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Extensions of the core of cooperative games and noisy auctions with resale



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Résumé :

Cette thèse, structurée en deux parties indépendantes, explore deux domaines distincts de la théorie des jeux: les jeux coopératifs et la théorie des enchères. Dans la première partie, je propose de nouveaux concepts de cœur pour les jeux coopératifs. Le premier concept se place dans un cadre de jeux coopératifs dynamiques, prenant en compte la faisabilité temporelle et les compensations entre les étapes. Les suivants sont des concepts dédiés aux jeux coopératifs avec externalités, fondés sur des notions de stabilité. Pour chacun de ces concepts, je donne des conditions de non-vacuité du cœur, et pour certains des caractérisations.

Dans la seconde partie, j'introduis un modèle d'enchères bruitées avec possibilité de revente. L'objectif initial étant de modéliser le mieux possible les comportements des enchérisseurs dans les enchères en ligne. Ce modèle ne vérifie pas le théorème d'équivalence du revenu, mais classe ce type d'enchères en deux catégories, aux seins desquelles des nouveaux résultats d'équivalence du revenu apparaissent.

Descripteurs :

Jeux coopératifs, Cœur, Externalités, Dynamique, Enchères

Title and Abstract:

This thesis, structured into two independent parts, explores two distinct domains of game theory: cooperative games and auction theory. In the first part, I introduce new core concepts for cooperative games. The first concept is situated within the framework of dynamic cooperative games, considering temporal feasibility and compensations between stages. Subsequent concepts are dedicated to cooperative games with externalities, based

on stability notions. For each of these concepts, I provide conditions for the non-emptiness of the core, and for some, characterizations.

In the second part, I present a model of noisy auctions with resale. The initial objective is to model the behaviors of bidders in online auctions as accurately as possible. While this model does not satisfy the revenue equivalence theorem, it categorizes this type of auctions into two groups, within which new income-equivalence results emerge.

Keywords:

Cooperative games, Core, Externalities, Dynamics, Auctions

Notations

$\mathbb{N}$: Set of the natural numbers

$\mathbb{R}$: Set of the real numbers

$\mathscr{M}_{n\times T}(\mathbb{R})$: Set of the real matrix with n lines and T columns

N: Set of players of a cooperative game

2^N: Power set of N

$\mid N \mid$: Cardinal of N

Π_N: Set of the partitions of N

$A \subset B$: The set A is included or equal to the set B

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Introduction

Game theory, born at the crossroads of mathematics and social sciences through the collaboration of John von Neumann and Oskar Morgenstern, has matured into a sophisticated framework with applications across diverse fields. Its historical journey, initially confined to abstract mathematical problems, expanded into economics, political science, biology, computer science, and beyond. John Nash's mid-20th-century work, particularly his concept of Nash equilibrium, further propelled game theory into the spotlight, earning him the Nobel Prize in Economics in 1994. In the contemporary landscape, game theory remains a dynamic field, blending mathematical elegance with practical utility, offering valuable insights into human behavior and strategic interactions for scientists in various disciplines.

As our world evolves, the introduction of time into non-cooperative game theory brings forth a new layer of analysis, giving rise to repeated games and the Folk Theorem. This theorem extends game theory's scope for repeated games, allowing a broader range of equilibria shaped by players' evolving strategies over time. Dynamic games further push the boundaries, explicitly incorporating time to study strategic interactions unfolding sequentially, enabling a refined understanding of adaptive player strategies in changing environments.

Auctions theory stands out as a crucial application of game theory to economics. This influential discipline, spotlighted by Myerson's work, especially in his in-depth exploration of mechanism design, has significantly enriched the field of modern economics. Auction theory goes beyond mere price-setting and focuses into the design of auctions and incentive mechanisms aimed at maximizing economic efficiency. Mechanism design, a critical branch of game theory, focuses on crafting auction rules and procedures to encourage participants to truthfully reveal their private information. This approach has profound implications for constructing more robust and effective economic institutions. As for game theory, introducing a temporal dimension into auction models brings new perspectives to have a better understanding of reality.

While game theory predominantly focuses into non-cooperative situations, there exists a distinct branch known as cooperative game theory, which focuses specifically on scenarios where players form coalitions. This branch diverges significantly in structure

from its non-cooperative counterpart, as it eschews the Nash equilibrium framework that characterizes the latter. In cooperative game theory, the emphasis shifts from individual strategies to the collaborative efforts of groups of players. The key concepts within cooperative game theory include coalition formation, payoff distribution, and solution concepts such as the Core, the Nash bargaining solution or the Shapley value. This shift in focus from individual rationality to cooperative stability brings a nuanced perspective to strategic interactions. By acknowledging the potential for collaboration and coalition-building, cooperative game theory captures scenarios where actors can achieve outcomes that are mutually beneficial. It provides a more realistic framework for situations where negotiation, cooperation, and the formation of alliances play pivotal roles. Therefore, this theory enriches the understanding derived from non-cooperative game theory. Moreover, the exploration of cooperative game theory not only broadens the analytical toolkit but also contributes valuable insights to the understanding of interactions between players.

Nash (1953) initiated the Nash program, as a research agenda to bridge the gap between the cooperative and non-cooperative approaches to the discipline. Many authors have contributed to the program since its beginnings. Historically, the first argument to justify the Nash bargaining solution and the Shapley value was the axiomatic approach, which consist in defining abstract principles, called axioms, that one would like the solution to have, and to pursue their logical consequences. However, some may think that the definition may be somewhat arbitrary, or one may object that the axiomatic approach is too abstract. By proposing non-cooperative games that specify the details of negotiation in cooperative games, the Nash program might help to understand better those mechanisms. Such a procedure should tell a story about how coalitions form and interaction bewteen players. In that process, a cooperative solution will be understood with the tools of non-cooperative games as the outcome of strategic problems facing individual players. As a consequence, the connections and differences bewteen cooperative solutions may be uncovered from the distinct negotiation procedures that lead to each of them. Therefore, a result in the Nash program, referred to as a non-cooperative implementation of a cooperative solution, highlights its significance from a non-cooperative point of view. In the first paper of the Nash program, Nash provides a non-cooperative approach to his ax-

iomatically derived solution of Nash (1950), as a continuum of Nash equilibria. Hart and Mas-Colell (1996) propose a non-cooperative approach of the Shapley value, and show that the latter can be seen as a unique stationary subgame perfect equilibrium payoff profile of a non-cooperative procedure. Likewise, Perry and Reny (1994), Moldovanu and Winter (1995) and Serrano (1995) implement the Core as a non-cooperative solution.

As the initial cooperative framework does not encapsulate externalities, Thrall and Lucas (1963) define partition function games, a model in which the benefits of coalitions depends on the behaviour of the outsiders' action. This model enriches a lot cooperative games theory, as it encapsulates both cooperative and non-cooperative dynamics in games. As the Nash program, this model creates a bridge between cooperative and non-cooperative games.

This thesis encapsulates three years of research during my Ph.D., focusing into two distinct domains: auction theory and cooperative game theory. My purpose in auction theory was to the explain bidders' behavior in online auctions, where there is the possibility of reselling the object at the auction's conclusion. Compounded by the uncertainty surrounding bidders' estimations of the object's value, my investigation aimed to provide a comprehensive description of their strategic interactions. Concurrently, my exploration in cooperative game theory revolved around discerning the conditions under which a group of players can sustain collaboration. Addressing this question involved adopting two distinct approaches: the first entailed developing a dynamic model of cooperative games, while the second incorporated externalities into the cooperative game model. Both approaches are based on the definitions of new notions of the Core and its non-emptiness, serving as a common point for understanding the sustainability of collaborative efforts. The dynamic model captures the evolution of cooperative interactions over time, while the externality-infused model accounts for the impact of external players on the stability of cooperative endeavors.

One common thread in all my models is that each embodies an idea of dynamic over time: my auction model is a two step auction in which the one who got the item in the first auction can decide to resell it, the dynamic model of cooperative games is obviously dynamic, while for cooperative games with externalities, I study the dynamics of the

remainng players from a group reacting to a deviation of a smaller group of players. What arises from these models is that dynamics enriches the frameworks, as the Folk Theorem for repeated games for example. In a constantly evolving world, considering only static models is highly restrictive, and one cannot limit themselves to this approach if they want to accurately model reality. Adding temporal dynamics to models is therefore necessary for precise representation of reality. It becomes interesting to analyze how the addition of temporal dynamics will complicate and enrich the models that already exist in the literature.

My contributions to the literature are in Chapters 2, 3, and 5.

Chapter 2: Dynamic Cooperative Games

In this chapter, the focus is on cooperative game theory, particularly exploring dynamic games with discrete time and a predetermined pattern of coalition valuation changes. Unlike the prevalent literature, which predominantly examines static games played only once, the study focuses into adding time in cooperative games, with each stage being independent, having no historical dependence, and players possessing complete information on the game. The central goal is to introduce two concepts: a coalitional $\vec{\varepsilon}$-core for static cooperative games, offering a less restrictive alternative to Shapley and Shubik's ε-core; and a core for dynamic coalitional games, allowing worth compensation within periods while ensuring that the total worth distributed at each stage equals the value of the grand coalition. These concepts are interconnected, as the coalitional $\vec{\varepsilon}$-core is used to characterize the non-emptiness of the dynamic core. The main contribution lies in providing a geometrical approach, using the coalitional $\vec{\varepsilon}$-core, to characterize the non-emptiness of the dynamic core. The chapter outlines the limitations of existing core concepts in dynamic settings and establishes a novel framework that aligns with real-world dynamics while explaining the evolution of the classical core over time. I also provide axiomatizations of the dynamic core and two other core concept that already exist in the literature, and define a version of the Shapley value in this context. This chapter opens questions about intergenerational equity and inheritance in a group of players.

Chapter 3: Cooperative Game with Externalities

This chapter introduces new core concepts for cooperative games with externalities,where the values of coalitions depends on the behaviour of the external players. The aim is to introduce strategic stability concepts in the model of partition form games. While not any core concept in the literature embodies an idea of stability of the remaining players after a deviation of a subgroup of players from the grand coalition, this chapter fills this void. As for the α-core, or the ω-core, which are the most respectively pessimistic and optimistic version of the core in this context, I define two notion of strategic stability for partition function games, called stable recursive cores and stable cores by successive deviations, and provide an optimistic and pessimistic versions for each. For stable recursive cores, I prove a characterization of non-emptiness as a Bondareva-Shapley result. I also give sufficient conditions of non-emptiness and Pareto efficiency for these core concepts.

Chapter 5: Noisy Auction with Resale

This chapter explores the dynamics introduced by post-auction resale in auction scenarios, particularly in situations where bidders possess partial information about the value of the auctioned object. Therefore, winners may later discover that their valuation was not the highest, prompting a potential resale. The study focuses on a model featuring multiple agents with symmetrically distributed values, where bidders have imperfect information, perceiving only a noisy valuation centered around their true value. The analysis reveals that without resale opportunities, agents behave as if they have perfect information about their valuations. However, the inclusion of resale options disrupts the Revenue Equivalence Theorem, leading to a categorization of standard auctions into two types based on the winner's ability to infer the second bidder's noisy valuation. This dichotomy arises from what the winner learns post-auction, determining the optimal approach for potential resale. The study demonstrates that the Revenue Equivalence Theorem no longer holds in the presence of resale options, offering insights into how the initial seller's expected revenue remains consistent within these two auction types.

The chapter distinguishes itself from existing literature by considering both uncertainty about private valuations and the possibility of resale. While similar conclusions regarding expected revenue for the seller emerge, the model's uniqueness lies in its symmetrical valuation distribution and the incorporation of resale dynamics.

Guidelines for the reader

This section details the organization of the remainder of the thesis. Part I is about cooperative games. In Chapter 1, I introduce all the basic notions about cooperative games, the models of characteristic function games and partition function games (which is a model of cooperative games with externalities), and their basic definitions. Within these two models, I introduce the most known solution concepts, as the core and the Shapley value. For partition function games, I explain what are the issues of extending the solution concepts for characteristic function games, and how the literature finds ways to solve those. Chapter 2 introduces my model of dynamic cooperative games and the coalitional $\vec{\varepsilon}$-core. Chapter 3 details my models of stable cores for partition function games and their characterization. Part II is about auction theory. Chapter 4 is a reminder of the basic definitions and results in auction theory for single objects auctions à la Vrickey. The second part of this chapter introduces the literature about noisy auction and auction with resale. Chapter 5 presents my model of noisy auction with resale for single objects. As the proofs of this chapter are long, for clarity, they are mainly in Appendix A.

To ensure that the chapters can be read independently of each other, most of the concepts necessary for their understanding are presented within them, even if they have been introduced previously, that is why some concepts and results are introduced several times.

Part I

Cooperative Game Theory

1 Theoretical Preleminaries in Cooperative Game Theory

This chapter introduces the framework of cooperative game theory.

In the first part, I lay the groundwork by examining the basic definitions and solution concepts of cooperative games. These concepts include the basic definitions of cooperative games and solution concepts like the Core and the Shapley value.

The second part is about partition function games, introducing externalities into the traditional cooperative games framework. By accounting for external factors, this model enhances the applicability of cooperative game theory to real-world scenarios. This extension allows to capture the complexities of cooperative decision-making in situations where the impact of external influences cannot be ignored.

1.1 TU-Games

In the annals of game theory, the pivotal work of John von Neumann and Oskar Morgenstern in their seminal 1953 book, « Theory of Games and Economic Behavior », stands as a foundational milestone. At a time when the intellectual landscape was evolving with the advent of mathematical rigor in economics, von Neumann and Morgenstern laid the groundwork for the formalization of strategic decision-making under uncertainty. The authors studied into the intricacies of strategic interactions, introducing concepts like utility theory and the famous minimax theorem. While their immediate focus was on non-cooperative settings, the seeds of cooperation were sown through their exploration of strategic decision-making. The very nature of rational players seeking optimal outcomes hinted at the potential for collaboration, a theme that would later be fully realized in

cooperative game theory.

Von Neumann and Morgenstern (1953) define the worth of a TU-game from a non cooperative game. In this context, the worth of a coalition S is defined by its maxmin value in a two person zero-sum game, where S plays against all the other players in $N \setminus S$.

In this chapter, I introduce the basic definitions of cooperative games, identify some well known families of cooperative games, and introduce the axiomatic method. I also introduce solution concepts, as the Shapley value and the Core.

1.1.1 Basic definitions

Let $N \subset \mathbb{N}$ be a non-empty and finite set of players. Every subset $S \in 2^N$ is called a coalition of cooperative players. By extension, the grand coalition N represents the situation in which all players cooperate, and $\emptyset$ the one in which none of them cooperate. For every coalition S, $\mid S \mid \in \mathbb{N}$ represents the cardinality of S, i.e. the number of players cooperating in S. In what follows, $\mid N \mid = n$ and $\mid S \mid = s$.

Definition 1. A coalitional game with transferable utility, also called **characteristic function game (CFG)**, or **TU-game**, is a pair (N, c), where $N \subset \mathbb{N}$ is a finite set of players, and $c : 2^N \longrightarrow \mathbb{R}$ a function which maps every possible coalition to a real value, with $c(\emptyset) = 0$.

In most applications, players are persons or groups of persons, like employees, companies, nations etc. However, some applications model situations where players are differents entities, as economic projects, raw materials or machines in a factory for example.

For each coalition $S \subset N$, the value $c(S)$ can be interpreted as the benefits generated by the players in S by cooperating: it is called the worth of S. The set of TU-games on N is denoted by $\mathbb{G}^N$, and the set of all TU-games is denoted $\mathbb{G} := \bigcup_{N \subset \mathbb{N} \text{ finite}} \mathbb{G}^N$.

Given a TU-game (N, c), one may want to consider the game restricted to a smaller set of players $S \subset N$. I denote by $(S, c_{|S})$ this subgame, defined with the characteristic function such that $c_{|S}(T) = c(T)$ for all $T \in 2^S$.

A TU-game (N, c) is **non-negative** if each coalition generates a non-negative worth in this game:

$$\forall S \in 2^N, c(S) \geq 0$$

There exists a class of TU-games, called **monotonic games**, for which, a larger coalition implies a higher worth. Formally, the CFG (N, c) is monotonic if:

$$\forall S \subset T \subset N, c(S) \leq c(T)$$

As $c(\emptyset) = 0$, a monotonic game is always non-negative.

In some situations, cooperation between players is inefficient, because they do not generate any surplus by cooperating: the benefits of a coalition S are only the sums of the benefits of the players in S. This case is modeled by **additive** CFG. Formally, (N, c) is additive if:

$$\forall S \in 2^N, c(S) = \sum_{i \in S} c(\{i\})$$

Additive games form a subset of **superadditive** games, for which the benefits of all partitions S are not smaller than the sum of the benefits of any split in two disjoint coalitions. Formally, the CFG (N, c) is superadditive if:

$$\forall S, T \in 2^N, S \cap T = \emptyset, c(S) + c(T) \leq c(S \cup T)$$

In this case, players have interest to merge and form the grand coalition, as its worth is always greater than the sum of benefits of any partition of N.

In a lot of applications, TU-games are superadditive, but it is very restrictive: large coalitions may be inefficient, because of the difficulties to act together. For example, several individuals facing a complex problem could take longer to solve it than just the two smartest ones, because of the time it takes to convince the others.

The fact that superadditivity can be too strong to model reality creates an incentive to define weaker classes for CFGs.

A CFG (N, c) is **weakly superadditive** if:

$$\forall S \in 2^N, i \in N \setminus S, c(S) + c(\{i\}) \leq c(S \cup \{i\})$$

It is straightforward to see that a superadditive game is weakly superadditive. Contrariwise, the following game shows that weakly superadditivity does not imply superadditivity:

Example 1. Let $N = \{1, 2, 3, 4\}$, and c be the characteristic function defined by the following table for $i \neq j \neq k \in N$:

S	$\{i\}$	$\{i,j\}$	$\{i,j,k\}$	N
$c(S)$	0	2	2	3

(N,c) is weakly superadditive. However, $c(\{1,2\}) + c(\{3,4\}) = 4 \geq v(N)$.

Convexity extends superadditivity to any pair of coalitions, not only disjoint ones. Formallly, a TU-game (N,c) is **convex** or **supermodular** if:

$$\forall S,T \in 2^N, c(S) + c(T) \leq c(S \cup T) + c(S \cap T)$$

The following inequalities characterize convexity: for all $i \in N$, $S \subset T \subset N \setminus \{i\}$, $c(S \cup \{i\}) - c(S) \leq c(T \cup \{i\}) - c(T)$.

This characterization leads to the following interpretation of convex games: a TU-game is convex if and only if, the marginal contribution of any player in a coalition grows if the coalition expands. Convex games appear in a lot of applications of cooperative game theory.

The CFG (N,c) is called **submodular** if the CFG $(N,-c)$ is convex.

A TU-game game (N,c) is called **symmetric** if the worth of a coalition S does not depend on the identity of the members of S, but only the number of players in S. Formally, (N,c) is symmetric if for all permutations σ of N, $S \subset N$, $c(S) = c(\sigma(S))$, where $\sigma(S) = \{\sigma(i) \mid i \in S\}$.

All these definitions having been given, I now focus on the objectives of cooperative games. In TU-games, determining how benefits resulting from cooperation should be allocated among the members of the coalition is a key objective. This involves finding a distribution that can be supported by a binding agreement. Hence the players focus on the choice of some particular payoff vectors and not on the choice of a stable strategy profile as in a noncooperative game.

Let me define the payoff vectors properly:

Definition 2. An **allocation** for a game (N,v) is a vector $x \in \mathbb{R}^n$, where for all $i \in N$, x_i represents the value allocated to player i.

In what follows, for all $S \subset N$, $\mathbf{x}(S)$ will denote the sum of the values allocated to players in S: $\mathbf{x}(S) := \sum_{i \in S} x_i$. Therefore, each allocation $x \in \mathbb{R}^n$ can define an additive game (N,c) such that $c(S) := \mathbf{x}(S)$ for all $S \subset N$.

For a monotonic game (N, c), an allocation $x \in \mathbb{R}^n$ is called **feasible** if the sum of the values of x does not exceed the worth of N: $\mathbf{x}(N) \leq c(N)$.

1.1.2 Some Families of Games

In this section, I introduce some important classes of TU-games in terms of applications or theoretical contribution.

1.1.2.1 Unanimity Games

Let $N \subset \mathbb{N}$ be a set of players. Since 2^N is a finite set, any characteristic function $c : 2^N \longrightarrow \mathbb{R}$ can be described by the vector $(c(S))_{S \subset N}$. Hence, the set $\mathbb{G}^N$ is a linear subspace of $\mathbb{R}^{2^n}$. Since the worth of $\emptyset$ is always 0 by convention, $\mathbb{G}^N$ admits a basis of cardinality $2^n - 1$. Shapley (1953) identifies a basis formed by what he calls unanimity games.

Definition 3. Let $N \subset \mathbb{N}$ a finite set of players. For any non-empty coalition $S \in 2^N$, the **unanimity game** with ruling coalition S is defined as the CFG (N, c_S) such that for all $T \subset N$:

$$c_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise} \end{cases}$$

Shapley (1953) shows that all games in $\mathbb{G}^N$ can be written as linear combinations of unanimity games:

Proposition 1 (Shapley (1953))**.** *Let (N, c) be a CFG. There exists a unique decomposition of c in term of unanimity games:*

$$c = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_c(S) c_S$$

where for all $S \in 2^N$, $\Delta_c(S)$ is called the Harsanyi dividend (Harsanyi (1958)). It represents the net surplus generated by S in the game (N, c), and is defined by $\Delta_c(S) = c(S) - \sum_{T \subset S} \Delta_c(T)$.

Example 2. Let $N = \{1, 2\}$, and c be the characteristic function defined by the following table:

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$c(S)$	2	3	7

The Harsanyi dividend of $\{1,2\}$ is the net surplus of this coalition, so $\Delta_c(\{1,2\}) = c(S) - \Delta_c(\{1\}) - \Delta_c(\{2\}) = 7 - 3 - 2 = 2$. Hence, the net surplus of the coalition $\{1,2\}$ compared to $\{1\}$ and $\{2\}$ alone is 2.

1.1.2.2 Cost Allocation Games

A **cost game** is a TU-game (N, c) where c is a cost function. The idea behind this concept is that N represents a set of customers of a service or facility. For each coalition S, $c(S)$ represents the least cost of serving the members in S.

Technically, a cost game is defined as a CFG, but cannot be considered as one: the cost function is not interpreted as a characteristic function. But it is still possible to define an associated « classical » TU-game, called a saving game, defined as (N, v) such that for all $S \subset N$:

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S)$$

Example 3. A group of factories use water from a river to produce items. This generates pollution, and the factories need to take charge for the treatment of water. They can make coalitions in order to reduce the costs of the treatment. That defines a cost function, that maps each coalition of factories to the costs of water treatment for the pollution they have generated.

These games have been widely studied, as detailed in the following surveys: Lucas (1981), Young (1985a), and Tijs and Driessen (1986).

1.1.3 Solution concepts

One of the most important questions in cooperative game theory is the following one: « if the players agree to form the grand coaltion, how to divide its worth ». The answer to that question is to find a rule to divide the value of the grand coalition, called a solution concept.

Formally, a **solution concept** on the set of players N is a map $\Phi : \mathbb{G}^N \longrightarrow 2^{\mathbb{R}^n}$ that associates to a given TU-game (N, c) a set of allocation vectors. Φ can also map

to a unique allocation vector. In this case, Φ is said to be single valued, and called an **allocation rule** to simplify, $\Phi : \mathbb{G}^N \longrightarrow \mathbb{R}^n$. Otherwise, it is said to be set valued (even if the set of allocation vector can be empty).

In this section, I introduce the most emblematic solution concepts in cooperative games, as the Core or the Shapley value.

1.1.3.1 The Axiomatic Method

This section is mainly inspired by Thomson (2001) and Béal and Gabuthy (2018).

Many economic situations may be conflicting, in particular resource allocation situations. In a cooperative framework, one way to deal with these situations is to rely on an external arbitration. Several rules may be introduced in this arbitration. The goal of the axiomatic method is to choose a solution concept proceeding by the following steps:

1. Define a set of rules, called axioms, considered as desirable for a solution concept. For example, one may argue that the allocation rule must not discriminate some players, that would lead to an egalitarian axiom.

2. Study the logical relations between these axioms. For example, if a subset of axioms are stronger than another one i.e. if a solution concept verify the first ones, it also verify the latter. In that case, the last axiom can be removed from the set of axioms.

3. Combine axioms in order to find a result among one of the following 3 types:

   - There is no solution concept that satisfies all the axioms. The axioms are then said to be incompatible, and it leads to an impossibility result.
   - There are many solution concepts which satisfy the axioms. In that case, the set of axioms let at least one degree of freedom that permits to a social planner to chose one solution concept.
   - There is only one solution concept satisfying all the axioms. This leads to a characterization of this concept. This is the case that appears the most in the literature on cooperative games.

4. Seek if the solution concepts is robust to small variations of the original axioms.

The axiomatic method is a cornerstone of the cooperative game theory, but its use goes far beyond this field of research. For example, the Arrow's impossibility theorem in social choice theory is a very well-known example using this method. The axiomatic method is also relevant for non cooperative game theory.

1.1.3.2 The Core

The Core, introduced by Gillies (1953) and Shapley (1955), is one of the best known set valued solution for TU-games. This notion focuses on the stability and sustainability of allocations. It identifies a set of payoff allocations that are not only feasible but also resistant to internal challenges.

When players or a social planner seek for a solution concept, there is no limitation in this choice, a priori. However, one can convince quite easily that certain conditions are desirable. One of the most natural is to search for a solution concept that gives allocations that are optimal in the sense of Pareto. In other words, such a solution shares the total worth of the grand coalition. Formally, for the game (N, c), this gives the efficiency axiom:

Efficiency (E): $x \in \mathbb{R}^n$ is efficient if $\mathbf{x}(N) = c(N)$.

Obviously, in terms of social justice, this axiom is very weak: one of the players could get the total worth of the grand coalition even if he does not contribute to any coalition in the game. In order to have a desirable solution concept, more axioms are necessary.

To introduce the next axiom, let consider the following example. Let $N = \{1, 2\}$, and c the characteristic function defined by the following table:

S	$\{1\}$	$\{2\}$	$\{1, 2\}$
$c(S)$	2	3	10

Here, the sum of the worth of players 1 and 2 alone is 5. Thus, the formation of the grand coalition generates a surplus of 5. While sharing the worth of the cooperation between 1 and 2, it seems natural to give at least 2 to the player 1, and 3 to the player 2. Otherwise, one of the player does not have interest in forming the grand coalition. In other words, it is desirable to give each player at least what he gets alone. This condition gives the axiom of individual rationality. Formally, for the CFG (N, c):

Individual Rationality (IR) $x \in \mathbb{R}^n$ is individually rational if for all $i \in N$, $x_i \geq c(\{i\})$.

If an allocation is both efficient and individually rational, it is called an **imputation**. For some games, the set of imputations is empty, as for the game (N, c) defined as follows:

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$c(S)$	3	3	5

Here, the sum of the worth of the players alone exceeds the worth of the grand coalition. In this situation, the two players have no interests in cooperating.

The definition of the Core of a TU-game extends the axiom of individual rationality to any coalition. This gives a condition of collective rationality. Formally, an allocation of the TU-game (N, c) is coalitionally rational if the following condition is verified:

Coalitional Rationality (CR): An allocation $x \in \mathbb{R}^n$ is coalitionally rational if for all coalitions $S \in 2^N$, $\mathbf{x}(S) \geq c(S)$.

The Core of a CFG is defined as the set of allocations that verify the three axioms **(E)**, **(IR)** and **(CR)**. **(IR)** being weaker than **(CR)**, it is not necessary in the definition of the core. This gives the following definition.

Definition 4. The Core of a TU-game (N, v) is the set $\mathscr{C}(N, c)$ is the set of allocation that are efficient and coalitionally rational:

$$\mathscr{C}(N, c) = \{x \in \mathbb{R}^n : \mathbf{x}(N) = c(N) \text{ and for all } S \subset N, \mathbf{x}(S) \geq c(S)\}$$

Geometrically, the Core is a bounded convex polyhedron of dimension at most $n-1$.

Let me give an illustration with the following example:

Example 4. Let $N = \{1, 2, 3\}$, and c the characteristic function defined by the following table:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i,j\}$	N
$c(S)$	2	2	2	5	9

for all $i \neq j \in N$. The core of this game is in the hyperplane $x_1 + x_2 + x_3 = 9$. An allocation of this core must verify the following inequalities: $x_i \geq 2$ for all $i \in N$, and $x_i + x_j \geq 5$ for all $i, j \in N$, $i \neq j$.

Here is an illustration of this Core:

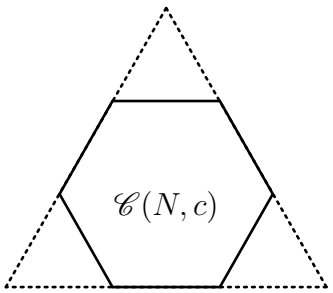


Figure 1.1: Core of the game of (N,c) in the plane $x_1+x_2+x_3=9$

For example, the allocation $(3,3,3)$ verifies all the inequations and is efficient. Therefore, it is an allocation of the Core of (N,c).

The Core is often viewed as the set of stable payoff vectors. The explanation is simple: each payoff vector in this set is efficient and coalitionally rational. Therefore, no feasible allocation can share a bigger value, and no smaller coalition than the grand coalition could get a bigger value if it splits from the grand coalition. Thus, strategically, an allocation in the Core does not allow any profitable deviation: if the Core is non-empty, any allocation inside must be accepted by the players, these are highly stable. Thus, the question of non-emptiness of the Core emerges as soon as one seeks if there is a strategically stable allocation vector.

In what follows, I give some properties of the Core of a CFG.

Definition 5. A solution concept Φ on $\mathbb{G}$ is:

1. **reasonable from above** if

$$\Big((N,c)\in\mathbb{G} \text{ and } x\in\Phi(N,c)\Big)\Rightarrow x_i\leq \max_{S\subset N\setminus\{i\}}\Big(c(S\cup\{i\})-c(S)\Big) \text{ for all } i\in N$$

2. **reasonable from below** if

$$\Big((N,c)\in\mathbb{G} \text{ and } x\in\Phi(N,c)\Big)\Rightarrow x_i\geq \min_{S\subset N\setminus\{i\}}\Big(c(S\cup\{i\})-c(S)\Big) \text{ for all } i\in N$$

3. **reasonable** if it is both reasonable from above and below.

The idea behind reasonability is the following one: let $i\in N$ be a player of the game. Other players may not want to pay i more than his maximal contribution to a coalition.That is, a priori, the biggest payoff he can ask to join a coalition. For reasonability

from below, i may legitimately ask to get at least his minimal contribution: she could refuse to join a coalition that offers her less than this value. Furthermore, i can ask to get his minimal contribution joining any coalition without causing the other members of the coalition to lose anything. Since minimal contribution is lower than that worth of i playing alone, (IR) implies reasonability from below.

Proposition 2. *The Core satisfies reasonability.*

This result gives a lower and an upper bound of the value that each player gets in all allocations of the Core if it is non-empty.

The following property of the Core shows that this concept is strong enough to ensure that its allocations are also allocations of the Core of a well chosen reduced game.

Definition 6 (Davis and Maschler (1965))**.** Let (N, c) be a game, $S \in 2^N$ a non-empty coalition, and $x \in \mathbb{R}^n$ a feasible allocation of this game. The **reduced game** with respect to S and x is the game $(S, v_{S,x})$ defined by the following characteristic function:

$$c_{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset \\ c(N) - x(N \setminus S) & \text{if } T = S \\ \max_{Q \subset N \setminus S} \Big(c(T \cup Q) - x(Q) \Big) & \text{otherwise} \end{cases}$$

The idea behind this definition is the following one: assume that all players in N agree that players in $N \setminus S$ will get a value given by the allocation vector x. Nevertheless, players in S continue to play in order to share what remains. For all $T \subset S$, the value $c_{S,x}$ is the maximal worth T expects to get.

Definition 7. A solution concept Φ on $\mathbb{G}$ satisfies the **reduced game property (RGP)** if it satisfies the following condition: for all $(N, c) \in \mathbb{G}$, $S \subset N$ on-empty and $x \in \Phi(N, c)$, $x_S \in \Phi(S, c_{S,x})$.

(RGP) ensures that, if $x \in \mathbb{R}^n$ solves the game (N, c), then for all $S \in 2^N$, x_S solves the game $(S, c_{S,x})$. Any solution concept that verifies (RGP) can be considered as self-consistent.

Proposition 3. *The Core satisfies the reduced game property on the set of games with a non-empty Core.*

The following condition ensures that, if all the members among a set of players N increase the worth of N, but the worth of all coalition remain unchanged, they should all benefits from this increase.

Definition 8. A solution concept Φ on $\mathbb{G}$ satisfies the **strong aggregate monotonicity (SAM)** if the following condition is met: for all CFGs (N, c), (N, d) on the same set of players N such that $d(N) > c(N)$ and $d(S) = c(S)$ for all $S \subsetneq N$, and $x \in \Phi(N, c)$, then there exists an allocation $y \in \Phi(N, d)$ such that $y_i \geq x_i$ for all $i \in N$.

Obviously, the core satisfies (SAM).

In what follows, I focus on the Bondareva-Shapley theorem, that gives a necessary and sufficient condition of non emptiness of the Core.

Definition 9. A collection of coalitions $\mathscr{B} \subset 2^N$ is called **balanced** if there exists $\lambda : \mathscr{B} \to \mathbb{R}_+$ that balances $\mathscr{B}$, i.e $\sum_{S \in \mathscr{B}} \lambda(S) \mathbb{1}_{i \in S} = 1$ for all $i \in N$. The idea is that each coalition $S \in \mathscr{B}$ is weighted so that each player $i \in N$ has a weight 1 distributed in the coalitions of $\mathscr{B}$. A CFG (N, c) is balanced if for every balanced set of coalitions $\mathscr{B}$ and $\lambda : \mathscr{B} \to \mathbb{R}_+$ that balances $\mathscr{B}$:

$$\sum_{S \in \mathscr{B}} \lambda(S) c(S) \leq c(N)$$

The balancing vector can be interpreted as the fraction of time that each player devotes to each coalition he is a member of. The fact that this fraction is the same for all the players in a coalition can be interpreted as an agreement between player to devote this time to the coalition.

Theorem 1 (Bondareva (1963), Shapley (1967))**.** *A TU-game (N, c) has a non-empty core if and only if (N, c) is balanced.*

Peleg (1965) showed that it is possible to construct minimal balanced collections, which exist in finite number for all $n \in \mathbb{N}$. If the condition of Bondareva-Shapley is met on the set of minimal balanced collections, it is also met on the set of balanced collections. Yet, the number of minimal balanced collection grows really fast with n (201 076 for $n = 6$). Still, this condition stay really practical from a theoretical point of view: for example to show that a class of games has a non-empty Core.

Definition 10. A TU-game (N, c) is called **totally balanced** if for all $S \subset N$, the subgame $(S, c_{|S})$ is balanced.

The previous definition is really strong: this ensures that totally balanced games have a non-empty core, but also with a non-empty core for all subgames. For those games, cooperation is always sustainable, even for a subset of players.

Shapley also proved the following sufficient condition of non-emptiness of the Core.

Proposition 4 (Shapley (1971))**.** *If the CFG (N, c) is convex, then $\mathscr{C}(N, c) \neq \emptyset$.*

Because the core may be empty, Shapley and Shubik (1966) introduced a generalization. For all $\varepsilon \in \mathbb{R}$, they define the ε-core as the set of efficient allocation vectors that are coalitionally rational, up to ε. Formally, it is defined as follows.

$$\mathscr{C}_\varepsilon(N, c) = \{x \in \mathbb{R}^n : \mathbf{x}(N) = c(N) \text{ and for all } S \subset N, \mathbf{x}(S) \geq c(S) - \varepsilon\}$$

The idea behind the definition of this core is as follows. A social planner wants the players to form the grand coalition. To encourage cooperation, he decides that if a coalition decides to leave the grand coalition, it receives a penalty of ε (ε may be negative, in that case, ε represents a bonus for leaving the grand coalition). Clearly, regardless of whether the core is empty or not, there exists ε large enough to make the ε-core non-empty, and another ε' small enough (maybe negative) to make the ε'-core empty.

With this definition, it is possible to define the **least-core** $\mathscr{C}_l(N, c)$ of a CFG (N, c), introduced by Maschler, Peleg and Shapley (1979), as the intersection of all non-empty ε-cores. Formally, $\mathscr{C}_l(N, c) := \bigcap_{\epsilon \text{ s.t.} \mathscr{C}_\varepsilon(N,c) \neq \emptyset} \mathscr{C}_\varepsilon(N, c)$. The fact that ε-cores are closed set and they are non-decreasing in ε for inclusion ensures the existence of $\varepsilon_0 \in \mathbb{R}$ such that $\mathscr{C}_l(N, c) = \mathscr{C}_{\varepsilon_0}(N, c)$. ε_0 represents the biggest penalty such that players in N have interest in cooperating. In a situation where players in N cooperate in two different games (N, c) and (N, d) such that $\mathscr{C}(N, c)$ is non-empty and $\mathscr{C}(N, d) = \emptyset$, a social planner may find $\varepsilon \in \mathbb{R}$ such that $\mathscr{C}_{-\varepsilon}(N, c) \neq \emptyset$ and $\mathscr{C}_\varepsilon(N, d) \neq \emptyset$ if it is possible to transfer the values between the games and force cooperation between players for each game.

1.1.3.3 The Shapley Value

The Shapley value, pioneered by Lloyd Shapley, is the most know solution concept in cooperative games. It provides a systematic approach to fairly distributing the value generated by a coalition of players, acknowledging the unique contributions each player brings to the table. This concept not only encapsulates the essence of equitable outcomes but also serves as a powerful tool for understanding the mechanics of collaboration.

The Shapley value Φ^{Sh} is an allocation rule that gives a unique allocation vector for each TU-game. This idea behind this concept is to provide a rule which, considering that the grand coalition will form, share thes worth in a way that seems fair for all the players. In what follows, $\Phi_i^{Sh}(N, c)$ will denote the payoff given to player i for his participation in the game (N, c). The Shapley value is characterized by four axioms that are detailed in what follows.

The first axiom that is natural to consider to define the Shapley value is the axiom of Efficiency. As for the Core, it seems natural to share all the worth of the grand coalition. This axiom also ensures that the chosen allocation is Pareto optimal.

The second axiom is an egalitarian axiom. In words, the idea is that, if two players contribute in the same way to all coalitions they join, it seems reasonable that they get the same payoff. Formally, an allocation rule verifies **equal treatment of equals (ETE)** if for each CFG (N, c), $i, j \in N$ distincts, and for all $S \subset N \setminus \{i, j\}$, $c(S \cup \{i\}) = c(S \cup \{j\})$, then $\Phi_i(N, c) = \Phi_j(N, c)$.

The third axiom ensures that, if a player does not contribute to any coalition he joins, he should get 0, as he does not create any surplus. Formally, a player i of the CFG (N, c) is called a **null player** if for all $S \subset N \setminus \{i\}$, $c(S \cup \{i\}) = c(S)$. An allocation rule Φ verifies the **null player property (NP)** if, for all $(N, c) \in \mathbb{G}$, if $i \in N$ is a null player, then $\Phi_i(N, c) = 0$.

The last axiom creates a link between two games on the same set of players. Let $N \subset \mathbb{N}$ a set of players, (N, c) and (N, d) two games in $\mathbb{G}^N$. The sum of the two games is the game $(N, c + d)$ such that for all $S \in 2^N$, $(c + d)(S) = c(S) + d(S)$. They idea is as follows: if the players in S can cooperate in both games, what they get by forming the coalition S is the sum of what they get in both games. This idea presupposes that the

problem of the summed game can be divided into two separeted games without changing the final allocation of the game. One interpretation is that if there is many problems that can be modelized into CFGs, the order of the resolution does not affect the result. This gives the **additivity (A)** axiom, which is formally as follows: for a pair of games on the same set of players $N \subset \mathbb{N}$, (N, c) and (N, d), then $\Phi(N, c + d) = \Phi(N, c) + \Phi(N, d)$.

Proposition 5 (Shapley (1953b), Shubik (1962))**.** *There exists a unique allocation rule on $\mathbb{G}$ that verifies the four axioms (E), (ETE), (NP) and (A). This allocation rule is called the Shapley value, denoted Φ^{Sh}, and gives to each player $i \in N$ the following payoff:*

$$\Phi_i^{Sh}(N, c) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \Big(v(S \cup \{i\}) - v(S) \Big)$$

These four axioms are logically independent, so there should exist other rules of allocation that satisfies three of the axioms, but not the last one.

The interpretation of this formula is the following: the last part of it represents the contribution of the player $i \in N$ in the coalition $S \subset N \setminus \{i\}$, so the Shapley value gives to each player a value that depends only of his contributions. Moreover, for all $i \in N$, the following result holds:

$$\sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} = 1$$

Therefore, the Shapley value is a weighted mean of contributions of players to coalitions. $\frac{s!(n-s-i)!}{n!}$ represents the weight of the contribution of i in S.

Example 5. Let (N, c) the game representing a vote between the members of the security Council of the United Nations Organization. Thus $N = \{1, \ldots, 15\}$ with $\{1, \ldots, 5\}$ representing the veto players, i.e. China, France, USA, UK and Russia. Therefore c is the characteristic function defined by:

$$c(S) = \begin{cases} 1 & \text{if } \{1, \ldots, 5\} \subset S \text{ and } s \geq 9 \\ 0 & \text{otherwise} \end{cases}$$

If $i \in N$ is not a veto player, then for all $S \subset N \setminus \{i\}$, $c(S \cup \{i\}) - c(S) = 1$ if and only if $s = 8$ and $\{1, \ldots, 5\} \subset S$. The number of those situations is as follows: the veto players must be in the coalition S, and $s = 8$, so there are 3 more players in S, taken in the

other players. Therefore, there are $C_9^3 = 84$ situations like this. Thus, a non-permanent member of the security Council has a power of $84 \times \frac{8!.6!}{15!} \simeq 0,00186$.

All the veto players have the same power, by the axiom of equal treatment of equals. Therefore, they all have a power approximately equals to $\frac{1-10\times0,00186}{5} \simeq 0.196$.

Thus, the permanent members alone possess approximately $5 \times 0.196 \simeq 98\%$ of the total power of the security council.

The Shapley value has been very widely used in real cases, for example for the distribution of the price of the installation of a new water distribution network between several municipalities in Young (1982).

The following proposition gives some properties on the Shapley value, and highlights a link with the Core of a CFG.

Proposition 6. *Let* $(N, c) \in \mathbb{G}$.

- *If* (N, c) *is non-negative and monotonic, then for all* $i \in N$, $\Phi_i^{Sh}(N, c) \geq 0$
- *If* (N, c) *is superadditive, then for all* $i \in N$, $\Phi_i^{Sh}(N, c) \geq c(\{i\})$
- *If* (N, c) *is convex, then* $\Phi^{Sh}(N, c) \in \mathscr{C}(N, c)$.

The last result comes from Shapley (1971). This property is sufficient, but not necessary: there are a lot of non-convex games for which the Shapley value is in the Core. In the convex case, the property is even stronger: the Shapley value is actually the weighted barycenter of the Core.

1.1.3.4 Bargaining set, Nucleolus, Prekernel and Kernel

In the expansive field of cooperative game theory, numerous solution concepts have been devised to address the complex issue of allocating values within coalitions. While a some of these concepts are beyond the scope of this thesis, it is still imposrtant to acknowledge the existence of various solution frameworks, each contributing to the understanding of cooperative decision-making.

Among these concepts, four notable solutions are the bargaining set, nucleolus, prekernel, and kernel, which are all linked together. While these concepts are of great importance in cooperative game theory, I refer to those who are interested in the comprehensive explanations provided in « Introduction to the Theory of Cooperative Games » (2007) by

Bezalel Peleg and Peter Sudhölter. This work offers an in-depth understanding of these concepts, their applications, and their significance in cooperative game theory.

1.1.4 Conclusion

In conclusion, the exploration of TU-games provides a nuanced understanding of how rational individuals strategically collaborate to achieve optimal outcomes within coalitions. However, it is imperative to recognize the current chapter's temporal constraint. The analysis herein has predominantly been static, focusing on the formation and stability of coalitions at a given point in time. The omission of temporal dynamics is intentional, because this point will be the main topic in the first paper that I present in this thesis. This part will focus on dynamic TU-games, where the strategic interactions between players varies over time, introducing new mechanisms in the formation of coalitions.

Furthermore, it is crucial to acknowledge the limitations inherent in the TU-games framework. One notable constraint is the assumption that the value of a coalition is most of the time independent of external factors—an assumption that does not account for externalities. In reality, the value generated by a coalition may indeed be influenced by external forces, challenging the applicability of pure cooperative game models in certain contexts. The consideration of externalities will be a focal point in the subsequent section, highlighting the need for more comprehensive models that capture the interplay between coalitions and their external environment.

The next part, focuses on externalities and their impact on cooperative games, providing a more comprehensive understanding of the intricacies that govern strategic collaborations.

1.2 Partition Function Games

Characteristic function games prove highly valuable for analyzing scenarios involving the formation of coalitions between entities. It is implicitly assumed that when players form a coalition, their worth is solely contingent upon the members within that coalition, with side effects being entirely disregarded. While this model is convenient for many situations, it falls short in describing cooperation when side effects come into play. In

today's globalized world, the benefits of cooperation between two countries are often influenced by the actions or reactions of other nations. For instance, if two countries collaborate in declaring war against a third, the outcome can significantly vary depending on whether the third country stands alone or receives support from the rest of the world.

The consideration of side effects necessitates the introduction of externalities into the model. Partition Function Games, a generalization of TU-games first introduced by Thrall and Lucas (1963), addresses this need by making the value of cooperation dependent on the actions of other players. While initially underused, the literature on Partition Function Games has experienced substantial growth since the 2000s.

This theory occupies a middle ground between cooperative and non-cooperative games. The incorporation of externalities into TU-games renders it a model that is both cooperative, through coalition formation, and strategic, through the inclusion of externalities. Consequently, there is a need to develop new solution concepts that account for both aspects. Additionally, literature exists that establishes connections with non-cooperative game theory, exploring the formations of player partitions.

This chapter aims to provide the basic definitions of Partition Function Games and introduce some specific games and solution concepts.

1.2.1 Basic definitions

As for CFGs, in this part S and N denote respectively coalitions and the grand coalition. A partition π of N is a splitting of N into disjoint coalitions. Π_S denotes the set of partitions of $S \subset N$, and for all $i \in N$, $\pi \in \Pi_N$, $\pi(i)$ is the cell of π containing i. I will use the following notation:

- Let $S, T \subset N$ such that $S \cap T = \varnothing$, $\pi_S = \{P_1, \dots, P_l\}$ a partition of S, $\pi_T = \{P_{l+1}, \dots, P_m\}$ a partition of T. I denote by $\pi_S \otimes \pi_T$ the partition $\{P_1, \dots P_m\}$ of $S \cup T$ obtained by concatening π_S and π_T.
- Let $\pi = \{S_1, \dots, S_l\} \in \Pi_N$, $S \subset N$. $\pi \setminus S := \{S_1 \setminus S, \dots, S_l \setminus S\}$.

Definition 11. A **partition function game**, thereafter PFG, is a pair (N, v) where $v : \{(S, \pi) \in 2^N \times \Pi_N \mid S \in \pi\} \longmapsto \mathbb{R}$. When it is defined, $v(S, \pi)$ is the value created by the coalition S when every $i \notin S$ belongs to the coalition $\pi(i)$. By convention, for all

$\pi \in \Pi_N$, $v(\emptyset, \pi) = 0$. The elements of $\mathscr{E}(N) := \{(S, \pi) \in 2^N \times \Pi_N \mid S \in \pi\}$ are called **embedded coalitions**.

The set of PFGs with the set of players N is denoted by $\mathbb{P}^N$, and the set of all PFGs is denoted $\mathbb{P} = \cup_{N \subset \mathbb{N}} \mathbb{P}^N$.

An **outcome** of a PFG (N, v) is a pair $(\pi, x) \in \Pi_N \times \mathbb{R}^n$ consisting of a partition of the players $\pi \in \Pi_N$, and an allocation vector x that is **Coalitionally Efficient (E)**, i.e. for each coalition $S \in \pi$, the wealth earned by the coalition is entirely distributed between the members of the coalition, that is:

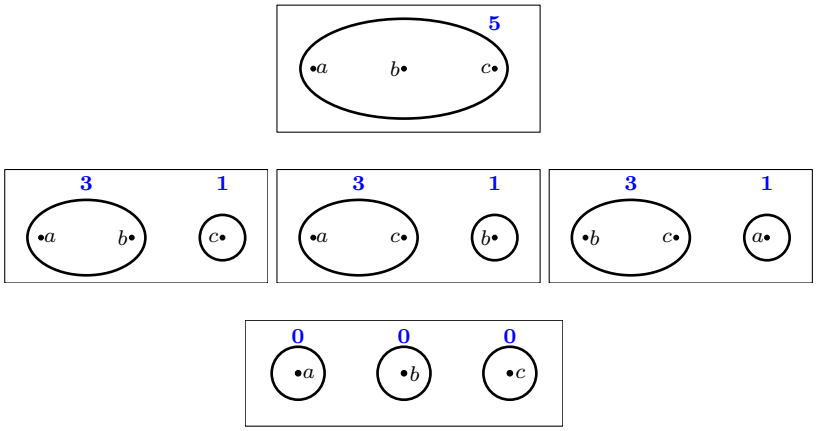
$$(E): \sum_{i \in S} x_i = v(S, \pi) \ \forall S \in \pi$$

The set of outcomes of a PFG (N, v) is denoted $\mathscr{O}(N, v)$.

By definition, for partition function games the worth of a coalition depends on the partition it is embedded in, thus on which partition the other players form.

To fix the ideas, consider the following example:

Example 6. Let (N, v) be the PFG with 3 players, a, b, c, defined as follows:



The coalitions are symbolised by the circles, with the supscripts indicating their valuations.

Here: $v(\{a\}, \{\{a\}, \{b\}, \{c\}\}) = 0$, $v(\{a\}, \{\{a\}, \{b, c\}\}) = 1$, but $v(\{a\}, N)$ is not defined because $\{a\}$ is not a cell of the partition $\{N\}$.

In this game, if a decides to stay on his own, b and c have interest in grouping up (and a prefers also when they group up).

1.2.2 Some Classes of Partition Function Games

For PFGs, side effects are called externalities. It is possible to define classes of games for which merging two coalitions or splitting a coalition generates benefits in the same way to the other ones.

TU-games can be seen as a subset of PFGs: for all CFGs $(N, c) \in \mathbb{G}$, it is possible to define an associated PFG $(N, v) \in \mathbb{P}$ such that for all embedded coalitions $(S, \pi) \in \mathscr{E}(N)$, $v(S, \pi) = c(S)$. Here, the worth of a coalition S does not depend of π. Reciprocally, if a PFG (N, v) can be described by a TU-game (N, c), such that for all $(S, \pi) \in \mathscr{E}(N)$, $v(S, \pi) = c(S)$, then (N, v) is called **externality-free**.

When the merge of two coalitions is always beneficial to any other coalition, a PFG has **positive externalities**. Formally, a PFG (N, v) has positive externalities when for all mutually disjoint coalitions $C, S, T \subset N$ and $\pi \in \Pi_{N\setminus(C\cup S\cup T)}$:

$$v(C, \{C, S \cup T\} \otimes \pi) \geq v(C, \{C, S, T\} \otimes \pi)$$

This means that each coalition has a higher value if two other coalitions decide to group up instead of remaining separated. Likewise, it has **negative externalities** if for all mutually disjoint coalitions $C, S, T \subset N$ and $\pi \in \Pi_{N\setminus(C\cup S\cup T)}$:

$$v(C, \{C, S \cup T\} \otimes \pi) \leq v(C, \{C, S, T\} \otimes \pi)$$

In other words, each coalition has a lower value if two other coalitions decide to group up instead of remaining separated.

A PFG (N, v) is **symmetric** if the identity of the players does not affect the payoffs. Formally, a PFG is symmetric if for every permutation σ of N, and $(S, \pi) \in \mathscr{E}(N)$, with $\pi = \{S_1, \ldots, S_l\}$, the following equality holds:

$$v(S, \pi) = v(\sigma(S), \{\sigma(S_1), \ldots, \sigma(S_l)\})$$

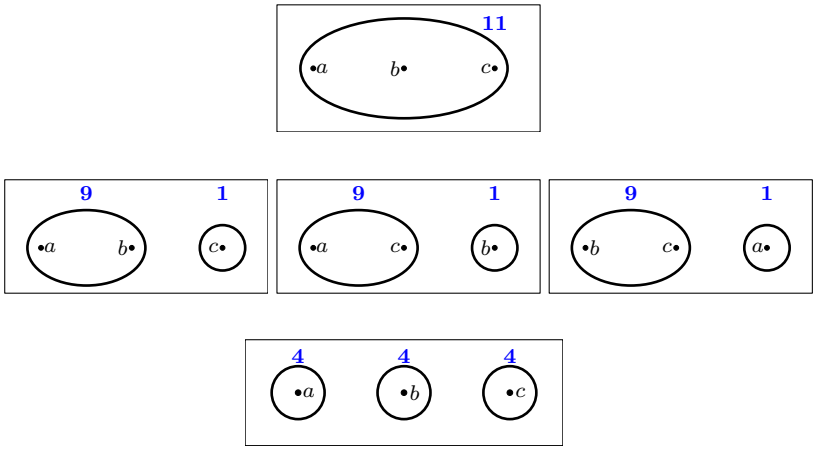
with $\sigma(T) := \{\sigma(i) \mid i \in S\}$

As for CFGs, Maskin (2003) and Hafalir (2007) define **superadditivity** for partition function games as follows. A PFG (N, v) is called superadditive if for all disjoint coalitions $S, T \subset N$ and any partition $\pi \in \Pi_{N\setminus S\cup T}$, the following inequality holds:

$$v(S \cup T, \{S \cup T\} \otimes \pi) \geq v(S, \{S, T\} \otimes \pi) + v(T, \{S, T\} \otimes \pi)$$

For PFGs, superadditivity is not strong enough to ensure full cooperation between the members of N, as shown in the following example.

Example 7 (Hafalir (2007))**.** Let (N, v) be the PFG with 3 players, a, b and c defined as follows:



While (N, v) is superadditive, the outcome $((4, 4, 4), \{\{a\}, \{b\}, \{c\}\})$ dominates any outcome where N is formed.

Since superadditivity is not strong enough to ensure cooperation, it becomes necessary to define a new class of games for which the players have always interest to group up. This is the case of **cohesive** games defined as the PFGs (N, v) those verify the following inequalities:

$$v(N, \{N\}) \geq \sum_{S \in \pi} v(S, \pi) \text{ for all } \pi \in \Pi_N$$

While for CFGs, superadditivity implies that the worth of N is always greater than the sum of the worthes of any partition π of N, it is not the case for PFGs, as shown in the previous example. As none of these concepts is stronger than the others, de Clippel and Serrano (2008) and Hafalir (2007) define a concept which is stronger, called **full cohesiveness**. Formally, a PFG (N, v) is fully cohesive if for any coalition $S \subset N$ and all partitions $\pi_S \in \Pi_S$ and $\pi_{N \setminus S} \in \Pi_{N \setminus S}$ the following inequality holds:

$$v(S, \{S\} \otimes \pi_{N\setminus S}) \geq \sum_{S_j \in \pi_S} v(S_j, \pi_S \otimes \pi_{N\setminus S})$$

Clearly, full cohesiveness implies both cohesiveness and superadditivity. Convex games also satisfy cohesiveness and superadditivity. A PFG (N, v) is **convex** if for all $S, T \subset N$ and $\pi \in \Pi_{N\setminus(S\cup T)}$:

$$v(S\cup T, \{S\cup T\}\otimes\pi)+v(S\cap T, \{S\cap T, S\setminus T, T\setminus S\}\otimes\pi) \geq v(S, \{S, T\setminus S\}\otimes\pi)+v(T, \{T, S\setminus T\}\otimes\pi)$$

This class of games includes all the games where any pair of coalitions (even if they are not disjoint) have interest in grouping up. If the condition $\mid S \setminus T \mid = \mid T \setminus S \mid = 1$ is added, the definition is equivalent to the former one.

As CFGs, for any set of players N, $\mathbb{P}^N$ is a linear subspace of $(\mathbb{R}^{\mathscr{E}(N)}, +, .)$, hence it is possible to define a basis of $\mathbb{P}^N$.

Definition 12 (Ju (2007))**.** Let $N \subset \mathbb{N}$ be a set of players. For each embedded coalition $(S, \pi) \in \mathscr{E}(N)$, the **standard basis game** $(N, B_{(S,\pi)}$ is given for all $(T, \rho) \in \mathscr{E}(N)$ by:

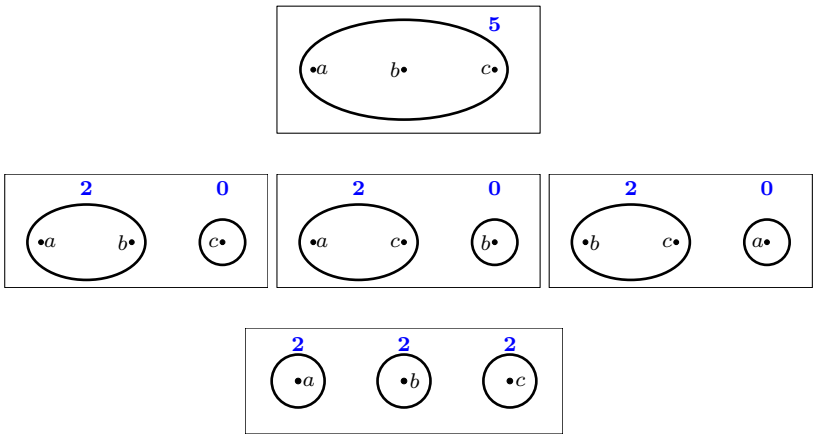
$$B_{S,\pi}(T, \rho) = \begin{cases} 1 & \text{if } (T, \rho) = (S, \pi) \\ 0 & \text{otherwise} \end{cases}$$

The set of standard basis games is a linear basis of $\mathbb{P}^N$ for all sets of players $N \subset \mathbb{N}$.

1.2.3 The Core

In this chapter, I introduce some core concepts for partition function games. The main issue in the definition of the core for PFGs is that the worth of a coalition depends on externalities. Therefore, it becomes difficult to generalize coalitional rationality. As a consequence, it is possible to define many different core concepts, based mainly on differences in what coalitions can expect if they deviate from the grand coalition. As for TU-games, any core is based on the idea that coalitions have a well defined value, and they will block any allocation that does not meet this demand. The main question is what is this value. The following example highlights the problem.

Example 8. Let (N, v) be the PFG with 3 players, a, b and c defined as follows:



The question here is what will happen if a player alone deviates from the grand coalition. From what he expects, it is possible to define the minimum payoff he will ask to cooperate in the grand coalition. It is straightforward to see that there are two options if a player deviates from the grand coalition: either the other players stay together, and the deviating player gets 0. In this case, giving $\frac{5}{3}$ to each player makes the grand coalition sustainable. Otherwise, the other players split. In this case, all players can ask at least to get 2 in the grand coalition. Since the worth of the grand coalition in this game is 5, the grand coalition cannot be sustainable.

In the previous example, what the remaining players will do after the deviation of a coalition is not clear, depending on their motives. They may want for instance to maximize their total payoff, punish or reward the deviating coalition. Each motive corresponds to a different concept of core. In what follows, I introduce the core concepts that appear in the literature about the core of a PFG. Since most of the literature admits that the action of the remaining players is not clear, I introduce most of the existing core concepts.

1.2.3.1 The s-Core

The first core-concept I define is the cornerstone of some results on non-emptiness of the other cores for PFG. It has been defined by Hafalir (2007). The main idea is that, if a coalition deviates, the remaining players will consider that cooperation is over, and, then, will deviate by splitting into singletons. This concept presupposes that, after the deviation of a coalition, the remaining players will not consider their benefits, or the

benefits of the deviating coalitions. Thus, the response to a deviation will be the same for every game. Formally, for all PFGs (N, v), one can define the singleton characteristic function c^s such that for all $S \subset N$, $c^s(S) = v(S, \{S\} \otimes [N \setminus S])$ where for all coalitions $A \subset N$, $[A]$ is the partion composed by the singletons of A: $[A] := \{i \mid i \in A\}$. The definition of the s-core follows.

Definition 13 (Hafalir (2007))**.** The **core with singleton expectations**, or **s-core** is the following set:

$$C^s(N, v) := \mathscr{C}(N, c^s) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N, \{N\}) \text{ and } \mathbf{x}(S) \geq c^s(S) \ \forall \ S \subset N\}$$

While this core concept uses a rule that does not depend on the game, it is mainly used to study games with positive or negative externalities, as I will show in following paragraphs. Hafalir (2007) gives a sufficient condition of non-emptiness of the s-core.

Proposition 7. *For all convex PFGs* (N, v)*,* $C^s(N, v) \neq \emptyset$*.*

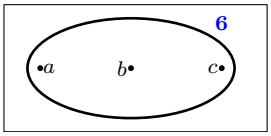
Proof. The proof is straightforward, and relies on the fact that if (N, v) is convex, then(N, c^s) is a convex CFG, thus has a non-empty core. □

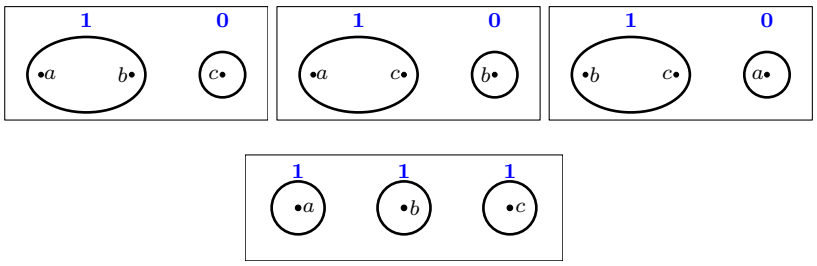
1.2.3.2 The α-Core

One of the first rule that one can think of about the reaction of the remaining players when a coalition deviates is that the latter may expect the worst situation. It is a very conservative rule: coalitions will deviate only if they cannot get in the grand cooperation at least what they have alone in the worst case scenario. In this situation, the players are pessimistic about what they can get. This idea is supported by Aumann and Peleg (1960).

Formally, for all PFGs (N, v), on can define the pessimistic characteristic function c^α such that for all $S \subset N$, $c^\alpha(S) = \min_{\pi \in \Pi_{N \setminus S}} v(S, \{S\} \otimes \pi)$. The following example illustrates this approach:

Example 9. Let (N, v) be the PFG with 3 players, a, b and c defined as follows:





In this game, if the players are pessimistic about what they can have if they split, they will ask to get at least 0 in the formation of the grand coalition, even if they could get 1 if is every player is alone.

This gives the following definition.

Definition 14. For all PFGs (N, v), the **pessimistic core**, or **α-core** is defined as:

$$C^{\alpha}(N,v) := \mathscr{C}(N, c^{\alpha}) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N, \{N\}) \text{ and } \mathbf{x}(S) \geq c^{\alpha}(S)\}$$

The fact that this concept presupposes that coalitions ask the minimum they can get alone makes it an upper bound for inclusion of all core-concepts that can be defined. As a first result, it is straightforward to see that $C^{s}(N,v) \subset C^{\alpha}(N,v)$. As a consequence, the following proposition holds.

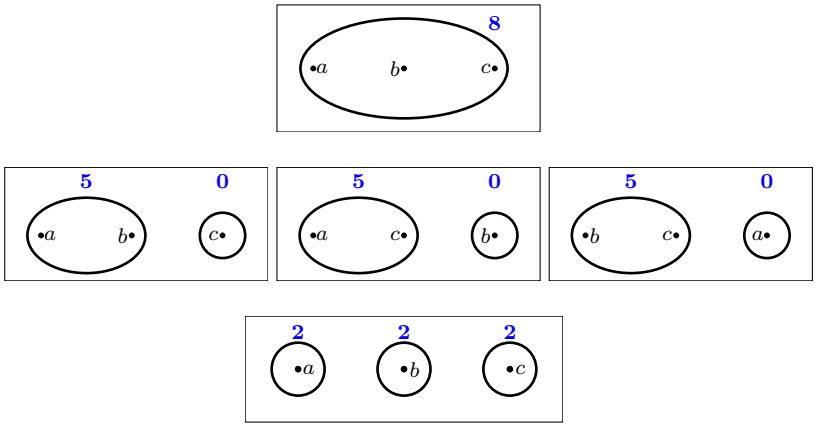
Proposition 8. *Let (N, v) be a convex PFG. Thus, $C^{\alpha}(N,v) \neq \emptyset$.*

This approach can be used to convert a partition function game into a CFG that is easier to solve. The main issue with this concept is that the players are too conservative: there exists situations in which after a deviation of a coalition, the threat of the remaining players to form the worst case partition is not credible. This creates a need to define other core concepts.

1.2.3.3 The ω-core

The ω-core is the opposite vision of the α-core. While for the formation of the α-core the coalitions ask for the minimum value they can get by deviating, in this concept, they will ask for the maximum. This is a really optimistic concept: if there is a chance that a coalition can get a huge worth by deviating, they will ask for it, as shown in the following example.

Example 10. Let (N, v) be the PFG with 3 players, a, b and c defined as follows:



In this game, if the players are optimistic about what they can have by splitting, they will ask to get a minimum of 2. The fact that after a deviation of a player alone, the other ones have interests to stay grouped does not affect that value.

Formally, for all PFGs (N, v), one can define the optimistic characteristic function c^ω such that for all $S \subset N$, $c^\omega(S) = \max_{\pi \in \Pi_{N \setminus S}} v(S, \{S\} \otimes \pi)$. The definition of the ω-core comes as follows:

Definition 15 (Shenoy (1979))**.** For all PFGs $(N, v) \in \mathbb{P}$, the **ω-core**, also called **optimistic core** is the following set:

$$C^\omega(N, v) := \mathscr{C}(N, c^\omega) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N, \{N\}) \text{ and } \mathbf{x}(S) \geq c^\omega(S)\}$$

While each coalition is given at least the maximum they could get by deviating in the formation of the grand coalition, it is straightforward to see that for all $(N, v) \in \mathbb{P}$, $C^\omega(N, v) \subset C^s(N, v) \subset C^\alpha(N, v)$. Hafalir (2007) gives the following sufficient condition of non-emptiness of the ω-core:

Proposition 9. *Let (N, v) be a convex PFG with negative externalities. Then, $C^\omega(N, v) \neq \emptyset$.*

Proof. Let (N, v) be a convex PFG with negative externalities. Thus, for all embedded coalitions $(S, \pi) \in \mathscr{E}(N)$, $v(S, \pi) \leq v(S, \{S\} \otimes [N \setminus S])$. Hence, for all coalitions $S \subset N$, $c^\omega(S) = \max_{\pi \in \Pi_{N \setminus S}} v(S, \{S\} \otimes \pi) = v(S, \{S\} \otimes [N \setminus S]) = c^s(S)$. Therefore, $C^\omega(N, v) = C^s(N, v)$. While (N, v) is convex, $C^s(N, v) = C^\omega(N, v) \neq \emptyset$. □

1.2.3.4 The m-core

While the s-core assumes that, after a deviation of a coalition, the remaining players split into singletons, Maskin (2003), Hafalir (2007), Ambec and Ehlers (2008), and McQuillin (2009) consider the opposite case. The m-core is defined as the core for which, after a deviation of a coalition from the grand coalition, the remaining players decide to stay grouped and form the complementary coalition.

Formally, for all PFGs (N, v), one can define the merge characteristic function c^m such that for all $S \subset N$, $c^m(S) = v(S, \{S, N \setminus S\})$. As for the s-core, the definition of c^m does not depend on the coalitions and externalities: the rule describing the reaction of the remaining players to a deviation does not depend on the game. The m-core can be defined as follows.

Definition 16 (Hafalir (2007))**.** For all PFGs $(N, v) \in \mathbb{P}$, the m**-core**, also called **merge core** is the following set:

$$C^m(N, v) := \mathscr{C}(N, c^m) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N, \{N\}) \text{ and } \mathbf{x}(S) \geq c^m(S) \ \forall \ S \subset N\}$$

Unlike for the s-core, it is not possible to infer a convex characteristic function game from the concept of merging players after a deviation. Anyway, the convexity is not a sufficient condition of non-emptiness of the m-core. However as for all PFGs $(N, v) \in \mathbb{P}$, the α-core and the ω-core are bounds for inclusion, so it is straightforward to see that $C^\omega(N, v) \subset C^m(N, v) \subset C^\alpha(N, v)$. As a consequence, the m-cores of convex PFGs with negative externalities are non-empty.

The previous definitions do not depend on the game for the s and m-core, unlike the α and ω ones. However, for the latter ones, the reaction of players after a deviation of a coalition S from the grand coalition is determined by the worth of S, and not by what happens for the remaining players, unlike the following concept.

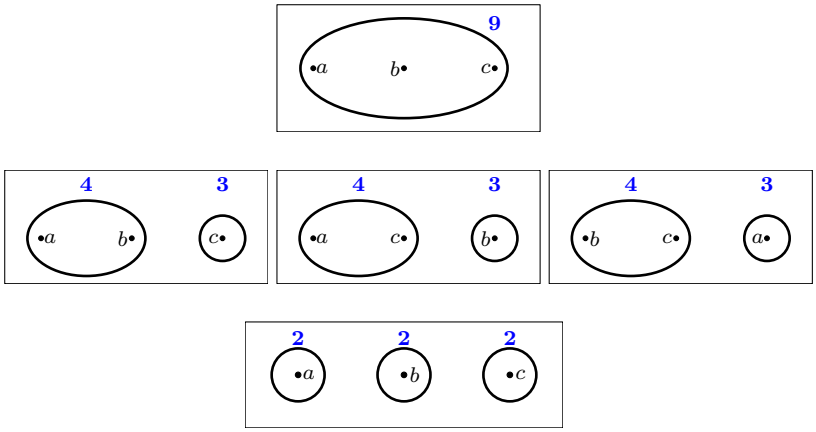
1.2.3.5 The Rational Expectation Core

In the previous concepts, deviating players assume deterministic behaviour of the remaining players, or have expectation based only on their own perspective without considering the benefits of the remaining players. Hafalir (2007) defines a concept in

which deviating players assume that the remaining players will choose the partition that maximizes their total payoff and that is the most harmful for the deviating players.

Formally, for all PFGs $(N, v) \in \mathbb{P}$, one can define a characteristic function c^{RE} such that for all $S \subset N$, $c^{RE}(S) = \min_{\pi(S) \in \Pi^*(S)} v(S, \{S\} \otimes \pi(S))$ where $\Pi^*(S) := \underset{\pi \in \Pi_{N \setminus S}}{\operatorname{argmax}} \sum_{T \in \pi} v(T, \{S\} \otimes \pi(S))$. Here, the remaining players choose, among the partitions which gives the best possible payoff, the worst one for the deviating players. The following example illustrates this concept.

Example 11. Let (N, v) be the PFG with 3 players, a, b and c defined as follows:



In this situation, if a player deviates alone from the grand coalition, the two remaining players can choose to split or to stay grouped. Since the two situations give them the same payoff, they choose the worst situation for the deviating player. Therefore, they decide to split.

From this idea, Hafalir (2007) defines the rational expectation core as follows:

Definition 17. For all PFGs $(N, v) \in \mathbb{P}$, the **rational expectation core** is the following set:

$$C^{RE}(N, v) := \mathscr{C}(N, c^{RE}) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N, \{N\}) \text{ and } \forall S \subset N,\ \mathbf{x}(S) \geq c^{RE}(S)\}$$

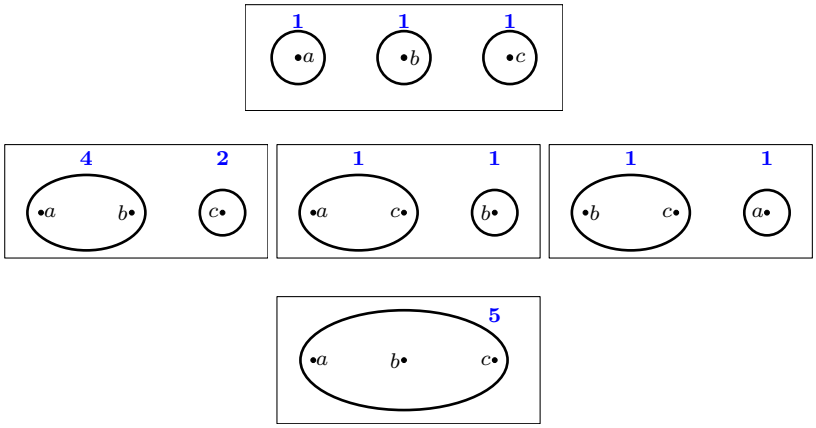
Although this approach brings to mind von Neumann's minimax theorem (1928) and seems to be relevant in terms of rationality of the remaining players, it is far from being perfect. On the one hand, the choice of the coalition can lead to disagreements between the players: since all possible partitions give the same payoff to the group of players,

choosing one over another would inevitably result in players having a smaller payoff than they could hope for. On the other hand, this concept presupposes full cooperation of the remaining players, which is unlikely to happen considering the fact that there is no mechanism to prevent a smaller group of players from trying to take advantage of the situation, even if it causes damage to the others.

1.2.3.6 The δ-core

In all the previous concepts, forming the grand coalition presupposes that the partition formed is the grand coalition. However, sometimes this is not the one that garantees the best total payoff for the players as shown in the following example.

Example 12. Let (N, v) be the PFG with 3 players, a, b and c defined as follows:



In this situation, it is straightforward to see that it is pointless to form $\{N\}$ when players prefer to form $\{\{a, b\}, \{c\}\}$. However, one may argue that a, b, and c can decide to cooperate anyway, while still forming the coalition $\{\{a, b\}, \{c\}\}$.

This idea gives rise to new types of core-concepts in the literature, containing not allocations but outcomes. Following this idea, Hart and Kurz (1983) define the δ-core. The main idea is that, for a given partition, after a coalition deviates, the remaining players decide to stay in the same configuration without the deviaiting players. This concept generalizes the concepts of m-core and s-core. It is defined as follows:

Definition 18. For all PFGs $(N, v) \in \mathbb{P}$, the **δ-core** is the following set:

$$C^{\delta}(N, v) := \{(x, \pi) \in \mathscr{O}(N, v) \mid \mathbf{x}(S) \geq v(S, \{S\} \otimes \pi \setminus S) \ \forall \ S \subset N\}$$

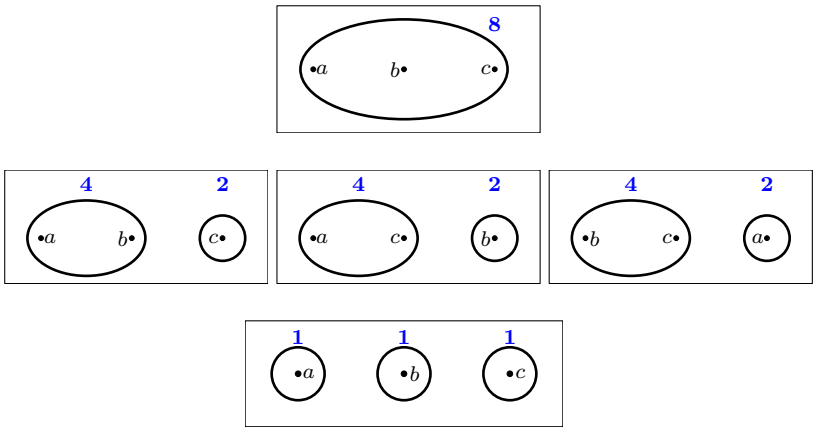
It is straightforward to see, considering $\pi = \{N\}$, that $C^m(N,v) \subset C^\delta(N,v)$ for all $(N,v) \in \mathbb{P}$. As a consequence, the following result holds:

Proposition 10. *Let (N,v) be a convex PFG with negative externalities. Then, $C^\delta(N,v) \neq \emptyset$.*

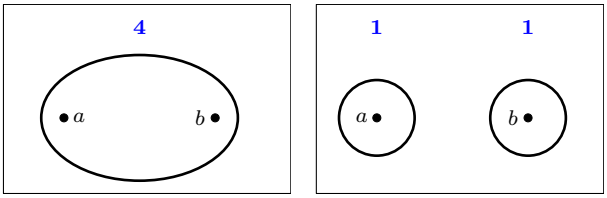
1.2.3.7 The Recursive Core

The recursive core, introduced by Kóczy (2007), is a core concept in which, as for the δ-core, the players can form the grand coalition but still can organize themselves into other partitions. The principle is recursive: if a coalition deviates from the grand coalition, the other players play in a smaller game, without the deviating coalition, and to seek for a core outome of this game. The following example illustrates the question at hand here:

Example 13. Let (N,v) be the PFG with 3 players, a, b and c defined as follows:



If c deviates from the grand coalition, the two other players a and b have to consider the subgame they are playing without c, which is the following one:



To define well the recursive cores, it is necessary to define residual games. The idea of this concept is to reduce a PFG (N,v) for a non-empty subset of players $S \subset N$.

However, considering that there are externalities, it is not trivial what PFG the players in S are playing. One answer to this question is to fix a partition for the other players. Once this partition is given, the definition of the game is clear.

Definition 19. Given a PFG (N, v), a coalition $S \subset N$, and a partition $\pi \in \Pi_{N\setminus S}$, the residual game $(S, \tilde{v}_\pi)$ is the PFG where:

$$\begin{aligned} \tilde{v}_\pi : 2^S \times \Pi_S &\longrightarrow \mathbb{R} \\ (Q, \tilde{\pi}) &\longmapsto v(Q, \tilde{\pi} \otimes \pi) \end{aligned}$$

$\tilde{v}_\pi(Q, \tilde{\pi})$ defined if $Q \in \tilde{\pi}$.

In this definition, the focus is on the game played by the members of S, assuming that the rest of the players form the partition π.

This definition gives all the possible subgames the players in a subset of players S can play, considering all the possible configuration of the other players. Given this definition, it becomes easier to define the PFG that the remaining players are playing after a coalition deviates from the grand coalition. In order to define the recursive cores, it is necessary to start defining the simple recursive cores. There are two alternative definitions, one optimistic, and one pessimistic.

Definition 20 (Pessimistic Simple Recursive Core, Kóczy (2007))**.** Let (N, v) be a PFG. The **pessimistic simple recursive core** $RC_s^\alpha(N, v)$ of (N, v) is defined recursively as follows.

- If $n = 1$, the core $RC_s^\alpha(N, v)$ is trivially defined via efficiency.
- If $n > 1$ and for all $k < n$ the core is defined, $RC_s^\alpha(N, v)$ is defined by extension as the set of undominated outcomes, i.e. $(x, \pi) \in RC_s^\alpha(N, v)$ if and only if, for all $S \subset N$ such that for all outcomes $(y, \{S\} \otimes \pi_{N\setminus S}) \in \mathscr{O}(N, v)$ verifying $(y_{N\setminus S}, \pi_{N\setminus S}) \in RC_s^\alpha(N \setminus S, \tilde{v}_{\{S\}})$ (if this set is non-empty), then $x_i \geq y_i$ for all $i \in S$.

Kóczy (2007) shows that the pessimistic simple recursive core is a generalization of the core of a TU-game.

Theorem 2 (Kóczy (2007))**.** *Let (N, c) be a CFG. Consider the associated PFG (N, v) such that for all $(S, \pi) \in \mathscr{E}(N)$, $v(S, \pi) = c(S)$, then $RC_s^\alpha(N, v) = \mathscr{C}(N, c)$.*

As for α and ω-cores, the simple recursive core admits an optimistic version.

Definition 21 (Optimistic Simple Recursive Core, Kóczy (2007))**.** Let (N, v) be a PFG. The **optimistic simple recursive core** $RC_s^{\omega}(N, v)$ of (N, v) is defined recursively as follows.

- If $n = 1$, the core $RC_s^{\omega}(N, v)$ is trivially defined via efficiency.
- If $n > 1$ and for all $k < n$ the core is defined, $RC_s^{\omega}(N, v)$ is defined by extension as the set of undominated outcomes, i.e. $(x, \pi) \in RC_s^{\omega}(N, v)$ if for all $S \subset N$, it does not exist an outcome $(y, \{S\} \otimes \pi_{N \setminus S})$ such that $x_i \geq y_i$ for all $i \in S$ and $(y_{N \setminus S}, \pi_{N \setminus S}) \in RC_s^{\omega}(N \setminus S, \tilde{v}_{\{S\}})$ if this latter is non-empty.

As for the ω and the α-core, the following result holds.

Proposition 11. *Let (N, v) be a PFG. $RC_s^{\omega}(N, v) \subset RC_s^{\alpha}(N, v)$.*

Kóczy (2007) emphasizes the fact that simple recursive cores can be inefficient, in the sense that some outcomes may be Pareto-dominated by others outcomes that are not in the core. As a consequence, he introduces another core concept that overcome the problem of inefficiency. Therefore, he defines the pessimistic and optimistic cores as follows:

Definition 22 (Pessimistic Recursive Core, Kóczy (2007))**.** Let (N, v) be a PFG. The **pessimistic recursive core** $RC^{\alpha}(N, v)$ of (N, v) is defined recursively as follows.

- If $n = 1$, the core $RC^{\alpha}(N, v)$ is trivially defined via efficiency.
- If $n > 1$ and for all $k < n$ the core is defined, $RC^{\alpha}(N, v)$ is defined by extension as the set of undominated outcomes, i.e. $(x, \pi) \in RC^{\alpha}(N, v)$ it does not exist a coalition S with a partition $\pi_S \in \Pi_S$ such that for all outcomes $(y, \pi_S \otimes \pi_{N \setminus S}) \in \mathscr{O}(N, v)$ satisfying $(y_{N \setminus S}, \pi_{N \setminus S}) \in RC^{\alpha}(N \setminus S, \tilde{v}_{\{\pi_S\}})$ (if it is non-empty), $y_i \geq x_i$ for all $i \in S$.

Definition 23 (Optimistic Recursive Core, Kóczy (2007))**.** Let (N, v) be a PFG. The **optimistic simple recursive core** $RC^{\omega}(N, v)$ of (N, v) is defined recursively as follows.

- If $n = 1$, the core $RC^{\omega}(N, v)$ is trivially defined via efficiency.
- If $n > 1$ and for all $k < n$ the core is defined, $RC^{\omega}(N, v)$ is defined by extension as the set of undominated outcomes, i.e. $(x, \pi) \in RC^{\omega}(N, v)$ it does not exist a

coalition S with a partition $\pi_S \in \Pi_S$ and an outcome $(y, \pi_S \otimes \pi_{N\setminus S}) \in \mathscr{O}(N,v)$ satisfying $(y_{N\setminus S}, \pi_{N\setminus S}) \in RC^{\omega}(N \setminus S, \tilde{v}_{\{\pi_S\}})$ (if it is non-empty), $y_i \geq x_i$ for all $i \in S$.

The only difference between simple recursive cores and the recursive cores is that, in the definition of recursive cores, the domination definition allows to take a partition of a coalition S instead of just taking $\{S\}$. This solves the issue of efficiency of simple recursive cores.

Proposition 12 (Kóczy (2007))**.** *Let (N,v) be a PFG. All allocations of $RC^{\omega}(N,v)$ and $RC^{\alpha}(N,v)$ are Pareto optimals.*

The following inclusions hold.

Proposition 13 (Kóczy (2007))**.** *For all PFGs (N,v), the following inclusions are verified:*

- $RC^{\omega}(N,v) \subset RC^{\alpha}(N,v)$
- $RC^{\omega}(N,v) \subset RC^{\omega}_{s}(N,v)$
- $RC^{\alpha}(N,v) \subset RC^{\alpha}_{s}(N,v)$

Kóczy (2007) also proves, as for simple recursive cores, that the recursive cores are generalizations of the core of TU-games.

Theorem 3 (Kóczy (2007))**.** *Let (N,c) be a CFG. Consider the associated PFG (N,v) such that for all $(S,\pi) \in \mathscr{E}(N)$, $v(S,\pi) = c(S)$, then $RC^{\alpha}(N,v) = RC^{\omega}(N,v) = \mathscr{C}(N,c)$.*

1.2.3.8 Axiomatizations of the cores

While there exists a lot of alternative definitions of the core for cooperative games with externalities, it becomes difficult to compare these concepts, and decide which ones are acceptable or good. Hence, there are two way to answer this question. The first one, is describing PFG as noncooperative games, and see if a core concept coincide with a solution of this game, like a Nash equilibrium for example. The other one is to fully characterize or axiomatize the concept. Therefore, if the concept can be axiomatized in a few elementary properties that seem desirable, then the concept can be considered as interesting.

Bloch and van den Nouweland (2014) give axioms of the rules of formation of partition of the remaining players after a deviation of a coalition from the grand coalition in core concepts. They call these rules expectation formation rules, which map situations of deviation with a new partition of the remaining players. In their paper, they give some axioms which can be defined on expectation formation rules, and provide an axiomatization of the projection rule, which is the one that defines the δ-core.

Abe (2019) also provides a characterization of the α and the ω-cores as a Bondareva-Shapley-like theorem using the linear programing's duality theorem to find the result.

1.2.4 The Shapley Value

While the Shapley value is one of the most popular solution for characteristic function games, it seems natural to try to extend this concept for partition function games. For CFGs, it is well defined, admits a solution for every game, and is axiomatized in a way that makes it simple to understand. With the introduction of externalities, it becomes necessary to redefine the concept, as well as its axioms. However, as for the core, the definition of the Shapley value is not clear for partition function games. In this part, I introduce one alternative definition of the Shapley value for PFGs, and provide some references for other definitions.

1.2.4.1 The Myerson Value

Myerson (1977) introduces an extension of the Shapley value for PFGs, verifying the three following axioms.

Let $N \subset \mathbb{N}$ be a set of players and $\Phi : \mathbb{P}^n \longrightarrow \mathbb{R}^N$ a solution concept.

Φ verifies **Anonimity (A)** if for any permutation σ of N, and player $i \in N$, $\Phi_{\sigma(i)}(\sigma(v)) = \Phi_i(v)$, where $\sigma(v)$ is the partition function defined by $\sigma(v)(S, \pi) := v(\sigma(S), \sigma(\pi))$ for all $(S, \pi) \in \mathscr{E}(N)$.

C is called a **carrier** of (N, v) if for any embedded coalition (S, π), $v(S, \pi) = v(S \cap C, \pi \cap \{C, C^c\})$, where $A \cap B := \{S \cap T, S \in A, T \in B\}$.

Φ verifies the **Carrier (C)** axiom if for any carrier C, $\sum_{i \in C} \Phi_i(v) = v(N, \{N\})$. This axiom is a natural extension of the Carrier-axiom for characteristic function games.

Φ verifies **Additivity (A2)** if for all PFGs (N, v) and (N, w), $\Phi(v+w) = \Phi(v)+\Phi(w)$.

With these three axioms, Myerson (1977) characterizes an extention of the Shapley value, defined as follows:

Theorem 4 (Myerson (1977))**.** *There exists a unique value function Φ^M that satisfies Anonimity, Carrier and Additivity. It is given for all PFGs (N, v), $i \in N$ as follows:*

$$\Phi_i^M(v) = \sum_{(S,\pi)\in\mathscr{E}(N)} (-1)^{|\pi|-1}\left(\frac{1}{n} - \sum_{\substack{i\notin C\in\pi \\ C\neq S}} \frac{1}{(\mid \pi \mid -1)(n- \mid C \mid)}\right) v(S,\pi)$$

This value is known as the Myerson value.

This Myerson value must not be confused with its homonym defined for characteristic function games over networks.

While this value is axiomatized and well defined, de Clippel and Serrano (2008) argue that this value is not desirable because it violates Monotonicity.

1.2.4.2 The Null Player Property Problem

For PFGs, what is a null player is not clear. While it is well defined for characteristic function games, the addition of externalities makes it really difficult to define. As a consequence, there is a wide variety of definitions of null players for PFGs. I refer to Kóczy (2018) for alternative definitions of this notion.

As for TU-games, literature ask if this is possible to find unique rules of allocation that verify Efficiency, Symmetry, Additivity and the Null player property. This latter being ambiguous, there are a lot of alternative definitions of the Shapley value, as the γ-Shapley value (Pham Do and Norde (2007)), or the Shapley value of Hu and Yang (2010) that is defined as the Shapley value of the δ-characteristic function.

As there are a lot of alternative definitions, I refer also to Kóczy (2018), which summarizes the different approaches existing in the literature in his chapter on the Shapley value.

1.2.5 The Problem of Computability of the Solutions in Partition Function Games

All the previous concepts have been given as theoretical solutions. In facts, game theorists are often satisfied by proving the existence of a solution, or giving its formula. Although these results have significant theoretical interest, for partition function games, they can become completely obsolete when it comes to use these in real situations. This problem comes from the evolution of the number of partitions with n.

The cardinality of Π_N is given by the Bell numbers, defined by $B_n = \mid \Pi_N \mid$ such that $B_0 = 1$, $B_1 = 1$ and recursively $B_{n+1} = \sum_{k=0}^{n} C_n^k B_k$.

Given these numbers, the number of embedded partition of the player set N grows faster than exponentially with n. The evolution of the Bell numbers for the first values of n is given in the following array.

n	B_n
0	1
1	1
2	2
3	5
4	15
5	52
6	203
7	877
8	4140
9	21147
10	115975

While for most of the core and Shapley value concepts one must compute the worthes of all embedded coalitions, even with a machine, a program computing such a value will have a very huge complexity, which makes the execution impossible, even for small values of n. As a consequence, PFGs have mainly a theoretical interests and will rarely be used for real applications as soon as the number of players becomes too large.

2 Core of Dynamic Coalitional Game

2.1 Introduction

In cooperative game theory, most of the literature considers static games: these are played only once and the solution is a set of acceptable allocations that satisfy some axioms.

In this chapter, I study dynamic games, with discrete time, where the value of a coalition changes over time with a predetermined pattern. The different stages are independent, do not depend on the history, and players have complete information on the game.

My aim is to provide two new concepts: the first one is a concept of coalitional $\vec{\varepsilon}$-core for static cooperative games that can be an answer to the problem of a game with an empty core and is less restrictive than the ε-core of Shapley and Shubik (1966) and the weak ε-core of Kannai (1992). The second one is a concept of core for dynamic coalitional games that allows worth compensation within periods, but for which the total worth distributed at each stage is the value of the grand coalition. In this context, coalitions are formed for all stages of the game: these cannot split or merge between stages. These concepts are linked by the fact that the coalitional $\vec{\varepsilon}$-core concept can be used to give a characterization of the non-emptiness of the dynamic core.

There exists many solutions to situations in which a game has an empty core. One of the most used is the ε-core of Shapley and Shubik (1966). The idea of this concept is to reduce the values of all coalitions by ε such that the core is non-empty. Kannai

(1992) defines a weaker version of this concept by reducing the values of the coalitions proportionally to their sizes. Guesnerie and Oddou (1979) define the c-core in which the allocation vectors distribute the value of partitions of the grand coalition that are not necesserily the grand coalition itself. Gómez (2003) introduces the extended core: he increases the value of the grand coalition such that the core in non-empty, and then renormalizes the allocations so that the sum equals the value of the grand coalition. Grabisch and Miranda (2008) define the k-additive core which is a generalization of the core considering allocation not to the players but to k-uplet of players. They show that for any cooperative game there exists a non-empty k-additive core for k high enough. Bennett (1983) defines the aspiration core as the set of allocations, verifying a weaker feasibility constraint than in the core definition, of minimal sum and shows this is a nonempty core extension. In games with an empty core, the aspiration core payoffs are not feasible for the grand coalition. This opens up the possibility for other coalitions to form.

Some generalizations of the core already exists in a dynamic context. I propose an alternative definition of the core which takes into account both compensation within stages and a feasibility constraint ensuring that, at each stage of the game, the value of the grand coalition is distributed among the players. I call it the dynamic core, and I give conditions under which it is non-empty. The main result of this chapter is a characterization of non-emptiness of the dynamic core based on a geometrical approach, using the coalitional $\vec{\varepsilon}$-core, which shows what happens when the dynamic core is non-empty, and how there can be a compensation within stages. I also provide an axiomatization of this concept.

In the literature, the core of dynamic cooperative games has already been widely studied. Filar and Petrosjan (2000) define the summed core as a sum of the cores of each stage game. This solution concept takes into account feasibility, but does not encompass the fact that there can be compensation within stages. The fact that the core at some stage can be really big is useless to maintain the grand coalition over time as soon as the core is empty at any date. Kranich, Perea and Peters (2005) define the classical core as the core of an associated static game. Unlike the summed core, this concept takes

into account the potential compensations of coalitions between stages, but there is no feasibility constraint on what players can share at each time. For instance, all the worth of the game could be distributed among players at the first stage. The problem here is that, in real situations, all the worth of the game cannot be created at this time. This does not modelize well the dynamics in the game, and it cannot explain what happens over time when the classical core is non-empty. However, these two papers consider that coalitions are formed for all stages of the game. Oviedo (2000) studies the core of a discounted repeated game that does not vary over time, and concludes that the non-emptiness of the dynamic core is equivalent to the non-emptiness of the static core. One interesting thing about this paper is that allocations of the dynamic core are discounted sums of allocations of the static core, so the agreement can be different at each stage of the dynamic game.

Kranich, Perea and Peters (2005) also define two other notions of core than the classical core: the weak and the strong sequential cores. Those, unlike the classical core, allow deviations of coalitions at each stage of the dynamic game. The difference between these two concepts is the fact that the weak sequential core allows only credible deviations of coalitions. Habis and Herings (2010) argue that there is a problem in the definition of the weak sequential core, and provide a new definition that maintain the results of Kranich, Perea and Peters (2005).

Some papers explore dynamic situations in which there is initial uncertainty about some parameters of the game, which will be revealed at later stages. It is thus necessary to extend the notion of core to this framework, requiring an allocation not only to be stable at the beginning, but in addition to remain stable against possible deviations after the resolution of uncertainty, or once the dynamic cooperation process is under way. Gale (1978) introduces the sequential core in a dynamic model of a monetary economy as the set of attainable allocations that cannot be improved on by any coalition either in the classical sense or by accepting an allocation to start with but improving on it at a later date. It takes into account the fact that players could split from the grand coalition even if they agreed to form it at first. Becker and Chakrabarti (1995), based on Becker (1982) consider infinite horizon capital allocation models and define recursive core allocations, the ones where no coalition can improve upon its consumption stream at any

time given its accumulation of assets up to that period. Predtetchinski (2007) considers infinite horizon dynamic games where the game of each stage is drawn in a finite set by an exogenous Markov chain that does not depend on past allocations. He studies the classical core and the strong sequential core in this context, and provides conditions of non-emptiness of the strong sequential one. Avrachenkov, Cottatellucci and Maggi (2012) study a cooperative Markov decision process, for which they define what they call the cooperation maintaining solution, which is a refinement of the concept of core in this context. Lehrer and Scarsini (2013) also study Markovian dynamic cooperative games and define the intertemporal core, ensuring that for each time h the allocations of the grand coalition ensures for the future at least what each coalition could get by splitting at h. They define also the ε-intertemporal core and provide conditions for the non-emptiness of the intertemporal core.

Predtetchinski, Herings, and Peters (2002) and Predtetchinski, Herings, and Perea (2006) use the strong and weak sequential cores for two stages games.

None of these papers consider a model with dynamic games with predetermined independent stages, and a core concept which take into account both compensation between stages and a feasibility constraint. Moreover, none of these consider a geometric characterization of the non-emptiness of the core. The main tool of the present chapter is the coalitional $\vec{\varepsilon}$-core, that is an extension of the ε-core. This latter has been used in many papers, as Wooders (1983), Wooders and Zame (1984), Shimomura (1997), Kovalenkov and Wooders (1999), Gómez (2003) and Askoura (2011).

This chapter is organized as follows. In Section 2.2, I define ε-core, and coalitional $\vec{\varepsilon}$-core with some basic properties, in Section 2.3, I illustrate dynamic coalitional games with an example, in Section 2.4, I define Dynamic Coalitional Games and recall some notions about cooperative games, in Section 2.5, I introduce the concept of Dynamic Core, in Section 2.6, I provide an axiomatization of the summed core, the classical core and the dynamic core, in Section 2.7 provides a characterization of non-emptiness of the dynamic core using the ε-core. Finally, in section 2.8, I seek for existence of the Shapley value in the different core concepts in the dynamic context.

2.2 The Coalitional $\vec{\varepsilon}$-Core

In this section, I extend the definition of ε-cores (Shapley and Shubik (1966)), as coalitional $\vec{\varepsilon}$-cores and show some of its properties.

The concept of the core provides a notion of stability for payoff allocations. However, there are situations where the core may be empty, prompting the introduction of a generalization by Shapley and Shubik (1966). This generalization is known as the ε-core, denoted for each TU-game (N, c) as $\mathscr{C}_\varepsilon(N, c)$, where $\varepsilon \in \mathbb{R}$ is a real number. Formally, it is defined as the following set of allocation vectors:

$$\mathscr{C}_\varepsilon(N,c) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N); \sum_{i \in S} x_i \geq c(S) - \varepsilon, \forall\ S \subseteq N \right\}.$$

Kannai (1992) defines a weaker version of the ε-core, called weak ε-core, defined as follows:

$$\mathscr{C}^\circ_\varepsilon(N,c) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N); \sum_{i \in S} x_i \geq c(S) - \varepsilon \mid S \mid, \forall\ S \subseteq N \right\}.$$

In economic terms, the ε-core represents the set of allocation where no coalition can improve its payoff by leaving the grand coalition, provided it incurs a penalty of ε for leaving. Notably, ε can be negative, implying a bonus for leaving the grand coalition. It is straightforward to see that, irrespective of whether the core is empty, the ε-core will be non-empty for a sufficiently large value of ε and empty for a small enough (possibly negative) value of ε.

Building upon this concept, the minimal ε-core, introduced by Maschler, Peleg, and Shapley (1979), is defined as the intersection of all non-empty ε-cores. Alternatively, it can be viewed as the ε-core for the smallest value of ε that renders the set non-empty (Bilbao 2000).

The idea of defining the ε-cores can be explained by the following situation. In the context of a developing society, a social planner has been given the task of allocating limited resources to different communities for growth projects. The planner wants full cooperation among the communities and adopts an approach based on the ε-core. To dissuade coalitions from separating from the grand coalition, he introduces a mechanism of punishment. A parameter $\varepsilon > 0$ is defined as the cost of deviation for coalitions. If

a coalition chooses to disband, ε is subtracted by the social planner from its worth. By communicating this policy to community representatives, the planner creates an incentive for stability and cooperation. Therefore, they are encouraged to form the grand coalition.

Although the ε-core can be a powerful tool to ensure cooperation, there are situations where it is too strong regarding of the reality of the game, as illustrated in the next example.

Example 14. Let $N = \{1, 2, 3\}$, and c the characteristic function defined as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$	N
$c(S)$	0	0	3	2	4

In this game, the grand coalition cannot be sustainable. Indeed, 3 asks for at least 3 to cooperate, so 1 and 2 must share a value of 1 to cooperate in the grand coalition, but they can have 2 if they split. If $\varepsilon = 1$ is set, and c_ε is defined such that $c_\varepsilon(S) = c(S) - \varepsilon$ for all $S \subsetneq N$ and $c_\varepsilon(N) = c(N)$, then the TU-game (N, c_ε) is defined as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$	N
$c_\varepsilon(S)$	-1	-1	2	1	4

Here, $\mathscr{C}_\varepsilon(N, c) = \mathscr{C}(N, c_\varepsilon)$ is non-empty, and the allocation $(-1, 2, 3)$ is in $\mathscr{C}_\varepsilon(N, c)$. From the point of view of a social planner, it can be considered as not fair that the player 1 must pay to join the grand coalition. Therefore, it becomes necessary to relax the definition of the ε-core to avoid these kind of situations to happen.

In light of the previous example I extend the notion of ε-core as follows:

Definition 24. For every $\vec{\varepsilon} \colon 2^N \setminus \{\emptyset, N\} \to \mathbb{R}$, the **coalitional $\vec{\varepsilon}$-core** of (N, c) is:

$$\mathscr{C}_{\vec{\varepsilon}}(N, c) = \{x \in \mathbb{R}^n \mid \ \forall S \subsetneq N,\ x(S) \geq c(S) - \vec{\varepsilon}(S) \text{ and } x(N) = c(N)\}$$

This notion is a lot more flexible as the penalties are personalised: the costs of leaving the grand coalition can be different for all coalitions, based on many considerations, as their cardinality, for instance. If a player alone blocks the formation of the grand coalition by asking a large portion of the total worth, it is possible to define a coalitional $\vec{\varepsilon}$-core that force him to join the grand coalition without penalizing the other players. Unlike the ε-core that can be seen as a dilatation of the original core, a coalitional $\vec{\varepsilon}$-core can have a completely different shape than the original core, as illustrated in the next example:

Example 15. Let $N = \{1, 2, 3\}$, and c the characteristic function defined as follows.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$	N
$c(S)$	2	2	2	5	9

for all $i \neq j \in N$. The core of this game is in the plane $x_1 + x_2 + x_3 = 9$. Different coalitional $\vec{\varepsilon}$-cores can be constructed. Consider for instance $\vec{\varepsilon}_1(S)$ and $\vec{\varepsilon}_2(S)$ defined by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$
$\vec{\varepsilon}_1(S)$	1	1	1	0

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$
$\vec{\varepsilon}_2(S)$	1	0	0	1

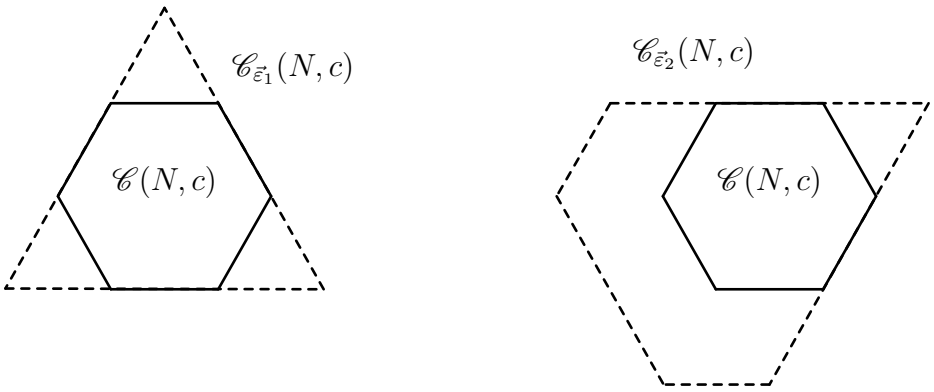


Figure 2.1: Differents coalitional $\vec{\varepsilon}$-cores of (N, c) in the plane $x_1 + x_2 + x_3 = 9$

The two coalitional $\vec{\varepsilon}$-core that are illustrated can be seen as the cores of $(N, c_{\vec{\varepsilon}_1})$ and $(N, c_{\vec{\varepsilon}_2})$ defined by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$	N
$c_{\vec{\varepsilon}_1}(S)$	1	1	1	5	9

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$	N
$c_{\vec{\varepsilon}_2}(S)$	1	2	2	4	9

The idea of using this concept is as follows: there can be a characteristic function game with an empty core, but, if just a coalition S has a lower worth, the core is non-empty. In this case, using an ε-core is not required. Only a coalitional $\vec{\varepsilon}$-core is necessary, with $\vec{\varepsilon}$ such that $\varepsilon_Q = 0$ for all $Q \subset N$, $Q \neq S$, and $\varepsilon_S > 0$. In this case, the only positive value of $\vec{\varepsilon}$ being S shows that only coalition S precludes the grand coalition to be sustainable.

The coalitional $\vec{\varepsilon}$-core can also be used to transform the core of a characteristic function game into another one, as follows:

Theorem 5. *Let N be a set of players, c and c' two characteristic function on N. There exists $\alpha, \beta \in \mathbb{R}$, and $\vec{\varepsilon}: 2^N \setminus \{\emptyset, N\} \to \mathbb{R}$ such that $\mathscr{C}_{\vec{\varepsilon}}(N, \alpha c + \beta) = \mathscr{C}(N, c')$.*

Proof. If $c(N) = 0$, set $\beta = c'(N)$ and $\alpha = 1$, otherwise, set $\beta = 0$ and $\alpha = \frac{c'(N)}{c(N)}$.

For all $S \in 2^N \subset \{\emptyset, N\}$, set $\vec{\varepsilon}(S) = \alpha c(S) + \beta - c'(S)$. Thus, for all $S \in 2^N \subset \{\emptyset, N\}$, $\alpha c(S) + \beta - \vec{\varepsilon}(S) = c'(S)$. Therefore, $\mathscr{C}_{\vec{\varepsilon}}(N, \alpha c + \beta) = \mathscr{C}(N, c')$. □

As for ε-cores, it is possible to define minimal coalitional $\vec{\varepsilon}$-cores.

Definition 25. A minimal coalitional $\vec{\varepsilon}$-core of a characteristic function game (N, c) is a coalitional $\vec{\varepsilon}$-core of (N, c) such that:

$$\vec{\varepsilon} \in \underset{\vec{\varepsilon} \in \mathscr{E}(N,c)}{\operatorname{argmin}} \left(\sum_{S \subsetneq N} \frac{\vec{\varepsilon}(S)}{\mid S \mid} \right) \text{ where } \mathscr{E}(N, c) = \{\vec{\varepsilon}: 2^N \setminus \{\emptyset, N\} \to \mathbb{R}_+ \text{ such that } \mathscr{C}_{\vec{\varepsilon}}(N, c) \neq \emptyset\}$$

This concept allows to determine which groups of coalition block the most the formation of the grand coaliton, as shown in the next example. The value $\min_{\vec{\varepsilon} \in \mathscr{E}(N,c)} \left(\sum_{S \subsetneq N} \frac{\vec{\varepsilon}(S)}{|S|} \right)$ represents the minimal weighted taxation a social planner has to charge to coalitions to ensure cooperation, then it can be used as a measure of the instability of a CFG. While Gómez (2003) proposes with the extended core to tax proportionally every coalition to ensure cooperation, this way of taxation charges only the coalitions which prevent cooperation.

Example 16. Let $N = \{1, 2, 3\}$, and c the characteristic function defined as follows.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{i, j\}$	N
$c(S)$	2	2	1	2	4

for all $i \neq j \in N$. In this game, $\mathscr{E}(N, c)$ is the set of vectors $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, 0, 0, 0) \in \mathbb{R}^6_+$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$. This example shows that the coalitions made up of single players all block the formation of the grand coalition in an equivalent way (even if player 3 has a lower value than the others). One can therefore distribute the cost of the blocking in any way whatsoever between the 3 players to allow the formation of the grand coalition.

In this case, $\min_{\vec{\varepsilon} \in \mathscr{E}(N,c)} \left(\sum_{S \subsetneq N} \frac{\vec{\varepsilon}(S)}{|S|} \right) = 1$: cooperation cannot be sustainable with a weighted taxation inferior than this value.

2.3 Motivating Example of Dynamic Coalitional Games

Two neighbouring villages A and B produce tomatoes. The season lasts for 2 months: July and August. Village A produces the majority of its tomatoes in July with a quantity of 10 units, and 2 units in August. The inhabitants of A eat up to 10 units a month. Village B produces the majority of its tomatoes in August with a production of 1 unit in July, and 45 in August. The inhabitants of village B eat up to 30 units of tomatoes a month. In this game, the value of a coalition for a month represents the total quantity of eaten tomatoes.

The two villages can share their tomatoes with each other, knowing that 20% of transported tomatoes rot because of the heat. Village B accepts to cooperate in August if, and only if, village A cooperates in July. For simplicity, I assume that there is no discount factor.

The villages, therefore, play the game given in the following table:

S	$\{A\}$	$\{B\}$	$\{A, B\}$
$c_J(S)$	10	1	10
$c_A(S)$	2	30	40

In July, if they cooperate, village A sent 5 units of tomatoes to village B. Thus, 1 unit rots during transport, and the two village consume $1 + 4 + 5 = 10$ units of tomatoes. Otherwise, each village consumes its production. In August, if they cooperate, village B consumes 30 units of tomatoes and send the 15 remaining units to village A. 3 units rot during transportation, but as village A cannot eat more than 10 units, there still is surplus. Therefore, the two village consume a total of 40 units of tomatoes this month.

One can observe that, in July, the coalition cannot be sustainable: the sum of the benefits of the villages if they stay apart is greater than the benefits if they cooperate. However, the coalition is sustainable in August. Overall, the benefits of cooperation are sufficiently significant in August to compensate for July. Thus, the villages may have an interest in cooperating.

2.4 Dynamic Coalitional Games

In this section, I define dynamic characteristic function games, and recall different notions of cores.

Let N be a set of players, referred to as the **grand coalition**, with cardinality n. A **coalition** S is a non-empty subset of players. The index i refers to the players in N. In this model, time is discrete, the horizon is $T \in \mathbb{N}$, and players discount time at a rate $\delta \in [0, 1]$.

A **dynamic characteristic function game**, or DCG, is a pair $(N, \vec{c})$ where $\vec{c} = (c_t)_{0 \leq t \leq T}$ is a sequence of characteristic functions. For each $t \in \{0, \dots, T\}$, $c_t : 2^N \longrightarrow \mathbb{R}$ associates each coalition $S \subseteq N$ with a characteristic function $c_t(S) \in \mathbb{R}$, where $c_t(\emptyset) = 0$.

For each coalition $S \subset N$, I denote $C_{\vec{c}}(S)$ the discounted characteristic function of S, defined by:

$$C_{\vec{c}}(S) := \sum_{t=0}^{T} \delta^t c_t(S)$$

An **allocation** is a matrix $(x_t^i)_{i \in N, t \in \{0,\dots,T\}} \in \mathscr{M}_{n \times (T+1)}(\mathbb{R})$ where x_t^i represents the value allocated to player i at time t. $x_t = (x_t^i)_{i \in N}$ denotes the allocation at time t and $x^i = (x_t^i)_{t \in \{0,\dots,T\}}$ represents the plan of values allocated to player i.

The **core** of a static characteristic function game (N, c) (the case where $T = 0$) is the set of allocations such that no coalition of players has an incentive to deviate from the grand coalition. It is defined as:

$$\mathscr{C}(N, c) = \{x \in \mathbb{R}^n \mid \ \forall S \subset N,\ \mathbf{x}(S) \geq c(S),\ x(N) = c(N)\}$$

where $\mathbf{x}(S) := \sum_{i \in S} x^i$.

A set of coalitions $\mathscr{B} \subset 2^N$ is **balanced** if there exists $\lambda : \mathscr{B} \to \mathbb{R}_+$ that balances $\mathscr{B}$, i.e $\sum_{S \subset \mathscr{B}} \lambda(S) \mathbb{1}_{i \in S} = 1$ for all $i \in N$. The idea is that each coalition $S \subset \mathscr{B}$ is weighted so that each player $i \in N$ has a weight 1 distributed among the coalitions of $\mathscr{B}$. A characteristic function game (N, c) is balanced if, for every balanced set of coalitions $\mathscr{B}$ and $\lambda : \mathscr{B} \to \mathbb{R}_+$ that balances $\mathscr{B}$:

$$\sum_{S \in \mathscr{B}} \lambda(S) c(S) \leq c(N)$$

Bondareva (1963) and Shapley (1967) showed that the core of a characteristic function game (N, c) is non-empty if and only if (N, c) is balanced.

The notion of the core has been extended for dynamic games in two ways:

The summed core (Filar and Petrosjan (2000)): It is defined as the sum of the cores of each stage game:

$$\mathscr{C}^s(N, \vec{c}) = \{x \in \mathscr{M}_{n\times(T+1)}(\mathbb{R}) \mid \ x_t \in \mathscr{C}(N, c_t) \ \forall \ 0 \leq t \leq T\}$$

This notion does not take into account the potential compensations between stages: players of a coalition S could accept forming the grand coalition, even if, at some stages, they prefer to form S if they can make up for the loss with the other periods. Nevertheless, all allocations of this core are feasible: the benefits of cooperation are distributed among the players at each stage of the game.

The classical core (Kranich, Perea, and Peters (2005)): It is a notion of core close to the core of the static game $(N, C_{\vec{c}})$, but the allocations can be distributed over time:

$$\mathscr{C}^c(N, \vec{c}) = \{x \in \mathscr{M}_{n\times(T+1)}(\mathbb{R}) \mid \forall S \subset N, \sum_{t=0}^{T} \delta^t \mathbf{x}_t(S) \geq C_{\vec{c}}(S) \text{ and } \sum_{t=0}^{T} \delta^t \mathbf{x}_t(N) = C_{\vec{c}}(N)\}$$

This notion does not take into account that some allocations of this core may not be feasible: for example, all the benefits of cooperation could be distributed among the players in the first stage of the game, even if these benefits do not already exist. Nevertheless, this notion encompasses the potential compensations between stages.

2.5 The Dynamic Core

In this section, I define the dynamic core of a DCG $(N, \vec{c})$, present a result of inclusion and relate it to the core of the associated static game $(N, C_{\vec{c}})$.

The **dynamic core** is similar to the classical core, with an additional feasibility condition ensuring that at each stage, the grand coalition must share the value created at that stage. It is defined by:

$$\vec{\mathscr{C}}(N, \vec{c}) = \{x \in \mathscr{M}_{n\times(T+1)}(\mathbb{R}) \mid \forall S \subset N, \sum_{t=0}^{T} \delta^t \mathbf{x}_t(S) \geq C_{\vec{c}}(S) \text{ and } \mathbf{x}_t(N) = c_t(N) \text{ for all } 0 \leq t \leq T\}$$

This notion also takes into account the potential compensations between periods. From the point of view of intergenerational equity this concept is interesting. It is legitimate to want in a society that a generation does not draw on the resources of future generations, and also that it does not sacrifice itself for the future either.

It is straightforward to establish the following result:

Lemma 1. *For every* $(N, \vec{c})$, $\mathscr{C}^s(N, \vec{c}) \subset \vec{\mathscr{C}}(N, \vec{c}) \subset \mathscr{C}^c(N, \vec{c})$.

Moreover, the following example shows that there exists $(N, \vec{c})$ such that $\mathscr{C}^c(N, \vec{c}) \not\subset \vec{\mathscr{C}}(N, \vec{c}) \not\subset \mathscr{C}^s(N, \vec{c})$.

Example 17. Consider $T = 1$, $\delta = 1$, and the DCG $(N, \vec{c})$ with $N = \{1, 2, 3\}$ and $\vec{c}$ defined by:

S	$\{i\}$	$\{i, j\}$	N
$c_0(S)$	1	4	5
$c_1(S)$	3	4	8

for every $i \neq j$. The discounted characteristic function is $C_{\vec{c}}(\{i\}) = 4$, $C_{\vec{c}}(\{i, j\}) = 8$, and $C_{\vec{c}}(N) = 13$. Thus, one can check that $\mathscr{C}(N, c_0) = \mathscr{C}(N, c_1) = \emptyset$. It follows that $\mathscr{C}^s(N, \vec{c}) = \emptyset$. It is straightforward to see that $\{(4, 1, 0), (0, 3, 5)\} \in \vec{\mathscr{C}}(N, \vec{c})$, and $\{(4, 4, 5), (0, 0, 0)\}$ belongs to $\mathscr{C}^c(N, \vec{c})$, but not to $\vec{\mathscr{C}}(N, \vec{c})$.

This example also shows that players can accept to lose benefits for a period of time by forming the grand coalition if they can make up for the loss at another period of time.

The following theorem shows that the non-emptiness of the dynamic core is equivalent to the non-emptiness of the associated static game.

Theorem 6. *Let* $(N, \vec{c})$ *be a dynamic characteristic function game, and* $C_{\vec{c}}$ *the associated dynamic discounted function. The following propositions are equivalent:*

(i) $\vec{\mathscr{C}}(N, \vec{c})$ *is non-empty*

(ii) $\mathscr{C}(N, C_{\vec{c}})$ *is non-empty*

Proof. $\boxed{(i) \Rightarrow (ii)}$ For every $x \in \vec{\mathscr{C}}(N, \vec{c})$, $y = (\sum_{t=0}^{T} \delta^t x_t^i)_{i \in N} \in \mathscr{C}(N, C_{\vec{c}})$ by definition.

$\boxed{(ii) \Rightarrow (i)}$ Let $y \in \mathscr{C}(C_{\vec{c}}, N)$. For every $i \in N$, let set $\alpha^i = \frac{y^i}{y(N)}$, and $x_t^i = \alpha^i c_t(N)$ for all t. Here, α represents the proportion of the value of the grand coalition given to each player, and then allows the latter to be distributed between the stages by defining x. Let show that $x \in \vec{\mathscr{C}}(N, \vec{c})$. $\sum_t \delta^t x_t^i = \alpha^i \sum_t \delta^t c_t(N) = \alpha^i C_{\vec{c}}(N) = \frac{y^i}{y(N)} C_{\vec{c}}(N) = y^i$. This ensures that $\sum_t \delta^t x_t(S) = y(S) \geq C_{\vec{c}}(S)$. Moreover, $\sum_i x_t^i = c_t(N) \sum_i \alpha^i = c_t(N).\frac{y(N)}{y(N)} = c_t(N)$. Therefore, $x \in \vec{\mathscr{C}}(N, \vec{c})$. □

As the only difference between the classical core of a DCG $(N, \vec{c})$ and the core of the associated static game $(N, C_{\vec{c}})$ is the fact that the allocation can be distributed over time, the following corollary holds.

Corollary 1. *Let $(N, \vec{c})$ be a dynamic characteristic function game. $\vec{\mathscr{C}}(N, \vec{c})$ is non-empty if, and only if $\mathscr{C}^c(N, \vec{c})$ is non-empty.*

2.6 Axiomatization of the Cores

In this section, I extend the standard axiomatization of the core for dynamic coalitional games and provide axiomatizations of the classical core, the summed core and the dynamic core in this context.

Let $(N, \vec{c})$ be a DCG. Let me define the following four axioms:

Efficiency (E): An allocation $x \in \mathscr{M}_{n \times (T+1)}(\mathbb{R})$ is efficient if $\sum_{t=0}^{T} \delta^t \mathbf{x}_t(N) = C_{\vec{c}}(N)$.

Coalitional Rationality (CR): An allocation $x \in \mathscr{M}_{n \times (T+1)}(\mathbb{R})$ is coalitionally rational if for all coalitions $S \in 2^N$, $\sum_{T=0}^{T} \delta^t \mathbf{x}_t(S) \geq C_{\vec{c}}(S)$.

Dynamically Efficiency (DE): An allocation $x \in \mathscr{M}_{n \times (T+1)}(\mathbb{R})$ is dynamically efficient if for all $0 \leq t \leq T$, $\mathbf{x}_t(N) = c_t(N)$.

Dynamically Coalitional Rationality (DCR): An allocation $x \in \mathscr{M}_{n \times (T+1)}(\mathbb{R})$ is dynamically coalitionally rational if for all coalitions $S \in 2^N$, $0 \leq t \leq T$, $\mathbf{x}_t(S) \geq c_t(S)$.

It is straightforward to see that (DE) implies (E) and (DCR) implies (CR).

Given these definitions, the following results hold:

Theorem 7.
- *The classical core is the set of allocations verifying (E) and (CR)*

- *The summed core is the set of allocations verifying (DE) and (DCR)*
- *The dynamic core is the set of allocations verifying (DE) and (CR)*

The proof of this theorem is straightforward to establish. One can see that the classical core does not verify (DE) and the dynamic core does not verify (DCR): that makes these concepts different.

One might wonder what a concept defined as the set of allocations verifying (E) and (DCR) could look like. Such a concept would permit value transfers between stages, but not as much as in the classical core. It may be an interesting concept from a cooperative point of view.

2.7 Non-emptiness Condition of the Dynamic Core

In this section, I employ the coalitional $\vec{\varepsilon}$-core concept to determine a sufficient and necessary condition of non-emptiness of the dynamic core.

Theorem 8. *Let $(N, \vec{c})$ be a dynamic characteristic function game. The following propositions are equivalent:*

(i) $\vec{\mathscr{C}}(N, \vec{c})$ is non-empty.

(ii) For all $t \in \{0, \dots T\}$, there exists $\vec{\varepsilon}_t : 2^N \backslash \{N, \emptyset\} \longmapsto \mathbb{R}$ such that for all $t, \mathscr{C}_{\vec{\varepsilon}_t}(N, c_t) \neq \emptyset$ and $\sum_{t=0}^{T} \delta^t \vec{\varepsilon}_t(S) \leq 0$ for all $S \subsetneq N$.

Proof. $\boxed{\textit{(i)} \Rightarrow \textit{(ii)}}$ Let $i \in N, t \in \{0, \dots, T\}, (x_t^i) \in \vec{\mathscr{C}}(N, \vec{c})$. For all $S \subsetneq N$, $\sum_{t=0}^{T} \delta^t x_t(S) \geq \sum_{t=0}^{T} \delta^t c_t(S)$. Thus, $\sum_{t=0}^{T} \delta^t (x_t(S) - c_t(S)) \geq 0$. Let set $\vec{\varepsilon}_t(S) = c_t(S) - x_t(S)$ for all $t \in \{0, \dots, T\}$, $S \subsetneq N$. Thus, $\sum_{t=0}^{T} \delta^t \vec{\varepsilon}_t(S) \leq \vec{0}$ for all $S \subsetneq N$.

Let me show that $C_{\vec{\varepsilon}_t}(N, c) \neq \emptyset$ for all $t \in \{0, \dots, T\}$. For $t \in \{0, \dots, T\}$ and $S \subsetneq N$, $x_t(S) - c_t(S) + \vec{\varepsilon}_t(S) = 0$, and $x_t(N) = c_t(N)$ by definition.

$\boxed{\textit{(ii)} \Rightarrow \textit{(i)}}$ Let take $x_t \in \mathscr{C}_{\vec{\varepsilon}_t}(N, c_t)$ for all $t \in \{0, \dots, T\}$. Set $x_t^i = x_t(i)$ for all $i \in N$, $t \in \{0, \dots, T\}$. Hence, for all $S \subsetneq N$, $\sum_{t=0}^{T} \delta^t x_t(S) \geq c(S) - \sum_{t=0}^{T} \delta^t \vec{\varepsilon}_t(S) \geq C_{\vec{c}}(S)$ and $\sum_{t=0}^{T} \delta^t x_t(N) = C_{\vec{c}}(N)$.

Therefore, $(x_{i,t})_{i \in N, t \leq T} \in \mathscr{C}(N, C_{\vec{c}}) \Rightarrow \vec{\mathscr{C}}(N, \vec{c}) \neq \emptyset$ by Theorem 6.

□

This result can be used as follows. Consider a DCG for which some of the stage games have an empty core while the dynamic core is non-empty. Given an element x of the dynamic core, one can construct $(\vec{\varepsilon}_t)_{t\in\{0,\dots T\}}$ such that for all $S \subsetneq N$ and t, $\vec{\varepsilon}_t(S) = c_t(S) - x_t(S)$ as in the previous proof. This reveals which coalitions block the formation of the grand coalition for the allocation x at each stage $t \in \{0, \dots T\}$, and which coalitions generate surplus that can be transfered to another stage. From Example 18, $x = (x_0, x_1) \in \vec{\mathscr{C}}(N, \vec{c})$ with $x_0 = (4, 1, 0)$, $x_1 = (0, 3, 5)$. Take $\vec{\varepsilon}$ as follows:

	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$
$\vec{\varepsilon}_0$	-3	0	1	-1	3	0
$\vec{\varepsilon}_1$	3	0	-2	1	-4	-1
$\vec{\varepsilon}_0 + \vec{\varepsilon}_1$	0	0	-1	0	-1	-1

At time $t = 0$, coalitions $\{3\}$ and $\{2, 3\}$ preclude the grand coalition to be sustainable. At time $t = 1$, this is $\{1\}$, and $\{1, 2\}$. But, with dynamics, there is compensation between stages , so the grand coalition is stable.

Consider another element of $\vec{\mathscr{C}}(N, \vec{c})$: $x' = (x'_0, x'_1)$ with $x'_0 = (2, 1, 2)$, $x'_1 = (2, 3, 3)$, and take $\vec{\varepsilon}$ as follows:

	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$
$\vec{\varepsilon}_0$	-1	0	-2	1	1	0
$\vec{\varepsilon}_1$	1	0	0	-1	-2	-1
$\vec{\varepsilon}_0 + \vec{\varepsilon}_1$	0	0	-2	0	-1	-1

At time $t = 0$, coalitions $\{1, 2\}$ and $\{2, 3\}$ preclude the grand coalition to be sustainable. At time $t = 1$, this is coalition $\{1\}$.

$\vec{\varepsilon}_0 + \vec{\varepsilon}_1$ is the same for x and x'. This example shows that the blocking coalitions at a given stage can depend on the choice of the element of the dynamic core.

Thus, this theorem provides a tool that allows to see in detail how the coalitions can compensate each others to allow the existence of elements in the dynamic core. One can link each dynamic core allocation x with a finite sequence $(\vec{\varepsilon}_t)_{t\in\{0,\dots T\}}$ such that for all $S \subsetneq N$ and $0 \leq t \leq T$, $\vec{\varepsilon}_t(S) < 0$ if S generates surplus at time t, and $\vec{\varepsilon}_t(S) > 0$ if S precludes the formation of the grand coalition at time t.

2.8 The Shapley Value of a Dynamic Coalitional Game

In this section, I define the summed Shapley value for DCGs, and consider its existence in the three different cores.

Let $(N, \vec{c})$ be a DFG. The **summed Shapley value** of $(N, \vec{c})$, denoted $Sh(N, \vec{c}) \in \mathscr{M}_{n\times(T+1)}(\mathbb{R})$, is defined for all $i \in N$ as follows:

$$Sh_t^i(N, \vec{c}) = \Phi_i^{Sh}(N, c_t)$$

While for every $0 \leq t \leq T$, $\Phi_i^{Sh}(N, c_t) \in \mathscr{C}(N, c_t)$ if this latter is non-empty, the following result holds:

Proposition 14. *Let $(N, \vec{c})$ be a DFG. If $\mathscr{C}^s(N, \vec{c})$ is non-empty, $Sh(N, \vec{c}) \in \mathscr{C}^s(N, \vec{c})$.*

Since $\mathscr{C}^s(N, \vec{c}) \subset \vec{\mathscr{C}}(N, \vec{c}) \subset \mathscr{C}^c(N, \vec{c})$, the following corollary holds:

Corollary 2. *Let $(N, \vec{c})$ be a DFG. If $\vec{\mathscr{C}}(N, \vec{c})$ (respectively $\mathscr{C}^c(N, \vec{c})$) is non-empty, $Sh(N, \vec{c}) \in \vec{\mathscr{C}}(N, \vec{c})$ (respectively $Sh(N, \vec{c}) \in \mathscr{C}^c(N, \vec{c})$).*

Concluding remarks

In this paper, I have introduced a new tool to answer the problem of non-empty cores, the coalitional $\vec{\varepsilon}$-core and a way to measure the instability of a CFG with the minimal coalitional $\vec{\varepsilon}$-core. I also define a new core concept for dynamic cooperative finite horizon games. While other papers have established results regarding the non-emptiness of the core in dynamic games, even with some stages featuring empty cores, no core concept took into account both compensation within stages and a time-feasibility constraint. The dynamic core addresses this issue.

Theorem 7 provides an axiomatization of the core concepts in the model of DCG. Theorem 8 provides a characterization of the non-emptiness of the dynamic core based on a geometric approach. This allows to understand how coalitions can mutually compensate across stages when the dynamic core is non-empty.

From the point of view of intergenerational equity, it could be interesting to explore a core concept verifying (E) and (DCR). Such a concept would ensure that every generation

has enough to sustain while it would be possible to transfer value between periods, so it should modelize well the inheritance dynamics for example.

As for the cores, one might define both a dynamic Shapley value and a classical one. Those concepts would permit compensations of the players between stages as for the core concepts, therefore would be sets rather than vectors. As for the summed Shapley value, the dynamic Shapley value should be in the dynamic core and the classical one should be in the classical core, as soon as they are non-empty.

Some interesting avenues to explore with the dynamic core include its practical applications and the consideration of variations of dynamic coalitional games with stochastic evolution, offering a more realistic modeling of reality.

3 Stability in Cooperative Games with Externalities

3.1 Introduction

A Characteristic Function Game (CFG), introduced by Von Neumann and Morgenstern (1947), is a pair (N, v), where N is a set of players, and v a map between the coalitions of players in N and their benefits. By definition, CFGs model situations where the benefits created by a coalition do not depend on the behavior of the players outside the coalition. For CFGs, the core is well-defined[1]: it is driven by the assumption that each possible coalition has a well-defined value and will deviate or block forming the grand coalition if this demand is not met.

However, in many economic situations, the value of a coalition of players depends on the way the rest of players cooperate. For example, when two companies A and B decide to team up, other companies can also decide to form coalitions. The benefits of the cooperation between A and B will depend on the structure of the other companies, while the market changes. These situations were modeled as Partition Function Games (PFGs) by Lucas and Thrall (1963). In contrast with CFGs, in a PFG the value is defined on the coalition of N, but also on the partitions of N.

The aim of this chapter is to introduce two notions of strategic stability in the definition of the core for PFGs, based on the same general idea. To form the grand coalition N, coalitions will ask to get at least a value which is credible, in the sense that if they

[1]Ray (1989) proposes an alternative definition of the core, considering only credible deviations, but ends up showing that the two concepts coincide.

deviate, they would get at least this value in a situation where the other players will have no incentive to form a thinner partition. The fact that other players have no incentive to form a thinner partition is due to strategic stability of the partition. There are many ways to define strategic stability of the remaining players. I introduce two notion of stability among those.

There exists several definitions of the core for a PFG: the α-core, (Aumann and Peleg (1960)) is the set of allocations such that a deviating coalition will expect from the remaining players that they form the partition that gives the former the worst possible value. An interpretation of the α-core is that players are pessimistic about what they would obtain after a deviation. On the contrary, the ω-core (Shenoy 1979) is the set of allocations such that a deviating coalition will expect from the remaining players that they form the partition that gives the former the best possible value. In other definitions, the behaviour of the players who did not deviate is fixed arbitrarily: the γ-core (Chander and Tulkens 1997) or s-core and the m-core (Hafalir 2007) are the sets of allocations where the remaining players split into singletons and stay together, respectively. Hart and Kurz (1983) propose a definition of different nature: the δ-core is the set of partitions and allocations for which, after a coalition deviation, the remaining players form the partition without the deviating players. The last two concepts consider a formation rule of partition independ of the game, unlike the rational expectation core (Hafalir 2007) which is the set of allocations such that the remaining players form one of the partitions that maximizes the sum of their benefits, without considering the value of the deviating coalition. Like the previous concepts, the rational expectation core presupposes full cooperation among the remaning players: none considers any kind of strategic stability of the partition formed. From this point of view, the strategic dimension introduced by externalities are not really taken into consideration. Most of the definitions of the core for a PFG just admit sufficient conditions of non emptiness, but Abe and Funaki (2016) provide a Bondareva-Shapley-like characterization for the α-core and the ω-core.

In this chapter, I introduce what I call stable recursive core and stable core by successive deviations. The former is based on the following strategic recursive stability: it is the set of allocation for which the remaining players form a partition in which all the

cells do not have incentive to deviate, in the sense that the stable recursive core of the subgame restricted on the cell is non-empty. This concept is a halfway between balanced games and totally balanced games for PFG, in the sense that to be non-empty, not all the subgames have to be non-empty but only th ecredible ones. The latter is based on the idea that, after a deviation, the remaining players will form a partition that is the result of a process of successive credible splits of the coalitions until no coalition of the partition have a beneficial split. For the two definitions, the considered partition may be unique, so I study an optimistic and a pessimistic version: the αr-core and the ωr-core for the first one, and the αD-core and the ωD-core for the second one.

Kòczy (2007) defines a core that is also recursive, but of a nature similar to the δ-core: it is the set of partitions and allocation for which after a deviation, the remaining players form the same partition but without the deviating players. If the core of the residual game composed by the deviating players in non-empty, they have to form a partition that is in the core, and if it is empty, they can form any outcome. The main difference between this concept and the αr and the ωr cores is that a coalition cannot split into partitions, and that the recursivity appear for the deviating players, not for the remaining ones. Moreover, the recursive core of Kòczy (2007) does not consider any strategic stability of the remaining players.

Grabisch and Funaki (2012) study sequential formations of coalitions for PFGs. Contrariwise to the αD-core and the ωD-core considering credible sequences of splittings from the grand coalition, they consider mergings of coalitions, starting from the partition of singletons. Based on this idea, they provide three ways to allocate the value of the grand coalition, and give two axiomatizations for one of those.

Although, less closely related, some papers study the strategic stability of partitions formation (Bloch (1996), Ray and Vohra (1999), Yi (1997), Konishi and Ray (2003) and Montero (2022)), highlighting that some bargaining procedure can result in finer partitions than the grand coalition. Bloch (1996) shows that core allocations of a cooperative game with externalities can be attained as a stationary perfect equilibrium of a non cooperative game. Ray and Vohra (1999) model a bargaining procedure for which players respond to proposals according to a fixed rule of order and the first player to reject a

proposal automatically becomes the next proposer. Yi (1997) compares stable coalition structures obtained by several rules of coalition formation. Konishi and Ray (2003) study a dynamic process of coalitions formation, using the non cooperative framework, by considering profitable deviations of the coalitions formed from a stage to the next one. The main result of this paper is that there is always a stable partition of the players, but the aim is not to form the grand coalition. Montero (2022) extends the work of Konishi and Ray to PFGs by considering a dynamic bargaining procedure where players alternatively propose coalitions with a payoff vector. The proccess ends when all players accept the coalition they belong to. She relates the formation of the grand coalition with the non-emptiness of the α-core and the ω-core without considering the stability of the remaining players if there is a deviation of a coalition from the grand coalition. Abe (2016) studies hedonic games with externalities, in which players have preferences over coalition structures, and defines several stability concepts in this context. As one can derivate a unique hedonic game out of a PFG, the concepts of stability defined in this paper might be used to analyze stability of PFGs.

The remainder of this chapter is organized as follows. In Section 3.2, I recall the existing definitions of core for PFGs. In Section 3.3, I introduce two definitions of stable recursive core and compare these with existing definitions, provide some conditions of non-emptiness and give two Bondareva-Shapley conditions. In Section 3.4, I define two definitions of core by stable successive deviations and highlight the differences with the recursive definition.

3.2 Preliminaries

Let N be a set of players, called the **grand coalition**, with cardinality n. A **coalition** S is a non-empty subset of players. The index i will refer to the players. A partition π of N is a splitting of N into disjoint coalitions. Π_S denotes the set of partitions of $S \subset N$, and for all $i \in N$, $\pi \in \Pi_N$, $\pi(i)$ is the cell of π containing i. I will use the following notations:

- Let $S, T \subset N$ such that $S \cap T = \varnothing$, $\pi_S = \{P_1, \ldots, P_l\}$ a partition of S, $\pi_T = \{P_{l+1}, \ldots, P_m\}$ a partition of T. I denote by $\pi_S \otimes \pi_T$ the partition $\{P_1, \ldots P_m\}$ of

$S \cup T$ obtained by concatening π_S and π_T.

- Let $\pi = \{S_1, \ldots, S_l\} \in \Pi_N$, $S \subset N$. $\pi \setminus S := \{S_1 \setminus S, \ldots, S_l \setminus S\}$.

A **partition function game**, thereafter PFG, is a pair (N, v) where $v : \{(S, \pi) \in 2^N \times \Pi_N \mid S \in \pi\} \longmapsto \mathbb{R}$. In what follows, the set $\{(S, \pi) \in 2^N \times \Pi_N \mid S \in \pi\}$ is denoted $\mathscr{E}(N)$ and its members are called **embedded coalitions**. When it is defined, $v(S, \pi)$ is the value created by the coalition S when every $i \notin S$ belongs to the coalition $\pi(i)$.

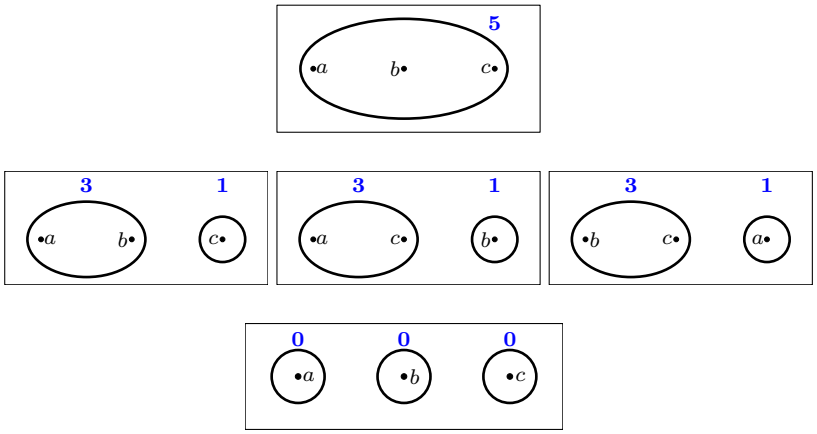
An **outcome** of a PFG (N, v) is a pair $(\pi, x) \in \Pi_N \times \mathbb{R}^n$ consisting of a partition of the players $\pi \in \Pi_N$, and an allocation vector x such that for each coalition $S \in \pi$, the wealth earned by the coalition in the configuration π is entirely distributed among the members of the coalition, that is:

$$(E) : \sum_{i \in S} x_i = v(S, \pi) \ \forall S \in \pi$$

In what follows, for all $x \in \mathbb{R}^n$, $S \subset N$, $\mathbf{x}(S) := \sum_{i \in S} x_i$.

To fix the ideas, let me introduce the following example:

Example 18. Let (N, v) be the PFG with 3 players, a, b, c, defined as follows:



The coalitions are symbolized by the circles, with the subscripts indicating their valuations.

Here: $v(\{a\}, \{\{a\}, \{b\}, \{c\}\}) = 0$, $v(\{a\}, \{\{a\}, \{b, c\}\}) = 1$, but $v(\{a\}, N)$ is not defined because $\{a\}$ is not a cell of the partition $\{N\}$.

In this game, if a decides to stay on his own, b and c have interest to group up (and a prefers also when they group up).

In the following paragraph, I introduce a theorem for cooperative games without externalities that I will use in Section 3.3:

If $v(S,\pi) = v(S,\pi')$ for all $\pi, \pi' \in \Pi_N$ such that $S \in \pi$, $S \in \pi'$, (N,v) is called a characteristic function game and can be denoted (N,c) with $c(S) = v(S,\pi)$ for all $S \subset N$, $S \in \pi$.

The **core** of a static characteristic function game game (N,c) is the set of allocations such that no coalition of players has incentive to deviate from the grand coalition. It is defined by:

$$\mathscr{C}(N,c) = \{x \in \mathbb{R}^n \mid \ \forall S \subset N,\ \mathbf{x}(S) \geq c(S),\ x(N) = c(N)\}$$

A set of coalitions $\mathscr{B} \subset 2^N$ is **balanced** if there exists $\lambda : \mathscr{B} \to \mathbb{R}_+$ that balances $\mathscr{B}$, i.e $\sum_{S \subset \mathscr{B}} \lambda(S) \mathbb{1}_{i \in S} = 1$ for all $i \in N$. The idea is that each coalition $S \subset \mathscr{B}$ is weighted so that each player $i \in N$ has a weight 1 distributed among the coalitions of $\mathscr{B}$. One way of explaining these weights is to consider that each $i \in N$ shares his time between the coalitions of $\mathscr{B}$ such he consumes all his time. A characteristic function game (N,c) is balanced if for every balanced set of coalitions $\mathscr{B}$, and $\lambda : \mathscr{B} \to \mathbb{R}_+$ that balances $\mathscr{B}$:

$$\sum_{S \in \mathscr{B}} \lambda(S) c(S) \leq v(N)$$

Bondareva (1963) and Shapley (1967) have shown that the core of a characteristic function game (N,v) is non-empty if and only if (N,v) is balanced.

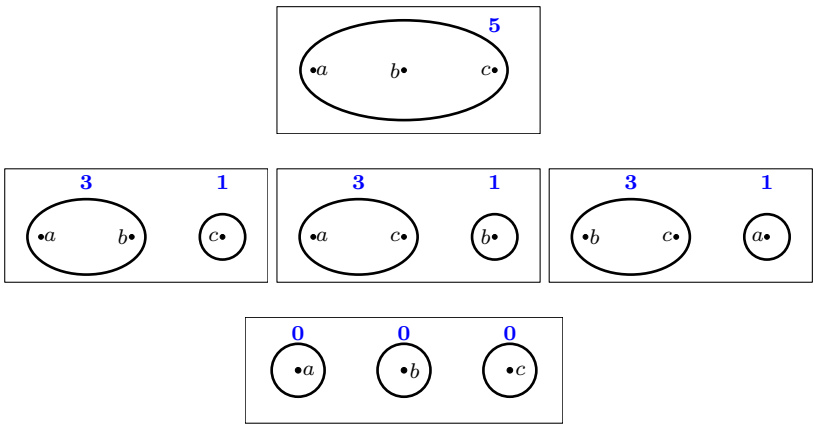
While the definition of PFGs is very simple, it is difficult to give theoretical results in the general case. As consequence, considering some classes of PFGs proves easier to provide some results. In the following paragraphs, I introduce the most used PFG classes. All the following definitions come from Kóczy (2018).

A PFG has positive externalities when for all pairwise disjoint coalitions $C, S, T \subset N$ and $\pi \in \Pi_{N \setminus (C \cup S \cup T)}$:

$$v(C, \{C, S \cup T\} \otimes \pi) \geq v(C, \{C, S, T\} \otimes \pi)$$

This means that each coalition has a higher value if two other coalitions decide to group up instead of remaining separated, as in the next example.

Example 19. Let (N, v) be the PFG with 3 players, a, b, c, defined as follows:



The 3 players a, b and c prefer when the other ones merge. This game has positive externalities.

Likewise, a PFG (N, v) has negative externalities if for all mutually disjoint coalitions $C, S, T \subset N$ and $\pi \in \Pi_{N\setminus(C\cup S\cup T)}$:

$$v(C, \{C, S \cup T\} \otimes \pi) \leq v(C, \{C, S, T\} \otimes \pi)$$

In other words, each coalitions has a lower value if two other coalitions decide to group up instead of remaining separated.

A PFG (N, v) is **convex** if for all $S, T \subset N$ and $\pi \in \Pi_{N\setminus(S\cup T)}$:

$$v(S\cup T, \{S\cup T\}\otimes\pi)+v(S\cap T, \{S\cap T, S\setminus T, T\setminus S\}\otimes\pi) \geq v(S, \{S, T\setminus S\}\otimes\pi)+v(T, \{T, S\setminus T\}\otimes\pi)$$

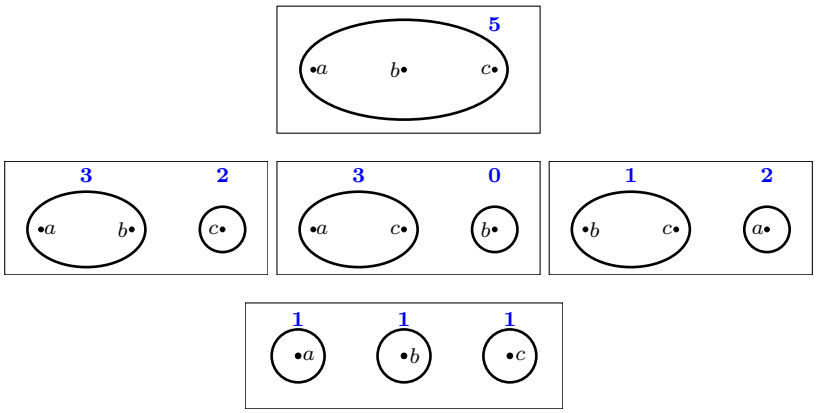
This means two coalitions, even if they are not disjoint have interest in merging.

A PFG (N, v) is **cohesive** if it verifies the following inequalities:

$$v(N, \{N\}) \geq \sum_{S\in\pi} v(S, \pi) \text{ for all } \pi \in \Pi_N$$

In partition function games, what a coalition obtains by deviating from the grand coalition depends on the partition of the remaining players. Consequently, there are multiple notions of core in partition function games. I will use the following example to illustrate the alternative definitions of the core.

Example 20. Let (N, v) be the PFG with 3 players, a, b, c, defined as follows:



The α-core (Aumann and Peleg (1960)). Also called **pessimistic core**, based on the idea that a deviating coalition must expect the worst from the rest of the players:

$$C^{\alpha}(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \forall S \subset N\ \mathbf{x}(S) \geq \min_{\pi \in \Pi_{N\setminus S}} v(S, \{S\} \otimes \pi)\}$$

In the Example 20, if c deviates from the grand coalition, he expects getting 1 while a and b split. Even if it is not profitable for them, a and b decide to punish c for deviating. The fact that coalitions consider deviating only if all the worst possible outcomes are profitable makes this core the biggest possible to define: given a PFG (N,v), all possible cores of (N,v) are included in $C^{\alpha}(N,v)$.

The ω-core (Shenoy (1979)). Also called **optimistic core**, based on the idea that a coalition wants to deviate as soon as there exists a partition of the complementary coalition ensuring a better payoff:

$$C^{\omega}(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \forall S \subset N\ \mathbf{x}(S) \geq \max_{\pi \in \Pi_{N\setminus S}} v(S, \{S\} \otimes \pi)\}$$

In the Example 20, if a deviates from the grand coalition, he expects getting 2, even if b and c would have interestss in splitting. The fact that coalitions consider deviating as soon as there exists a profitable outcome makes this core the smallest possible one: given a PFG (N,v), $C^{\omega}(N,v)$ is included in all possible cores of (N,v). As the coalitions expects the best for them if they deviate, the following result holds.

Proposition 15. *Let (N,v) be a PFG. All allocations of $C^{\omega}(N,v)$ are Pareto efficient.*

Proof. Let $x \in C^{\omega}(N,v)$. *Ad absurdum*: suppose that the outcome (y,π) pareto-dominates $(x,\{N\})$. Then $\sum_{S\in\pi} v(S,\pi) > v(N,\{N\})$. Therefore there exists $S \subset N$ such that $v(S,\pi) > \mathbf{x}(S)$. By definition of $C^{\omega}(N,v)$ it is impossible. □

Unlike the two previous core concepts, rule of deviation of the two following ones do not take into consideration the values of the coalition.

The s-core (Hafalir (2007)). Also called the core with singleton expectations, based on the idea that, if a coalition deviates from the grand coalition, there is no longer cooperation between the remaining players. It is defined by:

$$C^s(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \mathbf{x}(S) \geq v(S,\{S\} \otimes [N \setminus S]) \ \forall \ S \subset N\}$$

where $[A]$ denotes the partition of a set A into singletons. With this definition, in the Example 20, if a player deviates from the grand coalition, he expects that all the players will split, regardless of the potential benefits in merging.

The m-core (Hafalir (2007)). Also called the merge core, based on the idea that, if a coalition deviates from the grand coalition, the remaining players stay together:

$$C^m(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \mathbf{x}(S) \geq v(S,\{S, N \setminus S\}) \ \forall \ S \subset N\}$$

This core is the opposite vision of the s-core, but is similar in the sense that, after a deviation, the remaining players do not consider the valuations while deciding the partition they form.

The δ-core (Hart and Kurz (1983)). It is defined as the set of partitions and allocations for which, after a deviation, the remaining players form the partition without the deviating players:

$$C^\delta(N,v) = \{(x,\pi) \in \mathbb{R}^n \times \Pi_N \mid \mathbf{x}(N) = \sum_{T \in \pi} v(T,\pi) \text{ and } \mathbf{x}(S) \geq v(S,\{S\} \otimes \pi \backslash S) \ \forall \ S \subset N\}$$

This concept presupposes that full cooperation of the players is not restricted by the formation of the coalition, but all the players cooperate to form one of a partitions robust to deviations.

The rational expectation core (Hafalir (2007)). It is based on the idea that, if a coalition deviates from the grand coalition, the remaining players form the partition that maximizes the sum of the values of its cells. If there are several options, they choose one of the worst one for the deviating coalition:

$$C^{RE}(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \forall S \subset N,$$

$$\mathbf{x}(S) \geq \min_{\substack{\pi(S)\in \operatorname{argmax} \\ \pi\in\Pi_{N\setminus S}}} \sum_{T\in\pi} v(T,\{S\}\otimes\pi(S))\} v(S,\{S\}\otimes\pi(S))\}$$

In the Example 20, if c deviates from N, he expects getting 2, while a and b have interests in merging. Of all the cores defined here, this core is the one where the strategic interests of the remaining players are the most accounted for.

Hafalir (2007) uses the s-core to provide some results of inclusion of the different cores, and also sufficient conditions of non-emptiness of the cores.

Proposition 16 (Hafalir (2007))**.** *Let (N,v) be a PFG.*

- $C^{\omega}(N,v) \subset C^{s}(N,v) \subset C^{\alpha}(N,v)$.
- *If (N,v) is convex, then $C^{\alpha}(N,v) \neq \varnothing$ and $C^{s}(N,v) \neq \varnothing$.*
- *If (N,v) is convex and has negative externalities, then $C^{\omega}(N,v) \neq \varnothing$.*

There is no general results for the inclusion between $C^{RE}(N,v)$, $C^{\delta}(N,v)$, $C^{m}(N,v)$ and $C^{s}(N,v)$.

3.3 Stable recursive cores

I now introduce a notion of stable core, which encompasses the idea that if a coalition S deviates from the grand coalition, the players among S expect the remaining players to form a strategically stable partition, in a sense that will be clear in what follows. This notion is compatible with the idea of pessimistic and optimistic cores. As a preliminary step, I need to define the residual games of PFG (Kóczy (2018)).

Given a PFG (N,v), a coalition $S \subset N$, and a partition $\pi \in \Pi_{N\setminus S}$, the residual game $(S,\tilde{v}_{\pi})$ is the PFG where:

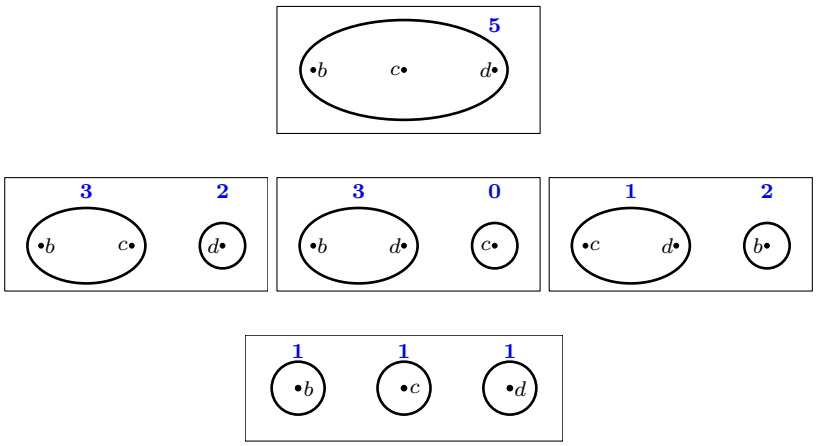
$$\begin{aligned} \tilde{v}_{\pi} : 2^{S} \times \Pi_{S} &\longrightarrow \mathbb{R} \\ (Q,\tilde{\pi}) &\longmapsto v(Q,\tilde{\pi}\otimes\pi) \end{aligned}$$

$\tilde{v}_{\pi}(Q,\tilde{\pi})$ defined if $Q \in \tilde{\pi}$.

The idea here is to isolate a cell of a partition and consider the subgame played by the players in this cell given that other cells will not deviate.

In order to introduce the concepts of stable recursive cores, I give the following example.

Example 21. Let (N, v) be a PFG with 4 players: a, b, c and d. a wonders how much he could get deviating from the grand coalition. This value is the minimum he will ask to participate in the grand coalition. He considers that, if he deviates, the other players will not consider his value, but instead play a residual game and hence, form a partition in which no cell have interests in deviating. After a deviates, the remaining players play the following residual game:



In this residual game, it is important to define what stability is for a partition. In stable recursive cores, a partition is considered stable when the residual games of all its cells have a non-empty stable recursive core while the other cells are fixed. In this residual game, the partition $\{\{b\}, \{c\}, \{d\}\}$ is stable since no player can form a smaller partition. The coalition $\{\{b, d\}, \{c\}\}$ are stable (so is $\{\{b, c\}, \{d\}\}$) since b and d have no interests in splitting.

In order to evaluate how much he can ask in the grand coalition to cooperate, a needs to find all stable partitions of b, c, and d, and then choose a rule of formation among these.

As for non recursive core concepts, I provide two extreme definitions of stable recursive cores: an optimistic and a pessimistic one.

3.3.1 The αr-core

The first stable recursive core I define is a concept in which a deviating coalition is pessimistic and asks in the grand coalition for the value it gets in the worst case scenario

among the stable partition of deviating players. The **pessimistic stable recursive core**, or αr-core, of (N, v) is given by:

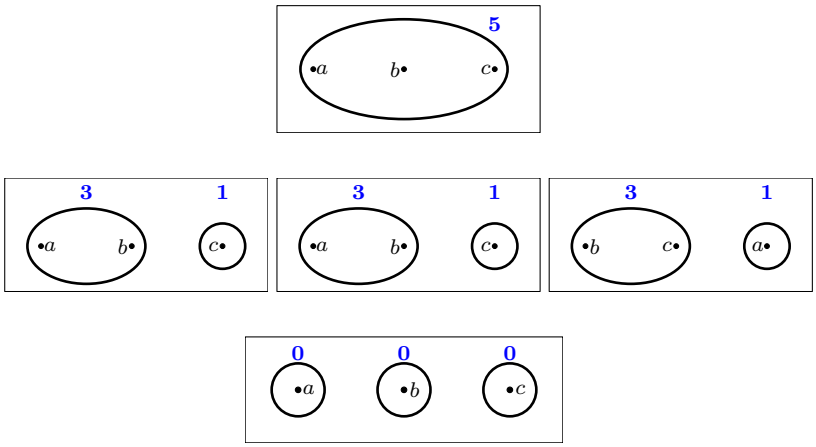
$$C^{\alpha r}(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \forall\, S \subset N, \min_{\pi \in \Pi_{N\setminus S} \text{ s.t. } \forall\, S' \in \pi, C^{\alpha r}(S', \tilde{v}_{(\pi \setminus S') \otimes \{S\}}) \neq \varnothing} v(S, \{S\} \otimes \pi) \leq \mathbf{x}(S)\}$$

with $C^{\alpha r}(\{i\}, \tilde{v}_\pi) = \{v(\{i\}, \{\{i\}\} \otimes \pi)\}$ for all $(i, \pi) \in N \times \Pi_{N\setminus\{i\}}$.

The idea here is that when a coalition deviates from the grand coalition, the remaining players want to form a partition which is stable, in the sense that for every cell S' of this partition, the pessimistic stable recursive core of the residual game generated by S' and $(\pi \setminus S') \otimes \{S\}$ is non-empty: in other words, every cell of the resulting partition will be sustainable if the partition is fixed. The fact that only stable partitions are considered ensures that, if S deviates, after chosing a stable partition π in $\Pi_{N\setminus S}$, the players of $N \setminus S$ cannot deviate. It is a pessimistic point of view so the players of the deviating coalition S considers only the worst stable partition for them.

The following example illustrates the formation of an αr core.

Example 22. Let me consider, as in Example 18, the following game with 3 players:



I seek for the worst stable partitions in response of deviations from the grand coalition, to obtain the inequalities that define the αr-core of this game. This game is symmetric, so it is enough to consider only the deviations of $\{a\}$ and $\{a, b\}$.

Case 1: $\{a\}$ deviates. By definition, the partition $\{\{b\}, \{c\}\}$ is stable. $\{\{b, c\}\}$ is also stable because $v(\{b, c\}, \{\{a\}, \{b, c\}\}) = 3$, and if $\{b\}$ or $\{c\}$ deviate, they get 0. Here,

the partition that gives the minimum value to $\{a\}$ is $\{\{b\},\{c\}\}$. Therefore, by definition of the αr-core, $x_a \geq 0$. As the game is symmetric: $x_i \geq 0$ for all $i \in \{a,b,c\}$.

Case 2: $\{a,b\}$ deviates. By definition, the partition $\{\{c\}\}$ is stable.
As $v(\{a,b\},\{\{a,b\},\{c\}\}) = 3$, for all $i,j \in \{a,b,c\}$, $x_i + x_j \geq 3$.

Hence, the αr-core is defined as follows: $C^{\alpha r}(N,v)$ is the set of $x \in \mathbb{R}^3$ verifying the following inequalities: $x_i \geq 0$, $x_i + x_j \geq 3$ for all $i,j \in N$ and $x_a + x_b + x_c = 5$. The allocation $x = (\frac{5}{3},\frac{5}{3},\frac{5}{3})$ satisfies the inequalities, then $x \in C^{\alpha r}(N,v)$.

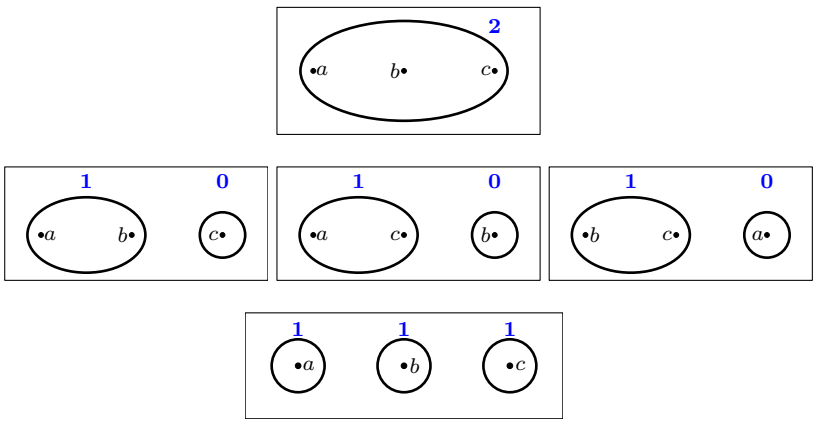
As a partition of singletons is always stable given the definition, the following proposition holds:

Proposition 17. *Let (N,v) be a PFG. $C^s(N,v) \subset C^{\alpha r}(N,v) \subset C^{\alpha}(N,v)$.*

Proof. It is straightforward to establish that $C^{\alpha r}(N,v) \subset C^{\alpha}(N,v)$. Let me show that $C^s(N,v) \subset C^{\alpha r}(N,v)$. Let $x \in C^s(N,v)$. By definition of $C^s(N,v)$, for all $S \subset N$, $\mathbf{x}(S) \geq v(S,\{S\} \otimes [N \setminus S])$. Let $S \subset N$. $[N \setminus S]$ is a partition of $N \setminus S$, and, for all $i \in [N \setminus S]$, $C^{\alpha r}(\{i\}, \tilde{v}_{([N\setminus S]\setminus\{i\})\otimes\{S\}}) \neq \varnothing$ by definition. Hence, for all $S \subset N$, there exists $\pi = [N \setminus S] \in \Pi_{N\setminus S}$ such that for all $S' \in \pi, C^{\alpha r}(S', \tilde{v}_{(\pi\setminus S')\otimes\{S\}}) \neq \varnothing$ and $v(S,\{S\} \otimes \pi) \leq \mathbf{x}(S)$. Consequently, by definition of $C^{\alpha r}(N,v)$, $x \in C^{\alpha r}(N,v)$. □

It follows from Proposition 17 and Hafalir (2007) that every convex PFG has a non-empty αr-core. However, non-convex PFG may have a non-empty α-core but no αr-core: the grand coalition may not be sustainable because the players have no interest to punish a deviating coalition, as shown in the following example:

Example 23. Let (N,v) be the symmetric PFG with 3 players, a,b,c, defined as follows:



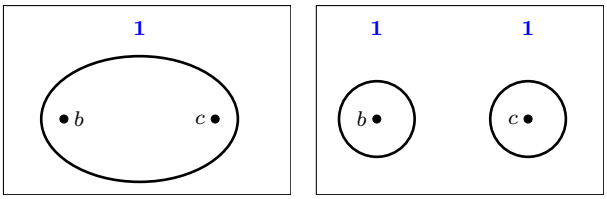
Let me show that $C^{\alpha}(N,v) \neq \varnothing$: if $x \in C^{\alpha}(N,v)$, $x_a + x_b + x_c = 2$, $x_i \geq 0$, and $x_i + x_j \geq 1$ for all $i, j \in N$. Thus, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C^{\alpha}(N,v)$.

Let me now prove that $C^{\alpha r}(N,v) = \varnothing$: in $C^{\alpha}(N,v)$, if i deviates, he considers that j and k will stay together. However, in this configuration, j and k have incentives to split. The notion of pessimistic stable recursive core ensures that, the reaction of the remaning players after one coalition decides to split will be stable.

Let $x \in C^{\alpha r}(N,v)$. By definition of the $C^{\alpha r}$ core, $x_a + x_b + x_c = 2$. By symmetry, $x_i + x_j \geq 1$ for all $i, j \in N$. Let me look at the worst stable partitions in response of deviations of one player from the grand coalition. This game is symmetric, then it is enough to just look at the deviation of $\{a\}$. There are two possibilities for b and c:

Case 1: b and c stay together

I study $C^{\alpha r}(\{b,c\}, \tilde{v}_{\{\{a\}\}})$. The residual game $(\{b,c\}, \tilde{v}_{\{\{a\}\}})$ is as follows:



Hence if $y \in C^{\alpha r}(\{b,c\}, \tilde{v}_{\{\{a\}\}})$, $y_b + y_c = 1$, and $y_b \geq 1$, $y_c \geq 1$. That is impossible. Then $C^{\alpha r}(\{b,c\}, \tilde{v}_{\{\{a\}\}}) = \varnothing$: $\{\{b,c\}\}$ is not a stable partition in reaction to $\{a\}$'s deviation.

Case 2: b and c split

By definition, $C^{\alpha r}(\{b\}, \tilde{v}_{\{\{a\},\{c\}\}}) \neq \varnothing$, $C^{\alpha r}(\{c\}, \tilde{v}_{\{\{a\},\{b\}\}}) \neq \varnothing$. Thus, $\{\{b\},\{c\}\}$ is a stable partition in reaction to i's deviation.

If $x \in C^{\alpha r}(N,v)$, $x_i \geq 1$ for all $i \in N$. Consequently $x_a + x_b + x_c \geq 3$: that contradicts the equality $x_a + x_b + x_c = 2$, so $C^{\alpha r}(N,v) = \varnothing$.

In what follows, I prove a Bondareva-Shapley-like theorem for the αr-core:

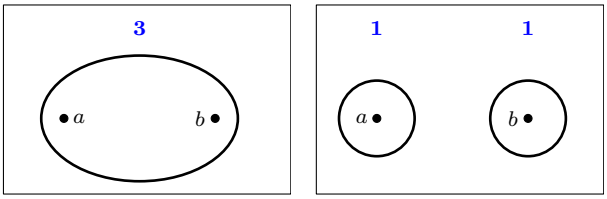
Let define α-recursively balanceness by induction: a PFG (N,v) is **α-recursively balanced** if $\mid N \mid = 1$. If $\mid N \mid > 1$, let define the set $\mathscr{L}^{\alpha RB} := \{(S,\pi) \mid S \in \pi$ and $\forall Q \in \pi \setminus \{S\}, \tilde{v}_{\pi\setminus\{Q\}}$ is α-recursively balanced$\}$.

(N,v) is α-recursively balanced if there exists $(\lambda_S^{\pi})_{(S,\pi)\in\mathscr{L}^{\alpha RB}}$ such that:

- $\lambda_S^{\pi} \in \{0,1\}$ for all $(S,\pi) \in \mathscr{L}^{\alpha RB}$ and $\sum_{(S,\pi)\in\mathscr{L}^{\alpha RB}} \lambda_S^{\pi} = 1$

- $v(N,\{N\}) \geq \sum_{(S,\pi)\in\mathscr{L}^{\alpha RB}} \delta_S^\pi \lambda_S^\pi v(S,\pi)$ for all $(\delta_S^\pi)_{(S,\pi)\in\mathscr{L}^{\alpha RB}}$ verifying the two following conditions:
  - $\delta_S^\pi \in [0,1]$ for all $(S,\pi)\in\mathscr{L}^{\alpha RB}$
  - $\sum_{(S,\pi)\in\mathscr{L}^{\alpha RB}} \delta_S^\pi \lambda_S^\pi \mathbb{1}_{i\in S} = 1$ for all $i \in N$

Example 24. Let (N,v) be the PFG with two players a and b, defined as follows:



In this game, $\mathscr{L}^{\alpha RB} = \{(\{a\},\{\{a\},\{b\}\}),(\{b\},\{\{a\},\{b\}\}),(N,\{N\})\}$.

Let define $\lambda_S^\pi := 1$ for all $(S,\pi)\in\mathscr{L}^{\alpha RB}$. The elements δ verifying the two conditions are such that $\delta_i^{\{\{a\},\{b\}\}} = p$ and $\delta_N^{\{N\}} = 1-p$ for $p\in[0,1]$.

Let $p\in[0,1]$. Therefore, $v(N,\{N\}) = 3 \geq \sum_{(S,\pi)\in\mathscr{L}^{\alpha RB}} \delta_S^\pi\lambda_S^\pi v(S,\pi) = p+p+3(1-p) = 3-p$. This game is α-recursively balanced.

The idea of α-recursively balanceness is similar to the Bondareva-Shapley one: λ gives a collection of embedded coalitions, and the δ are balancing vectors, so that all players S get a weight of 1. The only difference is that the members of the collection are chosen in order that the residual games that are considered have non-empty cores, so they satisfy the recursive condition of the αr-core. This definition allows me to provide a characterization of the αr-core.

Theorem 9. *A PFG (N,v) is α-recursively balanced if, and only if its αr-core is non-empty.*

Proof. I prove this result by induction. The case with $n=1$ is a direct consequence of the definition.

Let (N,v) a PFG such that $n>1$. For all $(S,\pi)\in\mathscr{L}^{\alpha RB}$, let

$$\lambda_S^\pi := \mathbb{1}_{v(S,\pi)=\min_{\pi' \text{ s.t. } (S,\pi')\in\mathscr{L}^{\alpha RB}} v(S,\pi')}$$

If for S, there are two π,π' such that $\lambda_S^\pi = \lambda_S^{\pi'} = 1$, randomly only one is set to 1, the other to 0.

By induction, $C^{\alpha r}(N,v) \neq \emptyset$ if, and only if the following linear program is feasible, and its value is lower than $v(N,\{N\})$:

$$\min \mathbf{x}(N) \text{ s.t. for all } S, \text{ there exists } \pi \ni S \text{ s.t. } \mathbf{x}(S) \geq v(S,\pi) \text{ and } (S,\pi) \in \mathscr{L}^{\alpha RB}$$

The fact that $\mathbf{x}(N)$ must be lower than $v(N,\{N\})$ comes from the efficiency condition of the core, and $\mathbf{x}(S) \geq v(S,\pi)$ is because for these embedded coalitions, all $Q \in \pi \setminus S$ are stable by induction and definition of $\mathscr{L}^{\alpha RB}$.

By multiplying by λ^{π}_{S}, this program is equivalent to:

$$\min \mathbf{x}(N) \text{ s.t. } \lambda^{\pi}_{S}\mathbf{x}(S) \geq \lambda^{\pi}_{S} v(S,\pi) \text{ for all } (S,\pi) \in \mathscr{L}^{\alpha RB}$$

The dual of this program is:

$$\max \sum_{(S,\pi)\in\mathscr{L}^{\alpha RB}} \delta^{\pi}_{S}\lambda^{\pi}_{S} v(S,\pi) \text{ s.t. } \sum_{(S,\pi)\in\mathscr{L}^{\alpha RB}} \delta^{\pi}_{S}\lambda^{\pi}_{S}\mathbb{1}_{i\in S} = 1 \text{ for all } i \in N$$

By the duality theorem, the linear program feasibility and its dual are equivalent, and if they have the same value if they are feasible. Therefore the condition holds.

Conversely, if the PFG (N,v) is α-recursively balanced, then the dual program (the maximization problem above) is feasible, and its maximum is equal to $v(N,\{N\})$. By the duality theorem, the minimization problem above is feasible, and its minimum is $v(N,\{N\})$. Therefore, there exists $x \in \mathbb{R}^n$ such that for all $S \subset N$, there exist $\pi \ni S$ such that $(S,\pi) \in \mathscr{L}^{\alpha RB}$ and $\mathbf{x}(S) \geq v(S,\pi)$. By induction and the definition of $C^{\alpha r}(N,v)$, it is non empty. □

3.3.2 The ωr-core

The general idea of the ωr-core is the same as for the αr-core up to the difference that deviating players consider the best case value for them to define their value. This gives the following definition.

The **optimistic stable recursive core**, or ωr-core of (N,v) is given by:

$$\begin{aligned} C^{\omega r}(N,v) = &\{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \forall\ S \subset N,\ \forall\ \pi \in \Pi_{N\setminus S} \\ &\text{s.t. } \forall\ S' \in \pi, C^{\omega r}(S', \tilde{v}_{(\pi\setminus S')\otimes\{S\}}) \neq \varnothing,\ v(S,\{S\}\otimes\pi) \leq \mathbf{x}(S)\} \end{aligned}$$

with $C^{\omega r}(\{i\}, \tilde{v}_\pi) = \{v(\{i\}, \{\{i\}\} \otimes \pi)\}$ for all $(i, \pi) \in N \times \Pi_{N\setminus\{i\}}$.

The idea here is the same as for the pessimistic stable recursive core: when a coalition deviates from the grand coalition, the remaining players want to form a partition which is stable, in the sense that for every cell S' of this partition, the optimistic stable recursive core of the residual game generated by S' and $(\pi \setminus S') \otimes \{S\}$ is non-empty. This is an optimistic point of view as the deviating coalition S considers the best case scenario of stable partition.

As for the αr-core, by definition, the partition of singletons is always stable. As a consequence, the following proposition holds:

Proposition 18. *Let (N, v) be a PFG. $C^{\omega}(N, v) \subset C^{\omega r}(N, v) \subset C^{s}(N, v)$.*

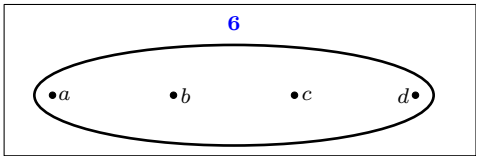
Proof. It is straightforward to establish that $C^{\omega}(N, v) \subset C^{\omega r}(N, v)$. Let me show that $C^{\omega r}(N, v) \subset C^{s}(N, v)$: consider $x \in C^{\omega r}(N, v)$. Hence, for all $S \subset N$, for all $\pi \in \Pi_{N\setminus S}$ such that for all $S' \in \pi$, $C^{\omega r}(S', \tilde{v}_{(\pi\setminus S')\otimes\{S\}}) \neq \varnothing$, the following inequality holds: $v(S, \{S\} \otimes \pi) \leq \mathbf{x}(S)$.

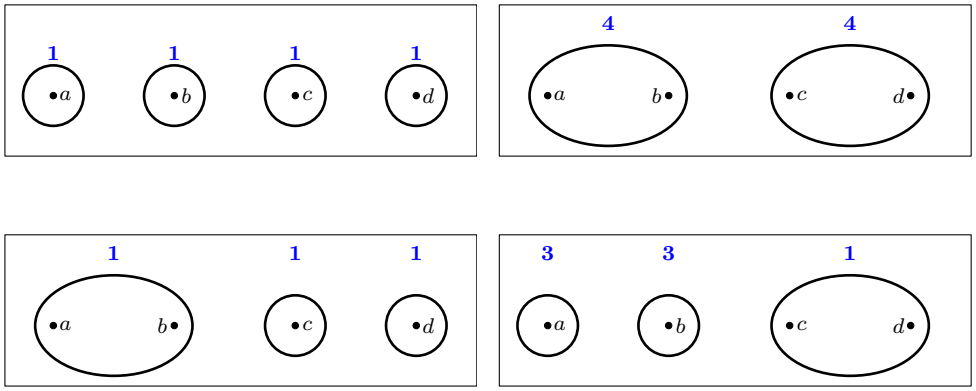
Let $S \subset N$. $[N \setminus S]$ is a partition of $N \setminus S$, and, for all $i \in [N \setminus S]$, by definition, $C^{\omega r}(\{i\}, \tilde{v}_{([N\setminus S]\setminus\{i\})\otimes\{S\}})$ cannot be empty. Therefore, for all $S \subset N$, $\mathbf{x}(S) \geq v(S, \{S\} \otimes [N \setminus S])$, that is $x \in C^{s}(N, v)$. □

It follows from Proposition 18 and Hafalir (2007) that every convex PFG with negative externalities has a non-empty optimistic stable recursive core.

Although the optimistic stable recursive core ensures that the reactions to deviations are stable, the strategic aspects of the formation of the core make outputs subject to global improvements. As a consequence $C^{\omega r}(N, v)$ may contain Pareto-dominated outcomes, as shown in the following example.

Example 25. Let (N, v) be the PFG such that $N = \{a, b, c, d\}$, defined as follows:





For all other embedded coalitions, $v(S, \pi) = 0$.

Let me find $x \in C^{\omega r}(N, v)$ that is Pareto-dominated. Let me find the best stable partitions in response to a deviation of a subcoalition from the grand partition: all the cases are listed below:

Case 1: a coalition of 3 players deviates

The remaining player form a stable coalition, and the deviating coalition gets 0: for all $i, j, k \in N$, $x_i + x_j + x_k \geq 0$.

Case 2: a coalition of 2 players deviates

There are 3 subcases that are listed below:

- if the coalition is $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, or $\{b, d\}$, they get 0 whatever happens. Thus, $x_a + x_c \geq 0$, $x_a + x_d \geq 0$, $x_a + x_c \geq 0$, $x_b + x_d \geq 0$.
- if the coalition is $\{a, b\}$, the remaining players form a stable partition anyway. Thus, $x_a + x_b \geq 4$.
- if the coalition is $\{c, d\}$, the only stable partition for $\{a, b\}$ is $\{a\}, \{b\}$. Thus, $x_c + x_d \geq 1$.

Case 3: a coalition of one player i deviates

The only stable partition of the remaining players is $\{\{j\}, \{k\}, \{l\}\}$. Thus, $x_i \geq 1$ for all $i \in N$.

Then $C^{\omega r}(N, v) = \{x \in \mathbb{R}^4 \mid x_i \geq 1, x_i + x_j + x_k \geq 0$ for all $i, j, k \in N, x_i + x_j \geq 0$ if $(i, j) \in \{(a, c), (a, d), (b, c), (b, d)\}, x_a + x_b \geq 4, x_c + x_d \geq 1$ and $x_a + x_b + x_c + x_d = 6\}$. Therefore $x = (2, 2, 1, 1) \in C^{\omega r}(N, v)$. However, the outcome $((2, 2, 2, 2), (\{a, b\}, \{c, d\}))$ Pareto-dominates the outcome $(x, \{N\})$, but $(2, 2, 2, 2)$ is not in the core.

As a corollary, using the fact that $C^{\omega r}(N,v) \subset C^{\alpha r}(N,v)$, $C^{\alpha r}(N,v)$ may also contain Pareto-dominated outcomes.

However, the fact that cohesiveness of a PFG ensures that all allocations are Pareto efficient gives the following result:

Proposition 19. *For all cohesive PFGs (N,v), the allocations of $C^{\omega r}(N,v)$ and $C^{\alpha r}(N,v)$ are Pareto efficient.*

The following proposition shows that, as for the ω-core, convexity is not sufficient to guarantee the non-emptiness of $C^{\omega r}(N,v)$.

Proposition 20. *Convexity is not a sufficient condition for the non-emptiness of $C^{\omega r}$.*

Proof. Ad absurdum: suppose that for all convex PFGs (N,v), $C^{\omega r}(N,v) \neq \varnothing$. Let (N,v) be a convex PFG. For all $S \subset N$, $\pi \in \Pi_{N\setminus S}$, $(S, \tilde{v}_\pi)$ is convex. Thus, for all $S \subset N$, $\pi \in \Pi_{N\setminus S}$, π is stable (because, as supposed: for all $S' \in \pi, C^{\omega r}(S', \tilde{v}_{\pi\setminus S')\otimes\{S\}}) \neq \varnothing$ due to the convexity of $\tilde{v}_{\pi\setminus S')\otimes\{S\}}$), then $C^{\omega r}(N,v) = C^{w}(N,v)$. The fact that $C^{\omega r}(N,v) \neq \varnothing$ would imply that $C^{w}(N,v) \neq \varnothing$ for all convex PFGs (N,v): that is absurd, as shown in Hafalir (2007). □

As for the αr-core, I prove a Bondareva-Shapley-like theorem for the ωr-core:

Let define ω-recursively balanceness by induction: a PFG (N,v) is **ω-recursively balanced** if $\mid N \mid = 1$. If $\mid N \mid > 1$, let define the set $\mathscr{L}^{RB} := \{(S,\pi) \mid S \in \pi \text{ and } \forall Q \in \pi \setminus \{S\}, \tilde{v}_{\pi\setminus\{Q\}} \text{ is } \omega\text{-recursively balanced}\}$. For all $(\delta^\pi_S)_{(S,\pi)\in\mathscr{L}^{RB}}$ such that

- $\delta^\pi_S \in [0,1]$ for all $(S,\pi) \in \mathscr{L}^{RB}$
- $\sum_{(S,\pi)\in\mathscr{L}^{RB}} \delta^\pi_S \mathbb{1}_{i\in S} = 1$ for all $i \in N$

(N,v) is ω-recursively balanced if $v(N,\{N\}) \geq \sum_{(S,\pi)\in\mathscr{L}^{RB}} \delta^\pi_S v(S,\pi)$.

Theorem 10. *A PFG (N,v) is ω-recursively balanced if, and only if its ωr-core is non-empty.*

Proof. I prove this result by induction. The case with $n = 1$ is a direct consequence of the definition.

Let (N, v) a PFG such that $n > 1$. By induction, for the same reasons as for the αr-core, $C^{\omega r}(N, v) \neq \emptyset$ if, and only if the following linear program is feasible, and its value is lower than $v(N, \{N\})$:

$$\min \sum_{j \in N} x_j \text{ s.t. } \mathbf{x}(S) \geq v(S, \pi) \text{ for all } (S, \pi) \in \mathscr{L}^{RB}$$

The dual of this program is:

$$\max \sum_{(S,\pi) \in \mathscr{L}^{RB}} \delta_S^\pi v(S, \pi) \text{ s.t. } \sum_{(S,\pi) \in \mathscr{L}^{RB}} \delta_S^\pi \mathbb{1}_{i \in S} = 1 \text{ for all } i \in N$$

By the duality theorem, the linear program feasibility and its dual are equivalent, and have the same value if they are feasible. Therefore the theorem works.

Conversely, if the PFG (N, v) is ω-recursively balanced, then the dual program (the maximization problem above) is feasible, and its maximum is equal to $v(N, \{N\})$. By the duality theorem, the minimization problem above is feasible, and its minimum is $v(N, \{N\})$. Therefore, there exists $x \in \mathbb{R}^n$ such that for all $(S, \pi) \in \mathscr{L}^{\omega RB}$, $\mathbf{x}(S) \geq v(S, \pi)$. By induction and the definition of $C^{\omega r}(N, v)$, it is non empty □

3.4 Stable cores by successive deviations

Another way to have a stable partition is to get it with successive deviations: this creates an iterative notion of stability. At the beginning, the grand coalition forms, and at each stage, if one of the coalitions that have formed has a subcoalition of players who have incentives to leave the coalition, there is a split. If no cell of a partition has a subcoalition of players that have incentives to split, the obtained partition is stable. This notion of core takes into account all stable partitions of the remaining players for the deviating coalitions.

The difference with stable recursive cores is that the stable partitions of the two concepts may be totally different: in successive deviations, the coalitions can just split into two coalitions, while in stable recursive cores, the grand coalition can split into all possible partitions of Π_N.

Let (N, v) be a PFG. Let $\pi, \pi' \in \Pi_N$. π' is a **rational decomposition** of π if there exists $S_1, S_2 \in \pi'$ such that:

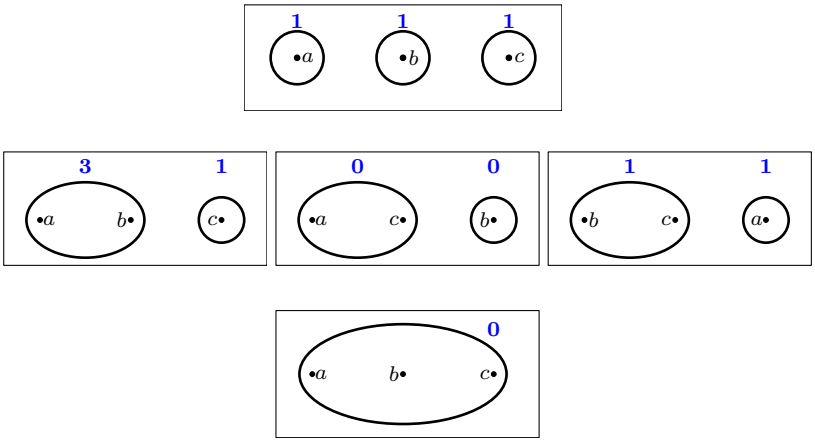
(i) $\pi = \{T_1, \dots T_l, S_1 \cup S_2\}$ and $\pi' = \{T_1, \dots, T_l, S_1, S_2\}$.

(ii) $v(S_1 \cup S_2, \pi) < v(S_1, \pi') + v(S_2, \pi')$.

A partition $\pi \in \Pi_N$ is a **stable decomposition of** N if there exists a sequence of partitions $\pi_1, \dots, \pi_p$, where $\pi_1 = \{N\}$, $\pi_p = \pi$ such that for all $i \in \{2, \dots p\}$, π_i is a rational decomposition of π_{i-1}, and π admits no rational decomposition. The set of stable decompositions of N is denoted by $S^D(N, v)$. The fact that N is a finite set implies that it cannot be subdivided indefinitely, hence for all PFGs (N, v), $S^D(N, v) \neq \varnothing$.

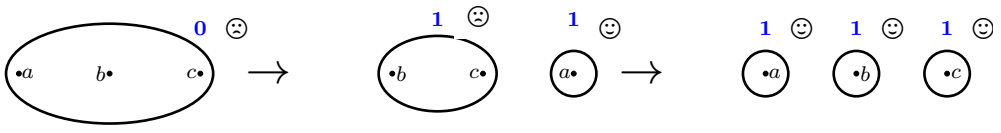
To illustrate, let me consider an example:

Example 26. Let (N, v) be the PFG with 3 players, a, b, c, defined as follows:



Let me find the set of stable decomposition of N: the rational decompositions of $\{N\}$ are $\{\{a, b\}, \{c\}\}$ and $\{\{a\}, \{b, c\}\}$. $\{\{a, b\}, \{c\}\}$ is stable, but $\{\{a\}, \{b, c\}\}$ is not, because $\{b, c\}$ have incentives to split. Thus $S^D(N, v) = \left\{\{\{a, b\}, \{c\}\}, \{\{a\}, \{b\}, \{c\}\}\right\}$.

Here is an illustration of the path from $\{N\}$ to $\{\{a\}, \{b\}, \{c\}\}$:



Here is an illustration of the path from $\{N\}$ to $\{\{a, b\}, \{c\}\}$:

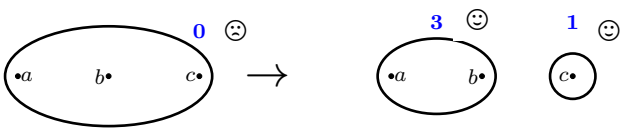


Figure 3.1: Illustration of different paths of rational decompositions of the grand coalition

Let (N, v) be a PFG. I define two notions of core using stable decompositions, based on the idea that, when a coalition S deviates, from the grand coalition the remaining players split until they reach a stable decomposition of $N \setminus S$. As for the stable recursive cores, the deviating players consider only the stable partition of the remaining players to fix their value. Although they may choose any stable partition, like for stable recursive cores, I define two extreme notions of stable cores by successive deviations as follows:

- The pessimistic stable core by successive deviations of (N, v) is the set:

$$C^{\alpha D}(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \forall\ S \subset N, \mathbf{x}(S) \geq \min_{\pi \in S^D(N\setminus S, \tilde{v}_{\{S\}})} v(S, \{S\} \otimes \pi)\}$$

- The optimistic stable core by successive deviations of (N, v) is the set:

$$C^{\omega D}(N,v) = \{x \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N,\{N\}) \text{ and } \forall\ S \subset N, \mathbf{x}(S) \geq \max_{\pi \in S^D(N\setminus S, \tilde{v}_{\{S\}})} v(S, \{S\} \otimes \pi)\}$$

By definition, $C^{\omega D}(N, v) \subset C^{\alpha D}(N, v)$.

The following condition is sufficient condition for the non-emptiness of stable cores by successive deviations:

A PFG (N, v) is **weakly convex** if for all $P \subset S \subset N$, $\pi \in \Pi_{N\setminus S}$, the following inequality holds:

$$\tilde{v}_\pi(S, \{S\}) > v(P, \{P, S \setminus P\} \otimes \pi) + v(S \setminus P, \{P, S \setminus P\} \otimes \pi)$$

It is straightforward to see that every weakly convex PFG is cohesive. The following results holds:

Proposition 21. *For all weakly convex PFGs (N, v), $S^D(N, v) = \{\{N\}\}$.*

Proof. Let (N, v) be a weakly convex PFG, then $\{N\}$ has no rational decompositions: Let $\pi = \{S_1, S_2\}$ be a partition of N. Weak convexity implies that $v(N, \{N\}) \geq v(S_1, \pi) + v(S_2, \pi)$.

Let

$$x = \left(\underbrace{x^1, \ldots, x^1}_{|S_1| \text{ times}}, \underbrace{x^2, \ldots, x^2}_{|S_2| \text{ times}} \right)$$

with $x^1 = \frac{v(N,\{N\}).v(S_1,\pi)}{|S_1|(v(S_1,\pi)+v(S_2,\pi))}$ and $x^2 = \frac{v(N,\{N\}).v(S_2,\pi)}{|S_2|(v(S_1,\pi)+v(S_2,\pi))}$.

For all $i \in \{1,2\}$, $\mathbf{x}(S_i) = v(N,\{N\}).\frac{v(S_i,\pi)}{v(S_1,\pi)+v(S_2,\pi)} \geq \frac{v(N,\{N\}}{v(N,\{N\}}.v(S_i,\pi) = v(S_i,\pi)$. Thus, π cannot be a rational decomposition of $\{N\}$. □

As a corollary, the following proposition holds:

Proposition 22. *For all weakly convex PFGs* (N,v), $C^{\omega D}(N,v) = C^{\alpha D}(N,v) = C^m(N,v)$.

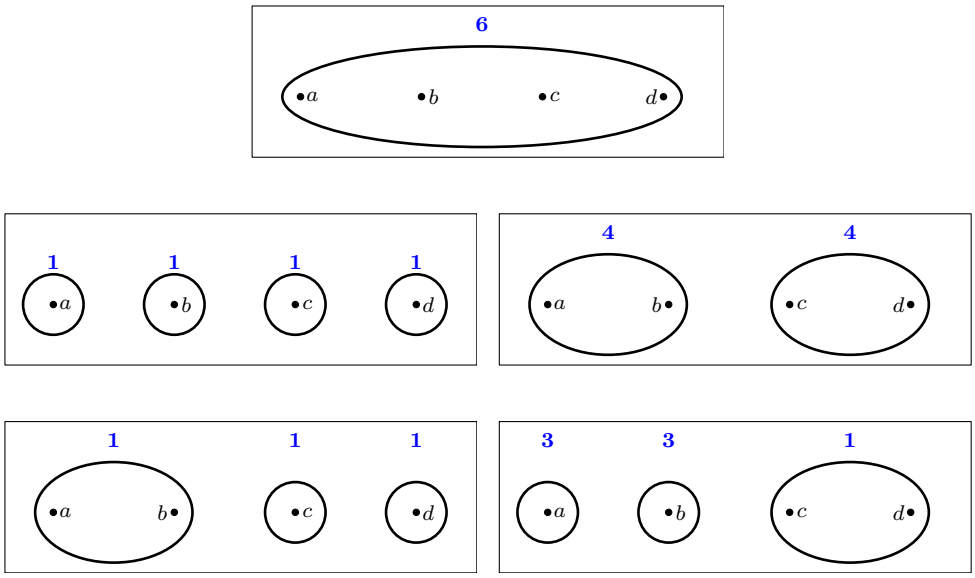
The fact that convexity implies weak convexity when the partition function is positive implies that:

Corollary 3. *For all positive convex PFGs* (N,v), $C^{\omega D}(N,v) = C^{\alpha D}(N,v) = C^m(N,v)$.

The following result shows that outcomes of stable cores by successive deviations are, as recursive cores, subjects to global improvements:

Proposition 23. $C^{\omega D}(N,v)$ *may contain Pareto-dominated outcomes.*

Proof. As in the Example 25, let (N,v) be the PFG such that $N = \{a,b,c,d\}$, defined as follows:



For all other embedded coalitions, $v(S,\pi) = 0$.

Let me find $x \in C^{\omega D}(N,v)$. Let me find the stable partitions in response of a deviation of a subcoalition from the grand coalition: all the cases are listed below:

Case 1: a coalition of 3 players deviates

The remaining player forms a stable coalition, and the deviating coalition gets 0. Therefore, for all $i, j, k \in N$, $x_i + x_j + x_k \geq 0$.

Case 2: a coalition of 2 players deviates

There are 3 subcases that are listed below:

1. if the coalition is $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, or $\{b, d\}$, they get 0 whatever happens. Thus, $x_a + x_c \geq 0$, $x_a + x_d \geq 0$, $x_b + x_c \geq 0$, $x_b + x_d \geq 0$

2. if the coalition is $\{a, b\}$, there is a unique stable partition of the remaining players: $\{c, d\}$. Thus, $x_a + x_b \geq 4$.

3. if the coalition is $\{c, d\}$, the only stable partition of the remaining players is $\{a\}, \{b\}$. Thus, $x_c + x_d \geq 1$

Case 3: a coalition of one player i deviates

The only stable partition is $\{\{j\}, \{k\}, \{l\}\}$. Thus, $x_i \geq 1$ for all $i \in N$.

Then $C^{\omega D}(N, v) = \{x \in \mathbb{R}^4 \mid x_i \geq 1, x_i + x_j + x_k \geq 0$ for all $i, j, k \in N, x_i + x_j \geq 0$ if $(i, j) \in \{(a, c), (a, d), (b, c), (b, d)\}, x_a + x_b \geq 4, x_c + x_d \geq 1$ and $x_a + x_b + x_c + x_d = 6\}$. Therefore $x = (2, 2, 1, 1) \in C^{\omega D}(N, v)$. However, the outcome $((2, 2, 2, 2), (\{a, b\}, \{c, d\}))$ Pareto-dominates the outcome $(x, \{N\})$, but $(2, 2, 2, 2)$ is not in the core. □

As $C^{\omega D}(N, v) \subset C^{\alpha D}(N, v)$, $C^{\alpha D}(N, v)$ may also contain Pareto-dominated outcomes.

As for stable recursive core, the following result holds:

Proposition 24. *For all cohesive PFGs (N, v), the allocations of $C^{\omega D}(N, v)$ and $C^{\alpha D}(N, v)$ are Pareto efficient.*

The following proposition shows that there is no order relation between stable cores by successive deviations and stable recursive cores.

Proposition 25. *Let (N, v) be a PFG. There is no general result of inclusion between $C^{\omega D}(N, v)$ and $C^{\omega r}(N, v)$.*

Proof. The proof consists in the following example. Let (N, v) the PFG defined as follows:

$N = \{1, 2, 3, 4\}$, $v(N, \{N\}) = 5$. For all $i, j, k, l \in N$:

$v(\{i\},\{\{i\},\{j,k,l\}\})=1$, $v(\{i\},\{\{i\},\{j\},\{k,l\}\})=1$, $v(\{i\},\{\{i\},\{j\},\{k\},\{l\}\})=2$

$v(\{i,j\},\pi)=1$ for all $\pi\in\Pi_N$

$v(\{j,k,l\},\{\{i\},\{j,k,l\}\})=3$

Let $x\in C^{\omega D}(N,v)$, $x'\in C^{\omega r}(N,v)$.

Let me find the best stable and recursive partitions in response of a deviation of a subcoalition from the grand coalition: all the deviations possible are listed below.

Case 1: $i\in N$ **deviates from the grand coalition**

$S^D(N\setminus\{i\},\tilde{v}_{\{\{i\}\}})=\{j,k,l\}$. Thus, $x_i\geq 1$.

$\pi=\{\{j\},\{k\},\{l\}\}$ is such that for all $S'\in\pi, C^{\omega r}(S',\tilde{v}_{(\pi\setminus S')\otimes\{S\}})\neq\varnothing$. Then $x'_i\geq 2$: therefore $x'_1+x'_2+x'_3+x'_4\geq 8$: $C^{\omega r}(N,v)=\varnothing$.

Case 2: $\{i,j\}\subset N$ **deviates from the grand coalition**

$S^D(N\setminus\{i,j\},\tilde{v}_{\{\{i,j\}\}})=\{\{k\},\{l\}\}$. Thus $x_i+x_j\geq 1$

Case 3: $\{i,j,k\}\subset N$ **deviates from the grand coalition**

$S^D(N\setminus\{i,j,k\},\tilde{v}_{\{\{i,j,k\}\}})=\{\{l\}\}$. Thus $x_i+x_j+x_k\geq 3$

Therefore $C^{\omega D}(N,v)=\{x\in\mathbb{R}^4\mid x_i\geq 1, x_i+x_j+x_k\geq 3, x_i+x_j\geq 1, x_i\geq 1 \text{ and } x_1+x_2+x_3+x_4=5\}$. Then $(\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4})\in C^{\omega D}(N,v)$.

With almost the same PFG:

$N=\{1,2,3,4\}$, $v(N,\{N\})=5$. For all $i,j,k,l\in N$:

$v(\{i\},\{\{i\},\{j,k,l\}\})=2$, $v(\{i\},\{\{i\},\{j\},\{k,l\}\})=1$, $v(\{i\},\{\{i\},\{j\},\{k\},\{l\}\})=\frac{5}{4}$

$v(\{i,j\},\pi)=1$ for all $\pi\in\Pi_N$

$v(\{j,k,l\},\{\{i\},\{j,k,l\}\})=3$

One can show that results is the opposite: $(\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4})\in C^{\omega r}(N,v)$, and $C^{\omega D}(N,v)=\varnothing$ □

The first PFG of the proof also shows that weak convexity is not a necessary condition for the non-emptiness of $C^{\omega D}(N,v)$. It is also possible to show that there is not any relation of inclusion between $C^{\alpha D}(N,v)$ and $C^{\alpha r}(N,v)$ with a similar proof.

Concluding remarks

While the stable cores by successive deviations consider sequential splits among members of the coalitions, the stable recursive cores takes into account splits of coalitions in more than two subcoalitions, that makes these two notions different. However, both concepts encompass a notion of stability, that does not exist the literature on the core of partition function games.

One can argue that those two notions of stability consider only the fact that coalitions can split, not the fact that coalitions can aggregate. The problem of considering aggregation could break the stability: it would be possible that, after a deviation of a coalition, the remaining players will not end up in a stable partition because of a cycle of profitable deviations. Therefore, it would be possible to define the same kind of core concepts but with a stability defined for aggregation, in the style of Grabisch and Funaki (2012). It may also be interesting to compare those concepts.

I restrict the analysis of these cores to a notion of stability to only pessimistic and optimistic visions, but it is possible to define other types of cores. A natural extension would be to define a kind of rational expectation core with both of these notions of stability for example. However, as for the rational expectation core, choosing the best partition of the remaining players is questionable while some partition among these may prefer another partition for them.

One interesting avenue to explore with these concepts would be to find some class of games for which one notion of core is stronger than the other for the inclusion.

Part II

Auction Theory

Auctions have been a fundamental part of human transactions for centuries. Originating in ancient Greek and Roman civilizations, auctions have served as a method for trading goods, land, and services. During the Roman Empire, auctions were used for the sale of assets seized in war or from bankruptcies. The applications have evolved, becoming a cornerstone in modern economies and strategic decision-making. This selling method has become ubiquitous in the modern world, used in diverse fields such as art markets, real estate, government procurement, and online platforms. For instance, eBay uses an ascending-bid format (similar to English auctions), whereas Google Ads employs a pay-per-click auction model in online advertising. Even spectrum auctions, conducted by governments for allocating radio frequencies to telecom companies, utilize auction theory for efficient allocation.

The formal theorization of auctions finds its roots in the 1960s, notably with the pioneering work of William Vickrey, who laid the foundation for auction theory. Vickrey's seminal contributions, particularly his exploration of auction formats and their implications for economic efficiency, mark the initial stages of rigorous academic inquiry into auctions. Since then, the field has experienced significant growth and expansion. It has evolved into a rich interdisciplinary domain, integrating insights from mathematics, economics, and game theory. The theoretical landscape has seen remarkable advancements, with contributions from scholars like Paul Milgrom, Robert Wilson, and Roger Myerson, among others, enriching our understanding of auctions, their complexities, and their practical applications across various industries. Auction theory, nestled at the intersection of mathematics and economics, serves as a cornerstone in understanding the intricate dynamics governing the exchange of goods and services through auction mechanisms. At its core, this theory dissects the strategic behaviors of bidders and auctioneers, exploring the nuances of diverse auction formats, their inherent rules, and the resulting economic implications. By focusing into the strategic interactions of participants with varying information and valuation structures, auction theory scrutinizes the links between competition, risk assessment, and profit maximization. Crucially, it elucidates the optimal design of auctions, shedding light on the most efficient and revenue-maximizing mechanisms while unraveling the subtleties of information asymmetry and its impact on outcomes. Central to this theory is the Roger Myerson's paper « Optimal Auction De-

sign » (1981). This work represents a pivotal milestone in defining the concept of an optimal auction mechanism for sellers aiming to maximize their revenues in scenarios where buyers hold private information. By establishing the principle of revenue equivalence Myerson showcased that, under specific conditions, different auction formats can yield equivalent revenues for the seller. Identifying the conditions under which a second-price auction with reserved price is optimal for maximizing seller revenues, this paper has had a significant impact on auction design, influencing domains ranging from art auctions to online auctions and the allocation processes, such as spectrum frequencies for telecommunications.

This chapter centers on core elements in auction theory, notably the definition of « single object auctions » and the traditional auction model. It begins by outlining the structure of single-object auctions and exploring bidder strategies and equilibrium in this context. Examining equilibrium strategies in first-price and second-price auctions follows, emphasizing symmetric equilibria. It provides a thorough analysis of revenue comparison among various auction formats, highlighting their efficiency and revenue implications. Additionally, it addresses the fragility of the revenue equivalence theorem, notably in scenarios where item resale undermines its applicability. This chapter then shifts focus to the literature on auctions with resale possibilities, introducing a domain where the revenue equivalence theorem doesn't hold, opening new avenues for auction theory exploration.

4 Single Object Auctions

4.1 Definition of a Single Object Auction

Let me consider the fundamental framework of single object auctions. These auctions are characterized by the sale of a unique item among multiple bidders. The model encompasses a range of auction formats, each defined by distinct rules governing bidder behavior and item allocation.

In a typical single object auction, there exists a seller, denoted as S, who offers an item for sale. A set of N bidders $B = \{1, 2, \ldots, N\}$ participate in the auction. Each bidder $i \in B$ is associated with a valuation θ_i representing their private valuation for the item. The valuations are generally assumed to be independently drawn from a known distribution function F on a set $[0, \omega]$, where ω can be infinite. It is assumed that F admits a continuous density f, and that f is integrable, so the valuations have a finite expected value.

The auction mechanism defines the rules guiding bidder interactions. For instance, in a first-price auction, each bidder submits a sealed bid, b_i, and the highest bidder wins the item, paying his submitted bid. Contrastingly, in a second-price auction (also known as a Vickrey auction), the highest bidder secures the item but pays an amount equivalent to the second-highest bid. These represent only a subset of the numerous auction formats available. Formally, the auction mechanism is a couple $(\star, p)$, where $\star : [0, \omega]^n \longrightarrow \{1, \ldots, N\}$ is an allocation function, which, given the different bids, choose the bidder who earns the object (most of the time it is an argmax), and $p : [0, w]^n \longmapsto \mathbb{R}^N$ a payment rule, which determines how much each bidder has to pay at the end of the auction: i pays the amount $p_i(b_1, \ldots, b_N)$.

The auction's outcome, given these bid submissions, generally aims to maximize the

seller's revenue and efficiently allocate the item among bidders. Understanding the strategies employed by bidders, the auction's equilibrium, and the revenue implications forms the core of analyzing these single object auctions. At the conclusion of the analysis of single object auctions, it becomes imperative to evaluate the expected payoffs for both buyers and the seller. Each bidder i faces a payoff determined by the expected payoff in the auction. This payoff is defined as the difference between their private valuation θ_i and the price they pay, $p_i(b)$, considering the auction mechanism, b, and their position in the auction.

The expected payoff for bidder i is given by the expression:

$$\mathbb{E}(\theta_i \mathbb{1}_{\{i=*(b)\}} - p_i(b))$$

On the other side of the transaction, the seller's gain is the expected sum of the prices obtained from the winning bidder(s). In an ideal scenario where there are no reserve prices or participation fees, the seller's gain is represented as :

$$\mathbb{E}(\sum_{i \in B} p_i(b))$$

However, in some auction formats, the seller may impose a reserve price, below which the sale will not proceed. Additionally, there might be instances where buyers are obligated to pay an entry fee to participate, which could be integrated into the payment rule. These nuances in auction design can significantly impact both buyer and seller payoffs, thus affecting the auction's efficiency and outcomes.

In what follows, I will center on two fundamental auction formats: the first-price sealed-bid auction and the second-price sealed-bid auction. These auctions offer a simple yet intricately strategic environment, allowing for a comprehensive analysis of bidder behavior, revenue implications, and resource allocation efficiency.

4.2 Symmetric Equilibria

In this analysis, the primary focus revolves around the examination of symmetric equilibria of first-price and second-price auctions. For both, the allocation function is an argmax. For the first-price auction, the one who gets the item pays for his bid, while for

the second-price auction, he pays for the second highest bid. Every other bidder pays nothing.

Each of these auction formats determines a game among the bidders. A strategy for a bidder is a function $\beta_i : [0, \omega] \to \mathbb{R}^+$, which determines his bid for any valuation. Given that bidders are symmetric, it seems natural to focus attention on symmetric equilibria. I will typically be interested in comparing the outcomes of a symmetric equilibrium – an equilibrium in which all bidders follow the same strategy – of one auction with a symmetric equilibrium of the other.

This analysis focuses on two primary inquiries regarding first-price and second-price auctions: identifying their symmetric equilibria and determining which yields the optimal expected revenue for the seller. By exploring and comparing the equilibria, the aim is to unveil the most efficient seller revenue outcomes.

4.3 The Second-Price auction

While the first-price auction format often holds more familiarity and intuitive appeal, let me start with an exploration of second-price auctions. The strategic complexity faced by bidders in second-price auctions is notably simpler than that encountered in first-price auctions, rendering them a good starting point for the analysis. Within the context of private values, second-price auctions equate to open ascending price (or English) auctions, bringing additional depth to their analysis.

The simplicity of analyzing this auction type arises from the existence of a weakly dominant strategy. Each player i has the following payoff after the auction :

$$\Pi_i(b) = (\theta_i - \max_{j \neq i} b_j)\mathbb{1}_{b_i > \max_{j \neq i} b_j}$$

It can be assumed that if multiple bidders have the maximum bid, the item is randomly assigned among them following a uniform distribution.

Equilibria are deduced from the following proposition:

Proposition 26. *In the case of a second-price auction, the strategy $\beta_i(\theta_i) = \theta_i$ is a weakly dominant strategy for player i.*

Proof. Consider player i. Let $p_i = \max_{j \neq i} b_j$, the highest bid of the other players. If player i bids θ_i, he wins the auction if $\theta_i > p_i$, and loses if $\theta_i < p_i$ (in the case of equality, player i is indifferent between winning and losing, resulting in a zero payoff in both cases).

Suppose he deviates and bids $z_i < \theta_i$:

- If $\theta_i > z_i > p_i$, he still wins and have the same payoff, which is $\theta_i - p_i$.
- If $p_i > \theta_i > z_i$, he still loses.
- If $\theta_i > p_i > z_i$, he loses while he would have made a profit by bidding θ_i.

With symmetric arguments, it can be shown that deviating and bidding $z_i > \theta_i$ is also not profitable. This ensures that the strategy $\beta_i(\theta_i) = \theta_i$ is weakly dominant. □

It should be noted that this proof disregards the random and independent nature of the values involved. The expected payoff for player i is thus:

$$\begin{aligned} m_i(\theta_i) &= \mathbb{P}(\text{win})\mathbb{E}(\max_{j \neq i} \theta_j \mid \theta_i > \max_{j \neq i} \theta_j) \\ &= F^{N-1}(\theta_i)\mathbb{E}(\max_{j \neq i} \theta_j \mid \theta_i > \max_{j \neq i} \theta_j) \end{aligned}$$

On of the main appeals of second-price auctions lies in their ability to induce bidders to truthfully reveal their private values, while it is a weakly dominant strategy. In essence, the second-price auction aligns individual incentives with truthful bidding. Therefore, the seller can learn from a second-price auction.

4.4 The First-Price auction

In contrast to second-price auctions, in first-price auctions, there is no weakly dominant strategy, which adds a layer of complexity to the analysis. In this scenario, player i receives the following payoff after the auction :

$$\Pi_i(b) = (\theta_i - b_i)\mathbb{1}_{b_i > \max_{j \neq i} b_j}$$

It is assumed that if multiple players bid the maximum, the item will be randomly assigned following a uniform distribution.

It can already be asserted that none of the players will bid more than their value, resulting in at best a payoff of 0, which is dominated by bidding less than their value.

Let me consider player 1, assuming all other players follow a symmetric equilibrium strategy β, assumed to be strictly increasing.

Here, the objective is to find player 1's optimal bid b. It is necessary to note that $\beta(0) = 0$, and it is always unprofitable to bid $b > \beta(w)$ as explained before. This leads to $b \in [0, \beta(w)]$. Let $u_1 = \max_{j \neq 1} \theta_j$, the highest value excluding θ_1. Player 1 wins the item by bidding b if $b > \beta(u_1)$. In this case, their expected payoff will be $G(\beta^{-1}(b)) \times (\theta_1 - b)$, where $G = F^{N-1}$ represents the distribution of u_1.

The aim is to find b that maximizes this payoff, leading to the consideration of the following first-order condition :

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(\theta_1 - b) - G(\beta^{-1}(b)) = 0$$

where $g = G'$. This equation holds at the symmetric equilibrium, where $b = \beta(\theta_1)$. This leads to the following equation :

$$G(\theta_1)\beta'(\theta_1) + g(\theta_1)\beta(\theta_1) = \theta_1 g(\theta_1)$$

Upon integration, considering $\beta(0) = 0$, the following bidding strategy is obtained :

$$\beta(\theta_1) = \frac{1}{G(\theta_1)} \int_0^{\theta_1} y g(y) dy = \mathbb{E}(u_1 \mid u_1 < \theta_1)$$

In this proof, it is assumed that β is differentiable, which isn't part of the hypotheses. In fact, it is directly feasible to show that $\beta(\theta_1) = \mathbb{E}(u_1 \mid u_1 < \theta_1)$ without differentiation.

4.5 Revenue Comparison and Revenue Equivalence Theorem

From the point of view of the seller, the issue to understand and compare the expected revenues generated by various auction formats becomes a central inquiry. The myriad possibilities for modeling auctions prompt a natural curiosity: how do different auction mechanisms impact the seller's anticipated revenue? Particularly intriguing are auctions characterized by a maximum allocation function, known as standard auctions, where non-winning participants pay the same value. It becomes clear that a deeper understanding of expected revenues can be gleaned by analyzing auctions through a comparative lens.

Such investigations provide insights into strategic considerations and economic efficiencies and their implications for sellers.

Example 27. All auctions are not standard. For instance, in a charity auction, participants not only compete for the acquisition of an item but also contribute to a philanthropic cause. In this altruistic context, each bidder is motivated not only by the desire to win the coveted item but also by the willingness to support a charitable endeavor. Formally, the payoff of player i in a charity auction is $\Pi_i(b) = \theta_i \mathbb{1}_{b_i > \max_{j \neq i} b_j} - b_i$, where $\mathbb{1}_{b_i > \max_{j \neq i} b_j}$ indicates whether bidder i wins the item.

Here, all non-winning bidders do not pay for the same value. Therefore, this auction format cannot be standard.

Although all auction formats are not standard, this part focuses only on standard auctions.

Definition 26. A buyer is considered **risk-neutral** if his payoff function is concave.

In his 1981 paper, Myerson undertook an exploration of standard auctions, seeking to compare their revenue implications. This research led to the Revenue Equivalence Theorem, a proposition asserting that, under certain conditions, all standard auctions are equivalent in terms of expected revenue for the seller. Myerson's theorem posits that despite the diversity in auction structures, the seller's expected revenue remains the same across standard auctions. Consequently, the theorem implies that the task of comparing different auction formats can be simplified to a singular focus on the second-price auction. Formally, this theorem is given by the following statement.

Theorem 11 (Revenue Equivalence, Myerson (1981))**.** *Suppose private values are randomly and independently drawn, and all buyers are risk-neutral. Then, any strictly increasing symmetric equilibrium in any standard auction, where the expected payoff for a buyer with a value of* 0 *is* 0*, generates the same expected revenue for the seller.*

Proof. Consider a standard auction satisfying the assumptions. Take β as a symmetric equilibrium for this auction. Let $m(x)$ be the expected payoff at equilibrium for a player with a private value of x. By assumption, $m(0) = 0$.

Suppose one of the players chooses to bid $\beta(z)$ instead of $\beta(x)$. His expected payoff will be $\Pi(z, x) = G(z)x - m(z)$, where $G = F^{N-1}$ represents the cumulative distribution function of the maximum of the other players' bids.

The first-order condition ensures that the player maximizes his expected revenue when z satisfies :

$$g(z)x = m'(z)$$

At equilibrium, $z = x$, leading to :

$$m'(x) = xg(x)$$

Thus, knowing that $m(0) = 0$, the following equality holds :

$$m(x) = \int_0^x yg(y)dy = G(x) \times \mathbb{E}(\max_{i \neq 1} x_i \mid \max_{i \neq 1} x_i < x)$$

Since this result is independent of the specific auction, the conclusion follows. □

In light of this equivalence, the second-price auction emerges as particularly compelling. Among all the auction formats, it has the strategic advantage of truthfully revealing bidder types further elevates its significance. This strategic alignment with truthfulness positions the second-price auction as not only theoretically elegant but also practically appealing.

4.6 Reserve Prices

In the previous analysis, the seller is passive, assuming he sells the object at whatever price it fetches. However, in practice, sellers often retain the right to decline the sale if the auction's determined price falls below a certain threshold, denoted as $r > 0$. This threshold is known as the reserve price. Let me explain how reserve price influences the anticipated revenue for the seller.

Reserve Prices in Second-Price Auctions

Consider a scenario where the seller sets a small reserve price $r > 0$. With this reserve price in place, serving as a lower limit for the selling price, any bidder i with a value

$\theta_i < r$ cannot generate a positive profit in the auction. In a second-price auction, the introduction of a reserve price does not alter the bidders' behavior. Bidding one's value remains a weakly dominant strategy. For a bidder with a value r, the expected payoff now simplifies to $rG(r)$, and for a bidder with a value $\theta \geq r$, the expected payoff is given as follows :

$$m(\theta, r) = rG(r) + \int_r^\theta yg(y)dy$$

This difference is induced by the fact that the winner will pay r whenever the second-highest bidder has a smaller value than r.

A similar reasoning shows that the expected payoff of other auctions formats is the same. Therefore, any strictly increasing symmetric equilibrium for a standard auction with a reserve price $r \geq 0$ yields a bidder with value θ the expected payoff $m(\theta, r)$.

Influence of a Reserve Price on the Revenue

Now that how a reserve price influences a bidder's expected payoff is clear, the next logical question is how it impacts the seller's expected revenue.

Calculation shows that the ex ante expected payoff of a bidder is given by the following formula :

$$\mathbb{E}(m^A(\theta, x)) = r(1 - F(r))G(r) + \int_r^\omega y(1 - F(y))g(y)dy$$

Consider that the seller's value for the object is $x_0 \in [0, \omega]$. If he sets a reserve price $r \geq x_0$, then his expected payoff is given by:

$$\Pi_0 = N \times \mathbb{E}(m^A(X, r)) + F^N(r)x_0$$

Knowing its expected revenue as a function of the reserve price, the seller seeks the reserve price that maximizes this revenue. The first order condition sets that the optimal reserve price $r^*(x_0)$ is such that :

$$\frac{d\Pi_0}{dr}(x_0, r^*) = 0 = NG(r^*)[1-F(r^*)-r^*f(r^*)+f(r^*)x_0] = NG(r^*)(1-F(r^*))[1-(r^*-x_0)\lambda(r^*)]$$

where $\lambda(r) = \frac{f(r)}{1-F(r)}$. This condition is equivalent to $r^* - \frac{1}{\lambda(r^*)} = x_0$.

Two interesting things emerge from this reasoning: firstly, the optimal reserve price $r^*(x_0)$ does not depend on the number on the number of bidders. Secondly, this value

is greater than x_0. The last phenomenon appears for situations where only one bidder has a value exceeding x_0. In this situation, setting a reserve price $r > x_0$ yields a better payoff for the seller.

It is important to notice that to be valid, this condition still requires lambda to be increasing, which is therefore often assumed in the literature.

4.7 Noisy Auctions and Resale Possibilities

The idea of resale adds a realistic dimension to auction models, reflecting the often volatile nature of real markets. Consider, for instance, an artwork auction. The question of resale becomes crucial in a market where collectors may be inclined to acquire a piece not only for its intrinsic appeal but also as a potential investment. This duality, between aesthetic value and resale opportunity, creates a subtle dynamic that extends beyond the traditional frameworks of auction theory.

Another concrete example can be found in the field of spectrum auctions in telecommunications. When an operator acquires a frequency license, the possibility of resale of that license becomes a determining factor in his strategic decisions. This added flexibility to the auction process significantly alters market participants' forecasts and incentives.

However, it is essential to note that introducing resale possibilities into the auction context can challenge well-established results such as the Revenue Equivalence Theorem. The latter is often shaken when resale is considered. Bidders' motivations, initially guided by the maximization of their individual payoff, may be reassessed in light of the future possibility of reselling the acquired item.

In this section, I will focus on giving an overview of existing literature dedicated to auctions with resale possibilities, which is a basis of my model on noisy auctions with resale possibility.

4.7.1 Asymmetric Auctions with Resale, Hafalir and Krishna (2008)

In this article, Isa Hafalir and Vijay Krishna investigate a two-player auction where both players have different types. After the auction, the player who has won the item can, if desired, make a posted-price offer to the other player.

They consider two risk-neutral players: a strong player, denoted by s, and a weak one, denoted by w. Both players have values θ_s and θ_w, independently distributed, following the cumulative distribution function F_i, with support $[0, a_i]$ ($a_s \geq a_w$) with a continuous density function $f_i = F_i'$ for each $i \in \{s, w\}$. The distinction between the two buyers lies in the assumption that F_s stochastically dominates F_w to order 1, i.e. for all $v \leq a_s$, $F_w(v) \leq F_s(v)$.

The main results of this paper are the following ones. For the second price auction, there still exists an equilibrium where each player bids his own value. However, it is not a weakly dominant strategy anymore. Therefore, while this equilibrium may occur, others may appear in this type of situations. For the first price auctions, the main result is that if β_s and β_w are strictly increasing and continuous bidding strategies, then for any b, $F_s(\beta_s^{-1}(b)) = F_w(\beta_w^{-1}(b))$. Therefore, the bidding distributions of the two players are identical. The idea of this result is that the weak bidder, knowing his type has incentive to bid more aggressively in order to compensate for his type.

From these results, Hafalir and Krishna explore the question of the revenue comparison of the two auctions for the seller. The outcome is that the revenue from the first-price auction is always higher than that of the second-price auction. Thus, the revenue equivalence theorem does not hold in this context.

4.7.2 First-price auctions with resale : the case of many bidders, Gábor Virág (2011)

In this paper, Virág extensively revisits the model of Hafalir and Krishna, extending it to the case where there are more than 2 bidders. This leads him into realize that not all the results for 2 bidders can be generalized to a context with many players. Virág focuses mainly on the case of a first-price auction with the possibility of resale. Notably,

the second-price case is equivalent to the 2-player auction: an equilibrium exists where each player bids their own value, which is equivalent to not allowing resale.

As Hafalir and Krishna, he considers two types of bidders. Formally, there are n risk-neutral players. He assumes that there are N_s strong players and N_w weak players, with cumulative distribution functions F_s and F_w, and that F_s stochastically dominates F_w to order 1. Their values $\theta_1, \ldots, \theta_N$ are independently drawn from the cumulative distribution functions which have strictly positive and continuous densities f_s, and f_w with support $[0, 1]$.

The auction unfolds in 2 stages as follows: a standard first-price auction is set: bidders submit bids; the highest bidder pays his bid and get the item. At the issue of this auction, the bids are not revealed. After this auction, the winner can choose to resell the item in a second price auction among the other $N - 1$ bidders, with an optimal reserve price. As there is no resale possibility at the issue of the second auction, setting it as a second price auction is optimal for the one who won the first auction. It is important to note that this model encompasses the Hafalir and Krishna model with $N_s = N_w = 1$.

Virág focuses only on the equilibria where there is symmetry among the bidders who have the same type. Therefore, He seeks for strategies β_s and β_w. He proves that, under some conditions of regularity, for all $y \in (0, 1)$, $\beta_s(y) > \beta_w(y)$, and moreover, if $N_s, N_w \geq 1$ and $N_s + N_w > 2$, then $F_s(\beta_s^{-1}(b)) < F_w(\beta_w^{-1}(b))$ for all b. This result highlights the fact that strong bidders have a bidding distribution that is more agressive than the weak ones. Thus, they win more often the first auction.

Virág concludes his paper by emphasizing the fact that numerical simulations seems to show that the more bidders there are, the more this model seems to tend toward the case without resale possibility. So the results of this paper seems to be more relevant when there are not a lot of players.

4.7.3 Asymmetric common value auctions with applications to private value auctions with resale, Cheng and Tan (2010)

This paper introduces the idea of noise in an auction format. In this model, there are 2 players, who has a common value for the item. However, any of them does know this value.

Instead, each player $i \in \{1, 2\}$ knows a signal s_i. s_1 and s_2 are drawn independently, from different cumulative distributions F_1 and F_2, admitting positive densities f_i for each i on a finite support $[0, a_i]$.

The results of this article are interesting on several levels. It demonstrates that, depending on circumstances, the type of auction that generates the maximum revenue can vary, so, obviously, the revenue equivalence theorem does not hold in this context. This implies that, depending on the market type and the information available to the seller, there are indeed considerations to be made regarding the choice of auction type. Moreover, it shows that it is possible, depending on the type of buyer one is dealing with, to design specific auctions that are more favorable in terms of revenue.

This is more or less directly correlated with allowing a resale option, which is often the case in real-world applications (as observed in online auctions or the art market). These findings suggest that in a real-world context, the revenue dynamics for the seller are potentially much more complex than what the « classic » theory asserts, justifying the need to address such questions.

5 Noisy Auctions with Resale

The present chapter presents my work on auction theory, as a model encompassing both resales possibility, and noise on the private values. All the models and results in this chapter are new in the auction literature. Most of the proofs in this chapter being long, calculative and not contributing to the understanding of the phenomena at work, I chose to put the longer ones in the appendix.

Summary

Auctions, serving as fundamental platforms in economic exchange, offer a dynamic stage where buyers and sellers converge to determine the value of goods. Real-world scenarios often reflect the complexities of auctions, extending beyond a single transaction. Platforms like eBay, Vinted, and various other auction-based systems frequently see buyers obtaining items, only to realize later that their valuation might not align with the actual market value. Such situations create an interesting dynamic where the winning bidder, upon gaining a deeper understanding of the item, may find an opportunity for a post-auction resale, altering the strategies and interactions in subsequent transactions.

In an auction scenario where participants possess only partial information regarding the value of the object being auctioned, the winning bidder might discover post-auction that his valuation is not the highest. This realization creates an incentive for a post-auction resale, significantly affecting bidders' strategies from the outset. The TV show Storage War, featuring auctions of abandoned furniture storage, highlights these mechanisms: the participants, having only incomplete visual information on the contents of the storage furniture, may discover post auction that they were wrong about their beliefs.

This chapter explores the impact of post-auction resale within a model featuring n

agents, each with privately and symmetrically distributed values. Bidders possess imperfect information about their valuations, perceiving only a signal as a random variable centered around their value, referred to as the « noisy valuation ». Following the auction, the winner learns his true valuation and decides whether to resell the object using an optimal mechanism. The optimal mechanism presents as a take-it-or-leave-it offer if the auction allows inference of the second-highest noisy valuation. Otherwise, in cases where the winner only discovers having the highest noisy valuation, a second-price auction with an optimal reserve price is optimal.

This study reveals that in the absence of resale opportunities, agents behave as if they possess perfect information about their valuations. However, with the inclusion of resale options, the Revenue Equivalence Theorem no longer holds, categorizing standard auctions into two distinct types based on the winner's ability to infer the second bidder's noisy valuation.

This dichotomy arises from what the winner learns post-auction, determining the optimal approach for potential resale. Revenue equivalence results demonstrate that, within these two auction types, the initial seller's expected revenue remains consistent.

Notably, these results diverge from standard auctions outcomes due to the presence of noise. Absent noise, the auction winner would infer no interest in reselling the object as he holds the highest valuation.

Existing literature on auction theory primarily considers situations with complete information about valuations, resulting in comparable outcomes without resale. Various studies, such as Garratt and Troger (2006), Hafalir and Krishna (2008), Cheng and Tan (2010), explore scenarios with different agent distributions, offering insights into symmetrization results and the effects of resale on auctions with varying bidder characteristics.

Virág (2013) expands on the ideas presented by Hafalir and Krishna, focusing on models with n agents and demonstrating that weak agents are less likely to win the initial auction at equilibrium. Additionally, Gupta and Lebrun (1998), Lebrun (2010), and Cheng and Tan (2010) assert that symmetrization in cases with more than two bidders is unattainable.

Furthermore, Virág (2013) observes that the larger the number of bidders, the closer

the initial auction's allocation resembles that of an auction without resale. This suggests that in larger markets, resale might hold less significance. Swinkels (2001) corroborates this by noting that, without resale, a first-price auction becomes increasingly efficient as the environment becomes more similar for all bidders, leading to comparable behaviors among bidders with similar valuations, irrespective of their initial type distributions.

The main contribution of the present chapter to the literature is that this is the first model considering both uncertainty about the private valuations and a the possibility to resell the item. Unlike the papers which allow a resale stage or consider noise about the valuations, in my model, the valuations of the players are drawn symetrically. Even though my model differs from existing ones, the conclusions are similar in terms of expected revenue for the seller.

The remainder of this chapter is organized as follows. In Section 5.1, I introduce noisy auctions and show that they are similar to auction without noise. In Section 5.2, I introduce resale possibilities in noisy auctions. In Section 5.2.2 and 5.2.3, I give the equilibria of noisy second price auction and first price auction. Finally, in Section 5.2.4, I study the expected revenues for the seller across various types of auctions, and define two type of auctions, within which a new revenue equivalence theorem holds.

5.1 Noisy Auctions Without Resale

Let me introduce first the model of noisy auctions without the resale possibility. Traditionally, auction theory assumes that bidders have perfect knowledge of their own value for an item. However, this assumption does not always capture the reality of the auction process. In this section, I will focus on noisy valuations.

Consider a scenario where bidders desire to get an item, but before taking possession, they are partially unaware of their personal value for that item. For instance, an auction for the allocation of an work of art. Before having the item in his possession, a bidder might have biaised estimation of his value for the item. This uncertainty can arise from various factors such as a lack of knowledge on the item, or even unforeseen circumstances altering perception about it. To model this uncertainty, I introduce a white noise into the private values of bidders. It can be conceptualized as a random disturbance reflecting

the inherent uncertainty in evaluating the value of an item. This means that before committing in the auction, participants are not certain about the exact value they attribute to the coveted item, they only perceives a signal.

Formally, the model of noisy auctions is the following: there is a seller, denoted as S, who offers an item for sale. A set of N bidders $B = \{1, 2, \ldots, N\}$ participating in the auction. Each bidder $i \in B$ is associated with a valuation θ_i representing their private valuation for the item. The valuations are generally assumed to be independently drawn from a common known distribution function F on a set $[0, \omega]$, where ω can be infinite. It is assumed that F admits a continuous density f, and that f is integrable, so the valuations have a finite expected value. However, the bidders do not perceive their value. Each of them has a noise $\varepsilon_i \in [-\overline{\varepsilon}, \overline{\varepsilon}]$, where $\overline{\varepsilon}$ can be infinite, drawn independently from a common known distribution function E, with continuous density e such that $\mathbb{E}(\varepsilon_i) = 0$ for all i. What bidders perceive is their noisy values $v_i = \theta_i + \varepsilon_i$ for all $i \in B$. To simplify notations, $g = f * e$ is the density of noisy valuations, with distribution G. The bidders take part in a standard auction, for instance a Vickrey auction. At the end of the auction, the bidder i who got the item learns his value θ_i. If he paid p_i, his payoff is $\theta_i - p_i$.

In what follows I focus on the second price noisy auction. I search for symmetric Perfect Bayesian Equilibria.

The auction unfolds as follows: each bidder i observes his noisy value, then places a bid b_i. The player submitting the highest bid is awarded the item and pays the second-highest bid. If many players have the highest bid, a random one gets the item and pays his bid. Player i has a payoff of the following form:

$$\Pi_i(b_i, b_{-i}) = \begin{cases} \theta_i - \max\limits_{j \neq i} b_j & \text{if } b_i \geq \max\limits_{j \neq i} b_j \text{ and } i \text{ get the item} \\ 0 & \text{otherwise} \end{cases}$$

A strategy of player i is a bidding function β_i, associating a noisy value with a positive real number. As in the case without noise, the following result holds.

Proposition 27. *For player i, bidding $\beta_i(v_i) = v_i$ is a weakly dominant strategy.*

Proof. By symmetry of roles, consider player 1. Let $p_1 = \max\limits_{j \neq 1} b_j$ and $j_0 = \operatorname{argmax}\limits_{j \neq 1} b_j$. By bidding $b_1 = \theta_1 + \varepsilon_1$, the player will get the item if $b_1 \geq p_1$, and loses otherwise.

Suppose that 1 deviates by playing $c_1 < b_1$. There are three situations:

- $b_1 > c_1 \geq p_1$: he gets the same as by bidding b_1.
- $p_1 > b_1 > c_1$: it does not change anything; the payoff remains 0.
- Otherwise, $b_1 > p_1 > c_1$: the player goes from a winning status by playing b_1 to a losing status (thus gaining 0) by playing c_1. While $\theta_1 + \varepsilon_1 > b_{j_0}$. Player 1's payoff being $\theta_1 - b_{j_0}$, and $\theta_1 + \varepsilon_1 - b_{j_0} > 0 \Leftrightarrow \theta_1 - b_{j_0} > -\varepsilon_1$. By monotonicity of expectation, the expected payoff of 1 is greater than 0 because $\mathbb{E}(-\varepsilon_1) = 0$. Therefore, playing b_1 is better: the deviation is unfavorable.

One can show with symmetrical arguments that bidding $c_1 > b_1$ is also not profitable.

□

This result would remain the same in cases where there is asymmetry among the noises, where some bidders have noise, others do not, and also in cases where bidders do not know if their observation of their value is noisy or not. The idea of the proof remains the same.

In this case, there is therefore no interest in considering noise, while bidders consider their expected payoff. As this model verifies the assumptions of the revenue equivalence theorem, one can deduce from the second-price auction what will happen for any other noisy auction mechanisms without resale. In the following sections, I introduce a resale stage into noisy auctions, that changes the dynamics of bidding.

5.2 Noisy Auctions with Resale

As seen in the previous section, introducting noise in the valuation process do not alter the auction dynamics. However, it becomes interesting when bidders are allowed to resell the item at the conclusion of the initial auction, subsequent to learning their private values. In this context, the introduction of noise gains relevance: after getting the item, a bidder, with the knowledge of his actual private value, can realize that the item is either more or less valuable than initially perceived. This opens the possibility for the winning bidder to make strategic decisions, such as reselling the item to others who may value it more. The introduction of noise thus transforms the auction landscape by incorporating

post-auction strategic considerations, adding a layer of complexity and strategic depth to the decision-making process in a way that is absent in traditional auction settings.

5.2.1 Setup and Equilibrium

As for noisy auction without resale, each bidder $i \in B$ perceives a noisy value $v_i = \theta_i + \varepsilon_i$. The timing of the game is as follows: first there is an auction. Then, the winner of this auction learns is own valuation, and may resell the object to one of the other $n-1$ bidders. I assume that, at the resale stage, the winner of the first auction resells the object with an optimal procedure which depends on the auction type. After the resale stage is over, the game ends without further resale possibilities. If a bidder with a private value θ owns the object at the end of the game (after the resale stage), and his overall payment was p, then his payoff is $\theta - p$.

My equilibrium concept is Perfect Bayesian Equilibrium. In such an equilibrium, in the first stage, each bidder i has a bidding function $\beta_i : [-\overline{\varepsilon}, \overline{\theta} + \overline{\varepsilon}] \to \mathbb{R}_+$ for the first stage that I suppose non decreasing, such that no other would yield a higher expected payoff, given the strategies of the other players. For the second stage, the strategy of player i depends on the resale procedure.

In that follows, I study the two cases of a second price auction, and then, the first price auction.

5.2.2 Noisy Second Price Auction with Resale

In this case, after the first stage, the one who got the object can deduce the noisy value of the one who has the second highest one from the symmetric equilibrium. Hence, he can infer that the highest price he can sell the item is the second highest value: the optimal procedure is a posted price that I can suppose the buyer will accept.

A strategy of buyer i is a tuple (β_i, p_i, D_i) such that:

- $\beta_i : [-\overline{\varepsilon}, \overline{\theta} + \overline{\varepsilon}] \longrightarrow \mathbb{R}_+$ is a bidding function for the stage 1.
- $p_i : [0, \overline{\theta}] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a function which gives the posted price, depending on the private value, and the value paid in the stage 1, if i wins at stage 1. If i doesn't want to resell, he can set a value that is too high.

- $D_i : [-\overline{\varepsilon}, \overline{\theta} + \overline{\varepsilon}] \times \mathbb{R}_+ \longrightarrow \{\text{accept, refuse}\}$ is a function which represent if the buyer want to accept the posted price of the stage 1's winner, depending on his noised value.

The payoffs of stage 1 and 2 are the following ones. Buyer i has a payoff in stage 1 as follows:

$$\Pi_i^1(b_i, b_{-i}) = \begin{cases} \theta_i - \max b_{-i} & \text{if } b_i > \max b_{-i} \\ 0 & \text{otherwise} \end{cases}$$

where b_i is the bid of player i. If he wins stage 1, he will have a payoff in stage 2 as follows:

$$\Pi_i^{2,W}(p_i, D_{-i}) = \begin{cases} p_i - \theta_i & \text{if there exists } j \neq i \text{ s.t.} D_j = \text{accept} \\ 0 & \text{else} \end{cases}$$

If he loses stage 1, he will has a payoff in stage 2 as follows:

$$\Pi_i^{2,L}(D_i, p_{-i}) = \begin{cases} \frac{\theta_i - p_j}{|\{k | D_k = \text{accept}\}|} & \text{if } D_i = \text{accept} \\ 0 & \text{else} \end{cases}$$

The non decreasing symmetric equilibrium is given in the following theorem:

Theorem 12. *The non decreasing symmetric equilibrium of the noisy second price auction with resale is:*

$$\beta(v_i) = v_i + \int_0^{\overline{\varepsilon}} \varepsilon d\varepsilon$$

Proof. The detailed proof is in Appendix A.1. □

The sketch of the proof is the following. The analysis is recursive: I detailed what happens in stage 2, and then, I use this result to find the equilibrium in stage 1.

One explanation of the difference with the noisy auction without resale is that bidders can underestimate their value. If it is the case, they have bigger benefits. In the opposite case, he can still resell the item, and then it is optimal to have a more aggressive biffing strategy.

Let me now compare the revenue for the seller with the revenue generated by a noisy second price auction. If R_{SPA} denotes the revenue of the seller in the classic second price

auction, and R_{SPAR} the revenue of the seller in the noisy second price auction with resale. The following result holds:

$$R_{SPAR} = R_{SPA} + \int_0^{\overline{\varepsilon}} \varepsilon d\varepsilon$$

Therefore, the resale stage generates a bigger expected revenue for the seller. Let me now compare the expected payoff of a bidder between the noisy second price auction and the noisy second price auction with resale:

Proposition 28. *The noisy second price auction with resale yields for bidders an inferior expected payoff than the the noisy second price auction.*

Proof. The detailed proof is in Appendix A.1. □

This result is a consequence of the fact that the noise and the resale stage create an incentive for the players to have more aggressive bidding strategies.

In the following section, I focus on the noisy first price auction with resale.

5.2.3 Noisy First Price Auction with Resale

In this situation, after the first stage, the one who got the object cannot deduce the noisy value of the one who has the second highest one. Hence, he just knows that he has the highest noisy value and can only reevaluate the distribution of the noisy values of the other players. Then, the optimal resale procedure is a second price auction with optimal reserve price.

A strategy of buyer i is a couple (β_i, p) such that:

- $\beta_i : [-\overline{\varepsilon}, \overline{\theta} + \overline{\varepsilon}] \longrightarrow \mathbb{R}_+$ is a bidding function for the stage 1.
- $p : [0, \overline{\theta}] \times \mathbb{R}_+ \to \mathbb{R}_+$ an optimal reserve price for the resale stage.

The non decreasing symmetric equilibrium is given in the following theorem:

Theorem 13. *If the following bidding function is non decreasing, this is the non decreasing symmetric equilibrium of the noisy first price auction with resale:*

$$\beta(V) = V + \int_{v<V} \frac{G^{N-1}(v)}{G^{N-1}(V)} dv - \frac{2(N-1)}{G^{N-1}(V)} \int_{v<V} g(v) \mathbb{E}\left(\int_p^v (G^{N-2}(y_2) dy_2)\right) dv$$

If β is not non decreasing, there are no such equilibria.

Proof. The detailed proof is in Appendix A.2. □

The sketch of the proof follows the same patterns than for noisy second price auctions with resale.

Remark 1. I did not succeed in showing that this function is always non decreasing in the general case, neither for particular cases.

5.2.4 A new revenue equivalence theorem

The previous analysis of noisy auctions, both first and second-price, coupled with resale opportunity, unveils a dynamic that challenges the traditional Revenue Equivalence Theorem. In this context, the application of this theorem no longer holds. However, two types of noisy auctions with resale emerge: the auctions with revelation, which permit of the winning player to deduce the second highest noisy value, and those without revelation. These findings allow for the classification of auction mechanisms into two distinct categories, within which the validity of a new revenue equivalence theorem holds.

Definition 27. A *standard auction with revelation* is a standard auction in which, at equilibrium, the winner of the auction can infer what is the type of the opponent who has the highest type.

In this context, the auction mechanism reveals the second highest noisy value.

Within standard auction with revelation, a revenue equivalence theorem holds:

Theorem 14. *Suppose that noisy values are independently and identically distributed, and all bidders are risk neutral. Then any symmetric and non decreasing equilibrium of any noisy standard auction with resale and revelation, such that the expected payoff of a bidder with noised value $-\bar{\varepsilon}$ is zero, yields the same expected revenue to the seller.*

Proof. The detailed proof is in Appendix A.3. □

A similar result can be proved for all noisy auctions with resale without revelation.

Conclusions

I have studied auctions with resale when there are many bidders who do not know exactly their values and derived that, in this context, the revenue equivalence theorem does not hold. Unlike most of the models in the literature, the noise allows resale even with symmetric distribution functions for the bidders. I proved that, provided that the equilibrium of the noisy first price auction with resale is valid, two type of auctions in this context emerge: noisy auctions with revelation, and the ones without, within which a revenue equivalence theorem holds.

An interesting avenue to explore to go further could be to consider models with more resale stages, and compare the noisy auction with and without revelation. The question of the evolution of the difference between these kind of auction may be an interesting one to explore.

It must also be interesting to compare the expected revenue for the seller bewteen noisy auction with and without revelation. However, I tried to find such a result, but considering the expression of the equilibrium of the first price auction with resale, I did not succeed, and it can be really hard to prove something in this context.

Conclusion

In this conclusion, I recall what were the objectives of chapters 2, 3 and 5, the main contributions to the literature and the issues the models do not solve. While each of those chapters has already a conclusion, I also give some new questions they highlight and what are the avenues to explore.

Core of Dynamic Coalitional Game

In this chapter, the main objective is to model situations in which a group of player may have interests to cooperate over time, even if they do not want to be together at some stages of a game. I provide a tool to deal with the situations in which a cooperative game has a non-empty core, which I call the coalitional $\vec{\varepsilon}$-core. This concept is a lot more flexible than the other alternatives already existing in the literature, and allows to measure the instability of a cooperative game with the minimal coalitional $\vec{\varepsilon}$-core. I also define dynamic coalitional games and the dynamic core, for which I provide an axiomatization and a characterization of non-emptiness using the coalitional $\vec{\varepsilon}$-cores of the stages games. This characterization provides a way to determine for each allocation of the dynamic core how the different coalitions can compensate each other over time to make the grand coalition sustainable. I also define the summed Shapley value in this context.

It could be interesting to provide other definitions of the Shapley value in this dynamic context,which would be sets, and not values, and to study their relationship with the different core concepts.

The axiomatization of the dynamic core opens a gate to define a new definition of the core in the dynamic context that would be efficient and dynamically coalitionally rational. Such a concept should be really interesting from the point of view of intergenerational equity, as every generation has enough to sustain the grand coalition at each stage of the game, but with the possibility to transfer value between stages. To see even further, from the point of view of inheritance, it would be interesting to define a new axiom of dynamic coalitional rationality, but adding the condition that the transfers of value are

possible only for the future. Such an axiom could explain for example, how a society can be sustainable even if in the present the grand coalition is not sustainable.

Stability in Cooperative Games with Externalities

This chapter explores the question of the formation of coalitions in cooperative games with externalities. While not any core concept in the literature considers stability of the formation of a partition of the remaining players after a coalition deviates from the grand coalition, I provide four core concepts, based on two main ideas of stability. Each of these concepts brings credibility in terms of stability of the core. I provide sufficient conditions of non-emptiness of the different core concepts, and conditions of Pareto efficiency. For the stable recursive cores, I prove characterizations of non-emptiness as Bondareva-Shapley results, using the results of Abe and Funaki (2016). I do not have a similar result for stable cores by successive deviations.

One avenue to explore is to define other types of stable cores, without considering only pessimistic and optimistic visions. For example, one may define a stable rational expectation recursive core, and compare this notion with the already existing ones.

It may be also be interesting to find a Bondareva-Shapley result for stable cores by successive deviations. Such a result might be a way to compare the stable recursive cores and the stable cores by successive deviations.

While in the definitions of stable cores by successive deviations and stable recursive cores, stability is only seen by below, in the sense that splitting is not possible, but merging could happen, it may be interesting to define variations of those definitions, but instead of considering splitting, define stability starting with singletons, and merging coalitions. Those concepts would be totally different than the ones I defined, but should be totally relevant in the literature.

Noisy Auctions with Resale

The main objective of this chapter is to modelize the dynamics in which, after someone get an item after participating in an auction, he might decide to resell the item if he discovers that he values the item less than expected. This model is inspired by online auctions like Ebay for example. From that purpose, I study the effects of post-auction resale in an auction model featuring n agents, each endowed with privately and symmetrically distributed noisy values. Bidders have information regarding their own valuations, perceiving only a signal as a random variable hovering around their value, called the noisy valuation. Subsequent to the auction, the winner discovers his true valuation and decides whether to resell the item using an optimal mechanism. The optimal mechanism is a take-it-or-leave-it proposition if the initial auction allows inference of the second-highest noisy value. Conversely, in scenarios in which the winner only has the highest noisy valuation, he can resell with an optimal second-price auction with a reserve price.

The primary contribution of this chapter to the literature lies in the fact that this is the first model considering both uncertainty regarding private values and the prospect of item resale.

This inquiry uncovers that without resale opportunities, agents have the same behavior than when they have perfect information about their valuations. Nevertheless, with the incorporation of resale options, the Revenue Equivalence Theorem does not hold anymore, and auctions split into two types, depending on the winner's capacity to deduce the second highest noisy valuation, within which a new revenue equivalence theorem holds.

What remain to do in this model is to prove, if this is true, that the noisy first price auction with resale always have a non decreasing symmetric equilibrium, and to compare the expected revenue for the seller between noisy auction with and without revelation. Considering the expressions of the equilibria, this may be hard to do.

A next step could be to explore with multi resale stages, in order to see the evolution of the difference between the auction types with the number of resale stages.

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Appendix

A Proofs of the Chapter 5

A.1 Symetric Equilibrium of the Noisy Second Price Auction with Resale

This section details the proof of the Theorem 12

As I seek for symmetric non decreasing equilibrium, I assume that buyers have a non decreasing bidding function β. Let me find the best response of i to β. The analysis recursive.

Stage 2

Suppose that i won in stage 1. Knowing that other buyers play β, he deduces that $v_i \geq \max_{j \neq i} v_j$. He will decide to resell if, and only if $\theta \leq \max_{j \neq i} \beta^{-1}(b_j) = \max_{j \neq i} v_j$, and the posted price will be $\max_{j \neq i} v_j$. Therefore, his payoff if he won in stage 1 is:

$$u_i^W(\theta_i, v_{-i}) = (\max v_{-i} - \theta_i)\mathbb{1}_{\max v_{-i} > \theta_i}$$

Thus, his expected payoff if he wins the first auction is the following one:

$$U_i^W(v_i, b_i) = \mathbb{E}\left((\max v_{-i} - \theta_i)\mathbb{1}_{\max v_{-i} > \theta_i} \mid v_i = \theta_i + \epsilon_i, b_i > \max \beta(v_{-i}))\right)$$

If he loses stage 1, the winner will set a posted price which is at least his noised value, so his payoff is:

$$U_i^L(b_i, v_i) = 0$$

Stage 1

The total expected payoff of i in this auction is:

$$\begin{aligned}\Pi_i(v_i, b_i) &= \mathbb{E}[\Pi_i^1(b_i, b_{-i}) + \Pi_i^{2,W}(p_i, D_{-i})\mathbb{1}_{b_i > \max b_{-i}} + \Pi_i^{2,L}(D_i, p_{-i})\mathbb{1}_{b_i < \max b_{-i}}] \\ &= \mathbb{E}((U_i^W(v_i, b_i) - \max \beta(v_{-i}) + v_i)\mathbb{1}_{b_i > \max \beta(v_{-i})}) \\ &= \mathbb{E}((U_i^W(v_i, b_i) - \beta(\max v_{-i}) + v_i)\mathbb{1}_{b_i > \beta(\max v_{-i})})\end{aligned}$$

The cumulative distribution of $\max v_{-i}$ is G^{N-1}, so $\max v_{-i}$ admits $(N-1) \times g \times G^{N-2}$ as density. Thus, the total expected payoff of i in this auction becomes:

$$\Pi_i(v_i, b_i) = \underbrace{U_i^G(v_i, b_i)G^{N-1}(\beta^{-1}(b_i))}_{A(v_i,b_i)} + \underbrace{v_i G^{N-1}(\beta^{-1}(b_i))}_{B(v_i,b_i)} - \underbrace{\int_0^{\beta^{-1}(b_i)} \beta(m)(N-1)g(m)G^{N-2}(m)dm}_{C(b_i)}$$

To find the best response of i, I need first order condition. Let me search for the partial derivatives of A, B, C:

$$\begin{aligned}A(v_i, b_i) &= \mathbb{E}\left((\max v_{-i} - \theta_i)\mathbb{1}_{\max v_{-i} > \theta_i}\mathbb{1}_{b_i > \max \beta(v_{-i}))} \mid v_i = \theta_i + \epsilon_i\right) \\ &= \mathbb{E}\left((\max v_{-i} - v_i + \epsilon_i)\mathbb{1}_{\max v_{-i} > v_i - \epsilon_i}\mathbb{1}_{\beta^{-1}(b_i) > \max v_{-i}}\right) \\ &= \mathbb{E}\left(\int_{v_i - \epsilon_i}^{\beta^{-1}(b_i)} (m - v_i + \epsilon_i)(N-1)g(m)G^{N-2}(m)dm\mathbb{1}_{v_i - \epsilon_i < \beta^{-1}(b_i)}\right) \\ &= \mathbb{E}\Bigg[\Bigg((\epsilon_i - v_i)(G^{N-1}(\beta^{-1}(b_i)) - G^{N-1}(\beta^{-1}(v_i - \epsilon_i))) \\ &+ (N-1)\int_{v_i - \epsilon_i}^{\beta^{-1}(b_i)} m g(m)G^{N-2}(m)dm\Bigg)\mathbb{1}_{\epsilon_i > v_i - \beta^{-1}(b_i)}\Bigg] \\ &= \int_{v_i - \beta^{-1}(b_i)}^{\bar{\epsilon}} (\epsilon_i - v_i)(G^{N-1}(\beta^{-1}(b_i)) - G^{N-1}(\beta^{-1}(v_i - \epsilon_i)))e(\epsilon)d\epsilon \\ &+ (N-1)\int_{v_i - \beta^{-1}(b_i)}^{\bar{\epsilon}} e(\epsilon)\int_{v_i - \epsilon_i}^{\beta^{-1}(b_i)} m g(m)G^{N-2}(m)dmd\epsilon \\ &= Y(v_i, b_i) + Z(v_i, b_i)\end{aligned}$$

Let me derive Y and Z:

$$\begin{aligned}
\frac{\partial Y}{\partial b_i}(v_i,b_i) &= \int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}(\epsilon-v_i)(N-1)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}G^{N-2}(\beta^{-1}(b_i))e(\epsilon)d\epsilon \\
&\quad \underbrace{-\frac{1}{\beta'(\beta^{-1}(b_i))}\beta^{-1}(b_i)(G^{N-1}(\beta^{-1}(b_i))-G^{N-1}(\beta^{-1}(b_i)))e(v_i-\beta^{-1}(b_i))}_{=0} \\
&= (N-1)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}G^{N-2}(\beta^{-1}(b_i))\int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}(\epsilon-v_i)e(\epsilon)d\epsilon \\
\frac{\partial Z}{\partial b_i}(v_i,b_i) &= (N-1)\int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}e(\epsilon)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}\beta^{-1}(b_i)G^{N-2}(\beta^{-1}(b_i))d\epsilon \\
&\quad + (N-1)\frac{e(v_i-\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}\underbrace{\int_{\beta^{-1}(b_i)}^{\beta^{-1}(b_i)}mg(m)G^{N-2}(m)dm}_{=0} \\
&= (N-1)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}\beta^{-1}(b_i)G^{N-2}(\beta^{-1}(b_i))\int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}e(\epsilon)d\epsilon
\end{aligned}$$

Therefore, the partial derivatives of A, B and C are the following ones:

$$\begin{aligned}
\frac{\partial A}{\partial b_i}(v_i,b_i) &= (N-1)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}G^{N-2}(\beta^{-1}(b_i))\int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}(e(\epsilon)-v_i+\beta^{-1}(b_i))d\epsilon \\
\frac{\partial B}{\partial b_i}(v_i,b_i) &= v_i(N-1)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}G^{N-2}(\beta^{-1}(b_i)) \\
\frac{\partial C}{\partial b_i}(v_i,b_i) &= b_i(N-1)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}G^{N-2}(\beta^{-1}(b_i))
\end{aligned}$$

Hence, the first order condition on P_i gives:

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial b_i}(v_i,b_i)=0 &\Leftrightarrow (N-1)\frac{g(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}G^{N-2}(\beta^{-1}(b_i))\left(v_i-b_i+\int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}(e(\epsilon)-v_i+\beta^{-1}(b_i))d\epsilon\right)=0 \\
&\Leftrightarrow v_i-b_i+\int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}(e(\epsilon)-v_i+\beta^{-1}(b_i))d\epsilon=0 \\
&\Leftrightarrow b_i=v_i+\int_{v_i-\beta^{-1}(b_i)}^{\overline{\epsilon}}(e(\epsilon)-v_i+\beta^{-1}(b_i))d\epsilon
\end{aligned}$$

I consider a symetric equilibrium, so $b_i=\beta(v_i)$. Hence, the non decreasing symmetric equilibrium of the noisy second price auction with resale is:

$$\beta(v_i)=v_i+\int_0^{\overline{\epsilon}}\epsilon d\epsilon$$

The theorem holds.

A.1.1 Payoffs Comparison Between the Noisy Second Price Auction and the Noisy Second Price Auction with Resale

The section details the proof of the Proposition 28.

Let me consider the expected payoff of a bidder, starting with the case without resale possibility. If $EP_{WR}(v_i)$ denotes the expected payoff of a player with type v_i in the case without resale, then:

$$\begin{aligned}
EP_{WR}(v_i) &= \mathbb{E}((v_i - \max v_{-i})\mathbb{1}_{V_i > \max v_{-i}}) \\
&= v_i G^{N-1}(v_i) - \int_0^{v_i} m(N-1)g(m)G^{N-2}(m)dm \\
&= v_i G^{N-1}(v_i) - [mG^{N-1}(m)]_0^{v_i} + \int_0^{v_i} G^{N-1}(m)dm \\
&= \int_0^{v_i} G^{N-1}(m)dm
\end{aligned}$$

If $EP_R(v_i)$ denotes the expected payoff of a bidder with type v_i, when bidders play the symetric equilibrium, then:

$$\begin{aligned}
&EP_R(v_i) = \mathbb{E}\left[\left(\theta_i - \max v_{-i} - \int_0^{\overline{\epsilon}} \epsilon d\epsilon + (\max v_{-i} - \theta_i)\mathbb{1}_{\theta_i < \max v_{-i}}\right)\mathbb{1}_{v_i > \max v_{-i}} \mid v_i = \theta_i + \epsilon_i\right] \\
&= \mathbb{E}\left[\left(v_i - \epsilon_i - \max v_{-i} - \int_0^{\overline{\epsilon}} \epsilon d\epsilon + (\max v_{-i} - v_i - \epsilon_i)\mathbb{1}_{v_i - \epsilon_i < \max v_{-i}}\right)\mathbb{1}_{v_i > \max v_{-i}}\right] \\
&= \mathbb{E}\left[\int_{-\overline{\epsilon}}^{v_i}\left(v_i - \epsilon_i - m - \int_0^{\overline{\epsilon}} \epsilon d\epsilon + (m - v_i - \epsilon_i)\mathbb{1}_{v_i - \epsilon_i < m}\right)(N-1)g(m)G^{N-2}(m)dm\right] \\
&= v_i G^{N-1}(v_i) - \underbrace{\mathbb{E}(\epsilon_i)G^{N-1}(v_i)}_{=0} - \int_{-\overline{\epsilon}}^{v_i} m(N-1)g(m)G^{N-2}(m)dm - G^{N-1}(v_i)\int_0^{\overline{\epsilon}} \epsilon d\epsilon \\
&+ \mathbb{E}\left[\int_{v_i - \epsilon_i}^{v_i} (m - v_i + \epsilon_i)\mathbb{1}_{\epsilon_i > 0}(N-1)g(m)G^{N-2}(m)dm\right] \\
&= \int_0^{v_i} G^{N-1}(m)dm - G^{N-1}(v_i)\int_0^{\overline{\epsilon}} \epsilon d\epsilon + \int_0^{\overline{\epsilon}}\int_{v_i - \epsilon}^{v_i} (m - v_i + \epsilon)((N-1)g(m)G^{N-2}(m)dmde(\epsilon) \\
&= \int_0^{v_i} G^{N-1}(m)dm - G^{N-1}(v_i)\int_0^{\overline{\epsilon}} \epsilon d\epsilon \\
&+ \int_0^{\overline{\epsilon}} (\epsilon - v_i)(G^{N-1}(v_i) - G^{N-1}(v_i - \epsilon)) + \int_{v_i - \epsilon}^{v_i} mg(m)(N-1)G^{N-2}(m)dmde(\epsilon) \\
&= \int_0^{v_i} G^{N-1}(m)dm - G^{N-1}(v_i)\int_0^{\overline{\epsilon}} \epsilon d\epsilon \\
&+ \int_0^{\overline{\epsilon}} (\epsilon - v_i)(G^{N-1}(v_i) - G^{N-1}(v_i - \epsilon)) \\
&+ (v_i G^{N-1}(v_i) - (v_i - \epsilon)G^{N-1}(v_i - \epsilon)) - \int_{v_i - \epsilon}^{v_i} G^{N-1}(m)dmde(\epsilon) \\
&= \int_0^{v_i} G^{N-1}(m)dm - G^{N-1}(v_i)\int_0^{\overline{\epsilon}} \epsilon d\epsilon + \int_0^{\overline{\epsilon}} (\epsilon - v_i)G^{N-1}(v_i) \\
&+ v_i G^{N-1}(v_i) - \int_{v_i - \epsilon}^{v_i} G^{N-1}(m)dmde(\epsilon)
\end{aligned}$$

$$= \int_0^{v_i} G^{N-1}(m)dm - G^{N-1}(v_i)\int_0^{\bar{\epsilon}} \epsilon d\epsilon + \int_0^{\bar{\epsilon}} \epsilon G^{N-1}(v_i) - \int_{v_i-\epsilon}^{v_i} G^{N-1}(m)dmde(\epsilon)$$
$$= \int_0^{v_i} G^{N-1}(m)dm - G^{N-1}(v_i)\int_0^{\bar{\epsilon}} \epsilon d\epsilon + G^{N-1}(v_i)\int_0^{\bar{\epsilon}} \epsilon d\epsilon - \int_0^{\bar{\epsilon}}\int_{v_i-\epsilon}^{v_i} G^{N-1}(m)dmde(\epsilon)$$
$$= \int_0^{v_i} G^{N-1}(m)dm - \int_0^{\bar{\epsilon}}\int_{v_i-\epsilon}^{v_i} G^{N-1}(m)dmde(\epsilon)$$

Therefore, the auction with resale yields for bidders an inferior expected payoff:

$$EP_{WR}(v_i) - EP_R(v_i) = \int_0^{\bar{\epsilon}}\int_{v_i-\epsilon}^{v_i} G^{N-1}(m)dmde(\epsilon) > 0$$

A.2 Symetric Equilibrium of the Noisy First Price Auction with Resale

This section details the proof the the Theorem 13.

I search for a symmetric equilibrium, with non decreasing bidding functions. I assume that bidders have all non decreasing bidding function β. let me find the best response of i to β. As for the second price auction, the analysis is done recursively:

Stage 2

Suppose that i won in stage 1. He learns his private value, and set the following optimal reserve price:

$$p(b,\theta_i) = \max(\theta_i, \underset{p<\beta^{-1}(b)}{\operatorname{argmax}}\,(\mathbb{E}(\mathbb{1}_{Y_1\geq p}(\mathbb{1}_{Y_2\geq p}(Y_2-\theta_i) + \mathbb{1}_{Y_2\leq p}(p-\theta_i)))$$

There, if he wins the stage 1, his expected payoff at the end of the stage 2 is the following one:

$$U_i^W(v_i,b) = \frac{1}{G^{N-1}}(\beta^{-1}(b)\mathbb{E}(\mathbb{1}_{Y_1\geq p}\mathbb{1}_{b\geq\beta(Y_1)}(\mathbb{1}_{Y_2\geq p}(Y_2-\theta_i) + \mathbb{1}_{Y_2\leq p}(p-\theta_i) \mid v_i = \theta_i + \epsilon_i)$$

Otherwise, it is:

$$U_i^L(b,v_i) = \frac{1}{1-G^{N-1}}\mathbb{E}(\mathbb{1}_{v_i>Y_2}\mathbb{1}_{v_i>p}\mathbb{1}_{Y_1>\beta^{-1}(b)}(\mathbb{1}_{Y_2>p}(v_i-Y_2) + \mathbb{1}_{Y_2<p}(v_i-p)))$$

To simplify the notations, let me denote these expected payoffs as follows:

$$U_i^W(v_i,b) = \frac{1}{G^{N-1}}A(b,v_i)$$

$$U_i^L(b,v_i) = \frac{1}{1-G^{N-1}} B(b,v_i)$$

with:

$$\begin{aligned} A(b,v_i) =& \mathbb{E}(\int_{y_1<\beta^{-1}(b)} \mathbb{1}_{y_1>p} \int [\mathbb{1}_{y_2<p}(y_2 - v_i + \epsilon_i) + \mathbb{1}_{y_2>p}(p - v_i + \epsilon_i)] \\ & \times (N-1)(N-2) g(y_1) g(y_2) \mathbb{1}_{y_2<\beta^{-1}(b)} G^{N-3}(y_2) dy_2 dy_1) \end{aligned}$$

$$\begin{aligned} B(b,v_i) =& \mathbb{E}(\int_{y_1>\beta^{-1}(b)} \int \mathbb{1}_{v_i>y_2}(\mathbb{1}_{y_2>p}(v_i - y_2) \\ & + \mathbb{1}_{y_2<p}(v_i - p))(N-1)(N-2) g(y_1) g(y_2) \mathbb{1}_{y_2<\beta^{-1}(b)} G^{N-3}(y_2) dy_2 dy_1) \end{aligned}$$

Stage 1

Let me find the best response of i. His expected payoff in this auction is:

$$\begin{aligned} \Pi_i(v_i,b) &= \mathbb{E}((U_i^W(v_i,b) - b + v_i)\mathbb{1}_{b>\beta(Y_1)}) + \mathbb{E}(U_i^L(v_i,b)\mathbb{1}_{b<\beta(Y_1)}) \\ &= A(b,v_i) + (v_i - b)G^{N-1}(\beta^{-1}(b)) + B(b,v_i) \end{aligned}$$

Let me search for the partial derivative of A, B to get a first order condition:

$$\begin{aligned} \frac{\partial A}{\partial b}(b,v_i) =& (N-1)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} \mathbb{E}(\underbrace{\mathbb{1}_{\beta^{-1}(b)>p}}_{=1} \int (\mathbb{1}_{y_2>p}(y_2 - v_i + \epsilon_i) \\ & + \mathbb{1}_{y_2<p}(p - v_i + \epsilon_i))(N-2) g(y_2) G^{N-3}(y_2) \mathbb{1}_{y_2<\beta^{-1}(b)} dy_2) \end{aligned}$$

$$\begin{aligned} \frac{\partial B}{\partial b}(b,v_i) =& -(N-1)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} \mathbb{E}(\mathbb{1}_{v_i>p} \int \mathbb{1}_{v_i>y_2}(\mathbb{1}_{y_2>p}(v_i - y_2) \\ & + \mathbb{1}_{y_2<p}(v_i - p))(N-2) g(y_2) G^{N-3}(y_2) \mathbb{1}_{y_2<\beta^{-1}(b)} dy_2 \end{aligned}$$

The first order consition gives:

$$\begin{aligned} &\frac{\partial \Pi_i}{\partial b}(v_i,b) = 0 = \frac{\partial A}{\partial b}(b,v_i) - G^{N-1}(\beta^{-1}(b)) + (N-1)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} G^{N-2}(\beta^{-1}(b))(v_i - b) + \frac{\partial B}{\partial b}(b,v_i) \\ &= (N-1)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} \mathbb{E}(\int (\mathbb{1}_{y_2>p}(y_2 - v_i + \epsilon_i) + \mathbb{1}_{y_2<p}(p - v_i + \epsilon_i))(N-2) g(y_2) G^{N-3}(y_2) \mathbb{1}_{y_2<\beta^{-1}(b)} dy_2) \\ &- G^{N-1}(\beta^{-1}(b)) + (N-1)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} G^{N-2}(\beta^{-1}(b))(v_i - b) \\ &- (N-1)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} \mathbb{E}(\mathbb{1}_{v_i>p} \int \mathbb{1}_{v_i>y_2}(\mathbb{1}_{y_2>p}(v_i - y_2) + \mathbb{1}_{y_2<p}(v_i - p))(N-2) g(y_2) G^{N-3}(y_2) \mathbb{1}_{y_2<\beta^{-1}(b)} dy_2 \end{aligned}$$

I consider a symetric equilibrium, so $b = \beta(v_i)$. Therefore:

$$
\begin{aligned}
0 &= (N-1)\frac{g(v_i)}{\beta'(v_i)}\mathbb{E}(\int(\mathbb{1}_{y_2>p}(y_2 - v_i + \epsilon_i) + \mathbb{1}_{y_2<p}(p - v_i + \epsilon_i))(N-2)g(y_2)G^{N-3}(y_2)\mathbb{1}_{y_2<v_i}dy_2) \\
&- G^{N-1}(v_i) + (N-1)\frac{g(v_i)}{\beta'(v_i)}G^{N-2}(v_i)(v_i - \beta(v_i)) \\
&- (N-1)\frac{g(v_i)}{\beta'(v_i)}\mathbb{E}(\int \mathbb{1}_{v_i>y_2}(\mathbb{1}_{y_2>p}(v_i - y_2) + \mathbb{1}_{y_2<p}(v_i - p))(N-2)g(y_2)G^{N-3}(y_2)\mathbb{1}_{y_2<v_i}dy_2 \\
\Leftrightarrow & G^{N-1}(v_i)\beta'(v_i) + \beta(v_i)(N-1)g(v_i)G^{N-2}(v_i) = (N-1)g(v_i)G^{N-2}(v_i)v_i \\
&+ (N-1)g(v_i)\mathbb{E}(\int(\mathbb{1}_{y_2>p}(y_2 - v_i + \epsilon_i) + \mathbb{1}_{y_2<p}(p - v_i + \epsilon_i))(N-2)g(y_2)G^{N-3}(y_2)\mathbb{1}_{y_2<v_i}dy_2) \\
&- (N-1)g(v_i)\mathbb{E}(\int \mathbb{1}_{v_i>y_2}(\mathbb{1}_{y_2>p}(v_i - y_2) + \mathbb{1}_{y_2<p}(v_i - p))(N-2)g(y_2)G^{N-3}(y_2)\mathbb{1}_{y_2<v_i}dy_2)
\end{aligned}
$$

By integrating, I get:

$$
\begin{aligned}
\beta(V) = V &+ \int_{v<V}\frac{G^{N-1}(v)}{G^{N-1}(V)}dv \\
&+ \frac{(N-1)}{G^{N-1}(V)}\int_{v<V} g(v)(\underbrace{\mathbb{E}(\int(\mathbb{1}_{y_2>p}(y_2 - v + \epsilon) + \mathbb{1}_{y_2<p}(p - v + \epsilon))(N-2)g(y_2)G^{N-3}(y_2)\mathbb{1}_{y_2<v}dy_2}_{\gamma(v)} \\
&- \underbrace{\mathbb{E}(\int \mathbb{1}_{v>y_2}(\mathbb{1}_{y_2>p}(v - y_2) + \mathbb{1}_{y_2<p}(v - p))(N_2)g(y_2)G^{N-3}(y_2)dy_2))}_{\delta(v)}dv
\end{aligned}
$$

Moreover:

$$
\begin{aligned}
\gamma(v) &= \mathbb{E}((\epsilon - v)(G^{N-2}(v) + pG^{N-2}(p) + \int_p^v y_2(N-2)g(y_2)G^{N-3}(y_2)dy_2) \\
&= -\mathbb{E}(\int_p^v (G^{N-2}(y_2)dy_2)) \\
&= -\delta(v)
\end{aligned}
$$

Therefore:

$$\beta(V) = V + \int_{v<V}\frac{G^{N-1}(v)}{G^{N-1}(V)}dv - \frac{2(N-1)}{G^{N-1}(V)}\int_{v<V} g(v)\mathbb{E}(\int_p^v (G^{N-2}(y_2)dy_2))dv$$

The theorem holds.

A.3 Revenue Equivalence Theorem for Noisy Auctions with Resale and Revelation

This section details the proof of the Theorem 14.

By symmetry, let me consider the buyer 1, of type $v = \theta + \epsilon$. All other buyers play a symmetric and non decreasing equilibrium β. I denote $m^A(v)$ the expected payoff in auction A by a bidder with value v. The analysis is recursive.

Stage 2

Suppose that 1 won in stage 1. Knowing that other buyers play β, he deduce that $v \geq \max_{j \neq 1} v_j$. He will decide to resell if, and only if $\theta \leq \max_{j \neq 1} \beta^{-1}(b_j) = \max_{j \neq 1} v_j$, and it's optimal for him to put a posted price that will be $\max_{j \neq 1} v_j$. Therefore, is payoff if he won in stage 1 is:

$$u^W(\theta, v_{-1}) = (\max v_{-1} - \theta)\mathbb{1}_{\max v_{-1} > \theta}$$

His expected payoff by bidding b is:

$$U_i^W(v, b) = \mathbb{E}\left((Y_1 - \theta)\mathbb{1}_{Y_1 > \theta} \mid v = \theta + \epsilon, b > \beta(Y_1)\right))$$

Where Y_1 is the highest competing bid. If he loses stage 1, the winners will set a posted price which is at least his noised value, so his expected payoff is:

$$U_i^L(v, b) = 0$$

Stage 1

If, instead of bidding $\beta(v)$, i plays like his type was z, his expected payoff is:

$$\Pi^A(z, v) = G(z)(v + U_i^G(v, \beta(z)) - m^A(z)$$

Where $G(z) = F^{n-1}(z)$ is the distribution of Y_1. The first order condition gives:

$$\frac{\partial}{\partial z}\Pi^A(z, v) = 0 = g(z)(v + \frac{\beta'(z)}{\beta'(z)}\mathbb{E}(\mathbb{1}_{z \geq v - \epsilon}(z - v + \epsilon)) - \frac{\partial}{\partial z}m^A(z)$$

At equilibrium, $z = v$, thus:

$$\frac{\partial}{\partial v}m^A(v) = g(v)(v + \mathbb{E}(\mathbb{1}_{\epsilon \geq 0}\epsilon)$$

Therefore:

$$m^A(V) = \int_{v \leq V} g(v).v dv + \mathbb{E}(\mathbb{1}_{\epsilon \geq 0}\epsilon)F(V) = F(V)(\mathbb{E}(Y_1 \mid Y_1 \leq X) + \mathbb{E}(\mathbb{1}_{\epsilon \geq 0}\epsilon))$$

Since $F(V)(\mathbb{E}(Y_1 \mid Y_1 \leq X) + \mathbb{E}(\mathbb{1}_{\epsilon \geq 0}\epsilon))$ does not depend of the auction type, I have the result.

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Résumé : Cette thèse, structurée en deux parties indépendantes, explore deux domaines distincts de la théorie des jeux: les jeux coopératifs et la théorie des enchères. Dans la première partie, je propose de nouveaux concepts de cœur pour les jeux coopératifs. Le premier concept se place dans un cadre de jeux coopératifs dynamiques, prenant en compte la faisabilité temporelle et les compensations entre les étapes. Les suivants sont des concepts dédiés aux jeux coopératifs avec externalités, fondés sur des notions de stabilité. Pour chacun de ces concepts, je donne des conditions de non-vacuité du cœur, et pour certains des caractérisations.

Dans la seconde partie, j'introduis un modèle d'enchères bruitées avec possibilité de revente. L'objectif initial étant de modéliser le mieux possible les comportements des enchérisseurs dans les enchères en ligne. Ce modèle ne vérifie pas le théorème d'équivalence du revenu, mais classe ce type d'enchères en deux catégories, aux seins desquelles des nouveaux résultats d'équivalence du revenu apparaissent.

Descripteurs : Jeux coopératifs, Cœur, Externalités, Dynamique, Enchères

Title and Abstract: This thesis, structured into two independent parts, explores two distinct domains of game theory: cooperative games and auction theory. In the first part, I introduce new core concepts for cooperative games. The first concept is situated within the framework of dynamic cooperative games, considering temporal feasibility and compensations between stages. Subsequent concepts are dedicated to cooperative games with externalities, based on stability notions. For each of these concepts, I provide conditions for the non-emptiness of the core, and for some, characterizations.

In the second part, I present a model of noisy auctions with resale. The initial objective is to model the behaviors of bidders in online auctions as accurately as possible. While

this model does not satisfy the revenue equivalence theorem, it categorizes this type of auctions into two groups, within which new income-equivalence results emerge.

Keywords: Cooperative games, Core, Externalities, Dynamics, Auctions

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