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Thurston theory in transcendental dynamics
Théorie de Thurston en dynamique transcendante

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I, undersigned, Nikolai PROCHOROV, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific supervision of Dierk SCHLEICHER, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with both the french national charter for Research Integrity and AMU charter on the fight against plagiarism.

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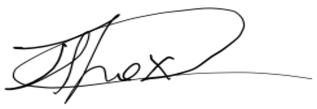

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Liste de publications et participation aux conférences

Liste des publications et/ou brevets réalisées dans le cadre du projet de thèse:

1. N. Prochorov. Finite and infinite degree Thurston maps with extra marked points. Preprint: arxiv.org/abs/2410.06206, hal.science/hal-04727048 (2024)

2. N. Prochorov. Finite and infinite degree Thurston maps with a small postsingular set. Preprint: arxiv.org/abs/2410.01146, hal.science/hal-04727045 (2024)

3. M. Mukundan, N. Prochorov, and B. Reinke. Dynamical approximation for postsingularly finite entire maps. Preprint: arxiv.org/abs/2305.17793, hal.science/hal-04526931 (2024) (submitted for publication).

4. M. Hlushchanka and N. Prochorov. Critically fixed Thurston maps: classification, recognition, and twisting. Preprint: arxiv.org/abs/2212.14759, hal.science/hal-04525890 (2022) (submitted for publication)

Participation aux conférences et écoles d'été au cours de la période de thèse:

1. 7ème école d'hiver en dynamique d'Aussois. December 2024. Centre Paul Langevin, Aussois, France.

2. Classical Function Theory in Modern Mathematics. July 2024. International Center for Mathematical Sciences, Edinburgh, UK.

   Poster presentation: "Dynamical line complexes and Thurston theory".

3. Summer school on Teichmüller theory. July 2024. University of Liverpool, Liverpool, UK.

4. Parameter spaces in complex dynamics and related topics. May 2024. Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy.

   Poster presentation: "Towards Transcendental Thurston theory".

5. 6ème école d'hiver en dynamique d'Aussois. December 2023. Centre Paul Langevin, Aussois, France.

   Talk: "La théorie de Thurston pour les revêtements ramifiés postcritiquement finis".

6. Quasiworld workshop. August 2023. University of Helsinki, Helsinki, Finland.

7. Renormalization and Visualization for packing, Billiard and Surfaces. July 2023. CIRM, Marseille, France.

8. Summer school: Facets of complex dynamics. June 2023. Institut de Mathématiques de Toulouse, Toulouse, France.

9. Topics in Complex Dynamics 2023. June 2023. Universitat de Barcelona and Centre de Recerca Matemàtica, Barcelona, Spain.

   Talk: "Dynamical approximations of postsingularly finite entire maps".

10. Around the Mandelbrot set: A conference celebrating the 60th birthday of Mitsuhiro Shishikura. May 2023. Research Institute for Mathematical Sciences, Kyoto, Japan.

11. Complex dynamics: connections to other fields. March 2023. European Centre for Geological Education, Chęciny, Poland.

12. On geometric complexity of Julia sets - IV. August 2022. Mathematical Research and Conference Center, Będlewo, Poland.

13. Rencontre autour des systèmes dynamiques. July 2022. Île de Porquerolles, Hyères, France.

    Talk: "Polynomial approximations of transcendental maps".

14. Complex Dynamics School: building a bridge to higher dimensions. March 2022. Indiana University – Purdue University Indianapolis, Indianapolis, Indiana, USA.

    Talk: "Potential theory and the computation of $dd^c \log|\cdot|$".

15. MSRI Semester Program "Complex Dynamics: from special families to natural generalizations in one and several variables", including Connections Workshop, Introductory Workshop, and "Adventurous Berkeley Complex Dynamics" workshop. January - June 2022. MSRI, Berkeley, California, USA.

    Talks "Critically fixed Thurston maps" and "Polynomial approximations in class $\mathscr{S}$", and poster presentation "Critically fixed Thurston maps".

16. 4ème école d'hiver en dynamique d'Aussois. December 2021. Centre Paul Langevin, Aussois, France.

17. Rencontre des jeunes chercheuses et jeunes chercheurs du GDR Platon. November 2021. CIRM, Marseille, France.

18. Advancing Bridges in Complex Dynamics. September 2021. CIRM, Marseille, France.

19. Rencontre autour des systèmes dynamiques. July 2021. Île de Porquerolles, Hyères, France.

Résumé et mots clés

Nous développons la théorie des fonctions de Thurston de degré infini, qui sont définies partout sur la sphère topologique S^2, en excluant potentiellement un ensemble fermé au plus dénombrable de singularités.

Nous établissons un analogue du célèbre théorème de caractérisation de W. Thurston pour une large classe de telles fonctions de Thurston ayant quatre valeurs postsingulières. Cela est particulièrement significatif car cela représente la première occurrence d'un critère à la Thurston dans le cadre transcendant, englobant un nombre indénombrable de fonctions de Thurston distinctes combinatoirement, à la fois réalisées et obstruées. Pour y parvenir, nous analysons les applications de tiré-en-arrière associées définies sur l'espace de Teichmüller de dimension complexe un, ce qui permet d'obtenir des informations sur la dynamique de tiré-en-arrière sur les espaces de Teichmüller et de modules. Cela met en lumière, à son tour, les propriétés supplémentaires des classes de Hurwitz et des espaces de paramètres correspondants des fonctions de Thurston considérées.

Dans la deuxième partie de cette thèse, nous nous concentrons sur les fonctions de Thurston marquées. Nous montrons que lorsqu'une fonction de Thurston non marquée f, qu'elle soit de degré fini ou infini, est réalisée, la fonction de Thurston marquée (f, A), où $A \subset S^2$ est un ensemble marqué fini, est réalisée si et seulement si elle ne possède pas de cycle de Levy dégénéré. Ce résultat est obtenu à travers l'étude de l'application de tiré-en-arrière correspondante définie sur l'espace de Teichmüller de dimension supérieure, montrant que certaines itérations de cette application admettent des sous-variétés complexes invariantes bien comportées. En appliquant des outils puissants de la dynamique complexe en dimension un et de la géométrie hyperbolique, nous obtenons une compréhension claire du comportement de l'application de tiré-en-arrière restreinte au sous-ensemble invariant correspondant de l'espace de Teichmüller.

Mots clés : fonctions de Thurston, fonctions holomorphes postsingulièrement finies, espaces de Teichmüller, espaces de modules, applications de tiré-en-arrière, cycles de Lévy.

Abstract and keywords

We develop the theory of Thurston maps of infinite degree, which are defined everywhere on the topological sphere S^2, potentially excluding at most a countable closed set of singularities.

We establish an analog of the celebrated W. Thurston's characterization theorem for a broad class of such Thurston maps having four postsingular values. This is particularly significant as it represents the first instance of a Thurston-like criterion in the transcendental setting that encompasses uncountably many combinatorially distinct Thurston maps, both realized and obstructed. To achieve this, we investigate the corresponding pullback maps defined on the one-complex-dimensional Teichmüller space, gaining insights into the pullback dynamics on Teichmüller and moduli spaces. This, in turn, sheds light on the further properties of Hurwitz classes and corresponding parameter spaces of the considered Thurston maps.

In the second part of this thesis, we focus on marked Thurston maps. We show that when an unmarked Thurston map f, whether of finite or infinite degree, is realized, the marked Thurston map (f, A), where $A \subset S^2$ is a finite marked set, is realized if and only if it has no degenerate Levy cycle. This result is obtained through a study of the corresponding pullback map defined on the multi-dimensional Teichmüller space, showing that certain iterates of this map admit well-behaved invariant complex submanifolds. By applying powerful machinery of one-dimensional complex dynamics and hyperbolic geometry, we gain a clear understanding of the behavior of the pullback map restricted to the corresponding invariant subset of the Teichmüller space.

Key words: Thurston maps, postsingularly finite holomorphic maps, Teichmüller spaces, moduli spaces, pullback maps, Levy cycles.

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Résumé substanciel

La théorie de Thurston en dynamique holomorphe de dimension un se développe autour de la question de savoir si une application topologique donnée peut être réalisée, dans un certain sens dynamique appelé équivalence combinatoire, par une application holomorphe. En termes simples, une application topologique est dite réalisée si une telle réalisation holomorphe existe, et obstruée sinon. Cette question peut être vue comme une tentative de promouvoir la topologie en géométrie, cherchant à comprendre comment les informations topologiques peuvent éclairer la structure géométrique correspondante.

La théorie classique de Thurston aborde cette question dans le contexte des revêtements ramifiés postcritiquement finis, préservant l'orientation, de la 2-sphère topologique S^2, connus sous le nom de fonctions de Thurston. On rappelle qu'un revêtement ramifié f de S^2 est appelé postcritiquement fini si chaque valeur critique de f a une orbite finie, c'est-à-dire qu'elle est périodique ou prépériodique sous f. Ces fonctions de Thurston servent de modèles topologiques pour la famille des fonctions rationnelles postcritiquement finies. Notamment, le célèbre théorème de caractérisation de Thurston [DH93] affirme que toute fonction de Thurston de ce type (à l'exception d'une petite famille bien comprise) est réalisée par une fonction rationnelle postcritiquement finie si et seulement si elle ne possède aucun obstruction de Thurston.

L'objectif de cette thèse est d'étendre la théorie de Thurston pour inclure les fonctions transcendantes, c'est-à-dire les fonctions de degré topologique infini. En particulier, nous élargissons la définition d'une fonction de Thurston pour inclure les fonctions topologiquement holomorphes, postsingulièrement finies, qui peuvent avoir un degré infini et être définies en dehors d'un ensemble fermé au plus dénombrable de singularités essentielles dans S^2. On rappelle qu'une fonction topologiquement holomorphe f est dite postsingulièrement finie si elle possède un nombre fini de valeurs singulières, et si chaque valeur singulière est soit éventuellement périodique, soit éventuellement envoyée sur un point où f n'est pas défini, par itération. Ces fonctions de Thurston sont conçues pour servir de modèles combinatoires ou topologiques pour les fonctions holomorphes postsingulièrement finies. Comme dans le cas classique, l'objectif principal est d'étudier si une telle fonction topologique peut être réalisée par une fonction holomorphe. Il convient de noter que des résultats partiels dans ce domaine ont été obtenus grâce aux travaux de Hubbard–Schleicher–Shishikura [HSS09] et de S. Shemyakov [She22], bien que le cas général reste largement non résolu.

L'outil clé pour déterminer si une fonction de Thurston donnée f admet une réalisation holomorphe est l'analyse d'un opérateur σ_f, appelé application de tiré-en-arrière,

qui est défini sur l'espace de Teichmüller $\mathcal{T}_{P_f}$ de la sphère épointée $S^2 - P_f$, où P_f est l'ensemble postsingulier de la fonction de Thurston f. De manière cruciale, il est connu que l'application de tiré-en-arrière σ_f possède un point fixe si et seulement si la fonction de Thurston f est réalisée. De plus, d'autres propriétés dynamiques de σ_f peuvent fournir des indications sur les caractéristiques supplémentaires de la fonction de Thurston f.

Cette thèse est principalement fondée sur les articles de l'auteur [Pro24a] et [Pro24b], et sa structure est la suivante. Le Chapitre 1 présente une introduction générale, tandis que le Chapitre 2 expose les préliminaires nécessaires. Les Chapitres 3, 4, et 5 apportent exclusivement des résultats nouveaux. Plus précisément, les Chapitres 3 et 4 suivent en grande partie le contenu de [Pro24a], tandis que le Chapitre 5 se base sur [Pro24b], avec une certaine dépendance aux résultats du Chapitre 3. Enfin, le Chapitre 6 traite de questions supplémentaires et de pistes de recherche futures découlant des résultats de cette thèse.

Dans le Chapitre 1, nous introduisons et motivons les concepts et approches clés, résumons les principaux résultats, et fournissons un aperçu informel de leurs démonstrations.

Le Chapitre 2 est consacré à l'établissement du cadre théorique. Dans ce chapitre, nous introduisons les fonctions de Thurston, qui peuvent avoir un degré infini et être définies en dehors d'un ensemble fermé, au plus dénombrable, de singularités essentielles dans S^2, ainsi que des concepts clés associés tels que les ensembles postsinguliers, les équivalences combinatoires et de Hurwitz, la réalisabilité, et plus encore. Nous passons également en revue les notions de base nécessaires en théorie de Teichmüller, ce qui nous permet de définir les applications de tiré-en-arrière, d'en discuter les propriétés principales, et de mettre en avant les liens entre leur dynamique et les caractéristiques des fonctions de Thurston correspondantes. Enfin, le chapitre se termine par une discussion sur les obstructions de Thurston et une liste des principaux résultats concernant les fonctions de Thurston de degré fini.

Le Chapitre 3 met en évidence l'importance de la géométrie hyperbolique pour le problème considéré. Dans la première partie, nous utilisons des outils de la géométrie hyperbolique et de la théorie de Teichmüller pour identifier les obstructions de Thurston et analyser leurs propriétés, en nous basant sur le comportement des applications de tiré-en-arrière correspondantes. La seconde partie est consacrée à l'étude de la dynamique des fonctions holomorphes de la disque unité qui satisfont à certaines hypothèses supplémentaires. De nombreux exemples naturels de telles fonctions émergeront des applications de tiré-en-arrière définies sur l'espace de Teichmüller de dimension complexe un, ou de restrictions de ces applications à des sous-ensembles invariants d'espaces de Teichmüller de dimension supérieure. Les outils développés dans ce chapitre joueront un rôle important dans l'établissement des résultats principaux de la thèse dans les deux chapitres suivants.

Dans le Chapitre 4, nous étudions une large famille de fonctions de Thurston avec quatre valeurs postsingulières. Plus précisément, nous considérons des fonctions de Thurston $f\colon S^2 \dashrightarrow S^2$ (où la flèche pointillée indique que f peut avoir des singularités essentielles dans S^2) telles qu'il existe un sous-ensemble $B \subset P_f$ avec $|B| = 3$, $S_f \subset B$,

et $|\overline{f^{-1}(B)} \cap P_f| = 3$, où $\overline{f^{-1}(B)}$ désigne la clôture de $f^{-1}(B)$ dans la topologie de S^2 et P_f est l'ensemble postsingulier de f. La condition $|P_f| = 4$ implique que l'application de tiré-en-arrière correspondante σ_f est définie sur un espace de Teichmüller de dimension un complexe. Le reste de la condition peut sembler un peu mystérieux, mais il garantit que l'application de tiré-en-arrière maintient un comportement bien structuré, bien que toujours complexe.

Il est important de noter que, pour toute fonction de Thurston f avec trois valeurs singulières, les conditions mentionnées ci-dessus peuvent être vérifiées en analysant simplement la dynamique de f sur l'ensemble fini P_f. Par exemple, dans le cas des fonctions de Thurston entières (celles qui peuvent être restreintes à des fonctions de $\mathbb{R}^2$ dans lui-même) avec trois valeurs singulières et quatre valeurs postsingulières, trois des sept portraits postsinguliers possibles possèdent ces propriétés. En particulier, malgré le caractère restrictif de ces conditions, le monde des fonctions transcendantes est si vaste qu'il existe encore une infinité non dénombrable de fonctions de Thurston, combinatoirement inéquivalentes par paires, à la fois réalisées et obstruées, qui les satisfont.

Nous commençons par analyser les applications de tiré-en-arrière correspondantes, qui sont définies sur l'espace de Teichmüller de dimension complexe un, biholomorphe au disque unité. En utilisant les résultats du Chapitre 3, nous arrivons à la conclusion suivante : toute fonction de Thurston qui satisfait les conditions mentionnées précédemment est réalisée si et seulement si elle ne présente aucune obstruction de Thurston d'un type très simple. Plus précisément, on dit qu'une courbe simple fermée $\gamma \subset S^2 \backslash P_f$ forme un cycle de Levy pour la fonction de Thurston f si elle ne peut pas être rétractée en un point par une homotopie dans $S^2 \backslash P_f$, et s'il existe $n \geqslant 1$ et une courbe simple fermée $\gamma' \subset f^{-n}(\gamma)$ telles que γ et γ' soient homotopes dans $S^2 \backslash P_f$ et $\deg(f^{\circ n}|\gamma' : \gamma' \to \gamma) = 1$. Si $n = 1$, on dit simplement que γ est une courbe fixe de Levy pour la fonction de Thurston f.

Theorem. *Soit $f : S^2 \dashrightarrow S^2$ une fonction de Thurston avec quatre valeurs postsingulières, telle qu'il existe un sous-ensemble $B \subset P_f$ avec $|B| = 3$, $S_f \subset B$, et $|\overline{f^{-1}(B)} \cap P_f| = 3$. Alors, f est réalisée si et seulement si elle ne possède pas de courbe fixe de Levy.*

Ce théorème présente le premier critère de caractérisation de type Thurston applicable à un nombre non dénombrable de fonctions de Thurston combinatoirement distinctes, qu'elles soient réalisées ou obstruées. De plus, ce résultat est obtenu grâce à des techniques assez explicites qui apportent de nouvelles perspectives sur la dynamique des applications de tiré-en-arrière dans l'espace de Teichmüller.

Le cadre développé nous permet également d'étudier simultanément toute la classe (d'équivalence) de Hurwitz $\mathscr{H}_f$ d'une fonction de Thurston f. Ici, deux fonctions de Thurston $f_1 : S^2 \dashrightarrow S^2$ et $f_2 : S^2 \dashrightarrow S^2$ sont dites équivalentes au sens de Hurwitz si $P_{f_1} = P_{f_2}$ et qu'il existe des homéomorphismes préservant l'orientation $\phi_1, \phi_2 : S^2 \to S^2$ tels que $\phi_1|P_{f_1} = \phi_2|P_{f_1}$ et $\phi_1 \circ f_1 = f_2 \circ \phi_2$. Par exemple, nous pouvons établir un critère simple pour déterminer si une fonction de Thurston donnée f est totalement sans obstruction, c'est-à-dire toute la classe de Hurwitz $\mathscr{H}_f$ est composée de fonctions

de Thurston qui admettent des réalisations holomorphes. Plus généralement, nous obtenons le résultat suivant.

Theorem. *Soit $f\colon S^2 \dashrightarrow S^2$ une fonction de Thurston avec quatre valeurs postsingulières telle qu'il existe un sous-ensemble $B \subset P_f$ avec $|B| = 3$, $S_f \subset B$, et $|\overline{f^{-1}(B)} \cap P_f| = 3$. Alors*

1. *f est totalement sans obstruction si et seulement s'il n'existe pas deux points $a, b \in P_f$ tels que $\deg(f,a) = \deg(f,b) = 1$ et que $f(\{a,b\})$ soit égal à $\{a,b\}$ ou à $P_f - \{a,b\}$;*

2. *si f n'est pas totalement sans obstruction, alors sa classe de Hurwitz $\mathcal{H}_f$ contient une infinité de fonctions de Thurston obstruées, combinatoirement non équivalentes entre elles ;*

3. *si f est de degré infini, alors sa classe de Hurwitz $\mathcal{H}_f$ contient une infinité de fonctions de Thurston réalisées, combinatoirement non équivalentes entre elles.*

Soit $f\colon S^2 \dashrightarrow S^2$ une fonction de Thurston et $A \subset S^2$ un ensemble fini tel que $P_f \subset A$ et que l'orbite de f de chaque $a \in A$ reste à A, c'est-à-dire soit $f(a) \in A$ ou a est une singularité essentielle de f. Nous pouvons alors considérer la paire (f, A) appelée fonction de Thurston marquée, souvent notée $f\colon (S^2, A) \circlearrowleft$. De manière similaire à la discussion précédente, nous pouvons introduire l'application de tiré-en-arrière $\sigma_{f,A}$ définie sur l'espace de Teichmüller $\mathcal{T}_A$ de la sphère épointée $S^2 - A$. De plus, les concepts d'équivalences de Hurwitz et combinatoire peuvent être étendus aux fonctions de Thurston marquées. Naturellement, plus l'ensemble marqué est grand, plus ces notions deviennent précises.

Les résultats discutés précédemment peuvent être étendus au contexte plus large des fonctions de Thurston marquées. Dans le Chapitre 5, nous développons davantage ces idées en étudiant les relations entre les propriétés des fonctions de Thurston $f\colon (S^2, A) \circlearrowleft$ et $f\colon (S^2, B) \circlearrowleft$, où B est un sous-ensemble de A. L'accent est principalement mis sur les cas où la fonction de Thurston $f\colon (S^2, B) \circlearrowleft$ est réalisée. Nous commençons par examiner des situations particulières, comme lorsque $A - B$ est constitué d'un seul point fixe de la fonction f. Dans ce cas, nous montrons que l'application de tiré-en-arrière $\sigma_{f,A}$ admet une sous-variété invariante de dimension complexe un au sein de l'espace de Teichmüller $\mathcal{T}_A$ de dimension supérieure. En utilisant les techniques développées dans le Chapitre 3, nous obtenons le résultat général suivant.

Theorem. *Soient $f\colon (S^2, A) \circlearrowleft$ et $f\colon (S^2, B) \circlearrowleft$ des fonctions de Thurston marquées, où B est un sous-ensemble de A. La fonction de Thurston $f\colon (S^2, A) \circlearrowleft$ est réalisée si et seulement si elle n'a pas de cycle de Levy dégénéré et que $f\colon (S^2, B) \circlearrowleft$ est réalisée.*

Ce théorème constitue le premier résultat vers le problème de caractérisation dans le contexte transcendant, sans imposer d'hypothèses sur les propriétés fonctionnelles des fonctions considérées ni sur la taille des ensembles postsinguliers ou marqués correspondants. De plus, il s'applique également dans le cas où $f\colon (S^2, A) \circlearrowleft$ appartient à

la famille exceptionnelle qui n'est pas couverte par la caractérisation de Thurston des fonctions rationnelles. En outre, ce théorème étend et propose une nouvelle démonstration d'un résultat similaire dans le contexte des fonctions de Thurston de degré fini, précédemment établi par Selinger–Yampolsky [SY15] et Bartholdi–Dudko [BD21].

Dans le Chapitre 6, nous proposons d'autres questions et pistes de recherche potentielles découlant du travail discuté dans cette thèse.

Extended abstract

Thurston's theory in one-dimensional holomorphic dynamics develops around the question of whether a given topological map can be realized, in a certain dynamical sense called combinatorial equivalence, by a holomorphic map. In simple terms, a topological map is called realized if such a holomorphic realization exists, and obstructed otherwise. This question can be seen as an attempt to promote topology to geometry, seeking to understand how topological information can shed light on the corresponding geometric structure.

The classical Thurston theory addresses this question in the context of orientation-preserving postcritically finite branched self-covers of the topological 2-sphere S^2, known as Thurston maps. We recall that a self-branched cover f of S^2 is called postcritically finite if every critical value of f has a finite orbit, i.e., it is periodic or pre-periodic under f. Thurston maps serve as topological models for the family of postcritically finite rational maps. Notably, the famous Thurston characterization theorem [DH93] states that any Thurston map of this type (with the exception of a well-understood small family) is realized by a postcritically finite rational map if and only if it has no Thurston obstruction.

The goal of this thesis is to extend Thurston's theory to encompass transcendental maps, that is, maps of infinite topological degree. In particular, we extend the definition of a Thurston map to include postsingularly finite topologically holomorphic maps that may have infinite degree and may be defined outside a closed at most countable set of essential singularities in S^2. We recall that a topologically holomorphic map f is referred to as postsingularly finite if it has a finite number of singular values, and each singular value is either eventually periodic or eventually maps to a point where f is undefined, under iteration. These Thurston maps are intended to serve as combinatorial or topological models for postsingularly finite holomorphic maps. As in the classical case, the main objective is to investigate whether such a topological map can be realized by a holomorphic one. It is worth noting that partial results in this area have been established through the works of Hubbard– Schleicher–Shishikura [HSS09], and S. Shemyakov [She22], though the general case remains largely unresolved.

The key tool for determining whether a given Thurston map f has a holomorphic realization is the analysis of an operator σ_f, known as pullback map, which is defined on the Teichmüller space $\mathcal{T}_{P_f}$ of the punctured sphere $S^2 - P_f$, where P_f is the postsingular set of the Thurston map f. Crucially, it is known that the pullback map σ_f has a fixed point if and only if the Thurston map f is realized. Furthermore, other dynamical properties of σ_f can provide insights into further characteristics of the Thurston map f.

We now present the structure of this thesis, which is primarily based on the author's

papers [Pro24a] and [Pro24b]. Chapter 1 provides an introduction, while Chapter 2 covers the necessary preliminaries. Chapters 3, 4, and 5 entirely consist of new results. Specifically, Chapters 3 and 4 are broadly aligned with the content of [Pro24a], and Chapter 5 corresponds to [Pro24b], with some dependence on the results from Chapter 3. In Chapter 6, we discuss further questions and potential research directions that arise from the results presented in this thesis.

In Chapter 1, we introduce and motivate the key concepts and approaches, summarize the main results, and provide an informal overview of their proofs.

Chapter 2 focuses on establishing the framework. In this chapter, we introduce Thurston maps, which may have infinite degree and are defined outside a closed, at most countable set of essential singularities in S^2, along with key related concepts such as postsingular sets, combinatorial and Hurwitz equivalence, realizability, and more. We also review the necessary background in Teichmüller theory, which enables us to introduce pullback maps, discuss their key properties, and highlight the connections between their dynamics and the characteristics of the corresponding Thurston maps. Finally, the chapter concludes with a discussion of Thurston obstructions and a review of major results in the context of Thurston maps of finite degree.

Chapter 3 highlights the importance of hyperbolic geometry for the considered problem. In the first part, we utilize tools from hyperbolic geometry and Teichmüller theory to identify Thurston obstructions and analyze their properties, based on the behavior of the corresponding pullback maps. The second part is dedicated to studying the dynamics of holomorphic self-maps of the unit disk satisfying certain additional assumptions. Many natural examples of such maps will emerge from pullback maps defined on the one-complex-dimensional Teichmüller space or from restrictions of these maps to invariant subsets of higher-dimensional Teichmüller spaces. The tools developed in this chapter will play an important role in establishing the main results of the thesis in the following two chapters.

In Chapter 4, we study a broad family of Thurston maps with four postsingular values. Specifically, we consider Thurston maps $f\colon S^2 \dashrightarrow S^2$ (where the dashed arrow indicates that f may have essential singularities in S^2) such that there is a subset $B \subset P_f$ with $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap P_f| = 3$, where $\overline{f^{-1}(B)}$ denotes the closure of $f^{-1}(B)$ in the topology of S^2 and P_f is the postsingular set of f. Condition $|P_f| = 4$ implies that the corresponding pullback map σ_f is defined on one-complex-dimensional Teichmüller space. The rest of the condition may seem a bit mysterious, but it guarantees that the pullback map maintains well-structured, though still complicated, behavior.

It is worth to note that for any Thurston map f with three singular values, the aforementioned conditions can be verified by simply analyzing the dynamics of f on the finite set P_f. For instance, in the case of entire Thurston maps (those that can be restricted to self-maps of the toplogical 2-plane $\mathbb{R}^2$) with three singular and four postsingular values, three out of seven possible postsingular portraits have these properties. In particular, despite the restrictive nature of these conditions, the world of transcendental maps is so large that there still exist uncountably many pairwise combinatorially inequivalent Thurston maps, both realized and obstructed, that

satisfy them.

We begin by analyzing the corresponding pullback maps, which are defined on the one-complex-dimensional Teichmüller space, biholomorphic to the unit disk. Using the results from Chapter 3, we arrive at the following conclusion: any Thurston map that meets the previously mentioned conditions is realized if and only if it has no Thurston obstruction of a very simple type. More precisely, we say that a simple closed curve $\gamma \subset S^2 \backslash P_f$ forms a Levy cycle for the Thurston map f if it cannot be shrinked to a point by a homotopy in $S^2 \backslash P_f$, and there exists $n \geqslant 1$ and a simple closed curve $\gamma' \subset f^{-n}(\gamma)$ such that γ and γ' are homotopic in $S^2 \backslash P_f$ and $\deg(f^{\circ n}|\gamma' \colon \gamma' \to \gamma) = 1$. If $n = 1$, we simply say that γ is a Levy fixed curve for the Thurston map f.

Theorem. *Let $f \colon S^2 \dashrightarrow S^2$ be a Thurston map with four postsingular values such that there exists a subset $B \subset P_f$ with $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap P_f| = 3$. Then f is realized if and only if it has no Levy fixed curve.*

This theorem presents the first Thurston-like characterization criterion applicable to uncountably many combinatorially distinct Thurston maps, both realized and obstructed. Additionally, this result is obtained through rather explicit techniques that provide new insights into the pullback dynamics on Teichmüller space.

Developed framework also allows us to simultaneously study the Hurwitz (equivalence) class $\mathscr{H}_f$ of a Thurston map f. Here, two Thurston maps $f_1 \colon S^2 \dashrightarrow S^2$ and $f_2 \colon S^2 \dashrightarrow S^2$ are said to be Hurwitz equivalent if $P_{f_1} = P_{f_2}$ and there exist orientation-preserving homeomorphisms $\phi_1, \phi_2 \colon S^2 \to S^2$ such that $\phi_1|P_{f_1} = \phi_2|P_{f_1}$ and $\phi_1 \circ f_1 = f_2 \circ \phi_2$. For example, we can establish a simple criterion to determine whether a given Thurston map f is totally unobstructed, meaning that the entire Hurwitz class $\mathscr{H}_f$ consists of Thurston maps that admit holomorphic realizations. More generally, we obtain the following result.

Theorem. *Let $f \colon S^2 \dashrightarrow S^2$ be a Thurston map with four postsingular values such that there exists a subset $B \subset P_f$ with $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap P_f| = 3$. Then*

1. *f is totally unobstructed if and only if there are no two points $a, b \in P_f$ such that $\deg(f, a) = \deg(f, b) = 1$ and $f(\{a, b\})$ equals $\{a, b\}$ or $P_f - \{a, b\}$;*

2. *if f is not totally unobstructed, then its Hurwitz class $\mathscr{H}_f$ contains infinitely many pairwise combinatorially inequivalent obstructed Thurston maps;*

3. *if f has infinite degree, then its Hurwitz class $\mathscr{H}_f$ contains infinitely many pairwise combinatorially inequivalent realized Thurston maps.*

Let $f \colon S^2 \dashrightarrow S^2$ be a Thurston map and $A \subset S^2$ be a finite set such that $P_f \subset A$ and the f-orbit of each $a \in A$ remains within A, i.e., either $f(a) \in A$ or a is an essential singularity of f. We can then consider the pair (f, A) called marked Thurston map, and often denoted by $f \colon (S^2, A) \circlearrowleft$. Similarly to the previous discussion, we can introduce the pullback map $\sigma_{f,A}$ defined on the Teichmüller space $\mathscr{T}_A$ of the punctured sphere $S^2 - A$. Furthermore, the concepts of Hurwitz and combinatorial equivalences can be

extended to marked Thurston maps. Naturally, the larger the marked set we consider, the finer these notions become.

The results discussed earlier can be extended to the broader context of marked Thurston maps. In Chapter 5, we further develop these ideas by investigating the relationships between properties of Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$, where B is a subset of A. The primary focus is on cases where the Thurston map $f\colon (S^2, B) \circlearrowleft$ is realized. We start with examining particular situations, such as when $A - B$ consists of a single fixed point of the map f. In this case, we show that the pullback map $\sigma_{f,A}$ admits a well-behaved one-complex-dimensional invariant submanifold within the higher-dimensional Teichmüller space $\mathcal{T}_A$. Using techniques developed in Chapter 3, we obtain the following general result.

Theorem. *Let $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$ be marked Thurston maps, where B is a subset of A. The Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized if and only if it has no degenerate Levy cycle and $f\colon (S^2, B) \circlearrowleft$ is realized.*

This theorem represents the first result towards the characterization problem in the transcendental context without imposing any assumptions on the function-theoretic properties of the functions involved or on the size of the corresponding postsingular or marked sets. Furthermore, it also applies in the case when $f\colon (S^2, A) \circlearrowleft$ belongs to the exceptional family that is not covered by Thurston's characterization of rational maps. Additionally, this theorem extends and offers a new proof for a similar result in the context of Thurston maps of finite degree, previously established by Selinger–Yampolsky [SY15] and Bartholdi–Dudko [BD21].

In Chapter 6, we provide further questions and potential research directions arising from the work discussed in this thesis.

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1 Introduction

Summary

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In this chapter, we introduce and motivate the fundamental concepts, approaches, and key questions, as well as summarize the main results and provide an informal overview of their proofs. Section 1.1 reviews the basic theory of iteration of one-dimensional holomorphic maps, highlighting the significance of postcritically finite rational maps and postsingularly finite holomorphic maps. Section 1.1.1 focuses on rational dynamics, while Section 1.1.2 addresses transcendental dynamics. In Section 2.2, we introduce concepts from Thurston theory, discuss major results in this field and their applications. Finally, Section 1.2.2 covers a key tool in Thurston theory: the iteration of pullback maps on Teichmüller space.

Finally, in Section 1.3, we present the main results of this thesis. Section 1.3.1 is devoted to a broad class of Thurston maps, both of finite and infinite degree, with four postsingular values. Here, we introduce the first two main results, Main Theorems A and B, along with sketches of their proofs. In Section 5.1.2, we explore marked Thurston maps and how their characteristics may change when the corresponding marked set is expanded. This section also introduces the third main result of the thesis, Main Theorem C.

1.1 Dynamics of rational and holomorphic finite type maps

The theory of one-dimensional complex dynamical systems investigates the iteration of holomorphic maps $g\colon S \to S$, where S is a Riemann surface. As discussed in [Mil06, Sections 5 and 6], the three most interesting cases are: $S = \widehat{\mathbb{C}}$ with g being a rational map, $S = \mathbb{C}$ with g being a polynomial or, more generally, an entire map, and $S = \mathbb{C}^*$ with g being a so-called *Rådström* function; see [Rad53] and [Mar16].

Holomorphic dynamics also studies maps where the range is larger than the domain of definition. A common example is the family of meromorphic maps $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$, where g has an essential singularity at ∞. It is important to note that such a map does not always define a discrete dynamical system in the usual sense, as some points in $\mathbb{C}$ may land at ∞ under iteration of g and can no longer be iterated. Nevertheless, the theory of iterations of meromorphic maps has been successfully developed and remains an active area of research (for an overview, see [Ber93]). One can extend this further by considering holomorphic maps $g\colon \widehat{\mathbb{C}} - E \to \widehat{\mathbb{C}}$, where $E \subset \widehat{\mathbb{C}}$ is a closed countable set, leading to maps of *class* $\mathscr{K}$ (see [Bol97]), or a closed totally disconnected set (see [BDH01]). Finally, Adam Epstein [Eps93] has developed the theory of iterations for certain classes of maps $g\colon W \to X$, where X is a connected compact Riemann surface and $W \subset X$ is an open subset.

1.1.1 Rational maps

The family of rational maps is one of the most extensively studied families of holomorphic maps, with the first major results tracing back to Pierre Fatou and Gaston Julia (see [Mil06]). Let $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with $\deg(g) \geqslant 2$. It is useful to decompose the Riemann sphere $\widehat{\mathbb{C}}$ into a disjoint union of two subsets, known as the *Fatou set* $\mathscr{F}_g$ and *Julia set* $\mathscr{J}_g$ of the map g, that are forward- and backward-invariant under g. The Fatou set of g is defined in one of two equivalent ways:

$$\begin{aligned}\mathscr{F}_g &:= \{z \in \widehat{\mathbb{C}} : \text{sequence of iterates } (f^{\circ n}) \text{ is equicontinuous in a neighborhood of } z\} \\ &= \{z \in \widehat{\mathbb{C}} : \text{sequence of iterates } (f^{\circ n}) \text{ is normal in a neighborhood of } z\},\end{aligned}$$

and the Julia set is then defined as the complement of the Fatou set, i.e., $\mathscr{J}_g := \widehat{\mathbb{C}} - \mathscr{F}_g$. The Fatou set $\mathscr{F}_g$ should be viewed as the part of the Riemann sphere where the dynamics of g is regular and stable under small perturbations. In contrast, the dynamics on the Julia set $\mathscr{J}_g$ exhibits chaotic and unstable behavior.

Every connected component of the Fatou set $\mathscr{F}_g$ is called a *Fatou component* of the rational map g. It is straightforward to verify that if U is a Fatou component of g, then $g(U)$ is also a Fatou component. In the renowned No-Wandering-Domains Theorem [Mil06, Appendix F], Dennis Sullivan proved that no Fatou component U can be *wandering*; that is, it cannot happen that $g^{\circ k}(U) \cap g^{\circ l}(U) = \varnothing$ for all distinct k and l. In other words, every Fatou component is either periodic or pre-periodic.

The classification of periodic Fatou components is simple: they are either cycles of *(super-)attracting basins*, cycles of *parabolic basins*, or cycles of *Siegel disks* or *Herman rings*. For more details, see [Mil06, Section 16].

This classification is closely related to the periodic points of the considered rational map. Every *(super-)attracting* periodic orbit gives rise to a cycle of (super-)attracting basins, while an *irrationally indifferent* periodic point within the Fatou set serves as a center of a cycle of Siegel disks. Similarly, a *rationally indifferent* periodic point indicates the presence of a cycle of parabolic domains. Additionally, there are two other types of periodic points that have not yet been mentioned: *repelling* periodic points, which are dense in the Julia set, and *Cremer* periodic points, i.e., irrationally indifferent periodic points that belong to the Julia set. Note that the number of non-repelling periodic orbits for a rational map g is bounded by $2d-2$, where $d = \deg(g)$, according to the Fatou-Shishikura inequality [Shi87]; see [Eps99] for a stronger version of these results.

All these aspects of the local and global dynamics of a rational map g can be uncovered by examining the behavior of finitely many orbits, specifically the orbits of the map's *critical values*. The points in these orbits are known as the *postcritical values* of g. For example, it is known that an attracting, rationally indifferent, or Cremer periodic point is an accumulation point of the postcritical set of g. Likewise, if $U_1, U_2, \dots, U_r$ is a cycle of Siegel disks or Herman rings, then postcritical values are dense in the boundary ∂U_i for each $i = 1, 2, \dots, r$.

These and numerous other observations emphasize the significance of dynamics of critical values for understanding the global behavior of a rational map. Consequently, one might expect that rational maps with simple dynamics of critical values exhibit particularly well-behaved global dynamics. This leads to the definition of *postcritically finite* (or *pcf* in short) rational maps, which are maps with every critical value being periodic or pre-periodic. For example, based on the previous discussion, every periodic Fatou component of a postcritically finite rational map g forms a cycle of superattracting basins, and every periodic orbit of g is either superattracting or repelling. Postcritically finite rational maps admit many other important properties, including some of combinatorial and algebraic nature, which will be discussed further in Section 1.2.1. In the following section, we will see that this family not only possesses strong properties but also plays a crucial role in one-dimensional rational dynamics.

1.1.1.1 Postcritically finite rational maps

Postcritically finite rational maps may seem to form a rather specific family, as up to Möbius conjugation, there are only countably many postcritically finite rational maps that are not *flexible Lattès maps*. For example, this holds true for all pcf rational maps with at least five postcritical values. However, they play a crucial role in the study of dynamics of rational maps. For example, every *hyperbolic* rational map with a connected Julia set exhibits strong dynamical similarities to a certain postcritically finite rational map; see [MSS83], [McM88], and [Mil12]. Recall that a rational map is called hyperbolic if the orbit of each critical value converges to a (super-)attracting

periodic point. It is worth noting that the famous conjecture of *density of hyperbolicity* states that the set of hyperbolic rational maps is open and dense in the space of all rational maps of any fixed degree; see [McM94, Conjecture 1.1 and Section 4].

To illustrate the ideas outlined in the previous paragraph, consider the famous *Mandelbrot set*:

$$\mathcal{M} := \{c \in \mathbb{C} : \text{the Julia set of } p_c(z) = z^2 + c \text{ is connected}\} = \{c \in \mathbb{C} : \text{the sequence } (p_c^{\circ n}(0)) \text{ is bounded}\}.$$

It is known that points $c \in M$ corresponding to postcritically finite polynomials with a strictly pre-periodic critical point 0 form a dense subset of the boundary of the Mandelbrot set. Furthermore, one of the corollaries of the famous *MLC* (Mandelbrot set is *locally connected*) conjecture states that every connected component D of the interior of $\mathcal{M}$ contains a unique *center* corresponding to a postcritically finite quadratic polynomial. Moreover, the dynamical properties of all polynomials corresponding to points within D share a strong dynamical similarity with the polynomial at the center of D. Furthermore, assuming the MLC conjecture holds, the entire intricate structure of the *Mandelbrot set* can be described in terms of postcritically finite polynomials; see [DH85], [Sch00], [Sch04b], [Sch17]. Finally, there is an elegant interplay between the points of the Mandelbrot set corresponding to postcritically finite polynomials and the associated *bifurcation measure*; see [Lev89] and [DF08].

These and the earlier observations demonstrate that postcritically finite rational maps not only possess particularly strong properties, but also hold structural significance for understanding the entire family of rational maps.

1.1.2 Transcendental maps

The Fatou-Julia theory covered in Section 1.1.1 can be extended to entire and even meromorphic maps; for overviews, see [Ber93] and [Sch10]. However, these families admit new dynamical phenomena. For example, they can have wandering Fatou components (see [Sch10, Examples 2.9-2.11] for a simple example), even under quite restrictive conditions; see [Bis15a, Section 17] and [MS20]. Additionally, there is a new type of periodic Fatou components known as *Baker domains*; see [Sch10, Example 2.8].

For a meromorphic map $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$, a point $z \in \widehat{\mathbb{C}}$ is called a *regular value* if there is a neighborhood U of z such that $g^{-1}(U) = \bigsqcup_{i\in I} U_i$ and the restriction $g|U_i\colon U_i \to U$ is a biholomorphism for each $i \in I$. If no such neighborhood exists for z, it is called a singular value of the map g. Obviously, any critical value of g is a singular value, but the opposite is not necessarily true. In this thesis, we restrict ourselves to entire and meromorphic maps of *finite type*, i.e., maps with finitely many *singular values*.

Entire or meromorphic maps of finite type do not exhibit the phenomena described in the first paragraph of this section. Furthermore, most of the theory discussed in Section 1.1.1 applies to these families, with one key adjustment: critical values should be replaced by singular values, and the postcritical set by the *postsingular set*. This holds true in even broader contexts. For example, the results analogous to ones in

Section 1.1.1 apply to holomorphic maps of finite type of class $\mathscr{K}$ (see [Bol97] and [BDH01]) and, more generally, to the family of finite type maps introduced by Adam Epstein; see [Eps93] and [Ast24]. It is important to note that, despite these families having particularly strong properties, already the families or meromorphic or entire maps of finite type are much larger than the whole family of rational maps, and exhibit many new phenomena.

It is natural to introduce the concept of a *postsingularly finite* map for any of the finite-type families discussed earlier. A map g is called postsingularly finite (*psf* in short) if each singular value of g either becomes eventually periodic or lands on a point where g is undefined, e.g., an essential singularity of g, under iteration. Similar to the rational case, it is easy to see that these maps exhibit particular dynamical properties.

The relationship between families of entire or meromorphic finite-type maps and postsingularly finite holomorphic maps is less well-understood than in the rational case. Nevertheless, there are several indications that psf maps play a structurally important role. For example, consider the family of exponential maps $E = \{\exp(z)+\kappa : \kappa \in \mathbb{C}^*\}$, which, despite its simplicity, exhibits rich and complex dynamical behavior. It is known that points κ corresponding to psf entire maps are dense in the *bifurcation locus* of the family E; see [Rem03], particularly [Rem03, Lemma 5.1.5], and [Sch04a], [RS08], [RS09] for further discussions on the bifurcation locus of this family. Additionally, the authors of [BL19] have demonstrated how to construct postsingularly finite meromorphic maps with prescribed postsingular dynamics using *quasiconformal folding*.

1.2 Thurston's theory in holomorphic dynamics

1.2.1 Key notions and motivations

As it was discussed in Section 1.1.1, in one-dimensional rational dynamics, a crucial role is played by the family of postcritically finite rational maps, i.e., maps with all critical points being periodic or pre-periodic. In this context, one of the most influential ideas has been to abstract from the rigid underlying complex structure and consider the more general setup of postcritically finite *branched self-coverings* of the topological 2-sphere S^2. Nowadays, orientation-preserving pcf branched covering maps $f\colon S^2 \to S^2$ of topological degree $\deg(f) \geqslant 2$ are called *Thurston maps* (*of finite degree*), in honor of William Thurston, who introduced them to deepen the understanding of the dynamics of postcritically finite rational maps on $\widehat{\mathbb{C}}$.

These ideas can be extended to the transcendental setting to explore the dynamics of holomorphic maps of infinite degree that are not defined everywhere on the Riemann sphere. In this thesis, we focus on postsingularly finite maps that are holomorphic everywhere on $\widehat{\mathbb{C}}$ except at the closed set of essential singularities, which can be at most countable. Recall, that such map is called postsingularly finite if it has finitely many singular values, and each of them eventually becomes periodic or lands on an essential

singularity under the iteration. We can generalize the notion of a Thurston map to postsingularly finite *topologically holomorphic* non-injective maps $f\colon S^2 - E \to S^2$, where $E \subset S^2$ is at most countable closed set, f does not extend continuously to a neighborhood of any of the points of E and meets a technical condition of being an *admissible type* map. The question of being an admissible type can be viewed as a more general version of the famous *type problem*, which arises from the fact that any simply connected non-compact Riemann surface is biholomorphic to either the unit disk $\mathbb{D}$ or the complex plane $\mathbb{C}$. In particular, if E is empty, then f is automatically of admissible type; when E consists of a single point, a topologically holomorphic finite-type map f is of admissible type if and only if it is of *parabolic type*; see Sections 2.1.2 and 2.2.1 for more details. It is important to note that the set E is non-empty if and only if f is *transcendental*, meaning that it has infinite topological degree. For simplicity, we will use the notation $f\colon S^2 \dashrightarrow S^2$ to indicate that the Thurston map f, whether finite or infinite degree, might not be defined on an at most countable closed set of S^2.

For a Thurston map $f\colon S^2 \dashrightarrow S^2$, the *postsingular set* P_f is defined as the union of all orbits of the singular values of the map f. It is important to note that some of these orbits might terminate after several iterations if a singular value reaches a point where the map f is not defined. The elements of the postsingular set P_f are called the *postsingular values* of the Thurston map f. If the map f is defined on the entire sphere S^2, we simply refer to its *postcritical set* and *postcritical values*, as the set of singular values of f coincides with the set of its *critical values*. Two Thurston maps are called *combinatorially* (or *Thurston*) *equivalent* if they are conjugate up to isotopy relative to their postsingular sets; see Definition 2.2.6.

A fundamental question in this context is whether a given Thurston map $f\colon S^2 \dashrightarrow S^2$ can be *realized* by a psf holomorphic map with the same combinatorics, that is, if $f\colon S^2 \dashrightarrow S^2$ is combinatorially equivalent to a psf holomorphic map. If the Thurston map f is not realized, then we say that f is *obstructed*. William Thurston answered this question for Thurston maps of finite degree in his celebrated *characterization of rational maps*: if a finite-degree Thurston map $f\colon S^2 \to S^2$ is not a $(2,2,2,2)$-*map*[1], then f is realized by a pcf rational map if and only if f has no *Thurston (or multicurve) obstruction* [DH93]. Such an obstruction is given by a finite collection of disjoint simple closed curves in $S^2 - P_f$ with certain invariance properties under the map f; see Section 2.2.5. In many instances, it suffices to restrict to simpler types of Thurston obstructions provided by *Levy cycles* that will be discussed a bit later; see [Hub16, Theorem 10.3.8], [Sha13, Proposition 1.1], [HP22, Corollary 1.5], or [Par23, Theorems 7.6 and 8.6] for examples of such cases. The family of $(2,2,2,2)$-maps that are not covered by Thurston's characterization result consists of all Thurston maps with four postcritical values, where each critical point is *simple* and does not belong to the postcritical set. Alternatively, for Thurston maps where each cycle of postcritical values contains a critical point, one can apply a recent positive characterization

[1] In the statement of this result, the entire family of Thurston maps with a *parabolic orbifold* is often excluded. However, a Thurston map with a parabolic orbifold that is not of $(2,2,2,2)$-type has at most three postcritical values. Consequently, such a map is always realized and has no Thurston obstructions for obvious reasons; see Proposition 2.2.10.

criterion developed by Dylan Thurston [Thu20].

The same characterization question can be also asked in the transcendental setting. The first breakthrough in this area was obtained in [HSS09], where it was shown that an *exponential* Thurston map is realized if and only if it has no Levy cycle. In this context, the exponential Thurston map is defined as a Thurston map with two singular values, both of which are *omitted*, and one of which is the only essential singularity of the considered Thurston map. Furthermore, in [She22] the author has studied the family of Thurston maps with a single essential singularity that are *structurally finite*, i.e., they have finitely many critical points and *asymptotic tracts*. Moreover, [MPR24] demonstrates that *entire* transcendental Thurston maps (i.e., those that can be restricted to topologically holomorphic self-maps of $\mathbb{R}^2$; see Section 2.1.2) can be understood as dynamically meaningful limits of finite-degree Thurston maps (see also [Muk23]). This observation suggests the possibility of a Thurston-like criterion for realizability in a broader setting. However, the problem of characterization in the transcendental case remains largely unresolved, as many of the techniques developed in Thurston theory for finite-degree maps do not directly apply in this context.

Thurston theory lays out the relationship between the topological properties of a map, its dynamics, and its geometry in terms of the existence of a holomorphic realization. Furthermore, it is strongly connected with the combinatorial and algebraic aspects of the dynamics of pcf rational and psf meromorphic maps. The results mentioned above have substantial applications for both rational and transcendental dynamics. For instance, Thurston's characterization result has allowed to classify various families of postcritically finite rational maps or finite-degree Thurston maps in terms of combinatorial models [Poi93; Poi10; Bel+22], [Dra+19; LMS22], [Cor+15; HP22], [Hlu17], [GH21; LLM22]. Building on the result of [HSS09], similar classifications were obtained in [LSV08] in terms of *kneading sequences* and in [PRS21] in terms of *homotopy Hubbard trees* for the family of postsingularly finite exponential maps. Moreover, the concept of a homotopic Hubbard tree was extended to general postsingularly finite entire maps in [Pfr19] (see also [PPS21]). It is plausible that a Thurston-like criterion is the final missing ingredient for the complete classification of the family of all psf entire maps; see [Pfr19, Chapter 5]. For application of Thurston's result for resolving dynamical questions of algebraic nature see [Flo+22] and [FG22].

1.2.1.1 Marked Thurston maps

Let $f\colon S^2 \dashrightarrow S^2$ be a Thurston map and A be a subset of S^2. We say that A is *f-pseudo-invariant* if the orbit of each point $a \in A$, whether it terminates after several iterations or not, remains within A; in other words, either $f(a) \in A$ or a is an essential singularity of f. Clearly, every f-invariant set is also f-pseudo-invariant; however, the converse does not necessarily hold for transcendental Thurston maps. If A is a finite f-pseudo-invariant subset of S^2 containing the set S_f of singular values of f (and therefore, the entire postsingular set P_f), we can consider the pair (f, A) called a *marked Thurston map*, where A is referred to as the *marked set*. For clarity in notation, we denote the marked Thurston map (f, A) by $f\colon (S^2, A) \circlearrowleft$. If no marked set is explicitly mentioned,

it is assumed to coincide with the postsingular set P_f, and in this case the Thurston map is sometimes called *unmarked* or *minimally marked*.

Two marked Thurston maps $f_1\colon (S^2, A_1) \circlearrowleft$ and $f_2\colon (S^2, A_2) \circlearrowleft$ are said to be *combinatorially equivalent* if they are conjugate up to isotopy relative to their marked sets; see Definition 2.2.6. It implies, for instance, that $f_1|A_1$ and $f_2|A_2$ are conjugate. Furthermore, for the unmarked Thurston map $f\colon S^2 \dashrightarrow S^2$, we can explore whether f is realized with respect to different marked sets. Naturally, the larger the marked set we consider, the finer the notion of combinatorial equivalence becomes. Finally, it is worth noting that the notion of a marked Thurston map naturally emerges even if we focus solely on unmarked Thurston maps. This phenomenon, for instance, arises in *Pilgrim's decomposition theory* for finite-degree Thurston maps; see [Pil03], [Sel13a], and [BD17].

This leads to the same realizability question as before, but now in the broader context of marked Thurston map. According to [BCT14] (see also [BMW23] for an alternative approach), Thurston's characterization criterion extends directly to the setting of finite degree marked Thurston maps that are not $(2,2,2,2)$-maps. Marked Thurston $(2,2,2,2)$-maps were studied in [SY15] and [BD21]. Additionally, although not explicitly stated, the result of [HSS09] also applies in the context of marked exponential Thurston maps.

1.2.2 Iteration on Teichmüller space

From this point forward, we assume that any marked set contains at least three points, as the case of fewer marked points is rather trivial; see Proposition 2.2.10. The key method in determining whether a given finite or infinite degree marked Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized by a psf holomorphic map is the analysis of the dynamics of an operator $\sigma_{f,A}$, known as the *pullback map*, defined on the *Teichmüller space* $\mathcal{T}_A$; see Sections 2.2.2 and 2.2.3. For simplicity, we will sometimes omit the marked set and use the notation σ_f for the pullback map $\sigma_{f,A}$. Here, roughly speaking, the Teichmüller space $\mathcal{T}_A$ encodes all possible ways to put a complex structure, which is biholomorphic to a punctured Riemann sphere, on the punctured topological sphere $S^2 - A$, up to a certain equivalence depending on the set A. Crucially, the Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized if and only if the pullback map $\sigma_{f,A}$ has a fixed point in $\mathcal{T}_A$; see Proposition 2.2.16. Moreover, the dynamics of the pullback map encodes many other properties of the corresponding Thurston map; see, for instance, [Buf+09], [KPS16], and [BDP24].

The characterization problem is closely related to two other key results developed by William Thurston. One of these is the renowned *Nielsen-Thurston classification* of *mapping classes* [FM12, Part 3]. This classification states that any orientation-preserving homeomorphism of an oriented topological surface of *finite type* is homotopic to one of the following: a periodic homeomorphism, which possesses an "invariant conformal structure"; a *pseudo-Anosov* homeomorphism, which admits a pair of well-behaved *transverse singular foliations*; or a homeomorphism that can be decomposed along an invariant multicurve into multiple "pieces", each of one of the

two aforementioned types.

The second key result related to this context is the celebrated *hyperbolization of Haken manifolds*. Specifically, W. Thurston proved that every *reducible atoroidal Haken 3-manifold* admits a hyperbolic structure (see [Thu22a], [Thu22b], and also [McM90]). This represented significant progress in the long-standing project toward the geometrization of 3-manifolds, a program that was later completed by Grigori Perelman; see [Bes+10] for an overview.

These three questions may look rather separated, but the unifying concept is an attempt to promote topology to geometry, to understand what topological data could tell about the geometric data. Additionally, iteration on Teichmüller space is central to the proof of all three results. In the characterization problem, we analyze the iteration of pullback maps; the group of homeomorphisms of a finite-type oriented surface acts by automorphisms on the corresponding Teichmüller space; and, finally, the crucial role in the proof of the hyperbolization theorem is played by *skinning maps*, which are also defined on a Teichmüller space.

1.2.2.1 Properties of pullback maps

Teichmüller space $\mathcal{T}_A$, with $|A| \geqslant 3$, has a structure of $(|A|-3)$-complex-dimensional manifold and can be equipped with a complete metric d_T, known as the *Teichmüller metric*. It is known that every pullback map σ_f, associated to a Thurston map $f\colon (S^2, A) \circlearrowleft$, is holomorphic and *1-Lipschitz* on $\mathcal{T}_A$, meaning $d_T(\sigma_f(\tau_1), \sigma_f(\tau_2)) \leqslant d_T(\tau_1, \tau_2)$ for all $\tau_1, \tau_2 \in \mathcal{T}_A$. In most cases, σ_f (or one of its iterates) is actually *distance-decreasing*, i.e., $d_T(\sigma_f(\tau_1), \sigma_f(\tau_2)) < d_T(\tau_1, \tau_2)$ for all distinct $\tau_1, \tau_2 \in \mathcal{T}_A$. Consequently, if such a pullback map σ_f has a fixed point, then it is necessarily unique.

For example, the pullback map σ_f is always distance-decreasing when f is transcendental. The same property holds for some iterate $\sigma_f^{\circ k}$ provided that f is not a $(2,2,2,2)$-map; see Proposition 2.2.19. However, these results no longer apply for $(2,2,2,2)$-maps. For instance, if f is a $(2,2,2,2)$-map, then σ_{f,P_f} is an isometry of $\mathcal{T}_{P_f}$, and it is even possible that $\sigma_{f,P_f} = \mathrm{id}_{\mathcal{T}_{P_f}}$; see [Hub16, Appendix C.6]. This creates the main challenge of working with marked Thurston $(2,2,2,2)$-maps, and one of the reasons why Thurston's result does not extend to this context.

Although the pullback maps of transcendental Thurston maps are always distance-decreasing, they exhibit more intricate behavior compared with those of finite-degree Thurston maps. To explain this further, we need to introduce the *moduli space* $\mathcal{M}_A$, which is a complex manifold that, roughly speaking, captures all possible complex structures on $S^2 - A$ that are biholomorhic to a punctured Riemann sphere; see Section 2.2.2 for the precise definition. There is a natural forgetful map $\pi_A\colon \mathcal{T}_A \to \mathcal{M}_A$ that is a holomorphic universal covering. However, the map σ_f rarely descends to a map on the moduli space $\mathcal{M}_A$. Nevertheless, for a finite-degree Thurston map $f\colon (S^2, A) \circlearrowleft$, there exists a complex manifold $\mathcal{W}_f$, known as the *Hurwitz space*, along with the holomorphic *X-map* $X_f\colon \mathcal{W}_f \to \mathcal{M}_A$, a holomorphic covering map $Y_f\colon \mathcal{W}_f \to \mathcal{M}_A$, called the *Y-map*, and a holomorphic covering $\omega_f\colon \mathcal{T}_A \to \mathcal{W}_f$, such that the

following diagram commutes [Koc13, Section 2]:

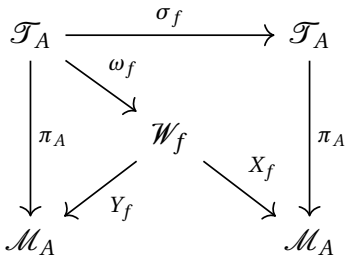

Figure 1.1: Fundamental diagram.

We note that $\mathscr{W}_f$, X_f, Y_f, and ω_f obviously depend on the marked set A, but for simplicity, we omit it from the notation. In other words, commutative diagram (1.1) indicates that the pullback map σ_f is semi-conjugate to a holomorphic *self-correspondence* $X_f \circ Y_f^{-1}$ of the moduli space $\mathscr{M}_A$. If $f\colon (S^2, A) \circlearrowleft$ is a finite-degree Thurston map, then the Y-map $Y_f\colon \mathscr{W}_f \to \mathscr{M}_A$ has a finite topological degree; see [Koc13, Theorem 2.6]. This observation plays a crucial role in the proof of Thurston's characterization of rational maps. In fact, it allows to conclude that for a Thurston map $f\colon (S^2, A) \circlearrowleft$ that is not a $(2,2,2,2)$-map, the σ_f-orbit $(\sigma_f^{\circ n}(\tau))$ of $\tau \in \mathscr{T}_A$ converges (indicating that $f\colon (S^2, A) \circlearrowleft$ is realized), if the projection $(\pi_A(\sigma_f^{\circ n}(\tau)))$ of this orbit visits some compact set of the moduli space $\mathscr{M}_A$ infinitely many times; see [Hub16, Section 10.9 and Lemma 10.11.9] and [Sel12, Proof of Theorem 2.3].

The objects introduced above, along with commutative diagram (1.1) and the fact that $Y_f\colon \mathscr{W}_f \to \mathscr{M}_A$ has a finite degree, have broad applications beyond the proof of Thurston's characterization of rational maps; see, for example, [BN06; Buf+09; Sel12; Sel13b; Koc13; Lod13; KPS16; Flo+17; BD21; Smi24a; Smi24b; BDP24]. These tools, for instance, allow to simultaneously study the entire *Hurwitz* (*equivalence*) *class* $\mathscr{H}_{f,A}$ of the finite-degree Thurston map $f\colon (S^2, A) \circlearrowleft$. Here, two Thurston maps $f_1\colon (S^2, A) \circlearrowleft$ and $f_2\colon (S^2, A) \circlearrowleft$, whether of finite or infinite degree, are said to be *Hurwitz equivalent* if there exist orientation-preserving homeomorphisms $\phi_1, \phi_2\colon S^2 \to S^2$ such that $\phi_1|A = \phi_2|A$ and $\phi_1 \circ f_1 = f_2 \circ \phi_2$; see [BN06; Lod13; KPS16] for examples of results on Hurwitz classes. For instance, one can pose a question whether a given Thurston map is *totally unobstructed*, i.e., $\mathscr{H}_{f,A}$ consists of only realized Thurston maps.

Commutative diagram (1.1) is particularly powerful in two specific cases. The first is when the Thurston map $f\colon (S^2, A) \circlearrowleft$ has the marked set A consisting of exactly four points. In this situation, the spaces $\mathscr{T}_A$, $\mathscr{M}_A$, and $\mathscr{W}_f$ are simply Riemann surfaces. In fact, the Teichmüller space is biholomorphic to the unit disk $\mathbb{D}$ and the moduli space $\mathscr{M}_A$ is biholomorphic to the three-punctured Riemann sphere $\Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$. This allows the use of powerful machinery of one-dimensional holomorphic dynamics to study pullback maps. For example, this approach was utilized in [Smi24a] to derive an alternative proof of Thurston's characterization of rational maps in the case of four postcritical values, as well as in [Smi24a; Smi24b; BHL24] to investigate the *global curve attractor conjecture*, which was ultimately resolved in [BDP24] for all pcf rational

maps with four postcritical values.

Secondly, when the X-map X_f is injective, the "inverse" of σ_f descends to the so-called *g-map* $g_f := Y_f \circ X_f^{-1}$, which is defined on the subset $X_f(\mathcal{W}_f)$ of the moduli space $\mathcal{M}_A$; see [Koc13, Section 5] for examples of finite-degree Thurston maps that satisfy this condition.

1.3 Main Results

1.3.1 Thurston maps with a small postsingular set

In the first part of this thesis, we study the family of Thurston maps $f\colon (S^2, A) \circlearrowleft$ satisfying the following conditions:

(A) the singular set S_f of the map f consists of at most three points;

(B) the marked set A consists of exactly four points;

(C) there exists a set $B \subset A$ such that $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap A| = 3$.

Here, $\overline{f^{-1}(B)}$ is the closure of the set $f^{-1}(B)$ in the topology of S^2. In particular, it coincides with $f^{-1}(B)$ if the Thurston map f has a finite degree; otherwise, it also includes all essential singularities of the map f. We note that one can also consider the case when $|\overline{f^{-1}(B)} \cap A| = 4$, i.e., $A \subset \overline{f^{-1}(B)}$, but this scenario is rather trivial due to Remark 2.2.17.

Conditions (A) and (B) are fairly straightforward. For example, condition (B) implies that the corresponding pullback map σ_f is defined on the one-complex-dimensional Teichmüller space $\mathcal{T}_A \sim \mathbb{D}$. On the other hand, condition (C) may appear more mysterious. In fact, when combined with the other conditions, it ensures that the pullback map maintains well-structured, though still complicated, behavior. For instance, σ_f admits an analog of commutative diagram (1.1), where the X-map is injective, the Y-map is a covering, potentially of infinite degree, and the analog of the Hurwitz space $\mathcal{W}_f$ is a finitely or countably punctured Riemann sphere (see Proposition 4.0.1 and Remark 4.0.2). Formal and informal explanations of this phenomenon are provided in Section 4.0.1.

Clearly, conditions (B) and (C) are independent of the function-theoretical properties of the map f. Specifically, if $F\colon S^2 \dashrightarrow S^2$ is any finite-type topologically holomorphic map of admissible type with at most three singular values, there exist two orientation-preserving homeomorphisms $\phi_1, \phi_2\colon S^2 \to S^2$ such that $\phi_1 \circ F \circ \phi_2$ becomes a Thurston map with four postsingular values that satisfies conditions (A)–(C); see the proof of Corollary 4.1.9 for a similar argument. Additionally, if the Thurston map f has at most three singular values, conditions (A)–(C) can be verified by simply examining the dynamics of f on the finite set A. For instance, in the case of entire Thurston maps (those that can be restricted to self-maps of $\mathbb{R}^2$) with three singular and four postsingular values, three out of seven possible *postsingular portraits* satisfy

condition (C); see Example 4.1.13. More examples of families of Thurston maps that meet these conditions can be found in Section 4.1.3.

Although conditions (A)–(C) are fairly restrictive, the world of transcendental maps is so large that there still exist uncountably many pairwise combinatorially inequivalent Thurston maps, both realized and obstructed, that satisfy them. This remains true even if we restrict to unmarked entire Thurston maps; see Remark 4.1.11. Lastly, we note these conditions are preserved under Hurwitz equivalence of Thurston maps.

1.3.1.1 Characterization problem

We establish an analog of Thurston's characterization result for the family of Thurston maps that satisfy conditions (A)–(C). In fact, we show that it is sufficient to consider one of the simplest types of Thurston obstructions — *Levy fixed curves* — to determine whether such a Thurston map is realized. For a Thurston map $f\colon (S^2, A) \circlearrowleft$, a Levy fixed curve is a simple closed curve $\gamma \subset S^2 - A$ such that γ is *essential*, i.e., it cannot be shrinked to a point by a homotopy in $S^2 - A$, and there exists another simple closed curve $\widetilde{\gamma} \subset f^{-1}(\gamma)$ such that γ and $\widetilde{\gamma}$ are homotopic in $S^2 - A$ and $f|\widetilde{\gamma}\colon \widetilde{\gamma} \to \gamma$ is a homeomorphism. If the map f is injective on one of the connected components of $S^2 - \widetilde{\gamma}$, then we say that the Levy fixed curve γ is *weakly degenerate*.

Main Theorem A (= Theorem 4.1.1)**.** *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree that satisfies conditions (A)–(C). Then $f\colon (S^2, A) \circlearrowleft$ is realized if and only if it has no weakly degenerate Levy fixed curve. Moreover, if $f\colon (S^2, A) \circlearrowleft$ is obstructed, then it has a unique Levy fixed curve up to homotopy in $S^2 - A$; otherwise, it is realized by a psf holomorphic map that is unique up to Möbius conjugation.*

To prove Main Theorem A, we start by showing that the corresponding pullback map σ_f defined on the Teichmüller space $\mathcal{T}_A \sim \mathbb{D}$, admits an analog of commutative diagram (1.1), where the X-map is injective, the Y-map is a covering, potentially of infinite degree, and the analog of the Hurwitz space $\mathcal{W}_f$ is a finitely or countably punctured Riemann sphere (see Proposition 4.0.1 and Remark 4.0.2). Further analysis reveals a crucial observation similar to that in the proof of Thurston's characterization result: if the sequence $(\pi_A(\sigma_f^{\circ n}(\tau)))$ with $\tau \in \mathcal{T}_A$ visits a certain compact set of the moduli space $\mathcal{M}_A$ infinitely many times, then the σ_f-orbit $(\sigma_f^{\circ n}(\tau))$ of τ converges to the unique fixed point of σ_f (see Claim 1 of the proof of Theorem 3.2.6).

Moreover, we establish a more refined result: if the pullback map σ_f does not have a fixed point (indicating that the Thurston map $f\colon (S^2, A) \circlearrowleft$ is obstructed), then the sequence $(\pi_A(\sigma_f^{\circ n}(\tau)))$ converges to the same "cusp" $x \in \partial\mathcal{M}_A \sim \partial\Sigma = \{0, 1, \infty\}$ of the moduli space $\mathcal{M}_A \sim \Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$, regardless of the choice of $\tau \in \mathcal{T}_A$. Furthermore, the map $g_f = Y_f \circ X_f^{-1}$, which is locally conjugate to a local inverse of σ_f, can be holomorphically extended to a neighborhood of this cusp and x becomes a repelling fixed point of g_f. It is worth noting that these results hold not only for pullback maps, but also in a broader class of holomorphic self-maps of the unit disk; see Theorem 3.2.6. Finally, this analysis provides a sufficient control over the dynamics of σ_f to derive

the existence of a Levy fixed curve for the obstructed Thurston map $f\colon (S^2, A) \circlearrowleft$; see Section 3.1.

Main Theorem A is the first result addressing the characterization problem in the transcendental setting that is established with minimal dependence on the function-theoretical properties of the considered Thurston maps. This differs from [HSS09], which focuses on exponential Thurston maps, and [She22], which is primarily devoted to structurally finite Thurston maps. In fact, Main Theorem A is the first result of its kind to encompass uncountably many pairwise combinatorially inequivalent Thurston maps, both realized and obstructed. Finally, note that the geometric and analytic properties of entire or meromorphic maps, even those with few singular values, can be highly varied and subtle; see [Bis15a; Bis15b; Bis17].

Main Theorem A also offers an alternative proof for the characterization of postsingularly finite exponential maps [HSS09, Theorem 2.4] for the case of four postsingular values; see Example 4.1.12. While the proof in [HSS09] relies on the intricate machinery of *integrable quadratic differentials* and their *thick-thin decompositions*, our approach uses more explicit techniques that shed the light on the geometry of the pull-back dynamics on Teichmüller and moduli spaces. At the same time, Main Theorem A provides a novel proof of Thurston's characterization of rational maps for Thurston maps of finite degree that satisfy conditions (A)–(C), and this proof does not rely on the fact that Y-map has a finite degree.

1.3.1.2 Hurwitz classes

The techniques described in Section 1.3.1.1 also allow us to establish several properties of Hurwitz classes. Recall that a Thurston map $f\colon (S^2, A) \circlearrowleft$ is called *totally unobstructed* if the corresponding Hurwitz class $\mathcal{H}_{f,A}$ consists entirely of realized Thurston maps.

Main Theorem B (= Theorem 4.1.7)**.** *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree that satisfies conditions (A)–(C). Then*

1. *f is totally unobstructed if and only if there are no two points $a, b \in A$ such that $\deg(f, a) = \deg(f, b) = 1$ and $f(\{a, b\})$ equals $\{a, b\}$ or $A - \{a, b\}$;*
2. *if $f\colon (S^2, A) \circlearrowleft$ is not totally unobstructed, then its Hurwitz class $\mathcal{H}_{f,A}$ contains infinitely many pairwise combinatorially inequivalent obstructed Thurston maps;*
3. *if $f\colon (S^2, A) \circlearrowleft$ has infinite degree, then its Hurwitz class $\mathcal{H}_{f,A}$ contains infinitely many pairwise combinatorially inequivalent realized Thurston maps.*

The main tool for proving Main Theorem B is the relationship between fixed points of the map $g_f = Y_f \circ X_f^{-1}$ and the elements of the Hurwitz class $\mathcal{H}_{f,A}$. Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map satisfying assumptions (A)–(C). We show that if the corresponding map g_f can be holomorphically extended to a neighborhood of a "cusp" $x \in \partial\mathcal{M}_A \sim \partial\Sigma = \{0, 1, \infty\}$ and x becomes a repelling fixed point of g_f, then the Hurwitz class $\mathcal{H}_{f,A}$ must contain an obstructed Thurston map. Moreover, $f\colon (S^2, A) \circlearrowleft$ is totally

unobstructed if and only if none of the "cusps" of the moduli space $\mathscr{M}_A$ exhibit this behavior; see Proposition 4.1.3. Using additional properties of the map g_f, we establish a simple criterion, as in item (1) of Main Theorem B, for determining whether a given Thurston map with properties (A)–(C) is totally unobstructed. Furthermore, with the understanding of possible obstructions provided by Main Theorem A, we can construct infinitely many pairwise combinatorially inequivalent obstructed Thurston maps within the Hurwitz class $\mathscr{H}_{f,A}$ starting from just one of them.

To prove item (3) of Main Theorem B, we show that a fixed point $x \in \mathscr{M}_A$ of the map g_f corresponds to a realized Thurston map within the Hurwitz class $\mathscr{H}_{f,A}$; see Proposition 4.1.2. Furthermore, only finitely many fixed points of g_f can correspond to the same Thurston map up to combinatorial equivalence (see the proof of Theorem 4.1.7). The desired result then follows because the map g_f has infinitely many fixed points when f is transcendental; see Proposition 4.0.1 and Lemma 4.1.5.

Similar connections between the fixed points of the map g_f and Thurston maps within the Hurwitz class $\mathscr{H}_{f,A}$ are already established in the context of finite-degree Thurston maps (cf. [Koc13, Propositions 4.3 and 4.4] and [KPS16, Theorem 9.1]). However, their extensions to the setting of transcendental Thurston maps are novel contributions. Additionally, as an application of Main Theorem B, we can obtain the following result regarding the structure of parameter spaces of finite-type meromorphic maps (see Definition 4.1.8):

Corollary 1.3.1 (= Corollary 4.1.9)**.** *Let $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic map such that $|S_g| \leqslant 3$. Then its parameter space $\mathrm{Par}(g)$ contains infinitely many pairwise (topologically or conformally) non-conjugate psf maps with four postsingular values.*

1.3.2 Thurston maps with extra marked points

In this subsection, we present our main results on the second topic, which is closely related to but distinct from the discussion of Section 1.3.1.

Let $f\colon (S^2, A) \circlearrowleft$ be a marked Thurston map, with finite or infinite degree, which may have an at most countable closed set of essential singularities. Suppose that $B \subset A$ is an f-pseudo-invariant subset containing the singular set S_f. Then, we can also consider the Thurston map $f\colon (S^2, B) \circlearrowleft$. In the second part of this thesis, we study how the dynamical properties of the Thurston map $f\colon (S^2, A) \circlearrowleft$ are related to those of $f\colon (S^2, B) \circlearrowleft$. For instance, it is clear that if $f\colon (S^2, A) \circlearrowleft$ is realized, then $f\colon (S^2, B) \circlearrowleft$ must also be realized, but the converse does not necessarily hold. The main result of this paper provides sufficient and necessary condition for when this happens and can be seen as a "relative" analog of Thurston's theorem.

Main Theorem C (= Theorem 5.2.5)**.** *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree. Suppose that $B \subset A$ is an f-pseudo-invariant set containing S_f. The Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized if and only if it has no degenerate Levy multicurve consisting of curves that are non-essential in $S^2 - B$, and $f\colon (S^2, B) \circlearrowleft$ is realized.*

In other words, if $f\colon (S^2, B) \circlearrowleft$ is realized, it is enough to verify that $f\colon (S^2, A) \circlearrowleft$ has no degenerate Levy multicurve consisting of curves that are non-essential in $S^2 - B$ to conclude that $f\colon (S^2, A) \circlearrowleft$ is also realized.

Here, a collection $\mathcal{G} = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$ of essential simple closed curves in $S^2 - A$ is called a *Levy cycle* for the Thurston map $f\colon (S^2, A) \circlearrowleft$ if there exists a cyclic permutation σ of $\{1, 2, \dots, r\}$ so that for every i there is a simple closed curve $\widetilde{\gamma}_i \in f^{-1}(\gamma_i)$ that is homotopic in $S^2 - A$ to $\gamma_{\sigma(i)}$ and $f|\widetilde{\gamma}_i\colon \widetilde{\gamma}_i \to \gamma_i$ is a homeomorphism. A Levy cycle $\mathcal{G}$ is *degenerate* if each γ_i bounds an open Jordan region D_i so that for every i some connected component $\widetilde{D}_i$ of $f^{-1}(D_i)$ is homotopic in $S^2 - A$ to $D_{\sigma(i)}$ and maps homeomorphically by the map f to D_i. Finally, Levy cycle $\mathcal{G}$ is called a *Levy multicurve* if all curves in $\mathcal{G}$ are pairwise disjoint and pairwise non-homotopic in $S^2 - A$; see Section 2.2.4 for further discussion. We note that in the case when $r = 1$, the notions of a Levy cycle and Levy multicurve coincide with the notion of a Levy fixed curve.

Main Theorem C addresses the following three cases, which often represent entirely different scenarios, for the map f in a unified way:

1. f is a finite degree map that is not a $(2,2,2,2)$-map;
2. f is a $(2,2,2,2)$-map;
3. f is an infinite degree map.

In scenario (1), the Main Theorem C can be derived as a direct consequence of Thurston's characterization criterion for marked Thurston map. However, for the other two cases, Thurston's result does not apply, and we have to address the challenges discussed in Section 1.2.2.1.

It is worth noting that Main Theorem C does not im fpose any assumptions on the function-theoretic properties of the considered Thurston maps, nor on the size of their postsingular or marked sets, which differs from other results in the same spirit, for instance, Main Theorems A and B or results of [HSS09] and [She22].

For simplicity, we will outline how to derive the existence of a Levy cycle for the obstructed Thurston map $f\colon (S^2, A) \circlearrowleft$ under assumption that $f\colon (S^2, B) \circlearrowleft$ is realized in the context of Main Theorem C. With further efforts, the complete version of Main Theorem C can be established.

We begin by considering the case where the set $C := A - B$ consists of a single fixed point c of the map f. Suppose that the Thurston map $f\colon (S^2, B) \circlearrowleft$ is realized by a postsingularly finite holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$, and let $\mu \in \mathcal{T}_B$ be the corresponding fixed point of the pullback map $\sigma_{f,B}$ defined on the Teichmüller space $\mathcal{T}_B$. There exists a holomorphic submersion $t_{A,B}\colon \mathcal{T}_A \to \mathcal{T}_B$ naturally induced by forgetting the marked point c. It is straightforward to show that the fiber $\Delta := t_{A,B}^{-1}(\mu)$ is an $\sigma_{f,A}$-invariant subset of the Teichmüller space $\mathcal{T}_A$. Moreover, Δ is a holomorphically and properly embedded unit disk in $\mathcal{T}_A$. In other words, although the pullback map $\sigma_{f,A}$ is defined on the $(|A| - 3)$-complex-dimensional manifold $\mathcal{T}_A$, it admits a one-complex-dimensional invariant subset, which allows us to use powerful tools

of one-dimensional complex dynamics and hyperbolic geometry to better understand the dynamical behavior of $\sigma_{f,A}$. Furthermore, the restriction $\sigma_{f,A}|\Delta\colon \Delta \to \Delta$ exhibits particularly well-behaved properties. Specifically, $\sigma_{f,A}|\Delta$ admits a commutative diagram similar to diagram (1.1), where the moduli space is replaced by a finitely punctured Riemann sphere, the analogue of the Y-map is a covering map of either finite or infinite degree, and the analogue of the X-map is injective; see Proposition 5.1.3. By applying similar reasoning as in the proof of Main Theorem A explained in Section 1.3.1.1, we conclude that the Thurston map $f\colon (S^2, A) \circlearrowleft$ is obstructed if and only if it has a Levy cycle.

These ideas also extend to the case when the set $C = B - A$ consists of several fixed points of the map f. We say that a point $c \in A$ is a *trivial marked point* of the Thurston map $f\colon (S^2, A) \circlearrowleft$ if $c \in A - P_f$ and either c is strictly pre-periodic, or it eventually lands to an essential singularity of the map f after several iterations (this includes the case when c is already an essential singularity of f). In other words, a marked point $c \in A - P_f$ is non-trivial if it is periodic under the map f. In Section 5.1.1, we demonstrate that extending the marked set by adding trivial marked points essentially does not change the properties of the Thurston map or the corresponding pullback map, allowing us to ignore them in the context of the proof of Main Theorem C. Therefore, from this point forward, we assume that every point $c \in C$ is periodic under the map f.

Given the previous assumptions, we can find an iterate $f^{\circ m}\colon (S^2, A) \circlearrowleft$ of the original Thurston map such that every point $c \in C$ becomes fixed, which allows us to apply previous considerations to the map $f^{\circ m}$. It is straightforward to see that the existence of a Levy cycle for the Thurston map $f^{\circ m}\colon (S^2, A) \circlearrowleft$ also implies the existence of one for the original Thurston map $f\colon (S^2, A) \circlearrowleft$. However, to complete the argument, we must show that $f^{\circ m}\colon (S^2, A) \circlearrowleft$ is obstructed if $f\colon (S^2, A) \circlearrowleft$ is also obstructed. This is a trivial result for Thurston maps that are not of $(2,2,2,2)$-type because of the contraction properties of the corresponding pullback maps discussed in Section 1.2.2.1. However, for the family of marked Thurston $(2,2,2,2)$-maps, this conclusion is non-trivial. Nevertheless, we confirm the positive answer, which finishes the proof of Main Theorem C; see Section 5.2.2, particularly Proposition 5.2.3.

In conclusion, we mention that the result of Main Theorem C is known for finite degree marked Thurston maps; see [SY15, Main Theorem II] and [BD21, Theorem E]. However, our approach differs significantly from these two works, and most crucially, applies to the infinite degree case.

1.4 Organization of the thesis

We now present the structure of this thesis, which is primarily based on the author's papers [Pro24a] and [Pro24b]. Chapter 2 mostly provides an overview of general background. Chapters 3, 4, and 5 entirely consist of new results. Specifically, Chapters 3 and 4 are broadly aligned with the content of [Pro24a], and Chapter 5 corresponds to [Pro24b], with some dependence on the results from Chapter 3. In Chapter 6, we

outline further questions and potential research directions stemming from the work discussed in this thesis.

In Chapter 2, we establish framework and provide a summary of known results. Section 2.1 introduces the notation and basic concepts. Section 2.1.2 focuses on topologically holomorphic maps. Moving forward, Section 2.2 introduces Thurston maps, covering both finite and infinite degree cases and maps with several essential singularities. This section roughly corresponds to Sections 1.2.1 and 1.2.2 from the introduction. In particular, Section 2.2.1 establishes key notions for Thurston maps, including combinatorial and Hurwitz equivalence, realizability, and Hurwitz classes. Section 2.2.2 reviews essential background on Teichmüller and moduli spaces, and in Section 2.2.3, we show how these concepts can be applied to study Thurston maps, with a specific focus on pullback maps and their core properties. Section 2.2.4 discusses particular type of Thurston obstructions — Levy cycles and Levy fixed curves. Lastly, Section 2.2.5 recalls Thurston's characterization of rational maps and several important corollaries from it.

Chapter 3 presents several results based on the application of hyperbolic geometry. In Section 3.1, we demonstrate how hyperbolic geometry and Teichmüller theory can be used to study obstructions for Thurston maps. We begin by proving that a Levy cycle creates an obstruction for any Thurston map, whether of finite or infinite degree, to be realized. Section 3.1.1 focuses on one of the simplest types of obstructions: Levy fixed curves. Then, in Section 3.1.2, we extend obtained results to general Levy cycles. In Section 3.2, we investigate dynamics of certain holomorphic self-maps of the unit disk, which will naturally arise later in the context of pullback maps. First, we provide several estimates related to hyperbolic contraction in Section 3.2.1, and then apply these estimates to dynamical questions in Section 3.2.2, with a particular emphasis on proving Theorem 3.2.6.

Chapter 4 corresponds to Section 1.3.1 of the introduction and focuses on Thurston maps with four marked points that meet conditions (A)–(C). We begin by studying the associated pullback maps on the one-dimensional Teichmüller space in Section 4.0.1. Next, in Section 4.1.1, we present Main Theorem A, providing a Thurston-like characterization for this class of maps. Section 4.1.2 investigates the related Hurwitz classes, particularly focusing on Main Theorem B and Corollary 4.1.9. Notably, Sections 4.1.1 and 4.1.2 correspond to Sections 1.3.1.1 and 1.3.1.2 from the introduction. Finally, Section 4.1.3 provides several examples of families of Thurston maps that satisfy conditions (A)–(C) and demonstrates how the developed framework applies to these rather concrete cases.

Finally, Chapter 5 corresponds to Section 1.3.2 of the introduction. In this chapter, we study relationships between properties of marked Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$, where $B \subset A$. We start by treating preliminary cases: when the set $A - B$ consists of trivial marked points, and when it consists of a single fixed point of the map f, see Sections 5.1.1 and 5.1.2, respectively. In Section 5.2, we explore more elaborate scenarios. In Section 5.2.1, we analyze the relationships between the fixed points of the maps $\sigma_{f,A}$ and $\sigma_{f,B}$. Section 5.2.2 highlights the connections between a Thurston map and its iterates. Finally, in Section 5.2.3, we present Main Theorem C.

In Chapter 6, we provide further questions and potential research directions arising from the work discussed in this thesis. This chapter provides a roadmap for future projects that could build on and extend the findings and contributions presented here.

2 Preliminaries

Summary

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In this section, we begin by establishing key notations and reviewing fundamental concepts, including topologically holomorphic maps in Section 2.1.2. Next, in Section 2.2, we present an overview of general Thurston theory. Section 2.2.1 introduces the notion of Thurston maps, extending to those of infinite degree and having several essential singularities. We demonstrate how concepts of the classical Thurston theory for finite degree maps (such as combinatorial or Hurwitz equivalence) can be generalized to this broader context. Section 2.2.2 provides the necessary background on Teichmüller and moduli spaces. In Section 2.2.3, we explore the connections between Thurston and Teichmüller theories. Section 2.2.4 is dedicated to discussing a family of obstructions that prevent a Thurston map from having a holomorphic realization. Lastly, in Section 2.2.5, we review significant results in the context of finite-degree Thurston maps, including Thurston's characterization criterion.

2.1 Background

2.1.1 Notation and basic concepts

The sets of (strictly) positive integers, non-zero integers, integers, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}^*$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. We use the notation $\mathbb{I} := [0,1]$ for the closed unit interval on the real line, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ for the open unit disk in the complex plane, $\mathbb{D}^* := \mathbb{D} - \{0\}$ for the punctured unit disk, $\mathbb{C}^* := \mathbb{C} - \{0\}$ for the punctured complex plane, $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ for the upper half-plane, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, and Σ for the three-punctured Riemann sphere

$\widehat{\mathbb{C}} - \{0,1,\infty\}$. The open and closed disks in $\mathbb{C}$ of radius $r > 0$ centered at 0 are denoted by $\mathbb{D}_r$ and $\overline{\mathbb{D}}_r$, respectively. For a complex number $z \in \mathbb{C}$, $\arg(z) \in [0, 2\pi)$ and $|z|$ represent its argument and absolute value, respectively. Lastly, for $m, n \in \mathbb{N}$, the greatest common divisor of m and n is denoted by $\gcd(m,n)$.

We denote the oriented 2-dimensional plane and sphere by $\mathbb{R}^2$ and S^2, respectively. In this paper, we treat them as purely topological objects. In particular, our convention is to write $g\colon \mathbb{C} \to \mathbb{C}$, $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$, or $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ to indicate that g is holomorphic, and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f\colon \mathbb{R}^2 \to S^2$, or $f\colon S^2 \to S^2$ if f is only continuous. The same rule applies to the notation $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ and $f\colon S^2 \dashrightarrow S^2$ (see Section 2.1.2 for the details).

The cardinality of a set A is denoted by $|A|$ and the identity map on A by id_A. If $f\colon U \to V$ is a map and $W \subset U$, then $f|W$ stands for the restriction of f to W. If U is a topological space and $W \subset U$, then $\overline{W}$ denotes the closure and ∂W the boundary of W in U.

A subset D of S^2 is called an *open* or *closed Jordan region* if there exists an injective continuous map $\varphi\colon \overline{\mathbb{D}} \to S^2$ such that $D = \varphi(\mathbb{D})$ or $D = \varphi(\overline{\mathbb{D}})$, respectively. In this case, $\partial D = \varphi(\partial\mathbb{D})$ is a simple closed curve in $\widehat{\mathbb{C}}$. A domain $U \subset \widehat{\mathbb{C}}$ is called an *annulus* if $\widehat{\mathbb{C}} - U$ has two connected components. The *modulus* of an annulus U is denoted by $\mathrm{mod}(U)$ (see [Hub06, Proposition 3.2.1] for the definition).

Let U and V be topological spaces. A continuous map $H\colon U \times \mathbb{I} \to V$ is called a *homotopy* from U to V. When $U = V$, we simply say that H is a homotopy *in* U. Given a homotopy $H\colon U \times \mathbb{I} \to V$, for each $t \in \mathbb{I}$, we associate the *time-t map* $H_t := H(t,\cdot)\colon U \to V$. Sometimes it is convenient to think of the homotopy H as a continuous family of its time maps $(H_t)_{t\in\mathbb{I}}$. The homotopy H is called an (*ambient*) *isotopy* if the map $H_t\colon U \to V$ is a homeomorphism for each $t \in \mathbb{I}$. Suppose A is a subset of U. An isotopy $H\colon U \times \mathbb{I} \to V$ is said to be an isotopy *relative to* A (abbreviated "H is an isotopy rel. A") if $H_t(p) = H_0(p)$ for all $p \in A$ and $t \in \mathbb{I}$.

Given $M, N \subset U$, we say that M *is homotopic* (*in* U) to N if there exists a homotopy $H\colon U \times \mathbb{I} \to U$ with $H_0 = \mathrm{id}_U$ and $H_1(M) = N$. Two homeomorphisms $\varphi_0, \varphi_1\colon U \to V$ are called *isotopic* (*rel.* $A \subset U$) if there exists an isotopy $H\colon U \times \mathbb{I} \to V$ (rel. A) with $H_0 = \varphi_0$ and $H_1 = \varphi_1$.

We assume that every topological surface X is oriented. We denote by $\mathrm{Homeo}^+(X)$ and $\mathrm{Homeo}^+(X,A)$ the group of all orientation-preserving self-homeomorphisms of X and the group of all orientation-preserving self-homeomorphisms of X fixing A pointwise, respectively. We use the notation $\mathrm{Homeo}_0^+(X,A)$ for the subgroup of $\mathrm{Homeo}^+(X,A)$ consisting of all homeomorphisms isotopic rel. A to id_X.

Lemma 2.1.1. *If A is a subset of S^2 containing at most three points, then any two orientation-preserving homeomorphisms $\varphi\colon S^2 \to S^2$ and $\psi\colon S^2 \to S^2$ are isotopic rel. A if and only if $\varphi|A = \psi|A$. In particular,* $\mathrm{Homeo}^+(S^2, A) = \mathrm{Homeo}_0^+(S^2, A)$.

Proof. This lemma follows directly from [FM12, Proposition 2.3]. □

Further, we will require the following two topological facts. Here, $\mathrm{Isol}(X)$ denotes the set of isolated points of the topological space X.

Lemma 2.1.2. *If E is at most countable closed set of S^2, then E coincides with the closure of its isolated points, i.e., $E = \overline{\mathrm{Isol}(E)}$, and its complement $S^2 - E$ is path-connected.*

Proof. Let $E - \mathrm{Isol}(E) = \{e_1, e_2, \ldots, e_n, \ldots\}$, and define $E_i := E - \{e_i\}$ for each $i \in \mathbb{N}$. Clearly, each E_i is open and dense in E. Therefore, by the Baire category theorem, $\mathrm{Isol}(E) = \bigcap_{i\in\mathbb{N}} E_i$ is dense in E.

The set $S^2 - E$ is path-connected because, for any two points $e_1, e_2 \in S^2 - E$, there are uncountably many arcs in S^2 connecting e_1 to e_2 and intersecting each other only at their endpoints. □

Lemma 2.1.3. *Let $\varphi\colon X \to Y$ be a homeomorphism, where $X = S^2 - E_X$ and $Y = S^2 - E_Y$, with E_X and E_Y being at most countable closed sets of S^2. Then φ has a unique continuous extension to S^2. Moreover, this extension is a self-homemorphism of S^2 with $\varphi(E_X) = E_Y$.*

If $X = \widehat{\mathbb{C}} - E_X$ and $Y = \widehat{\mathbb{C}} - E_Y$, where E_X and E_Y are at most countable closed sets of $\widehat{\mathbb{C}}$, and $\varphi\colon X \to Y$ is a biholomorphism, then φ extends to a Möbius transformation.

Proof. Let $p \in E_X$. We can select a sequence of closed Jordan regions (U_n) such that $p \in U_n$, $U_{n+1} \subset U_n$, and $\partial U_n \subset X$ for every $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} U_n = \{p\}$. Let $V_n := \overline{\varphi(U_n - X)}$ be a closed Jordan region with $\partial V_n = \varphi(\partial U_n)$. Denote $V := \bigcap_{n\in\mathbb{N}} V_n$.

Claim. The set V is a singleton.

Proof. Note that V is non-empty due to Cantor's intersection lemma. Next, we show that V is connected. Suppose, for the sake of contradiction, that $V = V' \cup V''$, where V' and V'' are both open and closed sets of V, and $V' \cap V'' = \varnothing$. Since V is closed in S^2, then V' and V'' must also be closed subsets of S^2. Since S^2 is a normal topological space, we can find two open sets $W' \supset V'$ and $W'' \supset V''$ such that $W' \cap W'' = \varnothing$.

Denote $W := W' \cup W''$. We aim to show that $V_n \subset W$ for sufficiently large n. Suppose, on the contrary, that it is not the case. Then the compact set $W_n := V_n - W$ is non-empty for every $n \in \mathbb{N}$. Since, $W_{n+1} \subset W_n$, Cantor's intersection lemma guarantees the existence a point $q \in \bigcap_{n\in\mathbb{N}} W_n$. However, it contradicts to the fact that $\bigcap_{n\in\mathbb{N}} W_n = V - W = \varnothing$. Therefore, $V_n \subset W$ for sufficiently large n. But then V_n can be written as a disjoint union of two open (in V_n) non-empty sets $V_n \cap W'$ and $V_n \cap W''$, and we obtain a contradiction with the fact that V_n is connected. Thus, V is a connected set.

Suppose there exists a point $q \in V \cap Y$. Then the point $\varphi^{-1}(q) \in X$ belongs to U_n for all $n \in \mathbb{N}$, which is impossible. Hence, $V \subset E_Y$. But E_Y is totally disconnected, and therefore, V must be a singleton. ■

Now let us define $\varphi(p)$ by setting $\varphi(p) := q$, where $\{q\} = V = \bigcap_{n\in\mathbb{N}} V_n$, and apply the same procedure to every point $p \in E_X$. It is straightforward to verify that this construction is independent of the choice of the sequence (U_n). Indeed, it easily follows from the fact that if D is a neighborhood of p, then $U_n \subset D$ for all sufficiently large n. At the same time, one can prove that φ is continuous on S^2 by showing that $\lim_{n\to\infty} f(p_n) = f(p)$ for any sequence (p_n) of points in S^2 converging to $p \in S^2$. Clearly, this is the unique continuous extensions of φ to S^2.

Suppose there are two points $p_1, p_2 \in S^2$ such that $f(p_1) = f(p_2)$. We can pick two disjoint open Jordan regions D_1 and D_2 such that $p_i \in D_i$ and $\partial D_i \subset X$ for $i = 1,2$. In this case, $f(D_1)$ and $f(D_2)$ should intersect at uncountably many points of S^2, and that would imply that the original map was not injective on X. Thus, φ is injective on S^2. Now, let $q \in S^2$ be an arbitrary point and (q_n) be a sequence of points in Y converging to q. Then the sequence $(\varphi^{-1}(q_n))$ has a converging subsequence in S^2. Without loss of generality, we assume that the entire sequence $(\varphi^{-1}(q_n))$ converges to $p \in S^2$. Since $\varphi\colon S^2 \to S^2$ is continuous, it follows that $\varphi(p) = q$. Thus, $\varphi\colon S^2 \to S^2$ is surjective and, because S^2 is both compact and Hausdorff, φ is a self-homeomorphism of S^2. Obviously, $\varphi(E_X) = E_Y$.

Now, suppose that $\varphi\colon X \to Y$ is a biholomorphism. As discussed earlier, it has a unique extension to a self-homeomorphism of $\widehat{\mathbb{C}}$. By post-composing φ with a Möbius transformation, we can assume that it restricts to a self-homeomorphism of $\mathbb{C}$ that is holomorphic outside of at most countable closed set. Then it can be shown, using Morera's theorem [Con95, Theorem 5.10], that φ must be holomorphic everywhere on $\widehat{\mathbb{C}}$. We leave this verification to the reader. □

2.1.2 Topologically holomorphic maps

In this section we recall the definition of a topologically holomorphic map and some of its basic properties (for other discussions see [LP20] and [MPR24, Section 2.3]).

Definition 2.1.4. Let X and Y be two connected topological surfaces. A non-constant map $f\colon X \to Y$ is called *topologically holomorphic* if it satisfies one of the following four equivalent conditions:

1. for every $p \in X$ there exist $d \in \mathbb{N}$, a neighborhood U of x, and two orientation-preserving homeomorphisms $\psi\colon U \to \mathbb{D}$ and $\varphi\colon f(U) \to \mathbb{D}$ such that $\psi(p) = \varphi(f(p)) = 0$ and $(\varphi \circ f \circ \psi^{-1})(z) = z^d$ for all $z \in \mathbb{D}$;

2. f is continuous, open, *discrete* (i.e., $f^{-1}(q)$ is discrete in X for very $q \in Y$), and for every $p \in X$ such that f is locally injective at p, there exists a neighborhood U of p for which $f|U\colon U \to f(U)$ is an orientation-preserving homeomorphism;

3. there exist Riemann surfaces S_X and S_Y and orientation-preserving homeomorphisms $\varphi\colon Y \to S_Y$ and $\psi\colon X \to S_X$ such that $\varphi \circ f \circ \psi^{-1}\colon S_X \to S_Y$ is a holomorphic map;

4. for every orientation-preserving homeomorphism $\varphi\colon Y \to S_Y$, where S_Y is a Riemann surface, there exist a Riemann surface S_X and an orientation-preserving homeomorphism $\psi\colon X \to S_X$ such that $\varphi \circ f \circ \psi^{-1}\colon S_X \to S_Y$ is a holomorphic map.

Remark 2.1.5. • In condition (1) of Definition 2.1.4, the integer $\deg(f,x) := d$ is independent of U, ψ and φ and is called the *local degree* of f at x.

- In condition (2) of Definition 2.1.4, the requirement of f being discrete can be replaced by the condition that f is *light* by [LP20, Theorem 1.1]. Here, the map $f\colon X \to Y$ is called light if $f^{-1}(q)$ is totally disconnected in X for every $q \in Y$.

- In condition (4) of Definition 2.1.4, the pair (ψ, S_X) is uniquely determined up to the following equivalence: $(\psi, S_X) \sim (\widetilde{\psi}, \widetilde{S}_X)$ if there exists a biholomorphism $\theta\colon S_X \to \widetilde{S}_X$ such that $\widetilde{\psi} = \theta \circ \psi$. In particular, the conformal type of the surface S_X is uniquely determined by the homeomorphism $\varphi\colon Y \to S_Y$. If φ and the surface S_X are fixed, then the homeomorphism $\psi\colon X \to S_X$ is uniquely determined up to post-composition with a biholomorphism of S_X.

Several classical notions of one-complex-dimensional function theory can be extended to the setting of topologically holomorphic maps.

Definition 2.1.6. Let X and Y be connected topological surfaces and $f\colon X \to Y$ be a topologically holomorphic map.

1. A point $y \in Y$ is a *regular value of* f if there exists a neighbourhood U of y such that $f^{-1}(U) = \bigsqcup_{i\in I} U_i$ and $f|U_i\colon U_i \to U$ is a homeomorphism for each $i \in I$. If such a neighborhood U does not exist, we call y a *singular value of* f. The set of all singular values of f is denoted by S_f. If S_f is finite, then we say that f is a map *of finite type* or f belongs to the *Speiser class* $\mathcal{S}$.

2. We say that $x \in X$ is a *critical point* of f if $\deg(f, x) > 1$. A point $y \in Y$ is called a *critical value* of f if there exists a critical point $x \in X$ such that $f(x) = y$.

3. We say that $y \in Y$ is an *asymptotic value* of f if there exists a continuous curve $\gamma\colon [0, \infty) \to X$ such that γ leaves every compact subset of X and $f(\gamma(t)) \to y$ as $t \to \infty$.

Remark 2.1.7. Suppose we are in the setting of Definition 2.1.6. Then:

- Critical and asymptotic values of the topologically holomorphic map $f\colon X \to Y$ are always singular;

- The restriction
$$f|X - f^{-1}(S_f)\colon X - f^{-1}(S_f) \to Y - S_f$$
is a covering map.

It is possible to derive the following isotopy lifting property for finite-type topologically holomorphic maps (cf. [ER15, Propostion 2.3]).

Proposition 2.1.8. *Let $f\colon X \to Y$ and $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ be topologically holomorphic maps of finite type, where X, Y, $\widetilde{X}$, and $\widetilde{Y}$ are connected topological surfaces. Suppose that $\varphi_0 \circ f = \widetilde{f} \circ \psi_0$ for some orientation-preserving homeomorphisms $\varphi_0\colon Y \to \widetilde{Y}$ and $\psi_0\colon X \to \widetilde{X}$. Let $A \subset Y$ be a finite set containing S_f and $\varphi_1\colon Y \to \widetilde{Y}$ be an orientation-preserving homeomorphism isotopic rel. A to φ_0. Then $\varphi_1 \circ f = \widetilde{f} \circ \psi_1$ for some orientation-preserving homeomorphism $\psi_1\colon X \to \widetilde{X}$ isotopic rel. $f^{-1}(A)$ to ψ_0.*

Proof. Let $(\varphi_t)_{t\in\mathbb{I}}$ be the corresponding isotopy. From the definition of a singular value, it follows that the restrictions $\varphi_t \circ f|Z\colon Z \to T$ are covering maps for each $t \in \mathbb{I}$, where $Z := X - f^{-1}(A)$ and $T := Y - A$. Therefore, [ABF21, Lemma 2.7] implies the existence of an isotopy $(\phi_t)_{t\in\mathbb{I}}$ in Z such that $\varphi_t \circ f = \varphi_0 \circ f \circ \phi_t$ for each $t \in \mathbb{I}$. Each homeomorphism $\phi_t\colon Z \to Z$ extends to a self-homeomorphism of the entire surface X since all points of the set $X - Z = f^{-1}(A)$ are isolated. Moreover, the homotopy $(\phi_t)_{t\in\mathbb{I}}$ can be viewed as an isotopy in X rel. $f^{-1}(A)$.

At the same time, $\varphi_0 \circ f = \varphi_0 \circ f \circ \phi_0$ and, therefore, we have the following:

$$\varphi_1 \circ f = \varphi_0 \circ f \circ \phi_1 = (\varphi_0 \circ f \circ \phi_0) \circ \phi_0^{-1} \circ \phi_1 = \varphi_0 \circ f \circ \phi_0^{-1} \circ \phi_1 = \widetilde{f} \circ (\psi_0 \circ \phi_0^{-1} \circ \phi_1).$$

Thus, we can set $\psi_1 := \psi_0 \circ \phi_0^{-1} \circ \phi_1$, and $(\psi_0 \circ \phi_0^{-1} \circ \phi_t)_{t\in\mathbb{I}}$ provides the required isotopy rel. $f^{-1}(A)$. Finally, ψ_1 is orientation-preserving since f and $\widetilde{f}$ are local orientation-preserving homeomorphisms outside the sets of their critical points. □

We say that a topological surface X is a *countably-punctured sphere*, if $X = S^2 - E$, where E is a closed subset of S^2 that is at most countable. It is worth noting that we allow the set E to be finite or even empty. In this paper, we mainly focus on topologically holomorphic maps $f\colon X \to S^2$ defined on a countably-punctured sphere X. Additionally, we always assume that f cannot be extended as a topologically holomorphic map to a neighborhood of any of the points of the set $S^2 - X$. For simplicity, we will use the notation $f\colon S^2 \dashrightarrow S^2$ in order to indicate that f might not be defined on an at most countable closed set of S^2. Similar to the holomorphic case, any point $e \in S^2 - X$ is referred as an *essential singularity* of the map f, which is *isolated* if it is an isolated point of the set $S^2 - X$. We say that a topologically holomorphic map $f\colon S^2 \dashrightarrow S^2$ is *transcendental* if it has an essential singularity. Based on our earlier assumptions about f, this is equivalent to saying that f has infinite *topological degree*, which we will simply refer to as the degree from this point onward. Likewise, for a holomorphic map g defined everywhere on $\widehat{\mathbb{C}}$, potentially excluding at most countable closed set of singularities, we use the notation $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$, and we say that $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic.

Let $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ be an orientation-preserving-homeomorphism and $f\colon X \to S^2$ be a topologically holomorphic map, where X is a countably-punctured sphere. According to item (4) of Definition 2.1.4, there exists an orientation-preserving homeomorphism $\psi\colon X \to R$, where R is a connected Riemann surface, so that the map $\varphi \circ f \circ \psi^{-1}\colon R \to \widehat{\mathbb{C}}$ is holomorphic. Moreover, according to Remark 2.1.5, the conformal type of R is uniquely determined for a given homeomorphism φ.

Definition 2.1.9. We define f as a topologically holomorphic map of *admissible type* if R, as described above, is biholomorphic to a *countably-punctured Riemann sphere* for some orientation-preserving homeomorphism $\varphi\colon S^2 \to \widehat{\mathbb{C}}$.

In fact, the following proposition shows that Definition 2.1.9 is independent of the choice of φ if we assume that f is a topologically holomorphic maps of finite type.

Proposition 2.1.10. *Suppose that $f\colon X \to S^2$ is a finite-type topologically holomorphic map, where X is a countably-punctured sphere. If f is of admissible type, then for every orientation-preserving homeomorphism $\varphi\colon S^2 \to \widehat{\mathbb{C}}$, there exists an orientation-preserving homeomorphism $\psi\colon X \to R$ such that R is a countably-punctured Riemann sphere and the map $\varphi \circ f \circ \psi^{-1}\colon R \to \widehat{\mathbb{C}}$ is holomorphic. Moreover, such ψ is unique up to post-composition with a Möbius transformation.*

Proof. Since f has an admissible type, there exist two orientation-preserving homeomorphisms $\varphi_1\colon S^2 \to \widehat{\mathbb{C}}$ and $\psi_1\colon X \to \widehat{\mathbb{C}} - E$, where E is an at most countable closed subset of $\widehat{\mathbb{C}}$, such that the map $g_1 := \varphi_1 \circ f \circ \psi_1^{-1}\colon \widehat{\mathbb{C}} - E \to \widehat{\mathbb{C}}$ is holomorphic. Now let $\varphi_2\colon S^2 \to \widehat{\mathbb{C}}$ be any other orientation-preserving homeomorphism. By item (4) of Proposition 2.1.4, there exists a connected Riemann surface R and an orientation-preserving homeomorphism $\psi_2\colon X \to R$ so that the map $g_2 := \varphi_2 \circ f \circ \psi_2^{-1}\colon R \to \widehat{\mathbb{C}}$ is holomorphic. In particular, $g_2 = \widetilde{\varphi} \circ g_1 \circ \widetilde{\psi}^{-1}$ on R, where $\widetilde{\varphi} := \varphi_2 \circ \varphi_1^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and $\widetilde{\psi} := \psi_2 \circ \psi_1^{-1}\colon \widehat{\mathbb{C}} - E \to R$ are orientation-preserving homeomorphisms. Note that $\widetilde{\varphi}$ is isotopic rel. S_f to some *diffeomorphism* of $\widehat{\mathbb{C}}$ (see, for instance, [FM12, Theorem 1.13]). According to Proposition 2.1.8, without loss of generality, we can assume that $\widetilde{\varphi}$ is itself a diffeomorphism of $\widehat{\mathbb{C}}$, and therefore, it is a *quasiconformal mapping*. Let $\mu_{\widetilde{\varphi}} := \widetilde{\varphi}^* \mu_0$ be the *Beltrami form* corresponding to $\widetilde{\varphi}$, where μ_0 is the zero Beltrami form on $\widehat{\mathbb{C}}$ (see [BF14, Sections 1.2 and 1.3]). We define another Beltrami form on $\widehat{\mathbb{C}}$ by

$$\mu(z) = \begin{cases} (g_1)^* \mu_{\widetilde{\varphi}}(z), \text{ if } z \in \widehat{\mathbb{C}} - E, \\ 0, \text{ otherwise.} \end{cases}$$

By a standard argument (see, for example, [BF14, Definition 1.34]) involving Measurable Riemann Mapping Theorem [BF14, Theorem 1.28], it is easy to establish the existence of a quasiconformal mapping $\theta \in \mathrm{Homeo}^+(\mathbb{C})$ such that $\mu_\theta = \mu$ almost everywhere on $\widehat{\mathbb{C}}$ and the map $g := \widetilde{\varphi} \circ g_1 \circ \theta^{-1}\colon \widehat{\mathbb{C}} - \theta(E) \to \widehat{\mathbb{C}}$ is holomorphic. Thus, applying Remark 2.1.5 to the map g_1 and pairs $(\widetilde{\psi}, R)$ and $(\theta, \widehat{\mathbb{C}} - \theta(E))$, we conclude that R must be biholomorphic to a countably-punctured Riemann sphere.

The uniqueness of ψ as stated in this proposition can be deduced from Lemma 2.1.3 and Remark 2.1.5. □

Remark 2.1.11. If $f\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is a *quasiregular* topologically holomorphic map of finite type, then by applying the Riemann Measurable Mapping Theorem [BF14, Theorem 1.28], it is straightforward to demonstrate that f is of admissible type.

Let $f\colon X \to S^2$ be a finite-type topologically holomorphic map. We illustrate how the defined concepts apply in the two simplest cases: when X is either the entire sphere S^2 or a once-punctured sphere.

Due to the Uniformization Theorem and item (4) of Definition 2.1.4, in the case when $X = S^2$, a topologically holomorphic map $f\colon X \to S^2$ can be written as $f = \varphi \circ g \circ \psi^{-1}$, where $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a non-constant rational map and $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ are orientation-preserving homeomorphisms. In fact, in this case $f\colon S^2 \to S^2$ is simply a *branched self-covering* of S^2, which is always of finite and admissible types and has finite topological degree.

Similarly, in the case when $X = S^2 - \{e\}$, we can write f as $\varphi \circ g \circ \psi^{-1}$ such that $g\colon R \to \widehat{\mathbb{C}}$ is a non-constant holomorphic map, $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ and $\psi\colon S^2 - \{e\} \to R$ are orientation-preserving homeomorphisms, where $R = \mathbb{C}$ or $R = \mathbb{D}$ by the Unifromization Theorem. Then the image of ψ above does not depend on the choice of the homeomorphism φ, according to Proposition 2.1.10. Thus, finite-type topologically holomorphic maps for which the image of ψ is $\mathbb{C}$ are referred to as *parabolic type* maps, while those for which the image of ψ is $\mathbb{D}$ are called *hyperbolic type* maps. In particular, in this case the notion of admissible type coincides with the concept of parabolic type, which has been extensively studied before in the context of famous *type problem*; see, for example, [Kak53], [Doy84], [Ere04], [Mer13].

The map $f\colon S^2 \dashrightarrow S^2$ is called *entire* if either f has finite topological degree and there exists a point $p \in S^2$ such that $f^{-1}(p) = \{p\}$ (in which case f is called a *topological polynomial*), or f has infinite topological degree and $f^{-1}(e) = \varnothing$, where e is the unique essential singularity of f. We can view entire topologically holomorphic maps as topologically holomorphic self-maps of $\mathbb{R}^2$.

Further, we will focus on finite-type topologically holomorphic maps $f\colon S^2 \dashrightarrow S^2$ of admissible type. These maps serve as topological models for the family of holomorphic maps $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ having finitely many singular values.

The main motivation for introducing maps with countably many essential singularities is that the family of topologically holomorphic maps with at most one (or even finitely many) essential singularities is not closed under composition. Indeed, it is straightforward to construct two such maps, each with a single essential singularity, whose composition has infinitely many of them. However, the class of finite-type topologically holomorphic maps $f\colon S^2 \dashrightarrow S^2$ of admissible type is closed under the operation of composition.

Proposition 2.1.12. *Let $f_1\colon X_1 \to S^2$ and $f_2\colon X_2 \to S^2$ be two finite-type topologically holomorphic maps, where X_1 and X_2 are countably-punctured spheres. Then the composition $f_1 \circ f_2\colon X_2 \cap f_2^{-1}(X_1) \to S^2$ is also a finite-type topologically holomorphic map, with $S_{f_1 \circ f_2} = S_{f_1} \cup f_1(S_{f_2})$. Moreover, if both f_1 and f_2 are of admissible type, then the composition $f_1 \circ f_2\colon S^2 \dashrightarrow S^2$ is of admissible type as well.*

Proof. The first part of this proposition easily follows from Definition 2.1.4 and the definition of a singular value.

Let $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ be an orientation-preserving homeomorphism. Then by Lemma 2.1.3 and Proposition 2.1.10, there exists another orientation-preserving homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $\varphi \circ f \circ \psi^{-1}\colon \psi(X_1) \to \widehat{\mathbb{C}}$ is holomorphic. Applying the same argument to the map f_2 and the homeomorphism ψ, we derive the existence of an orientation-preserving homeomorphism $\theta\colon S^2 \to \widehat{\mathbb{C}}$ such that $\psi \circ f \circ \theta^{-1}\colon \theta(X_2) \to \widehat{\mathbb{C}}$ is holomorphic. Hence,

$$\varphi \circ (f_1 \circ f_2) \circ \theta^{-1} = (\varphi \circ f_1 \circ \psi^{-1}) \circ (\psi \circ f_2 \circ \theta^{-1})\colon \theta(X_2 \cap f_2^{-1}(X_1)) \to \widehat{\mathbb{C}}$$

is holomorphic as well. Obviously, $\theta(X_2 \cap f_2^{-1}(X_1))$ is a countably-punctured Riemann sphere, and therefore, $f_1 \circ f_2$ is of admissible type. □

Let $f\colon S^2 \dashrightarrow S^2$ be a finite-type topologically holomorphic map of admissible type. Then Great Picard's Theorem and Lemma 2.1.2 imply that in any neighborhood of an essential singularity of such a map f, every value is attained infinitely often with at most two exceptions. In particular, f can have at most two *omitted values*, i.e., points p in S^2 such that the preimage $f^{-1}(p)$ is empty. Furthermore, each omitted value is an asymptotic value of f. Additionally, observe that if $A \subset S^2$ is a finite set with $|A| \geqslant 3$ and $S_f \subset A$, then the restriction

$$f|S^2 - \overline{f^{-1}(A)}\colon S^2 - \overline{f^{-1}(A)} \to S^2 - A$$

is a covering map. Note that the closure $\overline{f^{-1}(A)}$ consists of $f^{-1}(A)$ along with all essential singularities of the map f.

Finally, observe that Lemmas 2.1.2 and 2.1.3 allow us to state the following version of Proposition 2.1.8.

Corollary 2.1.13. *Let $f\colon X \to S^2$ and $\widetilde{f}\colon \widetilde{X} \to S^2$ be finite-type topologically holomorphic maps, where X and $\widetilde{X}$ are countably-punctured spheres. Suppose that $\varphi_0 \circ f = \widetilde{f} \circ \psi_0$ for some $\varphi_0, \psi_0 \in \mathrm{Homeo}^+(S^2)$. Let $A \subset S^2$ be a finite set containing S_f and $\varphi_1 \in \mathrm{Homeo}^+(S^2)$ is isotopic rel. A to φ_0. Then $\varphi_1 \circ f = \widetilde{f} \circ \psi_1$ for some $\psi_1 \in \mathrm{Homeo}^+(S^2)$ isotopic rel. $f^{-1}(A) \cup (S^2 - X) \supset \overline{f^{-1}(A)}$ to ψ_0.*

Corollary 2.1.14. *Let $f\colon X \to S^2$ be a finite-type topologically holomorphic map, where X is a countably-punctured topological sphere. Let $A \subset S^2$ be a finite set containing S_f, and suppose that γ_0 is a simple closed curve in $S^2 - A$. Let $\widetilde{\gamma}_0 \subset f^{-1}(\gamma)$ be a simple closed curve with $\deg(f|\widetilde{\gamma}_0\colon \widetilde{\gamma}_0 \to \gamma_0) = d$. If γ_1 is a simple closed curve that is homotopic in $S^2 - A$ to γ_0, then there exists a simple closed curve $\widetilde{\gamma}_1 \subset f^{-1}(\gamma_1)$ such that $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are homotopic in $X - f^{-1}(A) \subset S^2 - \overline{f^{-1}(A)}$ and $\deg(f|\widetilde{\gamma}_1\colon \widetilde{\gamma}_1 \to \gamma_1) = d$.*

Moreover, suppose that D_0 and $\widetilde{D}_0$ are connected components of $S^2 - \gamma_0$ and $S^2 - \widetilde{\gamma}_0$, respectively, such that $f|\widetilde{D}_0\colon \widetilde{D}_0 \to D_0$ is a homeomorphism. Then there exist connected components D_1 and $\widetilde{D}_1$ of $S^2 - \gamma_1$ and $S^2 - \widetilde{\gamma}_1$, respectively, such that D_0 is homotopic in $S^2 - A$ to D_1, $\widetilde{D}_0$ is homotopic in $X - f^{-1}(A) \subset S^2 - \overline{f^{-1}(A)}$ to $\widetilde{D}_1$, and $f|\widetilde{D}_1\colon \widetilde{D}_1 \to D_1$ is a homeomorphism.

Proof. According to [Bus10, Theorem A.3] (see also [FM12, Sections 1.2.5 and 1.2.6]), there exists an isotopy $(\varphi_t)_{t\in\mathbb{I}}$ rel. A in S^2 such that $\varphi_0 = \mathrm{id}_{S^2}$ and $\varphi_1(\gamma_0) = \gamma_1$. Since $\varphi_0 = \mathrm{id}_{S^2}$ is orientation-preserving, then φ_t is also orientation-preserving for each $t \in \mathbb{I}$. By Corollary 2.1.13, there exists a homeomorphism $\psi_1 \in \mathrm{Homeo}^+(S^2, f^{-1}(A) \cup (S^2 - X))$ such that $\varphi_1 \circ f = f \circ \psi_1$. Thus, we can take $\widetilde{\gamma}_1 := \psi_1(\widetilde{\gamma}_0)$. Finally, $X - f^{-1}(A) \subset S^2 - \overline{f^{-1}(A)}$, since any accumulation point $p \in S^2$ of the set $f^{-1}(A)$ cannot be in X.

Similar argument can be applied to derive the second part of this statement. □

2.2 Thurston theory

This section aims to introduce the key concepts, questions, and approaches that are well-established in the classical theory of finite-degree Thurston maps, and extend

them to the more general context of infinite-degree Thurston maps, which may have multiple essential singularities. Additionally, in Section 2.2.5, we revisit the most important results achieved for finite-degree Thurston maps.

2.2.1 Thurston maps

Let $f\colon X \to S^2$ be a topologically holomorphic map, where $X = S^2 - E$ is a countably-punctured sphere. Then we define the orbit $\mathcal{O}_f(p)$ of $p \in S^2$ under f as

$$\mathcal{O}_f(p) := \{q \in S^2 : q = f^{\circ n}(p) \text{ for some } n \geqslant 0\}.$$

It is worth to note that $f^{\circ n}(p)$ might be well-defined only for finitely many n if $f^{\circ m}(p) \in E$ for some $m \geqslant 0$. In this case, we say that the orbit of p *terminates* after m iterations of f.

The *postsingular set* P_f of the map f is defined as the union of all forward orbits of the singular values of f. We say that $f\colon S^2 \dashrightarrow S^2$ is *postsingularly finite* (*psf* in short) if the set P_f is finite, i.e., f has finitely many singular values and each of them eventually becomes periodic or lands on an essential singularity of f under the iteration. Postsingularly finite topologically holomorphic maps of finite degree are also called *postcritically finite* (*pcf* in short), and their *postsingular values* are called *postcritical*, as their singular values are always critical.

Definition 2.2.1. For a topologically holomorphic map $f\colon S^2 \dashrightarrow S^2$, the subset $A \in S^2$ is called *f-pseudo-invariant* if the orbit of each $a \in A$ belongs to the set A, i.e., either $f(a) \in A$ or a is an essential singularity of f.

Every f-invariant set is f-pseudo-invariant and, if the map f has no essential singularities, the converse holds as well. Note that if $|A| \geqslant 3$ and f is a finite-type topologically holomorphic map of admissible type, then the set A is f-pseudo-invariant if and only if $A \subset \overline{f^{-1}(A)}$. The postsingular set P_f provides an example of f-pseudo-invariant set.

Now we are ready to state one of the key definitions of this section.

Definition 2.2.2. A non-injective topologically holomorphic map $f\colon S^2 \dashrightarrow S^2$ is called a *Thurston map* if it is postsingularly finite and of admissible type.

Given a finite f-pseudo-invariant set $A \subset S^2$ such that $P_f \subset A$, we call the pair (f, A) a *marked Thurston map* and A its *marked set*.

We often consider marked Thurston maps in the same way as usual Thurston maps and use the notation $f\colon (S^2, A) \circlearrowleft$ while still assuming that f might not be defined on the entire sphere S^2. If no specific marked set is mentioned, we assume it to be P_f, and the corresponding Thurston map is often called *unmarked* or *minimally marked*.

The dynamics of a Thurston map on its marked set or some other finite subsets of S^2 can also be represented graphically, in a way that turns out to be useful in study. Suppose that $f\colon S^2 \dashrightarrow S^2$ is a Thurston map and $A \subset S^2$ is a finite f-pseudo-invariant subset. Then the *dynamical portrait* of the map f on the set A is a directed

abstract graph with the vertex set A, where for each vertex $v \in A$ that is not an essential singularity of f, there is a unique directed edge from v to $f(v)$, and if $v \in A$ is an essential singularity of f, there are no outgoing edges from v. If the set A coincides with the postsingular set P_f, the dynamical portrait of f on the set A is called the *postsingular portrait* of the Thurston map f.

Definition 2.2.3. Two Thurston maps $f_1 \colon (S^2, A) \circlearrowleft$ and $f_2 \colon (S^2, A) \circlearrowleft$ are called *isotopic* (*rel. A*) if there exists $\phi \in \mathrm{Homeo}_0^+(S^2, A)$ such that $f_1 = f_2 \circ \phi$.

Remark 2.2.4. Let $f_1 \colon (S^2, A) \circlearrowleft$ and $f_2 \colon (S^2, A) \circlearrowleft$ be two Thurston maps satisfying the relation $\phi_1 \circ f_1 = f_2 \circ \phi_2$ for some $\phi_1, \phi_2 \in \mathrm{Homeo}_0^+(S^2, A)$. Then it follows from Corollary 2.1.13 that f_1 and f_2 are isotopic rel. A.

Remark 2.2.5. We note that if $\varphi \in \mathrm{Homeo}^+(\widehat{\mathbb{C}})$ and A is a finite set, then φ is isotopic rel. A to a smooth, and therefore quasiconformal, orientation-preserving homeomorphism of $\widehat{\mathbb{C}}$; see [FM12, Theorem 1.13]. By applying Proposition 2.1.10 and Remark 2.2.4, one can show that every Thurston map is isotopic relative to its marked set to a smooth quasiregular Thurston map. This allows us to focus exclusively on quasiregular Thurston maps within the scope of this thesis. Furthermore, the condition of admissible type is automatically satisfied for these maps, as highlighted in Remark 2.1.11.

The notion of isotopy for Thurston maps depends on their common marked set. Consequently, we sometimes refer to isotopy *relative A* (or rel. A for short) to specify which marked set is being considered. This applies to other notions introduced below that also depend on the choice of the marked set.

We say that two (marked) Thurston maps are *combinatorially equivalent* if they are "topologically conjugate up to isotopy":

Definition 2.2.6. Two Thurston maps $f_1 \colon (S^2, A_1) \circlearrowleft$ and $f_2 \colon (S^2, A_2) \circlearrowleft$ are called combinatorially (or *Thurston*) equivalent if there exist two Thurston maps $\widetilde{f}_1 \colon (S^2, A_1) \circlearrowleft$ and $\widetilde{f}_2 \colon (S^2, A_2) \circlearrowleft$ such that:

- f_i and $\widetilde{f}_i$ are isotopic rel. A_i for each $i = 1, 2$, and
- $\widetilde{f}_1$ and $\widetilde{f}_2$ are conjugate via a homeomorphsim $\phi \in \mathrm{Homeo}^+(S^2)$, i.e., $\phi \circ \widetilde{f}_1 = \widetilde{f}_2 \circ \phi$, such that $\phi(A_1) = A_2$.

Remark 2.2.7. Definition 2.2.6 can be reformulated in a more classical way. Thurston maps $f_1 \colon (S^2, A_1) \circlearrowleft$ and $f_2 \colon (S^2, A_2) \circlearrowleft$ are combinatorially equivalent if and only if there exist two homeomorphisms $\phi_1, \phi_2 \in \mathrm{Homeo}^+(S^2)$ such that $\phi_1(A_1) = \phi_2(A_1) = A_2$, ϕ_1 and ϕ_2 are isotopic rel. A_1, and $\phi_1 \circ f_1 = f_2 \circ \phi_2$.

Remark 2.2.8. If $A_1 = P_{f_1}$ and $A_2 = P_{f_2}$, then the condition that $\phi(A_1) = A_2$ in Definition 2.2.6 and the condition $\phi_1(A_1) = \phi_2(A_1) = A_2$ in Remark 2.2.7 can be removed since they are automatically satisfied if all other conditions hold.

A Thurston map $f\colon (S^2, A) \circlearrowleft$ is said to be *realized* if it is combinatorially equivalent to a postsingularly finite holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$. If $f\colon (S^2, A) \circlearrowleft$ is not realized, we say that it is *obstructed*. One of the central questions in Thurston theory is the *characterization question*, which seeks to determine whether a given Thurston map is realized or obstructed.

We require one more notion of equivalence between Thurston maps.

Definition 2.2.9. We say that two Thurston maps $f_1\colon (S^2, A) \circlearrowleft$ and $f_2\colon (S^2, A) \circlearrowleft$ are *pure Hurwitz equivalent* (or simply *Hurwitz equivalent*) if there exist two homeomorphisms $\phi_1, \phi_2 \in \mathrm{Homeo}^+(S^2, A)$ such that $\phi_1 \circ f_1 = f_2 \circ \phi_2$.

For a more detailed discussion and the motivation behind this concept, refer to [Koc13, Section 2]. If $f\colon (S^2, A) \circlearrowleft$ is a Thurston map, then the *Hurwitz class* $\mathscr{H}_{f,A}$ of f is the union of all Thurston maps with the marked set A that are Hurwitz equivalent to f. If A coincides with the postsingular set P_f of f, we simply use the notation $\mathscr{H}_f$. We say that a Thurston map $f\colon (S^2, A) \circlearrowleft$ is *totally unobstructed* if every Thurston map in $\mathscr{H}_{f,A}$ is realized.

In the following statement we explore the case when the marked set of a Thurston map consists of at most three points.

Proposition 2.2.10. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. The marked set A must contain at least two points. Furthermore:*

1. *if $|A| = 2$, then $f\colon (S^2, A) \circlearrowleft$ is combinatorially equivalent to either $z \mapsto z^d\colon (\widehat{\mathbb{C}}, \{0, \infty\}) \circlearrowleft$ or $z \mapsto z^{-d}\colon (\widehat{\mathbb{C}}, \{0, \infty\}) \circlearrowleft$, where $d := \deg(f)$, and*
2. *if $|A| \leqslant 3$, then $f\colon (S^2, A) \circlearrowleft$ is realized and $\mathscr{H}_{f,A}$ consists of a single Thurston map up to combinatorial equivalence rel. A.*

Proof. Let X be the complement of the set of essential singularities of f. Then the restriction $f|X - f^{-1}(A)\colon X - f^{-1}(A) \to S^2 - A$ is a covering map. If $A = \varnothing$ or $|A| = 1$, this covering map must be a homeomorphism, since $S^2 - A$ is simply connected, and $X - f^{-1}(A)$ would either be S^2 or a once-punctured sphere, respectively. In either case, it is clear that f must be a homeomorphism of S^2, which is impossible for a Thurston map. Thus, the set A contains at least two points.

If $|A| = 2$, classical theory of covering map gives two possibilities: either $X - f^{-1}(A)$ is a once-punctured sphere and the covering $f|X - f^{-1}(A)\colon X - f^{-1}(A) \to S^2 - A$ has infinite degree, or $X - f^{-1}(A)$ is a twice-punctured sphere and the corresponding covering has a finite degree. In the first case, f has an essential singularity, implying that $S^2 - X$ consists of a single point and $f^{-1}(A)$ is empty. However, this leads to a contradiction, as there exists a point $a \in A$ that is not an essential singularity of f and, in particular, $f(a) \in A$.

In the second case, f has a finite degree and two critical values $a, b \in A = S_f = P_f$ with $f^{-1}(\{a, b\}) = \{a, b\}$. By item (4) of Definition 2.1.4, there exist two orientation-preserving homeomorphisms $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ and $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $f = \varphi \circ g \circ \psi^{-1}$, where g is a rational map. Clearly, $|S_g| = 2$, and therefore, we can post-compose

φ and ψ with Möbius transformations, to ensure that $S_g = \{0, \infty\}$, $\varphi(a) = \psi(a) = 0$, and $\varphi(b) = \psi(b) = \infty$. It follows that $g(z) = z^{\pm d}$, where $d = \deg(f)$. According to Lemma 2.1.1, φ and ψ are isotopic rel. A and the rest of item (1) follows.

Similarly, by Lemma 2.1.1, two orientation-preserving homeomorphisms $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ and $\psi\colon S^2 \to \widehat{\mathbb{C}}$ that agree on the set $A \subset S^2$, where $|A| \leqslant 3$, are isotopic rel. A. Just as in the case when $|A| = 2$, this observation implies that any Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized when the marked set A contains three or fewer points, leading to item (2). □

2.2.2 Teichmüller and moduli spaces

Let $A \subset S^2$ be a finite set containing at least three points. Then the *Teichmüller space of the sphere S^2 with the marked set A* is defined as

$$\mathcal{T}_A := \{\varphi\colon S^2 \to \widehat{\mathbb{C}} \text{ is an orientation-preserving homeomorphism}\}/\sim$$

where $\varphi_1 \sim \varphi_2$ if there exists a Möbius transformation M such that φ_1 is isotopic rel. A to $M \circ \varphi_2$; see [Hub16, Section 6.4] for the definition in a more general setting.

Similarly, we define the *moduli space of the sphere S^2 with the marked set A*:

$$\mathcal{M}_A := \{\eta\colon A \to \widehat{\mathbb{C}} \text{ is injective}\}/\approx,$$

where $\eta_1 \approx \eta_2$ if there exists a Möbius transformation M such that $\eta_1 = M \circ \eta_2$; see [FM12, Section 12.1] or [Hub16, Definition 6.4.14].

Further, $[\cdot]_A$ denotes an equivalence class corresponding to a point of either the Teichmüller space $\mathcal{T}_A$ or the moduli space $\mathcal{M}_A$. In situations when the considered marked set is obvious, we simply use the notation $[\cdot]$. Note that there is a natural map $\pi_A\colon \mathcal{T}_A \to \mathcal{M}_A$ defined as $\pi_A([\varphi]) = [\varphi|A]$. According to Lemma 2.1.1, when $|A| = 3$, both the Teichmüller space $\mathcal{T}_A$ and the moduli space $\mathcal{M}_A$ are just single points. Therefore, for the rest of this section, we assume that $|A| \geqslant 4$.

It is known that the Teichmüller space $\mathcal{T}_A$ admits a complete metric d_T, known as the *Teichmüller metric* [Hub06, Proposition 6.4.4]. Moreover, with respect to the topology induced by this metric, $\mathcal{T}_A$ is a contractible space [Hub06, Corollary 6.7.2]. At the same time, both $\mathcal{T}_A$ and $\mathcal{M}_A$ admit structures of $(|A|-3)$-complex manifolds (see [Hub06, Theorem 6.5.1]) so that the map $\pi_A\colon \mathcal{T}_A \to \mathcal{M}_A$ becomes a holomorphic universal covering map [Hub16, Section 10.9].

Moreover, the complex structure of $\mathcal{M}_A$ is quite explicit in the general case. Let $A = \{a_1, a_2, \dots, a_k, a_{k+1}, a_{k+2}, a_{k+3}\}$, $k \geqslant 1$, where the indexing of the points of A is chosen arbitrarily. Define the map $h\colon \mathcal{M}_A \to \mathbb{C}^k - \mathcal{L}_k$ by

$$h([\varphi]) = (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_k)),$$

where the representative $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi(a_{k+1}) = 0, \varphi(a_{k+2}) = 1$, and $\varphi(a_{k+3}) = \infty$, and where $\mathcal{L}_k$ is the subset of $\mathbb{C}^k$ defined by

$$\mathcal{L}_k := \{(z_1, z_2, \dots, z_k) \in \mathbb{C}^k : z_i = z_j \text{ for some } i \neq j, \text{ or } z_i = 0, \text{ or } z_i = 1\}.$$

$$\begin{array}{ccc} \mathcal{T}_A & \xrightarrow{t_{A,B}} & \mathcal{T}_B \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ \mathcal{M}_A & \xrightarrow{m_{A,B}} & \mathcal{M}_B \\ \downarrow{\scriptstyle h_A} & & \downarrow{\scriptstyle h_B} \\ \mathbb{C}^k - \mathcal{L}_k & \xrightarrow{\mathrm{proj}_{k,l}} & \mathbb{C}^l - \mathcal{L}_l \end{array}$$

Figure 2.1: Maps induced by forgetting marked points.

It is known that the map h provides a biholomorphism between $\mathcal{M}_A$ and $\mathbb{C}^k - \mathcal{L}_k$ (see [Hub16, Section 10.9]).

The case $|A| = 4$ is particularly simple. In this situation, the Teichmüller space $\mathcal{T}_A$ is biholomorphic to $\mathbb{D}$, with the metric d_T coinciding with the usual hyperbolic metric on the unit disk ([Hub06, Corollary 6.10.3 and Theorem 6.10.6]). Additionally, the moduli space $\mathcal{M}_A$ is biholomorphic to the three-punctured Riemann sphere $\Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

Suppose that A and B are finite subsets of S^2 such that $B \subset A$. There exist two maps $t_{A,B} \colon \mathcal{T}_A \to \mathcal{T}_B$ and $m_{A,B} \colon \mathcal{M}_A \to \mathcal{M}_B$ that are naturally induced by forgetting the marked points of the set $A - B$. More precisely, $t_{A,B}([\varphi]_A) = [\varphi]_B$ and $m_{A,B}([\eta]_A) = [\eta]_B$, where $[\varphi]_A \in \mathcal{T}_A$, $[\varphi]_B \in \mathcal{T}_B$, $[\eta]_A \in \mathcal{M}_A$, and $[\eta]_B \in \mathcal{M}_B$.

Let $A = \{a_1, a_2, \ldots, a_k, a_{k+1}, a_{k+2}, a_{k+3}\}$ and $B = \{a_1, a_2, \ldots, a_l, a_{k+1}, a_{k+2}, a_{k+3}\}$, where $l \leqslant k$. Define the maps $h_A \colon \mathcal{M}_A \to \mathbb{C}^k - \mathcal{L}_k$ and $h_B \colon \mathcal{M}_B \to \mathbb{C}^l - \mathcal{L}_l$ as follows:

$$h_A([\varphi]_A) = (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_k)), \qquad h_B([\psi]_B) = (\psi(a_1), \psi(a_2), \ldots, \psi(a_l)).$$

where the representatives $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ and $\psi \colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that $\varphi(a_{k+1}) = \psi(a_{k+1}) = 0, \varphi(a_{k+2}) = \psi(a_{k+2}) = 1$, and $\varphi(a_{k+3}) = \psi(a_{k+3}) = \infty$.

Then it easy to verify that diagram (2.1) commutes, where $\mathrm{proj}_{k,l} \colon \mathbb{C}^k \to \mathbb{C}^l$ is the projection onto the first l coordinates.

We note that $\mathrm{proj}_{k,l} \colon \mathbb{C}^k - \mathcal{L}_k \to \mathbb{C}^l - \mathcal{L}_l$ is a holomorphic submersion, and, as it was previously discussed, $\pi_A \colon \mathcal{T}_A \to \mathcal{M}_A$ and $\pi_B \colon \mathcal{T}_B \to \mathcal{M}_B$ are holomorphic coverings, and $h_A \colon \mathcal{M}_A \to \mathbb{C}^k - \mathcal{L}_k$ and $h_B \colon \mathcal{M}_B \to \mathbb{C}^l - \mathcal{L}_l$ are biholomorphisms. Therefore, diagram (2.1) indicates that both $t_{A,B}$ and $m_{A,B}$ are holomorphic submersions. In particular, according to the submersion lemma (see, for instance, [Lee12, Corollary 5.13]), if $\tau \in \mathcal{T}_B$, then its fiber $t_{A,B}^{-1}(\tau)$ is a properly embedded complex submanifold of the Teichmüller space $\mathcal{T}_A$ and $\dim_{\mathbb{C}}(t_{A,B}^{-1}(\tau)) = |A| - |B|$. Analogously, if $\nu \in \mathcal{M}_B$, then its fiber $m_{A,B}^{-1}(\nu)$ is a properly embedded complex submanifold of $\mathcal{M}_A$, also with complex dimension $|A| - |B|$.

When the sets A and B differ by only one point, a more refined result can be obtained from the *Bers isomorphism* between Teichmüller space and *Bers fiber space*; see [Nag82] for the proof.

Proposition 2.2.11. *Suppose that A and B are finite sets such that $B \subset A \subset S^2$, $|B| \geqslant 3$, and $|A| = |B| + 1$. If $\tau \in \mathcal{T}_B$, then $t_{A,B}^{-1}(\tau)$ is a properly and holomorphically embedded*

unit disk in the Teichmüller space $\mathcal{T}_A$.

2.2.3 Pullback maps

In this section we illustrate how the notions introduced in Section 2.2.2 can be applied for studying the properties of Thurston maps. Most importantly, we introduce the following crucial concept.

Proposition 2.2.12. *Suppose that $f\colon S^2 \dashrightarrow S^2$ is finite-type topologically holomorphic map of admissible type, and A and B are two finite subsets of S^2 such that $|A|, |B| \geqslant 3$, $S_f \subset A$, and $B \subset \overline{f^{-1}(A)}$. Let $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ be an orientation-preserving homeomorphism. Then there exists an orientation-preserving homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $g_\varphi := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. In other words, the following diagram commutes*

$$\begin{array}{ccc} (S^2, B) & \xrightarrow{\psi} & (\widehat{\mathbb{C}}, \psi(B)) \\ \big\downarrow f & & \big\downarrow g_\varphi \\ (S^2, A) & \xrightarrow{\varphi} & (\widehat{\mathbb{C}}, \varphi(A)) \end{array}$$

The homeomorphism ψ is unique up to post-composition with a Möbius transformation. Different choices of φ that represent the same point in $\mathcal{T}_A$ yield maps ψ that represent the same point in $\mathcal{T}_B$.

In other words, we have a well-defined map $\sigma_{f,A,B}\colon \mathcal{T}_A \to \mathcal{T}_B$[1] such that $\sigma_{f,A,B}([\varphi]_A) = [\psi]_B$, called the pullback map (or the σ-map). As φ ranges across all maps representing a single point in $\mathcal{T}_A$, the map g_φ is uniquely defined up to pre- and post-composition with Möbius transformations.

Proof. The existence of a homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$, as well as its uniqueness up to pre-composition by a Möbius transformation, is guaranteed by Lemma 2.1.3 and Proposition 2.1.10. If we modify φ by an isotopy rel. A, then ψ changes only by an isotopy rel. $\overline{f^{-1}(A)} \supset B$ according to Corollary 2.1.13. Post-composition of φ by a Möbius transformation does not affect the homeomorphism ψ. Thus, varying φ within its equivalence class in $\mathcal{T}_A$ does not affect $[\psi]_B$, showing that the pullback map $\sigma_{f,A,B}$ introduced above is well-defined. These arguments also show that g_φ is uniquely determined up to pre- and post-composition with a Möbius transformation. □

It is easy to derive the following property related to the behavior of pullback maps.

Proposition 2.2.13. *Suppose that we are in the setting of Proposition 2.2.12, and there exist subsets $C \subset A$ and $D \subset B$ such that $S_f \subset C$, $D \subset \overline{f^{-1}(C)}$, and $|C|, |D| \geqslant 3$. Then:*

[1] In the more classical convention, the roles of the sets A and B are usually reversed, meaning the domain of the pullback map is the Teichmüller space $\mathcal{T}_A$, while the range is the Teichmüller space $\mathcal{T}_B$; see [KPS16] and [Ast24, Definition 3.1]. However, in this paper, we adopt the opposite convention. Moreover, except for this section (i.e., Section 2.2.3), the sets B and C are always considered subsets of A, unless stated otherwise.

1. *the equivalence class* $[\psi]_D$ *depends only on* $[\varphi]_C$*, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{T}_A & \xrightarrow{\sigma_{f,A,B}} & \mathscr{T}_B \\ \downarrow{\scriptstyle t_{A,C}} & & \downarrow{\scriptstyle t_{B,D}} \\ \mathscr{T}_C & \xrightarrow{\sigma_{f,C,D}} & \mathscr{T}_D \end{array}$$

2. *if* $|C| = 3$ *and* $A \subset \overline{f^{-1}(C)}$*, then* $\sigma_{f,A,B}$ *is constant;*

3. *the map* g_φ *depends only on the isotopy class rel.* C *of* φ *and* $\psi|D$*;*

4. *if* $|C| = 3$*, then* g_φ *is uniquely determined by* $\varphi|C$ *and* $\psi|D$*.*

Proof. Item (1) directly follows from Corollary 2.1.13. Indeed, if $\varphi \circ f = g_\varphi \circ \psi$, then varying φ via isotopy rel. C, ψ only alters by isotopy rel. $\overline{f^{-1}(C)} \supset D$. Item (2) can then be deduced from item (1) and Lemma 2.1.1.

Now, let $\varphi_1, \varphi_2 \colon S^2 \to \widehat{\mathbb{C}}$ be two orientation-preserving homeomorphisms that are isotopic rel. C. For $i = 1,2$, let $\psi_i \colon S^2 \to \widehat{\mathbb{C}}$ be an orientation-preserving homeomorphism such that map $g_{\varphi_i} \colon \varphi_i \circ f \circ \psi_i^{-1} \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic for each $i = 1,2$.

Claim. We have $g_{\varphi_1} = g_{\varphi_2} \circ \theta$ for some Möbius transformation θ isotopic rel. $\psi_1(\overline{f^{-1}(C)})$ to $\psi_1 \circ \psi_2^{-1}$.

Proof. Clearly, $g_{\varphi_1} = \varphi \circ g_{\varphi_2} \circ \psi^{-1}$, where $\varphi := \varphi_1 \circ \varphi_2^{-1} \in \mathrm{Homeo}^+(\widehat{\mathbb{C}})$ and $\psi := \psi_1 \circ \psi_2^{-1} \in \mathrm{Homeo}^+(\widehat{\mathbb{C}})$. Given the previous assumptions, φ is isotopic rel. $\varphi_1(C)$ to $\mathrm{id}_{\widehat{\mathbb{C}}}$. Hence, by Corollary 2.1.13 we have that $g_{\varphi_1} = g_{\varphi_2} \circ \theta$, where $\theta \in \mathrm{Homeo}^+(\mathbb{C})$ is isotopic rel. $\overline{g_{\varphi_1}^{-1}(\varphi_1(C))} = \psi_1(\overline{f^{-1}(C)})$ to φ. Since $g_{\varphi_i} \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic for $i = 1,2$, and by Lemma 2.1.3, it follows that θ must be a Möbius transformation. ■

Item (3) follows directly from the claim above, as $\psi_1(D) \subset \psi_1(\overline{f^{-1}(C)})$. Likewise, item (4) follows due to Lemma 2.1.1. □

Using Proposition 2.1.12, it is straightforward to verify that the following composition rule holds in the context of pullback maps.

Proposition 2.2.14. *Let* $f_1 \colon S^2 \dashrightarrow S^2$ *and* $f_2 \colon S^2 \dashrightarrow S^2$ *be two finite-type topologically holomorphic maps of admissible type. Suppose that* A*,* B*, and* C *are finite subsets of* S^2*, each containing at least three points, and satisfying* $S_{f_1} \subset A$*,* $S_{f_2} \subset B$*,* $B \subset \overline{f_1^{-1}(A)}$*, and* $C \subset \overline{f_2^{-1}(B)}$*. Then* $\sigma_{f_1 \circ f_2, C, A} = \sigma_{f_2, B, C} \circ \sigma_{f_1, A, B}$*.*

Remark 2.2.15. Let $\phi \colon S^2 \to S^2$ be an orientation-preserving homeomorphism with $\phi(B) = A$ and $|A| \geqslant 3$. It is straightforward to verify that if $\tau = [\varphi] \in \mathscr{T}_A$, then $\sigma_{\phi,A,B}([\varphi]) = [\varphi \circ \phi]$.

In this paper, the primary focus is on the case when $A = B$. In this scenario, either $f\colon (S^2, A) \circlearrowleft$ is a Thurston map or f is an orientation-preserving homeomorphism such that $f(A) = A$. We will use the notation $\sigma_{f,A}$ for the pullback map $\sigma_{f,A,A}$. Note that the domain and co-domain of $\sigma_{f,A}$ are the same, allowing us to explore its dynamical properties. When the marked set is clear from the context, we will simply use the notation σ_f instead of $\sigma_{f,A}$.

The following observation provides the most crucial property of pullback maps.

Proposition 2.2.16. *A Thurston map $f\colon (S^2, A) \circlearrowleft$ with $|A| \geqslant 3$ is realized if and only if the pullback map σ_f has a fixed point in the Teichmüller space $\mathcal{T}_A$.*

Proof. Suppose that $f\colon (S^2, A) \circlearrowleft$ is realized by a postsingularly finite holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$. Then, as noted in Remark 2.2.7, there exist orientation-preserving homeomorphisms $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $\varphi(A) = \psi(A) = P$, φ and ψ are isotopic rel. A, and $\varphi \circ f = g \circ \psi$. Clearly, $\tau = [\varphi] = [\psi] \in \mathcal{T}_A$ is a fixed point of σ_f.

Now, suppose that $\tau = [\varphi] \in \mathcal{T}_A$ is a fixed point of σ_f. Since $A \subset \overline{f^{-1}(A)}$, by Proposition 2.2.12, there exists an orientation-preserving homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $g_\varphi := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic and $[\varphi]_A = [\psi]_A$. Therefore, there exists a Möbius transformation M such that φ and $M \circ \psi$ are isotopic rel. A and, in particular, $\varphi|A = M \circ \psi|A$. Hence, the holomorphic map $g := g_\varphi \circ M^{-1}\colon (\widehat{\mathbb{C}}, \varphi(A)) \circlearrowleft$ is postsingularly finite and combinatorially equivalent to $f\colon (S^2, A) \circlearrowleft$. □

To illustrate the principle formulated in Proposition 2.2.16, we present the following remark.

Remark 2.2.17. Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| \geqslant 3$, and suppose that there is a subset $B \subset A$ such that $S_f \subset B$, $|B| = 3$, and $A \subset \overline{f^{-1}(B)}$. Then f is realized rel. A because σ_f is a constant map according to Proposition 2.2.13.

However, Thurston maps that satisfy these conditions are somewhat artificial. For such a marked Thurston map $f\colon (S^2, A) \circlearrowleft$, it must hold $P_f \subset B$ and for every $a \in A - B$, either a is an essential singularity of f or $f(a) \in B$. For instance, this scenario is impossible for unmarked Thurston maps with at least four postsingular values.

Remark 2.2.18. In Remark 2.2.17, we noted that a pullback map defined on a higher-dimensional Teichmüller space can sometimes be constant. Interestingly, this phenomenon can also occur with somewhat non-trivial Thurston maps. For Thurston maps of finite degree, numerous examples can be obtained by applying *McMullen's composition trick* (see [Buf+09, Proposition 3]), and this construction can be extended to transcendental Thurston maps. For example, it can be shown that the pullback map of $g_n(z) = \exp(\pi i z^n)$ (with the postsingular set $P_{g_n} = \{-1, 0, 1, \infty\}$) is constant for every $n \geqslant 2$. However, this does not hold in the case $n = 1$; see Example 4.1.12.

We say that a Thurston map $f\colon (S^2, A) \circlearrowleft$ is a $(2,2,2,2)$-*map* if f has a finite degree, exactly four postcritical values, and each critical point is *simple* and does not belong to the postcritical set. The following proposition summarizes the most important properties of pullback maps.

Proposition 2.2.19. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| \geqslant 3$. Then*

1. *σ_f holomorphic;*
2. *σ_f is 1-Lipschitz, i.e., $d_T(\sigma_f(\tau_1), \sigma_f(\tau_2)) \leqslant d_T(\tau_1, \tau_2)$ for every $\tau_1, \tau_2 \in \mathcal{T}_A$;*
3. *σ_f is distance-decreasing, i.e., $d_T(\sigma_f(\tau_1), \sigma_f(\tau_2)) < d_T(\tau_1, \tau_2)$ for every distinct $\tau_1, \tau_2 \in \mathcal{T}_A$, if f is transcendental;*
4. *$\sigma_f^{\circ k}$ is distance-decreasing for some $k \geqslant 1$ if f is not a $(2,2,2,2)$-map;*
5. *σ_f is an isometry of $\mathcal{T}_A$ if $A = P_f$ and f is a $(2,2,2,2)$-map.*

Proof. This result is rather well-known in the context of finite-degree Thurston maps (see [BCT14, Section 1.3] and [Hub16, Sections 10.6 and 10.7, and Appendix C.6]), and it can be extended analogously to our setting. We provide only an outline of the argument here, leaving some of the details to the reader.

According to Lemma 2.1.3 and Proposition 2.1.10, there exist two orientation preserving homeomorphisms $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ and a holomorphic map $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ so that $f = \varphi^{-1} \circ g \circ \psi$. Let $B := \varphi(A)$ and $C := \psi(A)$. In this proof, we identify the topological sphere S^2 with the Riemann sphere $\widehat{\mathbb{C}}$. By Proposition 2.2.14, we have the following:

$$\sigma_{f,A} = \sigma_{\psi,C,A} \circ \sigma_{g,B,C} \circ \sigma_{\varphi^{-1},A,B}.$$

It is well-known that $\sigma_{\psi,C,A}\colon \mathcal{T}_C \to \mathcal{T}_A$ and $\sigma_{\varphi^{-1},A,B}\colon \mathcal{T}_A \to \mathcal{T}_B$ are holomorphic isometries (see [Hub06, Sections 6.4 and 7.4]).

Using approach from [Ast24, Lemma 3.3], we can show that the pullback g^* is holomorphic on the *space of Beltrami forms* $\mathrm{Bel}(\widehat{\mathbb{C}})$, which is a *complex Banach manifold*, and descends in a well-behaved manner to the pullback map $\sigma_{g,B,C}\colon \mathcal{T}_B \to \mathcal{T}_C$. This guarantees that $\sigma_{g,B,C}$ is holomorphic as well, thus verifying item (1).

As stated in [Eps93, Lemma 16] and [Ast24, Sections 2 and 3], the pullback map $\sigma_{g,B,C}$ is distance-decreasing if the map f is transcendental, leading to item (3). The general idea is as follows: the co-derivative of $\sigma_{g,B,C}$ corresponds to the *pushforward* g_* of *integrable holomorphic quadratic differentials* (see [Hub16, Section 10.7] and [Ast24, Lemma 3.3]). The *masses* of these quadratic differentials induce a *Finsler metric* on the Teichmüller space, which in turn determines the Teichmüller metric d_T [Hub16, Section 6.6]. Finally, it can be shown that g_* *cancels* masses of integrable meromorphic quadratic differentials when g is transcendental [Ast24, Lemma 2.4].

Item (2) follows from [Hub06, Corollary 6.10.7], and item (4) comes from the previous discussion and [BCT14, Lemma 2.9] (see also [Hub16, Corollary 10.7.8]). Lastly, item (5) is well-known; see [DH93, Section 9] or [Hub16, Appendix C.6]. □

Remark 2.2.20. Suppose that f is a $(2,2,2,2)$-map. According to items (1) and (5) of Proposition 2.2.19, the pullback map σ_f is an automorphism of the Teichmüller space $\mathcal{T}_{P_f}$, which is biholomorphic to the unit disk, as discussed in Section 2.2.2. Moreover, it is known there exist examples of Thurston maps f for which σ_f is the identity, or it

is conjugate to an elliptic, parabolic, or hyperbolic transformation of the unit disk; see [Hub16, Appendix C.6].

Remark 2.2.21. In many instances throughout this thesis, the results of Proposition 2.2.19 can be derived using more straightforward methods. For example, for a Thurston map $f\colon (S^2, A) \circlearrowleft$ that satisfies the conditions of Main Theorem A, we will demonstrate that the associated pullback map σ_f is holomorphic through direct computations; see item (5) of Proposition 4.0.1. This, together with Schwarz-Pick's lemma, implies that σ_f is distance-decreasing. Furthermore, Proposition 5.1.3 will explicitly show that certain pullback maps defined on higher-dimensional Teichmüller spaces have holomorphic restrictions on specific invariant submanifolds of the corresponding Teichmüller space.

Proposition 2.2.19, called *Thurston's rigidity* (cf. [Hub16, Corollary 10.7.8]), illustrates how the properties of a pullback map can be used to derive conclusions about dynamics of the corresponding Thurston map.

Proposition 2.2.22. *Suppose that $f\colon (S^2, A) \circlearrowleft$ is not a $(2,2,2,2)$-map. If $f\colon (S^2, A) \circlearrowleft$ is realized by two postsingularly finite holomorphic maps $g_1\colon (\widehat{\mathbb{C}}, P_1) \circlearrowleft$ and $g_2\colon (\widehat{\mathbb{C}}, P_2) \circlearrowleft$, then there exists a Möbius transformation M such that $M \circ g_1 = g_2 \circ M$ and $M(P_1) = P_2$.*

Proof. According to Remark 2.2.7, there exist orientation-preserving homeomorphisms φ_1, φ_2, ψ_1, $\psi_2\colon S^2 \to \widehat{\mathbb{C}}$ such that φ_i and ψ_i are isotopic rel. A, $\psi_i(A) = \varphi_i(A) = P_i$, and $g_i = \varphi_i \circ f \circ \psi_i^{-1}$ for each $i = 1,2$. Clearly, if $|A| \geqslant 3$, the points $\tau_i = [\varphi_i] = [\psi_i]$, $i = 1,2$ are fixed points of the pullback map σ_f. However, item (4) of Proposition 2.2.19 demonstrates that σ_f can have at most one fixed point in the Teichmüller space $\mathcal{T}_A$. Therefore, $\tau_1 = \tau_2$, and there exists a Möbius transformation M such that φ_2 is isotopic rel. A to $M \circ \varphi_1$. Consequently, ψ_2 must isotopic rel. A to $M \circ \psi_1$ and $M(P_1) = P_2$. Corollary 2.1.13 then implies that g_1 and g_2 must be conjugate by M. If $|A| = 2$, the same conclusion follows from Lemma 2.1.1. □

2.2.4 Levy cycles

A primary goal of this section is to explore specific objects that may indicate that a given Thurston map is obstructed (see, for example, Proposition 2.2.24).

Let A be a finite subset of S^2. We say that a simple closed curve $\gamma \subset S^2 - A$ is *essential* in $S^2 - A$ if each connected component of $S^2 - \gamma$ contains at least two points of the set A; otherwise γ is called *non-essential*. In other words, γ is essential in $S^2 - A$ if it cannot be shrinked to a point via a homotopy in $S^2 - A$. It is easy to see that this property is independent from the homotopy class of the curve in $S^2 - A$. Finally, we note that essential simple closed curves in $S^2 - A$ exist if and only if $|A| \geqslant 4$.

Now, let $f\colon S^2 \dashrightarrow S^2$ be a topologically holomorphic map of finite type, and let A and B be two finite sets with $S_f \subset A$ and $B \subset \overline{f^{-1}(A)}$. Suppose that $\gamma \subset S^2 - A$ and $\widetilde{\gamma} \subset S^2 - \overline{f^{-1}(A)}$ are two simple closed curves such that $\widetilde{\gamma} \subset f^{-1}(\gamma)$. If γ is non-essential in $S^2 - A$, then $\widetilde{\gamma}$ is also non-essential in $S^2 - B$ (see, for instance, [MPR24, Propositions 2.11 and 2.12]). However, if γ is essential in $S^2 - A$, then $\widetilde{\gamma}$ may or may

not be essential in $S^2 - B$. A *multicurve* $\mathcal{G}$ in $S^2 - A$ is a finite collection of essential simple closed curves in $S^2 - A$ that are pairwise disjoint and pairwise non-homotopic in $S^2 - A$.

Definition 2.2.23. Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. A collection $\mathcal{G} = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$ of essential simple closed curves in $S^2 - A$ is called a *Levy cycle*[2] for the Thurston map $f\colon (S^2, A) \circlearrowleft$ if there exists a cyclic permutation σ of $\{1, 2, \dots, r\}$ so that for every i there is a simple closed curve $\widetilde{\gamma}_i \in f^{-1}(\gamma_i)$ that is homotopic in $S^2 - A$ to $\gamma_{\sigma(i)}$ and $\deg(f|\widetilde{\gamma}_i\colon \widetilde{\gamma}_i \to \gamma_i) = 1$. A Levy cycle $\mathcal{G}$ is *degenerate* if each γ_i bounds an open Jordan region D_i so that for every i some connected component $\widetilde{D}_i$ of $f^{-1}(D_i)$ is homotopic in $S^2 - A$ to $D_{\sigma(i)}$ and maps with degree 1 by the map f to D_i.

The following observation is widely known in the context of finite-degree Thurston maps [Hub16, Exercise 10.3.6], and its proof extends to the case of transcendental Thurston maps as well (see Section 3.1 for the proof).

Proposition 2.2.24. *A Thurston map $f\colon (S^2, A) \circlearrowleft$ having a Levy cycle is obstructed.*

If $\mathcal{G} = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$ is a Levy cycle for a Thurston map $f\colon (S^2, A) \circlearrowleft$, then the number r is called the *period* of the Levy cycle $\mathcal{G}$. We say that an essential simple closed curve $\gamma \subset S^2 - P_f$ *forms a Levy cycle* for the Thurston map f if there exists a Levy cycle $\mathcal{G}$ such that $\gamma \in \mathcal{G}$.

We say that the Levy cycle $\mathcal{G}$ is *minimal*, if the curves in $\mathcal{G}$ are pairwise non-homotopic in $S^2 - A$. Clearly, if $\mathcal{G}$ is not minimal, we can extract $\mathcal{G}' \subset \mathcal{G}$ that provides a minimal Levy cycle for $f\colon (S^2, A) \circlearrowleft$.

We observe that if $\mathcal{G} = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$ is a Levy cycle for a Thurston map $f\colon (S^2, A) \circlearrowleft$ and $\mathcal{G}' = \{\gamma_1', \gamma_2', \dots, \gamma_r'\}$ is a collection of simple closed curves in $S^2 - A$ such that γ_i and γ_i' are homotopic in $S^2 - A$ for $i = 1, 2, \dots, r$, then by Corollary 2.1.14, $\mathcal{G}'$ is also a Levy cycle for $f\colon (S^2, A) \circlearrowleft$. Furthermore, if $\mathcal{G}$ is degenerate, then according to the second part of Corollary 2.1.14, $\mathcal{G}'$ must also be degenerate.

If additionally $\mathcal{G}$ is a multicurve, we say that $\mathcal{G}$ is a *Levy multicurve*. When the period of a Levy cycle $\mathcal{G}$ equals 1, then $\mathcal{G}$ consists of a single essential simple closed curve γ in $S^2 - A$ that is called a *Levy fixed* or *Levy invariant curve*. When the Levy cycle $\mathcal{G}$ of period 1 is degenerate, then γ is called *degenerate* Levy fixed (or invariant) curve. Levy fixed curve γ is called *weakly degenerate* if f is injective on one of the connected components of $S^2 - \widetilde{\gamma}$.

It is known that a Levy cycle, or even a Levy fixed curve, is not necessarily degenerate (see, for instance, [HP22, Section 3] for a large class of such examples). Additionally, it is not difficult to construct examples of weakly degenerate Levy fixed curves that are not degenerate. At the same time, it is easy construct an example of a degenerate Levy cycle $\mathcal{G}$ for a Thurston map $f\colon (S^2, A) \circlearrowleft$ that does not form a Levy multicurve even after replacing curves in $\mathcal{G}$ with simple closed curves homotopic to them in $S^2 - A$.

[2] Note that we do not require the collection of curves $\mathcal{G}$ to be a multicurve, which differs from the more classical version of this definition [DH93].

2.2.5 Thurston's characterization of rational maps

In this section, we recall Thurston's characterization criterion and several corollaries derived from it.

We say that a multicurve $\mathscr{G}$ in $S^2 - A$ is *invariant* for the Thurston map $f\colon (S^2, A) \circlearrowleft$ if, for every curve $\gamma \in \mathscr{G}$, each essential simple closed curve in $f^{-1}(\gamma)$ is isotopic rel. A to a curve in $\mathscr{G}$.

Let $\mathscr{G} = \{\gamma_1, \dots, \gamma_r\}$, be an invariant multicurve for the Thurston map $f\colon (S^2, A) \circlearrowleft$. We can now associate an $(r \times r)$-matrix $M(f, A, \mathscr{G}) = (m_{ij})$ with $\mathscr{G}$ as follows. Fix $i, j \in \{1, \dots, r\}$, and let $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \dots, \widetilde{\gamma}_K$, where $K = K(i,j) \geqslant 0$, be all the simple closed curves in of $f^{-1}(\gamma_j)$ that are isotopic to γ_i rel. A. Then the (i,j)-entry m_{ij} of the matrix $M(f, A, \mathscr{G})$ is given by

$$m_{ij} := \sum_{k=1}^{K(i,j)} \frac{1}{\deg(f|\widetilde{\gamma}_k\colon \widetilde{\gamma}_k \to \gamma_j)}.$$

If $K(i,j) = 0$, then the sum is empty and $m_{ij} = 0$.

Note that $M(f, A, \mathscr{G})$ depends only on the homotopy rel. A classes of curves in $\mathscr{G}$ (this easily follows from Corollary 2.1.14). The Perron-Frobenius theorem implies that the spectral radius of $M(f, A, \mathscr{G})$ is given by the largest non-negative (real) eigenvalue $\lambda(f, A, \mathscr{G})$ of this matrix. The invariant multicurve $\mathscr{G}$ is called a *Thurston (or multicurve) obstruction* for $f\colon (S^2, A) \circlearrowleft$ if $\lambda(f, A, \mathscr{G}) \geqslant 1$.

With these definitions in place, we can now present Thurston's theorem. The proof for unmarked Thurston maps can be found in [DH93]; see also [Hub16, Theorem 10.1.14]. The case for marked maps is addressed in [BCT14]. For an alternative approach, see also [BMW23].

Theorem 2.2.25 (W.Thurston, Douady-Hubbard, Buff-Tan-Cui)**.** *A finite-degree Thurston map $f\colon (S^2, A) \circlearrowleft$, which is not a $(2,2,2,2)$-map, is combinatorially equivalent to a rational map if and only if $f\colon (S^2, A) \circlearrowleft$ does not have a Thurston obstruction.*

Remark 2.2.26. If f is a $(2,2,2,2)$-map, it is possible for $f\colon (S^2, A) \circlearrowleft$ to be realized and still have a Thurston obstruction $\mathscr{G}$; see [BHI24, Section 3.2]. However, in this case, $\lambda(f, A, \mathscr{G})$ must be equal to 1; see [DH93, Theorem 4.1].

Remark 2.2.27. Using item (4) of Proposition 2.2.19 and following the proof of [Hub16, Theorem 10.8.2] word for word, one can demonstrate that an infinite-degree Thurston map $f\colon (S^2, A) \circlearrowleft$ with a Thurston obstruction is indeed obstructed. However, whether the converse is true remains unknown.

In many cases, it is sufficient to focus on a specific type of Thurston obstruction, particularly Levy cycles. For example, one such family is provided by Thurston maps that are topological polynomials; see [Hub16, Theorem 10.3.7] and [BMW23, Proposition B.1].

Theorem 2.2.28 (Levy-Rees-Tan-Berstein)**.** *A postcritically finite topological polynomial $f\colon (S^2, A) \circlearrowleft$ is realized if and only if f has no degenerate Levy cycle.*

The following corollary follows directly from Theorem 2.2.28; see [Hub16, Theorem 10.3.8].

Corollary 2.2.29 (Levy-Rees-Tan-Berstein)**.** *Let $f\colon S^2 \to S^2$ be a postcritically finite topological polynomial such that every periodic orbit in P_f contains a critical point of f. Then f is realized.*

Note that [Flo+22, Theorem 3] generalizes Corollary 2.2.29.
The following theorem from [Tan92] provides another example.

Theorem 2.2.30 (Tan)**.** *A finite-degree Thurston map f with two critical points is realized if and only if f has no Levy cycle.*

A finite-degree Thurston map $f\colon S^2 \to S^2$ is called *critically fixed* if all its critical points are fixed by f. According to [HP22, Corollary 3.27], for this family, it is sufficient to check for Levy fixed curves to determine whether the Thurston map is realized.

Theorem 2.2.31 (Hlushchanka-Prochorov)**.** *Let f be a critically fixed Thurston map. Then f is realized if and only if it has no Levy fixed curve.*

3 Hyperbolic tools

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Hyperbolic geometry plays a crucial role in one-dimensional complex dynamics (see, for example, [Mil06] and [Aba23]), and Thurston theory is no exception. In this section, we introduce several results and techniques rooted in hyperbolic geometry that will be valuable in later discussions. Specifically, in Section 3.1, we demonstrate how tools from hyperbolic geometry and Teichmüller theory can be used to identify and analyze Levy cycles for Thurston maps by investigating the behavior of the associated pullback maps. In Section 3.2, we explore the dynamics of holomorphic self-maps of the unit disk that satisfy certain additional conditions. Many such maps naturally arise from pullback maps defined on the one-complex-dimensional Teichmüller space or restricted to an invariant one-dimensional slice of higher-dimensional Teichmüller space (see Sections 4.0.1 and 5.1.2). Section 3.2.1 provides preliminary estimates for hyperbolic contraction of inclusion maps between two hyperbolic Riemann surfaces. Finally, in Section 3.2.2, we apply these estimates to gain a deeper understanding of the dynamics of certain holomorphic self-maps of the unit disk.

Before moving forward, it is necessary to introduce some classical concepts and notations from one-dimensional hyperbolic geometry. Let U be a hyperbolic Riemann surface, and let d_U denote the distance function of the hyperbolic metric on U. For any rectifiable curve α in U, we denote the length of α with respect to the hyperbolic metric by $\ell_U(\alpha)$. When we refer to γ as a geodesic in U, we always mean that γ is a geodesic with respect to the hyperbolic metric. Also, let $B_U(z,r)$ be the hyperbolic ball in U with center $z \in U$ and radius r. If U is a subset of $\mathbb{C}$, then the hyperbolic metric on U, as a conformal metric, can be written as $\rho_U(z)|dz|$, where $\rho_U \colon U \to [0,+\infty)$ is the *density* of the hyperbolic metric on U.

For a holomorphic map $g \colon U \to V$ between two hyperbolic Riemann surfaces, we denote by $\|\mathrm{D}g(z)\|_U^V$ the norm of the derivative of g with respect to the hyperbolic metrics on the domain U and the range V. More precisely, this norm is given by

$\|\mathrm{D}g(z)\|_U^V = \|\mathrm{D}_z g(v)\|_V \,/\, \|v\|_U$, where $v \in T_z U$ is any non-zero vector, and $\|\cdot\|_U$ represents the length of a tangent vector to U with respect to the hyperbolic metric. If $U = V$, we simply use the notation $\|\mathrm{D}g(z)\|_U$.

Schwarz-Pick's lemma [Hub06, Proposition 3.3.4] implies that for a holomorphic map $g\colon U \to V$ between two hyperbolic Riemann surfaces, we have $\|\mathrm{D}g(z)\|_U^V \leqslant 1$ for every $z \in U$. Furthermore, if g is a covering map, this inequality becomes an equality; otherwise, g is *locally uniformly contracting*, i.e., for every compact set $K \subset U$, there exists a constant $\lambda_K < 1$ such that $\|\mathrm{D}g(z)\|_U^V \leqslant \lambda_K$ for all $z \in K$. Suppose that α is a C^1-curve in U and $\|\mathrm{D}g(z)\|_U^V \leqslant \lambda$ for all $z \in \alpha$. Then it is straightforward to check that $\ell_V(g(\alpha)) \leqslant \lambda \ell_U(\alpha)$. In particular, Schwarz-Pick's lemma implies that the map $g\colon U \to V$ is always 1-Lipschitz and, if g is not a covering map, then g is *locally uniformly distance-decreasing*.

We will be particularly interested in the case when the map g mentioned above is simply the inclusion map $I\colon U \hookrightarrow V$. We are going to denote $\|\mathrm{D}I(z)\|_U^V$ by $c_U^V(z)$. Clearly, if $U \subset V \subset \mathbb{C}$, then $c_U^V(z) = \rho_V(z)/\rho_U(z)$.

3.1 Detecting obstructions

In this section, we show how tools from hyperbolic geometry and Teichmüller theory can be employed to detect and analyze Levy cycles for both finite and infinite-degree Thurston maps

Let δ be an essential simple closed curve in the hyperbolic Riemann surface X. According to [Hub06, Proposition 3.3.8], there exists a unique closed geodesic in X that is homotopic (in X) to δ. Note that this geodesic should be necessarily simple [Hub06, Proposition 3.3.9].

We start by proving that Levy cycles are obstructions for Thurston maps of both finite and infinite degree.

Proof of Proposition 2.2.24. Let γ be a simple closed curve that forms a Levy cycle $\mathscr{G}$ for the Thurston map $f\colon (S^2, A) \circlearrowleft$. To begin, assume that the Levy cycle $\mathscr{G}$ has period $r = 1$, meaning that γ is a Levy fixed curve.

Suppose that $f\colon (S^2, A) \circlearrowleft$ is combinatorially equivalent to a postsingularly finite holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$. Based on Definition 2.2.6 and Remark 2.2.7, it is easy to see that $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$ also has a Levy fixed curve $\delta \subset \widehat{\mathbb{C}} - P$. Note that $|P| \geqslant 4$ since, otherwise, δ would not be essential in $\widehat{\mathbb{C}} - P$. According to the previous discussion and Corollary 2.1.14, we can assume that δ is a simple closed geodesic in $\widehat{\mathbb{C}} - P$.

Let $\widetilde{\delta}$ be a simple closed curve in $g^{-1}(\delta)$ such that δ and $\widetilde{\delta}$ are homotopic in $\widehat{\mathbb{C}} - P$, and $\deg(g|\widetilde{\delta}\colon \widetilde{\delta} \to \delta) = 1$. By Schwarz-Pick's lemma and the fact that $g|\widehat{\mathbb{C}} - \overline{g^{-1}(P)}\colon \widehat{\mathbb{C}} - \overline{g^{-1}(P)} \to \widehat{\mathbb{C}} - P$ is a holomorphic covering, it follows that

$$\ell_{\widehat{\mathbb{C}}-P}(\delta) = \ell_{\widehat{\mathbb{C}}-\overline{g^{-1}(P)}}(\widetilde{\delta}) > \ell_{\widehat{\mathbb{C}}-P}(\widetilde{\delta}),$$

where the last inequality is strict since $\overline{g^{-1}(P)} - P$ is non-empty because of Great

Picard's theorem and the Riemann-Hurwitz formula [Hub06, Appendix A.3]. However, δ is the unique geodesic in its homotopy class in $\widehat{\mathbb{C}}-P$. Thus, $\ell_{\widehat{\mathbb{C}}-P}(\delta) \leqslant \ell_{\widehat{\mathbb{C}}-P}(\widetilde{\delta})$, and it leads to a contradiction.

Suppose that γ forms a Levy cycle $\mathcal{G}$ with period $r \geqslant 2$. By Corollary 2.1.14, γ is a Levy fixed curve for the Thurston map $f^{\circ r}\colon (S^2, A) \circlearrowleft$. Furthermore, since $f\colon (S^2, A) \circlearrowleft$ is realized, Proposition 2.2.16 guarantees that the pullback map σ_f has a fixed point in the Teichmüller space $\mathcal{T}_A$. From Proposition 2.2.14, it follows that $\sigma_{f^{\circ r}} = \sigma_f^{\circ r}$. Hence, $\sigma_{f^{\circ r}}$ also has a fixed point, which, by Proposition 2.2.16, implies that $f^{\circ r}$ is realized. However, since $f^{\circ r}$ has a Levy fixed curve γ, these conclusions result in a contradiction. □

3.1.1 Detecting Levy fixed curves

Let γ be an essential simple closed curve in $S^2 - A$, and $\tau = [\varphi]$ be a point in the Teichmüller space $\mathcal{T}_A$. We define $l_\gamma(\tau)$ as the length of the unique hyperbolic geodesic in $\widehat{\mathbb{C}}-\varphi(A)$ that is homotopic in $\widehat{\mathbb{C}}-\varphi(A)$ to $\varphi(\alpha)$. Additionally, we introduce $w_\gamma(\tau) := \log l_\gamma(\tau)$. It is known that $w_\gamma\colon \mathcal{T}_A \to \mathbb{R}$ is a 1-Lipschitz function [Hub06, Theorem 7.6.4].

Let X be a hyperbolic Riemann surface, and suppose $\alpha \subset X$ is a simple closed geodesic with $\ell_X(\alpha) < \ell^*$, where $\ell^* := \log(3+2\sqrt{2})$. In this case, we say that α is *short*. As stated in [Hub06, Proposition 3.3.8 and Corollary 3.8.7], two short simple closed geodesics on a hyperbolic Riemann surface X are either disjoint and non-homotopic in X, or they coincide. Therefore, [Hub06, Proposition 3.3.8] implies that a punctured Riemann sphere $\widehat{\mathbb{C}} - P$, where $3 \leqslant |P| < \infty$, can have at most $|P| - 3$ distinct short simple closed geodesics.

The following result allows us to identify a Levy fixed curve for a Thurston map based on the behavior of the corresponding pullback map. In essence, this proposition demonstrates that a "fat injective annulus" leads to a Levy fixed curve.

Proposition 3.1.1. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| = 4$, and $\tau = [\varphi]$ and $\sigma_f(\tau) = [\psi]$ be points in the Teichmüller space $\mathcal{T}_A$, where the representatives $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that the map $g := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. Suppose that there exists an annulus $U \subset \widehat{\mathbb{C}}$ such that:*

- *each connected component of $\widehat{\mathbb{C}} - U$ contains two points of $\psi(A)$;*
- *$\operatorname{mod}(U) > 5\pi e^{d_0}/\ell^*$, where $d_0 = d_T(\tau, \sigma_f(\tau))$;*
- *g is defined and injective on U.*

Then $f\colon (S^2, A) \circlearrowleft$ has a Levy fixed curve. Moreover, if g is defined and injective outside a single connected component of $\widehat{\mathbb{C}} - U$, then $f\colon (S^2, A) \circlearrowleft$ has a weakly degenerate Levy fixed curve.

Proof. Since g is defined and injective on U, the annulus U contains at most four points of the set $\overline{g^{-1}(\varphi(A))}$. Therefore, we can find a parallel subannulus V of U (i.e., U is homotopic in V to V) such that $\operatorname{mod}(V) \geqslant \operatorname{mod}(U)/5 > \pi e^{d_0} \ell^*$, and V does

not contain any points of $\overline{g^{-1}(\varphi(A))}$. Denote $X := \widehat{\mathbb{C}} - \psi(A)$, $Y := \widehat{\mathbb{C}} - \varphi(A)$, and $Z := \widehat{\mathbb{C}} - \overline{g^{-1}(\varphi(A))}$. In particular, $g|Z\colon Z \to Y$ is a holomorphic covering map.

Let α be a unique hyperbolic geodesic of V. It is known that α is a simple closed curve that forms a core curve of the annulus V, and its length in V is given by $\ell_V(\alpha) = \pi/\mathrm{mod}(V)$; see [Hub06, Proposition 3.3.7]. It is also easy to see that α is essential in X. Let β denote the curve $g(\alpha)$. Since α is a simple closed curve and $g|V$ is injective, then β is also a simple closed curve. At the same time, β must be essential in Y since, otherwise, α would not be essential in X (see, for instance, [For91, Theorems 5.10 and 5.11]). Define $\widetilde{\alpha} := \psi^{-1}(\alpha)$ and $\widetilde{\beta} := \varphi^{-1}(\beta)$, both of which are essential simple closed curves in $S^2 - A$. It is straightforward to verify that $f(\widetilde{\alpha}) = \widetilde{\beta}$ and $\deg(f|\widetilde{\alpha}\colon \widetilde{\alpha} \to \widetilde{\beta}) = 1$.

According to Schwarz-Pick's lemma and the choice of $\alpha \subset Z$, we have the following inequality:

$$\ell_X(\alpha) \leqslant \ell_Z(\alpha) \leqslant \ell_V(\alpha) = \pi/\mathrm{mod}(V) < e^{-d_0}\ell^*.$$

Therefore, since the function $w_{\widetilde{\alpha}}\colon \mathcal{T}_A \to \mathbb{R}$ is 1-Lipschitz, it follows that $l_{\widetilde{\alpha}}(\tau) < \ell^*$. Hence, there exists a simple closed geodesic δ in Y, homotopic in Y to $\varphi(\widetilde{\alpha})$, such that $\ell_Y(\delta) < \ell^*$. At the same time, $\ell_Y(\beta) = \ell_Z(\alpha) < e^{-d_0}\ell^*$. Therefore, since both β and δ are essential in Y, they are homotopic in Y to short simple closed geodesics. However, as discussed previously, the four-punctured Riemann sphere Y can have only one short simple closed geodesic. This implies that β and δ are homotopic in Y, which in turn means that the curves $\widetilde{\alpha}$ and $\widetilde{\beta}$ are homotopic in $S^2 - A$. Hence, $\widetilde{\beta}$ provides a Levy fixed curve for the Thurston map $f\colon (S^2, A) \circlearrowleft$. It is also straightforward to verify that if g is defined and injective outside a single connected component of $\widehat{\mathbb{C}} - U$, then this Levy fixed curve is weakly degenerate. □

The following result guarantees the uniqueness of a Levy fixed curve for a Thurston map with four marked points, given a specific technical condition.

Proposition 3.1.2. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| = 4$ that has a Levy fixed curve. Suppose that there exists a point τ in the Teichmüller space $\mathcal{T}_A$ such that the sequence $(\pi_A(\sigma_f^{\circ n}(\tau)))$ eventually leaves every compact subset of the moduli space $\mathcal{M}_A$. Then $f\colon (S^2, A) \circlearrowleft$ has a unique Levy fixed curve up to homotopy in $S^2 - A$.*

Proof. Let γ be a Levy fixed curve for the Thurston map $f\colon (S^2, A) \circlearrowleft$. Choose a point $\mu \in \mathcal{T}_A$ such that $l_\gamma(\mu) < \ell^*$. Define $\tau_n := \sigma_f^{\circ n}(\tau) = [\varphi_n]$ and $\mu_n := \sigma_f^{\circ n}(\mu) = [\psi_n]$ for all $n \geqslant 0$. According to Proposition 2.2.12, we can assume that $g_n := \psi_n \circ f \circ \psi_{n+1}^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic for every $n \geqslant 0$. We also define $X_n := \widehat{\mathbb{C}} - \varphi_n(A)$, $Y_n := \widehat{\mathbb{C}} - \psi_n(A)$, and $Z_n := \widehat{\mathbb{C}} - \overline{g_n^{-1}(\psi_n(A))} = \widehat{\mathbb{C}} - \overline{\psi_{n+1}(f^{-1}(A))}$ for all $n \geqslant 0$. In particular, $g_n|Z_n\colon Z_n \to Y_n$ is a holomorphic covering map.

Claim. We have $l_\gamma(\mu_n) < \ell^*$ for all $n \geqslant 0$.

Proof. Clearly, $l_\gamma(\mu_0) < \ell^*$ by the choice of μ. We will prove that $l_\gamma(\mu_1) < \ell^*$, and the rest easily follows by induction on n. There exists a short simple closed geodesic β in Y_0 such that β and $\psi_0(\gamma)$ are homotopic in Y_0, and $\ell_{Y_0}(\beta) = l_\gamma(\mu_0)$. Define $\widetilde{\beta} := \psi_0^{-1}(\beta)$. Since $\widetilde{\beta}$ and γ are homotopic in $S^2 - A$ and γ is a Levy fixed curve for $f\colon (S^2, A) \circlearrowleft$, then

by Corollary 2.1.14, there exists a simple closed curve $\widetilde{\alpha} \subset f^{-1}(\widetilde{\beta})$ that is homotopic to γ in $S^2 - A$ and $\deg(f|\widetilde{\alpha}\colon \widetilde{\alpha} \to \widetilde{\beta}) = 1$. Now, define $\alpha := \psi_1(\widetilde{\alpha})$, which is homotopic in Y_1 to $\psi_1(\gamma)$. Clearly, $g_0(\alpha) = \beta$, $\deg(g_0|\alpha\colon \alpha \to \beta) = 1$, and $\alpha \subset Z_0$. Therefore, by Schwarz-Pick's lemma, we have

$$\ell_{Y_1}(\alpha) \leqslant \ell_{Z_0}(\alpha) = \ell_{Y_0}(\beta) = l_\gamma(\mu_0) < \ell^*.$$

Finally, $l_\gamma(\mu_1) = l_{\widetilde{\alpha}}(\mu_1) < \ell^*$. ■

Since $(\pi_A(\tau_n))$ eventually leaves every compact subset of $\mathcal{M}_A$, Mumford's compactness theorem [Hub06, Theorem 7.3.3] states that the length of the shortest simple closed geodesic δ_n in X_n tends to zero. Then $w_{\varphi_n^{-1}(\delta_n)}(\mu_n) < \log \ell^*$ for any sufficiently large n, given that $w_{\varphi_n^{-1}(\delta)}\colon \mathcal{T}_A \to \mathbb{R}$ is 1-Lipschitz and $d_T(\mu_n, \tau_n) \leqslant d_T(\tau, \mu)$ by item (2) of Proposition 2.2.19. In particular, both $\psi_n(\gamma)$ and $\psi_n(\varphi_n^{-1}(\delta_n))$ are homotopic in Y_n to short simple closed geodesics. Since $Y_n = \widehat{\mathbb{C}} - \psi_n(A)$ is a four-punctured Riemann sphere, it follows that γ and $\varphi_n^{-1}(\delta_n)$ are homotopic in $S^2 - A$ for any n large enough.

This argument applies to any Levy fixed curve of $f\colon (S^2, A) \circlearrowleft$. Therefore, it has to be unique up to homotopy in $S^2 - A$. □

Remark 3.1.3. If an obstructed finite-degree Thurston map $f\colon (S^2, A) \circlearrowleft$, with $|A| \geqslant 3$, is not a $(2,2,2,2)$-map, then it is known (see [Hub16, Section 10.9 and Lemma 10.11.9] or [Sel12, Proof of Theorem 2.3]) that every point τ in the Teichmüller space $\mathcal{T}_A$ satisfies the condition of Proposition 3.1.2.

Remark 3.1.4. Suppose that we are in the setting of Proposition 3.1.2. It is clear that it suffices to require that the sequence $(\pi_A(\sigma_f^{\circ n}(\tau)))$ admits a subsequence eventually leaving every compact subset of the moduli space $\mathcal{M}_A$.

3.1.2 Detecting Levy cycles

The following result generalizes Proposition 3.1.1 and allows us to identify a Levy multicurve by examining the behavior of the associated pullback map. The proof is similar to that of Proposition 3.1.1, but we include it here for completeness. This proposition illustrates how "fat injective annuli" lead to a Levy cycle.

Proposition 3.1.5. *Suppose that $f\colon (S^2, A) \circlearrowleft$ is a Thurston map with $|A| = k+3, k \geqslant 1$. Let $\tau_1 \in \mathcal{T}_A$ and $\tau_i = [\varphi_i] = \sigma_f^{\circ(i-1)}(\tau)$ for $i = 1, 2, \ldots, k+1$, where the representatives $\varphi_i\colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that the map $g_i := \varphi_i \circ f \circ \varphi_{i+1}^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic for each $i = 1, 2 \ldots, k$. Assume that there exists an annulus $U \subset \widehat{\mathbb{C}}$ such that*

- *each connected component of $\widehat{\mathbb{C}} - U$ contains at least two points of $\varphi_{k+1}(A)$;*
- *$\operatorname{mod}(U) > (k+4)\pi e^{kd_0}/\ell^*$, where $d_0 := d_T(\tau_1, \tau_2)$;*
- *$g_1 \circ g_2 \circ \cdots \circ g_k$ is defined and injective on U.*

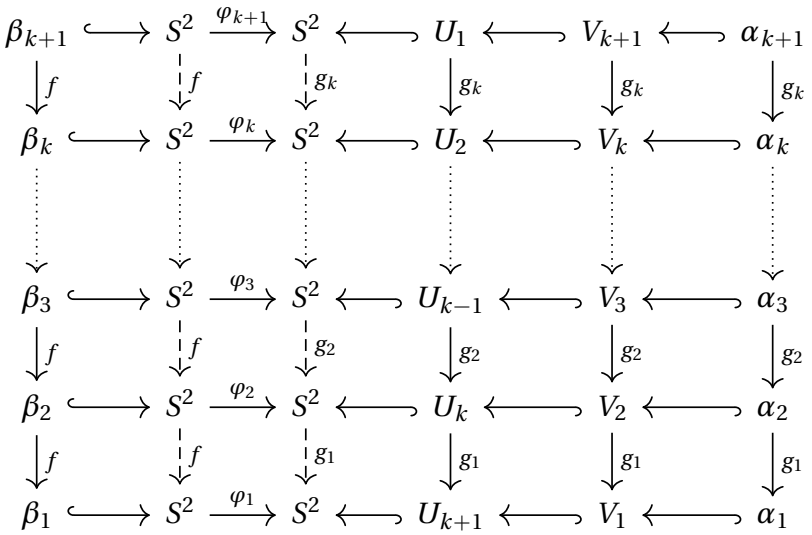

Figure 3.1: Diagram depicting the relationships among the introduced objects.

Then the Thurston map $f\colon (S^2, A) \circlearrowleft$ has a Levy multicurve $\mathscr{G}$. Moreover, for each $\gamma \in \mathscr{G}$, the inequality $l_\gamma(\tau_i) < (k+4)\pi e^{kd_0}/\mathrm{mod}(U) < \ell^*$ holds for each $i = 1,2,\dots,k+1$.

Proof. Define $U_1 := U$ and $U_{i+1} := g_{k-i+1}(U_i)$ for each $i = 1,2,\dots,k$. Notice that each U_i is an annulus with $\mathrm{mod}(U_i) = \mathrm{mod}(U)$. Therefore, we can find a parallel subannulus V of U_{k+1} such that $\mathrm{mod}(V) \geqslant \mathrm{mod}(U)/(k+4)$ and V does not contain any points of $\varphi_1(A)$. Next, define $V_1 := V$, and for each $i = 1,2,\dots,k$, let V_{i+1} be the annulus coinciding with the connected component of $g_i^{-1}(V_i)$ within U_{k-i+1}. Clearly, $V_i \cap \varphi_i(A) = \varnothing$ for each $i = 1,2,\dots,k+1$. It also can be easily verified that each connected component of $\widehat{\mathbb{C}} - V_i$ contains at least two points of $\varphi_i(A)$; otherwise, there would exist a connected component of $\widehat{\mathbb{C}} - U$ containing at most one point of $\varphi_{k+1}(A)$ (see, for instance, [For91, Theorems 5.10 and 5.11]).

Let α_{k+1} be the unique hyperbolic geodesic of V_{k+1}. It is known that α_{k+1} is a simple closed geodesic that forms a core curve of the annulus V_{k+1}, and its length in V_{k+1} is given by $\ell_{V_{k+1}}(\alpha_{k+1}) = \pi/\mathrm{mod}(V_{k+1})$; see [Hub06, Proposition 3.3.7]. For each $i = 2,3,\dots,k+1$, let $\alpha_{i-1} := g_{i-1}(\alpha_i)$. Next, define $\beta_i := \varphi_i^{-1}(\alpha_i)$ for each $i = 1,2,\dots,k+1$. It is straightforward to verify that β_i is an essential simple closed curve in $S^2 - A$, with $\beta_{i-1} = f(\beta_i)$ and $\deg(f|\beta_i\colon \beta_i \to \beta_{i-1}) = 1$; see Figure 3.1 for further clarifications.

Let $X_i := \widehat{\mathbb{C}} - \varphi_i(A)$ for each $i = 1,2,\dots,k+1$. Recall that $g_i|V_{i+1}\colon V_{i+1} \to V_i$ is a biholomorphism for each $i = 1,2,\dots,k$, and therefore, we have $\ell_{V_i}(\alpha_i) = \ell_{V_j}(\alpha_j)$ and $\mathrm{mod}(V_i) = \mathrm{mod}(V_j)$ for all i and j. Thus, according to Schwarz-Pick's lemma and the choice of α_{k+1}, we have:

$$\ell_{X_i}(\alpha_i) \leqslant \ell_{V_i}(\alpha_i) = \ell_{V_{k+1}}(\alpha_{k+1}) = \frac{\pi}{\mathrm{mod}(V_{k+1})} = \frac{\pi}{\mathrm{mod}(V)} \leqslant \frac{(k+4)\pi}{\mathrm{mod}(U)}.$$

By item (2) of Proposition 2.2.19, we have $d_T(\tau_i, \tau_j) \leqslant |i-j| \cdot d_0$ for each $i, j =$

$1,2,\dots,k+1$. Therefore, since the function $w_{\beta_i}\colon \mathcal{T}_A \to \mathbb{R}$ is 1-Lipschitz, it follows that $l_{\beta_i}(\tau_j) \leqslant (k+4)\pi e^{kd_0}/\mathrm{mod}(U) < \ell^*$. In particular, there exist short simple closed geodesics $\delta_1,\delta_2,\dots,\delta_{k+1}$ in X_1, with each δ_i homotopic in X_1 to $\varphi_1(\beta_i)$.

As it was discussed previously, any two short geodesics, δ_i and δ_j, are either disjoint and non-homotopic in X_1, or they coincide. However, the punctured Riemann sphere $X_1 = \widehat{\mathbb{C}} - \varphi_1(A)$ can contain at most k distinct short simple closed geodesics. Therefore, there must be distinct indices i_1 and i_2 such that $\delta_{i_1} = \delta_{i_2}$. Without loss of generality, assume that i_2 is the minimal index such that $i_2 > i_1$ and $\delta_{i_1} = \delta_{i_2}$. Since $\varphi_1^{-1}(\delta_i)$ is homotopic in $S^2 - A$ to β_i for each $i = 1,2,\dots k+1$, Corollary 2.1.14 implies that the multicurve $\{\varphi_1^{-1}(\delta_{i_1+1}),\varphi_1^{-1}(\delta_{i_1+2}),\dots,\varphi_1^{-1}(\delta_{i_2})\}$ forms a Levy multicurve for the Thurston map $f\colon (S^2,A) \circlearrowleft$. Finally, note that $l_{\varphi_1^{-1}(\delta_i)}(\tau) = l_{\beta_i}(\tau)$ for each $i = 1,2,\dots,k+1$ and $\tau \in \mathcal{T}_A$, and the rest follows from the previous discussion. □

The following proposition demonstrates that, under specific conditions, a Levy multicurve can be obtained from a Levy cycle by simply modifying the homotopy classes of the corresponding simple closed curves.

Proposition 3.1.6. *Let $\mathcal{G} = \{\gamma_1,\gamma_2,\dots,\gamma_r\}$ be a minimal Levy cycle for a Thurston map $f\colon (S^2,A) \circlearrowleft$. Suppose there exist a curve $\gamma \in \mathcal{G}$ and a point τ in the Teichmüller space $\mathcal{T}_A$ such that $l_\gamma(\tau) < \ell^* e^{-(r-1)d_0}$, where $d_0 := d_T(\tau,\sigma_f(\tau))$. Then, there exists a Levy multicurve $\mathcal{G}' = \{\gamma_1',\gamma_2',\dots,\gamma_r'\}$ for $f\colon (S^2,A) \circlearrowleft$ such that each γ_i' is homotopic in $S^2 - A$ to γ_i. Moreover, if $\mathcal{G}$ is a degenerate Levy cycle, then $\mathcal{G}'$ is a degenerate Levy multicurve.*

Proof. Without loss of generality, assume that $\gamma = \gamma_1$ and $\sigma(i) = i+1 \bmod r$, where σ is a cyclic permutation associated with the Levy cycle $\mathcal{G}$ (see Definition 2.2.23).

Let $\tau_i = [\varphi_i] = \sigma_f^{\circ(i-1)}(\tau)$ for $i \in \mathbb{N}$, where the representatives $\varphi_i\colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that $g_i := \varphi_i \circ f \circ \varphi_{i+1}^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. Additionally, define $X_i := \widehat{\mathbb{C}} - \varphi_i(A)$ and $Y_i := \widehat{\mathbb{C}} - \overline{g_i^{-1}(\varphi_i(A))} = \widehat{\mathbb{C}} - \overline{\varphi_{i+1}(f^{-1}(A))}$ for all $i \in \mathbb{N}$.

Claim. For each $i = 1,2,\dots,r$, we have the inequality $l_{\gamma_i}(\tau_i) \leqslant l_{\gamma_1}(\tau_1)$.

Proof. We assume that $r \geqslant 2$, and begin by showing that $l_{\gamma_2}(\tau_2) \leqslant l_{\gamma_1}(\tau_1)$. The rest then follows by induction. There exists a simple closed geodesic α_1 in X_1 such that $\ell_{X_1}(\alpha_1) = l_{\gamma_1}(\tau_1)$ and α_1 is homotopic in X_1 to $\varphi_1(\gamma_1)$. Denote $\beta_1 := \varphi_1^{-1}(\alpha_1)$. Since β_1 is homotopic in $S^2 - A$ to γ_1, by Corollary 2.1.14, there exists a simple closed curve $\beta_2 \subset f^{-1}(\beta_1)$ that is homotopic in $S^2 - A$ to γ_2 and satisfies $\deg(f|\beta_2\colon \beta_2 \to \beta_1) = 1$. Let $\alpha_2 := \varphi_2(\beta_2) \subset Y_1$. Obviously, $g_1(\alpha_2) = \alpha_1$ and $\deg(g_1|\alpha_2\colon \alpha_2 \to \alpha_1) = 1$. Therefore, since $g_1|Y_1\colon Y_1 \to X_1$ is a holomorphic covering map, Schwarz-Pick's lemma gives:

$$\ell_{X_2}(\alpha_2) \leqslant \ell_{Y_1}(\alpha_2) = \ell_{X_1}(\alpha_1) = l_{\gamma_1}(\tau_1).$$

At the same time, clearly, $l_{\gamma_2}(\tau_2) \leqslant \ell_{X_2}(\alpha_2)$, and the rest follows. ■

Item (2) of Proposition 2.2.19 implies that $d_T(\tau_i,\tau_j) \leqslant |i-j| \cdot d_0$ for each $i,j \in \mathbb{N}$. Therefore, based on the claim above and the fact that $w_{\gamma_i}\colon \mathcal{T}_A \to \mathbb{R}$ is 1-Lipschitz, we

conclude that $l_{\gamma_i}(\tau_1) < \ell^*$ for each $i = 1,2,\dots,r$. Therefore, there exist short simple closed geodesics $\delta_1,\delta_2,\dots,\delta_r$ in X_1, where each δ_i is homotopic in X_1 to $\varphi_1(\gamma_i)$.

Since $\mathscr{G}$ is a minimal Levy cycle, no two geodesics δ_i and δ_j are homotopic in X_1 for $i \neq j$. Therefore, as each δ_i is a short simple closed geodesic, δ_i and δ_j must be disjoint in X_1. Now, define $\gamma_i' := \varphi_1^{-1}(\delta_i)$ and $\mathscr{G}' := \{\gamma_1',\gamma_2',\dots,\gamma_r'\}$, which forms a multicurve in $S^2 - A$. Since γ_i and γ_i' are homotopic in $S^2 - A$, $\mathscr{G}'$ is a Levy multicurve for the Thurston map $f\colon (S^2, A) \circlearrowleft$, and if the original Levy cycle $\mathscr{G}$ is degenerate, then $\mathscr{G}'$ is also degenerate. □

3.2 Hyperbolic geometry and holomorphic maps on the unit disk

In Section 3.2.2, we examine the dynamics of holomorphic self-maps of the unit disk under specific additional conditions. However, prior to that, in Section 3.2.1, we must first investigate the hyperbolic contraction of inclusion maps between two hyperbolic Riemann surfaces.

3.2.1 Estimating contraction

The following proposition provides an estimate for the contraction of an inclusion between hyperbolic Riemann surfaces. Although this result is well-known (see [McM94, Theorem 2.25], [MR13, Proposition 3.4], or [Min17, Lemma 1]), we include the proof here for the sake of completeness.

Proposition 3.2.1. *Suppose that U and V are two hyperbolic Riemann surfaces such that $U \subsetneq V$. Let $z \in U$ and $s := d_V(z, V-U)$. Then*

$$c_U^V(z) \leqslant \frac{2r|\log r|}{1-r^2},$$

where $r := \tanh(s/2)$.

Proof. Suppose that $z_0 \in V - U$ is a point such that $s = d_V(z,z_0)$. Let $p\colon \mathbb{D} \to V$ be a holomorphic universal covering with $p(0) = z_0$. Let us chose $w \in p^{-1}(z)$ such that $d_{\mathbb{D}}(0,w) = s$. Finally, we denote by $\widetilde{U} \subset \mathbb{D}$ the connected component of w in $p^{-1}(U)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{D}^* & \hookleftarrow & \widetilde{U} & \xrightarrow{p|\widetilde{U}} & U \\ & \searrow & \downarrow & & \downarrow \\ & & \mathbb{D} & \xrightarrow{p} & V \end{array}$$

Applying Schwarz-Pick's lemma and recalling that $p\colon \mathbb{D} \to V$ and $p|\widetilde{U}\colon \widetilde{U} \to U$ are

holomorphic covering maps, we obtain the following:

$$c_U^V(z) = c_{\widetilde{U}}^{\mathbb{D}}(z) = \frac{\rho_{\mathbb{D}}(w)}{\rho_{\widetilde{U}}(w)} \leqslant \frac{\rho_{\mathbb{D}}(w)}{\rho_{\mathbb{D}^*}(w)}.$$

The rest easily follows since $\rho_{\mathbb{D}}(w) = 2/(1-|w|^2)$, $\rho_{\mathbb{D}^*}(w) = 1/|w\log|w||$ ([Hub06, Example 3.3.2]), and $s = d_{\mathbb{D}}(0,w) = 2\tanh^{-1}(|w|)$ due to [Hub06, Exercise 2.1.8]. □

Remark 3.2.2. Suppose that we are in the setting of Proposition 3.2.1. It shows that there exists an upper bound $\lambda(s) < 1$ for $c_U^V(z)$ depending only on $s = d_V(z, V-U)$ and not on U, V, or z. Moreover, $\lambda(s) \sim |s\log(s)|$ as $s \to 0$. In particular, $\lambda(s) \to 0$ as $s \to 0$.

Proposition 3.2.1 and Remark 3.2.2 allow us to estimate the contraction of certain inclusions between countably and finitely punctured Riemann spheres (see [Rem09, Lemma 2.1] for the result of a similar nature).

Proposition 3.2.3. *Suppose that (z_n) is a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} z_n = \infty$ and $\lim_{n\to\infty} \log|z_{n+1}|/\log|z_n| = 1$. Let P be a finite set consisting of at least two elements of (z_n). Then $c_U^V(z)$ converges to 0 as $|z|$ tends to ∞ for $z \in U$, where $U := \mathbb{C} - \{z_n : n \in \mathbb{N}\}$ and $V := \mathbb{C} - P$.*

Proof. Without loss of generality, we can assume that $P \subset \mathbb{D}$ and the sequence of absolute values $(|z_n|)$ is non-decreasing. Let $Q := \{z_n : n \in \mathbb{N}\}$, and let $p\colon \mathbb{H} \to \mathbb{C} - \overline{\mathbb{D}}$ be the holomorphic universal covering defined as $p(z) = \exp(-iz)$ for $z \in \mathbb{H}$.

According to Proposition 3.2.1 and Remark 3.2.2, it suffices to demonstrate that for any $r > 0$, the set $\bigcup_{x\in Q-P} B_V(x,r)$ covers some punctured neighborhood of ∞ in $\widehat{\mathbb{C}}$. Alternatively, it is sufficient to show that for any $r > 0$, there exists $t > 0$ so that

$$\bigcup_{x \in p^{-1}(Q-\overline{\mathbb{D}})} B_{\mathbb{H}}(x,r) \supset \mathbb{H}_t := \{z \in \mathbb{C} : \operatorname{Im}(z) > t\}. \tag{3.1}$$

Indeed, $p(\mathbb{H}_t) = \mathbb{C} - \overline{\mathbb{D}}_{e^t}$ and $p(B_{\mathbb{H}}(x,r)) \subset B_V(p(x),r)$ due to Schwarz-Pick's lemma.

Note that for sufficiently large n, we have that $|z_n| > 1$ and

$$p^{-1}(z_n) = \{2\pi k - \arg(z_n) + i\log(|z_n|) : k \in \mathbb{Z}\} =: \{z_{n,k} : k \in \mathbb{Z}\}. \tag{3.2}$$

Observation (3.2) implies that

$$d_{n,k} := \operatorname{diam}_{\mathbb{H}}\left(\{z : 2\pi k \leqslant \operatorname{Re}(z) \leqslant 2\pi(k+1), \operatorname{Im}(z_{n,k}) \leqslant \operatorname{Im}(z) \leqslant \operatorname{Im}(z_{n+1,k})\}\right) \leqslant \tag{3.3}$$

$$\leqslant \log\left(\frac{\log|z_{n+1}|}{\log|z_n|}\right) + \frac{2\pi}{\log|z_n|}.$$

Estimate (3.3) is derived from the following well-known facts about the hyperbolic distance between two points $z, w \in \mathbb{H}$:

- if $\operatorname{Re}(z) = \operatorname{Re}(w)$, then $d_{\mathbb{H}}(z,w) = |\log(\operatorname{Im}(z)/\operatorname{Im}(w))|$, and
- if $\operatorname{Im}(z) = \operatorname{Im}(w)$, then $d_{\mathbb{H}}(z,w) \leqslant |\operatorname{Re}(z) - \operatorname{Re}(w)|/\operatorname{Im}(z)$. In fact, this upper estimate is the hyperbolic length of the horizontal segment connecting z and w.

Finally, estimate (3.3) implies that $d_{n,k}$ tends to zero independently from k when $n \to \infty$. Thus, inclusion (3.1) holds for certain $t > 0$, and the desired result follows. □

Remark 3.2.4. Suppose that we are in the setting of Proposition 3.2.3. Assume that we know that $\log|z_{n+1}|/\log|z_n|$ is uniformly bounded from above but does not necessarily converge to 1 as $n \to \infty$. Following the proof of Proposition 3.2.3, one can show that there exists $\lambda < 1$ so that $c_U^V(z) < \lambda$ for any $z \in W \cap U$, where $W \subset \widehat{\mathbb{C}}$ is a neighborhood of ∞.

As an application of Proposition 3.2.3, we can obtain the following (see [BR20, Proposition 3.1] for a similar result in the setting of entire maps).

Proposition 3.2.5. *Suppose that $g\colon U \to V$ is a holomorphic covering map, where $U \subset V$ is a domain of $\widehat{\mathbb{C}}$ and $V = \widehat{\mathbb{C}} - P$ with $3 \leqslant |P| < \infty$. Let $x \in P$ be an accumulation point of the set U. Then exactly one of the following two possibilities is satisfied:*

1. *x is an isolated removable singularity of $g\colon U \to V$, or*
2. *$c_U^V(z)$ converges to 0 as the spherical distance $d_{\widehat{\mathbb{C}}}(z,x)$ tends to 0 for $z \in U$.*

Proof. Without loss of generality, we can assume that $x = \infty$. Suppose that condition (1) is not satisfied. We will prove that there exists a constant $a > 1$ such that for every sufficiently large $r > 0$, the annulus $A(r/a, ra) := \{z \in \mathbb{C} : r/a < |z| < ra\}$ contains a point of $\mathbb{C} - U$ (see [BR20, Lemma 3.2] for a similar result in the setting of entire maps). Suppose the contrary, i.e., that there exist sequences (a_n) and (r_n) such that $a_n > 1$, $a_n \to \infty$, $r_n \to \infty$, and $A_n := A(r_n/a_n, r_n a_n) \subset U$ for every $n \in \mathbb{N}$.

In particular, g is defined and meromorphic on A_n, allowing us to consider the sequence of meromorphic functions $g_n\colon A(1/a_n, a_n) \to \widehat{\mathbb{C}}$, $n \in \mathbb{N}$, where $g_n(z) = g(r_n z)$ for every $z \in A(1/a_n, a_n)$. By Montel's theorem, the family (g_n) is normal since each g_n omits the values in the set P. Thus, there exists a subsequence of (g_n) that converges locally uniformly on the punctured plane $\mathbb{C}^*$. Without loss of generality, we assume that this subsequence coincides with the original sequence (g_n). Let $h\colon \mathbb{C}^* \to \widehat{\mathbb{C}}$ be the limiting function. If h is not constant, then by Hurwitz's theorem [Gam01, p. 231], h also omits the values in the set P. This leads to a contradiction because either h have an isolated essential singularity at 0 or ∞, contradicting Great Picard's theorem, or both 0 and ∞ are removable singularities or poles of h, and thus, h is a rational map, which cannot omit any values.

Therefore, h is a constant map. Denote by $w \in \widehat{\mathbb{C}}$ its unique value. Choose a simply connected domain $W \ni w$ such that $\overline{W} \cap P = \{w\} \cap P$. From the previous discussion, $g(A(r_n/2, 2r_n)) \subset W$ for all n large enough. Note that every connected component of $g^{-1}(W) \subset U$ is either a simply connected domain or a simply connected domain with a single puncture (see, for instance, [For91, Theorems 5.10 and 5.11]). Hence,

for every sufficiently large n, either all but at most one of the points in $\mathbb{D}_{2r_n}$ belong to $g^{-1}(W)$, or the same is true for the set $\widehat{\mathbb{C}} - \overline{\mathbb{D}}_{r_n/2}$. The first case cannot happen for infinitely many n, so there must be some $n \in \mathbb{N}$ for which the second case occurs. It immediately leads to a contradiction, as we initially assumed that $x = \infty$ is not an isolated removable singularity of the map $g\colon U \to V$.

Now, we can take z_n to be a point of $\widehat{\mathbb{C}} - U$ in $A(a^{2n}, a^{2n+2})$ for every sufficiently large n. It is easy to see that the sequence (z_n) satisfies the conditions of Proposition 3.2.3. Given that

$$c_U^V(z) = \frac{\rho_V(z)}{\rho_U(z)} \leqslant \frac{\rho_V(z)}{\rho_{\mathbb{C}-Q}(z)},$$

where $Q := P \cup \{z_n : n \in \mathbb{N}\}$, item (2) follows from Proposition 3.2.3. □

3.2.2 Iteration on the unit disk

If $h\colon \mathbb{D} \to \mathbb{D}$ is a non-injective holomorphic map, the Denjoy-Wolff theorem [Aba23, Theorem 3.2.1] states that any point $z \in \mathbb{D}$ converges under iteration of h to a point $z_0 \in \overline{\mathbb{D}}$ that is independent of the initial choice of z. In this section, we explore holomorphic maps on the unit disk that satisfy stronger assumptions, allowing us to achieve more precise control on the behavior of their orbits. Many examples of such maps will appear in Section 2.2. Specifically, many pullback maps of (marked) Thurston maps with a marked set A, where $|A| = 4$, satisfy these conditions; see Section 4.0.1. For other family of such examples, see Section 5.1.2.

We say that $z \in \widehat{\mathbb{C}}$ is a *regular point* of a holomorphic map $g\colon U \to V$, where U and V are domains of $\widehat{\mathbb{C}}$, if either $z \in U$ and $\deg(g, z) = 1$, or z is an isolated removable singularity of g and, after extending g holomorphically to a neighborhood of z, $\deg(g, z) = 1$. A point $z \in \widehat{\mathbb{C}}$ is a *fixed point* of the map $g\colon U \to V$ if either $z \in U$ and $g(z) = z$, or z is an isolated removable singularity of g and, after extending g holomorphically to a neighborhood of z, we have $g(z) = z$. The concepts of repelling or attracting fixed points can be generalized in a similar way.

Theorem 3.2.6. *Let $h\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map, and $\pi\colon \mathbb{D} \to V$ and $g\colon U \to V$ are holomorphic covering maps, where $U \subset V$ is a domain of $\widehat{\mathbb{C}}$ and $V = \widehat{\mathbb{C}} - P$ with $3 \leqslant |P| < \infty$. Suppose that $\pi(h(\mathbb{D})) \subset U$ and $\pi = g \circ \pi \circ h$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{h} & h(\mathbb{D}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ V & \xleftarrow[g]{} & U \end{array}$$

If the map $g\colon U \to V$ is non-injective, then exactly one of the following two possibilities is satisfied:

1. *the map h has a unique fixed point z^* in $\mathbb{D}$, and the h-orbit of every z converges to z^*, or*

2. *the sequence* $(\pi(h^{\circ n}(z)))$ *converges to the same repelling fixed point* $x^* \in P$ *of the map* g*, regardless of the choice of* $z \in \mathbb{D}$.

Proof. Given that the maps $\pi\colon \mathbb{D} \to V$ and $g \circ \pi|\widetilde{U}\colon \widetilde{U} \to V$ are covering maps, where $\widetilde{U} := \pi^{-1}(U)$, it follows [Hat02, Section 1.3, Exercise 16] that $h\colon \mathbb{D} \to \widetilde{U}$ is also a covering map and, in particular, $\widetilde{U}$ is connected. Therefore, according to Schwarz-Pick's lemma, we have $\|\mathrm{D}h(z)\|_{\mathbb{D}} = c^{\mathbb{D}}_{\widetilde{U}}(h(z)) = c^{V}_{U}(\pi(h(z)))$ for every $z \in \mathbb{D}$. At the same time, Great Picard's theorem and the Riemann-Hurwitz formula [Hub06, Appendix A.3] imply that $V - U$ contains at least one point. Hence, the inclusion $I\colon U \hookrightarrow V$ is locally uniformly contracting with respect to the hyperbolic metrics on U and V. As a result, h is locally uniformly contracting with respect to the hyperbolic metric on $\mathbb{D}$. In particular, h has at most one fixed point, and if there exists a point $z \in \mathbb{D}$ such that its orbit $(h^{\circ n}(z))$ converges in $\mathbb{D}$, then every orbit of h converges to the unique fixed point of h.

Let us pick an arbitrary point $z_0 \in \mathbb{D}$. Define $z_n = h^{\circ n}(z_0)$ and $x_n = \pi(h^{\circ n}(z_0))$ for $n \geqslant 0$. Connect the points z_0 and z_1 by the hyperbolic geodesic arc $\delta_0 \subset \mathbb{D}$. We denote by $\delta_n \subset \mathbb{D}$ the arc $h^{\circ n}(\delta)$ that connects z_n and z_{n+1}. By Schwarz-Pick's lemma, the sequence $(\ell_{\mathbb{D}}(\delta_n))$ is non-increasing. In particular, if $d_0 = d_{\mathbb{D}}(z_0, z_1)$, then $d_{\mathbb{D}}(z_n, z_{n+1}) \leqslant \ell_{\mathbb{D}}(\delta_n) \leqslant d_0$ and $d_V(x_n, x_{n+1}) \leqslant \ell_V(\pi(\delta_n)) \leqslant d_0$ for every $n \geqslant 0$.

Further, we structure the proof as a series of claims.

Claim 1. If there exists a compact set $K \subset V$ such that $x_n \in K$ for infinitely many n, then the sequence (z_n) converges in $\mathbb{D}$.

Proof. Since K is compact, there exists a subsequence of (x_n) that converges to a point $x \in K$. Without loss of generality, we can enlarge K so that

- $\pi(\delta_n) \subset K$ for infinitely many n, which is possible since $\ell_V(\pi(\delta_n)) \leqslant d_0$, and
- $B_V(x, 2r) \subset K$ for a certain $r > 0$.

First, we demonstrate that there exists $\lambda_K < 1$ so that $c^V_U(z) \leqslant \lambda_K$ for all $z \in K \cap U$. Indeed, as we mentioned earlier, there exists a point $w \in U - V$. Therefore, for any $z \in K$, the distance $d_V(z, V - U) \leqslant d_V(z, w)$ is uniformly bounded from above. Hence, Proposition 3.2.1 and Remark 3.2.2 imply that such λ_K exists. Finally, from the previous discussions, if $\pi(\delta_{n+1}) \subset K$, then $\ell_{\mathbb{D}}(\delta_{n+1}) \leqslant \lambda_K \ell_{\mathbb{D}}(\delta_n)$. In particular, this shows that $(\ell_{\mathbb{D}}(\delta_n))$ converges to 0 as $n \to \infty$.

Now, we choose m such that $\ell_{\mathbb{D}}(\delta_m) < r(1 - \lambda_K)/2$ and $d_V(x_m, x) < r$. Notice that m is chosen so that $\ell_{\mathbb{D}}(\delta_m) \sum_{i=0}^{\infty} \lambda_K^i < r/2$. Using this fact and applying induction (see [Sel12, Proof of Theorem 2.3] for a similar argument), it can be shown that $\pi(\delta_n) \subset B_V(x, 2r)$ and $\ell_V(\pi(\delta_{n+1})) \leqslant \lambda_K \ell_V(\pi(\delta_n))$ for all $n \geqslant m$. In other words, the distance between z_n and z_{n+1} decreases exponentially, and the convergence follows from the completeness of the hyperbolic metric on $\mathbb{D}$. ■

Claim 2. If the sequence (x_n) converges to $x \in P$ along some subsequence, then the entire sequence (x_n) converges to x.

Proof. For any $x \in P$, we choose a neighborhood $V_x \subset \widehat{\mathbb{C}}$ so that if $x' \neq x''$, then $d_V(y_1, y_2) > d_0$ for any $y_1 \in V_{x'} - P$ and $y_2 \in V_{x''} - P$. Now, suppose that the sequence (x_n) has subsequences that converge to different limits $y' \in \widehat{\mathbb{C}}$ and $y'' \in \widehat{\mathbb{C}}$, respectively. If either of y' or y'' does not belong to P, then by Claim 1, the entire sequence (z_n) must converge to a limit in $\mathbb{D}$, which leads to a contradiction. If we instead assume that $y', y'' \in P$, then it follows that $x_n \in V - \bigcup_{x \in P} V_x$ for infinitely many n. Once again, Claim 1 implies that (z_n) must converge to a limit in $\mathbb{D}$, and it also leads to a contradiction. Thus, the entire sequence (x_n) converges to $x \in P$ as $n \to \infty$. ■

Claim 3. If the sequence (x_n) converges to $x \in P$, then x is a fixed point of the map $g\colon U \to V$.

Proof. Suppose $x \in P$ is not an isolated removable singularity of g. Observe that for sufficiently large n, $\pi(\delta_n)$ lies within any given neighborhood of x. Therefore, based on Proposition 3.2.5 and the fact that $\|\mathrm{D}h(z)\|_{\mathbb{D}} = c_U^V(\pi(h(z)))$ for all $z \in \mathbb{D}$, we have $\ell_{\mathbb{D}}(\delta_{n+1}) \leqslant \lambda \ell_{\mathbb{D}}(\delta_n)$ for n large enough and some $\lambda < 1$. This means that the distance between z_n and z_{n+1} decreases exponentially, so the sequence (z_n) must have a limit in $\mathbb{D}$, leading to a contradiction. Therefore, $x \in P$ is an isolated removable singularity of g. Finally, since $x_n = g(x_{n+1})$ for all $n \geqslant 0$, x must be a fixed point of the map g. ■

Claim 4. If the sequence (x_n) converges to a fixed point $x \in P$ of the map g, then x is a repelling fixed point of g.

Proof. Without loss of generality, we assume $x = 0$. Let us choose a continuous parametrization $\delta_0\colon \mathbb{I} \to \mathbb{D}$ for the arc δ_0. We then define a continuous curve $\delta\colon [0, +\infty) \to \mathbb{D}$ by setting $\delta(t+n) = h^{\circ n}(\delta_0(t))$ for any non-negative integer n and $t \in \mathbb{I}$. It is straightforward to see that $\pi(\delta(t))$ converges to 0 as t tends to ∞, since $\pi(\delta(n)) = x_n \to x = 0$ as $n \to \infty$ and $\ell_V(\pi(\delta([n, n+1]))) \leqslant d_0$ for all $n \geqslant 0$. Furthermore, note that $\pi(\delta(t)) = g(\pi(\delta(t+1))$ for $t \geqslant 0$. Applying the Snail Lemma [Mil06, Lemma 13.2, Corollary 13.3] to the curve $\pi \circ \delta$ and the map g in a neighborhood of 0, we obtain that either $|g'(0)| > 1$ or $g'(0) = 1$.

Suppose $g'(0) = 1$. Further, we assume that g is extended holomorphically to a neighborhood of 0, meaning that 0 is included in the domain of definition of g from now. If we choose an arbitrary simply connected domain $D \subset \widehat{\mathbb{C}}$ such that $D \cap P = \{0\}$, then $g^{-1}(D)$ has a connected component D' containing 0. Moreover, $D' \subset U \cup \{0\}$ is simply connected and $g|D'\colon D' \to D$ is a biholomorphism that fixes 0. Indeed, it easily follows from the fact that 0 is a regular point of g and $g|U\colon U \to V$ is a covering map. Similarly, we can define the local inverse branches $\widetilde{g}_n\colon D \to U \cup \{0\}$ of $g^{\circ n}$ in a neighborhood of 0. In other words, $\widetilde{g}_n := (g^{\circ n}|D_n)^{-1}$, where D_n is the connected component of 0 in $g^{-n}(D)$. In particular, $\widetilde{g}_n$ is a biholomorphism, $\widetilde{g}_n(0) = 0$, and $(\widetilde{g}_n)'(0) = 1/(g^{\circ n})'(0) = 1$ for every $n \in \mathbb{N}$.

Due to the previous discussions, $\widehat{\mathbb{C}} - (U \cup \{0\})$ contains at least three points of $\widehat{\mathbb{C}}$. Hence, Montel's theorem implies that the family $(\widetilde{g}_n)$ is normal. Therefore, up to a subsequence, it converges locally uniformly on D to some holomorphic map $\widetilde{g}$ such that $\widetilde{g} = 0$ and $\widetilde{g}'(0) = 1$. Since $\widetilde{g}'(0) = 1$, $\widetilde{g}$ is injective in a neighborhood of 0.

Thus, the iterates $(g^{\circ n})$ converge uniformly in a neighborhood of 0 to $\widetilde{g}^{-1}$, up to a subsequence. However, if

$$g(z) = z + a_k z^k + \ldots,$$

where $a_k \neq 0$, $k \geqslant 2$, and three dots represent higher order terms, then

$$g^{\circ n}(z) = z + n a_k z^k + \ldots,$$

so $(g^{\circ n})^{(k)}(0)$ diverges as $n \to \infty$. This leads to a contradiction ruling out the possibility $g'(0) = 1$. Thus, $x = 0$ must be a repelling fixed point of the map g. ■

Claims 1-4 imply that if the sequence $(h^{\circ n}(z_0))$ diverges in $\mathbb{D}$, then the sequence $(\pi(h^{\circ n}(z_0)))$ converges to a repelling fixed point $x^* \in P$ of the map g. Furthermore, in this case $(\pi(h^{\circ n}(z)))$ converges to x for every $z \in \mathbb{D}$. Indeed, this follows easily since $d_V(\pi(h^{\circ n}(z_0)), \pi(h^{\circ n}(z)))$ is bounded from above by $d_{\mathbb{D}}(z_0, z)$ due to Schwarz-Pick's lemma. □

4 Thurston maps with a small postsingular set

Summary

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In this section, we focus on the study of Thurston maps $f\colon (S^2, A) \circlearrowleft$ that satisfy the following two conditions:

(I) the marked set A contains exactly four points, and

(II) there exists a set $B \subset A$ such that $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap A| = 3$.

It is evident that conditions (I) and (II) are equivalent to conditions (A)–(C) from Section 1.3.1. Also, it is worth noting that the case when $|\overline{f^{-1}(B)} \cap A| = 4$ or, equivalently, $A \subset \overline{f^{-1}(B)}$, is rather trivial since the corresponding pullback map σ_f is constant due to Remark 2.2.17. Finally, we note that conditions (I) and (II) are preserved under Hurwitz equivalence of Thurston maps.

Using the tools developed Section 3.2 and framework of holomorphic dynamics, we analyze the corresponding pullback maps defined on the one-complex-dimensional Teichmüller space. It allows us to derived several properties of the corresponding Thurston maps and their Hurwitz classes. In particular, in Section 4.1.1 we prove Main Theorem A (see Theorem 4.1.1), and in Section 4.1.2 we prove Main Theorem B (see Theorem 4.1.7) and Corollary 1.3.1 (see Corollary 4.1.9).

4.0.1 One-dimensional pullback maps

In this section, we examine pullback maps associated with Thurston maps that meet conditions (I) and (II), and offer a detailed description of their behavior in Proposition 4.0.1.

Let $A = \{a_1, a_2, a_3, a_4\}$, $B = \{a_{i_1}, a_{i_2}, a_{i_3}\}$, and $C := \overline{f^{-1}(B)} \cap A = \{a_{j_1}, a_{j_2}, a_{j_3}\}$, where $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$. Additionally, assume that i and j are the indices so

that $a_i \in A - B$ and $a_j \in A - C$. Under this conventions, we have that $f(a_j) = a_i$ and $\deg(f, a_j) = 1$.

It is important to note that there may be multiple choices for the set B. However, when $|S_f| = 3$, the set B is uniquely determined by the properties described in condition (II). However, there is another choice that will be relevant in our subsequent calculations: the indexing of the set A.

Now we are ready to introduce the following objects:

- the map $p_B \colon \mathcal{T}_A \to \Sigma$, where $\Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$ represents the moduli space $\mathcal{M}_A$ (see Section 2.2.2), is defined by $p_B([\varphi]) = \varphi(a_i)$, where the representative $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi(a_{i_1}) = 0$, $\varphi(a_{i_2}) = 1$, and $\varphi(a_{i_3}) = \infty$;
- the map $p_C \colon \mathcal{T}_A \to \Sigma$ is defined by $p_C([\varphi]) = \varphi(a_j)$, where the representative $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi(a_{j_1}) = 0$, $\varphi(a_{j_2}) = 1$, and $\varphi(a_{j_3}) = \infty$;
- the map $\omega_f \colon \mathcal{T}_A \to \Sigma$ is defined by $\omega_f = p_C \circ \sigma_f$;
- the map $F_f \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is a unique (which follows from item (3) of Proposition 2.2.13) holomorphic map such that $F_f = \varphi \circ f \circ \psi^{-1}$, where $\varphi, \psi \colon S^2 \to \widehat{\mathbb{C}}$ are orientation-preserving homeomorphisms satisfying $\varphi(a_{i_1}) = \psi(a_{j_1}) = 0$, $\varphi(a_{i_2}) = \psi(a_{j_2}) = 1$, and $\varphi(a_{i_3}) = \psi(a_{j_3}) = \infty$;
- the map $M_{i,j} \colon \Sigma \to \Sigma$ is defined as $h_i \circ h_j^{-1}$, where $h_i \colon \mathcal{M}_A \to \Sigma$ is defined by $h_i([\eta]) = \eta(a_i)$, with $\eta \colon A \to \widehat{\mathbb{C}}$ chosen so that $\eta(a_{i_1}) = 0$, $\eta(a_{i_2}) = 1$, $\eta(a_{i_3}) = \infty$, and the map $h_j \colon \mathcal{M}_A \to \Sigma$ is defined analogously.
- the set $W_f := \widehat{\mathbb{C}} - \overline{F_f^{-1}(\{0, 1, \infty\})}$ is a domain of $\widehat{\mathbb{C}}$;
- the map $G_f := F_f \circ M_{i,j}^{-1} \colon M_{i,j}(W_f) \to \Sigma$.

Of course, the maps F_f and G_f, as well as the domain W_f, also depend on the choice of the subset B, and many other objects defined above depend on the indexing of A. However, for the simplicity, we are going to exclude these dependencies from the notation. Throughout this section, we will maintain the notation and conventions established above. Specifically, if $\widetilde{f} \colon (S^2, A) \circlearrowleft$ is any Thurston map that is Hurwitz equivalent to $f \colon (S^2, A) \circlearrowleft$, then we use the same set B and the same indexing of the set A when we work with the map $\widetilde{f}$ as we do it with f.

Before proceeding to Proposition 4.0.1, we will provide an informal explanation of the significance of condition (II), which may seem unclear at this stage, for the behavior of pullback maps. Let $f \colon (S^2, A) \circlearrowleft$ be a Thurston map satisfying conditions (I) and (II). Choose an arbitrary point $\tau = [\varphi]$ in the Teichmüller space $\mathcal{T}_A$, and let $\sigma_f(\tau) = [\psi] \in \mathcal{T}_A$, where the representatives $\varphi, \psi \colon S^2 \to \widehat{\mathbb{C}}$ are selected so that $g := \varphi \circ f \circ \psi^{-1} \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. By the definition of Teichmüller space, we are free to post-compose φ and ψ with Möbius transformations. Specifically, we assume that $\varphi(a_{i_1}) = \psi(a_{j_1}) = 0$, $\varphi(a_{i_2}) = \psi(a_{j_2}) = 1$, and $\varphi(a_{i_3}) = \psi(a_{j_3}) = \infty$. Of course, one could replace 0, 1, and ∞ with any other distinct points in $\widehat{\mathbb{C}}$ without altering

the conceptual framework, though actual calculations would need to be adjusted accordingly.

With this choice of representatives, as previously discussed, item (3) of Proposition 2.2.13 ensures that g coincides with F_f, which, in particular, does not depends on $\tau \in \mathcal{T}_A$. In other words, condition (II) guarantees a natural and well-behaved choice of representatives φ and ψ for the points $\tau \in \mathcal{T}_A$ and $\sigma_f(\tau) \in \mathcal{T}_A$, respectively

Now, let $x := \psi(a_j)$ and $y := \varphi(a_i)$. It is straightforward to see that $y = F_f(x)$, $p_B(\tau) = y$, and $p_C(\sigma_f(\tau)) = x$. Since $y \neq 0, 1, \infty$, it follows that $x \notin \overline{F_f^{-1}(\{0,1,\infty\})}$. This leads to the following commutative diagram for the pullback map σ_f:

$$\begin{array}{ccc} \mathcal{T}_A & \xrightarrow{\sigma_f} & \sigma_f(\mathcal{T}_A) \\ \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle p_C} \\ \Sigma & \xleftarrow[F_f]{} & W_f \end{array}$$

In other words, knowing the projection $\pi_A(\tau) \in \mathcal{M}_A$ of the point $\tau \in \mathcal{T}_A$ does not allow us to immediately determine $\pi_A(\sigma_f(\tau)) \in \mathcal{M}_A$. However, we do know that $\pi_A(\sigma_f(\tau))$ must lie within the "fiber of the map F_f". Essentially, while $\sigma_f \colon \mathcal{T}_A \to \mathcal{T}_A$ does not descend to a map on the moduli space $\mathcal{M}_A$, its "inverse" behaves in this manner. After this informal explanation, let us proceed to formally state and prove the discussed properties.

Proposition 4.0.1. *The objects introduced above satisfy the following properties:*

1. *diagram (4.1) commutes.*

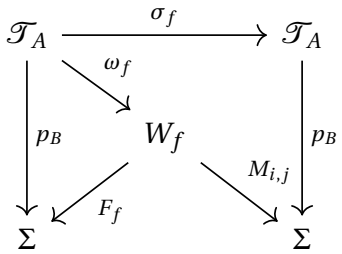

Figure 4.1: Fundamental diagram for Thurston maps satisfying conditions (I) and (II).

2. $S_{F_f} \subset \{0, 1, \infty\}$ *and* $F_f \colon W_f \to \Sigma$ *is a holomorphic covering map;*
3. $p_B \colon \mathcal{T}_A \to \Sigma$, $p_C \colon \mathcal{T}_A \to \Sigma$, *and* $\omega_f \colon \mathcal{T}_A \to W_f$ *are holomorphic covering maps;*
4. $M_{i,j}$ *extends to a Möbius transformation such that* $\{0, 1, \infty\}$ *is an* $M_{i,j}$*-invariant subset;*

5. *$\sigma_f(\mathcal{T}_A) \subsetneq \mathcal{T}_A$ is open and dense in $\mathcal{T}_A$ and $\sigma_f\colon \mathcal{T}_A \to \sigma_f(\mathcal{T}_A)$ is a holomorphic covering map;*

6. *the maps F_f and G_f, as well as the domain W_f, depend only on the Hurwitz equivalence class of the Thurston map f.*

Proof. Let $\tau = [\varphi] \in \mathcal{T}_A$ and $\sigma_f(\tau) = [\psi] \in \mathcal{T}_A$, where the representatives $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that $\varphi(a_{i_1}) = \psi(a_{j_1}) = 0$, $\varphi(a_{i_2}) = \psi(a_{j_2}) = 1$, $\varphi(a_{i_3}) = \psi(a_{j_3}) = \infty$, and the map $\varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. In particular, $\varphi \circ f \circ \psi^{-1}$ coincides with the map F_f. Since, $\omega_f([\varphi]) = p_C([\psi]) = \psi(a_j)$, we have the following

$$(F_f \circ \omega_f)([\varphi]) = F_f(\psi(a_j)) = \varphi(f(a_j)) = \varphi(a_i) = p_B([\varphi]).$$

Thus, $p_B = F_f \circ \omega_f$. At the same time, it is straightforward to verify that $p_B = M_{i,j} \circ p_C$. Finally, $M_{i,j} \circ \omega_f = M_{i,j} \circ p_C \circ \sigma_f = p_B \circ \sigma_f$, and item (1) follows.

Item (2) directly follows from the definition of a singular set. Maps p_B and p_C are holomorphic coverings, as discussed in Section 2.2.2. The map $\omega_f\colon \mathcal{T}_A \to W_f$ is also a holomorphic covering since both $p_B\colon \mathcal{T}_A \to \Sigma$ and $F_f\colon W_f \to \Sigma$ are holomorphic covering maps, and $p_B = F_f \circ \omega_f$ (see [Hat02, Section 1.3, Excercise 16]). Hence, item (3) follows.

The discussion of Section 2.2.2 shows that the maps $h_i, h_j\colon \mathcal{M}_A \to \Sigma$ are holomorphic. Consequently, $M_{i,j}$ is a conformal automorphism of Σ that extends to a Möbius transformation of $\widehat{\mathbb{C}}$ permuting 0, 1, and ∞. Alternative way to prove item (4) is through direct computation. For example, when $i = j$, then $M_{i,j} = \mathrm{id}_{\widehat{\mathbb{C}}}$; if $i = 1$ and $j = 2$, then $M_{1,2}(z) = z/(z-1)$, and so on.

Since $M_{i,j} \circ \omega_f = p_B \circ \sigma_f$, where ω_f and p_B are holomorphic covering maps and $M_{i,j}$ is a Möbius transformation, it follows that $\sigma_f\colon \mathcal{T}_A \to \sigma_f(\mathcal{T}_A)$ is also a holomorphic covering map, where $\sigma_f(\mathcal{T}_A) = p_B^{-1}(M_{i,j}(W_f))$. Note that by Great Picard's Theorem and the Riemann-Hurwitz formula [Hub06, Appendix A.3], the set $\widehat{\mathbb{C}} - W_f$ contains at least one point different from the points $\{0, 1, \infty\}$. Therefore, $\sigma_f(\mathcal{T}_A)$ is different from $\mathcal{T}_A$. Since W_f is open and dense in $\widehat{\mathbb{C}}$, it follows that $\sigma_f(\mathcal{T}_A)$ is open and dense in $\mathcal{T}_A$, establishing item (5).

Finally, let $\widetilde{f}$ be a Thurston map Hurwitz equivalent rel. A to f. Suppose that $\phi_1 \circ \widetilde{f} = f \circ \phi_2$, where $\phi_1, \phi_2 \in \mathrm{Homeo}^+(S^2, A)$. Then $F_f = \widetilde{\varphi} \circ \widetilde{f} \circ \widetilde{\psi}^{-1}$, where $\widetilde{\varphi} = \varphi \circ \phi_1$ and $\widetilde{\psi} = \psi \circ \phi_2$. In particular, $\widetilde{\varphi}(a_{i_1}) = \widetilde{\psi}(a_{j_1}) = 0$, $\widetilde{\varphi}(a_{i_2}) = \widetilde{\psi}(a_{j_2}) = 1$, and $\widetilde{\varphi}(a_{i_3}) = \widetilde{\psi}(a_{j_3}) = \infty$. Therefore, F_f and $F_{\widetilde{f}}$ coincide, as well as W_f and $W_{\widetilde{f}}$, and item (6) follows. □

Remark 4.0.2. Proposition 4.0.1 shows that for Thurston maps satisfying conditions (I) and (II), the corresponding pullback maps admit a commutative diagram analogous to diagram (1.1) from Section 1.2.2. In this context, the Möbius transformation $M_{i,j}$ serves the role of the X-map, the map F_f takes the place of the Y-map, and the domain W_f is an analog of the Hurwitz space. In particular, σ_f has the "g-map" $G_f = F_f \circ M_{i,j}^{-1}\colon M_{i,j}(W_f) \to \Sigma$.

In contrast to the finite degree case, when $f\colon (S^2, A) \circlearrowleft$ is transcendental, the map $F_f\colon W_f \to \Sigma$ is a covering of infinite degree and W_f is a countably punctured Riemann sphere.

Remark 4.0.3. Suppose that we are in the setting of Proposition 4.0.1. We have observed that $\sigma_f(\mathcal{T}_A) = p_B^{-1}(M_{i,j}(W_f))$ and the complement of the set $M_{i,j}(W_f)$ in Σ contains at least one point. Given that $p_B\colon \mathcal{T}_A \to \Sigma$ is a covering map of infinite degree, the pullback map σ_f has infinitely many omitted values, i.e., the points of $\mathcal{T}_A - \sigma_f(\mathcal{T}_A)$, and they are not compactly contained in $\mathcal{T}_A$. Moreover, the set of omitted values of σ_f is not discrete in the Teichmüller space $\mathcal{T}_A$ if there is an essential singularity of the map f that lies within the set $S^2 - A$.

4.1 Thurston theory

4.1.1 Characterization problem

In this section, we present and prove a slightly stronger version of Main Theorem A utilizing the tools developed in Chapter 3 along with the properties of pullback maps established in Proposition 4.0.1.

We recall that the Teichmüller space $\mathcal{T}_A$, where $|A| = 4$, is biholomorphic to the unit disk $\mathbb{D}$, and the metric d_T defined in Section 2.2.2 coincides with the hyperbolic metric $d_{\mathbb{D}}$. If $f\colon (S^2, A) \circlearrowleft$ is a Thurston map satisfying assumptions (I) and (II), then the corresponding pullback map $\sigma_f\colon \mathcal{T}_A \to \mathcal{T}_A$ is holomorphic. It can be established in two ways: either through the general approach outlined in item (1) of Proposition 2.2.19 or by more elementary methods as in item (5) of Proposition 4.0.1. It is worth mentioning that item (5) of Proposition 4.0.1 and Schwarz-Pick's lemma imply that σ_f is distance-decreasing on the Teichmüller space $\mathcal{T}_A$. In particular, item (5) of Proposition 2.2.19 implies that Thurston maps satisfying properties (I) and (II) are never $(2,2,2,2)$-maps.

Theorem 4.1.1. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree that satisfies properties (I) and (II). Then f is realized rel. A if and only if it has no weakly degenerate Levy fixed curve. Moreover,*

1. *if f is realized rel. A by postsingularly finite holomorphic maps $g_1\colon (\widehat{\mathbb{C}}, P_1) \circlearrowleft$ and $g_2\colon (\widehat{\mathbb{C}}, P_2) \circlearrowleft$, then g_1 and g_2 are conjugate by a Möbius transformation M, i.e., $M \circ g_1 = g_2 \circ M$, such that $M(P_1) = P_2$;*

2. *if f is obstructed rel. A, then it has a unique Levy fixed curve up to homotopy in $S^2 - A$.*

Proof. Suppose that the Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized. According to Proposition 2.2.24, $f\colon (S^2, A) \circlearrowleft$ cannot have a Levy fixed curve. Item (1) immediately follows from Proposition 2.2.22, as f is not a $(2,2,2,2)$-map according to the previous discussion.

Now, suppose that the Thurston map $f\colon (S^2, A) \circlearrowleft$ is obstructed. Choose an arbitrary point $\tau_0 \in \mathcal{T}_A$ and define $\tau_n := \sigma_f^{\circ n}(\tau_0)$, $x_n := p_C(\tau_{n+1})$, and $y_n := p_B(\tau_n)$ for every $n \geqslant 0$. Let $\tau_n = [\varphi_n] \in \mathcal{T}_A$, where the representative $\varphi_n\colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi_n(a_{i_1}) = 0$, $\varphi_n(a_{i_2}) = 1$, and $\varphi_n(a_{i_3}) = \infty$. Denote by $\psi_n\colon S^2 \to \widehat{\mathbb{C}}$ the unique (due to Proposition 2.2.12) orientation-preserving homeomorphism so that $g_n := \varphi_n \circ f \circ \psi_n^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic and $\psi_n(a_{j_1}) = 0$, $\psi_n(a_{j_2}) = 1$, and $\psi_n(a_{j_3}) = \infty$. According to the definition of the map F_f, it must coincide with the map g_n for every $n \geqslant 0$. Moreover, it follows that $\tau_{n+1} = [\psi_n]$. From this, we observe that $\varphi_n(a_i) = y_n$, $\psi_n(a_j) = x_n$, and $F_f(x_n) = y_n$ for every $n \geqslant 0$.

According to item (1) of Proposition 4.0.1, we have $p_B(\sigma_f(\mathcal{T}_A)) \subset M_{i,j}(W_f)$ and $p_B = G_f \circ p_B \circ \sigma_f$. Since G_f is not injective, items (2)-(5) of Proposition 4.0.1 allow us to apply Theorem 3.2.6 to the pullback map σ_f. This shows that the sequence (y_n) converges to a repelling fixed point $y \in \{0, 1, \infty\}$ of the map G_f. Given that $y_{n+1} = M_{i,j}(x_n)$, the sequence (x_n) converges to a regular point $x \in \{0, 1, \infty\}$ of the map F_f due to item (4) of Proposition 4.0.1.

We assume that the map F_f is extended holomorphically to a neighborhood of x, meaning that x is included in the domain of definition of F_f from now. Then there exists an open Jordan region $D \subset \widehat{\mathbb{C}}$ such that $D \cap \{0, 1, \infty\} = \{x\}$ and F_f is injective on D. Consider another open Jordan region D' such that $\overline{D'} \subset D$ and the annulus $D - \overline{D'}$ has modulus greater than $5\pi e^{d_0}/\ell^*$, where $d_0 = d_T(\tau_0, \tau_1)$ and $\ell^* = \log(3+2\sqrt{2})$. Observe that $x_n \in D'$ for all n large enough and, in particular, each connected component of $\widehat{\mathbb{C}} - (D - \overline{D'})$ contains two points of the set $\psi_n(A)$.

Finally, by Schwarz-Pick's lemma, we have $d_T(\tau_n, \tau_{n+1}) \leqslant d_T(\tau_0, \tau_1)$ for every $n \geqslant 0$. The existence of a weakly degenerate Levy fixed curve for the Thurston map $f\colon (S^2, A) \circlearrowleft$ then follows from Proposition 3.1.1, applied to $\tau_n = [\varphi_n]$ and $\tau_{n+1} = [\psi_n]$, where n is taken sufficiently large, and the annulus $D - \overline{D'}$ as above. The uniqueness part follows from Proposition 3.1.2 since the sequence $(\pi_A(\sigma_f^{\circ n}(\tau_0)))$ clearly leaves every compact set of the moduli space $\mathcal{M}_A$. □

4.1.2 Hurwitz classes

In Section 4.1.1, we demonstrated how Proposition 4.0.1, especially commutative diagram (4.1), can be helpful for studying Thurston maps that satisfy assumptions (I) and (II). In this section, we further develop this idea by showing the significance of the dynamical properties of the map G_f in understanding the Hurwitz class $\mathcal{H}_{f,A}$ of a Thurston map $f\colon (S^2, A) \circlearrowleft$ that satisfies properties (I) and (II). In particular, we prove Main Theorem B and Corollary 1.3.1. However, before proceeding with their proofs, we present two propositions that relate the fixed points of the map G_f to the Thurston maps in the Hurwitz class $\mathcal{H}_{f,A}$, which are either obstructed or realized depending on the properties of the corresponding fixed point.

Proposition 4.1.2. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map that satisfies conditions (I) and (II). Suppose that $x \in \Sigma$ is a fixed point of the map G_f. Then there exists a homeomorphism $\phi \in \mathrm{Homeo}^+(S^2, A)$ such that the Thurston map $\widetilde{f} := \phi \circ f\colon (S^2, A) \circlearrowleft$ is*

realized by a holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$, *where* $P = \{0, 1, \infty, x\}$.

Proof. Proposition 4.0.1 suggests that there exist points τ_0 and τ_1 in the Teichmüller space $\mathcal{T}_A$ so that $\sigma_f(\tau_0) = \tau_1$ and $p_B(\tau_0) = p_B(\tau_1) = x$. Indeed, choose an arbitrary point τ_1 of $p_B^{-1}(x)$. Since $x \in M_{i,j}(W_f)$, $\sigma_f^{-1}(\tau_1)$ is non-empty, and moreover, $p_B(\sigma_f^{-1}(\tau_1)) = \{x\}$. Thus, we can take τ_0 to be any point of $\sigma_f^{-1}(\tau_1)$.

Let $\tau_0 = [\varphi]$ and $\tau_1 = [\psi]$, where the representatives $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ are chosen such that $g := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic, $\varphi(a_{i_1}) = \psi(a_{i_1}) = 0$, $\varphi(a_{i_2}) = \psi(a_{i_2}) = 1$, $\varphi(a_{i_3}) = \psi(a_{i_3}) = \infty$, and $\varphi(a_i) = \psi(a_i) = x$. It is straightforward to verify that $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$ is a postsingularly finite holomorphic map, where $P = \varphi(A) = \psi(A) = \{0, 1, \infty, x\}$.

Now, define $\phi := \psi^{-1} \circ \varphi \in \mathrm{Homeo}^+(S^2, A)$ and $\widetilde{f} := \phi \circ f\colon (S^2, A) \circlearrowleft$. It is easy to see that $\psi \circ \widetilde{f} \circ \psi^{-1} = g$. Therefore, the Thurston map $\widetilde{f}\colon (S^2, A) \circlearrowleft$ is combinatorially equivalent to $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$. □

Proposition 4.1.3. *Let* $f\colon (S^2, A) \circlearrowleft$ *be a Thurston map that satisfies conditions* (I) *and* (II). *If* G_f *has a repelling fixed point* $x \in \{0, 1, \infty\}$, *then there exists a homeomorphism* $\phi \in \mathrm{Homeo}^+(S^2, A)$ *such that the Thurston map* $\widetilde{f} := \phi \circ f\colon (S^2, A) \circlearrowleft$ *is obstructed. Moreover,* f *is totally unobstructed rel.* A *if and only if none of the points* 0, 1, *or* ∞ *is a repelling fixed point of the map* G_f.

Proof. Suppose that $x \in \{0, 1, \infty\}$ is a repelling fixed point of the map G_f. Assume that G_f is extended holomorphically to a neighborhood of x, meaning that x is included in the domain of definition of G_f from now. Let $U \subset M_{i,j}(W_f) \cup \{x\}$ be a neighborhood of x where G_f is injective and $\overline{U} \subset G_f(U)$. Define the local inverse branch $h\colon G_f(U) \to U$ of G_f at x, i.e., $h := (G_f|U)^{-1}$. Note that every orbit of h converges to x, since h is uniformly distance-decreasing with respect to the hyperbolic metric on U according to Proposition 3.2.1 and Remark 3.2.2.

Claim. The distance $d_\Sigma(y, h(y))$ converges to 0 as $y \in G_f(U)$ tends to x.

Proof. Without loss of generality, assume that $x = 0$, $y \in \mathbb{D}$, and $h(y) \in \mathbb{D}$. Since $x = 0$ is a repelling fixed point of the map G_f, by choosing y sufficiently small, we can ensure that $\lambda|y| \leqslant |h(y)| \leqslant |y|$ for some λ, where $0 < \lambda < 1$. Let $p\colon \mathbb{H} \to \mathbb{D}$ be the holomorphic universal covering defined as $p(z) = \exp(iz)$ for $z \in \mathbb{H}$. Define $y_1 := \arg(y) - i\log(|y|) \in p^{-1}(y)$ and $y_2 := \arg(h(y)) - i\log(|h(y)|) \in p^{-1}(h(y))$. Similarly to the proof of Proposition 3.2.3, we have

$$d_{\mathbb{H}}(y_1, y_2) \leqslant \log\left(\frac{\log|h(y)|}{\log|y|}\right) + \frac{2\pi}{|\log|y||} \leqslant \log\left(\frac{\log|y| + \log\lambda}{\log|y|}\right) + \frac{2\pi}{|\log|y||}.$$

This shows that the distance $d_{\mathbb{H}}(y_1, y_2)$ converges to 0 as y approaches $x = 0$. According to Schwarz-Pick's lemma, the same holds for $d_{\mathbb{D}}(y, h(y))$ and $d_\Sigma(y, h(y))$. ■

Therefore, by making U even smaller, we can assume that $d_\Sigma(y, h(y)) < d_\Sigma(y, y')$, where $y \in G_f(U)$ is any point other than x and $y' \in G_f^{-1}(y)$ with $y \neq h(y)$.

Now, we choose $x_1 \in U - \{x\}$ and let $x_0 = G_f(x_1)$. Similarly to the proof of Proposition 4.1.2, there exists two points τ_1 and τ_2 of the Teichmüller space $\mathcal{T}_A$ so that $\sigma_f(\tau_0) = \tau_1$, $p_B(\tau_0) = x_0$, and $p_B(\tau_1) = x_1$. Since $p_B \colon \mathcal{T}_A \to \Sigma$ is a holomorphic covering map, then there also exists a point $\tau \in \mathcal{T}_A$ so that $d_T(\tau, \tau_1) = d_\Sigma(x_0, x_1)$ and $p_B(\tau) = x_0$.

Let $\tau = [\varphi]$ and $\tau_0 = [\psi]$, where the representatives $\varphi, \psi \colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that $\varphi|A = \psi|A$. Define the homeomorphism $\phi := \varphi^{-1} \circ \psi \in \mathrm{Homeo}^+(S^2, A)$. According to Remark 2.2.15, $\sigma_\phi(\tau) = \tau_0$, and according to Proposition 2.2.14, $\sigma_{\widetilde{f}}(\tau) = \sigma_f(\sigma_\phi(\tau)) = \sigma_f(\tau_0) = \tau_1$, where $\widetilde{f} := \phi \circ f \colon (S^2, A) \circlearrowleft$ is a Thurston map.

Schwarz-Pick's lemma, along with items (3) and (5) of Proposition 4.0.1, implies that

$$d_\Sigma(p_B(\sigma_{\widetilde{f}}^{\circ n}(\tau)), p_B(\sigma_{\widetilde{f}}^{\circ (n+1)}(\tau))) \leqslant d_T(\tau, \sigma_{\widetilde{f}}(\tau)) = d_T(\tau, \tau_1) = d_\Sigma(x_0, x_1).$$

At the same time, it follows from items (1) and (6) of Proposition 4.0.1 that

$$p_B\left(\sigma_{\widetilde{f}}^{\circ (n+1)}(\tau)\right) \in G_f^{-1}\left(p_B(\sigma_{\widetilde{f}}^{\circ n}(\tau))\right).$$

Since $p_B(\tau) = x_0 \in G_f(U)$, and based on the previous assumptions, we have $p_B(\sigma_{\widetilde{f}}^{\circ n}(\tau)) = h^{\circ n}(x_0)$. Thus, $p_B(\sigma_{\widetilde{f}}^{\circ n}(\tau))$ converges to x as $n \to \infty$. Given that $\sigma_{\widetilde{f}}$ is 1-Lipschitz, it cannot have a fixed point. Hence, by Proposition 2.2.16, the Thurston map $\widetilde{f} \colon (S^2, A) \circlearrowleft$ must be obstructed.

Suppose none of points 0, 1, or ∞ is a repelling fixed point of the map G_f. Let $\widehat{f}$ be any Thurston map Hurwitz equivalent rel. A to f. Then, according to item (6) of Proposition 4.0.1, we have $W_f = W_{\widehat{f}}$ and $G_f = G_{\widehat{f}}$. Taking into account Proposition 4.0.1 and applying Theorem 3.2.6, we see that $\sigma_{\widehat{f}}$ must have a fixed point. Thus, it follows from Proposition 2.2.16 that $\widehat{f}$ is realized rel. A. □

Before proceeding, let us recall two folklore results from one-dimensional holomorphic dynamics. The following lemma is well-known in the context of dynamics of rational maps [Mil06]. According to [Eps93, Proposition 16], it also applies more broadly to finite type maps (cf. [Ber93, Theorem 7] and [BDH01, Theorem C]).

Lemma 4.1.4. *Let $g \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ be a postsingularly finite holomorphic map. Then g has only finitely many superattracting periodic orbits, and all other periodic orbits are repelling.*

The next result we need states that a transcendental holomorphic function of finite type, with an at most countable closed set of essential singularities, has infinitely many repelling fixed points. This was proven in the broader setting of general finite type maps in [Eps93, Proposition 14]. Furthermore, in [LZ98, Theorem 1 and the discussion after it] it was proved that the same holds for transcendental meromorphic functions of *bounded type* (cf. [Ben16] and [Pro24a, Lemma A.2] for an alternative approach).

Lemma 4.1.5. *Every transcendental holomorphic map $g \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ of finite type has infinitely many repelling fixed points.*

Remark 4.1.6. It is clear that the map G_f extends to a postsingularly finite holomorphic map having at most countable closed set of essential singularities and a postsingular set contained within $\{0,1,\infty\}$; see Section 4.0.1. Therefore, by Lemma 4.1.4, every fixed point of G_f is either superattracting or repelling. Furthermore, the only possible superattracting fixed points are 0, 1, and ∞. If f is transcendental, then G_f is also transcendental. In this case, by Lemma 4.1.5, G_f has infinitely many repelling fixed points.

Now we are ready to state and prove Main Theorem B.

Theorem 4.1.7. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree that satisfies conditions (I) and (II). Then*

1. *f is totally unobstructed rel. A if and only if there are no two points $a, b \in A$ such that $\deg(f,a) = \deg(f,b) = 1$ and $f(\{a,b\})$ equals $\{a,b\}$ or $A - \{a,b\}$;*
2. *if f is not totally unobstructed rel. A, then its Hurwitz class $\mathcal{H}_{f,A}$ contains infinitely many pairwise combinatorially inequivalent obstructed Thurston maps;*
3. *if f has infinite degree, then its Hurwitz class $\mathcal{H}_{f,A}$ contains infinitely many pairwise combinatorially inequivalent realized Thurston maps.*

Proof. Without loss of generality, we assume that A is indexed so that $B = \{a_1, a_2, a_3\}$, and therefore, $i = 4$ (see the beginning of Section 4.0.1). We then analyze four different cases based on the value of $j, 1 \leqslant j \leqslant 4$, to find out when one of the points 0, 1, or ∞ is a repelling fixed point of the map G_f (this analysis will be needed to apply Proposition 4.1.3). We also recall that $\deg(f, a_j) = 1$ and $f(a_j) = a_i$.

- For the case $j = 1$, we have $M_{4,1}(z) = 1/z$. Therefore,

  $$G_f(0) = F_f(\infty) = (\varphi \circ f \circ \psi^{-1})(\infty) = \varphi(f(a_4)).$$

  This means that 0 is a fixed point of G_f if and only if $f(a_4) = \varphi^{-1}(0) = a_1$. Furthermore, according to Remark 4.1.6, 0 is a repelling fixed point of G_f if and only if $f(a_4) = a_1$ and a_4 is a regular point of f. Similarly, 1 is a repelling fixed point of G_f if and only if $f(a_3) = a_2$ and $\deg(f, a_3) = 1$. Lastly, ∞ is a repelling fixed point of the map G_f if and only if $f(a_2) = a_3$ and $\deg(f, a_2) = 1$.

- For the case $j = 2$, we have $M_{4,2}(z) = (z-1)/z$. Similarly to the previous case, one of the points 0, 1, and ∞ is a repelling fixed point of G_f if and only if $f(a_3) = a_1$ and $\deg(f, a_3) = 1$, or $f(a_4) = a_2$ and $\deg(f, a_4) = 1$, or $f(a_1) = a_3$ and $\deg(f, a_1) = 1$.

- For the case $j = 3$, we have $M_{4,3}(z) = 1 - z$. Here, one of the points 0, 1, and ∞ is a repelling fixed point of G_f if and only if $f(a_2) = a_1$ and $\deg(f, a_2) = 1$, or $f(a_1) = a_2$ and $\deg(f, a_1) = 1$, or $f(a_4) = a_3$ and $\deg(f, a_4) = 1$.

- For the case $j = 4$, we have $M_{4,4} = \mathrm{id}_{\widehat{\mathbb{C}}}$. In this case, one of the points 0, 1, and ∞ is a repelling fixed point of G_f if and only if $f(a_k) = a_k$ and $\deg(f, a_k) = 1$ for some $k = 1, 2, 3$.

Summarizing the calculations above and applying Proposition 4.1.3, we obtain item (1).

To establish item (2), we largely follow the approach used in the proof of [KPS16, Theorem 9.2(V)]. Suppose that f is not totally unobstructed rel. A, i.e., there exists an obstructed Thurston map in $\mathscr{H}_{f,A}$. Without loss of generality, we assume that this map is $f\colon (S^2, A) \circlearrowleft$. Theorem 4.1.1 shows that there exists a Levy fixed curve γ for $f\colon (S^2, A) \circlearrowleft$. Define $f_n := T_\gamma^{\circ n} \circ f\colon (S^2, A) \circlearrowleft$, where $n \in \mathbb{Z}$ and $T_\gamma \in \mathrm{Homeo}^+(S^2, A)$ is the *Dehn twist* about a curve γ. Clearly, each Thurston map f_n has a Levy fixed curve γ, and therefore, is obstructed rel. A by Proposition 2.2.24. We will show that these Thurston maps are pairwise combinatorially inequivalent rel. A.

Suppose the contrary. Then there exist two homeomorphisms $\phi_1, \phi_2 \in \mathrm{Homeo}^+(S^2)$ such that $\phi_1(A) = \phi_2(A) = A$, ϕ_1 is isotopic rel. A to ϕ_2, and $f_n = \phi_1 \circ f_m \circ \phi_2^{-1}$ for some $m \neq n$. It is straightforward to see that $f_n\colon (S^2, A) \circlearrowleft$ has $\phi_1(\gamma)$ as a Levy fixed curve. However, Theorem 4.1.1 states that this Levy fixed curve is unique up to homotopy in $S^2 - A$. This implies that γ and $\phi_1(\gamma)$ are homotopic in $S^2 - A$, and thus, ϕ_1 and ϕ_2 are isotopic rel. A to $T_\gamma^{\circ k}$ for some $k \in \mathbb{Z}$.

One can easily see that f commutes with the Dehn twist T_γ up to isotopy rel. A, meaning $T_\gamma \circ f$ is isotopic rel. A to $f \circ T_\gamma$. Indeed, up to isotopy rel. A, we can assume that f is the identity on a certain annulus in $S^2 - A$ with a core curve γ. Considering the previous discussion, we conclude that f is isotopic rel. A to $f \circ T_\gamma^{\circ(m-n)}$. The following claim proves that it is not possible, and item (2) follows.

Claim. Suppose that f is isotopic rel. A to $f \circ \phi$, where $\phi \in \mathrm{Homeo}^+(S^2, A)$. Then ϕ is isotopic rel. A to id_{S^2}.

Proof. According to Definition 2.2.3, we can assume without loss of generality that $f = f \circ \phi$. There exist orientation-preserving homeomorphisms $\varphi, \psi\colon \widehat{\mathbb{C}} \to S^2$ such that the map $g := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. One can easily check that $g = g \circ h$, where $h := \psi \circ \phi \circ \psi^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Since h must be a Möbius transformation and h fixes the points of the set $\psi(A)$, it follows that $h = \mathrm{id}_{\widehat{\mathbb{C}}}$. Thus, ϕ is also the identity map, proving the claim. ∎

Lemma 4.1.5 implies that the map G_f has infinitely many fixed points when the map f is transcendental. According to Proposition 4.1.2, every such fixed point, apart from 0, 1, and ∞, corresponds to a realized Thurston map in $\mathscr{H}_{f,A}$. However, some of these maps might be combinatorially equivalent rel. A. Nevertheless, we will show that only finitely many of them can be pairwise combinatorially equivalent rel. A.

Consider two Thurston maps f_1 and f_2 realized rel. A by postsingularly finite holomorphic maps $g_1\colon (\widehat{\mathbb{C}}, P_1) \circlearrowleft$ and $g_2\colon (\widehat{\mathbb{C}}, P_2) \circlearrowleft$, respectively, where $P_1 = \{0, 1, \infty, x_1\}$ and $P_2 = \{0, 1, \infty, x_2\}$, with x_1 and x_2 being distinct fixed points of the map G_f (see Proposition 4.1.2). If f_1 and f_2 are combinatorially equivalent rel. A, then f_1 is realized rel. A by both $g_1\colon (\widehat{\mathbb{C}}, P_1) \circlearrowleft$ and $g_2\colon (\widehat{\mathbb{C}}, P_2) \circlearrowleft$. By item (1) of Theorem 4.1.1, there

exists of a Möbius transformation M such that $M \circ g_1 = g_2 \circ M$ and $M(P_1) = P_2$. In particular, M is not the identity map and $\{0,1,\infty\} \subset M(\{0,1,\infty,x_1\})$. Since a Möbius transformation is uniquely determined by its values at three distinct points of $\widehat{\mathbb{C}}$, there can be at most 24 (total number of permutations of the set with four elements) such Thurston maps that are pairwise combinatorially equivalent rel. A, and item (3) follows. □

Let us now proceed to prove Corollary 1.3.1 from Section 1.3.1.2. First of all, we recall the definition of a *parameter space*.

Definition 4.1.8. Let $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic map of finite type. Then the parameter space of g is defined as follows:

$$\mathrm{Par}(g) := \{\varphi \circ g \circ \psi\colon \mathbb{C} \to \widehat{\mathbb{C}} \text{ holomorphic for some } \varphi \in \mathrm{Homeo}^+(\widehat{\mathbb{C}}) \text{ and } \psi \in \mathrm{Homeo}^+(\mathbb{C})\}.$$

Corollary 4.1.9. *Let $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic map such that $|S_g| \leqslant 3$. Then its parameter space $\mathrm{Par}(g)$ contains infinitely many pairwise (topologically or conformally) non-conjugate postsingularly finite maps with four postsingular values.*

Proof. Note that g should have at least two singular values and, according to Great Picard's Theorem, at most two *exceptional* values, i.e., points $w \in \widehat{\mathbb{C}}$ such that the preimage $g^{-1}(w)$ is finite. Clearly, every exceptional value is an asymptotic value of g. Therefore, by post-composing g with a Möbius transformation, we can assume that $\{0,\infty\} \subset S_g \subset \{0,1,\infty\}$ and $g^{-1}(1)$ is infinite. By pre-composing g with an affine transformation, we can assume that $g(0) = 1$ and $g(1) = 1$. Let $x, y \neq 0, 1, \infty$ be two points in $\widehat{\mathbb{C}}$ so that $g(x) = y$.

Next, choose four distinct point a, b, c, and d in S^2, and two orientation-preserving homeomorphisms $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ such that:

- $\varphi(a) = 0$, $\varphi(b) = 1$, $\varphi(c) = y$, and $\varphi(d) = \infty$, and
- $\psi(a) = 0$, $\psi(b) = x$, $\psi(c) = 1$, and $\psi(d) = \infty$.

Then $f := \varphi^{-1} \circ g \circ \psi\colon S^2 \dashrightarrow S^2$ is a topologically holomorphic map of admissible type. Moreover, $\{a,d\} \subset S_f \subset \{a,b,d\}$, and $f(a) = b$, $f(b) = c$, and $f(c) = b$, while $d \in S_f$ is the unique essential singularity of the map f. In other words, f is a Thurston map with the postsingular set $P_f = \{a,b,c,d\}$. Moreover, it is easy to see that f satisfies conditions (I) and (II). Indeed, by setting $B = \{a,b,d\}$, we have $S_f \subset B$ and $\overline{f^{-1}(B)} \cap P_f = \{a,c,d\}$.

Item (3) of Theorem 4.1.7 implies that the Hurwitz class $\mathscr{H}_f$ contains infinitely many realized Thurston maps that are pairwise combinatorially inequivalent. Clearly, each of these maps is realized by a postsingularly finite map from $\mathrm{Par}(g)$. Obviously, these maps must be pairwise (topologically or conformally) non-conjugate, leading to the desired result. □

Remark 4.1.10. If $g\colon \mathbb{C} \to \mathbb{C}$ is a non-constant entire map of finite type, its *entire parameter space* is defined by

$$\mathrm{Par}_E(g) := \{\varphi \circ g \circ \psi\colon \mathbb{C} \to \mathbb{C} \text{ holomorphic for some } \varphi, \psi \in \mathrm{Homeo}^+(\mathbb{C})\}.$$

Following the proof of Corollary 4.1.9, one can show that if g is a transcendental and $|S_g| \leqslant 3$, then $\mathrm{Par}_E(g)$ contains infinitely many postsingularly finite entire maps with four postsingular values that are pairwise non-conjugate.

Remark 4.1.11. Using the framework of *line complexes* (see [GO08, Section XI] or [MPR24, Section 2.7]), it can be shown that there are uncountably many distinct parameter spaces of entire or meromorphic maps having three singular values. Therefore, proof of Corollary 4.1.9 implies that conditions (I) and (II) are met by uncountably many pairwise combinatorially inequivalent realized Thurston maps with four postsingular values. Furthermore, by applying item (1) of Theorem 4.1.1, it is easy to verify that the Thurston maps constructed in the proof Corollary 4.1.9 are not totally unobstructed if $z = 1$ is a regular point of the map g. This shows that the family of Thurston maps under consideration also includes uncountably many pairwise combinatorially inequivalent obstructed Thurston maps. According to Remark 4.1.10, the same observations holds even if we restrict to the class of entire Thurston maps.

4.1.3 Examples

In this section, we provide examples of several families of Thurston maps that satisfy conditions (I) and (II). We also demonstrate how the framework of Sections 4.1.1 and 4.1.2 applies to these rather concrete cases.

Example 4.1.12 (Exponential maps). Let $f\colon S^2 \dashrightarrow S^2$ be an entire transcendental Thurston map with $P_f = \{a, b, c, d\}$ and $S_f = \{a, d\}$, where d is the essential singularity of f. We recall that Thurston maps of this type are called *exponential* Thurston maps. It is easy to see that a must be an omitted value of the map f. In particular, f has one of the following two dynamical portraits on the set $\{a, b, c\}$ as illustrated in Figure 4.2: either the singular value a has pre-period 1 and period 2, or it has pre-period 2 and period 1.



Figure 4.2: Possible orbit of the singular value of an exponential Thurston map with four postsingular values.

In both cases, the map f satisfies properties (I) and (II). In particular, Theorem 4.1.1 shows that f is realized if and only if it has no Levy fixed curve, which must, in fact, be degenerate since f is entire. This, in particular, provides a new proof for the more general result [HSS09, Theorem 2.4] in the case of four postsingular values.

According to item (1) of Theorem 4.1.7, if the singular value a has pre-period 2 and period 1, then f it totally unobstructed. However, if a has pre-period 1 and period 2, the Thurston map f is never totally unobstructed since $f(\{b,c\}) = \{b,c\}$ and $\deg(f,b) = \deg(f,c) = 1$ because otherwise either b or c would be a singular value of f. Moreover, in this case the Hurwitz class $\mathscr{H}_f$ contains infinitely many pairwise combinatorially inequivalent obstructed Thurston maps by item (2) of Theorem 4.1.7. In both cases, item (3) of Theorem 4.1.7 states that the Hurwitz class of f contains infinitely many pairwise combinatorially inequivalent realized Thurston maps.

We further assume that $a = a_1$, $b = a_2$, $c = a_3$, and $d = a_4$, and adopt the notation introduced at the beginning of Section 2.2. Our goal is to derive an explicit formula for the map G_f, analyze its dynamics, and observe the phenomena described in Propositions 4.1.2 and 4.1.3, as well as in the proof of Theorem 4.1.7.

First, we consider the case when the singular value a has pre-period 1 and period 2. Let $B = \{a,b,d\} = \{a_1,a_2,a_4\}$, and then $C = \overline{f^{-1}(B)} \cap P_f = \{a,c,d\} = \{a_1,a_3,a_4\}$. In particular, here $i = 3$ and $j = 2$.

Let us compute the map F_f. It is evident that F_f is a transcendental entire function. Moreover, $S_{F_f} = \{0,\infty\}$. By the classical theory of covering maps, $F_f(z) = \alpha\exp(\lambda z)$ for some $\alpha,\lambda \in \mathbb{C}^*$. Given that $F_f(0) = 1$ and $F_f(1) = 1$, it follows that $\alpha = 1$ and $\lambda = 2\pi i k, k \in \mathbb{Z}^*$, where k is determined by the Hurwitz equivalence class of f, as stated in item (6) of Proposition 4.0.1. At the same time, $W_f = \mathbb{C} - \{l/k : l \in \mathbb{Z}\}$ and $M_{3,2}(z) = 1/z$. Therefore, $G_f(z) = \exp(2\pi i k/z)$.

In particular, 0 is the unique essential singularity of G_f, 1 is a fixed repelling fixed point of G_f of multiplier $2\pi i k$, and $G_f(\infty) = 1$. Moreover, by Lemma 4.1.5, the map G_f has infinitely many repelling fixed points. Thus, Propositions 4.1.2 and 4.1.3 already imply that the Hurwitz class $\mathscr{H}_f$ contains both realized and obstructed Thurston maps.

Now, let the singular value a has pre-period 2 and period 1. Let $B = \{a,c,d\} = \{a_1,a_3,a_4\}$ and then $C = \{b,c,d\} = \{a_2,a_3,a_4\}$. Here, $i = 2$ and $j = 1$. Similarly to the previous case, we find that $F_f(z) = \exp(2\pi i k z)$ with $k \in \mathbb{Z}^*$, $W_f = \mathbb{C} - \{l/k : l \in \mathbb{Z}\}$, $M_{2,1}(z) = z/(z-1)$, and $G_f(z) = \exp(2\pi i k/(z-1))$. In particular, 1 is the essential singularity of G_f, and $G_f(0) = G_f(\infty) = 1$. Thus, none of the points 0, 1, or ∞ is a fixed point of the map G_f. Therefore, Proposition 4.1.3 implies that f is indeed totally unobstructed.

Example 4.1.13 (Entire maps with three singular values). Let $f\colon S^2 \dashrightarrow S^2$ be an entire Thurston map with the postsingular set $P_f = \{a,b,c,d\}$, where $S_f = \{a,b,d\}$ and d is the essential singularity of f. If f satisfies condition (II), then it should have one of the three (up to relabeling) possible dynamical portraits on the set $\{a,b,c\}$ as illustrated in Figure 4.3. Additionally, there are four more dynamical portraits when condition (II) is not satisfied.

Theorem 4.1.1 states that a Thurston map with one of the dynamical portraits as in Figure 4.3 is realized if and only if it has no weakly degenerate Levy fixed curve, which must, in fact, be degenerate since f is entire. Furthermore, according to item (1) of Theorem 4.1.7, f is totally unobstructed for the first two dynamical portraits (from left

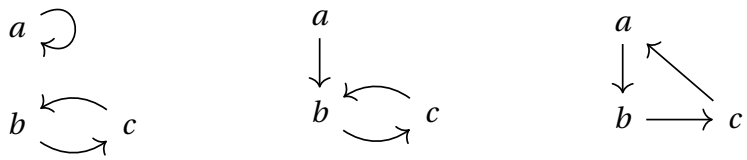

Figure 4.3: Possible dynamical portraits for an entire Thurston map with three singular and four postsingular values that satisfies condition (II).

to right) in Figure 4.3 if and only if $\deg(f,c)=1$, and f is always totally unobstructed for the third dynamical portrait.

Let $a=a_1$, $b=a_2$, $c=a_3$, and $d=a_4$. Then we take $B=S_f=\{a,b,d\}=\{a_1,a_2,a_4\}$, and then $C=\{a,c,d\}=\{a_1,a_3,a_4\}$. In particular, $i=3$ and $j=2$. It can can be verified that F_f is an entire function with $S_{F_f}=\{0,1,\infty\}$. At the same time, $M_{3,2}(z)=1/z$, and therefore $G_f(z)=F_f(1/z)$. In particular, 0 is the unique essential singularity of G_f. Thus, we have the following behavior of the map G_f on the "cusps" 0 and 1 of the moduli space $\mathscr{M}_A\sim\Sigma$:

- for the first dynamical portrait, $G_f(\infty)=0$ and 1 is a fixed point of G_f that, according to Lemma 4.1.4, is repelling if $\deg(f,c)=1$, or superattracting otherwise;
- for the second dynamical portrait, $G_f(\infty)=1$ and 1 is a fixed point of G_f that is repelling if and only if $\deg(f,c)=1$, or superattracting otherwise;
- for the third dynamical portrait, $G_f(1)=0$ and $G_f(\infty)=1$. In particular, neither of 0, 1, and ∞ is a fixed point of the map G_f.

Example 4.1.14 (Non-entire examples). Most of the observations in Example 4.1.13 do not depend on the condition $f^{-1}(d)=\varnothing$, i.e., that the Thurston map f is entire. Furthermore, there are more non-entire examples of Thurston maps that satisfy conditions (I) and (II). For instance, if $f\colon S^2\dashrightarrow S^2$ is a Thurston map with $|S_f|\leqslant 3$, where the postsingular set $P_f=\{a,b,c,d\}$ does not contain an essential singularity and f has one of the postsingular portraits shown in Figure 4.4.

We note that the postsingular portraits depicted on Figure 4.4 can appear for both transcendental and finite-degree Thurston maps. In particular, Theorem 4.1.1 provides a novel proof of celebrated Thurston's characterization theorem [DH93, Theorem 1] for a specific family of finite-degree Thurston maps with four postcritical values.

Of course, there are more examples, e.g., a Thurston map f with $P_f=\{a,b,c,d\}$ satisfies condition (II) if $S_f=\{a,b,c\}$, d is an essential singularity of the map f, $f(a)=d$, and b and c form a 2-cycle for the map f.

Example 4.1.15 (Maps with three postsingular values). Suppose that $f\colon (S^2,A)\circlearrowleft$ is a Thurston map such that $|A|=4$ and $|P_f|\leqslant 3$. It is easy to see that if there exists a marked point $a\in A-P_f$ that is not periodic (i.e., it is either strictly pre-periodic or lands to an essential singularity of f under the iteration), then the pullback map σ_f

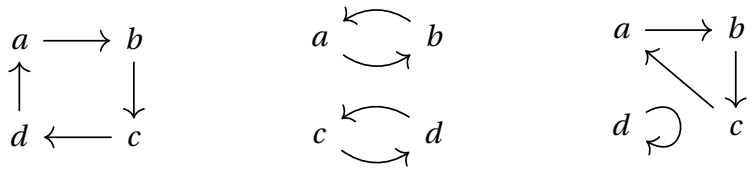

Figure 4.4: Examples of postsingular portraits of non-entire Thurston maps that satisfy conditions (I) and (II).

is constant by Proposition 2.2.13 and the Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized (see also Remark 2.2.17).

On the other hand, if every marked point $a \in A - P_f$ is periodic, then conditions (I) and (II) are clearly satisfied. For instance, if $|P_f| = 3$, we can simply take $B = P_f$. If $|P_f| = 2$, we can take P_f along with one of the points from the set $A - P_f$ to form B. According to Proposition 2.2.10, the case where $|P_f| \leqslant 1$ is not possible.

5 Thurston maps with extra marked points

Summary

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In this chapter, we explore relationships between the properties of Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$, where B is a subset of A. Section 5.1 focuses on two specific cases: when the set $A - B$ consists of either trivial marked points, i.e., points that are not periodic under f (see Section 5.1.1), or a single fixed point of f (see Section 5.1.2).

In Section 5.2, we build on these results to address the more general case. Specifically, in Section 5.2.1, we establish connections between the fixed points of the pullback maps $\sigma_{f,A}$ and $\sigma_{f,B}$, while in Section 5.2.3, we investigate the relation between the realizabilities of Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$, proving Main Theorem C (see Theorem 5.2.5). Additionally, in Section 5.2.2, we apply these results to analyze the relationships between a marked Thurston map and its iterates.

5.1 Preliminary results

In this section, we examine the relationships between the properties of Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$, where B is a subset of A. We focus on two specific cases: when $A - B$ consists of trivial marked points, and when $A - B$ consists of a single fixed point of the map f.

5.1.1 Trivial marked points

Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. We say that a marked point $a \in A$ is *trivial* for $f\colon (S^2, A) \circlearrowleft$ if $a \in A - P_f$ and either a is strictly pre-periodic, or it lands to an

essential singularity of the map f after several iterations (this includes the case when a is already an essential singularity of f). It is easy to see that if $a \in A - P_f$ is not a trivial marked point of $f\colon (S^2, A) \circlearrowleft$, then a must be periodic under f. The following proposition demonstrates that adding trivial marked points to the marked set does not affect the essential properties of a Thurston map (cf. [Par22, Proposition 2.8]).

Proposition 5.1.1. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map, where the set A can be decomposed as $A = B \sqcup C$, where $S_f \subset B$, B is f-pseudo-invariant, and C consists of trivial marked points. Then*

1. *$f\colon (S^2, B) \circlearrowleft$ is realized if and only if $f\colon (S^2, A) \circlearrowleft$ is realized. Moreover, if $|B| \geqslant 3$ and $\tau \in \mathcal{T}_B$ is a fixed point of the pullback map $\sigma_{f,B}$, then the pullback map $\sigma_{f,A}$ has a unique fixed point in $t_{A,B}^{-1}(\tau)$;*

2. *if $f\colon (S^2, B) \circlearrowleft$ has a Levy cycle (or degenerate Levy cycle, Levy multicurve, or degenerate Levy multicurve) $\mathcal{G} = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$, then $f\colon (S^2, A) \circlearrowleft$ has a Levy cycle (or degenerate Levy cycle, Levy multicurve, or degenerate Levy multicurve, respectively) $\widetilde{\mathcal{G}} = \{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \dots, \widetilde{\gamma}_r\}$ such that γ_i and $\widetilde{\gamma}_i$ are homotopic in $S^2 - B$ for each $i = 1, 2, \dots, r$;*

3. *if $\mathcal{G}$ is a Levy cycle (or a degenerate Levy cycle) for $f\colon (S^2, A) \circlearrowleft$, then $\mathcal{G}$ is also a Levy cycle (degenerate Levy cycle, respectively) for $f\colon (S^2, B) \circlearrowleft$.*

Proof. Let $m \geqslant 1$ be an integer such that every point of C lands either to a point in B within at most m iterations, or to an essential singularity of f after at most $m-1$ iterations. It is straightforward to see that $A \subset \overline{f^{-m}(B)}$.

For item (1), we can assume that $|B| \geqslant 3$. If $|B| \leqslant 2$, this can be easily adjusted by moving points from C to B and applying item (2) of Proposition 2.2.10. If the Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized, then naturally $f\colon (S^2, B) \circlearrowleft$ is also realized. Now, assume that $f\colon (S^2, B) \circlearrowleft$ is realized, and the pullback map $\sigma_{f,B}$ has a fixed point $\tau \in \mathcal{T}_B$. According to item (1) of Proposition 2.2.13 and Proposition 2.2.14, it follows that $[\psi]_A = \sigma_{f,A}^{\circ m}([\varphi]_A) = \sigma_{f^{\circ m},A}([\varphi]_A)$ depends only on $[\varphi]_B$ for any $[\varphi]_A \in \mathcal{T}_A$, and that $[\psi]_B = [\varphi]_B$ if $[\varphi]_A \in t_{A,B}^{-1}(\tau)$. Therefore, $\sigma_{f,A}^{\circ m}(t_{A,B}^{-1}(\tau))$ consists of a single point $\mu \in t_{A,B}^{-1}(\tau)$, making μ a unique fixed point of $\sigma_{f,A}$ within $t_{A,B}^{-1}(\tau)$. In particular, Proposition 2.2.16 implies that $f\colon (S^2, A) \circlearrowleft$ is realized.

For items (2) and (3), we assume that $m = 1$, i.e., for every $c \in C$, either $f(c) \in B$ or c is an essential singularity of f. Equivalently, this means that $A \subset \overline{f^{-1}(B)}$. The general case can be obtained through induction on m, and we leave this to the reader.

Suppose $f\colon (S^2, B) \circlearrowleft$ has a Levy cycle $\mathcal{G} = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$, and let σ be the corresponding cyclic permutation of the set $\{1, 2, \dots, r\}$. Then, for each $i = 1, 2, \dots, r$, there exists a simple closed curve $\widetilde{\gamma}_i \subset f^{-1}(\gamma_i)$ such that $\widetilde{\gamma}_i$ is homotopic in $S^2 - B$ to $\gamma_{\sigma(i)}$ and $\deg(f|\widetilde{\gamma}_i\colon \widetilde{\gamma}_i \to \gamma_i) = 1$. Now, consider the collection of essential simple closed curves $\widetilde{\mathcal{G}} := \{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \dots, \widetilde{\gamma}_r\}$ in $S^2 - A$. We aim to show that $\widetilde{\mathcal{G}}$ provides a Levy cycle for the Thurston map $f\colon (S^2, A) \circlearrowleft$.

There exists a simple closed curve $\widetilde{\gamma}_{\sigma(i)} \subset f^{-1}(\gamma_{\sigma(i)})$ such that $\deg(f|\widetilde{\gamma}_{\sigma(i)} \colon \widetilde{\gamma}_{\sigma(i)} \to \gamma_{\sigma(i)}) = 1$ for each $i = 1, 2, \ldots, r$. Since $\widetilde{\gamma}_i$ and $\gamma_{\sigma(i)}$ are homotopic in $S^2 - B$, Corollary 2.1.14 guarantees the existence of a simple closed curve $\delta_i \subset f^{-1}(\widetilde{\gamma}_i)$ such that δ_i is homotopic in $S^2 - \overline{f^{-1}(B)} \subset S^2 - A$ to $\widetilde{\gamma}_{\sigma(i)}$ and $\deg(f|\delta_i \colon \delta_i \to \widetilde{\gamma}_i) = 1$. Thus, $\widetilde{\mathscr{G}}$ provides a Levy cycle for $f \colon (S^2, A) \circlearrowleft$. Moreover, by the second part of Corollary 2.1.14, if $\mathscr{G}$ is a degenerate Levy cycle for $f \colon (S^2, B) \circlearrowleft$, then $\widetilde{\mathscr{G}}$ is also a degenerate Levy cycle for $f \colon (S^2, A) \circlearrowleft$. Similarly, if $\mathscr{G}$ is a multicurve, then $\widetilde{\mathscr{G}}$ is a multicurve as well, establishing item (2).

Suppose that $\mathscr{G} = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ is a Levy cycle for the Thurston map $f \colon (S^2, A) \circlearrowleft$. If $\mathscr{G}$ were not a Levy cycle for $f \colon (S^2, B) \circlearrowleft$, it would mean that some curve $\gamma_j \in \mathscr{G}$ is non-essential in $S^2 - B$. That implies that if $\delta \subset f^{-1}(\gamma_j)$ is a simple closed curve, then δ is non-essential in $S^2 - \overline{f^{-1}(B)} \subset S^2 - A$. However, there exists an essential simple closed curve $\widetilde{\gamma}_j \subset f^{-1}(\gamma_j)$ in $S^2 - A$, which leads to a contradiction. Finally, it is clear that if $\mathscr{G}$ is a degenerate Levy cycle for the Thurston map $f \colon (S^2, A) \circlearrowleft$, then it is also degenerate for the Thurston map $f \colon (S^2, B) \circlearrowleft$. □

Remark 5.1.2. Suppose we are in the setting of Proposition 5.1.1. Let $m \geqslant 1$ be the minimal integer such that every point of C lands either to a point in B within at most m iterations, or to an essential singularity of f after at most $m-1$ iterations. Based on the ideas from the proof of item (1) of Proposition 5.1.1, and Propositions 2.2.10 and 2.2.13, it can be shown that if $|B| = 3$, then $\sigma_{f,A}^{\circ m}$ is constant, although $\sigma_{f,A}^{\circ n}$ is never constant for $1 \leqslant n \leqslant m-1$ (cf. Remark 2.2.17).

5.1.2 Additional marked fixed point

Let $f \colon (S^2, B) \circlearrowleft$ be a Thurston map with $|B| \geqslant 3$. If $c \in S^2 - B$ is a fixed point of the map f, then we can analyze the Thurston map $f \colon (S^2, A) \circlearrowleft$, where $A := B \cup \{c\}$, using the information about the original Thurston map $f \colon (S^2, B) \circlearrowleft$.

In this section, we assume that $f \colon (S^2, B) \circlearrowleft$ is realized. This allows us to gain a deeper understanding of the behavior of the pullback map $\sigma_{f,A}$, which is defined on the Teichmüller space $\mathscr{T}_A$.

According to Proposition 2.2.16, there exists a fixed point $\mu = [\theta_1] = [\theta_2] \in \mathscr{T}_B$ of the pullback map $\sigma_{f,B}$. We assume that the representatives $\theta_1, \theta_2 \colon S^2 \to \widehat{\mathbb{C}}$ are chosen such that θ_1 and θ_2 are isotopic rel. B (in particular, $\theta_1|B = \theta_2|B$) and $g := \theta_1 \circ f \circ \theta_2^{-1} \colon (\widehat{\mathbb{C}}, P) \circlearrowleft$ is a postsingularly finite holomorphic map, where $P := \theta_1(B) = \theta_2(B)$.

Now, we are ready to introduce the following objects:

- $\Delta := t_{A,B}^{-1}(\mu)$, which is a $\sigma_{f,A}$-invariant subset of $\mathscr{T}_A$ according to item (1) of Proposition 2.2.13;
- $\pi \colon \Delta \to \widehat{\mathbb{C}} - P$, defined by $\pi([\varphi]) = \varphi(c)$, where the representative $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi|B = \theta_1|B = \theta_2|B$;
- $\omega \colon \Delta \to \widehat{\mathbb{C}} - P$, defined as $\omega := \pi \circ \sigma_{f,A}|\Delta$;

- $W := \widehat{\mathbb{C}} - \overline{g^{-1}(P)}$, which is a domain of $\widehat{\mathbb{C}}$.

Naturally, the objects introduced above depend on the map f, the marked set B, the fixed point c, as well as the choice of the representatives θ_1 and θ_2 for the fixed point $\mu \in \mathcal{T}_B$ (which might not be unique if f is a $(2,2,2,2)$-map; see Remark 2.2.20). However, for simplicity, we exclude these dependencies from the notation.

Proposition 5.1.3. *The objects defined above satisfy the following properties:*

1. *diagram (1.1) commutes.*

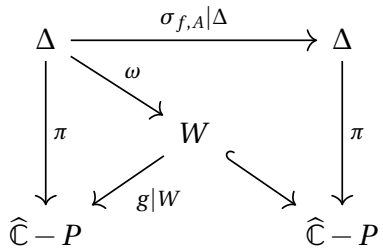

Figure 5.1: Fundamental diagram for a Thurston map with an additional marked fixed point.

2. *Δ is a properly and holomorphically embedded unit disk in $\mathcal{T}_A$, which is $\sigma_{f,A}$-invariant;*

3. *$\pi\colon \Delta \to \widehat{\mathbb{C}} - P$, $g|W\colon W \to \widehat{\mathbb{C}} - P$, and $\omega\colon \Delta \to W$ are holomorphic covering maps;*

4. *$\sigma_{f,A}(\Delta) \subsetneq \Delta$ is open and dense in Δ, and $\sigma_{f,A}|\Delta\colon \Delta \to \sigma_{f,A}(\Delta)$ is a holomorphic covering map.*

Proof. Let $\tau = [\varphi] \in \Delta$, where the representative $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ is chosen such that $\varphi|B = \theta_1|B = \theta_2|B$. By Propositions 2.2.12 and 2.2.13, there exists an orientation-preserving homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $g = \varphi \circ f \circ \psi^{-1}$ and $\psi|B = \theta_1|B = \theta_2|B$. Clearly, $g(\psi(c)) = \varphi(c)$. At the same time, $\pi(\tau) = \varphi(c)$, and $\omega(\tau) = \psi(c)$ because $\sigma_{f,A}(\tau) = [\psi]$. Thus, we have $\pi = \omega \circ g$. Using this, we conclude that diagram (1.1) commutes, establishing item (1).

Item (2) follows directly from Proposition 2.2.11 and the previous discussion. The map $g|W\colon W \to \widehat{\mathbb{C}} - P$ is a covering because $S_g \subset P$. Similarly, the map $\pi\colon \Delta \to \widehat{\mathbb{C}} - P$ is a holomorphic covering map due to the discussion of Section 2.2.2, and particularly, commutative diagram (2.1) and Proposition 2.2.11. Therefore, using the relation $\pi = \omega \circ g$ and [Hat02, Section 1.3, Excercise 16], we deduce that $\omega\colon \Delta \to W$ is also a holomorphic covering map, thereby completing item (3).

Similarly, since $\omega = \pi \circ \sigma_{f,A}|\Delta$, the restriction $\sigma_{f,A}|\Delta\colon \Delta \to \sigma_{f,A}(\Delta)$ is a holomorphic covering map. Note that by Great Picard's Theorem and the Riemann-Hurwitz formula [Hub06, Appendix A.3], the set $\widehat{\mathbb{C}} - W$ contains at least one point different from the points of the set P. Therefore, $\sigma_{f,A}(\Delta) = \pi^{-1}(W)$ is different from Δ. Since W is open and dense in $\widehat{\mathbb{C}}$, it follows that $\sigma_{f,A}(\Delta)$ is open and dense in Δ, establishing item (4). □

Remark 5.1.4. Proposition 5.1.3 shows that the restriction $\sigma_{f,A}|\Delta$ of the pullback map $\sigma_{f,A}$ admits a commutative diagram (5.1) similar to diagram (1.1) from Section 1.2.2 and diagram (4.1) from Proposition 4.0.1.

5.2 Expanding marked set

In this section, we continue exploring the relationships between the properties of Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$, where B is a subset of A.

5.2.1 Fixed points of pullback maps

The following proposition established the relationship between the fixed points of the pullback maps $\sigma_{f,A}$ and $\sigma_{f,B}$, where $f\colon (S^2, A) \circlearrowleft$ is a Thurston map and B is a subset of A.

Proposition 5.2.1. *Let $f\colon (S^2, A) \circlearrowleft$ be a realized Thurston map, and let $B \subset A$ be an f-pseudo-invariant set with $|B| \geqslant 3$ and $S_f \subset B$. If $\mu \in \mathcal{T}_B$ is a fixed point of the pullback map $\sigma_{f,B}$, then the pullback map $\sigma_{f,A}$ has a unique fixed point in $t_{A,B}^{-1}(\mu)$. Furthermore, every fixed point of $\sigma_{f,A}$ belongs to the $t_{A,B}$-fiber of a fixed point of $\sigma_{f,B}$.*

Proof. First of all, if τ is a fixed point of $\sigma_{f,A}$, then by item (1) of Proposition 2.2.13, $t_{A,B}(\tau)$ is a fixed point of $\sigma_{f,B}$. Next, we will show that the pullback map $\sigma_{f,A}$ has a unique fixed point in $t_{A,B}^{-1}(\mu)$, where μ is a fixed point of the pullback map $\sigma_{f,B}$.

The answer is straightforward for Thurston maps that are not $(2,2,2,2)$-maps due to item (4) of Proposition 2.2.19, which implies that the associated pullback maps can have at most one fixed point. However, the case of $(2,2,2,2)$-maps is non-trivial.

First, we demonstrate that trivial marked points of the set $C := A - B$ can be ignored.

Claim 1. Let $C = C_1 \sqcup C_2$, where C_2 consists of trivial marked points of the Thurston map $f\colon (S^2, A) \circlearrowleft$. If the pullback map $\sigma_{f,B\cup C_1}$ has a unique fixed point in $t_{B\cup C_1,B}^{-1}(\mu)$, then the pullback map $\sigma_{f,A}$ also has a unique fixed point in $t_{A,B}^{-1}(\mu)$.

Proof. Let τ' be the unique fixed point of $\sigma_{f,B\cup C_1}$ such that $\tau' \in t_{B\cup C_1,B}^{-1}(\mu)$. According to item (1) of Proposition 5.1.1, there exists a unique fixed point of $\sigma_{f,A}$ within $t_{A,B\cup C_1}^{-1}(\tau')$.

Now, suppose that τ is any fixed point of $\sigma_{f,A}$ in $t_{A,B}^{-1}(\mu)$. By item (1) of Proposition 2.2.13, $t_{A,B\cup C_1}(\tau)$ is a fixed point of $\sigma_{f,B\cup C_1}$. Since $t_{A,B} = t_{B\cup C_1,C_1} \circ t_{A,B\cup C_1}$, it follows that $t_{A,B\cup C_1}(\tau) \in t_{B\cup C_1,B}^{-1}(\mu)$, and therefore, $t_{A,B\cup C_1}(\tau) = \tau'$. Thus, any fixed point of $\sigma_{f,A}$ in $t_{A,B}^{-1}(\mu)$ must belong to $t_{A,B\cup C_1}^{-1}(\tau')$, and the rest follows. ■

Next, we address the case when all points of the set $C = A - B = \{c_1, c_2, \ldots, c_l\}$ are fixed under the map f.

Claim 1. Suppose that every point $c \in C$ is fixed point of f. Then the pullback map $\sigma_{f,A}$ has a unique fixed point in $t_{A,B}^{-1}(\mu)$.

Proof. Define $B_i = B \cup \{c_1, c_2, \dots, c_i\}$ for each $i = 0, 1, \dots, l$. In particular, $B_0 = B$ and $B_l = A$. We prove by induction on i with $0 \leqslant i \leqslant l$ that σ_{f,B_i} has a unique fixed point in $t_{B_i,B}^{-1}(\mu)$. The base case is obvious, since $t_{B_0,B}^{-1}(\mu) = \{\mu\}$ and μ is a fixed point of σ_{f,B_0}.

Now suppose that for some j with $0 \leqslant j < l$, the pullback map σ_{f,B_j} has a unique fixed point $\tau_j \in t_{B_j,B}^{-1}(\mu)$. Define $\Delta := t_{B_{j+1},B_j}^{-1}(\tau_j)$. According to Proposition 2.2.11, Δ is a properly and holomorphically embedded unit disk in the Teichmüller space $\mathcal{T}_{B_{j+1}}$. Furthermore, by item (1) of Proposition 2.2.13, Δ is $\sigma_{f,B_{j+1}}$-invariant, and according to item (4) of Proposition 5.1.3, the restriction $\sigma_{f,B_{j+1}}|\Delta\colon \Delta \to \Delta$ is holomorphic but not an automorphism of Δ.

Note that the $\sigma_{f,B_{j+1}}$-orbit of any point in Δ forms a pre-compact set within $\mathcal{T}_{B_{j+1}}$, and thus within Δ as well. This follows easily, for example, from the facts that $\sigma_{f,B_{j+1}}$ has a fixed point in $\mathcal{T}_{B_{j+1}}$ due to Proposition 2.2.16, and that $\sigma_{f,B_{j+1}}$ is 1-Lipschitz on $\mathcal{T}_{B_{j+1}}$, as noted in item (2) of Proposition 2.2.19. As a result, by the Denjoy-Wolff theorem [Aba23, Theorem 3.2.1], the pullback map $\sigma_{f,B_{j+1}}$ has a unique fixed point in Δ.

Now, suppose that τ is any fixed point of the pullback map $\sigma_{f,B_{j+1}}$ in $t_{B_{j+1},B}^{-1}(\mu)$. By item (1) of Proposition 2.2.13, it is straightforward to see that $t_{B_{j+1},B_j}(\tau)$ is a fixed point of σ_{f,B_j}. Moreover, since $t_{B_{j+1},B} = t_{B_j,B} \circ t_{B_{j+1},B_j}$, we have $t_{B_{j+1},B_j}(\tau) \in t_{B_j,B}^{-1}(\mu)$. As τ_j is unique fixed point of σ_{f,B_j} in $t_{B_j,B}^{-1}(\mu)$, it follows $t_{B_{j+1},B_j}(\tau) = \tau_j$, or equivalently, $\tau \in t_{B_{j+1},B_j}^{-1}(\tau_j) = \Delta$. Since we have already shown that $\sigma_{f,B_{j+1}}$ has a unique fixed point in Δ, the uniqueness of the fixed point of $\sigma_{f,B_{j+1}}$ within $t_{B_{j+1},B}^{-1}(\mu)$ follows. ■

Now, consider the general case. By Claim 1, we may assume that the set C does not contain any trivial marked points of the Thurston map $f\colon (S^2, A) \circlearrowleft$. Therefore, every point $c \in C$ is periodic under f, and we can choose $m \geqslant 1$ such that every $c \in C$ is fixed under $f^{\circ m}$. Applying Claim 2 to the Thurston map $f^{\circ m}\colon (S^2, A) \circlearrowleft$, we find that there is a unique fixed point τ of $\sigma_{f^{\circ m},A}$ in $t_{A,B}^{-1}(\mu)$. By Proposition 2.2.14, we know that $\sigma_{f^{\circ m},A} = \sigma_{f,A}^{\circ m}$, and in particular, τ is a periodic point of $\sigma_{f,A}$.

If τ is a fixed point of $\sigma_{f,A}$, we are done, because any other fixed point of $\sigma_{f,A}$ in $t_{A,B}^{-1}(\mu)$ would lead to a fixed point of $\sigma_{f^{\circ m},A} = (\sigma_{f,A})^{\circ m}$, contradicting Claim 2. Now, suppose that τ is not a fixed point of $\sigma_{f,A}$. Then τ and $\sigma_{f,A}(\tau)$ would be distinct fixed points of $\sigma_{f^{\circ m},A}$. However, both τ and $\sigma_{f,A}(\tau)$ lie in $t_{A,B}^{-1}(\mu)$, as $t_{A,B}^{-1}(\mu)$ is $\sigma_{f,A}$-invariant by item (1) of Proposition 2.2.13. This leads to a contradiction with Claim 2. □

Corollary 5.2.2. *Let $f\colon (S^2, A) \circlearrowleft$ be a realized Thurston map. Suppose that $B \subset A$ is an f-pseudo-invariant set containing S_f, and there exist two orientation-preserving homeomorphisms $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ such that φ and ψ are isotopic rel. B and $g = \varphi \circ f \circ \psi^{-1}\colon (\widehat{\mathbb{C}}, \varphi(B)) \circlearrowleft$ is a holomorphic postsingularly finite map. Then, there exist two orientation-preserving homeomorphisms $\widetilde{\varphi}, \widetilde{\psi}\colon S^2 \to \widehat{\mathbb{C}}$ such that*

1. *$\widetilde{\varphi}$ and $\widetilde{\psi}$ are isotopic rel. A;*
2. *φ and $\widetilde{\varphi}$ are isotopic rel. B, and ψ and $\widetilde{\psi}$ are isotopic rel. B;*

3. $g = \widetilde{\varphi} \circ f \circ \widetilde{\psi}^{-1} \colon (\widehat{\mathbb{C}}, \widetilde{\varphi}(A)) \circlearrowleft$.

Moreover, if $|B| \geqslant 3$, *the orientation-preserving homeomorphisms* $\widetilde{\varphi}$ *and* $\widetilde{\psi}$ *that satisfy properties* (1)-(3) *are unique up isotopy rel.* A.

Proof. This result is follows directly from Proposition 5.2.1, along with Proposition 2.2.13 and Lemma 2.1.1. □

5.2.2 Passing to an iterate

The following proposition establishes a connection between certain properties of a marked Thurston map and of those of its iterates:

Proposition 5.2.3. *Let* $f \colon (S^2, A) \circlearrowleft$ *be a Thurston map and* $m \geqslant 1$. *Then:*

1. $f \colon (S^2, A) \circlearrowleft$ *is realized if and only if* $f^{\circ m} \colon (S^2, A) \circlearrowleft$ *is realized;*
2. *if* $f^{\circ m} \colon (S^2, A) \circlearrowleft$ *has a Levy cycle (or a degenerate Levy cycle)* $\mathscr{G}$ *of period* r, *then* $f \colon (S^2, A) \circlearrowleft$ *has a Levy cycle (or a degenerate Levy cycle, respectively)* $\mathscr{F}$ *of period* mr *such that* $\mathscr{G} \subset \mathscr{F}$;
3. $f^{\circ m} \colon (S^2, A) \circlearrowleft$ *has a Levy cycle (or a degenerate Levy cycle)* $\mathscr{G}$ *and there exist* $\gamma \in \mathscr{G}$ *and* $\tau \in \mathcal{T}_A$ *such that* $l_\gamma(\tau) < \ell^* e^{-(mr-1)d_0}$, *where* $d_0 := d_T(\tau, \sigma_{f,A}(\tau))$, *then* $f \colon (S^2, A) \circlearrowleft$ *has a Levy multicurve (or a degenerate Levy multicurve, respectively)* $\mathscr{F}$ *of period at most* mr;
4. *if* $f \colon (S^2, A) \circlearrowleft$ *has a Levy cycle (or a degenerate Levy cycle, Levy multicurve, or degenerate Levy multicurve)* $\mathscr{G}$ *of period* r, *then* $f^{\circ m} \colon (S^2, A) \circlearrowleft$ *has a Levy cycle (or degenerate Levy cycle, Levy multicurve, or degenerate Levy multicurve, respectively)* $\mathscr{F}$ *of period* $r/\gcd(m, r)$ *such that* $\mathscr{F} \subset \mathscr{G}$.

Proof. For item (1), we can assume that $|A| \geqslant 3$ due to item (2) of Proposition 2.2.10. Suppose the Thurston map $f \colon (S^2, A) \circlearrowleft$ is realized. According to Proposition 2.2.16, the pullback map $\sigma_{f,A}$ has a fixed point $\tau \in \mathcal{T}_A$. By Proposition 2.2.14, τ is also a fixed point of $\sigma_{f^{\circ m},A}$, since $\sigma_{f^{\circ m},A} = (\sigma_{f,A})^{\circ m}$. From Proposition 2.2.16, we conclude that $f^{\circ m} \colon (S^2, A) \circlearrowleft$ is also realized.

Now, suppose that $f^{\circ m} \colon (S^2, A) \circlearrowleft$ is realized. As previously discussed, this implies the existence of a periodic point of period m for the pullback map $\sigma_{f,A}$. If f is not a $(2,2,2,2)$-map, then by item (4) of Proposition 2.2.19, every periodic point of $\sigma_{f,A}$ must be fixed, and the desired result follows from Proposition 2.2.16.

Suppose that f is a $(2,2,2,2)$-map. As discussed in Section 2.2.2, the Teichmüller space $\mathcal{T}_{P_f}$ is biholomorphic to the unit disk. Since σ_{f,P_f} has a periodic point by item (1) of Proposition 2.2.13, then Schwarz's lemma and the classification of automorphisms of $\mathbb{D}$ imply that σ_{f,P_f} must have a fixed point μ. Clearly, μ is also a fixed point of $\sigma_{f^{\circ m},P_f}$. Therefore, according to Proposition 5.2.1, $\sigma_{f^{\circ m},A}$ has a unique fixed point τ in $t^{-1}_{A,P_f}(\mu)$. Note that both τ and $\sigma_{f,A}(\tau)$ are fixed points of $\sigma_{f^{\circ m},A}$. Moreover, $\sigma_{f,A}(\tau) \in t^{-1}_{A,P_f}(\mu)$,

since $t^{-1}_{A,P_f}(\mu)$ is $\sigma_{f,A}$-invariant according to item (1) of Proposition 2.2.13. Therefore, τ and $\sigma_{f,A}(\tau)$ must coincide, and item (1) follows from Proposition 2.2.16.

Suppose $f^{\circ m}\colon (S^2,A)\circlearrowleft$ has a Levy cycle $\mathscr{G}=\{\gamma_1,\gamma_2,\dots,\gamma_r\}$. Let σ denote the corresponding cyclic permutation of the set $\{1,2,\dots,r\}$, and for each $i=1,2,\dots,r$, let $\widetilde{\gamma}_i\subset f^{-m}(\gamma_i)$ be a simple closed curve such that $\widetilde{\gamma}_i$ is homotopic in S^2-A to $\gamma_{\sigma(i)}$ and $\deg(f^{\circ m}|\widetilde{\gamma}_i\colon \widetilde{\gamma}_i\to\gamma_i)=1$. Denote $\gamma_{i,k}:=f^{\circ(m-k)}(\widetilde{\gamma}_i)$ for each $i=0,1,\dots m-1$. In particular, $\gamma_{i,0}=\gamma_i$. It is clear that each $\gamma_{i,k}$ must be an essential curve in S^2-A, as $\widetilde{\gamma}_i$ would not be essential otherwise. It is straightforward to verify that

$$\mathscr{F}:=\{\gamma_{i,k}\colon i=1,2,\dots,r \text{ and } k=0,1,\dots,m-1\}$$

provides a Levy cycle for the map $f\colon (S^2,A)\circlearrowleft$. Similarly, if $\mathscr{G}$ is a degenerate Levy cycle for $f^{\circ m}\colon (S^2,A)\circlearrowleft$, then $\mathscr{F}$ is a degenerate Levy cycle for $f\colon (S^2,A)\circlearrowleft$, and this establishes item (2).

Suppose that additionally there exist $\gamma\in\mathscr{G}$ and $\tau\in\mathcal{T}_A$ such that $l_\gamma(\tau)<\ell^* e^{-(mr-1)d_0}$. Let us choose a minimal Levy cycle $\mathscr{F}'$ such that $\gamma\in\mathscr{F}'$ and $\mathscr{F}'\subset\mathscr{F}$. With this, item (3) follows from Proposition 3.1.6 applied to the minimal Levy cycle $\mathscr{F}'$.

Finally, item (4) can be easily derived from the definition of a Levy cycle (degenerate Levy cycle, Levy multicurve, degenerate Levy multicurve, respectively); see Section 2.2.4. □

5.2.3 Realizability question

In this section, we investigate the properties of a Thurston map $f\colon (S^2,A)\circlearrowleft$ under the assumption that the Thurston map $f\colon (S^2,B)\circlearrowleft$ is realized, where B is a subset of A. In particular, we establish Main Theorem C. We begin by proving a result that demonstrates that Levy cycles for the map $f\colon (S^2,A)\circlearrowleft$ must exhibit particularly nice properties.

Proposition 5.2.4. *Let $f\colon (S^2,A)\circlearrowleft$ be a Thurston map, and let $B\subset A$ be an f-pseudo-invariant subset containing S_f. Suppose that the Thurston map $f\colon (S^2,B)\circlearrowleft$ is realized, and the Thurston map $f\colon (S^2,A)\circlearrowleft$ has a Levy cycle $\mathscr{G}$. Then $\mathscr{G}$ is degenerate, and every $\gamma\in\mathscr{G}$ is non-essential in S^2-B.*

Proof. Let $\mathscr{G}=\{\gamma_1,\gamma_2,\dots,\gamma_r\}$, and let σ denote the corresponding cyclic permutation of the set $\{1,2,\dots,r\}$. Then, for each $i=1,2,\dots,r$, there exists a simple closed curve $\widetilde{\gamma}_i\subset f^{-1}(\gamma_i)$ such that $\widetilde{\gamma}_i$ is homotopic in S^2-B to $\gamma_{\sigma(i)}$ and $\deg(f|\widetilde{\gamma}_i\colon\widetilde{\gamma}_i\to\gamma_i)=1$.

Suppose that some γ_j is essential in S^2-B. Then $\widetilde{\gamma}_{\sigma^{-1}(j)}$ is also essential in S^2-B, since it is homotopic in S^2-A to γ_j. This implies that $\gamma_{\sigma^{-1}(j)}$ is essential in S^2-B. Repeatedly applying this argument, we show that each $\gamma_i\in\mathscr{G}$ is essential in S^2-B, and therefore, $\mathscr{G}$ is a Levy cycle for the Thurston map $f\colon (S^2,B)\circlearrowleft$. However, this leads to a contradiction with Proposition 2.2.24.

Now, let us demonstrate that $\mathscr{G}$ forms a degenerate Levy cycle for $f\colon (S^2,A)\circlearrowleft$. First, we suppose that $|B|\geqslant 3$. For each $i=1,2,\dots,r$, since γ_i is non-essential in S^2-B, we

can select the unique connected component D_i of $S^2 - \gamma_i$ that contains at most one point of B. Similarly, we introduce $\widetilde{D}_i$ as the unique connected component of $S^2 - \widetilde{\gamma}_i$ that contains at most one point of B.

Suppose that some D_i contains a single point q_i from the set B. Since $S_f \subset B$, by the classical theory of covering maps (see, for instance, [MPR24, Propositions 2.11 and 2.12]), there exists a unique connected component U_i of $S^2 - \widetilde{\gamma}_i$ that contains a single point p_i of $\overline{f^{-1}(B)} \supset B$. Moreover, $f(p_i) = q_i$, and $f|U_i - \{p_i\} \colon U_i - \{p_i\} \to D_i - \{q_i\}$ is a covering map of degree $\deg(f|\widetilde{\gamma}_i \colon \widetilde{\gamma}_i \to \gamma_i) = \deg(f, p_i)$. Clearly, U_i must coincide with $\widetilde{D}_i$. Since $\deg(f|\widetilde{\gamma}_i \colon \widetilde{\gamma}_i \to \gamma_i) = 1$, we also have $\deg(f|\widetilde{D}_i \colon \widetilde{D}_i \to D_i) = 1$. Similarly, when $D_i \cap B = \varnothing$, we can also conclude that $f(\widetilde{D}_i) = D_i$ and $\deg(f|\widetilde{D}_i \colon \widetilde{D}_i \to D_i) = 1$.

Finally, it is easy to verify that $D_{\sigma(i)}$ and $\widetilde{D}_i$ are homotopic in $S^2 - A$. Indeed, there exists an isotopy $(\varphi_t)_{t\in\mathbb{I}}$ in S^2 rel. A with $\varphi_0 = \mathrm{id}_{S^2}$ and $\varphi_1(\partial D_{\sigma(i)}) = \varphi_1(\gamma_{\sigma(i)}) = \widetilde{\gamma}_i = \partial\widetilde{D}_i$ ([Bus10, Theorem A.3]; see also [FM12, Sections 1.2.5 and 1.2.6]). Since both $D_{\sigma(i)}$ and $\widetilde{D}_i$ contain at most one point of B and $|B| \geqslant 3$, we have $\varphi_1(D_{\sigma(i)}) = \widetilde{D}_i$. Thus, $D_{\sigma(i)}$ and $\widetilde{D}_i$ are homotopic in $S^2 - A$, implying that $\mathscr{G}$ is a degenerate Levy cycle for the Thurston map $f \colon (S^2, A) \circlearrowleft$.

In the case when the set B contains fewer then three points, we can use item (1) of Proposition 2.2.10 to prove that $\mathscr{G}$ is a degenerate Levy cycle for the Thurston map $f \colon (S^2, A) \circlearrowleft$. In particular, open Jordan regions D_i and $\widetilde{D}_i$ can be defined as the unique connected components of $S^2 - \gamma_i$ and $S^2 - \widetilde{\gamma}_i$, respectively, that contain no points of the set B. □

Theorem 5.2.5. *Let $f \colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree. Suppose that $B \subset A$ is an f-pseudo-invariant set containing S_f. The Thurston map $f \colon (S^2, A) \circlearrowleft$ is realized if and only if it has no degenerate Levy multicurve consisting of curves that are non-essential in $S^2 - B$, and $f \colon (S^2, B) \circlearrowleft$ is realized.*

Proof. As stated in Proposition 2.2.24, if the Thurston map $f \colon (S^2, A) \circlearrowleft$ has a Levy cycle, then $f \colon (S^2, A) \circlearrowleft$ is obstructed. At the same time, it is clear that if $f \colon (S^2, B) \circlearrowleft$ is obstructed, then $f \colon (S^2, A) \circlearrowleft$ must also be obstructed, thus confirming one direction of the desired result. Therefore, from this point onward, we assume that $f \colon (S^2, A) \circlearrowleft$ is obstructed while $f \colon (S^2, B) \circlearrowleft$ is realized. According to Proposition 5.2.4, it remains to prove that $f \colon (S^2, A) \circlearrowleft$ has a Levy multicurve.

At the same time, items (1) and (2) of Proposition 5.1.1 allow us to assume that the set $C := A - B$ contains no trivial marked points, meaning that every point $c \in C$ is periodic. We will focus on the case when every point $c \in C$ is fixed under f. We aim to show that for any such obstructed Thurston map $f \colon (S^2, A) \circlearrowleft$ (satisfying the assumptions of this theorem) and for every $\varepsilon > 0$, there exists a Levy cycle $\mathscr{G}$, a curve $\gamma \in \mathscr{G}$, and a point τ in the Teichmüller space $\mathscr{T}_A$ such that $l_\gamma(\tau) < \varepsilon$. Note that all marked points in C can be made fixed by passing to an appropriate iterate of the original Thurston map. Therefore, items (1) and (3) of Proposition 5.2.3 imply that the scenario described above is sufficient to conclude the desired result in the general case. Since under these assumptions, every point $c \in C$ is fixed under f, we can suppose

without loss of generality that there is no point $c \in C$ such that the Thurston map $f\colon (S^2, B \cup \{c\}) \circlearrowleft$ is realized. This assumption and item (2) of Proposition 2.2.10 also ensure that $|B| \geqslant 3$.

According to Proposition 2.2.16, there exists a fixed point $\mu = [\varphi_1]_B = [\psi_1]_B \in \mathcal{T}_B$ of the pullback map $\sigma_{f,B}$. We assume that the representatives $\varphi_1, \psi_1\colon S^2 \to \widehat{\mathbb{C}}$ are chosen such that φ_1 and ψ_1 are isotopic rel. B (in particular, $\varphi_1|B = \psi_1|B$) and $g := \varphi_1 \circ f \circ \psi_1^{-1}\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$ is a postsingularly finite holomorphic map, where $P := \varphi_1(B) = \psi_1(B)$.

Now, let $\tau_1 = [\varphi_1]_A \in \mathcal{T}_A$ and define $\tau_n := \sigma_{f,A}^{\circ(n-1)}(\tau_1) = [\varphi_n]_A$ for all $n \in \mathbb{N}$, where the representatives $\varphi_n\colon S^2 \to \widehat{\mathbb{C}}$ are chosen such that $\varphi_n|B = \varphi_1|B$ and the map $\varphi_n \circ f \circ \varphi_{n+1}^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. Proposition 2.2.13 ensures that such choice of representatives is possible and implies that $\varphi_n \circ f \circ \varphi_{n+1}^{-1} = g$ for all $n \in \mathbb{N}$. Let $C = \{c_1, c_2, \ldots, c_l\}$, and define $x_{n,i} := \varphi_n(c_i)$ for each $n \in \mathbb{N}$ and $i = 1, 2, \ldots, l$.

Claim. For each $i = 1, 2, \ldots, l$, the sequence $(x_{n,i})$ converges to a limit $x_i \in P$, which is a repelling fixed point of the map g.

Proof. Let $A_i := B \cup \{c_i\}$ for each $i = 1, 2, \ldots, l$. Since $f(c_i) = c_i$, we can study the Thurston map $f\colon (S^2, A_i) \circlearrowleft$. Specifically, we consider the σ_{f,A_i}-orbit of the point $[\varphi_1]_{A_i}$ in the Teichmüller space $\mathcal{T}_{A_i}$. By item (1) of Proposition 2.2.13, this orbit coincides with the sequence $([\varphi_n]_{A_i})$.

Define $\Delta_i := t_{A_i,B}^{-1}(\mu)$, which, according to item (1) of Proposition 2.2.13, is a σ_{f,A_i}-invariant subset of $\mathcal{T}_{A_i}$. Also, let $\pi_i\colon \Delta_i \to \widehat{\mathbb{C}} - P$ be defined by $\pi_i([\varphi]_{A_i}) = \varphi(c_i)$, where the representative $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi|B = \varphi_1|B$, and let $\omega_i = \pi_i \circ \sigma_{f,A_i}|\Delta_i$. Recall that these objects were discussed in Section 5.1.2. Additionally, we notice that $[\varphi_n]_{A_i} = \sigma_{f,A_i}^{\circ(n-1)}([\varphi_1]_{A_i}) \in \Delta_i$ for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, l$.

According to item (2) of Proposition 5.1.3, Δ_i is biholomorphic to the unit disk, and by item (1) of the same proposition, we have $\pi_i(\sigma_{f,A_i}(\Delta_i)) = W$, where $W = \widehat{\mathbb{C}} - \overline{g^{-1}(P)}$, and $\pi_i = g \circ \pi_i \circ \sigma_{f,A_i}$. Since g is not injective, items (3) and (4) of Proposition 5.1.3 allow us to apply Theorem 3.2.6 to the map $\sigma_{f,A_i}|\Delta_i$. Since the Thurston map $f\colon (S^2, A_i) \circlearrowleft$ is obstructed, $\sigma_{f,A_i}|\Delta_i$ has no fixed points, as stated in Proposition 2.2.16. Therefore, by Theorem 3.2.6, the sequence $x_{n,i} = \pi_i([\varphi_n]_{A_i}) = \pi_i(\sigma_{f,A_i}^{\circ(n-1)}([\varphi_1]_{A_i}))$ converges to a point $x_i \in P$, which is a repelling fixed point of the map g. ■

Let $x \in P$ be a point such that there exists a sequence $(x_{n,j})$ converging to x as n tends to ∞. According to the previous claim, such a point x always exists and must be a repelling fixed point of the map g. Given this, and the fact that every sequence $(x_{n,i})$ converges as $n \to \infty$, we can find an annulus $U \subset \widehat{\mathbb{C}}$ such that, for all sufficiently large n, the following conditions are satisfied:

- one of the connected components of $\widehat{\mathbb{C}} - U$ contains points x and x_n^j, while the other connected component contains all points of $P - \{x\}$;
- $g^{\circ(|A|-3)}$ is defined and injective on U.

It is clear that we can take the modulus of the annulus U as large as desired, assuming that we consider n large enough. Therefore, we can apply Proposition 3.1.5 to derive the existence of a Levy cycle $\mathscr{G}$ for the Thurston map $f\colon (S^2, A) \circlearrowleft$. Moreover, Proposition 3.1.5 also shows that, by taking $\operatorname{mod}(U)$ sufficiently large, we can ensure that for some n and any $\gamma \in \mathscr{G}$, $l_\gamma(\tau_n)$ is as small as we want. This completes the proof. □

Remark 5.2.6. One of the key tools of the proof of Theorem 3.2.6 is the use of the hyperbolic metric on the unit disk. In the proof of Main Theorem 4.1.1, Theorem 3.2.6 was applied to the restrictions $\sigma_{f,A_i}|\Delta_i\colon \Delta_i \to \Delta_i$, where $\Delta_i = t^{-1}_{A_i,B}(\mu)$ is a properly and holomorphically embedded unit disk in the Teichmüller space $\mathscr{T}_{A_i}$, where $|A_i| \geqslant 4$ and $|A_i| - |A| = 1$. However, it is worth noting, that the hyperbolic distance on Δ_i does not coincide with the metric inherited from the Teichmüller metric d_T on $\mathscr{T}_{A_i}$, except when $|A_i| = 4$ and $\Delta_i = \mathscr{T}_{A_i}$, due to [Nag82, Theorem 1]. The same observation applies to properly and holomorphically embedded unit disk $\Delta = t^{-1}_{B_{j+1},B_j}(\tau_j)$ in $\mathscr{T}_{B_{j+1}}$ and the map $\sigma_{f,B_{j+1}}|\Delta\colon \Delta \to \Delta$ from the proof of Proposition 5.2.1.

Example 5.2.7. Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map, and assume that $B \subset A$ is an f-pseudo-invariant subset containing S_f.

Now, suppose $|A| = 4$ and $|B| \leqslant 3$. In this case, by item (2) of Proposition 2.2.10, the Thurston map $f\colon (S^2, B) \circlearrowleft$ is clearly realized. Consequently, Theorem 5.2.5 directly implies that $f\colon (S^2, A) \circlearrowleft$ is realized if and only if it has no degenerate Levy multicurve. Since any multicurve on the four-punctured sphere $S^2 - A$ consists of just one simple closed curve, $f\colon (S^2, A) \circlearrowleft$ is realized if and only if it has a degenerate Levy fixed curve, confirming the result of Theorem 4.1.1; see Example 4.1.15.

6 Outlook

Summary

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In this chapter, we provide further questions and potential research directions arising from the work discussed in this thesis. This chapter provides a roadmap for future projects that could build on and extend the findings and contributions presented here.

6.1 Further questions on Thurston maps with extra marked points

Although, in Chapter 5, we have explored the relationships between the properties of Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$, where B is a subset of A, several questions in this context remain unresolved. We plan to address these in the forthcoming work [Proa].

Firstly, based on Proposition 2.2.19, it can be shown that the global dynamics of the pullback map $\sigma_{f,A}$ on the Teichmüller space $\mathcal{T}_A$ is relatively simple if f is not a $(2,2,2,2)$-map: either $\sigma_{f,A}$ has a unique *globally attracting* fixed point, or every orbit eventually escapes any compact subset of $\mathcal{T}_A$. However, this dichotomy does not hold for $(2,2,2,2)$-maps. Still, when $A = P_f$, the corresponding pullback maps are well-understood (see [Hub16, Appendix C.6]). However, the global dynamics of a pullback map $\sigma_{f,A}$ for a $(2,2,2,2)$-map f and $|A| \geqslant 5$ remains somewhat unclear. We plan to fully describe it in the forthcoming [Proa], utilizing tools from the dynamics of holomorphic self-maps on complete Kobayashi-hyperbolic complex manifolds (see [Aba15]) and methods developed in Chapter 5.

Another question is inspired by the following classical result. Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite degree that is not of the $(2,2,2,2)$-type. According to [Pil01], it is known that for every essential simple closed curve γ in $S^2 - A$, exactly one of the following holds:

1. for every $\tau \in \mathcal{T}_A$, $l_\gamma(\sigma_{f,A}^{\circ n}(\tau)) \to 0$ as $n \to \infty$, or

2. for every $\tau \in \mathcal{T}_A$, $l_\gamma(\sigma_{f,A}^{\circ n}(\tau)) \geqslant l$, where l depends on τ and γ.

Clearly, this behavior is independent of the homotopy class of γ in $S^2 - A$. In particular, for an obstructed Thurston map $f\colon (S^2, A) \circlearrowleft$, by selecting a representative from each homotopy class in $S^2 - A$ that satisfies condition (1), one constructs a Thurston obstruction known as a *canonical Thurston obstruction*. Canonical Thurston obstructions have several significant properties and play a crucial role in Pilgrim's decomposition theory. For more details, see [Pil03], [Sel13a], and [BD17].

The results of Chapter 4 demonstrate that this observation also holds for Thurston maps $f\colon (S^2, A) \circlearrowleft$ of infinite degree that meet conditions (I) and (II). In this case, the canonical Thurston obstruction is provided by a unique Levy fixed curve. The proof of Theorem 5.2.5 suggests that similar observations may also apply to a Thurston map $f\colon (S^2, A) \circlearrowleft$ of infinite degree, provided that the Thurston map f is realized. In the forthcoming work [Proa], we aim to confirm this and further investigate the properties of canonical Thurston obstructions within this context.

6.2 Thurston theory and dynamics in several complex variables

In this section, we consider the family of Thurston maps $f\colon (S^2, A) \circlearrowleft$ that satisfy condition (II), but we allow the cardinality of the marked set A to be arbitrary. This condition encompasses many well-known examples of Thurston maps. For instance, it holds for the family of postcritically finite *unicritical* topological polynomials and the family of exponential Thurston maps. However, this class is not limited to these families; for example, it also includes the famous example by Mitsuhiro Shishikura and Tan Lei (see [Ché12]).

In the forthcoming [Prob], we show that if a Thurston map $f\colon (S^2, A) \circlearrowleft$ satisfying condition (II) has a finite degree, then analogously to Proposition 4.0.1, the corresponding X-map X_f is injective and the Hurwitz space $\mathcal{W}_f$ is biholomorphic to $\mathbb{C}^k - \mathcal{L}$, where $k = |A| - 3$ and $\mathcal{L}$ is a union of finitely many hyperplanes (see commutative diagram (1.1) from Section 1.2.2.1). In particular, the family of these Thurston maps provides a new broad class of examples, where the X-map is injective (cf. [Koc13, Section 5]). Moreover, in this case, both X- and Y-maps admit simple explicit formulas. Due to these facts, one can try to apply methods from the complex dynamics in several variables to better understand the pullback dynamics on Teichmüller and moduli spaces by studying the dynamics of the g-map $g_f = Y_f \circ X_f^{-1}$.

Similar observations hold in the case when a Thurston map $f\colon (S^2, A) \circlearrowleft$ satisfying condition (II) has infinite degree, with the distinction that the analogue of the Hurwitz space is now biholomorphic to $\mathbb{C}^k - \mathcal{L}$, where $\mathcal{L}$ is an infinite countable union of hyperplanes in $\mathbb{C}^k$. In particular, one can attempt to derive an analogue of Theorem 3.2.6 in the higher-dimensional setting to analyze the corresponding pullback maps and potentially establish an analogue of Thurston's characterization criterion for Thurston maps satisfying only condition (II). Additionally, the investigation of fixed

points of the g-map for such a Thurston map could yield more refined results on the parameter spaces of meromorphic maps with at most three singular values, similar to Corollary 1.3.1. However, for these methods to work, the concept of hyperbolicity for Riemann surfaces needs to be replaced with the concept of *Kobayashi hyperbolicity* for complex manifolds. For instance, one would need to derive estimates similar to those in Section 3.2.1 for inclusion maps between two *complete Kobayashi-hyperbolic* complex manifolds.

6.3 Uncountably many parameter spaces

Let $g\colon \mathbb{C} \to \mathbb{C}$ be an entire map of finite type. Then we define a *topological parameter space* (cf. Definition 4.1.8 and Remark 4.1.10) of g as

$$\mathrm{TopPar}(g) := \{\varphi^{-1} \circ g \circ \psi : \varphi, \psi\colon \mathbb{R}^2 \to \mathbb{C} \text{ are orientation-preserving homeomorphisms}\}.$$

It is evident that every $f \in \mathrm{TopPar}(g)$ is a topologically holomorphic map of parabolic type. The topological parameter space $\mathrm{TopPar}(g)$ can be viewed as a collection of topologically holomorphic maps that share similar function-theoretic properties with g. However, we can also study Thurston maps within a given topological parameter space. Notably, it can be shown that each topological parameter space contains countably many Thurston maps, up to combinatorial equivalence.

Certain realizability criteria can be applied to all Thurston maps within a given topological parameter space. For example, the result of [HSS09] characterizes Thurston maps within $\mathrm{TopPar}(\exp)$, and the result in [She22] applies to all parameter spaces of structurally finite entire maps. However, Main Theorem A does not possess this generality, as it is restricted to Thurston maps with only four postsingular values.

It is not hard to demonstrate that there are countably many distinct topological parameter spaces of structurally finite entire maps. Furthermore, the same is true if we consider all topological parameter spaces $\mathrm{TopPar}(g)$, where g is a composition of finitely many structurally finite entire maps. In other words, all known to this moment realizability criteria cover only countably many distinct parameter spaces

In the forthcoming work [Proc], we plan to use the theory of line complexes and the methods from [HSS09], including thick-thin decompositions of integrable meromorphic quadratic differentials, to construct uncountably many topological parameter spaces, where Thurston maps are realized if and only if they have no Levy cycles.

In other words, this result would confirm that a Thurston-like criterion holds for set-theoretically large family of transcendental Thurston maps. Moreover, these parameter spaces can be selected to consist of topologically holomorphic maps with infinitely many asymptotic tracts, which presents an example of a challenge in extending the results of [She22].

6.4 Thurston theory for general finite type maps

In [Eps93] (see also [Ast24] and [ABF24]), Adam Epstein introduced the concept of finite type maps:

Definition 6.4.1. Let X be a connected compact Riemann surface and W be an open subset of X. We say that a holomorphic map $g\colon W \to X$ is of *finite type* if:

1. g is non-constant on any connected component of W;
2. g has no isolated removable singularities;
3. the singular set S_g is finite.

Although the map $g\colon W \to X$ does not represent a discrete dynamical system when $g(W) - W$ is non-empty, one can still define the orbit of a point z in X, which may terminate after several iterations if $f^{\circ m}(z) \in X - W$ for $m \geqslant 0$. In [Eps93], it was shown that general finite type maps appear as natural objects already in the dynamics of rational maps, and many key results from the dynamics of rational and meromorphic maps of finite type can be extended to this much broader context. These results include the density of repelling fixed points in the Julia set, the No-Wandering-Domains Theorem, and the classification of periodic Fatou components. In the forthcoming [Prod], we aim to extend Thurston theory to finite type maps.

More specifically, let $f\colon W \to X$ be a quasiregular map, where X is a connected compact Riemann surface and W is an open subset of X. We say that f is a Thurston map if it satisfies the conditions of Definition 6.4.1 and f is postsingularly finite, i.e., every singular value w of f is either periodic, pre-periodic, or there exists $m \geqslant 0$ such that $f^{\circ m}(w) \in X - W$. In this definition, a Thurston map is always considered quasiregular, but the assumption of admissible type is not included as part of the definition.

For any such Thurston map with a postsingular set P_f containing at least three points, it is possible to define a pullback map σ_f that acts on the Teichmüller space $\mathcal{T}_{X,P_f}$ of the hyperbolic Riemann surface $X - P_f$. This observation has already been demonstrated in [Ast24, Section 3.1], specifically in the case where $X = \widehat{\mathbb{C}}$, f is holomorphic, and the singular values of f do not eventually land on a point where f is undefined. However, similar methods can be applied to extend this concept to more general settings. Furthermore, one can demonstrate that a pullback map σ_f has a fixed point if and only if the corresponding Thurston map $f\colon W \to X$ has a holomorphic realization with respect to the following notion of combinatorial equivalence (cf. Proposition 2.2.16).

Definition 6.4.2. Two Thurston maps $f_1\colon W_1 \to X_1$ and $f_2\colon W_2 \to X_2$ are said to be combinatorially equivalent if there exist two quasiconformal homeomorphisms $\phi_1, \phi_2\colon X_1 \to X_2$ such that ϕ_1 and ϕ_2 are isotopic rel. P_{f_1}, $\phi_1 \circ f_1 = f_2 \circ \phi_2$, and $\partial_{\overline{z}}\phi_2$ equals to zero almost everywhere on $X_1 - W_1$.

It is worth noting that when $X_1 - W_1$ has measure zero, Definition 6.4.2 closely resembles the definition provided in Remark 2.2.7.

These observations allow us to pose the realization question in the context of general finite type maps. In the forthcoming work [Prod], we plan to develop Thurston theory in this setting and extend the results of Main Theorems A and C.

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