.23). Then we get:*

$$\exp\left(\alpha u\right)\exp\left(\epsilon\beta v\right) = \exp\left(\alpha u + \epsilon\frac{\beta\alpha}{2}\left(\cot\left(\frac{\alpha}{2}\right)v + w\right)\right) + \mathcal{O}\left(\epsilon^2\right). \tag{3.28}$$

More generally we have:

$$\exp(\alpha u)\exp(\beta v) = \exp\left(\beta(\cos\left(\alpha\right)v + \sin\left(\alpha\right)w)\right)\exp\left(\alpha u\right). \tag{3.29}$$

Proof. Let us deal with (3.28). Using the Campbell Hausdorff formula we have

$$\exp(\alpha u)\exp(\epsilon\beta v) = \exp\left(\alpha u + \epsilon\beta\psi(-\mathrm{ad}_{\alpha u})(v)\right) + \mathcal{O}(\epsilon^2)$$

with $\psi(z) = \frac{z}{e^z - 1} = \sum_{n\geq 0}\frac{B_n}{n!}z^n$, B_n denoting the Bernoulli coefficients.
We have: $\mathrm{ad}_{(\alpha u)}(v) = \alpha u \wedge v = \alpha w$ and $\mathrm{ad}^{(2)}_{(\alpha u)}(v) = \alpha^2 u \wedge w = -\alpha^2 v$.
So we obtain:

$$\mathrm{ad}^{(k)}_{(\alpha u)}(v) = \begin{cases} (-1)^n\alpha^{2n}v & \text{if } k = 2n,\ n \geq 0; \\ (-1)^n\alpha^{2n+1}w & \text{if } k = 2n+1,\ n \geq 0. \end{cases}$$

Thus:

$$\begin{aligned}\psi\left(-\mathrm{ad}_{(\alpha u)}\right) &= \sum_{n\geq 0}\frac{B_{2n}}{(2n)!}(-1)^n\alpha^{2n}v - \sum_{n\geq 0}\frac{B_{2n+1}}{(2n+1)!}(-1)^n\alpha^{2n+1}w \\ &= Re\left(\psi(i\alpha)\right)v - Im\left(\psi(i\alpha)\right)w.\end{aligned}$$

To obtain the announced equality, we just need to notice that:

$$\psi(i\alpha) = \frac{i\alpha}{e^{i\alpha} - 1} = \frac{i\alpha e^{-i\frac{\alpha}{2}}}{e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}}} = \frac{\alpha e^{-i\frac{\alpha}{2}}}{2\sin(\frac{\alpha}{2})} = \frac{\alpha}{2}\left(\cot\left(\frac{\alpha}{2}\right) - i\right).$$

Note also that $\exp(A)\exp(B)\exp(-A) = \exp\left(\mathrm{Ad}\left(\exp(A)\right)(B)\right) = \exp\left(e^{\mathrm{ad}_A}(B)\right)$ for any matrices A, B with real or complex coefficients. Thus:

$$\begin{aligned}\exp(\alpha u)\exp(\beta v) &= \exp(\alpha u)\exp(\beta v)\exp(-\alpha u)\exp(\alpha u)\\ &= \exp\left(e^{\mathrm{ad}_{\alpha u}}(\beta v)\right)\exp(\alpha u).\end{aligned}$$

We finally obtain (3.29) as

$$\begin{aligned}e^{\mathrm{ad}_{\alpha u}}(\beta v) &= \sum_{k\geq 0}\frac{1}{k!}\mathrm{ad}_{\alpha u}^{(k)}(\beta v) = \beta\left(\sum_{n\geq 0}\frac{\alpha^{2n}}{(2n)!}(-1)^n v + \sum_{n\geq 0}\frac{\alpha^{2n+1}}{(2n+1)!}(-1)^n w\right)\\ &= \beta\left(\cos(\alpha)v + \sin(\alpha)w\right).\end{aligned}$$

□

Lemma 3.2.3. *Let (u, v, w) be a basis of the Lie algebra $\mathfrak{su}(2)$ satisfying (3.23) and $t \in \mathfrak{su}(2)$ such that $u \wedge t = \rho w$ with $\rho \neq 0$. Then we get:*

$$\exp(\alpha u)\exp(\epsilon t) = \exp\left(\alpha u + \epsilon\left(t + \rho\left(-v + \frac{\alpha}{2}\cot\left(\frac{\alpha}{2}\right)v + \frac{\alpha}{2}w\right)\right)\right) + \mathcal{O}(\epsilon^2). \quad (3.30)$$

Proof. We use exactly the same strategy as for the previous case using the Campbell Hausdorff formula. This time we have: $\mathrm{ad}_{(\alpha u)}(t) = \alpha u \wedge t = \alpha\rho w$; $\mathrm{ad}_{(\alpha u)}^{(2)}(t) = -\alpha^2\rho v$; $\mathrm{ad}_{(\alpha u)}^{(3)}(v) = -\alpha^3\rho w$.
Thus we obtain:

$$\mathrm{ad}_{(\alpha u)}^{(k)}(t) = \begin{cases} t & \text{if } k = 0\\ (-1)^n\alpha^{2n}\rho v & \text{if } k = 2n, n \geq 1\\ (-1)^n\alpha^{2n+1}\rho w & \text{if } k = 2n+1,\ n \geq 0\end{cases}$$

and

$$\begin{aligned}\psi\left(-\mathrm{ad}_{\alpha u}\right)(t) &= t + \rho\left(\mathrm{Re}\left(\psi(i\alpha)\right) - 1\right)v - \rho\mathrm{Im}\left(\psi(i\alpha)\right)w\\ &= t + \rho\left(\frac{\alpha}{2}\cot\left(\frac{\alpha}{2}\right) - 1\right)v + \rho\frac{\alpha}{2}w.\end{aligned}$$

□

Remark 3.2.4. *Note that the two lemmas stay true if the coefficients α and β are complex.*

Now we can find the expressions of the invariant vectors on $SU(2)$ in cylindrical coordinates. We take (φ, θ, z) some cylindrical coordinates of an element of $SU(2)$. We will denote $x = \varphi\cos(\theta)$ and $y = \varphi\sin(\theta)$. We look for the cylindrical coordinates of $\exp(xX + yY)\exp(zZ)\exp(\epsilon(aX + bY))$, with a,$b \in \mathbb{R}$ and ϵ small.

- First, we use Lemma 3.2.2 with $u = Z$, $v = \frac{aX+bY}{r}$, $\alpha = z$, $\beta = r$ and $r = \sqrt{a^2+b^2}$. Note that $w = \frac{-bX+aY}{r}$. We get:

  $$\begin{aligned}\exp(zZ)\exp(\epsilon(aX + bY)) &= \exp(\epsilon r(\cos(z)v + \sin(z)w))\exp(zZ)\\ &= \exp(\epsilon(\cos(z)(aX + bY) + \sin(z)(-bX + aY)))\exp(zZ)\\ &= \exp(\epsilon(A(z)X + B(z)Y))\exp(zZ)\end{aligned}$$

  with $A(z) = a\cos(z) - b\sin(z)$ and $B(z) = a\sin(z) + b\cos(z)$.

  If $x = y = 0$, we obtain the expected expression in cylindrical coordinates. For what will follows, we will suppose that $(x, y) \neq (0, 0)$. We recall that $\varphi = \sqrt{x^2+y^2}$.

- To obtain the expression of $\exp(xX + yY)\exp(\epsilon(A(z)X + B(z)Y))$ in cylindrical coordinates, we use Lemma 3.2.3 with $u = \frac{xX+yY}{\varphi}$, $\alpha = \varphi$ and $t = A(z)X + B(z)Y$. We have $u \wedge t = \frac{xB(z)-yA(z)}{\varphi} Z$ thus we take $w = Z$ and $\rho = \frac{xB(z)-yA(z)}{\varphi}$. Taking $v = w \wedge u = \frac{-yX+xY}{\varphi}$, we obtain a basis (u, v, w) satisfying (3.23). We get:

$$\begin{aligned}\exp(xX + yY)\exp(\epsilon(A(z)X + B(z)Y)) &= \exp\Bigg(xX + yY + \epsilon\Bigg(A(z)X + B(z)Y \\ &\quad + \rho\Bigg(-\frac{-yX + xY}{\varphi} + \frac{\varphi}{2}\cot\left(\frac{\varphi}{2}\right)\frac{-yX + xY}{\varphi} + \frac{\varphi}{2}Z\Bigg)\Bigg)\Bigg) + \mathcal{O}(\epsilon^2) \\ &= \exp\left((x + \epsilon C)X + (y + \epsilon D)Y + \epsilon E \cdot Z\right) + \mathcal{O}(\epsilon^2)\end{aligned}$$

with $C = A(z) + \rho y\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)$, $D = B(z) + x\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)$, $E = \rho\frac{\varphi}{2}$.

- In cylindrical coordinates we can write:

$$\begin{aligned}&\exp\left((x + \epsilon C)X + (y + \epsilon D)Y + \epsilon EZ\right) + \mathcal{O}(\epsilon^2) \\ &\qquad = \exp(x(\epsilon)X + y(\epsilon)Y)\exp(\epsilon\gamma(\epsilon)Z) + \mathcal{O}(\epsilon^2).\end{aligned}$$

We use the first equality of Lemma 3.2.2 with $u = \frac{x(\epsilon)X+y(\epsilon)Y}{\varphi(\epsilon)}$, $\alpha = \varphi(\epsilon) = \sqrt{x(\epsilon)^2 + y(\epsilon)^2}$, $v = Z$ and $\beta = \gamma(\epsilon)$. We also have $w = \frac{y(\epsilon)X - x(\epsilon)Y}{\varphi(\epsilon)}$. Then we obtain:

$$\begin{aligned}&\exp(x(\epsilon)X + y(\epsilon)Y)\exp(\epsilon\gamma(\epsilon)Z) \\ &= \exp\left(x(\epsilon)X + y(\epsilon)Y + \epsilon\frac{\gamma(\epsilon)\varphi(\epsilon)}{2}\left(\cot\left(\frac{\varphi(\epsilon)}{2}\right)Z + \frac{y(\epsilon)X - x(\epsilon)Y}{\varphi(\epsilon)}\right)\right) + \mathcal{O}(\epsilon^2) \\ &= \exp\left(\left(x(\epsilon) + \epsilon\frac{\gamma(\epsilon)}{2}y(\epsilon)\right)X + \left(y(\epsilon) - \epsilon\frac{\gamma(\epsilon)}{2}x(\epsilon)\right)Y + \epsilon\frac{\gamma(\epsilon)\varphi(\epsilon)}{2}\cot\left(\frac{\varphi(\epsilon)}{2}\right)Z\right) \\ &+ \mathcal{O}(\epsilon^2).\end{aligned}$$

Thus we take $x(\epsilon), y(\epsilon)$ and $\gamma(\epsilon)$ such that:

$$\begin{cases} x + \epsilon C = x(\epsilon) + \epsilon\dfrac{\gamma(\epsilon)}{2}y(\epsilon) & \text{(3.31a)} \\ y + \epsilon D = y(\epsilon) - \epsilon\dfrac{\gamma(\epsilon)}{2}x(\epsilon) & \text{(3.31b)} \\ E = \dfrac{\varphi(\epsilon)\gamma(\epsilon)}{2}\cot(\dfrac{\varphi(\epsilon)}{2}) & \text{(3.31c)} \end{cases}$$

Note that $x(0) = x$, $y(0) = y$ and $\varphi(0) = \varphi$. Finally we get:

$$\begin{aligned}&\exp(xX + yY)\exp(zZ)\exp(\epsilon(aX + bY)) \\ &\qquad = \exp(x(\epsilon)X + y(\epsilon)Y)\exp(\epsilon\gamma(\epsilon)Z)\exp(zZ) + \mathcal{O}(\epsilon^2) \\ &\qquad = \exp(x(\epsilon)X + y(\epsilon)Y)\exp(z(\epsilon)Z) + \mathcal{O}(\epsilon^2) \\ &\qquad \text{with } z(\epsilon) = z + \epsilon\gamma(\epsilon) \\ &\qquad = \exp(\varphi(\epsilon)(\cos(\theta(\epsilon))X + \sin(\theta(\epsilon))Y)\exp(z(\epsilon)Z) + \mathcal{O}(\epsilon^2).\end{aligned}$$

- To obtain the values of the left invariant vector, we now have to study $\theta'(0)$, $\varphi'(0)$ and $z'(0)$.
  Using (3.31c), we directly have $z'(0) = \gamma(0) = \frac{2}{\varphi}\tan\left(\frac{\varphi}{2}\right) E = \rho \times \tan\left(\frac{\varphi}{2}\right)$ with

$$\begin{aligned}
\rho &= \frac{xB(z) - yA(z)}{\varphi} = \cos(\theta)B(z) - \sin(\theta)A(z) \\
&= \cos(\theta)\left(a\sin(z) + b\cos(z)\right) - \sin(\theta)\left(a\cos(z) - b\sin(z)\right) \\
&= a\left(\cos(\theta)\sin(z) - \sin(\theta)\cos(z)\right) + b\left(\cos(\theta)\cos(z) + \sin(\theta)\sin(z)\right) \\
&= a\sin(z-\theta) + b\cos(z-\theta).
\end{aligned}$$

  Thus $z'(0) = \tan\left(\frac{\varphi}{2}\right)\left(a\sin(z-\theta) + b\cos(z-\theta)\right)$.
  Then, summing the squares of (3.31a) and (3.31b), we get $\varphi(\epsilon)^2 = x(\epsilon)^2 + y(\epsilon)^2 = x^2 + y^2 + 2\epsilon\left(xC + yD\right) + \mathcal{O}(\epsilon^2)$, hence $2\varphi(0)\varphi'(0) = 2\left(xC + yD\right)$ and

$$\begin{aligned}
\varphi'(0) &= \frac{xC + yD}{\varphi} = \cos(\theta)C + \sin(\theta)D \\
&= \cos(\theta)\left(A(z) + \rho y\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right) \\
&+ \sin(\theta)\left(B(z) + x\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right) \\
&= \cos(\theta)A(z) + \sin(\theta)B(z) + \rho\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\left(y\cos(\theta) - x\sin(\theta)\right) \\
&= \cos(\theta)A(z) + \sin(\theta)B(z) \\
&= \cos(\theta)\left(a\cos(z) - b\sin(z)\right) + \sin(\theta)\left(a\sin(z) + b\cos(z)\right) \\
&= a\cos(\theta - z) + b\sin(\theta - z).
\end{aligned}$$

  Now note that $\tan\left(\theta(\epsilon)\right) = \frac{y(\epsilon)}{x(\epsilon)}$.
  Thus $\frac{\theta'(\epsilon)}{\cos^2(\theta(\epsilon))} = \frac{y'(\epsilon)x(\epsilon) - x'(\epsilon)y(\epsilon)}{x(\epsilon)^2}$ and $\theta'(0) = \cos^2(\theta)\frac{y'(0)x - x'(0)y}{r^2\cos^2(\theta)} = \frac{y'(0)x - x'(0)y}{r^2}$.
  Using (3.31a) and then (3.31b), we get:

$$\begin{aligned}
x'(0) &= C - \frac{\gamma(0)}{2}y(0) = C - \frac{\rho}{2}\tan(\frac{\varphi}{2})y \\
y'(0) &= D + \frac{\gamma(0)}{2}x(0) = D + \frac{\rho}{2}\tan(\frac{\varphi}{2})x.
\end{aligned}$$

  We obtain:

$$\begin{aligned}
\theta'(0) &= \frac{\left(D + \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right)x\right)x - \left(C - \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right)y\right)y}{\varphi^2} = \frac{Dx - Cy}{\varphi^2} + \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right) \\
&= \frac{D\cos(\theta) - C\sin(\theta)}{\varphi} + \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right).
\end{aligned}$$

We have:

$$
\begin{aligned}
&D\cos(\theta) - C\sin(\theta)\\
&= \cos(\theta)\left(B(z) + x\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right) - \sin(\theta)\left(A(z) + \rho y\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right)\\
&= \cos(\theta)B(z) - \sin(\theta)A(z) + \rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)(x\cos(\theta) + y\sin(\theta))\\
&= \rho + \varphi\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\\
&= \frac{\varphi\rho}{2}\cot\left(\frac{\varphi}{2}\right).
\end{aligned}
$$

Hence: $\theta'(0) = \frac{\rho}{2}\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right) = \frac{a\sin(z-\theta)+b\cos(z-\theta)}{2}\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)$.

- Finally, taking $(a,b) = (1,0)$, we get $\bar{X} = \begin{pmatrix} \cos(\theta - z) \\ \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right) \\ \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta) \end{pmatrix}$ and, taking $(a,b) = (0,1)$, we get $\bar{Y} = \begin{pmatrix} \sin(\theta - z) \\ \frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right) \\ \tan\left(\frac{\varphi}{2}\right)\cos(z-\theta) \end{pmatrix}$.

We can now calculate the sub-Laplacian operator $L = \frac{1}{2}\left(\bar{X}^2 + \bar{Y}^2\right)$ (also made in [20]).

$$
\begin{aligned}
\bar{X}^2 = \cos(\theta-z)\partial_\varphi \Bigg[&\cos(\theta-z)\partial_\varphi + \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta\\
&+ \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z\Bigg]\\
+\frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta \Bigg[&\cos(\theta-z)\partial_\varphi + \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right)\right.\\
&\left.+ \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta + \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z\Bigg]\\
+\tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z \Bigg[&\cos(\theta-z)\partial_\varphi + \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta\\
&+ \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{Y}^2 = \sin(\theta-z)\partial_\varphi \Bigg[&\sin(\theta-z)\partial_\varphi + \frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta +\\
&+ \tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z\Bigg]\\
+\frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta \Bigg[&\sin(\theta-z)\partial_\varphi + \frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right)\right.\\
&\left.+ \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta + \tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z\Bigg]\\
+\tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z \Bigg[&\sin(\theta-z)\partial_\varphi + \frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta\\
&+ \tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z\Bigg].
\end{aligned}
$$

We obtain:

$$\begin{aligned}\bar{X}^2+\bar{Y}^2 &= \partial^2_{\varphi,\varphi}+\frac{1}{4}\left(\cot\left(\frac{\varphi}{2}\right)+\tan\left(\frac{\varphi}{2}\right)\right)^2\partial^2_{\theta,\theta}+\left(1+\tan^2\left(\frac{\varphi}{2}\right)\right)\partial^2_{\theta,z}\\ &\quad+\frac{1}{2}\left(\cot\left(\frac{\varphi}{2}\right)-\tan\left(\frac{\varphi}{2}\right)\right)\partial_\varphi+\tan^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z}\\ &= \partial^2_{\varphi,\varphi}+\frac{1}{\sin^2(\varphi)}\partial^2_{\theta,\theta}+\tan^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z}+\frac{1}{\cos^2\left(\frac{\varphi}{2}\right)}\partial^2_{\theta,z}+\cot(\varphi)\partial_\varphi.\end{aligned}$$

3.2.3 Computation of the sub-Laplacian on $SL(2,\mathbb{R})$

We now consider the case of $SL(2,\mathbb{R})$. In this case, (X,Y,Z) is satisfying (3.24). In order to use our previous results, we define $(\tilde{X},\tilde{Y},\tilde{Z}) := (iX,iY,-Z)$. This way we get $(\tilde{X},\tilde{Y},\tilde{Z})$ a basis of $\mathfrak{su}(2)$ satisfying (3.23). We have to study

$$\begin{aligned}&\exp(xX+yY)\exp(zZ)\exp(\epsilon(aX+bY))\\ &\qquad=\exp\left(-ix\tilde{X}-iy\tilde{Y}\right)\exp\left(-z\tilde{Z}\right)\exp\left(\epsilon\left(-ia\tilde{X}-ib\tilde{Y}\right)\right).\end{aligned}$$

In the previous part, as said Remark 3.2.4, the computation can be made using complex coefficients. Then we obtain:

$$\begin{aligned}&\exp(xX+yY)\exp(zZ)\exp(\epsilon(aX+bY))\\ &\qquad=\exp\left(\varphi(\epsilon)\left(\cos(\theta(\epsilon))\tilde{X}+\sin(\theta(\epsilon))\tilde{Y}\right)\right)\exp\left(z(\epsilon)\tilde{Z}\right)+\mathcal{O}(\epsilon^2)\\ &\qquad=\exp\left(i\varphi(\epsilon)\left(\cos(\theta(\epsilon))\tilde{X}+\sin(\theta(\epsilon))\tilde{Y}\right)\right)\exp\left(-z(\epsilon)\tilde{Z}\right)+\mathcal{O}(\epsilon^2),\end{aligned}$$

with $\varphi(0)=-i\varphi$, $\theta(0)=\theta$ and $z(0)=-z$.

We have:

$$\frac{d}{d\epsilon}\left(i\varphi(\epsilon)\right)_{|\epsilon=0}=i\varphi'(0)=i\left(-ia\cos(\theta-(-z))-ib\sin(\theta-(-z))\right)=a\cos(\theta+z)+b\sin(\theta+z);$$

$$\begin{aligned}\frac{d}{d\epsilon}\left(-z(\epsilon)\right)_{|\epsilon=0}=-z'(0)&=-\tan\left(\frac{-i\varphi}{2}\right)\left(-ia\sin(-z-\theta)-ib\cos(-z-\theta)\right)\\ &=-\tanh\left(\frac{\varphi}{2}\right)\left(a\sin(z+\theta)-b\cos(\theta+z)\right);\end{aligned}$$

$$\begin{aligned}\theta'(0)&=\frac{-ia\sin(-z-\theta)-ib\cos(\theta-(-z))}{2}\left(\cot\left(\frac{-i\varphi}{2}\right)+\tan\left(\frac{-i\varphi}{2}\right)\right)\\ &=-i\frac{-a\sin(z+\theta)+b\cos(\theta+z)}{2}\left(i\coth\left(\frac{\varphi}{2}\right)-i\tanh\left(\frac{\varphi}{2}\right)\right)\\ &=\frac{-a\sin(z+\theta)+b\cos(\theta+z)}{2}\left(\coth\left(\frac{\varphi}{2}\right)-\tanh\left(\frac{\varphi}{2}\right)\right).\end{aligned}$$

We get:

$$\bar{X}=\begin{pmatrix}\cos(\theta+z)\\ -\frac{\sin(z+\theta)}{2}\left(\coth\left(\frac{\varphi}{2}\right)-\tanh\left(\frac{\varphi}{2}\right)\right)\\ -\tanh\left(\frac{\varphi}{2}\right)\sin(z+\theta)\end{pmatrix}\text{ and }\bar{Y}=\begin{pmatrix}\sin(\theta+z)\\ \frac{\cos(\theta+z)}{2}\left(\coth\left(\frac{\varphi}{2}\right)-\tanh\left(\frac{\varphi}{2}\right)\right)\\ \tanh\left(\frac{\varphi}{2}\right)\cos(z+\theta)\end{pmatrix}.$$

As before, we can calculate the sub-Laplacian operator:

$$\begin{aligned} L = \frac{1}{2}(\bar{X}^2 + \bar{Y}^2) &= \partial^2_{\varphi,\varphi} + \frac{1}{4}\left(\coth\left(\frac{\varphi}{2}\right) - \tanh\left(\frac{\varphi}{2}\right)\right)^2 \partial^2_{\theta,\theta} + \left(1 - \tanh^2\left(\frac{\varphi}{2}\right)\right)\partial^2_{\theta,z} \\ &\quad + \frac{1}{2}\left(\coth\left(\frac{\varphi}{2}\right) + \tanh\left(\frac{\varphi}{2}\right)\right)\partial_\varphi + \tanh^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z} \\ &= \partial^2_{\varphi,\varphi} + \frac{1}{\sinh^2(\varphi)}\partial^2_{\theta,\theta} + \tanh^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z} + \frac{1}{\cosh^2\left(\frac{\varphi}{2}\right)}\partial^2_{\theta,z} + \coth(\varphi)\partial_\varphi. \end{aligned}$$

3.2.4 Norms of gradients on $SU(2)$ and $SL(2,\mathbb{R})$

Let $k \in \{-1, 1\}$. We recall that $E_1 = SU(2)$ and $E_{-1} = SL(2,\mathbb{R})$.

As in subsection 3.1.6, for any function f on E_k and $g \in E_k$, we can define the norm of the gradient $||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} := \sqrt{(\bar{X}f)^2(g) + (\bar{Y}f)^2(g)}$ and the gradient length $|\nabla f|(g)$ associated to the Carnot-Carathéodory distance. As for the homogeneous Carnot groups, E_k is a regular subRiemannian manifold. Thus, for all u upper gradient of f, and in particular for u the gradient length:

$$||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} \leq u(g) \text{ a.e. in } E_k. \tag{3.32}$$

In particular, this is true for the upper-gradient defined by the gradient length.

3.2.5 The Brownian motion on $SU(2)$ and $SL(2,\mathbb{R})$

The following proposition gives a characterization and a formal geometrical interpretation of the Brownian motion on $SU(2)$ and $SL(2,\mathbb{R})$ due to Baudoin and Bonnefont (see [10, 21, 20]):

Proposition 3.2.5 ([10, 21, 20])**.** *Let us consider a Brownian motion $\mathbb{B}_t$ on $SU(2)$ (resp. $SL(2,\mathbb{R})$). There exist three continuous processes $(\varphi_t)_t$, $(\theta_t)_t$ and $(z_t)_t$ such that*

$$\mathbb{B}_t = \exp(\varphi_t(\cos(\theta_t)\bar{X} + \sin(\theta_t)\bar{Y}))\exp(z_t\bar{Z}) \text{ and } \begin{cases} d\varphi_t = dB^1_t + \frac{1}{2}\sqrt{k}\cot(\sqrt{k}\varphi_t)dt \\ d\theta_t = \frac{\sqrt{k}}{\sin(\sqrt{k}\varphi_t)}dB^2_t \\ dz_t = \frac{\tan\left(\frac{\sqrt{k}\varphi_t}{2}\right)}{\sqrt{k}}dB^2_t \end{cases} \tag{3.33}$$

with B^1_t and B^2_t two independent Brownian motions in $\mathbb{R}$ (in the above formula, we take $k = 1$ for the case of $SU(2)$ and $k = -1$ for the case of $SL(2,\mathbb{R})$).

Moreover, we have:

- *The process $(X_t := \Pi_1(\mathbb{B}_t))_t$ (resp. $(\Pi_{-1}(\mathbb{B}_t))_t$) is a Brownian motion on the sphere S^2 (resp. the hyperbolic plane $\mathbf{H}^2$). Its spherical (resp. polar) coordinates are given by (φ_t, θ_t) with respect to the pole N_0 fixed by the submersion.*
- *$z_t - z_0$ is the signed swept area (modulo 4π) of X_t with respect to the fixed pole N_0.*

Remark 3.2.6. *Using the geometrical interpretation stated by this proposition, we can define a Brownian motion on $SU(2)$ (resp. $SL(2,\mathbb{R})$) by defining a Brownian motion $(X_t)_t$ on S^2 (resp. $\mathbf{H}^2$) and by considering its lifting on $SU(2)$ (resp. $SL(2,\mathbb{R})$.*

Let us explain this result.

Proof of Proposition 3.2.5. We define the Brownian motion on $SU(2)$ (resp. $SL(2,\mathbb{R})$ as the diffusion process $\mathbb{B}_t$ with infinitesimal generator L. Using the expression of L in the cylindrical coordinates obtained in (3.26) and (3.27), we can state that there exist φ_t, θ_t and z_t three real diffusion processes, such that:

$$\mathbb{B}_t = \exp(\varphi_t(\cos(\theta_t)X + \sin(\theta_t)Y))\exp(z_t Z) \text{ and } \begin{cases} \langle d\varphi_t, d\varphi_t\rangle = dt \\ \langle d\theta_t, d\theta_t\rangle = \frac{k}{\sin^2(\sqrt{k}\varphi_t)}dt \\ \langle dz_t, dz_t\rangle = \frac{1}{k}\tan^2\left(\frac{\sqrt{k}\varphi_t}{2}\right)dt \\ \langle d\theta_t, dz_t\rangle = \frac{1}{2\cos^2\left(\frac{\sqrt{k}\varphi_t}{2}\right)}dt \\ \langle d\varphi_t, d\theta_t\rangle = \langle d\varphi_t, dz_t\rangle = 0 \\ Drift(d\varphi_t) = \frac{1}{2}\sqrt{k}\cot(\varphi_t)dt \\ Drift(d\theta_t) = 0 \\ Drift(dz_t) = 0. \end{cases}$$

This leads to (3.33).

Because of the properties of Π_k described in Subsection 3.2.1, it is immediate that the spherical (resp. polar) coordinates of X_t with respect to the pole N_0 are given by (φ_t, θ_t). Using the two first equations of (3.33), we get the infinitesimal generator of the diffusion $(\varphi_t, \theta_t)_t$:

$$\frac{1}{2}\left(\partial^2_{\varphi,\varphi} + \frac{k}{\sin^2(\sqrt{k}\varphi)}\partial^2_{\theta,\theta} + \sqrt{k}\cot(\sqrt{k}\varphi)\partial_\varphi\right).$$

This is the Laplace Beltrami operator on the sphere S^2 (resp. the hyperbolic plane $\mathbf{H}^2$) in spherical (resp. polar) coordinates. Moreover, the signed area swept by the path of $(\varphi_t, \theta_t)_t$ on S^2 (resp. $\mathbf{H}^2$) with respect of the pole N_0, is given by:

$$\int_0^t \frac{1}{k}(1-\cos(\sqrt{k}\varphi_s))d\theta_s = \int_0^t \frac{2}{k}\sin^2\left(\frac{\sqrt{k}\varphi_s}{2}\right)d\theta_s = \int_0^t \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}\varphi_t}{2}\right)dB^2_t.$$

The last integral is exactly equal to $z_t - z_0$. □

Remark 3.2.7. *Note that this swept area is a signed value depending of the orientation of the curve $(\varphi_t, \theta_t)_t$. For example, if θ_t is decreasing for all t, then we get $z_t - z_0 < 0$.*

Remark 3.2.8. *Let consider a Brownian motion on the Heisenberg group (which is the free Carnot group with step 2 and rank 2) starting at $(x, z) \in \mathbb{R}^2 \times \mathbb{R}$. According to Subsection 3.1.3, it can be defined by $\mathbb{B}_t := (X_t, z_t)$ with $X_t = (X^1_t, X^2_t)$ a 2-dimensional Brownian motion starting at 0 and $z_t = z + A_t$ with A_t denoting the Lévy area associated to (X^1_t, X^2_t). We thus get a similar geometrical interpretation of the Brownian motion on the Heisenberg group. Moreover, by denoting $(\varphi_t, \theta_t)_t$ the continuous process describing the usual polar coordinates of $(X_t)_t$, $(\varphi_t, \theta_t, z_t)_t$ satisfies the system (3.33) for $k \to 0$, that is:*

$$\begin{cases} d\varphi_t = dB^1_t + \frac{1}{2\varphi_t}dt \\ d\theta_t = \frac{1}{\varphi_t}dB^2_t \\ dz_t = \frac{\varphi_t}{2}dB^2_t \end{cases}$$

with $(B^1_t)_t$ and $(B^2_t)_t$ two real independent Brownian motions.

3.2.6 Comparison of two elements in $SU(2)$ and $SL(2,\mathbb{R})$

The results given in this subsection come from the contribution:

[25] Magalie Bénéfice. Couplings of Brownian motions on $SU(2)$ and $SL(2,\mathbb{R})$. *Stochastic Processes and their Applications*, page 104434, 2024

To compare (with respect to the subRiemannian structure) two elements expressed in the cylindrical coordinates $g = (\varphi, \theta, z)$ and $g' = (\varphi', \theta', z')$ on $SU(2)$ (resp. $SL(2,\mathbb{R})$), we would like to use the estimates of the Carnot-Carathéodory distance given by (3.25). To use this relation, we thus need to compute the cylindrical coordinates (ρ, Θ, ζ) of $g^{-1}g'$, or at least ρ and ζ which are the only coordinates appearing in the estimate of $d_{cc}(g, g') = d_{cc}(0, g^{-1}g')$. In the following proposition, we give an expression of ρ and ζ in function of the projections of g and g' on S^2 (resp. $\mathbf{H}^2$) and of the last coordinates z and z'.

Proposition 3.2.9. *Let $g = (\varphi, \theta, z)$ and $g' = (\varphi', \theta', z')$ be two elements of $SU(2)$ (resp. $SL(2,\mathbb{R})$) written in cylindrical coordinates. Using the submersions defined in Subsection 3.2.1, we denote $x := \Pi_1(g)$ and $x' := \Pi_1(g')$ (resp. $x := \Pi_{-1}(g)$ and $x' := \Pi_{-1}(g')$). The cylindrical coordinates of $g^{-1}g'$ are given by (ρ, Θ, ζ) with for ρ and ζ:*

- *ρ equals to the usual Riemannian distance on S^2 (resp. on $\mathbf{H}^2$) between x and x'.*
- *$\zeta \in]-2\pi, 2\pi]$ and $\zeta \equiv z' - z + sign(\theta - \theta')\mathcal{A}_{x',x,N_0} \mod (4\pi)$ with $\mathcal{A}_{a,b,c}$ the area of the spherical (resp. hyperbolic) triangle of vertices a, b and c and N_0 a pole induced by the submersion Π_1 (resp. Π_{-1}). Note that $sign(\theta-\theta')\mathcal{A}_{x',x,N_0}$ is in fact the signed area of the oriented triangle of vertices x', x and N_0.*

In particular, we directly obtain the estimate:

$$c_1(\rho^2 + |\zeta|) \leq d_{cc}^2(g, g') \leq c_2(\rho^2 + |\zeta|) \tag{3.34}$$

with c_1 and c_2 two positive constants only depending on the subRiemannian structure of $SU(2)$ (resp. $SL(2,\mathbb{R})$).

Proof. We first begin with the case of $SU(2)$.

Let $g = (\varphi, \theta, z)$ and $g' = (\varphi', \theta', z')$ be two elements of $SU(2)$ written in cylindrical coordinates. We denote by (ρ, Θ, ζ) the cylindrical coordinates of $g^{-1}g'$. Using matrix computations, we have:

$$\begin{aligned} g^{-1}g' &= \big(\exp\big(\varphi(\cos(\theta)X + \sin(\theta)Y)\big)\exp(zZ)\big)^{-1} \exp\big(\varphi'(\cos(\theta')X + \sin(\theta')Y)\big)\exp(z'Z) \\ &= \exp\big(-\varphi(\cos(\theta - z)X + \sin(\theta - z)Y)\big) \\ &\quad \exp\big(\varphi'(\cos(\theta' - z)X + \sin(\theta' - z)Y)\big)\exp\big((z' - z)Z\big) \end{aligned}$$

Then $\exp\big(\rho(\cos(\Theta)X + \sin(\Theta)Y)\big)\exp(\zeta Z)$ is equal to the matrix product:

$$\begin{pmatrix} \cos\left(-\frac{\varphi}{2}\right) & e^{i(\theta - z)}\sin\left(-\frac{\varphi}{2}\right) \\ -e^{-i(\theta-z)}\sin\left(-\frac{\varphi}{2}\right) & \cos\left(-\frac{\varphi}{2}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\varphi'}{2}\right) & e^{i(\theta' - z)}\sin\left(\frac{\varphi'}{2}\right) \\ -e^{-i(\theta'-z)}\sin\left(\frac{\varphi'}{2}\right) & \cos\left(\frac{\varphi'}{2}\right) \end{pmatrix}.$$

In particular we have:

$$\cos\left(\frac{\rho}{2}\right)e^{i\frac{\zeta}{2}} = \cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi'}{2}\right) + e^{i(\theta-\theta')}\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi'}{2}\right). \tag{3.35}$$

From this equation we can get the expected geometrical interpretation on the values ρ and ζ.

- We begin with ρ: we are going to prove that ρ is the usual distance on S^2 between x and x' with $x := \Pi_1(g)$ and $x' := \Pi_1(g')$. Let us denote by $\rho(x, x')$ this distance. We note that:

  $$\begin{aligned}\cos(\rho(x,x')) &= < x, x' >_{\mathbb{R}^3} \\ &= \sin(\theta)\sin(\theta')\sin(\varphi)\sin(\varphi') + \cos(\theta)\cos(\theta')\sin(\varphi)\sin(\varphi') + \cos(\varphi)\cos(\varphi') \\ &= \cos(\varphi)\cos(\varphi') + \sin(\varphi)\sin(\varphi')\cos(\theta-\theta').\end{aligned}$$

  From (3.35), we get

  $$\begin{aligned}\cos^2\left(\frac{\rho}{2}\right) &= \left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi'}{2}\right) + \cos(\theta-\theta')\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi'}{2}\right)\right)^2 \\ &\quad + \sin^2(\theta-\theta')\sin^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\varphi'}{2}\right) \\ &= \frac{1+\cos(\varphi)\cos(\varphi')}{2} + \frac{\sin(\varphi)\sin(\varphi')\cos(\theta-\theta')}{2} \\ &= \frac{1+\cos(\rho(x,x'))}{2} = \cos^2\left(\frac{\rho(x,x')}{2}\right).\end{aligned}$$

  We obtain the announced result as ρ and $\rho(\Pi_1(x), \Pi_1(x'))$ live in $[0, \pi]$. Moreover, we get that

  $$\cos(\rho) = \cos(\varphi)\cos(\varphi') + \sin(\varphi)\sin(\varphi')\cos(\theta-\theta'). \tag{3.36}$$

- We now deal with ζ. We consider $\mathcal{A}_{x',x,N_0}$ the area of the spherical triangle with vertices x', x and N_0, where N_0 still denotes the north pole. The lengths of the opposite sides are respectively φ, φ' and ρ. Using an equivalent of the Heron formula for the spherical triangle (see [35]), we have:

  $$\cos\left(\frac{\mathcal{A}_{x',x,N_0}}{2}\right) = \frac{1}{4\cos\left(\frac{\rho}{2}\right)\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi'}{2}\right)}\left(1+\cos(\varphi)+\cos(\varphi')+\cos(\rho)\right).$$

  Because of (3.35) and (3.36), we get:

  $$\begin{aligned}&\cos\left(\frac{\zeta}{2}\right) \\ &= \frac{1}{\cos\left(\frac{\rho}{2}\right)}\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi'}{2}\right) + \frac{\cos(\rho)-\cos(\varphi)\cos(\varphi')}{\sin(\varphi)\sin(\varphi')}\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi'}{2}\right)\right) \\ &= \frac{1}{4\cos\left(\frac{\rho}{2}\right)\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi'}{2}\right)}\left(\left(1+\cos(\varphi)\right)\left(1+\cos(\varphi')\right)+\cos(\rho)-\cos(\varphi)\cos(\varphi')\right) \\ &= \cos\left(\frac{\mathcal{A}_{x',x,N_0}}{2}\right)\end{aligned}$$

Thus we have $|\zeta| \equiv \mathcal{A}_{x',x,N_0} \mod (4\pi)$.

Moreover, still using (3.35), we get that $\sin\left(\frac{\zeta}{2}\right) = \sin(\theta-\theta')\frac{\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi'}{2}\right)}{\cos\left(\frac{\rho}{2}\right)}$, and so $\zeta < 0$ if and only if $\theta < \theta'$. This gives the expected result.

As the Carnot-Carathéodory distance is left invariant, we have $d_{cc}(g,g') = d_{cc}(0, g^{-1}\cdot g')$ and we directly deduce (3.34) from (3.25).

We now deal with the case of $SL(2,\mathbb{R})$. The proof is quite similar as for the case of $SU(2)$. We use the same notations as before. Let $g, g' \in SL(2,\mathbb{R})$. This time, the matrix computations give:

$$\begin{aligned} g^{-1}\cdot g' = &\exp\big(-\varphi(\cos(\theta+z)X + \sin(\theta+z)Y)\big) \\ &\exp\big(\varphi'(\cos(\theta'+z)X + \sin(\theta'+z)Y)\big)\exp\big((z'-z)Z\big) \end{aligned}$$

If we denote $M := \exp\big(\rho(\cos(\Theta)X + \sin(\Theta)Y)\big)\exp(\zeta Z)$, we have:

$$M = \begin{pmatrix} \cosh\left(\frac{\rho}{2}\right)\cos\left(\frac{\zeta}{2}\right) + \sinh\left(\frac{\rho}{2}\right)\cos\left(\Theta+\frac{\zeta}{2}\right) & -\cosh\left(\frac{\rho}{2}\right)\sin\left(\frac{\zeta}{2}\right) - \sinh\left(\frac{\rho}{2}\right)\sin\left(\Theta+\frac{\zeta}{2}\right) \\ \cosh\left(\frac{\rho}{2}\right)\sin\left(\frac{\zeta}{2}\right) - \sinh\left(\frac{\rho}{2}\right)\sin\left(\Theta+\frac{\zeta}{2}\right) & \cosh\left(\frac{\rho}{2}\right)\cos\left(\frac{\zeta}{2}\right) - \sinh\left(\frac{\rho}{2}\right)\cos\left(\Theta+\frac{\zeta}{2}\right) \end{pmatrix}$$

The matrix M is also equal to the product:

$$\exp\big(-\varphi(\cos(\theta+z)X + \sin(\theta+z)Y)\big)\exp\big(\varphi'(\cos(\theta'+z)X + \sin(\theta'+z)Y)\big).$$

In particular we get:

$$\begin{aligned} M_{1,1} = &\cosh\left(\frac{\varphi}{2}\right)\cosh\left(\frac{\varphi'}{2}\right) + \cosh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\cos(\theta'+z) \\ &- \cosh\left(\frac{\varphi'}{2}\right)\sinh\left(\frac{\varphi}{2}\right)\cos(\theta+z) - \sinh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\cos(\theta'-\theta) \\ M_{1,2} = &-\cosh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\sin(\theta'+z) + \sinh\left(\frac{\varphi}{2}\right)\cosh\left(\frac{\varphi'}{2}\right)\sin(\theta+z) \\ &+ \sinh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\sin(\theta'-\theta) \\ M_{2,1} = &-\cosh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\sin(\theta'+z) + \sinh\left(\frac{\varphi}{2}\right)\cosh\left(\frac{\varphi'}{2}\right)\sin(\theta+z) \\ &- \sinh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\sin(\theta'-\theta) \\ M_{2,2} = &\cosh\left(\frac{\varphi}{2}\right)\cosh\left(\frac{\varphi'}{2}\right) - \cosh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\cos(\theta'+z) \\ &+ \cosh\left(\frac{\varphi'}{2}\right)\sinh\left(\frac{\varphi}{2}\right)\cos(\theta+z) - \sinh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\cos(\theta'-\theta) \end{aligned}$$

As $\cosh\left(\frac{\rho}{2}\right)\cos\left(\frac{\zeta}{2}\right) = \frac{M_{1,1}+M_{2,2}}{2}$ and $\cosh\left(\frac{\rho}{2}\right)\sin\left(\frac{\zeta}{2}\right) = \frac{M_{2,1}-M_{1,2}}{2}$, we get:

$$\begin{cases} \cosh\left(\frac{\rho}{2}\right)\cos\left(\frac{\zeta}{2}\right) = \cosh\left(\frac{\varphi}{2}\right)\cosh\left(\frac{\varphi'}{2}\right) - \sinh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\cos(\theta'-\theta) & \text{(3.37a)} \\ \cosh\left(\frac{\rho}{2}\right)\sin\left(\frac{\zeta}{2}\right) = -\sinh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)\sin(\theta'-\theta). & \text{(3.37b)} \end{cases}$$

- This time, the distance between $x := \Pi_{-1}(g)$ and $x' := \Pi_{-1}(g')$ satisfies:

$$\cosh(\rho(x,x')) = \cosh(\varphi)\cosh(\varphi') + \sinh(\varphi)\sinh(\varphi')\cos(\theta-\theta').$$

  Using the identity $\cosh^2(\frac{\rho}{2}) = \frac{1+\cosh(\rho)}{2}$ we obtain:

$$\cosh(\rho) = \cosh(\varphi)\cosh(\varphi') + \sinh(\varphi)\sinh(\varphi')\cos(\theta-\theta') \tag{3.38}$$

  and thus: $\rho = \rho(x,x')$.

- Using the same notations as before (this time N_0 is the pole used to define the polar coordinates on $\mathbf{H}^2$) and the equivalent of the Heron formula for hyperbolic triangles, we have:

$$\begin{aligned}\cos\left(\frac{\mathcal{A}_{x',x,N_0}}{2}\right) &= \frac{1}{4\cosh\left(\frac{\rho}{2}\right)\cosh\left(\frac{\varphi}{2}\right)\cosh\left(\frac{\varphi'}{2}\right)}(1+\cosh(\varphi)+\cosh(\varphi')+\cosh(\rho)) \\ &= \cos\left(\frac{\zeta}{2}\right)\end{aligned}$$

  and thus $\zeta = \text{sign}(\zeta)\mathcal{A}_{x',x,N_0}$. Moreover, we have $\sin\left(\frac{\zeta}{2}\right) = \sin(\theta-\theta')\frac{\sinh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi'}{2}\right)}{\cosh\left(\frac{\rho}{2}\right)}$, and so $\zeta < 0$ if and only if $\theta < \theta'$.

□

From Proposition 3.2.9, we obtain the following geometrical interpretation of the distance between two Brownian motions. Using the notations E_k, M_k from Remark 3.2.1, we have:

Corollary 3.2.10. *Let us take $\mathbb{B}_t$ and $\mathbb{B}'_t$ two Brownian motions on E_k starting from g and g' respectively, with (φ_t,θ_t,z_t) and $(\varphi'_t,\theta'_t,z'_t)$ their cylindrical coordinates. We denote $X_t := \Pi_k(\mathbb{B}_t)$ and $X'_t := \Pi_k(\mathbb{B}'_t)$. The cylindrical coordinates of $(\mathbb{B}_t)^{-1}\mathbb{B}'_t$ are given by $(\rho_t,\Theta_t,\zeta_t)$ with:*

- *ρ_t the Riemannian distance between X_t and X'_t on M_k;*
- *$\zeta_t \in]-2\pi, 2\pi]$ and*

$$\begin{aligned}\zeta_t &\equiv z'_t - z_t + sign(\theta_t-\theta'_t)\mathcal{A}_{X_t,X'_t,N_0} \mod (4\pi) &(3.39)\\ &\equiv \zeta_0 + A_t \mod (4\pi) &(3.40)\end{aligned}$$

  with A_t the signed swept area between $(X_s)_{s\le t}$ and $(X'_s)_{s\le t}$ as defined in the following Definition 3.2.1.

As previously, we directly obtain the estimate:

$$c_1(\rho_t^2 + |\zeta_t|) \le d_{cc}^2(\mathbb{B}_t,\mathbb{B}'_t) \le c_2(\rho_t^2+|\zeta_t|) \tag{3.41}$$

with c_1 and c_2 as in Proposition 3.2.9.

The continuous process A_t announced in this corollary can be defined as follows:

Definition 3.2.1. *Let $(X_t)_t$ and $(X'_t)_t$ be two Brownian motions on M_k such that X'_s is not in the cut locus of X_s for all $s \leq t$. We define the signed swept area A_t as the signed area of the oriented loop starting from X'_0, following:*

- *the path $s \mapsto X'_s, s \leq t$ from X'_0 to X'_t;*
- *the geodesic joining X'_t to X_t;*
- *the path $s \mapsto X_{t-s}, s \leq t$ joining X_t to X_0;*
- *finally the geodesic joining X_0 to X'_0.*

Note that this definition makes sense for any oriented Riemannian manifold. Here the sign of the quantity A_t changes when the paths of X'_t and X_t are crossing. Note also that A_t takes values in $\mathbb{R}$ whereas ζ_t takes values in $]-2\pi, 2\pi]$.

Proof of Corollary 3.2.10. Applying Proposition 3.2.9, we get the result for ρ_t. We also get $\zeta_t \equiv z'_t - z_t + \text{sign}(\theta_t - \theta'_t)\mathcal{A}_{X'_t,X_t,N_0} \mod (4\pi)$. In particular

$$\zeta_0 \equiv z'_0 - z_0 + \text{sign}(\theta_0 - \theta'_0)\mathcal{A}_{X'_t,X_t,N_0} \mod (4\pi).$$

To finish the proof Corollary 3.2.10, we then just need to remark that $(z'_t - z'_0) + \text{sign}(\theta_t - \theta'_t)\mathcal{A}_{X'_t,X_t,N_0} - (z_t - z_0) - \text{sign}(\theta_0 - \theta'_0)\mathcal{A}_{X'_0,X_0,N_0}$ is equal to A_t. □

Remark 3.2.11. *Using the polar coordinates and the pseudo norm 3.1 defined in Section 3.1.2, Proposition 3.2.9 and Proposition 3.2.10 also make sense (and are easier to obtain) for the Heisenberg group with the use of planar triangles.*

3.2.7 Stability of $z'_t - z_t$ under changes of coordinates

The results given in this subsection come from the contribution:

[24] Magalie Bénéfice. Non co-adapted couplings of Brownian motions on subRiemannian manifolds. `https://arxiv.org/abs/2312.14512`, 2023

As before we use the notations E_k, M_k and Π_k for $k \in \{-1, 1\}$.

In Corollary 3.2.10, the quantities ρ_t, ζ_t and A_t are intrinsic to $\mathbb{B}_t^{-1}\mathbb{B}_t$ and do not depend on the choice of the projection Π_k. As seen in this same proposition, this does not seem to be the case for the quantity $z'_t - z_t$ as the quantity $sign(\theta_t - \theta'_t)\mathcal{A}_{X_t,X'_t,N_0} \mod (4\pi)$ depends on the choice of the pole N_0 and the vector e_0. However we will need to study this non intrinsic quantity when we will define couplings (in particular non co-adapted couplings) of Brownian motions defined with spherical/polar coordinates. For various reasons we need to change the system of spherical/polar coordinates on M_k induced by Π_k, in the sense that we change the pole and the vector of reference. Thus it is interesting to see how $z'_t - z_t$ reacts. Let us consider $(X_t)_t$ and $(X'_t)_t$ as in Proposition 3.2.5. We chose $(N, e) \in TM_k$. We also introduce some notations:

- We denote by $(\varphi_t^{(N,e)}, \theta_t^{(N,e)})$ (resp. $(\varphi'^{(N,e)}_t, \theta'^{(N,e)}_t)$) the spherical/polar coordinates of X_t (resp. X'_t) relative to N and e.

- We denote by $I_t(N,e)$ (resp. $I'_t(N,e)$) the signed area swept by $(X_s)_{s\leq t}$ (resp. $(X'_s)_{s\leq t}$) relative to N and e and starting at point z_0 (resp. z'_0). More precisely, it is defined such that $(\varphi_t^{(N,e)}, \theta_t^{(N,e)}, I_t(N,e))_t$ satisfies the stochastic differential system (3.33).

In particular, we have $(\varphi_t, \theta_t) = (\varphi_t^{(N_0,e_0)}, \theta_t^{(N_0,e_0)})$ and $z_t = I_t(N_0,e_0)$. Note that $z_t \neq I_t(N,e)$ in general for $(N,e) \neq (N_0,e_0)$. Then we have the following results:

Lemma 3.2.12. *For all $(N,e) \in TM_k$,we have:*

$$\begin{aligned} A_t = I'_t(N,e) - I_t(N,e) - (z'_0 - z_0) + sign\left(\theta_t^{(N,e)} - \theta'^{(N,e)}_t\right)\mathcal{A}_{X_t,X'_t,N} \\ - sign\left(\theta_0^{(N,e)} - \theta'^{(N,e)}_0\right)\mathcal{A}_{X_0,X'_0,N}. \end{aligned} \tag{3.42}$$

In particular, we have:

$$\begin{aligned} \zeta_t \equiv I'_t(N,e) - I_t(N,e) + sign\left(\theta_t^{(N,e)} - \theta'^{(N,e)}_t\right)\mathcal{A}_{X_t,X'_t,N} \\ - sign\left(\theta_0^{(N,e)} - \theta'^{(N,e)}_0\right)\mathcal{A}_{X_0,X'_0,N} \\ + sign\left(\theta_0^{(N_0,e_0)} - \theta'^{(N_0,e_0)}_0\right)\mathcal{A}_{X_0,X'_0,N_0} \mod (4\pi). \end{aligned} \tag{3.43}$$

Proof. Relation (3.42) is an immediate geometric result. We look at the second relation (3.43). From Corollary 3.2.10, we have:

$$\zeta_t \equiv I'_t(N_0,e_0) - I_t(N_0,e_0) + \text{sign}\left(\theta_t^{(N_0,e_0)} - \theta'^{(N_0,e_0)}_t\right)\mathcal{A}_{X_t,X'_t,N_0} \mod (4\pi).$$

Then using (3.42) with N_0, we get:

$$\zeta_t \equiv z'_0 - z_0 + A_t + \text{sign}\left(\theta_0^{(N_0,e_0)} - \theta'^{(N_0,e_0)}_0\right)\mathcal{A}_{X_0,X'_0,N_0} \mod (4\pi).$$

Using (3.42) this time with N, we obtain the expected result. □

Remark 3.2.13. *If we have $X_0 = X'_0$ and $X_T = X'_T$, then for any $(N,e) \in TM_k$:*

$$\zeta_t \equiv I'_t(N,e) - I_t(N,e) \mod (4\pi).$$

Remark 3.2.14. *The impact of these change of coordinates can also be seen directly in E_k on the process $\mathbb{B}_t$. Let $(N,e) \in TM_k$. We claim that the process:*

$$J_t := \exp\left(\varphi_t^{(N,e)}\left(\cos(\theta_t^{(N,e)})X + \sin(\theta_t^{(N,e)})Y\right)\right)\exp\left(I_t(N,e)Z\right)$$

can be obtained from $\mathbb{B}_t$ by looking at $g^{-1}\mathbb{B}_t\exp(zZ)$ for g depending only on the choice (N,e) and z depending on N, e but also on X_0. Let explain this fact. Taking $g = (\varphi_g, \theta_g, z_g) \in E_k$, it is possible to prove that for all $h \in E_k$, $\Pi_k(g^{-1}h) = m_g\left(\Pi_k(h)\right)$ with m_g a direct isometry in M_k that can be decomposed as follows:

- *We first make a rotation of angle $-z_g$ and of axis directed by $(0,0,1)$ for $SU(2)$ (resp. of center N_0 for $SL(2,\mathbb{R})$). It keeps the pole N_0 invariant but acts on the vector of reference $e_0 = T_{I_2}\Pi_k(X)$;*

- *We then apply a direct isometry which acts by translation on the geodesic from $\Pi_k(g)$ to N_0. In particular this isometry transports the new vector obtained above parallelly along this geodesic. The vector obtained finally is equal to $T_g\Pi_k(\bar{X})$.*

Thus, for $(N,e) \in TM_k$, we can find $g \in E_k$ such that $\Pi_k(g) = N$ and $T_g\Pi_k(\bar{X}) = e$. For all t, $\Pi_k(g^{-1}\mathbb{B}_t)$ gives the polar coordinates of $\Pi_k(\mathbb{B}_t)$ relative to N and e, that is, $\left(\varphi_t^{(N,e)}, \theta_t^{(N,e)}\right)$. With similar results as in Proposition 3.2.9, we have $g^{-1}\mathbb{B}_t$ equal to:

$$\exp\left(\varphi_t^{(N,e)}\left(\cos(\theta_t^{(N,e)})X + \sin(\theta_t^{(N,e)})Y\right)\right) \times \exp\left(\left(z_t - z_g + sign\left(\theta_g - \theta_t^{(N_0,e_0)}\right)\mathcal{A}_{N,X_t,N_0}\right)Z\right).$$

Using some geometric comparisons, we can obtain:

$$z_t - z_g + sign\left(\theta_g - \theta_t^{(N_0,e_0)}\right)\mathcal{A}_{N,X_t,N_0} = I_t(N,e) - z_g + sign\left(\theta_g - \theta_0^{(N_0,e_0)}\right)\mathcal{A}_{N,X_0,N_0}.$$

The process J_t is then equal to:

$$J_t = g^{-1}\mathbb{B}_t \exp\left(\left(z_g - sign\left(\theta_g - \theta_0^{(N_0,e_0)}\right)\mathcal{A}_{N,X_0,N_0}\right)Z\right).$$

We now consider this change of coordinates for the two processes $(\mathbb{B}_t)_t$ and $(\mathbb{B}'_t)_t$ that we want to compare. As before, we define

$$J'_t := \exp\left({\varphi'}_t^{(N,e)}\left(\cos({\theta'}_t^{(N,e)})X + \sin({\theta'}_t^{(N,e)})Y\right)\right)\exp\left(I'_t(N,e)Z\right).$$

We have

$$J'_t = g^{-1}\mathbb{B}'_t \exp\left(\left(z_g - sign\left(\theta_g - {\theta'}_0^{(N_0,e_0)}\right)\mathcal{A}_{N,X'_0,N_0}\right)Z\right)$$

Here it seems evident that, in general $d_{cc}(\mathbb{B}_t, \mathbb{B}'_t) \neq d_{cc}(J_t, J'_t)$. However, if $X_0 = X'_0$ and $X_T = X'_T$, as $\exp(\alpha Z)$ and $\exp(\beta Z)$ commute for all α, $\beta \in \mathbb{R}$, we have $\mathbb{B}_T^{-1}\mathbb{B}'_T = \exp(-z_T Z)\exp(z'_T Z)$ and:

$$J_T^{-1}J'_T = \exp\left(\left(-z_g + sign\left(\theta_g - \theta_0^{(N_0,e_0)}\right)\mathcal{A}_{N,X_0,N_0}\right)Z\right)\exp(-z_T Z)\exp(z'_T Z)$$
$$\exp\left(\left(z_g - sign\left(\theta_g - {\theta_0}^{(N_0,e_0)}\right)\mathcal{A}_{N,X_0,N_0}\right)Z\right) = \exp(-z_T Z)\exp(z'_T Z).$$

Thus at time T, $d_{cc}(\mathbb{B}_T, \mathbb{B}'_T) = d_{cc}(J_T, J'_T)$. This gives an echo of Remark 3.2.13.
In general, by left invariance of the Carnot Carathéodory distance, we have $d_{cc}(\mathbb{B}_t, \mathbb{B}'_t) = d_{cc}(g^{-1}\mathbb{B}_t, g^{-1}\mathbb{B}'_t)$. In fact, $\left(g^{-1}\mathbb{B}_t\right)^{-1} g^{-1}\mathbb{B}'_t = \mathbb{B}_t^{-1}\mathbb{B}_t$ and thus, the third cylindrical coordinate of $\left(g^{-1}\mathbb{B}_t\right)^{-1} g^{-1}\mathbb{B}'_t$ is equal to ζ_t as defined in Corollary 3.2.10. Applying the same Corollary to $g^{-1}\mathbb{B}'_t$ and $g^{-1}\mathbb{B}_t$, we obtain the following equality modulo 4π:

$$\zeta_t = I'_t(N,e) - I_t(N,e) + sign\left(\theta_g - {\theta'_0}^{(N_0,e_0)}\right)\mathcal{A}_{N,X'_0,N_0} - sign\left(\theta_g - \theta_0^{(N_0,e_0)}\right)\mathcal{A}_{N,X_0,N_0} + sign\left(\theta_t^{(N,e)} - {\theta'}_t^{(N,e)}\right)\mathcal{A}_{m_g(X'_t),m_g(X_t),N_0}.$$

As we have:

$$sign\left(\theta_g - {\theta'_0}^{(N_0,e_0)}\right)\mathcal{A}_{N,X'_0,N_0} - sign\left(\theta_g - \theta_0^{(N_0,e_0)}\right)\mathcal{A}_{N,X_0,N_0}$$
$$= -sign\left(\theta_0^{(N_0,e_0)} - {\theta'_0}^{(N_0,e_0)}\right)\mathcal{A}_{X_0,X'_0,N} + sign\left(\theta_0^{(N_0,e_0)} - {\theta'_0}^{(N_0,e_0)}\right)\mathcal{A}_{X_0,X'_0,N_0},$$

and $\mathcal{A}_{m_g(X'_t),m_g(X_t),N_0} = \mathcal{A}_{X'_t,X_t,m_g^{-1}(N_0)}$ with $m_g^{-1}(N_0) = N$, we obtain the relation (3.43) from Lemma 3.2.12.

Chapter 4

Co-adapted couplings on $SU(2)$ and $SL(2,\mathbb{R})$

This chapter is devoted to the study of co-adapted couplings on $SU(2)$ and $SL(2,\mathbb{R})$. In particular, we construct a co-adapted successful couplings but only on $SU(2)$ as announced in Subsection 2.3.3. We recall that, as explained in Chapter 2 (Subsection 2.3.5), there is no co-adapted successful coupling on $SL(2,\mathbb{R})$.

These results were announced in :

[18] Magalie Bénéfice, Marc Arnaudon, and Michel Bonnefont. Couplings of Brownian motions on $SU(2,\mathbb{C})$. In *Geometric science of information. Part I*, volume 14071 of *Lecture Notes in Comput. Sci.*, pages 592–600. Springer, Cham, [2023] ©2023

and proven in :

[25] Magalie Bénéfice. Couplings of Brownian motions on $SU(2)$ and $SL(2,\mathbb{R})$. *Stochastic Processes and their Applications*, page 104434, 2024.

Please note that Section 3.2 and more precisely Subsection 3.2.6 are necessary to apprehend what follows. In all this chapter, to construct a coupling $(\mathbb{B}_t, \mathbb{B}'_t)$ of Brownian motions on the subRiemannian manifold $SU(2)$ (resp. $SL(2,\mathbb{R})$), we just construct a coupling (X_t, Y_t) of Brownian motions on the Riemannian manifold S^2 (resp. $\mathbf{H}^2$) and consider its lifting on $SU(2)$ (resp. $SL(2,\mathbb{R})$) as in Remark 3.2.6. According to Corollary 3.2.10, to compare $(\mathbb{B}_t)_t$ and $(\mathbb{B}'_t)_t$, we just need to study the processes R_t and A_t with $R_t := \rho(X_t, Y_t)$ the Riemannian distance between X_t and Y_t and with A_t the area swept by $(X_s, Y_s)_{s\leq t}$ as described in Definition 3.2.1. To do that, we make good use of the Itô depiction of the Brownian motions $(X_t)_t$ and $(Y_t)_t$ in a moving frame as in the first strategy described in Subsection 2.3.2.

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4.1 Couplings on the simply connected Riemannian surfaces M_k of constant curvature k and variation of the swept area

In this section we want to describe a model of co-adapted coupling for every simply connected Riemannian surface endowed with the Riemannian metric of constant sectional curvature k. By the Uniformization Theorem, we can reduce our study to one of the following spaces (endowed with the usual Riemannian metric):

- if $k = 0$, $M_0 = \mathbb{R}^2$;
- if $k > 0$, M_k is the sphere of dimension 2 and of radius $\frac{1}{k}$;
- if $k < 0$, M_k is the hyperbolic space of sectional curvature k.

In particular for $M_1 = S^2$, $M_{-1} = \mathbf{H}^2$ and even $M_0 = \mathbb{R}^2$, this will induce a model of co-adapted couplings on $SU(2)$, $SL(2, \mathbb{R})$ and the Heisenberg group $\mathbb{H}$ respectively.

In the remainder we denote by $i(M_k)$ the injectivity radius of M_k. Note that $i(M_k) = +\infty$ if $k \leq 0$ and $i(M_k) = \frac{\pi}{\sqrt{k}}$ if $k > 0$.

4.1.1 Itô depiction and co-adapted couplings for the distance and the swept area

We first give a formal definition of a co-adapted coupling as stated in [39].

Definition 4.1.1. *Let consider two continuous Markov processes $(X_t)_t$ and $(Y_t)_t$ on a manifold M. The process $(\tilde{X}_t, \tilde{Y}_t)_t$ is said a co-adapted coupling of $(X_t)_t$ and $(Y_t)_t$ if there exists a filtered probability space $(\mathcal{F}_t)_t$ such that:*

- *the processes $(\tilde{X}_t)_t$ and $(\tilde{Y}_t)_t$ are both adapted to $(\mathcal{F}_t)_t$;*
- *for all bounded measurable function f, $z \in M$, s, $t > 0$, the functions $z \mapsto \mathbb{E}[f(\tilde{X}_{t+s}) \mid \mathcal{F}_s,\ \tilde{X}_s = z]$ and $z \mapsto \mathbb{E}[f(X_{t+s}) \mid X_s = z]$ (resp. $z \mapsto \mathbb{E}[f(\tilde{Y}_{t+s}) \mid \mathcal{F}_s,\ \tilde{Y}_s = z]$ and $z \mapsto \mathbb{E}[f(Y_{t+s}) \mid Y_s = z]$) are equal $\mathbb{P}_{X_s}$-a.s. (resp. $\mathbb{P}_{Y_s}$-a.s.).*

In fact, this means that the stochastic dynamic of $(\tilde{X}_{s+t})_s$ agree with the one from $(X_{s+t})_s$ even when the past behaviour of $(Y_u)_{u \leq t}$ is taken into account.

In particular, a coupling of Brownian motions $(X_t, Y_t)_t$ is co-adapted if there exists a filtration $(\mathcal{F}_t)_t$ such that $(X_t)_t$ (resp. $(Y_t)_t$) is a Brownian motion for the filtration $(\mathcal{F}_t)_t$.

As explained previously, we are going to use the Itô depiction of a Brownian motion in a frame in the sense of Emery (see [30] for a basic introduction) to describe a large class of co-adapted couplings in M_k.

Proposition 4.1.1. *We consider $(X_t)_t$ a semi-martingale on M_k and $(e_1(t), e_2(t))_t$ a continuous semi-martingale (adapted to the same filtration than $(X_t)_t$) in the orthonormal frame bundle $\mathcal{O}M_k$ above $(X_t)_t$. If $(X_t)_t$ is a Brownian motion on M_k, there exists (U_1, U_2) a Brownian motion on $\mathbb{R}^2$ such that:*

$$d^\nabla X_t = dU_1(t)e_1(t) + dU_2(t)e_2(t). \tag{4.1}$$

Conversely, if X_t satisfies (4.1) with (U_1, U_2) a Brownian motion on $\mathbb{R}^2$, then X_t is a Brownian motion on M_k.

Even if the proof of this Proposition can be found in the literature (see [30]), we give it here for the reader.

Proof. Let us begin with the direct implication. For $i \in \{1, 2\}$, taking U_i such that $dU_i(t) = \langle e_i^*(t), d^\nabla X_t\rangle$, with (e_1^*, e_2^*) the dual basis of (e_1, e_2) in T^*M_k, we directly get that U_i are martingales. Moreover, as X_t is a Brownian motion and $(e_1, e_2)(t)$ forms an orthonormal basis of $T_{X_t}M_k$, for $i, j \in \{1, 2\}$:

$$dU_i(t) \cdot dU_j(t) = (e_i^* \otimes e_j^*)(t)(dX_t, dX_t) = \mathrm{Tr}(e_i^* \otimes e_j^*)(t)dt = \delta_{i,j}dt.$$

We obtain the expected result.

For the converse implication, as

$$\int_0^\cdot \langle \phi(X_t), d^\nabla X_t\rangle = \int_0^\cdot \langle \phi(X_t), e_1^*(t)\rangle dU_1(t) + \int_0^\cdot \langle \phi(X_t), e_2^*(t)\rangle dU_2(t)$$

is a martingale for all $\phi \in \Gamma(T^*M_k)$, we obtain that $(X_t)_t$ is a martingale. Then for f a smooth function on M_k, by Itô's formula, we get:

$$d(f(X_t)) = df(X_t) \cdot d^\nabla X_t + \frac{1}{2}\,\mathrm{Hess}(f)(X_t)(d^\nabla X_t, d^\nabla X_t)$$

with:

$$\begin{aligned}
\mathrm{Hess}(f)(X_t)(d^\nabla X_t, d^\nabla X_t) &= \sum_{i,j} \langle \mathrm{Hess}(f)(X_t), (e_i^* \otimes e_j^*)(t)\rangle (e_i^* \otimes e_j^*)(t)(d^\nabla X_t, d^\nabla X_t) \\
&= \sum_{i,j} \langle \mathrm{Hess}(f)(X_t), (e_i^* \otimes e_j^*)(t)\rangle \langle e_i^*(t), d^\nabla X_t\rangle \cdot \langle e_j^*(t), d^\nabla X_t\rangle \\
&= \sum_{i,j} \langle \mathrm{Hess}(f)(X_t), (e_i^* \otimes e_j^*)(t)\rangle dU_i(t) \cdot dU_j(t) \\
&= \mathrm{Tr}(\mathrm{Hess}(f))(X_t)dt = \Delta f(X_t)dt.
\end{aligned}$$

Thus, $(X_t)_t$ is a Brownian motion on M_k. □

Let us go back to the coupling problem. Using Itô depiction, two Brownian motions $(X_t)_t$ and $(Y_t)_t$ on the Riemannian manifold M_k can be described by the equations

$$d^\nabla X_t = dU_1(t)e_1^X(t) + dU_2(t)e_2^X(t) \text{ and } d^\nabla Y_t = dV_1(t)e_1^Y(t) + dV_2(t)e_2^Y(t) \tag{4.2}$$

with:

- $U(t) := (U_1(t), U_2(t))$ and $V(t) := (V_1(t), V_2(t))$ two Brownian motions in $\mathbb{R}^2$;
- $e^X(t) := (e_1^X(t), e_2^X(t))$ a continuous semi-martingale (adapted to the same filtration than $(X_t)_t$) in the orthonormal frame bundle $\mathcal{O}M_k$ above $(X_t)_t$.
- $e^Y(t) := (e_1^Y(t), e_2^Y(t))$ a continuous semi-martingale (adapted to the same filtration than $(Y_t)_t$) in the orthonormal frame bundle $\mathcal{O}M_k$ above $(Y_t)_t$.

Thus a coupling (X_t, Y_t) is characterized by its starting points and the joint law of

$$\left(\left(U(t), e^X(t)\right), \left(V(t), e^Y(t)\right)\right)_t.$$

By choosing a co-adapted coupling of $\left(\left(U(t), e^X(t)\right), \left(V(t), e^Y(t)\right)\right)_t$, we can describe and study a wide range of co-adapted couplings $(X_t, Y_t)_t$ on M_k.

In the next subsection, we compute the first and second order derivatives of the signed swept area between two smooth curves in some specific basis of the tangent bundle (see Lemma 4.1.3). This will allow us to obtain a general stochastic equation for A_t in the co-adapted coupling described above. Using some well-known similar results about the distance in M_k, we will also obtain a stochastic equation for R_t (Lemma 4.1.2).

4.1.2 First and second order derivatives

Let x, $y \in M_k$ such that $0 < \rho(x, y) < i(M_k)$. We can define:

- $e_1^x(x, y) = \frac{\exp_x^{-1}(y)}{\rho(x,y)} \in T_xM_k$
- $e_1^y(x, y)$ the parallel transport of e_1^x along the geodesic joining x and y;
- $e_2^x(x, y) \in T_xM_k$ such that $(e_1^x(x, y), e_2^x(x, y))$ is an orthonormal positive basis of T_xM_k;
- $e_2^y(x, y) \in T_yM_k$ such that $(e_1^y(x, y), e_2^y(x, y))$ is an orthonormal positive basis of T_yM_k.

We first give a local expression of the oriented swept area between some paths on M_k which coincides with the one used to define the process $(A_t)_t$ in Definition 3.2.1. We consider:

- γ_x, $\gamma_y : [0, +\infty[\mapsto M_k$ two curves $\mathcal{C}^1$ on M_k starting at x and y respectively;
- $(s, t) \in [0, 1] \times [0, +\infty[\mapsto c(s, t) \in M_k$ such that for all $t \geq 0$, $s \mapsto c(s, t)$ is a geodesic starting at $\gamma_x(t)$ and ending at $\gamma_y(t)$.

- Provided that $0 < \rho(\gamma_x(t), \gamma_y(t)) < i(M_k)$ we can define $e_1(s,t) := \frac{\partial_s c(s,t)}{\rho(\gamma_x(t),\gamma_y(t))}$ and $e_2(s,t)$ such that $(e_1(s,t), e_2(s,t))$ is an orthonormal positive basis of $T_{c(s,t)}M_k$ for each s,t. In particular $e_1(0,0) = e_1^x(x,y)$ and $e_1(1,0) = e_1^y(x,y)$.

Considering the positive local chart (x^1, x^2) induced by the parametrization $(s,\varsigma) \mapsto c(s,\varsigma)$ on M_k, the volume form is given by $\sqrt{\det(G)}dx^1 \wedge dx^2$ where $\mathbb{G}$ is the positive definite symmetric matrix representing the metric in these local coordinates. In particular, we have:

$$\det(G) = ||\partial_s c(s,\varsigma)||^2 ||\partial_\varsigma c(s,\varsigma)||^2 - \langle \partial_s c(s,\varsigma), \partial_\varsigma c(s,\varsigma)\rangle^2 = \det\left(\partial_s c(s,\varsigma), \partial_\varsigma c(s,\varsigma)\right)^2$$

where the determinant is calculated in the basis $(e_1(s,\varsigma), e_2(s,\varsigma))$ of $T_{c(s,\varsigma)}M$ for each ς, t. Note that $x_1(c(s,\varsigma)) = s$ if and only if $\det\left(\partial_s c(s,\varsigma), \partial_\varsigma c(s,\varsigma)\right) > 0$. Then, the signed swept area can be locally defined by:

$$\mathcal{A}(t) := \int_0^1 \int_0^t \det(\partial_s c(s,\varsigma), \partial_\varsigma c(s,\varsigma)) d\varsigma ds. \tag{4.3}$$

In particular, there exists a vector field V (that will be described just after in Lemma 4.1.3) on $M_k \times M_k$ such that:

$$\mathcal{A}(t) = \int_0^t \left\langle V_{(\gamma_1(\varsigma),\gamma_2(\varsigma))}, (\dot\gamma_1(\varsigma), \dot\gamma_2(\varsigma))\right\rangle d\varsigma. \tag{4.4}$$

The following lemma gives the expressions of the first order derivative and the Hessian of the distance using the basis $(e_1^x(x,y), e_2^x(x,y))$ and $(e_1^y(x,y), e_2^y(x,y))$ defined above. Even though this result is well known, we will give its proof later in this subsection as it uses the same elements as to prove Lemma 4.1.3.

Lemma 4.1.2. *Let x, $y \in M_k$, $r = \rho(x,y)$ with $0 < r < i(M_k)$, $u \in T_xM_k$ and $v \in T_yM_k$. Then, we have:*

$$d\rho_{(x,y)}(u,v) = v_1 - u_1$$

and

$$\mathrm{Hess}(\rho)_{(x,y)}((u,v),(u,v)) = \sqrt{k}(u_2^2 + v_2^2)\cot(\sqrt{k}r) - 2\sqrt{k}u_2v_2\frac{1}{\sin(\sqrt{k}r)}.$$

with $u_i = \langle u, e_i^x(x,y)\rangle$ and $v_i = \langle v, e_i^y(x,y)\rangle$ for $i \in \{1,2\}$.

We also give a result for what could be seen as the first order derivative and hessian of the signed swept area (if it was a function on $M_k \times M_k$). As far as the author knows, this has never been computed before.

Lemma 4.1.3. *With the same hypothesis and notations than in Lemma 4.1.2 and V defined as in (4.4), we have:*

$$\left\langle V_{(x,y)}, (u,v)\right\rangle = \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}r}{2}\right)(u_2 + v_2),$$

$$\left\langle \nabla_{(u,v)}V_{(x,y)}, (u,v)\right\rangle = \frac{u_2v_1 - v_2u_1}{\cos^2(\frac{\sqrt{k}r}{2})} + \tan^2\left(\frac{\sqrt{k}r}{2}\right)(v_2v_1 - u_2u_1).$$

Note that the above results are given as general formulas for the three cases $k > 0$, $k < 0$ and $k = 0$ (taking the limits when k tends to 0 in the last case).

To prove these two lemmas, we need to evaluate in zero the first and second derivatives of the functions $r : t \mapsto \rho(\gamma_x(t), \gamma_y(t))$ and $\mathcal{A}$ with $\gamma_x(t) := \exp_x(tu)$ and $\gamma_y(t) := \exp_y(tv)$. We use the notations $c(s,t)$, $e_1(s,t)$, $e_2(s,t)$ to denote the same objects as previously. In particular, as $0 < \rho(x,y) < i(M_k)$, there exists $t_0 > 0$ such that $(e_1(s,t), e_2(s,t))$ is well defined for $t \in [0, t_0]$.

Let define $s \mapsto J(s,t) := \partial_t c(s,t)$. With this choice of γ_x and γ_y, $s \mapsto J(s,t) := \partial_t c(s,t)$ is a Jacobi field for all $t \geq 0$. For $i \in \{1,2\}$, we denote:

$$u_i(t) := \langle J(0,t), e_i(0,t)\rangle,\ u_i := u_i(0) \text{ and } v_i(t) := \langle J(1,t), e_i(1,t)\rangle,\ v_i := v_i(0).$$

In the basis $(e_1(s,t), e_2(s,t))$ we have the following decomposition of this Jacobi field J:

Lemma 4.1.4. *With the above notations and hypothesis we can write $J(s,t) = j_1(s,t)e_1(s,t) + j_2(s,t)e_2(s,t)$ with:*

$$j_1(s,t) = u_1(t)(1-s) + v_1(t)s; \tag{4.5}$$

$$j_2(s,t) = u_2(t)\cos(\sqrt{k}r(t)s) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\sin(\sqrt{k}r(t)s); \tag{4.6}$$

$$u_2'(0) = \sqrt{k}\frac{u_2\cos(\sqrt{k}r) - v_2}{\sin(\sqrt{k}r)}u_1;\ v_2'(0) = \sqrt{k}\frac{u_2 - v_2\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}v_1. \tag{4.7}$$

Note that for all the following computations, we will use the notations ∇_s for $\nabla_{\partial_s c(s,t)}$ and ∇_t for $\nabla_{\partial_t c(s,t)}$. This lemma is the main result to prove Lemma 4.1.2 and Lemma 4.1.3.

Proof of Lemma 4.1.4. For t constant, $s \mapsto e_i(s,t)$ is defined by parallel transport along $s \mapsto c(s,t)$. Thus $\nabla_s e_i(s,t) = 0$ and we have:

$$\begin{aligned}\nabla_s J(s,t) &= \partial_s j_1(s,t)e_1(s,t) + \partial_s j_2(s,t)e_2(s,t)\\ \text{and } \nabla_s\nabla_s J(s,t) &= \partial_{ss}^2 j_1(s,t)e_1(s,t) + \partial_{ss}^2 j_2(s,t)e_2(s,t).\end{aligned}$$

By definition of the Jacobi fields, we also have:

$$\begin{aligned}\nabla_s\nabla_s J(s,t) &= -R(J(s,t), \partial_s c(s,t))\partial_s c(s,t)\\ &= -r(t)^2(j_1(s,t)R(e_1(s,t), e_1(s,t))e_1(s,t) + j_2(s,t)R(e_2(s,t), e_1(s,t))e_1(s,t))\\ &= -kr(t)^2 j_2(s,t)e_2(s,t).\end{aligned}$$

By solving the differential equations

$$\begin{cases}\partial_{ss}^2 j_1(s,t) = 0\\ j_1(0,t) = u_1(t)\\ j_1(1,t) = v_1(t)\end{cases} \text{ and } \begin{cases}\partial_{ss}^2 j_2(s,t) = -kr(t)^2 j_2(s,t)\\ j_2(0,t) = u_2(t)\\ j_2(1,t) = v_2(t)\end{cases}$$

we obtain (4.5) and (4.6).
We now just have to compute $u_2'(0)$ and $v_2'(0)$. As $u_2(t) = \langle \partial_t c(0,t), e_2(0,t)\rangle$ and $t \mapsto c(0,t)$ is a geodesic, we have:

$$u_2'(t) = \langle \nabla_t \partial_t c(0,t), e_2(0,t)\rangle + \langle \partial_t c(0,t), \nabla_t e_2(0,t)\rangle = \langle \partial_t c(0,t), \nabla_t e_2(0,t)\rangle.$$

Thus, $u_2'(0) = \langle u, \nabla_t e_2(0,t)_{|t=0}\rangle$ and, in the same way, we have $v_2'(0) = \langle v, \nabla_t e_2(1,t)_{|t=0}\rangle$.

Note that, as $\langle e_2(s,t), e_2(s,t)\rangle = 1$ and $\langle e_2(s,t), e_1(s,t)\rangle = 0$, we have:

$$\langle \nabla_t e_2(s,t), e_2(s,t)\rangle = \frac{1}{2}\partial_t \langle e_2(s,t), e_2(s,t)\rangle = 0 \text{ and}$$
$$\langle \nabla_t e_2(s,t), e_1(s,t)\rangle = \partial_t(\langle e_2(s,t), e_1(s,t)\rangle) - \langle e_2(s,t), \nabla_t e_1(s,t)\rangle = -\langle e_2(s,t), \nabla_t e_1(s,t)\rangle.$$

Then, $\nabla_t e_2(s,t) = -\langle \nabla_t e_1(s,t), e_2(s,t)\rangle e_1(s,t)$. We have:

$$\begin{aligned}\nabla_t e_1(s,t) &= \nabla_t\left(\partial_s c(s,t) \times \frac{1}{r(t)}\right) = -\frac{r'(t)}{r(t)^2}\partial_s c(s,t) + \nabla_t(\partial_s c(s,t)) \times \frac{1}{r(t)} \\ &= -\frac{r'(t)}{r(t)} e_1(s,t) + \frac{\nabla_s J(s,t)}{r(t)} \text{ since } \nabla \text{ is torsion free.}\end{aligned}$$

Thus:

$$\begin{aligned}\langle \nabla_t e_1(s,t), e_2(s,t)\rangle &= \frac{1}{r(t)}\partial_s j_2(s,t) \\ &= \sqrt{k}\left(-u_2(t)\sin(\sqrt{k}r(t)s) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\cos(\sqrt{k}r(t)s)\right).\end{aligned} \tag{4.8}$$

We obtain:

$$\nabla_t e_2(0,t) = -\sqrt{k}\frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))} e_1(0,t)$$
$$\text{and } \nabla_t e_2(1,t) = -\sqrt{k}\frac{v_2(t)\cos(\sqrt{k}r(t)) - u_2(t)}{\sin(\sqrt{k}r(t))} e_1(1,t).$$

This gives (4.7).

□

We can now give the proofs of the two Lemmas.

Proof of Lemma 4.1.2. The general formulas for $L'(0)$ and $L''(0)$ are classical results that can be found in numerous books. As an example they are given in *Lemma 6.1.1* and *Theorem 6.1.1* in [36]. With our notations, we obtain:

$$r'(0) = \left(\int_0^1 \frac{\frac{d}{ds}\langle J(s,t), r(t)e_1(s,t)\rangle}{r(t)} - \frac{\langle J(s,t), r(t)\nabla_s e_1(s,t)\rangle}{r(t)} ds\right)_{|t=0} \tag{4.9}$$

$$\begin{aligned}r''(0) = &\frac{1}{r}\left(\int_0^1 \partial_s j_2(s,t)^2 - r(t)^2 j_2(s,t)^2 \langle R(e_1(s,t), e_2(s,t))e_2(s,t), e_1(s,t)\rangle\, ds\right)_{|t=0} \\ &- \frac{1}{r}\left[\langle \nabla_t \partial_t c(s,t), r(t)e_1(s,t)\rangle_{t=0}\right]_0^1.\end{aligned} \tag{4.10}$$

In particular, as said before, $\nabla_s e_1(s,0) = 0$, thus $r'(0) = j_1(1,0) - j_1(0,0) = v_1 - u_1$.
We now look at $r''(0)$. We note that $t \mapsto c(s,t)$ is a geodesic for $s = 0$ and $s = 1$. Thus, we have $\nabla_t \partial_t c(s,t) = 0$. As $(e_1(s,t), e_2(s,t))$ is an orthonormal basis,

$$\langle R(e_1(s,t), e_2(s,t))e_2(s,t), e_1(s,t)\rangle = k.$$

Thus, $r''(0) = \frac{1}{r}\int_0^1 \partial_s j_2(s,0)^2 - kr^2 j_2(s,0)^2 ds$. Using, (4.6) and (4.8) evaluated in $t=0$, we get:

$$r''(0) = kr\int_0^1 \left(\sin^2(\sqrt{k}rs) - \cos^2(\sqrt{k}rs)\right)\left(u_2^2 - \frac{(v_2 - u_2\cos(\sqrt{k}r))^2}{\sin^2(\sqrt{k}r)}\right)$$
$$- 4u_2\frac{v_2 - u_2\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}\cos(\sqrt{k}rs)\sin(\sqrt{k}rs)ds.$$

By integrating we finally obtain $r''(0) = \sqrt{k}\left((u_2^2+v_2^2)\cot(\sqrt{k}r) - 2\frac{u_2v_2}{\sin(\sqrt{k}r)}\right)$. □

Proof of Lemma 4.1.3. We first explain why we just need to compute $\mathcal{A}'(0)$ and $\mathcal{A}''(0)$. As γ_x and γ_y are chosen to be two geodesics, we have $\nabla_t(\dot\gamma_x(t),\dot\gamma_y(t)) = 0$. Using relation (4.4), we then obtain:

$$\mathcal{A}'(t) = \langle V_{(\gamma_1(t),\gamma_2(t))}, (\dot\gamma_1(t),\dot\gamma_2(t))\rangle \text{ and } \mathcal{A}''(t) = \langle \nabla_{(\dot\gamma_1(t),\dot\gamma_2(t))}V_{(\gamma_1(t),\gamma_2(t))}, (\dot\gamma_1(t),\dot\gamma_2(t))\rangle.$$

For $t=0$, we obtain $\left\langle V_{(x,y)}, (u,v)\right\rangle$ and $\left\langle \nabla_{(u,v)}V_{(x,y)}, (u,v)\right\rangle$.

We now use relation (4.3) to compute $\mathcal{A}'(0)$ and $\mathcal{A}''(0)$. We have:

$$\mathcal{A}'(t) = \int_0^1 \det(\partial_s c(s,t), \partial_t c(s,t))ds = \int_0^1 \det(r(t)e_1(s,t), J(s,t))ds = r(t)\int_0^1 j_2(s,t)ds$$
$$= r(t)\left(\int_0^1 u_2(t)\cos(\sqrt{k}r(t)s) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\sin(\sqrt{k}r(t)s)ds\right)$$
$$= (u_2(t)+v_2(t))\frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}r(t)}{2}\right).$$

We obtain the first expected result. This also leads to:

$$\mathcal{A}''(t) = (u_2'(t)+v_2'(t))\frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}r(t)}{2}\right) + (u_2(t)+v_2(t))\frac{r'(t)}{2\cos^2(\frac{\sqrt{k}r(t)}{2})}.$$

Note that $r'(0)$ has been calculated for the proof of Lemma 4.1.2, so:

$$\mathcal{A}''(0) = (u_2'(0)+v_2'(0))\frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}r}{2}\right) + (u_2+v_2)\frac{v_1-u_1}{2\cos^2(\frac{\sqrt{k}r}{2})}.$$

Finally:

$$\mathcal{A}''(0) = \left(\frac{u_2\cos(\sqrt{k}r) - v_2}{\sin(\sqrt{k}r)}u_1 + \frac{u_2 - v_2\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}v_1\right)\tan\left(\frac{\sqrt{k}r}{2}\right) + (u_2+v_2)\frac{v_1-u_1}{2\cos^2(\frac{\sqrt{k}r}{2})}$$
$$= -(u_2u_1 - v_2v_1)\tan^2\left(\frac{\sqrt{k}r}{2}\right) + (u_2v_1 - v_2u_1)\frac{1}{\cos^2(\frac{\sqrt{k}r}{2})}.$$

□

4.1.3 Description of the model for co-adapted couplings on M_k

Using the model from Subsection 4.1.1 with the orthonormal frame used in Subsection 4.1.2, we obtain the relations announced in Subsection 2.3.2: Let M_k, $i(M_k)$ and ρ be defined as above.

Proposition 4.1.5. *Let us consider $x, y \in M_k$ such that $0 < \rho(x,y) < i(M_k)$ and $(U(t), V(t))_t$ a co-adapted coupling of Brownian motions on $\mathbb{R}^2$.*
For $t \in [0,T]$, we construct $(X_t)_t$ and $(Y_t)_t$ two processes on M_k starting from x and y with $T := \inf\{t > 0 \mid \rho(X_t, Y_t) = 0\}$ and such that $(X_t)_t$ and $(Y_t)_t$ satisfy (2.33) with:

- $e_1^X(t) = \frac{\exp_{X_t}^{-1}(Y_t)}{\rho(X_t,Y_t)}$ *a unitary vector on $T_{X_t}M_k$;*
- $e_2^X(t)$ *such that $(e_1^X(t), e_2^X(t))$ is a positive orthonormal basis on $T_{X_t}M_k$;*
- $(e_1^Y(t), e_2^Y(t))$ *the parallel transport of $(e_1^X(t), e_2^X(t))$ along the geodesic joining X_t and Y_t (this defines a positive orthonormal basis on $T_{Y_t}M_k$).*

Then $(X_t, Y_t)_t$ is a co-adapted coupling of Brownian motions.

Let denote by R_t the Riemannian distance $\rho(X_t, Y_t)$ and by A_t the area swept by $(X_s, Y_s)_{s \leq t}$ as in Corollary 3.2.10. Then $(R_t)_t$ and $(A_t)_t$ satisfy:

$$dR_t = dV_1(t) - dU_1(t) + \sqrt{k}\cot(\sqrt{k}R_t)dt - \frac{\sqrt{k}}{\sin(\sqrt{k}R_t)}dU_2(t)\cdot dV_2(t) \tag{4.11}$$

$$dA_t = \frac{\tan(\frac{\sqrt{k}R_t}{2})}{\sqrt{k}}(dU_2(t) + dV_2(t)) + \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}(dU_2 \cdot dV_1(t) - dV_2(t)\cdot dU_1(t)) \tag{4.12}$$

$$dR_t \cdot dA_t = \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}R_t}{2}\right)(dV_1(t)\cdot dU_2(t) - dU_1(t)\cdot dV_2(t)) \tag{4.13}$$

with $dU_i(t)\cdot dV_j(t)$ denoting the derivative of the joint quadratic variation of U_i and V_j.

Note that, if k tends to 0, we obtain some well known relations for the Heisenberg group.

Remark 4.1.6. *Let denote by $(\mathbb{B}_t, \mathbb{B}'_t)_t$ the coupling of Brownian motion induced (by lifting) by the above construction on $SU(2)$, $SL(2,\mathbb{R})$ and $\mathbb{H}$. Using Proposition 4.1.5 together with Corollary 3.2.10, we have a way to obtain an estimate of $d_{cc}(\mathbb{B}_t, \mathbb{B}'_t)$.*

Proof of Proposition 4.1.5. As we define the processes until the first time T where $R_t \in \{0, i(M_k)\}$, $e_1^X(t)$ and $e_1^Y(t)$ are well defined on $[0,T[$. By construction U, V, e_1^X, e_2^X, e_1^Y and e_2^Y, X and Y are adapted to the same filtration. Thus the processes $(X_t)_{0\leq t\leq T}$ and $(Y_t)_{0\leq t\leq T}$ are Brownian motions on M_k and we obtain a co-adapted coupling.

To deal with the second part of the proof, we just need to notice that $e_i^X(t) = e_i^{X_t}(X_t, Y_t)$ and $e_i^Y(t) = e_i^{Y_t}(X_t, Y_t)$ for all $i \in \{1,2\}$. We obtain Equation (4.11) by using Lemma 4.1.2 and Itô's formula:

$$dR_t = d\rho_{(X_t,Y_t)}(d^\nabla X_t, d^\nabla Y_t) + \frac{1}{2}\mathrm{Hess}(\rho)(X_t,Y_t)((d^\nabla X_t, d^\nabla Y_t),(d^\nabla X_t, d^\nabla Y_t))$$

We now deal with Equation (4.12). As (4.4) is true for every $\mathcal{C}^1$ paths γ_1 and γ_2, we have:

$$\begin{aligned} A_t &= \int_0^t \left\langle V_{(X_\varsigma,Y_\varsigma)}, \circ d(X_\varsigma, Y_\varsigma) \right\rangle \text{ with } \circ\, d \text{ the Stratonovich derivative} \\ &= \int_0^t \left\langle V_{(X_\varsigma,Y_\varsigma)}, d^\nabla(X_\varsigma, Y_\varsigma) \right\rangle + \frac{1}{2}\int_0^t \left\langle \nabla_{d^\nabla(X_\varsigma,Y_\varsigma)} V_{(X_\varsigma,Y_\varsigma)}, d^\nabla(X_\varsigma, Y_\varsigma) \right\rangle. \end{aligned}$$

Using Lemma 4.1.3, we obtain the expected result. The last relation in Proposition 4.1.5 is immediate. □

Remark 4.1.7. *In fact, with some computations similar to the ones from Subsection 4.1.2, we can describe the covariant derivative of $(e_i^X(t))_t$ (resp. $(e_i^Y(t))_t$) along $(X_t)_t$ (resp. $(Y_t)_t$):*

$$\begin{cases} D_t e_1^X(t) = \sqrt{k}\left(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right) e_2^X(t) - \frac{k}{2}\left(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_1^X(t) \\ D_t e_2^X(t) = \sqrt{k}\left(\frac{-dV_2(t) + dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right) e_1^X(t) - \frac{k}{2}\left(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_2^X(t) \end{cases}$$

and

$$\begin{cases} D_t e_1^Y(t) = -\sqrt{k}\left(\frac{dU_2(t) - dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right) e_2^Y(t) - \frac{k}{2}\left(\frac{dU_2(t) - dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_1^Y(t) \\ D_t e_2^Y(t) = -\sqrt{k}\left(\frac{-dU_2(t) + dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right) e_1^Y(t) - \frac{k}{2}\left(\frac{dU_2(t) - dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_2^Y(t) \end{cases}$$

See Appendix B for a proof.

*In particular, we can define our moving frames as the solutions of a system of stochastic differential equations depending on U_1, U_2, V_1, V_2 and R_t. In the case where **no singularities appear in this system**, our coupling keeps sense even if $R_t \in \{0, i(M_k)\}$ (if this is the case for $t = 0$, we just need to chose an arbitrary value for $e_1^X(0)$). Then we can consider R_t as a signed distance, $Y_t = exp_{X_t}(R_t e^X(t))$ and the results of Proposition 4.1.5 are still true.*

In $\mathbb{R}^2$, there exists a general method to construct co-adapted couplings. The proof of the following proposition can be found in [38] for example.

Proposition 4.1.8. *We consider the filtered probability space $(\Omega, (\mathcal{F}_t)_t, \mathbb{P})$. The following assertions are equivalent:*

(i) (U, V) is a co-adapted coupling of Brownian motions in $\mathbb{R}^2$;

(ii) Enlarging the filtration if needed, there exists a two dimensional Brownian motion W adapted to the filtration and independent of U, and $K(t)$, $\hat{K}(t) \in M_{2,2}(\mathbb{R})$ with $K(t)K(t)^T + \hat{K}(t)\hat{K}(t)^T = I_2$, $K(t)$, $\hat{K}(t) \in \mathcal{F}_t$ such that :

$$dV(t) = K(t)dU(t) + \hat{K}(t)dW(t). \tag{4.14}$$

Using (4.14), we can rewrite our relations for R_t and A_t:

Proposition 4.1.9. *For a co-adapted coupling satisfying Proposition 4.1.5 and relation (4.14), we have:*

$$dR_t \cdot dR_t = 2(1 - K_{1,1}(t))dt;$$
$$dR_t \overset{(m)}{=} \sqrt{k}\cot(\sqrt{k}R_t)dt - \frac{\sqrt{k}}{\sin(\sqrt{k}R_t)}K_{2,2}(t)dt = \sqrt{k}\frac{\cos(\sqrt{k}R_t) - K_{2,2}(t)}{\sin(\sqrt{k}R_t)}dt;$$
$$dA_t \cdot dA_t = 2\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}(1 + K_{2,2}(t))dt;$$
$$dA_t \overset{(m)}{=} \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}(K_{1,2}(t) - K_{2,1}(t));$$
$$dR_t \cdot dA_t = \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}R_t}{2}\right)(K_{1,2}(t) - K_{2,1}(t))dt.$$

4.1.4 Some examples of co-adapted couplings on the Riemannian manifold M_k

In this subsection, we give some examples of co-adapted couplings on M_k constructed by using Proposition 4.1.8. As explained in the Proposition 3.2.5, all these constructions induce co-adapted couplings of Brownian motions on the subRiemannian manifolds $\mathbb{H}$, $SU(2)$ and $SL(2,\mathbb{R})$.

Example 4.1.10 (Synchronous coupling)**.**

We take $K(t) = I_2$, $\hat{K}(t) = 0$, $dV(t) = dU(t)$. We get:

$$dR_t \cdot dR_t = 0;$$
$$Drift(dR_t) = \sqrt{k}\frac{\cos(\sqrt{k}R_t) - 1}{\sin(\sqrt{k}R_t)}dt = -\sqrt{k} \times \tan\left(\frac{\sqrt{k}R_t}{2}\right)dt.$$

Thus R_t is deterministic and $R_t = \frac{2}{\sqrt{k}}\arcsin\left(e^{-\frac{kt}{2}}\sin(\frac{\sqrt{k}R_0}{2})\right)$. We also have:

$$dA_t \cdot dA_t = 4\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}dt \text{ and } Drift(dA_t) = 0.$$

In particular, A_t is a martingale.

Example 4.1.11 (Reflection coupling)**.**

For $K(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{K}(t) = 0$, $dV_1(t) = -dU_1(t)$ and $dV_2(t) = dU_2(t)$, we get:

$$dR_t \cdot dR_t = 4dt \text{ and } Drift(dR_t) = -\sqrt{k} \times \tan\left(\frac{\sqrt{k}R_t}{2}\right).$$

As before, we have:

$$dA_t \cdot dA_t = 4\frac{\tan^2\left(\frac{\sqrt{k}R_t}{2}\right)}{k}dt \text{ and } Drift(dA_t) = 0.$$

Note that, using Remark 4.1.7, we can show that R_t keeps sense even if $R_t \leq 0$ and thus obtain a coupling with a signed distance.

Example 4.1.12 (Perverse coupling)**.**

We use here the terminology introduced by Kendall in [38] to name some shy couplings of Brownian motions on $\mathbb{R}^2$*.*
For $K(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{K}(t) = 0$, $dV_1(t) = dU_1(t)$ *and* $dV_2(t) = -dU_2(t)$*, we get:*

$$dR_t \cdot dR_t = 0;$$
$$Drift(dR_t) = \sqrt{k}\frac{\cos(\sqrt{k}R_t) + 1}{\sin(\sqrt{k}R_t)}dt = \sqrt{k} \times \cot\left(\frac{\sqrt{k}R_t}{2}\right).$$

Thus R_t *is deterministic and* $R_t = \frac{2}{\sqrt{k}} \arccos\left(e^{-\frac{kt}{2}} \cos\left(\frac{\sqrt{k}R_0}{2}\right)\right)$ *for* $k \neq 0$*. We also have* A_t *constant.*

In all these examples, $dR_t \cdot dA_t = 0$.
We can also add a noise to these couplings in order to remove the drift part. In particular, by doing it for the synchronous coupling, we obtain a new coupling with constant distance between the Brownian motions:

Example 4.1.13 (Fixed-distance coupling)**.**

Taking $K(t) = \begin{pmatrix} 1 & 0 \\ 0 & \cos(\sqrt{k}R_t) \end{pmatrix}$, $\hat{K}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \sin(\sqrt{k}R_t) \end{pmatrix}$*, we get* R_t *constant. We also have:*

$$dA_t \cdot dA_t = 2\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}(1 + \cos(\sqrt{k}R_t))dt = \frac{4}{k}\sin^2\left(\frac{\sqrt{k}R_t}{2}\right)dt = \frac{4}{k}\sin^2\left(\frac{\sqrt{k}R_0}{2}\right)dt;$$
$$Drift(dA_t) = 0.$$

Note that A_t *is a Brownian motion up to a multiplicative constant.*

We can do the same for the reflection coupling.

Example 4.1.14 (Reflection coupling with noise:)**.**

For $K(t) = \begin{pmatrix} -1 & 0 \\ 0 & \cos(\sqrt{k}R_t) \end{pmatrix}$, $\hat{K}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \sin(\sqrt{k}R_t) \end{pmatrix}$*, we get:*

$$dR_t \cdot dR_t = 4dt \text{ and } Drift(dR_t) = 0.$$

Thus $\frac{1}{2}R_t$ *is a Brownian motion. We also have:*

$$dA_t \cdot dA_t = \frac{4}{k}\sin^2\left(\frac{\sqrt{k}R_t}{2}\right)dt \text{ and } Drift(dA_t) = 0.$$

For these two couplings, $dR_t \cdot dA_t = 0$ too.

Remark 4.1.15.
- *Let* $x, y \in M_k$ *with* $0 < \rho(x, y) < i(M_k)$ *if* $k > 0$*. Let* $f : [0, \infty[\to [0, \infty[$ *be a positive (deterministic) function such that* $f(0) = \rho(x, y)$*. In [48], Pascu and Popescu proved that there exists a co-adapted coupling of Brownian motions*

$(X_t, Y_t)_t$ starting at (x, y) such that $R_t = f(t)$ if and only if f is continuous a.s. and satisfies, a.e. for $t \geq 0$, the differential inequality:

$$-\sqrt{k}\tan\left(\sqrt{k}\frac{f(t)}{2}\right) \leq f'(t) \leq \sqrt{k}\cot\left(\frac{\sqrt{k}f(t)}{2}\right).$$

In particular the synchronous coupling and perverse coupling described above are the couplings realizing the extrema of this inequality.

- *In the case where $k = 0$ we find all the expected results for the Heisenberg group. See for example [37, 38, 22].*

4.2 Successful couplings on $SU(2)$

4.2.1 The results

We use the same notations as in Section 4.1.

We want to describe the construction of a successful coupling on $SU(2)$. To begin with, for $k > 0$, we construct a successful coupling of Brownian motions on M_k and their swept areas for the areas lying in $\mathbb{R}$. Note that, since $k > 0$, we have $i(M_k) = \frac{\pi}{\sqrt{k}}$. Note that the two following results have been previously stated as Theorem 2.3.1.

Theorem 4.2.1. *For $k > 0$, let $x, y \in M_k$ and $a \in \mathbb{R}$. There exists a coupling (X_t, Y_t) of Brownian motions on M_k starting from (x, y) such that $\tau := \inf\{t > 0 | R_t = 0 \text{ and } A_t = a\}$ is a.s. finite with A_t the signed swept area (started from 0) defined in Definition 3.2.1.*

The proof will be given in Subsection 4.2.2. For now we set $k = 1$. As $\zeta \equiv A_t + z'_0 - z_0 + \text{sign}(\theta_0 - \theta'_0)\mathcal{A}_{Y_0,X_0,N} \mod (4\pi)$, the coupling of Theorem 4.2.1 for $a = z'_0 - z_0 + \text{sign}(\theta_0 - \theta'_0)\mathcal{A}_{Y_0,X_0,N_0}$ also induces a successful coupling on $SU(2)$. We thus obtain the following coupling result:

Corollary 4.2.2. *There exists a co-adapted successful coupling in $SU(2)$.*

In fact, using the compactness of $SU(2)$, the last coordinate z_t of any Brownian motion lies in $\mathbb{R}/]-2\pi, 2\pi]$. Then we can change some steps of the strategy of Theorem 4.2.1 to obtain a faster coupling. This will be commented at the end of this section in Subsection 4.2.3.

4.2.2 Coupling Brownian motions on the sphere together with their swept area

In this subsection, we give the proof of Theorem 4.2.1. We first explain the global strategy. As said before, our method is based on the idea from Kendall [37] for coupling two dimensional real Brownian motions and their swept areas. The original idea is to switch between reflection and synchronous coupling, using reflection coupling to make R_t decrease, and synchronous coupling to keep the swept area comparable to R_t^2 and decreasing as well. Here, in comparison to Kendall's original proposition, we will need to use the fixed-distance coupling defined in Example 4.1.13 instead of the synchronous coupling in order

to keep R_t constant. If not, we would have a strictly positive probability to be trapped in a "synchronous coupling step" without returning in a "reflection coupling step". Note that, for $k \to 0$, i.e., for real two dimensional Brownian motion, fixed-distance coupling and synchronous coupling describe in fact the same coupling.

We first make and explain some hypothesis on the initial parameters.

- We can first suppose that $0 < R_0 < i(M_k)$. Indeed, by taking $t_0 > 0$ any deterministic positive real, we can define the coupling $(X_t, Y_t)_{t\in[0,t_0]}$ such that X_t and Y_t are independent. Then we have $0 < R_{t_0} < i(M_k)$ a.s..
- Without loss of generality, we can suppose that $a = 0$ by considering the process $A_t - a$ instead of A_t.
- Using fixed-distance coupling if necessary, we can suppose that $A_0 = 0$ without changing the value of R_0 (indeed, A_t is a changed-time Brownian motion during the fixed-distance coupling).

Let $\eta > 0$ such that $i(M_k) - 2\eta > R_0 > 0$. We define another stopping time:

$$\tau_\eta := \inf\{t > 0 | R_t \geq i(M_k) - \eta\}.$$

We are going to study first $\tau \wedge \tau_\eta$ instead of τ: this way we will have $R_t < i(M_k) - \eta$ for all $t < \tau \wedge \tau_\eta$. We define κ and ϵ such that $\kappa > \epsilon > 0$.

We will construct the coupling as follows on $[0, \tau \wedge \tau_\eta]$:

1. We use the reflection coupling until the process $\frac{|A_t|}{R_t^2}$ starting at 0 takes the value κ;
2. While the process $\frac{|A_t|}{R_t^2}$, starting at κ, satisfies $\frac{|A_t|}{R_t^2} > \kappa - \epsilon$ we use the fixed-distance coupling;
3. While the process $\frac{|A_t|}{R_t^2}$, starting at $\kappa - \epsilon$, satisfies $\frac{|A_t|}{R_t^2} < \kappa$ we use the reflection coupling.

We iterate the steps 2. and 3. until $R_t = A_t = 0$ or $R_t = i(M_k) - \eta$.

Proposition 4.2.3. *Under the hypothesis $i(M_k) - 2\eta > R_0 > 0$, the co-adapted coupling described below satisfies $\tau \wedge \tau_\eta < +\infty$ a.s. for $k > 0$. Moreover, we get $\mathbb{P}(\tau > \tau_\eta) < 1$.*

Proof of Proposition 4.2.3. Let us denote $\tau' := \tau \wedge \tau_\eta$. We define $N^{(\epsilon)} : [0, \tau'] \to \{0, 1\}$ such that $N^{(\epsilon)}(t) = 0$ during fixed-distance coupling and $N^{(\epsilon)}(t) = 1$ during reflection coupling. We get, for all $t > 0$:

$$dR_t = 2N^{(\epsilon)}(t)dC_t - \sqrt{k}\tan\left(\frac{\sqrt{k}R_t}{2}\right)N^{(\epsilon)}(t)dt$$

$$dA_t = \frac{2}{\sqrt{k}} \times \frac{\sin(\frac{\sqrt{k}R_t}{2})}{\cos(N^{(\epsilon)}(t)\frac{\sqrt{k}R_t}{2})}d\tilde{C}_t$$

with C_t and $\tilde{C}_t$ two independent Brownian motions in $\mathbb{R}$. Note that, during reflection coupling (i.e., when $FN^{(\epsilon)}(t) = 1$) we have $\frac{|A_t|}{R_t^2} \leq \kappa$, whereas during fixed-distance coupling (i.e., $N^{(\epsilon)}(t) = 0$) we have $\frac{|A_t|}{R_t^2} \geq \kappa - \epsilon$. As R varies only during reflection couplings,

if $R_t \to 0$, we have $R_t^2 \geq \frac{1}{\kappa}|A_t|$, thus $A_t \to 0$. In fact, with this strategy we have $\tau' = \inf\{t \mid R_t \in \{0, i(M_k) - \eta\}\}$. For the following computation, we take $t < \tau'$. This way we have $0 < R_t < i(M_k) - \eta$.

Let us define $\sigma(t) := \int_0^t \frac{4}{R_s^2} ds$ and $K_{\sigma(t)} = \ln(R_t)$. Using Itô's formula, we get:

$$dK_{\sigma(t)} \cdot dK_{\sigma(t)} = \frac{4N^{(\epsilon)}(t)}{R_t^2} dt = N^{(\epsilon)}(t) d\sigma(t)$$

$$\begin{aligned} Drift(dK_{\sigma(t)}) &= -\frac{\sqrt{k}}{R_t} \tan\left(\frac{\sqrt{k}R_t}{2}\right) N^{(\epsilon)}(t)dt - \frac{2N^{(\epsilon)}(t)}{R_t^2} dt \\ &= -N^{(\epsilon)}(t) \left(\frac{\sqrt{k}R_t \tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2} \right) d\sigma(t). \end{aligned}$$

With this change of time, excluding the times when $N^{(\epsilon)}(t) = 0$ (fixed-distance coupling) during which K stays constant, $\sigma \mapsto K_\sigma$ acts like a Brownian motion with a negative drift.
Let us also define $W_{\sigma(t)} = \frac{A_t}{R_t^2}$. Using Itô's formula, we get:

$$\begin{aligned} dW_{\sigma(t)} \cdot dW_{\sigma(t)} &= \left(\frac{4\sin^2\left(\frac{\sqrt{k}R_t}{2}\right)}{kR_t^4 \cos^2\left(N^{(\epsilon)}(t)\frac{\sqrt{k}R_t}{2}\right)} + \frac{16N^{(\epsilon)}A_t^2}{R_t^6} \right) dt \text{ as } \langle dC_t, d\tilde{C}_t \rangle = 0 \\ &= \left(\frac{1}{4} \left(\frac{2}{\sqrt{k}R_t} \right)^2 \frac{\sin^2\left(\frac{\sqrt{k}R_t}{2}\right)}{\cos^2\left(N^{(\epsilon)}\frac{\sqrt{k}R_t}{2}\right)} + 4N^{(\epsilon)}(t)W_{\sigma(t)}^2 \right) d\sigma(t); \\ Drift(dW_{\sigma(t)}) &= \left(\frac{2\sqrt{k}}{R_t^3} \tan\left(\frac{\sqrt{k}R_t}{2}\right) N^{(\epsilon)}(t)A_t + 3 \times 4N^{(\epsilon)}(t)\frac{A_t}{R_t^4} \right) dt \\ &= N^{(\epsilon)}(t)W_{\sigma(t)} \left(\frac{\sqrt{k}R_t}{2} \tan\left(\frac{\sqrt{k}R_t}{2}\right) + 3 \right) d\sigma(t). \end{aligned}$$

Finally, $dK_{\sigma(t)} \cdot dW_{\sigma(t)} = -2\frac{A_t}{R_t^4} \times 4N^{(\epsilon)}(t)dt = -2N^{(\epsilon)}(t)W_{\sigma(t)}\, d\sigma(t)$.

Let $S := \sigma(\tau')$. For the following part, in order to simplify the notations, we will denote σ instead of $\sigma(t)$ and $N^{(\epsilon)}(\sigma)$ instead of $N^{(\epsilon)}(t)$. We want to show that $\tau' < +\infty$ a.s.. As $\int_0^S e^{2K_\sigma} d\sigma = \int_0^\tau 4e^{2\log(R_t)} \frac{dt}{R_t^2} = 4\tau'$, this is the same as showing that $\int_0^S e^{2K_\sigma} d\sigma < +\infty$ a.s..
First of all let us note that the number of changes of types of coupling (reflection coupling/ fixed-distance coupling) is countable as it is finite on all closed and bounded interval of time $[S_1, S_2]$ such that $S_2 < \tau'$. Actually if there were an infinite number of changes of types of coupling, we could define (considering the time scale induced by σ) two sequences $(\sigma_n^s)_n$ and $(\sigma_n^r)_n$ such that $\sigma_n^s < \sigma_n^r < \sigma_{n+1}^s$ for all $n \geq 0$ and such that $N^{(\epsilon)}(\sigma) = 0$ on $[\sigma_n^s, \sigma_n^r[$ and $N^{(\epsilon)}(\sigma_n^r) = 1$. As $S_2 < \tau'$, we would have $R_t \neq 0$ and W_σ well defined on $[S_1, S_2]$ such that $|W_{\sigma_n^s}| = \kappa$ and $|W_{\sigma_n^r}| = \kappa - \epsilon$. As $(\sigma_n^s)_n$ and $(\sigma_n^r)_n$ would converge to the same limit and as $|W|$ is continuous, this would lead to a contradiction.

Thus we have a countable number of changes of type of coupling. Using the previous notations, let us denote by $[\sigma_n^s, \sigma_n^r[$ the intervals during which $N^{(\epsilon)}(\sigma) = 0$ and by $[\sigma_n^r, \sigma_{n+1}^s[$ the intervals during which $N^{(\epsilon)}(\sigma) = 1$. The same way we denote t_n^s and t_n^r such that $\sigma(t_n^s) = \sigma_n^s$ and $\sigma(t_n^r) = \sigma_n^r$.
As seen before, $(R_t)_t$ is constant on intervals $[t_n^s, t_n^r[$ and acting as a two times Brownian motion with negative drift out of these intervals. As the time needed to exit a bounded open interval for a real Brownian motion with negative drift is a.s. finite, we have $\sum_{n\geq 0}(t_{n+1}^s - t_n^r) < +\infty$ a.s.. Note that:

$$\int_0^S N^{(\epsilon)}(\sigma)e^{2K_\sigma}d\sigma = \sum_{n\geq 0}\int_{\sigma_n^r}^{\sigma_{n+1}^s} R_t^2 d\sigma = \sum_{n\geq 0}\int_{t_n^r}^{t_{n+1}^s} 4dt = 4\sum_{n\geq 0}(t_{n+1}^s - t_n^r).$$

Then, this quantity is a.s. finite and, in order to show that $\tau' < +\infty$, it is enough to show that:

$$\int_0^S (1 - N^{(\epsilon)})e^{2K_\sigma}d\sigma < +\infty.$$

As $(R_t)_t$ and $(K_\sigma)_\sigma$ are constant during fixed-distance coupling, we have:

$$\int_0^S (1 - N^{(\epsilon)})e^{2K_\sigma}d\sigma = \sum_{n\geq 0}\int_{\sigma_n^s}^{\sigma_n^r} e^{2K_\sigma}d\sigma = \sum_{n\geq 0} e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s).$$

Thus it is enough to show that $\sum_{n\geq 0} e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s) < +\infty$ a.s..

- We first show the equality:

$$\mathbb{E}\left[\exp\left(-\sum_{n\geq 0} e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)\right)\Big|(K_{\sigma_m^s})_m\right] = \exp\left(-\sum_{n\geq 0} e^{K_{\sigma_n^s}} \times \frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)} \times 2\sqrt{2}\epsilon\right). \tag{4.15}$$

As our coupling is the fixed-distance one on $[\sigma_n^s, \sigma_n^r]$, there exists V_σ, a real Brownian motion, such that:

$$dW_\sigma = \frac{1}{\sqrt{k}R_t}\sin\left(\frac{\sqrt{k}R_t}{2}\right)dV_\sigma = \frac{1}{\sqrt{k}R_{t_n^s}}\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)dV_\sigma.$$

In particular $|W_\sigma|$ only depends on $R_{t_n^s}$ for $\sigma_n^r \geq \sigma \geq \sigma_n^s$ and so depends on $K_{\sigma_n^s}$ only.
Then, knowing $\left(K_{\sigma_m^s}\right)_m$, $\sigma_n^r - \sigma_n^s$ is the first hitting time of $k - \epsilon$ by the process $|W_\sigma|$. As $|W_{\sigma_n^s}| = \kappa$, by continuity, W_σ keeps the same sign all along the interval $[\sigma_n^s, \sigma_n^r]$. Then we have:

$$\begin{aligned}
\sigma_n^r - \sigma_n^s &= \inf\{\sigma > 0 \ |W_{\sigma+\sigma_n^s} = \text{sign}(W_{\sigma_n^s})(\kappa - \epsilon)\} \\
&= \inf\{\sigma > 0 \ |(W_{\sigma+\sigma_n^s} - W_{\sigma_n^s}) = \text{sign}(W_{\sigma_n^s})(\kappa - \epsilon - \kappa)\} \\
&= \inf\left\{\sigma > 0 \ \left| \frac{\sqrt{k}R_{t_n^s}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}(W_{\sigma+\sigma_n^s} - W_{\sigma_n^s}) = -\frac{\sqrt{k}R_{t_n^s}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}\text{sign}(W_{\sigma_n^s})\epsilon\right.\right\}.
\end{aligned}$$

Moreover, on $[\sigma_n^s, \sigma_n^r]$ and conditional on $K_{\sigma_n^s}$, $\frac{\sqrt{k}R_{t_n^s}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}(W_{\sigma+\sigma_n^s} - W_{\sigma_n^s})_\sigma$ is a real Brownian motion starting from 0. Thus, conditional on $K_{\sigma_n^s}$, $\sigma_n^r - \sigma_n^s$ has the same law than T_{a_n}, the first hitting time of $a_n = \frac{\sqrt{k}R_{t_n^s}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}\epsilon$ for a real Brownian motion starting at 0. Using the Laplace transform we get:

$$\mathbb{E}[\exp(-\mu T_{a_n})] = \exp(-a_n\sqrt{2\mu})\ \forall \mu > 0$$

$$\text{and } \mathbb{E}[\exp\left(-\mu(\sigma_n^r - \sigma_n^s)\right)|K_{\sigma_n^s}] = \exp\left(-\frac{\sqrt{k}R_{t_n^s}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}\epsilon\sqrt{2\mu}\right).$$

Finally:

$$\mathbb{E}\left[\exp\left(-e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)\right)|(K_{\sigma_m^s})_m\right] = \exp\left(-\frac{\sqrt{k}R_{t_n^s}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}\epsilon\sqrt{2}e^{K_{\sigma_n^s}}\right).$$

Furthermore, conditional to $(K_{\sigma_m^s})_m$, $(\sigma_n^r - \sigma_n^s)_n$ are independent. Thus:

$$\mathbb{E}\left[\exp\left(-\sum_{n=0}^{N} e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)\right)\middle|(K_{\sigma_m^s})_m\right] = \exp\left(-\sum_{n=0}^{N}\frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}2\epsilon\sqrt{2}e^{K_{\sigma_n^s}}\right).$$

Using the dominated convergence theorem we get the announced equality.

- We now deal with the quantity $\sum_{n\geq 0}\frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}e^{K_{\sigma_n^s}}$ occurring in the previous result.

  Let us first notice that $\int_0^{\sigma_n^s} N^{(\epsilon)}(\sigma)d\sigma = \sum_{m=0}^{n-1}(\sigma_{m+1}^s - \sigma_m^r)$. During reflection times $([\sigma_m^r, \sigma_{m+1}^s])$, the quantities $\sigma_{m+1}^s - \sigma_m^r$ are the first exit times from the open $]-\kappa, \kappa[$ of the diffusion W_σ starting at $\pm(\kappa-\epsilon) \in]-\kappa, \kappa[$. Note that, unlike the fixed-distance coupling case, here the sign of W_t can change.

  During these times we have:

$$\begin{cases} dW_{\sigma(t)} \cdot dW_{\sigma(t)} = \left(\frac{1}{4}\left(\frac{\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{\frac{\sqrt{k}R_t}{2}}\right)^2 + 4W_{\sigma(t)}^2\right)d\sigma(t) \\ Drift(dW_{\sigma(t)}) = W_{\sigma(t)}\left(\frac{\sqrt{k}R_t}{2}\tan\left(\frac{\sqrt{k}R_t}{2}\right) + 3\right)\ d\sigma(t). \end{cases} \tag{4.16}$$

  We now take m some positive integer. We define a new time-change: $\zeta_m(\sigma) := \int_0^\sigma dW_{s+\sigma_m^r} \cdot dW_{s+\sigma_m^r}$. As R_t is upper-bounded by $i(M_k) - \eta$ on $[0, \tau']$, there exists M a positive constant such that $\frac{\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{\frac{\sqrt{k}R_t}{2}} < M$ and $\frac{\sqrt{k}R_t}{2}\tan\left(\frac{\sqrt{k}R_t}{2}\right) < M$. Thus,

$$\frac{\sigma}{4} \leq \zeta_m(\sigma) \leq \left(\frac{M^2}{4} + 4\kappa^2\right)\sigma \text{ and } \left|\frac{Drift(dW_\sigma)}{d\zeta_m(\sigma)}\right| \leq 4\kappa(M+3).$$

Then there exists a one dimensional Brownian motion B^m, starting at 0 and independent of ζ_m, such that for all $\sigma \in [0, \sigma^s_{m+1} - \sigma^r_m]$:

$$B^m_{\zeta_m(\sigma)} - 4\kappa(3+M)\zeta_m(\sigma) \leq W_{\sigma+\sigma^r_m} - W_{\sigma^r_m} \leq B^m_{\zeta_m(\sigma)} + 4\kappa(3+M)\zeta_m(\sigma).$$

We now obtain

$$\begin{aligned}\zeta_m(\sigma^s_{m+1} - \sigma^r_m) \geq \inf\{\zeta > 0 \mid B^m_\zeta + 4\kappa(3+M)\zeta = \kappa - W_{\sigma^r_m}\} \\ \wedge \inf\{\zeta > 0 \mid B^m_\zeta - 4\kappa(3+M)\zeta = -\kappa - W_{\sigma^r_m}\}.\end{aligned}$$

As $W_{\sigma^r_m}$ can only take the two values $\kappa - \epsilon$ and $-(\kappa - \epsilon)$, we get $\zeta_m(\sigma^s_{m+1} - \sigma^r_m) \geq T_m$ with:

$$T_m := \inf\{\zeta > 0 \mid B^m_\zeta + 4\kappa(3+M)\zeta = \epsilon\} \wedge \inf\{\zeta > 0 \mid B^m_\zeta - 4\kappa(3+M)\zeta = -\epsilon\}.$$

In particular, we have:

$$\sigma^s_{m+1} - \sigma^r_m \geq \frac{1}{\frac{M^2}{4} + 4\kappa^2} T_m. \tag{4.17}$$

For all m, $(T_m)_m$ is a sequence of independent and equally distributed variables with non-negative and finite mean. Then, the strong law of large numbers yields:

$$\left(\frac{M^2}{4} + 4\kappa^2\right) \frac{1}{n} \sum_{m=0}^{n-1} (\sigma^s_{m+1} - \sigma^r_m) \geq \frac{1}{n} \sum_{m=0}^{n-1} T_m \xrightarrow[n\to+\infty]{a.s.} \mathbb{E}[T_0].$$

Thus, a.s. for n large enough, we get :

$$\frac{\int_0^{\sigma^s_n} N^{(\epsilon)}(\sigma)d\sigma}{n} = \frac{1}{n} \sum_{m=0}^{n-1} (\sigma^s_{m+1} - \sigma^r_m) \geq \frac{\mathbb{E}[T_0]}{2(\frac{M^2}{4} + 4\kappa^2)} > 0. \tag{4.18}$$

Moreover, we obtain $\int_0^{\sigma^s_n} N^{(\epsilon)}(\sigma)d\sigma \xrightarrow[n\to+\infty]{a.s} +\infty$.

Let us now recall that $K_\sigma = K_0 + \int_0^\sigma N^{(\epsilon)}(s)dC_s - \int_0^\sigma N^{(\epsilon)} \left(\frac{\sqrt{k}R_t \tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right) ds$.

Thus $K_\sigma - K_0 + \int_0^\sigma N^{(\epsilon)} \left(\frac{\sqrt{k}R_t \tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right) ds$ is a Brownian motion for the change of time $\int_0^\sigma N^{(\epsilon)}(s)ds$.

Note that:

$$\frac{1}{2} \leq \frac{\int_0^{\sigma^s_n} N^{(\epsilon)} \left(\frac{\sqrt{k}R_t \tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right) ds}{\int_0^{\sigma^s_n} N^{(\epsilon)}(s)ds}.$$

By the strong law of large number for Brownian motions, we also have:

$$\frac{K_{\sigma^s_n} - K_0 + \int_0^{\sigma^s_n} N^{(\epsilon)} \left(\frac{\sqrt{k}R_t \tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right) ds}{\int_0^{\sigma^s_n} N^{(\epsilon)}(s)ds} \xrightarrow[n\to+\infty]{a.s} 0.$$

Finally, a.s. for n large enough, we obtain:

$$\frac{K_{\sigma^s_n}}{\int_0^{\sigma^s_n} N^{(\epsilon)}(s)ds} \leq -\frac{1}{4}. \tag{4.19}$$

By combining the results (4.18) and (4.19), we get a.s. for n large enough:

$$\frac{1}{n}K_{\sigma_n^s} = \frac{K_{\sigma_n^s}}{\int_0^{\sigma_n^s} N^{(\epsilon)}(s)ds} \times \frac{\int_0^{\sigma_n^s} N^{(\epsilon)}(s)ds}{n} \leq -\frac{c_0}{4} < 0$$

with $c_0 := \frac{\mathbb{E}[T_0]}{2\left(\frac{M^2}{4}+4\kappa^2\right)}$. It remains to notice that a.s., for all n, we have $0 < \frac{\sqrt{k}R_{t_n^s}}{2} < \frac{\pi}{2}$. Thus:

$$\frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)} \leq \frac{\pi}{2} \text{ and } \sum_{n\geq M} \frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)} e^{K_{\sigma_n^s}} \leq \frac{\pi}{2} \sum_{n\geq M} (e^{-\frac{c_0}{4}})^n < +\infty.$$

Finally we get:

$$\sum_{n\geq 0} \frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)} e^{K_{\sigma_n^s}} < +\infty \ a.s.. \tag{4.20}$$

- Using (4.15) and (4.20), we have $\mathbb{E}[\exp(-\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s))|(K_{\sigma_m^s})_m] > 0$ a.s..
  Thus, still conditional to $(K_{\sigma_m^s})_m$, the event $\exp(-\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)) > 0$ has a non-zero probability. Equivalently the event $\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s) < +\infty$ has a non-zero probability. As $\left(e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)\right)_n$ are independent, using the Kolmogorov zero-one law, we get $\mathbb{P}(\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s) < +\infty|(K_{\sigma_m^s})_m) = 1$ a.s..
  Finally, $\sum_{m\geq 0} e^{2K_{\sigma_m^s}}(\sigma_m^r - \sigma_m^s) < +\infty$ a.s. and so $\tau' < +\infty$ a.s..

To show that $\mathbb{P}(\tau > \tau_\eta) < 1$, we just need to remark that this event only depends on the evolution of R_t. As R_t is acting as a time-changed Brownian motion with negative drift omitting the times where it stays constant, we directly obtain our result. □

We can now give the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. To construct a successful coupling, we just need to start the coupling described in Theorem 4.2.3.

(i) If $\tau \wedge \tau_\eta = \tau$, we obtain $R_t = A_t = 0$.

(ii) If $\tau \wedge \tau_\eta = \tau_\eta$, we use a synchronous coupling until $R_t = R_0$ and then a fixed distance coupling until $A_t = 0$ and we re-start the coupling of Theorem 4.2.3.

At step (ii), as R_t is deterministic and decreasing during synchronous coupling, each "synchronous step" will take a constant finite time. It also takes an a.s. finite time to obtain $A_t = 0$ with fixed distance coupling. Thus we repeat the same experiment independently. As the probability that $\tau < \tau_\eta$ is non-zero during the coupling of Theorem 4.2.3, it will take a finite number of change of coupling to be in this event and then have an a.s. finite time of success. Let us remind that, if $R_0 \in \{0, i(M_k)\}$, we need to use coupling of independent Brownian motions on any finite deterministic interval of time before our successful coupling to start with $R_t \in]0, i(M_k)[$. This does not change our result. □

4.2.3 Improvement of the strategy for a successful coupling in SU(2)

We have seen previously that Theorem 4.2.1 can be used directly to prove Theorem 4.2.2. However, using the compactness of $SU(2)$, we can make two improvements for the initial strategy to obtain a co-adapted successful coupling in $SU(2)$. We take $k = 1$.

- Considering $A_t \mod (4\pi)$ instead of a value in $\mathbb{R}$, we can suppose that A_t stays in $]-2\pi, 2\pi]$. The idea is to take κ small enough such that fixed-distance coupling is stopped when the process W_σ starting at $\pm\kappa$ hits $\pm(\kappa-\epsilon)$ with this time a non zero probability to reach one or another of these values. Then there will be less time spent in fixed-distance coupling than in Theorem 4.2.3. To be more precise, for $n \geq 0$, there exists $(V_\sigma)_\sigma$ a Brownian motion starting at 0 such that:

$$\begin{aligned}\sigma_n^r - \sigma_n^s = \inf\Bigg\{\sigma \;\Big|\; \operatorname{sign}(W_{\sigma_n^s})\kappa + \frac{\sin\left(\frac{R_{t_n^s}}{2}\right)}{R_{t_n^s}} V_\sigma \in \Big\{\operatorname{sign}(W_{\sigma_n^s})(\kappa-\epsilon) \\ ; \operatorname{sign}(W_{\sigma_n^s})\left(\frac{4\pi}{R_{t_n^s}^2} - (\kappa-\epsilon)\right)\Big\}\Bigg\} \\ = \inf\Big\{\sigma \mid \operatorname{sign}(W_{\sigma_n^s})\frac{\sin\left(\frac{R_{t_n^s}}{2}\right)}{R_{t_n^s}} V_\sigma \in \Big\{-\epsilon; \frac{4\pi}{R_{t_n^s}^2} - 2\kappa + \epsilon\Big\}\Big\}.\end{aligned}$$

  If $\frac{4\pi}{R_{t_n^s}^2} - 2\kappa + \epsilon > 0$, which is the case for $\kappa < \frac{2}{\pi} + \frac{\epsilon}{2}$, then the above stopping time is less than T_{a_n} with a_n defined as before. This way, instead of (4.15), we obtain:

$$\mathbb{E}\left[\exp\left(-\sum_{n\geq 0} e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)\right)\middle|\,(K_{\sigma_m^s})_m\right] > \exp\left(-\sum_{n\geq 0} e^{K_{\sigma_n^s}} \times \frac{\frac{R_{t_n^s}}{2}}{\sin\left(\frac{R_{t_n^s}}{2}\right)} \times 2\sqrt{2}\epsilon\right).$$

- In addition to decreasing the time spent in fixed-distance coupling, having bounded values for A_t prevents from using synchronous coupling when R_t is too close of $i(M_1) = \pi$. Let us explain this.
  During reflection coupling, W_σ stays continuous and still satisfies (4.16). Supposing that $R_t \leq \pi - \eta$ for all t in $[t_m^r, t_{m+1}^s]$, we still have (4.17). Moreover, supposing that $R_t > \pi - \eta$ for some $t \in [t_m^r, t_{m+1}^s]$, we get $|W_{\sigma(t)}| = \frac{|A_t|}{R_t^2} \leq \frac{2\pi}{(\pi-\eta)^2}$. Choosing η, κ and δ such that $\frac{2\pi}{(\pi-\eta)^2} < \kappa - \delta < 2\pi$, we obtain $|W_{\sigma(t)}| < \kappa$. This is the case for $\kappa - \delta > \frac{2}{\pi}$. Note that, in order to keep the previous improvement, we take $\delta < \frac{\epsilon}{2}$. Thus reflection coupling will not end while R_t stays up to $\pi - \eta$. Moreover, there will exist $\tilde{t}_m^r \in]t_m^r, t_{m+1}^s[$ such that $|W_{\sigma(\tilde{t}_m^r)}| = \kappa - \delta$ and $R_t \leq \pi - \eta$ for all t in $[\tilde{t}_m^r, t_{m+1}^s]$. As before, we obtain:

$$\sigma_{m+1}^s - \sigma_m^r > \sigma_{m+1}^s - \sigma(\tilde{t}_m^r) \geq \frac{1}{\frac{M^2}{4} + 4\kappa^2}\tilde{T_m}$$

  with, $\tilde{T}_m$ defined as T_m but with δ instead of ϵ:

$$\tilde{T}_m := \inf\{\zeta > 0 \mid B_\zeta^m + 4\kappa(3+M)\zeta = \delta\} \wedge \inf\{\zeta > 0 \mid B_\zeta^m - 4\kappa(3+M)\zeta = -\delta\}.$$

  Thus, using $(\tilde{T}_m)_m$ instead of $(T_m)_m$, we get again (4.18). Following the rest of the proof of Theorem 4.2.3, we directly obtain $\tau < +\infty$ a.s..

Chapter 5

Non co-adapted couplings on $SU(2)$ and $SL(2,\mathbb{R})$

In this chapter, we continue the generalisation of the successful couplings from the Heisenberg group to $SU(2)$ and, in a weaker sense, to $SL(2,\mathbb{R})$. We take inspiration from the non co-adapted coupling due to Banerjee, Gordina and Mariano [6]. Contrary to Chapter 4, here, the successful coupling we construct is non co-adapted and its coupling rate can be estimated. We then give estimates of the coupling rate for $SU(2)$ and, if the processes start from the same fiber, for $SL(2,\mathbb{R})$. These estimates leads to some gradient estimate.

The work presented here, together with Subsection 3.2.7, can be found in the publication:

[24] Magalie Bénéfice. Non co-adapted couplings of Brownian motions on subRiemannian manifolds. `https://arxiv.org/abs/2312.14512`, 2023.

Note that, more recently, Luo and Neel [44] also studied non co-adapted successful couplings of subelliptic Brownian motions on $\mathbb{H}$, $SU(2)$, $SL(2,\mathbb{R})$ and also on the universal cover of $SL(2,\mathbb{R})$. By using a different strategy, they obtained estimates of the same order for the coupling rate $\mathbb{P}(\tau > t)$.

Please note that Section 3.2 and more precisely Lemma 3.2.12 are necessary to apprehend what follows.

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5.1 The main coupling theorems

The results stated here have been briefly presented in Subsections 2.3.4 and 2.3.5. The main result is the existence and the construction of a non co-adapted coupling with an exponentially decreasing coupling rate proportional to the distance between the starting points of the processes. We remind that, using the decomposition given by the fibration, every element of $SU(2)$ (resp. $SL(2,\mathbb{R})$) can be written on the form $g = (x, z)$ with x an element of the sphere S^2 (resp. of the hyperbolic plane $\mathbf{H}^2$) and $z \in]-2\pi, 2\pi]$.

If the starting points of the Brownian motions are in a same fiber, we obtain a successful coupling on $SU(2)$ and on $SL(2,\mathbb{R})$.

Theorem 5.1.1. *Let $g = (x, z)$, $g' = (x', z') \in SU(2)$ (resp. $SL(2,\mathbb{R})$). We suppose that $x = x'$.*
There exists a non co-adapted successful coupling of Brownian motions $(\mathbb{B}_t, \mathbb{B}'_t)_t$ on $SU(2)$ (resp. $SL(2,\mathbb{R})$) starting at (g, g'). Moreover, denoting $\tau := \inf\{t \mid \mathbb{B}_t = \mathbb{B}'_t\}$, for all $0 < q < 1$, there exist C_q and c some non negative constants that do not depend on the starting points of the process, such that, for all $t > 1$:

$$\mathbb{P}(\tau > t) \leq \left(C_q e^{-ct} d_{cc}(g, g')^{2q}\right) \wedge 1. \tag{5.1}$$

This Theorem will be proven in Section 5.2.

In the case of $SU(2)$, we can first use a reflection coupling to bring back the processes in the same fiber and then the construction from Theorem 5.1.1. We get a successful coupling whatever the starting points of the processes:

Theorem 5.1.2. *Let g, $g' \in SU(2)$. There exists a non co-adapted successful coupling of Brownian motions $(\mathbb{B}_t, \mathbb{B}'_t)$ on $SU(2)$ starting at (g, g'). Moreover, denoting $\tau := \inf\{t \mid \mathbb{B}_t = \mathbb{B}'_t\}$, there exist C, c some non negative constants that do not depend on the starting points of the process, such that, for all $t > 2$:*

$$\mathbb{P}(\tau > t) \leq \left(Ce^{-ct} d_{cc}(g, g')\right) \wedge 1. \tag{5.2}$$

This Theorem will be proven in Subsection 5.3.2. Please note that, with Theorem 5.1.1 and Theorem 5.1.2, we improve and give a proof of the results announced during the GSI'23 Conference ([18], *Theorem 3*).

All along this chapter we use the following notations (previously defined in Subsection 3.2). For $k \in \{-1, 1\}$, we have:

- $E_{-1} = SL(2,\mathbb{R})$, $E_1 = SU(2)$;
- $M_{-1} = \mathbf{H}^2$, $M_1 = S^2$.

Let consider $(\mathbb{B}_t = (X_t, z_t))_t$ and $(\mathbb{B}'_t = (X'_t, z'_t)_t$ two Brownian motions on E_k. We recall that, according to Proposition 3.2.5, the cylindrical coordinates $(\varphi_t, \theta_t, z_t)_t$ and $(\varphi'_t, \theta'_t, z'_t)_t$ of these processes satisfies the systems:

$$\begin{cases} d\varphi_t = dB_t^1 + \frac{\sqrt{k}}{2}\cot(\sqrt{k}\varphi_t)dt \\ d\theta_t = \frac{\sqrt{k}}{\sin(\sqrt{k}\varphi_t)}dB_t^2 \\ dz_t = \frac{\tan\left(\frac{\sqrt{k}\varphi_t}{2}\right)}{\sqrt{k}}dB_t^2 \end{cases} \quad \text{and} \quad \begin{cases} d\tilde{\varphi}_t = d\tilde{B}_t^1 + \frac{\sqrt{k}}{2}\cot(\sqrt{k}\tilde{\varphi}_t)dt \\ d\tilde{\theta}_t = \frac{\sqrt{k}}{\sin(\sqrt{k}\tilde{\varphi}_t)}d\tilde{B}_t^2 \\ d\tilde{z}_t = \frac{\tan\left(\frac{\sqrt{k}\tilde{\varphi}_t}{2}\right)}{\sqrt{k}}d\tilde{B}_t^2 \end{cases}.$$

As in Corollary 3.2.10, we denote by $(\rho_t, \Theta_t, \zeta_t)$ the cylindrical coordinates of $\mathbb{B}_t^{-1}\mathbb{B}'_t$.

5.2 The Brownian motion start from the same fiber

As announced we first deal with the coupling in the case where $x = x'$, that is, with the proof of the Theorem 5.1.1. Let first consider some deterministic constants $T > 0$, $\varphi_0 \in]0, i(M_k)[$ and $\theta_0 \in]0, 2\pi[$ with $i(M_k)$ the injectivity radius of M_k. We begin with the construction of a coupling strategy on $[0, T]$:

Proposition 5.2.1. *Let $k \in \{\pm 1\}$. We fix $(N, e) \in TM_k$. We consider $(X_t)_t$ and $(X'_t)_t$ two Brownian motions on M_k and $\left(\varphi_t^{(N,e)}, \theta_t^{(N,e)}\right)_t$, $\left(\varphi_t^{(N,e)'}, \theta_t^{(N,e)'}\right)_t$ their spherical/polar coordinates relative to (N, e). We suppose that $\varphi_0^{(N,e)} = \varphi_0^{(N,e)'} = \varphi_0$ and $\theta_0^{(N,e)} = \theta_0^{(N,e)'} = \theta_0$. We also consider the swept areas $(I_t(N, e))_t$ and $(I'_t(N, e))_t$, as defined in Subsection 3.2.7, starting from z_0 and z'_0 respectively such that $z_0 - z'_0 \in]-4\pi, 4\pi[$.*
There exists a coupling of $(X_t)_t$ and $(X'_t)_t$ such that $X_T = X'_T$ a.s. and such that, for T small enough, we have:

$$\min_{z_0, z'_0} \left(\mathbb{P}\left(I_T(N, e) - I'_T(N, e) \equiv 0 \mod (4\pi)\right)\right) > 0.$$

5.2.1 Construction of the coupling on $[0, T]$

Proof of Proposition 5.2.1. To simplify the notations and as the change of coordinates induced by (N, e) doesn't intervene in this section, during this proof we will simply denote $\left(\varphi_t^{(N,e)}, \theta_t^{(N,e)}, I_t(N, e)\right)$ by $(\varphi_t, \theta_t, I_t)$ and $\left(\varphi_t^{(N,e)'}, \theta_t^{(N,e)'}, I'_t(N, e)\right)$ by $(\varphi'_t, \theta'_t, I'_t)$. By exchanging the roles of X_t and X'_t if needed, we can suppose that $z_0 - z'_0 > 0$.

We first chose $B_t^1 = B_t^{1'}$ and thus $\varphi_t = \varphi'_t$ for all $t \in [0, T]$. We define the change of time $\sigma(t) = \int_0^t \frac{k}{\sin^2(\sqrt{k}\varphi_s)}ds$. There exist two Brownian motions $(\boldsymbol{\beta}_\sigma)_\sigma$ and $(\boldsymbol{\beta}'_\sigma)_\sigma$ adapted to the filtration $\left(\mathcal{F}_{\sigma^{-1}(\sigma)}\right)_\sigma$ such that:

$$\begin{cases} \theta_t = \theta_0 + \boldsymbol{\beta}_{\sigma(t)} \\ \theta'_t = \theta_0 + \boldsymbol{\beta}'_{\sigma(t)} \end{cases}.$$

As in the coupling described in [6], we are going to couple $(\boldsymbol{\beta}_\sigma)_\sigma$ and $(\boldsymbol{\beta}'_\sigma)_\sigma$ using Brownian bridges. Knowing all the path of $(\varphi_t)_{t\in[0,T]}$ we define, for $\sigma \in [0, \sigma(T)]$:

$$\begin{cases} \boldsymbol{\beta}_\sigma = B_\sigma^{br} + \frac{\sigma}{\sigma(T)}\xi_0 \\ \boldsymbol{\beta}'_\sigma = B_\sigma^{br'} + \frac{\sigma}{\sigma(T)}\xi_0 \end{cases}$$

with $(B^{br}_\sigma)_\sigma$ and $(B^{br'}_\sigma)_\sigma$ two Brownian bridges on $[0,\sigma(T)]$ and ξ_0 a Gaussian variable with mean 0 and variance $\sigma(T)$, independent of the Brownian bridges. This way we will be able to define $(B^2_t)_t$ (resp. $(B^{2'}_t)_t$) such that $dB^2_t = \frac{\sin(\sqrt{k}\varphi_t)}{\sqrt{k}} d\boldsymbol{\beta}_{\sigma(t)}$ (resp. $dB^{2'}_t = \frac{\sin(\sqrt{k}\varphi_t)}{\sqrt{k}} d\boldsymbol{\beta}'_{\sigma(t)}$). Using the Karhunen Loève decomposition of the Brownian bridges, for $\sigma \in [0,\sigma(T)]$, we can write:

$$B^{br}_\sigma = \sqrt{\sigma(T)} \sum_{j\geq 1} \xi_j \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi\sigma}{\sigma(T)}\right) \tag{5.3}$$

$$\left(\text{resp. } B^{br'}_\sigma = \sqrt{\sigma(T)} \sum_{j\geq 1} \xi'_j \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi\sigma}{\sigma(T)}\right)\right) \tag{5.4}$$

with $(\xi_j)_j$ (resp. $(\xi'_j)_j$) a sequence of independent standard Gaussian variables, independent of $(B^1_t)_t$. Note that, because of this independence with $(B^1_t)_t$, knowing $(B^1_s)_{s\in[0,T]}$, $B^2_t := \int_0^t \frac{\sqrt{k}}{\sin(\sqrt{k}\varphi_s)} d\boldsymbol{\beta}_{\sigma(s)}$ defines an almost surely continuous process with independents increments and such that $B^2_t \overset{\mathcal{L}}{\sim} \mathcal{N}(0,t)$, that is, a Brownian motion. As that distribution doesn't depend of the conditioning, $(B^1_t)_t$ and $(B^2_t)_t$ are two independent Brownian motions and our coupling is well defined.
We now explain how we chose $(\xi_j)_j$ and $(\xi'_j)_j$. If we take $\xi_j = \xi'_j$ for all $j \geq 2$, we get:

$$\boldsymbol{\beta}_\sigma - \boldsymbol{\beta}'_\sigma = (\xi_1 - \xi'_1) \frac{\sqrt{2\sigma(T)}}{\pi} \sin\left(\frac{\pi\sigma}{\sigma(T)}\right).$$

Note that, with this choice of $(\xi_j, \xi'_j)_{j\geq 2}$, X_t and X'_t are equal only for $t \in \{0,T\}$. When we look at the impact of the choice of (ξ_1,ξ'_1) on the swept areas, we have:

$$\begin{aligned}
I_t - I'_t &= z_0 - z'_0 + \int_0^t \frac{\tan\left(\frac{\sqrt{k}\varphi_s}{2}\right)}{\sqrt{k}} \frac{\sin(\sqrt{k}\varphi_s)}{\sqrt{k}} (\xi_1 - \xi'_1) \frac{\sqrt{2\sigma(T)}}{\pi} d\left(\sin\left(\frac{\pi\sigma(s)}{\sigma(T)}\right)\right) \\
&= z_0 - z'_0 + \int_0^t \frac{1}{k}\left(1 - \cos\left(\sqrt{k}\varphi_s\right)\right)(\xi_1 - \xi'_1) \frac{\sqrt{2\sigma(T)}}{\pi} \cos\left(\frac{\pi\sigma(s)}{\sigma(T)}\right) \frac{\pi d\left(\sigma(s)\right)}{\sigma(T)} \\
&= z_0 - z'_0 + \int_0^t \left(1 - \cos(\sqrt{k}\varphi_s)\right)(\xi_1 - \xi'_1) \sqrt{\frac{2}{\sigma(T)}} \cos\left(\frac{\pi\sigma(s)}{\sigma(T)}\right) \frac{ds}{\sin^2(\sqrt{k}\varphi_s)} \\
&= z_0 - z'_0 + K(t)\frac{\xi_1 - \xi'_1}{2} \\
&\text{with } K(t) = 2\sqrt{\frac{2}{\sigma(T)}} \int_0^t \frac{1}{1+\cos(\sqrt{k}\varphi_s)} \cos\left(\frac{\pi\sigma(s)}{\sigma(T)}\right) ds.
\end{aligned}$$

In order to obtain a successful coupling at time T, we need to have $I_T - I'_T \equiv 0 \mod (4\pi)$, that is, $K(T)\frac{\xi_1-\xi'_1}{2} \equiv -(z_0 - z'_0) \mod (4\pi)$. Let's take $(W_t)_t$ a Brownian motion independent of $(B^1_t)_t$, ξ_0 and $(\xi_j)_{j\geq 2}$. We define $\varsigma := \inf\{t | W_t \notin]-\frac{z_0-z'_0}{K(T)}, \frac{-(z_0-z'_0)+4\pi}{K(T)}[\}$ and $W'_t := \begin{cases} -W_t & \text{if } t \leq \varsigma \\ W_t - 2W_\varsigma & \text{else} \end{cases}$. Note that, by the strong Markov property, $(W'_t)_t$ is a real Brownian motion starting at 0 and independent of $K(T)$. We then chose $\xi_1 = W_1 \overset{\mathcal{L}}{\sim} \mathcal{N}(0,1)$ and $\xi'_1 = W'_1 \overset{\mathcal{L}}{\sim} \mathcal{N}(0,1)$.

In fact, with this construction we have: $\frac{\xi_1-\xi'_1}{2} = W_{1\wedge\varsigma}$. Thus, we get two cases:

- If $\varsigma \leq 1$, then $K(T)\frac{\xi_1 - \xi_1'}{2} = K(T)W_\varsigma \equiv -(z_0 - z_0')(4\pi)$.
- If $\varsigma > 1$, then $K(T)\frac{\xi_1 - \xi_1'}{2} = K(T)W_1 \not\equiv -(z_0 - z_0') \mod (4\pi)$.

We have $\mathbb{P}(I_T - I_T' \equiv 0 \mod (4\pi)) = \mathbb{P}(\varsigma \leq 1)$. For this probability to be positive, we need to ensure that $K(T) = 0$ does not occur a.s.. This can be obtained from the following Lemma:

Lemma 5.2.2. *Set $k \in \{\pm 1\}$. Let define $A(T) := \sqrt{\frac{2}{T}} \int_0^T \sin\left(\frac{\pi t}{T}\right) dB_t^1$. Then:*

$$K(T) = -\frac{2T}{\pi} A(T) + o\left(T^{\frac{3}{2}} \ln\left(\frac{1}{T}\right)\right) \tag{5.5}$$

with o the Landau's notation for an a.s. convergence with T close to 0. In particular, as $A(T)$ has a standard Gaussian distribution, we have $\frac{\pi K(T)}{2T} \xrightarrow[T\to 0]{\mathcal{L}} \mathcal{N}(0,1)$ and, thus, $\mathbb{P}(K(T) = 0) \xrightarrow[T\to 0]{} 0$.

The proof of Lemma 5.2.2 will be given in Subsection 5.2.3. From Lemma 5.2.2, we get that $-\frac{z_0 - z_0'}{K(T)}$ and $\frac{-(z_0 - z_0') + 4\pi}{K(T)}$ are finite with a non zero probability for T small enough. Moreover, by construction, $K(T)$ is independent of $(W_t)_t$. Thus we have:

$$0 < \mathbb{P}(\varsigma \leq 1).$$

Note that, with this strategy, as $(X_t)_t$ and $(X_t')_t$ only meet at time 0 or T, the coupling is successful after time T if and only if $\varsigma > 1$. Note also that, by defining $\tilde{\varsigma} := \inf\left\{t |\ |W_t| = \frac{4\pi}{|K(T)|}\right\}$, we have $\tilde{\varsigma} \geq \varsigma$ and thus $\mathbb{P}(\varsigma \leq 1)$ is bounded below by $\mathbb{P}(\tilde{\varsigma} \leq 1)$ that does not depend of the starting points (z_0, z_0').

This ends the proof of Proposition 5.2.1. □

Remark 5.2.3. *In fact, $K(T)$ can also be defined for $k = 0$ which would be the case where E_0 is the Heisenberg group $\mathbb{H}$ and $M_0 = \mathbb{R}^2$. Then Lemma 5.2.2 is also true for $k = 0$. Obviously, this is not the case for Proposition 5.2.1 as the "area coordinates" takes values in all $\mathbb{R}$ on the Heisenberg group.*

5.2.2 Proof of Theorem 5.1.1

We now have all the tools to construct the successful coupling if the starting points are in the same fiber. We first begin with the construction of an exponentially decreasing successful coupling without dependence with the starting points of the Brownian motion. We now only take $k \in \{-1, 1\}$. We recall that E_k denotes $SU(2)$ and $SL(2, \mathbb{R})$ depending of the value of k.

Proposition 5.2.4. *Let $g = (x, z)$, $g' = (x', z') \in E_k$. We suppose that $x = x'$.*
There exists a non co-adapted successful coupling of Brownian motions $(\mathbb{B}_t, \mathbb{B}_t')_t$ on E_k starting at (g, g') and T, $\tilde{C}$, $\tilde{c}$ some non negative constants that do not depend on the starting points of the processes, such that, for all $t > T$:

$$\mathbb{P}(\tau > t) \leq \tilde{C} e^{-\tilde{c}t}. \tag{5.6}$$

Proof of the Proposition 5.2.4. To define the coupling on $[0,+\infty[$, we divide the time in intervals $[t_n, t_{n+1}[$ with length T_n small enough, as in Lemma 5.2.2, and we repeat the coupling from Proposition 5.2.1. As we proved that the probability of success at time T_n is non zero, reproducing this strategy identically and independently on each interval $[t_n, t_{n+1}[$ should be efficient.
With this in mind, we consider $(T_n)_n$ constant with $T_n = T$. We define $K^n(T)$, $(W_t^n)_t$ and ς^n the objects used in the construction of the coupling from Proposition 5.2.1 for each interval $[t_n, t_{n+1}[$. It is true that the experiments will not be identical non independent as $\left(\varphi_{t_{n+1}}^{(N_0,e_0)}, \theta_{t_{n+1}}^{(N_0,e_0)}, I_{t_{n+1}}(N_0,e_0)\right)$ is, in general, not constant and dependent of $\left(\varphi_{t_n}^{(N_0,e_0)}, \theta_{t_n}^{(N_0,e_0)}, I_{t_n}(N_0,e_0)\right)$. To avoid this problem, the idea is to change the spherical/polar coordinate system on each interval of time $]t_n, t_{n+1}[$ by considering a sequence $(N_n, e_n)_n$ in TM_k such that the new sequence of coordinates $\left(\varphi_{t_n}^{(N_n,e_n)}, \theta_{t_n}^{(N_n,e_n)}\right)_n$ stays constant equal to (φ_0, θ_0).

To obtain a successful coupling on $SU(2)$ (resp. $SL(2,\mathbb{R})$), we need to obtain $\zeta_{t_n} \equiv 0 \mod (4\pi)$ for some n. It is true that, for any $(N,e) \neq (N_0,e_0)$, we have in general $\zeta_t \neq I_t(N,e) - I_t'(N,e)$. However, using Remark 3.2.13, we have $\zeta_t \equiv I_t'(N_n,e_n) - I_t(N_n,e_n) \mod (4\pi)$ at last at times $t = t_n$ for all n (because $X_{t_n} = X'_{t_n}$ for all n). Thus, the coupling is successful if and only if there exists n such that $I'_{t_n}(N_n,e_n) - I_{t_n}(N_n,e_n) \equiv 0 \mod (4\pi)$. We consider the variables $\tilde{\varsigma}^n := \inf\left\{t|\ |W_t| \leq \frac{4\pi}{|K^n(T)|}\right\}$ as introduced in the last part of Proposition 5.2.1. By choice of $(T_n)_n$ and $(N_n,e_n)_n$, we have $(K^n(T))_n$ independently and identically distributed and thus $(\tilde{\varsigma}^n)_n$ is too. In particular $\tilde{\varsigma}^n \geq \varsigma^n$ for all n. This way we get:

$$\begin{aligned}\mathbb{P}(\tau > t_n) &= \mathbb{P}(\varsigma^i > 1 \ \forall\ 0 \leq i \leq n-1) \\ &\leq \mathbb{P}(\tilde{\varsigma}^i > 1 \ \forall\ 0 \leq i \leq n-1) = \mathbb{P}(\tilde{\varsigma}^0 > 1)^n. \end{aligned} \tag{5.7}$$

This last quantity tends to zero when $n \to +\infty$, thus τ is a.s. finite, the coupling is successful and the coupling rate is clearly exponentially decreasing. More precisely, we obtain for $t \in [t_n, t_{n+1}[$:

$$\begin{aligned}\mathbb{P}(\tau > t) \leq \mathbb{P}(\tau > t_n) &\leq \exp\left(-n \ln\left(\frac{1}{\mathbb{P}(\tilde{\varsigma}^0 > 1)}\right)\right) \\ &= \frac{1}{\mathbb{P}(\tilde{\varsigma}^0 > 1)} \exp\left(-(n+1)T \frac{\ln\left(\frac{1}{\mathbb{P}(\tilde{\varsigma}^0>1)}\right)}{T}\right) \leq \tilde{C} \exp\left(-t\tilde{c}\right)\end{aligned}$$

with $\tilde{C} = \frac{1}{\mathbb{P}(\tilde{\varsigma}^0>1)}$ and $\tilde{c} = \frac{1}{T}\ln\left(\frac{1}{\mathbb{P}(\tilde{\varsigma}^0>1)}\right)$.
Note that, if we change the system of coordinates (N_0,e_0) on M_k at the first step, we can chose $(\varphi_{t_n}^{(N_n,e_n)}, \theta_{t_n}^{(N_n,e_n)})_n$ constant equal to a value chosen independent of the initial position (φ_0,θ_0) of the Brownian motions. Thus the random variables $K^n(T)$ and $\tilde{\varsigma}^n$ do not depend of these starting points and $\mathbb{P}(\tilde{\varsigma}^0 > 1)$ neither. The coupling rate obtained in this case does not depend of the starting points. □

We now want to study how the coupling built above depends on the starting points of the Brownian motions.

Proposition 5.2.5. *Set $k \in \{\pm 1\}$. Let $g = (x, z)$, $g' = (x', z') \in E_k$. We suppose that $x = x'$. There exists a non co-adapted coupling of Brownian motions $(\mathbb{B}_t, \mathbb{B}'_t)_t$ on E_k starting at (g, g') and some non negative constant C that does not depend on g and g' such that for $|\zeta_0|$small enough (with (ζ_t) defined as in Corollary 3.2.10):*

$$\mathbb{P}(\tau > 1) \leq C|\zeta_0| \ln\left(\frac{1}{|\zeta_0|}\right). \tag{5.8}$$

In particular, for all $0 < q < 1$, there exists a coupling and some non negative constant $\tilde{C}_q$ (still not depending on the starting points g and g') such that for all $\zeta_0 > 0$:

$$\mathbb{P}(\tau > 1) \leq \left(\tilde{C}_q \times |\zeta_0|^q\right) \wedge 1. \tag{5.9}$$

Proof of Proposition 5.2.5. Without loosing generality, we can still suppose that $z_0 - z'_0 \in]-4\pi, 4\pi]$. Then, from Corollary 3.2.10, as $X_0 = X'_0$, we have exactly $\zeta_0 = z_0 - z'_0$, thus, (5.8) can be written:

$$\mathbb{P}(\tau > 1) \leq C|z_0 - z'_0| \ln\left(\frac{1}{|z_0 - z'_0|}\right).$$

Let n be an integer that we will specify later, we consider $T = \frac{1}{n}$. We construct on $[0, 1]$ the coupling described in Proposition 5.2.4 using the decomposition in the intervals $[t_j, t_{j+1}]$ with $t_j := jT$ and $t_{j+1} := (j+1)T$. We simply denote by $I_t - I'_t$ the concatenation of all the $(I_t(N_{j+1}) - I'_t(N_{j+1}))_{t\in[t_j,t_{j+1}[}$. Observing that, for $t \in [0, 1]$, we have:

$$I_t - I'_t = z_0 - z'_0 + \sum_{j=0}^{n-1} K^j(t \wedge T) W^j_{1\wedge\varsigma^j}, \tag{5.10}$$

we define:

$$M_t := z_0 - z'_0 + \sum_{j=0}^{n-1} K^j(T) W^j_{1\wedge\left(\frac{t-t_j}{T}\right)} \mathbb{1}_{\{t\geq t_j\}}. \tag{5.11}$$

On the event $\tau > 1$, we have $\varsigma^j > 1$ for all $j \leq n-1$, and $M_t = I_t - I'_t$ at times t_j. Moreover, by construction of ς_i, $M_t \not\equiv 0 \mod (4\pi)$ for all $t \in [0, 1]$. As a consequence, $\tau > 1$ if and only if $M_t \not\equiv 0 \mod (4\pi)$ for all $t \leq 1$.
As $(M_t)_t$ is a martingale, for all $t \in [0, 1]$, using the change of time defined by $S(t) := \sum_{j=0}^{n-1} K^j(T)^2 \left(\frac{t-t_j}{T}\right) \mathbb{1}_{\{t\geq t_j\}}$, we can write $M_t = z_0 - z'_0 + C_{S(t)}$ with $(C_s)_s$ a Brownian motion starting at 0. As in Lemma A.0.1, we denote $D_{-(z_0-z'_0)} := \inf\{s > 0 | C_s = -(z_0 - z'_0)\}$ and we get:

$$\begin{aligned}\mathbb{P}(\tau > 1) &= \mathbb{P}(z_0 - z'_0 + C_{S(t)} \in]0, 4\pi[\text{ for all } t \leq 1) \\ &\leq \mathbb{P}(D_{-(z_0-z'_0)} > S(nT)) = \mathbb{P}\left(D_{-(z_0-z'_0)} > \sum_{j=0}^{n-1} K^j(T)^2\right).\end{aligned} \tag{5.12}$$

We separate the cases where $|K^j(T)|$ is large enough and the cases where it is not. We choose $0 < p_0 < 1$ such that $p_0 > \mathbb{P}(|G| \leq \frac{1}{2})$ with $G \overset{\mathcal{L}}{\sim} \mathcal{N}(0, 1)$. From Lemma 5.2.2, using the convergence in law, there exists $T_0 > 0$ such that for all $T \leq T_0$, $\mathbb{P}\left(\frac{\pi|K(T)|}{2T} \leq \frac{1}{2}\right) < p_0$.

If $T \leq T_0$, we get:

$$\begin{aligned}
\mathbb{P}(\tau > 1) &= \mathbb{P}\left(\{\tau > 1\} \cap \left\{\exists j \in \{0, ..., n-1\} \;\middle|\; \frac{\pi|K^j(T)|}{2T} > \frac{1}{2}\right\}\right) \\
&\quad + \mathbb{P}\left(\{\tau > 1\} \cap \left\{\forall j \in \{0, ..., n-1\},\ \frac{\pi|K^j(T)|}{2T} \leq \frac{1}{2}\right\}\right) \\
&\leq \mathbb{P}\left(D_{-(z_0 - z_0')} > \sum_{j=0}^{n-1} K^j(T)^2 > \frac{T^2}{\pi^2}\right) + \mathbb{P}\left(\frac{\pi|K^j(T)|}{2T} \leq \frac{1}{2}\right)^n \\
&\leq \sqrt{2\pi}\frac{|z_0 - z_0'|}{T} + p_0^n \qquad (5.13)
\end{aligned}$$

where we use Lemma A.0.1 to get the left hand side term in (5.13). Finally, we have

$$\mathbb{P}(\tau > 1) \leq \sqrt{2\pi}|z_0 - z_0'|n + p_0^n.$$

If we chose n such that $\frac{\ln(|z_0 - z_0'|)}{\ln(p_0)} \leq n \leq \frac{\ln(|z_0 - z_0'|)}{\ln(p_0)} + 1$, we get:

- $p_0^n < |z_0 - z_0'|$;
- $|z_0 - z_0'|n \leq |z_0 - z_0'|\left(\frac{\ln(|z_0 - z_0'|)}{\ln(p_0)} + 1\right)$.

Moreover, if $|z_0 - z_0'| \leq p_0^{\frac{1}{T_0}}$, we have $\frac{1}{n} \leq \frac{\ln(p_0)}{\ln(|z_0 - z_0'|)}$, and thus, $T \leq T_0$. Then we can use (5.13) and we obtain Inequality (5.8) and Inequality (5.9).

For $|z_0 - z_0'| > p_0^{\frac{1}{T_0}}$, inequality (5.9) can be obtained directly from (5.6). □

Remark 5.2.6. *In (5.8), for $|\zeta_0| < p_0^{\frac{1}{T_0}}$ (with p_0 and T_0 defined as in the proof), we can replace $\mathbb{P}(\tau > 1)$ by $\mathbb{P}(\tau > t)$ for all $t > 0$. The coupling strategy is slightly different since it depends on the given time $t > 0$. In particular, the obtained constant C will be of order $\frac{1}{t}$ for t close to 0.*

Remark 5.2.7. *Note here that, if $\frac{1}{\sqrt{\sum_{j=0}^{n-1} K^j(T)^2}}$ is integrable, then,*

$$\mathbb{P}(\tau > nT) \leq C|z_0 - z_0'|$$

(with C independent of the starting points) which would be better than the obtained inequality. Here, contrary to the coupling in [6] we have not been able to prove this integrability.

As discussed before in Remark 2.3.4, if we had this integrability, then this coupling would be successful even for some areas living in $\mathbb{R}$. In particular, this could lead to a successful coupling on the universal covering of $SL(2, \mathbb{R})$. This could also lead to a new strategy of successful couplings on the Heisenberg group defining the coupling with polar coordinates instead of Cartesian coordinates as it is made in [6] .

Remark 5.2.8. *The process $(M_t)_t$ introduced in the above proof is the one used in [6] to deal with the case of the Heisenberg group. We can also use it to obtain a proof for Proposition 5.2.4. Denoting $H = \inf\{t > 0 | C_t \notin]-(z_0 - z_0'), 4\pi - (z_0 - z_0')[\}$, we get:*

$$\begin{aligned}
\mathbb{P}(\tau > t_n) &= \mathbb{P}(z_0 - z_0' + C_{S(s)} \in (0, 4\pi) \text{ for all } s \leq t_n) \\
&= \mathbb{P}(H > S(t_n)) = \mathbb{P}\left(H > \sum_{k=0}^{n-1} K^k(T)^2\right).
\end{aligned}$$

Taking some $\delta > 0$ such that $\sqrt{\frac{\delta}{2}} \times 4\pi \not\equiv \frac{\pi}{2} \mod(\pi)$, and using Lemma A.0.2:

$$\mathbb{P}(H > u) = \mathbb{E}[e^{\delta H} e^{-\delta H} \mathbb{1}_{\{H>u\}}] \leq e^{-\delta u} \mathbb{E}[e^{\delta H}] \leq e^{-\delta u} \frac{\cos\left(\sqrt{\frac{\delta}{2}}(4\pi - 2(z_0 - z_0'))\right)}{\cos\left(\sqrt{\frac{\delta}{2}} \times 4\pi\right)}.$$

Then, $\mathbb{P}(\tau > t) \leq \mathbb{E}[e^{-\delta S(t_n)}] \frac{1}{\cos\left(\sqrt{\frac{\delta}{2}} \times 4\pi\right)}$. As $(K^k(T))_k$ is a sequence of independent and identically distributed variables, we get:

$$\mathbb{E}\left[e^{-\delta S(t_n)}\right] = \prod_{k=0}^{n-1} \mathbb{E}\left[e^{-\delta (K^k(T))^2}\right] = \mathbb{E}\left[e^{-\delta (K^0(T))^2}\right]^n \leq e^{nT \frac{\ln\left(\mathbb{E}\left[\exp\left(-\delta (K^0(T))^2\right)\right]\right)}{T}}.$$

As $\mathbb{P}\left(K^0(T)^2 = 0\right) < 1$, we have $\mathbb{E}\left[\exp\left(-\delta K^0(T)^2\right)\right] < 1$ and $\mathbb{E}\left[e^{-\delta S(t_n)}\right] \leq e^{-nT \frac{c(\delta,T)}{T}}$ with $c(\delta, T) = -\ln\left(\mathbb{E}\left[\exp\left(-\delta K^0(T)^2\right)\right]\right) > 0$. This gives the expected rate of convergence.

We can now give the final construction of the successful coupling from Theorem 5.1.1:

Proof of Theorem 5.1.1. We first use the coupling from Proposition 5.2.5 on $[0, 1]$ and, then we construct the rest of the coupling using Proposition 5.2.4 on $[1, \tau]$. We have, for $t > 1$:

$$\begin{aligned}\mathbb{P}(\tau > t) &= \mathbb{P}(\tau > 1)\mathbb{P}(\tau > t | \tau > 1) \\ &\leq \tilde{C}_q \times |\zeta_0|^q \tilde{C} \exp(-(t-1)\tilde{c}).\end{aligned}$$

As $X_0 = X_0'$, we have $c_1\sqrt{|\zeta_0|} \leq d_{cc}(\mathbb{B}_0, \mathbb{B}_0') \leq c_2\sqrt{|\zeta_0|}$. This gives the expected inequality. □

5.2.3 Behaviour of $K(T)$ for T close to 0

In this subsection we prove the asymptotic behaviour of $K(T)$ stated in Lemma 5.2.2.

Proof. The proof is using series expansion for T close to 0. In all that follows Landau's notations o and $\mathcal{O}$ are used for an a.s. convergence with T close to 0. We give the proof for $k \neq 0$ but note that the same method can be used for $k = 0$. Let $t \in [0, T]$. We first claim that:

$$\begin{aligned}\sigma(t) :&= \int_0^t \frac{k}{\sin^2(\sqrt{k}\varphi_s)} ds \\ &= \frac{kt}{\sin^2(\sqrt{k}\varphi_0)} \left(1 - 2\sqrt{k}\cot(\sqrt{k}\varphi_0)\frac{1}{t}\int_0^t B_s^1 ds + o\left(T \ln\left(\frac{1}{T}\right)\right)\right). \qquad (5.14)\end{aligned}$$

Indeed for $s \in [0, t]$, using the Itô formula, we have:

$$\begin{aligned}\sin(\sqrt{k}\varphi_s) = \sin(\sqrt{k}\varphi_0) &+ \int_0^s \sqrt{k}\cos(\sqrt{k}\varphi_u) dB_u^1 \\ &+ \frac{k}{2}\int_0^s \left(-\sin(\sqrt{k}\varphi_u) + \frac{\cos^2(\sqrt{k}\varphi_u)}{\sin(\sqrt{k}\varphi_u)}\right) du.\end{aligned}$$

We remind that, using the law of the iterated logarithm, we have, for s small enough: $B_s = o\left(\sqrt{s\ln\left(\frac{1}{s}\right)}\right)$. More generally, if we consider the martingale $M_s := \int_0^s v(\omega,u)dB_u^1$, for $s \to 0$, we have:

$$M_s = o\left(\sqrt{\langle M_s, M_s\rangle \ln\left(\frac{1}{\langle M_s, M_s\rangle}\right)}\right). \tag{5.15}$$

Indeed, we just have to use Dambis-Dubins-Schwartz Theorem to write $(M_s)_s$ as a time-changed Brownian motion. Then the law of iterated logarithm gives the expected result. Thus, for $s \to 0$, using (5.15) for $v(u) = \sqrt{k}\cos(\sqrt{k}\varphi_u)$ and remarking that $\int_0^s v(u)^2 du = \mathcal{O}(s)$ (we use the continuity of φ and the compactness of $[0,T]$), we get:

$$\int_0^s \sqrt{k}\cos(\sqrt{k}\varphi_u)dB_u^1 = o\left(\sqrt{s\ln\left(\frac{1}{s}\right)}\right).$$

Thus:

$$\sin^2(\sqrt{k}\varphi_s) = \sin^2(\sqrt{k}\varphi_0) + 2\sin(\sqrt{k}\varphi_0)\int_0^s \sqrt{k}\cos(\sqrt{k}\varphi_u)dB_u^1 + o\left(s\ln\left(\frac{1}{s}\right)\right)$$

The same way, using the Itô formula and relation (5.15), we have:

$$\begin{aligned}\cos(\sqrt{k}\varphi_u) &= \cos(\sqrt{k}\varphi_0) - \int_0^u \sqrt{k}\sin(\sqrt{k}\varphi_v)dB_v^1 - k\int_0^u \cos(\sqrt{k}\varphi_v)dv \\ &= \cos(\sqrt{k}\varphi_0) + \epsilon(u).\end{aligned} \tag{5.16}$$

with $\epsilon(u) = o\left(\sqrt{u\ln\left(\frac{1}{u}\right)}\right)$. In particular, $\int_0^s \epsilon(u)^2 du = o\left(s^2\ln\left(\frac{1}{s}\right)\right)$. Thus, applying (5.15) to $\int_0^s \epsilon(u)dB_u^1$, we get:

$$\int_0^s \sqrt{k}\cos(\sqrt{k}\varphi_u)dB_u^1 = \sqrt{k}\cos(\sqrt{k}\varphi_0)B_s^1 + o\left(s\ln\left(\frac{1}{s}\right)\right).$$

Finally we obtain:

$$\begin{aligned}\sin^2(\sqrt{k}\varphi_s) &= \sin^2(\sqrt{k}\varphi_0) + 2\sqrt{k}\sin(\sqrt{k}\varphi_0)\cos(\sqrt{k}\varphi_0)B_s^1 + o\left(s\ln\left(\frac{1}{s}\right)\right) \\ &= \sin^2(\sqrt{k}\varphi_0)\left(1 + 2\sqrt{k}\cot(\sqrt{k}\varphi_0)B_s^1 + o\left(T\ln\left(\frac{1}{T}\right)\right)\right) \text{ as } s \le T.\end{aligned}$$

and:

$$\frac{1}{\sin^2(\sqrt{k}\varphi_s)} = \frac{1}{\sin^2(\sqrt{k}\varphi_0)}\left(1 - 2\sqrt{k}\cot(\sqrt{k}\varphi_0)B_s^1 + o\left(T\ln\left(\frac{1}{T}\right)\right)\right).$$

We just have to integrate this expression to obtain (5.14). We can then deduce an estimate for $\cos\left(\frac{\pi\sigma(t)}{\sigma(T)}\right)$. Indeed, we have:

$$\frac{1}{\sigma(T)} = \frac{\sin^2(\sqrt{k}\varphi_0)}{kT}\left(1 + 2\sqrt{k}\cot(\sqrt{k}\varphi_0)\frac{1}{T}\int_0^T B_s^1 ds + o\left(T\ln\left(\frac{1}{T}\right)\right)\right). \tag{5.17}$$

Thus, as $\frac{1}{t}\int_0^t B_s^1 ds = o\left(\sqrt{T\ln\left(\frac{1}{T}\right)}\right)$ for all $0 \le t \le T$, we obtain:

$$\frac{\pi\sigma(t)}{\sigma(T)} = \frac{\pi t}{T}\left(1 - 2\sqrt{k}\cot(\sqrt{k}\varphi_0)\left(\frac{1}{t}\int_0^t B_s^1 ds - \frac{1}{T}\int_0^T B_s^1 ds\right) + o\left(T\ln\left(\frac{1}{T}\right)\right)\right).$$

Finally:

$$\cos\left(\frac{\pi\sigma(t)}{\sigma(T)}\right) = \cos\left(\frac{\pi t}{T}\right) + \sin\left(\frac{\pi t}{T}\right)\frac{\pi t}{T} \times 2\sqrt{k}\cot(\sqrt{k}\varphi_0)\left(\frac{1}{t}\int_0^t B_s^1 ds - \frac{1}{T}\int_0^T B_s^1 ds\right) + o\left(T\ln\left(\frac{1}{T}\right)\right).$$

Using the same methods as in the first part of the proof, we have:

$$\sqrt{k}\sin(\sqrt{k}\varphi_u) = \sqrt{k}\sin(\sqrt{k}\varphi_0) + \epsilon(u) \text{ with } \epsilon(u) = o\left(\sqrt{u\ln\left(\frac{1}{u}\right)}\right)$$

$$\text{and so: } \cos(\sqrt{k}\varphi_t) = \cos(\sqrt{k}\varphi_0) - \int_0^t \sqrt{k}\sin(\sqrt{k}\varphi_u)dB_u^1 - k\int_0^u \cos(\sqrt{k}\varphi_u)du$$

$$= \cos(\sqrt{k}\varphi_0) - \sqrt{k}\sin(\sqrt{k}\varphi_0)B_t^1 + o\left(T\ln\left(\frac{1}{T}\right)\right).$$

Then we get:

$$\begin{aligned}\frac{1}{1+\cos(\sqrt{k}\varphi_t)} &= \frac{1-\cos(\sqrt{k}\varphi_t)}{\sin^2(\varphi_t)} \\ &= \left(1-\cos(\sqrt{k}\varphi_0) + \sqrt{k}\sin(\sqrt{k}\varphi_0)B_t^1 + o\left(T\ln\left(\frac{1}{T}\right)\right)\right) \\ &\times \frac{1}{\sin^2(\sqrt{k}\varphi_0)}\left(1 - 2\sqrt{k}\cot(\sqrt{k}\varphi_0)B_t^1 + o\left(T\ln\left(\frac{1}{T}\right)\right)\right) \\ &= \frac{1-\cos(\sqrt{k}\varphi_0) + \sqrt{k}\left(\sin(\sqrt{k}\varphi_0) - 2\cot(\sqrt{k}\varphi_0)(1-\cos(\sqrt{k}\varphi_0))\right)B_t^1}{\sin^2(\sqrt{k}\varphi_0)} \\ &+ o\left(T\ln\left(\frac{1}{T}\right)\right).\end{aligned}$$

We can now finalize the computation of $K(T) = \sqrt{\frac{2}{\sigma(T)}}\int_0^T \frac{2}{1+\cos(\sqrt{k}\varphi_t)}\cos\left(\frac{\pi\sigma(t)}{\sigma(T)}\right)dt$. Using the previous results, we get:

$$\frac{2}{1+\cos(\sqrt{k}\varphi_t)}\cos\left(\frac{\pi\sigma(t)}{\sigma(T)}\right) = \frac{2}{\sin^2(\sqrt{k}\varphi_0)}\big(R_1(t)+R_2(t)+R_3(t)\big) + o\left(T\ln\left(\frac{1}{T}\right)\right).$$

With

$$R_1(t) := \left(1-\cos(\sqrt{k}\varphi_0)\right)\cos\left(\frac{\pi t}{T}\right)$$

$$R_2(t) := \sqrt{k}\left(\sin(\sqrt{k}\varphi_0) - 2\cot(\sqrt{k}\varphi_0)\left(1-\cos(\sqrt{k}\varphi_0)\right)\right)B_t^1\cos\left(\frac{\pi t}{T}\right)$$

$$R_3(t) := \sqrt{k}\left(1-\cos(\sqrt{k}\varphi_0)\right) \times 2\sin\left(\frac{\pi t}{T}\right)\frac{\pi t}{T}\cot(\sqrt{k}\varphi_0)\left(\frac{1}{t}\int_0^t B_s^1 ds - \frac{1}{T}\int_0^T B_s^1 ds\right).$$

In particular $\int_0^T R_1(t)dt$ vanishes. We also have:

$$\begin{aligned}
&\int_0^T \sin\left(\frac{\pi t}{T}\right)\frac{\pi t}{T}\left(\frac{1}{t}\int_0^t B_s^1 ds - \frac{1}{T}\int_0^T B_s^1 ds\right)dt \\
&= \int_0^T \frac{\pi}{T}\sin\left(\frac{\pi t}{T}\right)\int_0^t B_s^1 ds dt - \frac{1}{T}\int_0^T B_s^1 ds\left(\left[-t\cos\left(\frac{\pi t}{T}\right)\right]_0^T + \int_0^T \cos\left(\frac{\pi t}{T}\right)dt\right) \\
&= \left(\left[-\cos\left(\frac{\pi t}{T}\right)\int_0^t B_s^1 ds\right]_0^T + \int_0^T \cos\left(\frac{\pi t}{T}\right)B_t^1 dt\right) - \int_0^T B_s^1 ds \\
&= \int_0^T \cos\left(\frac{\pi t}{T}\right)B_t^1 dt.
\end{aligned}$$

Thus $\int_0^T (R_2(t)+R_3(t))\,dt = \sqrt{k}\sin(\sqrt{k}\varphi_0)\int_0^T B_t^1\cos\left(\frac{\pi t}{T}\right)dt$.
As $\int_0^T B_t^1\cos\left(\frac{\pi t}{T}\right)dt = -\frac{T}{\pi}\int_0^T \sin\left(\frac{\pi t}{T}\right)dB_t^1$, we obtain:

$$\int_0^T \frac{2\cos\left(\frac{\pi\sigma(t)}{\sigma(T)}\right)}{1+\cos(\sqrt{k}\varphi_t)}dt = -\frac{2\sqrt{k}T}{\sin(\sqrt{k}\varphi_0)\pi}\int_0^T \sin\left(\frac{\pi t}{T}\right)dB_t^1 + o\left(T^2\ln\left(\frac{1}{T}\right)\right).$$

Using (5.17) we also have $\sqrt{\frac{1}{\sigma(T)}} = \frac{\sin(\sqrt{k}\varphi_0)}{\sqrt{kT}}\left(1+o\left(\sqrt{T\ln\left(\frac{1}{T}\right)}\right)\right)$ and:

$$K(T) = \frac{-2\sqrt{2T}}{\pi}\int_0^T \sin\left(\frac{\pi t}{T}\right)dB_t^1 + o\left(T^{\frac{3}{2}}\ln\left(\frac{1}{T}\right)\right).$$

As $\int_0^T \sin^2\left(\frac{\pi t}{T}\right)dt = \frac{T}{2}$, the distribution of $A(T) := \sqrt{\frac{2}{T}}\int_0^T \sin\left(\frac{\pi t}{T}\right)dB_t^1$ is a standard Gaussian and $K(T) = -\frac{2T}{\pi}A(T) + o\left(T^{\frac{3}{2}}\ln\left(\frac{1}{T}\right)\right)$. □

5.2.4 A study of the event $\{K(T)=0\}$

As said in Remark 5.2.7, we are not really satisfied about the knowledge we have of the random variable $K(T)$. According to Lemma 5.2.2, we have information for T small enough only. During the preparation of this work we however developed another strategy based on the use of a simplified version of Malliavin calculus as can been found in [46].Taking account of Remark 5.2.3, we show the following result for $k \in \{-1,0,1\}$:

Lemma 5.2.9. *Supposing that $\varphi_0 > 0$ for $k \le 0$ and $\varphi_0 = \frac{\pi}{2\sqrt{k}}$ for $k > 0$, we have $K(T) \neq 0$ a.s..*

In particular, this allows us to prove Proposition 5.2.1 and Proposition 5.2.4 without any restriction on T. However, we did not succeed to obtain information about $\frac{1}{\sqrt{\sum_{j=0}^{n-1} K^j(T)^2}}$ with this strategy.

Proof of Lemma (5.2.9). We consider a real deterministic variable a and a functional $h : \Omega\times[0,T] \mapsto \mathbb{R}$ that we suppose admissible in the sense that $(h(t))_{0\le t\le T}$ is adapted to the filtration induced by $(B_t^1)_{0\le t\le T}$ and that the process M_t^{ah} defined by $M_t^{ah} = e^{-a\int_0^t h(s)dB_s - \frac{a^2}{2}h(s)^2 ds}$ is a martingale on $[0,T]$ at least for a near 0. To simplify the notations, as T is fixed, we will denote $K(T)$ by K. We then define:

- $B_t^{ah} := B_t^1 + \int_0^t ah(s)ds$;
- φ^{ah} such that: $\begin{cases} d\varphi_t^{ah} = dB_t^{ah} + \frac{\sqrt{k}}{2}\cot(\sqrt{k}\varphi_t^{ah})dt \\ \varphi_0^{ah} = \varphi_0 \end{cases}$;
- $\sigma^{ah}(t) := \int_0^t \frac{k}{\sin^2(\sqrt{k}\varphi_s^{ah})}ds$;
- $K^{ah} := 2\sqrt{\frac{2}{\sigma^{ah}(T)}}\int_0^T \frac{1}{1+\cos(\sqrt{k}\varphi_s^{ah})}\cos\left(\frac{\pi\sigma^{ah}(s)}{\sigma^{ah}(T)}\right)ds$.

Using Girsanov theorem, as M^{ah} is a martingale on $[0,T]$, we can define a new probability measure $d\mathbb{P}^{ah}_{|\mathcal{F}_t} := M_t^{ah}d\mathbb{P}_{|\mathcal{F}_t}$ for $0 \leq t \leq T$, such that $(B^{ah}, \varphi^{ah}, \sigma^{ah}, K^{ah})$ has the same law under $\mathbb{P}^{ah}$ than $(B^1, \varphi, \sigma, K)$ under $\mathbb{P}$. Let us first suppose that we have the two following strong hypothesis:

- Hypothesis $\mathscr{H}_1$: h is bounded;
- Hypothesis $\mathscr{H}_2$: $\sup_{|a|\leq m}\frac{d}{da}K^{ah} \in L^2(d\mathbb{P})$ for all $m > 0$.

For every $f : \mathbb{R} \mapsto \mathbb{R}$, bounded and $\mathcal{C}^1$, we have:

$$\frac{d}{da}\mathbb{E}_{\mathbb{P}^{ah}}[f(K^{ah})] = \frac{d}{da}\mathbb{E}\left[M_T^{ah}f(K^{ah})\right] \tag{5.18}$$

$$= \mathbb{E}\left[\left(-\int_0^T h(s)dBs - a\int_0^T h(s)^2ds\right)M_T^{ah}f(K^{ah})\right] + \mathbb{E}\left[M_T^{ah}f'(K^{ah})\frac{d}{da}(K^{ah})\right]. \tag{5.19}$$

Indeed, for any $m > 0$, we have

$$\sup_{|a|\leq m}|M_T^{ah}| \leq \exp\left(m\left|\int_0^T h(s)dB_s\right|\right) \text{ with } \int_0^T h(s)dB_s \overset{\mathcal{L}}{\sim} \mathcal{N}\left(0, \int_0^T h(s)^2ds\right).$$

In particular, as h is bounded, we have $\exp\left(m\left|\int_0^T h(s)dB_s\right|\right) \in L^2(d\mathbb{P})$. With the same arguments,

$$\sup_{|a|\leq m}|\frac{d}{da}M_T^{ah}| \leq \left(\left|\int_0^T h(s)dB_s\right| + m\|h\|_\infty\right)\exp\left(m\left|\int_0^T h(s)dB_s\right|\right) \in L^2(d\mathbb{P}).$$

Thus, under the hypothesis $\mathscr{H}_1$ and $\mathscr{H}_2$, we can differentiate under the expectation sign which leads to (5.18).

As $\mathbb{E}_{\mathbb{P}^{ah}}[f(K^{ah})]$ is constant, we obtain:

$$\mathbb{E}[M_T^{ah}f'(K^{ah})\frac{d}{da}(K^{ah})] = \mathbb{E}\left[\left(\int_0^T h(s)dBs + a\int_0^T h(s)^2ds\right)M_t^{ah}f(K^{ah})\right].$$

In particular for $a = 0$:

$$\mathbb{E}[f'(K)\frac{d}{da}_{|a=0}(K^{ah})] = \mathbb{E}\left[\int_0^T h(s)dB_s f(K)\right]. \tag{5.20}$$

For $\epsilon > 0$, we define f_ϵ smooth such that $f_\epsilon(x) := \begin{cases} 2\epsilon & \text{if } 2\epsilon < x \\ x & \text{if } -\epsilon < x < \epsilon \\ -2\epsilon & \text{if } x < -2\epsilon \end{cases}$ and such that $|f'_\epsilon(x)| \leq 1$ for all $x \in \mathbb{R}$. Applying (5.20) to f we have:

$$\left|\mathbb{E}\left[f'_\epsilon(K)\mathbb{1}_{\{|K|\leq\epsilon\}}\frac{d}{da}_{|a=0}(K^{ah})\right]\right| \leq \mathbb{E}\left[\int_0^T h^2(s)ds\right]^{\frac{1}{2}}\mathbb{E}\left[f_\epsilon(K)^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\left|f'_\epsilon(K)\mathbb{1}_{\{|K|>\epsilon\}}\frac{d}{da}_{|a=0}(K^{ah})\right|\right].$$

We then get:

$$\left|\mathbb{E}\left[\mathbb{1}_{\{|K|\leq\epsilon\}}\frac{d}{da}_{|a=0}(K^{ah})\right]\right| \leq 2\epsilon\mathbb{E}\left[\int_0^T h^2(s)ds\right]^{\frac{1}{2}} + \mathbb{E}\left[\mathbb{1}_{\{2\epsilon>|K|>\epsilon\}}\frac{d}{da}_{|a=0}(K^{ah})\right].$$

By passing to the limit for $\epsilon \to 0$, we obtain:

$$\mathbb{E}\left[\mathbb{1}_{\{K=0\}}\frac{d}{da}_{|a=0}(K^{ah})\right] = 0. \tag{5.21}$$

Our strategy is now to compute $\frac{d}{da}_{|a=0}(K^{ah})$ and choose a good h such that relation (5.21) can be true only for $K \neq 0$ a.s.
As the differential stochastic equation defining φ^{ah} is smooth in a, $a \mapsto \varphi^{ah}$ is smooth in a too. Denoting $X_t^{ah} := \frac{d}{da}\varphi_t^{ah}$, we have:

$$\begin{cases} dX_t^{ah} = h(t)dt - \frac{k}{2\sin^2(\sqrt{k}\varphi_t^a)}X_t^{ah}dt \\ X_0^{ah} = 0 \end{cases}.$$

Thus using the method of variation of the constant, we obtain an explicit expression for X_t^{ah}:

$$X_t^{ah} = \int_0^t h(u)e^{-\frac{k}{2}\int_u^t \frac{1}{\sin^2(\sqrt{k}\varphi_s^{ah})}ds}du = \int_0^t h(u)e^{-\frac{1}{2}(\sigma^{ah}(t)-\sigma^{ah}(u))}du.$$

Note that, providing $\sin(\sqrt{k}\varphi_0) \neq 0$, we have $\sin(\sqrt{k}\varphi_s) \neq 0$ for all $s \in [0,T]$ a.s. By absolute continuity of $\mathbb{P}$ with respect to $\mathbb{P}^{ah}$, we also get $\sin(\sqrt{k}\varphi_s^{ah}) \neq 0$ for all $s \in [0,T]$ a.s. Using this together with the continuity of φ^{ah}, we obtain:

$$\begin{aligned} \frac{d}{da}\sigma^{ah}(t) &= \int_0^t \frac{-2k\sqrt{k}\cos(\sqrt{k}\varphi_s^{ah})X_s^{ah}}{\sin^3(\sqrt{k}\varphi_s^{ah})}ds \\ &= \int_0^t \frac{-2k\sqrt{k}\cos(\sqrt{k}\varphi_s^{ah})}{\sin^3(\sqrt{k}\varphi_s^{ah})}\int_0^s h(u)e^{-\frac{1}{2}(\sigma^{ah}(s)-\sigma^{ah}(u))}duds \\ &= \int_0^t h(u)\int_u^t e^{-\frac{1}{2}(\sigma^{ah}(s)-\sigma^{ah}(u))}\frac{-2k\sqrt{k}\cos(\sqrt{k}\varphi_s^{ah})}{\sin^3(\sqrt{k}\varphi_s^{ah})}dsdu \end{aligned} \tag{5.22}$$

We then have:

$$\frac{d}{da}K^{ah} = -\frac{1}{2}\frac{d}{da}\sigma^{ah}(T)\frac{1}{\sigma^{ah}(T)}K^{ah} + 2\sqrt{\frac{2}{\sigma^{ah}(T)}}\left(\int_0^T \cos\left(\frac{\pi\sigma^{ah}(s)}{\sigma^{ah}(T)}\right)\frac{\sin\left(\sqrt{k}\varphi_s^{ah}\right)\sqrt{k}X_s^{ah}}{\left(1+\cos\left(\sqrt{k}\varphi_s^{ah}\right)\right)^2}ds\right.$$
$$\left. - \int_0^T \frac{\frac{d}{da}\sigma^{ah}(s)\sigma^{ah}(T) - \sigma^{ah}(s)\frac{d}{da}\sigma^{ah}(T)}{(\sigma^{ah}(T))^2}\frac{\pi\sin\left(\frac{\pi\sigma^{ah}(s)}{\sigma^{ah}(T)}\right)}{1+\cos\left(\sqrt{k}\varphi_s^{ah}\right)}ds\right)$$

By combining this with (5.22), switching the orders of integration to isolate h, and using the definition of $\sigma(s) := \int_0^s \frac{k}{\sin^2(\sqrt{k}\varphi_\rho)} d\rho$, we obtain

$$\frac{d}{da}K^{ah} = \int_0^T h(u) \int_u^T G(\varphi^{ah})(s) e^{-\frac{1}{2}(\sigma^{ah}(s)-\sigma^{ah}(u))} ds du$$

with:

$$G(\varphi^{ah})(s) = \frac{k\sqrt{k}\cos(\sqrt{k}\varphi_s^{ah})}{\sin^3(\sqrt{k}\varphi_s^{ah})}\frac{K^{ah}}{\sigma^{ah}(T)} + \frac{2\sqrt{2k}}{\sqrt{\sigma^{ah}(T)}}\frac{\cos\left(\frac{\pi\sigma^{ah}(s)}{\sigma^{ah}(T)}\right)}{\left(1+\cos(\sqrt{k}\varphi_s^{ah})\right)^2}\sin(\sqrt{k}\varphi_s^{ah})$$
$$+ \sqrt{\frac{2}{\sigma^{ah}(T)}}\frac{2k\sqrt{k}\cos(\sqrt{k}\varphi_s^{ah})}{\sin^3(\sqrt{k}\varphi_s^{ah})}\frac{2\pi}{\sigma(T)^2}\int_0^T \frac{k}{\sin^2(\sqrt{k}\varphi_\rho^{ah})}\int_s^\rho \frac{\sin\left(\frac{\pi\sigma^{ah}(v)}{\sigma^{ah}(T)}\right)}{1+\cos(\sqrt{k}\varphi_v^{ah})} dv d\rho.$$

In particular, if $K = 0$, $a = 0$ and $s = 0$:

$$G(\varphi)(0) = \frac{2\sqrt{2k}}{\sqrt{\sigma(T)}}\frac{1}{\left(1+\cos(\sqrt{k}\varphi_0)\right)^2}\sin(\sqrt{k}\varphi_0)$$
$$+ \sqrt{\frac{2}{\sigma(T)}}\frac{2k\sqrt{k}\cos(\sqrt{k}\varphi_0)}{\sin^3(\sqrt{k}\varphi_0)}\frac{2\pi}{\sigma(T)^2}\int_0^T \frac{k}{\sin^2(\sqrt{k}\varphi_\rho)}\int_0^\rho \frac{\sin\left(\frac{\pi\sigma(v)}{\sigma(T)}\right)}{1+\cos(\sqrt{k}\varphi_v)} dv d\rho.$$

- For $k > 0$, choosing $\varphi_0 = \frac{\pi}{2\sqrt{k}}$, we get: $G(\varphi)(0) = \frac{2\sqrt{2k}}{\sqrt{\sigma(T)}} > 0$.
- For $k = 0$, choosing $\varphi_0 > 0$, we have $\varphi_s > 0$ for all $s \in [0, T]$ and thus $G(\varphi)(0) > 0$ a.s..

For $k > 0$, taking $K = 0$, $a = 0$ and $s = T$, and supposing that $\varphi_0 > 0$, we get $G(\varphi)(T) < 0$ a.s. because $\varphi_s > 0$ a.s. for all $s \in [0, T]$.

For the heuristic of the proof, let us suppose that $G(\varphi^{ah})(s) \in L^2(d\mathbb{P})$ for all $s \in [0, T]$. Under this hypothesis, using the fact that h is adapted to the filtration $(\mathcal{F}_u)_{u\in[0,T]}$, we can write:

$$\mathbb{E}[\mathbb{1}_{\{|K|=0\}}\frac{d}{da}_{|a=0}(K^{ah})] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{|K|=0\}}\int_0^T h(u)\int_u^T G(\varphi)(s)e^{-\frac{1}{2}(\sigma(s)-\sigma(u))}ds du\middle|\mathcal{F}_u\right]\right]$$
$$= \int_0^T \mathbb{E}\left[h(u)e^{\frac{1}{2}\sigma(u)}\mathbb{E}\left[\mathbb{1}_{\{|K|=0\}}\int_u^T G(\varphi)(s)e^{-\frac{1}{2}\sigma(s)}ds\middle|\mathcal{F}_u\right]\right] du$$

Then we could define $\tilde{h}(u) := \mathbb{E}\left[\mathbb{1}_{\{|K|=0\}}\int_u^T G(\varphi)(s)e^{-\frac{1}{2}\sigma(s)}ds\middle|\mathcal{F}_u\right]$ and choose

$$h(u) = \frac{\tilde{h}(u)\mathbb{1}_{\{\tilde{h}(u)\neq 0\}}}{|\tilde{h}(u)| + |\tilde{h}(u)|^2}e^{-\frac{1}{2}\sigma(u)}.$$

As it is bounded and adapted, such a h would be admissible and would satisfy Hypothesis $\mathscr{H}_1$. Supposing that $\mathscr{H}_2$ is also true, by using (5.21) we would obtain:

$$\int_0^T \mathbb{E}\left[\frac{\mathbb{1}_{\{\tilde{h}(u)\neq 0\}}\tilde{h}(u)^2}{|\tilde{h}(u)| + |\tilde{h}(u)|^2}\right] du = 0 \tag{5.23}$$

In particular we would have a.s. $\tilde{h}(u) = 0$ for all $u \in [0,T]$ (by continuity) and then, for all $u \in [0,T]$:

$$\mathbb{E}\left[\mathbb{1}_{\{|K|=0\}} \int_u^T G(\varphi)(s) e^{-\frac{1}{2}\sigma(s)} ds\right] = 0.$$

Thus, by differentiating in u, we would get $\mathbb{E}\left[\mathbb{1}_{|K|=0} G(\varphi)(u) e^{-\frac{1}{2}\sigma(u)}\right] = 0$ for all u (in particular for $u = 0$ and for $u = T$). As we have noticed that $G(\varphi)$ is not identically zero, this would lead to $K \neq 0$ a.s.

It seems difficult to directly obtain $G(\varphi^{ah})(s) \in L^2(d\mathbb{P})$ for all $s \in [0,T]$ as well as $\mathscr{H}_2$. In fact, problems could only occur if φ_s^{ah} is too close to zero or $i(M_k)$.

Let first suppose that $k > 0$. In this case, $i(M_k) = \frac{\pi}{\sqrt{k}}$. For $\eta > 0$ small enough, we choose $\chi_\eta : \left]0, \frac{\pi}{\sqrt{k}}\right[\to \mathbb{R}$ smooth such that:

- $\chi_\eta(x) = x$ if $x \in \left[\frac{\eta}{\sqrt{k}}, \frac{\pi}{\sqrt{k}} - \frac{\eta}{\sqrt{k}}\right[$;
- $0 < \chi_\eta'(x) \leq 1 \ \forall x \in \left]0, \frac{\pi}{\sqrt{k}}\right[$;
- $\chi_\eta(x) \in \left[\frac{\eta}{2\sqrt{k}}, \frac{\pi}{\sqrt{k}} - \frac{\eta}{2\sqrt{k}}\right] \ \forall x \in \left]0, \frac{\pi}{\sqrt{k}}\right[$.

We then define generalisations of $(\varphi^{ah}, \sigma^{ah}, K^{ah})$:

$$\begin{cases} \varphi(a,\eta)(s) := \chi_\eta(\varphi_s^{ah}) \\ \sigma(a,\eta)(t) = \int_0^t \frac{k}{\sin^2(\sqrt{k}\varphi(a,\eta)(s))} ds \\ K(a,\eta) = 2\sqrt{\frac{2}{\sigma(a,\eta)(T)}} \int_0^T \frac{\cos\left(\frac{\pi\sigma(a,\eta)(s)}{\sigma(a,\eta)(T)}\right)}{1+\cos(\sqrt{k}\varphi(a,\eta)_s)} ds \end{cases}.$$

Let denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space used here. By nature of φ, for all $\omega \in \Omega$ there exists $\eta_0(\omega) > 0$ such that $\chi_{\eta_0(\omega)}(\varphi_t(\omega)) = \varphi_t(\omega)$ for all $t \in [0,T]$. Thus, $K(0, \eta_0(\omega))(\omega) = K(\omega)$.

Let now consider $\eta > 0$ deterministic. We are going to work with $K(0,\eta)$ instead of K. Noticing that $\frac{d}{da}_{|a=0} \varphi(a,\eta)(t) = \chi_\eta'(\varphi_t) X_t^a{}_{|a=0}$, defining G as before, we can show that:

$$\frac{d}{da} K(a,\eta) = \int_0^T h(u) \int_u^T \chi_\eta'\left(\varphi(a,\eta)(s)\right) G(\varphi(a,\eta))(s) e^{-\frac{1}{2}(\sigma(a,\eta)(s) - \sigma(a,\eta)(u))} ds du.$$

This time we have $G(\varphi(a,\eta))(s)$ bounded for all $s \in [0,T]$. In particular, $G(\varphi(a,\eta))(s) \in L^2(d\mathbb{P})$ for all $s \in [0,T]$ and the hypothesis $\mathscr{H}_2$ is satisfied too. We define:

$$\tilde{h}^\eta(u) := \mathbb{E}\left[\mathbb{1}_{\{|K(0,\eta)|=0\}} \int_u^T G(\varphi(0,\eta))(s) e^{-\frac{1}{2}\sigma(s)} ds | \mathcal{F}_u\right]$$

and then we can choose $h(u) = \dfrac{\tilde{h}^\eta(u) \mathbb{1}_{\{\tilde{h}^\eta(u) \neq 0\}}}{|\tilde{h}^\eta(u)| + |\tilde{h}^\eta(u)|^2} e^{-\frac{1}{2}\sigma(u)}$ with h bounded and admissible. Using (5.21), as $(\chi_n'(\varphi_u))_u$ is bounded, adapted to the filtration, we have:

$$\int_0^T \mathbb{E}\left[\frac{\tilde{h}^\eta(u)^2}{|\tilde{h}^\eta(u)| + |\tilde{h}^\eta(u)|^2} \mathbb{1}_{\{\tilde{h}^\eta(u) \neq 0\}} \chi_\eta'(\varphi_u)\right] du = 0 \tag{5.24}$$

We get $\tilde{h}^\eta(u) = 0$ for all $u \in [0,T]$ a.s. and $\mathbb{E}\left[\mathbb{1}_{\{|K(0,\eta)|=0\}} \int_u^T G(\varphi(0,\eta))(s) e^{-\frac{1}{2}\sigma(s)} ds\right] = 0$ for all $u \in [0,T]$. Thus, by differentiating in u, we get:

$$\mathbb{E}\left[\mathbb{1}_{\{|K(0,\eta)|=0\}} G(\varphi(0,\eta))(u) e^{-\frac{1}{2}\sigma(u)}\right] = 0 \text{ for all } u \text{ and in particular for } u = 0 \text{ and } u = T. \tag{5.25}$$

Using the same arguments as before, Relation (5.25) is true only if $K(0,\eta) \neq 0$ a.s. Finally:

$$\mathbb{P}(K=0) = \mathbb{P}\left(\bigcup_{m\geq 1}\left\{K\left(0,\frac{1}{m}\right)=0\right\}\right) \leq \sum_{m\geq 1}\mathbb{P}\left(K\left(0,\frac{1}{m}\right)=0\right) = 0.$$

For $k \leq 0$, $i(M_0) = +\infty$ and for $\eta < 1$, we choose $\chi_\eta :]0,\pi[\to \mathbb{R}$ smooth such that:

- $\chi_\eta(x) = x$ if $x \in [\eta, \frac{1}{\eta}]$;
- $0 < \chi'_\eta(x) \leq 1 \ \forall x \in]0,+\infty[$;
- $\chi_\eta(x) \in \left[\frac{\eta}{2}, \frac{2}{\eta}\right] \ \forall x \in]0,\pi[$.

The rest of the proof runs the same way as for $k > 0$.

□

5.3 Successful coupling in $SU(2)$

We now want to prove Theorem 5.1.2, that is the successful coupling result for $SU(2)$. We then take $k = 1$, that is, $E_1 = SU(2)$, $M_1 = S^2$ and $i(M_1) = \pi$.

5.3.1 Description of the reflection coupling in S^2 by using the spherical coordinates

We first explicit one possible construction for a successful coupling in S^2. As explained in [40], the reflection coupling is maximal in S^2. It can be constructed by different ways. See for example [40, 27] for a construction using projections. As seen in Chapter 4, it can be done using covariant derivatives (see Example 4.1.11). We can also obtain this coupling directly with the spherical coordinates $(\varphi_t, \theta_t)_t$ and $(\varphi'_t, \theta'_t)_t$:

$$\begin{cases} d\varphi_t = dB_t^1 + \frac{1}{2}\cot(\varphi_t)dt \\ d\theta_t = \frac{1}{\sin(\varphi_t)} dB_t^2 \end{cases} \quad \text{and} \quad \begin{cases} d\varphi'_t = dB_t^{1\prime} + \frac{1}{2}\cot(\varphi'_t)dt \\ d\theta'_t = \frac{1}{\sin(\varphi'_t)} dB_t^{2\prime} \end{cases}.$$

Changing the pole and vector of reference in S^2 if needed,we can suppose that $\varphi_0 = \pi - \varphi'_0$, $\varphi_0 \in]0, \frac{\pi}{2}[$ and $\theta_0 = \theta'_0$. To construct the reflection coupling, we take $B_t^1 = -B_t^{1\prime}$ and $B_t^2 = B_t^{2\prime}$, we get $\varphi_t = \pi - \varphi'_t$ and $\theta_t = \theta'_t$ for all t. With this construction the paths of the two Brownian motions in S^2 have a symmetry with respect to the equator. In particular for $t \in [0, \tau_1]$, defining $\rho_t := \rho\left((\varphi_t, \theta_t), (\varphi'_t, \theta'_t)\right)$, we have $\rho_t = \pi - 2\varphi_t$. We obtain:

$$d\left(\frac{\rho_t}{2}\right) = -dB_t^1 - \frac{1}{2}\tan\left(\frac{\rho_t}{2}\right) dt. \tag{5.26}$$

Note that we obtain the same equation as for the reflection coupling obtained in Example 4.1.11.

Proposition 5.3.1. *We consider the reflection coupling described above. For any* $0 \leq u < \frac{\pi}{2}$*, let denote* $T_u := \inf\{t \mid \frac{\rho_t}{2} = u\}$*. If* $\frac{\rho_0}{2} < u < \frac{\pi}{2}$*, then:*

$$\mathbb{E}[T_0 \wedge T_u] \leq \frac{\rho_0}{2}\left(2u - \frac{\rho_0}{2}\right). \tag{5.27}$$

Moreover, the reflection coupling is successful. If we denote by τ_1 *its first coupling time, we have:*

- $\mathbb{E}[\tau_1] \leq \frac{\rho_0}{2}\left(\pi - \frac{\rho_0}{2}\right)$.
- *there exist some constants* $C, c > 0$ *independent of the distance between the starting points such that:*

$$\mathbb{P}(\tau_1 > t) \leq C\rho_0 \frac{e^{-ct}}{t}.$$

Proof. We first prove Inequality (5.27). Let $\frac{\pi}{2} > u > \frac{\rho_0}{2}$. Let define three processes:

- $W_t := \frac{\rho_0}{2} + B_t$, with $(B_t)_t$ a standard Brownian motion;
- $U_t := u - |u - W_t| = \frac{\rho_0}{2} + \beta_t - L_t^u(W)$ with $\beta_t := -\int_0^t \operatorname{sign}(u - W_s)dB_s$ defining a standard Brownian motion and $L_t^u(W)$ the local time of $(W_t)_t$ in u.
- $(V_t)_t$ starting at $\frac{\rho_0}{2}$ satisfying: $dV_t = d\beta_t - \frac{1}{2}\tan(V_t)dt$. In particular $(\rho_t)_t$ and $(V_t)_t$ have the same distribution.

We also define the stopping time $\tilde{T}_v := \inf\{t > 0 \mid V_t = v\}$ for $v \in \mathbb{R}$. We claim that $V_t \leq U_t$ for all $0 \leq t \leq \tilde{T}_0 \wedge \tilde{T}_u$. Let explain this fact.
We consider $\underline{T} := \inf\{t > 0 \mid U_t = V_t\}$. As $\frac{\rho_0}{2} < u$, for t small enough, we have $U_t < u$. Thus, $d(U_t - V_t) = \frac{1}{2}\tan(V_t)dt > 0$ and $\underline{T} > 0$ a.s.. We suppose that $\underline{T} < \tilde{T}_0 \wedge \tilde{T}_u$. Then, we have $U_{\underline{T}} = V_{\underline{T}} < u$ and, by continuity of U, there exists $\epsilon > 0$ such that, for all $t \in [\underline{T} - \epsilon, \underline{T}]$, $U_t < u$. In particular, $L^u_{\underline{T}-\epsilon}(W) = L^u_{\underline{T}}(W)$. We obtain:

$$\begin{aligned} 0 &= U_{\underline{T}} - V_{\underline{T}} \\ &= U_{\underline{T}-\epsilon} - V_{\underline{T}-\epsilon} + \frac{1}{2}\int_{\underline{T}-\epsilon}^{\underline{T}} \tan(V_s)ds \end{aligned}$$

with $U_{\underline{T}-\epsilon} - V_{\underline{T}-\epsilon} > 0$ and with $\frac{1}{2}\int_{\underline{T}-\epsilon}^{\underline{T}} \tan(V_s)ds > 0$ since $\underline{T} < \tilde{T}_0$. We get a contradiction, thus $\underline{T} \geq \tilde{T}_0 \wedge \tilde{T}_u$ so $V_t < U_t$ on $]0, \tilde{T}_0 \wedge \tilde{T}_u[$.
As $U_t = 0$ if and only if $W_t \in \{0, 2u\}$, we get:

$$\tilde{T}_0 \wedge \tilde{T}_u \leq \inf\{t > 0 \mid W_t \notin]0, 2u[\} := H_{-\frac{\rho_0}{2}, 2u - \frac{\rho_0}{2}}.$$

Finally, as $T_0 \wedge T_u$ has the same distribution as $\tilde{T}_0 \wedge \tilde{T}_u$ we get:

$$\mathbb{E}[T_0 \wedge T_u] \leq \mathbb{E}[H_{-\frac{\rho_0}{2}, 2u - \frac{\rho_0}{2}}] = \frac{\rho_0}{2}\left(2u - \frac{\rho_0}{2}\right).$$

We can now look at the coupling rate of the reflection coupling. As, $\frac{\rho_t}{2} < \frac{\pi}{2}$ a.s., we have $\tau_1 = T_0 \wedge T_{\frac{\pi}{2}}$. Using the previous results, we have $\underline{T}_0 \wedge \underline{T}_{\frac{\pi}{2}} \leq H_{a,b}$, with $a = -\frac{\rho_0}{2}$ and

$b = \pi - \frac{\rho_0}{2}$. From Lemma A.0.2, we obtain:

$$\begin{aligned}\mathbb{P}(\tau_1 > t) &\leq \mathbb{E}\left[\frac{e^{-\delta H_{a,b}}}{H_{a,b}} H_{a,b} e^{\delta H_{a,b}} \mathbb{1}_{\{H_{a,b}>t\}}\right] \\ &\leq \frac{e^{-\delta t}}{t} \times \frac{(\pi - \frac{\rho_0}{2})\frac{\rho_0}{2}}{\cos^2\left(\sqrt{\frac{\delta}{2}}\pi\right)}\end{aligned}$$

with δ such that $0 < \sqrt{\frac{\delta}{2}} < \frac{1}{2}$, that is, $0 < \delta < \frac{1}{2}$. Taking $C = \frac{\pi}{2\cos^2\left(\sqrt{\frac{\delta}{2}}\pi\right)}$ and $c = \delta$, we have:

$$\mathbb{P}(\tau_1 > t) \leq \mathbb{P}(S_1 > t) \leq \rho_0 C \frac{e^{-ct}}{t}.$$

with c and C not depending on the initial distance between the Brownian motions. In particular, the coupling is successful. □

5.3.2 Proof of Theorem 5.1.2

We now deal with the coupling $(\mathbb{B}_t = (X_t, z_t), \mathbb{B}'_t = (X_t, z_t))_t$ on $SU(2)$ announced in Theorem 5.1.2. As explained before, the idea is to use the reflection coupling until the first time τ_1 such that $X_{\tau_1} = X'_{\tau_1}$ and then to use the Brownian bridges coupling to couple the "area parts". According to Theorem 5.1.1, we need to have an estimate of the quantity $\mathbb{E}[|\zeta_{\tau_1}|^q \wedge 1]$ for at least one $q \in]0,1[$. Thus we need the following Proposition:

Proposition 5.3.2. *At the end of the reflection coupling we have, for* $0 < p < \frac{1}{2}$*:*

$$\mathbb{E}[|\zeta_{\tau_1}|^{\frac{1}{2}+p}] \leq \tilde{C}_p d_{cc}(\mathbb{B}_0, \mathbb{B}'_0).$$

with $\tilde{C}_p$ *some constant independent of* $\mathbb{B}_0$ *and* $\mathbb{B}'_0$*.*

Proof of Proposition 5.3.2. By construction of the reflection coupling using the change of pole (N_1, e_1), the quantity $\mathcal{A}_{X_t, X'_t, N_1}$ is equal to 0 for all t. Then by Lemma 3.2.12, we have $\zeta_t \equiv I'_t(N_1, e_1) - I_t(N_1, e_1) \mod (4\pi)$. As $\frac{\rho_t}{2} = \frac{\pi}{2} - \varphi_t$, using the equation of $(I'_t(N_1, e_1) - I_t(N_1, e_1))_t$ we get $\zeta_t \equiv \zeta_0 - 2\int_0^t \tan\left(\frac{\rho_s}{2}\right) dB_s^2 \mod (4\pi)$. We define $\phi_p : x \in]0, +\infty[\mapsto x^{\frac{1}{2}+p}$ with $0 < p < \frac{1}{2}$. We are interested in the quantity $\mathbb{E}\left[\phi_p\left(|\zeta_0 - 2\int_0^{\tau_1} \tan\left(\frac{\rho_s}{2}\right) dB_s^2| \wedge 4\pi\right)\right]$. As $\phi_p(x+y) \leq \phi_p(x) + \phi_p(y)$ for all $x, y > 0$,

$$\begin{aligned}\mathbb{E}\left[\phi_p\left(\left|\zeta_0 - 2\int_0^{\tau_1} \tan\left(\frac{\rho_s}{2}\right) dB_s^2\right| \wedge 4\pi\right)\right] &\leq \phi_p\left(|\zeta_0|\right) \\ &\quad + \mathbb{E}\left[\phi_p\left(\left|2\int_0^{\tau_1} \tan\left(\frac{\rho_s}{2}\right) dB_s^2\right| \wedge 4\pi\right)\right].\end{aligned}$$

Since $|\zeta_0|$ is bounded by 2π, we have $\phi_p(|\zeta_0|) \leq \delta_p\sqrt{|\zeta_0|} \leq \tilde{\delta}_p d_{cc}(\mathbb{B}_0, \mathbb{B}'_0)$ with δ_p and $\tilde{\delta}_p$ two constants independent of $\mathbb{B}_0$ and $\mathbb{B}'_0$. We now just need to prove that the quantity $\mathbb{E}\left[\phi_p\left(\left|2\int_0^{\tau_1} \tan\left(\frac{\rho_s}{2}\right) dB_s^2\right| \wedge 4\pi\right)\right]$ is upper bounded by ρ_0 up to a multiplicative constant. It is obvious for ρ_0 large enough, thus we consider $\rho_0 \leq m$, with $m > 0$ chosen later in

the proof. We can write:

$$\begin{aligned}\mathbb{E}\left[\phi_p\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|\wedge 4\pi\right)\right]&\\ &=\int_0^{\phi_p(4\pi)}\mathbb{P}\left(\phi_p\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|\wedge 4\pi\right)>y\right)dy\\ &=\int_0^{4\pi}\mathbb{P}\left(\phi_p\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|\right)>\phi_p(x)\right)\phi_p'(x)dx\\ &=\int_0^{4\pi}\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x\right)\left(\frac{1}{2}+p\right)x^{p-\frac{1}{2}}dx.\end{aligned}$$

We set $0<\alpha<1$. We are going to split the integral into two parts:

$$\int_0^{\left(\frac{\rho_0}{2}\right)^{\frac{1}{\alpha}}}\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x\right)\phi_p'(x)dx \tag{5.28}$$

and

$$\int_{\left(\frac{\rho_0}{2}\right)^{\frac{1}{\alpha}}}^{4\pi}\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x\right)\phi_p'(x)dx. \tag{5.29}$$

By simply upper-bounding the probability by 1, the quantity (5.28) can be upper-bounded by $\phi_p\left(\left(\frac{\rho_0}{2}\right)^{\frac{1}{\alpha}}\right)\leq\left(\frac{\rho_0}{2}\right)^{\frac{1+2p}{2\alpha}}$. For $\alpha\leq\frac{1}{2}+p$ and m small enough, we have $\left(\frac{\rho_0}{2}\right)^{\frac{1+2p}{2\alpha}}\leq\frac{\rho_0}{2}$. To deal with the second quantity we look at $\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x\right)$. Using the same notations as in Proposition 5.3.1, we consider $\frac{\rho_0}{2}<u<\frac{\pi}{2}$. We have:

$$\begin{aligned}\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x\right)&=\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x,T_u<T_0\right)\\ &\quad+\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x,T_u>T_0\right)\\ &\leq\mathbb{P}(T_u<T_0)+\mathbb{P}\left(\left|2\int_0^{T_u\wedge T_0}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x\right)\\ &\leq\mathbb{P}(T_u<T_0)+\frac{1}{x^2}\mathbb{E}\left[4\int_0^{T_u\wedge T_0}\tan^2\left(\frac{\rho_s}{2}\right)ds\right]\\ &\leq\mathbb{P}(T_u<T_0)+\frac{4\tan^2(u)}{x^2}\mathbb{E}\left[T_u\wedge T_0\right].\end{aligned}$$

As the drift part in the equation of $(\rho_t)_t$ (given by (5.26)) is negative, we have $T_u\geq\inf\{t>0\mid\frac{\rho_0}{2}-B_t^1=u\}$ and $T_0\leq\inf\{t>0\mid\frac{\rho_0}{2}-B_t^1=0\}$. Thus, using relation (A.5) from Lemma A.0.2 with $a=-\frac{\rho_0}{2}$ and $b=u-\frac{\rho_0}{2}$, we get $\mathbb{P}(T_u<T_0)\leq\frac{\rho_0}{2u}$. From Proposition 5.3.2, we have $\mathbb{E}\left[T_u\wedge T_0\right]\leq\frac{\rho_0}{2}\left(2u-\frac{\rho_0}{2}\right)$. Thus:

$$\mathbb{P}\left(\left|2\int_0^{\tau_1}\tan\left(\frac{\rho_s}{2}\right)dB_s^2\right|>x\right)\leq\frac{\rho_0}{2}\left(\frac{1}{u}+4\frac{\tan^2(u)}{x^2}\left(2u-\frac{\rho_0}{2}\right)\right). \tag{5.30}$$

We introduce β such that $0<\beta<1$. We first chose $u(x):=x^\beta$. If $\left(\frac{\rho_0}{2}\right)^{\frac{1}{\alpha}}<x<1$ and $\beta<\alpha$, we have $\frac{\rho_0}{2}<u(x)<\frac{\pi}{2}$. Then we use inequality (5.30) with $u=u(x)$. We get:

$$\int_{\left(\frac{\rho_0}{2}\right)^{\frac{1}{\alpha}}}^{1} \mathbb{P}\left(\left| 2\int_0^{\tau_1} \tan\left(\frac{\rho_s}{2}\right) dB_s^2 \right| > x \right) \phi_p'(x)dx$$
$$\leq \frac{\rho_0}{2} \int_{\left(\frac{\rho_0}{2}\right)^{\frac{1}{\alpha}}}^{1} \left(x^{-\beta} + 8\frac{\tan^2(x^\beta)}{x^2} x^\beta \right) dx$$
$$\leq \delta_\beta \frac{\rho_0}{2} \int_0^1 \left(x^{-\beta} + x^{3\beta-2} \right) x^{p-\frac{1}{2}} dx,$$

with δ_β independent of ρ_0. The last integral is finite if and only if $p + \frac{1}{2} > \beta > \frac{1}{2} - \frac{p}{3}$. In particular, this is the case for $\beta = \frac{1}{2}$. If $x > 1$, we use inequality (5.30) with $u \equiv 1$. We get:

$$\int_1^{4\pi} \mathbb{P}\left(\left| 2\int_0^{\tau_1} \tan\left(\frac{\rho_s}{2}\right) dB_s^2 \right| > x \right) \phi_p'(x)dx \leq \frac{\rho_0}{2} \int_1^{4\pi} \left(1 + 8\frac{\tan^2(1)}{x^2} \right) \phi_p'(x)dx.$$

We thus have (5.29) upper bounded by ρ_0 up to a multiplicative constant only depending of m, β and p. □

We can now give the proof of Theorem 5.1.2:

Proof of Theorem 5.1.2. To construct our successful coupling on $SU(2)$, we first construct the reflection coupling described above until time τ_1, that is, until $X_{\tau_1} = X'_{\tau_1}$. Then, we use the coupling from Theorem 5.1.1 to deal with the swept area coordinates. As the coupling rate from the two couplings have an exponential decay, we obtain a successful coupling with an exponential decay on $SU(2)$. We want to obtain the initial distance-control from inequality (5.2). We denote $\tau_2 := \inf\{t > \tau_1 \mid \mathbb{B}_t = \mathbb{B}'_t\}$. The first meeting time τ of the Brownian motions in $SU(2)$ satisfies $\tau = \tau_1 + \tau_2$. Then we have, for $\frac{1}{2} < q < 1$ and $t > 2$:

$$\mathbb{P}(\tau > t) = \mathbb{P}\left(\tau > t, \tau_1 \leq \frac{t}{2}\right) + \mathbb{P}\left(\tau > t, \tau_1 > \frac{t}{2}\right) \leq \mathbb{P}\left(\tau_2 > \frac{t}{2}, \tau_1 \leq \frac{t}{2}\right) + \mathbb{P}\left(\tau_1 > \frac{t}{2}\right)$$
$$\leq \mathbb{E}\left[\mathbb{P}\left(\tau_2 > \frac{t}{2} \;\middle|\; |\zeta_{\tau_1}| \right) \mathbb{1}_{\{\tau_1 \leq \frac{t}{2}\}} \right] + C\rho_0 \frac{e^{-c\frac{t}{2}}}{\frac{t}{2}}$$
$$\leq \mathbb{E}\left[C_q e^{-\tilde{c}\frac{t}{2}} |\zeta_{\tau_1}|^q \mathbb{1}_{\{\tau_1 \leq \frac{t}{2}\}} \right] + 2C\rho_0 \frac{e^{-c\frac{t}{2}}}{t}$$
$$\leq C_q e^{-\tilde{c}\frac{t}{2}} \mathbb{E}\left[|\zeta_{\tau_1}|^q \right] + 2C\rho_0 \frac{e^{-c\frac{t}{2}}}{t}.$$

As $\rho_0 \leq \rho_0 + \sqrt{|\zeta_0|}$ with $\rho_0 + \sqrt{|\zeta_0|}$ equivalent to $d_{cc}(\mathbb{B}_0, \mathbb{B}'_0)$ (in the sense of (3.41)), using Proposition 5.3.2, we obtain the expected inequality. □

5.4 Applications to gradient estimates

In this section, we show the gradient inequalities previously stated in Subsection 2.3.6. They involve the heat semi group $(P_t)_t$, i.e., the semi-group with infinitesimal generator $\frac{1}{2}L$, the sub-Laplacian operator. We recall that, for all $g \in \mathbb{G}$ and f a measurable bounded

function, considering a Brownian motion $(\mathbb{B}_t)_t$ starting at g, we have $P_t f(g) = \mathbb{E}[f(\mathbb{B}_t)]$. As in Subsection 3.2.4, we denote by $\nabla_{\mathcal{H}}$ the subgradient induced by the sub-Laplacian operator and by $||\cdot||_{\mathcal{H}}$ the norm induced by the subRiemannian structure. Then, we can use the coupling rate of Theorem 5.1.2, to obtain the following gradient inequality on $SU(2)$:

Corollary 5.4.1. *There exists $C > 0$ such that, for every function $f \in \mathcal{C}(SU(2))$ (or simply bounded measurable) and $t > 2$:*

$$||\nabla_{\mathcal{H}} P_t f(g)||_{\mathcal{H}} \leq 2||f||_\infty C e^{-ct} \text{ a.e..} \tag{5.31}$$

In particular, if f is harmonic on $SU(2)$, then it is constant.

Proof of Corollary 5.4.1. Let $g, g' \in SU(2)$. We consider $(\mathbb{B}_t, \mathbb{B}'_t)_t$ the coupling constructed in Theorem 5.1.2 starting from (g, g') and τ its first coupling time. As $SU(2)$ is compact and f continuous, then it is bounded and we get for $t > 2$:

$$\begin{aligned} |P_t f(g) - P_t f(g')| &= |\mathbb{E}[f(\mathbb{B}_t) - f(\mathbb{B}'_t)]| \\ &\leq \mathbb{E}[|f(\mathbb{B}_t) - f(\mathbb{B}'_t)|] = \mathbb{E}[|f(\mathbb{B}_t) - f(\mathbb{B}'_t)|\mathbb{1}_{\{\tau > t\}}] \\ &\leq 2||f||_\infty \mathbb{P}(\tau > t) \\ &\leq 2||f||_\infty C e^{-ct} d_{cc}(g, g'). \end{aligned} \tag{5.32}$$

In particular, $g \mapsto P_t f(g)$ is Lipschitz and $|\nabla P_t f|(g) \leq 2||f||_\infty C e^{-ct}$. We just use (3.32) to obtain: $||\nabla_{\mathcal{H}} P_t f(g)||_{\mathcal{H}} \leq 2||f||_\infty C e^{-ct}$ a.e..
If f is harmonic on all $SU(2)$, we have $P_t f = f$ and so $||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} \leq 2||f||_\infty C_1 e^{-C_2 t}$ for all t a.e.. Then, letting t tend to $+\infty$, we obtain $||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} = 0$ and so $\bar{X} f = \bar{Y} f = 0$ a.e.. Using the Lie bracket generating property of $\mathcal{H} = \text{Span}\{\bar{X}, \bar{Y}\}$, we get f constant a.e. and, by continuity, f is constant on $SU(2)$. □

Using Theorem 5.1.1, we also get some (weaker) results for $SL(2, \mathbb{R})$:

Corollary 5.4.2. *Let $g = (x, z)$, $g' = (x', z') \in SL(2, \mathbb{R})$. We suppose that $x = x'$ and we consider a bounded measurable function f on $SL(2, \mathbb{R})$. For all $q \in]0, 1[$, there exist C_q, c some constants that do not depend on g, g' such that for $t > 1$:*

$$|P_t f(g) - P_t f(g')| \leq 2||f||_\infty C_q e^{-ct} d_{cc}(g, g')^{2q}.$$

Moreover, if f is harmonic and bounded, it is constant on each fiber above x, that is, on the sets of the form $\{(x, z) \in SL(2, \mathbb{R}) \mid z \in [-2\pi, 2\pi]\}$.

The computations to prove Corollary 5.4.2 are quite similar than for Corollary 5.4.1:

Proof of Corollary 5.4.2. We suppose that $g = (x, z)$, $g' = (x', z') \in SL(2, \mathbb{R})$ with $x = x'$. The same way as for inequality (5.32), using Theorem 5.1.1, for $t > 1$ we obtain $|P_t f(g) - P_t f(g')| \leq 2||f||_\infty C_q e^{-ct} d_{cc}(g, g')^{2q}$.
Moreover, if f is harmonic, $P_t f = f$. Then $|f(g) - f(g')| \leq 2||f||_\infty C_q e^{-ct} d_{cc}(g, g')^{2q}$. Taking $t \to 0$, we get $f(g) = f(g')$. Thus, $z \mapsto f(x, z)$ is constant for all $x \in \mathbf{H}^2$. □

Chapter 6

Successful couplings on the homogeneous, step 2, Carnot groups

The present chapter is devoted to the construction of a non co-adapted successful coupling on $\mathbb{G}_n$, the free, step 2 Carnot groups of rank n for all $n \geq 3$. By an extensive study of the properties of the obtained coupling we also present several applications to gradients inequalities. We remind that, when $n = 2$, $\mathbb{G}_2$ is the Heisenberg group. The idea is then to generalise the results from Banerjee, Gordina and Mariano [6] to higher dimension. Please note that the generalised strategy and its induced estimates still makes sense for $n = 2$ and that, in this case, it is exactly following the strategy from Banerjee, Gordina and Mariano. We thus make the analysis keeping generality for $n \geq 2$.

Note also that a co-adapted successful strategy can be found in the work from Kendall in [37] without any estimate of the coupling rate.

The content of this chapter will make the object of a publication which is currently in construction:

[26] Magalie Bénéfice. Non co-adapted successful couplings of Brownian motions on the free, step 2 carnot groups. `https://arxiv.org/abs/2407.06593`, 2024.

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6.1 The non co-adapted coupling strategy for $\mathbb{G}_n$

Using the notations from Subsection 3.1.5, we consider $\mathbb{G}_n = \mathbb{R}^n \times \mathfrak{so}(n)$. We recall that, on $\mathbb{G}_n$, the Brownian motion starting at (x, z) can be decomposed under the form $(\mathbb{B}_t)_t = (X_t, z_t)_t$ where $(X_t)_t$ is a $\mathbb{R}^n$ Brownian motion starting at x and $z_t = z + \frac{1}{2}\int_0^t X_s \odot dX_s$. Moreover, for two Brownian motions $(\mathbb{B}_t)_t = (X_t, z_t)_t, (\tilde{\mathbb{B}}_t)_t = (\tilde{X}_t, \tilde{z}_t)_t$, we have $\mathbb{B}_t^{-1}\tilde{\mathbb{B}}_t = (\tilde{X}_t - X_t, \zeta_t)$ with $\zeta_t = \tilde{z}_t - z_t - \frac{1}{2}X_t \odot \tilde{X}_t \in \mathfrak{so}(n)$. In particular, for any $1 \leq i \neq j \leq n$, using Itô formula, we can write:

$$\begin{aligned} d\zeta_t^{i,j} &= \frac{1}{2}\tilde{X}_t^i d\tilde{X}_t^j - \frac{1}{2}\tilde{X}_t^j d\tilde{X}_t^i - \left(\frac{1}{2}X_t^i dX_t^j - \frac{1}{2}X_t^j dX_t^i\right) - \frac{1}{2}d\left(X_t^i\tilde{X}_t^j - X_t^j\tilde{X}_t^i\right) \\ &= \tilde{X}_t^i d\tilde{X}_t^j - X_t^i dX_t^j + \frac{1}{2}d\left(\left(\tilde{X}_t^i + X_t^i\right)\left(X_t^j - \tilde{X}_t^j\right)\right). \end{aligned} \tag{6.1}$$

Let also notice that if $X_T = \tilde{X}_T$, $\zeta_T = \tilde{z}_T - z_T$.

The main Theorem of this section is the successful non co-adapted coupling result stated previously as as Theorem 2.4.1:

Theorem 6.1.1. *Set $n \geq 2$. Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z})$ be two points in $\mathbb{G}_n$ and $\zeta \in \mathfrak{so}(n)$ such that $g^{-1} \star \tilde{g} = (\tilde{x} - x, \zeta)$. There exists a successful coupling of Brownian motions $\left(\mathbb{B}_t, \tilde{\mathbb{B}}_t\right)_t$ on $\mathbb{G}_n$ starting from $(g, \tilde{g})$. Moreover, for all $t \geq \beta_n \|x - \tilde{x}\|_2^2$ we have:*

$$\mathbb{P}(\tau > t) \leq \left(C_1(n)\frac{||x - \tilde{x}||_2}{\sqrt{t}} + C_2(n)\frac{||\zeta||_2}{t}\right) \tag{6.2}$$

with:

- $\beta_n = 2\pi(n-1)^2\sqrt{2(n-1)}$;
- $C_1(n) = 4\sqrt{\beta_n}(n-1) + \frac{1}{\sqrt{\pi}} = 4\sqrt{2\pi}(n-1)^2(2(n-1))^{\frac{1}{4}} + \frac{1}{\sqrt{\pi}}$;
- $C_2(n) = 2\beta_n = 4\pi(n-1)^2\sqrt{2(n-1)}$.

To prove this theorem we describe a strategy for the case when $x = \tilde{x}$ and then deal with the general case.

6.1.1 The Brownian motion start from the same fiber

Let us describe the strategy for the case when $x = \tilde{x}$, that is, when the Brownian motions start from the same fiber. The main idea is to work on the matrix $(\zeta_t)_t$ line by line. For each line we will applied the following Proposition:

Proposition 6.1.2. *Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z})$ be two points in $\mathbb{G}_n$ such that $x = \tilde{x}$. Let $1 \leq i_0 \leq n-1$. For two Brownian motions $\mathbb{B}_t$ and $\tilde{\mathbb{B}}_t$ we denote $v_t^{i_0} := \left(\zeta_t^{i_0,i}\right)_{i \in \{1,\dots,n\}\setminus\{i_0\}} \in \mathbb{R}^{n-1}$ and*

$$\bar{\tau}_{i_0} := \inf\{t \geq 0 \mid X_t = \tilde{X}_t \text{ and } z_t^{i_0,i} = \tilde{z}_t^{i_0,i} \ \forall 1 \leq i \leq n\}.$$

There exists a non co-adapted coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ such that $\bar{\tau}_{i_0}$ is a.s. finite and satisfies for all $t \geq \|v_0^{i_0}\|_2$:

$$\mathbb{P}\left(\bar{\tau}_{i_0} > t\right) \leq \frac{\|v_0^{i_0}\|_2}{t} \times 2\pi(n-1) \tag{6.3}$$

Moreover, for $l, i \in \{1, \dots n\} \setminus \{i_0\}$, we have $\zeta_t^{l,i} = \zeta_0^{l,i}$ for all $t \in [0, \bar{\tau}_{i_0}]$.

Remark 6.1.3. *As $(\zeta_t)_t$ is skew-symmetric, at time $\bar{\tau}_{i_0}$ the i_0^{th} line* ***and*** *the i_0^{th} column are null.*

Remark 6.1.4. *As $2\pi(n-1) > 1$ for all $n \geq 2$, Inequality (6.3) can be replaced by*

$$\mathbb{P}\left(\bar{\tau}_{i_0} > t\right) \leq \mathbb{E}\left[\left(\frac{\|v_0^{i_0}\|_2}{t} \times 2\pi(n-1)\right) \wedge 1\right]$$

for all $t \geq 0$.

Proof of Proposition 6.1.2. For the proof we can suppose without lost of generality that $i_0 = 1$. Then $v_t^1 = \begin{pmatrix} \zeta_t^{1,2} \\ \vdots \\ \zeta_t^{1,n} \end{pmatrix}$. Let $T > 0$ and $v := \frac{v_0^1}{\|v_0^1\|_2}$. We choose:

- $X_t^i = \tilde{X}_t^i$ for $2 \leq i \leq n$. Then $\zeta_t^{l,i}$ stays constant equal to 0 for $2 \leq l < i \leq n$.
- We define the coupling $(X_t^1, \tilde{X}_t^1)_t$ by using the Karhunen-Loève expansion of the Brownian bridges as in the looking-ahead coupling introduced briefly in Section 2.1 and used for the study of $SU(2)$ and $SL(2,\mathbb{R})$ in Section 5.2.1. We have for $t \in [0,T]$:

$$\tilde{X}_t^1 - X_t^1 = \sum_{j \geq 1} \frac{\tilde{\xi}_j - \xi_j}{2} \times 2\frac{\sqrt{2T}}{j\pi} \sin\left(\frac{j\pi t}{T}\right) + \left(\tilde{\xi}_0 - \xi_0\right) \frac{t}{\sqrt{T}}$$

  with $(\xi_j)_{j\geq 0}$ (resp. $(\tilde{\xi}_j)_{j\geq 0}$) a sequence of independent variables with the same distribution $\mathcal{N}(0,1)$.

We choose $\xi_j = \tilde{\xi}_j$ for $j = 0$ and for $j \geq n$. With these choices, for all $2 \leq i \leq n$, using (6.1) we get:

$$\zeta_t^{1,i} = \zeta_0^{1,i} + \int_0^t \left(\tilde{X}_s^1 - X_s^1\right) dX_s^i = \zeta_0^{1,i} + \sum_{j=1}^{n-1} \frac{\tilde{\xi}_j - \xi_j}{2} K_{(i-1),j}(t)$$

with: $K_{(i-1),j}(t) := 2\frac{\sqrt{2T}}{j\pi} \int_0^t \sin\left(\frac{j\pi s}{T}\right) dX_s^i$. Note that, as we consider the coordinates $\zeta_t^{1,i}$ for $2 \leq i \leq n$, we introduce a shift of the index i to define $K_{(i-1),j}(t)$. This will simplify the notation in the remainder of the proof. For $j, l > 0$, we have:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \sin\left(\frac{j\pi s}{T}\right) dX_s^i \int_0^T \sin\left(\frac{l\pi s}{T}\right) dX_s^i\right] &= \int_0^T \sin\left(\frac{j\pi s}{T}\right) \sin\left(\frac{l\pi s}{T}\right) ds \\
&= \frac{1}{2} \int_0^T \left(\cos\left(\frac{(j-l)\pi s}{T}\right) - \cos\left(\frac{(j+l)\pi s}{T}\right)\right) ds \\
&= \begin{cases} 0 & \text{if } j \neq l \\ \frac{T}{2} & \text{if } j = l. \end{cases}
\end{aligned}$$

Thus, we have $K_{(i),j}(t) = \frac{2T}{j\pi} G_{(i),j}(t)$ for all $1 \leq i, j \leq n-1$ with $(G_{(i),j}(T))_{i,j}$ some independent standard Gaussian variables.

In particular, denoting $A(t) := (K_{(i),j}(t))_{1\leq i,j\leq n-1} \in M_{n-1,n-1}(\mathbb{R})$, $\xi := (\xi_j)_{1\leq j\leq n-1} \in \mathbb{R}^{n-1}$ and $\tilde{\xi} := (\tilde{\xi}_j)_{1\leq j\leq n-1} \in \mathbb{R}^{n-1}$, we have:

$$v_t^1 = v_0^1 + A(t)\frac{\tilde{\xi} - \xi}{2}.$$

Note that $A(T)$ is measurable with respect to the σ-field $\sigma\left(X_t^j, 0 \leq t \leq T, 2 \leq j \leq n\right)$ and thus will be independent of $(X_t^1)_t$ and $(\tilde{X}_t^1)_t$. As the entries $K_{(i),j}(T)$ of the matrix $A(T)$ are independent random variables with a density, $A(T)$ is a.s. invertible. Indeed, the Lebesgue measure of any hyperplane is null. Thus the Lebesgue measure of the kernel of the determinant is also null and $\mathbb{P}\big(\det(A(T)) = 0\big) = 0$.

By considering the inverse of $A(T)$, we can now see that the coupling is successful at time T if and only if:

$$\frac{\tilde{\xi} - \xi}{2} = -A(T)^{-1} v_0^1 = -\|v_0^1\|_2 A(T)^{-1} v. \tag{6.4}$$

We define $f_1 := \frac{A(T)^{-1}v}{||A(T)^{-1}v||_2}$ and $f_2, \ldots, f_{n-1}$ such that $(f_1, f_2, \ldots, f_{n-1})$ is an orthonormal basis of $\mathbb{R}^{n-1}$. Let W_t be a $n-1$ dimensional Brownian motion starting from 0 and $\sigma := \inf\{t \geq 0 \mid W_t^1 = \|v_0^1\|_2 ||A(T)^{-1}v||_2\}$. We define $\tilde{W}_t$ such that $W_t^j = \tilde{W}_t^j$ for all $j \geq 2$ and

$$\tilde{W}_t^1 := \begin{cases} -W_t^1 & \text{if } t \leq \sigma \\ W_t^1 - 2W_\sigma^1 & \text{else} \end{cases}.$$

In particular, $\frac{\tilde{W}_t^1 - W_t^1}{2} = -W_{\sigma \wedge t}^1$.

We can now define the coupling for the Gaussians vectors $(\xi, \tilde{\xi})$. We choose: $\xi = \sum_{l=1}^{n-1} W_1^l f_l$ and $\tilde{\xi} = \sum_{l=1}^{n-1} \tilde{W}_1^l f_l$. As $(W_t)_t$ is independent of $(f_l)_{1 \leq l \leq n-1}$, this provides a well defined coupling for $\left(\mathbb{B}_t, \tilde{\mathbb{B}}_t\right)_t$. In particular, $\frac{\tilde{\xi} - \xi}{2} = -W_{\sigma \wedge 1}^1 f_1 = -W_{\sigma \wedge 1}^1 \frac{A(T)^{-1}v}{||A(T)^{-1}v||_2}$. According to (6.4) and by definition of σ, the coupling is successful if and only if $\sigma < 1$.
Note that we have:

$$v_t^1 = \|v_0^1\|_2 v - \frac{W_{\sigma \wedge 1}^1}{||A(T)^{-1}v||_2} A(t) A(T)^{-1} v$$

and at time $t = T$:

$$v_T^1 = \left(\|v_0^1\|_2 - \frac{W_{\sigma \wedge 1}^1}{||A(T)^{-1}v||_2}\right) v. \tag{6.5}$$

For rest of the proof, we will need to study an upper bound of $||A(T)^{-1}v||_2^2 = v^t \left(A(T)A(T)^t\right)^{-1} v$. We denote $\Sigma := \text{Diag}\left(1, \frac{1}{4}, \ldots, \frac{1}{(n-1)^2}\right)$ the $(n-1) \times (n-1)$ diagonal matrix with $\left(1, \frac{1}{4}, \ldots, \frac{1}{(n-1)^2}\right)$ on the diagonal. We have $A(T) := \frac{2T}{\pi} R(T) \Sigma^{\frac{1}{2}}$ with $R(T) = \big(G_{(i),j}(T)\big)_{1 \leq i,j \leq n-1}$. We get:

$$\begin{aligned} \|A(T)^{-1}v\|_2^2 &= \left(\frac{\pi}{2T}\right)^2 v^t \left(\Sigma^{-\frac{1}{2}} R(T)^{-1}\right)^t \left(\Sigma^{-\frac{1}{2}} R(T)^{-1}\right) v \\ &= \left(\frac{\pi}{2T}\right)^2 (R(T)^{-1}v)^t \left(\Sigma^{-1} R(T)^{-1} v\right) \\ &\leq \left(\frac{\pi}{2T}\right)^2 (n-1)^2 \|R(T)^{-1}v\|_2^2. \end{aligned} \tag{6.6}$$

As the entries of $R(T)$ are independent with distribution $\mathcal{N}(0,1)$, $R(T)R(T)^t$ is a Wishart matrix $\mathcal{W}(n-1, n-1)$ of size $(n-1) \times (n-1)$ and with $n-1$ degree of freedom. Moreover v is trivially independent of $A(T)$ and thus on $R(T)$. Then the distribution of $\frac{\|v\|_2^2}{\|R(T)^{-1}v\|_2^2}$ is a chi-square with 1 degree of freedom $\chi^2(1)$ (see for example [45], Chapter 3).

Let now give the global construction of our successful coupling. As for the Heisenberg group, we define a partition of the time line with geometrically growing lengths: $t_0 = 0 < t_1 < ... < t_k < ...$ such that for all $k \geq 0$, $T_k := t_{k+1} - t_k = \frac{\|v_0^1\|_2}{3} 2^k$. We reproduce the above construction on each interval $[t_k, t_{k+1}]$ and we use the exponent "(k)" to distinguish the objects define above. Note that we do not update the vector v at each iteration but keep it equal to $\frac{v_0^1}{\|v_0^1\|_2}$ all the time. The obtained matrices $A^{(k)}(T_k)$ are independent and the same is true for the Brownian motions $\left(W_t^{1,(k)}\right)_t$. Then we get $\tau > t_N$ if and only if $v_t^1 \neq 0$ for all $t = t_k$, $k \leq N$ which is equivalent to $\sigma^{(k)} > 1$ for all $k \leq N-1$. We define the process $(M_t)_t$ at least on $[0, t_N]$ such that:

$$\begin{cases} M_0 = \|v_0^1\|_2 \\ M_t = M_{t_k} - W^{1,(k)}_{\frac{t-t_k}{T_k}} \times \frac{1}{\|\left(A^{(k)}(T_k)\right)^{-1} v\|_2} \text{ for } t_k \leq t \leq t_{k+1} \end{cases}$$

In particular, from (6.5), for all $k \leq M$, we have $v_{t_k} = M_{t_k} v$.

As $(M_t)_t$ is a martingale, it can be written $M_t = \|v_0^1\|_2 + \boldsymbol{\beta}_{S(t)}$ with $\boldsymbol{\beta}$ a Brownian motion starting from 0 and $S(t) = \sum_{k=0}^{N-1} \frac{1}{\|\left(A^{(k)}(T_k)\right)^{-1} v\|_2^2} \left(\frac{t-t_k}{T_k} \wedge 1\right) \mathbb{1}_{\{t \geq t_k\}}$. By construction of $(M_t)_t$, we have $\bar{\tau}_1 > t_N$ if and only if $D_{-\|v_0^1\|_2} > S(t_N)$ with $D_{-\|v_0^1\|_2} := \inf\{s > 0 \mid \boldsymbol{\beta}_s = -\|v_0^1\|_2\}$. Thus, using an estimate of the first hitting time of a Brownian motion (see Lemma A.0.1 in the appendix), we get $\mathbb{P}(\bar{\tau}_1 > t_N) \leq \frac{\sqrt{2}\|v_0^1\|_2}{\sqrt{\pi}} \mathbb{E}\left[\frac{1}{\sqrt{S(t_N)}}\right]$. For $N \geq 2$, using (6.6), we have:

$$\begin{aligned} S(t_N) &\geq \frac{1}{\|\left(A^{(N)}(T_N)\right)^{-1} v\|_2^2} + \frac{1}{\|\left(A^{(N-1)}(T_{N-1})\right) v\|_2^2} \\ &\geq \left(\frac{2T_N}{\pi(n-1)}\right)^2 \frac{1}{\|\left(R^{(N)}(T_N)\right)^{-1} v\|_2^2} + \left(\frac{2T_{N-1}}{\pi(n-1)}\right)^2 \frac{1}{\|\left(R^{(N-1)}(T_{N-1})\right)^{-1} v\|_2^2} \\ &\geq \left(\frac{2\|v_0^1\|_2 2^{N-1}}{3\pi(n-1)}\right)^2 \left(\frac{1}{\|\left(R^{(N)}(T_N)\right)^{-1} v\|_2^2} + \frac{1}{\|\left(R^{(N-1)}(T_{N-1})\right)^{-1} v\|_2^2}\right). \end{aligned}$$

In particular $\chi := \frac{1}{\|\left(R^{(N)}(T_N)\right)^{-1} v\|_2^2} + \frac{1}{\|\left(R^{(N-1)}(T_{N-1})\right)^{-1} v\|_2^2}$ is Chi-squared distributed with two degrees of freedom and $\mathbb{E}\left[\frac{1}{\sqrt{\chi}}\right] = \sqrt{\frac{\pi}{2}}$. We get:

$$\mathbb{P}(\bar{\tau}_1 > t_N) \leq 2\|v_0^1\|_2 \frac{3\pi(n-1)}{\|v_0^1\|_2 2^{N+1}}.$$

Thus, for $t \in [t_N, t_{N+1}[= \left[\|v_0^1\|_2 \frac{2^N - 1}{3}, \|v_0^1\|_2 \frac{2^{N+1}-1}{3}\right[$ with $N \geq 2$ we have:

$$\mathbb{P}(\bar{\tau}_1 > t) = \mathbb{P}(\bar{\tau}_1 > t_N) < 2\|v_0^1\|_2 \frac{\pi(n-1)}{t_{N+1}} < 2\|v_0^1\|_2 \frac{\pi(n-1)}{t}.$$

By construction $t_2 = \|v_0^1\|_2$, thus the inequality is true for all $t \geq \|v_0^1\|_2$. □

We can now give the global construction for the case when the Brownian motions start from the same fiber, that is $X_0 = \tilde{X}_0$:

Theorem 6.1.5. *We set $n \geq 2$. Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z})$ be two points in $\mathbb{G}_n$ and $\zeta \in \mathfrak{so}(n)$ such that $g^{-1} \star \tilde{g} = (\tilde{x} - x, \zeta)$. We suppose that $x = \tilde{x}$. Then there exists a successful non co-adapted coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)$ starting from $(g, \tilde{g})$ such that, for all $t \geq (n-1)\|\zeta\|_2$:*

$$\mathbb{P}(\tau > t) \leq ||\zeta||_2 \frac{\beta_n}{t} \text{ with } \beta_n = 2\pi(n-1)^2\sqrt{2(n-1)}. \tag{6.7}$$

Remark 6.1.6. *As $\beta_n \geq 1$ for all $n \geq 2$, (6.7) is trivially still true if $t < (n-1)\|\zeta\|_2$ and can be replaced by:*

$$\mathbb{P}(\tau > t) \leq \left(||\zeta||_2 \frac{\beta_n}{t} \right) \wedge 1 \text{ for all } t > 0.$$

Proof of Theorem 6.1.5. Let g, $\tilde{g} \in \mathbb{G}_n$.

Let denote $\tau_0 := 0$ and, for $1 \leq i \leq n-1$:

$$\tau_i := \inf \left\{ t > \tau_{i-1} \mid X_t = \tilde{X}_t \text{ and } \zeta_t^{i,j} = \tilde{\zeta}_t^{i,j} \ \forall j \in \{1, \dots n-1\} \setminus \{i\} \right\}.$$

For all $1 \leq i \leq n-1$, we apply the coupling from Proposition 6.1.5 for $i_0 = i$ on $[\tau_{i-1}, \tau_i]$. As ζ_t is skew-symmetric and by construction of the coupling, we have $\sum_{i=1}^{n-1} \tau_i = \tau$.

If $t \geq (n-1)\|\zeta_0\|_2$, then $\frac{t}{n-1} \geq \|v_0^i\|_2$ for all $1 \leq i \leq n-1$. We obtain:

$$\begin{aligned}
\mathbb{P}(\tau > t) &\leq \sum_{i=2}^{n-1} \mathbb{P}\left(\tau_i - \tau_{i-1} > \frac{t}{n-1}, \tau_l - \tau_{l-1} \leq \frac{t}{n-1} \ \forall 1 \leq l \leq i-1 \right) \\
&\quad + \mathbb{P}\left(\tau_1 > \frac{t}{n-1} \right) \\
&\leq \sum_{i=1}^{n-1} \|v_0^i\|_2 \frac{2\pi(n-1)^2}{t} \leq \sqrt{2(n-1)} \|\zeta_0\|_2 \frac{2\pi(n-1)^2}{t}.
\end{aligned}$$

□

Remark 6.1.7. *As ζ is skew-symmetric, there exists a matrix P_v such that $(P_v^t \zeta P_v)_{i,j} \neq 0$ only if i is odd and $j = i+1$. In particular, $\|P_v^t \zeta P_v\|_2 = \|\zeta\|_2$. Then applying the coupling from Proposition 6.1.5 to this new matrix, we can reduce the number of line to be treated. In particular, we obtain the previous result with $2\pi(n-1)\lfloor \frac{n}{2} \rfloor^{\frac{3}{2}}$ instead of β_n.*

6.1.2 General case

We can now deal with the general case, that is when $x \neq \tilde{x}$.

Proof of Theorem 6.1.1. To obtain a successful coupling on $\mathbb{G}_n$, we are going to use first the reflection coupling until the time $\tau_0 := \inf\{t \geq 0 \mid X_t = \tilde{X}_t\}$ and then the coupling from Theorem 6.1.5. Let us remind how we define the reflection coupling. There exists $P_h \in \mathcal{O}(n)$ such that $(P_h x)_i = (P_h \tilde{x})_i$ for all $2 \leq i \leq n$, and $(P_h x)_1 - (P_h \tilde{x})_1 = ||x - \tilde{x}||_2$. We first define a coupling $\left(\underline{X}_t, \underline{\tilde{X}}_t\right)$ of Brownian motions on $\mathbb{R}^n$ starting from $(P_h x, P_h \tilde{x})$: we choose $\underline{X}_t^i = \underline{\tilde{X}}_t^i$ for $2 \leq i \leq n$ and $d\underline{X}_t^1 = -d\underline{\tilde{X}}_t^1$. We are then able to deduce a

coupling of Brownian motion on $\mathbb{G}_n$ such that $(X_t, \tilde{X}_t) := \left(P_h^t \underline{X}_t, P_h^t \underline{\tilde{X}}_t\right)$. In particular, we have $\tau_0 = \inf\{t \geq 0 \mid \underline{X}_t = \underline{\tilde{X}}_t\}$. Thus τ_0 has the distribution of the first hitting time D_a of a Brownian motion with $a = \frac{1}{2}||x - \tilde{x}||_2$ and $\mathbb{P}(\tau_0 > t) \leq \left(\frac{||x-\tilde{x}||_2}{\sqrt{2\pi t}}\right) \wedge 1$ (see Lemma A.0.1). At time τ_0, the Brownian motions are in the same fiber and we can apply the coupling from Theorem 6.1.5 on $[\tau_0, \tau]$. For $t > 0$, we get:

$$\begin{aligned}
\mathbb{P}(\tau > t) &= \mathbb{P}\left(\tau > t, \tau_0 \leq \frac{t}{2}\right) + \mathbb{P}\left(\tau > t, \tau_0 > \frac{t}{2}\right) \leq \mathbb{P}\left(\tau - \tau_0 > \frac{t}{2}, \tau_0 \leq \frac{t}{2}\right) + \mathbb{P}\left(\tau_0 > \frac{t}{2}\right) \\
&\leq \mathbb{E}\left[\mathbb{P}\left(\tau - \tau_0 > \frac{t}{2} \;\middle|\; \|\zeta_{\tau_0}\|_2\right) \mathbb{1}_{\{\tau_0 \leq \frac{t}{2}\}}\right] + \frac{||x - \tilde{x}||_2}{\sqrt{\pi t}} \\
&\leq \mathbb{E}\left[\left(||\zeta_{\tau_0}||_2 \frac{2\beta_n}{t}\right) \wedge 1\right] + \frac{||x - \tilde{x}||_2}{\sqrt{\pi t}}. \qquad (6.8)
\end{aligned}$$

We now need a good estimate of $\mathbb{E}\left[||\zeta_{\tau_0}||_2\right]$. This can be done by using the following Lemma which will be proven just after the end of the current proof:

Lemma 6.1.8. *For every $m \geq \frac{1}{2}\|x - \tilde{x}\|_2^2$, we have:*

$$\mathbb{E}\left[\frac{\|\zeta_{\tau_0}\|_2}{m} \wedge 1\right] \leq \frac{\|\zeta\|_2}{m} + \frac{2\sqrt{2}}{\sqrt{m}}(n-1)||x - \tilde{x}||_2.$$

If we choose $m := \frac{t}{2\beta_n}$ with t such that $\frac{t}{2\beta_n} \geq \frac{1}{2}\|x - \tilde{x}\|_2^2$, we can use Lemma 6.1.8 together with (6.8) to obtain the expected result. This ends the proof of Theorem 6.1.1. □

We now proceed with the proof of Lemma 6.1.8. We use the results from the coupling on the Heisenberg group. In particular *Lemma 3.4* from [6] can be written this way:

Lemma 6.1.9. *Let $\left((X_t, z_t), (\tilde{X}_t, \tilde{z}_t)\right)_t$ be a coupling of two Brownian motion on $\mathbb{G}_n$ such that:*

- *$X_t^j = \tilde{X}_t^j$ for all t and $j \geq 2$;*
- *$dX_t^1 = -d\tilde{X}_t^1$ on $[0, S]$ with $S = \inf\{t \geq 0 | X_t^1 = \tilde{X}_t^1\}$.*

For all $m \geq \frac{1}{2}|X_0^1 - \tilde{X}_0^1|^2$, we have:

$$\mathbb{E}\left[\left|\zeta_S^{1,j} - \zeta_0^{1,j}\right| \wedge m\right] \leq 2\sqrt{2m}|X_0^1 - \tilde{X}_0^1|.$$

The original proof can be found in [6] for $\zeta_S^{1,j}$ instead of $\zeta_S^{1,j} - \zeta_0^{1,j}$. This slight modification allows us to release the hypothesis on m and to obtain better estimates for $C_1(n)$ and $C_2(n)$ in Theorem 6.1.1 (better order for n). We write here the detail of the proof for convenience for the reader but the method is the same than in the original work.

Proof of Lemma 6.1.9. For $t \in [0, S]$, we have $\tilde{X}_t^1 - X_t^1 = -2X_t^1 + X_0^1 + \tilde{X}_0^1$ and $d\zeta_t^{1,j} = -2(X_t^1 - X_0^1 + \frac{X_0^1 - \tilde{X}_0^1}{2})dX_t^j$. We are going to use the following characterization of the mean:

$$\mathbb{E}\left[\left|\zeta_S^{1,j} - \zeta_0^{1,j}\right| \wedge m\right] = \int_0^m \mathbb{P}\left(|\zeta_S^{1,j} - \zeta_0^{1,j}| > t\right) dt \qquad (6.9)$$

Let denote $a = \frac{X_0^1 - \tilde{X}_0^1}{2}$. Exchanging the roles of $(X_t)_t$ and $(\tilde{X}_t)_t$ if needed we can make the hypothesis $a > 0$. For any $b \in \mathbb{R}$ we denote $D_b := \inf\{t \geq 0 \mid X_t^1 - X_0^1 = b\}$. In particular, we have $S = D_{-a}$. Then, for $y > a^2$, by comparing the stopping times S and $D_{\sqrt{y}-a}$ and by using the Markov inequality we get:

$$\begin{aligned}
\mathbb{P}\left(\left|\zeta_S^{1,j} - \zeta_0^{1,j}\right| > 2y\right) &= \mathbb{P}\left(\left|\int_0^S (a + X_t^1 - X_0^1) dX_t^j\right| > y\right) \\
&\leq \mathbb{P}\left(D_{\sqrt{y}-a} < S\right) + \mathbb{P}\left(\left|\int_0^S (a + X_t^1 - X_0^1) dX_t^j\right| > y, D_{\sqrt{y}-a} \geq S\right) \\
&\leq \mathbb{P}\left(D_{\sqrt{y}-a} < S\right) + \mathbb{P}\left(\left|\int_0^{S \wedge D_{\sqrt{y}-a}} (a + X_t^1 - X_0^1) dX_t^j\right| > y\right) \\
&\leq \mathbb{P}\left(D_{\sqrt{y}-a} < S\right) + \frac{1}{y^2}\mathbb{E}\left[\left|\int_0^{S \wedge D_{\sqrt{y}-a}} (a + X_t^1 - X_0^1) dX_t^j\right|^2\right] \\
&\leq \mathbb{P}\left(D_{\sqrt{y}-a} < S\right) + \frac{1}{y^2}\mathbb{E}\left[\int_0^{S \wedge D_{\sqrt{y}-a}} (a + X_t^1 - X_0^1)^2 dt\right].
\end{aligned}$$

Using some known results on the first exit time of a Brownian motion (see Lemma A.0.2), we get:

$$\begin{aligned}
\mathbb{P}\left(\left|\zeta_S^{1,j} - \zeta_0^{1,j}\right| > 2y\right) &\leq \mathbb{P}\left(D_{\sqrt{y}-a} < D_{-a}\right) + \frac{1}{y^2}\mathbb{E}\left[D_{-a} \wedge D_{\sqrt{y}-a}\right](a + \sqrt{y} - a)^2 \\
&\leq \frac{a}{\sqrt{y}} + \frac{a(\sqrt{y} - a)}{y} \leq \frac{2a}{\sqrt{y}}.
\end{aligned}$$

Finally for $t > \frac{1}{2}|X_0^1 - \tilde{X}_0^1|^2$ we obtain:

$$\mathbb{P}\left(\left|\zeta_S^{1,j} - \zeta_0^{1,j}\right| > t\right) \leq \sqrt{2}\frac{|X_0^1 - \tilde{X}_0^1|}{\sqrt{t}}. \tag{6.10}$$

We come back to Equality (6.9):

$$\begin{aligned}
\mathbb{E}\left[\left|\zeta_S^{1,j} - \zeta_0^{1,j}\right| \wedge m\right] &= \int_0^{\frac{|X_0^1 - \tilde{X}_0^1|^2}{2}} \mathbb{P}\left(|\zeta^{1,j} - \zeta_0^{1,j}| > t\right) dt + \int_{\frac{|X_0^1 - \tilde{X}_0^1|^2}{2}}^m \mathbb{P}\left(|\zeta^{1,j} - \zeta_0^{1,j}| > t\right) dt \\
&\leq \frac{|X_0^1 - \tilde{X}_0^1|^2}{2} + \sqrt{2}|X_0^1 - \tilde{X}_0^1| \int_{\frac{|X_0^1 - \tilde{X}_0^1|^2}{2}}^m \frac{1}{\sqrt{t}} dt \\
&\leq \frac{|X_0^1 - \tilde{X}_0^1|^2}{2} + 2\sqrt{2m}|X_0^1 - \tilde{X}_0^1| - 2|X_0^1 - \tilde{X}_0^1|^2 \\
&\leq 2\sqrt{2m}|X_0^1 - \tilde{X}_0^1|.
\end{aligned}$$

□

We can now give the proof of Lemma 6.1.8:

Proof of Lemma 6.1.8. We use the same notations as in the proof of Theorem 6.1.1. In particular we use the notation $\left(\underline{\mathbb{B}}_t, \underline{\tilde{\mathbb{B}}}_t\right)_t$ to describe the couplings of Brownian motions induced by $\left(\underline{X}_t, \underline{\tilde{X}}_t\right)_t$ and $\underline{\zeta}_t$ such that $\underline{\mathbb{B}}_t^{-1}\underline{\tilde{\mathbb{B}}}_t = (\underline{\tilde{X}}_t - X_t, \underline{\zeta}_t)$. Note that $\underline{\zeta}_t = P_h \zeta_t P_h^t$. As

$\left(\underline{X}_t, \underline{\tilde{X}}_t\right)_t$ is defined by using the reflection coupling, the conditions of Lemma 6.1.9 are satisfied.

Note that $|\underline{X}_0^1 - \underline{\tilde{X}}_0^1|^2 = \|x - \tilde{x}\|_2^2$. Then for $m \geq \frac{1}{2}\|x - \tilde{x}\|_2^2$ and for every $j \geq 2$, by applying Lemma 6.1.9, we obtain:

$$\mathbb{E}\left[\frac{\left|\underline{\zeta}_{\tau_0}^{1,j} - \underline{\zeta}_0^{1,j}\right|}{m} \wedge 1\right] = \frac{1}{m}\mathbb{E}\left[\left|\underline{\zeta}_{\tau_0}^{1,j} - \underline{\zeta}_0^{1,j}\right| \wedge m\right] \leq \frac{2\sqrt{2}}{\sqrt{m}}\|x - \tilde{x}\|_2. \tag{6.11}$$

Moreover, for $t \in [0, \tau_0]$, $\underline{X}_t^j = \underline{\tilde{X}}_t^j$ for $2 \leq j \leq n$, thus, $(\underline{\zeta}_t)^{i,j}$ is constant on $[0, \tau_0]$ for all $2 \leq i \neq j \leq n$. Using equivalences of norms and the fact that $\|\zeta_{\tau_0} - \zeta_0\|_2 = \|\underline{\zeta}_{\tau_0} - \underline{\zeta}_0\|_2$, we get:

$$\begin{aligned}
\mathbb{E}\left[\frac{\|\zeta_{\tau_0}\|_2}{m} \wedge 1\right] &\leq \frac{\|\zeta_0\|_2}{m} + \mathbb{E}\left[\frac{\|\zeta_{\tau_0} - \zeta_0\|_2}{m} \wedge 1\right] = \frac{\|\zeta\|_2}{m} + \mathbb{E}\left[\frac{\|\underline{\zeta}_{\tau_0} - \underline{\zeta}_0\|_2}{m} \wedge 1\right] \\
&\leq \frac{\|\zeta\|_2}{m} + \sum_{j=2}^{n} \mathbb{E}\left[\frac{|\underline{\zeta}_{\tau_0}^{1,j} - \underline{\zeta}_0^{1,j}|}{m} \wedge 1\right] \\
&\leq \frac{\|\zeta\|_2}{m} + \frac{2\sqrt{2}}{\sqrt{m}}(n-1)||x - \tilde{x}||_2.
\end{aligned}$$

□

6.1.3 Coupling on homogeneous Carnot groups and distance in total variation

From Theorem 6.1.1, we can obtain a successful coupling of Brownian motions for any homogeneous step 2 Carnot group $\mathbb{G}$.

Corollary 6.1.10. *Let $\mathbb{G}$ be an homogeneous step 2 Carnot group of rank n. For all $g = (x, z), \tilde{g} = (\tilde{x}, \tilde{z}) \in \mathbb{G}$ we can construct a successful coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)$ of Brownian motions starting from $(g, \tilde{g})$. Moreover, for all $t \geq 4\beta_n\|x - \tilde{x}\|_2^2$:*

$$\mathbb{P}(\tau > t) \leq C_1(n)\frac{\|x - \tilde{x}\|}{\sqrt{t}}\mathbb{1}_{\{x \neq \tilde{x}\}} + \frac{C_2(n)}{m_1(n)^2}\frac{d_{cc}(g, \tilde{g})^2}{t} \tag{6.12}$$

$$\leq C_1(n)\frac{d_{cc}(g, \tilde{g})}{\sqrt{t}}\mathbb{1}_{\{x \neq \tilde{x}\}} + \frac{C_2(n)}{m_1(n)^2}\frac{d_{cc}(g, \tilde{g})^2}{t} \tag{6.13}$$

with $C_1(n)$ and $C_2(n)$ the explicit constants from Theorem 6.1.1.

Proof. Let $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z}) \in \mathbb{G}$. Using Proposition 3.1.4, we can build a surjective morphism $\phi : \mathbb{G}_n \to G$ that preserves the horizontal coordinates. Moreover, there exists $a = (x, v), \tilde{a} = (\tilde{x}, \tilde{v}) \in \mathbb{G}_n$ such that:

$$\phi(a) = g, \ \phi(\tilde{a}) = \tilde{g} \text{ and } d_{cc}(g, \tilde{g}) = d_{cc}(a, \tilde{a}). \tag{6.14}$$

Using the construction from Theorem 6.1.1, we construct a successful coupling $(\mathbf{B}_t, \tilde{\mathbf{B}}_t)_t$ of Brownian motions starting from $(a, \tilde{a})$ on $\mathbb{G}_n$. Denoting by S the first coupling time of $(\mathbf{B}_t, \tilde{\mathbf{B}}_t)_t$ we have for $t \geq 4\beta_n\|x - \tilde{x}\|_2^2$:

$$\mathbb{P}(S > t) \leq C_1(n)\frac{\|x - \tilde{x}\|_2}{\sqrt{t}}\mathbb{1}_{\{x \neq \tilde{x}\}} + C_2(n)\frac{\|\zeta\|_2}{t}.$$

On $\mathbb{G}_n$, we have: $\|x-\tilde{x}\|_2 \leq d_{cc}(a,\tilde{a})$ and $\|\zeta\|_2 \leq \frac{1}{m_1(n)^2} d_{cc}(a,\tilde{a})^2$ (see Relations (3.7) and (3.15)), thus:

$$\mathbb{P}(S>t) \leq C_1(n)\frac{\|x-\tilde{x}\|_2}{\sqrt{t}}\mathbb{1}_{\{x\neq\tilde{x}\}} + \frac{C_2(n)}{m_1(n)^2}\frac{d_{cc}(a,\tilde{a})^2}{t}. \tag{6.15}$$

We can then define a coupling of Brownian motions $\left(\phi(\mathbf{B}_t), \phi(\tilde{\mathbf{B}}_t)\right)_t$ starting at $(g,\tilde{g})$. Denoting by τ its first coupling time, we have $\mathbb{P}(S>t) \leq \mathbb{P}(\tau>t)$. Using (6.15) together with (6.14), we obtain the expected result. □

From these results, by applying the Coupling Inequality (also called Aldous Inequality), we directly obtain an estimate for the total variation distance between the laws of the Brownian motion. Please note that the constants obtained above are in fact improved by the coupling constructed in Chapter 7.

Let $\mu_t^{(x,z)}$ denote the law at t of the subRiemannian Brownian motion started at (x,z).

Corollary 6.1.11. *Let $n \geq 2$, $g=(x,z)$, $\tilde{g}=(\tilde{x},\tilde{z})$ two points in $\mathbb{G}_n$ and $\zeta \in \mathfrak{so}(n)$ such that $g^{-1}\star\tilde{g} = (\tilde{x}-x,\zeta)$. For $t \geq \beta_n\|x-\tilde{x}\|_2^2$:*

$$d_{TV}\left(\mu_t^g, \mu_t^{\tilde{g}}\right) \leq C_1(n)\frac{\|\tilde{x}-x\|_2}{\sqrt{t}} + C_2(n)\frac{\|\zeta\|_2}{t}. \tag{6.16}$$

with $C_1(n)$ and $C_2(n)$ the constants given in Theorem 6.1.1.
Replacing $\mathbb{G}_n$ by any homogeneous, step 2 Carnot group of rank n, the same property is true by replacing (6.16) by:

$$d_{TV}\left(\mu_t^g, \mu_t^{\tilde{g}}\right) \leq C_1(n)\frac{d_{cc}(g,\tilde{g})}{\sqrt{t}} + \frac{C_2(n)}{m_1(n)^2}\frac{d_{cc}(g,\tilde{g})^2}{t}.$$

6.2 Comparison between the first coupling time and the first exit time from a domain

In this section, we consider a domain D on $\mathbb{G}_n$. We remind some notations introduced in Chapter 2. For any process $(Y_t)_t$ on $\mathbb{G}_n$ we define $\tau_D(Y) := \inf\{t \geq 0 \mid Y_t \notin D\}$. Using the pseudo-norm $\boldsymbol{\delta}$ introduced in Subsection 3.1.5, for any $g \in D$, we denote $\delta_g = \boldsymbol{\delta}(g, D^c)$. For the coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ constructed in Theorem 6.1.1, we are going to compare the coupling time τ with $\tau_D(\mathbb{B}) \wedge \tau_D(\tilde{\mathbb{B}})$ in order to obtain gradient inequalities as in [28, 27, 6]. As previously we adapt the proof used for the Heisenberg group to all free, step 2 Carnot groups [6].

6.2.1 Main results

For $n \geq 2$ and $g=(x,z)$, $\tilde{g}=(\tilde{x},\tilde{z}) \in \mathbb{G}_n$ and $\zeta \in \mathfrak{so}(n)$ such that $g^{-1}\star\tilde{g} = (\tilde{x}-x,\zeta)$, we define $\hat{x} := \frac{x+\tilde{x}}{2} \in \mathbb{R}^n$ and $\hat{z} := \frac{z+\tilde{z}}{2} \in \mathfrak{so}(n)$. In particular, $x-\hat{x} = \frac{x-\tilde{x}}{2}$ and $z-\hat{z}-\frac{1}{2}\hat{x}\odot x = \frac{\zeta}{2}$. For any $\alpha>0$ and $\gamma>0$ real, we also define the set $Q(\alpha,\gamma) \subset \mathbb{G}_n$ such that:

$$Q(\alpha,\gamma) := \{(y,v) \in \mathbb{G}_n \mid \|y-\hat{x}\|_2 < \alpha, \|v-\hat{z}-\frac{1}{2}\hat{x}\odot y\|_2 < \gamma^2\}. \tag{6.17}$$

Let consider $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ the coupling starting at $(g, \tilde{g})$ described in Theorem 6.1.1. We are first dealing with the problem for $D = Q(\alpha, \gamma)$. We have the following result:

Proposition 6.2.1. *We suppose that $\|x - \tilde{x}\|_2 \leq \frac{2}{\sqrt{n-1}}$. Then, for any $0 < \alpha \leq \gamma$, there exist some constants D_1 and D_2, that depend neither on α, γ nor on the starting points of the coupling, such that:*

$$\mathbb{P}\left(\tau > \tau_{Q(\alpha,\gamma)}(\mathbb{B})\right) \leq D_1(n,\gamma,\alpha)\|x - \tilde{x}\|_2 + D_2(n,\gamma,\alpha)\max\left(\|\zeta\|_2, \|\zeta\|_2^2\right)$$

with:

$$D_1(n,\gamma,\alpha) \leq D_1\left(n\sqrt{n} + \frac{n\sqrt{n}}{\alpha} + \frac{\alpha^3}{\gamma^4}n + \frac{n^{9+\frac{1}{4}}}{\gamma^4} + \frac{n^{11+\frac{1}{4}}}{\alpha^4}\right);$$
$$D_2(n,\gamma,\alpha) \leq D_2\left(1 + \frac{n^{8+\frac{1}{2}}}{\gamma^4} + \frac{n^{10+\frac{1}{2}}}{\alpha^4}\right).$$

The same inequality is true for $\mathbb{P}\left(\tau > \tau_{Q(\alpha,\gamma)}(\tilde{\mathbb{B}})\right)$.

Proposition 6.2.1 will be proven in Subsection 6.2.2. For now we give a direct consequence for the exit time of any domain D.

Theorem 6.2.2. *Let $g = (x, z), \tilde{g} = (\tilde{x}, \tilde{z}) \in \mathbb{G}_n$ and $g^{-1} \star \tilde{g} = (\tilde{x} - x, \zeta)$ such that $\boldsymbol{\delta}(g, \tilde{g}) \leq \frac{\sqrt{2}\delta_g}{3}$ and $\|x - \tilde{x}\|_2 \leq \frac{2}{\sqrt{n-1}}$. Denoting by $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ the coupling starting at $(g, \tilde{g})$ described in Theorem 6.1.1, we have:*

$$\mathbb{P}\left(\tau > \tau_D(\mathbb{B}) \wedge \tau_D(\tilde{\mathbb{B}})\right) \leq D_1 \times \left(n\sqrt{n} + \frac{n\sqrt{n}}{\delta_g} + \frac{n^{11+\frac{1}{4}}}{\delta_g^4}\right)\|x - \tilde{x}\|_2$$
$$+ D_2 \times \left(1 + \frac{n^{10+\frac{1}{2}}}{\delta_g^4}\right)\max\left(\|\zeta\|_2, \|\zeta\|_2^2\right)$$

with D_1 and D_2 two constant that depend neither on $g, \tilde{g}$ nor on n.

Proof of Theorem 6.2.2. By construction of $(\hat{x}, \hat{z})$, we have:

$$\boldsymbol{\delta}\left((\hat{x}, \hat{z}), (x, z)\right) = \boldsymbol{\delta}\left((\hat{x}, \hat{z}), (\tilde{x}, \tilde{z})\right) \leq \frac{1}{\sqrt{2}}\boldsymbol{\delta}\left((\tilde{x}, \tilde{z}), (x, z)\right) \leq \frac{\delta_g}{3}.$$

Using the pseudo triangular inequality (3.13), for any $(y, v) \in \mathbb{G}_n$, we have:

$$\boldsymbol{\delta}\left((y, v), (x, z)\right) \leq \frac{3}{2}\Big(\boldsymbol{\delta}\left((y, v), (\hat{x}, \hat{z})\right) + \boldsymbol{\delta}\left((\hat{x}, \hat{z}), (x, z)\right)\Big). \tag{6.18}$$

Thus, if $(y, v) \in Q\left(\frac{\delta_g}{3\sqrt{2}}, \frac{\delta_g}{3\sqrt{2}}\right)$, then $\boldsymbol{\delta}\left((y, v), (x, z)\right) < \frac{3}{2}\left(\sqrt{2} \times \frac{\delta_g}{3\sqrt{2}} + \frac{\delta_g}{3}\right) = \delta_g$. Thus, $Q := Q\left(\frac{\delta_g}{3\sqrt{2}}, \frac{\delta_g}{3\sqrt{2}}\right) \subset D$. We have:

$$\mathbb{P}\left(\tau > \tau_D(\mathbb{B}) \wedge \tau_D(\tilde{\mathbb{B}})\right) \leq \mathbb{P}\left(\tau > \tau_Q(\mathbb{B}) \wedge \tau_Q(\tilde{\mathbb{B}})\right)$$
$$\leq \mathbb{P}\left(\tau > \tau_Q(\mathbb{B})\right) + \mathbb{P}\left(\tau > \tau_Q(\mathbb{B})\right).$$

As $\|x - \tilde{x}\|_2 \leq \frac{2}{\sqrt{n-1}}$, we can apply Proposition 6.2.1. We get the expected result. □

6.2.2 Proof of Proposition 6.2.1

Proof of Proposition 6.2.1. It is enough to prove the proposition for $(\mathbb{B}_t)_t$. We first notice that

$$\tau_{Q(\alpha,\gamma)}(\mathbb{B}) = \inf\{t > 0 \mid \|X_t - \hat{x}\|_2 \geq \alpha \text{ or } \|U_t\|_2 \geq \gamma^2\}$$

with $U_t := z_t - \hat{z} - \frac{1}{2}\hat{x} \odot X_t$. In particular, we have $dU_t = \frac{1}{2}(X_t - \hat{x}) \odot dX_t$ and $U_0 = \frac{z-\tilde{z}}{2} - \frac{x-\tilde{x}}{4} = \frac{\zeta}{2}$. We use the notations from the the proofs of Section 6.1:

- As in the proof of Theorem 6.1.1, we denote: $\tau_0 := \inf\{t \geq 0 \mid X_t = \tilde{X}_t\}$, in particular on $[0, \tau_0]$, the processes $(X_t)_{t\in[0,\tau_0]}$ and $(\tilde{X}_t)_{t\in[0,\tau_0]}$ are coupled by reflection.
- As in the proof of Theorem 6.1.5, we denote for $1 \leq i \leq n-1$:

$$\tau_i := \inf\left\{t > \tau_{i-1} \mid \underline{X}_t = \underline{\tilde{X}}_t \text{ and } \underline{\zeta}^{i,j}_t = \underline{\tilde{\zeta}}^{i,j}_t \ \forall j \in \{1, \dots n-1\} \setminus \{i\}\right\}.$$

We remind that $\tau := \inf\{t > 0 \mid \mathbb{B}_t = \tilde{\mathbb{B}}_t\}$ is exactly equal to τ_{n-1}. We also denote $\tau_{-1} := 0$. As we have a different description of the coupling on each of the intervals of time $[0, \tau_0]$, $[\tau_0, \tau_1]$, ..., $[\tau_{n-2}, \tau_{n-1}]$, it will be convenient to decompose the process $(\mathbb{B}_t)_t = (X_t, z_t)_t$ following this time decomposition. One of the main tool that will be used is the Burkholder-Davis-Gundy Inequality that we will denote afterward "B-D-G Inequality". Then it will be convenient to consider martingales starting from 0 on each of these intervals of time. Keeping these constraints in mind, we define:

- for $t \in [0, \tau_0]$, $Y_0(t) := X_t - \hat{x}$ and $V_0(t) := U_t$;
- for $1 \leq i \leq n-1$ and $t \in [\tau_{i-1}, \tau_i]$, $Y_i(t) := X_t - X_{\tau_{i-1}}$ and $V_i(t) := \frac{1}{2}\int_{\tau_{i-1}}^t (X_s - X_{\tau_{i-1}}) \odot dX_s \in \mathfrak{so}(n)$. In particular, we have:

$$V_i^{j,l}(t) = \frac{1}{2}\left(\int_{\tau_{i-1}}^t \left(X_s^j - X_{\tau_{i-1}}^j\right) dX_s^l - \int_{\tau_{i-1}}^t \left(X_s^l - X_{\tau_{i-1}}^l\right) dX_s^j\right). \tag{6.19}$$

Proceeding by induction one can then check that, for all $0 \leq i \leq n-1$ and $t \in [\tau_{i-1}, \tau_i]$, we have:

$$(X_t - \hat{x}, U_t) = \big(Y_0(\tau_0), V_0(\tau_0)\big) \star \dots \star \big(Y_{i-1}(\tau_{i-1}), V_{i-1}(\tau_{i-1})\big) \star \big(Y_i(t), V_i(t)\big). \tag{6.20}$$

Moreover, using the fact that we iterate $i+1$ times the operation $\star$, we have for all $1 \leq i \leq n-1$:

$$\begin{aligned}(X_t - \hat{x}, U_t) = \Bigg(&\sum_{j=0}^{i-1} Y_j(\tau_j) + Y_i(t), \\ &\sum_{j=0}^{i-1} V_j(\tau_j) + V_i(t) + \frac{1}{2}\sum_{j=0}^{i-1} Y_j(\tau_j) \odot \left(\sum_{l=j+1}^{i-1} Y_l(\tau_l) + Y_i(t)\right)\Bigg).\end{aligned} \tag{6.21}$$

We consider two sequences $(\alpha(i))_{0\leq i\leq n-1}$ and $(\gamma(i))_{0\leq i\leq n-1}$ satisfying:

$$\sum_{i=0}^{n-1} \alpha(i) < \alpha \text{ and } \sum_{i=0}^{n-1} \gamma(i)^2 + \frac{1}{2}\sum_{0\leq i<j\leq n-1} \alpha(i)\alpha(j) < \gamma^2. \tag{6.22}$$

For all $0 \leq i \leq n-1$, we define the events:

$$A_i := \{\omega, \sup_{\tau_{i-1} \leq t \leq \tau_i} \|Y_i(t)\|_2 > \alpha(i)\} \text{ and } \Gamma_i := \{\omega, \sup_{\tau_{i-1} \leq t \leq \tau_i} \|V_i(t)\|_2 > \gamma(i)^2\}.$$

Using (6.21), we remark that under the event $\bigcap_{i=0}^{n-1} (A_i^c \cap \Gamma_i^c)$, we have for all $0 \leq i \leq n-1$ and for all $t \in [\tau_{i-1}, \tau_i]$:

$$\|X_t - \hat{x}\|_2 \leq \sum_{j=0}^{i} \alpha(j) < \alpha \text{ and } \|U_t\|_2 \leq \sum_{j=0}^{i} \gamma(j)^2 + \frac{1}{2} \sum_{0 \leq j < l \leq i} \alpha(j)\alpha(l) < \gamma^2. \tag{6.23}$$

In other words, $\mathbb{B}_t \in Q(\alpha, \gamma)$ for all $t \in [0, \tau]$. Thus $\mathbb{P}\left(\tau > \tau_{Q(\alpha,\gamma)}(\mathbb{B})\right) \leq \mathbb{P}\left(\bigcup_{i=0}^{n-1} (A_i \cup \Gamma_i)\right)$.

To study these probabilities we will use the three following lemmas.

The first one contains results on the reflection coupling in $\mathbb{R}^n$ and is due to Cranston. These results can be found in the proof of *Theorem 1* in [28]. Using the Euclidean norm $\|\cdot\|_2$ instead of the maximum norm $\|\cdot\|_\infty$ the results can be presented as follows:

Lemma 6.2.3. *For any $a > 0$, let denote $S(a) := \inf\{t \geq 0 \mid \|X_t - \hat{x}\|_2 > a\}$. With the previous notations, we have:*

$$\mathbb{P}\left(\tau_0 > S(a)\right) \leq \frac{n\sqrt{n}\|x - \tilde{x}\|_2}{2a} \text{ and } \mathbb{E}\left[\tau_0 \wedge S(a)\right] \leq \frac{\|x - \tilde{x}\|_2}{2} a.$$

The second lemma can be seen as the generalisation in higher dimension of *Lemma 4.1* from [6]:

Lemma 6.2.4. *For any $1 \leq i \leq n-1$ and $M > 0$ there exists a constant C (depending neither on the starting points of the processes, nor on the rank n of the group) such that:*

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_i - \tau_{i-1} < M\}} \sup_{\tau_{i-1} \leq t \leq \tau_i} \|Y_i(t)\|_2^4\right] \leq Cn^2 \mathbb{E}\left[\left((\tau_i - \tau_{i-1}) \wedge M\right)^2\right]; \tag{6.24}$$

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_i - \tau_{i-1} < M\}} \sup_{\tau_{i-1} \leq t \leq \tau_k} \|V_i(t)\|_2^2\right] \leq Cn(n-1) \mathbb{E}\left[\left((\tau_i - \tau_{i-1}) \wedge M\right)^2\right]. \tag{6.25}$$

The third lemma gives an estimate for the coupling time after time τ_0:

Lemma 6.2.5. *Using the previous notations, for any $M \geq \frac{\beta_n}{2}\|x - \tilde{x}\|_2^2$, we have:*

$$\mathbb{E}\left[(\tau - \tau_0)^2 \wedge M^2\right] \leq \frac{4}{3} M^{\frac{3}{2}} 2\sqrt{2\beta_n}(n-1)||x - \tilde{x}||_2 + 2M\beta_n\|\zeta\|_2.$$

Please note that Lemmas 6.2.4 and 6.2.5 will be proven in Subsection 6.2.3. For now, we pursue the proof of Proposition 6.2.1.

From Lemma 6.2.3, we obtain:

$$\mathbb{P}(A_0) \leq \mathbb{P}\big(\tau_0 > S\left(\alpha(0)\right)\big) \leq \frac{n\sqrt{n}\|x - \tilde{x}\|_2}{2\alpha(0)}. \tag{6.26}$$

Using the Markov Inequality, we also have:

$$\begin{aligned}\mathbb{P}(A_0^c \cap \Gamma_0) &\leq \mathbb{P}\left(\sup_{0\leq t\leq \tau_0\wedge S(\alpha(0))} \|U_t\|_2 > \gamma(0)^2\right) \leq \frac{1}{\gamma(0)^4}\mathbb{E}\left[\sup_{0\leq t\leq \tau_0\wedge S(\alpha(0))} \|U_t\|_2^2\right] \\ &\leq \frac{1}{\gamma(0)^4}\left(\mathbb{E}\left[\sup_{0\leq t\leq \tau_0\wedge S(\alpha(0))} \|U_t - U_0\|_2^2\right] + \frac{\|\zeta\|_2^2}{4}\right).\end{aligned}$$

We remind that, for $1 \leq j \neq l \leq n$, $dU_t^{j,l} = \frac{1}{2}\left((X_t - \hat{x})^j dX_t^l - (X_t - \hat{x})^l dX_t^j\right)$. As the reflection coupling is a co-adapted coupling, $\tau_0 \wedge S(\alpha(0))$ is a stopping time for the common filtration of $(X_t)_t$ and $(\tilde{X}_t)_t$. Using the B-D-G Inequality, we obtain:

$$\begin{aligned}&\sum_{1\leq j<l\leq n} \mathbb{E}\left[\sup_{0\leq t\leq \tau_0\wedge S(\alpha(0))} \left(U_t^{j,l} - U_0^{j,l}\right)^2\right] \\ &\qquad\leq \sum_{1\leq j<l\leq n} \frac{c_2}{4}\mathbb{E}\left[\int_0^{\tau_0\wedge S(\alpha(0))} |X_t^j - \hat{x}^j|^2 + |X_t^l - \hat{x}^l|^2 dt\right] \\ &\qquad\leq \frac{c_2(n-1)}{4}\mathbb{E}\left[\int_0^{\tau_0\wedge S(\alpha(0))} \|X_t - \hat{x}\|_2^2 dt\right] \\ &\qquad\leq \frac{c_2(n-1)}{4}\alpha(0)^2\mathbb{E}\left[\tau_0 \wedge S(\alpha(0))\right] \\ &\qquad\leq \frac{c_2(n-1)}{8}\alpha(0)^3\|x - \tilde{x}\|_2.\end{aligned}$$

Finally we have:

$$\mathbb{P}\left(A_0^c \cap \Gamma_0\right) \leq \frac{c_2 n}{8}\frac{\alpha(0)^3}{\gamma(0)^4}\|x - \tilde{x}\|_2 + \frac{\|\zeta\|_2^2}{4\gamma(0)^4}. \tag{6.27}$$

We now look at $\mathbb{P}\left(\bigcup_{i=1}^{n-1}(A_i \cup \Gamma_i)\right)$. We have:

$$\begin{aligned}\mathbb{P}\left(\bigcup_{i=1}^{n-1}(A_i \cup \Gamma_i)\right) &\leq \sum_{i=1}^{n-1}\mathbb{P}\left(\tau_i - \tau_{i-1} > \frac{\beta_n}{n-1}\right) \\ &\quad + \sum_{i=1}^{n-1}\mathbb{P}\left(\tau_i - \tau_{i-1} \leq \frac{\beta_n}{n-1}, \sup_{\tau_{i-1}\leq t\leq \tau_i} \|Y_i(t)\|_2 > \alpha(i)\right) \\ &\quad + \sum_{i=1}^{n-1}\mathbb{P}\left(\tau_i - \tau_{i-1} \leq \frac{\beta_n}{n-1}, \sup_{\tau_{i-1}\leq t\leq \tau_i} \|V_i(t)\|_2 > \alpha(i)\right).\end{aligned}$$

As in Subsection 6.1.1, for all $1 \leq i \leq n-1$, we denote $v_{\tau_0}^i = (\zeta_{\tau_0}^{i,j})_{j\in\{1,\dots,n\}\setminus\{i\}} \in \mathbb{R}^{n-1}$. We also recall that $\beta_n = 2\pi(n-1)^2\sqrt{2(n-1)}$. Iterating $(n-1)$-times Proposition 6.1.2 together with Remark 6.1.4, we have:

$$\sum_{i=1}^{n-1}\mathbb{P}\left(\tau_i - \tau_{i-1} > \frac{\beta_n}{n-1}\right) \leq \sum_{i=1}^{n-1}\mathbb{E}\left[\left(\|v_{\tau_0}^i\|_2\frac{2\pi(n-1)^2}{\beta_n}\right) \wedge 1\right] \leq \mathbb{E}\left[\|\zeta_{\tau_0}\|_2 \wedge (n-1)\right].$$

By hypothesis, we have $\frac{1}{2}\|x - \tilde{x}\|_2^2 \leq n-1$. Thus, using Lemma 6.1.8, we get:

$$\sum_{i=1}^{n-1}\mathbb{P}\left(\tau_i - \tau_{i-1} > \frac{\beta_n}{n-1}\right) \leq \|\zeta\|_2 + 2\sqrt{2}(n-1)^{\frac{3}{2}}||x - \tilde{x}||_2. \tag{6.28}$$

Using Markov's Inequality and Lemma 6.2.4, we obtain:

$$\mathbb{P}\left(\tau_i - \tau_{i-1} \leq \frac{\beta_n}{n-1}, \sup_{\tau_{i-1}\leq t\leq \tau_i} \|Y_i(t)\|_2 > \alpha(i)\right) \leq \frac{Cn^2}{\alpha(i)^4}\mathbb{E}\left[(\tau_i - \tau_{i-1})^2 \wedge \left(\frac{\beta_n}{n-1}\right)^2\right]$$

$$\mathbb{P}\left(\tau_i - \tau_{i-1} \leq \frac{\beta_n}{n-1}, \sup_{\tau_{i-1}\leq t\leq \tau_i} \|V_i(t)\|_2 > \gamma(i)^2\right) \leq \frac{Cn(n-1)}{\gamma(i)^4}\mathbb{E}\left[(\tau_i - \tau_{i-1})^2 \wedge \left(\frac{\beta_n}{n-1}\right)^2\right].$$

Thus:

$$\mathbb{P}\left(\bigcup_{i=1}^{n-1}(A_i \cup \Gamma_i)\right) \leq \|\zeta\|_2 + 2\sqrt{2}(n-1)^{\frac{3}{2}}||x-\tilde{x}||_2 + C\left(\frac{n^2}{\min\limits_{1\leq i\leq n-1}(\alpha(i))^4} + \frac{n(n-1)}{\min\limits_{1\leq i\leq n-1}(\gamma(i))^4}\right)\mathbb{E}\left[(\tau-\tau_0)^2 \wedge \frac{\beta_n^2}{n-1}\right]. \tag{6.29}$$

Still using the fact that $\|x-\tilde{x}\|_2^2 \leq \frac{2}{\sqrt{n-1}}$ we can also use Lemma 6.2.5 with $M = \frac{\beta_n}{\sqrt{n-1}}$ to get:

$$\mathbb{E}\left[(\tau-\tau_0)^2 \wedge \frac{\beta_n^2}{n-1}\right] \leq \frac{4}{3}\beta_n^2(n-1)^{\frac{1}{4}}||x-\tilde{x}||_2 + \frac{2\beta_n^2}{\sqrt{n-1}}\|\zeta\|_2. \tag{6.30}$$

To construct two sequences $(\alpha(i))_{0\leq i\leq n-1}$ and $(\gamma(i))_{0\leq i\leq n-1}$ satisfying (6.22), we can choose $\alpha(0) = \frac{\alpha}{2}$, $\gamma(0) = \frac{\gamma}{\sqrt{2}}$ and, for $1 \leq i \leq n-1$, $\alpha(i) = \frac{\alpha}{4(n-1)}$ and $\gamma(i) = \frac{\gamma}{2\sqrt{n-1}}$. Indeed, we have:

$$\sum_{i=0}^{n-1}\alpha(i) < \frac{\alpha}{2} + \sum_{i=1}^{n-1}\frac{\alpha}{4(n-1)} < \alpha$$

and

$$\begin{aligned}\sum_{i=0}^{n-1}\gamma(i)^2 + \frac{1}{2}\sum_{0\leq i<j\leq n-1}\alpha(i)\alpha(j) &\leq \frac{\gamma^2}{2} + \sum_{i=1}^{n-1}\frac{\gamma^2}{4(n-1)} + \frac{1}{2}\left(\frac{\alpha}{2}\sum_{j=1}^{n-1}\frac{\alpha}{4(n-1)} + \sum_{1\leq i<j\leq n-1}\frac{\alpha^2}{16(n-1)^2}\right)\\ &< \frac{\gamma^2}{2} + \frac{\gamma^2}{4} + \frac{\alpha^2}{16} + \frac{\alpha^2}{32} < \gamma^2.\end{aligned}$$

The last inequality follows from the hypothesis $\alpha \leq \gamma$. We thus obtain inequalities (6.27), (6.29) and (6.30) which ends the proof of Proposition 6.2.1. □

6.2.3 Proofs of the Lemmas

In this subsection we give the announced proofs for Lemma 6.2.4 and Lemma 6.2.5.

We begin with the proof of Lemma 6.2.5.

Proof of Lemma 6.2.5. From Theorem 6.1.5 and Lemma 6.1.8, for $\sqrt{t} \geq m_0 := \frac{\beta_n}{2}\|x-\tilde{x}\|_2^2$ we get:

$$\begin{aligned}\mathbb{P}\left(\tau - \tau_0 > \sqrt{t}\right) &\leq \mathbb{E}\left[\left(\|\zeta_{\tau_0}\|_2 \frac{\beta_n}{\sqrt{t}}\right) \wedge 1\right] \\ &\leq t^{-\frac{1}{2}}\beta_n\|\zeta\|_2 + t^{-\frac{1}{4}}2\sqrt{2\beta_n}(n-1)||x-\tilde{x}||_2.\end{aligned}$$

Then:

$$\begin{aligned}\mathbb{E}\left[(\tau-\tau_0)^2 \wedge M^2\right] &= \int_0^{m_0^2} \mathbb{P}((\tau-\tau_0)^2 > t)dt + \int_{m_0^2}^{M^2} \mathbb{P}((\tau-\tau_0)^2 > t)dt \\ &\leq m_0^2 + \int_{m_0^2}^{M^2} \mathbb{P}\left(\tau - \tau_0 > \sqrt{t}\right) dt \\ &\leq m_0^2 + \int_{m_0^2}^{M^2} t^{-\frac{1}{4}}dt \times 2\sqrt{2\beta_n}(n-1)||x-\tilde{x}||_2 + \int_0^{M^2} t^{-\frac{1}{2}}dt \times \beta_n\|\zeta\|_2 \\ &= m_0^2 + \frac{4}{3}\left(M^{\frac{3}{2}} - m_0^{\frac{3}{2}}\right)2\sqrt{2\beta_n}(n-1)||x-\tilde{x}||_2 + 2M\beta_n\|\zeta\|_2.\end{aligned}$$

As $m_0^2 - \frac{4}{3}m_0^{\frac{3}{2}}2\sqrt{2\beta_n}(n-1)||x-\tilde{x}||_2 = m_0^{\frac{3}{2}}\sqrt{\frac{\beta_n}{2}}\|x-\tilde{x}\|_2\left(1-\frac{16}{3}(n-1)\right) < 0$ for all $n \geq 2$, we obtain the expected result. □

We now deal with the proof of Lemma 6.2.4.

Proof. By construction of the coupling, it is enough to prove these results for $i = 1$. To simplify the notations we also suppose that the Brownian motions start from the same fiber, that is, $\tau_0 = 0$. For all $t \in [0, \tau_0]$, we have:

$$\|Y_1(t)\|_2^4 \leq n\sum_{j=1}^n |X^j(t) - X^j(0)|^4 \text{ and } \|V_1(t)\|_2^2 \leq \frac{1}{2}\sum_{1\leq j\neq l\leq n}\left(\int_0^t (X_s^j - X_0^j)dX_s^l\right)^2.$$

Thus, the lemma will be proven if we can show that there exists a constant $\tilde{C}$ (not depending on the starting points of the processes or the rank n), such that, for any $1 \leq j \neq l \leq n$:

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_1<M\}}\sup_{0\leq t\leq\tau_1}\left(X^j(t)-X^j(0)\right)^4\right] \leq \tilde{C}\mathbb{E}\left[(\tau_1\wedge M)^2\right] \tag{6.31}$$

and

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_1<M\}}\sup_{0\leq t\leq\tau_1}\left(\int_0^t \left(X_s^j - X_0^j\right)dX^l(s)\right)^2\right] \leq \tilde{C}\mathbb{E}\left[(\tau_1\wedge M)^2\right]. \tag{6.32}$$

These two results can be seen as a generalisation of *Lemma 4.1* from [6] to higher dimensions. Although the method is quite similar, it needs some adjustments due to the change of the dimension. We give here the detail of the proof for the convenience of the reader and will point the major element needing extra care.

As in the case of the Heisenberg group, we want to use the B-D-G Inequality. The main difficulty is due to the fact that the coupling is not co-adapted. Thus, τ_1 is not a stopping time for the usual filtration induced by $(X_s)_s$ and we need to consider an enlarged

filtration. The second difficulty comes from the fact that the considered processes (or most of them) have to be martingales for this new filtration.

Using the same notations as in the proof of Proposition 6.1.2, we define:

$$\chi_2(t) := \sum_{k\geq 0} \mathbb{1}_{\{t_k\leq t\}} \left(\sum_{m\geq n} \xi_m^{(k)} \frac{\sqrt{2T_k}}{m\pi} \sin\left(\frac{m\pi\left((t-t_k)\wedge T_k\right)}{T_k} \right) + \xi_0^{(k)} \frac{\left((t-t_k)\wedge T_k\right)}{\sqrt{T_k}} \right); \tag{6.33}$$

$$\text{and } \chi_1(t) := \sum_{k\geq 0} \mathbb{1}_{\{t_k\leq t\leq t_{k+1}\}} \sum_{m=1}^{n-1} \xi_m^{(k)} \frac{\sqrt{2T_k}}{m\pi} \sin\left(\frac{m\pi(t-t_k)}{T_k} \right) \tag{6.34}$$

such that $X_t^1 = X_0^1 + \chi_1(t) + \chi_2(t)$.

We remind that, for all $k \geq 0$, $\left(\xi_m^{(k)}\right)_{1\leq m\leq n-1} = \sum_{l=1}^{n-1} W_1^{l,(k)} f_l^{(k)}$ with $W^{(k)}$ a $\mathbb{R}^{n-1}$-Brownian motion independent of the basis $\left(f_l^{(k)}\right)_{1\leq l\leq n-1}$. In particular, $\left(\xi_m^{(k)}\right)_{1\leq m\leq n-1}$ has the distribution $\mathcal{N}(0, I_{n-1})$ because of this independence. With the same argument, we can define a Brownian motion $(C_t)_{0\leq t\leq \tau_1}$ starting from 0 by taking $C_t := \sum_{k\geq 0} \sqrt{T_k} \sum_{l=1}^{n-1} \mathbb{1}_{\{t\geq t_k\}} W^{l,(k)}_{\frac{(t-t_k)\wedge T_k}{T_k}} f_l^{(k)}$. In particular $(C_t)_{0\leq t\leq\tau_1}$ is independent of all the bases $\left(f_l^{(k)}\right)_{1\leq l\leq n-1}$. We also have $\left(\xi_m^{(k)}\right)_{1\leq m\leq n-1} = \frac{C_{t_{k+1}}-C_{t_k}}{\sqrt{T_k}}$. For $t \geq 0$, we define two filtrations:

$$\mathcal{F}_t^\star := \sigma\Big(\{X_s^j, s\leq t\wedge\tau_1, 2\leq j\leq n\} \cup \{C_s, 0\leq s\leq \tau_1\} \\ \cup \{\xi_0^{(k)}, k\geq 0\} \cup \{\xi_m^{(k)}, m\geq n, k\geq 0\} \Big) \tag{6.35}$$

and

$$\mathcal{F}_t^{\star\star} := \sigma\Big(\{X_s^j, s\leq t\wedge\tau_1, 2\leq j\leq n\} \cup \{C_s, 0\leq s\leq t\wedge\tau_1\} \\ \cup \{\xi_0^{(k)}, k\geq 0\} \cup \{\xi_m^{(k)}, m\geq n, k\geq 0\} \Big). \tag{6.36}$$

Note here, that, contrary to the case of the Heisenberg group, our coupling strategy needs the introduction, for all $k \geq 0$, of the basis $\left(f_l^{(k)}\right)_{1\leq l\leq n-1}$ which depends on $(X_s^j)_{t_k\leq t\leq t_{k+1}}$ for all $2 \leq j \leq n$. We then have to take extra care to define the filtration $(\mathcal{F}_t^\star)_t$ with $(C_{t\wedge\tau_1})_t$ and not with a direct concatenation of $(W_s^{(k)})_{0\leq s\leq 1}$. Then $(X^1_{t\wedge\tau_1} - X_0^1)_t$ is $\mathcal{F}_0^\star$-measurable which will be important in the future to deal with the quantity $\mathbb{E}\left[\mathbb{1}_{\{\tau_1\leq M\}} \sup_{0\leq t\leq\tau_1} \left(\int_0^t (X_s^1 - X_0^1)\, dX_s^l\right)^2\right]$ with $2 \leq l \leq n$ (upcoming inequality (6.43)).

As for the proof in [6], we have $\{\sigma^{(k-1)} > 1\} \in \mathcal{F}_{t_k}^{\star\star}$, and thus $\{\tau_1 > t_k\} = \bigcap_{l=0}^{k-1} \{\sigma^{(l)} > 1\} \in \mathcal{F}_{t_k}^{\star\star}$. Since τ_1 only takes its values in $\{t_k,\ k \geq 0\}$, τ_1 is a stopping time for the filtration $(\mathcal{F}_t^{\star\star})_t$. As $\mathcal{F}_t^{\star\star} \subset \mathcal{F}_t^\star$, we have the same result for the filtration $(\mathcal{F}_t^\star)_t$.

For $2 \leq j \leq n$, $(X^j_{t\wedge\tau_1})_t$ is clearly a (stopped) Brownian motion adapted to the two filtrations $(\mathcal{F}_t^{\star\star})_t$ and $(\mathcal{F}_t^\star)_t$, thus, using the B-D-G Inequality, we obtain (6.31):

$$\mathbb{E}\left[\sup_{0\leq t\leq\tau_1\wedge M} \left(X^j(t) - X^j(0)\right)^4 \right] \leq c_4 \mathbb{E}\left[(\tau_1\wedge M)^2 \right], \tag{6.37}$$

with c_4 the universal constant for the B-D-G Inequality. With the same arguments, using two times the B-D-G Inequality, we obtain (6.32) for $2 \leq j \neq l \leq n$:

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq \tau_1\wedge M}\left(\int_0^t \left(X_s^j - X_0^j\right) dX_s^l\right)^2\right] &\leq c_2\mathbb{E}\left[\int_0^{\tau_1\wedge M}\left(X_s^j - X_0^j\right)^2 ds\right] \\
&\leq c_2\mathbb{E}\left[(\tau_1\wedge M)\sup_{0\leq t\leq \tau_1\wedge M}\left(X_s^j - X_0^j\right)^2\right] \\
&\leq c_2\mathbb{E}\left[(\tau_1\wedge M)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\sup_{0\leq t\leq \tau_1\wedge M}\left(X_s^j - X_0^j\right)^4\right]^{\frac{1}{2}} \\
&\leq c_2\sqrt{c_4}\mathbb{E}\left[(\tau_1\wedge M)^2\right]. \qquad (6.38)
\end{aligned}$$

Again, c_2 and c_4 are the two universal constants intervening in the B-D-G Inequality. Note that these universal constant can be estimated (see [50]).

We now look for (6.31) for $j = 1$. Let $k_M \geq 0$ such that $M \in [t_{k_M}, t_{k_M+1}[$. Let $t_{k+1} \leq \tau_1 \wedge t_{k_M}$ and $t \in [t_k, t_{k+1}]$. We denote by $(e_1, \ldots, e_{n-1})$ the canonic basis in $\mathbb{R}^{n-1}$. We get:

$$\begin{aligned}
|\chi_1(t)|^2 &= \left|\sum_m^{n-1}\langle C_{t_{k+1}} - C_{t_k}, e_m\rangle\frac{\sqrt{2}}{m\pi}\sin\left(\frac{m\pi(t-t_k)}{T_k}\right)\right|^2 \leq \left\|C_{t_{k+1}} - C_{t_k}\right\|_2^2\frac{2}{\pi^2}\sum_{m=1}^{n-1}\frac{1}{m^2} \\
&\leq \frac{4}{3}\sup_{t_k\leq t\leq t_{k+1}}\|C_t\|_2^2 \leq \frac{4}{3}\sup_{0\leq t\leq \tau_1\wedge M}\|C_t\|_2^2.
\end{aligned}$$

As τ_1 takes its values in $\{t_k, k \geq 0\}$, there exists $0 \leq k \leq k_M$ such that $\tau_1 \wedge t_{k_M} = t_k$. Thus we have:

$$\sup_{0\leq t\leq \tau_1\wedge t_{k_M}}|\chi_1(t)|^2 \leq \frac{4}{3}\sup_{0\leq t\leq \tau_1\wedge M}\|C_t\|_2^2.$$

Since $(C_{t\wedge\tau_1})_t$ is a (stopped) Brownian motion adapted to the filtration $(\mathcal{F}_t^{\star\star})_t$, we can use the B-D-G Inequality to obtain:

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau_1\wedge t_{k_M}}|\chi_1(t)|^4\right] \leq \frac{16}{9}\left[\sup_{0\leq t\leq \tau_1\wedge M}\|C_t\|^4\right] \leq \frac{16c_4}{9}\mathbb{E}\left[(\tau_1\wedge M)^2\right]. \qquad (6.39)$$

We now deal with $\mathbb{E}\left[\sup_{0\leq t\leq \tau_1\wedge t_{k_M}}|\chi_2(t)|^4\right]$. Let first consider any deterministic time T. We have:

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge t_{k_M}}|\chi_2(t)|^4\right] \leq 8\mathbb{E}\left[\sup_{0\leq t\leq T\wedge t_{k_M}}|\chi_1(t)|^4\right] + 8\mathbb{E}\left[\sup_{0\leq t\leq T\wedge t_{k_M}}|X_t^1 - X_0^1|^4\right]$$

We can apply the B-D-G inequality to $(X_t^1 - X_0^1)_t$ for the deterministic stopping time $T \wedge t_{k_M}$ by considering the natural filtration of $(X_t^1)_t$:

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge t_{k_M}}|\chi_2(t)|^4\right] \leq 8\mathbb{E}\left[\sup_{0\leq t\leq T\wedge t_{k_M}}|\chi_1(t)|^4\right] + 8c_4(T\wedge t_{k_M})^2. \qquad (6.40)$$

By construction of the coupling, χ_2 is independent of τ_1, thus, conditioning by τ_1 and using (6.40), we obtain:

$$\mathbb{E}\left[\sup_{0\le t\le\tau_1\wedge t_{k_M}}|\chi_2(t)|^4\right]\le 8\mathbb{E}\left[\sup_{0\le t\le\tau_1\wedge t_{k_M}}|\chi_1(t)|^4\right]+8c_4\mathbb{E}[(\tau_1\wedge t_{k_M})^2].$$

Finally, using (6.39) we get:

$$\mathbb{E}\left[\sup_{0\le t\le\tau_1\wedge t_{k_M}}|\chi_2(t)|^4\right]\le 8c_4\frac{25}{9}\mathbb{E}\left[(\tau_1\wedge M)^2\right]. \tag{6.41}$$

From (6.39) and (6.41), we obtain (6.31) for $j=1$:

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\{\tau_1\le M\}}\times\sup_{0\le t\le\tau_1}|X_t^1-X_0^1|^4\right]&=\mathbb{E}\left[\mathbb{1}_{\{\tau_1\le t_{k_M}\}}\times\sup_{0\le t\le\tau_1}|X_t^1-X_0^1|^4\right]\\&\le\mathbb{E}\left[\sup_{0\le t\le\tau_1\wedge t_{k_M}}|\chi_1(t)+\chi_2(t)|^4\right]\le 3\times 8^2c_4\mathbb{E}\left[(\tau_1\wedge M)^2\right].\end{aligned} \tag{6.42}$$

Let us deal with (6.32) for $j=1$ and $2\le l\le n$. For the filtration $(\mathcal{F}_t^\star)_t$, $(X^l_{t\wedge\tau_1})_t$ is an adapted Brownian motion. Moreover $\left(X^1_{t\wedge\tau_1}-X^1_0\right)_t$ is $\mathcal{F}_0^\star$-measurable. Then, using the B-D-G inequality and (6.42):

$$\begin{aligned}&\mathbb{E}\left[\mathbb{1}_{\{\tau_1\le M\}}\sup_{0\le t\le\tau_1}\left(\int_0^t\left(X_s^1-X_0^1\right)dX_s^l\right)^2\right]\le\mathbb{E}\left[\sup_{0\le t\le\tau_1\wedge t_{k_M}}\left(\int_0^t\left(X_s^1-X_0^1\right)dX_s^l\right)^2\right]\\&\le c_2\mathbb{E}\left[(\tau_1\wedge t_{k_M})^2\right]^{\frac12}\mathbb{E}\left[\sup_{0\le t\le\tau_1\wedge t_{k_M}}\left(X_t^1-X_0^1\right)^4\right]^{\frac12}\\&\le 8c_2\sqrt{3c_4}\mathbb{E}\left[(\tau_1\wedge M)^2\right].\end{aligned} \tag{6.43}$$

Finally, to obtain (6.32) with $j>2$ and $l=1$, we can use Itô's formula to get

$$\int_0^t\left(X_s^j-X_0^j\right)dX^1(s)=(X_t^1-X_0^1)(X_t^j-X_0^j)-\int_0^t\left(X_s^1-X_0^1\right)dX^j(s).$$

Using the previous results, we then have:

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_1\le M\}}\sup_{0\le t\le\tau_1}\left(\int_0^t\left(X_s^j-X_0^j\right)dX^1(s)\right)^2\right]\le 16\sqrt{3c_4}(c_2+\sqrt{c_4})\mathbb{E}\left[(\tau_1\wedge M)^2\right].$$

□

6.3 Gradient estimates

In this section, we present several gradient estimates. The first ones gives estimates for the heat semi-group and can be directly deduced from the coupling rate obtained in Section 6.1. Note that these estimates can be improved with the One sweep coupling strategy in Chapter 7. The second ones are about harmonic functions and come from the comparison between the first coupling time and the first exit time from a domain dealt with in Section 6.2.

6.3.1 Estimates of the horizontal and vertical gradient for the heat semi-group

We first obtain an upper-bound (with an explicit estimate) for the horizontal gradient of the heat semi-group for any homogeneous step 2 Carnot groups. Let $g \in \mathbb{G}$ and $(\mathbb{B}_t)_t$ be a Brownian motion starting at g. We recall that, for all measurable bounded function f on $\mathbb{G}$, $P_t f(g) = \mathbb{E}[f(\mathbb{B}_t)]$.

Corollary 6.3.1. *Let $\mathbb{G}$ be a homogeneous Carnot group of step 2 and of rank $n \geq 2$. For any bounded measurable function f on $\mathbb{G}$, for any $g \in \mathbb{G}$ and $t \geq 1$:*

$$||\nabla_{\mathcal{H}} P_t f||_{\mathcal{H}} \leq 2||f||_\infty \frac{C_1(n)}{\sqrt{t}} \ a.e. \tag{6.44}$$

with $C_1(n)$ the explicit constant from Theorem 2.4.1.

Proof. We consider f a bounded measurable function on $\mathbb{G}$ and $g, \tilde{g} \in \mathbb{G}$. Let consider the following coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$:

- still denoting by τ the first coupling time, for $t \in [0, \tau]$, we take $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_{0 \leq t \leq \tau}$ the coupling constructed in Corollary 6.1.10;
- for $t \geq \tau$, we take $\mathbb{B}_t = \tilde{\mathbb{B}}_t$.

We have:

$$\begin{aligned} |P_t f(g) - P_t f(\tilde{g})| &= |\mathbb{E}[f(\mathbb{B}_t) - f(\tilde{\mathbb{B}}_t)]| \leq \mathbb{E}[|f(\mathbb{B}_t) - f(\tilde{\mathbb{B}}_t)|] \\ &= \mathbb{E}[|f(\mathbb{B}_t) - f(\tilde{\mathbb{B}}_t)| \mathbb{1}_{\{\tau > t\}}] \\ &\leq 2||f||_\infty \mathbb{P}(\tau > t). \end{aligned} \tag{6.45}$$

Using the estimate of the coupling rate (6.12), we get, for t not to close to 0:

$$|P_t f(g) - P_t f(\tilde{g})| \leq 2\|f\|_\infty \left(C_1(n) \frac{d_{cc}(g, \tilde{g})}{\sqrt{t}} \mathbb{1}_{x \neq \tilde{x}} + \frac{C_2(n)}{m_1(n)^2} \frac{d_{cc}(g, \tilde{g})^2}{t} \right). \tag{6.46}$$

We have:

$$|\nabla P_t f|(g) = \lim_{r \downarrow 0} \sup_{\substack{g \neq g' \\ d_{cc}(g,g') < r}} \left| \frac{P_t f(g) - P_t f(g')}{d_{cc}(g, g')} \right| \leq 2||f||_\infty \frac{C_1(n)}{\sqrt{t}}. \tag{6.47}$$

In particular, for all $t > 0$ and f bounded measurable, $g \mapsto P_t f(g)$ is Lipschitz on $(\mathbb{G}, d_{cc})$. As the gradient length $|\nabla P_t f|$ is an upper-gradient of $P_t f$, from Inequality (3.18), we obtain the expected result. □

We can also obtain estimates on $\mathbb{G}_n$ for the vertical gradient as defined in Subsection 3.1.6.

Corollary 6.3.2. *Let $n \geq 3$. For any bounded measurable function f on $\mathbb{G}_n$ and for any $t \geq 1$:*

$$\|\nabla_{\mathfrak{v}} P_t f\| \leq ||f||_\infty \frac{C_2(n)}{t} \ a.e. \tag{6.48}$$

with $C_2(n)$ the explicit constant from Theorem 2.4.1.

Proof. Let denote $g = (x, z) \in \mathbb{G}_n$. We set $\tilde{z} \in \mathfrak{so}(n)$ and $\tilde{g} = (x, \tilde{z}) \in \mathbb{G}_n$. Applying (6.45) together with the coupling rate estimate (6.2), we get:

$$|P_t f(g) - P_t f(\tilde{g})| \leq 2\|f\|_\infty \beta_n \frac{\|z - \tilde{z}\|_2}{t} \text{ with } \beta_n = \frac{C_2(n)}{2}.$$

As previously, we obtain an inequality for the gradient length defined on the fiber $\{(x, z) \in \mathbb{G}_n \mid z \in \mathfrak{so}(n)\}$. As for Inequality (3.22), we obtain the expected result. □

6.3.2 Estimates of the gradient for the harmonic functions on $\mathbb{G}_n$

By using the results from Theorem 6.2.2, we can obtain the same results than in Corollaries 4.4, 4.5 and 4.6 from [6]. As the proofs are exactly the same, we just give here an overview of the method and refer the reader to [6] for more details.

As in Section 6.2, we consider a domain D on $\mathbb{G}_n$ and $\bar{D}$ its closure. Let f be a continuous function on $\bar{D}$, smooth and such that f is harmonic on D, that is, $Lf(g) = 0$ for $g \in D$. For any set $A \subset D$, we define $\mathrm{osc}_A(f) := \sup_A f - \inf_A f$. In particular, supposing $\bar{D}$ compact, by continuity of f on $\bar{D}$, $\mathrm{osc}_D(f)$ is finite. Let $g = (x, z), \tilde{g} = (\tilde{x}, \tilde{z}) \in D$, as f is harmonic, by Itô's formula, for any successful coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)_t$ (defined such that $\mathbb{B}_t = \tilde{\mathbb{B}}_t$ after the first coupling time τ as previously) we get:

$$\begin{aligned} |f(g) - f(\tilde{g})| &= \left| \mathbb{E}\left[f(\mathbb{B}_{\tau_D(\mathbb{B})}) - f(\tilde{\mathbb{B}}_{\tau_D(\tilde{\mathbb{B}})}) \right] \right| \\ &\leq \mathrm{osc}_D(f) \mathbb{P}\left(\tau > \tau_D(\mathbb{B}) \wedge \tau_D(\tilde{\mathbb{B}}) \right) \end{aligned}$$

In particular, using the results from Theorem 6.2.2, for $\tilde{g}$ close enough to g, we get:

$$\begin{aligned} |f(g) - f(\tilde{g})| \leq \mathrm{osc}_D(f) \Bigg(& D_1 \times \left(n\sqrt{n} + \frac{n\sqrt{n}}{\delta_g} + \frac{n^{11+\frac{1}{4}}}{\delta_g^4} \right) \|x - \tilde{x}\|_2 \\ & + D_2 \times \left(1 + \frac{n^{10+\frac{1}{2}}}{\delta_g^4} \right) \max\left(\|\zeta\|_2, \|\zeta\|_2^2 \right) \Bigg). \end{aligned}$$

Finally, using the gradient length associated to the pseudo-distance $\boldsymbol{\delta}$, we get:

$$|\nabla P_t f|_{\boldsymbol{\delta}}(g) \leq \mathrm{osc}_D(f) D_1 \times \left(n\sqrt{n} + \frac{n\sqrt{n}}{\delta_g} + \frac{n^{11+\frac{1}{4}}}{\delta_g^4} \right). \tag{6.49}$$

Using (3.21), we have: $|\nabla P_t f|_{\boldsymbol{\delta}}(g) \geq m_1(n) |\nabla P_t f|$. Then we obtain:

Corollary 6.3.3. *Let $n \geq 2$ and D a domain on $\mathbb{G}_n$ such that its closure $\bar{D}$ is compact. For any $f \in \mathcal{C}(\bar{D}) \cap \mathcal{C}^2(D)$ harmonic on D:*

$$||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} \leq \mathrm{osc}_D(f) \frac{D_1}{m_1(n)} \left(n\sqrt{n} + \frac{n\sqrt{n}}{\delta_g} + \frac{n^{11+\frac{1}{4}}}{\delta_g^4} \right) \text{ a.e. on } D$$

with D_1 the same constant as in Theorem 6.2.2 which depend neither on n, nor on D.

We now consider the result from Corollary 6.3.3 when we take $D = B(g, r)$ with $r > 0$ and $B(g, r)$ the usual open ball for the Carnot Carathéodory distance d_{cc}. Using the homogeneous invariant Harnack Inequality (see for example Theorem 5.7.2 from [19]), there exists a constant $\bar{C}$ that does not depend on g and r but only on the metric d_{cc} such that $\text{osc}_{B(g,r)}(f) \leq \bar{C} f(g)$. This leads to the following result:

Corollary 6.3.4. *Let $n \geq 2$. There exists a constant C depending on $\mathbb{G}_n$ and d_{cc} such that, for all $g \in \mathbb{G}_n$, $r > 0$ and for any smooth non-negative function $f \in \mathcal{C}(\bar{B}(g, r))$ harmonic on $B(g, r)$ we have:*

$$||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} \leq C \left(n\sqrt{n} + \frac{n\sqrt{n}}{\delta_g} + \frac{n^{11+\frac{1}{4}}}{\delta_g^4} \right) f(g) \text{ a.e..}$$

Finally, as $\frac{||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}}}{f(g)} = ||\nabla_{\mathcal{H}} \log f(g)||_{\mathcal{H}}$, we can obtain the following Cheng-Yau estimate:

Corollary 6.3.5. *Let $n \geq 2$ and $g_0 \in \mathbb{G}_n$. We consider a real $r > 0$ and a smooth non-negative function f harmonic on $B(g_0, 2r)$. Then:*

$$\sup_{g \in B(g_0, r)} ||\nabla_{\mathcal{H}} f(g)||_{\mathcal{H}} \leq \frac{C}{r} \text{ a.e.}$$

with C a constant that does depend only on $\mathbb{G}_n$ and d_{cc} (neither on g_0 nor on r).

Note that this last result has been proven for all Carnot groups by Baudoin, Gordina and Mariano in [15] by using analytic methods.

Chapter 7

One-sweep coupling on the free, step 2 Carnot groups

In this chapter, we present a second coupling strategy on the free, step 2 Carnot groups. Contrary to Chapter 6, the strategy presented here is not successful. We define the coupling only on a finite interval of time $[0, T]$ and evaluate it only at the final time T. The construction is based on a Legendre expansion of the standard Brownian motion which is well adapted to the computation of the Lévy area. As in Chapter 6, we obtain estimates for the distance in total variation and for the horizontal and vertical gradient of the heat semi-group.

In order to enlighten the simplicity of the method, and since some small complication arises in the case of higher dimensional Carnot groups, we chose to present first the construction and the total variation distance estimate in the case of the Heisenberg group. The case of higher dimensional step 2 Carnot groups on $\mathbb{R}^n$, $n \geq 3$ will be investigated in a second time.

We also use a change of probability technique to obtain a Bismut type formula for the derivative of the semi-group and from which we derive some reverse Poincaré or reverse Sobolev inequality for $p > 1$. We also obtain a log Harnack inequality.

The content of this chapter will make the object of a publication currently in construction:

[3] Marc Arnaudon, Magalie Bénéfice, Michel Bonnefont, and Delphine Féral. A coupling strategy for Brownian motions at fixed time on Carnot groups using Legendre expansion. `https://arxiv.org/abs/2407.04321`, 2024

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7.1 The coupling method on the Heisenberg group $\mathbb{H}$

Below, let us denote $\mu_t^{x_1,x_2,z}$ to be the law of the subelliptic Brownian motion on the Heisenberg group starting from (x_1, x_2, z). The first main result is the estimate of the Total variation distance between the laws of the Brownian motions in the Heisenberg group. This result already appears in [6]. One of our improvement here is that we obtain some explicit constants.

Theorem 7.1.1. *There exist two constants $C_1, C_2 \geq 0$ such that for all $t \geq 0$ and all (x_1, x_2, z) and $(\tilde{x}_1, \tilde{x}_2, \tilde{z})$ in $\mathbb{H}$,*

$$d_{TV}\left(\mu_t^{(x_1,x_2,z)}, \mu_t^{(\tilde{x}_1,\tilde{x}_2,\tilde{z})}\right) \leq C_1 \frac{\|(\tilde{x}_1 - x_1, \tilde{x}_2 - x_2)\|_2}{\sqrt{t}} + C_2 \frac{|\tilde{z} - z - \frac{1}{2}(x_1\tilde{x}_2 - x_2\tilde{x}_1)|}{t}. \quad (7.1)$$

Moreover, one can take

$$C_1 = \frac{1}{\sqrt{2\pi}}\left(1 + \sqrt{14}\right) \text{ and } C_2 = \sqrt{\frac{21}{2}}.$$

As noticed in [6], Theorem 7.1.1 provides the sharp order of decreasing. In this sense, the associated coupling is called *efficient*. It is also noted in [6] that any Markovian or co-adapted coupling can not reach the sharp estimate when the initial point are in the same fiber, that is when $(\tilde{x}_1, \tilde{x}_2) = (x_1, x_2)$.

We recall (see Section 3.1 for more details) that the Heisenberg group $\mathbb{H}$ can be identified with $(\mathbb{R}^2 \times \mathbb{R}, \star)$ equipped with the law:

$$(x, z) \star (x, z') = \left(x + x', z + z' + \frac{1}{2} x \cdot x'\right),$$

where

$$x \cdot x' := x_1 x_2' - x_2 x_1' \text{ for all } x, x' \in \mathbb{R}^2.$$

The Brownian motion starting in (x, z) can be written as the continuous process:

$$\begin{aligned} \mathbb{B}_t^{(x,z)} &:= (x, z) \star \left(B_t, \frac{1}{2}\left(\int_0^t B_s^1 dB_s^2 - \int_0^t B_s^2 dB_s^1\right)\right) \\ &= \left(x + B_t, z + \frac{1}{2}(x_1 B_t^2 - x_2 B_t^1) + \frac{1}{2}\left(\int_0^t B_s^1 dB_s^2 - \int_0^t B_s^2 dB_s^1\right)\right) \end{aligned}$$

where $(B_t^1, B_t^2)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^2$ and

$$A_t := \frac{1}{2}\left(\int_0^t B_s^1 B_s^2 - \int_0^t B_s^2 dB_s^1\right) \quad (7.2)$$

is the Lévy area of the 2-dimensional Brownian motion $(x_1 + B_t^1, x_2 + B_t^2)$. In the remainder, we will often denote

$$\mathbb{B}_t^{(x,z)} = (X_t, z_t) \text{ where } z_t = z + \frac{1}{2}(x_1 B_t^2 - x_2 B_t^1) + A_t.$$

7.1.1 The description of the Brownian motion on $\mathbb{H}$ with Legendre polynomials

Let $T > 0$ and consider the scalar product defined for $f, g \in \mathcal{C}([0,T], \mathbb{R})$ by

$$\langle f, g \rangle = \int_0^T f(t)g(t)dt.$$

Let take Q_k to be the normalized orthogonal polynomials; that is such that $\|Q_k\|^2 = 1$. By dilation and translation, one sees that

$$Q_k(x) = \sqrt{\frac{2}{T}} P_k\left(-1 + \frac{2x}{T}\right)$$

where $(P_k)_k$ are the standard (normalized) Legendre polynomials, which are orthogonal for the Lebesgue measure on $[-1, 1]$.

We first consider the following representation of a standard one-dimensional Brownian motion $(B_t)_{0 \le t \le T}$ starting in 0. This representation is somehow close to the standard Karhunen-Loève decomposition of the Brownian motion but as noticed in [42], it is well adapted to the computation of the Lévy area.

Lemma 7.1.2. *Let $(\xi_k)_{k \ge 1}$ be a sequence of independent and identically distributed random variables of law $\mathcal{N}(0,1)$. Define*

$$B_t = \sum_{k \ge 0} \xi_k \int_0^t Q_k(s)ds, \ 0 \le t \le T. \tag{7.3}$$

Then the process $(B_t)_{0 \le t \le T}$ is a standard Brownian motion on $[0,T]$.

The proof is done in [42], but let us recall the main ideas for the reader's convenience.

Proof. Let $T \ge 0$ and let $(B_t)_{0 \le t \le T}$ be defined by (7.3). The process $(B_t)_{0 \le t \le T}$ is clearly a centered Gaussian process. To prove it is a standard Brownian motion, let us compute its covariance. For $0 \le s, t \le T$, one has:

$$\begin{aligned}
\mathbb{E}[B_s B_t] &= \sum_{k \ge 0} \left(\int_0^t Q_k(u)du \right) \left(\int_0^s Q_k(u)du \right) \\
&= \sum_{k \ge 0} \langle \mathbb{1}_{[0,t]}, Q_k \rangle \langle \mathbb{1}_{[0,s]}, Q_k \rangle \\
&= \langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle = s \wedge t.
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $L^2([0,T])$ and the result follows. □

Lemma 7.1.3. *Let $(\xi_k)_{k \ge 0}$ be a sequence of independent and identically distributed random vectors with common law $\mathcal{N}(0, I_2)$. Write $\xi_k = (\xi_k^1, \xi_k^2)^t$ and for $0 \le t \le T$ and $i = 1, 2$ let*

$$B_t^i = \sum_{k \ge 0} \xi_k^i \int_0^t Q_k(s)ds. \tag{7.4}$$

Then $(B_t^1, B_t^2)_{0\le t\le T}$ *is a standard two-dimensional Brownian motion and its associated Lévy area* $A_t := \frac{1}{2}\Big(\int_0^t B_s^1 dB_s^2 - \int_0^t B_s^2 dB_s^1\Big)$ *at the given time* T *may be written as*

$$A_T = T\sum_{k\ge 0} \alpha_k\, \xi_k \cdot \xi_{k+1} \tag{7.5}$$

with

$$\alpha_k = \frac{1}{2\sqrt{4(k+1)^2-1}}, \quad k \ge 0. \tag{7.6}$$

As before the proof is done in [42], but we shall recall the main ideas for the reader's convenience.

Proof. With the notation of Lemma 7.1.3, one has

$$\int_0^T B_s^1 dB_s = \sum_{k,l\ge 0} \xi_k^1 \xi_l^2 c_{k,l} \text{ with } c_{k,l} = \int_0^T \left(\int_0^t Q_k(s)ds\right) Q_l(t)dt.$$

Now by integration by parts, one has for $k, l \ge 0$,

$$c_{k,l} = \left(\int_0^T Q_k(u)du\right)\left(\int_0^T Q_l(u)du\right) - c_{l,k}.$$

Since Q_k is a family of orthogonal polynomials, one infers that for $(k,l) \neq (0,0)$, $c_{k,l} = -c_{l,k}$ and thus

$$c_{k,l} = 0 \text{ if } |k-l| \ge 2 \text{ or } k = l \ge 1.$$

Therefore

$$\int_0^T B_s^1 dB_s = c_{0,0}\xi_0^1\xi_0^2 + \sum_{k\ge 0} c_{k,k+1}(\xi_k^1\xi_{k+1}^2 - \xi_{k+1}^1\xi_k^2).$$

and thus the Lévy area at the final time T writes:

$$A_T = \sum_{k\ge 0} c_{k,k+1}(\xi_k^1\xi_{k+1}^2 - \xi_{k+1}^1\xi_k^2).$$

The result follows by an explicit computation of the constant $c_{k,k+1}$. □

We recall that here in the case of the Heisenberg group:

$$\xi_k \cdot \xi_{k+1} = \xi_k^1\xi_{k+1}^2 - \xi_{k+1}^1\xi_k^2. \tag{7.7}$$

As a direct application of Lemma 7.1.3, the Brownian motion on $\mathbb{H}$ starting in (x_1, x_2, z) at time T may be represented by

$$\mathbb{B}_T^{(x_1,x_2,z)} = \begin{pmatrix} x_1 + \sqrt{T}\xi_0^1 \\ x_2 + \sqrt{T}\xi_0^2 \\ z + \frac{\sqrt{T}}{2}(x_1\xi_0^2 - x_2\xi_0^1) + T\sum_{k\ge 0}\alpha_k\,(\xi_k^1\xi_{k+1}^2 - \xi_{k+1}^1\xi_k^2) \end{pmatrix}$$

or equivalently with $x = (x_1, x_2)$,

$$\mathbb{B}_T = \begin{pmatrix} x + \sqrt{T}\xi_0 \\ z + \frac{\sqrt{T}}{2}x\cdot\xi_0 + T\sum_{k\ge 0}\alpha_k\,\xi_k\cdot\xi_{k+1}. \end{pmatrix}$$

7.1.2 Proof of Theorem 7.1.1

Before we turn to the proof of Theorem 7.1.1, we recall the standard estimate for Gaussian vectors on $\mathbb{R}^d$, $d \geq 1$ (with the same identity covariance matrix).

Lemma 7.1.4. *Let $d \geq 1$ an integer, $m, m' \in \mathbb{R}^d$, there exists a random couple (X, Y) whose marginals are Gaussian random variables $\mathcal{N}(m, I_d)$ and $\mathcal{N}(m', I_d)$ and such that*

$$\mathbb{P}(X \neq Y) \leq \left(\frac{\|m - m'\|_2}{\sqrt{2\pi}} \right) \wedge 1.$$

Proof of Theorem 7.1.1. For any choice of two Heisenberg valued subRiemannian Brownian motions $((X_t, z_t))_{t\geq 0}$ and $((\tilde{X}_t, \tilde{z}_t))_{t\geq 0}$ started respectively at (x, z) and $(\tilde{x}, \tilde{z})$, we have

$$d_{TV}\left(\mu_T^{(x,z)}, \mu_T^{(\tilde{x},\tilde{z})}\right) \leq \mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right). \tag{7.8}$$

Consequently, to establish the estimate (7.1) it is sufficient, for each $T > 0$, to find $((X_t, z_t))_{t\geq 0}$ and $((\tilde{X}_t, \tilde{z}_t))_{t\geq 0}$ started respectively at (x, z) and $(\tilde{x}, \tilde{z})$, satisfying

$$\mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right) \leq C_1 \frac{\|\tilde{x} - x\|_2}{\sqrt{T}} + C_2 \frac{|\tilde{z} - z - \frac{1}{2} x \cdot \tilde{x}|}{T} \tag{7.9}$$

for $C_1, C_2 > 0$ not depending on T.

To perform the construction of the coupling, we construct the Brownian motions $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ with the Legendre polynomials as in Lemma 7.1.2.

So let us fix $T > 0$. We write

$$\forall\ 0 \leq t \leq T,\ X_t = x + B_t \quad \text{with} \quad B_t = \sum_{k=0}^{\infty} \xi_k \int_0^t Q_k(s)\, ds, \tag{7.10}$$

where $\left(\xi_k = \begin{pmatrix} \xi_k^1 \\ \xi_k^2 \end{pmatrix}\right)_{k\geq 0}$ is a sequence of independent $\mathbb{R}^2$-valued random vectors with law $\mathcal{N}(0, I_2)$. We do the same with $(\tilde{X}_t)_{0\leq t\leq T}$, using independent $\mathbb{R}^2$-valued random variables $\left(\tilde{\xi}_k\right)_{k\geq 0}$ with law $\mathcal{N}(0, I_2)$. Equation (7.9) will be obtained thanks to a well-chosen coupling of $(\xi_k)_{k\geq 0}$ and $\left(\tilde{\xi}_k\right)_{k\geq 0}$.

At time T, using Lemma 7.1.3, we get

$$X_T = x + \sqrt{T}\xi_0, \qquad z_T = z + \frac{1}{2}\sqrt{T} x \cdot \xi_0 + T \sum_{k\geq 0} \alpha_k \xi_k \cdot \xi_{k+1}, \tag{7.11}$$

$$\tilde{X}_T = \tilde{x} + \sqrt{T}\tilde{\xi}_0, \qquad \tilde{z}_T = \tilde{z} + \frac{1}{2}\sqrt{T} \tilde{x} \cdot \tilde{\xi}_0 + T \sum_{k\geq 0} \alpha_k \tilde{\xi}_k \cdot \tilde{\xi}_{k+1}, \tag{7.12}$$

where for $k \geq 0$, α_k is given by (7.6).

From (7.11) and (7.12), we find that the coupling equation $(X_T, z_T) = (\tilde{X}_T, \tilde{z}_T)$ is equivalent to

$$\begin{cases} \tilde{\xi}_0 - \xi_0 &= \frac{x - \tilde{x}}{\sqrt{T}} \\ z - \tilde{z} + \frac{\sqrt{T}}{2}\left(x \cdot \xi_0 - \tilde{x} \cdot \tilde{\xi}_0\right) &= T \sum_{k\geq 0} \alpha_k \left(\tilde{\xi}_k \cdot \tilde{\xi}_{k+1} - \xi_k \cdot \xi_{k+1}\right). \end{cases} \tag{7.13}$$

Replacing $\tilde{\xi}_0$ by $\xi_0 + \frac{x-\tilde{x}}{\sqrt{T}}$ in the second equation we get

$$-\zeta + (x-\tilde{x}) \cdot \left(\frac{\sqrt{T}}{2} \xi_0 - \sqrt{T} \alpha_0 \tilde{\xi}_1 \right) = T\alpha_0 \xi_0 \cdot (\tilde{\xi}_1 - \xi_1) + T \sum_{k \geq 1} \alpha_k \left(\tilde{\xi}_k \cdot \tilde{\xi}_{k+1} - \xi_k \cdot \xi_{k+1} \right). \tag{7.14}$$

where $\zeta = \tilde{z} - z - \frac{1}{2}(x \cdot \tilde{x})$ is the last coordinate in the Heisenberg group of $(x,z)^{-1} \cdot (\tilde{x}, \tilde{z})$ We are in position to start the coupling. We take

$$\xi_k = \tilde{\xi}_k \quad \text{for all} \quad k \notin \{0,3\}. \tag{7.15}$$

so that we are left to couple

$$(\xi_0, \tilde{\xi}_0), \quad (\xi_3, \tilde{\xi}_3). \tag{7.16}$$

If (7.15) is satisfied we have the simplification

$$\begin{aligned} & T\alpha_0 \xi_0 \cdot (\tilde{\xi}_1 - \xi_1) + T \sum_{k \geq 1} \alpha_k \left(\tilde{\xi}_k \cdot \tilde{\xi}_{k+1} - \xi_k \cdot \xi_{k+1} \right) \\ &= T\alpha_2 \left(\xi_2 \cdot \tilde{\xi}_3 - \xi_2 \cdot \xi_3 \right) + T\alpha_3 \left(\tilde{\xi}_3 \cdot \xi_4 - \xi_3 \cdot \xi_4 \right) \\ &= T\sqrt{\alpha_2^2 + \alpha_3^2} \left(\tilde{\xi}_3 - \xi_3 \right) \cdot \frac{\alpha_3 \xi_4 - \alpha_2 \xi_2}{\sqrt{\alpha_2^2 + \alpha_3^2}}. \end{aligned}$$

Define

$$W = -\zeta + (x-\tilde{x}) \cdot \left(\frac{\sqrt{T}}{2} \xi_0 - \sqrt{T} \alpha_0 \xi_1 \right) \in \mathbb{R}, \; V = \frac{\alpha_3 \xi_4 - \alpha_2 \xi_2}{\sqrt{\alpha_2^2 + \alpha_3^2}} \in \mathbb{R}^2. \tag{7.17}$$

With these definitions, Equation (7.14) becomes

$$T\sqrt{\alpha_2^2 + \alpha_3^2} \left(\tilde{\xi}_3 - \xi_3 \right) \cdot V = W, \tag{7.18}$$

and where the random vector V is of law $\mathcal{N}(0, I_2)$ and is independent of W. Consider (E_1, E_2) to be a direct orthonormal basis of $\mathbb{R}^2$ and such that E_1 is proportional to V. Writing $U = U^1 E_1 + U^2 E_2$, and since $E_1 \cdot E_1 = 0$ and $E_1 \cdot E_2 = 1$, the solutions of equation

$$U \cdot V = W \tag{7.19}$$

are precisely the vectors $U \in \mathbb{R}^2$ such that

$$U^2 = -\frac{W}{\|V\|_2}.$$

Note that nothing is imposed on the coordinate U^1. A solution of (7.14) is thus obtained if

$$\tilde{\xi}_3 - \xi_3 = -\frac{1}{T\sqrt{\alpha_2^2 + \alpha_3^2}} \frac{W}{\|V\|_2} E_2. \tag{7.20}$$

We also denote (F_1, F_2) to be the direct orthonormal basis of $\mathbb{R}^2$ and such that F_1 is proportional to $\tilde{x} - x$. We emphasize that W depends only on $\langle \xi_0, F_2 \rangle$ (and on $\langle \xi_1, F_2 \rangle$) and thus we will also take $\langle \tilde{\xi}_0, F_2 \rangle = \langle \xi_0, F_2 \rangle$.

Now by Lemma 7.1.4, given the values of ξ_k for $k \in \mathbb{N} \setminus \{0,3\}$ and the value of $\langle \xi_0, F_2 \rangle$, it is possible to construct a coupling of the three dimensional Gaussian random vectors $(\langle \xi_0, F_1 \rangle, \xi_3)$ and $(\langle \tilde{\xi}_0, F_2 \rangle, \tilde{\xi}_3)$ such that

$$\mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T) | (\xi_k)_{k \in \mathbb{N} \setminus \{0,3\}}, \langle \xi_0, F_2 \rangle\right) \leq \frac{1}{\sqrt{2\pi}} \left(\frac{\|\tilde{x} - x\|}{\sqrt{T}} + \frac{1}{T\sqrt{\alpha_2^2 + \alpha_3^2}} \frac{|W|}{\|V\|_2} \right)$$

and thus since V and W are independent

$$\mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right) \leq \frac{1}{\sqrt{2\pi}} \left(\frac{\|x - \tilde{x}\|_2}{\sqrt{T}} + \frac{1}{T\sqrt{\alpha_2^2 + \alpha_3^2}} \mathbb{E}\left[\frac{1}{\|V\|_2}\right] \mathbb{E}[|W|] \right).$$

Now since V is a random vector with law $\mathcal{N}(0, I_2)$, one has

$$\mathbb{E}\left[\frac{1}{\|V\|_2}\right] = \sqrt{\frac{\pi}{2}}.$$

Denoting $\hat{\xi}_0 = \frac{1}{\sqrt{\frac{1}{4} + \alpha_0^2}} \left(\frac{\sqrt{T}}{2} \xi_0 - \sqrt{T} \alpha_0 \xi_1 \right) \sim \mathcal{N}(0, I_2)$ and with the same orthonormal basis (F_1, F_2) of $\mathbb{R}^2$, one has

$$\begin{aligned}
\mathbb{E}[|W|] &\leq |\zeta| + \sqrt{T} \sqrt{\frac{1}{4} + \alpha_0^2}\, \mathbb{E}\left[\left|(\tilde{x} - x) \cdot \hat{\xi}_0\right|\right] \\
&= |\zeta| + \sqrt{T} \sqrt{\frac{1}{4} + \alpha_0^2}\, \|\tilde{x} - x\|_2\, \mathbb{E}\left[|\langle F_2, \hat{\xi}_0 \rangle|\right] \\
&= |\zeta| + \sqrt{\frac{T}{3}} \sqrt{\frac{2}{\pi}}\, \|\tilde{x} - x\|_2
\end{aligned}$$

since $\langle F_2, \hat{\xi}_0 \rangle \sim \mathcal{N}(0,1)$. Thus the conclusion (7.1) holds with

$$C_1 = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{1}{\sqrt{3}\sqrt{\alpha_2^2 + \alpha_3^2}} \right)$$

and

$$C_2 = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\alpha_2^2 + \alpha_3^2}} \sqrt{\frac{\pi}{2}}$$

and the result follows by considering the explicit values of the α_k given in (7.6). □

Remark 7.1.5. *The above explicit constant C_1 and C_2 are not optimal. In some sense, we try to use the less noise possible in the coupling. It should be possible to decrease their values by allowing more random Gaussian vectors to be different in* (7.15).

Remark 7.1.6. *Using the left invariance and the rotational invariance of the Heisenberg group, it is enough to consider the total variation between the measures $\mu_T^{(a,0,c)}$ and $\mu_T^{(0,0,0)}$. In this case, we can take*

$$\tilde{\xi}_0^2 = \xi_0^2, \tilde{\xi}_2^1 = \xi_2^1, \tilde{\xi}_3^1 = \xi_3^1 \text{ and } \tilde{\xi}_k = \xi_k \text{ for } k = 1 \text{ and } k \geq 4.$$

so that we are left to couple

$$(\xi_0^1, \xi_2^2, \xi_3^2), \quad (\tilde{\xi}_0^1, \tilde{\xi}_2^2, \tilde{\xi}_3^2).$$

In this case, by rewriting carefully the above proof, one can replace the constant

$$\frac{1}{\sqrt{\alpha_2^2+\alpha_3^2}}\mathbb{E}\left[\frac{1}{\|V\|_2}\right] \text{ by } \mathbb{E}\left[\frac{1}{\sqrt{(\alpha_1^2+\alpha_2^2)Z_1^2+(\alpha_2^2+\alpha_3^2)Z_2^2}}\right]$$

where Z_1 and Z_2 are two independent $\mathcal{N}(0,1)$ random variables. In fact, in view of Remark 7.1.5, if one allows to couple,

$$\left(\xi_0^1,(\xi_2^2,\xi_3^2),(\xi_5^2,\xi_6^2),(\xi_8^2,\xi_9^2),\dots\right), \quad \left(\tilde{\xi}_0^1,(\tilde{\xi}_2^2,\tilde{\xi}_3^2),(\tilde{\xi}_5^2,\tilde{\xi}_6^2),(\tilde{\xi}_8^2,\tilde{\xi}_9^2),\dots\right),$$

the previous constant may even be replaced by

$$\mathbb{E}\left[\frac{1}{\sqrt{\sum_{k\geq 1}c_k^2Z_k^2}}\right]$$

where $(Z_n)_{n\geq 1}$ is an independent sequence of $\mathcal{N}(0,1)$ random variables and where c^2 is the sequence:

$$c^2=\left(\alpha_1^2+\alpha_2^2,\alpha_2^2+\alpha_3^2,\alpha_4^2+\alpha_5^2,\alpha_5^2+\alpha_6^2,\alpha_7^2+\alpha_8^2,\alpha_8^2+\alpha_9^2,\dots\right).$$

7.2 Distance in total variation of two Brownian motions on $\mathbb{G}_n$

The aim of this section is to extend Theorem 7.1.1 to free, step 2 Carnot groups $\mathbb{G}_n = \mathbb{R}^n \times \mathfrak{so}(n)$ for all $n \geq 2$. The method developed here is slightly different, but the estimates will be similar.

We briefly recall the notations from Subsection 3.1.5. A subRiemannian Brownian motion in $\mathbb{G}_n$ started at (x,z) is a process $((X_t,z_t))_{t\geq 0}$ such that $(X_t)_{t\geq 0}$ is a $\mathbb{R}^n$-valued Brownian motion started at x and $(z_t)_{t\geq 0}$ is the $\mathfrak{so}(n)$-valued process satisfying

$$\forall\, t\geq 0, \quad z_t = z+\frac{1}{2}\int_0^t X_s\odot dX_s. \tag{7.21}$$

We state the following theorem:

Theorem 7.2.1. *For $T>0$ and $(x,z)\in\mathbb{G}_n$ let $\mu_T^{(x,z)}$ be the law at T of the subRiemannian Brownian motion started at (x,z). We have for all $T>0$ and all $((x,z),(\tilde{x},\tilde{z}))\in\mathbb{G}_n^2$,*

$$d_{TV}\left(\mu_T^{(x,z)},\mu_T^{(\tilde{x},\tilde{z})}\right)\leq C_1(n)\frac{\|\tilde{x}-x\|_2}{\sqrt{T}}+C_2(n)\frac{\|\tilde{z}-z-\frac{1}{2}x\odot\tilde{x}\|_{HS}}{T} \tag{7.22}$$

where $\|\cdot\|_{HS}$ denote the Hilbert-Schmidt norm and

$$C_2(n):=\frac{1}{\sqrt{\pi}}\left(6\sqrt{n}+\frac{4}{\sqrt{n}}\right) \quad \text{and} \quad C_1(n):=\frac{1}{\sqrt{2\pi}}+\sqrt{\frac{2(n-1)}{3}}C_2(n). \tag{7.23}$$

Remark 7.2.2. *Theorem 7.2.1 also applies when $n = 2$, that is, in the case of the Heisenberg group. In order to compare the constants in Theorems 7.1.1 and 7.2.1, note that $\|\tilde{z} - z - \frac{1}{2}x \odot \tilde{x}\|_{HS} = \sqrt{2}|\tilde{z} - z - \frac{1}{2}(x_1\tilde{x}_2 - x_2\tilde{x}_1)|$.*

We also recall that, for all $v \in \mathfrak{so}(n)$, $\||\tilde{z} - z - \frac{1}{2}x \odot \tilde{x}\|_{HS} = \sqrt{2}\|\tilde{z} - z - \frac{1}{2}x \odot \tilde{x}\|_2$. We can then compare the results from Theorem 7.2.1 and Corollary 6.1.11. In particular, for n large, the estimates from Theorem 7.2.1 are clearly better.

Remark 7.2.3. *As in Corollary 6.1.11, Theorem 7.2.1 can be extended to any homogeneous, step 2 Carnot group $\mathbb{G}$ of rank n. For $g, \tilde{g} \in \mathbb{G}$ and $T > 0$, we obtain*

$$d_{TV}\left(\mu_T^g, \mu_T^{\tilde{g}}\right) \leq C_1(n)\frac{d_{cc}(g,\tilde{g})}{\sqrt{t}} + \frac{\sqrt{2}C_2(n)}{m_1(n)}\frac{d_{cc}(g,\tilde{g})^2}{t}$$

where μ_t^g (resp. $\mu_t^{\tilde{g}}$) denotes the law of the subRiemannian Brownian motion on $\mathbb{G}$ started at $g \in \mathbb{G}$ (resp. $\tilde{g} \in \mathbb{G}$).

Before giving the proof of this theorem, we make a preliminary study of an inverse problem.

7.2.1 Some preliminaries

Consider an integrable random variable W taking its values in $\mathfrak{so}(n)$, and for $m \geq n+2$, $V_1, \ldots, V_m$, m independent random vectors taking their values in $\mathbb{R}^n$, with law $\mathcal{N}(0, I_n)$ and independent of W. Our next aim is to solve in $U_1, \ldots, U_m$ random variables with values in $\mathbb{R}^n$, the equation

$$\sum_{k=1}^{m} U_k \odot V_k = W. \tag{7.24}$$

Clearly the solution is not unique. We will make a specific choice which will together give uniqueness and allow explicit computations. Letting $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n = M_{n,1}(\mathbb{R})$, we have the canonical decomposition

$$W = \sum_{i,j=1}^{n} W^{i,j} e_i e_j^t = \frac{1}{2}\sum_{i,j=1}^{n} W^{i,j} e_i \odot e_j, \quad \text{in particular} \quad W^{j,i} = -W^{i,j}. \tag{7.25}$$

We will denote

$$U_k = \sum_{j=1}^{n} U_k^j e_j, \qquad k = 1, \ldots, m, \tag{7.26}$$

$$V_k = \sum_{j=1}^{n} V_k^j e_j, \qquad k = 1, \ldots, m, \tag{7.27}$$

$$\mathscr{U} = \begin{pmatrix} U_1^1 & U_2^1 & \cdots & U_m^1 \\ U_1^2 & U_2^2 & \cdots & U_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ U_1^n & U_2^n & \cdots & U_m^n \end{pmatrix} = \left(U_k^j\right)_{1\leq j\leq n,\ 1\leq k\leq m}, \tag{7.28}$$

$$\mathscr{V} = \begin{pmatrix} V_1^1 & V_2^1 & \cdots & V_m^1 \\ V_1^2 & V_2^2 & \cdots & V_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ V_1^n & V_2^n & \cdots & V_m^n \end{pmatrix} = \left(V_k^i\right)_{1\leq i\leq n,\ 1\leq k\leq m} \tag{7.29}$$

and

$$\mathscr{W} = \begin{pmatrix} 0 & W^{1,2} & W^{1,3} & \cdots & W^{1,n} \\ -W^{1,2} & 0 & \ddots & \cdots & W^{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & W^{n-1,n} \\ -W^{1,n} & \cdots & \cdots & -W^{n-1,n} & 0 \end{pmatrix} = \left(W^{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}, \tag{7.30}$$

Equation (7.24) rewrites as

$$\mathscr{U}\mathscr{V}^t - \mathscr{V}\mathscr{U}^t = \mathscr{W}. \tag{7.31}$$

In particular we have a solution of Equation (7.31) if

$$\mathscr{V}\mathscr{U}^t = -\frac{1}{2}\mathscr{W}. \tag{7.32}$$

Proposition 7.2.4. *A solution to Equation* (7.24) *is given by*

$$\mathscr{U}^t = -\frac{1}{2}\mathscr{V}^t\left(\mathscr{V}\mathscr{V}^t\right)^{-1}\mathscr{W}. \tag{7.33}$$

Moreover, we have

$$\mathbb{E}[\|\mathscr{U}\|_{HS}] \leq \frac{1}{2\sqrt{m-n-1}}\mathbb{E}\left[\|\mathscr{W}\|_{HS}\right]. \tag{7.34}$$

Proof. We easily check that $\mathscr{U}$ given by (7.33) is a solution of (7.32) and thus also of (7.24). We are left to prove that

$$\mathbb{E}[\|\mathscr{U}\|_{HS}] \leq \frac{1}{2\sqrt{m-n-1}}\mathbb{E}\left[\|\mathscr{W}\|_{HS}\right] \tag{7.35}$$

with $\mathscr{U}$, $\mathscr{W}$ defined in (7.28) and (7.30). From (7.33) we get

$$\mathscr{U}\mathscr{U}^t = \frac{1}{4}\mathscr{W}^t\left(\mathscr{V}\mathscr{V}^t\right)^{-1}\mathscr{W}. \tag{7.36}$$

This yields

$$\|\mathscr{U}\|_{HS} = \sqrt{\mathrm{Tr}\left(\mathscr{U}\mathscr{U}^t\right)} = \frac{1}{2}\sqrt{\mathrm{Tr}\left(\mathscr{W}^t\left(\mathscr{V}\mathscr{V}^t\right)^{-1}\mathscr{W}\right)}. \tag{7.37}$$

On the other hand, $\mathscr{V}$ is independent of $\mathscr{W}$ and $\mathscr{V}\mathscr{V}^t$ is a Wishart matrix $\mathcal{W}(n,m)$ of size $n\times n$ and with m degree of freedom and thus can be written in singular value decomposition as

$$\mathscr{V}\mathscr{V}^t = \mathscr{S}^t\mathscr{D}^2\mathscr{S} \tag{7.38}$$

where $\mathscr{S}$ and $\mathscr{D}$ are two independent random variables taking their values respectively in $O(n)$ and $M_{n,n}(\mathbb{R})$, $\mathscr{S}$ having uniform law and $\mathscr{D}$ being diagonal with positive eigenvalues

$0 < d_1 < \ldots < d_n$. From this we get

$$\begin{aligned}
2\mathbb{E}\left[\sqrt{\mathrm{Tr}\left(\mathscr{U}\mathscr{U}^t\right)}\right] &= \mathbb{E}\left[\sqrt{\mathrm{Tr}\left(\mathscr{W}^t\left(\mathscr{V}\mathscr{V}^t\right)^{-1}\mathscr{W}\right)}\right] = \mathbb{E}\left[\sqrt{\mathrm{Tr}\left(\mathscr{W}^t\mathscr{S}^t\mathscr{D}^{-2}\mathscr{S}\mathscr{W}\right)}\right] \\
&= \mathbb{E}\left[\sqrt{\mathrm{Tr}\left(\mathscr{S}\mathscr{W}\mathscr{W}^t\mathscr{S}^t\mathscr{D}^{-2}\right)}\right] = \mathbb{E}\left[\sqrt{\sum_{i=1}^n d_i^{-2} e_i^t \mathscr{S}\mathscr{W}\mathscr{W}^t\mathscr{S}^t e_i}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\sqrt{\sum_{i=1}^n d_i^{-2} e_i^t \mathscr{S}\mathscr{W}\mathscr{W}^t\mathscr{S}^t e_i}|\mathscr{W},\mathscr{D}\right]\right] \\
&\leq \mathbb{E}\left[\sqrt{\mathbb{E}\left[\sum_{i=1}^n d_i^{-2} e_i^t \mathscr{S}\mathscr{W}\mathscr{W}^t\mathscr{S}^t e_i|\mathscr{W},\mathscr{D}\right]}\right].
\end{aligned}$$

Now for all $1 \leq i \leq n$,

$$\mathbb{E}\left[e_i^t\mathscr{S}\mathscr{W}\mathscr{W}^t\mathscr{S}^t e_i|\mathscr{W},\mathscr{D}\right] = \frac{1}{n}\mathrm{Tr}(\mathscr{W}\mathscr{W}^t)$$

since $\mathscr{S}$ is uniformly distributed and independent of $\mathscr{W}$ and $\mathscr{D}$. We get from this

$$\mathbb{E}\left[\sqrt{\mathbb{E}\left[\sum_{i=1}^n d_i^{-2} e_i^t \mathscr{S}\mathscr{W}\mathscr{W}^t\mathscr{S}^t e_i|\mathscr{W},\mathscr{D}\right]}\right] = \frac{1}{\sqrt{n}}\mathbb{E}\left[\sqrt{\mathrm{Tr}(\mathscr{W}\mathscr{W}^t)\mathrm{Tr}\left(\mathscr{D}^{-2}\right)}\right]$$

yielding since $\mathrm{Tr}\left(\mathscr{D}^{-2}\right) = \mathrm{Tr}\left(\left(\mathscr{V}\mathscr{V}^t\right)^{-1}\right)$ and is independent of $\mathscr{W}$,

$$\mathbb{E}[\|\mathscr{U}\|_{HS}] \leq \frac{1}{2\sqrt{n}}\mathbb{E}\left[\sqrt{\mathrm{Tr}\left(\left(\mathscr{V}\mathscr{V}^t\right)^{-1}\right)}\right]\mathbb{E}[\|\mathscr{W}\|_{HS}]. \tag{7.39}$$

By [49] Example 3.1 we have

$$\mathbb{E}\left[\mathrm{Tr}\left(\left(\mathscr{V}\mathscr{V}^t\right)^{-1}\right)\right] = \frac{n}{m-n-1} \tag{7.40}$$

so finally by Cauchy-Schwarz,

$$\mathbb{E}[\|\mathscr{U}\|_{HS}] \leq \frac{1}{2\sqrt{m-n-1}}\mathbb{E}[\|\mathscr{W}\|_{HS}]. \tag{7.41}$$

□

7.2.2 Proof of Theorem 7.2.1

Proof of Theorem 7.2.1. For any choice of two $\mathbb{G}_n$-valued subRiemannian Brownian motions $((X_t, z_t))_{t\geq 0}$ and $((\tilde{X}_t, \tilde{z}_t))_{t\geq 0}$ started respectively at (x, z) and $(\tilde{x}, \tilde{z})$, we have

$$d_{TV}\left(\mu_T^{(x,z)}, \mu_T^{(\tilde{x},\tilde{z})}\right) \leq \mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right). \tag{7.42}$$

Consequently, to establish the estimate (7.22) it is sufficient, for each $T > 0$, to find $((X_t, z_t))_{t\geq 0}$ and $((\tilde{X}_t, \tilde{z}_t))_{t\geq 0}$ started respectively at (x, z) and $(\tilde{x}, \tilde{z})$, satisfying

$$\mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right) \leq C_1(n)\frac{\|\tilde{x} - x\|_2}{\sqrt{T}} + C_2(n)\frac{\|\tilde{z} - z - \frac{1}{2}x \odot \tilde{x}\|_{HS}}{T}. \tag{7.43}$$

Adopting the same strategy as in Section 7.1, we construct the Brownian motions $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ with Legendre polynomials.

So let us fix $T > 0$. Similarly to Equation (7.4) but now in dimension n, we write

$$\forall\ 0 \leq t \leq T,\ X_t = x + B_t \quad \text{with} \quad B_t = \sum_{k=0}^{\infty} \xi_k \int_0^t Q_k(s)\, ds, \tag{7.44}$$

where $\left(\xi_k = \begin{pmatrix} \xi_k^1 \\ \vdots \\ \xi_k^n \end{pmatrix} \right)_{k\geq 0}$ is a sequence of independent $\mathbb{R}^n$-valued random vectors with law $\mathcal{N}(0, I_n)$. We do the same with $(\tilde{X}_t)_{0\leq t\leq T}$, using independent $\mathbb{R}^n$-valued random variables $\left(\tilde{\xi}_k\right)_{k\geq 0}$ with law $\mathcal{N}(0, I_n)$. Equation (7.43) will be obtained thanks to a well-chosen coupling of $(\xi_k)_{k\geq 0}$ and $\left(\tilde{\xi}_k\right)_{k\geq 0}$.

At time T we get

$$X_T = x + \sqrt{T}\xi_0, \qquad z_T = z + \frac{1}{2}\sqrt{T}x \odot \xi_0 + T\sum_{k\geq 0} \alpha_k \xi_k \odot \xi_{k+1}, \tag{7.45}$$

$$\tilde{X}_T = \tilde{x} + \sqrt{T}\tilde{\xi}_0, \qquad \tilde{z}_T = \tilde{z} + \frac{1}{2}\sqrt{T}\tilde{x} \odot \tilde{\xi}_0 + T\sum_{k\geq 0} \alpha_k \tilde{\xi}_k \odot \tilde{\xi}_{k+1}, \tag{7.46}$$

where $(\alpha_k)_{k\geq 0}$ is defined in (7.6).

From (7.45) and (7.46), we find that the coupling equation $(X_T, z_T) = (\tilde{X}_T, \tilde{z}_T)$ is equivalent to

$$\begin{cases} \tilde{\xi}_0 - \xi_0 &= \frac{x-\tilde{x}}{\sqrt{T}} \\ z - \tilde{z} + \frac{\sqrt{T}}{2}\left(x \odot \xi_0 - \tilde{x} \odot \tilde{\xi}_0\right) &= T\sum\limits_{k\geq 0} \alpha_k \left(\tilde{\xi}_k \odot \tilde{\xi}_{k+1} - \xi_k \odot \xi_{k+1}\right). \end{cases} \tag{7.47}$$

Replacing $\tilde{\xi}_0$ by $\xi_0 + \frac{x-\tilde{x}}{\sqrt{T}}$ in the second equation we get

$$-\zeta + (x-\tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right) = T\alpha_0\xi_0 \odot (\tilde{\xi}_1 - \xi_1) + T\sum_{k\geq 1} \alpha_k \left(\tilde{\xi}_k \odot \tilde{\xi}_{k+1} - \xi_k \odot \xi_{k+1}\right) \tag{7.48}$$

where $\zeta = \tilde{z} - z - \frac{1}{2}x \odot \tilde{x}$. We are in position to start the coupling. As in previous section we let $m \geq n + 2$, that we will choose at the end. We take

$$\xi_k = \tilde{\xi}_k \quad \text{for all} \quad k \notin \{0, 3, 6, \ldots 3m\}, \tag{7.49}$$

so that we are left to couple

$$(\xi_0, \tilde{\xi}_0), \quad (\xi_3, \tilde{\xi}_3), \ldots (\xi_{3(n-1)}, \tilde{\xi}_{3m}). \tag{7.50}$$

If (7.49) is satisfied we have the simplification

$$\begin{aligned}
&T\alpha_0\xi_0 \odot (\tilde{\xi}_1 - \xi_1) + T\sum_{k\geq 1} \alpha_k \left(\tilde{\xi}_k \odot \tilde{\xi}_{k+1} - \xi_k \odot \xi_{k+1}\right) \\
&= T\sum_{k=1}^{m} \left(\alpha_{3k-1}\left(\xi_{3k-1} \odot \tilde{\xi}_{3k} - \xi_{3k-1} \odot \xi_{3k}\right) + \alpha_{3k}\left(\tilde{\xi}_{3k} \odot \xi_{3k+1} - \xi_{3k} \odot \xi_{3k+1}\right)\right) \\
&= \sum_{k=1}^{m} \left(\tilde{\xi}_{3k} - \xi_{3k}\right) \odot T\left(\alpha_{3k}\xi_{3k+1} - \alpha_{3k-1}\xi_{3k-1}\right) \\
&= \sum_{k=1}^{m} T\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}\left(\tilde{\xi}_{3k} - \xi_{3k}\right) \odot \frac{\alpha_{3k}\xi_{3k+1} - \alpha_{3k-1}\xi_{3k-1}}{\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}}.
\end{aligned}$$

Define

$$W = -\zeta + (x - \tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right), \tag{7.51}$$

$$V_k = \frac{\alpha_{3k}\xi_{3k+1} - \alpha_{3k-1}\xi_{3k-1}}{\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}}, \quad k = 1, \ldots, m. \tag{7.52}$$

With these definitions, Equation (7.48) becomes

$$\sum_{k=1}^{m} T\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}\left(\tilde{\xi}_{3k} - \xi_{3k}\right) \odot V_k = W, \tag{7.53}$$

and the random vectors V_k, $1 \leq k \leq m$ are independent with the same law $\mathcal{N}(0, I_n)$.

Let $(U_1, \ldots, U_m)$ be the solution given by (7.33) to Equation (7.24) $\sum_{k=1}^{m} U_k \odot V_k = W$. Using (7.53) we see that a solution to (7.48) is given by

$$\tilde{\xi}_{3k} - \xi_{3k} = \frac{U_k}{T\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}} =: \hat{U}_k, \quad k = 1, \ldots, m. \tag{7.54}$$

Define

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix}, \quad \tilde{\xi} = \begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \vdots \\ \tilde{\xi}_m \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} \hat{U}_1 \\ \hat{U}_2 \\ \vdots \\ \hat{U}_m \end{pmatrix} \tag{7.55}$$

with the $\hat{U}_k$ defined in (7.54). The random vectors ξ, $\tilde{\xi}$ and $\hat{U}$ take their values in $M_{nm,1}(\mathbb{R})$ and ξ, $\tilde{\xi}$ have law $\mathcal{N}(0, I_{nm})$. Recalling the system (7.47), we obtain with (7.54) that

$$\mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right) \leq \mathbb{P}\left(\tilde{\xi}_0 - \xi_0 \neq \frac{x - \tilde{x}}{\sqrt{T}}\right) + \mathbb{P}\left(\tilde{\xi} - \xi \neq \hat{U}\right). \tag{7.56}$$

Observing that the random vector $\hat{U}$ is independent of ξ, and using Lemma 7.1.4, we get the estimate

$$\mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right) \leq \frac{\|x - \tilde{x}\|_2}{\sqrt{2\pi T}} + \frac{\mathbb{E}[\|\hat{U}\|_{HS}]}{\sqrt{2\pi}}. \tag{7.57}$$

By (7.6) the sequence $(\alpha_k)_{k\geq 0}$ is decreasing, consequently the sequence $\left(\frac{1}{\sqrt{\alpha_{3k}^2+\alpha_{3k-1}^2}}\right)_{k\geq 0}$ is increasing and $\mathbb{E}[\|\hat{U}\|_{HS}] \leq \frac{\mathbb{E}[\|U\|_{HS}]}{T\sqrt{\alpha_{3m}^2+\alpha_{3m-1}^2}}$. On the other hand using (7.6) we have for $k \geq 1$

$$\begin{aligned}\frac{1}{\sqrt{\alpha_{3k}^2+\alpha_{3k-1}^2}} &= 2\left(\frac{(4(3k+1)^2-1)(4(3k)^2-1)}{(4(3k+1)^2-1)+(4(3k)^2-1)}\right)^{1/2} \\ &\leq 2\left(\frac{4^2(3k+1)^2(3k)^2}{8(3k)^2}\right)^{1/2} \leq 2\sqrt{2}(3k+1).\end{aligned} \tag{7.58}$$

Recalling that by Proposition 7.2.4,

$$\mathbb{E}[\|U\|_{HS}] = \mathbb{E}[\|\mathscr{U}\|_{HS}] \leq \frac{1}{2\sqrt{m-n-1}}\mathbb{E}\left[\|W\|_{HS}\right]$$

we get

$$\mathbb{E}[\|\hat{U}\|_{HS}] \leq \frac{\sqrt{2}(3m+1)}{T\sqrt{m-n-1}}\mathbb{E}\left[\|W\|_{HS}\right] \tag{7.59}$$

On the other hand, writing from (7.51)

$$W = -\zeta + (x-\tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right),$$

we get

$$\mathbb{E}[\|W\|_{HS}^2] = \|\zeta\|_{HS}^2 + \mathbb{E}\left[\left\|(x-\tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right)\right\|_{HS}^2\right].$$

We will do the computation in an orthonormal basis $(E_1,\ldots,E_n)$ of $\mathbb{R}^n$ such that $x-\tilde{x} = \|x-\tilde{x}\|_2 E_1$. Since $\alpha_0 = \frac{1}{2\sqrt{3}}$ we have $\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1 = \sqrt{\frac{T}{3}}\hat{\xi}_0$ where $\hat{\xi}_0$ is a $\mathbb{R}^n$-valued Gaussian random variable with law $\mathcal{N}(0,I_n)$. Writing $\hat{\xi}_0 = \sum_{i=1}^n \hat{\xi}_0^i E_i$ we obtain

$$(x-\tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right) = \sqrt{\frac{T}{3}}\|x-\tilde{x}\|_2 \sum_{i=2}^n \hat{\xi}_0^i E_1 \odot E_i.$$

The matrices $E_1 \odot E_i = E_1E_i^t - E_iE_1^t$ being orthogonal each with norm $\sqrt{2}$ we obtain

$$\mathbb{E}\left[\left\|(x-\tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right)\right\|_{HS}^2\right] = \|x-\tilde{x}\|_2^2\,\frac{2T(n-1)}{3}.$$

We get

$$\mathbb{E}\left[\|W\|_{HS}^2\right] = \|\zeta\|_{HS}^2 + \frac{2T(n-1)}{3}\|x-\tilde{x}\|_2^2. \tag{7.60}$$

Using this estimate in (7.59)) yields

$$\mathbb{E}[\|\hat{U}\|_{HS}] \leq \frac{\sqrt{2}(3m+1)}{T\sqrt{m-n-1}}\left(\|\zeta\|_{HS} + \sqrt{\frac{2T(n-1)}{3}}\|x-\tilde{x}\|_2\right). \tag{7.61}$$

We can easily prove that the best choice for an integer m is

$$m = 2n+1 \quad \text{implying} \quad \frac{\sqrt{2}(3m+1)}{\sqrt{m-n-1}} = 6\sqrt{2n} + \frac{4\sqrt{2}}{\sqrt{n}}. \tag{7.62}$$

So together with (7.57),

$$\begin{aligned} &\mathbb{P}\left((X_T, z_T) \neq (\tilde{X}_T, \tilde{z}_T)\right) \leq \\ &\frac{\|x-\tilde{x}\|_2}{\sqrt{2\pi T}} + \frac{1}{T\sqrt{\pi}}\left(6\sqrt{n} + \frac{4}{\sqrt{n}}\right)\left(\|\zeta\|_{HS} + \sqrt{\frac{2T(n-1)}{3}}\|x-\tilde{x}\|_2\right). \end{aligned} \tag{7.63}$$

We obtain the wanted inequality (7.22) with

$$C_2(n) = \frac{1}{\sqrt{\pi}}\left(6\sqrt{n} + \frac{4}{\sqrt{n}}\right) \quad \text{and} \quad C_1 = \frac{1}{\sqrt{2\pi}} + \sqrt{\frac{2(n-1)}{3}}C_2. \tag{7.64}$$

□

7.3 Application to gradients inequalities

7.3.1 Direct estimates for the horizontal and vertical gradient

As in Chapter 6, we can obtain direct estimates for the horizontal and vertical gradient of the heat semi group. Indeed, using the coupling from Theorem 7.2.1, we have:

$$\begin{aligned} |P_t f(g) - P_t f(\tilde{g})| &\leq 2||f||_\infty \mathbb{P}\left(\mathbb{B}_t^g \neq \mathbb{B}_t^{\tilde{g}}\right) \\ &\leq 2||f||_\infty \left(\frac{C_1(n)}{\sqrt{t}}\|x-\tilde{x}\|_2 + C_2(n)\frac{\sqrt{2}\|\tilde{z} - z - \frac{1}{2}x \odot \tilde{x}\|_2}{t}\right). \end{aligned} \tag{7.65}$$

Using the same arguments as in the proof of Corollaries 6.3.1 and 6.3.2, we obtain:

Corollary 7.3.1. *Let $\mathbb{G}_n$ be the free step 2 Carnot group of of rank $n \geq 2$. For any bounded measurable function f on $\mathbb{G}_n$, for any $g \in \mathbb{G}_n$ and $t > 0$,*

$$\|\nabla_{\mathcal{H}} P_t f(g)\|_{\mathcal{H}} \leq \frac{2C_1(n)}{\sqrt{t}}||f||_\infty \ a.s. \tag{7.66}$$

and

$$\|\nabla_{\mathfrak{v}} P_t f(g)\| \leq \frac{2\sqrt{2}C_2(n)}{t}||f||_\infty \ a.s.. \tag{7.67}$$

where $C_1(n)$ and $C_2(n)$ are the constant appearing in Theorem 7.2.1 (or 7.1.1).

Remark 7.3.2. *As for Corollary 6.3.1, we can also show that* (7.66) *is true on all the homogeneous step* 2 *Carnot groups.*

7.3.2 Coupling with change of probability and applications

In this section we will construct couplings at time T with probability one, but the price to pay will be to make changes of probabilities for the second process. The distance between semi-groups will be measured by the change of probability. The main results are a log Harnack inequality (Proposition 7.3.7, an integration by parts formula (Proposition 7.3.8) which expresses the spatial derivative $dP_T f$ of the semi-group $P_T f$ of the Brownian motion by a formula which does not make appear derivative of f, and a reverse Sobolev inequality (Proposition 7.3.8).

The notations are the same as in the previous section. The processes $(\mathbb{B}_t^g) := (X_t, z_t)$ and $(\mathbb{B}_t^{\tilde{g}}) := ((\tilde{X}_t, \tilde{z}_t))$ started respectively at $g = (x, z)$ and $\tilde{g} = (\tilde{x}, \tilde{z})$ are defined with equation (7.44). The sequence $(\xi_k)_{k\geq 0}$ will be identically distributed will law $\mathcal{N}(0, I_n)$ under the probability $\mathbb{P}$. The difference will be that we will look for a sequence $(\tilde{\xi}_k)_{k\geq 0}$ which is independent and identically distributed with law $\mathcal{N}(0, I_n)$ under another probability $\mathbb{P}(\tilde{g})$, and so that at time T, a.s. $\mathbb{B}_T^g = \mathbb{B}_T^{\tilde{g}}$.

So let us fix $K \in \{n+1, \ldots\} \cup \{\infty\}$ and let

$$J_K := \{\ell \in \mathbb{N},\ \ell \leq K\} \quad \text{if} \quad K < \infty, \quad J_\infty := \mathbb{N} \quad \text{and} \quad J_K^* := J_K \backslash \{0\}\ \forall\ K. \tag{7.68}$$

We will take

$$\xi_k = \tilde{\xi}_k \quad \text{for all} \quad k \notin 3J_K \tag{7.69}$$

so that we are left to couple

$$(\xi_\ell, \tilde{\xi}_\ell), \quad \ell \in 3J_K. \tag{7.70}$$

Now we consider the sequence $(V_k)_{k\in J_K^*}$ defined in (7.52), of independent random vectors taking their values in $\mathbb{R}^n$, with the same law $\mathcal{N}(0, I_n)$. We solve in $(U_k)_{k\in J_K^*}$ the equation

$$\sum_{k\in J_K^*} U_k \odot V_k = W \tag{7.71}$$

with W given by Equation (7.51). Then we will choose $(\tilde{\xi}_k)_{k\geq 0}$ such that almost surely

$$\tilde{\xi}_0 - \xi_0 = \frac{x - \tilde{x}}{\sqrt{T}} =: \hat{U}_0 \tag{7.72}$$

and

$$\forall\ k \in J_K^*,\ \tilde{\xi}_{3k} - \xi_{3k} = \frac{U_k}{T\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}} =: \hat{U}_k. \tag{7.73}$$

We denote

$$\forall k \in J_K^*, \quad V_k = \sum_{i=1}^n V_k^i e_i, \quad U_k = \sum_{j=1}^n U_k^j e_j, \tag{7.74}$$

$$\beta_k = \frac{1}{T\sqrt{\alpha_{3k}^2 + \alpha_{3k-1}^2}} \tag{7.75}$$

$$\hat{\mathscr{V}} = \hat{\mathscr{V}}_K = \left(\frac{V_k^i}{\beta_k}\right)_{1\leq i\leq n,\ k\in J_K^*}, \quad \hat{\mathscr{U}} = \hat{\mathscr{U}}_K = (\beta_k U_k^i)_{1\leq j\leq n,\ k\in J_K^*}, \tag{7.76}$$

the upper index representing the lines and the lower index representing the columns. With these notations and similarly as before, Equation (7.71) is equivalent to

$$\hat{\mathscr{U}}\hat{\mathscr{V}}^t - \hat{\mathscr{V}}\hat{\mathscr{U}}^t = \mathscr{W} \tag{7.77}$$

with $\mathscr{W}$ defined by (7.30). In particular, we have a solution of Equation (7.71) if

$$\hat{\mathscr{V}}\hat{\mathscr{U}}^t = -\frac{1}{2}\mathscr{W}. \tag{7.78}$$

The $n \times n$ matrix

$$\hat{\mathscr{V}}\hat{\mathscr{V}}^t = \sum_{k \in J_K^*} \frac{1}{\beta_k^2} V_k V_k^t \tag{7.79}$$

is a.s. well-defined since $\mathbb{E}\left[\sum_{k=1}^{\infty} \frac{1}{\beta_k^2}\mathrm{Tr}(V_k V_k^t)\right] < \infty$ (the computation (7.58) proves that β_k is of order k). It is a.s. symmetric positive since $K \geq n$. Consequently it is invertible, and a solution to (7.78) is given by

$$\hat{\mathscr{U}}^t = -\hat{\mathscr{V}}^t(\hat{\mathscr{V}}\hat{\mathscr{V}}^t)^{-1}\mathscr{W}. \tag{7.80}$$

Let us make a specific choice of probability space, which will be very convenient for our computations. This probability space is $(\Omega, \mathscr{A}, \mathbb{P})$, where $\Omega := \ell^2(\mathbb{R}^n)$ is the Hilbert space of square integrable $\mathbb{R}^n$-valued sequences, $\mathscr{A}$ is the smallest σ-field for which the projections are measurable, completed with respect to the probability measure $\mathbb{P}$ for which the canonical projections

$$\begin{aligned} \xi_k : \Omega &\to \mathbb{R}^n \\ \omega = (\omega_0, \omega_1, \ldots, \omega_k, \ldots) &\mapsto \omega_k =: \xi_k(\omega) \end{aligned}$$

are i.i.d. and $\mathcal{N}(0, I_n)$. We will need to split Ω into two supplementary orthogonal spaces: $\Omega = \Omega_a \oplus \Omega_b$. Let us now describe these spaces. For $k \geq 1$ and $1 \leq i \leq n$, we denote by e_k^i the element of Ω which satisfies $\xi_\ell(e_k^i) = 0$ if $\ell \neq k$ and $\xi_k(e_k^i) = e_i$, the i-th element of the canonical basis of $\mathbb{R}^n$. Letting $(f_1, \ldots, f_n)$ be an orthonormal basis of $\mathbb{R}^n$ such that $\|x - \tilde{x}\| f_1 = x - \tilde{x}$, for $1 \leq i \leq n$ we denote by f_0^i the element of Ω such that $\xi_\ell(f_0^i) = 0$ if $\ell \neq 0$ and $\xi_0(f_0^i) = f_i$. Notice that the $(e_k^i),\ k \geq 1,\ 1 \leq i \leq n$ together with the $(f_0^i),\ 1 \leq i \leq n$ form an Hilbertian basis of Ω and that the random variables $\langle e_k^i, \omega\rangle$, $\langle f_0^i, \omega\rangle$ are i.i.d and

$$\Omega_a = \mathrm{Span}\left\{f_0^1,\ e_k^i,\ k \in 3J_K^*,\ 1 \leq i \leq n\right\}, \tag{7.81}$$

$$\Omega_b = \Omega_a^\perp = \mathrm{Span}\left\{f_0^i,\ 2 \leq i \leq n\right\} \oplus \mathrm{Span}\left\{\ e_k^i, k \notin 3J_K,\ 1 \leq i \leq n\right\}. \tag{7.82}$$

For the sequel, we will denote ω_a (resp. ω_b) the projection of ω on Ω_a (resp. Ω_b). Let $\mathscr{A}_a$ and $\mathscr{A}_b$ be the canonical σ-fields and $\mathbb{P}_a$ (resp. $\mathbb{P}_b$) be such that the $\langle \omega_a, e_k^i\rangle$, $k \geq 1, 1 \leq i \leq n$, $\langle \omega_a, f_0^1\rangle$ (resp. $\langle \omega_b, e_\ell^j\rangle$, $\langle \omega_b, f_0^i\rangle$ $\ell \notin 3J_K, 1 \leq j \leq n, 2 \leq i \leq n$) are independent $\mathscr{N}(0,1)$ random variables. Then

$$\begin{aligned} (\Omega_a \times \Omega_b, \mathscr{A}_a \times \mathscr{A}_b, \mathbb{P}_a \times \mathbb{P}_b) &\to (\Omega, \mathscr{A}, \mathbb{P}) \\ (\omega_a, \omega_b) &\mapsto \omega_a + \omega_b \end{aligned} \tag{7.83}$$

is an isometry.

Recall that $\tilde{\xi}_k = \xi_k$ if $k \notin 3J_K$ and $\tilde{\xi}_{3k} = \xi_{3k} + \hat{U}_k$ if $k \in J_K$. Let $\mathbb{P}(\tilde{g})$ be the probability on Ω such that all $\tilde{\xi}_k$ are i.i.d. and $\mathcal{N}(0,1)$.

Lemma 7.3.3. *The probability $\mathbb{P}(\tilde{g})$ is equivalent to $\mathbb{P}$, and*

$$R(u) := \frac{d\mathbb{P}(\tilde{g})}{d\mathbb{P}}(\omega) = e^{\frac{1}{2}\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle^2 - \left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle + \|u\|\right)^2\right)} = e^{-\langle \omega, u\rangle - \frac{1}{2}\|u\|^2} \tag{7.84}$$

where $u = u(\tilde{g})(\omega) \in \Omega$ is defined by

$$u_k = 0 \ \forall\ k \notin 3J_K \quad \textit{and} \quad u_{3k} = \hat{U}_k(\omega)\ \forall k \in J_K, \tag{7.85}$$

and $\langle \omega, u\rangle = \sum_{k=0}^{\infty} \langle \omega_k, u_k\rangle_{\mathbb{R}^n}$.

In particular,

$$dR(u(\cdot))|_{\tilde{g}=g} = -\sum_{k=0}^{\infty} \left\langle \xi_{3k}, d\hat{U}_k(\cdot)|_{\tilde{g}=g} \right\rangle. \tag{7.86}$$

Moreover, for all measurable $F : \Omega \to \mathbb{R}$, we have that F is $\mathbb{P}$-integrable if and only if $\omega \mapsto F(\omega + u(\omega))$ is $R(u)\cdot\mathbb{P}$-integrable, and in this case

$$\mathbb{E}[F(\omega)] = \mathbb{E}\left[F(\omega + u(\omega))R(u(\omega))\right]. \tag{7.87}$$

Proof. First observe that for a fixed nonzero vector $u \in \Omega_a$, we can make the orthogonal decomposition

$$\omega_a = \left\langle \omega_a, \frac{u}{\|u\|} \right\rangle \frac{u}{\|u\|} + P^{\Omega_a}_{(u)^\perp}(\omega_a) \tag{7.88}$$

where $\left\langle \omega_a, \frac{u}{\|u\|} \right\rangle$ is an $\mathcal{N}(0,1)$ real-valued random variable independent of $P^{\Omega_a}_{(u)^\perp}(\omega_a)$. Now remark that the random vector $u(\omega)$ satisfies $u(\omega) = u(\omega_b)$ in the decomposition $\omega = \omega_a + \omega_b$ of (7.83). This is due to the fact that the $\hat{U}_k$ do not change when one replaces ξ_0 by $\xi_0 + \frac{x-\tilde{x}}{\sqrt{T}}$ in the expression of

$$W = -\zeta + (x - \tilde{x}) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1 \right).$$

In other words, u is measurable with respect to σ-field $\mathscr{G} := \sigma(\xi_k, k \notin 3J_K^*) \vee \sigma(P^{\mathbb{R}^n}_{(x-\tilde{x})^\perp}(\xi_0))$ ($P^{\mathbb{R}^n}_{(x-\tilde{x})^\perp}$ denoting the projection in $\mathbb{R}^n$ orthogonal to $x - \tilde{x}$).

A second important fact is that $\omega \mapsto u(\omega)$ takes its values in Ω_a. In other words $u_\ell = 0$ if $\ell \notin 3J_K^*$ and u_0 is collinear to $x - \tilde{x}$. Consequently, conditioned to $\mathscr{G}$, u is a Ω_a-valued constant. So we can make the same decomposition as in (7.88):

$$\omega_a = \left\langle \omega_a, \frac{u(\omega)}{\|u(\omega)\|} \right\rangle \frac{u(\omega)}{\|u(\omega)\|} + P^{\Omega_a}_{(u(\omega))^\perp}(\omega_a)$$

where conditioned to $\mathscr{G}$, $\left\langle \omega_a, \frac{u(\omega)}{\|u(\omega)\|} \right\rangle$ is an $\mathscr{N}(0,1)$ random variable independent of $P^{\Omega_a}_{(u(\omega))^\perp}(\omega_a)$. Adding ω_b which is $\mathscr{G}$-measurable and orthogonal to Ω_a we get

$$\omega = \left\langle \omega, \frac{u}{\|u\|} \right\rangle \frac{u}{\|u\|} + P_{(u)^\perp}(\omega) \tag{7.89}$$

where conditioned to $\mathscr{G}$, $\left\langle \omega, \frac{u}{\|u\|} \right\rangle$ is an $\mathcal{N}(0,1)$ real-valued random variable independent of $P_{(u)^\perp}(\omega)$.

Let $F : \Omega \to \mathbb{R}$ a bounded measurable function.

$$\begin{aligned}\mathbb{E}[F(\omega)] &= \mathbb{E}\left[\mathbb{E}\left[F\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle \frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)\middle| P_{(u)^\perp}(\omega),\ \mathscr{G}\right]\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} F\left(x\frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)\varphi(x)\,dx\right]\end{aligned}$$

where φ is the density of $\mathcal{N}(0,1)$. But

$$\int_{\mathbb{R}} F\left(x\frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)\varphi(x)\,dx = \int_{\mathbb{R}} F\left((x+\|u\|)\frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)\varphi(x+\|u\|)\,dx$$

yielding

$$\begin{aligned}&\mathbb{E}[F(\omega)] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} F\left((x+\|u\|)\frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)\varphi(x+\|u\|)\,dx\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} F\left((x+\|u\|)\frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)\frac{\varphi(x+\|u\|)}{\varphi(x)}\varphi(x)\,dx\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\frac{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle + \|u\|\right)}{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle\right)} F\left(\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle + \|u\|\right)\frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)|P_{(u)^\perp}(\omega),\ \mathscr{G}\right]\right]\end{aligned}$$

recalling that conditioned to $\mathscr{G}$, $\left\langle \omega, \frac{u}{\|u\|}\right\rangle$ is an $\mathcal{N}(0,1)$ real-valued random variable independent of $P_{(u)^\perp}(\omega)$. So

$$\begin{aligned}&\mathbb{E}[F(\omega)] \\ &= \mathbb{E}\left[\frac{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle + \|u\|\right)}{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle\right)} F\left(\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle + \|u\|\right)\frac{u}{\|u\|} + P_{(u)^\perp}(\omega)\right)\right] \\ &= \mathbb{E}\left[\frac{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle + \|u\|\right)}{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle\right)} F\left(\omega + u\right)\right].\end{aligned}$$

Observing that

$$\frac{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle + \|u\|\right)}{\varphi\left(\left\langle \omega, \frac{u}{\|u\|}\right\rangle\right)} = e^{-\langle \omega, u\rangle - \frac{1}{2}\|u\|^2}$$

yields (7.84) via (7.87). Equation (7.86) is a direct consequence. □

Corollary 7.3.4. *Let us take $K = 2n+1$. Let $R = R(u)$ be as in Lemma 7.3.3. Then $R \ln R$ is integrable and*

$$\begin{aligned}&\mathbb{E}[R\ln R] = \frac{1}{2}\mathbb{E}\left[\|u\|^2\right] \\ &\leq \frac{\|x-\tilde{x}\|_2^2}{2T} + \left(6\sqrt{n} + \frac{4}{\sqrt{n}}\right)^2\left(\frac{1}{T^2}\left\|z - \tilde{z} - \frac{1}{2}x\odot\tilde{x}\right\|_{HS}^2 + \frac{2(n-1)}{3T}\|x-\tilde{x}\|_2^2\right).\end{aligned} \tag{7.90}$$

Proof. Recall that $\|u\|^2 = \|\hat{U}_0^2\| + \|\hat{U}\|^2 = \frac{x-\tilde{x}^2}{T} + \frac{\|\hat{U}\|^2}{T}$, $\hat{U}$ being defined in (7.55). First observe that the inequality in (7.90) comes from (7.59) and (7.60). Secondly, we have $u(\omega) = u(\omega - u(\omega))$ since $u(\omega) \in V$ and u depends only on $P_{V^\perp}(\omega)$, see the proof of Lemma 7.3.3. So using (7.87),

$$\begin{aligned}
&\mathbb{E}\left[R(u(\omega))(\omega) \ln R(u(\omega))(\omega)\right] \\
&= \mathbb{E}\left[\ln R(u(\omega - u(\omega)))(\omega - u(\omega))\right] \\
&= \mathbb{E}\left[\ln R(u(\omega))(\omega - u(\omega))\right] \\
&= \mathbb{E}\left[-\langle \omega - u(\omega), u(\omega)\rangle - \frac{1}{2}\|u(\omega)\|^2\right] \\
&= \mathbb{E}\left[-\mathbb{E}[\langle \omega, u(\omega)\rangle | \mathscr{G}] + \frac{1}{2}\|u(\omega)\|^2\right] \quad \text{with } \mathscr{G} \text{ defined in the proof of Lemma 7.3.3} \\
&= \mathbb{E}\left[\frac{1}{2}\|u(\omega)\|^2\right]
\end{aligned}$$

since $\mathbb{E}[\langle \omega, u(\omega)\rangle | \mathscr{G}] = 0$: $u(\omega)$ is $\mathscr{G}$-measurable and conditioned to $\mathscr{G}$ $\langle \omega, u(\omega)\rangle$ is Gaussian and centered. □

We will need two preparatory lemmas.

Lemma 7.3.5. *Let $(Y_\ell)_{\ell \geq 1}$ be a sequence of independent real-valued random variables with law $\chi^2(2)$. Let $a > 0$, for any $q > a$ such that q is an integer, we have*

$$\mathbb{E}\left[\left(\sum_{\ell=1}^{\infty} \frac{Y_\ell}{\ell^2}\right)^{-a}\right] \leq q^{2a} \frac{\Gamma(q-a)}{2^a \Gamma(q)}. \tag{7.91}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{\ell=1}^{\infty} \frac{Y_\ell}{\ell^2}\right)^{-a}\right] &\leq \mathbb{E}\left[\left(\sum_{\ell=1}^{q} \frac{Y_\ell}{\ell^2}\right)^{-a}\right] \\
&\leq q^{2a} \mathbb{E}\left[\left(\sum_{\ell=1}^{q} Y_\ell\right)^{-a}\right] \\
&= q^{2a} \int_0^\infty y^{-a} \frac{y^{q-1} e^{-y/2}}{2^q \Gamma(q)}\, dy \\
&= q^{2a} \frac{2^{q-a}\Gamma(q-a)}{2^q \Gamma(q)} = q^{2a} \frac{\Gamma(q-a)}{2^a \Gamma(q)}
\end{aligned}$$

where we used in the third line that $\sum_{\ell=1}^q Y_\ell$ has law $\chi^2(2q)$ with density

$$y \mapsto 1_{(0,\infty)}(y) \frac{y^{q-1} e^{-y/2}}{2^q \Gamma(q)}.$$

□

Lemma 7.3.6. *Let $(V_k)_{k \geq 1}$ be a sequence of $\mathbb{R}^n$-valued independent random variables with law $\mathcal{N}(0, I_n)$. Then for any $p \in (0,1)$,*

$$\forall\ \lambda \in \mathbb{R}, \quad \mathbb{E}\left[\exp\left(\lambda \left(\mathrm{Tr}\left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2}\right)^{-1}\right)^p\right)\right] \leq 1 + \sum_{q=1}^{\infty} \left(\frac{C\lambda}{T^{2p}}\right)^q \frac{q^{2pq}\Gamma(q(1-p))}{q!\Gamma(q)} < \infty \tag{7.92}$$

where

$$C = C(p) = \left(\frac{n^2}{2} \left(2\sqrt{2}(3n+4) \right)^2 \right)^p. \tag{7.93}$$

Proof. It is enough to make the proof with $\lambda > 0$. We have

$$\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \geq \sum_{\ell=1}^{\infty} \frac{1}{\beta_{\ell(n+1)}^2} \mathscr{M}_\ell \quad \text{with} \quad \mathscr{M}_\ell := \sum_{\ell'=(\ell-1)(n+1)+1}^{\ell(n+1)} V_{\ell'} V_{\ell'}^t.$$

The matrices $\mathscr{M}_\ell$ are Wishart $\mathscr{W}(n, n+1)$ with smallest eigenvalue $\lambda_{\min}(\mathscr{M}_\ell)$ having an exponential law with parameter $n/2$, or equivalently a law $\frac{1}{n}\chi^2(2)$. Consequently, by independence, we have

$$\lambda_{\min} \left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right) \geq \sum_{\ell=1}^{\infty} \frac{Y_\ell}{n \beta_{\ell(n+1)}^2} \tag{7.94}$$

with Y_ℓ independent $\chi^2(2)$ random variables. Then using $\beta_k \leq \frac{1}{T} 2\sqrt{2}(3k+1)$ we can write

$$\lambda_{\min} \left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right) \geq T^2 \sum_{\ell=1}^{\infty} \frac{n}{C_3 \ell^2} Y_\ell \tag{7.95}$$

with $C_3 = n^2 \left(2\sqrt{2}(3n+4) \right)^2$. We have

$$\begin{aligned} \operatorname{Tr} \left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right)^{-1} &\leq n \lambda_{\max} \left(\left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right)^{-1} \right) \\ &= n \left(\lambda_{\min} \left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right) \right)^{-1} \end{aligned}$$

Consequently, for $p \in (0,1)$,

$$\left(\operatorname{Tr} \left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right)^{-1} \right)^p \leq \frac{C_3^p}{T^{2p}} \left(\sum_{\ell=1}^{\infty} \frac{Y_\ell}{\ell^2} \right)^{-p} \tag{7.96}$$

implying

$$\begin{aligned} \mathbb{E} \left[\exp \left(\lambda \left(\operatorname{Tr} \left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right)^{-1} \right)^p \right) \right] &\leq \mathbb{E} \left[\exp \left(\frac{C_3^p}{T^{2p}} \lambda \left(\sum_{\ell=1}^{\infty} \frac{Y_\ell}{\ell^2} \right)^{-p} \right) \right] \\ &= 1 + \sum_{q=1}^{\infty} \frac{C_3^{pq} \lambda^q}{q! T^{2pq}} \mathbb{E} \left[\left(\sum_{\ell=1}^{\infty} \frac{Y_\ell}{\ell^2} \right)^{-pq} \right] \\ &\leq 1 + \sum_{q=1}^{\infty} \frac{C_3^{pq} \lambda^q}{q! T^{2pq}} q^{2pq} \frac{\Gamma(q(1-p))}{2^{pq} \Gamma(q)} \end{aligned}$$

where we used Lemma 7.3.5 with $a = pq$. We finally get

$$\mathbb{E} \left[\exp \left(\lambda \left(\operatorname{Tr} \left(\sum_{k=1}^{\infty} \frac{V_k V_k^t}{\beta_k^2} \right)^{-1} \right)^p \right) \right] \leq 1 + \sum_{q=1}^{\infty} \left(\left(\frac{C_3}{2T^2} \right)^p \lambda \right)^q \frac{q^{2pq} \Gamma(q(1-p))}{q! \Gamma(q)} \tag{7.97}$$

which is the first inequality in (7.92). We are left to prove that the right hand side is finite. Using $\ln\Gamma(a) \sim a\ln(a)$ as $a \to \infty$ we get

$$\ln\left(\frac{q^{2pq}\Gamma(q(1-p))}{q!\Gamma(q)}\right) \sim (2pq + q(1-p) - 2q)\ln(q) = q(p-1)\ln(q) < -\varepsilon q\ln(q)$$

with $\varepsilon = \frac{1-p}{2}$. Letting $\alpha = \left(\frac{C_3}{2T^2}\right)^p \lambda$ we have

$$\left(\left(\frac{C_3}{2T^2}\right)^p \lambda\right)^q \frac{q^{2pq}\Gamma(q(1-p))}{q!\Gamma(q)} \leq \alpha^q q^{-\varepsilon q} \quad \text{for } q \text{ sufficiently large}$$

and $\sum_{q=1}^{\infty} \alpha^q q^{-\varepsilon q} < \infty$, proving the finiteness of the right hand side of (7.92). □

Let $f : \mathbb{G}_n \to \mathbb{R}$ a bounded measurable function. We have

$$P_T f(g) = \mathbb{E}[f(\mathbb{B}_T^g)] \quad \text{together with} \quad P_T f(\tilde{g}) = \mathbb{E}[f(\mathbb{B}_T^{\tilde{g}})R(u)] \tag{7.98}$$

$\tilde{g}$ and u being related as in Lemma 7.3.3. But with our construction, we have a.s $\mathbb{B}_T^{\tilde{g}} = \mathbb{B}_T^g$, yielding

$$P_T f(\tilde{g}) = \mathbb{E}[f(\mathbb{B}_T^g)R(u)]. \tag{7.99}$$

From this and Corollary 7.3.4 we get the following log Harnack inequality.

Proposition 7.3.7. *Let f be a positive function in $\mathbb{G}_n$, $T > 0$ and $g = (x, z), \tilde{g} = (\tilde{x}, \tilde{z}) \in \mathbb{G}_n$. Then*

$$\begin{aligned} &P_T(\ln f)(\tilde{g}) \leq \ln(P_T f(g)) \\ &+ \frac{\|x - \tilde{x}\|_2^2}{2T} + \left(6\sqrt{n} + \frac{4}{\sqrt{n}}\right)^2 \left(\frac{1}{T^2}\left\|z - \tilde{z} - \frac{1}{2}x \odot \tilde{x}\right\|_{HS}^2 + \frac{2(n-1)}{3T}\|x - \tilde{x}\|_2^2\right). \end{aligned} \tag{7.100}$$

Proof. Again take $K = 2n + 1$. By Equation (7.98) applied to $\ln f$ and Young inequality,

$$\begin{aligned} P_T(\ln f)(\tilde{g}) &= \mathbb{E}[\ln f(\mathbb{B}_T^g)R(u)] \\ &\leq \mathbb{E}[R(u)\ln R(u)] + \ln \mathbb{E}\left[\exp \ln f(\mathbb{B}_T^g)\right] \\ &= \mathbb{E}[R(u)\ln R(u)] + \ln(P_T f(g)). \end{aligned}$$

We conclude with (7.90). □

The next proposition aims at establishing an integration by parts formula, together with a reverse Poincaré inequality.

Proposition 7.3.8. *Fix $K \geq n + 2$ (and possibly infinite). Let $f : \mathbb{G}_n \to \mathbb{R}$ be a bounded continuous function, $g = (x, z), h = (h_x, h_z) \in \mathbb{G}_n$. Denote $\tilde{g} = g + h$ we have*

$$d_g P_T f(h) = \mathbb{E}\left[f(\mathbb{B}_T^g)\left(-\sum_{k \in J_K}\left\langle \xi_{3k}, \hat{U}_k\right\rangle\right)\right], \tag{7.101}$$

where $(\hat{U}_k)_{k\geq 0}$ is given by (7.73). Consequently, for any $p \in (1, \infty]$, denoting $q \in [1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|d_g P_T f(h)\| \leq (P_T|f|^p)^{1/p}\, m_q \mathbb{E}\left[\left(\sum_{k \in J_K}\left\|\hat{U}_k\right\|^2\right)^{q/2}\right]^{1/q}. \tag{7.102}$$

with $m_q^q = \mathbb{E}[|Z|^q]$ the q-th moment of a $\mathcal{N}(0,1)$-variable Z. The right hand side is finite for all $q \geq 1$ when $K = \infty$.

In the special case $p = q = 2$, we get the reverse Poincaré inequality

$$\begin{aligned}&\|d_g P_T f(h)\|^2 \\ &\leq \left(P_T|f|^2\right)\left(\frac{\|h_x\|_2^2}{T} + \left(6\sqrt{2n} + \frac{4\sqrt{2}}{\sqrt{n}}\right)^2\left(\frac{1}{T^2}\left\|h_z - \frac{1}{2}x \odot h_x\right\|_{HS}^2 + \frac{2(n-1)}{3T}\|h_x\|_2^2\right)\right).\end{aligned} \tag{7.103}$$

Proof. Considering a vector $h = (h_x, h_z) \in \mathbb{G}_n$, we will compute

$$\lim_{a\to 0} \frac{1}{a}\left(P_T f(g + ah) - P_T f(g)\right). \tag{7.104}$$

Denote $\tilde{g}(a) = (\tilde{x}(a), \tilde{z}(a)) = g + ah$. The matrix $W(\tilde{g}(a))$ defined in (7.51) rewrites as

$$W(\tilde{g}(a)) = z - \tilde{z}(a) - \frac{1}{2}\tilde{x}_a \odot x + (x - \tilde{x}(a)) \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right) \tag{7.105}$$

and since $x \odot x = 0$,

$$\forall a \in \mathbb{R}, \quad d_{\tilde{g}(a)}W(h) = \frac{d}{da}W(\tilde{g}(a)) = -h_z - \frac{1}{2}h_x \odot x - h_x \odot \left(\frac{\sqrt{T}}{2}\xi_0 - \sqrt{T}\alpha_0\tilde{\xi}_1\right) \tag{7.106}$$

not depending on a. Consequently, with the notation of (7.76),

$$d_{\tilde{g}(a)}\hat{\mathscr{U}}^t(h) = \hat{\mathscr{V}}^t(\hat{\mathscr{V}}\hat{\mathscr{V}}^t)^{-1}d_{\tilde{g}(a)}\mathscr{W}^t(h) = \hat{\mathscr{V}}^t(\hat{\mathscr{V}}\hat{\mathscr{V}}^t)^{-1}\mathscr{W}^t(\tilde{g}(1)) \tag{7.107}$$

does not depend on a, and we have, letting $\hat{\mathscr{U}}^t = \hat{\mathscr{U}}^t(\tilde{g}(1))$ and $\mathscr{W}^t = \mathscr{W}^t(\tilde{g}(1))$

$$\hat{\mathscr{U}}^t(\tilde{g}(a)) = a\hat{\mathscr{U}}^t = a\hat{\mathscr{V}}^t(\hat{\mathscr{V}}\hat{\mathscr{V}}^t)^{-1}\mathscr{W}^t. \tag{7.108}$$

also, $\hat{U}_0(\tilde{g}(a)) = a\frac{\tilde{x}-x}{\sqrt{T}}$ yielding $\frac{d}{da}\hat{U}_0(\tilde{g}(a)) = u_0$. Then using (7.84) and the fact that $\hat{\mathscr{U}} = (u_3, u_6, \ldots)$ we get

$$\begin{aligned}\frac{1}{a}\left(P_T f(g + ah) - P_T f(g)\right) &= \frac{1}{a}\mathbb{E}\left[f(\mathbb{B}_T^g)\left(R(au) - 1\right)\right] \\ &= \frac{1}{a}\mathbb{E}\left[f(\mathbb{B}_T^g)\left(\int_0^a \frac{d}{da'}R(a'u)\,da'\right)\right] \\ &= -\frac{1}{a}\mathbb{E}\left[f(\mathbb{B}_T^g(\omega))\left(\int_0^a R(a'u)\langle\omega + a'u, u\rangle\,da'\right)\right]\end{aligned}$$

By definition of $R(a'u)$ we have as soon as $\omega \mapsto F(\omega)$ is $\mathbb{P}$-integrable, that $\omega \mapsto F(\omega + a'u)$ is $R(a'u)\mathbb{P}$-integrable and

$$\mathbb{E}[R(a'u)F(\omega + a'u)] = \mathbb{E}[F(\omega)]. \tag{7.109}$$

In our situation f is bounded and $\langle \omega, u\rangle = \left\langle \omega, \frac{u}{\|u\|}\right\rangle \|u\|$ is $\mathbb{P}$-integrable since, since, conditioned to $\mathscr{G}$ $\left\langle \omega, \frac{u}{\|u\|}\right\rangle$ has law $\mathcal{N}(0,1)$, $\|u\| \leq \left\| \frac{h_x}{\sqrt{T}} \right\|_2 + \|\hat{\mathscr{U}}\|_{HS}$,

$$\|\hat{\mathscr{U}}\|_{HS} = \sqrt{\mathrm{Tr}\left(\hat{\mathscr{U}}\hat{\mathscr{U}}^t\right)} = \sqrt{\mathrm{Tr}\left(\mathscr{W}^t(\hat{\mathscr{V}}\hat{\mathscr{V}}^t)^{-1}\mathscr{W}\right)} \leq \|\mathscr{W}\|_{HS}\left(\mathrm{Tr}\left(\left(\hat{\mathscr{V}}\hat{\mathscr{V}}^t\right)^{-1}\right)\right)^{1/2},$$

$\mathscr{W}$ is Gaussian and independent of $\mathscr{V}$ and

- if $K = \infty$ then by Equation (7.92) $\left(\mathrm{Tr}\left(\left(\hat{\mathscr{V}}\hat{\mathscr{V}}^t\right)^{-1}\right)\right)^{1/2}$ has exponential moments,
- if $K < \infty$ then $\left(\mathrm{Tr}\left(\left(\hat{\mathscr{V}}\hat{\mathscr{V}}^t\right)^{-1}\right)\right)^{1/2} \leq \beta_K \left(\mathrm{Tr}\left((\mathscr{V}\mathscr{V}^t)^{-1}\right)\right)^{1/2}$ which is integrable by (7.40), since we chose $K \geq n+2$.

So we can apply equality (7.109) after exchanging the orders of integration (which is allowed here for the same integrability reasons), and we get

$$\begin{aligned}
\frac{1}{a}\left(P_T f(g+ah) - P_T f(g)\right) &= -\frac{1}{a}\int_0^a \mathbb{E}\left[f(\mathbb{B}_T^g(\omega))\left(R(a'u)\left\langle \omega + a'u, u\right\rangle\right)\right] da' \\
&= -\frac{1}{a}\int_0^a \mathbb{E}\left[f(\mathbb{B}_T^g(\omega - a'u))\left\langle \omega, u\right\rangle\right] da' \\
&= -\mathbb{E}\left[\left(\frac{1}{a}\int_0^a f(\mathbb{B}_T^g(\omega - a'u))\, da'\right)\left\langle \omega, u\right\rangle\right].
\end{aligned}$$

Since f is bounded and continuous, and a.s. $\mathbb{B}_T^g(\omega - a'u) \to \mathbb{B}_T^g(\omega)$ as $a' \to 0$ we can use the dominated convergence theorem to obtain

$$\lim_{a\to 0}\frac{1}{a}\left(P_T f(g+ah) - P_T f(g)\right) = -\mathbb{E}\left[f(\mathbb{B}_T^g(\omega))\left\langle \omega, u\right\rangle\right] \tag{7.110}$$

which yields (7.101). To establish (7.102) we first use Hölder inequality which yields

$$\|d_g P_T f(h)\| \leq \mathbb{E}\left[|f|^p(\mathbb{B}_T^g)\right]^{1/p}\mathbb{E}\left[\left|-\sum_{k\in J_K}\left\langle \xi_{3k}, \hat{U}_k\right\rangle\right|^q\right]^{1/q}. \tag{7.111}$$

As in the proof of Corollary 7.3.4, conditioning with respect to $\mathscr{G}$ we get

$$\begin{aligned}
\mathbb{E}\left[\left|-\sum_{k\in J_K}\left\langle \xi_{3k}, \hat{U}_k\right\rangle\right|^q\right] &= \mathbb{E}\left[\mathbb{E}\left[\left|-\sum_{k\in J_K}\left\langle \xi_{3k}, \hat{U}_k\right\rangle\right|^q \middle| \mathscr{G}\right]\right] \\
&= \mathbb{E}\left[\left(\sum_{k\in J_K}\left\|\hat{U}_k\right\|^2\right)^{q/2} m_q^q\right]
\end{aligned}$$

with $m_q^q = \mathbb{E}[|Z|^q]$ the q-th moment of a $\mathcal{N}(0,1)$-variable Z. This proves (7.102). Notice that when $K = \infty$ the last term is finite thanks to Lemma 7.3.6 which implies that all moments of $(\hat{\mathscr{V}}\hat{\mathscr{V}}^t)^{-1}$ are finite. Finally, to prove (7.103) we apply (7.102) with $K = 2n+1$ which allows to use (7.90) with $\tilde{g} = g + h$. □

As a final corollary we provide estimates of the horizontal and vertical differential of the heat kernel $(g,h) \mapsto p_t(g,h)$ on $\mathbb{G}_n$. Note that it will be useful in Chapter 8.

Corollary 7.3.9. *There exist three positive constants $K(n)$, $K_1(n)$ and $K_2(n)$ only depending on n such that:*

$$|d_g p_t(0,\cdot)(h)| \leq t^{-\frac{n^2}{2}} e^{-\frac{K(n)}{t} d_{cc}(0,g)^2} \left(K_1(n) \frac{\|h_x\|_2}{\sqrt{t}} + K_2(n) \frac{\|h_z - \frac{1}{2} x \odot h_x\|_{HS}}{t} \right). \quad (7.112)$$

Proof. From [53], see also [12], there exist some positive constants $\tilde{K}(n)$ and $\tilde{K}_1(n)$ depending on n such that:

$$p_t(g,h) \leq \frac{\tilde{K}_1(n)}{t^{\frac{n^2}{2}}} e^{-\frac{\tilde{K}(n)}{t} d_{cc}(g,h)^2}. \quad (7.113)$$

Set $g, h \in \mathbb{G}_n$ then $p_t(0,g) = P_{\frac{t}{2}}(p_{\frac{t}{2}}(0,\cdot))(g)$. Using the reverse Poincaré inequality from Proposition 7.3.8 with $f = p_{\frac{t}{2}}(0,\cdot)$:

$$\begin{aligned} |d_g p_t(0,\cdot)(h)|^2 &\leq \mathbb{E}\left[p_{\frac{t}{2}}(0, \mathbb{B}^g_{\frac{t}{2}})^2 \right] \\ &\times \left(\frac{2\|h_x\|_2^2}{t} + \left(6\sqrt{2n} + \frac{4\sqrt{2}}{\sqrt{n}} \right)^2 \left(\frac{4}{t^2} \|h_z - \frac{1}{2} x \odot h_x\|_{HS}^2 + \frac{4(n-1)}{3t} \|h_x\|_2^2 \right) \right). \end{aligned} \quad (7.114)$$

We now examine $\mathbb{E}\left[p_{\frac{t}{2}}(0, \mathbb{B}^g_{\frac{t}{2}})^2 \right]$. Using (7.113):

$$\begin{aligned} \mathbb{E}\left[p_{\frac{t}{2}}(0, \mathbb{B}^g_{\frac{t}{2}})^2 \right] &= \int_{\mathbb{G}_n} p_{\frac{t}{2}}(0,l)^2 p_{\frac{t}{2}}(g,l) dl \\ &\leq \int_{\mathbb{G}_n} \tilde{K}_1(n)^3 \left(\frac{t}{2} \right)^{-\frac{3n^2}{2}} e^{-\frac{2\tilde{K}(n)}{t}(2d_{cc}(0,l)^2 + d_{cc}(g,l)^2)} dl \\ &\leq \int_{\mathbb{G}_n} \tilde{K}_1(n)^3 \left(\frac{t}{2} \right)^{-\frac{3n^2}{2}} e^{-\frac{2\tilde{K}(n)}{t}\left(2d_{cc}(0,l)^2 + (d_{cc}(g,0) - d_{cc}(0,l))^2\right)} dl \\ &\leq \tilde{K}_1(n)^3 \left(\frac{t}{2} \right)^{-\frac{3n^2}{2}} \int_{\mathbb{G}_n} e^{-\frac{2\tilde{K}(n)}{t} d_{cc}(0,l)^2} dl \, e^{-\frac{\tilde{K}(n)}{t} d_{cc}(0,g)^2} \end{aligned} \quad (7.115)$$

where the last expression is obtained by using the inequality

$$\begin{aligned} 2a^2 + (a-b)^2 = 3a^2 + b^2 - 2ab &\geq 3a^2 + b^2 - \frac{a^2}{\lambda} - \lambda b^2 \\ &= a^2 + \frac{b^2}{2} \text{ with } \lambda = 1/2. \end{aligned}$$

Using the property of the dilation on $(\mathbb{G}_n, d_{cc})$, $\frac{1}{\sqrt{t}} d_{cc}(0,l) = d_{cc}(0, \mathrm{dil}_{\frac{1}{\sqrt{t}}}(l))$, and since the homogeneous dimension of $\mathbb{G}_n$ is n^2, we have:

$$\left(\frac{t}{2} \right)^{-\frac{n^2}{2}} \int_{\mathbb{G}_n} e^{-\frac{2\tilde{K}(n)}{t} d_{cc}(0,l)^2} dl = \int_{\mathbb{G}_n} e^{-2\tilde{K}(n) d_{cc}(0,l)^2} dl$$

which is finite and does not depend on t . The expected result follows. □

Remark 7.3.10. *If in the above proof, using the reverse Poincaré inequality* (7.102) *with* $p = 1 + v$ *for* $v > 0$ *(and with* $K = +\infty$*), it is possible to obtain* (7.112) *with some constants* $K(n, v)$*,* $K_1(n, v)$ *and* $K_2(n, v)$ *depending only on* n *and on* v *with*

$$K(n, v) = \frac{\tilde{K}(n)}{1 + v}$$

and where $K_1(n, v)$ *and* $K_2(n, v)$ *tend to infinity as* $v \to 0$*.*

Chapter 8

Couplings and Wasserstein distance

8.1 Introduction to the topic

In all this manuscript, we have looked for successful couplings or, at least, couplings that give estimates of the total variation distance. They are part of the first category of coupling presented in Subsection 2.1.2. We recall that the second category proposed in the same subsection is about couplings $(X_t, \tilde{X}_t)_t$ satisfying inequalities of the type:

$$d(X_t, \tilde{X}_t) \leq C(t) d(X_0, \tilde{X}_0) \text{ for all } t \geq 0 \, a.s.. \tag{8.1}$$

This result implies for all $t > 0$:

$$\mathbb{E}\left[d(X_t, \tilde{X}_t)^p\right]^{\frac{1}{p}} \leq C(t) d(X_0, \tilde{X}_0) \tag{8.2}$$

and thus the Wasserstein distance inequality for all $p \in [1, +\infty]$ and $t > 0$:

$$W_p\left(\mathcal{L}(X_t), \mathcal{L}(\tilde{X}_t)\right) \leq C(t) d(X_0, \tilde{X}_0). \tag{8.3}$$

As explained previously, results on Riemannian manifolds with Ricci curvature bounded below by k are well known: using the synchronous coupling, (8.1) is satisfied for $C(t) = e^{-\frac{kt}{2}}$; see [54], (the reader can also look at the results for the synchronous coupling in Subsection 4.1.4 in the case of Riemannian manifold with constant curvature).

Under strong conditions on the considered manifold and for $t \geq 0$ and $p \in [1, +\infty]$ fixed, Kuwada proved in [41] the existence of a relation of duality between the inequality (8.3) for all $x, \tilde{x}$ and the following gradient inequalities for any f bounded and Lipschitz:

- if $p \in]1, +\infty]$, by denoting by $q \geq 1$ the conjugate of p ($q = 1$ if $p = +\infty$), for all x:

$$|\nabla P_t f|(x) \leq C(t) \left(P_t\left(|\nabla f|^q\right)\right)^{\frac{1}{q}}(x) \tag{8.4}$$

- if $p = 1$:

$$||\nabla P_t f||_\infty \leq C(t) \| P_t\left(|\nabla f|\right) \|_\infty. \tag{8.5}$$

In these relations, for any f bounded and lipschitz, $|\nabla f|(x)$ is the gradient length associated to the metric defined the same way as in Subsection 3.1.6 and $||\nabla P_t f||_\infty$ is the supremum norm associated.

As far as we know, there is few studies on subRiemannian manifolds. Using properties of the carré du champs opérateur and the duality relation, Kuwada, proved that (8.3) and thus (8.2) is true on the following subRiemannian manifold (still considering the Brownian motion and the heat semi group):

- On the Heisenberg group for $p = +\infty$ (and thus for all $p \in [1, +\infty]$) with $C(t) > 0$ constant.
- On the nilpotent Lie groups, including the free, step 2 Carnot groups, for $p \in [1, +\infty[$ with $C_p(t) > 0$ constant.
- On $SU(2)$ for $p \in [1, +\infty[$ with $C_p(t) = C_p e^{-t}$.

One of the question that can be considered is then the following one: can we obtain explicit coupling satisfying (8.2) or even (8.1)?

A thorough study was made by Bonnefont and Juillet for the Heisenberg group in [22]. In particular they proved that, for $C(t) \equiv C > 0$ constant, (8.1) cannot be obtain with any co-adapted couplings of Brownian motions. They also gave an explicit construction of a coupling $(\mathbb{B}_t, \tilde{\mathbb{B}}_t)$ at each time $t > 0$ satisfying (8.2) for $p = 1$. As the obtained construction is not dynamic, it is called *static coupling*. This coupling is constructed by considering the heat kernel and thus considering the problem as a transport problem. In fact, this method can be well generalised to free, step 2 Carnot groups and then to all homogeneous, step 2 Carnot groups as it will be proven in Theorem 8.2.1.

If the method used to prove these results is different from the one developed in this manuscript (decomposition of the Brownian motion in a basis of $L^2[0,T]$), the common message seems to be that non co-adapted couplings should be the good way to obtain gradient inequalities for hypoelliptic operator. In the future, it would be interesting to see if the method of decomposition of the Brownian motion in a basis of $L^2[0,T]$ provides result to obtain (8.1) or at least (8.2).

8.2 The static coupling on homogeneous Carnot groups

We give here the generalisation of the static coupling from [22] to all homogeneous, step 2 Carnot groups.

Let $\mathbb{G}$ be an homogeneous, step 2 Carnot group. We denote by μ_t^g the law of the Brownian motion at time $t \geq 0$, starting at $g \in \mathbb{G}$.

Theorem 8.2.1. *Let $\mathbb{G}$ be an homogeneous, step 2 Carnot group, $t \geq 0$ and $g = (x, z)$, $\tilde{g} = (\tilde{x}, \tilde{z}) \in \mathbb{G}$. There exists a random vector $\left((X, Z), (\tilde{X}, \tilde{Z})\right)$ such that:*

- $\mathcal{L}(X, Z) = \mu_t^g$ *and* $\mathcal{L}\left(\tilde{X}, \tilde{Z}\right) = \mu_t^{\tilde{g}}$;
- $\tilde{X} - X = \tilde{x} - x$;
- *we have the Wasserstein distance inequality:*

$$W_1(\mu_t^g, \mu_t^{\tilde{g}}) \leq \mathbb{E}\left[d_{cc}\left((X, Z), (\tilde{X}, \tilde{Z})\right)\right] \leq C d_{cc}(g, \tilde{g})$$

with C a constant that only depends on the rank n of $\mathbb{G}$.

As explained above, this theorem has been proven on the Heisenberg group, that is for $\mathbb{H} \approx \mathbb{G}_2$, by Bonnefont and Juillet in [22]. The construction is using upper-bounds for the heat kernel and its differential as well as a Transport Lemma for real density measures with increasing concave cost functions (*Lemma 5.2* [22]). We first present the generalisation in all dimension of this Transport Lemma:

Lemma 8.2.2. *Consider $d \in \mathbb{N}$, η a probability measure on $\mathbb{R}^d$. For $v \in \mathbb{R}^d$, define the probability measure $(trans_v)_{\#}\eta : A \mapsto \eta(A - v)$. Suppose that η has a smooth density f rapidly decreasing and such that, $x \mapsto d_x f(v)$ is rapidly decreasing too. Then:*

$$\inf_{\pi \in \Pi(\eta, (trans_v)_{\#}\eta)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \sqrt{|y - x|} d\pi \leq \int_{\mathbb{R}^d} |\langle df(x), v\rangle| \sqrt{\|x\|_2} dx$$

Proof. We are looking for a upper bound of the 1-Wasserstein distance between $(\eta, (trans_v)_{\#}\eta)$ using the cost function $(x, y) \mapsto \sqrt{\|x - y\|_2}$. We denote it here d_W^1 to avoid confusions with the Wasserstein distance W_1 associated to the Carnot-Carathéodory metric on $\mathbb{G}_n$. As the cost function is a distance, d_W^1 defines a distance on the set of the positive measures. Using the Rubinstein-Kantorovich formula recall that, for any probability measure σ^+, σ^-, $d_W^1(\sigma^+, \sigma^-) = ||\sigma^+ - \sigma^-||_1$ with $|| \cdot ||_1$ the Kantorovich norm defined on the set of the signed Radon measure with mass 0.

Set $\sigma = \eta - (trans_v)_{\#}\eta$. Note that $(trans_v)_{\#}\eta$ is a measure with density $x \mapsto f(x - v)$. Then, the densities of η and $(trans_v)_{\#}\eta$ are both rapidly decreasing and σ is a signed Radon measure with mass 0. We then just need to upper-bound $\|\sigma\|_1$.

By smoothness of f, for any Borel set A:

$$\begin{aligned}\sigma(A) &= (trans_0)_{\#}\eta(A) - (trans_v)_{\#}\eta(A) = \int_A (f(x) - f(x - v))dx \\ &= \int_A \int_0^1 \langle df(x - sv), v\rangle ds dx = \int_0^1 \gamma_{s,v}(A) ds\end{aligned}$$

with $\gamma_{s,v} := \langle df(x - sv), v\rangle dx$. As previously, the mass of $\gamma_{s,v}$ is 0. Then:

$$||\sigma||_1 = || \int_0^1 \gamma_{s,v} ds||_1 \leq \int_0^1 ||\gamma_{s,v}||_1 ds. \tag{8.6}$$

We now study the quantity $||\gamma_{s,v}||_1$. For any $s \in [0, 1]$, $p \in \mathbb{R}^d$, the signed measure $\gamma_{s,v}$ has a unique decomposition (Jordan decomposition) as the difference of two positive measures: $\gamma_{s,v} = \gamma_{s,v}^+ - \gamma_{s,v}^-$. In particular the density of $\gamma_{s,v}^+$ is given by

$$\langle df(x - sv), v\rangle^+ = \max(0, \langle df(x - sv), v\rangle)$$

and the density of $\gamma_{s,v}^-$ is given by

$$\langle df(x - sv), v\rangle^- = -\min(0, \langle df(x - sv), v\rangle).$$

As $\gamma_{s,v}$ has mass 0, $\gamma_{s,v}^+$ and $\gamma_{s,v}^-$ have the same mass $m_{s,v}$. Using the Wasserstein distance:

$$\begin{aligned}||\gamma_{s,v}||_1 &\leq m_{s,v} d_W^1 \left(\frac{\gamma_{s,v}^+}{m_{s,v}}, \frac{\gamma_{s,v}^-}{m_{s,v}}\right) \\ &\leq m_{s,v} \left(d_W^1 \left(\frac{\gamma_{s,v}^+}{m_{s,v}}, \delta_{sv}\right) + d_W^1 \left(\delta_{sv}, \frac{\gamma_{s,v}^-}{m_{s,v}}\right)\right).\end{aligned}$$

Then:

$$\begin{aligned} d_W^1\left(\frac{\gamma_{s,v}^-}{m_{s,v}}, \delta_{sv}\right) &= \inf_{\pi \in \Pi\left(\frac{\gamma_{s,v}^+}{m_{s,v}}, \delta_{sv}\right)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \sqrt{\|y-x\|_2} d\pi(x,y) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \sqrt{\|y-x\|_2} d\left(\frac{\gamma_{s,v}^-}{m_{s,v}} \otimes \delta_{sv}\right)(x,y) \\ &= \frac{1}{m_{s,v}} \int_{\mathbb{R}^d} \sqrt{\|sv-x\|_2} \langle df(x-sv), v\rangle^- dx \\ &= \frac{1}{m_{s,v}} \int_{\mathbb{R}^d} \sqrt{\|x\|_2} \langle df(x), v\rangle^- dx. \end{aligned}$$

We also get $d_W^1\left(\frac{\gamma_{s,v}^+}{m_{s,v}}, \delta_{sv}\right) = \frac{1}{m_{s,v}} \int_{\mathbb{R}^d} \sqrt{\|x\|_2} \langle df(x), v\rangle^+ dx$. Finally,

$$||\gamma_{s,v}||_1 \leq \int_{\mathbb{R}^d} \sqrt{\|x\|_2} |\langle df(x), v\rangle| dx. \tag{8.7}$$

The result ensues from (8.6) and (8.7). □

The following Proposition is the key to construct the static coupling:

Proposition 8.2.3. *Set $a \in \mathbb{G}_n$. There exists a coupling $((X,U),(X,V))$ on $\mathbb{G}_n \times \mathbb{G}_n$ such that:*

- $\mathcal{L}(X,U) = \mu_t^0$;
- $\mathcal{L}(X,V) = \mathcal{L}(X, U + a \odot X)$;
- $\mathbb{E}\left[\sqrt{\|U-V\|_2}\right] \leq C(n)\|a\|_2$ *with $C(n)$ a constant depending neither on a nor on t.*

Proof of Proposition 8.2.3. Let $(X,Z) \in \mathbb{G}_n$ be a random vector with distribution μ_t^0. For $x, a \in \mathbb{R}^n$, define $\eta(x) := \mathcal{L}(Z|X=x)$ and, $\eta(x,a) := (trans_{a \odot x})_\# \eta(x)$. Identifying $\mathfrak{so}(n)$ with $\mathbb{R}^{n(n-1)}$, we apply Lemma 8.2.2 to $\eta(x)$ with $v = a \odot x$. Then there exists (U,V) on $\mathfrak{so}(n) \times \mathfrak{so}(n)$ such that $\mathcal{L}(U|X=x) = \eta(x)$, $\mathcal{L}(V|X=x) = \eta(x,a)$ and

$$\mathbb{E}\left[\sqrt{\|U-V\|_2}\,|X=x\right] \leq \int_{\mathfrak{so}(n)} |\langle df_{Z|X=x}(z), a \odot x\rangle| \sqrt{\|z\|_2} dz.$$

In particular, $\mathcal{L}(X,U) = \mu_t^0$ and $\mathcal{L}(X,V) = \mathcal{L}(X, U + a \odot X)$. Moreover:

$$\begin{aligned} \mathbb{E}\left[\sqrt{\|U-V\|_2}\right] &= \int_{\mathbb{G}_n} \left|\langle d\left(\frac{f_{(X,Z)}(x,z)}{f_X(x)}\right), (0, a \odot x)\rangle\right| \sqrt{\|z\|_2} f_X(x) d(x,z) \\ &= \int_{\mathbb{G}_n} \left|\langle df_{(X,Z)}(x,z), (0, a \odot x)\rangle\right| \sqrt{\|z\|_2} d(x,z). \end{aligned}$$

As $f_{(X,Z)} = p_t(0,\cdot)$, using Proposition 7.3.9:

$$\begin{aligned} \mathbb{E}\left[\sqrt{\|U-V\|_2}\right] &\leq K_2(n) t^{-\frac{n^2}{2}} \int_{\mathbb{G}_n} \sqrt{\|z\|_2} e^{-\frac{K(n)}{t} d_{cc}((0,0),(x,z))^2} \frac{\|a \odot x\|_{HS}}{t} d(x,z) \\ &\leq \sqrt{2}\|a\|_2 K_2(n) t^{-\frac{n^2}{2}} \int_{\mathbb{G}_n} \sqrt{\frac{\|z\|_2}{t}} e^{-K(n) d_{cc}((0,0), \mathrm{dil}_{\frac{1}{\sqrt{t}}}(x,z))^2} \frac{\|x\|_2}{\sqrt{t}} d(x,z) \\ &= \sqrt{2}\|a\|_2 K_2(n) \int_{\mathbb{G}_n} \sqrt{\|z\|_2} e^{-K(n) d_{cc}((0,0),(x,z))^2} \|x\|_2 d(x,z). \end{aligned}$$

In particular the integral in the last term is finite and does not depend on t. □

We can now give the proof of Theorem 8.2.1.

Proof of Theorem 8.2.1. We first give the proof for the free, step 2 Carnot groups $\mathbb{G}_n$ for all $n \geq 2$. As for the proof on the Heisenberg group, we can use the left-invariance of the subRiemannian structure to suppose that $g = 0$ and $\tilde{g} = (a, b)$.

Let $(X, Z) \in \mathbb{G}_n$ a random vector having the distribution μ_t^0 and consider $g' = (a, 0) \in \mathbb{G}_n$. Note that

$$g' \star (X, Z) \star (g')^{-1} = (X, Z + a \odot X).$$

We define (U, V) as in Proposition 8.2.3, $\mathbb{B} := (X, U) \in \mathbb{G}_n$ and $\tilde{\mathbb{B}} := (0, b) \star (X, V) \star g' \in \mathbb{G}_n$. By construction $\mathcal{L}(X, V) = \mathcal{L}(g' \star (X, Z) \star (g')^{-1})$. Thus:

$$\mathcal{L}(\tilde{\mathbb{B}}) = \mathcal{L}\left((0, b) \star g' \star (X, Z) \star (g')^{-1} \star g'\right)$$

with $(0, b) \star g' = (a, b) = \tilde{g}$. Thus, $\mathcal{L}(\mathbb{B}) = \mu_t^0$ and $\mathcal{L}(\tilde{\mathbb{B}}) = \mu_t^{\tilde{g}}$. Using the left-invariant properties of the Carnot-Carathéodory distance and the estimates (3.6) and (3.16) of the Carnot-Carathéodory distance, we have:

$$\begin{aligned}
\mathbb{E}\left[d_{cc}(\mathbb{B}, \tilde{\mathbb{B}})\right] &\leq \mathbb{E}\left[d_{cc}\left((X, U), (X, V)\right)\right] + \mathbb{E}\left[d_{cc}\left((X, V), (0, b) \star (X, V)\right)\right] \\
&\quad + \mathbb{E}\left[d_{cc}\left((0, b) \star (X, V), (0, b) \star (X, V) \star g'\right)\right] \\
&\leq \mathbb{E}\left[d_{cc}\left((0, 0), (0, V - U)\right)\right] + \mathbb{E}\left[d_{cc}\left((0, 0), (0, b)\right)\right] + d_{cc}\left(0, g'\right) \\
&\leq \sqrt{2\pi}(2n)^{\frac{3}{4}}\left(\mathbb{E}\left[\sqrt{\|V - U\|_2}\right] + \sqrt{\|b\|_2}\right) + \|a\|_2.
\end{aligned}$$

By construction of (U, V):

$$\mathbb{E}\left[d_{cc}(\mathbb{B}, \tilde{\mathbb{B}})\right] \leq \sqrt{2\pi}(2n)^{\frac{3}{4}}\left(C(n)\|a\|_2 + \sqrt{\|b\|_2}\right) + \|a\|_2$$

with $C(n)$ as in Corollary 8.2.3. As $\boldsymbol{\delta}(0, g') = \sqrt{\|a\|_2^2 + \|b\|_2}$, using the equivalence between the homogeneous norms we obtain the expected result for free, step 2 homogeneous Carnot groups.

Consider now a homogeneous, step 2, Carnot group $\mathbb{G}$ of rank n and $a, \tilde{a} \in \mathbb{G}$. Using the epimorphism ϕ from Proposition 3.1.4 , there exist $g, \tilde{g} \in \mathbb{G}_n$ such that $d_{cc}(a, \tilde{a}) = d_{cc}(g, \tilde{g})$. In particular, if we denote by $\left((X, Z), (\tilde{X}, \tilde{Z})\right)$ the static coupling of Brownian motions on $\mathbb{G}_n$ at time t and starting from $(a, \tilde{a})$, then, by Proposition 3.1.4, $\left(\phi(X, Z), \phi(\tilde{X}, \tilde{Z})\right)$ has the same distribution as a Brownian motion on $\mathbb{G}$ at time t starting from $(g, \tilde{g})$. Moreover:

$$\mathbb{E}\left[d_{cc}\left(\phi(X, Z), \phi(\tilde{X}, \tilde{Z})\right)\right] \leq d_{cc}\left((X, Z), (\tilde{X}, \tilde{Z})\right) \leq C d_{cc}(a, \tilde{a}) = d_{cc}(g, \tilde{g}).$$

This ends the proof of the Theorem. □

With this static coupling, we can obtain an estimate for the gradient of the heat semi-group:

Corollary 8.2.4. *Let $\mathbb{G}$ be an homogeneous, step 2 Carnot group. There exists a constant $C > 0$ only depending of the rank n of $\mathbb{G}$, such that for any f bounded and Lipschitz on $\mathbb{G}$ and for all $t \geq 0$:*

$$||\nabla P_t f||_\infty \leq C P_t(||\nabla f||_\infty).$$

Here $\|\nabla f\|_\infty = \sup_{g\in\mathbb{G}} |\nabla f|(g)$ with $|\nabla f|(g)$ the gradient length associated to the sub-Riemannian metric as defined in 3.19.

Proof. Let $t > 0$ et $((X, Z), (\tilde{X}, \tilde{Z}))$ the static coupling of the Brownian motions at time t starting at $(g, \tilde{g})$ constructed in Theorem 8.2.1.
Note that, by Chow's Theorem, for any a, $\tilde{a} \in \mathbb{G}$, there exists a horizontal curves γ with constant length such that $\gamma(0) = a$, $\gamma(1) = \tilde{a}$ and minimising the Carnot Carathéodory distance $d_{cc}(a, \tilde{a})$ (that is, a geodesic). Then:

$$\begin{aligned}
|f(a) - f(\tilde{a})| = |f(\gamma(0)) - f(\gamma(1))| &= \int_0^1 |\nabla f(\gamma(t)) \cdot \dot{\gamma}(t)| dt \\
&\leq \int_0^1 | \sum_{1\leq i\leq n} \nabla f(\gamma(t)) \cdot \bar{X}_i(\gamma(t)) \dot{\gamma}_i(t)| dt \\
&\leq ||\nabla f||_\infty \int_0^1 \sqrt{\sum_{1\leq i\leq n} \dot{\gamma}_i(t)^2} dt = ||\nabla f||_\infty \int_0^1 ||\dot{\gamma}(t)||_{\mathcal{H}} dt \\
&= ||\nabla f||_\infty d_{cc}(a, a')
\end{aligned}$$

Finally:

$$\begin{aligned}
|P_t f(g) - P_t f(\tilde{g})| = |\mathbb{E}\left[f(X, Z) - f\left(\tilde{X}, \tilde{Z}\right)\right]| &\leq \mathbb{E}\left[||\nabla f||_\infty d_{cc}\left((X, Z), (\tilde{X}, \tilde{Z})\right)\right] \\
&\leq C||\nabla f||_\infty d_{cc}(g, \tilde{g})
\end{aligned}$$

with C the constant from Theorem 8.2.1.
Then $\frac{|P_t f(g) - P_t f(\tilde{g})|}{d_{cc}(g,\tilde{g})} \leq C||\nabla f||_\infty$. By passing to the limit, we obtain the expected inequality. □

We get here the duality between the gradient estimates and the Wasserstein control for $p = 1$ as explained in Section 8.1.

Chapter 9

Conclusion and outcomes

The first aim of this thesis was the study of successful couplings in $SU(2)$ and in a weaker sense on $SL(2,\mathbb{R})$. We have obtained the existence and the explicit construction of two methods on $SU(2)$: one adapted and one co-adapted. The non co-adapted strategy is also successful on $SL(2,\mathbb{R})$ for Brownian motion starting from the same fiber. For the non co-adapted strategy on $SU(2)$ and on $SL(2,\mathbb{R})$, we have computed the coupling rate and obtained the estimate $\mathbb{P}(\tau > t) \leq Ce^{-ct}d_{cc}(g,\tilde{g})$ with $C, c > 0$ some constant independent of the starting points (with g and $\tilde{g}$ in the same fiber in the case of $SL(2,\mathbb{R})$). From this result we have deduced some gradient inequalities.

The results are promising but not entirely satisfying as we have not been able to obtain estimates for the constants C and c. Here the constants are mostly obtained by using the compactness of the fibers. We hope that, in the future, a better understanding of the random variable $K(T)$ used in Chapter 5 could solve this problem. In particular this could help us to state on the efficiency of this coupling method. Moreover, if $K(T)$ have good properties of concentration around the value 0 (for instance something comparable to the behaviour of a standard Gaussian variable), we should be able to obtain a successful coupling on the universal covering of $SL(2,\mathbb{R})$ which could be more relevant for the study of the model spaces in subRiemannian manifolds.

We recall that, for the cases of $SU(2)$ and $SL(2,\mathbb{R})$, in [44], Luo and Neel developed another non co-adapted coupling strategy and obtained comparable results for the coupling rate. If they have been able to state that this alternative method provides an efficient coupling, they have not obtained explicit constants neither, as they also used the compactness of $SU(2)$ and $SL(2,\mathbb{R})$. Note that they method provides an efficient coupling with estimates of the constant on the universal covering of $SL(2,\mathbb{R})$, which, for instance, is better than our results.

The second part of this thesis is about the free, step 2 Carnot groups $\mathbb{G}_n$, $n \geq 2$. We have obtained two non co-adapted strategies inducing similar estimates of the total variation distance with explicit constants. We also obtained different gradient inequalities from these two strategy. In the two cases, the solution come from the knowledge on the random matrices with independent Gaussian entries and on the Wishart matrices.

For now, the One sweep coupling offers better coupling rates. Aside for the horizontal and vertical gradient estimates for the heat semi group directly obtained from the coupling rate, the other results are quite different and, then not really comparable.

Be that as it may, the gradient results induced by the One Sweep coupling are quite promising and should be more explored. In the future, it could be interesting to see if this

method could be applied on $SU(2)$ and $SL(2,\mathbb{R})$ even though the difficulty brought by the random variable $K(T)$ in the present work should likely appear with the One sweep strategy.

In this work we have only dealt with subRiemannian manifold whose horizontal bundle are 2-step bracket generating. It could then be interesting to see if the coupling strategies can be generalised with higher steps, starting with the Brownian motions on the Engel group for example. In this case the Brownian motion can be written under the form $(X_t, z_t)_t$ where $(X_t)_t = (X_t^1, X_t^2)_t$ is a 2-dimensional Brownian motion and where the driven process is $z_t := \left(\int_0^t X_s^1 dX_s^2, \frac{1}{2}\int_0^t (X_s^1)^2 dX_s^2\right)$. As for the free, step 2 Carnot group, the difficulty in the construction of a coupling comes from the necessity to deal with a multiple dimensional driven process. Thus, a similar method would lead to an inverse problem that seems, however, more difficult to solve.

Another lead for future research could be to look at other types of hypoelliptic diffusions. As an example, a finite-looking-ahead coupling have been constructed to obtain efficient couplings for Kolmogorov diffusions (see [8]). With the same idea, it could be interesting to study efficient or One sweep couplings of the kinetic Brownian motion and/or of the kinetic Langevin diffusion. In this case, the Brownian motion can be written $(X_t, z_t)_t$ with X_t the velocity of a particle and z_t its position. In particular, contrary to the subRiemannian case considered here, z_t has finite variations. Note that, in the case of the kinetic Langevin diffusion, under good conditions on a chosen semi-metric ρ, a previous work from Eberle (see [29]) uses a co-adapted coupling to obtain Wasserstein distance inequalities. As such inequalities have a lot of practical applications, it is a big issue to improve them. However, even in the case of the kinetic Brownian motion, the definition of a One sweep coupling is not easy as it seems to be reduced to a non linear inverse problem.

Appendix A

Standard Brownian motion - First hitting time, first exit time

Let $(W_t)_t$ be a one dimensional Brownian motion starting at 0. The results we give here are well known and can be found in numerous references. As they are often used in this work to obtain estimates of the coupling rates, we give their poof.

We first begin with relations about first hitting time.

Lemma A.0.1. *Let $a \in \mathbb{R}$. We denote $D_a := \inf\{t > 0 \mid W_t = a\}$, the first hitting time of a by $(W_t)_t$. We have, for all $t > 0$:*

$$\mathbb{P}(D_a > t) \leq \left(\sqrt{\frac{2}{\pi}}\frac{|a|}{\sqrt{t}}\right) \wedge 1.$$

Proof. The density of D_a is well known, given by $g_a(u) = \frac{|a|}{\sqrt{2\pi u^3}} \exp(-\frac{a^2}{2u}) \times \mathbb{1}_{[0,+\infty[}(u)$. Upper bounding the exponential part in this density we get for all $t > 0$:

$$\begin{aligned}
\mathbb{P}(D_a > t) &= \int_t^{+\infty} \frac{|a|}{\sqrt{2\pi}} \exp(-\frac{a^2}{2u}) u^{-3/2} du \\
&\leq \int_t^{+\infty} \frac{|a|}{\sqrt{2\pi}} u^{-3/2} du \\
&= \frac{|a|}{\sqrt{2\pi}} [-2u^{-1/2}]_t^{+\infty} \\
&= 2\frac{|a|}{\sqrt{2\pi t}}
\end{aligned}$$

□

We now list some relations involving the first exit time of a Brownian motion from an open set:

Lemma A.0.2. *We set two reals a and b such that $a < 0 < b$ and*

$$H_{a,b} = \inf\{t > 0 | W_t \notin]a, b[\}.$$

Then, for $\delta > 0$, we get:

$$\mathbb{E}[e^{-\delta H_{a,b}}] = \frac{\cosh\left(\sqrt{\frac{\delta}{2}}(a+b)\right)}{\cosh\left(\sqrt{\frac{\delta}{2}}(b-a)\right)}; \tag{A.1}$$

$$\mathbb{E}[e^{\delta H_{a,b}}] = \frac{\cos\left(\sqrt{\frac{\delta}{2}}(a+b)\right)}{\cos\left(\sqrt{\frac{\delta}{2}}(b-a)\right)} \text{ if } \sqrt{\frac{\delta}{2}}(b-a) \in \left]0, \frac{\pi}{2}\right[; \tag{A.2}$$

$$\mathbb{E}[H_{a,b}e^{\delta H_{a,b}}] \leq \frac{-ab}{\cos^2\left(\sqrt{\frac{\delta}{2}}(b-a)\right)} \text{ if } \sqrt{\frac{\delta}{2}}(b-a) \in \left]0, \frac{\pi}{2}\right[; \tag{A.3}$$

$$\mathbb{E}[H_{a,b}] = -ab; \tag{A.4}$$

$$\mathbb{P}(H_{a,b} = D_b) = \frac{-a}{b-a}. \tag{A.5}$$

Proof of Lemma (A.0.2). For $\lambda \in \mathbb{C}$, let's define $M_t^\lambda := \exp(\lambda(\frac{a+b}{2} - W_t) - \frac{\lambda^2}{2}t)$ and $N_t^\lambda := \exp(\lambda(W_t - \frac{a+b}{2}) - \frac{\lambda^2}{2}t)$. By Ito's formula, we get $dM_t^\lambda = -\lambda M_t dW_t$ and $dN_t^\lambda = \lambda N_t dW_t$, thus $\left(M_t^\lambda\right)_t$ and $\left(N_t^\lambda\right)_t$ are two local martingales. For all $T > 0$, we obtain:

$$\mathbb{E}[M^\lambda_{H_{a,b}\wedge T} + N^\lambda_{H_{a,b}\wedge T}] = \mathbb{E}[M_0^\lambda + N_0^\lambda] = e^{\lambda(\frac{a+b}{2})} + e^{-\lambda(\frac{a+b}{2})} = 2\cosh\left(\frac{a+b}{2}\right). \tag{A.6}$$

On the other hand, we have:

$$\mathbb{E}[M^\lambda_{H_{a,b}}] = e^{\lambda\frac{b-a}{2}}\mathbb{E}[e^{-\frac{\lambda^2}{2}H_{a,b}}\mathbb{1}_{\{W_{H_{a,b}}=a\}}] + e^{-\lambda\frac{b-a}{2}}\mathbb{E}[e^{-\frac{\lambda^2}{2}H_{a,b}}\mathbb{1}_{\{W_{H_{a,b}}=b\}}]$$
$$\text{and } \mathbb{E}[N^\lambda_{H_{a,b}}] = e^{-\lambda\frac{b-a}{2}}\mathbb{E}[e^{-\frac{\lambda^2}{2}H_{a,b}}\mathbb{1}_{\{W_{H_{a,b}}=a\}}] + e^{\lambda\frac{b-a}{2}}\mathbb{E}[e^{-\frac{\lambda^2}{2}H_{a,b}}\mathbb{1}_{\{W_{H_{a,b}}=b\}}].$$

Summing the two results, we obtain: $\mathbb{E}[M^\lambda_{H_{a,b}} + N^\lambda_{H_{a,b}}] = 2\cosh\left(\lambda\frac{b-a}{2}\right)\mathbb{E}[e^{-\frac{\lambda^2}{2}H_{a,b}}]$.

- To obtain the first relation (A.1) of the Lemma, we choose $\lambda = \sqrt{2\delta}$. We can then apply the dominated convergence Theorem to (A.6) to obtain $\mathbb{E}[M^\lambda_{H_{a,b}} + N^\lambda_{H_{a,b}}] = 2\cosh\left(\frac{a+b}{2}\right)$. This gives the attended result.
- To obtain the second relation (A.2), we choose $\lambda = i\sqrt{2\delta}$. This time we obtain the attended result by using the convergence monotone Theorem.

To deal with (A.3), we just need to differentiate the two sides of equality (A.2) in δ. We deduce (A.4) by evaluating the limit of (A.3) in $\delta = 0$.
Finally, for the last equality, using martingale properties of the Brownian motion we have $\mathbb{E}[W_{H_{a,b}}] = 0$. We compare it to:

$$\mathbb{E}[W_{H_{a,b}}] = a\mathbb{P}(H_{a,b} = D_a) + b\mathbb{P}(H_{a,b} = D_b) = a + (b-a)\mathbb{P}(H_{a,b} = D_b).$$

□

Appendix B

Derivatives of the moving frames for co-adapted couplings in Riemannian manifolds

In this appendix, we aim to compute the covariant Itô differential for the moving frames $(e_1^X(t), e_2^X(t))_t$ and $(e_1^Y(t), e_2^Y(t))_t$ to obtain the System from Remark 4.1.7. For an introduction to Covariant stochastic calculus in a manifold, we refer the reader to [47].

We can define the covariant derivative of the moving frame $e_1^X(t)$ defined above $(X_t)_t$ by $D_t e_1^X(t) := /\!\!/_t d(/\!\!/_t^{-1} e_1^X(t))$ with $/\!\!/_t^{-1}$ the horizontal lift of $(X_t)_t$, that can be seen as the parallel transport along $(X_t)_t$. We then get:

Proposition B.0.1. *Under the hypothesis of Proposition 4.1.5 and using the same notations, we have:*

$$\begin{cases} D_t e_1^X(t) = \sqrt{k}(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)})e_2^X(t) - \frac{k}{2}\left(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_1^X(t) \\ D_t e_2^X(t) = \sqrt{k}(\frac{-dV_2(t) + dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)})e_1^X(t) - \frac{k}{2}\left(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_2^X(t) \end{cases} \tag{B.1}$$

and

$$\begin{cases} D_t e_1^Y(t) = -\sqrt{k}(\frac{dU_2(t) - dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)})e_2^Y(t) - \frac{k}{2}\left(\frac{dU_2(t) - dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_1^Y(t) \\ D_t e_2^Y(t) = -\sqrt{k}(\frac{-dU_2(t) + dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)})e_1^Y(t) - \frac{k}{2}\left(\frac{dU_2(t) - dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\right)^2 e_2^Y(t) \end{cases} \tag{B.2}$$

To prove this proposition we prove and use the following Lemma:

Lemma B.0.2. *Let $x, y \in M_k$, $r = \rho(x,y)$ with $0 < r < i(M)$, $u \in T_x M_k$ and $v \in T_y M_k$. Using the same notations as in Section 4.1.2, we have:*

$$\nabla_{(u,v)} e_1^x(x,y) = \sqrt{k}\left(\frac{\langle v, e_2^y\rangle - \langle u, e_2^x\rangle\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}\right)e_2^x$$

and

$$\nabla_{(u,v)} e_1^y(x,y) = \sqrt{k}\left(\frac{-\langle u, e_2^x\rangle + \langle v, e_2^y\rangle\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}\right)e_2^y.$$

Lemma B.0.2. Using the same objects as in Lemma 4.1.4, we notice that $\nabla_{(u,v)} e_1^x(x,y) = \nabla_t e_1(0,t)_{|t=0}$ and $\nabla_{(u,v)} e_1^y(x,y) = \nabla_t e_1(1,t)_{|t=0}$. We recall that $e_1(s,t) = \frac{\partial_s c(s,t)}{r(t)}$. From Lemma 4.1.2, we also notice that $r'(t) = v_1(t) - u_1(t)$. Using the results from Lemma 4.1.4, we then get:

$$\begin{aligned}
\nabla_t e_1(s,t) &= \nabla_t \left(\frac{\partial_s c(s,t)}{r(t)} \right) = -\frac{r'(t)}{r(t)^2} \partial_s c(s,t) + \frac{1}{r(t)} \nabla_t \partial_s c(s,t) \\
&= -\frac{v_1(t) - u_1(t)}{r(t)^2} r(t) e_1(s,t) + \frac{1}{r(t)} \nabla_s J(s,t) \\
&= -\frac{v_1(t) - u_1(t)}{r(t)} e_1(s,t) + \frac{\partial_s j_1(s,t) e_1(s,t) + \partial_s j_2(s,t) e_2(s,t)}{r(t)} \\
&= \sqrt{k} \left(-u_2(t) \sin(\sqrt{k} r(t) s) + \frac{v_2(t) - u_2(t) \cos(\sqrt{k} r(t))}{\sin(\sqrt{k} r(t))} \cos(\sqrt{k} r(t) s) \right) e_2(s,t).
\end{aligned}$$

By evaluating this value for $t \in \{0,1\}$, we get the expected result. □

To prove Proposition B.0.1, we make the choice here to use the Stratonovich integral. The covariant Stratonovich differential is defined by $\circ D_t e_1^X(t) := /\!\!/_t \circ d\, /\!\!/_t^{-1}\, e_1^X(t)$. Moreover, considering the process $E(t) := \int_0^T /\!\!/_t^{-1} D_t e_1^X(t)$ on $\mathbb{R}^2$, we have

$$E(t) = \int_0^T /\!\!/_t^{-1} \circ D_t e_1^X(t). \tag{B.3}$$

This enables us to switch from the Stratonovich depiction $\circ D_t e_1^X(t)$ to the Itô depiction $D_t e_1^X(t)$.

Proof of Proposition B.0.1. Using the chain rule and Lemma B.0.2, we directly obtain:

$$\begin{cases} \circ D_t e_1^X(t) = \sqrt{k} \left(\frac{\langle \circ dY_t, e_2^Y(t) \rangle - \langle \circ dX_t, e_2^X(t) \rangle \cos(\sqrt{k} R_t)}{\sin(\sqrt{k} R_t)} \right) e_2^X(t) \\ \circ D_t e_1^Y(t) = \sqrt{k} \left(\frac{-\langle \circ dX_t, e_2^X(t) \rangle + \langle \circ dY_t, e_2^Y(t) \rangle \cos(\sqrt{k} R_t)}{\sin(\sqrt{k} R_t)} \right) e_2^Y(t) \end{cases}. \tag{B.4}$$

As (e_1^X, e_2^X) is an orthonormal moving frame, we have:

$$\begin{aligned}
0 &= \circ d \langle e_2^X(t), e_i^X(t) \rangle = \circ d \langle /\!\!/_t^{-1} e_2^X(t), /\!\!/_t^{-1} e_i^X(t) \rangle \\
&= \langle \circ d\, /\!\!/_t^{-1}\, e_2^X(t), /\!\!/_t^{-1} e_i^X(t) \rangle + \langle /\!\!/_t^{-1} e_2^X(t), \circ d\, /\!\!/_t^{-1}\, e_i^X(t) \rangle \\
&= \langle \circ D_t e_2^X(t), e_i^X(t) \rangle + \langle e_2^X(t), \circ D_t e_i^X(t) \rangle.
\end{aligned}$$

Then: $\langle \circ D_t e_2^X(t), e_1^X(t) \rangle = -\langle e_2^X(t), \circ D_t e_1^X(t) \rangle$ and $\langle \circ D_t e_2^X(t), e_2^X(t) \rangle = 0$. Using the same argument on (e_1^Y, e_2^Y), we get:

$$\begin{cases} \circ D_t e_2^X(t) = -\sqrt{k} \left(\frac{\langle \circ dY_t, e_2^Y(t) \rangle - \langle \circ dX_t, e_2^X(t) \rangle \cos(\sqrt{k} R_t)}{\sin(\sqrt{k} R_t)} \right) e_1^X(t) \\ \circ D_t e_2^Y(t) = -\sqrt{k} \left(\frac{-\langle \circ dX_t, e_2^X(t) \rangle + \langle \circ dY_t, e_2^Y(t) \rangle \cos(\sqrt{k} R_t)}{\sin(\sqrt{k} R_t)} \right) e_1^Y(t) \end{cases}. \tag{B.5}$$

For now we just have the (Itô) covariant derivatives of $(X_t)_t$ and $(Y_t)_t$:

$$d^\nabla X_t = dU_1(t) e_1^X(t) + dU_2(t) e_2^X(t) \text{ and } d^\nabla Y_t = dV_1(t) e_1^Y(t) + dV_2(t) e_2^Y(t)$$

To complete the previous calculus, we need to compute the the Stratonovich differential of $(X_t)_t$ and $(Y_t)_t$. Let us describe the computations for X_t.

Defining the two dimensional real semi martingale $Z_t := \int_0^t /\!\!/_t^{-1} \circ dX_t$, we have $Z_t = \int_0^t /\!\!/_t^{-1} d^\nabla X_t$. In particular, $dZ_t = dU_1(t)\, /\!\!/_t^{-1}\, e_1^X(t) + dU_2(t)\, /\!\!/_t^{-1}\, e_2^X(t)$. Using the relations between Itô and Stratonovich differential in $\mathbb{R}^2$, we get:

$$\circ dZ_t = \circ dU_1(t) /\!\!/_t^{-1} e_1^X(t) + \circ dU_2(t) /\!\!/_t^{-1} e_2^X(t) - \frac{1}{2}(dU_1(t)\cdot d(/\!\!/_t^{-1} e_1^X(t)) + dU_2(t)\cdot d(/\!\!/_t^{-1} e_2^X(t))).$$

Then:

$$\begin{aligned}\circ dX_t &= /\!\!/_t \circ dZ_t \\ &= \circ dU_1(t) e_1^X(t) + \circ dU_2(t) e_2^X(t) - \frac{1}{2}(dU_1(t)\cdot D_t e_1^X(t) + dU_2(t)\cdot D_t e_2^X(t)).\end{aligned}$$

As the martingale parts of Itô and Stratonovich differentials are the same, the martingale part of $D_t e_1^X(t)$ is given by $\sqrt{k}\big(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\big) e_2^X(t)$ and the one for $D_t e_2^X(t)$ is given by $-\sqrt{k}\big(\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\big) e_1^X(t)$. We get:

$$\begin{aligned}\circ dX_t &= \circ dU_1(t) e_1^X(t) + \circ dU_2(t) e_2^X(t) - \frac{1}{2}\frac{\sqrt{k}}{\sin(\sqrt{k}R_t)} dU_1(t)\cdot dV_2(t) e_2^X(t) \\ &\quad + \frac{1}{2} dU_2(t)\cdot \left(\frac{\sqrt{k}(dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t))}{\sin(\sqrt{k}R_t)}\right) e_1^X(t) \\ &= \left(\circ dU_1(t) + \sqrt{k}\frac{dU_2\cdot dV_2 - \cos(\sqrt{k}R_t)dt}{2\sin(\sqrt{k}R_t)}\right) e_1^X(t) \\ &\quad + \left(\circ dU_2(t) - \frac{\sqrt{k}}{2\sin(\sqrt{k}R_t)} dU_1(t)\cdot dV_2(t)\right) e_2^X(t).\end{aligned}$$

With the same method we get:

$$\begin{aligned}\circ dY_t &= \left(\circ dV_1(t) - \sqrt{k}\frac{dU_2\cdot dV_2 - \cos(\sqrt{k}R_t)dt}{2\sin(\sqrt{k}R_t)}\right) e_1^Y(t) \\ &\quad + \left(\circ dV_2(t) + \frac{\sqrt{k}}{2\sin(\sqrt{k}R_t)} dV_1(t)\cdot dU_2(t)\right) e_2^Y(t).\end{aligned}$$

Then we obtain :

$$\begin{cases} \circ D_t e_1^X(t) = \left(\sqrt{k}\big(\frac{\circ dV_2(t) - \circ dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\big) + \frac{k}{2}\times \frac{\cos(\sqrt{k}R_t) dU_1(t)\cdot dV_2(t) + dV_1(t)\cdot dU_2(t)}{\sin^2(\sqrt{k}R_t)}\right) e_2^X(t) \\ \circ D_t e_1^Y(t) = \left(\sqrt{k}\big(\frac{-\circ dU_2(t) + \circ dV_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)}\big) + \frac{k}{2}\times \frac{\cos(\sqrt{k}R_t) dV_1(t)\cdot dU_2(t) + dU_1(t)\cdot dV_2(t)}{\sin^2(\sqrt{k}R_t)}\right) e_2^Y(t) \end{cases}. \tag{B.6}$$

To obtain the Itô depiction of the covariant derivative, we consider $E(t) := \int_0^T /\!\!/_t^{-1} D_t e_1^X(t)$. From B.6 we have the expression of $\circ dE(t)$. As $(E(t))_t$ is a process in $\mathbb{R}^2$, we can use the

usual formulas to obtain its Itô derivative:

$$\begin{aligned}
dE(t) = &\sqrt{k}\frac{dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t)}{\sin(\sqrt{k}R_t)} \, /\!\!/_t^{-1} \, e_2^X(t) \\
&+ \frac{k}{2}\frac{\cos(\sqrt{k}R_t)dU_1(t)\cdot dV_2(t) + dV_1(t)\cdot dU_2(t)}{\sin^2(\sqrt{k}R_t)} \, /\!\!/_t^{-1} \, e_2^X(t) \\
&- \frac{k}{2}\left(\frac{-\cos(\sqrt{k}R_t)dV_2(t) + dU_2(t)}{\sin^2(\sqrt{k}R_t)}\cdot dR_t\right) /\!\!/_t^{-1} \, e_2^X(t) \\
&+ \frac{\sqrt{k}(dV_2(t) - dU_2(t)\cos(\sqrt{k}R_t))}{2\sin(\sqrt{k}R_t)} \cdot d\left(/\!\!/_t^{-1} e_2^X(t)\right).
\end{aligned}$$

Using the stochastic equation of $(R_t)_t$ (see 4.11, we get $dV_2(t)\cdot dR_t = -dV_2(t)\cdot dU_1(t)$ and $dU_2(t)\cdot dR_t = dU_2(t)\cdot dV_1(t)$. Moreover, we have $/\!\!/_t d(/\!\!/_t^{-1} e_2^X(t)) = D_t e_2^X(t)$, with $D_t e_2^X(t)$ and $\circ D_t e_2^X(t)$ having the same martingale part. Using Relation (B.6), we obtain the expected equation for $(D_t e_1^X(t))_t$. We use the same method to obtain the second equation in (B.1). Finally, using (B.5) we also get (B.2). □

Bibliography

[1] A. Agrachev and D. Barilari. Sub-Riemannian structures on 3D Lie groups. *J. Dyn. Control Syst.*, 18(1):21–44, 2012.

[2] Andrei Agrachev, Davide Barilari, and Ugo Boscain. *A comprehensive introduction to sub-Riemannian geometry*, volume 181 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020. From the Hamiltonian viewpoint, With an appendix by Igor Zelenko.

[3] Marc Arnaudon, Magalie Bénéfice, Michel Bonnefont, and Delphine Féral. A coupling strategy for Brownian motions at fixed time on Carnot groups using Legendre expansion. `https://arxiv.org/abs/2407.04321`, 2024.

[4] Marc Arnaudon and Anton Thalmaier. The differentiation of hypoelliptic diffusion semigroups. *Illinois J. Math.*, 54(4):1285–1311, 2010.

[5] Søren Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.

[6] Sayan Banerjee, Maria Gordina, and Phanuel Mariano. Coupling in the Heisenberg group and its applications to gradient estimates. *Ann. Probab.*, 46(6):3275–3312, 2018.

[7] Sayan Banerjee and Wilfrid Kendall. Coupling polynomial Stratonovich integrals: the two-dimensional Brownian case. *Electron. J. Probab.*, 23:Paper No. 24, 43, 2018.

[8] Sayan Banerjee and Wilfrid S. Kendall. Coupling the Kolmogorov diffusion: maximality and efficiency considerations. *Adv. in Appl. Probab.*, 48(A):15–35, 2016.

[9] Sayan Banerjee and Wilfrid S. Kendall. Rigidity for Markovian maximal couplings of elliptic diffusions. *Probab. Theory Related Fields*, 168(1-2):55–112, 2017.

[10] Fabrice Baudoin and Michel Bonnefont. The subelliptic heat kernel on SU(2): representations, asymptotics and gradient bounds. *Math. Z.*, 263(3):647–672, 2009.

[11] Fabrice Baudoin and Michel Bonnefont. Log-Sobolev inequalities for subelliptic operators satisfying a generalized curvature dimension inequality. *J. Funct. Anal.*, 262(6):2646–2676, 2012.

[12] Fabrice Baudoin and Michel Bonnefont. Reverse Poincaré inequalities, isoperimetry, and Riesz transforms in Carnot groups. *Nonlinear Anal.*, 131:48–59, 2016.

[13] Fabrice Baudoin, Nizar Demni, and Jing Wang. Stochastic areas, horizontal Brownian motions, and hypoelliptic heat kernels. `https://arxiv.org/abs/2212.07483`, 2023.

[14] Fabrice Baudoin and Nicola Garofalo. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. *J. Eur. Math. Soc. (JEMS)*, 19(1):151–219, 2017.

[15] Fabrice Baudoin, Maria Gordina, and Phanuel Mariano. On the Cheng-Yau gradient estimate for Carnot groups and sub-Riemannian manifolds. *Proceedings of the American Mathematical Society*, 147(7):3181–3189, 2019.

[16] Fabrice Baudoin, Maria Gordina, and Phanuel Mariano. Gradient bounds for Kolmogorov type diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(1):612–636, 2020.

[17] Gérard Ben Arous, Michael Cranston, and Wilfrid S. Kendall. Coupling constructions for hypoelliptic diffusions: two examples. In *Stochastic analysis (Ithaca, NY, 1993)*, volume 57 of *Proc. Sympos. Pure Math.*, pages 193–212. Amer. Math. Soc., Providence, RI, 1995.

[18] Magalie Bénéfice, Marc Arnaudon, and Michel Bonnefont. Couplings of Brownian motions on $SU(2, \mathbb{C})$. In *Geometric science of information. Part I*, volume 14071 of *Lecture Notes in Comput. Sci.*, pages 592–600. Springer, Cham, [2023] ©2023.

[19] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[20] Michel Bonnefont. *Functional inequalities for subelliptic heat kernels*. Theses, Université Paul Sabatier - Toulouse III, November 2009.

[21] Michel Bonnefont. The subelliptic heat kernels on $\mathrm{SL}(2, \mathbb{R})$ and on its universal covering $\widetilde{\mathrm{SL}(2, \mathbb{R})}$: integral representations and some functional inequalities. *Potential Anal.*, 36(2):275–300, 2012.

[22] Michel Bonnefont and Nicolas Juillet. Couplings in L^p distance of two Brownian motions and their Lévy area. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(1):543–565, 2020.

[23] Ugo Boscain and Francesco Rossi. Invariant Carnot-Caratheodory metrics on S^3, $\mathrm{SO}(3)$, $\mathrm{SL}(2)$, and lens spaces. *SIAM J. Control Optim.*, 47(4):1851–1878, 2008.

[24] Magalie Bénéfice. Non co-adapted couplings of Brownian motions on subRiemannian manifolds. `https://arxiv.org/abs/2312.14512`, 2023.

[25] Magalie Bénéfice. Couplings of Brownian motions on $SU(2)$ and $SL(2, \mathbb{R})$. *Stochastic Processes and their Applications*, page 104434, 2024.

[26] Magalie Bénéfice. Non co-adapted successful couplings of Brownian motions on the free, step 2 carnot groups. `https://arxiv.org/abs/2407.06593`, 2024.

[27] M. Cranston. Gradient estimates on manifolds using coupling. *J. Funct. Anal.*, 99(1):110–124, 1991.

[28] M. Cranston. A probabilistic approach to gradient estimates. *Canad. Math. Bull.*, 35(1):46–55, 1992.

[29] Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. Couplings and quantitative contraction rates for Langevin dynamics. *Ann. Probab.*, 47(4):1982–2010, 2019.

[30] Michel Émery. *Stochastic calculus in manifolds*. Universitext. Springer-Verlag, Berlin, 1989. With an appendix by P.-A. Meyer.

[31] Arnaud Guillin and Feng-Yu Wang. Degenerate Fokker-Planck equations: Bismut formula, gradient estimate and Harnack inequality. *J. Differential Equations*, 253(1):20–40, 2012.

[32] Piotr Hajłasz and Pekka Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.

[33] Elton P. Hsu and Karl-Theodor Sturm. Maximal coupling of Euclidean Brownian motions. *Commun. Math. Stat.*, 1(1):93–104, 2013.

[34] Yonghong Huang and Shanzhong Sun. Non-embedding theorems of nilpotent Lie groups and sub-Riemannian manifolds. *Front. Math. China*, 15(1):91–114, 2020.

[35] Svante Janson. Euclidean, spherical and hyperbolic trigonometry. *Notes*, 2015.

[36] Jürgen Jost. *Riemannian geometry and geometric analysis*. Universitext. Springer, Cham, seventh edition, 2017.

[37] Wilfrid S. Kendall. Coupling all the Lévy stochastic areas of multidimensional Brownian motion. *Ann. Probab.*, 35(3):935–953, 2007.

[38] Wilfrid S. Kendall. Brownian couplings, convexity, and shy-ness. *Electron. Commun. Probab.*, 14:66–80, 2009.

[39] Wilfrid S. Kendall. Coupling time distribution asymptotics for some couplings of the Lévy stochastic area. In *Probability and mathematical genetics*, volume 378 of *London Math. Soc. Lecture Note Ser.*, pages 446–463. Cambridge Univ. Press, Cambridge, 2010.

[40] Kazumasa Kuwada. On uniqueness of maximal coupling for diffusion processes with a reflection. *J. Theoret. Probab.*, 20(4):935–957, 2007.

[41] Kazumasa Kuwada. Duality on gradient estimates and Wasserstein controls. *J. Funct. Anal.*, 258(11):3758–3774, 2010.

[42] D.F. Kuznetsov. New simple method of expansion of iterated ito stochastic integrals of multiplicity 2 based on expansion of the brownian motion using legendre polynomials and trigonometric functions. *arXiv*, 2018.

[43] Torgny Lindvall and L. C. G. Rogers. Coupling of multidimensional diffusions by reflection. *Ann. Probab.*, 14(3):860–872, 1986.

[44] Liangbing Luo and Robert W. Neel. Non-markovian maximal couplings and a vertical reflection principle on a class of sub-riemannian manifolds. `https://arxiv.org/abs/2402.13976`, 2024.

[45] Robb J. Muirhead. *Aspects of multivariate statistical theory*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1982.

[46] James Norris. Simplified Malliavin calculus. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 101–130. Springer, Berlin, 1986.

[47] James R. Norris. A complete differential formalism for stochastic calculus in manifolds. *Séminaire de probabilités de Strasbourg*, 26:189–209, 1992.

[48] Mihai N. Pascu and Ionel Popescu. Couplings of Brownian motions of deterministic distance in model spaces of constant curvature. *J. Theoret. Probab.*, 31(4):2005–2031, 2018.

[49] Jolanta Pielaszkiewicz and Thomas Holgersson. Mixtures of traces of Wishart and inverse Wishart matrices. *Comm. Statist. Theory Methods*, 50(21):5084–5100, 2021.

[50] Yao-Feng Ren. On the Burkholder-Davis-Gundy inequalities for continuous martingales. *Statist. Probab. Lett.*, 78(17):3034–3039, 2008.

[51] Michael Ruzhansky and Ville Turunen. *Pseudo-differential operators and symmetries*, volume 2 of *Pseudo-Differential Operators. Theory and Applications.* Birkhäuser Verlag, Basel, 2010. Background analysis and advanced topics.

[52] M. Yu. Sverchkov and S. N. Smirnov. Maximal coupling for processes in $D[0, \infty]$. *Dokl. Akad. Nauk SSSR*, 311(5):1059–1061, 1990.

[53] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1992.

[54] Max-K. von Renesse and Karl-Theodor Sturm. Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm. Pure Appl. Math.*, 58(7):923–940, 2005.

[55] Feng-Yu Wang. Liouville theorem and coupling on negatively curved Riemannian manifolds. *Stochastic Process. Appl.*, 100:27–39, 2002.

[56] Feng-Yu Wang. *Analysis for diffusion processes on Riemannian manifolds*, volume 18 of *Advanced Series on Statistical Science & Applied Probability.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.