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Categorical variations on Frobenius quantales

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Résumé et mots clés

L'objectif premier de cette thèse est d'étudier d'un point de vue catégorique le théorème suivant : un treillis est complètement distributif si et seulement si le quantale des endomorphismes de ce treillis est un quantale de Frobenius.

Nous avons ainsi rencontré une définition des quantales de Frobenius qui, contrairement à la définition usuelle, n'implique pas d'unité. La première contribution de cette thèse est une étude des quantales de Frobenius sans unité. Nous généralisons la théorie des nuclei à ces quantales permettant ainsi d'obtenir des "négations classiques" à partir de "négations intuitionnistes". Grâce à cette théorie nous donnons un théorème de représentation de ces quantales en terme de quantales de phase, c'est-à-dire comme quotient du quantale libre sur un semi-groupe. Nous fournissons des exemples de quantales de Frobenius sans unité dont le quantale des endomorphismes serrés sur un treillis complet L. Ce quantale est toujours un quantale de Frobenius et nous montrons qu'il possède une unité si et seulement si le treillis L est complètement distributif.

La seconde partie répond directement à notre premier objectif. Nous donnons une définition des structures de Frobenius dans une catégorie monoïdale symétrique quelconque en suivant de très près la définition des quantales de Frobenius sans unité. Pour ce faire, nous introduisons les paires duales qui généralisent le dual d'un objet. Cette notion permet notamment d'étudier le dual d'un objet à isomorphisme près. Nous montrons précisément comment, et à quelles conditions il est possible de transiter entre les structures de Frobenius, telles que nous les avons définies, et les différentes définitions d'algèbre de Frobenius bien établies dans la littérature. Surtout, nous démontrons que, dans une catégorie étoile-autonome, le monoïde des endomorphismes d'un objet nucléaire peut toujours être doté d'une structure de Frobenius. Nous montrons aussi que, sous certaines conditions, une "négation intuitionniste" induit une structure de Frobenius sur son image. De plus, nous exposons une condition suffisante pour que, si l'objet des endomorphismes est une structure de Frobenius, alors cet objet est nucléaire. Nous appelons pseudo-affines les catégories qui vérifient cette condition. Afin d'explorer cette condition nous construisons une catégorie étoile-autonome qui n'est pas pseudoaffine.

Le troisième et dernier moment de cette thèse étudie et généralise la construction de Shalk et de Paiva. Nous nous demandons à quelles conditions la structure monoïdale, monoïdale fermée ou étoile-autonome d'une catégorie C peut être élevée à la catégorie totale d'un foncteur de domaine C et de codomaine Pos, la catégorie des ensembles partiellement ordonnés. Pour ce faire, nous fournissons une bijection entre les élévations de foncteurs et certaines transformations lax-naturelles. Nous donnons aussi les conditions exactes pour que les adjonctions soient elles aussi élevées. Notam-

ment lorsque le foncteur Q est monoïdal et se factorise de façon monoïdale via SLatt, la catégorie des sup-treillis complets, la catégorie totale sur Q est monoïdale fermée si et seulement si la catégorie de base C l'est. Nous fournissons ensuite une caractérisation précise pour que la catégorie totale soit étoile-autonome. Finalement nous développons une théorie des nuclei pour les foncteurs monoïdaux dont le codomaine est SLatt, analogue à celle des quantales. Cette théorie nous permet de constuire un foncteur dont la catégorie totale est étoile-autonome à partir d'un foncteur monoïdal de codomaine Slatt quelconque. Tout comme pour les quantales de Frobenius, cette théorie des nuclei nous permet d'obtenir un théorème de représentation des foncteurs dont la catégorie totale est étoile autonome. Nous terminons cette thèse en montrant que ces foncteurs sont exactement les structures de Frobenius dans la catégorie étoile-autonome des foncteurs dont le domaine est étoile-autonome et le codomaine est SLatt.

Mots clés : Treillis complets, Logique linéaire, Quantales de Frobenius, Algèbres de Frobenius, Catégories étoile-autonomes, Objets nucléaires, Construction de Grothendieck

Abstract and keywords

The primary objective of this thesis is to study from a categorical point of view the following theorem: a lattice is completely distributive if and only if the quantale of endomorphisms of this lattice is a Frobenius quantale.

In doing so, we have encountered a definition of Frobenius quantales which, unlike the usual definition, does not imply unity. The first contribution of this thesis is a study of unitless Frobenius quantales. We generalize the theory of nuclei to these quantales, making it possible to obtain "classical negations" from "intuitionistic negations". Thanks to this theory, we give a representation theorem for these quantales in terms of phase quantales, i.e. as quotients of the free quantale over a semigroup. We provide several examples of unitless Frobenius quantales, including the quantale of tight endomorphisms on a complete lattice L. This quantale is always a Frobenius quantale, and we show that it has a unit if and only if the lattice L is completely distributive.

The second part directly addresses our first objective. We give a definition of Frobenius structures in any symmetric monoidal category, following very closely the definition of unitless Frobenius quantales. To this end, we introduce dual pairs, which generalize the dual of an object. In particular, this notion makes possible the study of duality up to isomorphism. We show precisely how, and under what conditions, it is possible to transit between Frobenius structures as we have defined them, and various definitions of Frobenius algebras well-established in the literature. Above all, we show that, in a star-autonomous category, the monoid of endomorphisms of a nuclear object can always be endowed with a Frobenius structure. We also show that, under certain conditions, an "intuitionistic negation" induces a Frobenius structure on its image. Furthermore, we expose a sufficient condition for the following statement to be true: if the object of endomorphisms is a Frobenius structure, then this object is nuclear. A category that verifies this condition is named pseudo-affine. In order to explore this condition, we construct a star-autonomous category that is not pseudoaffine.

The third and final part of this thesis studies and generalizes Shalk's and Paiva's construction. We ask under what conditions the monoidal, closed monoidal or star-autonomous structure of a category C can be raised to the total category—the Grothendieck construction—of a functor of domain C and codomain Pos, the category of partially ordered sets. To this end, we provide a bijection between liftings of functors and certain lax-natural transformations. We also give the exact conditions under which adjunctions can also be lifted. In particular, when the functor Q is monoidal and factorizes monoidally through SLatt the category of complete sup-lattices, the total category over Q is closed monoidal if and only if the base category C is. We then provide a precise characterization for the total category to be star-autonomous.

Finally, we develop a theory of nuclei for monoidal functors whose co-domain is SLatt, analogous to the one for quantales. This theory allows us to construct a functor whose total category is star-autonomous from monoidal functor whose co-domain is Slatt. Just like for Frobenius quantales, this theory of nuclei allows us to obtain a representation theorem for functors whose total category is star-autonomous. We conclude this thesis by showing that these functors are exactly the Frobenius structures in the star-autonomous category of functors whose domain is star-autonomous and whose co-domain is SLatt.

Keywords: Complete lattices, Linear logic, Frobenius quantales, Frobenius algebras, Star-autonomous categories, Nuclear objects, Grothendieck construction

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Introduction

We first encountered Heyting algebras while studying intuitionistic logic and toposes. Similarly, we were investigating linear logic and $*$-autonomous categories when Frobenius quantales crossed our paths. Accordingly, we would like to initiate our journey in the world of Frobenius quantales as it began: with the correspondence between categories, ordered algebraic structures and logics.

Intuitionistic logic does not accept a classical negation, meaning one cannot infer, in general, that a formula A holds under the only hypothesis of $\neg\neg A$. Not allowing this rule—or any equivalent one—makes it a constructive logic. Indeed, one always knows how a set of hypotheses produces a conclusion. This idea of production is important, a proof in intuitionistic logic is seen as a typed program producing a result from some arguments. Intuitionistic logic also has an algebraic counterpart: Heyting algebras. Those are bounded lattices for which we have a binary operation $- \to -$ that satisfies

$$x \wedge y \leq z \text{ iff } y \leq x \to z\,. \tag{1}$$

The adjunction $x \wedge - \dashv x \to -$ notably implies that meets distribute over the joins:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\,. \tag{2}$$

Cartesian closed categories categorify Heyting algebras: they have binary categorical product and $X \times -$ has a right adjoint, the exponentiation. A morphism $f : X \to Y$ in such categories corresponds to a proof of a sequent $X \vdash Y$ up to cut-eliminations—or β-equivalence. This triple correspondence between proofs, programs and morphisms is known as the Curry-Howard-Lambek correspondence. However, a classical proof cannot be interpreted as a morphism in a category, or rather, forcing a cartesian closed category to have a classical negation makes the cartesian closed category a poset, that is, a boolean algebra.

A major motivation for giving birth to linear logic was to restore a classical negation $(-)^\perp$ in a constructive setting—see [**Gir87**]. This was achieved on the proof-theoretic side, by controlling the structural rules of weakening and contraction. The logical connectives of conjunction and disjunction have

been split into their multiplicative—$\otimes$ and $\parr$—and additive versions—& and $\oplus$. The multiplicative connectives endorse the linearity of linear logic, for instance, it is not possible to prove the sequent $A \vdash A \otimes A$, while $A \vdash A\&A$ is provable. This allows us to regard formulas as "resources" that are "consumed" when being used. If the negation is not classical, then we talk about *intuitionistic multiplicative and additive linear logic* (*IMALL*). Commutative unital quantales are standard posetal models of *IMALL*. They are complete lattices with a monoid operation $- * -$ and a binary operation $-\backslash -$ defined by

$$x * y \leq z \text{ iff } y \leq x \backslash z\,. \tag{3}$$

Note the resemblance between the equivalence (1) defining Heyting algebras and the equivalence (3). In the former, the multiplication is the meet operation while it is any associative and commutative multiplication in the second. The operation $\to$ interprets the intuitionistic implication while $\backslash$ the linear implication. Just like Heyting algebras are posetal cartesian closed categories, commutative unital quantales are posetal symmetric monoidal closed categories—which are moreover complete. Of course, cartesian closed categories are special cases of symmetric monoidal closed categories.

As we mentioned, linear logic is particularly interesting when it features a classical negation, this logic is called *MALL*. In order to interpret the negation in a quantale, one usually asks the existence of an element $0 \in Q$ which will play the role of "false", that is, it must respect the equality

$$(x \backslash 0) \backslash 0 = x\,. \tag{4}$$

Such elements are called *dualizing*, and a commutative quantale with a dualizing element is a *commutative Frobenius quantale*. Those quantales are actually $*$-autonomous categories, and $*$-autonomous categories are symmetric monoidal closed categories equipped with a linear negation.

Controlling further structural rules gave birth to intuitionistic and classical *noncommutative* linear logic (*INCLL* and *NCLL*). This is done by restraining the use of the exchange rule, meaning the sequent $A \otimes B \vdash B \otimes A$ is not provable in general—see for instance [**Yet90, Ros90a, Abr91**]. Non commutativity leads to a notion of time: intuitively, A comes before B in $A \otimes B$. This is notably motivated by a computer science perspective: resources may not be all accessible independently of the others. The semantics of intuitionistic and classical *noncommutative* linear logic (*INCLL* and *NCLL*) is roughly similar to what we have briefly described in the commutative case. The main difference is that now, we define two implications, and two negations nicely working together. More precisely, in a (non necessarily commutative) quantale, $- * y$ also has a right adjoint $- \mathbin{/} y$. Of course, in the commutative case, the two implications are the same. A dualizing element 0 in such quantales must

respect the equations

$$0 \diagup (x \diagdown 0) = (0 \diagup x) \diagdown 0 = x \,. \tag{5}$$

A quantale equipped with a dualizing 0 is a *Frobenius quantale*. There is also a similar notion of nonsymmetric $*$-autonomous categories—see [**Bar95**]. They are an interesting object of study, but for practical reasons, we shall keep noncommutativity only for the posetal case—*i.e.*, Frobenius quantales.

Although our interest in quantales, Frobenius quantales and $*$-autonomous categories is mainly motivated by a linear logic semantics perspective, those structures are worth studying for themselves. The history of quantales goes back to the end of the thirties when Dilworth—together with other lattice theorists—studied residuated lattices. Lattices of ideals of a ring were the main motivating example for that theory—see for instance [**Dil39**]. They have gained popularity in the middle of the eighties when it was noticed that several important properties of a C*-algebra can be recovered from the quantale of its right closed ideals—works going in that direction started with [**Mul86**]. For instance, the multiplication of this quantale is the meet operation—making it a locale—if and only if the C*-algebra is commutative. Frobenius quantales are also related to C*-algebra, for instance $\mathbf{MAX}_{M_n(\mathbb{C})}$, the closed linear subspaces of square $\mathbb{C}$-matrices, is a quantale and the kernel of the trace operation is a dualizing element of that quantale—see for instance [**KP08**]. Motivation for researching on $*$-autonomous categories is not restricted by linear logic semantics—although this has motivated many works including the present one. These categories were already introduced by Barr in the late seventies in [**Bar79**] when researching on self-dual categories. Among usual $*$-autonomous categories we can mention: $\textsc{finVect}_k$, the category of finite-dimensional vector spaces and linear maps, and Rel, the category of sets and relation. Our favorite $*$-autonomous category is SLatt, the category of complete lattices and sup-preserving maps between them.

The category SLatt has many useful properties that we have been exploring along this thesis. First, quantales are exactly the semigroup objects in SLatt. Indeed, the existence of the two residuations implies that $- * -$ is sup-preserving in both variables. This makes its linear extension $- * - : Q \otimes Q \to Q$ an associative multiplication in SLatt. Therefore, properties of quantales may be derived from general theory of semigroups and monoids in symmetric monoidal closed categories and more specifically in $*$-autonomous categories—this has been done for instance in [**EGGHK18**]. A canonical example of monoid object in a symmetric monoidal closed categories is the internal hom of an object with itself $[X, X]$ whose multiplication is given by the internal composition. In SLatt, quantales of the form $[L, L]$ of sup-preserving endomaps of a complete lattice L naturally arise in logic, computer science, and quantum mechanics. For example, for a set

X and its powerset lattice $P(X)$, the quantale $[P(X), P(X)]$ is isomorphic to the quantale $P(X \times X)$ of relations on X and it is the main object of study in relation algebra [**Mad06**]. Use of these quantales to model concurrency and quantum mechanics/computation abound in the literature—see *e.g.*, [**AV93, ACS98, Res01**]. Our interest for those quantales stems from generalized linear orders [**GS18, San19, GS21, San21**]. Notably, paths in a d-dimensional cube are in bijection with elements of $[L, L]^{[d]_2}$, where $[d]_2 := \{(i, j) \mid 1 \leq i < j \leq d\}$, for a certain complete lattice L. These elements are seen as generalized relations—where we identify a relation with its characteristic map. These relations need to satisfy constraints which are expressed using a negation, hence the quantale $[L, L]$ must be a Frobenius quantale. The theorem we will mention in the next paragraph was used in [**GS21**] to show that L must be the unit interval of the real numbers.

The following theorem gives a sufficient and necessary condition for the quantale on $[L, L]$ to be a Frobenius quantale. This theorem was the starting point of this thesis. A proof of it appears in [**KP08**, Theorem 3.3.6]—see also [**EGGHK18**, Example 2.6.16]—and, in the wider context of quantaloids, in [**San20a**]. The study of dualizing elements of $[L, L]$ was also carried out in [**San20b**].

THEOREM 0.1. *Let L be a complete lattice. The following statements are equivalent:*

- *L is a completely distributive lattice;*
- *There exists a sup-preserving map $0 : L \to L$ making $([L, L], \circ, 0)$ a Frobenius quantale.*

Complete distributivity is an infinitary generalization of the distributive law—see in Eq. (2). Basic examples of completely distributive lattices are the complete linear orders, the powerset lattices and, of course, the finite distributive lattices. Important properties of completely distributive lattices were pointed out by Raney in the fifties—see [**Ran52, Ran53**]. In particular, the relation between tight Galois connections and complete distributive lattices described in [**Ran60**] is the key tool to prove Theorem 0.1. As logicians, the above theorem struck us, since it shows that a model theoretic approach to restoring a classical negation fundamentally diverges from the proof-theoretic one. Indeed, the theorem has the following reading: *if a classical negation is enforced on the most standard class of models of non-commutative intuitionistic linear logic* (the sup-preserving endomaps of a complete lattice)*, then also the additive connectives of the logic* (the suprema and infima of the lattice) *have a classical behaviour* (that is, they distribute over each other).

Category theory yields another perspective on this theorem. Indeed, both Frobenius quantales and completely distributive lattices have been independently characterized in categorical terms. In [**Egg10**], Egger defined Frobenius monoids in linearly distributive categories, and thus in $*$-autonomous categories. Notably, in the same sense that quantales are exactly semigroup objects in SLatt, Frobenius quantales—quantales with a dualizing element—are exactly Frobenius monoids in SLatt. On the other hand, Higgs and Rowe in [**HR89**] have shown that completely distributive lattices are exactly the nuclear objects in the category SLatt. Tight Galois connections defined by Raney in [**Ran60**] are once again mobilized to prove this result. An object X in a symmetric monoidal closed category is *nuclear* if the object $X^* := [X, I]$ is its adjoint object, or, equivalently, if the `mix` map $\mathtt{mix}_X : X^* \otimes X \to [X, X]$ is invertible. We can therefore reformulate Theorem 0.1:

Theorem 0.2. *Let L be an object in* SLatt. *The following statements are equivalent:*

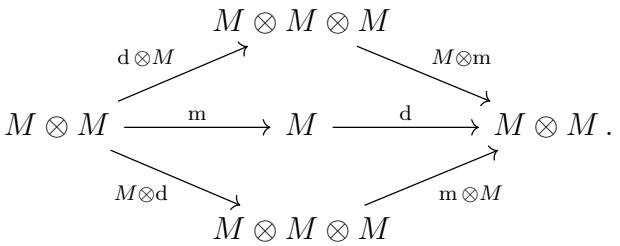
- *L is nuclear;*
- *The object of endomorphisms of L is a Frobenius monoid.*

With this discussion in mind, we naturally formulate the following conjecture:

Conjecture 0.1. Let A be an object of symmetric monoidal closed category. The following are equivalent:

- A is nuclear.
- The object of endomorphisms of A is a Frobenius monoid.

Proving this conjecture was the first objective of this thesis—and the main topic of Chapter 3. We had good hopes in doing so as relations between nuclear objects and Frobenius algebras in a symmetric monoidal category are already well-established. Usually, a *Frobenius algebra* in a monoidal category is defined as a monoid-comoid object $(M, \mathrm{m}, \mathrm{u}, \mathrm{d}, \mathrm{tr})$ such that the following diagram commutes —see for instance [**Koc03**, Subsection 3.6.8]



The definition of Egger's Frobenius monoid is actually quite similar, only the codomain of the comultiplication is not $M \otimes M$ but $M \oplus M$, and the last

diagram requires the use of the two linear strength of a linearly distributive category. Of course, Frobenius algebras are Frobenius monoids in the sense of Egger since a monoidal category is a linearly distributive category for which the cotensor is the tensor. In a symmetric monoidal category, if $(M, \mathrm{m}, \mathrm{u}, \mathrm{d}, \mathrm{tr})$ is a Frobenius algebra, then M is a nuclear object—see for instance [**Hyl04**]. This property is actually not true if we consider general Frobenius monoids—for which the codomain of d is $M \oplus M$ and not $M \otimes M$. Indeed, there are Frobenius quantales that are not completely distributive lattices.

While exploring Conjecture 0.1, we (re)discovered another definition of Frobenius quantale. We were looking for a categorical definition of Frobenius quantale that would easily permit us to translate the proof of Theorem 0.1 in categorical terms. The definition of Frobenius monoid given by Egger is undoubtedly a nice generalization of Frobenius algebra, but is quite distant from the definition of Frobenius quantales. Moreover, Frobenius quantales are usually defined using a dualizing element, when the use of elements is not very practical in category theory. Furthermore, the existence of a dualizing element implies the existence of a unit of the quantale. Precisely, if 0 is a dualizing element of $(Q, *)$, then $0 \diagdown 0$ is the neutral element of $*$. Intuitively, a dualizing element 0 plays the role of "false". Hence, $0 \diagdown 0$ means "false implies false" which is always "true", that is the unit of the conjunction. That every Frobenius quantale is unital seems a very strong property, since we like to consider quantales as semigroups in SLatt. Indeed, the existence of the unit has consequences on other properties of the quantale. For instance, a unital idempotent left-sided—*i.e.*, $\forall x, x * x = x$ and $\top * x \leq x$—quantale is necessarily a frame, hence commutative—see [**EGGHK18**, Section 2.4]. This phenomenon also appears with Frobenius algebras: the property that Frobenius algebras are necessarily nuclear is not true if one considers *non-unital* Frobenius algebras. This was the main motivation for Abramsky and Heunen in [**AH12**] to consider non-unital Frobenius algebras and H^*-algebras. All these reasons led us to refine the elementary theory of Frobenius quantale to *unitless Frobenius quantale*. Chapter 2 develops this axis of research. The categorical version of unitless Frobenius quantale we shall define and develop in Chapter 3 will be named *Frobenius Structures*.

The third line of research we have been focusing on is the relation between Frobenius Structures and $*$-autonomous categories. Some work in this direction has already been done by Day and Street in [**DS04, Str04**]—see also [**Mel13**], namely that Frobenius pseudomonoids in $\mathcal{V}$-Cat are exactly $\mathcal{V}$-enriched $*$-autonomous categories. Our research explores a different path. It is based on the fact that Frobenius structures in SLatt are specific examples of non-symmetric posetal $*$-autonomous categories—that is enriched over *2*. Since SLatt is itself a $*$-autonomous category, they are provability models of *NCLL* inside a model of proofs of *MLL*. The ambient category we

will be working with is not $\mathcal{V}$-Cat but an unspecified $*$-autonomous category. A relation between categorical semantics and posetal semantics of a logic already appears in the case of intuitionistic logic: the internal logic of a topos is completely determined by the Heyting algebras of its subobjects. We believe it is pertinent to wonder whether such a relation exists between Frobenius structures and $*$-autonomous categories. In particular, is it possible to construct a $*$-autonomous category from a Frobenius structure in some symmetric monoidal category?

The category $P_{\mathtt{id}}$-Set of poset-valued sets introduced by Schalk and de Paiva in [**SdP04**] were, for us, a major source of inspiration. Given a poset P, the objects of $P_{\mathtt{id}}$-Set are pairs (X, α) with X a set and α a map $X \to P$, arrows $R : (X, \alpha) \to (Y, \beta)$ in that category are relations $R : X \to Y$ such that xRy implies $\alpha(x) \leq \beta(y)$. In [**SdP04**] it is shown that, for a unital commutative Frobenius quantale Q, the category $Q_{\mathtt{id}}$-Set has a canonical $*$-autonomous structure. Note that $Q_{\mathtt{id}}$-Set is actually the total category $\int \mathsf{Q}$ of the functor $\mathsf{Q} : \textsc{Rel} \to \textsc{SLatt}$ which sends a set X to Q^X. Many important properties of $\int \mathsf{Q}$ are derived from properties of the quantale Q. For instance, the unit object of $\int \mathsf{Q}$ is dualizing if and only if the unit of Q is a dualizing element.

The Grothendieck construction allows us to build a $*$-autonomous category from a given $*$-autonomous category $\mathcal{C}$, usually a degenerate one such as Rel. Intuitively, it attaches to each object a structure and requires the maps to be compatible with the structures. The new category is usually better behaved, at least, for the semantics of proofs. Most often, the collection of all the structures that we can attach to an object is a poset. This leads to a functor $\mathsf{Q} : \mathcal{C} \longrightarrow \textsc{Pos}$, where Pos is the category of posets and order-preserving functions. Many examples of Grothendieck constructions appear in [**SKDH18**] where $\mathcal{C}$ is the category of sets and functions. However, we shall see that $\mathcal{C}$ must be a $*$-autonomous category in order for $\int \mathsf{Q}$ to be a $*$-autonomous category. If $\mathcal{C}$ and Q are symmetric monoidal, then the monoidal structure of $\mathcal{C}$ is lifted to the total category and the canonical opfibration strictly preserves it. A quantale is actually an example of such functor since the total category of a monoidal functor from the terminal category to SLatt is exactly a quantale. Actually, one can think of a monoidal functor $\mathsf{Q} : \mathcal{C} \to \textsc{SLatt}$ as a generalized quantale. Indeed, if $\mathcal{C}$ is a monoidal locally small category, the functor category $[\mathcal{C}, \textsc{SLatt}]$ is a monoidal category under the Day convolution, and monoids in that category are exactly monoidal functors—see [**Day70a, Egg08**]. We contribute to the theory of monoidal fibrations—see *e.g.*, [**Shu08, MV20**]— by answering variants of the following questions: given a functor $\mathsf{Q} : \mathcal{C} \to \textsc{Pos}$ which is symmetric monoidal—thus ensuring that $\int \mathsf{Q}$ is symmetric monoidal—when is $\int \mathsf{Q}$ closed, and when is it a $*$-autonomous category?

The first chapter of this thesis provides most of the categorical background needed to read the following chapters. Its main objectives are to settle our notation and to recall some elementary notions and examples. Even though the thesis is not completely self-contained, we tried to guide the reader through the important concepts. In this chapter, most of the statements will come without proofs since they are either stated in the literature or part of the folklore.

In Chapter 2, we introduce and study unitless Frobenius quantale. Most of the material comes from the journal paper [**dLS22**]. We enriched this article with intuitions and remarks; many statements and proofs are also reformulated. Frobenius quantales are usually defined as quantales with a dualizing element. As we mentioned, a quantale with a dualizing element is necessarily unital. However, non-unital quantales are important in quantale theory. In order to allow non-unital Frobenius quantales we redefine Frobenius quantales as follows. A *Frobenius quantale* is now a tuple $(Q, *, {}^{\perp}(-), (-)^{\perp})$ where $(Q, *)$ is a quantale and ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$ are two sup-preserving maps inverse of each other such that for every $x, y \in Q$, we have

$$x \backslash {}^{\perp} y = x^{\perp} / y \,. \tag{Serre equality}$$

This equation already appears in the literature, notably in [**GJKO07**, Section 3.3] under the name contraposition law, and such invertible maps were called Serre duality in [**Rum21**]. Examples of non-unital Frobenius quantales are easily provided by the Chu construction over a non-unital quantale.

In Heyting algebras, the set of fixed points of the double intuitionistic negation $(- \to \perp) \to \perp$ is a boolean algebra. This construction actually corresponds to the double negation translation from intuitionistic logic to classical logic. A similar statement is known for dualizing element of a quantale: if 0 is cyclic—*i.e.*, $(- \backslash 0) = (0 / -)$, then the set of fixed points of $(- \backslash 0) \backslash 0$ is a Frobenius quantale—see [**Ros90a**, Theorem 1]. Maps of the form $(- \backslash 0) \backslash 0$ are examples of nuclei which correspond exactly to congruences of quantale. We adapt this construction for unitless Frobenius quantales. We do not use a cyclic element but a *Serre Galois connection*: a Galois connection (l, r) which satisfies Eq. (Serre equality) and $l \circ r = r \circ l$. More precisely, we have the following proposition:

Proposition 0.1. *If (l, r) is a Serre Galois connection, then $\mathrm{j} := l \circ r = r \circ l$ is a nucleus, and the restriction of (l, r) to Q_{j}, the set of fixed point of* j*, yields a Frobenius quantale structure on Q_{j}.*

Rosenthal in [**Ros90a**, Theorem 2] has also shown that any quantale with a cyclic dualizing element is isomorphic to a phase quantale—*i.e.*, a Frobenius quantale which is a quotient of the powerset of a semigroup. We also show

that any (unitless) Frobenius quantale is isomorphic to a phase quantale. To do so, we use the fact that Galois connections on the free lattice—given by the powerset functor—over a set are in bijection with binary relations of that set. We then characterize those binary relations that correspond to Serre Galois connections under this isomorphism. This allows us to demonstrate the following representation theorem—see Theorem 2.3 for the formal statement:

THEOREM 0.3. *Let $(Q, *, {}^{\perp}(-), (-)^{\perp})$ be a Frobenius quantale. There exists a Serre Galois connection (l, r) on the powerset quantale of Q such that the set of fixed point of $l \circ r$ is isomorphic to $(Q, *, {}^{\perp}(-), (-)^{\perp})$.*

The characterization of Serre Galois connections as special binary relations is also very useful to construct examples of unitless Frobenius quantales. We will show that the powerset of a pregroup, and the closed subspaces of the algebra of trace class operators are non-unital Frobenius quantales. The main example of non-unital Frobenius quantales we have developed and also the one that prompted this research is the quantale of tight endomaps. As we shall see, this quantale is closely related to Theorem 0.1. For a function $f : L \longrightarrow L$, where L is a complete lattice, its Raney transforms $f^{\vee}$ and $f^{\wedge}$ are defined by

$$f^{\vee}(x) := \bigvee_{x \not\leq t} f(t)\,, \qquad\qquad f^{\wedge}(x) := \bigwedge_{t \not\leq x} f(t)\,.$$

Note that the two Raney transforms may be applied to any map f but $f^{\vee}$ is always sup-preserving while $f^{\wedge}$ is always inf-preserving. An endomap $f : L \longrightarrow L$ is *tight* if $f = f^{\wedge\vee}$ and the set of tight endomaps of L is written $[L, L]^{t}_{\vee}$. Surprisingly, any complete lattice L leads to a (unitless) Frobenius quantale—see Theorem 2.4 for the precise statement:

THEOREM 0.4. *There is a duality $(-)^{\perp}$ making $([L, L]^{t}_{\vee}, \circ, (-)^{\perp}, (-)^{\perp})$ a Frobenius quantale, where $\circ$ is the function composition.*

Moreover, Raney has shown in [**Ran60**] that a complete lattice L is a completely distributive lattice if and only if $[L, L] = [L, L]^{t}_{\vee}$, and this equality holds if and only if the identity map $\mathtt{id}_L$ is tight. Therefore, we can formulate an even more precise statement than Theorem 0.1:

THEOREM 0.5. *The Frobenius quantale $([L, L]^{t}_{\vee}, \circ, (-)^{\perp}, (-)^{\perp})$ is unital if and only if L is a completely distributive lattice.*

We end this chapter by showing that a unitless Frobenius quantale cannot be embedded into a unital Frobenius quantale if the embeding preserves the two negations. That is: if one takes the negations as primitive data of a Frobenius quantales, and hence wants to preserve them, the study of Frobenius quantale shall be done without assuming the existence of a unit.

Chapter 3 deals directly with Conjecture 0.1. The main results and ideas come from the conference paper [**dLS23**]. We have increased this article with the idea to make it a journal paper in the future. Several sections have been substantially rewritten, and some are completely new. This notably means that some results have not been peer-reviewed. We are, of course, thankful to anyone pointing out our errors.

The first step in order to prove Conjecture 0.1 is to give a suitable definition that generalizes Frobenius quantales in any symmetric monoidal category and facilitates the proof. The key ingredient of a Frobenius quantale is the isomorphism between Q and its opposite lattice Q^{op}. The opposite of a lattice is usually identified with the dual of a lattice. If 0 is a dualizing object in a $*$-autonomous category $\mathcal{C}$, then the dual of an object A is $A^* := [A, 0]$. In the case of SLatt, there is a natural isomorphism $(-)^* \cong (-)^{\mathrm{op}}$, allowing the identification between the dual and the opposite lattice. Moreover, in a $*$-autonomous category where the unit of the tensor I is the dualizing object, the domain of $\mathtt{mix}_A : A^* \otimes A \to [A, A]$ is canonically isomorphic to the dual of $[A, A]$. Hence, if A is nuclear, that is when $\mathtt{mix}_A$ is invertible, then $[A, A]$ is isomorphic to its dual. These two remarks encouraged us to develop the notion of *dual pair* in order to work on the dual of an object up to isomorphism. Given a fixed object 0, a map $\epsilon : A \otimes B \longrightarrow 0$ in $\mathcal{C}$ is said to be a *dual pairing* (w.r.t. the object 0) if the induced natural transformations

$$
\begin{aligned}
{}^{\flat}(-) : \hom(X, A) &\to \hom(X \otimes B, 0)\,, \\
f &\mapsto \epsilon \circ (f \otimes B)
\end{aligned}
\qquad\qquad
\begin{aligned}
(-)^{\flat} : \hom(Y, B) &\to \hom(A \otimes Y, 0)\,. \\
g &\mapsto \epsilon \circ (A \otimes g)
\end{aligned}
$$

are isomorphisms. We give several examples of dual pairings. Notably, for a complete lattice L, the dual pairing $\epsilon : L \otimes L^{\mathrm{op}} \longrightarrow \mathbb{2}^{\mathrm{op}}$ encodes the order of the lattice. Moreover, for any Hilbert space H, the (linear extension of the) inner product $\langle -, - \rangle : H \otimes \overline{H} \longrightarrow \mathbb{C}$ is a dual pairing in Hilb, the category of Hilbert spaces and bounded linear maps between them. An interesting outcome of dual pairings is the generalization of adjoints. Meaning, if $\epsilon : A \otimes B \to 0$ and $\epsilon' : A' \otimes B' \to 0$ are dual pairs, then for every map $f : A \to A'$ there exists a unique map $f^\dagger : B' \to B$ such that the diagram

$$
\begin{array}{ccc}
A \otimes B' & \xrightarrow{f \otimes B} & A' \otimes B' \\
{\scriptstyle A \otimes f^\dagger}\downarrow & & \downarrow{\scriptstyle \epsilon'} \\
A \otimes B & \xrightarrow[\epsilon]{} & 0
\end{array}
$$

commutes. In the case of SLatt and of Hilb, they are the usual adjoints maps: $f(x) \leq y \iff x \leq f^\dagger(y)$, and $\langle f(x), y \rangle = \langle x, f^\dagger(y) \rangle$. We will give several properties on dual pairings. Notably, we will relate dual pairings with other notions such as dagger categories and linear adjoints. Along the three first sections of Chapter 3, we develop an informal circuit calculus based on

the circuit calculus established by Cockett and Seely in [**BCST96**] for linearly distributive categories.

An important feature of a dual pair (A, B) appears when one asks that A is equipped with an associative multiplication $\mathrm{m}_A : A \otimes A \to A$. In that case, A acts on the left and on the right over its dual B. These actions are the only maps making the two following diagrams commute

$$\begin{array}{ccc} A \otimes A \otimes B & \xrightarrow{A \otimes \alpha^\ell_A} & A \otimes B \\ {\scriptstyle \mathrm{m}_a \otimes B}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ A \otimes B & \xrightarrow[\epsilon]{} & 0 \end{array} \qquad \begin{array}{ccc} B \otimes A \otimes A & \xrightarrow{\alpha^\rho_A \otimes A} & B \otimes A \\ {\scriptstyle B \otimes \mathrm{m}_a}\downarrow & & \downarrow{\scriptstyle \epsilon \circ \sigma} \\ B \otimes A & \xrightarrow[\epsilon \circ \sigma]{} & 0\,. \end{array}$$

For instance, in SLatt, the left diagram should be read: $\mathrm{m}_q(x, y) \leq z \iff x \leq \alpha^\ell_A(z, y)$. With the usual infix notation, this gives $x * y \leq z \iff x \leq z \mathbin{/} y$. Intuitively, the two actions α^ℓ_A and α^ρ_A are generalized linear implications.

We can now define *Frobenius structures* in symmetric monoidal categories, following closely the definition of Frobenius quantales:

Frobenius quantales	*Frobenius structures*
$(Q, Q^{\mathrm{op}}, \epsilon)$ is a dual pair;	(A, B, ϵ) is a dual pair;
$(Q, *)$ is a semigroup;	(A, m_A) is a semigroup;
${}^\perp(-), (-)^\perp : Q \longrightarrow Q^{\mathrm{op}}$;	$l, r : A \longrightarrow B$;
${}^\perp(x^\perp) = x = ({}^\perp x)^\perp$	(l, r) is an invertible adjoint pair
$x \mathbin{\backslash} {}^\perp y = x^\perp \mathbin{/} y$	$\begin{array}{ccc} A \otimes A & \xrightarrow{A \otimes r} & A \otimes B \\ {\scriptstyle l \otimes A}\downarrow & & \downarrow{\scriptstyle \alpha^\ell_A} \\ B \otimes A & \xrightarrow[\alpha^\rho_A]{} & B. \end{array}$

Note that, in SLatt, if an antitone map is invertible, then its inverse is necessarily its adjoint. However, in the general context of dual pairings, the adjoint of a map $A \to B$ is also typed $A \to B$. Hence, for obvious typing reasons, it is not its inverse. That is why we asked for $l, r : A \to B$ to form an adjoint pair and to be invertibles.

Among the properties of Frobenius structures we provide in Section 3.3, we would like to emphasize that the object B also has a canonical semigroup structure $\mathrm{m}_B : B \otimes B \to B$. In the case of Frobenius quantales this multiplication is important as it interprets the connective par of linear logic. We

also give the correspondence between Frobenius structures and associative bracketed semigroups, which are semigroups (S, m_S) equipped with a pairing $\pi : S \otimes S \to 0$ that makes the following diagram commutes:

$$S \otimes S \otimes S \underset{S\otimes \mathrm{m}_S}{\overset{\mathrm{m}_S \otimes S}{\rightrightarrows}} S \otimes S \xrightarrow{\pi} 0\,.$$

This allows us to relate Frobenius structures with usual Frobenius algebras in monoidal categories. We give examples from the literature and some less explored examples, such as Frobenius structures in Rel.

The section coming next did not appear in [**dLS23**]. In this section, we connect the definition of Frobenius structures to two existing definitions: Frobenius monoids defined by Egger in [**Egg10**], and H^*-algebras mainly developed in [**AH12**]. We show exactly how, and under which hypotheses, it is possible to move from one definition to another. More precisely, we show that every unital Frobenius structure in a $*$-autonomous category leads to a Frobenius monoid. This proof is a nice example of the use of the circuit calculus developed in the previous sections. In order to relate our definition with H^*-algebras, we had to extend the definition given in [**AH12**] to the more general case of symmetric monoidal categories. More precisely, we redefined axiom (H) by separating a multiplication on A and a multiplication on B, whereas [**AH12**] deals with the case where A is self-dual and the two multiplications are the same. Under the hypothesis that the ambient symmetric monoidal category is well-pointed, we show that Frobenius structures correspond to H^*-algebras.

In Sections 3.5 and 3.6, we establish the main results of the chapter. The first part of Conjecture 0.1 can be seen as an instance of [**Str04**, Corollary 3.3]. We provide another proof using the correspondence between Frobenius stuctures and associative bracketed semigroups. Precisely, we have:

Theorem 0.6. *If A is a nuclear object in a $*$-autonomous category $\mathcal{V}$, then there is a map l such that $([A, A], [A, A]^*, \mathrm{ev}_{[A,A],I}, \circ, l, l)$ is a Frobenius structure.*

The `mix` map plays an important role in our proof. As we mentioned, there exists a dual pairing $\epsilon : A^* \otimes A \otimes [A, A] \to I$. Moreover, $A^* \otimes A$ is a well-known semigroup, and if $\mathtt{mix}_A$ is invertible—*i.e.*, if A is nuclear, then $(A^* \otimes A, [A, A], \epsilon, \mathrm{m}_{A^*\otimes A}, \mathtt{mix}_A, \mathtt{mix}_A)$ is a Frobenius structure. Furthermore, the `mix` map in SLatt is exactly the Raney transform $(-)^\vee$, and the quantale of tight endomaps is exactly its image. That is, the main result of Section 2.5

can be formulated in categorical terms: in SLatt, the image of $\mathtt{mix}$ is always a Frobenius structure. This actually holds in general if the ambient $*$-autonomous category has a suitable epi-mono factorization system.

Theorem 0.7. *Let $\mathcal{C}$ be a $*$-autonomous category such that its category of semigroups has an epi-mono factorization system, and A an object of $\mathcal{C}$. The image of $\mathtt{mix}_A$ can always be endowed with a Frobenius structure:*

Of course, if $\mathtt{mix}_A$ is invertible, then $\mathsf{Img}(\mathtt{mix}_A) \cong [A, A]$, and we obtain Theorem 0.6 as a corollary of the previous theorem. We have proven it by showing a more general statement, similar to Proposition 0.1—see Theorem 3.2 for the exact theorem.

Showing the converse of Theorem 0.6, *i.e.*, that a Frobenius structure on $[A, A]$ implies that A is nuclear is actually trickier. In order to do so, we had to append another hypothesis: that I must embed into A as a retract. We call such an object A *pseudo-affine* and a category $\mathcal{C}$ in which every object is pseudo-affine is itself called *pseudo-affine*. This hypothesis is used in the proof of Theorem 0.1, and under this hypothesis we were able to show the sought-after theorem:

Theorem 0.8. *If A is pseudo-affine and $([A, A], \circ)$ carries a Frobenius structure, then A is nuclear.*

This pseudoaffine hypothesis seems technical, and it is legitimate to wonder whether it is possible to show the last statement without it. After many unfruitful attempts, we started looking for a counterexample. That is, we searched for an object A in a $*$-autonomous category which is not nuclear but such that $([A, A], \circ)$ carries a Frobenius structure. This is not trivial since our usual $*$-autonomous categories—SLatt, Rel, $\mathrm{finVect}_k$ ...—are pseudo-affine. In order to construct this counter-example, we used an instance of the category of poset-valued sets introduced by Shalk and de Paiva in [**SdP04**] and named $Q_{\mathtt{id}}$-Set. Roughly, several important properties of the category $Q_{\mathtt{id}}$-Set depend on a given quantale Q. Notably, if Q is a commutative unital Frobenius quantale, then $Q_{\mathtt{id}}$-Set is a $*$-autonomous category. Investigations on $Q_{\mathtt{id}}$-Set has enabled us to construct a quantale Q and an object A in $Q_{\mathtt{id}}$-Set which is not nuclear but whose object of endomorphisms has a Frobenius structure. Thus, Conjecture 0.1 does not hold without the condition that A is pseudo-affine. This work has not been unpublished yet but is expressed in detail in Section 3.7

This inquiry on the pseudo-affine hypothesis and the study of $Q_{\mathtt{id}}$-Set was the starting point of our third axis of research. The category of poset-valued set actually comes as the Grothendieck construction $\int \mathsf{Q}$ over a functor

$\mathsf{Q} : \textsc{Rel} \longrightarrow \textsc{SLatt}$. It appeared that under certain hypotheses regarding the functor Q, the structure of $\textsc{Rel}$ is lifted to the category $\int \mathsf{Q}$. Surprisingly, the category $\int \mathsf{Q}$ has a "better behaviour" than $\textsc{Rel}$, dependent on the functor Q. Indeed, even though $\textsc{Rel}$ is a compact closed category, $Q_{\mathtt{id}}$-$\textsc{Set}$ can be a $*$-autonomous category which is not compact closed. This is actually the main point of the work done in [**SdP04**]. In Chapter 4, we continue this work by considering functors $\mathsf{Q} : \mathcal{C} \to \textsc{Pos}$, and we provide exact conditions for the total category $\int \mathsf{Q}$ to be symmetric monoidal, symmetric monoidal closed or a $*$-autonomous category. Of course, we do not want any kind of structure on $\int \mathsf{Q}$: this structure must be preserved "on the nose" by the opfibration $\pi : \int \mathsf{Q} \longrightarrow \mathcal{C}$. This notably implies that $\mathcal{C}$ must have the same kind of structure as $\int \mathsf{Q}$. We formalize this problem as follows.

> Assume the existence of three functors $\mathsf{Q}_{\mathcal{A}} : \mathcal{A} \longrightarrow \textsc{Pos}$, $\mathsf{Q}_{\mathcal{B}} : \mathcal{B} \longrightarrow \textsc{Pos}$ and $F : A \longrightarrow B$. Is there a *lifting* of F, that is a functor $\overline{F} : \int \mathsf{Q}_{\mathcal{A}} \longrightarrow \int \mathsf{Q}_{\mathcal{B}}$ which makes the following diagram commute?
>
> $$\begin{array}{ccc} \int \mathsf{Q}_{\mathcal{A}} & \xrightarrow{\overline{F}} & \int \mathsf{Q}_{\mathcal{B}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

We call this kind of problem a *lifting problem*. Such liftings are in bijection with specific families of maps in $\textsc{Pos}$—see Theorem 4.1 for details. We notably apply this result to $\mathcal{A} := \mathcal{C} \times \mathcal{C}$ and $F := \otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor which makes $\mathcal{C}$ a symmetric monoidal category. Interestingly, such a lifting exactly corresponds to lax natural maps $\mu_{X,Y} : \mathsf{Q}(X) \times \mathsf{Q}(Y) \to \mathsf{Q}(X \otimes Y)$ making Q a symmetric lax monoidal functor. By "lax" we mean that the usual square of naturality is not an equality but an inequality, meaning the following diagram semi-commutes:

$$\begin{array}{ccc} \mathsf{Q}(X) \times \mathsf{Q}(Y) & \xrightarrow{\mathsf{Q}(f) \times \mathsf{Q}(g)} & \mathsf{Q}(Z) \times \mathsf{Q}(W) \\ \downarrow{\scriptstyle \mu_{X,Y}} & \swarrow & \downarrow{\scriptstyle \mu_{Z,W}} \\ \mathsf{Q}(X \otimes Y) & \xrightarrow[\mathsf{Q}(f \otimes g)]{} & \mathsf{Q}(Z \otimes W)\,. \end{array} \tag{6}$$

We also provide a similar result for liftings of a contravariant functor $\mathcal{A}^{op} \longrightarrow \mathcal{B}$ and for liftings of a bifunctor, contravariant on its first variable and covariant on its second—see Theorems 4.2 and 4.3 for details. Our main objective with the latter result is to lift the closed structure of a symmetric monoidal closed category $\mathcal{C}$. Of course, the internal hom functor of $\mathcal{C}$ shall not be lifted alone, one must lift the whole adjunction $X \otimes - \dashv [X, -]$. General lifting of adjunctions is carried out in Section 4.2. In order to lift

the closed structure of $\mathcal{C}$, we must lift the tensor, the internal hom and the adjunction. Such a lifting corresponds to a collection of order-preserving maps

$$\{\, \iota_{X,Y} : Q(X)^{op} \times Q(Y) \longrightarrow Q([X,Y]) \mid X, Y \in \textsc{Obj}(\mathcal{C}) \,\},$$

such that the following diagram semi-commutes:

$$\begin{array}{ccc} Q(X)^{op} \times Q(Y) & \xrightarrow{\mathtt{id} \times Q(g)} & Q(X)^{op} \times Q(Y') \\ {\scriptstyle Q(f)^{op} \times \mathtt{id}}\downarrow & & \\ Q(X')^{op} \times Q(Y) & \swarrow & \downarrow{\scriptstyle \iota_{X,Y'}} \\ {\scriptstyle \iota_{X',Y}}\downarrow & & \\ Q([X',Y]) & \xrightarrow{Q([f,g])} & Q([X,Y']), \end{array}$$

and the following inclusions hold:

$$\begin{aligned} Q(\eta_{X,Y})(y) &\leq \iota_{X,X\otimes Y}(x, \mu_{X,Y}(x,y)), \\ Q(\mathrm{ev}_{X,Y})(\mu_{X,X\iota Y}(x, \iota_{X,Y}(x,y)) &\leq y\,. \end{aligned}$$

Those two inequalities simply state that the unit and counit of the adjunction in $\mathcal{C}$ are maps in $\int \mathsf{Q}$. The category SLatt has a special property—see Theorem 4.7 for the exact statement:

Theorem 0.9. *Let $\mathcal{C}$ be a symmetric monoidal closed category. If $\mathsf{Q} : \mathcal{C} \longrightarrow$ SLatt is a symmetric monoidal functor, then $\int \mathsf{Q}$ is a symmetric monoidal closed category.*

We moreover give necessary and sufficient conditions for $\int \mathsf{Q}$ to be a $*$-autonomous category with $\mathsf{Q} : \mathcal{C} \longrightarrow$ Pos. This means that $\int \mathsf{Q}$ admits a dualizing object $(0, \omega)$ with 0 a dualizing object of $\mathcal{C}$ and ω an element of $\mathsf{Q}(0)$. Every element $\omega \in \mathsf{Q}(0)$ induces a family of maps $\omega_X : \mathsf{Q}(X)^{\mathrm{op}} \longrightarrow \mathsf{Q}(X^*)$ defined by $\omega_X(\alpha) := \iota_{X,0}(\alpha, \omega)$. This family of maps is the key for $\int \mathsf{Q}$ to be a $*$-autonomous category. It is the principal object of study of Section 4.3. The main result of this section is the following theorem:

Theorem 0.10. *For each $\omega \in \mathsf{Q}(0)$, $(0,\omega)$ is dualizing if and only if 0 is dualizing in $\mathcal{C}$ and, for each object X of $\mathcal{C}$, ω_X is invertible.*

There may not be an element $\omega \in \mathsf{Q}(0)$ making $(0,\omega)$ a dualizing object of $\int \mathsf{Q}$. However, given a monoidal functor $\mathsf{Q} : \mathcal{C} \longrightarrow$ SLatt and element $\omega \in \mathsf{Q}(0)$, we show how to construct a functor $\mathsf{Q}^{\neg^\omega} : \mathcal{C} \longrightarrow$ SLatt such that $(0,\omega)$ is a dualizing object of $\int \mathsf{Q}^{\neg^\omega}$. This construction generalizes the double negation construction of a Frobenius quantale that we mentioned in Proposition 0.1. We base this construction on the theory of nuclei for monoidal functors whose codomain is SLatt. This theory is developed in Section 4.4. We actually closely follow the theory of nuclei for quantales. Since unital

quantales are in bijection with monoidal functors from the terminal category to SLATT, nuclei on quantales are particular cases of nuclei on monoidal functors into SLATT. We also provide a representation theorem analogous to the representation theorem for Frobenius quantale. Briefly, for a functor $Q : \mathcal{C} \longrightarrow \text{SLATT}$, define another functor

$$\mathsf{PUQ} := \mathcal{C} \xrightarrow{Q} \text{SLATT} \xrightarrow{U} \text{SET} \xrightarrow{\mathcal{P}(-)} \text{SLATT}\,.$$

Using the theory of nuclei on a monoidal functor into SLATT, we were able to show the following representation theorem—see Theorem 4.10 for the precise statement:

THEOREM 0.11. *Let* $\mathsf{Q} : \mathcal{C} \longrightarrow \text{SLATT}$ *be a monoidal functor and suppose that* $(0, \omega)$ *is a dualizing element of* $\int \mathsf{Q}$. *There exists a set* $\Omega \subset \mathsf{Q}(0)$ *and a monoidal natural isomorphism between the monoidal functors* $\mathsf{Q} : \mathcal{C} \longrightarrow \text{SLATT}$ *and* $\mathsf{PUQ}^{\neg^\Omega} : \mathcal{C} \longrightarrow \text{SLATT}$. *Consequently,* $\int \mathsf{Q}$ *and* $\int \mathsf{PUQ}^{\neg^\Omega}$ *are isomorphic.*

After giving some examples of applications, we end by connecting the results of this chapter to Frobenius Structures. More precisely, if $\mathcal{C}$ a small $*$-autonomous category, functors $\mathsf{Q} : \mathcal{C} \longrightarrow \text{SLATT}$ such that there exists an element $\omega \in \mathsf{Q}(0)$ making $(0, \omega)$ a dualizing object of $\int \mathsf{Q}$ exactly correspond to some Frobenius structures of $[\mathcal{C}, \text{SLATT}]$, the category of functors between $\mathcal{C}$ and SLATT. Since SLATT is a complete-cocomplete $*$-autonomous category, $[\mathcal{C}, \text{SLATT}]$ is itself a $*$-autonomous category where the tensor is given by the Day's convolution—see [**Egg08**]. Day had pointed out in [**Day70b**, Example 3.2.2] that monoidal functors are exactly the monoid objects in that category, hence by Theorem 0.9 monoid objects of $[\mathcal{C}, \text{SLATT}]$ lift the monoidal closed structure. We demonstrate a similar statement—see Theorem 4.11:

THEOREM 0.12. *Let* $\mathsf{Q} : \mathcal{C} \to \text{SLATT}$ *be a monoidal functor, and let* ω *be an element of* $\mathsf{Q}(0)$. *The object* $(0, \omega)$ *is a dualizing object of* $\int \mathsf{Q}$ *if and only if* $(\mathsf{Q}, \mathsf{Q}^\perp, \mathrm{m}_\mathsf{Q}, \epsilon, l^\omega, r^\omega)$ *is a Frobenius structure in* $[\mathcal{C}, \text{SLATT}]$.

CHAPTER 1

Categorical background

For most of the basic notions, facts, and notation of category theory we will rely on the monograph [**Mac78**]. Even though this book is the usual main reference, the beginner on category theory can for instance look at ([**Bor94**], [**Lei16**], [**Awo10**]...).

We are mainly interested in the connections category theory holds with logic. These have firstly been studied between cartesian closed categories and intuitionistic logic. One of the objectives has been to interpret a proof of intuitionistic logic as a morphism in a certain cartesian closed category. In this case, the negation of a formula A is interpreted as the object $A \to \bot$ where $\bot$ is the initial object of the category. This perfectly works with intuitionistic logic. However, in classical logic, we have $A \Leftrightarrow (A \to \bot) \to \bot$, and a cartesian closed category that respects this equivalence is necessarily a poset [**LS86**]. This "problem" comes from the universal property of the cartesian product. To work with a category which has a classical duality but is not a poset, we have to consider monoidal categories and especially $*$-autonomous categories.

The objective of this chapter is to provide some basic definitions, to settle some notations, and start working with examples. Most of the theorems given in this chapter have already been well established and we shall not always give their proof.

1.1. Symmetric monoidal closed categories

Even though we assume some familiarity of the reader with symmetric monoidal closed categories, we give some definitions in order to settle our notations and to discuss these notions.

1.1.1. Monoidal categories.

Definition 1.1. A *monoidal* category is a tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ where:

- $\mathcal{C}$ is a category;
- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor called the *tensor*;
- I is an object of $\mathcal{C}$ called the *unit of the tensor*;
- α, λ, ρ are natural isomorphisms whose components are typed

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z\,, \quad \lambda_X : I \otimes X \cong X\,, \quad \rho_X : X \otimes I \cong X\,,$$

respectively called the *associator*, the *left unitor* and the *right unitor*.

The three natural maps α, λ, ρ must respect Mac Lane's pentagon and triangles. We shall not give these diagrams as they are easily found in many references, notably in the Chapter 7 of Mac Lane's book [**Mac78**].

A monoidal category $\mathcal{C}$ is *strict* if the associator and the two unitors are the identities. A *monoidal functor* between monoidal categories $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \otimes, I, \alpha, \lambda, \rho)$ is a tuple (F, μ, η) where

- $F : \mathcal{C} \to \mathcal{D}$ is a functor;
- μ is a natural transformation called the *multiplication* of F, and whose components are typed

$$\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)\,;$$

- $\eta : I \to F(I)$ is a morphism in $\mathcal{D}$ called the *unit* of F.

The unit and the multiplication must make the three following diagrams commute.

$$\begin{array}{ccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha} & F(X) \otimes (F(Y) \otimes F(Z)) \\
{\scriptstyle \mu \otimes F(Z)}\downarrow & & \downarrow{\scriptstyle F(X) \otimes \mu} \\
F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(\alpha)]{} & F((X \otimes Y) \otimes Z)
\end{array}$$

$$\begin{array}{ccc}
F(X) \otimes I & \xrightarrow{F(X) \otimes \eta} & F(X) \otimes F(I) \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \mu} \\
F(X) & \xrightarrow[F(\lambda)]{} & F(X \otimes I)
\end{array}
\qquad\qquad
\begin{array}{ccc}
I \otimes F(X) & \xrightarrow{\eta \otimes F(X)} & F(I) \otimes F(X) \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \mu} \\
F(X) & \xrightarrow[F(\rho)]{} & F(I \otimes X)
\end{array}$$

FIGURE 1.1. Coherence diagrams for a monoidal functor.

In Section 2 of Chapter 7 of [**Mac78**], Mac Lane shows that the pentagon and the two triangle diagrams are necessary and sufficient to make every commutative every well-formed diagram that only uses the natural maps α, λ ρ. That is called the coherence theorem.

NOTATION 1.1. We will simply write $\mathcal{C}$ for the category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$. For the reader's (and the author's) convenience, we shall not write the components' subscripts of a natural transformation when they are obvious.

Another version of the coherence theorem, given in Theorem 1 of Section XI.3 of [**Mac78**], is

THEOREM 1.1. *Every monoidal category is monoidally equivalent—i.e., the equivalence of category is a monoidal functor—to a strict one.*

More precisely, Mac Lane demonstrate that every monoidal category $\mathcal{C}$ is monoidally equivalent to some subcategory of the free strict monoidal category over $\mathcal{C}$. This theorem also appears in Theorem 1.4 of [**JS93**], this time the strict monoidal category is a subcategory of the monoidal category of endofunctors over $\mathcal{C}$.

This theorem is very important, as we can reformulate it as: "a property that holds in a strict monoidal category holds in any monoidal category". For practical reasons, we shall develop the theory assuming that the associator of the monoidal category we are working with is the identity. The unitors have quite a special role: they involve the unit of the tensor, and we will often consider maps whose domain or codomain involve the unit (for instance, the (co)unit of a (co)monoid, of an ajunction...). That is why we will often express the unitors but not the associator. Nonetheless it is important to keep in mind that most examples of monoidal categories we will be considering are not strict.

REMARK 1.1. The idea of categorical semantics is to interpret a formula A of a logic L as an object $|A|$ in a category $\mathcal{C}$, and proofs as morphisms of this category. This is done inductively, using the logical operators of L and their interpretations in $\mathcal{C}$. For instance, the conjunction $\bigwedge$ of intuitionistic logic is interpreted as the cartesian product in $\mathcal{C}$. In Linear Logic the tensor does not have a particular universal property and is therefore interpreted as a tensor in $\mathcal{C}$. A formula of the form $A \otimes B$ in will be interpreted as $|A| \otimes |B|$ in the target monoidal category. Throughout this chapter, we will expose what structure is needed to interpret the classical multiplicative fragment of linear logic (MLL).

Let us now give some examples of monoidal categories that we will study along our thesis.

EXAMPLE 1.1. A poset $(P, \leq)$ is a category in which the objects are the elements of P, and in which there exists an arrow from x to y if and only if $x \leq y$. Such categories are called *posetal*. The simplest example of a monoidal category comes from any ordered monoid $(M, \leq, *)$. The map $- * -$ is a bifunctor as it is order-preserving in both variables. This algebraic structure is the basis of many algebraic models of provability of substructural logics (see for instance [**GJKO07**]). Provability is closely related to the categorical semantics of proofs, as we shortly described in Remark 1.1. A class $\mathbf{S}$ of posetal structures is *sound* and *complete* with respect to a logic L (that we assume defined by a Gentzen's style sequent calculus) if for any $S \in \mathbf{S}$ and possible interpretation of sequence of formulas $|-|$ we have

$$\Gamma \vdash_L \Delta \iff |\Gamma| \leq_S |\Delta| .$$

The right side of the equivalence is trivially equivalent to demand the existence of a morphism from $|\Gamma|$ to $|\Delta|$ in the corresponding category. The relation between proof semantics and provability comes from the change of base functor $U : \textsc{Set} \to \mathit{2}$. $\textsc{Set}$ is the category of sets and functions and $\mathit{2}$ is the two-element boolean algebra with the usual order. The functor U sends a set X to $\top$ if it is non-empty and to $\bot$ otherwise. It is immediate to check that U is monoidal and therefore induces a $\mathbf{2}$-functor $U_* : \textsc{Set}\text{-}\textsc{Cat} \longrightarrow \mathit{2}\text{-}\textsc{Cat}$, that is, from usual categories to posets[1]. The general case of change of base in $\mathcal{V}$-enriched categories was studied by Eilenberg and Kelly in [**EK66**] and explained in detail in Section 4.2 of Cruttwell's thesis [**Cru08**]. ◊

EXAMPLE 1.2. Any category that has binary cartesian product and a terminal object is monoidal with the tensor given by the cartesian product. This kind of monoidal category are called *cartesian monoidal*. The examples that we are interested in are the category $\textsc{Set}$, the category $\textsc{Pos}$ of posets and monotone maps—*i.e.*, $x \leq y \Rightarrow f(x) \leq f(y)$—between them; and any poset that has binary meet and a top element—*i.e.*, a meet-semilattice. ◊

EXAMPLE 1.3. The usual example of a monoidal category where the tensor is not the cartesian product is the category $\textsc{Vect}_k$ of vector spaces over a field k and linear maps between them. The tensor is the usual tensor product of vector spaces, and the unit is the ground field. Note that this tensor is different from the cartesian product of two vector spaces. However, the tensor product is defined by the universal property: there exists a map $u : X \times Y \to X \otimes Y$

[1] Be aware of the difference between $\mathit{2}$, the boolean algebra of two elements, and $\mathbf{2}$ which serves as a notation for higher category theory.

such that for every bilinear map $f : X \times Y \to Z$, there exists a linear map $\tilde{f} : X \otimes Y \to Z$ called the *linear extension* of f such that $f = \tilde{f} \circ u$. ◊

EXAMPLE 1.4. The category REL of sets and relations plays an important role in the proof semantics of linear logic. Its tensor is the set theoretic cartesian product which is not the categorical cartesian product of REL. ◊

EXAMPLE 1.5. Our main example, and the one we will investigate the most is the category SLATT of complete lattices and sup-preserving maps between them. As it is our main example, we will give some basic facts and definitions. We recommend [**DP90**] for an introduction to lattices and [**EGGHK18**] for a categorical approach to this theory.

A complete lattice L is a poset that has all joins—*i.e.*, it is a posetal complete category. A morphism in SLATT is a sup-preserving map $f : L \to M$: for any set I, we have

$$f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)\,.$$

Notably, this implies that the bottom element $\bot := \bigvee \emptyset$ is preserved—*i.e.*, $f(\bot) = \bot$. A complete lattice has all meets, and they are given by

$$\bigwedge_{i \in I} x_i = \bigvee \{y \in L \mid y \leq x_i \text{ for all } i \in I\}\,.$$

The top element is $\top := \bigwedge \emptyset$. We denote by $[L, M]$ the complete lattice of sup-preserving maps between L and M. The order and the joins are computed pointwise. A sup-preserving map is not necessarily inf-preserving. Each complete lattice L has its opposite L^{op} where the order has been reversed. This operation defines a functor $(-)^{\mathrm{op}} : \mathrm{SLATT} \to \mathrm{SLATT}^{\mathrm{op}}$ which sends a sup-preserving map $f : L \to M$ to its *right adjoint* $\rho(f) : M^{\mathrm{op}} \to L^{\mathrm{op}}$: the unique map determined by the equivalence for each $x \in L$ and $y \in M$

$$f(x) \leq y \iff x \leq \rho(f)(y)\,.$$

The typing of the right adjoint $\rho(f)$ essentially means that it is an inf-preserving map from M to L. This gives a bijection $[L, M] \cong [M^{\mathrm{op}}, L^{\mathrm{op}}]$. Note that a map between complete lattices is sup-preserving if and only if it has a right adjoint. Also, the adjoint is unique and is explicitly given by

$$\rho(f)(y) = \bigvee \{x \in L \mid f(x) \leq y\}\,.$$

We now define the tensor by:

$$- \otimes - := [-, (-)^{\mathrm{op}}]^{\mathrm{op}}\,.$$

This means that the elements of $L \otimes M$ are maps from L to M that reverse the sups in infs with the reverse point-wise order. They correspond to the Galois connections between those two lattices. That is, they are maps (f, g) such that $x \leq g(y) \iff y \leq f(x)$ for all $x \in L$ and $y \in M$. The two maps f and g, forming a Galois connection, detemine each other; hence we only refer to one of them. The order of this lattice is given by $f \leq g \iff f(x) \leq g(x)$ for each $x \in L$. The unit of the tensor is the lattice with two elements $2 := \{\bot, \top\}$. We refer to the [**EGGHK18**, Section 2.1] for the remaining monoidal structure of SLatt, that is the associator and the two unitors. [**EGGHK18**, Theorem 2.1.8] notably gives a proof of the universal property of the tensor, similar to the one of the tensor product in Vect_k: there is a bijection between maps $L \times M \to N$ which are sup-preserving in both variables, called *bilinear maps*, and sup-preserving maps $L \otimes M \to N$. More precisely, there exists a bilinear map $u : L \times M \to L \otimes M$ such that for every bilinear map $f : L \times M \to N$, there exists a unique sup-preserving $\tilde{f} : L \otimes M \to N$ called the *linear extension* of f such that $f = \tilde{f} \circ u$. This tensor product have been studied in section 1.5 of [**JT84**]. To do so, the authors use the more general result of [**BN76**]. ◊

Example 1.6. The category of endofunctor $[\mathcal{C}, \mathcal{C}]$ over a category $\mathcal{C}$ is a strict monoidal category with the tensor given by the composition of functors. This category plays a key role in the theory of monoidal categories as Theorem 1.1 is proven by constructing a subcategory of $[\mathcal{C}, \mathcal{C}]$. We shall treat this monoidal category in Chapter 2 in the case where $\mathcal{C}$ is a complete lattice. ◊

Example 1.7. The powerset functor $\mathcal{P}(-) : \text{Set} \to \text{SLatt}$ is the left adjoint to forgetful functor $U : \text{SLatt} \to \text{Set}$, leading to the free complete sup-lattice over a set. It is a monoidal functor whose multiplication is the linear extension of the obvious map $\mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$. ◊

Example 1.8. A similar monoidal powerset functor is $\mathcal{P}(-) : \text{Rel} \to \text{SLatt}$. Note that Rel is enriched over SLatt—*i.e.*, it is a quantaloid. Moreover, we have a natural isomorphism $\mathcal{P}(-) \cong \hom(\{*\}, -)$. More generally, if $\mathcal{A}$ is a monoidal enriched category over $\mathcal{V}$, then the functor $\hom(I, -)$ is a monoidal enriched functor $\mathcal{A} \to \mathcal{V}$. ◊

1.1.2. Symmetric monoidal categories. Most of the categories we have considered are actually symmetric monoidal categories.

Definition 1.2. A *symmetric monoidal category* is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism σ called the *symmetry* whose components are typed

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X$$

such that

$$\sigma_{X,Y}^{-1} = \sigma_{Y,X} .$$

Again, usually the monoidal category $\mathcal{C}$ is not strict, and one should add two other commutative diagrams for coherence: one for the associator and one for the two unitors.

Even though many of our results hold for non symmetric monoidal categories, it is usually simpler to work with symmetries. In most of this thesis, monoidal categories will be considered symmetric unless they are posetal. We quickly discuss which of the categories we have already considered are symmetric.

EXAMPLE 1.9. A posetal monoid is symmetric if and only if the underlying monoid is commutative. If the tensor is actually the cartesian product of the category, then it is symmetric. VECT_k, REL and SLATT are symmetric monoidal categories. The category of endofunctor is usually not symmetric as the composition of functors is not a commutative operation. ◊

1.1.3. Symmetric monoidal closed categories. In order to prove an implication of the form $A \Rightarrow B$ from the assumption Γ, one adds A to the hypotheses and then proves B: this is the usual introduction rule of the implication in sequent calculus. The Curry-Howard-Lambek correspondence relates this rule with the curryfication of a lambda term and transposition in cartesian closed category. The main idea of Linear Logic is to work with a weaker conjunction, and indeed, one does not need the tensor of the category to be cartesian in order to have such a correspondence. In order to interpret the multiplicative fragment of Linear Logic with the linear implication in a monoidal category, this must be closed.

DEFINITION 1.3. A symmetric monoidal category $\mathcal{C}$ is *closed* if for each X in $\mathcal{C}$, the functor $X \otimes -$ has a right adjoint written $[X, -]$. The co-unit of this adjunction $\text{ev}_{X,Y} : X \otimes [X, Y] \to X$ is called the *evaluation map*.

The adjunction induces an isomorphism $\hom(X \otimes Y, Z) \cong \hom(Y, [X, Z])$ that is natural in X, Y, and Z. We use the following notation for the natural isomorphism between the two hom with the horizontal line:

$$\sharp \downarrow \frac{X \otimes Y \longrightarrow Z}{Y \longrightarrow [X, Z]} \uparrow \flat \tag{7}$$

The universal property of the co-unit of the adjunction means that for every arrow $\phi : X \otimes Y \to Z$, $\phi^\sharp$ is the only arrow such that the diagram

$$\begin{array}{ccc} X \otimes Y & & \\ {\scriptstyle X\otimes\phi^\sharp}\downarrow & \searrow^{\phi} & \\ X \otimes [X, Z] & \xrightarrow[\text{ev}]{} & Z \end{array}$$

commutes.

We only defined closedness with the functor $X \otimes -$. There is also the notions of right, left, and bi-closedness where one also looks at $- \otimes X$. When we look at symmetric monoidal categories, those notions are not relevant: in that case, the two functors $X \otimes -$ and $- \otimes X$ are naturally isomorphic, which implies that one has a right adjoint if and only if the other has one. The functor $[X, -]$ extends to a bifunctor $[-, -] : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$, whose first coordinate is defined on arrows of the form $f : X \to X'$ by the commutative diagram:

$$\begin{array}{ccc} X \otimes [X', Y] & \xrightarrow{X\otimes[f,Y]} & X \otimes [X, Y] \\ {\scriptstyle f\otimes[X',Y]}\downarrow & & \downarrow{\scriptstyle \text{ev}} \\ X' \otimes [X', Y] & \xrightarrow[\text{ev}]{} & Y\,. \end{array}$$

Notably, for a fixed object 0 of $\mathcal{C}$, this defines a contravariant functor

$$(-)^* := [-, 0] : \mathcal{C} \to \mathcal{C}^{\text{op}}\,.$$

This functor will play the role of an intuitionistic negation. There is a natural map $\mathtt{j} : \mathtt{id}_{\mathcal{C}} \to (-)^{**}$ whose components are defined by:

$$\begin{array}{ccc} X^* \otimes X & \xrightarrow{X^*\otimes\mathtt{j}} & X^* \otimes X^{**} \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \text{ev}} \\ X \otimes X^* & \xrightarrow[\text{ev}]{} & Y\,. \end{array} \tag{8}$$

Remark 1.2. The *name* of a map $f : X \to Y$ in $\mathcal{C}$ is a morphism $\ulcorner f \urcorner : I \to [X, Y]$ defined as the only map making the following diagram commute:

$$\begin{array}{ccc} X \otimes I & \xrightarrow{\rho} & X \\ {\scriptstyle X\otimes\ulcorner f\urcorner}\downarrow & & \downarrow{\scriptstyle f} \\ X \otimes [X, Y] & \xrightarrow[\text{ev}]{} & Y\,. \end{array}$$

Remark 1.3. The linear implication of *MLL* is interpreted in a symmetric monoidal closed category by the internal hom functor. The contravariant functor $(-)^*$ interprets the intuitionistic negation—*i.e.*, $|A^\perp| = A^*$. Therefore, j corresponds to a proof of $A \vdash A^{\perp\perp}$ for every formula A.

Once again, we will quickly discuss which, of the categories we have introduced before, is a symmetric monoidal closed category.

EXAMPLE 1.10. A commutative monoidal poset $(P, *, \leq)$ that is closed is usually called a commutative residuated partially ordered monoid. It respects the following equivalences:

$$x * y \leq z \iff y \leq x \backslash z$$

where $- \backslash -$ will be our notation of the internal hom in the posetal case, as we will understand it as an implication. In Chapter 2 we will study a similar structure called *quantale*. It is an important structure in our work. We emphasize that, from a categorical point of view, they are biclosed monoidal posetal and complete categories. This is not completely correct: they often do not have a unit object. Moreover, note that this structure is usually not symmetric. This notably means we have to distinguish between the right and the left implications, that is, we have:

$$x * y \leq z \iff y \leq x \backslash z \iff x \leq z / y\,.$$

We will come back to this example very soon. ◊

EXAMPLE 1.11. SET and POS are usual examples of cartesian closed categories. ◊

EXAMPLE 1.12. The category VECT_k is a symmetric monoidal closed category with respect to the usual tensor product. The internal hom of two vector spaces X and Y is the usual vector space of linear maps between them. However, this category is not closed with respect to the cartesian product. More generally, a cartesian category with a initial object which is also terminal, and with at least one non-invertible arrow, is not closed. Indeed, if 0 is initial in a cartesian closed category, then we have $X \times 0 \cong 0$. This comes from the fact that $X \times -$ preserves the colimit since it has a right adjoint. Therefore, if 1 is terminal and initial, we always have $X \cong X \times 1 \cong X \times 0 \cong 0$. The zero-dimensional vector space is initial and terminal, hence VECT_k is not a cartesian closed category. ◊

EXAMPLE 1.13. Just like in VECT_k, the internal hom in SLATT is given by the lattice of sup-preserving maps. ◊

EXAMPLE 1.14. In REL, it is immediate to see that we have

$$\hom(X \times Y, Z) \cong \hom(X, Y \times Z)\,,$$

which means that the internal hom is given by $[X, Y] = X \times Y$. This example shows that the intuition of the internal hom as the object of morphism is not always accurate. ◊

1.2. Semigroups and monoids in monoidal categories

In a monoidal category $\mathcal{C}$, one can define algebraic structures much like we usually do in the category of sets with respect to the cartesian product.

DEFINITION 1.4. A *magma* in a monoidal category $\mathcal{C}$ is a tuple (X, m) where X is a $\mathcal{C}$-object and $\mathrm{m} : X \otimes X \to X$ is a $\mathcal{C}$-morphism. A *semigroup* (S, m) is a magma where m is associative, meaning the following diagram commutes:

$$\begin{array}{ccc} S \otimes S \otimes S & \xrightarrow{S \otimes \mathrm{m}} & S \otimes S \\ {\scriptstyle \mathrm{m} \otimes S}\downarrow & & \downarrow{\scriptstyle \mathrm{m}} \\ S \otimes S & \xrightarrow[\mathrm{m}]{} & S\,. \end{array}$$

A *monoid* in $\mathcal{C}$ is a tuple $(M, \mathrm{m}, \mathrm{u})$ where (M, m) is a semigroup and $\mathrm{u} : I \to M$ is a $\mathcal{C}$-morphism such that the diagram

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{\mathrm{u} \otimes M} & M \otimes M & \xleftarrow{M \otimes \mathrm{u}} & M \otimes I \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \mathrm{m}} & \swarrow{\scriptstyle \rho} & \\ & & M & & \end{array}$$

commutes.

The category $\mathrm{SEM}_{\mathcal{C}}$ has semigroups as objects and morphisms of semigroups as arrows, that is arrows of $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} S_0 \otimes S_0 & \xrightarrow{f \otimes f} & S_1 \otimes S_1 \\ {\scriptstyle \mathrm{m}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{m}_1} \\ S_0 & \xrightarrow[f]{} & S_1 \end{array}$$

commmutes.

The category $\mathrm{MON}_{\mathcal{C}}$ has monoids as objects and monoid morphisms as arrows, that is a $\mathcal{C}$-arrow which is a semigroup homomorphism such that the following diagram commutes:

$$\begin{array}{ccc} I & \xrightarrow{\mathrm{u}_1} & M_1 \\ & {\scriptstyle \mathrm{u}_0}\searrow \quad \nearrow{\scriptstyle f} & \\ & M_0 & \end{array}$$

A *bracketed magma* in $\mathcal{C}$ is a triple (X, m, π) with (X, m) a magma and an arrow $\pi : X \otimes X \longrightarrow 0$, where 0 is a fixed object. A bracketed magma (S, m, π) is *associative* if (S, m) is a semigroup in $\mathcal{C}$ and π is associative, meaning that the following two horizontal arrows are equal:

$$S \otimes S \otimes S \underset{S\otimes \mathrm{m}}{\overset{\mathrm{m}\otimes S}{\rightrightarrows}} S \otimes S \xrightarrow{\ \pi\ } 0\,.$$

If $\mathcal{C}$ is symmetric monoidal, and if (S, m) is a semigroup, then a pairing π is said to be *co-associative* when $(S, \mathrm{m} \circ \sigma, \pi)$ is an associative bracketed magma.

FACT 1.1. *Given a semigroup (S, m), a pairing $\pi : S \otimes S \to 0$ is associative if and only if the pairing $\pi \circ \sigma$ is co-associative.*

PROOF. We show that if π is associative, then $\pi \circ \sigma$ is co-associative.

$$\begin{aligned}
\pi \circ \sigma \circ ((\mathrm{m} \circ \sigma) \otimes S) &= \pi \circ (S \otimes (\mathrm{m} \circ \sigma)) \circ \sigma_{S\otimes S,S} \\
&= \pi \circ (S \otimes \mathrm{m}) \circ (S \otimes \sigma) \circ \sigma_{S\otimes S,S} \\
\pi \circ \sigma \circ (S \otimes (\mathrm{m} \circ \sigma)) &= \pi \circ ((\mathrm{m} \circ \sigma) \otimes S) \circ \sigma_{S,S\otimes S} \\
&= \pi \circ (\mathrm{m} \otimes S) \circ (\sigma \otimes S) \circ \sigma_{S,S\otimes S}\,,
\end{aligned}$$

thus the statement follows from the equality

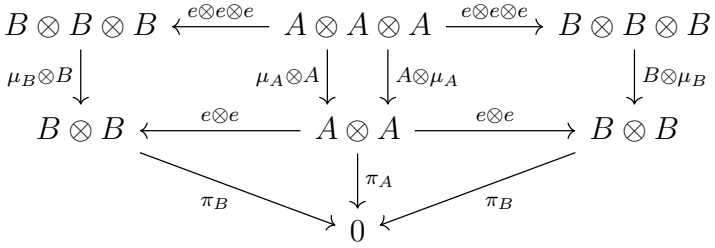
$$(S \otimes \sigma) \circ \sigma_{S\otimes S,S} = (\sigma \otimes S) \circ \sigma_{S,S\otimes S}$$

and associativity of π. □

Notice that if (S, m) is a semigroup and $\mathrm{tr} : S \longrightarrow 0$ is any arrow, then $\mathrm{tr} \circ \mathrm{m} : S \otimes S \longrightarrow 0$ is associative.

LEMMA 1.1. *Let $\mathcal{C}$ be symmetric monoidal closed category and let $e : (A, \mu_A, \pi_A) \longrightarrow (B, \mu_B, \pi_B)$ be an epimorphism of bracketed magmas—morphisms of magma are defined as expected. If (A, μ_A, π_A) is associative, then so is (B, μ_B, π_B).*

PROOF. Let us recall that since $\mathcal{C}$ is biclosed, then $X \otimes e$ (resp. $e \otimes X$) is epi if e is epi. From this, it follows that an arrow which is a tensor of epic arrows is epi. We argue below that π_B is associative.



Since the two vertical parallel arrows from $A \otimes A \otimes A$ are equal and all the squares and triangles commute, the left leg of the above diagram equals its

right leg. Since $e \otimes e \otimes e$ is epi, we deduce that $\pi_B \circ (\mu_B \otimes B) = \pi_B \circ (B \otimes \mu_B)$. That is, μ_B is associative. □

Let us give some examples of semigroups and monoids in the monoidal categories we have introduced. We shall discuss more about (associative) pairings in Chapter 3.

EXAMPLE 1.15. A magma in a posetal monoid $(M, *, 1, \leq)$ is an element s such that $s * s \leq s$. Since $(M, \leq)$ is a posetal category, every diagram commutes and therefore every magma is a semigroup in that category. It is a monoid if, furthermore, $1 \leq s$ and in that case s is idempotent. That is, monoids in $(M, *, 1, \leq)$ are exactly the idempotent elements that are above the unit. ◊

EXAMPLE 1.16. Usual examples of semigroups and monoids are the following. A semigroup (resp. monoid) in SET is a usual semigroup (resp. monoid). A semigroup in VECT_k is a k-algebra. A monoid in the category of endofunctor $[\mathcal{C}, \mathcal{C}]$ is usually called a monad (or a triple). ◊

EXAMPLE 1.17. Let X be an object in a symmetric monoidal closed category. The internal composition $\circ : [X, X] \otimes [X, X] \to [X, X]$, defined as the transpose of

$$X \otimes [X, X] \otimes [X, X] \xrightarrow{\text{ev} \otimes [X,X]} X \otimes [X, X] \xrightarrow{\text{ev}} X \,,$$

leads to a monoid $([X, X], \circ, \ulcorner \mathtt{id} \urcorner)$. The arrow $\ulcorner \mathtt{id} \urcorner$ is the name of the identity, meaning it is the transpose of

$$I \otimes X \cong X \xrightarrow{\mathtt{id}} X \,.$$

We will explore this canonical example of a monoid in the next two chapters. As we shall see, for a complete lattice L, Theorem 2.1 relates a property of $[L, L]$—namely that it is a Frobenius quantale— to the nuclearity of L. ◊

EXAMPLE 1.18. If we set $0 := I$ in a symmetric monoidal closed category, the object $X^* \otimes X$ is a semigroup with the multiplication given by

$$\text{m} : X^* \otimes X \otimes X^* \otimes X \xrightarrow{X^* \otimes \text{ev} \otimes X} X^* \otimes I \otimes X \cong X^* \otimes X.$$

Note that in VECT_k, $(X^* \otimes X, \text{m})$ corresponds to the algebra of matrices while in SLATT it is the opposite of the lattice of inf-preserving endomaps. ◊

We will come back frequently to Examples 1.17 and 1.18 since the `mix` map that defines nuclear objects has $X^* \otimes X$ as domain and $[X, X]$ as codomain. We shall describe nuclear objects in Section 1.3. For now we continue our discussion about semigroups and monoids.

Monoidal functors preserve monoids, more precisely, we have the following property:

PROPOSITION 1.1. *Let (F, μ, η) be a monoidal functor between two monoidal categories $\mathcal{C}$ and $\mathcal{V}$. F induces a functor $F_* : \mathrm{MON}_{\mathcal{C}} \to \mathrm{MON}_{\mathcal{V}}$ such that:*

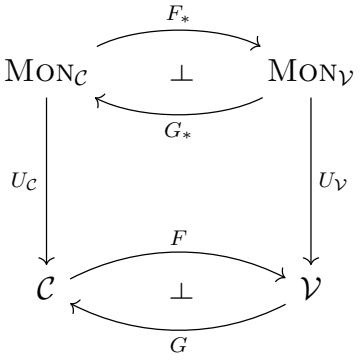
$$\begin{array}{ccc} \mathrm{MON}_{\mathcal{C}} & \xrightarrow{F_*} & \mathrm{MON}_{\mathcal{V}} \\ {\scriptstyle U_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle U_{\mathcal{V}}} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{V}\,, \end{array}$$

where $U_{\mathcal{C}}$, and $U_{\mathcal{V}}$ are the obvious first projection functors.

If $(M, \mathrm{m}, \mathrm{u})$ is a monoid in $\mathcal{C}$, then the tuple $(F(M), F(\mathrm{m}) \circ \mu_{M,M}, F(\mathrm{u}) \circ \eta)$ is a monoid in $\mathcal{V}$. Moreover, $F_*(f) := F(f)$ preserves monoid homomorphisms on the nose. It is direct to check that it is enough to define the functor $F_* : \mathrm{MON}_{\mathcal{C}} \to \mathrm{MON}_{\mathcal{V}}$. This proposition is actually an instance of the change of base induced by the monoidal functor F. Indeed, a monoid in $\mathcal{C}$ is exactly a $\mathcal{C}$-enriched category with one object; the functor F_* is the change of base functor reduced to one-object $\mathcal{C}$-categories. Note, moreover, that we have stated those two propositions for monoids; of course, similar statements hold for semigroups.

REMARK 1.4. The expression $(-)_*$ that sends F to F_* builds a functor whose domain is $\mathrm{MON}_{\mathrm{CAT}}$, the category of monoidal categories; and whose codomain is SET−CAT, the category of categories. Those two categories are actually **2**-categories, and $(-)_*$ is actually a **2**-functor that we can type $(-)_* : \mathrm{MON}_{\mathrm{CAT}} \to \mathrm{SET}{-}\mathrm{CAT}$. This notably implies that it preserves adjunctions—see [**Cru08**, Section 4.4]. Again, as monoid objects in $\mathcal{C}$ are one-object $\mathcal{C}$-categories, every monoidal adjunction $F \dashv G$ between $\mathcal{C}$ and $\mathcal{V}$ induces an adjunction $F_* \dashv G_*$ between $\mathrm{MON}_{\mathcal{C}}$ and $\mathrm{MON}_{\mathcal{V}}$. That is, the following diagram

commutes:

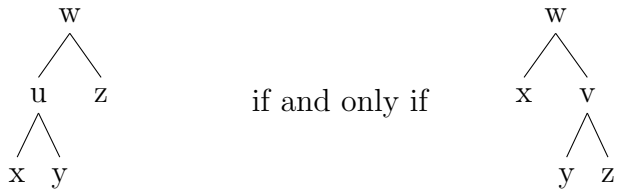


Proposition 1.1 leads to more examples of monoids and semigroups in monoidal categories.

EXAMPLE 1.19. A magma (S, M) in REL is given by a set S and a ternary relation $M \subset S \times S \times S$. We write $xyMz$ for $(x, y, z) \in M$. This multiplication is associative if for all $x, y, z, w \in S$,

$$\exists u \in M, xyMu \text{ and } uzMw \quad \text{if and only if} \quad \exists v \in S, xvMw \text{ and } yzMv\,. \tag{9}$$

Using the binary tree notation where a node z has a left child x and a right child y if and only if $xyMz$, we obtain the following equivalence:



A unit $\eta : I \to S$ to the multiplication M is a subset of $\eta \subset S$ such that for every $e \in \eta$ and $x \in S$, we have

$$exMz \iff xeMz \iff x = z\,.$$

Let us give a some concrete examples of a semigroup in REL. The category SET and the category of sets and partial functions embeds monoidally into REL. Therefore, by Proposition 1.1, any semigroup $(S, *)$ in SET is a semigroup in REL with the same object and multiplication. A partial semigroup is also a semigroup in REL. The partial order of an ordered semigroup is also an associative multiplication in REL.

Moreover, for any semigroup $(S, *)$ in Set, there is a semigroup object (S, M) in Rel, with the multiplication given for each $x, y, z \in S$ by:

$$xyMz \iff \exists a \in S \text{ s.t. } z = xay\,. \tag{10}$$

Let us check that this multiplication is indeed associative. Assume that $xyMu$ and $uzMw$. That is, there exists $a, b \in S$, such that

$$u = xay \qquad \text{and} \qquad w = ubz\,.$$

This implies that $w = xaybz$. Then, set $v := ybz$—which implies that $yzMv$—and we have that $w = xav$—which implies that $xvMw$. The other implication is given with the same argument.

Those examples of semigroups in Rel are quite useful as one obtain a semigroup in SLatt by applying the powerset functor defined in Example 1.8. ◊

Example 1.20. Semigroup in SLatt are exactly quantales. We already mentioned this structure in Example 1.10 and it will be the main object of Chapter 2. A quantale is *unital* if it is a monoid in SLatt. Recall from Example 1.5, that given a complete lattice Q, a join-preserving map $Q \otimes Q \to Q$, corresponds exactly to a bilinear map $Q \times Q \to Q$. We have already mentioned in Example 1.8 that the powerset functor $\mathcal{P}(-) : \text{Rel} \to \text{SLatt}$ is monoidal. Therefore, by Proposition 1.1 all the examples of semigroups or monoid we have given in Example 1.19 lead to quantales. In Chapter 2 we will study some specific quantales and we shall see many concrete examples during this exploration. ◊

Definition 1.5. Let (S, m) be a semigroup in $\mathcal{C}$ and X a $\mathcal{C}$-object. A $\mathcal{C}$-arrow $\alpha^\ell : S \otimes X \to X$ is a *left action* if the diagram

$$\begin{array}{ccc} S \otimes S \otimes X & \xrightarrow{S \otimes \alpha^\ell} & S \otimes X \\ {\scriptstyle \mathrm{m} \otimes X}\downarrow & & \downarrow{\scriptstyle \alpha^\ell} \\ S \otimes X & \xrightarrow[\alpha^\ell]{} & X \end{array}$$

commutes. The tuple (X, α^ℓ) is called a *left S-module*. For (X, α^ℓ) and (Y, α^ℓ) two left S-modules, a map $f : X \to Y$ is *linear* if the diagram

$$\begin{array}{ccc} S \otimes X & \xrightarrow{S \otimes f} & S \otimes Y \\ {\scriptstyle \alpha^\ell}\downarrow & & \downarrow{\scriptstyle \alpha^\ell} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes. We denote by Mod_S the category of (left) modules and linear maps between them.

One defines a *right action* α^ρ and a *right module* in a similar way. If (S, m) is a commutative semigroup, then a left action induces a right action. In that case we do not need to distinguish them as they are equal.

EXAMPLE 1.21. The multiplication of a semigroup (S, m) trivially acts over itself with $\alpha^\ell_A = \alpha^\rho_A = \mathrm{m}$.

EXAMPLE 1.22. A quantale $(Q, *)$ acts over its dual Q^{op} on the left and on the right. Indeed, the two implications $-\!\!\diagup\!\!- : Q \otimes Q^{\mathrm{op}} \to Q^{\mathrm{op}}$ and $-\!\!\diagdown\!\!- : Q^{\mathrm{op}} \otimes Q \to Q^{\mathrm{op}}$ are respectively left and right actions. Indeed, the implications of a quantale validate the two equations

$$z \diagup (y * x) = (z \diagup y) \diagup x \qquad \text{and} \qquad (x * y) \diagdown z = x \diagdown (y \diagdown z)\,.$$

These two equations can be understood as the internalization of the currying.

EXAMPLE 1.23. We will study the category $\textsc{Mod}_Q$ of modules over a commutative unital quantale $(Q, *, 1)$. Since we assume Q commutative, we do not distinguish between right and left actions. Note that any lattice L is a module over the boolean algebra $(\mathbb{2}, \wedge, \top)$. The category SLATT is actually equivalent to $\textsc{Mod}_{\mathbb{2}}$. In a sense, the category $\textsc{Mod}_Q$ generalizes the SLATT and many properties of SLATT hold in $\textsc{Mod}_Q$. Notably, $\textsc{Mod}_Q$ is a symmetric monoidal closed category with unit $(Q, *)$. The construction of the tensor is similar to SLATT and it enjoys the same universal property—see Example 1.5. We refer the reader to [**JT84**, **KP08**] and [**EGGHK18**, Chapter 3] for elementary properties of $\textsc{Mod}_Q$. For a module (M, α), we usually refer to the bilinear map induced by α with the infix notation $q \cdot x$. Morphisms $f : M \to M'$ are sup-preserving map such that $q \cdot f(a) = f(q \cdot a)$ for all $q \in Q$ and $a \in M$. We shall not always write the action of Q when mentionning a Q-module. We denote by $[M, N]_{\mathrm{Lin}}$ the internal hom of $\textsc{Mod}_Q$ which is given by the Q-module of linear maps between M and N. For any Q-module (M, α), there is a map $\langle -, - \rangle : M \otimes M^{\mathrm{op}} \to Q$ given by the following chain of natural isomorphisms:

$$\dfrac{\dfrac{Q \otimes M \xrightarrow{\alpha} M}{Q \xrightarrow{\alpha^\sharp} [M, M]_{\mathrm{Lin}}}}{M \otimes M^{\mathrm{op}} := [M, M]_{\mathrm{Lin}}{}^{\mathrm{op}} \xrightarrow{\rho(\alpha^\sharp)} Q^{\mathrm{op}}}\,.$$

Explicitly, $\langle -, - \rangle$ is given by the linear extension of

$$\langle x, y \rangle = \bigvee \{ q \mid q \cdot x \leq y \}\,.$$

In the case where $Q := \mathbb{2}$, we have

$$\langle x, y \rangle = \begin{cases} \top \text{ if } x \leq y \\ \bot \text{ otherwise,} \end{cases}$$

which corresponds to the order of the lattice. Therefore $\langle -, - \rangle$ generalizes the order of lattices and indeed it makes any Q-module a Q^{op}-enriched category.

We even have an isomorphism of category $\textsc{Mod}_Q \cong \textsc{Sup}_{Q^{\mathrm{op}}} \cong \textsc{Inf}_{Q^{\mathrm{op}}}$ where $\textsc{Sup}_{Q^{\mathrm{op}}}$—respectively $\textsc{Inf}_{Q^{\mathrm{op}}}$—is the category of co-complete—respectively complete—Q^{op}-enriched categories and co-continuous—respectively continuous—maps. This was shown in the wider context of quantaloid in [**Stu06**, Corollary 4.10]—and explained in detail for the case of one-object quantaloid, *i.e.*, quantales, in [**EGGHK18**, Section 3.3].

1.3. Nuclear objects in symmetric monoidal closed categories

Nuclear objects have been introduced by Rowe in [**Row88**]. The idea was to generalize the case of finite-dimensional vector spaces. This notion is important for us as it characterizes completely distributive lattices in categorical terms—see Example 1.27 in the sequel of this section. Rowe gave several characterizations of this notion; and we will use the following one:

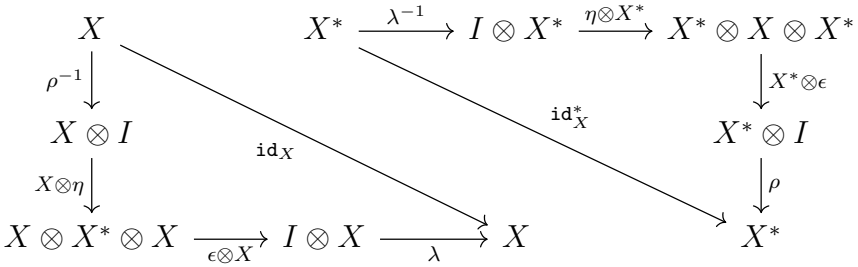
Definition 1.6. Let us settle $0 := I$, we define $\mathtt{mix}_X : X^* \otimes X \to [X, X]$ as the only map making the following diagram commute

$$\begin{array}{ccc} X \otimes X^* \otimes X & \xrightarrow{X \otimes \mathtt{mix}} & X \otimes [X, X] \\ {\scriptstyle \mathrm{ev} \otimes X}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ I \otimes X & \xrightarrow[\lambda]{} & X\,. \end{array}$$

An object X is *nuclear* iff $\mathtt{mix}_X$ is invertible. A category whose objects are all nuclear is called a *compact closed category*.

Remark 1.5. It is possible to give a more general definition of the $\mathtt{mix}$ map in a symmetric monoidal closed category. If there exists an arrow $0 \to I$, then one defines $\mathtt{mix}_{X,Y} : Y^* \otimes X \to [X, Y]$ just like we did for $\mathtt{mix}_X$. This kind of map was studied in by Cockett and Seely in [**CS97b**], in the more general framework of linearly distributive categories. We will not need such a general setting for our purpose, and when studying nuclearity we will always assume that the object 0 is the unit of the tensor.

Remark 1.6. A nuclear object X necessarily has an adjoint X^*, so that there always exists a unit $\eta : I \to X^* \otimes X$ and a counit $\epsilon : X \otimes X^* \to I$ such that the two diagrams below commute.



The unit of the adjunction is given by the name of the identity—see Remark 1.2—composed with $\mathtt{mix}_X$, while the counit is the evaluation map. Conversely, if X has an adjoint, then the map $\mathtt{mix}_X$ is necessarily invertible, and X is nuclear. This was already noticed by Kelly in [**Kel72**, p.103]. By "compact closed category", Kelly was refering to a category for which every object has an adjoint, that is, a nuclear category. Since we are working with symmetric monoidal categories, we do not have to distinguish between right and left adjoints. We will use these two characterizations of nuclear objects in the sequel of this thesis.

EXAMPLE 1.24. Every vector space of finite dimension over a field k is nuclear. The idea behind the work of Rowe in [**Row88**] was to generalize this situation in order to find a notion of *finiteness* in monoidal categories. The subcategory $\textsc{finVect}_k$ of finite-dimensional vector spaces is the main example of a compact closed category. ◊

EXAMPLE 1.25. In a unital quantale, nuclear objects are exactly the invertible elements. ◊

EXAMPLE 1.26. The category REL is a compact closed category. ◊

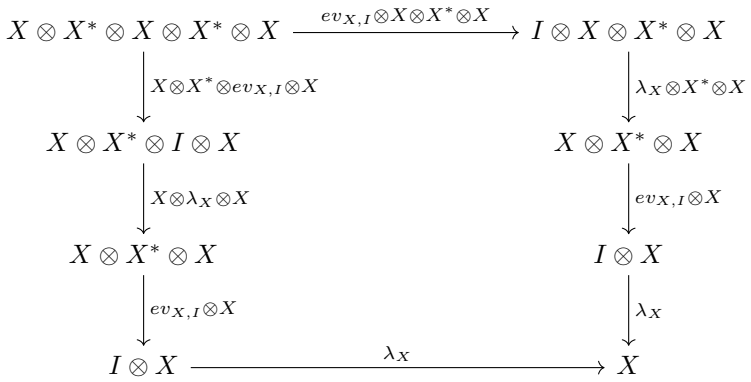
EXAMPLE 1.27. In SLATT, nuclear objects are exactly the completely distributive lattices. This result has been shown in [**HR89**] and is an important motivation for our research. We will give the definition of completely distributive lattices in Definition 2.4. ◊

FACT 1.2. *The map* $\mathtt{mix}$ *is a semigroup homomorphism between* $(X^* \otimes X, \mathrm{m})$ *and* $([X,X], \circ)$.

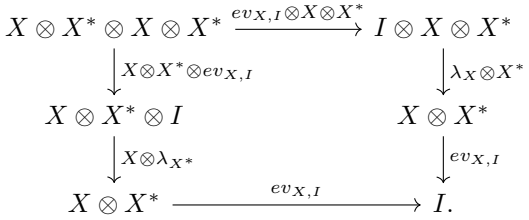
PROOF. We must check the commutativity of

$$\begin{array}{ccc} X^* \otimes X \otimes X^* \otimes X & \xrightarrow{\mathtt{mix}\otimes\mathtt{mix}} & [X,X] \otimes [X,X] \\ \downarrow{\scriptstyle \mu_{X^*\otimes X}} & & \downarrow{\scriptstyle \circ} \\ X^* \otimes X & \xrightarrow{\qquad\mathtt{mix}\qquad} & [X,X]. \end{array}$$

Transposing and developing this diagram, we obtain



which reduces to the commutativity of



□

1.4. ∗-autonomous categories

We now introduce ∗-autonomous categories. First recall that in a symmetric monoidal closed category, $\mathtt{j}_X : X \to X^{**}$ is the only map making Diagram (8) comutes.

Definition 1.7. Let 0 be an object in a symmetric monoidal closed category $\mathcal{C}$. We say that an object X of $\mathcal{C}$ is *reflexive with respect* to 0—or just *reflexive*—if the map $\mathtt{j}_X$ is an isomorphism. An object 0 is a *dualizing object* if every object of $\mathcal{C}$ is reflexive—*i.e.*, j is a natural isomorphism.

There are many equivalent definitions of ∗-autonomous categories—see [**Bar79**]. We will focus on the following ones:

Definition 1.8. A symmetric monoidal category $\mathcal{C}$ is a *∗-autonomous category* if one the following equivalent conditions hold:

- The category $\mathcal{C}$ is closed, and the natural transformation j is a natural isomorphism.

- There exists a functor $(-)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ which is an equivalence of category, such that:
$$\hom(X \otimes Y, Z^*) \cong \hom(Y \otimes Z, X^*)\,. \tag{11}$$

We define the bifunctor $- \oplus -$ by
$$X \oplus Y := [X^*, Y]\,, \tag{12}$$
and this functor is called the *cotensor* of $\mathcal{C}$.

Remark 1.7. The natural isomorphism (11) actually corresponds to what we will name for Frobenius quantales the *shift relation*—see Eq. (Shift relation). It is not difficult to check, via Yoneda, that this natural isomorphism exists if and only if there exists a natural isomorphism $[X, Y^*] \cong [Y, X^*]$. In the posetal case, this isomorphism is an identity—it already appeared in the introduction, more precisely in Eq. (Serre equality), to define unitless Frobenius quantales. We will define Frobenius structures in Chapter 3 using a commutative square that will generalize the latter identity.

Note also that we have the following natural isomorphisms:
$$X \oplus Y = [X^*, Y] \cong (Y^* \otimes X^*)^*\,.$$
This implies that the cotensor $- \oplus -$ makes $\mathcal{C}^{\mathrm{op}}$ a symmetric closed monoidal category. Its internal hom is given by the natural isomorphism of $\mathcal{C}$-morphisms:
$$\frac{Y \longrightarrow X \oplus Z}{X^* \otimes Y \longrightarrow Z}$$
$*$-autonomous categories are closely related to compact closed categories. Indeed, we have that:

Lemma 1.2. *Every nuclear object is reflexive. Every compact closed category is a $*$-autonomous category.*

One can think of a compact closed category as a degenerate $*$-autonomous category. Indeed, in a compact closed category, the tensor and the cotensor are isomorphic.

There are many examples of $*$-autonomous categories in the literature. Notably, for a symmetric monoidal closed category $\mathcal{V}$ and 0 a $\mathcal{V}$-object, the Chu category $\mathrm{Chu}_{\mathcal{V},0}$ is a $*$-autonomous category—see [**Bar79**]. We emphasize the following examples of $*$-autonomous categories.

Example 1.28. A $*$-autonomous poset $(P, \leq, *)$ is a commutative residuated monoid with an element 0 such that
$$(x \backslash 0) / 0 = x\,.$$

When $(P, \leq)$ is a complete lattice, this structure is called a commutative *Frobenius quantale* or a commutative *Girard quantale*. We will be interested in the case where $(Q, *)$ is a noncommutative quantale, and even non-unital. This is the whole subject of Chapter 2—and of most of this thesis. ◊

Example 1.29. By Lemma 1.2, the categories Rel and finVect_k are ∗-autonomous categories ◊

Example 1.30. The category SLatt is a ∗-autonomous category. More generally, given a commutative unital quantale $(Q, *, 1)$, the category Mod_Q is a ∗-autonomous category and Q^{op} is a dualizing object of this category—this is proven in [**JT84**, Section 2.2, Proposition 1,] and [**EGGHK18**, Section 3.3.1]. We believe it is instructive to sketch of proof of that fact. For a Q-module (M, α), the tuple (M^{op}, β) is a Q-module where $\beta(q, x) := \rho(\alpha_q)(x)$ with α_q the join-preserving map defined by $\alpha_q(x) = \alpha(q, x)$. Using the infix notation $q \blacktriangleright x := \beta(q, x)$, we have

$$q \cdot x \leq z \iff x \leq q \blacktriangleright z \,.$$

Note that the explicit definition of $\blacktriangleright$ is given by:

$$q \blacktriangleright y = \bigvee \{x \in M \mid q \cdot x \leq y\} \,.$$

A map is linear—*i.e.*, it is action-invariant—if and only if its right adjoint is linear. Indeed, we have the following chain of equivalences—where we use an horizontal line just like in Eq. (7):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \leq q \blacktriangleright \rho(f)(y)}{q \cdot x \leq \rho(f)(y)}}{f(q \cdot x) \leq y}}{q \cdot f(x) \leq y}}{f(x) \leq q \blacktriangleright y}}{x \leq \rho(f)(q \blacktriangleright y) \,.}$$

That is, $q \blacktriangleright \rho(f)(y) = \rho(f)(q \blacktriangleright y)$. This implies that the functor $(-)^{\text{op}} : \text{SLatt} \to \text{SLatt}^{\text{op}}$ lifts to a functor $(-)^{\text{op}} : \text{Mod}_Q \to \text{Mod}_Q^{\text{op}}$. Since $M^{\text{opop}} = M$, and $[M, N^{\text{op}}]_{\text{Lin}} \cong [N, M^{\text{op}}]_{\text{Lin}}$ is given by the right adjoint operation, by Remark 1.7, Mod_Q is a ∗-autonomous category. Moreover, we have the following lemma:

Lemma 1.3. *There is a natural isomorphism*

$$\alpha : (M^{\text{op}}, \blacktriangleright) \cong ([M, Q^{\text{op}}]_{\text{Lin}}, \cdot) \,.$$

PROOF. Define $\alpha : M^{\mathrm{op}} \to [M, Q^{\mathrm{op}}]_{\mathrm{Lin}}$ and $\beta : [M, Q^{\mathrm{op}}]_{\mathrm{Lin}} \to M^{\mathrm{op}}$ by:

$$\alpha(y)(x) = \langle x, y\rangle \qquad\qquad \beta(f) = \rho(f)(1)\,.$$

That $(\alpha \circ \beta(f))(x) = \langle x, \rho(f)(1)\rangle = f(x)$ is given by the chain of equivalences

$$\dfrac{\dfrac{\dfrac{\dfrac{q \leq \langle x, \rho(f)(1)\rangle}{q \cdot x \leq \rho(f)(1)}}{1 \leq f(q \cdot x)}}{1 \leq q \diagdown f(x)}}{q \leq f(x)\,.}$$

That is, $\beta \circ \alpha = 1_{[M,Q]_{\mathrm{Lin}}}$. Moreover, as $- \blacktriangleright x$ is the adjoint of $\langle x, -\rangle$, we have

$$\beta \circ \alpha(x) = \rho(\langle x, -\rangle)(1) = 1 \blacktriangleright x = x\,.$$

Therefore, $\alpha \circ \beta = 1_{M^{\mathrm{op}}}$ and $\beta = \alpha^{-1}$.

The map α is linear as it is the transpose of $\langle -, -\rangle : M \otimes_Q M^{\mathrm{op}} \to Q^{\mathrm{op}}$. The inverse of a linear map is also linear. □

This lemma directly implies that Q^{op} is a dualizing object of MOD_Q. The standard isomorphism $2 \cong 2^{\mathrm{op}}$ makes 2 a dualizing object in SLATT. However, since in general the base object is not isomorphic to its dual, we shall keep 2^{op} as the dualizing object in SLATT. We moreover characterize unital commutative quantales Q which are dualizing objects of mod Q that respect the canonical duality, *i.e.*, such that $(-)^{\mathrm{op}} \cong [-, Q]_{\mathrm{Lin}}$. We have not found the following proposition in the litterature, and this is probably folklore.

PROPOSITION 1.2. *Let $(Q, *, 1)$ be a unital commutative quantale. We have a natural isomorphism $[-, Q]_{\mathrm{Lin}} \cong (-)^{\mathrm{op}}$ if and only if Q carries a structure of Girard quantale.*

PROOF. By Lemma 1.3, we have $[-, Q]_{\mathrm{Lin}} \cong (-)^{\mathrm{op}}$ if and only if there exists an isomorphim $(-)^{\perp} : Q \to Q^{\mathrm{op}}$ in MOD_Q, that is $(-)^{\perp}$ is invertible, sup-reversing and linear. Its linearity means that for every $x, y \in Q$, we have

$$(x * y)^{\perp} = x \diagdown y^{\perp}\,. \tag{13}$$

By the commutativity of Q, we also have

$$(x * y)^{\perp} = (y * x)^{\perp} = y \diagdown x^{\perp}\,.$$

That is $x \diagdown y^{\perp} = y \diagdown x^{\perp}$ for each $x, y \in Q$, that is $(Q, *, 1, (-)^{\perp})$ is a unital commutative Girard quantale—by Remark 1.7. That the equation 13 holds

in such quantales is given by the two following chains of equivalences:

$$\dfrac{\dfrac{(xy)^\perp \leq x \diagdown y^\perp}{x * (xy)^\perp \leq y^\perp}}{y * x = x * y \leq xy\,,} \qquad \dfrac{\dfrac{x \diagup y^\perp \leq (xy)^\perp}{xy \leq (x \diagup y^\perp)^\perp}}{(x \diagup y^\perp) * x \leq y^\perp\,.}$$

□

◊

Before closing this chapter, we shall mention that ∗-autonomous categories are special cases of linearly distributive categories. Linearly distributive categories were introduced by Cockett and Seely in [**CS97a**]. The main motivation for introducing this structure was the categorical interpretation of the sequent calculus of classical linear logic. Such a sequent is of the form $A_0, \ldots, A_n \vdash B_0, \ldots, B_m$ where commas on the left part of the sequent are interpreted with $\otimes$ and commas on the right by $\oplus$. As we have mentioned, in a ∗-autonomous category, $\otimes$ and $\oplus$ define each other via $(-)^*$. Linearly distributive categories take those two tensors as primitive datas with a linear distribution law between them. More precisely,

Definition 1.9. A *linearly distributive category* is a category $\mathcal{C}$ equipped with two monoidal structures $(\otimes, I)$ and $(\oplus, 0)$, and two natural maps called the *left* and *right distributors*; or the *left* and *right linear strenghts*

$$\kappa^\lambda : X \otimes (Y \oplus Z) \longrightarrow (X \otimes Y) \oplus Z\,,$$
$$\kappa^\rho : (X \oplus Y) \otimes Z \longrightarrow X \oplus (Y \otimes Z)\,.$$

They must respect several diagrams relating the two associators and the four unitors with the two distributors—see [**CS97a**] for the coherence diagrams.

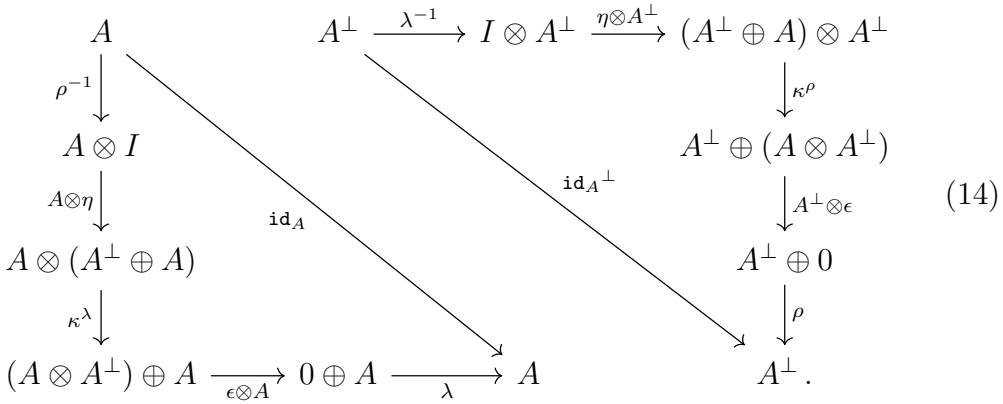
Note that a monoidal category is actually a linearly distributive category in which the two tensors are the same. General linearly distributive categories are of course much richer. They can notably be seen as categories whose morphisms have several inputs and several outputs—*i.e.*, as polycategories. Indeed, there is an equivalence of **2**-categories between the category of linearly distributive categories and the category of two-tensor-polycategories—see [**CS97a**, Theorem 2.1]. In that setting, an input $A_0, \ldots, A_n$ of a morphism in a polycategory corresponds to an object $A_0 \otimes \ldots \otimes A_n$; while its output $B_0, \ldots, B_m$ corresponds to an object $B_0 \oplus \ldots \oplus B_m$. This makes them close to the syntax of the sequent calculus of interest. Actually, morphisms in linearly distributive categories can be expressed with sorts of proof nets called *circuits*—see [**BCST96**]. The inputs on top of a circuit are implicitely connected by a $\otimes$, while the outputs are connected by a $\oplus$. Of course, not every circuit comes from a morphism in a linearly distributive category; for that,

circuits must respect a correctness criterion—just like proof structures and actual proofs in Linear Logic. We will use the circuit calculus introduced by Cockett *et al.* in Chapter 3 mainly as an heuristic to find proofs, and we will not check that the circuits we are producing validate a correctness criterion.

Removing the negation $(-)^*$ is also a way of better understanding its role. In [**CS97a**, Theorem 4.5], it is notably stated that $*$-autonomous categories are exactly *symmetric linearly distributive with negation*. Those are linearly distributive categories in which the two monoidal structures are symmetric and equipped with an object function $A \mapsto A^\perp$ and two arrows

$$\epsilon : A \otimes A^\perp \longrightarrow 0$$
$$\eta : I \longrightarrow A^\perp \oplus A$$

for each object A such that the two following diagrams commute:



(14)

Note the ressemblance with the adjoints we mentioned in Remark 1.5; indeed, $(A, A^\perp, \epsilon, \eta)$ is actually a *linear adjoint*. We will encounter them again in Chapter 3, where we will consider their definition by means of circuits.

CHAPTER 2

Unitless Frobenius quantales

In this chapter, we define unitless Frobenius quantales, and we develop the basic theory of this structure. Unitless Frobenius quantales does not take a dualizing element as the basic of the structure, but rather two negations ${}^{\perp}(-)$ and $(-)^{\perp}$ working nicely together. As the name suggests, units are not necessarily present in a unitless Frobenius quantale. Although this structure is more general than unital Frobenius quantales, many important properties can be recovered, with some variations. Notably, we show how to construct a unitless Frobenius quantale from two "intuitionistic negations"—via a double negation nucleus—and we provide a representation theorem from phase quantales. We give several examples of non-unital Frobenius quantales, arising notably from pregroups and from C*-algebras. Any complete lattice induces a unitless Frobenius quantale: the quantale of tight endomaps over a complete lattice. In the continuity of Theorem 0.1, the quantale of tight endomaps yields another equivalent condition for a lattice L to be a completely distributive lattice.

Unitless Frobenius quantales have widely inspirited us to define Frobenius structures as specific semigroups in symmetric monoidal categories. This chapter provides some intuitions, before going into the abstraction of Chapter 3. We will mainly follow the journal paper [**dLS22**].

2.1. Elementary quantale theory

In this section, we recall some important facts and notions about quantales and lattice theory. Some of them have already been mentioned in Examples 1.1, 1.5, 1.10 and 1.20 and we recommend [**DP90**, **Ros90b**, **KP08**, **GJKO07**, **EGGHK18**] for a further introduction to these notions. We also recall some facts regarding Frobenius and Girard quantales.

Notation 2.1. In this chapter, we will focus on SLatt, the category of complete lattices and sup-preserving maps between them. The letter L will denote a lattice and Q a quantale. We shall work with sup-preserving maps and we will "play" with the op operation to obtain sup-preserving maps from inf-preserving or sup-reversing maps.

DEFINITION 2.1. A *quantale* is a pair $(Q, *)$, with Q a complete lattice and $*$ a semigroup operation on Q, which preserves sups in both variables: we have for any set I

$$(\bigvee_{i\in I} x_i) * y = \bigvee_{i\in I} x_i * y\,, \qquad\qquad x * (\bigvee_{i\in I} y_i) = \bigvee_{i\in I} x * y_i\,.$$

If the quantale has a unit 1, then $(Q, *)$ is a *unital* quantale. A *quantale homomorphism* f between two quantales $(Q_0, *_0)$ and $(Q_1, *_1)$ is a sup-preserving map that preserves the multiplication:

$$f(x *_0 y) = f(x) *_1 f(y)\,.$$

We denote by QUANT the category of quantales and quantales homomorphism.

We have already observed in Example 1.20 that quantales are exactly the semigroups in SLATT. More precisely, we have a category isomorphism $\mathrm{QUANT} \cong \mathrm{SEM}_{\mathrm{SLATT}}$. Moreover, a unital quantale is a biclosed monoidal category—as mentioned in Example 1.10: we have two implications, $-\backslash-$ and $-/-$, given by the equivalences:

$$\frac{\dfrac{x * y \leq z}{y \leq x\backslash z}}{x \leq z/y\,.} \tag{15}$$

They also have the following explicit expressions:

$$x\backslash z = \bigvee\{\, y \mid x * y \leq z \,\} \qquad \text{and} \qquad z/y = \bigvee\{\, x \mid x * y \leq z \,\}\,.$$

EXAMPLE 2.1. A complete Heyting algebra, or *frame*, is a commutative unital quantale where the multiplication is given by the meet. In this sense, quantales generalize frames. Indeed, the category FRAME of frames and maps that preserve finite meets and arbitrary joins is a full subcategory of QUANT. The opposite category of FRAME is the category of *locales*, which is closely related to topological spaces. The set of open subsets of a topological space is a frame. The inverse image of a continuous map is a frame homomorphism, and this map goes in the opposite direction. The closed ideals of a commutative C*-algebra is also a frame and the C*-algebra is completely determined by this frame.

The name "quantale" derives from locale and quantum mechanics—as it is usually not commutative, and was introduced by Mulvey in [**Mul86**]. He noticed that right closed ideals of (non-necessarily commutative) C*-algebras are quantales, and that several important properties of a C*-algebra can be recovered from this quantale. For instance, right closed ideals of a C*-algebra

is a locale if and only if the C*-algebra is commutative. The relation between quantales and C*-algebras has motivated many works, see for instance [**PR00**]. ◊

The covariant powerset functor $\mathcal{P}(-) : \textsc{Set} \to \textsc{SLatt}$ is the left adjoint of the forgetful functor, giving rise to the free complete lattice over a set. This adjunction lifts to the category of semigroups as explained in Remark 1.4. This gives the following class of examples:

Example 2.2. Let $(S, \cdot)$ be a semigroup in Set. The tuple $(\mathcal{P}(S), \bullet)$ is a quantale in which the multiplication is defined by:

$$X \bullet Y := \{x \cdot y \mid x \in X, y \in Y\}\,.$$

This is the free quantale over the semigroup $(S, \cdot)$. The two implications are given by:

$$X \backslash Y = \{s \in S \mid x \cdot s \in Y, \text{for all } x \in X\}$$

and

$$Y / X = \{s \in S \mid s \cdot x \in Y, \text{for all } x \in X\}\,.$$

This structure is rather common. For instance, when considering the free monoid over an alphabet Σ, this is exactly the usual concatenation of languages. ◊

Example 2.3. We noticed in Example 1.5 that the collection $[L, L]$ of sup-preserving endomaps of a complete lattice L is a complete lattice where the sups are computed pointwise, *i.e.*,

$$(\bigvee_{i\in I} f_i)(x) := \bigvee_{i\in I}(f_i(x))\,.$$

This complete lattice is endowed with a multiplication given by the composition of maps—just as in Example 1.6. The composition is associative and preserves the joins in both variables, as we have:

$$f \circ (\bigvee_{i\in I} g_i)(x) = f(\bigvee_{i\in I} g_i(x)) = \bigvee_{i\in I} f(g_i(x)) = \bigvee_{i\in I}(f \circ g_i)(x)\,,$$

and

$$(\bigvee_{i\in I} f_i)(g(x)) = \bigvee_{i\in I} f_i(g(x)) = \bigvee_{i\in I}(f_i \circ g)(x)\,.$$

That is, $([L, L], \circ)$ is a quantale. Such quantales appear quite often. For instance, the quantale of relations over a set X is isomorphic to the quantale $[\mathcal{P}(X), \mathcal{P}(X)]$, see [**KP08**, Example 1.2.4]. In the case where $L := [0, 1]$, the element of $[L, L]$ may be considered as paths, see [**San19**]. This example will

appear regularly in this thesis as Theorem 2.1 relates quantales of the form $[L, L]$ with completely distributive lattices. ◊

As expected, quantales are residuated lattices. The only difference between such structures—quantales and residuated lattices—is the requirement that sups of arbitrary sets exist. This constraint may seem too strong, as many important lattices are not complete. However, the category SLatt is quite convenient, in particular it is a $*$-autonomous category. Since, as we shall see, unital Frobenius quantales are posetal non symmetric $*$-autonomous categories, they are $*$-autonomous categories inside a $*$-autonomous category. This interesting categorical setting is not observed with residuated lattices. Moreover, Theorem 2.1, which was the main motivation to develop this thesis, is a statement which concerns SLatt. This theorem is closely related to the semantics of linear logic. For those reasons we shall study complete residuated lattices—*i.e.*, quantales. Since the end of the eighties, quantales were studied in the perspective of linear logic. We develop this perspective in the following.

As early as in the first chapter of [**Gir87**], Girard introduces the syntax of Linear Logic together with its semantics: the phase semantics. Models are constructed as follows: consider a commutative monoid $(M, \cdot)$ and a subset $\mathbf{0} \subset M$. Then, writing $(-)^{\perp} := - \backslash \mathbf{0}$ with the implication given in Example 2.2, define

$$\mathcal{P}(M)_{\mathbf{0}} := \{X \in \mathcal{P}(M) \mid X^{\perp\perp} = X\},$$

and

$$X \bullet_{\mathbf{0}} Y := (X \bullet Y)^{\perp\perp}$$

This construction always leads to a quantale $(\mathcal{P}(M)_{\mathbf{0}}, \bullet_{\mathbf{0}})$. Such quantales $(\mathcal{P}(M)_{\mathbf{0}}, \bullet_{\mathbf{0}})$ are called *phase quantales*. Yetter has introduced in [**Yet90**] the notion of Girard quantale, generalizing phase quantales; let us introduce this notion now.

Definition 2.2. Let $(Q, *)$ be a quantale and let $0 \in Q$. We use the notations

$${}^{\perp}(-) := - \backslash 0\,, \qquad\qquad (-)^{\perp} := 0 / -\ .$$

The element 0 is *dualizing* if, for every x in Q, we have

$${}^{\perp}(x^{\perp}) = ({}^{\perp}x)^{\perp} = x\,.$$

The element 0 is *cyclic* if, for every x in Q, we have

$${}^{\perp}x = x^{\perp}\,.$$

A *unital Frobenius quantale* is a tuple $(Q, *, 0)$ where $(Q, *)$ is a quantale and $0 \in Q$ is dualizing. A *unital Girard quantale* is a Frobenius quantale $(Q, *, 0)$ whose dualizing element 0 is cyclic.

We explicitely added the adjective "unital" to the usual name of Frobenius and Girard quantale. The existence of the unit is usually given as a property—for instance in [**EGGHK18**, proposition 2.6.2]. However, in the rest of this chapter we will explore another definition of Frobenius and Girard quantales that might not have units. The name "Frobenius quantales" was given by Egger in [**Egg10**]. He actually studied unital Frobenius quantales in the categorical setting of linearly distributive categories, and noticed their similitude with Frobenius algebras. In Chapter 3 we study another description of this structure in monoidal symmetric categories. Several properties given in the sequel of this chapter will be generalized for Frobenius structures in any $*$-autonomous categories.

NOTATION 2.2. The notations we are using for the operations in a Frobenius quantales mainly come from the usual residuated lattice theory and may confuse the reader used to linear logic. To help him, we provide a table relating the usual non commutative linear logic connectives to their interpretation in a Frobenius quantale and in a $*$-autonomous category. Note that, since we have not introduced non symmetric $*$-autonomous categories, we do not distinguish between right and left internal homs in a $*$-autonomous category.

Linear logic	Meaning	Quantales	Categories
$\otimes$	multipicative conjunction	$*$	$\otimes$
$\parr$	multipicative disjunction	$\oplus$	$\oplus$
$\multimap$	left to right linear implication	$\backslash$	$[-,-]$
$\multimap$	right to left linear implication	$/$	$[-,-]$
1	unit of the multipicative conjunction	u	I
$\bot$	unit of the multipicative disjonction	0	0
$(-)^\bot$	Right negation	$(-)^\bot$	$(-)^*$
${}^\bot(-)$	Left negation	${}^\bot(-)$	$(-)^*$
&	additive conjunction	$\wedge$	$\times$
$\oplus$	additive disjunction	$\vee$	$\sqcup$
$\top$	unit of the additive conjunction	$\top$	1
0	unit of the additive disjunction	$\bot$	0

If $(Q, *)$ is a quantale and $0 \in Q$ is cyclic, then the double negation map ${}^\bot((-)^\bot) = ({}^\bot(-))^\bot$ is a nucleus—[**Ros90a**]. This notion is defined as follows.

DEFINITION 2.3. Let L be a complete lattice. A monotone function $\mathrm{j} : L \to L$ is a *closure operator*, if it is increasing and idempotent. That is, for

each $x \in L$, we have:

$$x \leq \mathrm{j}(x)\,, \tag{16}$$
$$\mathrm{j}(\mathrm{j}(x)) = \mathrm{j}(x)\,. \tag{17}$$

An element $x \in L$ is j-*closed* if it is a fixed point of j—*i.e.*, $x = \mathrm{j}(x)$. The set of j-closed elements is denoted L_{j}.

A closure operator j over a quantale $(Q, *)$ is a *nucleus* if we have:

$$\mathrm{j}(x) * \mathrm{j}(y) \leq \mathrm{j}(x * y)\,. \tag{18}$$

It is easily seen that a closure operator j is a nucleus if and only if the two conditions below hold:

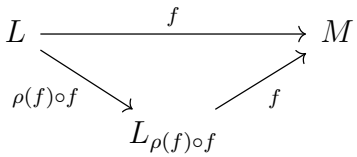
$$x * \mathrm{j}(y) \leq \mathrm{j}(x * y)\,, \qquad \mathrm{j}(x) * y \leq \mathrm{j}(x * y), \qquad \text{for all } x, y \in Q\,.$$

In categorical term, a closure operator $\mathrm{j} : L \to L$ is a monad, and a j-closed element is a j-algebra. Indeed Eq. (16) induces its unit, and Eq. (17) its multiplication. This was the first example of a monad in Section VI.1 of [**Mac78**]. A nucleus is a monoidal monad—meaning the endofunctor is monoidal by Eq. (18). The set L_{j} is a complete lattice in which the sups—denoted by $\bigsqcup$ to contrast from the sups in L—are computed by:

$$\bigsqcup \{\, x_i \in L_{\mathrm{j}} \mid i \in I \,\} := \mathrm{j}\Big(\bigvee \{\, x_i \in L_{\mathrm{j}} \mid i \in I \,\}\Big)\,. \tag{19}$$

Closure operators were studied in Section 1.4 of [**JT84**]. The authors notably show that closure operators are in bijection with quotient object—see also [**KP08**, Section 2.3] or [**EGGHK18**, Subsection 2.1.1]. A closure operator $\mathrm{j} : L \to L$ is not a sup-preserving function in general. However, it a surjective sup-preserving map if we restrain its codomain to L_{j}—with the sups given by Eq. (19). Moreover, for every sup-preserving map $f : L \to M$, denoting by $\rho(f)$ its right adjoint, $\rho(f) \circ f : L \to L$ is a closure operator—this is just the correspondence between adjoints and monads.

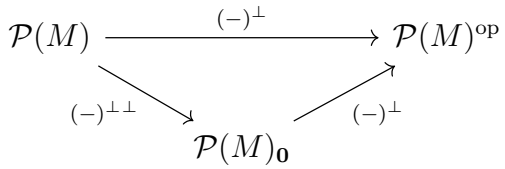
Remark 2.1. This implies that $L_{\rho(f)\circ f}$ is actually the image of f, and leads to an epi-mono factorization system:



If j is a nucleus over a quantale $(Q, *)$, then the tuple $(Q_{\mathrm{j}}, *_{\mathrm{j}})$ is a quantale where $*_{\mathrm{j}}$ is defined by:

$$x *_{\mathrm{j}} y := \mathrm{j}(x * y)\,.$$

The two implications $(Q_{\mathrm{j}}, *_{\mathrm{j}})$ are computed exactly as in $(Q, *)$. Moreover $j : Q \to Q_{\mathrm{j}}$ is a surjective quantale homomorphism. Just as closure operators lead to an epi-mono factorization system in SLATT, nucleus lead to one in QUANT. Note that this is exactly what happens in the construction of phase quantales: for a monoid $(M, \cdot, 1)$ and $\mathbf{0} \subset M$, we have the following commutative diagram.



Rosenthal, in [**Ros90a**, Theorem 2], has proved that every Girard quantale has a representation as a phase quantale. The construction is given as follows. For a Girard quantale $(Q, *, 0)$, the phase quantale associated is $(P(Q)_{\mathbf{0}}, \bullet_{\mathbf{0}})$ with the same notation as above and

$$\mathbf{0} :=\!\downarrow 0 = \{x \in Q \mid x \leq 0\}\,.$$

In Theorem 2.3 we give an analogous representation theorem for unitless Frobenius quantales—see Definition 2.5.

Surprisingly, completely distributive lattices yield Girard quantales. Completely distributive lattices generalize distributive lattices to the possibly infinite case.

DEFINITION 2.4. A complete lattice L is *completely distributive*, if for every doubly indexed family, $\{x_{ij}\}_{i \in I j \in J}$ of L, we have

$$\bigwedge_{i \in I} (\bigvee_{j \in J} x_{ij}) = \bigvee_{\alpha : I \to J} (\bigwedge_{i \in I} x_{i\alpha(i)})\,.$$

EXAMPLE 2.4. The main examples of completely distributive lattices are complete chains such as a closed interval of the real numbers, and powerset lattices. ◊

Kruml and Paseka in Theorem 3.3.6 of [**KP08**] have shown the following result:

THEOREM 2.1. *Let L be a complete lattice. The following statements are equivalent:*

- *L is a completely distributive lattice;*
- *There exists a sup-preserving map $0 : L \to L$ making $([L, L], \circ, 0)$ a Girard quantale.*

This theorem have been further studied in [**EGGHK18**, Example 2.6.16], and, in the wider framework of quantaloid, in [**San20a**] and [**San20b**]. While further exploring this result in a categorical setting, we (re)discovered another definition of Frobenius and Girard quantales that does not entail the existence of the unit. Using this definition we notably give, in Theorem 2.7, a further condition which is equivalent to the two statements of Theorem 2.1. The rest of the chapter is a study of this definition. We generalize several properties we have given for unital Frobenius quantales to the unitless case. Most of its material comes from [**dLS22**].

2.2. Unitless Frobenius quantales

As already mentioned, a Frobenius quantale is usually defined as a quantale with a dualizing element 0—see Definition 2.2. Such a quantale is necessarily unital, see [**Ros90a**, Corollary to Proposition 1] and [**EGGHK18**, Proposition 2.6.3]. Indeed, $0 \backslash 0 = 0 / 0$ turns out to be the unit of the quantale. Intuitively, in a logical sense, this means that "false implies false is true". This property led us to call them unital Frobenius quantales. The forced existence of the unit contrasts with the theory of quantales, that was originally motivated by *unitless* quantales. Indeed, under certain hypotheses, adding the existence of the unit leads to a commutative quantale. For instance, a unital idempotent left-sided—*i.e.*, $\forall x, x * x = x$ and $\top * x \leq x$—quantale is necessarily a frame. This implies that the study of non-commutative quantale should be done in the non-unital case.

We propose a slightly different definition of Frobenius and Girard quantales, which does not rely on dualizing elements. The definition is more general in that, as we shall see, it allows us to consider Frobenius unitless quantales.

Definition 2.5. Let $(Q, *)$ be a quantale.

We say that a pair of sup-preserving maps ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q^{\mathrm{op}}$ is *Serre* if for every $x, y \in Q$, we have

$$x \backslash {}^{\perp} y = x^{\perp} / y \,. \tag{Serre equality}$$

A Serre pair of maps ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q^{\mathrm{op}}$ is a *Serre duality*, if they are inverse to each other, *i.e.*, we have

$${}^{\perp}(x^{\perp}) = ({}^{\perp}x)^{\perp} = x \,.$$

A *Frobenius quantale* is a tuple $(Q, *, {}^{\perp}(-), (-)^{\perp})$ where $(Q, *)$ is a quantale and ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q^{\mathrm{op}}$ is a Serre duality.

The map $(-)^\perp$ is called the *right negation*, while the map ${}^\perp(-)$ ia called the *left negation*.

A *Girard quantale* is a Frobenius quantale for which right and left negations coincide.

We informally call a structure as defined in Definition 2.5 a *unitless Frobenius* (or *Girard*) *quantale*. This naming, however, is slightly imprecise, as these quantales might in some cases actually have a unit. Let us make straight the relation between Definition 2.2 and Definition 2.5. This has already been studied in [**GJKO07**, Lemma 3.18]. The Lemma has been restated in our framework, so it becomes clearer.

LEMMA 2.1. *The Frobenius quantale structures defined in Definition 2.2 and those that are unital among the ones defined in Definition 2.5, are in bijection.*

PROOF. Let $(Q, *, 0)$ be a unital Frobenius quantale as defined in Definition 2.2. By letting $x^\perp := x \backslash 0$ and ${}^\perp x := 0 / x$ one obtains a Frobenius quantale as defined in Definition 2.5, in view of the well known identity $x \backslash (z / y) = (x \backslash z) / y$.

In the opposite direction, let $(Q, *, {}^\perp(-), (-)^\perp)$ be a Frobenius quantale as defined in Definition 2.5 which, moreover, is unital. Observe that ${}^\perp 1 = 1 \backslash {}^\perp 1 = 1^\perp / 1 = 1^\perp$. Thus, let $0 := {}^\perp 1 = 1^\perp$ and observe that

$$ {}^\perp x = 1 \backslash {}^\perp x = 1^\perp / x = 0 / x $$

and, similarly, $x^\perp = x \backslash 0$.

It is then immediate that 0 is a dualizing element. □

REMARK 2.2. In Definition 2.5, we do not make explicit reference to any element. This will come in handy when we will abstract Frobenius quantales in monoidal categories in Chapter 3.

Eq. (Serre equality) was already studied in Section 3.3 of [**GJKO07**], under the name of *contraposition* in the more general setting of involutive division posets. This identity is—under the assumption that negations are inverse to each other—equivalent to the following ones:

$$ x \backslash y = x^\perp / y^\perp \,, \qquad x / y = {}^\perp x \backslash {}^\perp y \,, \qquad {}^\perp x \backslash y = x / y^\perp \,. \tag{20} $$

Invertible maps satisfying the above kind of conditions were called Serre dualities in [**Rum21**]. The last of the three identities above was considered in [**EK10**].

The following Lemma immediately follows from the previous definition.

LEMMA 2.2. *A pair of sup-preserving maps* ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q^{\mathrm{op}}$ *is Serre if and only if the equivalence below holds (for each* $x, y, z \in Q$*):*

$$x * z \leq {}^{\perp}y \;\; \textit{iff} \;\; z * y \leq x^{\perp} . \tag{Shift relation}$$

Notice that the equivalence Eq. (Shift relation) amounts to what were called elsewhere (see e.g. [**San21**]) the *shift relations*:

$$x * y \leq z \;\text{ iff }\; {}^{\perp}z * x \leq {}^{\perp}y \;\text{ iff }\; y * z^{\perp} \leq x^{\perp} ,$$

which also appear in [**GJKO07**, §3.3.2] under the naming of (IGP) for involutive po-groupoid.

If Q is unital, then a Serre pair ${}^{\perp}(-), (-)^{\perp}$ necessarily forms a Galois connection—take $z := 1$ in Eq. (Shift relation). If Q is not unital, then a Serre pair need not to be a Galois connection. For example, if $x * y = \bot$ for each $x, y \in Q$, then $x \backslash y = y / x = \top$, and the constant map $x^{\perp} = \bot$, $x \in Q$, is Serre with itself, but it is not self-adjoint if Q has at least two elements. We shall restrict ourself, notably in the next section, to consider Serre pairs forming a Galois connection. Of course, if ${}^{\perp}(-), (-)^{\perp}$ are inverse to each other, then they are also adjoint—but the converse is false.

Given a quantale $(Q, *)$ and pair of sup-preserving maps ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q^{\mathrm{op}}$ inverse to each other, we can define two distinct dual operators:

$$x \oplus_{\perp} y := {}^{\perp}(y^{\perp} * x^{\perp}) , \qquad x \,{}_{\perp}\!\oplus\, y := ({}^{\perp}y * {}^{\perp}x)^{\perp} . \tag{21}$$

PROPOSITION 2.1 (cf. [**GJKO07**, Lemma 3.17]). *Let* $(Q, *, {}^{\perp}(-), (-)^{\perp})$ *be a Frobenius quantale.*

The two dual multiplications coincide and they are determined by the implication and the negations as follows

$$x \oplus_{\perp} y = {}^{\perp}x \backslash y = x / y^{\perp} = x \,{}_{\perp}\!\oplus\, y .$$

PROOF. The chain of equivalences

$$z \leq ({}^{\perp}y * {}^{\perp}x)^{\perp} \;\text{ iff }\; {}^{\perp}y * {}^{\perp}x \leq {}^{\perp}z \;\text{ iff }\; {}^{\perp}x * z \leq y \;\text{ iff }\; z \leq {}^{\perp}x \backslash y$$

yields the identity $x \,{}_{\perp}\!\oplus\, y = {}^{\perp}x \backslash y$. In a similar way, we have

$$z \leq {}^{\perp}(y^{\perp} * x^{\perp}) \;\text{ iff }\; y^{\perp} * x^{\perp} \leq z^{\perp} \;\text{ iff }\; z * y^{\perp} \leq x \;\text{ iff }\; z \leq x / y^{\perp}$$

yielding $x \oplus_{\perp} y = x / y^{\perp}$.

The equality $x \oplus_{\perp} y = x \,{}_{\perp}\!\oplus\, y$ then follows from (20). □

We therefore define the multiplication on Q^{op}.

$$x \oplus y := x \oplus_{\perp} y = x \perp\!\!\!\oplus y\,.$$

LEMMA 2.3. *The maps* ${}^{\perp}(-)$ *and* $(-)^{\perp}$ *are quantales isomorphisms between* $(Q, *)$ *and* $(Q^{\mathrm{op}}, \overline{\oplus})$ *where* $\overline{\oplus}$ *is the multiplication defined by*

$$x \overline{\oplus} y := y \oplus x\,.$$

PROOF. This is directly derived by the definition of $\oplus$ in Eq. (21). □

The notion of Frobenius quantale is stable by the op duality.

PROPOSITION 2.2. *Let* $(Q, *, {}^{\perp}(-), (-)^{\perp})$ *be a Frobenius quantale. The tuple* $(Q^{\mathrm{op}}, \oplus, (-)^{\perp}, {}^{\perp}(-))$ *is a Frobenius quantale.*

PROOF. The shift relation is given by the following equivalences:

$$\dfrac{\dfrac{\dfrac{\dfrac{x \oplus z \leq_{Q^{\mathrm{op}}} y^{\perp}}{y^{\perp} \leq x \oplus z = ({}^{\perp}z * {}^{\perp}x)^{\perp}}}{{}^{\perp}z * {}^{\perp}x \leq y}}{{}^{\perp}x \leq {}^{\perp}z \backslash y = z \oplus y}}{z \oplus y \leq_{Q^{\mathrm{op}}} {}^{\perp}x.}$$

□

REMARK 2.3. Note that the multiplication $\oplus$ corresponds to the "par" of linear logic. In the unital and commutative case, this is the cotensor we introduced for $*$-autonomous categories in Section 1.4.

Next, we provide simple examples of unitless Frobenius quantales that indeed lack units.

EXAMPLE 2.5 (Chu construction). Our first example of unitless Girard quantale arises form the usual Chu construction—see the appendix of [**Bar79**] for the categorical version and [**Ros90a**, Theorem 6.1.3] for the unital Girard quantale case. This construction actually does not require a quantale to be unital and yields a Girard quantale as defined in Definition 2.5. For a quantale $(Q, *)$, the quantale $C(Q) = (Q \times Q^{\mathrm{op}}, \star)$ is defined by

$$(x_1, x_2) \star (y_1, y_2) := (x_1 * y_1, y_1 \backslash x_2 \wedge y_2 / x_1)\,.$$

One can directly check the associativity of $\star$ and that it distributes in both arguments over joins of $Q \times Q^{\mathrm{op}}$.

Recall that

$$(x_1, x_2)\backslash(z_1, z_2) = (x_1\backslash z_1 \wedge x_2/z_2, z_2 * x_1)\,,$$

and

$$(z_1, z_2)/(y_1, y_2) = (z_1/y_1 \wedge z_2\backslash y_2, y_1 * z_2)\,,$$

since the conditions $(x_1, x_2) \star (y_1, y_2) \leq (z_1, z_2)$, $(y_1, y_2) \leq (x_1, x_2)\backslash(z_1, z_2)$, and $(x_1, x_2) \leq (z_1, z_2)/(y_1, y_2)$, are all equivalent to the conjuction of the three conditions:

$$x_1 * y_1 \leq z_1\,, \quad y_1 * z_2 \leq x_2\,, \quad z_2 * x_1 \leq y_2\,.$$

The duality being given by

$$(x_1, x_2)^\perp := (x_2, x_1)\,,$$

the computations

$$(x_1, x_2)\backslash(y_1, y_2)^\perp = (x_1, x_2)\backslash(y_2, y_1) = (x_1\backslash y_2 \wedge x_2/y_1, y_1 * x_1)$$
$$(x_1, x_2)^\perp/(y_1, y_2) = (x_2, x_1)/(y_1, y_2) = (x_2/y_1 \wedge x_1\backslash y_2, y_1 * x_1)$$

exhibit $(-)^\perp$ as Serre self-dual. Hence $(Q \times Q^{\mathrm{op}}, \star, (-)^\perp)$ is a Girard quantale.

Proposition 2.3. *The quantale $C(Q)$ is unital if and only if Q is unital.*

Indeed, the first projection is a surjective semigroup homomorphism and so, if $C(Q)$ has a unit (u_1, u_2), then u_1 is a unit of Q. On the other hand, it is well known that if u_1 is a unit of Q, then $(u_1, \top)$ is a unit of $C(Q)$. ◊

Example 2.6 (Couples of quantales). In [**EK10**] the authors define a couple of quantales as a pair of quantales C, and Q that are related by a sup-preserving map $\phi : C \longrightarrow Q$; moreover, C is asked to be a Q-bimodule and the following equations

$$\phi(c_1) \cdot c_2 = c_1 \cdot \phi(c_2) = c_1 * c_2 \tag{22}$$

with $c_1, c_2 \in C$, are to be satisfied.

If Q is a quantale, then $C = Q^{\mathrm{op}}$ is a canonical Q-bimodule, with, for $x, y \in Q$ and $q \in Q^{\mathrm{op}}$,

$$y \cdot q = q/y\,, \qquad q \cdot x = x\backslash q\,.$$

Thinking of $\phi : Q^{\mathrm{op}} \longrightarrow Q$ as a sort of negation, the first equation in (22) yields

$$x/\phi(y) = \phi(x)\backslash y\,,$$

that is, the last equation in (20).

Notice that, by taking the above equality as definition of a binary operator on Q^{op}, we obtain an operator that is necessarily associative. If, moreover, ϕ is an involution, then Q becomes a unitless Girard quantale. Vice versa, unitless Girard quantale structures on Q give rise to couples of quantales $\phi : C \longrightarrow Q$ with $C = Q^{\mathrm{op}}$ and ϕ an antitone involution. Let us notice, however, that *Girard* couples of quantales, in contrast to couples of quantales, turn out to have some unit [**EK10**, Proposition 8]. ◊

We gave two examples of Girard quantales, one with a unit and one without. Other examples of unitless Frobenius quantale will arise from the theory developed in the next section.

2.3. Double negation construction

In this section, we show how to generalise the elementary theory of nuclei from unital Girard quantales to unitless Girard quantales and to unitless Frobenius quantales.

Definition 2.6. Let $(Q, *)$ be a quantale. A pair of sup-preserving maps $l, r : Q \longrightarrow Q^{\mathrm{op}}$ is a *Serre Galois connection* if it is

- Serre,
- a Galois connection,
- and $l \circ r = r \circ l$.

We say that $r : Q \to Q^{\mathrm{op}}$ is a *Serre map* if (r, r) is a Serre Galois connection.

Recall that (l, r) is Serre if we have the shift relation

$$x * y \leq r(z) \text{ iff } z * x \leq l(y)\,.$$

or equivalently, the contraposition law

$$r(z) \diagup y = z \diagdown l(y)\,.$$

Note that $l \circ r, r \circ l : Q \to Q$ are closure operators. Those functions are not maps in SLatt but in Pos as, in general, those are not sup-preserving maps. Restraining their codomain to their fixed-points lattice, we obtain surjective sup-preserving maps.

The definition of a Serre Galois connection is weaker than the one of a Serre duality as defined in Definition 2.5, since the maps are not necessarily invertible. However, a Serre Galois connection induces a Serre duality as

we will demonstrate. One can think of a Serre Galois connection (l, r) as two intuitionistic negations working together harmoniously. In a complete Heyting algebra, the set of fixed points of the double negation $(- \to \bot) \to \bot$ forms a Boolean algebra. For (l, r) a Serre Galois connection over $(Q, *)$, $\mathrm{j} := l \circ r = r \circ l$ plays the role of the double negation. It is, of course, a closure operator and we shall see that it is actually a nucleus. We have mentioned in Remark 2.1 that the set of fixed point of j is a quantale $(Q^{\mathrm{j}}, *_{\mathrm{j}})$. We show in the following that $(Q^{\mathrm{j}}, *_{\mathrm{j}})$ is actually a Frobenius quantale, thus generalizing Theorem 1 of [**Ros90a**]—which already generalizes the double negation construction of Heyting algebras.

PROPOSITION 2.4. *If (l, r) is a Serre Galois connection, then $\mathrm{j} := l \circ r = r \circ l$ is a nucleus, and the restriction of (l, r) to Q_{j} yields a Frobenius quantale structure on $(Q_{\mathrm{j}}, *_{\mathrm{j}})$.*

PROOF. We first prove that $x * \mathrm{j}(y) \leq \mathrm{j}(x * y)$, that is, $x * rl(y) \leq rl(x * y)$. Using the shift relation, this inclusion is equivalent to $l(x * y) * x \leq lrl(y)$. Since $lrl(y) = l(y)$, the latter inclusion is $l(x * y) * x \leq l(y)$ and equivalent, by the shift relation, to $x * y \leq rl(x * y)$, where the last inclusion holds since $r \circ l$ is a closure operator. The inclusion $\mathrm{j}(x) * y \leq \mathrm{j}(x * y)$ is proved similarly, using now $\mathrm{j} = l \circ r$. Therefore,

Next, the restrictions of (l, r) to Q_{j} have Q_{j} as codomain: if $x \in Q_{\mathrm{j}}$, then $\mathrm{j}(r(x)) = r(l(r(x))) = r(x)$, so $r(x) \in Q_{\mathrm{j}}$. Similarly, $l(x) \in Q_{\mathrm{j}}$. These restrictions are also inverse to each other: for $x \in Q_{\mathrm{j}}$, $l(r(x)) = r(l(x)) = \mathrm{j}(x) = x$. Let us argue that these restrictions are Serre: for $x, y, z \in Q_{\mathrm{j}}$, $x *_{\mathrm{j}} y = \mathrm{j}(x * y) \leq r(z)$ iff $x * y \leq r(z)$ (since $r(z) \in Q_{\mathrm{j}}$) iff $z * x \leq l(y)$ iff $z *_{\mathrm{j}} x = \mathrm{j}(z * x) \leq l(y)$, so the shift relation holds in Q_{j}. □

PROPOSITION 2.5. *Let j be a nucleus on $(Q, *)$ and suppose that $l, r : Q_{\mathrm{j}} \longrightarrow Q_{\mathrm{j}}$ is a Serre duality on $(Q_{\mathrm{j}}, *_{\mathrm{j}})$. Then $(l \circ \mathrm{j}, r \circ \mathrm{j})$ is a Serre Galois connection on $(Q, *)$ and the Frobenius quantale structure $(Q_{\mathrm{j}}, *_{\mathrm{j}}, l, r)$ is induced from $(l \circ \mathrm{j}, r \circ \mathrm{j})$ as described in Proposition 2.4.*

PROOF. Let j, l, r be as stated. For $x, y, z \in Q$, we have $x \leq r(\mathrm{j}(y))$ iff $\mathrm{j}(x) \leq r(\mathrm{j}(y))$ iff $\mathrm{j}(y) \leq l(\mathrm{j}(x))$ iff $y \leq l(\mathrm{j}(x))$. That is, $(l \circ \mathrm{j}, r \circ \mathrm{j})$ is a Galois connection on Q. Moreover, the pair is Serre as we have:

$$\begin{aligned} x * y \leq r(\mathrm{j}(z)) \;\text{ iff }\; & \mathrm{j}(x * y) = \mathrm{j}(x) *_{\mathrm{j}} \mathrm{j}(y) \leq r(\mathrm{j}(z)) \\ \text{iff }\; & \mathrm{j}(z * x) = \mathrm{j}(z) *_{\mathrm{j}} \mathrm{j}(x) \leq l(\mathrm{j}(y)) \;\text{ iff }\; z * x \leq l(\mathrm{j}(y)) \,. \end{aligned}$$

Notice now that $\mathrm{j} \circ r = r$ and $\mathrm{j} \circ l = l$, since by assumption the codomain of r and l is Q_{j}. Then

$$(l \circ \mathrm{j}) \circ (r \circ \mathrm{j}) = l \circ r \circ \mathrm{j} = \mathrm{j}$$

and, similarly, $(r\circ\mathrm{j})\circ(l\circ\mathrm{j})=\mathrm{j}$. Finally, the restriction of $r\circ\mathrm{j}$ (resp., $l\circ\mathrm{j}$) to Q_{j} is r (resp., l), for example, for $x\in Q_{\mathrm{j}}$, we have $(r\circ\mathrm{j})(x)=r(\mathrm{j}(x))=r(x)$. □

We add next some remarks.

Definition 2.7. Let 0 be an element of a quantale $(Q,*)$, we say that 0 is *weakly cyclic* if

$$0/(x\backslash 0)=(0/x)\backslash 0\,,\quad \text{for all } x\in Q.$$

We say that a Serre Galois connection (l,r) on $(Q,*)$ is *representable by* 0 if

$$r(x)=x\backslash 0\,,\qquad l(x)=0/x\,,\qquad \text{for all } x\in Q.$$

Observe that if a Serre Galois connection (l,r) is representable by $0\in Q$, then 0 is a weakly cyclic element, as $r\circ l=l\circ r$.

Lemma 2.4. *If $(Q,*)$ is unital, then every Serre Galois connection is representable.*

Proof. If 1 is the unit of $(Q,*)$, then

$$r(x)=r(x)/1=x\backslash l(1)\qquad\text{and}\qquad l(x)=1\backslash l(x)=r(1)/x\,.$$

Moreover

$$r(1)=r(1)/1=1\backslash l(1)=l(1)\,,$$

and therefore (l,r) is representable by $0:=r(1)=l(1)$. □

Lemma 2.5. *Let 0 be a weakly cyclic element of a quantale $(Q,*)$ and set $r(x):=x\backslash 0$ and $l(x):=0/x$. Then (l,r) is a representable Serre Galois connection. If 0 is* j*-closed (with $\mathrm{j}=r\circ l=l\circ r$), then $(Q_{\mathrm{j}},*_{\mathrm{j}})$ is unital. If $(Q,*)$ is unital, then 0 is* j*-closed.*

Proof. The first statement is obvious. If 0 is j-closed, then the relations $r(x)=x\backslash 0$ and $l(x)=0/x$ hold within Q_{j}, whence 0 is a dualizing element of Q_{j} which therefore is unital. Finally, if 1 is the unit of $(Q,*)$, then $0=r(1)=rlr(1)=l(1)=lrl(1)$, showing that 0 is j-closed. □

Remark 2.4. Let $(Q,*)$ be a commutative quantale. If $0\in Q$ is a dualizing element, then clearly 0 is cyclic, and $(Q,*,0)$ is a unital Girard quantale. However, there may exist a Serre duality $({}^{\perp}(-),(-)^{\perp})$ with ${}^{\perp}(-)\neq(-)^{\perp}$—unless the Serre duality is representable. The trivial quantale in Example 2.7 structure will give such an example.

Example 2.7 (Trivial quantales). As we do not require units, each complete lattice Q can be endowed with the trivial quantale structure, defined by:

$$x * y := \bot\,, \quad \text{for each } x, y \in Q.$$

The trivial quantale structure clearly is not unital, unless Q is a singleton. Notice that, for the trivial structure,

$$x \backslash y = y / x = \top\,, \quad \text{for each } x, y \in Q.$$

If Q is autodual, say with $l, r : Q \longrightarrow Q$ inverse to each other, then $(Q, *, l, r)$ is a unitless Frobenius quantale. $\Diamond$

The trivial structure is a source of counter-examples to quick conjectures. Consider $(\mathbf{2}, *)$, the two-element Boolean algebra equipped with the trivial quantale structure. Then we have $x * y \leq r(z)$ if and only if $y * z \leq l(x)$ no matter how we choose (l, r). Thus, we might chose (l, r) not antitone and yet satisfying the shift relation (take $r = l$ be the identity of $\mathbf{2}$), or choose (l, r) antitone but not a Galois connection (take $r = l$ be the constant function taking $\bot$ as its unique value).

Also, consider the diamond lattice M_3 with 3 atoms-coatoms and endow it with the trivial quantale structure. Take for r a duality that cyclically permutes the 3 atoms and let l be its inverse, so to have $l \neq r$. Then (l, r) is a Serre duality on the commutative quantale over M_3. On the other hand, every representable Serre duality on a commutative quantale is necessarily self-adjoint (*i.e.*, we have $r = l$).

It might be asked whether (l, r) representable on Q implies that (l, r) is representable on Q_{j}. This is not the case. On the chain $0 < 1 < 2$, consider the following commutative quantale structure:

$*$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

By commutativity, 0 is weakly cyclic and, letting $r(x) = x \backslash 0$, we have $r(0) = r(1) = 2$ and $r(2) = 1$. Therefore, $Q_{\mathrm{j}} = \{\, 1, 2 \,\}$ and the $*_{\mathrm{j}}$ is the trivial quantale structure on Q_{j}. As the implication of the trivial structure is constant, it cannot yield the (unique) duality of $\{\, 1, 2 \,\}$. Considering Lemma 2.5, notice here that 0 is not j-closed.

It is not difficult to reproduce similar counter-examples with the multiplication being non-trivial (*i.e.*, $x * y \neq \bot$, for some $x, y \in Q$).

2.4. Phase quantales and representation theorem

We explore next the construction described in Proposition 2.4 and Proposition 2.5 when $(Q, *)$ is the free quantale $(\mathcal{P}(S), \bullet)$ over a semigroup $(S, \cdot)$—see Example 2.2.

It is well-known (see e.g. [**Ore44**]) that Galois connections on a powerset lattice $\mathcal{P}(A)$ bijectively correspond to binary relations over the set A. The correspondence is given by the following. Let A be a set and (l, r) a Galois connection on $\mathcal{P}(A)$, define $R \subset A \times A$ by

$$xRy \text{ iff } x \in l(\{ y \}) \text{ iff } y \in r(\{ x \}).$$

For a binary relation R over A, define $l, r : P(A) \to P(A)^{\text{op}}$ by:

$$l(Y) = \{ u \in S \mid uRy, \text{ for all } y \in Y \},$$

and

$$r(Z) = \{ u \in S \mid zRu, \text{ for all } z \in Z \}.$$

In the following $(S, \cdot)$ is a semigroup. We aim to characterize Serre Galois connections on the quantale $(\mathcal{P}(S), \bullet)$ via properties of their corresponding relation on S.

Proposition 2.6. *A Galois connection (l, r) on $(\mathcal{P}(S), \bullet)$ satisfies the shift relation* (Shift relation) *if and only if its corresponding binary relation R satisfies*

$$x \cdot yRz \text{ iff } xRy \cdot z, \quad \text{for all } x, y, z \in S. \tag{23}$$

Proof. If (l, r) satisfies (Shift relation), then the following is a chain of equivalent statements (where X, Y, Z are subsets of S):

$$\begin{aligned} \forall x \in X, y \in Y, z \in Z,\ zRx \cdot y &\text{ iff } X \bullet Y \subseteq r(Z) \\ &\text{iff } Z \bullet X \subseteq l(Y) \\ &\text{iff } \forall x \in X, y \in Y, z \in Z,\ z \cdot xRy. \end{aligned}$$

Considering singletons we obtain that R satisfies

$$\forall x, y, z \in S \ \ zRx \cdot y \text{ iff } z \cdot xRy,$$

and clearly the latter condition suffices for (Shift relation) to hold. □

We call *associative* a relation satisfying (23). To complete the characterisation of Serre Galois connections, we need to characterise the condition

$l \circ r = r \circ l$ in terms of the corresponding relation R. Notice first that the condition trivially holds if R is symmetric, in which case $r = l$. Let us state a general lemma about Galois connection satisfying $r \circ l = r \circ l$.

LEMMA 2.6. *For (l, r) a Galois connection on a lattice L, the condition $l \circ r = r \circ l$ holds if and only if the images of l and r in L coincide.*

PROOF. Denote the image of a map f by $\mathsf{Img}(f)$. Notice that $\mathsf{Img}(r) = L_{r \circ l}$, since $r(x) = r(l(r(x)))$ and similarly $\mathsf{Img}(l) = L_{l \circ r}$. Therefore, if $r \circ l = l \circ r$, then $\mathsf{Img}(r) = \mathsf{Img}(l)$. Conversely, if $\mathsf{Img}(r) = \mathsf{Img}(l)$, then $L_{r \circ l} = L_{l \circ r}$ and since a closure operator is determined by the set of its fixed points, then $l \circ r = r \circ l$. □

When applying Lemma 2.6 to the lattice $\mathcal{P}(S)$, we obtain that the condition $l \circ r = r \circ l$ is equivalent to

$$\forall X \exists Y \quad \text{s.t.} \quad r(X) = l(Y)\,, \quad \text{and} \quad \forall Y \exists X \quad \text{s.t.} \quad r(X) = l(Y)\,.$$

Clearly, these two conditions hold if and only if they hold with X (in the first case, or Y in the second) restricted to singleton, yielding

$$\forall x \exists Y_x \forall z \; (xRz \;\text{ iff }\; zRy, \forall y \in Y_x)\,, \tag{24}$$

$$\forall y \exists X_y \forall z \; (zRy \;\text{ iff }\; xRz, \forall x \in X_y)\,. \tag{25}$$

We call *weakly-symmetric* a relation satisfying (24) and (25).

We exemplify our previous observations, in particular the use of Proposition 2.4, in different ways.

EXAMPLE 2.8. Our first example illustrates how to construct ad-hoc unitless Frobenius quantales with almost no effort.

Consider $(\Sigma^+, \cdot)$, the free semigroup over the alphabet Σ. Let R be the binary relation total on letters and empty otherwise: wRu iff $w, u \in \Sigma$. Due to the absence of units, condition (23) is trivially satisfied. Notice also that R is symmetric. The closed subsets of Σ^+ are $\emptyset$, Σ, and Σ^+ and the table for the multiplication is as follows:

	$\emptyset$	Σ	Σ^+
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Σ	$\emptyset$	Σ^+	Σ^+
Σ^+	$\emptyset$	Σ^+	Σ^+

This quantale does not have units but it has a Serre duality, the unique duality of the chain $\emptyset \subseteq \Sigma \subseteq \Sigma^+$. ◊

EXAMPLE 2.9 (Unital Frobenius quantales from pregroups). Here we illustrate how the same tools can be used to build Frobenius quantales that are not Girard quantales. Recall that a pregroup, see e.g. [**Bus01**], is an ordered monoid $(M, 1, \cdot, \leq)$ coming with functions $(-)^l, (-)^r : M \longrightarrow M$ satisfying

$$x \cdot x^r \leq 1\,, \qquad x^l \cdot x \leq 1\,, \qquad 1 \leq x^r \cdot x\,, \qquad 1 \leq x \cdot x^l\,,$$

for all $x \in M$. That is, a pregroup is a posetal rigid category. Consider the relation R defined by xRy iff $x \cdot y \leq 1$. Clearly, R is associative. Observe now that $x \cdot z \leq 1$ if and only if $z \cdot x^{rr} \leq 1$. That is, the condition in (24) is satisfied by letting $Y_x = \{\, x^{rr} \,\}$. Similarly, (25) is satisfied by letting $X_y = \{\, y^{ll} \,\}$. Since M has a unit, both $\mathcal{P}(M)$ and $\mathcal{P}(M)_{\mathrm{j}}$ are unital. ◊

EXAMPLE 2.10 (Unitless Girard quantales from C*-algebras). Let us consider an algebra A coming with an associative symmetric pairing (*i.e.*, a bilinear form) $\langle -, - \rangle$ into the base field $\mathbb{K}$. We do not assume that A has a unit. We recall that a pairing is said to be associative when it satisfies $\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle$, for each $x, y, z \in A$ (in which case A is called a Frobenius algebra).

The binary relation R, defined by xRy if and only if $\langle x, y \rangle = 0$, is then an associative relation on the semigroup reduct $(A, \cdot)$ of A, so we can consider the powerset quantale $(\mathcal{P}(A), \bullet)$ and the Serre Galois connection (l, r) the relation R gives rise to. Since R is symmetric, we denote $l(X) = r(X) = X^{\perp}$, as usual from standard algebra, and so $\mathrm{j}(X) = X^{\perp\perp}$. Clearly, if A has a unit, then $\mathcal{P}(A)$ and its quotient $\mathcal{P}(A)_{\mathrm{j}}$ have a unit as well.

We argue next that the converse holds, if we can transform the pairing into a sort of inner product (as for example with C*-algebras). Namely, suppose that A comes with an involution $(-)^* : A \longrightarrow A$ such that, for each $f \in A$,

$$\langle f, f^* \rangle = 0 \;\text{ implies }\; f = 0\,. \tag{26}$$

In particular, assuming (26), we have $A^{\perp} = \{\, 0 \,\}$. Under these assumptions, the following statement holds:

PROPOSITION 2.7. *If $\mathcal{P}(A)_{\mathrm{j}}$ has a unit, then A has a unit.*

PROOF. Obviously, every set of the form $X^{\perp}$ is a subspace of A. Let us say that a subpace is closed if it is of the form $\mathrm{j}(Y)$ or, equivalently, $X^{\perp}$ (for some X or some Y).

We firstly observe that every one dimensional subspace of A is closed. That is, we have $\mathrm{j}(\{\, f \,\}) = \mathbb{K}f$, for each $f \in A \setminus \{\, 0 \,\}$. Indeed, for $g \in A$ and $k = \frac{\langle g, f^* \rangle}{\langle f, f^* \rangle}$, we can write $g = (g - kf) + kf$ with $g - kf \in \{\, f^* \,\}^{\perp}$. Because

of (26), we also have $f \notin \{ f^* \}^\perp$. That is, we have and $\mathbb{K}f \vee \{ f^* \}^\perp = A$ and $\mathbb{K}f \wedge \{ f^* \}^\perp = \{ 0 \}$ in the lattice of subspaces of A. The relation $\mathbb{K}f \vee \{ f^* \}^\perp = A$ a fortiori implies the relation $\mathrm{j}(\mathbb{K}f) \vee_\mathrm{j} \{ f^* \}^\perp = A$, taken in the lattice of closed subpaces of A. Using the duality, we derive $\{ 0 \} = A^\perp = (\mathrm{j}(\mathbb{K}f) \vee_\mathrm{j} \{ f^* \}^\perp)^\perp = \mathrm{j}(\mathbb{K}f)^\perp \wedge_\mathrm{j} \{ f^* \}^{\perp\perp} = \{ f \}^\perp \wedge \{ f^* \}^{\perp\perp}$, where we have used that both meets $\wedge$ (the meet in the lattice of subspaces of A) and $\wedge_\mathrm{j}$ (the meet in the lattice of closed subspace of A) are computed as intersection. Then we also have $\{ 0 \} = \{ f^* \}^\perp \wedge \{ f^{**} \}^{\perp\perp} = \{ f^* \}^\perp \wedge \{ f \}^{\perp\perp}$. It follows that both $\mathbb{K}f$ and $\mathrm{j}(\{ f \}) = \{ f \}^{\perp\perp}$ are complements of $\{ f^* \}^\perp$ in the lattice of subspaces of A. Since $\mathbb{K}f \subseteq \mathrm{j}(\{ f \})$, we derive $\mathbb{K}f = \mathrm{j}(\{ f \})$ using modularity of this lattice.

Let now U be the unit of $\mathcal{P}(A)_\mathrm{j}$. Then $U \bullet \{ f \} = \mathrm{j}(U) \bullet \{ f \} \subseteq \mathrm{j}(U \bullet \{ f \}) \subseteq \mathrm{j}(U \bullet \mathrm{j}(\{ f \})) = U \bullet_\mathrm{j} \mathrm{j}(\{ f \}) = \mathrm{j}(\{ f \}) = \mathbb{K}f$ and, similarly, $\{ f \} \bullet U \subseteq \mathbb{K}f$. Thus, for each $u \in U$ and $f \in A$, $u \cdot f = kf$ and $f \cdot u = k'f$ for some $k, k' \in \mathbb{K}$. In particular, for $u, v \in U$, then $ku = u \bullet v = k'v$, showing that U has dimension 1 (the unit U cannot be $\{ 0 \}$ unless $A = \{ 0 \}$). We claim now that, given $u \in U$, there exists $k \in \mathbb{K}$ such that, for all $f \in A$, $u \cdot f = kf$. Indeed, let $f, g \in A \setminus \{ 0 \}$ and write $u \cdot f = k_f f$ and $u \cdot g = k_g g$. If $f = kg$, then $k_f f = u \cdot f = u \cdot kg = k(u \cdot g) = kk_g g = k_g kg = k_g f$, thus $k_f = k_g$. Suppose now that f, g are linearly independent. Then, for some $k \in \mathbb{K}$, $kf + kg = k(f + g) = u \cdot (f + g) = u \cdot f + u \cdot g = k_f f + k_g g$ which yields the relation $(k - k_f)f = (k_g - k)g$ and $k_f = k = k_g$. We have proved the claim from which it readily follows that $u' := \frac{1}{k}u$ is a left unit of A. Similarly, we can find a right unit $\tilde{u}$ and, as usual, $\tilde{u} = u'$. Thus A is unital. □

The C*-algebra of $n \times n$ matrices over the complex numbers $\mathcal{C}$ comes with the pairing $\langle A, B \rangle = tr(A \cdot B)$, where A^* is the conjugate transpose of A. This algebra is unital and gives rise to a well known Girard quantale, see e.g [**EGGHK18**, §2.6.15]. We can adapt this construction to consider classes of linear operators on an infinite dimensional Hilbert space H. A continuous linear mapping $f : H \longrightarrow H$ is trace class, see e.g. [**Con90**, §18], if

$$\sum_{e \in \mathcal{E}} \langle |f| e, e \rangle_H < \infty \,,$$

where $\langle -, - \rangle_H$ above is the inner product of the Hilbert space H, $\mathcal{E}$ is an orthonormal basis, and $|f|$ is the unique positive operator such that $f^* \circ f = |f|^2$. For such an operator, we can define

$$tr(f) := \sum_{e \in \mathcal{E}} \langle f(e), e \rangle_H \,, \qquad \langle f, g \rangle := tr(f \cdot g) \,, \tag{27}$$

yielding an associative symmetric pairing. Now, f is trace class if and only if its adjoint f^* is trace class. With respect to the involution given by the adjoint, this pairing satisfies (26). The algebra of trace class operators does

not have a unit. First of all, it is easily seen that the identity is not trace class. Indeed, this algebra is a well-known proper (when H is infinite dimensional) ideal of the algebra of bounded linear operators on H. Most importantly, this algebra cannot have a unit. In fact, for each $h \in H$ there exists a trace class operator c_h and $p_h \in H$ such that $c_h(p_h) = h$. For example, if $\mathcal{E}$ is a basis for H, we can let $e_0 \in \mathcal{E}$, and define $c_h(e_0) = h$, and $c_h(e) = 0$, for $e \in \mathcal{E} \setminus \{ e_0 \}$. Then a unit u is forcedly the identity:

$$u(h) = u(c_h(p_h)) = (u \circ c_h)(p_h) = c_h(p_h) = h\,, \qquad \text{for all } h \in H.$$

We collect these observations into a formal statement.

THEOREM 2.2. *The collection of closed subspaces of the algebra of trace class operators is a unitless Girard quantale.*

Similar considerations can be developed for Hilbert-Schmidt operators, see [**Con90**]. Indeed, the composition of two such operators is trace class, showing that the formula for the pairing in (27) also applies to these operators. ♢

Our next goal is to give a representation theorem, analogous to the representation theorem for Girard quantales, see e.g. [**Ros90a**, Theorem 2]. The representation theorem is here extended to Frobenius and unitless quantales.

THEOREM 2.3. *Given a Frobenius quantale $(Q, *, {}^{\perp}(-), (-)^{\perp})$, define xRy iff $x \leq {}^{\perp}y$. Then R is an associative weakly symmetric relation yielding a Serre Galois connection (l, r) on $(\mathcal{P}(Q), \bullet)$, whose quotient $(\mathcal{P}(Q)_{\mathrm{j}}, \bullet_{\mathrm{j}}, l, r)$ is isomorphic to $(Q, *, {}^{\perp}(-), (-)^{\perp})$.*

PROOF. First, we verify that R is associative:

$$zRx * y \;\text{ iff }\; z \leq {}^{\perp}(x * y) \;\text{ iff }\; x * y \leq z^{\perp} \;\text{ iff }\; z * x \leq {}^{\perp}y \;\text{ iff }\; z * xRy\,.$$

Now let (l, r) be the Galois connection corresponding to R. Let us argue that $r \circ l = l \circ r$ by showing that the images of l and r coincide—see Lemma 2.6. By definition, a subset $X \subseteq Q$ in the image of l is of the form $\{ y \mid y \leq {}^{\perp}x, \text{ for all } x \in X \}$. As the condition $y \leq {}^{\perp}x$, for all $x \in X$, is equivalent to $y \leq \bigwedge_{x \in X} {}^{\perp}x = {}^{\perp}(\bigvee X)$, we deduce that such set equals $\downarrow {}^{\perp}(\bigvee X)$. Since every element $y \in Q$ is of the form ${}^{\perp}z$ for some $z \in Q$, we deduce that the image of l is the set of principal downsets of Q. Similalry, a subset $X \subseteq Q$ in the image of r is of the form $\{ y \mid y \leq x^{\perp}, \text{ for all } x \in X \} = \downarrow (\bigvee X)^{\perp}$ and the image of r is again the set of principal downsets of Q.

Finally, $\mathrm{j}(X) = \downarrow \bigvee X$, since

$$\mathrm{j}(X) = l(r(X)) = \downarrow ({}^{\perp}\bigvee(\downarrow (\bigvee X)^{\perp})) = \downarrow ({}^{\perp}(\bigvee X)^{\perp}) = \downarrow (\bigvee X)\,.$$

It is then easily seen that the mapping $\downarrow\colon Q \longrightarrow \mathcal{P}(Q)_{\mathrm{j}}$ is inverted by $\bigvee : \mathcal{P}(Q)_{\mathrm{j}} \longrightarrow Q$. These maps are quantale homomorphisms, indeed we have, for all x and y in Q,

$$\downarrow x \bullet_{\mathrm{j}} \downarrow y = \downarrow \left(\bigvee (\downarrow x \bullet \downarrow y)\right) = \{z \in Q \mid z \leq x * y\} = \downarrow (x * y). \qquad \square$$

2.5. The Girard quantale of tight maps

We present in this section our main example of a unitless Girard quantale. This example prompted this research as it is closely related to Theorem 2.1.

Notation 2.3. In this section, we will use the following notations:

- L shall be an arbitrary but fixed complete lattice,
- L^L is the lattice of all function from L to L,
- $[L, L]_\vee$ is the lattice of all sup-preserving endomaps of L,
- and $[L, L]_\wedge$ is the lattice of all inf-preserving endomaps of L.

For a function $f : L \longrightarrow L$, its Raney transforms $f^\vee$ and $f^\wedge$ are defined by

$$f^\vee(x) := \bigvee_{x \not\leq t} f(t)\,, \qquad\qquad f^\wedge(x) := \bigwedge_{t \not\leq x} f(t)\,.$$

Let us remark that, in the definition above, we do not require that f has any property, such as being monotone for instance. As we shall see, the Raney transform $(-)^\vee$ makes any function $f : L \to L$ into a sup-preserving map while $(-)^\wedge$ transforms it into an inf-preserving map.

Definition 2.8. An endomap $f : L \longrightarrow L$ is *tight* if $f = f^{\wedge\vee}$. We write $[L, L]^t_\vee$ for the set of tight endomaps of L. We say that $f : L \longrightarrow L$ is *cotight* if $f = f^{\vee\wedge}$ and write $[L, L]^t_\wedge$ for the set of cotight maps from L to L.

We aim at demonstrating the following theorem.

Theorem 2.4. *For any complete lattice L, the tuple $([L, L]^t_\vee, \circ, (-)^\perp)$ is a Girard quantale (as defined in Definition 2.5), where $\circ$ is the function composition and the duality $(-)^\perp$ is defined by $f^\perp := \rho(f)^\vee$.*

Remark 2.5. Before we prove the theorem, let us note that the Raney transform $(-)^\vee : [L, L]_\wedge \longrightarrow [L, L]_\vee$ is actually the `mix` map that we defined in Definition 1.6. This has been shown in [**HR89**] in order to prove that nuclear objects of SLatt are exactly the completely distributive lattices.

In order to prove Theorem 2.4, let us recall some elementary properties of Raney transforms, most of which are stated and proved in [**HR89**].

LEMMA 2.7. *For any function $f : L \longrightarrow L$, $f^\vee$ has a right-adjoint, and, in particular $f^\vee$ is sup-preserving and monotone. If f is inf-preserving, then the following relation holds:*

$$\rho(f^\vee) = (\ell(f))^\wedge . \tag{28}$$

For $f : L \longrightarrow L$, $f^\wedge$ is inf-preserving and, when f is sup-preserving, his left-adjoint satisfies

$$\ell(f^\wedge) = (\rho(f))^\vee . \tag{29}$$

PROOF. For each $x, y \in L$, we have

$$\begin{aligned} f^\vee(x) = \bigvee_{x \not\leq t} f(t) \leq y \;\text{ iff }\; & \text{for all } t\,,\ x \not\leq t \;\text{ implies }\; f(t) \leq y\,, \\ & \text{iff }\; \text{for all } t\,,\ f(t) \not\leq y \;\text{ implies }\; x \leq t\,, \\ & \text{iff }\; x \leq \bigwedge_{f(t) \not\leq y} t\,. \end{aligned}$$

Therefore, if we define $g(y) := \bigwedge_{f(t)\not\leq y} t$, then g is right-adjoint to $f^\vee$. Let us argue that equality (28) holds if f is inf-preserving. We shall show that, for all x and y in L, we have

$$f^\vee(x) \leq y \;\text{ iff }\; x \leq (\ell(f))^\wedge(y)\,.$$

On the one hand we have

$$\bigvee_{x \not\leq t} f(t) \leq y \;\text{ iff }\; \text{for all } t \in L \text{ s.t. } x \not\leq t,\ f(t) \leq y\,. \tag{a}$$

On the other hand we have

$$x \leq \bigwedge_{t \not\leq y} \ell(f)(y) \;\text{ iff }\; \text{for all } t \in L \text{ s.t. } t \not\leq y,\ x \leq \ell(f)(t). \tag{b}$$

For the implication (a) $\Rightarrow$ (b), let $t \in L$ be such that $t \not\leq y$. Suppose that $x \not\leq \ell(f)(t)$. Then, by (a), $f(\ell(f)(t)) \leq y$ and, by the adjunction, $t \leq f(\ell(f)(t))$, whence we deduce $t \leq y$, a contradiction. Therefore, $x \leq \ell(f)(t)$. The implication (b) $\Rightarrow$ (a) is proved in a similar way. By duality, it follows that $f^\wedge$ is inf-preserving and that $\ell(f^\wedge) = (\rho(f))^\vee$ when f is sup-preserving. □

Recall that the set L^L is pointwise ordered, i.e. we have $f \leq g$ iff $f(t) \leq g(t)$, for all $t \in L$.

PROPOSITION 2.8 (cf. [**HR89**, Proposition 4.6]). *The operation $(-)^\wedge : L^L \longrightarrow L^L$ is right-adjoint to $(-)^\vee : L^L \longrightarrow L^L$.*

PROOF. Let f and g be two endomaps of L, we have

$$\begin{aligned} f^\vee \leq g \;\text{ iff }\; & \text{for all } x \in L,\; f^\vee(x) = \bigvee_{x \not\leq t} f(t) \leq g(x) \\ \text{iff }\; & \text{for all } x, t \in L,\; x \not\leq t \;\text{ implies }\; f(t) \leq g(x) \\ \text{iff }\; & \text{for all } t \in L,\; f(t) \leq \bigwedge_{x \not\leq t} g(x) = g^\wedge(t) \\ \text{iff }\; & f \leq g^\wedge . \end{aligned}$$

□

Let us briefly analyse the typing of the Raney transforms. Lemma 2.7 exhibits the Raney transforms as overloaded operators: for example, the codomain of $(-)^\vee$ can be taken as any of the complete lattices among L^L, $[L, L]_\vee$, $[L, L]^t_\vee$, as suggested in the left diagram in Figure 2.1.

When we restrict the domain of $(-)^\vee$ to the lattice of inf-preserving maps $[L, L]_\wedge$, the adjunction $(-)^\vee \dashv (-)^\wedge$ of Lemma 2.8 leads to an adjunction between $[L, L]_\wedge$ and $[L, L]_\vee$. Notably, $(-)^{\wedge\vee} : [L, L]_\vee \to [L, L]_\vee$ is a closure operator while $(-)^{\wedge\vee} : [L, L]_\wedge \to [L, L]_\wedge$ is an interior operator. This is further illustrated in the diagram on the right of the figure, where the vertical inclusions on the left are inf-preserving and the vertical inclusions on the right are sup-preserving. The two Raney transforms in the bottom row are inverse isomorphisms, as a standard consequence of the characterisation of factorization systems in the category of sup-lattices and sup-preserving maps, see [**EGGHK18**].

We recall next some usual consequences of the adjunction $(-)^\vee \dashv (-)^\wedge$ stated in Proposition 2.8.

LEMMA 2.8. *A map $f : L \longrightarrow L$ is tight if and only if it lies in the image of the Raney transform $(-)^\vee$.*

PROOF. By general properties of adjunctions, $f^\vee = f^{\vee\wedge\vee}$, showing that if f lies in the image of $(-)^\vee$, then it is tight. The converse is obvious. □

LEMMA 2.9. *A map $f : L \longrightarrow L$ is tight if and only if its right adjoint $\rho(f)$ is cotight.*

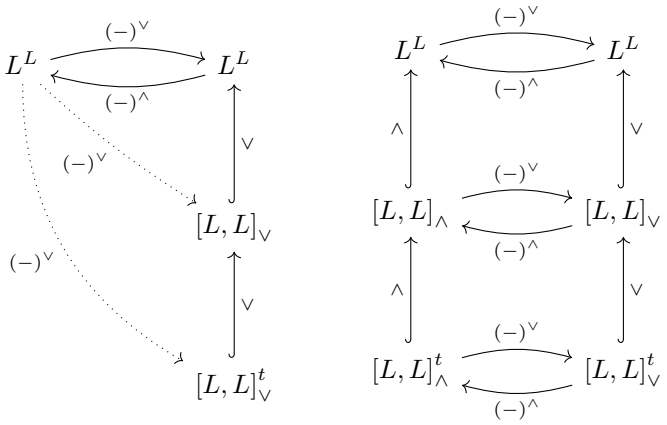


FIGURE 2.1. Typing of the Raney transforms

PROOF. Assume that f is tight, notably it is a sup-preserving map and it has a right adjoint $\rho(f)$. We have the following equalities:

$$\begin{aligned}
\rho(f)^{\vee\wedge} &= \rho(f^{\wedge\vee})^{\vee\wedge} \\
&= \ell(f^\wedge)^{\wedge\vee\wedge} \qquad \text{(by Lemma 2.7)} \\
&= \ell(f^\wedge)^\wedge \\
&= \rho(f^{\wedge\vee}) \qquad \text{(by Lemma 2.7)} \\
&= \rho(f)\,.
\end{aligned}$$

This shows that $\rho(f)$ is cotight. The converse directly follows by duality. □

LEMMA 2.10. *For each map $f : L \longrightarrow L$, $f^{\wedge\vee}$ is the greatest tight map below f, and $f^{\vee\wedge}$ is the least cotight map above f.*

PROOF. We have $f^{\wedge\vee} \leq f$, and if g is tight and below f, then $g = g^{\wedge\vee} \leq f^{\wedge\vee}$. □

COROLLARY 2.1. *For every complete lattice L, the set $[L,L]^t_\vee$ is closed under arbitrary suprema.*

PROOF. Proposition 2.8 implies that the operation $(-)^{\wedge\vee}$ is an interior operator, that is, the dual of a closure operator. As the set of fixed-points of a closure operator is closed under arbitrary infima, the set of fixed-points of an interior operator is closed under arbitrary suprema. □

LEMMA 2.11. *If f is sup-preserving and g is any function, then*

$$(f \circ g)^\vee = f \circ (g)^\vee\,. \tag{30}$$

PROOF. Let f and g be as stated, and compute as follows:

$$(f \circ g^\vee)(x) = f(g^\vee(x)) = f(\bigvee_{x \not\leq t} g(t)) = \bigvee_{x \not\leq t} (f \circ g)(t) = (f \circ g)^\vee(x)\,. \qquad \square$$

COROLLARY 2.2. *Tight maps are closed under composition.*

PROOF. If f and g are tight maps, then f is sup-preserving, and so equation (30) ensures that $f \circ g = f \circ g^{\wedge\vee} = (f \circ g^\wedge)^\vee$, so $f \circ g$ belongs to the image of $(-)^\vee$. As we have seen in Lemma 2.8, this is enough to ensure that this map is tight. $\square$

Corollaries 2.1 and 2.2 ensure that $([L, L]^t_\vee, \circ)$ is a subquantale of $([L, L], \circ)$. The following proposition suffices to prove that $([L, L]^t_\vee, \circ, (-)^\perp)$ is a Girard quantale, as stated in Theorem 2.4, where we recall that $f^\perp := \rho(f)^\vee$.

PROPOSITION 2.9. *The map $(-)^\perp$ is an involutive Serre duality on $([L, L]^t_\vee, \circ)$.*

PROOF. First the map $(-)^\perp$ is an involution. Indeed using the equality (28), we have

$$f^{\perp\perp} = \rho(\rho(f)^\vee)^\vee = \ell(\rho(f))^{\wedge\vee} = f^{\wedge\vee} = f\,.$$

Since this map is the composition of a monotone map with an antitone one, it is antitone. Let us now show that the shift relation (Shift relation) holds, so $(-)^\perp$ is a Serre duality. We have

$$f \circ g \leq h^\perp \text{ iff } g \leq \rho(f) \circ h^\perp\,.$$

Since $k^{\wedge\vee}$ is the greatest tight map below k, see Lemma 2.10, the last inequality is equivalent to

$$g \leq (\rho(f) \circ h^\perp)^{\wedge\vee}\,.$$

We have the following equalities

$$\begin{aligned}
(\rho(f) \circ h^\perp)^{\wedge\vee} &= (\rho(f) \circ (h^{\perp\wedge}))^\vee && \text{(by the dual of Lemma 2.11)}\\
&= (\rho(f) \circ \rho(h))^\vee && \text{(by Lemma 2.9, } \rho(f) \text{ is cotight)}\\
&= \rho(h \circ f)^\vee\\
&= (h \circ f)^\perp\,.
\end{aligned}$$

That is, we have

$$f \circ g \leq h^\perp \text{ iff } g \leq (h \circ f)^\perp\,.$$

Since $(-)^\perp$ is an antitone involution, we obtain

$$f \circ g \leq h^\perp \text{ iff } h \circ f \leq g^\perp\,.$$

$\square$

Let us recall at this point Raney's theorem:

THEOREM 2.5 (Raney [**Ran60**]). *A complete lattice is completely distributive if and only if id_L is tight.*

The following statement appears in [**KP08, EK10**] and, in a more detailed study, in [**San20b**]:

THEOREM 2.6. *The quantale $([L,L]_\vee, \circ)$ is a Girard quantale if and only if L is a completely distributive lattice.*

REMARK 2.6. We can sketch an argument to show that nuclear objects in SLATT are exactly the completely distributive lattices. We refer to Higgs and Rowe paper [**HR89**] for the complete proof. We asserted that `mix` in SLATT is actually $(-)^\vee$—see Remark 2.5. If $(-)^\vee$ is invertible, then $(-)^\wedge$ is necessarily its inverse—since it is its right adjoint. That is, we would have for each $f \in [L,L]_\vee$, $f = f^{\wedge\vee}$, and therefore, L is completely distributive by Theorem 2.6.

We refine Theorem 2.6 by arguing that the Girard quantale $([L,L]^t_\vee, \circ, (-)^\perp)$ has a unit if and only if L is a completely distributive lattice, see Theorem 2.7. If the quantale of tight maps is unital, then its unit necessarily is the identity, and then every map $f \in [L,L]_\vee$ is tight, $f = f \circ id_L = f \circ (id_L)^{\wedge\vee} = (f \circ (id_L)^\wedge)^\vee$. We obtain therefore Theorem 2.6 as a corollary.

To achieve our goals, we recall some elementary properties of the quantale $([L,L]_\vee, \circ)$. In particular, a characterisation of tights maps shall be given using the fundamental morphisms c_y and a_x. For given $x, y \in L$, these maps are defined by

$$c_y(t) := \begin{cases} y\,, & t \neq \bot\,, \\ \bot\,, & t = \bot\,, \end{cases} \qquad \text{and} \qquad a_x(t) := \begin{cases} \top\,, & t \not\leq x\,, \\ \bot\,, & t \leq x\,. \end{cases}$$

These two maps are tight. Indeed, $c_y = (K_y)^\vee$, where $K_y : L \longrightarrow L$ is the constant function with value $y \in L$, and $a_x = (\delta_x)^\vee$ where $\delta_x(t) = \top$ if $t = x$ and, otherwise, $\delta_x(t) = \bot$. Therefore, c_y and a_x are tight and then, using Corollary 2.2, $c_y \circ a_x$ is tight as well. Let us remind the following fact:

LEMMA 2.12. *The set of tight maps is generated under (arbitrary) joins by the maps $c_y \circ a_x$, for x and y in L.*

PROOF. Recall that f is a tight map if and only if there exists $g \in L^L$ such that $g^\vee = f$. Now for such a g, we have

$$g^\vee(x) = \bigvee_{x \not\leq t} g(t) = \bigvee \{\, (c_{g(t)} \circ a_t)(x) \mid t \in L \,\}\,, \tag{31}$$

since

$$(c_{g(t)} \circ a_t)(x) = \begin{cases} c_{g(t)}(\top) = g(t)\,, & x \not\leq t\,, \\ \bot\,, & x \leq t\,. \end{cases}$$ □

PROPOSITION 2.10. *If u is a left or right unit of the quantale $([L,L]^t_\vee, \circ)$, then u is the identity of L.*

PROOF. Let $u \in [L,L]^t_\vee$ be a left unit, then, for each $y \in L$,

$$u(y) = u(c_y(\top)) = (u \circ c_y)(\top) = c_y(\top) = y\,,$$

so u is the identity of L. In a similar way, if $u \in [L,L]^t_\vee$ is a right unit, then, for each $x, t \in L$,

$$t \leq x \;\text{ iff }\; \bot = a_x(t) = (a_x \circ u)(t) \;\text{ iff }\; u(t) \leq x\,.$$

Thus $u(t) = t$, for each $t \in L$, and we conclude again that u is the identity of L. □

THEOREM 2.7. *The Girard quantale $([L,L]^t_\vee, \circ, (-)^\perp)$ is unital if and only if L is a completely distributive lattice.*

PROOF. If L is a completely distributive lattice, then $[L,L]^t_\vee = [L,L]_\vee$ and $[L,L]_\vee$ is a Girard quantale [**KP08**, **EGGHK18**, **San20b**]. Conversely, If $[L,L]^t_\vee$ has a unit, then by the previous proposition this unit is the identity. Thus $\mathtt{id}_L \in [L,L]^t_\vee$ and, by Raney's theorem, L is a completely distributive lattice. □

Finally we summarize some equivalent condition for a lattice L to be completely distributive.

RECAPITULATION 2.1. *Let L be a complete lattice. The following statements are equivalent:*

(1) The lattice L is completely distributive;
(2) The equality $[L,L]^t_\vee = [L,L]$ holds [**Ran60**]*;*
(3) L is a nuclear object of SLATT [**HR89**]*;*
(4) There is a unique sup-preserving map $0 : L \to L$ such that $([L,L], \circ, 0)$ is a unital Girard quantale [**KP08**, **San20a**, **San20b**]*;*
(5) The Girard quantale $([L,L]^t_\vee, \circ, (-)^\perp)$ has a unit [**dLS22**].

2.6. No unital extensions

Let L be a complete lattice which is not completely distributive. In view of Theorem 2.7, implying that the Girard quantale $[L, L]^t_\vee$ has no unit, it might be asked whether it is possible to embed $[L, L]^t_\vee$ into a unital Girard quantale. Without further constraints, this is always possible, by firstly adding a unit, see e.g. [**KP08**, Lemma 1.1.11], and then by embedding the resulting quantale into its Chu construction. Such an embedding, however, does not preserve the duality.

We defined unitless Frobenius quantales as certain semigroups coming with a notion of duality. Given this choice of elementary operations, a more appropriate demand is to find an embedding that preserve sups, the binary multiplication, and the dualities. Given that

$$x \backslash y = ({}^\perp y * x)^\perp \,, \qquad x / y = {}^\perp(y * x^\perp) \,, \qquad \bigwedge_{i\in I} x_i = (\bigvee_{i\in I} {}^\perp x_i)^\perp$$

such an embedding necessarily preserves the implications and the infima.

Definition 2.9. We say that a quantale embedding $\iota : (Q_0, *_0) \longrightarrow (Q_1, *_1)$ is *strongly continuous* if it preserves infima and the two implications.

Theorem 2.8 proves that such an embedding into a unital quantale never exists, neither for $[L, L]^t_\vee$, nor for any Frobenius quantale with no unit.

Theorem 2.8. *If a Frobenius quantale has a strongly continuous embedding into a unital Frobenius quantale, then it has a unit.*

Proof. Let $(Q, *, {}^\perp(-), (-)^\perp)$ be a Frobenius quantale and define

$$u := \bigwedge \{\, x \backslash x \mid x \in Q \,\} \,.$$

Then, for each $x \in Q$, $u \leq x \backslash x$ and so $x * u \leq x$. Also, $u \leq {}^\perp x \backslash {}^\perp x = x / x$, so $u * x \leq x$.

Let us suppose next that this quantale has a strongly continuous embedding into a unital quantale $(Q', 1, *)$. Let us regard Q as a subset of Q' closed under suprema, infima, the binary multiplication, and the two implications. Then, for each $x \in Q$, $1 \leq x \backslash x$ and so $1 \leq u$. It follows that, for each $x \in Q$, $x = x * 1 \leq x * u$, thus $x = x * u$. Similarly, $x \leq u * x$ and $x = u * x$. □

Notice in the previous proof that we might have $1 < u$. That is, strongly continuous embeddings need not to preserve units. Examples of strongly continuous embeddings among unital Girard quantales arise from chain refinements. Let C_0, C_1 be two complete chains with $C_0 \subseteq C_1$ and C_0 being closed under meets and suprema of C_1. If C_1 is finite, C_0 being closed under meets and suprema simply means that the endpoints of C_1 belongs to C_0. Under these assumptions, there is a strongly continuous embedding of $[C_0, C_0]_\vee$ into $[C_1, C_1]_\vee$ which does not preserve the unit, while it preserves composition and the duality, see [**GS21**, Proposition 81].

CHAPTER 3

Frobenius structures in $*$-autonomous categories

The main objective of this chapter is to provide a generalization of Theorem 2.1 in any $*$-autonomous categories. We have already mentioned in Example 1.27 that completely distributive lattices are exactly the nuclear objects of SLatt—see [**HR89**], and Remarks 2.5 and 2.6. Moreover, it has been shown in [**Egg10**] that Frobenius quantales are exactly the Frobenius monoids in SLatt. The definition of *Frobenius structures* we introduce follows closely the one of unitless Frobenius quantales—see Definition 2.5. It is actually slightly different from the one given in [**Egg10**], notably, we do not impose the existence of a unit. The generalization we are going to explore may be formulated by the following conjecture:

Conjecture 3.1. Let A be an object of $*$-autonomous category. The following are equivalent:

- A is nuclear.
- The object of endomorphisms of A is a Frobenius structure.

This conjecture as stated is actually false. However, it is a nice source of inspiration, and we will provide the technical hypothesis needed for the equivalence to hold.

The first four section of this section provide the ingredients needed to define Frobenius structures in symmetric monoidal category, some basic properties, examples, and a comparison with other similar stuctures already defined in the literature. The rest of this chapter is the result of our inquiry about the conjecture.

Most of this chapter follows [**dLS23**]. It has been extended with several properties, intuitions, and examples with the idea to make it a journal paper in the future. Notably, the following items did not appear in the conference paper:

- An informal circuit calculus suitable for Frobenius structures;
- Several examples of dual pairings and Frobenius structures;

- The relation of dual pairings with linear adjoints and with Dagger categories;

Moreover, Sections 3.4 and 3.7 are completely new.

3.1. Dual pairings

3.1.1. Definition and first properties. In a ∗-autonomous category with a dualizing object 0, each object A comes with a canonical dual $A^* := [A, 0]$. It is useful to study the duals of an object up to isomorphism. For instance, in the ∗-autonomous category SLatt, one usually prefers to look at the opposite lattice L^{op} instead of L^* since $(-)^{\mathrm{op}}$ is naturally isomorphic to $(-)^*$—see Lemma 1.3 for details. This indicates a general definition of the dual of an object in any monoidal category $\mathcal{V}$.

Definition 3.1. A map $\epsilon : A \otimes B \longrightarrow 0$ in $\mathcal{V}$ is said to be a *dual pairing* (w.r.t. the object 0) if the induced natural transformations

$$\begin{array}{ll} {}^\flat(-) : \hom(X, A) \to \hom(X \otimes B, 0)\,, & (-)^\flat : \hom(Y, B) \to \hom(A \otimes Y, 0)\,. \\ \qquad f \mapsto \epsilon \circ (f \otimes B) & \qquad\qquad g \mapsto \epsilon \circ (A \otimes g) \end{array}$$

are isomorphisms with inverses ${}^\sharp(-)$ and $(-)^\sharp$ respectively.

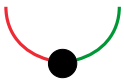
Notation 3.1. We will always name the left object of a dual pair A and the right B. We encountered this notation in [**Mel13**].

We now introduce a representation of dual pairing using the circuits calculus developed in [**BCST96**, **CS99**]. We will not provide a formal calculus but a tool we used as an heuristic in order to prove our results.

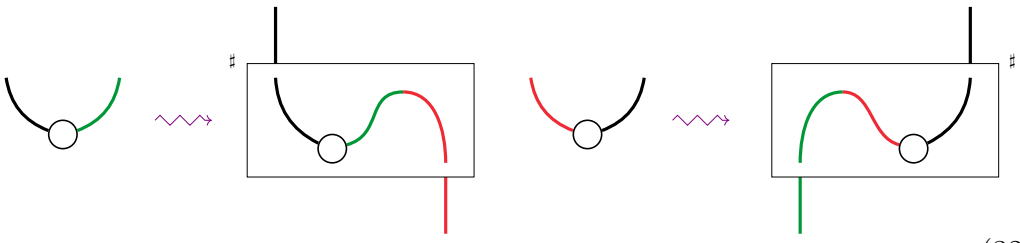
Notation 3.2. To express a dual pairing using circuits we will keep the following convention:

- Red wires ——— will denote the object on the left of the pairing (usually called A);
- Green wires ——— will denote the object on the right of the pairing (usually called B);
- Black wires ——— will denote an arbitrary object X;
- Arrows with domain I are not represented.
- Arrows with codomain 0 are not represented.
- Circuits with one input and one output will be reprensented by a node □
- Circuits with two inputs and one output to 0—*i.e.*, bilinear forms—will be reprensented by a node ◯

We extend the circuit calculus with a new node, representing a dual pairing $\epsilon : A \otimes B \longrightarrow 0$:

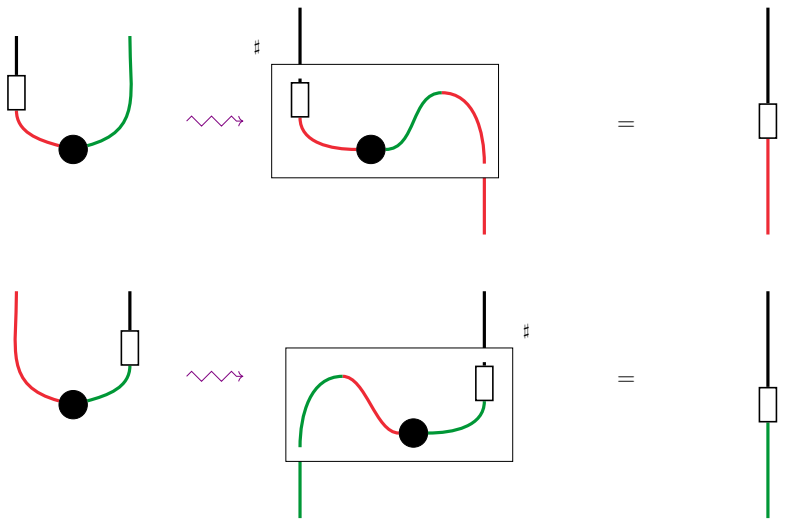


which comes with two circuit constructions:



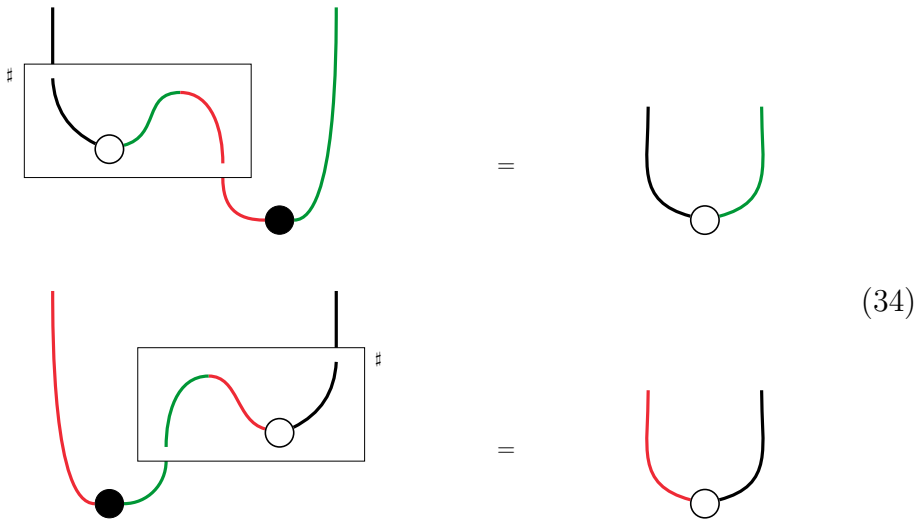
(32)

The inside of the box is purely formal and is not part of the calculus. By introducing this notation we have two objectives. First we want to underline that the circuit produced by the box depends on the circuit which takes two entry and return 0. Secondly, this construction must respect certain equalities that recall the ones for adjoints—see for instance [**BCST96**, Section 5]. These constructions are related to the dual pairing node with the following equalities:



(33)

and:



(34)

EXAMPLE 3.1. Let L be a complete lattice. The map $\epsilon : L \otimes L^{\mathrm{op}} \to \mathcal{Z}^{\mathrm{op}}$ defined by (the linear expansion of)

$$\epsilon(x,y) = \begin{cases} \top \text{ if } x \leq y \\ \bot \text{ otherwise,} \end{cases}$$

is a dual pairing—we already mentioned this map in Example 1.23. Let $\phi : X \otimes L^{\mathrm{op}} \to \mathcal{Z}^{\mathrm{op}}$ and $\psi : L \otimes Y \to \mathcal{Z}^{\mathrm{op}}$ be sup-preserving maps. It is direct to verify that ${}^{\sharp}(-)$ and $(-)^{\sharp}$ defined by

$${}^{\sharp}\phi(x) = \bigwedge \{z \in M \mid \phi(x,z) = \top\},$$

and

$$\psi^{\sharp}(y) = \bigvee \{z \in M \mid \psi(z,y) = \top\},$$

are the inverses of ${}^{\flat}(-)$ and $(-)^{\flat}$ respectively. Note that those maps are given by the following natural isomorphisms:

$$\frac{\dfrac{X \otimes L^{\mathrm{op}} \to \mathcal{Z}^{\mathrm{op}}}{X \to [L^{\mathrm{op}}, \mathcal{Z}^{\mathrm{op}}]}}{X \to L^{\mathrm{opop}} = L} \qquad \frac{\dfrac{L \otimes Y \to \mathcal{Z}^{\mathrm{op}}}{Y \to [L, \mathcal{Z}^{\mathrm{op}}]}}{Y \to L^{\mathrm{op}}}.$$

We have used the canonical isomorphism $L^* := [L, \mathcal{Z}^{\mathrm{op}}] \cong L^{\mathrm{op}}$ which sends a $f : L \to \mathcal{Z}^{\mathrm{op}}$ to $\rho(f)(\top)$—see Lemma 1.3 for the general case of Q-modules. More generaly, the maps $\langle -,- \rangle : M \otimes M^{\mathrm{op}} \to Q^{\mathrm{op}}$ defined for Q-modules in Example 1.23 are dual pairings in MOD_Q. We shall see that if 0 is a dualizing object, then (A,B) is a dual pair if and only if we have $B \cong A^*$. ◊

In SLatt, the right adjoint of a sup-preserving map $f : L \to M$ is typed $\rho(f) : M^{\mathrm{op}} \to L^{\mathrm{op}}$. Let us recall that it exists, it is unique, and it is defined by the following equivalence for each $x \in L$ and $y \in M$,

$$f(x) \leq y \text{ iff } x \leq \rho(f)(y)\,.$$

This equivalence may be expressed diagrammatically by use of dual pairings:

$$\begin{array}{ccc} L \otimes M^{\mathrm{op}} & \xrightarrow{f \otimes M^{\mathrm{op}}} & M \otimes M^{\mathrm{op}} \\ {\scriptstyle L\otimes\rho(f)}\downarrow & & \downarrow{\scriptstyle \epsilon_M} \\ L \otimes L^{\mathrm{op}} & \xrightarrow[\epsilon_L]{} & 0 := \mathscr{Q}^{\mathrm{op}}\,. \end{array}$$

This suggests that a similar notion can be defined for dual pairings in arbitrary monoidal categories.

Definition 3.2. Let (A, B, ϵ) and (A', B', ϵ') be two dual pairings. For a morphism $f : A \longrightarrow A'$, we denote by $f^\dagger : B' \longrightarrow B$ the arrow given by the two transposes below:

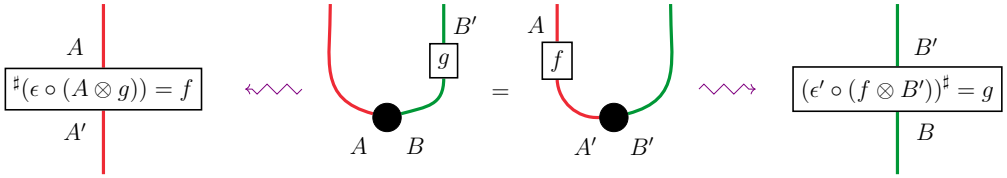
$$\dfrac{\dfrac{A \longrightarrow A'}{A \otimes B' \longrightarrow 0}}{B' \longrightarrow B\,.}$$

We say that $g : B' \longrightarrow B$ is *adjoint* to $f : A \longrightarrow A'$ if $f^\dagger = g$.

This means that $f^\dagger$ is the only map making the diagram

$$\begin{array}{ccc} A \otimes B' & \xrightarrow{f \otimes B'} & A' \otimes B' \\ {\scriptstyle A\otimes f^\dagger}\downarrow & & \downarrow{\scriptstyle \epsilon'} \\ A \otimes B & \xrightarrow[\epsilon]{} & 0\,. \end{array}$$

commute. This can also be expressed with circuits: g is the adjoint of f if and only if the following equalities hold:



Note that we have used the same notation ● for two different pairings. We will write down the types of the wires when needed.

Notation 3.3. It is not easy to find the right notation for the adjoint of an arrow $f : A \to A'$. In the case of SLatt, we have denoted the usual right adjoint of $f : L \to M$ by $\rho(f) : M^{\mathrm{op}} \to L^{\mathrm{op}}$. It is given by the functorial action of the functor $(-)^{\mathrm{op}} : \mathrm{SLatt} \to \mathrm{SLatt}^{\mathrm{op}}$, so we could have written it f^{op}. However, there is also a functor $(-)^{\mathrm{op}} : \mathrm{Pos} \to \mathrm{Pos}$ which acts on monotone

maps by $f^{\mathrm{op}} := f$, hence the notation f^{op} is ambiguous. We also want to keep the notation f^* for the functorial action of $[-,0]$, and the notation $f^\perp$ will appear for the mate of an arrow $X \to Y^*$. Moreover, adjoints depend on the dual pairing ϵ... We have adopted the notation $f^\dagger$ which recall the one for dagger categories. We shall see in the sequel of this section some relations between dagger categories and dual pairings. This notation is not completely pleasing since we will mainly be working on ∗-autonomous categories, not on dagger categories.

A dual pairing (A, B, ϵ) is a special object of the category $\textsc{Chu}_{\mathcal{V},0}$— introduced in [**Bar79**, Appendix]. An object in $\textsc{Chu}_{\mathcal{V},0}$ is a tuple (X_0, X_1, p_X) such that $p_X : X_0 \otimes X_1 \to 0$ is an arrow in $\mathcal{V}$. A $\textsc{Chu}_{\mathcal{V},0}$-morphism between two $\textsc{Chu}_{\mathcal{V},0}$-objects (X_0, X_1, p_X) and (Y_0, Y_1, p_Y) of $\textsc{Chu}_{\mathcal{V},0}$ is a pair of $\mathcal{V}$-morphisms $(f : X_0 \to Y_0, g : Y_1 \to X_1)$ such that the square below commutes.

$$\begin{array}{ccc} X_0 \otimes Y_1 & \xrightarrow{f \otimes Y_1} & Y_0 \otimes Y_1 \\ {\scriptstyle X_0 \otimes g}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X_0 \otimes X_1 & \xrightarrow[p_X]{} & 0\,. \end{array}$$

If (A, B, ϵ) and (A', B', ϵ') are dual pairings, then a $\textsc{Chu}_{\mathcal{V},0}$ morphism between them is necessarily of the form $(f, f^\dagger)$ and, also, a pair $(f, f^\dagger)$ is obviously a morphism in $\textsc{Chu}_{\mathcal{V},0}$. Denoting by $\textsc{Pair}_{\mathcal{V},0}$ the full subcategory of $\textsc{Chu}_{\mathcal{V},0}$ whose objects are dual pairings, these remarks amount to the following statement, whose most relevant consequences is presented immediately after.

Lemma 3.1. *The obvious projection functors, $\mathcal{P}_0 : \textsc{Pair}_{\mathcal{V},0} \longrightarrow \mathcal{V}$ and $\mathcal{P}_1 : \textsc{Pair}_{\mathcal{V},0} \longrightarrow \mathcal{V}^{\mathrm{op}}$, are full and faithful.*

Proposition 3.1. *Let (A_i, B_i, ϵ_i), $i = 0, 1, 2$, be dual pairings. We have*

$$\mathtt{id}_{A_i}{}^\dagger = \mathtt{id}_{B_i}\,,$$

and for every $f : A_0 \to A_1$ and $g : A_1 \longrightarrow A_2$, we have

$$(f \circ g)^\dagger = g^\dagger \circ f^\dagger\,.$$

In particular, if $f : A_0 \to A_1$ is inverted by $g : A_1 \to A_0$, then $f^\dagger$ is inverted by $g^\dagger$.

A dual pairing $\epsilon : A \otimes B \longrightarrow 0$ is unique up to unique isomorphism, as stated next.

Proposition 3.2. *If $(A, B_1, \epsilon_{A,B_1})$ is a dual pairing and $\psi : B_0 \longrightarrow B_1$ is an isomorphism, then $(A, B_0, \epsilon_{A,B_1} \circ (A \otimes \psi))$ is a dual pairing. Conversely, if $(A, B_0, \epsilon_{A,B_0})$ and $(A, B_1, \epsilon_{A,B_1})$ are two dual pairings, then there exists a unique isomorphism $\psi : B_0 \longrightarrow B_1$ such that $\epsilon_{A,B_0} = \epsilon_{A,B_1} \circ (A \otimes \psi)$.*

PROOF. We have to check that $\epsilon_{A,B_0} := \epsilon_{A,B_1} \circ (A \otimes \psi)$ is a dual pairing. Indeed, we have two natural isomorphisms

$$\hom(X, A) \xrightarrow{\flat(-)} (X \otimes B_1, 0) \xrightarrow{\hom(X\otimes\psi,0)} \hom(X \otimes B_0, 0)\,,$$
$$\hom(Y, B_0) \xrightarrow{\hom(Y,\psi)} \hom(Y, B_1) \xrightarrow{(-)^\flat} \hom(A \otimes Y, 0)\,,$$

sending both $\mathtt{id}_A$ and, respectively, $\mathtt{id}_{B_0}$ to ϵ_{A,B_0}.

Conversely, if $(A, B_0, \epsilon_{A,B_0})$ and $(A, B_1, \epsilon_{A,B_1})$ are two dual pairings, we have by definition two natural isomorphisms

$$\hom(Y, B_0) \xrightarrow{(-)^\flat} \hom(A \otimes Y, 0) \xrightarrow{(-)^\sharp} \hom(Y, B_1)\,,$$

whose composition, using the Yoneda Lemma, is induced by post-composition with an invertible map $\psi : B_0 \longrightarrow B_1$. Taking the image of $\mathtt{id}_{B_0}$, we see that ${\epsilon_{A,B_0}}^\sharp = \psi$, that is, we have $\epsilon_{A,B_0} = \epsilon_{A,B_1} \circ (A \otimes \psi)$. □

Obviously, similar statements also hold with respect to the object on the left, for example: if $(A_0, B, \epsilon_{A_0,B})$ and $(A_1, B, \epsilon_{A_1,B})$ are two dual pairings, then there exists a unique iso $\chi : A_0 \longrightarrow A_1$ such that $\epsilon_{A_0,B} = \epsilon_{A_1,B} \circ (\chi \otimes B)$.

EXAMPLE 3.2. Every Hilbert space H leads a dual pairing $(H, \overline{H}, \langle -, - \rangle)$ where $\overline{H}$ is the complex conjugate of H and $\langle -, - \rangle$ is—the abusive notation for—the linear extension of the inner product. Before proving this statement, recall that $\overline{H}$ has the same element as H—we will denote them by $\overline{x}$ for practical reasons, but its external multiplication $\star$ is defined by $\lambda \star \overline{x} = \overline{\lambda} \cdot x$, and its inner poduct is given by $\langle \overline{x}, \overline{y} \rangle_{\overline{H}} = \langle y, x \rangle_H$. Note that a linear map $f : K \to H$ induces a linear map $\overline{f} : \overline{K} \to \overline{H}$—defined by $\overline{f}(\overline{x}) = f(x)$ for all $\overline{x} \in \overline{H}$. Also an antilinear map $f : K \to H$ is equivalently a linear map $f : \overline{K} \to H$. Therefore, a sesquilinear form on H—*i.e.*, linear on the first argument and antilinear on the second—$\langle -, - \rangle : H \times H \to \mathcal{V}$, is a bilinear form $\langle -, - \rangle : H \times \overline{H} \to \mathcal{V}$. The category of Hilbert spaces and bounded linear maps between them is denoted by HILB. It is a symmetric monoidal category, and the construction of the tensor is described in detail in [**KR97**, section 2.6].

In order to show that $(H, \overline{H})$ is a dual pair in HILB, we will use the following lemma:

LEMMA 3.2. *Let H and K be Hilbert spaces and let $\phi : H \otimes \overline{K} \to \mathbb{C}$ be a linear continuous map. There exist two linear continuous maps $f : H \to K$*

and $g : \overline{K} \to \overline{H}$ *such that the diagram*

$$\begin{array}{ccc} H \otimes \overline{K} & \xrightarrow{f \otimes \overline{K}} & K \otimes \overline{K} \\ {\scriptstyle H \otimes g}\downarrow & \swarrow^{\phi} & \downarrow{\scriptstyle \langle -,- \rangle_K} \\ H \otimes \overline{H} & \xrightarrow[\langle -,- \rangle_H]{} & \mathbb{C} \end{array} \tag{35}$$

commutes.

Proof. By Riez theorem—see for instance [**KR97**, Theorem 2.3.1], there exists $\alpha \in H \otimes \overline{K}$ such that $\phi(z) = \langle z, \alpha \rangle_{H \otimes \overline{K}}$ for all $z \in H \otimes \overline{K}$. Let z be an elementary tensor of the form $x \otimes \overline{y}$. We shall reason by case, first if α is an elementary tensor, then if it is an element of the algebraic tensor product $H \otimes_0 \overline{K}$ and finally if α is in the completion of $H \otimes_0 \overline{K}$. Assume that $\alpha = a \otimes \overline{b}$ for some $a \in H$ and $\overline{b} \in \overline{K}$. By the definition of the inner product of tensors, we have:

$$\phi(x \otimes y) = \langle x \otimes \overline{y}, a \otimes \overline{b} \rangle_{H \otimes \overline{K}} = \langle x, a \rangle_H \langle \overline{y}, \overline{b} \rangle_{\overline{K}} .$$

From this we can deduce

$$\phi(x \otimes y) = \langle \overline{y}, \overline{\langle x, a \rangle}_H \star \overline{b} \rangle_{\overline{K}} = \langle \langle x, a \rangle_H \cdot b, y \rangle_K , \tag{36}$$

and

$$\phi(x \otimes y) = \langle x, \overline{\langle \overline{y}, \overline{b} \rangle_{\overline{K}}} \cdot a \rangle_H = \langle x, \overline{\langle b, y \rangle_K} \cdot a \rangle_H . \tag{37}$$

where we have used the sesquilinearity axiom. That is, we have

$$f(x) := \langle x, a \rangle_H \cdot b , \qquad g(\overline{y}) := \overline{\langle b, y \rangle_K} \cdot a .$$

Now, assume that α is an element of $H \otimes_0 \overline{K}$, that is, $\alpha = \sum_{i=1}^n a_i \otimes \overline{b}_i$. Since the inner product preserves the addition in both variables, we have

$$\phi(x \otimes y) = \langle x \otimes \overline{y}, \sum_{i=1}^n a_i \otimes \overline{b}_i \rangle_{H \otimes \overline{K}} = \sum_{i=1}^n \langle x, a_i \rangle_H \langle \overline{y}, \overline{b}_i \rangle_{\overline{K}} .$$

From this, we deduce using Eqs. (36) and (37), that

$$\phi(x \otimes y) = \sum_{i=1}^n \langle \langle x, a_i \rangle_H \cdot b_i, y \rangle_K = \langle \sum_{i=1}^n \langle x, a_i \rangle_H \cdot b_i, y \rangle_K , \tag{38}$$

and

$$\phi(x \otimes y) = \sum_{i=1}^n \langle x, \overline{\langle b_i, y \rangle_K} \cdot a_i \rangle_H = \langle x, \sum_{i=1}^n \overline{\langle b_i, y \rangle_K} \cdot a_i \rangle_H . \tag{39}$$

That is, we have

$$f(x) := \sum_{i=1}^{n} \langle x, a_i \rangle_H \cdot b_i \,, \qquad\qquad g(\overline{y}) := \sum_{i=1}^{n} \overline{\langle b_i, y \rangle_K} \cdot a_i \,.$$

Finally, assume that α is an element of the completion of $H \otimes_0 \overline{K}$—*i.e.*, an element of the tensor of Hilbert spaces; that is, α is the limit of a Cauchy sequence $(\alpha_m)_{m\in\mathbb{N}}$ with $\alpha_m = \sum_{i=1}^{n_m} a_{mi} \otimes \overline{b_{mi}} \in H \otimes_0 \overline{K}$ where $a_{mi} \in H$, and $\overline{b_{mi}} \in \overline{K}$ for each $i = 1, \cdots, n_m$. We have by continuity of the inner product:

$$\phi(x \otimes y) = \langle x \otimes \overline{y}, \lim_{m\to\infty} (\alpha_m)_{m\in\mathbb{N}} \rangle_{H\otimes\overline{K}} = \lim_{m\to\infty} \langle x \otimes \overline{y}, \sum_{i=1}^{n_m} a_{mi} \otimes \overline{b_{mi}} \rangle_{H\otimes\overline{K}} \,.$$

From this, we deduce, using Eqs. (38) and (39), that

$$\phi(x \otimes y) = \lim_{m\to\infty} \langle \sum_{i=1}^{n_m} \langle x, a_{mi} \rangle_H \cdot b_{mi}, y \rangle_K = \langle \lim_{m\to\infty} \sum_{i=1}^{n_m} \langle x, a_{mi} \rangle_H \cdot b_{mi}, y \rangle_K \,, \tag{40}$$

and

$$\phi(x \otimes y) = \lim_{m\to\infty} \langle x, \sum_{i=1}^{n_m} \overline{\langle b_{mi}, y \rangle_K} \cdot a_{mi} \rangle_H = \langle x, \lim_{m\to\infty} \sum_{i=1}^{n_m} \overline{\langle b_{mi}, y \rangle_K} \cdot a_{mi} \rangle_H \,. \tag{41}$$

Note that we are not working with infinite sums but rather with limits of sequences whose terms are finite sums. Hence, there is no issue of convergence.

We can now define $f : H \to K$ and $g : \overline{K} \to \overline{H}$ from Eqs. (40) and (41):

$$f(x) := \lim_{m\to\infty} \sum_{i=1}^{n_m} \langle x, a_{mi} \rangle_H \cdot b_{mi} \qquad g(\overline{y}) := \lim_{m\to\infty} \sum_{i=1}^{n_m} \overline{\langle b_{mi}, y \rangle_K} \cdot a_{mi} \,. \tag{42}$$

They are obviously linear and continuous, and by Riez theorem, they make Diagram (35) commute. □

Note that in the proof the map g is actually the adjoint of f. We have typed it $g : \overline{H} \to \overline{K}$ but we can equivalently type this map $g : H \to K$ and this is the usual Hermitian adjoint of f. We denote this map $f^\dagger$.

We now deduce that, for every Hilbert space H, $(H, \overline{H}, \langle -, - \rangle_H)$ is a dual pairing. Indeed, as for any Hilbert space X, we have $X = \overline{\overline{X}}$, every bilinear form $\phi : H \otimes X \to \mathbb{C}$ may be written as $\phi : H \otimes \overline{\overline{X}} \to \mathbb{C}$. Now take $K := \overline{X}$ in the previous lemma. It follows that there exists an arrow $f^\dagger : \overline{\overline{X}} \to \overline{H}$—or

equivalently typed $X \to \overline{H}$—, defined in Eq. (42), which makes the triangle commutative:

$$\begin{array}{ccc} H \otimes X & \xrightarrow{H \otimes f^\dagger} & H \otimes \overline{H} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \langle -,- \rangle_H} \\ & & \mathbb{C}\,. \end{array}$$

That is, ${}^\sharp\phi = f^\dagger$ and ${}^{\flat\sharp}\phi = \phi$. Similarly, for $\psi : Y \otimes \overline{H} \to \mathbb{C}$ an arrow in Hilb, $\psi^\sharp$ is given by the arrow f we have constructed above. The unicity of adjoint between Hilbert spaces implies that the operations ${}^\sharp(-)$—respectively $(-)^\sharp$—is indeed the inverse of ${}^\flat(-)$—respectively of $(-)^\flat$.

This ends Example 3.2. ◊

Remark 3.1. Recall that Hilb is a symmetric monoidal category but not a symmetric monoidal closed category. However, every object still has a dual in the sense of dual pairings. That is, dual pairings may be used as a categorical abstraction of Hilb. Usually this is done with dagger categories—see for instance [**HV19**]. The notion of dagger categories is related to dual pairings as we discuss next.

Definition 3.3. A *dagger category* is a pair $(\mathcal{D}, (-)^\dagger)$ where $\mathcal{D}$ is a category, $(-)^\dagger : \mathcal{D}^{\mathrm{op}} \to \mathcal{D}$ is a contravariant functor which is an involution and the identity on objects. That is, we have

$$\begin{aligned} (g \circ f)^\dagger &= f^\dagger \circ g^\dagger \\ f^{\dagger\dagger} &= f \\ A^\dagger &= A \end{aligned}$$

for all $f : A \to B$ and $g : B \to C$. A *monoidal dagger category* is a tuple $(\mathcal{D}, \otimes, (-)^\dagger)$ where $(\mathcal{D}, \otimes)$ is a monoidal category, $(\mathcal{D}, (-)^\dagger)$ is a dagger category and furthermore $(-)^\dagger$ is a strict monoidal functor. That is, we have

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$$

for all $f : A \to B$ and $g : C \to D$.

Proposition 3.3. *A monoidal category $\mathcal{V}$ for which every object is self-dual—i.e., (A, A) is a dual pair for every object A—is a dagger category.*

Proof. We define the functor $(-)^\dagger : \mathcal{V}^{\mathrm{op}} \to \mathcal{V}$ by $A^\dagger = A$ and for $f : A \to B$, $f^\dagger$ is the adjoint to f. Proposition 3.1 notably implies $(-)^\dagger$ is indeed a contravariant functor. It is also direct to see that it is an involution. □

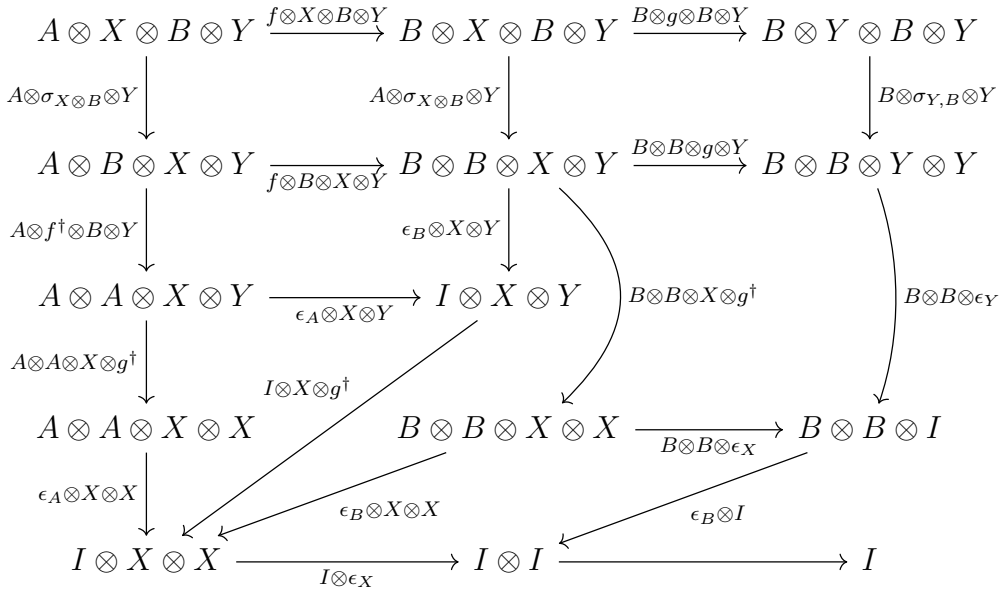
Proposition 3.4. *If, under the condition of Proposition 3.3, $\mathcal{V}$ is furthermore:*

(1) a symmetric monoidal category;
(2) the object 0 *is the unit of the tensor;*
(3) the dual pairing $\epsilon_{A\otimes X} : A \otimes X \otimes A \otimes X \to I$ *is given by*

$$\epsilon_{A\otimes X} := \rho \circ (\epsilon_A \otimes \epsilon_X) \circ (A \otimes \sigma_{A,X} \otimes X)\,,$$

then $\mathcal{V}$ *is a dagger symmetric monoidal category—i.e.,* $(-)^\dagger$ *is a monoidal strict functor.*

PROOF. Assuming the conditions of the proposition, let us show that $(-)^\dagger$ is monoidal strict, *i.e.*, that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for every $f : A \to B$ and $g : X \to Y$. This equality is given by the commutativity of the following diagram.



Indeed, using the naturality of σ, the diagram above gives the following equation:

$$\epsilon_{B\otimes Y} \circ (f \otimes g \otimes \mathtt{id}_B \otimes \mathtt{id}_Y) = \epsilon_{A\otimes X} \circ (\mathtt{id}_A \otimes \mathtt{id}_X \otimes f^\dagger \otimes g^\dagger)\,.$$

That is, by the unicity of the adjoint, we have

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger\,.$$

□

3.1.2. Dual pairing in symmetric monoidal closed categories. The definition of a dual pairing only supposes the ambient category to have a tensor. We shall work later within symmetric and monoidal closed categories. This imposes a few remarks.

LEMMA 3.3. *If $\mathcal{V}$ is symmetric and (A, B, ϵ) is a dual pairing, then so is $(B, A, \epsilon \circ \sigma_{B,A})$.*

PROOF. Clearly we have natural isomorphisms

$$\hom(X, A) \longrightarrow \hom(X \otimes B, 0) \xrightarrow{\hom(\sigma_{B,X},0)} \hom(B \otimes X, 0)\,,$$
$$\hom(Y, B) \longrightarrow \hom(A \otimes Y, 0) \xrightarrow{\hom(\sigma_{Y,A},0)} \hom(Y \otimes A, 0)\,,$$

with universal arrows equal to $\epsilon \circ \sigma_{B,A} : B \otimes A \longrightarrow 0$. □

REMARK 3.2. In a symmetric monoidal category $\mathcal{V}$, there is a monoidal isomorphism $\mathcal{V} \cong \mathcal{V}'$ where $\mathcal{V}'$ has the same objects and arrows as $\mathcal{V}$, where, however, we have $X \otimes' Y = Y \otimes X$. By the previous lemma, if (A, B) is a dual pair in a symmetric monoidal closed category, then any property we state for A holds dualy for B.

Let $\mathcal{V}$ be a monoidal closed category and let 0 be a fixed object of $\mathcal{V}$. Just like in Section 1.1.3, $(-)^*$ denotes the contravariant functor $[-, 0]$. For an arrow $f : X \to Y^*$ in $\mathcal{V}$, we let $f^\perp : Y \to X^*$ be the only map making the following diagram commute:

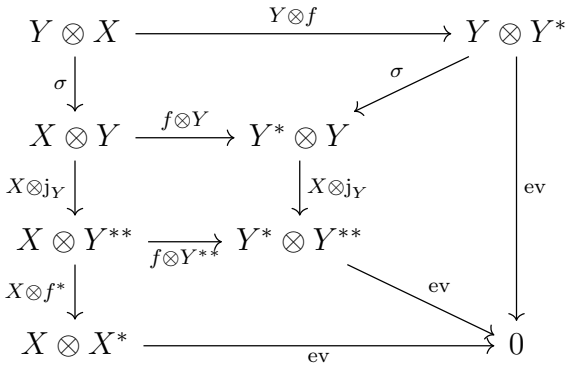
$$\begin{array}{ccccc} Y \otimes X & & \xrightarrow{\quad Y \otimes f \quad} & & Y \otimes Y^* \\ {\scriptstyle\sigma}\downarrow & & & & \downarrow{\scriptstyle \mathrm{ev}} \\ X \otimes Y & \xrightarrow[X \otimes f^\perp]{} & X \otimes X^* & \xrightarrow[\mathrm{ev}]{} & 0\,. \end{array}$$

Notice that $(-)^\perp : \hom(X, Y^*) \to \hom(Y, X^*)$ is natural in X and Y. If $g = f^\perp$, then we say that f and g are *mates*. Mates are different from adjoints as defined above: the evaluation arrow $\mathrm{ev}_{X,0}$ is not necessarily a dual pairing. In fact, as we shall see, it is a dual pairing if and only if X is a reflexive object—see Definition 1.7.

First, let us remark that the canonical arrow $\mathrm{j}_X : X \to X^{**}$—defined by Diagram (8)—is the mate of $\mathtt{id}_{X^*}$, from which the following statement easily follows:

LEMMA 3.4. *For $f : X \to Y^*$, $f^\perp = f^* \circ \mathrm{j}_Y$.*

PROOF. We have the following commutative diagram.

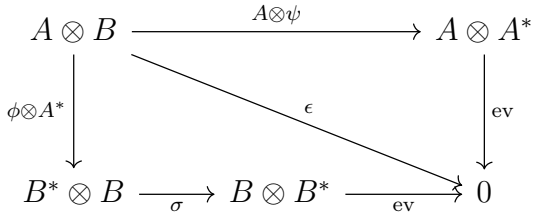


Which implies by the unicity of the transpose that $f^\perp = f^* \circ \mathrm{j}_Y$. □

Mates allow us to precisely characterize dual pairs in symmetric monoidal closed categories, as follows.

PROPOSITION 3.5. *If $\mathcal{V}$ is a symmetric monoidal closed category, then*

(1) (A, B) is a dual pair if and only if there are invertible mates $\phi : A \to B^*$ and $\psi : B \to A^*$ such that the two triangles commute



(43)

(2) If (A, B) is a dual pair, then (A^*, B^*) is also a dual pair.
(3) If (A, B) is a dual pair, then both A and B are reflexive objects of $\mathcal{V}$.
(4) If A is reflexive, then $(A, A^*, \mathrm{ev}_{A,0})$ is a dual pairing.
(5) (A, B) is a dual pair if and only if A is reflexive and there is some iso $\psi : B \to A^*$.

PROOF. 1. If (A, B) is a dual pair, then we have natural isomorphisms

$$\hom(X, B) \longrightarrow \hom(A \otimes X, 0) \longrightarrow \hom(X, A^*)\,,$$

whose composition is induced, by the Yoneda Lemma, by post-composition with an invertible map $\psi : B \longrightarrow A^*$. Taking the image of $\mathtt{id}_B$, we see that $(\epsilon)^\sharp = \psi$. Similalry, the chain of natural isomorphisms

$$\hom(X, A) \longrightarrow \hom(X \otimes B, 0) \longrightarrow \hom(B \otimes X, 0) \longrightarrow \hom(X, B^*)$$

yields an invertible map $\phi : A \longrightarrow B^*$ satisfying $(\epsilon \circ \sigma_{B,A})^\sharp = \phi$. We have therefore $\phi = \psi^\perp$.

Conversely, given those isomorphisms ϕ and ψ, we obtain the two natural isomorphisms as follows:

$$\hom(X, A) \longrightarrow \hom(X, B^*) \longrightarrow \hom(B \otimes X, 0) \longrightarrow \hom(X \otimes B, 0)\,,$$
$$\hom(Y, B) \longrightarrow \hom(Y, A^*) \longrightarrow \hom(A \otimes Y, 0)\,.$$

The two universal arrows are, respectively, $\psi^\flat$ and $\phi^\flat \circ \sigma_{A,B}$ which are equal since we suppose $\phi = \psi^\perp$.

2. If (A, B) is dual, then (B, A) is dual, by Lemma 3.3. By Proposition 3.2, dual pairs are closed under isos. Since $B \simeq A^*$ and $A \simeq B^*$, the pair (A^*, B^*) is dual.

3. Let $\phi : A \to B^*$ and $\psi : B \to A^*$ be the two isomorphisms stated in 1. Using Lemma 3.4, we obtain $\phi = \psi^\perp = \psi^* \circ \mathrm{j}_A$. Since ψ is invertible, ψ^* is invertible by functoriality, therefore j_A is also an isomorphism and A is a reflexive object. By Lemma 3.3, B is reflexive as well.

4. By definition, j_A is the transpose of $\mathrm{ev}_{A,0} \circ \sigma_{A^*,A}$, that is, $\mathrm{j}_A = \mathtt{id}_{A^*}{}^\perp$. Therefore, if j_A is invertible then $(A, A^*, ev_{A,0})$ is a dual pair.

5. The direct implication is induced By 1 and 3. If A is reflexive, then (A, A^*) is a dual pair and if $\psi : B \to A^*$ is invertible then by lemma 3.2, (A, B) is also a dual pair. □

Those properties easily provides many examples.

Example 3.3. Proposition 3.5.5 shows that in a ∗-autonomous category $\mathcal{V}$, where all the objects are reflexive, any isomorphism $B \longrightarrow A^*$ is enough to build a dual pair (A, B). This is a key observation for the coherence theorem in [**CHS06**]. We have already mentioned in Example 3.1 that, in SLatt, (L, L^{op}) is a dual pairing. This is actually immediate from the fact that $L^{\mathrm{op}} \simeq L^*$—see Example 1.30. ◊

Example 3.4. In a ∗-autonomous category where 0 is a dualizing object, we have $I^* \cong 0$, meaning $(I, 0)$ is a dual pair with pairing $\lambda : I \otimes 0 \to 0$. ◊

Example 3.5. For any object A of a ∗-autonomous category $\mathcal{V}$, the pair $(A^* \otimes A, [A, A])$ is a dual pair with pairing given by

$$\epsilon := \mathrm{ev}_{A,0} \circ \sigma_{A^*,A} \circ (A^* \otimes \mathrm{ev}_{A,A})\,.$$

Indeed, every object of $\mathcal{V}$ is reflexive, so $(A^* \otimes A, (A^* \otimes A)^*, \mathrm{ev}_{A^* \otimes A,0})$ is a dual pairing by Proposition 3.5.4. It is well known that in a $*$-autonomous category the transpose of $\mathrm{ev}_{A,0} \circ \sigma_{A^*,A} \circ (A^* \otimes \mathrm{ev}_{A,A})$ is an isomorphism $\psi : [A, A] \longrightarrow (A^* \otimes A)^*$. Then, by Proposition 3.2, $\mathrm{ev}_{A^* \otimes A,0} \circ (A^* \otimes A \otimes \psi) = \mathrm{ev}_{A,0} \circ \sigma_{A^*,A} \circ (A^* \otimes \mathrm{ev}_{A,A})$ is a dual pairing.

The dual pair $(A^* \otimes A, [A, A])$ is important in our work since $A^* \otimes A$ is the domain of the arrow `mix`, and $[A, A]$ its codomain—whenever `mix` is defined, see Definition 1.6. ◊

Example 3.6. If $\epsilon : A \otimes B \to 0$ is a dual pairing in a $*$-autonomous category, then $(A \otimes B, B \oplus A)$ is a dual pair. Indeed, since $A^* \cong B$ and $B^* \cong A$, we have $A \otimes B \cong A^* \otimes A$, and $B \oplus A = [B^*, A] \cong [A, A]$. Therefore, by Proposition 3.2 and Example 3.5, $(A \otimes B, B \oplus A)$ forms a dual pair. ◊

There is a dual notion of a dual pairing. The formal definition is the following.

Definition 3.4. A *codual pairing* in a monoidal category $(\mathcal{V}, \otimes)$ is a dual-pairing in the category $(\mathcal{V}^{\mathrm{op}}, \otimes)$.

The construction rules are, of course, duals to the ones in 32—meaning they are simply upside-down. This notion becomes more interesting when two monoidales tensors occur in the ambient category (like in a linearly distributive category).

Definition 3.5. Let $\mathcal{V}$ be a category equipped with two monoidal tensors $\otimes$ and $\oplus$ whose unit are respectively I and 0. A *dual-codual pairing* is a tuple (A, B, ϵ, η) where

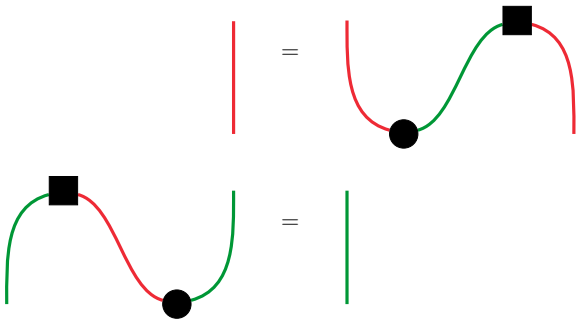
- A, B are $\mathcal{V}$-objects;
- $\epsilon : A \otimes B \to 0$ makes (A, B, ϵ) a dual pairing with respect to 0;
- $\eta : I \to B \oplus A$ makes (B, A, η) a codual pairing with respect to I.

Dual-codual pairings are closely related to what have been called negations in [**CS97a**], complemented objects in [**CS99**] or linear adjoint in [**CKS00, Egg10**]. We will use either the expression complemented objects or linear adjoint. We have already mentioned linearly distributive categories and linear adjoints at the very end of Section 1.4. Linear adjoints are tuples (A, B, ϵ, η) in a linearly distributive category $(\mathcal{V}, \otimes, I, \oplus, 0)$ with $\epsilon : A \otimes B \longrightarrow 0$ and $\eta : I \longrightarrow B \oplus A$, such that diagram Eq. (14) commutes—where B replaces

all the occurences of $A^\perp$. This can be expressed by two nodes:



such that the two following equations hold:

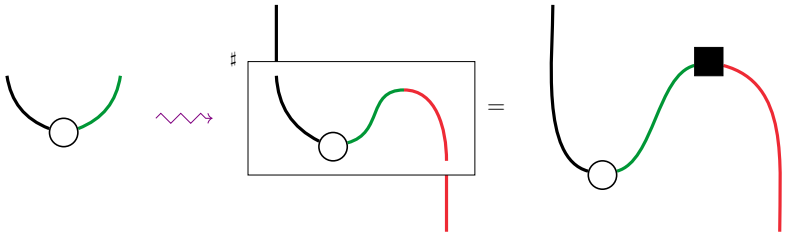


(44)

REMARK 3.3. A monoidal category is a special case of a linearly distributive category where the two monoidal structures are the same. A complemented object (A, B, ϵ, η) in a monoidal category is just an adjunction—where one regards a monoidal category as a 2-category with one object, see [**Mac78**, XII,4].

LEMMA 3.5. *If (A, B, ϵ, η) is a complemented object in a linearly distributive category, then (A, B, ϵ) is a dual pairing with respect to 0 and (A, B, η) is a codual pairing with respect to I.*

PROOF. We define the following construction rule:



It is immediate from the definition of complemented object that this construction indeed leeds to a natural isomorphism $\hom(X \otimes B, 0) \cong \hom(X, A)$. The other three natural isomorphisms needed to conclude the proof are constructed similarly. □

With linear adjoints, the box is realised with an explicit construction. Our formal notation of boxe actually comes from linear adjoints.

PROPOSITION 3.6. *In a ∗-autonomous category, every dual pair leads to a linear adjunction.*

PROOF. If (A, B) is a dual pair, then B and A^* are isomorphic. By [**CS97a**, Theorem 4.5] (A, A^*) are part of a linear adjunction. Moreover, linear adjunction are closed under isomorphisms. Therefore, (A, B) is also part of a linear adjunction. □

REMARK 3.4. It is not true that a dual-codual pairing always yields a linear adjunction. Infinite-dimensional Hilbert spaces provide a counter-example. Consider the category HILB as a linearly distributive category for which the two monoidal structures are exactly the same. We have shown in Example 3.2 that for every Hilbert spaces H, $(H, \overline{H}, \langle -, - \rangle)$ is a dual pair. Since HILB is a dagger category, $\langle -, - \rangle^\dagger : \mathbb{C} \to H \otimes \overline{H}$ is a codual pairing, hence $(H, \overline{H}, \langle -, - \rangle, \langle -, - \rangle^\dagger)$ is a dual-codual pairing where the two monoidal structures are the same. Now if H had a linear adjoint, then it would be an adjoint as defined in Remark 1.5 since we are considering the trivial linearly distributive structure on HILB. However every hilbert space with an adjoint is of finite-dimension. A similar argument is given in [**AH12**, Section 3.1] to show that HILB is not a closed.

Finally, let us mention that we shall focus on maps $f : A \longrightarrow B$, where (A, B, ϵ) is a dual pairing and, consequently, $(B, A, \epsilon \circ \sigma_{B,A})$ as well. The following statement is then easily verified.

LEMMA 3.6. *For (A, B, ϵ) a dual pairing and $f : A \longrightarrow B$, the transposes of f and $f^\dagger$ differ by a symmetry:*

$$\epsilon \circ (A \otimes f^\dagger) = \epsilon \circ (A \otimes f) \circ \sigma_{A,A} \,.$$

3.2. Action of a semigroup over its dual

Since SLATT is a $*$-autonomous category, a quantale is a dual pair (Q, Q^{op}) with an associative multiplication $- * - : Q \otimes Q \longrightarrow Q$. Moreover, there are two implications $\diagdown$ and $\diagup$ defined by the adjunctions:

$$x * y \leq z \iff y \leq x \diagdown z \iff x \leq z \diagup y \,.$$

These equivalences can be encoded by the two commutative diagrams:

$$\begin{array}{ccc} Q \otimes Q \otimes Q^{\mathrm{op}} & \xrightarrow{Q \otimes (- \diagup -)} & Q \otimes Q^{\mathrm{op}} \\ {\scriptstyle (- * -) \otimes Q^{\mathrm{op}}} \downarrow & & \downarrow {\scriptstyle \epsilon} \\ Q \otimes Q^{\mathrm{op}} & \xrightarrow[\epsilon]{} & 2^{\mathrm{op}} \end{array} \qquad \begin{array}{ccc} Q^{\mathrm{op}} \otimes Q \otimes Q & \xrightarrow{(- \diagdown -) \otimes Q} & Q^{\mathrm{op}} \otimes Q \\ {\scriptstyle Q^{\mathrm{op}} \otimes (- * -)} \downarrow & & \downarrow {\scriptstyle \epsilon \circ \sigma} \\ Q^{\mathrm{op}} \otimes Q & \xrightarrow[\epsilon \circ \sigma]{} & 2^{\mathrm{op}} \end{array}$$

As we mentioned in Example 1.22, the implication $\diagup$ is a left action while $\diagdown$ is a right action.

In this section, we show that two canonical actions arise from a dual pair when one of its object is a semigroup. We refer to Section 1.2 for some definitions, examples and properties regarding semigroups and actions in monoidal categories. From now on, $\mathcal{V}$ will be a symmetric monoidal category. Let (A, μ_A) be a magma, and let $\epsilon : A \otimes B \longrightarrow 0$ be a dual pairing in $\mathcal{V}$. That is, we have another node in our circuit calculus :



By adjunction, we define two morphisms

$$\alpha^{\ell}_A : A \otimes B \longrightarrow B \qquad \text{and} \qquad \alpha^{\rho}_A : B \otimes A \longrightarrow B\,,$$

as the only morphisms making these two diagrams commute:

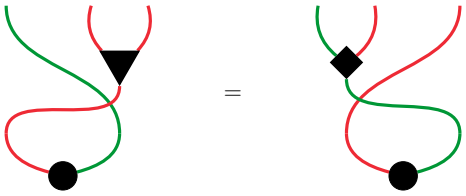
$$\begin{array}{ccc} A \otimes A \otimes B & \xrightarrow{A \otimes \alpha^{\ell}_A} & A \otimes B \\ {\scriptstyle \mathrm{m} \otimes B}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ A \otimes B & \xrightarrow[\epsilon]{} & 0 \end{array} \qquad \begin{array}{ccc} B \otimes A \otimes A & \xrightarrow{\alpha^{\rho}_A \otimes A} & B \otimes A \\ {\scriptstyle B \otimes \mathrm{m}}\downarrow & & \downarrow{\scriptstyle \epsilon \circ \sigma} \\ B \otimes A & \xrightarrow[\epsilon \circ \sigma]{} & 0\,. \end{array} \tag{45}$$

We represent the left and right actions of A over B with two nodes:



They must respect the two equalities:





By the naturality of σ, the previous equality is equivalent to:



EXAMPLE 3.7. For a quantale $(Q, *)$, commutativity of the left (resp. right) diagram in (45) amounts to the relation

$$x * y \leq z \text{ iff } x \leq \alpha^{\ell}_{Q}(y, z) \qquad (\text{resp., } x * y \leq z \text{ iff } y \leq \alpha^{\rho}_{Q}(z, x)) .$$

Therefore, by the uniqueness of the adjoint, we have $\alpha^{\ell}_{Q}(y, z) = z \diagup y$ and $\alpha^{\rho}_{Q}(z, x) = x \diagdown z$. ◊

EXAMPLE 3.8. Let A be a finite-dimensional k-algebra. As $\mathrm{FINVECT}_k$ is a $*$-autonomous category, (A, A^*) is a dual pair and A is a semigroup in $\mathrm{FINVECT}_k$— see Examples 1.16 and 1.29. For $x, y \in A$, and $l \in A^* = [A, k]$, we have

$$\alpha^{\ell}_{A}(y, l) : x \mapsto l(xy) , \qquad \text{and} \qquad \alpha^{\rho}_{A}(l, x) : y \mapsto l(xy) .$$

◊

EXAMPLE 3.9. In REL, for any set X, $(X, X, \mathtt{id}_X)$ is a dual pair—see Example 1.29. If (X, M) is a semigroup—see Example 1.19, the left and the right action are given by

$$\alpha^{\ell} = \{(x, y, z) \mid zxMy\}$$
$$\alpha^{\rho} = \{(x, y, z) \mid yzMx\} .$$

◊

Let us give some elementary properties of α^{ℓ}_{A} and α^{ρ}_{A}.

LEMMA 3.7. *Let (A, B, ϵ) be a dual pairing and (A, μ_A) be a magma.*

(1) The following equation, relating α^{ℓ}_{A} and α^{ρ}_{A}, holds:

$$\epsilon \circ (A \otimes \alpha^{\rho}_{A}) = \epsilon \circ \sigma_{B,A} \circ (\alpha^{\ell}_{A} \otimes A) . \tag{46}$$

(2) The left and right action of the symmetric magma $(A, \mathrm{m}^{\sigma}_{A})$ where $\mathrm{m}^{\sigma}_{A} := \mathrm{m}_A \circ \sigma$ are given by

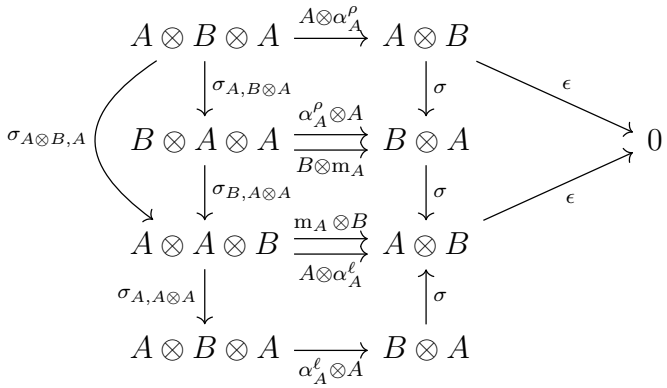
$$\alpha^{\ell}_{A^{\sigma}} = \alpha^{\rho}_{A} \circ \sigma$$
$$\alpha^{\rho}_{A^{\sigma}} = \alpha^{\ell}_{A} \circ \sigma .$$

(3) If (A, m_A) is a semigroup, then the map α^{ℓ}_{A} is a left action, and the map α^{ρ}_{A} is a right action.

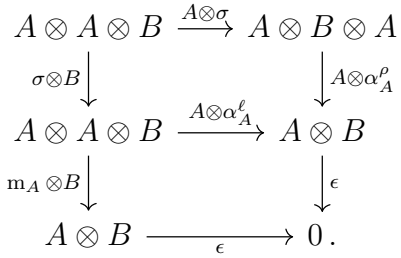
(4) If (A, μ_A) is a semigroup, then the two actions are equivariant to each other, meaning the following diagram commutes:

$$\begin{array}{ccc} A \otimes B \otimes A & \xrightarrow{A \otimes \alpha^{\rho}_{A}} & A \otimes B \\ {\scriptstyle \alpha^{\ell}_{A} \otimes A}\downarrow & & \downarrow{\scriptstyle \alpha^{\ell}_{A}} \\ B \otimes A & \xrightarrow{\alpha^{\rho}_{A}} & B . \end{array}$$

PROOF. 1: The equation is given by the commutativity of the diagram

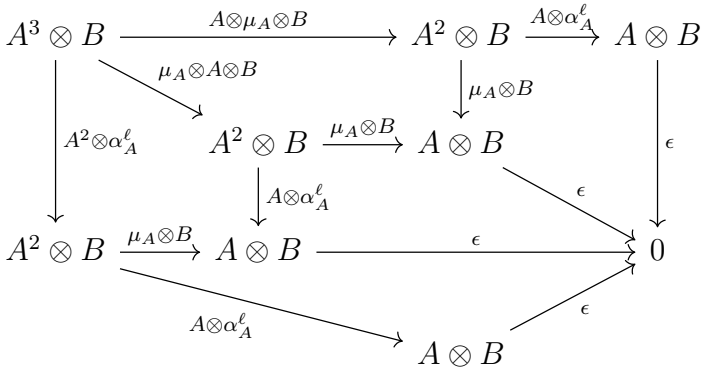


2: we obtain the left action of $(A, \mathrm{m}^{\sigma}_{A})$ by the following diagram which commutes by 1:



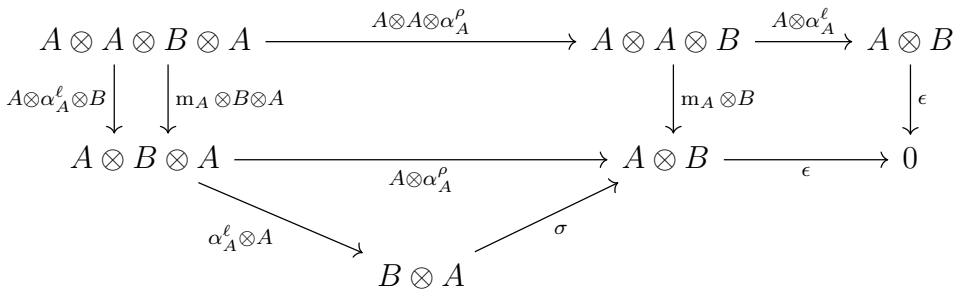
We also deduce the other inequality using Remark 3.2.

3: In the following diagram



the upper leg and lower leg are the transposes of the two parallel maps that are required to be equal for α^{ℓ}_{A} to be a left action. If (A, μ_A) is a semigroup, then the above diagram is commutative, and so these two maps are indeed equal. The second statement follows by Remark 3.2

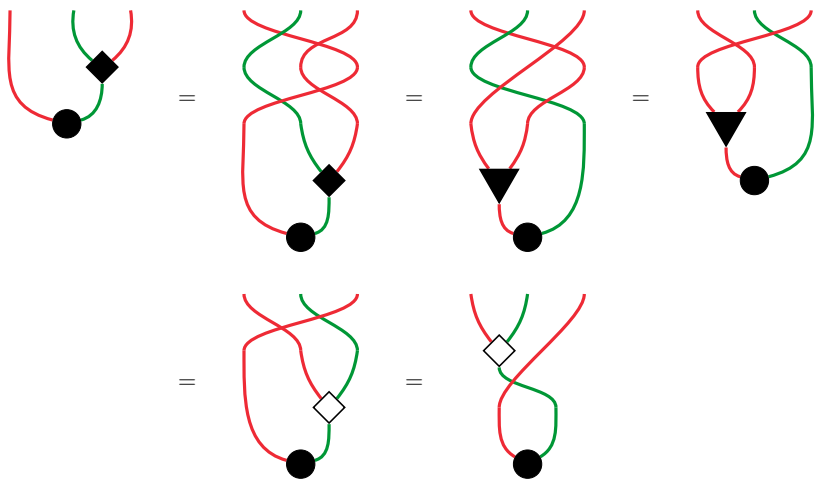
4: By 3, α_A^ρ and α_A^ℓ are two actions, therefore the following diagram commutes.



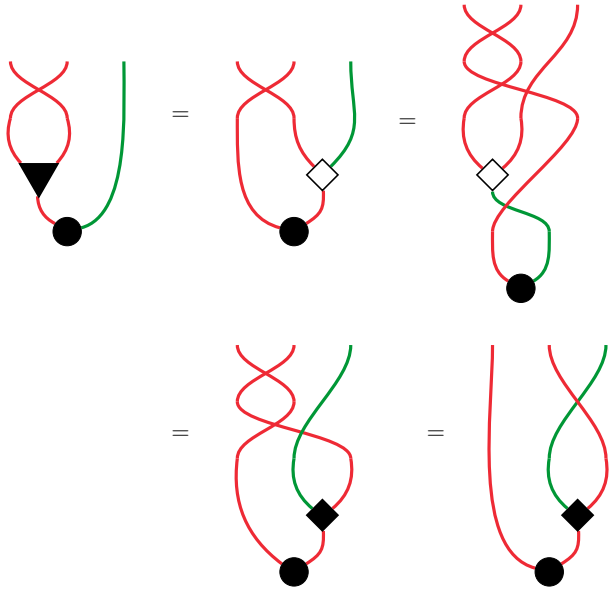
□

We illustrate the use of the circuit calculus we have introduced by giving another proof of Lemma 3.7.

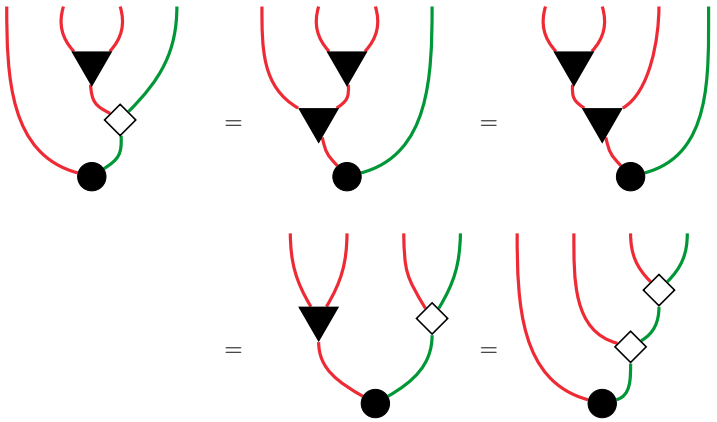
PROOF. For 1, we use the equation defining ◆, some combinatories of σ and the naturality of σ. This leads to the following computation:



To prove Item 2, we compute as follows—using the definition of the right action, the naturality of σ and the previous statement:

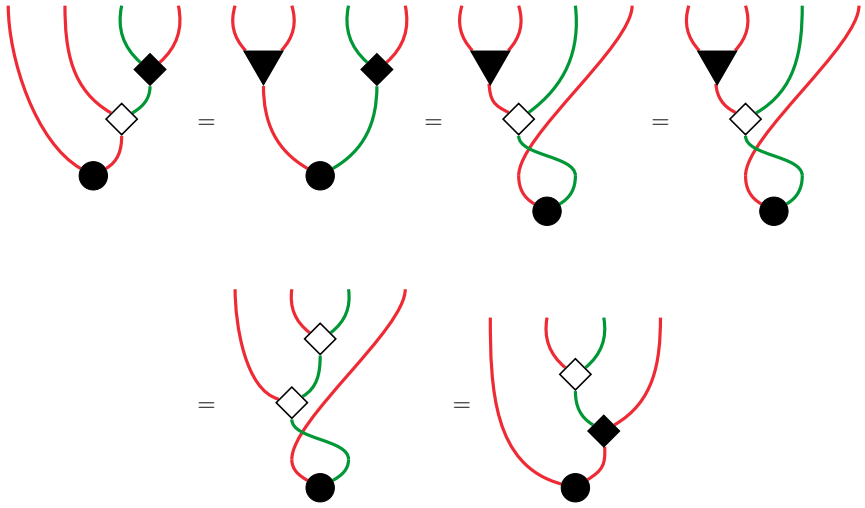


To show Item 3, that α^{ℓ}_{A} is indeed a left action, we compute as follows:



We used the definition of the left action, the associativity of the multiplication and twice again the definition of α^{ℓ}_{A}. By Remark 3.2, the map α^{ρ}_{A} is a right action.

To prove 4, we use the definition of α^{ℓ}_{A}, the first property of this lemma and the fact that α^{ℓ}_{A} is an action. We obtain



□

One of the interest of this proof using circuits is that they visually ressemble to the proofs one makes for complete lattices. For instance, to show that $\forall x, y, z \in Q$, we have

$$z/(y * x) = (z/y)/x$$

we equivalently show that $\forall w, x, y, z \in Q$, we have

$$w \leq z/(y * x) \text{ iff } w \leq (z/y)/x$$

which is exactly what we have done in the general setting for the latter proof of 3.

In view of Section 3.6, we give the action of $[A, A]$ over its dual $A^* \otimes A$.

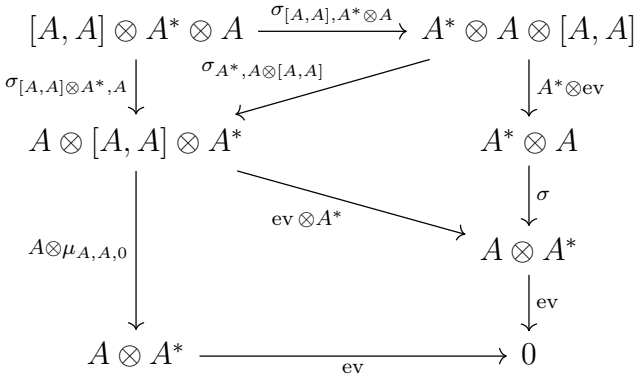
Proposition 3.7. *In a $*$-autonomous category $\mathcal{V}$, we have*

$$\alpha^{\rho}_{[A,A]} = A^* \otimes \mathrm{ev}_{A,A}\,, \qquad \alpha^{\ell}_{[A,A]} = \mu_{A,A,0} \otimes A\,.$$

Proof. First recall from Example 3.5, that the pairing of $(A^* \otimes A, [A, A])$ is given by

$$\epsilon_{A^* \otimes A} := \mathrm{ev}_{A,0} \circ \sigma_{A^*,A} \circ (A^* \otimes \mathrm{ev}_{A,A})\,.$$

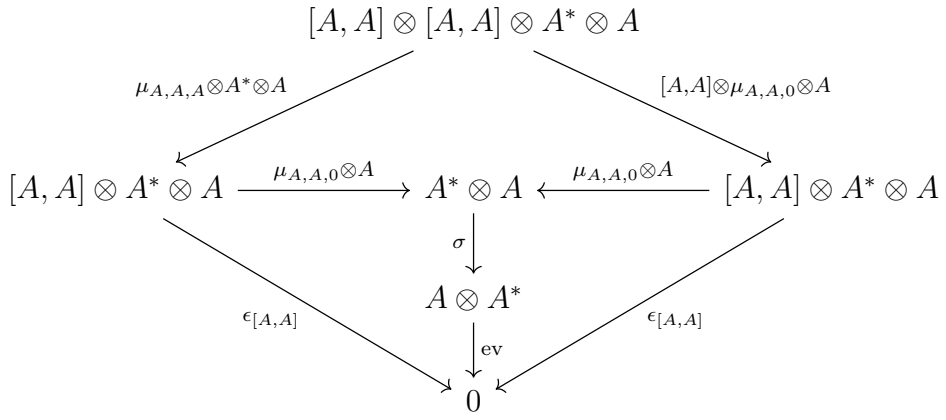
Note that the following diagram commutes:



This implies that we have

$$\epsilon_{[A,A]} = \mathrm{ev}\circ(A\otimes\mu_{A,A,0})\circ\sigma_{[A,A]\otimes A^*,A}\,.$$

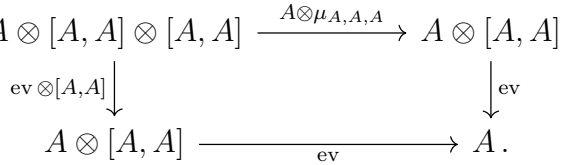
By the usual associativity of the internal composition, the following diagram commutes



Hence we have

$$\alpha^{\ell}_{[A,A]} = \mu_{A,A,0}\otimes A\,.$$

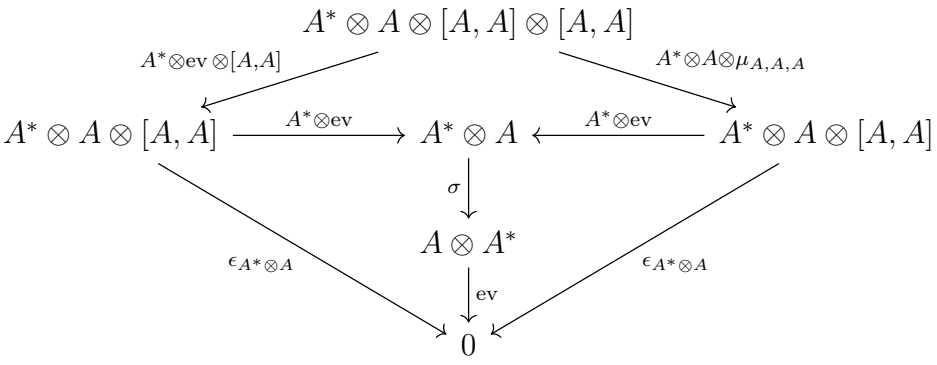
The arrow $\mu_{A,A,A}$ is, by definition, the only arrow that makes the following diagram commute.



By tensoring on the left with A^* and then by postcomposing with

$$\mathrm{tr} := \mathrm{ev}_{A,0}\circ\sigma_{A^*,A}\,,$$

we obtain the commutativity of the diagram



showing that $\alpha^{\rho}_{[A,A]} = A^* \otimes \mathrm{ev}_{A,A}$.

□

3.3. Frobenius structures

We finally define in this section the notion of Frobenius structures in a symmetric monoidal category $\mathcal{V}$. While other approaches are possible, notably [**Str04**, **Egg10**, **AH12**], they are not strictly equivalent, starting from the assumptions on the environment category.

Definition 3.6. A *Frobenius structure* is a tuple $(A, B, \epsilon, \mathrm{m}_A, l, r)$ where (A, B, ϵ) is a dual pairing, (A, m_A) is a semigroup, and l and r are adjoint invertible maps from A to B such that diagram (Serre equality) commutes.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{A\otimes r} & A \otimes B \\ {\scriptstyle l\otimes A}\downarrow & & \downarrow{\scriptstyle \alpha^{\ell}_A} \\ B \otimes A & \xrightarrow[\alpha^{\rho}_A]{} & B. \end{array} \qquad \text{(Serre equality)}$$

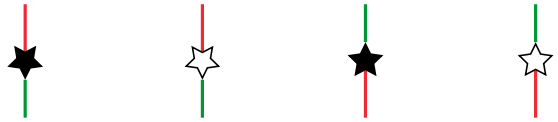
Example 3.10. Definition 3.6 mimics the definition of (unitless) Frobenius quantales given in Definition 2.5. A Frobenius quantale is defined there as a tuple $(Q, *, {}^{\perp}(-), (-)^{\perp})$ such that $(Q, *)$ is a quantale, and where $({}^{\perp}(-), (-)^{\perp})$ form an invertible Galois connection satisfying the equation

$$y \backslash\!\!\!\backslash {}^{\perp} x = y^{\perp} /\!\!\!/ x\,. \tag{47}$$

We have deeply studied this structure in Chapter 2. Recall from Example 3.7 that the right action corresponds to implication $(-\backslash\!\!\!\backslash -)$ while the left action corresponds to $(-/\!\!\!/-)$. The commutative diagram (Serre equality) is therefore a direct translation of Eq. (47)—which we had called "Serre equality"

in Chapter 2. It follows that Frobenius quantales are exactly the Frobenius structures $(A, B, \epsilon, \mathrm{m}, l, r)$ in SLatt for which $B = A^{\mathrm{op}}$. ◊

Before giving other examples of Frobenius structures, we develop our circuit calculus for Frobenius structures. To do so, we add four nodes:



such that

- ★ and ☆ form a Galois connection;
- ★ and ☆ are respectively inverted by ★ and ☆;
- ★ and ☆ must respect the following equality:



(48)

The Galois connection means that the following equality holds:



(49)

Let us give some characterization of these structures. In a quantale, equation (47) is equivalent to the shift/associative relations—see Eq. (Shift relation):

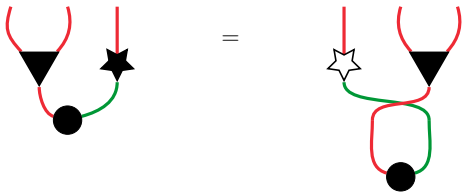
$$x * y \leq {}^{\perp}z \qquad \text{iff} \qquad x^{\perp} \geq y * z\,.$$

This also holds in the general setting, simply by transposing diagram (Serre equality) we have:

Lemma 3.8. *A pair of morphisms (l, r) makes diagram* (Serre equality) *commute if and only if the* (Shift diagram) *below commutes:*

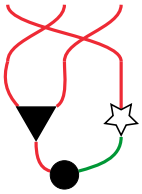
$$\begin{array}{ccccc} A \otimes A \otimes A & & \xrightarrow{\quad \mathrm{m}_A \otimes l \quad} & & A \otimes B \\ \downarrow{\scriptstyle r \otimes \mathrm{m}_A} & & & & \downarrow{\scriptstyle \epsilon} \\ B \otimes A & \xrightarrow{\sigma_{B,A}} & A \otimes B & \xrightarrow{\epsilon} & 0\,. \end{array} \qquad \text{(Shift diagram)}$$

In circuit's terms, the shifting diagram is given by the equation below.



(50)

Of course, by naturality of σ, the right side of the former equality is equal to

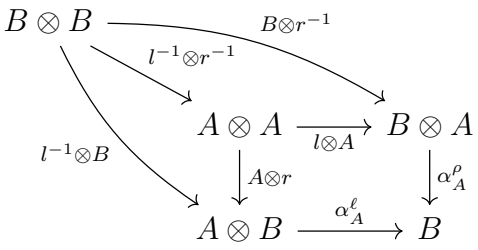


Our next goal is to show that, for a Frobenius structure $(A, B, \epsilon, \mathrm{m}_A, l, r)$, the dual pair (B, A) also carries a Frobenius structure, in particular B carries a canonical semigroup structure.

LEMMA 3.9. *Let (A, B, ϵ) be a dual pair, (A, m_A) a semigroup, and (l, r) invertible adjoint maps from A to B. The tuple $(A, B, \epsilon, \mathrm{m}_A, l, r)$ is a Frobenius structure if and only if diagram (51) commutes.*

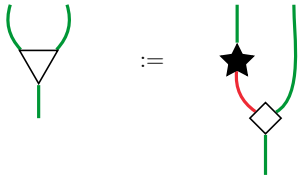
$$\begin{array}{ccc} B \otimes B & \xrightarrow{B \otimes r^{-1}} & B \otimes A \\ {\scriptstyle l^{-1} \otimes B}\downarrow & \overset{\mathrm{m}_B}{\searrow} & \downarrow{\scriptstyle \alpha^{\rho}_A} \\ A \otimes B & \xrightarrow[\alpha^{\ell}_A]{} & B \end{array} \tag{51}$$

PROOF. By Lemma 3.1, l and r are adjoint maps iff l^{-1} and r^{-1} are adjoints. Assuming that diagram (Serre equality) commutes, we prove commutation of diagram (51) with the following commutative diagram:



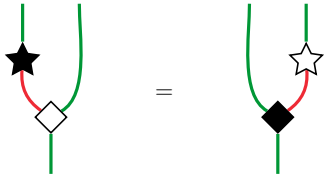
A similar argument shows that commutation of diagram (51) implies that of diagram (Serre equality). □

In circuits, this means that we define a magma structure $\bigtriangledown$ on B from the nodes $\bigstar$ and $\star$ by



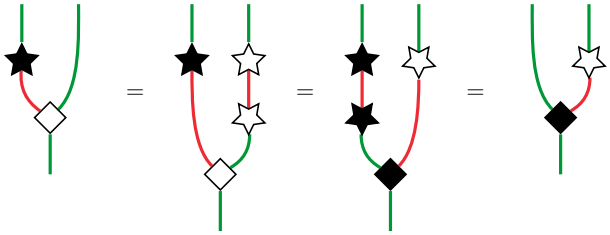
(52)

If Eq. (48) holds, then the following equality holds.



(53)

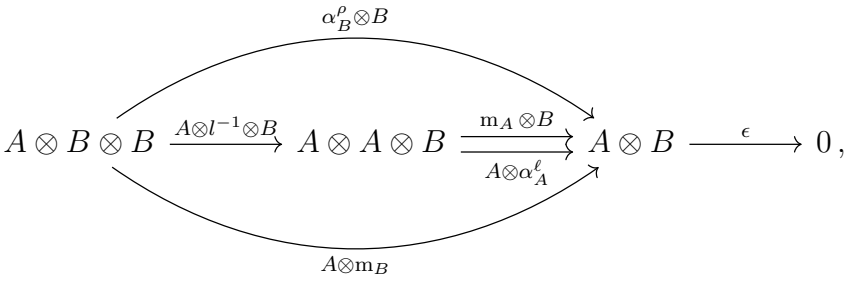
This is given by the computation:



For a Frobenius structure $(A, B, \epsilon, \mathrm{m}_A, l, r)$, let us denote by m_B the diagonal of diagram (51). We next consider the magma (B, m_B). Note that from the definition of m_B and of the actions of a magma over its dual, we obtain two pairs of parallel equal arrows

$$\begin{tikzcd}
B \otimes B \otimes A \arrow[r, "B\otimes r^{-1}\otimes A"] \arrow[rr, bend left, "B\otimes\alpha^{\ell}_B"] \arrow[rr, bend right, "\mathrm{m}_B\otimes A"'] & B \otimes A \otimes A \arrow[r, shift left, "B\otimes \mathrm{m}_A"] \arrow[r, shift right, "\alpha^{\rho}_A\otimes A"'] & B \otimes A \arrow[r, "\sigma_{B,A}"] & A \otimes B \arrow[r, "\epsilon"] & 0\,,
\end{tikzcd}$$

and



showing that the actions of the magma (B, m_B) on the dual A depends on m_A by:

$$\alpha^\ell_B = \mathrm{m}_A \circ (r^{-1} \otimes A)\,, \qquad \alpha^\rho_B = \mathrm{m}_A \circ (A \otimes l^{-1})\,. \tag{54}$$

Moreover, the multiplication m_A is obtained from the action of (B, m_B) over A by:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{A \otimes l} & A \otimes B \\ {\scriptstyle r \otimes A}\downarrow & \overset{\mathrm{m}_A}{\searrow} & \downarrow{\scriptstyle \alpha^\rho_B} \\ B \otimes A & \xrightarrow[\alpha^\ell_B]{} & A\,. \end{array} \tag{55}$$

Note that in a Frobenius quantale, the right negation is inverted by the left negation and we have:

$$\dfrac{\dfrac{y \oplus z \leq_{Q^{\mathrm{op}}} x}{x \leq y \oplus z := {}^\perp y \diagdown z}}{\dfrac{{}^\perp y * x \leq z}{z \leq_{Q^{\mathrm{op}}} {}^\perp y * x}}$$

Thus showing that $\alpha^\ell_{Q^{\mathrm{op}}}(y, x) = {}^\perp y * x$ which is exactly the left equation of (54).

Next, we collect the properties of Frobenius structures that we want to emphasize.

PROPOSITION 3.8. *For a Frobenius structure $(A, B, \epsilon, \mathrm{m}_A, l, r)$, the following statements hold:*

(1) $(A, B, \epsilon, \mathrm{m}_A \circ \sigma_{A,A}, r, l)$ is a Frobenius structure.
(2) The magma (B, m_B) is a semigroup.

(3) The following diagrams commute:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\mathrm{m}_A} & A \\ \downarrow{\scriptstyle A\otimes l} & & \downarrow{\scriptstyle l} \\ A\otimes B & \xrightarrow{\alpha^\ell_A} & B \end{array} \qquad\qquad \begin{array}{ccc} A\otimes A & \xrightarrow{\mathrm{m}_A} & A \\ \downarrow{\scriptstyle r\otimes A} & & \downarrow{\scriptstyle r} \\ A\otimes B & \xrightarrow{\alpha^\rho_A} & B \end{array} \tag{56}$$

(4) The maps l and r are both semigroup homomorphisms from (A, m_A) to (B, m_B).
(5) The tuple $(B, A, \epsilon\circ\sigma_A, \mathrm{m}_B, r^{-1}, l^{-1})$ is a Frobenius structure.

Proof. 1 This immediately follows from Item 2 of Lemma 3.7 where we have shown that the left action of the symmetric magma (A, m^σ_A) is given by the right action of (A, m_A) precomposed by σ.

2. In view of Lemma 3.7, the following diagram commutes:

$$\begin{array}{ccccc} B\otimes B\otimes B & & \xrightarrow{B\otimes(\alpha^\rho_A\circ(B\otimes r^{-1}))} & & B\otimes B \\ & \searrow^{l^{-1}\otimes B\otimes r^{-1}} & & & \downarrow{\scriptstyle l^{-1}\otimes B} \\ \downarrow{\scriptstyle (\alpha^\ell_A\circ(l^{-1}\otimes B))\otimes B} & & A\otimes B\otimes A & \xrightarrow{A\otimes\alpha^\rho_A} & A\otimes B \\ & & \downarrow{\scriptstyle \alpha^\ell_A\otimes A} & & \downarrow{\scriptstyle \alpha^\ell_A} \\ B\otimes B & \xrightarrow{B\otimes l^{-1}} & B\otimes A & \xrightarrow{\alpha^\rho_A} & B \end{array}$$

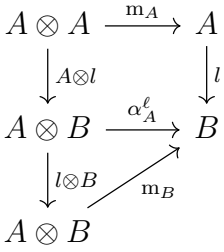
3. Using Lemma 3.8, we remark that the following diagram commutes:

$$\begin{array}{ccccc} A\otimes A\otimes A & \underset{\mathrm{m}_A\otimes A}{\overset{A\otimes\mathrm{m}_A}{\rightrightarrows}} & A\otimes A & \xrightarrow{A\otimes l} & A\otimes B \\ \downarrow{\scriptstyle A\otimes A\otimes l} & & & & \downarrow{\scriptstyle \epsilon} \\ A\otimes A\otimes A & \underset{A\otimes\alpha^\ell_A}{\overset{\mathrm{m}_A\otimes A}{\rightrightarrows}} & A\otimes B & \xrightarrow{\epsilon} & 0 \end{array}$$

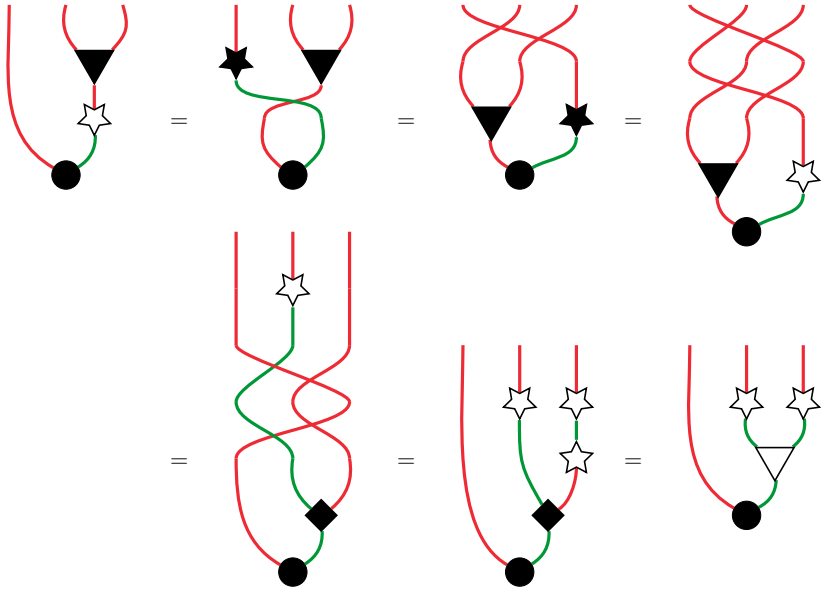
We obtain therefore the equality $\epsilon\circ(A\otimes(\alpha^\ell_A\circ(A\otimes l))) = \epsilon\circ(A\otimes(l\circ\mathrm{m}_A))$, which, by uniqueness, implies $\alpha^\ell_A\circ(A\otimes l) = l\circ\mathrm{m}_A$.

In a similar way we obtain that $\alpha^\ell_{A^\sigma}\circ(A\otimes r) = r\circ\mathrm{m}_{A^\sigma}$. Considering that $\alpha^\ell_{A^\sigma} = \alpha^\rho_A\circ\sigma_{A,B}$, we infer $\alpha^\rho_A\circ(r\otimes A)\circ\sigma_{A,A} = r\circ\mathrm{m}_A{}^\sigma = r\circ\mathrm{m}_A\circ\sigma_{A,A}$, whence $\alpha^\rho_A\circ(r\otimes A) = r\circ\mathrm{m}_A$.

4. Immediate from diagrams (56). For example:



We give the proof for the negation r using the circuit calculus. Recall that r corresponds to the node ☆. We compute as follows.



We first used the fact that the two negation form a Galois connection, then the naturality of σ, then the shifting lemma Eq. (50). We next used the naturality of σ, the definition of the right action, the invertibility of σ and of the right negation, and finally Eq. (53).

5. By Lemma 3.3, $(B, A, \epsilon \circ \sigma_{B,A})$ is a dual pair. By Item 2 of this proposition, (B, m_B) is a semigroup. By Proposition 3.1, (r^{-1}, l^{-1}) are adjoint. By diagram (55) and Lemma 3.9, the tuple $(B, A, \epsilon \circ \sigma_{B,A}, \mathrm{m}_B, r^{-1}, l^{-1})$ is a Frobenius structure. □

Item 5 of Proposition 3.8 exhibits the expected duality. Indeed, using diagram (55), it is easily seen that the dual of $(B, A, \epsilon \circ \sigma_{B,A}, \mathrm{m}_B, r^{-1}, l^{-1})$ is $(A, B, \epsilon, \mathrm{m}_A, l, r)$ itself. Notice that having such duality is not an obvious result, since our definition of Frobenius structure is not self dual. *E.g.*, the

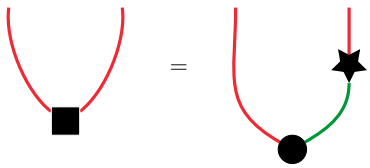
dual B of A is not required to be a semigroup itself and the requirement that diagram (Serre equality) commutes is a condition on the semigroup (A, m_A).

Finally, let us establish the connection with associative bracketed magmas.

PROPOSITION 3.9. *Let $(A, B, \epsilon, \mathrm{m}_A, l, r)$ be a Frobenius structure and define $\pi^l_A := \epsilon \circ (A \otimes l)$ and $\pi^r_A := \epsilon \circ (A \otimes r)$. Then $(A, \mathrm{m}_A, \pi^l_A)$ is an associative bracketed magma, $(A, \mathrm{m}_A, \pi^r_A)$ is a co-associative bracketed magma and π^l_A and π^r_A are dual pairings. Conversely, given an associative bracketed magma (A, m_A, π_A) for which π_A is a dual pairing, and given another dual pairing (A, B, ϵ), define l so that $\epsilon \circ (A \otimes l) = \pi_A$, $r = \widetilde{l}$, then $(A, B, \epsilon, \mathrm{m}_A, l, r)$ is a Frobenius structure.*

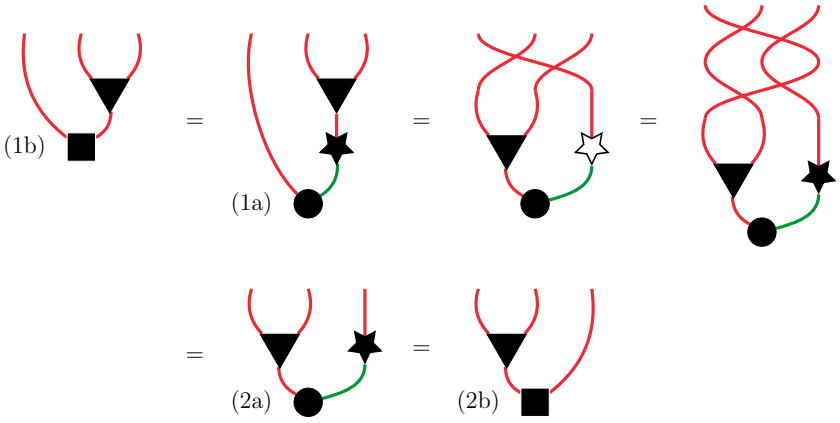
PROOF. We give the proof of this proposition using our (informal)circuit calculus.

We define the bracket π_A with a node ■ by:



(57)

To show it is an associative bracket, we use mainly the Galois connection and the shifting lemma. We compute as follows.



Conversely, if π_A is an associative bracket which is a dual pairing, and (A, B, ϵ) is another pairing, then Eq. (57) is given by Proposition 3.2. Moreover, in our previous computation, the equality between circuit (1a) and circuit (2a) corresponds to the commutativity of the right diagram of Eq. (56)—using the definition of the left action in the circuit (2a). Since Eq. (57) holds, circuit (1a) is equal to circuit (1b), and circuit (2a) is equal to circuit (2b). Furthermore, the associativity of the bracket π_A exactly means that circuits

(1b) and (2b) are equal. Briefly, circuit (1a) is equal to circuit (1b) which is equal to circuit (2b) which is equal to circuit (2a), Therefore, circuit (1a) is equal to circuit (2a). One can prove the left diagram of Eq. (56) with the same argument. The fact that l is the adjoint of r is directly induced by their definition. □

From this correspondence between Frobenius structures and associative bracketed magmas, it is easy to find some examples from the literature.

EXAMPLE 3.11. In $\mathrm{finVect}_k$, Frobenius algebra is a finite-dimensional k-algebra (A, m_A) coming with a nondegenerate form $\langle -, - \rangle$ such that $\langle xy, z\rangle = \langle x, yz\rangle$. That is, it is a bracketed semigroup $(A, \mathrm{m}_A, \langle -, -\rangle)$ and $\langle -, -\rangle$ is a dual pairing as its transpose is an isomorphism between A and A^*. Hence, $(A, A^*, \mathrm{ev}, \mathrm{m}_A, r, l)$ is a Frobenius structure in the category of finite-dimensional k-vector spaces, where l and r are defined as stated in Proposition 3.9. We refer the reader to [**Koc03**] for a recap on Frobenius Algebras and their relation with topological quantum field theory. We are not directly concerned with quantum theory, although we are aware of the importance of Frobenius algebras for this theory—see [**HV19**]. ◊

EXAMPLE 3.12. A Frobenius algebra $(A, \mathrm{m}, \mathrm{u}, \mathrm{d}, \mathrm{tr})$ in a symmetric monoidal category, as defined in [**Koc03**] or in [**Hyl04**], yields an associative bracketed magma $(A, \mathrm{m}, \mathrm{tr} \circ \mathrm{m})$. Moreover, in [**Hyl04**], it is noticed that $(A, A, \mathrm{tr} \circ \mathrm{m}, \mathrm{d} \circ \mathrm{u})$ is an adjunction. This implies, by Lemma 3.5, that $(A, A, \mathrm{tr} \circ \mathrm{m})$ is a dual pairing. Whence, a Frobenius structure on the semigroup (A, m) is obtained as stated in Proposition 3.9. ◊

EXAMPLE 3.13. A Frobenius monoid $(A, \mathrm{m}, \mathrm{u}, \mathrm{d}, \mathrm{tr})$ in a linearly distributive category as defined in [**Egg10**] also yields a Frobenius structure, when the ambient category is actually a $*$-autonomous category. Indeed, one obtains an associative bracketed magma just like in the last two examples. Moreover, Theorem 3.1 of [**Egg10**] ensures that the pairing is the co-unit of a linear adjunction. That is, $(A, A, t \circ \mathrm{m})$ is a dual pairing—see Lemma 3.5—and we obtain a Frobenius structure just like before. In the next section we will show that unital Frobenius structures in a $*$-autonomous category are Frobenius monoid as defined in [**Egg10**]. ◊

EXAMPLE 3.14. A Frobenius structure in the category Rel amounts to a tuple (X, M, l, r) with (X, M) a semigroup—see Example 1.19—and with

l, r inverse bijections on X satisfying $xyMl(z)$ if and only if $yzMr(x)$, for all $x, y, z \in X$. Indeed, using the action computed in Example 3.9, we have

$$\begin{aligned} xy(\alpha^{\ell} \circ A \otimes r)z &\iff xr(y)\alpha^{\ell} z \iff zxMr(y)\,, \\ xy(\alpha^{\rho} \circ l \otimes A)z &\iff l(x)y\alpha^{\rho} z \iff yzMl(x)\,. \end{aligned}$$

A concrete example of a Frobenius structure in Rel is obtained by the following construction. Let $(A, *, 1, (-)^{-1})$ be an abelian group. The tuple $(A, M, (-)^{-1})$ is a Frobenius structure in Rel where M is the multiplication defined in Eq. (10) of Example 1.19. It is clear that $(-)^{-1}$ is a self-inverted function. We only need to show that for all $x, y, z \in A$ we have $xyMz^{-1}$ if and only if $yzMx^{-1}$. This is given by the following chain of equivalences:

$$\begin{aligned} xyMz^{-1} &\iff z^{-1} = xay \\ &\iff 1 = xayz \\ &\iff x^{-1} = ayz = yaz \\ &\iff yzMx^{-1}\,. \end{aligned}$$

◊

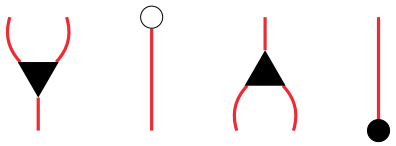
3.4. Comparison with other definitions

We now explore in some depth the relations between our definition of Frobenius Structures in Definition 3.6 and the Frobenius Monoid as defined by Egger in [**Egg10**] and axiom (H) introduced by Abramsky and Heunen in [**AH12**].

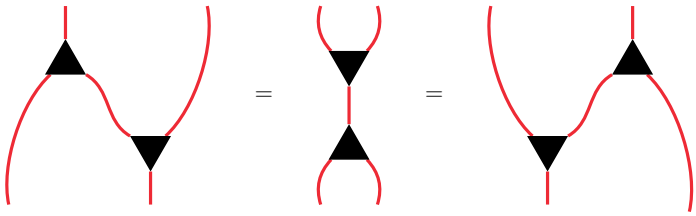
3.4.1. Frobenius Monoid as defined by Egger. Egger in [**Egg10**] gave a definition of a Frobenius monoid in linearly distributive categories. A Frobenius monoid in $(\mathcal{V}, \otimes, I, \oplus, 0)$ is defined there as a tuple $(A, \mathrm{m}, \mathrm{u}, \mathrm{d}, \mathrm{tr})$ where $(A, \mathrm{m}_A, \mathrm{u})$ is a monoid with respect to the monoidal structure $(\mathcal{V}, \otimes, I)$ and $(A, \mathrm{d}, \mathrm{tr})$ is a co-monoid with respect to $(\mathcal{V}, \oplus, 0)$ such that m is co-equivariant and d is equivariant. The last conditions mean that axiom (F)—also called *Frobenius relation*—holds:

$$\begin{array}{ccccc} A \otimes (A \oplus A) & & \xrightarrow{\quad K \quad} & & (A \otimes A) \oplus A \\ {\scriptstyle A\otimes \mathrm{d}}\uparrow & & & & \downarrow{\scriptstyle \mathrm{m}\oplus A} \\ A \otimes A & \xrightarrow{\mathrm{m}} & A & \xrightarrow{\mathrm{d}} & A \oplus A \\ {\scriptstyle \mathrm{d}\otimes A}\downarrow & & & & \uparrow{\scriptstyle A\oplus \mathrm{m}} \\ (A \oplus A) \otimes A & & \xrightarrow[K]{} & & A \oplus (A \otimes A)\,. \end{array} \tag{F}$$

We informaly use the circuit calculus introduced by Cockett to represent a Frobenius Monoid as defined by Egger. We will use the following data:



Note that we are currently working in a linearly distributive category, thus there is an implicit $\otimes$ between the wires in top while there is a implicit $\oplus$ in the bottom of the circuit. The two circuits on the left form a monoid while the two on the right form a co-monoid. The multiplication and the co-multiplication must respect the two equations:



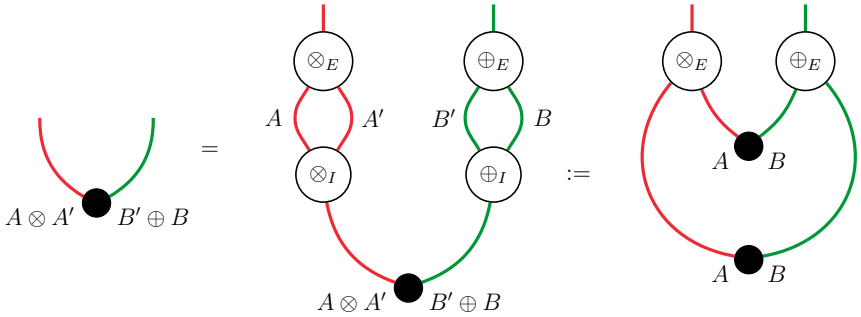
(F)

REMARK 3.5. In the "degenerate" case, where the two monoidal structures coincides—*i.e.*, $\mathcal{V}$ is a usual monoidal category, a Frobenius monoid is just a Frobenius algebra as for instance defined in [**Koc03**].

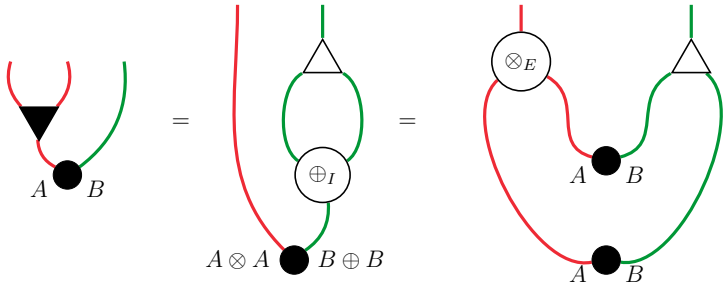
The objective of this subsection is to show that the definition given in Section 3.3 is equivalent to the one given by Egger. We have already noticed in Example 3.13 that a Frobenius monoid leads to a Frobenius with structure. It remains to show that if $(A, B, \epsilon, \mathrm{m}_A, l, r)$ is a Frobenius structure in a $*$-autonomous category $\mathcal{C}$, then A can be endowed with an associative comultiplication which respects axiom (F). To do so, we will mainly reason with circuits, leaving aside the formal proof. We first construct the comultiplication.

In a $*$-autonomous category, every object A has a dual A^*. Moreover, we have a natural isomorphism $(A \otimes A')^* \cong A'^* \oplus A^*$. This implies that, given two dual pairs (A, B) and (A', B'), $(A \otimes A', B' \oplus B)$ is a dual pair. The dual pairing $\epsilon_{A\otimes A'}$ is defined using the pairings ϵ_A and ϵ'_A as follows—for the reader

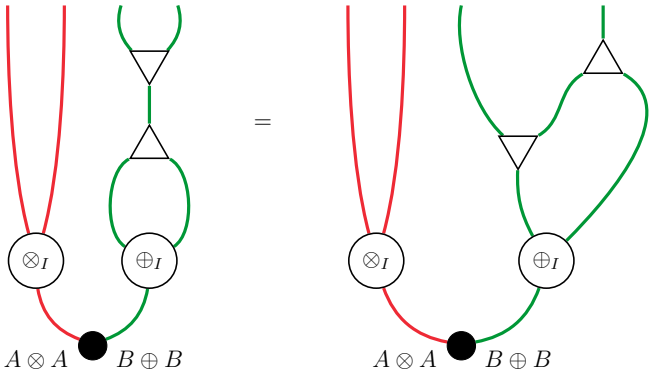
convenience we will keep track of the dual pairings involved:



We now assume that $A' := A$. Since (A, B) and $(A \otimes A, B \oplus B)$ are dual pairs, the adjoint—in the sense of Definition 3.2—of a multiplication $\mathrm{m}_A : A \otimes A \to A$ is a comultiplication $\mathrm{d}_B := \mathrm{m}_A{}^\dagger : B \to B \oplus B$ which we represent with the node $\triangle$. This node is defined by adjunction, meaning it must respect the following equations



Recall that B is itself a semigroup—the multiplication was defined in Eq. (52). We will show that the equation (F) holds for the multiplication m_B and the co-multiplication d_B. By the unicity of the adjoint this is equivalent to show that



To prove this equation we have adapted the proof of [**AH12**, Lemma 6] in the context of a Frobenius structure in a ∗-autonomous category. The computation is given in Fig. 3.1.

For the first line, we used the definition of m_B and of d_B. For the second line, we used the definition of the left action and the associativity of m_A. For

the third line we used the definition of d_B and of the pairing of $A \otimes A$. For the final line, we used the definition of the left action, of m_B and of the pairing of $A \otimes A$.

We have proven this equality for m_B and d_B. We can also construct d_A from m_B. Since a Frobenius structure is autodual, this gives the equality (F) for m_A and d_A.

3.4.2. Axiom H. Another related notion we shall study was developed by Abramsky and Heunen.

Axiom (H) appears in the work [**AH12**]. In this paper, the authors work in monoidal dagger categories—see Definition 3.3. They are mainly interested in the category Hilb of (possibly infinite-dimensional) Hilbert spaces and continuous maps, and in the category Rel. Their objective is to generalize, to the infinite-dimensional case, the result from [**CPV08**], that states a bijection between commutative Frobenius algebras and orthogonal[1] basis in finite-dimensional Hilbert spaces. The problem is that a commutative Frobenius algebra in a monoidal dagger category is unital if and only if it is in the compact-closed core of the monoidal category—[**AH12**, Lemma 3 and Proposition 7]. In particular any unital Frobenius algebra in Hilb is of finite dimension. In order to define a reasonable notion of Frobenius algebra in infinite dimension, they study a relaxed axiomatization, introducing axiom (H). In a monoidal dagger category, they show that this axiom is related to axiom (F) as follows :

- (F) and having a unit implies (H)
- (H) and associativity implies (F).

Their main result is that semigroups which respect axiom (H) correspond to orthogonal basis, thus extending the result of [**CPV08**] to infinite-dimensional Hilbert spaces.

We present axiom (H) as follows. Let (A, m_A) be a semigroup in a monoidal dagger category. By "currying" the multiplication m_A for points, one obtains a map $R_A : \hom(I, A) \to \hom(A, A)$ defined for $a : I \to A$ by $R_A(a) = \mathrm{m}_A \circ (a \otimes \mathtt{id}_A) \circ {\lambda_A}^{-1}$ which corresponds to $\lambda b.ab$. In the non-commutative case, there is another curryfication $L_A : \hom(I, A) \to \hom(A, A)$ defined similarly for $a : I \to A$ by $L_A(a) = \mathrm{m}_A \circ (\mathtt{id}_A \otimes a) \circ {\rho_A}^{-1}$. That is, it corresponds to $\lambda b.ba$.

[1]This theorem is stated in [**CPV08**] and [**AH12**] for orthonormal basis. However, this corresponds to axiom (M), also called "special" in [**HV19**] which does not make sense in general $*$-autonomous categories, and we therefore do not mention it.

We say that a commutative semigroup (A, m_A) respects axiom (H) if there exists and involution $(-)^\perp : \hom(I, A) \to \hom(I, A)$ such that

$$R_A(a^\perp) = R_A(a)^\dagger$$

where $(-)^\dagger$ is the usual operation of Dagger categories. That is, the following diagram must commute:

$$\begin{array}{ccc} \hom(I,A) & \xrightarrow{R_A} & \hom(A,A) \\ {\scriptstyle (-)^\perp}\downarrow & & \downarrow{\scriptstyle (-)^\dagger} \\ \hom(I,A) & \xrightarrow[R_A]{} & \hom(A,A)\,. \end{array} \tag{58}$$

As our intuition has mainly been developed by means of the equational theory of residuated lattices, we express diagram (58) with the operations of this theory. By Example 3.2 and Proposition 3.4, we identify the dagger operation with the adjoint. The adjoint of R_A corresponds to $a \backslash -$ so for all $a, b \in A$, we have

$$a \backslash b = a^\perp * b\,. \tag{59}$$

Note that a unital commutative residuated lattice with an involution respecting Eq. (59) is a compact closed posetal category, that is, every element is invertible, see Example 1.25. If a quantale respects Eq. (59), then $\top = \bot$ and therefore it is the singleton quantale.

To relate axiom (H) with our work, we have to make a few changes. First the non-commutative case is important for us as many quantales are not commutative. Secondly, in a monoidal category, an object A is usually not dual with itself but with another object B, thus, in our case, we must type the adjoint operation $(-)^\dagger : \hom(A, A) \to \hom(B, B)$. We obtain the following definition.

Definition 3.7. Let (A, B, ϵ) be a dual pair, and (A, m_A) and (B, m_B) be semigroups in a symmetric monoidal category $\mathcal{V}$. We say that the tuple $(A, B, \epsilon, \mathrm{m}_A, \mathrm{m}_B)$ *respects axiom (H)* if there are two invertible morphisms $r : A \to B$ and $l : A \to B$ which form a Galois connection, and such that the two commutative diagram below commute.

$$\begin{array}{ccc} \hom(I,A) & \xrightarrow{R_A} & \hom(A,A) \\ {\scriptstyle \hom(I,r)}\downarrow & & \downarrow{\scriptstyle (-)^\dagger} \\ \hom(I,B) & \xrightarrow[R_B]{} & \hom(B,B) \end{array} \tag{60}$$

$$\begin{array}{ccc} \hom(I,A) & \xrightarrow{L_A} & \hom(A,A) \\ {\scriptstyle \hom(I,l)}\downarrow & & \downarrow{\scriptstyle (-)^\dagger} \\ \hom(I,B) & \xrightarrow[L_B]{} & \hom(B,B) \end{array} \tag{61}$$

Remark 3.6. Note that our definition is slightly stronger than the one in [**AH12**]: the existence of $l, r : A \to B$ implies negations typed $\hom(I,A) \to \hom(I,B)$ on the set of points, but the converse is not true. This means our definition is more "internal"; however, diagrams (60) and (61) still use the external hom. The next definition—found in [**AH12**]—relates external and internal structures of a monoidal category.

Definition 3.8. We say that a category is *monoidally well-pointed* if for every $f, g : A \otimes A' \to B \otimes B'$, we have $f = g$ if and only if $f \circ (a \otimes a') = g \circ (a \otimes a')$ for all $a : I \to A$ and $a' : I \to A'$.

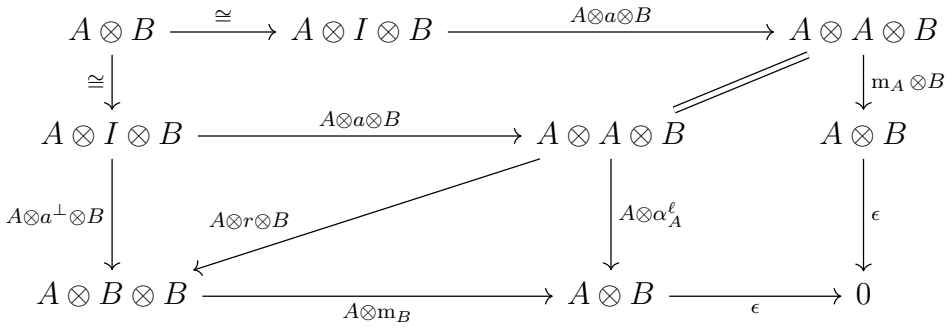
We can now develop the relation between axiom (H) and Frobenius Structures.

Proposition 3.10. *Let $\mathcal{V}$ be a monoidally well-pointed category. If a tuple $(A, B, \epsilon, \mathrm{m}_A, l, r)$ is a Frobenius structure, then $(A, B, \epsilon, \mathrm{m}_A, \mathrm{m}_B)$ respects axiom (H) where m_B is the induced multiplication on B defined in Eq. (52). Conversely, if a tuple $(A, B, \epsilon, \mathrm{m}_A, \mathrm{m}_B)$ respects axiom (H), then there exists l, r making $(A, B, \epsilon, \mathrm{m}_A, l, r)$ a Frobenius structure.*

Proof. Let $(A, B, \epsilon, \mathrm{m}_A, l, r)$ be a Frobenius structure in $\mathcal{V}$. Recall that $\mathrm{m}_B = \alpha^\ell_A \circ (\mathtt{id}_B \otimes r^{-1})$. Let $a : I \to A$; we wright $a^\perp := r \circ a$. Let us show that the two multiplications respect axiom (H), that is $R_A(a)^\dagger = R_B(a^\perp)$ which is equivalent to the commutativity of

$$\begin{array}{ccc} A \otimes B & \xrightarrow{R_A(a) \otimes B} & A \otimes B \\ {\scriptstyle A \otimes R_B(a^\perp)}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ A \otimes B & \xrightarrow[\epsilon]{} & 0 \end{array} \tag{62}$$

Developing this diagram, we obtain:



The pentagon on the right commutes by definition of α^{ρ}_{A} and the square on the bottom left commutes by definition of m_B. Therefore, the whole diagram commutes. One proves commutativity of diagram (61) with a similar argument.

Now suppose that we are given two multiplications m_A and m_B and two isomorphisms $l, r : A \to B$ respecting axiom (H), that is diagram (62) and the one for l commute for all $a : I \to A$. Again, developing (62), and since $\mathcal{V}$ is monoidally well-pointed, the diagram

$$\begin{array}{ccccc} A \otimes A \otimes B & & \xrightarrow{\quad\mathrm{m}_A \otimes B\quad} & & A \otimes B \\ {\scriptstyle A\otimes r\otimes B}\downarrow & & & & \downarrow{\scriptstyle \epsilon} \\ A \otimes B \otimes B & \xrightarrow[A\otimes \mathrm{m}_B]{} & A \otimes B & \xrightarrow[\epsilon]{} & 0 \end{array} \tag{63}$$

commutes. This exactly means that

$$\alpha^{\ell}_{A} = \mathrm{m}_B \circ (r \otimes B) \iff \mathrm{m}_B = \alpha^{\ell}_{A} \circ (r^{-1} \otimes B)\,.$$

With a similar argument, one can infer that

$$\alpha^{\rho}_{A} = \mathrm{m}_B \circ (B \otimes l) \iff \mathrm{m}_B = \alpha^{\rho}_{A} \circ (B \otimes l^{-1})\,.$$

From Lemma 3.9, we know the two previous equalities are sufficient to make $(A, B, \epsilon, \mathrm{m}_A, l, r)$ a Frobenius structure. □

3.5. From nuclear to Frobenius

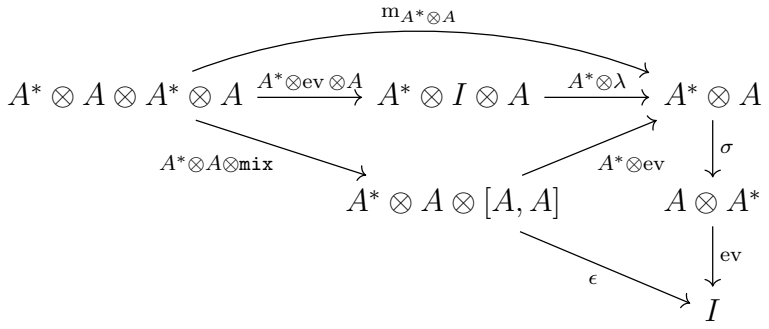
The next two sections present the main results of this chapter in view of Conjecture 3.1. First we show how nuclearity—see Section 1.3—yields Frobenius structures on the objects of the internal hom $[A, A]$. We will work in a ∗-autonomous category $\mathcal{V}$ where the dualizing object 0 is the unit of the tensor I.

THEOREM 3.1. *If A is a nuclear object in a ∗-autonomous category $\mathcal{V}$, then there is a map l such that $([A, A], [A, A]^*, \mathrm{ev}_{[A,A],I}, \circ, l, l)$ is a Frobenius structure.*

PROOF. As mentioned in Example 1.18, the object $A^* \otimes A$ has a canonical semigroup structure given by the multiplication

$$\mathrm{m}_{A^*\otimes A} := (A^* \otimes \lambda_A) \circ (A^* \otimes \mathrm{ev}_{A,I} \otimes A)\,.$$

By the definition of $\mathtt{mix}$, the following diagram commutes



meaning, the equation

$$\epsilon \circ (A^* \otimes A \otimes \mathtt{mix}) = \mathrm{ev}_{A,I} \circ \sigma_{A^*,A} \circ \mathrm{m}_{A^*\otimes A}$$

holds, where ϵ was defined in Example 3.5. This immediately ensures that the map $\epsilon \circ (A^* \otimes A \otimes \mathtt{mix})$, that is, the transpose of $\mathtt{mix}$, is an associative pairing. Moreover, it is also verified that $\epsilon \circ (A^* \otimes A \otimes \mathtt{mix}) \circ \sigma_{A^*\otimes A, A^*\otimes A} = \epsilon \circ (A^* \otimes A \otimes \mathtt{mix})$, whence $\mathtt{mix}$ is self-adjoint.

It follows from Proposition 3.9 that, if $\mathtt{mix}$ is invertible, then the tuple $(A^* \otimes A, [A, A], \epsilon, \mathrm{m}_{A^*\otimes A}, \mathtt{mix}, \mathtt{mix})$ is a Frobenius structure. By Item 5 of Proposition 3.8, the tuple $([A, A], A^* \otimes A, \epsilon \circ \sigma_{[A,A],A^*\otimes A}, \mathrm{m}_{[A,A]}, \mathtt{mix}^{-1}, \mathtt{mix}^{-1})$ is also a Frobenius structure. Since $\mathtt{mix}$ is a semigroup homomorphism from $(A^*\otimes A, \mathrm{m}_{A^*\otimes A})$ to $([A, A], \circ)$—see Fact 1.2, then $\mathrm{m}_{[A,A]}$ is the standard monoid structure $\circ$ induced from composition in $[A, A]$—see Example 1.17. Finally, there is a canonical isomorphism $\psi : A^*\otimes A \cong [A, A]^*$ such that $\epsilon\circ([A, A]\circ\psi) = \mathrm{ev}_{[A,A],I}$, see Example 3.5. This implies that $([A, A], [A, A]^*, \mathrm{ev}_{[A,A],I}, \circ, \psi \circ \mathtt{mix}^{-1}, \psi \circ \mathtt{mix}^{-1})$ is a Frobenius structure. □

Let us notice that, in view of Remark 1.5 identifying nuclear objects and adjoints, the previous statement can be seen as an instance of [**Str04**, Corollary 3.3].

We continue this section by generalizing the theory of Serre Galois connection we have developed in Section 2.3. First, let us recall some facts we have developed in Section 2.5. In Remark 2.5 we have argued that the $\mathtt{mix}$ map in SLatt is actually the Raney's transform $(-)^\vee$. Moreover, the image of $(-)^\vee$, $[L, L]^t_\vee$, always has a canonical structure of (unitless) Girard quantale—see Theorem 2.4 for the exact statement. This notably implies that the subquantale $[L, L]^t_\vee$ of $[L, L]$ has a unit if and only if $[L, L]^t_\vee = [L, L]$ if and only if L is a completely distributive lattice if and only if L is nuclear—see Recapitulation 2.1.

Secondly, as we mentioned in the proof of Theorem 3.1, the tuple $(A^* \otimes A, [A, A], \epsilon, \mathrm{m}_{A^*\otimes A}, \mathtt{mix}, \mathtt{mix})$ is a Frobenius structure if and only if `mix` is invertible—*i.e.*, if A is nuclear. Hence, intuitively, `mix` is a sort of an intuitionistic negation. We abstract this situation and we look at the image of an "intuitionistic negation" $f : A \longrightarrow B$ with (A, B) a dual pair and (A, m_A) a semigroup. This inquiry has given the following result.

THEOREM 3.2. *Let $\mathcal{V}$ be a $*$-autonomous category with an epi-mono factorization system, (A, B, ϵ) be a dual pairing, and (A, m_A) be a semigroup in $\mathcal{V}$. Let $f : A \longrightarrow B$ be a map, put $\psi_A := \epsilon \circ (A \otimes f)$ and suppose that $\psi_A = \psi_A \circ \sigma_{A,A}$. Factor f as $f = m \circ e$ with $e : A \longrightarrow C$ epi and $m : C \longrightarrow B$ mono. If C is a magma with multiplication m_C and e is a magma homomorphism, then there exists a map $g : C \longrightarrow C^*$ making $(C, C^*, \mathrm{ev}_{C,I}, \mathrm{m}_C, g, g)$ into a Frobenius structure.*

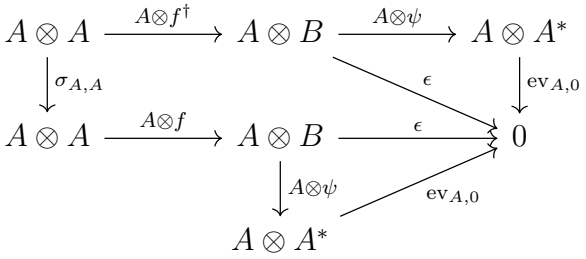
To prove this theorem, we need a lemma relating factorization systems to dual pairs. If $f = m_f \circ e_f$ with e_f epi and m_f mono, then we denote by $\mathsf{Img}(f)$ the codomain of e_f—which is also the domain of m_f.

LEMMA 3.10. *Let $\mathcal{V}$ be $*$-autonomous with an epi-mono factorization system. If (A, B) is a dual pair and $f : A \longrightarrow B$ then $(\mathsf{Img}(f), \mathsf{Img}(f^\dagger))$ is a dual pair.*

PROOF. As from Proposition 3.5.1, let $\phi : A \longrightarrow B^*$ and $\psi : B \longrightarrow A^*$ be isomorphisms such that $\phi = \psi^\perp$. We pretend that, up to these isomorphisms, $f^\dagger$ equals f^*, as stated below.

FACT 3.1. *We have $\psi \circ f^\dagger = f^* \circ \phi$.*

PROOF. By transposing $\psi \circ f^\dagger$, we obtain the upper leg in the following diagram:

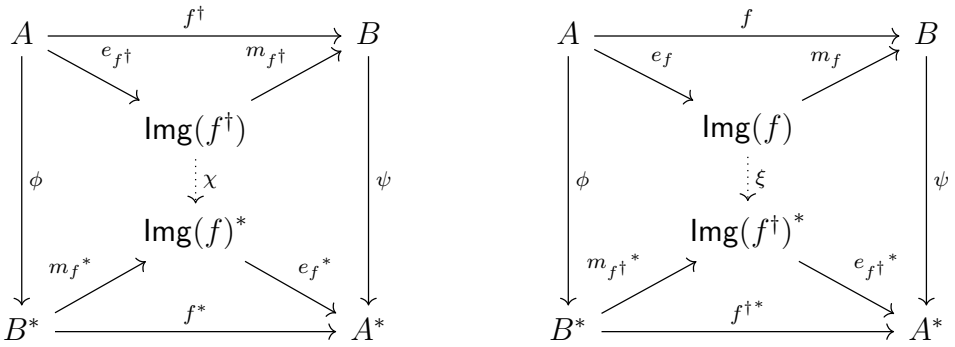


The lower leg of this diagram being the transpose of $(\psi \circ f)^\perp$, we deduce $\psi \circ f^\dagger = (\psi \circ f)^\perp$. By the naturality of the operation $(-)^\perp$, we have

$$(f^* \circ \phi)^\perp = \phi^\perp \circ f = \psi \circ f$$

whence $f^* \circ \phi = (\psi \circ f)^\perp = \psi \circ f^\dagger$. □

Let now (e_f, m_f) and $(e_{f^\dagger}, m_{f^\dagger})$ be the epi-mono factorizations of f and $f^\dagger$, respectively. By applying the duality functor $(-)^*$, $({m_f}^*, {e_f}^*)$ (resp. $({m_{f^\dagger}}^*, {e_{f^\dagger}}^*)$) is an epi-mono factorization of f^* (resp. $f^{\dagger *}$). Therefore, we obtain two different epi-mono factorizations of $\psi \circ f^\dagger = f^* \circ \phi$ (resp. $\psi \circ f = f^{\dagger *} \circ \phi$), as follows:



and therefore isomorphisms $\chi : \mathsf{Img}(f^\dagger) \to \mathsf{Img}(f)^*$ and $\xi : \mathsf{Img}(f) \to \mathsf{Img}(f^\dagger)^*$. In order to conclude that we have a dual pair, we need to argue that $\xi = \chi^\perp$. To this goal, compute as follows:

$$(\chi \circ e_{f^\dagger})^\perp \circ e_f = ({e_f}^* \circ \chi \circ e_{f^\dagger})^\perp = (f^* \circ \phi)^\perp = \phi^\perp \circ f = \psi \circ f = {e_{f^\dagger}}^* \circ \xi \circ e_f \,.$$

From this, considering that e_f is epi, it follows that $(\chi \circ e_{f^\dagger})^\perp = {e_{f^\dagger}}^* \circ \xi$. By applying the operator $(-)^\perp$ to both sides of this equality, we obtain

$$\chi \circ e_{f^\dagger} = ({e_{f^\dagger}}^* \circ \xi)^\perp = \xi^\perp \circ e_{f^\dagger} \,.$$

from which the desired equality $\chi = \xi^\perp$ follows—since $e_{f^\dagger}$ is epi. □

We can now prove Theorem 3.2.

Proof of Theorem 3.2. Let (A, B) be a dual pair and let $f : A \longrightarrow B$ be such that $f^\dagger = f$. Let $e_f : A \longrightarrow C$ and $m_f : C \longrightarrow B$ be a factorization of f. Lemma 3.10 exhibits (C, C) as a dual pair with isomorphism $\chi, \xi : C \longrightarrow C^*$ such that $\chi^\perp = \xi = \chi$. Define therefore the pairing $\pi_C := \mathrm{ev}_{C,I} \circ (C \otimes \chi) : C \otimes C \longrightarrow I$ and observe that it is symmetric. The diagram in the proof of Lemma 3.10 also exhibits the equality

$$\psi \circ f = {e_f}^* \circ \chi \circ e_f : A \longrightarrow A^* \,.$$

Transposing this relation, we obtain the pairing

$$\pi_A = (\psi \circ f)^\flat = ({e_f}^* \circ \chi \circ e_f)^\flat = \chi^\flat \circ (e_f \otimes e_f) = \pi_C \circ (e_f \otimes e_f)\,.$$

Thus, assuming that $f : A \longrightarrow B$ is a semigroup homomorphism factoring as $f = m_f \circ e_f$ with $e_f : (A, \mathrm{m}_A) \longrightarrow (C, \mathrm{m}_C)$ a magma homomorphisms, we have that $e : (A, \mathrm{m}_A, \pi_A) \longrightarrow (C, \mathrm{m}_C, \pi_C)$ is a bracketed magma homomorphism. As argued in Lemma 1.1, (C, m_C, π_C) is an associative bracketed magma, and, as mentioned in Proposition 3.9, this is enough to have a Frobenius structure on the dual pair (C, C^*). □

Just like $[L, L]^t_\vee$ is always a Girard quantale, under some assumption on the ambient category, the image of $\mathtt{mix}$ is endowed with a Frobenius structure.

THEOREM 3.3. *Let $\mathcal{V}$ be $*$-autonomous category such that the semigroup category over $\mathcal{V}$, $\mathrm{SEM}_\mathcal{V}$, has an epi-mono factorization system. There exists a map $g : \mathsf{Img}(\mathtt{mix}) \to \mathsf{Img}(\mathtt{mix})^*$ making $(\mathsf{Img}(\mathtt{mix}), \mathsf{Img}(\mathtt{mix})^*, \mathrm{ev}, g, g)$ a Frobenius structure.*

PROOF. Since $\mathtt{mix}$ is an homomorphism of semigroup—see Fact 1.2, we can apply Theorem 3.2. □

Of course, if $\mathtt{mix}$ is invertible—*i.e.*, A is nuclear, then $\mathsf{Img}(\mathtt{mix}) \cong [A, A]$ and we obtain Theorem 3.1 as corollary of the previous theorem.

3.6. From Frobenius to nuclear

We finally give the converse of Theorem 3.1. However, to do so, we impose additional conditions on the carrier object A of a Frobenius structure on $[A, A]$.

DEFINITION 3.9. We say that an object A of a monoidal category $\mathcal{V}$ is *pseudo-affine* if the tensor unit I embeds into A as a retract. If every object of $\mathcal{V}$ which is not terminal nor initial is pseudo-affine, then $\mathcal{V}$ is said to be *pseudo-affine*.

REMARK 3.7. If A is pseudo-affine, then we can pick a split monomorphism $p_A : I \longrightarrow A$ and a split epimorphism $c_A : A \longrightarrow I$ such that $c_A \circ p_A = \mathtt{id}_I$. We shall assume that any pseudo-affine A comes with a canonical choice of such p and c. Then, for A pseudo-affine, and for each object X we can define

$$\begin{aligned}
p^\rho_X &: X \longrightarrow X \otimes A\,, & p^\rho_X &:= (X \otimes p) \circ \rho_X^{-1}\,,\\
c^\rho_X &: X \otimes A \longrightarrow X & c^\rho_X &:= \rho_X \circ (X \otimes c)\,,\\
p^\ell_X &: X \longrightarrow A \otimes X & p^\ell_X &:= (p \otimes X) \circ \lambda_X^{-1}\,,\\
c^\ell_X &: A \otimes X \longrightarrow X & c^\ell_X &:= \lambda_X \circ (c \otimes X)\,.
\end{aligned}$$

EXAMPLE 3.15. In SLATT, for any lattice L, the map $p_L : 2 \to L$ is given by $p_L(\top) = \top$ and $p_L(\bot) = \bot$ and $c_L : L \to 2$ is defined by:

$$c_L(x) = \begin{cases} \bot \text{ if } x = \bot\,; \\ \top \text{ otherwise.} \end{cases}$$

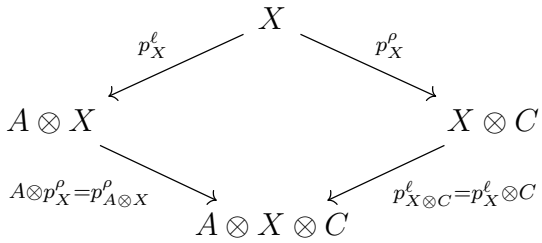
Of course, c_L is not a retract when L is the singleton. That is why we have asked in the definition of a pseudo-affine category—Definition 3.9–that any object which is not initial nor terminal must be pseudo-affine. This makes SLATT a pseudo-affine category. The fact that SLATT is pseudo-affine was actually used in [**San20a**] to prove that if $[L, L]$ is endowed with a Frobenius quantale structure, then L is completely distributive. This example led us to Definition 3.9.

It is not difficult to check out that many categories we have given as examples in Chapter 1 are pseudo-affine. This is the case for the categories SET, POS, REL, VECT_k ... If $\mathcal{C}$ is pseudo-affine, then the endofunctor category $[\mathcal{C}, \mathcal{C}]$ is pseudo-affine as well. ◊

Let us introduce some useful observations.

LEMMA 3.11. *Tensoring with a pseudo-affine object yields a faithful functor.*

PROOF. Let A be pseudo-affine and suppose that $f, g : X \longrightarrow Y$ and consider the following diagram:



If $f \otimes A = g \otimes A : X \otimes A \longrightarrow Y \otimes A$, then also $p^\rho_Y \circ f = p^\rho_Y \circ g$ and then $f = g$, since p^ρ_Y is monic. □

LEMMA 3.12. *Identity morphisms of pseudo-affine objects are orthogonal to each other, in the following sense: if A, C are pseudo-affine, $f : A \otimes B_0 \longrightarrow A \otimes B_1$, $g : B_0 \otimes C \longrightarrow B_1 \otimes C$, and $f \otimes C = A \otimes g : A \otimes B_0 \otimes C \longrightarrow A \otimes B_1 \otimes C$, then there exists $h : B_0 \longrightarrow B_1$ such that $f = A \otimes h$ and $g = h \otimes C$.*

PROOF. First, using that C is pseudo-affine, we have

$$\begin{aligned} f &= c^{\rho}_{A\otimes B_0} \circ (f \otimes C) \circ p^{\rho}_{A\otimes B_0} = c^{\rho}_{A\otimes B_1} \circ (A \otimes g) \circ p^{\rho}_{A\otimes B_0} \\ &= (A \otimes c^{\rho}_{B_1}) \circ (A \otimes g) \circ A \otimes (p^{\rho}_{B_0}) = A \otimes (c^{\rho}_{B_0} \circ g \circ p^{\rho}_{B_0}) \,. \end{aligned}$$

Using this equation and various properties of naturality, we obtain

$$\begin{aligned} c^{\ell}_{B_1} \circ f \circ p^{\ell}_{B_0} &= c^{\ell}_{B_1} \circ (A \otimes (c^{\rho}_{B_0} \circ g \circ p^{\rho}_{B_0})) \circ p^{\ell}_{B_0} \\ &= c^{\ell}_{B_1} \circ (A \otimes c^{\rho}_{B_1}) \circ (A \otimes g) \circ (A \otimes p^{\rho}_{B_0}) \circ p^{\ell}_{B_0} \\ &= c^{\ell}_{B_1} \circ c^{\rho}_{A\otimes B_1} \circ (A \otimes g) \circ p^{\rho}_{A\otimes B_0} \circ p^{\ell}_{B_0} \\ &= c^{\rho}_{B_1} \circ c^{\ell}_{B_1\otimes C} \circ (A \otimes g) \circ p^{\ell}_{B_0\otimes C} \circ p^{\rho}_{B_0} \\ &= c^{\rho}_{B_1} \circ g \circ p^{\rho}_{B_0} \,. \end{aligned}$$

Therefore, we can define $h := c^{\ell}_{B_1} \circ f \circ p^{\ell}_{B_0} = c^{\rho}_{B_1} \circ g \circ p^{\rho}_{B_0}$ which complete the proof of the statement. □

The following is the main result in this section.

THEOREM 3.4. *If A is pseudo-affine and the semigroup $([A, A], \circ)$ can be completed to a Frobenius structure, then A is part of an adjunction.*

PROOF. We can suppose that the dual of $[A, A]$ is $A^* \otimes A$. Assume that $([A, A], A^* \otimes A, \epsilon_{A^*\otimes A} \circ \sigma_{[A,A],A^*\otimes A}, \circ, l, r)$ is a Frobenius structure where $\epsilon^*_A \otimes A$ was defined in Example 3.5. By definition, the dual multiplication $\mathrm{m}_{A^*\otimes A} : A^*\otimes A\otimes A^*\otimes A \longrightarrow A^*\otimes A$ arises from the maps $l^{-1}, r^{-1} : A^*\otimes A \longrightarrow [A, A]$ as the diagonal of the diagram below.

$$\begin{array}{ccc} A^* \otimes A \otimes A^* \otimes A & \xrightarrow{A^*\otimes A\otimes r^{-1}} & A^* \otimes A \otimes [A, A] \\ \downarrow{\scriptstyle l^{-1}\otimes A^*\otimes A} & \searrow^{\mathrm{m}_{A^*\otimes A}} & \downarrow{\scriptstyle \alpha^{\rho}_{[A,A]}} \\ [A, A] \otimes A^* \otimes A & \xrightarrow[\alpha^{\ell}_{[A,A]}]{} & A^* \otimes A \end{array}$$

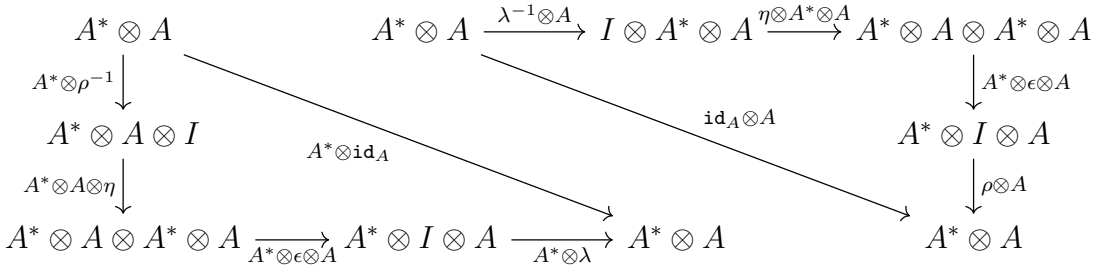
We already computed the maps $\alpha^{\ell}_{[A,A]}$ and $\alpha^{\rho}_{[A,A]}$ in Proposition 3.7. Hence, we obtain the following commutative diagram.

$$\begin{array}{ccc} A^* \otimes A \otimes A^* \otimes A & \xrightarrow{A^*\otimes A\otimes r^{-1}} & A^* \otimes A \otimes [A, A] \\ \downarrow{\scriptstyle l^{-1}\otimes A^*\otimes A} & \searrow^{\mathrm{m}_{A^*\otimes A}} & \downarrow{\scriptstyle A^*\otimes \mathrm{ev}} \\ [A, A] \otimes A^* \otimes A & \xrightarrow[\mu_{A,A,I}\otimes A]{} & A^* \otimes A \end{array}$$

Since A is pseudo-affine, then, by Lemma 3.12, we necessarily have

$$\mu_{A^*\otimes A} = (A^* \otimes \lambda_A) \circ (A^* \otimes \epsilon \otimes A)$$

for a map $\epsilon : A \otimes A^* \longrightarrow I$. Moreover, since $([A, A], \circ)$ is unital and $r : [A, A] \to A^* \otimes A$ is an isomorphism, $\mu_{A^*\otimes A}$ has a unit $\eta : I \longrightarrow A^* \otimes A$. This means that the the following diagrams commute:



Since tensoring with A and A^* is faithful—see Lemma 3.11—commutativity of the above diagrams is equivalent to the equalities

$$\lambda_A \circ (\epsilon \otimes A) \circ (A \otimes \eta) \circ \rho_A^{-1} = id_A\,, \quad id_{A^*} = \rho_A \circ (A^* \otimes \epsilon) \circ (\eta \otimes A^*) \circ \lambda_A^{-1}\,,$$

showing that (A, A^*, η, ϵ) is an adjunction. □

In view of Remark 1.5, we can restate Theorem 3.4 as follows:

THEOREM 3.5. *If A is pseudo-affine and $([A, A], \circ)$ carries a Frobenius structure, then A is nuclear.*

3.7. The pseudo-affine condition, a counterexample

In the previous section, we have shown that Conjecture 3.1 holds under the assumption that A is pseudo-affine. It is natural to wonder whether this theorem may be shown without any hypothesis on A. After some unsuccessful attempt at proving the result, we started searching for a counterexample. That is, we were looking for non pseudo-affine object A in a $*$-autonomous category $\mathcal{V}$ such that $[A, A]$ is a Frobenius structure but A is not nuclear. This section present such a counterexample.

3.7.1. The category Q-Set. Since most of our main examples of monoidal category are pseudo-affines, we have to look at non-standard examples of $*$-autonomous categories. To do so, we use the category P_F-SET of poset-valued sets introduced by Shalk and de Paiva in [**SdP04**]. This category is defined as follows.

Let $(P, \leq)$ be a poset and $F : \textsc{Rel} \to \textsc{Rel}$ be an endofunctor of Rel. We define the category P_F-Set by:

- An object of P_F-Set is a pair (X, α), with X a set and α a function as $\alpha : FX \to P$;
- An arrow (or map) from (Y, α) to (X, β) is a relation $R \subseteq X \times Y$ such that, for all $x, y \in X$, $xFRy$ implies $\alpha(x) \leq \beta(y)$.

We refer to the original paper [**SdP04**] for the full categorical structure, details, examples and many properties of P_F-Set. The idea is to play on the structure of P and of F in order to obtain interesting structures on the category P_F-Set. They notably give sufficient conditions to obtain a $*$-autonomous structure on P_F-Set. In Chapter 4 we shall see how this category arises as an example of the Grothendieck construction over a certain functor $Q : \mathcal{C} \to \textsc{Pos}$ with $\mathcal{C} := \textsc{Rel}$, and we will give a method to lift structures from $\mathcal{C}$ to the total category $\int \mathsf{Q}$.

To complete the objective of this section, we need to find a poset P and an endofunctor F which makes P_F-Set a $*$-autonomous category that is not pseudo-affine. By various properties given in [**SdP04**], for P_F-Set to be a $*$-autonomous category, it is sufficient to assume that P is a commutative unital Girard quantale $(Q, *, 0)$—see Definition 2.2—and $F : \textsc{Rel} \to \textsc{Rel}$ is the identity functor. We name the resulting category Q-Set. In the sequel of this section, we will exhibit non-pseudo-affine objects in this category. Some concrete examples of Q-Set are the following.

Example 3.16. For the trivial quantale $\mathit{1} := \{*\}$, $\mathit{1}$-Set is isomorphic to the category Rel of sets and relations. Thus Q-Set generalizes Rel. For the boolean algebra $\mathit{2} = \{\top, \bot\}$ and $* = \wedge$ then $\mathit{2}$-Set is isomorphic to the category whose objects are pairs (X, α) with $\alpha \subseteq X$ and whose arrows from (X, α) to (Y, β) are relations $R \subseteq X \times Y$ such that, if $x \in \alpha$ and xRy, then $y \in \beta$. ◊

The following Proposition is stated in [**SdP04**, Proposition 6].

Proposition 3.11. *The obvious forgetful functor from Q-Set to* Rel *strictly preserves all the star-autonomous structure.*

Concretely, the $*$-autonomous structure of Q-Set is given by the following tensor, unit, internal hom and dualizing object:

- The tensor product is given by:
$$(X,\alpha)\otimes(Y,\beta) := (X\times Y, (-*-)\circ(\alpha\times\beta))\,,$$
that is, $X\times Y \xrightarrow{\alpha\times\beta} Q\times Q \xrightarrow{-*-} Q$.
- The unit is given by $I := (\{*\}, 1 : * \mapsto 1)$.
- The internal hom is given by
$$[(X,\alpha),(Y,\beta)] := (X\times Y, (-\backslash -)\circ(\alpha\times\beta))\,,$$
that is, $X\times Y \xrightarrow{\alpha\times\beta} Q\times Q \xrightarrow{-\backslash -} Q$.
- The object $(\{*\}, 0 : * \mapsto 0)$ is a dualizing object of Q-Set.

In order to study nuclear objects of Q-Set, the unit of this category must be a dualizing object. Therefore, we want the objects $(\{*\}, 1 : * \mapsto 1)$ and $(\{*\}, 0 : * \mapsto 0)$ to be equal, so we request that $0 = 1$ is Q.

The following statement immediately follows from Proposition 3.11.

Fact 3.2. *An arrow $R : (X,\alpha) \to (Y,\beta)$ in Q-Set is an isomorphism if and only if R is a bijection and $\alpha = \beta \circ R$.*

We shall identify a function $f : X \to Y$ with its graph $G_f := \{\,(x, f(x)) \mid x \in X\,\}$. Thus, a function $f : X \to Y$ is a map from (X,α) to (Y,β) if $\alpha(x) \le \beta(f(x))$, for each $x \in X$.

Proposition 3.11 allows us to immediately compute the expression of `mix` in Q-Set:
$$\texttt{mix} = \{(x_0, x_1, x_2, x_3) \mid x_0 = x_2, x_1 = x_3\}$$

From this expression and Fact 3.2, we see that an object (X,α) in Q-Set is nuclear if and only if, for all $x, y \in X$,
$$\alpha(x)^\perp * \alpha(y) = \alpha(x)\backslash\alpha(y)\,. \tag{64}$$
Recall from Example 1.25 that nuclear objects of a quantale are its invertible elements. This gives the following caracterization of nuclear objects in Q-Set.

Lemma 3.13. *A Q-Set (X,α) is nuclear if and only if the image of α is a subset of the invertible elements of Q.*

Proof. If (X,α) is nuclear, then, for all $x \in X$,
$$1 \le \alpha(x)\backslash\alpha(x) = \alpha(x)^\perp * \alpha(x) \le 1\,.$$

That is, $\alpha(x)$ is invertible. On the other hand, if $\alpha(x)$ is invertible, then, necessarily $\alpha(x)^{-1} = \alpha(x)^{\perp}$. Then, it is easily seen that $\alpha(x)^{-1} * q = \alpha(x) \backslash q$, for all $q \in Q$. □

LEMMA 3.14. *A Q-SET (X, α) is pseudo-affine if and only if there exists $x \in X$ such that $\alpha(x) = 1$.*

PROOF. Let (X, α) be a pseudo-affine object of Q-SET, that is, the unit of the tensor embeds into it as a retract. This means there exists $p : I \to A$ and $c : A \to I$ such that $c \circ p = \mathtt{id}_I$. For c and p to be morphism in Q-SET, the following inclusions must hold:

$$p \subset \{\, (*, x) \mid x \in X,\ 1 \leq \alpha(x) \,\}\,,$$

and

$$c \subset \{\, (x, *) \mid x \in X,\ \alpha(x) \leq 1 \,\}.$$

The fact that $c \circ p = \mathtt{id}_I$ implies there exists x such that $x \in c$ and $x \in p$, that $\alpha(x) = 1$. Assuming there exists $x \in X$ such that $\alpha(x) = 1$, pick $p = (*, x)$ and $c = (x, *)$. It is direct to see that that p is a retract and c is its left inverse. □

By Proposition 3.11, if $A := (X, \alpha)$ is the carrier of a Frobenius structure in Q-SET, then X is a Frobenius structure in REL. This implies—see Example 3.14 for details— that there are a multiplication $M : X \times X \to X$ and inverses isomorphisms $l, r : X \to X$ such that for all $x, y, z \in X$,

$$xyMl(z) \ \text{ iff } \ yzMr(x)\,.$$

Again by Proposition 3.11, the multiplication on the monoid $[A, A]$ in Q-SET is given by the multiplication on $[X, X]$ in REL. Therefore, we first study Frobenius structures on monoids $[X, X]$ in REL. We recall the explicit definition of the multiplication of the monoid $([X, X], M)$ in REL:

$$(x_0, x_1)(y_0, y_1)M(z_0, z_1) \iff (z_0, z_1) = (x_0, y_1) \text{ and } x_1 = y_0\,.$$

PROPOSITION 3.12. *Frobenius structures in* REL *on the carrier $([X, X], M)$ bijectively correspond to pairs (X, f) where f is a bijection of X.*

PROOF. First, given such an f with g as inverse, let us define

$$l(x, y) := (f(y), x)\,, \qquad\qquad r(x, y) = (y, g(x))\,,$$

so l and r are inverse to each other. Observe now that

$$\begin{aligned}(x,y),(z,w)Ml(u,v) = (f(v),u) \text{ iff } & x = f(v), y = z, w = u\\ \text{iff } & z = y, w = u, v = g(x)\\ \text{iff } & (z,w),(u,v)M(y,g(x))\\ \text{iff } & (z,w),(u,v)Mr(x,y)\,.\end{aligned}$$

Conversely, suppose there exists $l, r : X \times X \to X \times X$ inverse of each other such that,

$$(x,y),(z,w)Ml(u,v) \text{ iff } (z,w),(u,v)Mr(x,y)\,.$$

Write $l(u,v) = (a,b)$. For each $k \in X$, we have

$$(a,k),(k,b)Ml(u,v) \text{ iff } (k,b),(u,v)Mr(a,k)\,,$$

and so $b = u$. This implies that $l(u,v)$ is of the form $(f(u,v),u)$ for a certain map $f : X \times X \to X$. Since l is inverted by r, we have that

$$r(l(u,v)) = r(f(u,v),u) = (u,v)\,.$$

Similarly, for each $k \in X$, we have $(u,k),(k,v)M(u,v) = r(f(u,v),u)$ and therefore $(f(u,v),u),(u,k)Ml(k,v) = (f(k,v),k)$. This implies that $f(u,v) = f(k,v)$, for each $k \in X$, and so f does not depend on its first variable.

We have, therefore, $l(x,y) = (f(y),x)$ for a certain $f : X \to X$. Necessarily, if l has an inverse, then so has f. □

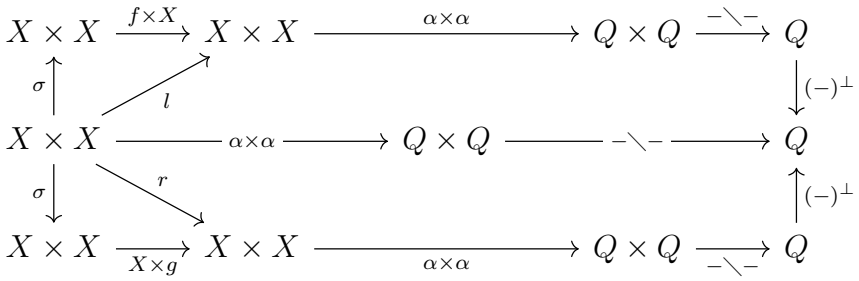
Let $A = (X,\alpha)$ and consider a Frobenius structure $([A,A],[A,A]^*,M,l,r)$ in Q-Set. By the previous Propositon $l(x,y) = (f(y),x)$ and $r(x,y) = (y,g(x))$ for some $f, g : X \to X$ inverse to each other. Moreover, since $l, r : [A,A] \to [A,A]^*$ are isomorphisms in Q-Set, by Fact 3.2, we have, for all $x, y \in X$,

$$\alpha(x)\backslash\alpha(y) = (\alpha(fy)\backslash\alpha(x))^{\perp}\,, \tag{65}$$

and

$$\alpha(x)\backslash\alpha(y) = (\alpha(y)\backslash\alpha(gx))^{\perp}\,. \tag{66}$$

or written differently, the following diagram commutes in SET.



Notice that Eq. (65) holds if and only if Eq. (66) holds. Moreover, Eq. (65) may be reformulated by

$$\alpha(x) \backslash \alpha(y) = \alpha(x)^{\perp} * \alpha(fy)\,.$$

If f is the identity, then the previous equation amounts to Eq. (64), meaning A is nuclear.

3.7.2. Prenuclear structures. We call *prenuclear* an object A such that $[A, A]$ may be endowed with a Frobenius structure. In Q-SET, we will use the following definition.

DEFINITION 3.10. A *prenuclear structure* in Q-SET is a tuple (X, α, f) where (X, α) is a Q-SET and $f : X \longrightarrow X$ is a bijection such that for all $x, y \in X$, one of the equivalent equation holds

$$\alpha(x) \backslash \alpha(y) = \alpha(x)^{\perp} * \alpha(fy)\,, \tag{67}$$

$$\alpha(x) \backslash \alpha(y) = \alpha(f^{-1}x)^{\perp} * \alpha(y)\,. \tag{68}$$

. We say that (X, α) is a *prenuclear object* if there exists f which makes (X, α, f) a prenuclear structure.

NOTATION 3.4. For a prenuclear structure (X, α, f), we will always name g the inverse of f. The n-th iteration of f is written f^n. It is defined by induction on n: $f^0(x) := x$, and $f^{n+1}(x) := f(f^n(x))$.

We now study some properties of prenuclear objects.

FACT 3.3. *If (X, α, f) is a prenuclear structure, then the following inqualities hold for each $x \in X$:*

$$\alpha(x) \leq \alpha(fx)\,,$$
$$\alpha(gx) \leq \alpha(x)\,.$$

PROOF. We have the following chain of equivalences:

$$\dfrac{\dfrac{\alpha(x) \leq \alpha(x)}{1 \leq \alpha(x) \backslash \alpha(x)}}{1 \leq \alpha(x)^{\perp} * \alpha(fx)}$$

Recall that 1 is the unit and a dualizing element of the commutative quantale $(Q, *)$. By multiplying both side of the last inequality by $\alpha(x)$, we obtain

$$\alpha(x) \leq \alpha(x) * \alpha(x)^{\perp} * \alpha(f(x)) \leq 1 * \alpha(fx) = \alpha(fx)\,.$$

The second inequality is obtained with a similar computation. □

LEMMA 3.15. *Let (X, α, f) be a prenuclear structure and let x be in X. The following statements are equivalent:*

(1) $\alpha(fx) \leq \alpha(x)$;
(2) $\alpha(x) \leq \alpha(gx)$;
(3) $\alpha(x) = \alpha(fx)$;
(4) $\alpha(gx) = \alpha(x)$;
(5) $\alpha(x)$ is an invertible element;
(6) $\alpha(y)$ is an invertible element for all $y \in X$;
(7) (X, α) is a nuclear object.

PROOF. Item 1 is equivalent to Item 2, indeed we have the following chain of equivalences

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha(fx) \leq \alpha(x)}{1 \leq \alpha(fx) \backslash \alpha(x)}}{1 \leq (\alpha(x) \backslash \alpha(gfx))^{\perp} = (\alpha(x) \backslash \alpha(x))^{\perp}}}{1 \leq (\alpha(x) \backslash \alpha(gx))^{\perp\perp} = \alpha(x) \backslash \alpha(gx)}}{\alpha(x) \leq \alpha(gx)}$$

By Fact 3.3, the inequalities $\alpha(x) \leq \alpha(fx)$ and $\alpha(gx) \leq \alpha(x)$ always hold. Therefore, Item 1 is equivalent to Item 3, to Item 2 and to Item 4.

Item 3 implies Item 5: If $\alpha(f(x)) = \alpha(x)$, then

$$1 \leq \alpha(x) \backslash \alpha(x) = \alpha(x)^{\perp} * \alpha(fx) = \alpha(x)^{\perp} * \alpha(x) \leq 1\,,$$

meaning $\alpha(x)$ is inverted bu $\alpha(x)^{\perp}$.

5 implies 2: If $\alpha(x)$ is invertible, then we have

$$1 \leq \alpha(x)^{\perp} * \alpha(x) = \alpha(g(fx))^{\perp} * \alpha(x) = \alpha(fx) \backslash \alpha(x)\,,$$

which is equivalent to $\alpha(fx) \leq \alpha(x)$.

5 implies 6: Recall that, when $u \in Q$ is invertible, then $u^{-1} = u^{\perp}$ and, moreover, $u \backslash w = u^{\perp} * w$, for each $w \in Q$. From this also follows (assuming u invertible) that

$$w \backslash u = u^{\perp} \backslash w^{\perp} = u * w^{\perp} .$$

Therefore, if $\alpha(x)$ is invertible, then we have for all $y \in X$,

$$\alpha(x) * \alpha(y)^{\perp} = \alpha(y) \backslash \alpha(x) = \alpha(x) * \alpha(gy)^{\perp} .$$

By cancelling on the left this relation, we obtain $\alpha(y)^{\perp} = \alpha(gy)^{\perp}$, whence $\alpha(y) = \alpha(gy)$. As 4 implies 5, we have that $\alpha(y)$ is invertible for all $y \in X$.

Obviously 6 implies 5.

The equivalence between 6 and 7 is given by Lemma 3.13. □

LEMMA 3.16. *If for some $x \in X$ and $n \in \mathbb{Z}\backslash\{0\}$, $\alpha(f^n x) = \alpha(x)$, then (X, α) is nuclear whenever the monoid of endomorphisms of (X, α) has a Frobenius structure.*

PROOF. The case $n = 1$ corresponds to Item 3 of Lemma 3.15. We have already shown it is equivalent to (X, α) being nuclear. For $n \geq 2$, if $\alpha(f^n x) = \alpha(x)$, then $\alpha(f^{n-1} x) = \alpha(gx)$. A simple induction on n is enough to prove that for every x and for every $n \geq 0$, we have $\alpha(x) \leq \alpha(f^n x)$. Therefore, we obtain the inequality

$$\alpha(x) \leq \alpha(f^{n-1}(x)) = \alpha(gx) .$$

By lemma 3.15, the previous inequality implies that (X, α) is nuclear. □

PROPOSITION 3.13. *In Q-SET, a prenuclear object (A, f) is nuclear if one of the following conditions is true:*

(1) A is pseudo-affine;
(2) The two negations of the Frobenius structure in $[A, A]$ are the same;
(3) The Frobenius quantale Q has no infinite chain;
(4) The underlyng set X is finite.

PROOF. The Item 1 was already proven in Theorem 3.5. In the case of Q-SET, there is another proof which goes as follow. If A is pseudo-affine, that

is if there is $u \in X$ such that $\alpha(u) = 1$, then we have for all $x \in X$,

$$\begin{aligned}\alpha(x)^\perp &= \alpha(x) \backslash 1 \\ &= \alpha(x) \backslash \alpha(u) \\ &= (\alpha(u) \backslash \alpha(fx))^\perp \\ &= (1 \backslash \alpha(fx))^\perp \\ &= \alpha(fx)^\perp .\end{aligned}$$

Therefore, we have

$$\alpha(x) \backslash \alpha(y) = \alpha(fx)^\perp * \alpha(y) = \alpha(x)^\perp * \alpha(y) ,$$

and A is nuclear in Q-Set.

For Item 2, if $r = l$, then for each $x, y \in X$, we have

$$(fy, x) = l(x, y) = r(x, y) = (y, gx) .$$

Therefore, $f(y) = y$, for each $y \in X$, and Eq. (67) amounts to the nuclearity of A.

Asumme now that Item 3 or Item 4 holds. Let x be in X and consider the increasing sequence $\alpha(f^n x)$, $n \geq 0$. By our hypothesis, this sequence takes only a finite number of values, so for some n, $\alpha(f^n x) = \alpha(f^{n+1} x)$ and we obtain the result by Lemma 3.16. □

3.7.3. A non-trivial prenuclear structure. We now construct a non-trivial prenuclear structure (X, α, f) in Q-Set which is not nuclear. From Proposition 3.13, the counterexample we are looking for must fulfill the following properties:

- The object (X, α) is not be pseudo-affine in Q-Set, meaning, for every $x \in X$, $\alpha(x) \neq 1$;
- The inverse of f is not f itself;
- The quantale Q has an infinite chain;
- The underlying set X is infinite.

The first idea that comes in mind is to take the embedding of $\mathbb{Z}$ in itself and to settle $f(n) := n + 1$. This, however, does not work for several reasons. First, $\mathbb{Z}$ is not a quantale. This is not a real difficulty. Indeed, we can consider the following quantale $(\mathcal{Z}^+, *)$, whose underlying set is $\mathbb{Z} \cup \{ -\infty, +\infty \}$, with the

standard ordering, and whose multiplication table is given by:

$*$	$-\infty$	$m \in \mathbb{Z}$	∞
$-\infty$	$-\infty$	$-\infty$	$-\infty$
$n \in \mathbb{Z}$	$-\infty$	$n+m$	∞
∞	$-\infty$	∞	∞

FACT 3.4. *$(\mathcal{Z}^+, *)$ is a unital commutative quantale whose unit is a dualizing element.*

PROOF. We prove this using the theory developed in Section 2.4. On $P(\mathbb{Z})$ define xRy iff $x + y \leq 0$. This relation is symmetric and associative. By defining $X^\perp := \{\, y \mid xRy, \forall x \in X \,\}$, we have that $X^\perp$ is downset of $\mathbb{Z}$. Moreover, every downset arises in this way, since $\{\, -x \,\}^\perp = {\downarrow} x$.

The quantale $(\mathcal{Z}^+, *)$ is isomorphic to the Girard quantale $(P(Z)/\mathrm{j}, \bullet_\mathrm{j})$ induced by the Serre Galois connection of R. □

However, the embedding of $\mathbb{Z}$ into $\mathcal{Z}^+$ is not a pseudo-affine in $\mathcal{Z}^+$-SET since 0 is the unit of $\mathcal{Z}^+$. Another attempt would be for instance to take the set of odd numbers and set $f(n) := n + 2$ but this also fails since we have $m \backslash n = n - m$ but $n^\perp + f(n) = n + 2 - m$. All this research led to the following construction.

We add to $\mathcal{Z}^+$ a new unit, thus obtaining the following multiplication table:

$*$	$-\infty$	u	$m \in \mathbb{Z}$	∞
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
u	$-\infty$	u	m	∞
$n \in \mathbb{Z}$	$-\infty$	n	$n+m$	∞
∞	$-\infty$	∞	∞	∞

That a unit can be added to quantale is well known. Here, however, we set u to be an upper cover of -1 and a lower cover of 0. We obtain therefore an ordered monoid, for which, the only thing we need to verify is that multiplication by the new unit is order preserving. This is fairly trivial. We call $\mathcal{Z}_u^+$ the ordered monoid so obtained.

PROPOSITION 3.14. *The ordered monoif $(\mathcal{Z}_u^+, *)$ is a quantale.*

PROOF. Obviously, $\mathcal{Z}_u^+$ is a complete lattice. Let us argue that multiplication by $x \in \mathcal{Z}_u^+$ has a right adjoint. Observe that the quantale $\mathcal{Z}^+$ embeds into $\mathcal{Z}_u^+$ and that this embedding, call it ι, has both a right and a left adjoint. If $\{\, x, y, z \,\} \cap \{\, u \,\} = \emptyset$, then the relation $x * y \leq z$ holds in $\mathcal{Z}_u^+$, iff it holds in $\mathcal{Z}^+$, if $x \leq y \backslash z$ in $\mathcal{Z}^+$, so $\iota(y \backslash x)$ exhibits the existence of the

right-adjoint. Suppose $y = u$, then clearly we can set $u \diagdown z = z$. Next, if $z = u$ and $\{ x, y \} \cap \{ u \} = \emptyset$, then $\iota(x) * \iota(y) \leq u$ if and only if $x * y \leq -1$. Thus the following table completes the proof of the existence of the right-adjoint $x \diagdown -$ to $x * -$.

	$x \diagdown u$
$-\infty$	∞
u	u
$n \in \mathbb{Z}$	$-n - 1 \in \mathbb{Z}$
∞	$-\infty$

.

□

Notice that, since $(x \diagdown u) \diagdown u = x$, the unit is a dualzing element of the quantale.

We define next an object (X, α) of the category $\mathcal{Z}_u^+$-Set. Take $X = \mathbb{Z}$, and α the inclusion of $\mathbb{Z}$ into $\mathcal{Z}_u^+$. Since $u \notin \mathbb{Z}$, (X, α) is not pseudo-affine. For $n \in \mathbb{Z}$, let us define

$$f(n) := n + 1 \,.$$

Clealry, f has

$$g(n) := n - 1$$

as inverse.

Proposition 3.15. *The object (X, α) is prenuclear, but not nuclear.*

Proof. For each $n, m \in \mathbb{Z}$, we have

$$n \diagdown m = (n * m^{\perp})^{\perp} = (n + -m - 1)^{\perp} = -(n - m - 1) - 1 = m - n \,.$$

and

$$n^{\perp} * f(m) = -n - 1 + m + 1 = m - n \,.$$

Thus (X, α) is prenuclear. It is not nuclear, as the only invertible element in $\mathcal{Z}_u^+$ is its unit. □

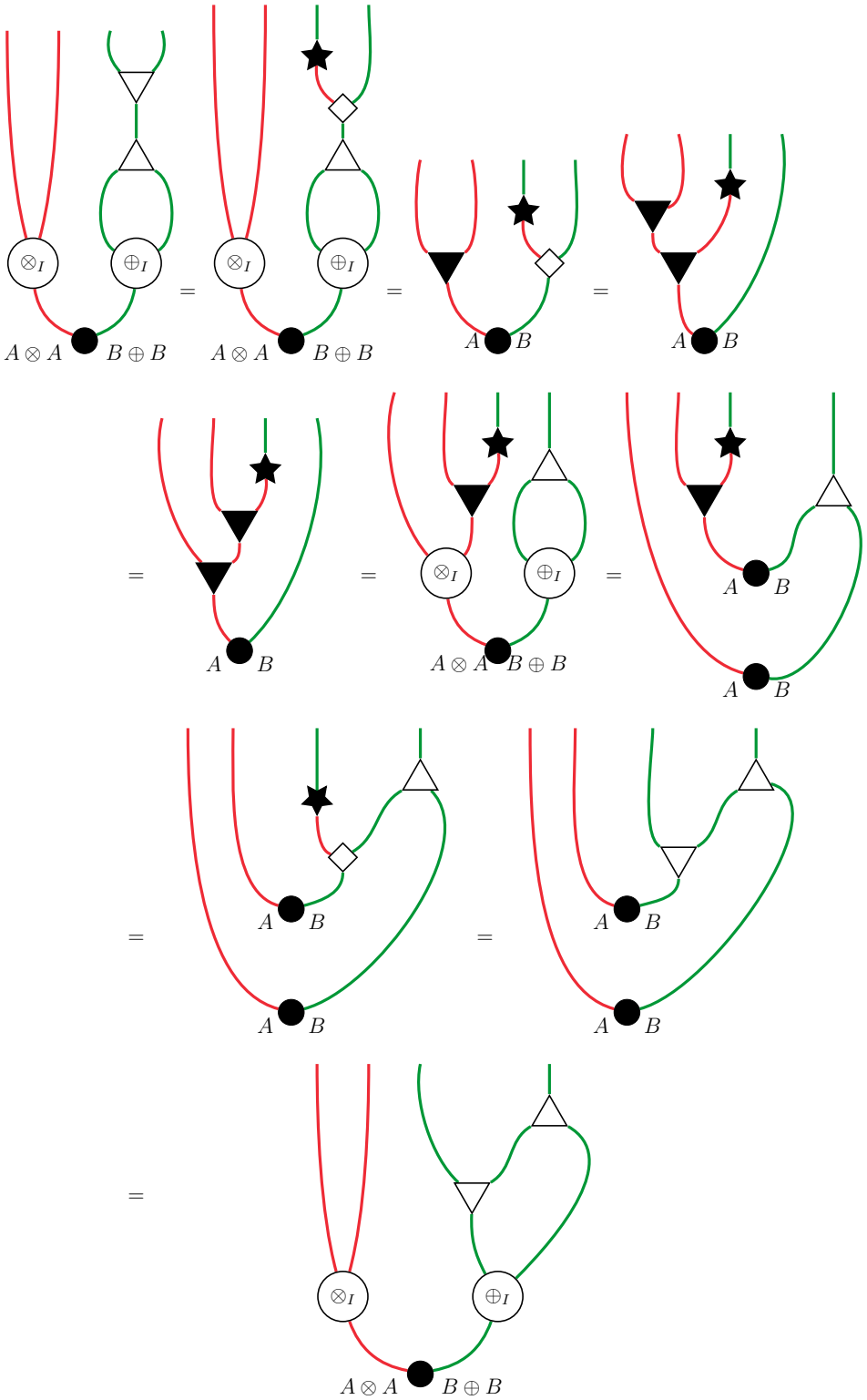


FIGURE 3.1. Any Frobenius structure in a ∗-autonomous category respects axiom (F)

CHAPTER 4

Lifting star-autonomy

In the previous section, we have used the category P_F-SET introduced by Schalk and de Paiva in [SdP04] in order to construct a counterexample of a conjecture. We actually have used in the case where P is a unital commutative Frobenius quantale $(Q, *, 1)$ where 1 is a dualizing element of Q. Under those conditions, the category $Q_{\mathtt{id}}$-SET has a canonical $*$-autonomous structure, where the unit of the tensor is a dualizing objet. In the last section, we have considered this category as a tool to test our conjecture. Indeed, many properties of $Q_{\mathtt{id}}$-SET depend on the quantale Q. For instance, if Q has no infinite chain, then prenuclear objects are nuclear—see Proposition 3.13. But the relation that struck us the most is between the Frobenius structure of the quantale Q and the $*$-autonomous structure of $Q_{\mathtt{id}}$-SET. The category of $Q_{\mathtt{id}}$-SET actually arises as the Grothendieck construction over the functor $\mathsf{Q} : \textsc{Rel} \longrightarrow \textsc{Pos}$ defined for on a set X by $\mathsf{Q}(X) = Q^X$ and on a relation $R : X \to Y$ by

$$\mathsf{Q}(R)(\alpha)(y) := \bigvee_{xRy} \alpha(x)\,, \quad \text{for each } \alpha \in Q^X \text{ and } y \in Y.$$

The Grothendieck (or total) category of a functor $\mathsf{Q} : \mathcal{C} \to \textsc{Pos}$, noted $\int \mathsf{Q}$, is defined as follows:

- objects are pairs (X, α) with $X \in \textsc{Obj}(\mathcal{C})$ and $\alpha \in \mathsf{Q}(X)$,
- arrows $(X, \alpha) \to (Y, \beta)$ are $\mathcal{C}$-arrow $f : X \to Y$ such that

$$\mathsf{Q}(f)(\alpha) \leq \beta\,.$$

Intuitively, an element $\alpha \in \mathsf{Q}(X)$ is a structure attached to the object X of $\mathcal{C}$. Morphisms between objects of $\int \mathsf{Q}$ must preserve the structure. The fact that the codomain is POS means that the set of structures over an object of $\mathcal{C}$ is a poset. The Grothendieck construction $\int$ is actually a major concept of category theory. We have defined it for functors whose codomain is POS but it is usually defined for functors whose codomain is CAT. It is actually a **2**-equivalence $\int : [\mathcal{C}, \textsc{Cat}] \cong \textsc{Fib}(\mathcal{C})$ between contravariant—resp. covariant—functors into CAT and fibrations—resp. opfibration—see for instance [Her94, Proposition 2.10]. We will not go into such generalities in this work, and we shall consider covariant functors going into POS, the subcategory of CAT

of posets and monotone maps. The corresponding opfibrations of a functor $\mathsf{Q} : \mathcal{C} \longrightarrow \textsc{Pos}$ is given by the first projection functor $\pi : \int \mathsf{Q} \to \mathcal{C}$.

The objective of this chapter is to provide conditions to *lift* a structure in $\mathcal{C}$ into $\int \mathsf{Q}$. Mainly we are interested in the cases where $\mathcal{C}$ is a monoidal category, a symmetric monoidal closed category and a $*$-autonomous category. Of course, the lifting we are considering must preserve the opfibration $\pi : \int \mathsf{Q} \to \mathcal{C}$ "on the nose".

This chapter mainly follows the conference paper [**SCdLar**]. The last section, relating the lifting of the $*$-autonomous structure and Frobenius structures in the category $[\mathcal{C}, \textsc{SLatt}]$, is a completely new one.

4.1. Lifting functors from the base categories

In this section, we show how to lift the monoidal structure of a category $\mathcal{C}$ to $\int \mathsf{Q}$ where $\mathsf{Q} : \mathcal{C} \to \textsc{Pos}$. To do so, we give a general result relating liftings of functor $F : \mathcal{A} \to \mathcal{B}$ to what we call lax (extra)natural transformations.

Definition 4.1. Let $H, K : \mathcal{C} \longrightarrow \textsc{Pos}$ be two functors. A *lax natural transformation* $\psi : H \longrightarrow K$ is a collection of order-preserving maps $\psi_X : H(X) \longrightarrow K(X)$, indexed by objects X of $\mathcal{C}$, such that, for each map $f : X \longrightarrow Y$ of $\mathcal{C}$, Diagram (69) semi-commutes.

$$\begin{array}{ccc} H(X) & \xrightarrow{H(f)} & H(Y) \\ \downarrow{\scriptstyle \psi_X} & \swarrow & \downarrow{\scriptstyle \psi_Y} \\ K(X) & \xrightarrow{K(f)} & K(Y) \end{array} \tag{69}$$

By saying that a diagram in Pos semi-commutes we mean that the lower leg of the diagram is below the upper leg in the pointwise ordering—as suggests the symbol $\swarrow$ in the middle of the diagram.

Let us observe, however, the following fact.

Lemma 4.1. *If $\psi : H \longrightarrow K$ is lax natural and $f : X \longrightarrow Y$ is invertible, then a diagram as in (69) commutes.*

Proof. It will be enough to argue for the inclusion

$$\psi_Y(H(f)(x)) \leq K(f)(\psi_X(x)) \,,$$

since the opposite inclusion holds by definition of lax natural transformation. Let f^{-1} be the inverse of f and remark that

$$\begin{aligned} K(f^{-1})(\,\psi_Y(H(f)(x))\,) &\leq \psi_X(\,H(f^{-1})(H(f)(x))\,) \\ &= \psi_X(x) \\ &= K(f^{-1})(\,K(f)(\psi_X(x))\,)\,. \end{aligned}$$

Since $K(f^{-1}) : K(Y) \longrightarrow K(X)$ is an embedding, we obtain

$$\psi_Y(H(f)(x)) \leq K(f)(\psi_X(x))$$

as desired. □

Before we start lifting functors, let us state an important lemma that generalizes Fact 3.2. It has an easy proof that we skip.

LEMMA 4.2. *A morphism $f : (X, \alpha) \longrightarrow (Y, \beta)$ is invertible if and only if $f : X \longrightarrow Y$ is invertible and $Q(f)(\alpha) = \beta$.*

4.1.1. Liftings and lax (extra)natural transformations. Let $\mathcal{A}$ and $\mathcal{B}$ be categories coming with functors $\mathsf{Q}_\mathcal{A} : \mathcal{A} \longrightarrow \textsc{Pos}$ and $\mathsf{Q}_\mathcal{B} : \mathcal{B} \longrightarrow \textsc{Pos}$, and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. A *lifting* of F is a functor $\overline{F} : \int \mathsf{Q}_\mathcal{A} \longrightarrow \int \mathsf{Q}_\mathcal{B}$ such that the diagram on the left of (70) commutes. If $\alpha : F \longrightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$ is a natural transformation and $\overline{F}, \overline{G} : \int \mathsf{Q}_\mathcal{A} \longrightarrow \int \mathsf{Q}_\mathcal{B}$ are liftings of F and G, respectively, then a lifiting of α is a natural transformation $\overline{\alpha} : \overline{F} \longrightarrow \overline{G}$ such that $\pi(\overline{\alpha}) = \alpha_\pi$, that is, such that the diagram on the right of (70) commutes for 1-cells and 2-cells.

$$\begin{array}{ccc} \int \mathsf{Q}_\mathcal{A} & \xrightarrow{\overline{F}} & \int \mathsf{Q}_\mathcal{B} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \qquad\qquad \begin{array}{ccc} \int \mathsf{Q}_\mathcal{A} & \overset{\overline{F}}{\underset{\overline{G}}{\Downarrow \overline{\alpha}}} & \int \mathsf{Q}_\mathcal{B} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{A} & \overset{F}{\underset{G}{\Downarrow \alpha}} & \mathcal{B} \end{array} \tag{70}$$

THEOREM 4.1. *There is a bijection between liftings of a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ and lax natural transformations $\psi : \mathsf{Q}_\mathcal{A} \longrightarrow \mathsf{Q}_\mathcal{B} \circ F$.*

PROOF. Suppose that we have a lifting $\overline{F}$ of F, so $\overline{F}(X, x) = (F(X), z)$ with $z \in \mathsf{Q}_\mathcal{B}(F(X))$. Calling then $z = \psi_X(x)$, this defines a map $\psi_X : \mathsf{Q}_\mathcal{A}(X) \longrightarrow \mathsf{Q}_\mathcal{B}(F(X))$.

We claim that ψ_X is monotone. If $x, x' \in \mathsf{Q}_\mathcal{A}(X)$ and $x \leq x'$, then $\mathtt{id}_X$ is a map in $\int \mathsf{Q}_\mathcal{A}$ from (X, x) to (X, x'). By functoriality, $\mathtt{id}_{F(X)} = \overline{F}(\mathtt{id}_X)$ is a map in $\int \mathsf{Q}_\mathcal{B}$ from $\overline{F}(X, x)$ to $\overline{F}(X, x')$, which amounts to the inequality

$\psi_X(x) \leq \psi_X(x')$. Next, consider the follwing statement: if $f : (X, x) \longrightarrow (Y, y)$ is a map in $\int \mathsf{Q}_{\mathcal{A}}$, then $\overline{F}(f) : \overline{F}(X, x) \longrightarrow \overline{F}(Y, y)$ is a map of $\int \mathsf{Q}_{\mathcal{B}}$. This statement is equivalent to saying that $\mathsf{Q}_{\mathcal{A}}(f)(x) \leq y$ implies $\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_X(x)) \leq \psi_Y(y)$, for all $x \in \mathsf{Q}_{\mathcal{A}}(X)$ and $y \in \mathsf{Q}_{\mathcal{A}}(Y)$. Letting $y = \mathsf{Q}_{\mathcal{A}}(f)(x)$, we deduce the inclusion

$$\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_X(x)) \leq \psi_Y(\mathsf{Q}_{\mathcal{A}}(f)(x))\,,$$

which amounts to the semi-commutativity of (69). On the other hand, this inclusion is sufficient as well: if $\mathsf{Q}_{\mathcal{A}}(f)(x) \leq y$, then, by the order properties of ψ, we have

$$\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_X(x)) \leq \psi_Y(\mathsf{Q}_{\mathcal{A}}(f)(x)) \leq \psi_Y(y)\,.$$

Consequently, the construction of ψ from $\overline{F}$ can be reversed, thus building a lifting of F from a collection ψ with the stated properties. □

Given the bijection, if $\overline{F}$ corresponds to the lax natural transformation ψ via the bijection, then we shall write it $\overline{F}^{\psi}$. As the codomain of $\mathsf{Q}_{\mathcal{A}}$ and $\mathsf{Q}_{\mathcal{B}}$ is Pos, given lax natural transformations $\psi : \mathsf{Q}_{\mathcal{A}} \longrightarrow \mathsf{Q}_{\mathcal{B}} \circ F$ and $\chi : \mathsf{Q}_{\mathcal{A}} \longrightarrow \mathsf{Q}_{\mathcal{B}} \circ G$, a natural transformation $\alpha : F \longrightarrow G$ has at most one lifting $\overline{\alpha} : \overline{F}^{\psi} \longrightarrow \overline{G}^{\chi}$. It is easily seen that α has such a lifting if and only if, for any object X of $\mathcal{A}$ and $x \in \mathsf{Q}_{\mathcal{A}}(X)$,

$$\mathsf{Q}_{\mathcal{B}}(\alpha_X)(\psi_X(x)) \leq \chi_X(x)\,. \tag{71}$$

Remark 4.1. Let $\mathsf{INat}(F)$ denote the class of lax natural transformations $\psi : \mathsf{Q}_{\mathcal{A}} \longrightarrow \mathsf{Q}_{\mathcal{B}} \circ F$. This class is naturally ordered: $\psi \leq \chi$ if and only if $\psi_X(x) \leq \chi_X(x)$, for all objects X of A and $x \in \mathsf{Q}_{\mathcal{A}}(X)$.

If $\alpha : F \longrightarrow G$ is a natural transformation and $\psi : \mathsf{Q}_{\mathcal{A}} \longrightarrow \mathsf{Q}_{\mathcal{B}} \circ F$ is lax natural, then the expression $\mathsf{Q}_{\mathcal{B}}(\alpha_X) \circ \psi_X : \mathsf{Q}_{\mathcal{A}}(X) \longrightarrow \mathsf{Q}_{\mathcal{B}}(G(X))$ yields a lax natural transformation $\mathsf{INat}(\alpha)(\psi) := \mathsf{Q}_{\mathcal{B}}(\alpha) \circ \psi$; the correspondence $\mathsf{INat}(\alpha)$, sending ψ to $\mathsf{Q}_{\mathcal{B}}(\alpha) \circ \psi$, is order preserving. Thus INat is a functor from the functor category $[\mathcal{A}, \mathcal{B}]$ to the category of partially ordered classes and their maps. Equation (71) exhibits an isomorphisms between the obvious category of liftings of functors from $\mathcal{A}$ to $\mathcal{B}$ with the total category $\int \mathsf{INat}$.

Next, we consider liftings of a contravariant functor $F : \mathcal{A}^{op} \longrightarrow \mathcal{B}$. These are functors $\overline{F} : (\int \mathsf{Q}_{\mathcal{A}})^{op} \longrightarrow \int \mathsf{Q}_{\mathcal{B}}$ making commutative the diagram on the left of (72). A *lax extranatural transformation* $\psi : Q_{\mathcal{A}}^{op} \longrightarrow \mathsf{Q}_{\mathcal{B}} \circ F$ is a collection of order-preserving maps $\psi_X : \mathsf{Q}_{\mathcal{A}}(X)^{op} \longrightarrow \mathsf{Q}_{\mathcal{B}}(F(X))$, indexed by objects X of $\mathcal{A}$, such that, for each map $f : X \longrightarrow Y$ of $\mathcal{A}$, the diagram

on the right of (72) semi-commutes.

$$\begin{array}{ccc} (\int \mathsf{Q}_{\mathcal{A}})^{op} & \xrightarrow{\overline{F}} & \int \mathsf{Q}_{\mathcal{B}} \\ \downarrow{\scriptstyle \pi^{op}} & & \downarrow{\scriptstyle \pi} \\ \mathcal{A}^{op} & \xrightarrow{F} & \mathcal{B} \end{array} \qquad \begin{array}{ccc} \mathsf{Q}_{\mathcal{A}}(X)^{op} & \xrightarrow{\psi_X} & \mathsf{Q}_{\mathcal{B}}(F(X)) \\ \downarrow{\scriptstyle \mathsf{Q}_{\mathcal{A}}(f)^{op}} & \swarrow & \uparrow{\scriptstyle \mathsf{Q}_{\mathcal{B}}(F(f))} \\ \mathsf{Q}_{\mathcal{A}}(Y)^{op} & \xrightarrow{\psi_Y} & \mathsf{Q}_{\mathcal{B}}(F(Y)) \end{array} \tag{72}$$

THEOREM 4.2. *There is a bijection between liftings of a functor $F : \mathcal{A}^{op} \longrightarrow \mathcal{B}$ and lax extranatural transformations $\psi : Q_{\mathcal{A}}^{op} \longrightarrow \mathsf{Q}_{\mathcal{B}} \circ F$.*

PROOF. Suppose that we have a lifting $\overline{F}$ of F, so $\overline{F}(X, x) = (F(X), z)$ with $z \in \mathsf{Q}_{\mathcal{B}}(F(X))$. Calling then $z = \psi_X(x)$, this defines a map $\psi_X : \mathsf{Q}_{\mathcal{A}}(X) \longrightarrow \mathsf{Q}_{\mathcal{B}}(F(X))$. We claim that ψ_X is antitone: if $x, x' \in \mathsf{Q}_{\mathcal{A}}(X)$ and $x' \leq x$, then $\mathtt{id}_X$ is a map in $\int \mathsf{Q}_{\mathcal{A}}$ from (X, x') to (X, x); then, by functoriality, $\mathtt{id}_{F(X)} = \overline{F}(\mathtt{id}_X)$ is a map in $\int \mathsf{Q}_{\mathcal{B}}$ from $\overline{F}(X, x)$ to $\overline{F}(X, x')$, which amounts to the inequality $\psi_X(x) \leq \psi_X(x')$. Next, consider the following statement: if $f = (X, x) \longrightarrow (Y, y)$ is a map in $\int \mathsf{Q}_{\mathcal{A}}$, then $\overline{F}(f) : \overline{F}(Y, y) \longrightarrow \overline{F}(X, x)$ is a map of $\int \mathsf{Q}_{\mathcal{B}}$. This statement is equivalent to saying that $\mathsf{Q}_{\mathcal{A}}(f)(x) \leq y$ implies $\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_Y(y)) \leq \psi_X(x)$, for all $x \in \mathsf{Q}_{\mathcal{A}}(X)$ and $y \in \mathsf{Q}_{\mathcal{A}}(Y)$. Letting $y = \mathsf{Q}_{\mathcal{A}}(f)(x)$, we deduce the inclusion

$$\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_Y(\mathsf{Q}_{\mathcal{A}}(f)(x))) \leq \psi_X(x)\,,$$

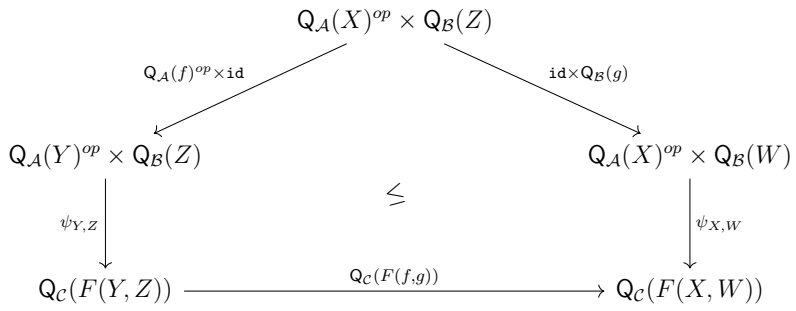
which amounts to the semi-commutativity of (72). On the other hand, this inclusion is sufficient as well: if $\mathsf{Q}_{\mathcal{A}}(f)(x) \leq y$, then, by the order properties of ψ, we have

$$\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_Y(y)) \leq \mathsf{Q}_{\mathcal{B}}(F(f))(\psi_Y(\mathsf{Q}_{\mathcal{A}}(f)(x))) \leq \psi_X(x)\,.$$

Consequently, the contruction of ψ from $\overline{F}$ can be reversed, thus building a lifting of F from a collection ψ with the stated properties. □

Similarly to lax natural transformations, lax extranatural transformations of F are elements of a functorial partially ordered class $\mathsf{lEnat}(F)$, yielding an isomorphism between a category of lifings of functors and natural transformations from $\mathcal{A}^{op}$ to $\mathcal{B}$ and the total category $\int \mathsf{lEnat}$. Let us remark that Theorem 4.1 and Theorem 4.2 can be combined into just one statement, as follows:

THEOREM 4.3. *Liftings of functors $F : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathcal{C}$ to $\overline{F} : (\int \mathsf{Q}_{\mathcal{A}})^{op} \times \int \mathsf{Q}_{\mathcal{B}} \longrightarrow \int \mathsf{Q}_{\mathcal{C}}$ bijectively correspond to collections of order preserving maps $\psi_{X,Z} : \mathsf{Q}_{\mathcal{A}}(X)^{op} \times \mathsf{Q}_{\mathcal{B}}(Z) \longrightarrow \mathsf{Q}_{\mathcal{C}}(F(X, Z))$, such that, for $f : X \longrightarrow Y$ and $g : Z \longrightarrow W$, the following diagram semi-commutes:*



Noticing that the categories $\int \mathsf{Q} \times \int \mathsf{Q}$ and $\int(\mathsf{Q} \times \mathsf{Q})$ are isomorphic, similar statements can be devised for liftings of functors of the form $F : (\mathcal{C}^{op})^n \times \mathcal{C}^m \longrightarrow \mathcal{C}$.

We shall say that Q *factors through* SLatt if

(1) each $\mathsf{Q}(X)$ is complete lattice,
(2) for $f : X \longrightarrow Y$, $\mathsf{Q}(f) : \mathsf{Q}(X) \longrightarrow \mathsf{Q}(Y)$ is sup-preserving.

The reader will have recognised that Q factors through SLatt means indeed that we can write $\mathsf{Q} = U \circ \mathsf{Q}_0$ for a functor $\mathsf{Q}_0 : \mathcal{C} \longrightarrow$ SLatt, where $U :$ SLatt $\longrightarrow$ Pos is the inclusion functor. If Q factors through SLatt, then (lax) extranaturality can be replaced by (lax) naturality, as stated in the next lemma.

Lemma 4.3. *Let $F : \mathcal{A}^{op} \longrightarrow \mathcal{B}$ be a contravariant functor. Suppose that $\mathsf{Q}_{\mathcal{A}}$ (resp., $\mathsf{Q}_{\mathcal{B}}$) factors through* SLatt. *Then, for a lax extranatural transformations $\psi : \mathsf{Q}_{\mathcal{A}}^{op} \longrightarrow \mathsf{Q}_{\mathcal{B}} \circ F$ and $f : X \longrightarrow Y$, the diagram below on the left (resp., on the right) semi-commutes:*

$$\begin{array}{ccc} \mathsf{Q}_{\mathcal{A}}(Y)^{op} & \xrightarrow{\rho(\mathsf{Q}_{\mathcal{A}}(f))} & \mathsf{Q}_{\mathcal{A}}(X)^{op} \\ \downarrow{\scriptstyle \psi_Y} & \swarrow & \downarrow{\scriptstyle \psi_X} \\ \mathsf{Q}_{\mathcal{B}}(F(Y)) & \xrightarrow{\mathsf{Q}_{\mathcal{B}}(F(f))} & \mathsf{Q}_{\mathcal{B}}(F(X)) \end{array} \qquad \begin{array}{ccc} \mathsf{Q}_{\mathcal{A}}(X) & \xrightarrow{\mathsf{Q}_{\mathcal{A}}(f)} & \mathsf{Q}_{\mathcal{A}}(Y) \\ \downarrow{\scriptstyle \psi_X^{op}} & \swarrow & \downarrow{\scriptstyle \psi_Y^{op}} \\ \mathsf{Q}_{\mathcal{B}}(F(X))^{op} & \xrightarrow{\rho(\mathsf{Q}_{\mathcal{B}}(F(f)))} & \mathsf{Q}_{\mathcal{B}}(F(Y))^{op}, \end{array} \tag{73}$$

where ψ_X^{op} is the action of the functor $(-)^{op}$ in Pos.

Proof. We provide a first argument for the Lemma. Suppose that $\mathsf{Q}_{\mathcal{A}}$ factors through SLatt. If $f : (X, x) \longrightarrow (Y, y)$ is in $\int \mathsf{Q}_{\mathcal{A}}$, then $\mathsf{Q}_{\mathcal{A}}(f)(x) \leq y$ is equivalent to $x \leq \rho(\mathsf{Q}_{\mathcal{A}}(f))(y)$, since $\mathsf{Q}_{\mathcal{A}}(f)$ is a sup-preserving map. Thus, the functoriality property of $\overline{F}^{\psi} : (\int \mathsf{Q}_{\mathcal{A}})^{op} \longrightarrow \int \mathsf{Q}_{\mathcal{B}}$ can be stated as follows: if $x \leq \rho(\mathsf{Q}_{\mathcal{A}}(f))(y)$, then $\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_Y(y)) \leq \psi_X(x)$. By instantiating

in this statement x with $\rho(\mathsf{Q}_{\mathcal{A}}(f))(y)$, we obtain

$$\mathsf{Q}_{\mathcal{B}}(F(f))(\psi_Y(y)) \leq \psi_X(\rho(\mathsf{Q}_{\mathcal{A}}(f))(y))\,.$$

By using the order properties of ψ_X, we have that functoriality is actually equivalent to the above inequation. Suppose now that $\mathsf{Q}_{\mathcal{B}}$ factors through SLatt. The functoriality property of $\overline{F}^{\psi} : (\int \mathsf{Q}_{\mathcal{A}})^{op} \longrightarrow \int \mathsf{Q}_{\mathcal{B}}$ can then be stated as follows: if $\mathsf{Q}_{\mathcal{A}}(f)(x) \leq y$, then $\psi_Y(y) \leq \rho(\mathsf{Q}_{\mathcal{B}}(F(f)))(\psi_X(x))$. This gives the inclusion

$$\psi_Y(\mathsf{Q}_{\mathcal{A}}(f)(x)) \leq \rho(\mathsf{Q}_{\mathcal{B}}(F(f)))(\psi_X(x))$$

which, using the order properties of ψ_Y, is again realized to be equivalent to functoriality. Finally, notice that the inclusion above can be understood as an inclusion in the opposite direction within $\mathsf{Q}_{\mathcal{B}}(F(Y))^{op}$. □

Next, we give a more conceptual proof of the lemma.

Another proof of Lemma 4.3. If $\mathsf{Q} : \mathcal{C} \longrightarrow$ SLatt, then, by post-composing with the duality of SLatt, we obtain the functor

$$\mathcal{C}^{op} \xrightarrow{\mathsf{Q}^{op}} \text{SLatt}^{op} \xrightarrow{(-)^{\text{op}}} \text{SLatt}\,.$$

Denoting by Q^* this functor, we have an isomorphism of categories $\int Q^* \simeq (\int \mathsf{Q})^{op}$ making the diagram below commutative:

$$\begin{array}{ccc} (\int \mathsf{Q})^{op} & \xrightarrow{\simeq} & \int Q^* \\ & {\scriptstyle \pi^{op}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathcal{C}^{op} & \end{array}$$

Let now $\overline{F}^{\psi}$ be the lifting of F by the lax natural ψ to $\int \mathsf{Q}_{\mathcal{A}}^{op} \longrightarrow \int \mathsf{Q}_{\mathcal{B}}$. Suppose that $\mathsf{Q}_{\mathcal{A}}$ factors through SLatt. This lifting induces a lifting of $F : \mathcal{A}^{op} \longrightarrow \mathcal{B}$ as follows:

$$\begin{array}{ccccc} \int \mathsf{Q}_{\mathcal{A}}{}^* & \xrightarrow{\simeq} & (\int \mathsf{Q}_{\mathcal{A}})^{op} & \xrightarrow{(\overline{F}^{\psi})} & (\int \mathsf{Q}_{\mathcal{B}})^{op} \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle \pi^{op}} & & \downarrow {\scriptstyle \pi} \\ & & \mathcal{A}^{op} & \xrightarrow{F} & \mathcal{B} \end{array}$$

It is verified that this lifting is induced by the collection of order preserving maps $\psi_X : \mathsf{Q}_{\mathcal{A}}{}^*(X) = \mathsf{Q}_{\mathcal{A}}(X)^{op} \longrightarrow \mathsf{Q}_{\mathcal{B}}(F(X))$, which therefore is lax natural, as in the diagram on the left of (73). Suppose now that $\mathsf{Q}_{\mathcal{B}}$ factors through SLatt. The lifting $\overline{F}^{\psi}$ induces now a lifting of $F^{op} : \mathcal{A} \longrightarrow \mathcal{B}^{op}$ as

follows:

$$\begin{array}{ccccc} \int \mathsf{Q}_{\mathcal{A}} & \xrightarrow{(\overline{F}^{\psi})^{op}} & (\int \mathsf{Q}_{\mathcal{B}})^{op} & \xrightarrow{\simeq} & \int \mathsf{Q}_{\mathcal{B}}{}^{*} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^{op}} & \swarrow{\scriptstyle \pi} & \\ \mathcal{A} & \xrightarrow{F^{op}} & \mathcal{B}^{op} & & \end{array}$$

Recalling that the typing of the collection $\psi_X : \mathsf{Q}_{\mathcal{A}}(X)^{op} \longrightarrow \mathsf{Q}_{\mathcal{B}}(F(X))$, it is verified that this lifting is induced by the collection of order preserving maps $\psi_X^{op} : \mathsf{Q}_{\mathcal{A}}(X) \longrightarrow \mathsf{Q}_{\mathcal{B}}(F(X))^{op} = \mathsf{Q}_{\mathcal{B}}{}^{*}(F^{op}(X))$, which therefore is lax natural, as in the diagram on the right of (73). □

Note that, in the statement of Lemma 4.3, the maps ψ_X are not required to preserve any kind of suprema or infima.

4.1.2. Example: lifting a tensor. Considering the isomorphism $\int \mathsf{Q} \times \int \mathsf{Q} \simeq \int \mathsf{Q} \times \mathsf{Q}$ over $\mathcal{C}$, Theorem 4.1 immediately tells how to lift a tensor structure from $\mathcal{C}$ to $\int Q$.

Proposition 4.1. *A lifting of a symmetric monoidal structure $(I, \otimes, \lambda, \rho, \alpha, \sigma)$ on $\mathcal{C}$ to $\int \mathsf{Q}$ consists of a map (or element) $u : 1 \longrightarrow \mathsf{Q}(I)$ and of a collection μ of order-preserving maps*

$$\{\, \mu_{X,Y} : \mathsf{Q}(X) \times \mathsf{Q}(Y) \longrightarrow \mathsf{Q}(X \otimes Y) \mid X, Y \in \textsc{Obj}(\mathcal{C}) \,\},$$

lax natural in X and Y, satisfying the coherence requirements for a monoidal functor (see Figure 1.1).

Let us recall that lax naturality of μ means that, for $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$, the following diagram semi-commutes:

$$\begin{array}{ccc} \mathsf{Q}(X) \times \mathsf{Q}(Y) & \xrightarrow{\mathsf{Q}(f) \times \mathsf{Q}(g)} & \mathsf{Q}(Z) \times \mathsf{Q}(W) \\ \downarrow{\scriptstyle \mu_{X,Y}} & \swarrow & \downarrow{\scriptstyle \mu_{Z,W}} \\ \mathsf{Q}(X \otimes Y) & \xrightarrow[\mathsf{Q}(f \otimes g)]{} & \mathsf{Q}(Z \otimes W) \end{array} \tag{74}$$

Moreover, the maps $\mu_{X,Y}$ are natural (i.e. diagram (74) fully commutes) if and only if the lifted bifunctor preserves opcartesian arrows, that is the projection functor $\int \mathsf{Q} \longrightarrow \mathcal{C}$ is a monoidal opfibration as defined in [**MV20**].

It is instructive to understand how commutativity of the usual diagrams for a monoidal functor arises. Lifting the natural maps $\lambda, \rho, \alpha, \sigma$ to $\mathcal{C}$ simply

amounts to requiring the following inclusions:

$$\begin{aligned} \mathsf{Q}(\lambda)(\mu_{I,Y}(u,y)) &\leq y\,, & \mathsf{Q}(\rho)(\mu_{X,I}(x,u)) &\leq x\,,\\ \mathsf{Q}(\alpha)(\mu_{X\otimes Y,Z}(\mu_{X,Y}(x,y),z)) &\leq \mu_{X,Y\otimes Z}(x,\mu_{Y,Z}(y,z))\,,\\ \mathsf{Q}(\sigma)(\mu_{X,Y}(x,y)) &\leq \mu_{Y,X}(y,x)\,. \end{aligned}$$

Considering that we actually need these lifted maps to be invertible, by Lemma 4.2 we require the above inclusions to be equalities, thus yielding the commutativity of the diagrams of coherence for monoidal functors—given in Fig. 1.1.

4.2. Lifting adjunctions

We would like to lift the monoidal closed structure of a category $\mathcal{C}$. We have already mentioned how to lift the monoidal structure and the same work can be done to lift the internal hom functor $[-,-]:\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\longrightarrow\mathcal{C}$ with Theorem 4.3. But in order for $\int\mathsf{Q}$ to be monoidal closed, we have to lift the whole adjunction $X\otimes-\dashv[X,-]$. This section works out this problem.

4.2.1. General theory. We suppose next that two functors $F:\mathcal{A}\longrightarrow\mathcal{B}$ and $G:\mathcal{B}\longrightarrow\mathcal{A}$ are part of an adjunction (F,G,η,ϵ) and have liftings $\overline{F}^{\psi}:\int\mathsf{Q}_{\mathcal{A}}\longrightarrow\int\mathsf{Q}_{\mathcal{B}}$ and $\overline{G}^{\iota}:\int\mathsf{Q}_{\mathcal{B}}\longrightarrow\int\mathsf{Q}_{\mathcal{A}}$. Let us recall that

$$\eta_X : X\longrightarrow G(F(X))\,,\qquad \epsilon_Y : F(G(Y))\longrightarrow Y\,,$$

are, respectively, the unit and the counit of the adjunction (natural in X and Y) satisfying the usual pasting diagrams for adjunctions:

$$G(\epsilon_Y)\circ\eta_{G(Y)}=\mathtt{id}_{G(Y)}\,,\qquad \epsilon_{F(X)}\circ F(\eta_X)=\mathtt{id}_{F(X)}\,. \tag{75}$$

In order to lift this adjunction to the total categories, the unit and counit should satisfy

$$\mathsf{Q}_{\mathcal{A}}(\eta_X)(x)\leq\iota_{F(X)}(\psi_X(x))\,, \tag{76}$$

$$\mathsf{Q}_{\mathcal{B}}(\epsilon_Y)(\psi_{G(Y)}(\iota_Y(y)))\leq y\,, \tag{77}$$

for each object X of $\mathcal{A}$, each $x\in\mathsf{Q}_{\mathcal{A}}(X)$, each object Y of $\mathcal{B}$, and each $y\in\mathsf{Q}_{\mathcal{B}}(Y)$.

Let us introduce an explicit notation for the map

$$\ell_Y:=\mathsf{Q}_{\mathcal{A}}(G(Y))\xrightarrow{\psi_{G(Y)}}\mathsf{Q}_{\mathcal{B}}(F(G(Y)))\xrightarrow{\mathsf{Q}_{\mathcal{B}}(\epsilon_Y)}\mathsf{Q}_{\mathcal{B}}(Y)\,. \tag{78}$$

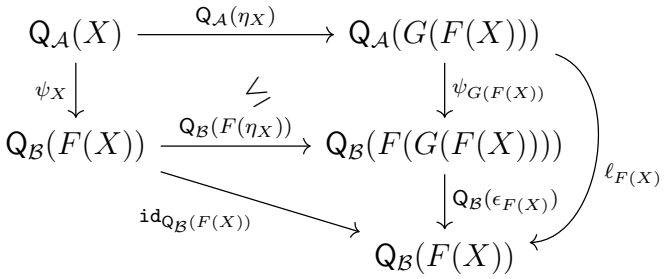
Notice that $\ell = \mathsf{INat}(\epsilon)(\psi)$—see Remark 4.1, so ℓ is itself lax natural, thus, for $f : Y \longrightarrow Y'$ a map of $\mathcal{B}$, it satisfies

$$\mathsf{Q}_{\mathcal{B}}(f) \circ \ell_Y \leq \ell_{Y'} \circ \mathsf{Q}_{\mathcal{A}}(G(f))\,.$$

Using this notation, the inclusion (77) is written as $\ell_Y(\iota_Y(y)) \leq y$, suggesting the presence of a posetal adjunction. Also, the inclusion

$$\psi_X \leq \ell_{F(X)} \circ \mathsf{Q}_{\mathcal{A}}(\eta_X)$$

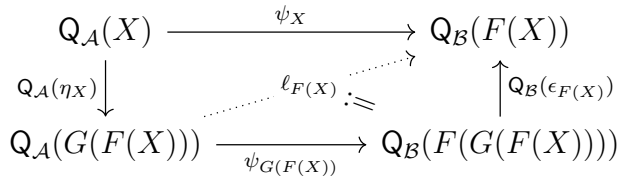
holds. This follows from



(79)

THEOREM 4.4. *The following conditions are equivalent:*

(1) The adjunction (F, G, η, ϵ) lifts to an adjunction $(\overline{F}^{\psi}, \overline{G}^{\iota}, \eta, \epsilon)$.
(2) For each object X of $\mathcal{A}$, the following diagram



(80)

commutes and ι_Y is right adjoint to $\ell_Y : \mathsf{Q}_{\mathcal{A}}(G(Y)) \longrightarrow \mathsf{Q}_{\mathcal{B}}(Y)$.

PROOF. Firstly, let us recall that the map $\iota_Y : \mathsf{Q}_{\mathcal{B}}(Y) \longrightarrow \mathsf{Q}_{\mathcal{A}}(G(Y))$ is the right adjoint of ℓ_Y if and only if the usual unit-counit relations

$$\alpha \leq \iota_Y(\,\ell_Y(\alpha)\,)\,, \qquad\qquad \ell_Y(\iota_Y(\beta)) \leq \beta\,, \tag{81}$$

hold, for each $\alpha \in \mathsf{Q}_{\mathcal{A}}(G(Y))$ and $\beta \in \mathsf{Q}_{\mathcal{B}}(Y)$.
Assume (1), so $\iota_Y : \mathsf{Q}_{\mathcal{B}}(Y) \longrightarrow \mathsf{Q}_{\mathcal{A}}(G(Y))$ is given and satisfies (76) and (77). Let us show first that (80) holds. We have observed that $\psi_X \leq \ell_{F(X)} \circ \mathsf{Q}_{\mathcal{A}}(\eta_X)$. For the opposite inclusion, observe that, using (76) and (77),

$$\ell_{F(X)}(\mathsf{Q}_{\mathcal{A}}(\eta_X)(x)) \leq \ell_{F(X)}(\iota_{F(X)}(\psi_X(x))) \leq \psi_X(x)\,.$$

Next, (77) is the counit in (81). For the unit in (81), we argue as follows:

$$\begin{aligned}\alpha &= \mathsf{Q}_{\mathcal{A}}(G(\epsilon_Y)\circ\eta_{G(Y)})(\alpha) = \mathsf{Q}_{\mathcal{A}}(G(\epsilon_Y))(\mathsf{Q}_{\mathcal{A}}(\eta_{G(Y)})(\alpha))\\ &\leq \mathsf{Q}_{\mathcal{A}}(G(\epsilon_Y))(\iota_{F(G(Y))}(\psi_{G(Y)}(\alpha)))\,, && \text{using (76),}\\ &\leq \iota_Y(\mathsf{Q}_{\mathcal{B}}(\epsilon_Y)(\psi_{G(Y)}(\alpha)))\,, && \text{since } \iota \text{ is lax natural,}\\ &= \iota_Y(\ell_Y(\alpha))\,.\end{aligned}$$

Let assume now (2) and verify that (77) and (76) hold. Again, the co-unit in (81) is exactly the inclusion (77). Thus, we only need to derive (76) . Letting $\alpha := \mathsf{Q}_{\mathcal{A}}(\eta_X)(x) \in \mathsf{Q}_{\mathcal{A}}(G(F(X)))$ in the unit of (81), we have

$$\begin{aligned}\mathsf{Q}_{\mathcal{A}}(\eta_X)(x) &\leq \iota_{F(X)}(\,\ell_{F(X)}(\mathsf{Q}_{\mathcal{A}}(\eta_X)(x))\,)\\ &= \iota_{F(X)}(\psi_X(x))\,, && \text{using (80).}\end{aligned}$$ □

Let us notice that if ψ is natural (and not merely lax natural), then Diagram (80) commutes. Consequently, Theorem 4.4 immediately implies the following statement.

THEOREM 4.5. *Suppose that* $\mathsf{Q}_{\mathcal{A}}, \mathsf{Q}_{\mathcal{B}}$ factor through SLATT *and that* $\psi_X : \mathsf{Q}_{\mathcal{A}}(X) \longrightarrow \mathsf{Q}_{\mathcal{B}}(F(X))$ *is natural (and not merely lax natural) and is a map in* SLATT. *If* (F, G, η, ϵ) *is an adjunction, then* $\overline{F}^{\psi}$ *has a right adjoint that lifts* G.

Let us notice that it could be the case that $\overline{F}^{\psi}$ has a right adjoint, yet this right adjoint fails to lift G. An example of this comes from the closed tensor product of an opfibration described in [**SKDH18**].

4.2.2. Example: lifting the closed structure. We suppose next that $\mathcal{C}$ is symmetric monoidal closed, so for each object X of $\mathcal{C}$, we have the adjunction $X \otimes - \dashv [X, -]$ with unit and counit

$$\eta_{X,Y} : Y \longrightarrow [X, X \otimes Y]\,, \qquad \mathrm{ev}_{X,Y} : X \otimes [X, Y] \longrightarrow Y\,.$$

We suppose that the $\otimes$ has been lifted to $\int \mathsf{Q}$ by means of μ. By Theorem 4.3, a lifting of $[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ amounts to a collection of order-preserving maps

$$\{\, \iota_{X,Y} : Q(X)^{op} \times Q(Y) \longrightarrow Q([X, Y]) \mid X, Y \in \mathrm{OBJ}(\mathcal{C})\,\}\,,$$

such that, for $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$, the following diagram semi-commutes:

$$\begin{array}{ccc} Q(X)^{op} \times Q(Y) & \xrightarrow{\mathtt{id}\times Q(g)} & Q(X)^{op} \times Q(Y') \\ {\scriptstyle Q(f)^{op}\times\mathtt{id}}\downarrow & & \\ Q(X')^{op} \times Q(Y) & \swarrow & \downarrow{\scriptstyle \iota_{X,Y'}} \\ {\scriptstyle \iota_{X',Y}}\downarrow & & \\ Q([X',Y]) & \xrightarrow{Q([f,g])} & Q([X,Y']) \end{array} \tag{82}$$

Therefore, the adjunction $X \otimes - \dashv [X, -]$ lifts to an adjunction

$$(X, x)\overline{\otimes}^{\mu} - \dashv \overline{[X, x, -]}^{\iota}$$

if and only if the following inclusions hold:

$$Q(\eta_{X,Y})(y) \leq \iota_{X,X\otimes Y}(x, \mu_{X,Y}(x, y))\,, \tag{83}$$

$$Q(\mathrm{ev}_{X,Y})(\mu_{X,X\iota Y}(x, \iota_{X,Y}(x, y)) \leq y\,. \tag{84}$$

Analogously to ℓ_Y defined in (78), we introduce the explicit notation

$$\langle -, - \rangle_{X,Y} := Q(\mathrm{ev}_{X,Y}) \circ \mu_{X,[X,Y]} : Q(X) \times Q([X, Y]) \longrightarrow Q(Y)\,.$$

The following theorem is essentially a restatement of Theorem 4.4.

THEOREM 4.6. *The following conditions are equivalent:*

(1) $\mathcal{C}$ *and* $\int \mathsf{Q}$ *are symmetric monoidal closed and* $\pi : \int \mathsf{Q} \longrightarrow \mathcal{C}$ *strictly preserves this structure.*

(2) For each pair of objects X, Y *of* $\mathcal{C}$*, and each* $x \in Q(X)$ *and* $y \in Q(Y)$*, the equality*

$$\mu_{X,Y}(x, y) = \langle\, x,\, Q(\eta_{X,Y})(y)\,\rangle_{X,X\otimes Y} \tag{85}$$

holds and, for each $\alpha \in Q(X)$*, the map*

$$\langle \alpha, - \rangle_{X,Y} : Q([X, Y]) \longrightarrow Q(Y) \tag{86}$$

has a right adjoint $\iota_{X,Y}(\alpha, -)$*.*

Let us recall that if μ is natural (and not merely lax natural) in its right variable, then equation (85) automatically holds.

We say that Q *monoidally factors through* SLATT if Q factors through SLATT and, moreover, $\mu_{X,Y} : \mathsf{Q}(X) \times \mathsf{Q}(Y) \longrightarrow \mathsf{Q}(X \otimes Y)$ is natural (and not merely lax natural) and bilinear, that is, sup-preserving in each variable, separately. That is, writing $\mathsf{Q} = U \circ \mathsf{Q}_0$, the extension of $\mu_{X,Y}$ to the tensor

product $\mathsf{Q}_0(X) \otimes \mathsf{Q}_0(Y)$ in SLATT makes $\mathsf{Q}_0 : \mathcal{C} \longrightarrow$ SLATT into a monoidal functor. Theorem 4.6 immediately implies the following statement.

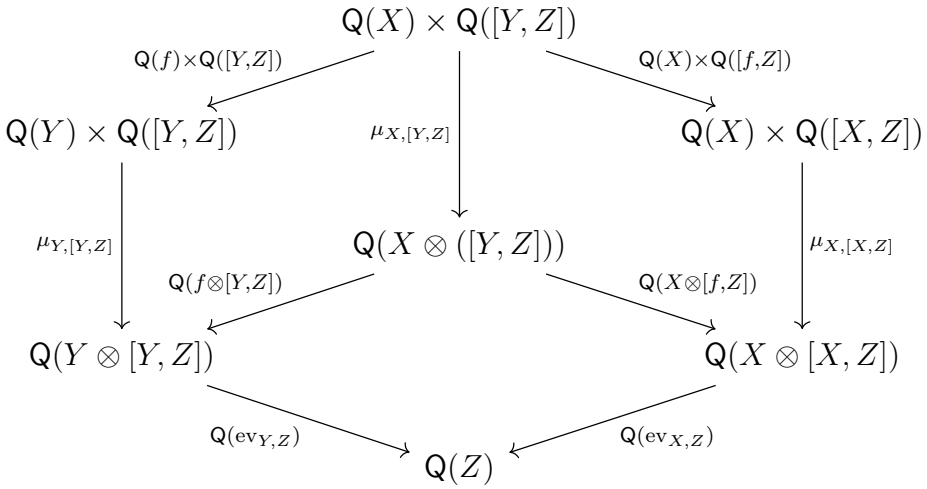
THEOREM 4.7. *If* Q *monoidally factors through* SLATT*, then* $\int \mathsf{Q}$ *is symmetric monoidal closed and the projection functor* $\pi : \int \mathsf{Q} \longrightarrow \mathcal{C}$ *strictly preserves this structure.*

It can be expected that if μ is natural, then ι has also some kind of naturality. This is the content of the next lemma and corollary.

LEMMA 4.4. *Let* $\mathcal{C}$ *and* $\int \mathsf{Q}$ *be symmetric monoidal closed categories such that the first projection preserves the structure on the nose. Suppose that* μ *is natural. Let* $f : X \longrightarrow Y$ *and suppose that* $\mathsf{Q}([f, Z])$ *has a right adjoint* $\rho(\mathsf{Q}([f, Z]))$*. Then, for* $\alpha \in \mathsf{Q}(X)$ *and* $\gamma \in \mathsf{Q}(Z)$*, the following holds:*

$$\iota_{Y,Z}(\mathsf{Q}(f)(\alpha), \gamma) = \rho(\mathsf{Q}([f, X]))(\iota_{X,Z}(\alpha, \gamma))\,. \tag{87}$$

PROOF. Observe first that the following diagram commutes:



Moreover, by Item 2 of Theorem 4.6, for each $y \in Q(Y)$, the map $\iota_{Y,Z}(y, -)$ is the right adjoint of $\langle y, - \rangle_{Y,Z}$. Now, for $\alpha \in Q(X)$,$\beta \in \mathsf{Q}([Y, Z])$, $\gamma \in Q(Z)$, we have

$$\begin{aligned}
\beta \leq \iota_{Y,Z}(\mathsf{Q}(f)(\alpha), \gamma) \;\; &\text{iff} \;\; \mathsf{Q}(\mathrm{ev}_{Y,Z})(\mu_{Y,[Y,Z]}(\mathsf{Q}(f)(\alpha), \beta)) \leq \gamma \\
&\text{iff} \;\; \mathsf{Q}(\mathrm{ev}_{X,Z})(\mu_{X,[X,Z]}(\alpha, \mathsf{Q}([f, Z])(\beta))) \leq \gamma \\
&\text{iff} \;\; \mathsf{Q}([f, Z])(\beta) \leq \iota_{X,Z}(\alpha, \gamma) \\
&\text{iff} \;\; \beta \leq \rho(\mathsf{Q}([f, Z]))(\iota_{X,Z}(\alpha, \gamma))\,.
\end{aligned}$$

□

Notice that Lemma 4.4 applies when $f : X \longrightarrow Y$ is invertible, in which case $\rho(\mathsf{Q}([f, X])) = \mathsf{Q}([f^{-1}, X])$.

In the followings, $0 \in \mathrm{Obj}(\mathcal{C})$ and $\omega \in \mathsf{Q}(0)$ shall be fixed. We let $X^* := [X, 0]$ and $\omega_X : \mathsf{Q}(X) \longrightarrow \mathsf{Q}(X^*)^{op}$ be defined by $\omega_X(\alpha) := \iota_{X,0}(\alpha, \omega)$. Notice that, if Q monoidally factors through SLatt, then ω_X so typed is a map in SLatt, as a consequence of the following computation:

$$\beta \leq \omega_X(\bigvee_{i\in I} \alpha_i) = \iota_{X,0}(\bigvee_{i\in I} \alpha_i, \omega) \text{ iff } \langle \bigvee_{i\in I} \alpha_i, \beta\rangle_{X,0} = \bigvee_{i\in I} \langle \alpha_i, \beta\rangle_{X,0} \leq \omega\,,$$

by the adjunction and since μ is bilinear and $\mathsf{Q}(\mathrm{ev}_{X,0})$ is sup-preserving,

$$\begin{aligned} &\text{iff } \forall i \quad \langle \alpha_i, \beta\rangle_{X,0} \leq \omega \\ &\text{iff } \forall i \quad \beta \leq \omega_X(\alpha_i) \\ &\text{iff } \beta \leq \bigwedge_{i\in I} \omega_X(\alpha_i)\,. \end{aligned}$$

Using $(-)^{\mathrm{op}} : \mathrm{SLatt}^{op} \longrightarrow \mathrm{SLatt}$ to denote the functor sending a complete lattice to its opposite and a sup-preserving map to its right adjoint, Eq. (87) implies commutativity of the following diagram.

$$\begin{array}{ccc} \mathsf{Q}(X) & \xrightarrow{\mathsf{Q}(f)} & \mathsf{Q}(Y) \\ {\scriptstyle \omega_X = \iota_{X,0}(-,\omega)}\downarrow & & \downarrow{\scriptstyle \omega_Y = \iota_{Y,0}(-,\omega)} \\ \mathsf{Q}(X^*)^{\mathrm{op}} & \xrightarrow{\rho(\mathsf{Q}(f^*))} & \mathsf{Q}(Y^*)^{\mathrm{op}} \end{array}$$

Corollary 4.1. *Suppose that* Q *monoidally factors through* SLatt. *Then, considering* Q *as a functor* $\mathcal{C} \longrightarrow$ SLatt, ω_X *is a natural transformation from* $\mathsf{Q}(X)$ *to* $\mathsf{Q}(X^*)^{\mathrm{op}}$.

4.3. Lifting dualizing objects

We incrementally assume that $\int \mathsf{Q}$ and $\mathcal{C}$ are symmetric monoidal closed categories and that the projection strictly preserves all the structure. We use the notation introduced before. Namely, for a fixed object 0 of $\mathcal{C}$, $\omega \in \mathsf{Q}(0)$ and X a $\mathcal{C}$-object, $\omega_X : \mathsf{Q}(X)^{\mathrm{op}} \longrightarrow \mathsf{Q}(X^*)$ is defined by $\omega_X(\alpha) := \iota_{X,0}(\alpha, \omega)$. As the map ω_X is a map in Pos, we shall take the freedom to type it either as $\omega_X : \mathsf{Q}(X)^{\mathrm{op}} \longrightarrow \mathsf{Q}(X^*)$, or as $\omega_X : \mathsf{Q}(X) \longrightarrow \mathsf{Q}(X^*)^{\mathrm{op}}$.

We devise in this section conditions for $(0, \omega)$ to be a dualizing object of $\int \mathsf{Q}$—see Definition 1.7.

Recall from Section 4.2.2 that $(X, \alpha)^* := [(X, \alpha), (0, \omega)] = (X^*, \omega_X(\alpha))$, and notice that, since $\int \mathsf{Q}$ is closed, the canonical arrow $\mathtt{j}_{(X,\alpha)}$ is part of this structure and projects to $\mathtt{j}_X$. As this might not be evident, we explicitly record this fact.

LEMMA 4.5. *For each object X of $\mathcal{C}$ and each $\alpha \in \mathsf{Q}(X)$, the canonical arrow $\mathtt{j}_X : X \longrightarrow X^{**}$ lifts to an arrow $\mathtt{j}_{(X,\alpha)} : (X,\alpha) \longrightarrow (X,\alpha)^{**}$ where $(X,\alpha)^{**} = (X^{**}, \omega_{X^*}(\omega_X(\alpha)))$ in $\int \mathsf{Q}$. That is, we have the inequality $\mathsf{Q}(\mathtt{j}_X)(\alpha) \leq \omega_{X^*}(\omega_X(\alpha))$ for each $\alpha \in \mathsf{Q}(X)$. Equivalently, the following diagram semi-commutes.*

$$\begin{array}{ccc} \mathsf{Q}(X) & \xrightarrow{\omega_X} & \mathsf{Q}(X^*)^{\mathrm{op}} \\ & \searrow^{\mathsf{Q}(\mathtt{j}_X)} & \downarrow{\scriptstyle \omega_{X^*}} \\ & & \mathsf{Q}(X^{**}) \end{array} \tag{88}$$

PROOF. Writing $\mathtt{j}_X = [X^*, \mathrm{ev}_{X,0} \circ \sigma] \circ \eta_{X^*,X}$—-we use here the general formula for transpose in an adjunction—we have for $\alpha \in \mathsf{Q}(X)$

$$\mathsf{Q}(\mathtt{j}_X)(\alpha) = \mathsf{Q}([X^*, \mathrm{ev}_{X,0} \circ \sigma] \circ \eta_{X^*,X})(\alpha) = \mathsf{Q}([X^*, \mathrm{ev}_{X,0} \circ \sigma])(\mathsf{Q}(\eta_{X^*,X})(\alpha))\,.$$

By the inclusion (76),for each $x \in \mathsf{Q}(X^*)$, we have

$$\mathsf{Q}(\mathtt{j}_X)(\alpha) \leq \mathsf{Q}([X^*, \mathrm{ev}_{X,0} \circ \sigma])(\iota_{X^*,X^* \otimes X}(x, \mu_{X^*,X}(x,\alpha)))\,.$$

By instancing Diagram (82) with $f := \mathtt{id}^*_X$, we have for each $x \in \mathsf{Q}(X^*)$

$$\begin{aligned} \mathsf{Q}(\mathtt{j}_X)(\alpha) &\leq \iota_{X^*,0}(x, \mathsf{Q}(\mathrm{ev}_{X,0} \circ \sigma)(\mu_{X^*,X}(x,\alpha))) \\ &= \iota_{X^*,0}(x, \mathsf{Q}(\mathrm{ev}_{X,0})(\mu_{X,X^*}(\alpha,x)))\,, \end{aligned}$$

where the last equality comes from the functoriality of Q and the fact that $\int \mathsf{Q}$ is a symmetric monoidal category. Now, for $x := \iota_{X,0}(\alpha,\omega)$ and by the inequality (77), we have $\mathsf{Q}(\mathrm{ev}_{X,0})(\mu_{X,X^*}(\alpha, \iota_{X,0}(\alpha,\omega))) \leq \omega$. Therefore, we have

$$\begin{aligned} \mathsf{Q}(\mathtt{j}_X)(\alpha) &\leq \iota_{X^*,0}(\iota_{X,0}(\alpha,\omega),\omega) \\ &= \omega_{X^*}(\omega_X(\alpha))\,. \end{aligned}$$

□

We study next under which conditions the maps $\mathtt{j}_{(X,\alpha)}$ are isomorphisms. Using Lemma 4.2 and assuming that $\mathsf{Q}(X) \neq \emptyset$, for each object X of $\mathcal{C}$, we infer the following:

PROPOSITION 4.2. *An object $(0,\omega)$ of $\int \mathsf{Q}$ is dualizing if and only if 0 is a dualizing object of $\mathcal{C}$ and, for each object X of $\mathcal{C}$, the diagram (88) fully commutes.*

PROOF. Asking for the object $(0,\omega)$ in $\int \mathsf{Q}$ to be dualizing means that for each object X of $\mathcal{C}$ and $\alpha \in \mathsf{Q}(X)$, the arrow $\mathtt{j}_{(X,\alpha)} : (X,\alpha) \longrightarrow (X,\alpha)^{**}$ is invertible. By Lemma 4.2 and since we can always find $\alpha \in \mathsf{Q}(X)$, this is equivalent to asking $\mathtt{j}_X$ to be invertible, for each object X of $\mathcal{C}$, and that $\mathsf{Q}(\mathtt{j}_X)(\alpha) = \omega_{X^*}(\omega_X(\alpha))$, for each object X of $\mathcal{C}$ and $\alpha \in \mathsf{Q}(X)$. □

LEMMA 4.6. *If $(0,\omega)$ is dualizing, then, for each object X of $\mathcal{C}$, $\omega_X : \mathsf{Q}(X)^{\mathrm{op}} \longrightarrow \mathsf{Q}(X^*)$ is an isomorphism.*

PROOF. In diagram (88), since $\mathsf{Q}(\mathtt{j}_X)$ is invertible, ω_X is split mono, and ω_{X^*} is split epi, for each object X. Thus, ω_{X^*} is both split epi and split mono and therefore it is invertible. It follows that ω_X is inverted by $\mathsf{Q}(\mathtt{j}_X)^{-1} \circ \omega_{X^*}$. □

As 0 is fixed, we shall write $\langle -,-\rangle_X$ for $\langle -,-\rangle_{X,0}$. To show the converse of Lemma 4.6, we need the following observation.

LEMMA 4.7. *If $\mathcal{C}$ is $*$-autonomous, then the following relation holds*

$$\langle \alpha, \beta\rangle_X = \langle \beta, \mathsf{Q}(\mathtt{j}_X)(\alpha)\rangle_{X^*}\,, \tag{89}$$

for each $\alpha \in \mathsf{Q}(X)$ and $\beta \in \mathsf{Q}(X^*)$. Consequently, if ω_{X^*} is invertible, then diagram (88) commutes.

PROOF. We obtain Eq. (89) by diagram chasing:

$$\begin{array}{ccccc}
\mathsf{Q}(X)\times\mathsf{Q}(X^*) & \xrightarrow{\sigma_{X,X^*}} & \mathsf{Q}(X^*)\times\mathsf{Q}(X) & \xrightarrow{\mathtt{id}\times\mathsf{Q}(\mathtt{j}_X)} & \mathsf{Q}(X^*)\times\mathsf{Q}(X^{**}) \\
\downarrow{\scriptstyle \mu_{X,X^*}} & & \downarrow{\scriptstyle \mu_{X^*,X}} & & \downarrow{\scriptstyle \mu_{X^*,X^{**}}} \\
\mathsf{Q}(X\otimes X^*) & \xrightarrow{\mathsf{Q}(\sigma_{X,X^*})} & \mathsf{Q}(X^*\otimes X) & \xrightarrow{\mathsf{Q}(\mathtt{id}\otimes\mathtt{j}_X)} & \mathsf{Q}(X^*\otimes X^{**}) \\
& {\scriptstyle \mathsf{Q}(\mathrm{ev}_{X,0})}\searrow & \downarrow & \swarrow{\scriptstyle \mathsf{Q}(\mathrm{ev}_{X^*,0})} & \\
& & \mathsf{Q}(0) & &
\end{array}$$

Here the square on the top right commutes since $\mathtt{j}_X$ is invertible and using Lemma 4.1. The lower triangle commutes since it amounts to the definition of $\mathtt{j}_X : X \longrightarrow X^{**}$ as the transpose of $\mathrm{ev}_{X,0}\circ\sigma_{X^*,X} : X^*\otimes X \longrightarrow X\otimes X^* \longrightarrow 0$, and by functoriality of Q. We now argue as follows:

$$\begin{aligned}
\beta \le \omega_X(\alpha) \;\text{ iff }\; & \langle \alpha,\beta\rangle_X \le \omega \\
\text{iff }\; & \langle \beta, \mathsf{Q}(\mathtt{j}_X)(\alpha)\rangle_X \le \omega \\
\text{iff }\; & \mathsf{Q}(\mathtt{j}_X)(\alpha) \le \omega_{X^*}(\beta) \\
\text{iff }\; & \beta \le \omega_{X^*}^{-1}(\mathsf{Q}(\mathtt{j}_X)(\alpha))\,,
\end{aligned}$$

showing the identities $\omega_X = \omega_{X^*}^{-1}\circ\mathsf{Q}(\mathtt{j}_X)$ and $\mathsf{Q}(\mathtt{j}_X) = \omega_{X^*}\circ\omega_X$. □

Assume that $\mathcal{C}$ is $*$-autonomous, so each $\mathtt{j}_X$ has an inverse. Define $\omega^X : \mathsf{Q}(X^*) \longrightarrow \mathsf{Q}(X)$ by

$$\omega^X := \mathsf{Q}(\mathtt{j}_X^{-1}) \circ \omega_{X^*} .$$

The proof of Lemma 4.7 is easily adapted to prove that

$$\beta \leq \omega_X(\alpha) \text{ iff } \alpha \leq \omega^X(\beta) , \tag{90}$$

for all $\alpha \in \mathsf{Q}(X)$ and $\beta \in \mathsf{Q}(X^*)$. That is, ω_X and ω^X form a Galois connection.

THEOREM 4.8. *For each $\omega \in \mathsf{Q}(0)$, $(0, \omega)$ is dualizing if and only if 0 is dualizing in $\mathcal{C}$ and, for each object X of $\mathcal{C}$, ω_X is invertible.*

PROOF. We only need to prove the converse of Lemma 4.6, namely that diagram (88) commutes.
If ω_X is invertible, then it is inverted by its adjoint ω^X. That is, we have $\mathsf{Q}(\mathtt{j}_X^{-1}) \circ \omega_{X^*} = \omega^X = \omega_X^{-1}$, from which commutativity of (88) follows. □

Define now

$$\neg\neg_X^\omega := \omega^X \circ \omega_X : \mathsf{Q}(X) \longrightarrow \mathsf{Q}(X) .$$

As the composite of two maps forming a Galois connection, $\neg\neg_X^\omega$ is a closure operator. We give next a variant of Proposition 4.2 emphasizing the role of the closure operators $\neg\neg_X^\omega$.

PROPOSITION 4.3. *$(0, \omega)$ is a dualizing element of $\int \mathsf{Q}$ if and only if 0 is a dualizing element of $\mathcal{C}$ and, for each object X of $\mathcal{C}$ and each $\alpha \in \mathsf{Q}(X)$, $\neg\neg_X^\omega(\alpha) = \alpha$.*

PROOF. Clearly, the statement that 0 is a dualizing element and, for each object X of $\mathcal{C}$, $\neg\neg_X^\omega = \mathsf{Q}(\mathtt{j}_X^{-1}) \circ \omega_{X^*} \circ \omega_X = \mathtt{id}_{\mathsf{Q}(X)}$, is equivalent to asserting that 0 is a dualizing element and, for each object X of $\mathcal{C}$, diagram (88) commutes. □

4.4. Nuclei on monoidal functors into SLatt

Let us recall that a closure operator j on a unital commutative quantale $(Q, e, *)$ is a *nucleus* if the inclusion $j(j(\alpha) * \beta) \leq j(\alpha * \beta)$ holds. This inclusion is the key property to construct a unital commutative quantale structure on the set $Q_j := \{ \alpha \in Q \mid j(\alpha) = \alpha \}$ of fixed point of j, the multiplication $*_j$ on Q_j being defined by $\alpha *_j \beta := j(\alpha * \beta)$. We refer to Section 2.1 for elementary properties of closure operators and nucleus.

We argue in this section that a similar notion of nucleus can be defined for a symmetric monoidal functor $\mathsf{Q} : \mathcal{C} \longrightarrow \text{SLatt}$, yielding a new symmetric monoidal functor $\mathsf{Q}^{\mathsf{j}} : \mathcal{C} \longrightarrow \text{SLatt}$. As we shall be dealing with two monoidal functors, Q and Q^{j}, we distinguish the structures making $\int \mathsf{Q}$ and $\int \mathsf{Q}^{\mathsf{j}}$ into symmetric monoidal closed categories by adding j in superscript for the second.

Definition 4.2. Let $(\mathsf{Q}, \mu, \eta) : \mathcal{C} \longrightarrow \text{SLatt}$ be a symmetric monoidal functor. A *nucleus* on Q is a collection $j_X : \mathsf{Q}(X) \longrightarrow \mathsf{Q}(X)$ of closure operators, indexed by objects X of $\mathcal{C}$, such that the relations

$$\mathsf{Q}(f)(j_X(\alpha)) \leq j_Y(\mathsf{Q}(f)(\alpha))\,, \tag{91}$$

$$\mu_{X,Y}(\, j_X(\alpha),\, \beta\,) \leq j_{X\otimes Y}(\, \mu_{X,Y}(\, \alpha,\, \beta\,)\,)\,, \tag{92}$$

holds, for each pair of objects X, Y of $\mathcal{C}$, each $f : X \longrightarrow Y$, each $\alpha \in \mathsf{Q}(X)$ and $\beta \in \mathsf{Q}(Y)$. Equivalently, the two following diagram semi-commute.

$$\begin{array}{ccc} \mathsf{Q}(X) & \xrightarrow{\mathsf{Q}(f)} & \mathsf{Q}(Y) \\ {\scriptstyle j_X}\downarrow & \leq & \downarrow{\scriptstyle j_Y} \\ \mathsf{Q}(X) & \xrightarrow[\mathsf{Q}(f)]{} & \mathsf{Q}(Y) \end{array} \qquad \begin{array}{ccc} \mathsf{Q}(X)\times\mathsf{Q}(Y) & \xrightarrow{\mu_{X,Y}} & \mathsf{Q}(X\otimes Y) \\ {\scriptstyle j_X\otimes\mathsf{Q}(Y)}\downarrow & \leq & \downarrow{\scriptstyle j_{X\otimes Y}} \\ \mathsf{Q}(X)\times\mathsf{Q}(Y) & \xrightarrow[\mu_{X,Y}]{} & \mathsf{Q}(X\otimes Y) \end{array}$$

Note that the inequality (91) means that j is lax natural.

Proposition 4.4. *For a nucleus j on the monoidal functor (Q, μ, η), define*

$$\mathsf{Q}^{\mathsf{j}}(X) := \{\, \alpha \in \mathsf{Q}(X) \mid j_X(\alpha) = \alpha \,\}\,, \qquad \mathsf{Q}^{\mathsf{j}}(f)(\alpha) := j_Y(\mathsf{Q}(f)(\alpha))\,,$$

$$\mu^{\mathsf{j}}_{X,Y}(\alpha,\beta) := j_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta))\,, \qquad \eta^{\mathsf{j}} := j_I \circ \eta\,.$$

Then, $(\mathsf{Q}^{\mathsf{j}}, \eta^{\mathsf{j}}, \mu^{\mathsf{j}})$ is a symmetric monoidal functor $\mathcal{C} \longrightarrow \text{SLatt}$, and $j : \mathsf{Q} \longrightarrow \mathsf{Q}^{\mathsf{j}}$ is a monoidal and surjective natural transformation.

Proof. By standard theory, $\mathsf{Q}^{\mathsf{j}}(X)$ is a complete lattice where the supremum of a family $\{\, \alpha_i \mid i \in I \,\}$ is $j_X(\bigvee_i \alpha_i)$ (we shall denote this supremum by $\bigsqcup_{i\in I} \alpha_i$) and, moreover, $j_X : \mathsf{Q}(X) \longrightarrow \mathsf{Q}^{\mathsf{j}}(X)$ is sup-preserving and surjective. It is obvious that Q^{j} preserves the identities. The fact that it preserves the composition and naturality of j are direct consequences of the equality

$$j_Y(\mathsf{Q}(f)(j_X(\alpha))) = j_Y(\mathsf{Q}(f)(\alpha))\,, \tag{93}$$

valid for each $\alpha \in \mathsf{Q}(X)$ and $f : X \longrightarrow Y$, and easily derivable from (91). The equality in (93) also allows us to prove that $\mathsf{Q}^{\mathsf{j}}(f)$ is sup-preserving:

$$\mathsf{Q}^{\mathsf{j}}(f)(\bigsqcup_i \alpha_i) = j_Y(\mathsf{Q}(f)(j_X(\bigvee_i \alpha_i)))] = j_Y(\mathsf{Q}(f)(\bigvee_i \alpha_i)) = j_Y(\bigvee_i \mathsf{Q}(f)(\alpha_i))$$
$$\leq j_Y(\bigvee_i j_Y(\mathsf{Q}(f)(\alpha_i))) = \bigsqcup_i \mathsf{Q}^{\mathsf{j}}(f)(\alpha_i)\,,$$

where the opposite inclusion follows from monotonicity of $\mathsf{Q}^{\mathsf{j}}(f)$. Let us verify that $\mu^{\mathsf{j}} := j \circ \mu$ makes Q^{j} into a monoidal functor. Since μ is compatible with the symmetry, from (92) we derive

$$\mu_{X,Y}(\alpha, j_Y(\beta)) \leq j_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta))\,.$$

Moreover, by monotonicity, $j_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta)) \leq j_{X\otimes Y}(\mu_{X,Y}(j_X(\alpha), j_Y(\beta)))$. Altogether we deduce:

$$\mu^{\mathsf{j}}_{X,Y}(j_X(\alpha), j_Y(\beta)) = j_{X\otimes Y}(\mu_{X,Y}(j_X(\alpha), j_Y(\beta))) = j_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta))\,. \quad (94)$$

From $\mu^{\mathsf{j}} = j \circ \mu$, it immediately follows that μ^{j} is bilinear (as we assume that μ is bilinear) and that it makes commutative the diagrams required for monoidality, see Figure (1.1). Also, (94) yields naturality of μ^{j}:

$$\begin{aligned}
\mathsf{Q}^{\mathsf{j}}(f\otimes g)(\mu^{\mathsf{j}}_{X,Y}(\alpha,\beta)) &= j_{X'\otimes Y'}(\mathsf{Q}(f\otimes g)(j_{X,Y}(\mu_{X,Y}(\alpha,\beta))))\\
&= j_{X'\otimes Y'}(\mathsf{Q}(f\otimes g)(\mu_{X,Y}(\alpha,\beta)))\,, \quad \text{by equation (93)},\\
&= j_{X'\otimes Y'}(\mu_{X',Y'}(\mathsf{Q}(f)(\alpha), \mathsf{Q}(g)(\beta))),\\
&\qquad\qquad \text{since } \mu \text{ is natural},\\
&= j_{X'\otimes Y'}(\mu_{X',Y'}(j_{X'}(\mathsf{Q}(f)(\alpha)), j_{Y'}(\mathsf{Q}(g)(\beta))))\,,\\
&\qquad\qquad \text{by (94)},\\
&= \mu^{\mathsf{j}}_{X',Y'}(\mathsf{Q}^{\mathsf{j}}(f)(\alpha), \mathsf{Q}^{\mathsf{j}}(g)(\beta))\,.
\end{aligned}$$

Because of (94) and $\eta^{\mathsf{j}} = j \circ \eta$, j is a monoidal natural transformation. □

The following remark will be used in the next section.

LEMMA 4.8. *Let j be a nucleus on a monoidal functor $\mathsf{Q} : \mathcal{C} \longrightarrow$ SLATT. If $f : X \longrightarrow Y$ is invertible, then $\mathsf{Q}(f)$ restricts to a map $\mathsf{Q}^{\mathsf{j}}(X) \longrightarrow \mathsf{Q}^{\mathsf{j}}(Y)$. Thus, for such f, $\mathsf{Q}^{\mathsf{j}}(f) = \mathsf{Q}(f)$.*

PROOF. Equation (91) expresses lax naturality of $j : \mathsf{Q} \longrightarrow \mathsf{Q}$. The statement is then an instance of Lemma 4.1. □

An important feature of the quantale $(Q_j, j(e), *_j)$ is that its implication is computed exactly as in Q. We argue next that a similar fact holds for the functors Q and Q^{j}. Let us recall that since Q^{j} is monoidal into SLATT,

$\int \mathsf{Q}^{\mathsf{j}}$ is symmetric monoidal closed, and the projection π strictly preserves the structure. Thus, in $\int \mathsf{Q}^{\mathsf{j}}$, the closed structure is of the form

$$[(X,\alpha),(Y,\beta)] = ([X,Y], \iota^{j}_{X,Y}(\alpha,\beta))\,,$$

where $\iota^{j}_{X,Y} : \mathsf{Q}^{\mathsf{j}}(X)^{op} \times \mathsf{Q}^{\mathsf{j}}(Y) \longrightarrow \mathsf{Q}^{\mathsf{j}}([X,Y])$ is uniquely determined as the right adjoint of some other map—see Theorems 4.6 and 4.7.

PROPOSITION 4.5. *The map $\iota^{j}_{X,Y}$ is the restriction of $\iota_{X,Y}$ to $\mathsf{Q}^{\mathsf{j}}(X)^{op} \times \mathsf{Q}^{\mathsf{j}}(Y)$.*

More in detail, this last proposition is a consequence of the following observation.

LEMMA 4.9. *For a nucleus j on the monoidal functor $\mathsf{Q} : \mathcal{C} \longrightarrow$ SLATT, the following inclusion holds:*

$$j_{[X,Y]}(\iota_{X,Y}(\alpha,\beta)) \leq \iota_{X,Y}(j_X(\alpha), j_Y(\beta))\,. \tag{95}$$

Consequently, if $\alpha \in \mathsf{Q}(X)$ and $\beta \in \mathsf{Q}^{\mathsf{j}}(Y)$, then $\iota_{X,Y}(\alpha,\beta) \in \mathsf{Q}^{\mathsf{j}}([X,Y])$. In particular, $\iota_{X,Y}$ restricts to a map $\mathsf{Q}^{\mathsf{j}}(X)^{op} \times \mathsf{Q}^{\mathsf{j}}(Y) \longrightarrow \mathsf{Q}^{\mathsf{j}}([X,Y])$.

PROOF. We have

$$\begin{aligned}
&\langle j_X(\alpha), j_{[X,Y]}(\iota_{X,Y}(\alpha,\beta))\rangle_{X,Y} \\
&\qquad = \mathsf{Q}(\mathrm{ev}_{X,Y})(\mu_{X,[X,Y]}(j_X(\alpha), j_{[X,Y]}(\iota_{X,Y}(\alpha,\beta)))) \\
&\qquad \leq \mathsf{Q}(\mathrm{ev}_{X,Y})(j_{X\otimes([X,Y])}(\mu_{X,[X,Y]}(j_X(\alpha), j_{[X,Y]}(\iota_{X,Y}(\alpha,\beta))))) \\
&\qquad = \mathsf{Q}(\mathrm{ev}_{X,Y})(j_{X\otimes([X,Y])}(\mu_{X,[X,Y]}(\alpha, \iota_{X,Y}(\alpha,\beta)))) && \text{using equation (94)} \\
&\qquad \leq j_Y(\mathsf{Q}(\mathrm{ev}_{X,Y})(\mu_{X,X\iota Y}(\alpha, \iota_{X,Y}(\alpha,\beta)))) && \text{by inequality (91)} \\
&\qquad \leq j_Y(\beta)\,, && \text{by inequality (77).}
\end{aligned}$$

From the above inequality, the inclusion (95) follows by adjointness. Next, assume $\beta \in \mathsf{Q}^{\mathsf{j}}(Y)$, let $\alpha \in \mathsf{Q}(X)$, and observe that

$$\begin{aligned}
\iota_{X,Y}(\alpha,\beta) &\leq j_{[X,Y]}(\iota_{X,Y}(\alpha,\beta)) \\
&\leq \iota_{X,Y}(j_X(\alpha), j_Y(\beta))\,, && \text{by inequality (95),} \\
&= \iota_{X,Y}(j_X(\alpha), \beta)\,, && \text{since } \beta \in \mathsf{Q}^{\mathsf{j}}(Y), \\
&\leq \iota_{X,Y}(\alpha,\beta)\,, && \text{since } \iota \text{ reverses the order in its first variable,}
\end{aligned}$$

and therefore $\iota_{X,Y}(\alpha,\beta) = j_{[X,Y]}(\iota_{X,Y}(\alpha,\beta))$. □

PROOF OF PROPOSITION 4.5. We have the following characterization of $\langle -,-\rangle^{\mathsf{j}}_{X,Y}$:

$$\begin{aligned}\langle \alpha,\beta\rangle^{\mathsf{j}}_{X,Y} &= \mathsf{Q}^{\mathsf{j}}(\mathrm{ev}_{X,Y})(\,\mu^{\mathsf{j}}_{X,Y}(\alpha,\beta)\,)\\ &= j_Y(\,\mathsf{Q}(\mathrm{ev}_{X,Y})(j_{X\otimes(X\imath Y)}(\mu_{X,X\imath Y}(\alpha,\beta)))\,)\\ &= j_Y(\mathsf{Q}(\mathrm{ev}_{X,Y})(\mu_{X,X\imath Y}(\alpha,\beta))) \qquad \text{by equation (93)}\\ &= j_Y(\langle\alpha,\beta\rangle_{X,Y})\,.\end{aligned}$$

Using this, we see that the right adjoint to $\langle\alpha,\cdot\rangle^{\mathsf{j}}$ is $\iota_{X,Y}(\alpha,\cdot)$. Indeed, for $\alpha\in\mathsf{Q}^{\mathsf{j}}(X)$, $\beta\in\mathsf{Q}^{\mathsf{j}}(X\imath Y)$, and $\gamma\in\mathsf{Q}^{\mathsf{j}}(Y)$, we have $\iota_{X,Y}(\alpha,\gamma)\in\mathsf{Q}^{\mathsf{j}}(Y)$ and

$$\langle\alpha,\beta\rangle^{\mathsf{j}}_{X,Y}=j_Y(\langle\alpha,\beta\rangle_{X,Y})\le\gamma \text{ iff } \langle\alpha,\beta\rangle_{X,Y}\le\gamma \text{ iff } \beta\le\iota_{X,Y}(\alpha,\gamma)\,. \qquad \square$$

4.5. The double negation nucleus

We already mentioned in Section 2.1 that, given an element ω of a unital commutative quantale $(Q,e,*)$, the map $\neg^{\omega}(x):=(x\backslash\omega)\backslash\omega$ is a nucleus, that $\omega\in Q_{\neg^\omega}$, and that the quantale $(Q_{\neg^\omega},\neg^{\omega}(e),*_{\neg^\omega})$ is a Girard quantale—and, therefore, a $*$-autonomous category. We show here that, for a symmetric monoidal functor Q and $\omega\in\mathsf{Q}(0)$, a collection of analogous double negation maps yields a nucleus $\neg^{\omega}$ on Q, that $\omega\in\mathsf{Q}^{\neg^\omega}(0)$, and, finally, that $\int\mathsf{Q}^{\neg^\omega}$ is $*$-autonomous.

4.5.1. Construction of a $*$-autonomous category from any functor monoidally factorizing through SLatt. Let us fix our assumptions. We suppose that $\mathcal{C}$ is $*$-autonomous with 0 as dualizing element, that $\mathsf{Q}:\mathcal{C}\longrightarrow$ SLATT is a symmetric monoidal functor, in particular that μ is natural. Consequently, $\int\mathsf{Q}$ is symmetric monoidal closed according to Theorem 4.7. We fix $\omega\in\mathsf{Q}(0)$. Recall from Section 4.3 that $\omega^X:=\mathsf{Q}(\mathsf{j}_X^{-1})\circ\omega_{X^*}$.

We define, for each object X of $\mathcal{C}$ the maps $\neg^{\omega}_X:Q(X)\longrightarrow Q(X)$ by

$$\neg^{\omega}_X:=\omega^X\circ\omega_X\,.$$

PROPOSITION 4.6. *The collection of maps $\neg^{\omega}_X$ is a nucleus on Q.*

The proof of the proposition amounts to establishing the following 3 lemmas.

LEMMA 4.10. *The collection $\neg^{\omega}$ is lax natural, meaning it satisfies Eq. (91).*

PROOF. Consider a commuting diagram $g \circ f = k \circ h$ in POS

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \underset{\rho(f)}{\leftarrow} & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\rho(k)]{k} & D \end{array}$$

where $\rho(f)$ and $\rho(k)$ are the right adjoints of f and k, respectively. It is easy to see that

$$h(\rho(f)(b)) \leq \rho(k)(g(b))\,, \tag{96}$$

holds for each $b \in B$. We apply now this observation to the following diagram

$$\begin{array}{ccc} Q(X) & \xrightarrow{\omega_X} & Q(X^*)^{\mathrm{op}} \\ {\scriptstyle Q(f)}\downarrow & \underset{\omega^X}{\leftarrow} & \downarrow{\scriptstyle Q(f^*)^{\mathrm{op}}} \\ Q(Y) & \xrightarrow[\omega^Y]{\omega_Y} & Q(Y^*)^{\mathrm{op}} \end{array}$$

where $f : X \longrightarrow Y$ in $\mathcal{C}$, which commutes by Corollary 4.1. Indeed, the Galois connection expressed in (90) can be rephrased by saying that $\omega^X : Q(X^*)^{\mathrm{op}} \longrightarrow Q(X)$ is right adjoint to $\omega_X : Q(X) \longrightarrow Q(X^*)^{\mathrm{op}}$. Thus, applying (96) to this diagram, we obtain

$$Q(f)(\omega^X(\beta)) \leq \omega^Y(Q(f^*)^{\mathrm{op}}(\beta)))$$

for each $\beta \in Q(X^*)$. Therefore, using naturality of ω,

$$\begin{aligned} Q(f)(\neg^\omega_X(\alpha)) &= Q(f)(\omega^X(\omega_X(\alpha))) \\ &\leq \omega^Y(((Q(f^*)^{\mathrm{op}})(\omega_X(\alpha)))) \\ &= \omega^Y(\omega_Y(Q(f)(\alpha))) = \neg^\omega_Y(Q(f)(\alpha))\,. \end{aligned}$$

□

Before arguing that $\neg^\omega$ satifsfies equation (92), we need to develop a few remarks. In the following, let $\psi : [X \otimes Y, Z] \longrightarrow [Y, [X, Z]]$ be the (natural) isomorphism determined by commutativity of

$$\begin{array}{ccc} X \otimes Y \otimes [X \otimes Y, Z] & \xrightarrow{X \otimes Y \otimes \psi} & X \otimes Y \otimes [Y, [X, Z]] \\ & \searrow^{\mathrm{ev}_{X\otimes Y, Z}} & \downarrow{\scriptstyle X \otimes \mathrm{ev}_{Y,[X,Z]}} \\ & & X \otimes ([X, Z]) \\ & & \downarrow{\scriptstyle \mathrm{ev}_{X,Z}} \\ & & Z \end{array}$$

LEMMA 4.11. *The following relation holds:*

$$\langle\, \mu_{X,Y}(\alpha, \beta),\, \gamma \rangle_{X\otimes Y, Z} = \langle\, \alpha,\, \langle \beta, Q(\psi)(\gamma)\, \rangle_{Y,[X,Z]}\, \rangle_{X,Z}\,, \tag{97}$$

for each $\alpha \in Q(X)$, $\beta \in Q(Y)$, *and* $\gamma \in Q([X \otimes Y, Z])$.

PROOF. Recall that we assume in this section that μ is natural. Equation (97) is argued via the diagram chasing in Fig. 4.1. □

LEMMA 4.12. *The lax natural map* $\neg\!\!\urcorner^{\omega}$ *satisfies Eq.* (92). *That is,* $\neg\!\!\urcorner^{\omega}$ *is a nucleus.*

PROOF. We have for $\alpha \in \mathsf{Q}(X)$ and $\beta \in \mathsf{Q}(Y)$, and since $\mathtt{j}$ is invertible,

$$\begin{aligned} & \mu_{X,Y}(\,\neg\!\!\urcorner^{\omega}_X(\alpha),\,\beta\,) \leq \neg\!\!\urcorner^{\omega}_{X\otimes Y}(\,\mu_{X,Y}(\,\alpha,\,\beta\,)\,) \\ \text{iff}\ \ & \mathsf{Q}(\mathtt{j}_{X\otimes Y})(\mu_{X,Y}(\,\neg\!\!\urcorner^{\omega}_X(\alpha),\,\beta\,)) \leq \omega_{X\otimes Y^*} \circ \omega_{X\otimes Y}(\,\mu_{X,Y}(\,\alpha,\,\beta\,)\,) \\ & \qquad\qquad = \iota_{X\otimes Y^*}(\omega_{X\otimes Y}(\,\mu_{X,Y}(\,\alpha,\,\beta\,)\,),\omega) \end{aligned}$$

By the adjunction $\langle \alpha, - \rangle \dashv \iota(\alpha, -)$—see Theorem 4.6, the inequality is equivalent to

$$\langle \omega_{X\otimes Y}(\,\mu_{X,Y}(\,\alpha,\,\beta\,)\,), \mathsf{Q}(\mathtt{j}_{X\otimes Y})(\mu_{X,Y}(\,\neg\!\!\urcorner^{\omega}_X(\alpha),\,\beta\,))\rangle_{X\otimes Y^*} \leq \omega$$

Using Eq. (89), this relation is equivalent to

$$\langle\, \mu_{X,Y}(\neg\!\!\urcorner^{\omega}_X(\alpha),\beta),\, \omega_X(\mu_{X,Y}(\alpha,\beta))\,\rangle_{X\otimes Y} \leq \omega\,.$$

We derive the latter relation by means of the following computations:

$$\begin{aligned} \langle\, \mu_{X,Y}(\neg\!\!\urcorner^{\omega}_X(\alpha),\beta), \omega_X(\mu_{X,Y}(\alpha,\beta))\,\rangle_{X\otimes Y,0} & \\ = \langle \neg\!\!\urcorner^{\omega}_X(\alpha), \langle \beta,\, Q(\psi)(\omega_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta)))\,\rangle_{Y,X^*}\rangle_X\,, & \\ \text{by equation (97)}, & \\ = \langle \neg\!\!\urcorner^{\omega}_X(\alpha), \langle \beta, \iota_{Y,X^*}(\beta,\omega_X(\alpha))\rangle_{Y,X^*}\rangle_X\,, & \\ \text{by equation (98), to be argued for}, & \\ \leq \langle \neg\!\!\urcorner^{\omega}_X(\alpha), \omega_X(\alpha)\rangle_X\,, \qquad \text{by equation (77)}, & \\ = \langle \omega^X(\omega_X(\alpha))), \omega_X(\alpha)\rangle_X & \end{aligned}$$

Again, by Eq. (89) and Eq. (84), we obtain

$$\langle \omega^X(\omega_X(\alpha)), \omega_X(\alpha)\rangle_X = \langle \omega_X(\alpha), \omega_{X^*}(\omega_X(\alpha))\rangle_{X^*} \leq \omega\,.$$

To complete the proof, we need to asses the equality

$$Q(\psi)(\omega_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta))) = \iota_{Y,X^*}(\beta,\omega_X(\alpha))\,. \tag{98}$$

The canonical isomorphism $\psi : [X \otimes Y, Z] \longrightarrow [Y, [X, Z]]$ lifts from $\mathcal{C}$ to $\int \mathsf{Q}$, which means that the following diagram commutes:

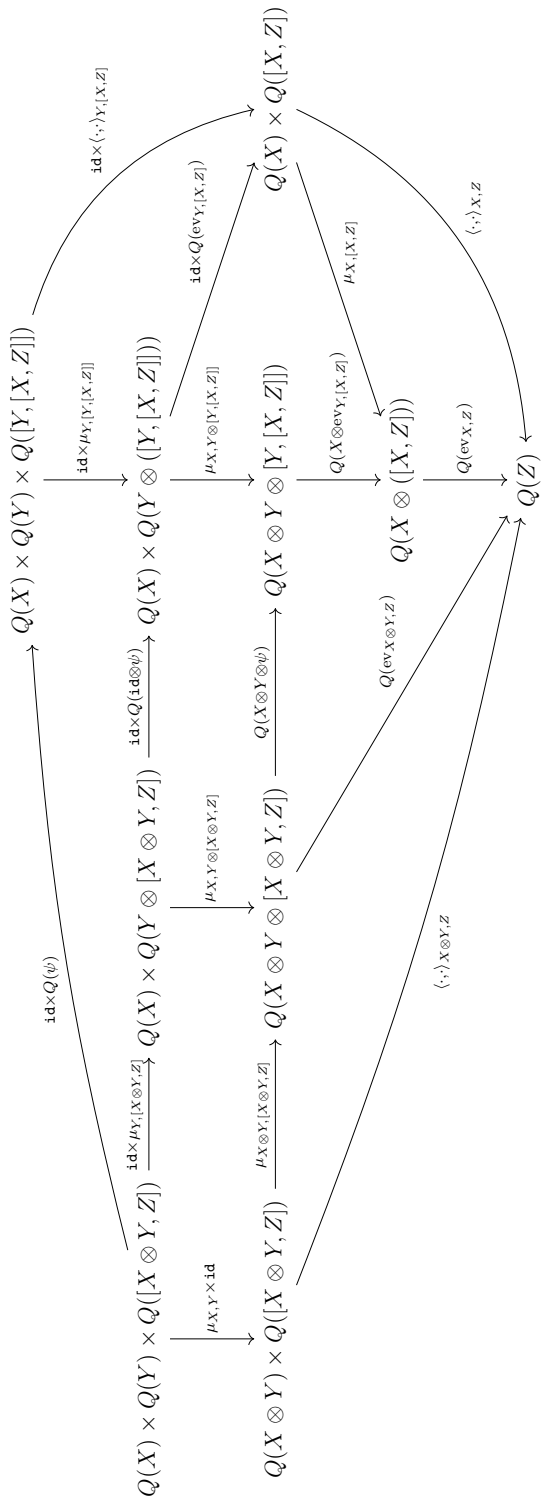


FIGURE 4.1. Proof of Eq. (97)

$$\begin{array}{ccc}
Q(X)^{\mathrm{op}} \times Q(Y)^{\mathrm{op}} \times Q(Z) & \xrightarrow{\sigma_{Q(X)^{\mathrm{op}},Q(Y)^{\mathrm{op}} \times Q(Z)}} & Q(Y)^{\mathrm{op}} \times Q(X)^{\mathrm{op}} \times Q(Z) \\
\downarrow{\scriptstyle \mu^{\mathrm{op}}_{X,Y} \times Q(Z)} & & \downarrow{\scriptstyle Q(Y)^{\mathrm{op}} \times \iota_{X,Z}} \\
Q(X \otimes Y)^{\mathrm{op}} \times Q(Z) & & Q(Y)^{\mathrm{op}} \times Q([X,Z]) \\
\downarrow{\scriptstyle \iota_{X \otimes Y,Z}} & & \downarrow{\scriptstyle \iota_{Y,X\iota Z}} \\
Q([X \otimes Y, Z]) & \xrightarrow{\qquad Q(\psi) \qquad} & Q([Y,[X,Z]])
\end{array}$$

Eq. (98) is just an instance of this commutative diagram, with $Z = 0$. □

We prove now that $\omega \in \mathtt{Q}^{\neg^\omega}(0)$. Before doing this, let us recall a fact on symmetric monoidal closed categories.

Lemma 4.13. *The relation*

$$\mathtt{id}_{X^*} = (\mathtt{j}_X)^* \circ \mathtt{j}_{X^*} \tag{99}$$

hold in any symmetric monoidal closed category.

Proof. The natural isomorphisms of homsets

$$\mathcal{C}^{\mathrm{op}}(X^*, Y) = \mathcal{C}(Y, X^*) \simeq \mathcal{C}(X \otimes Y, 0) \simeq \mathcal{C}(Y \otimes X, 0) \simeq \mathcal{C}(X, Y^*)$$

exhibit $(-)^* : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ as being left adjoint to $(-)^* : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}$. The map $\mathtt{j}_X : X \longrightarrow X^{**}$ is easily seen to be the unit of this adjunction. Thought as a map in $\mathcal{C}^{\mathrm{op}}$ it is also the counit of this adjunction. Therefore, the usual triangle identities of an adjunction (F, G, η, ϵ), namely $G(\epsilon_X) \circ \eta_{G(X)} = \mathtt{id}_{G(X)}$ and $\epsilon_{F(X)} \circ F(\eta_X) = \mathtt{id}_{F(X)}$, hold. The first of these identities takes place in $\mathcal{C}$ and yields (99). The second of these indentities takes place in $\mathcal{C}^{\mathrm{op}}$, and yields (99) as well. □

Lemma 4.14. *Every element of the form $\omega_X(\alpha)$, $\alpha \in Q(X)$, belongs to $\mathtt{Q}^{\neg^\omega}(X^*)$. As a consequence, $\omega \in Q(0)$ belongs to $\mathtt{Q}^{\neg^\omega}(0)$.*

Proof. By Corollary 4.1, the following diagram commutes:

$$\begin{array}{ccc}
Q(X) & \xrightarrow{\omega_X} & Q(X^*)^{\mathrm{op}} \\
\downarrow{\scriptstyle Q(\mathtt{j}_X)} & & \downarrow{\scriptstyle \rho(Q((\mathtt{j}_X)^*))} \\
Q(X^{**}) & \xrightarrow{\omega_{X^{**}}} & Q(X^{***})^{\mathrm{op}}
\end{array}$$

Next, since $\mathtt{j}_X$ is invertible, then so is $Q((\mathtt{j}_X)^*)$, thus, its right adjoint $\rho(Q((\mathtt{j}_X)^*))$ is necessarily its inverse. We have, therefore,

$$\rho(Q((\mathtt{j}_{X^*})^*)) = Q((\mathtt{j}_X)^*)^{-1} = Q((\mathtt{j}_X)^{*-1}) = Q(\mathtt{j}_{X^*})\,.$$

where the equality $(\mathtt{j}_X)^{*-1} = \mathtt{j}_{X^*}$ follows from the identity $(\mathtt{j}_X)^* \circ \mathtt{j}_{X^*} = \mathtt{id}_{X^*}$, valid in any symmetric monoidal closed category. This yields commutativity of the two diagrams [1]

$$\begin{array}{ccc} Q(X) & \xrightarrow{\omega_X} & Q(X^*)^{\mathrm{op}} \\ \downarrow{\scriptstyle Q(\mathtt{j}_X)} & & \downarrow{\scriptstyle Q(\mathtt{j}_{X^*})} \\ Q(X^{**}) & \xrightarrow{\omega_{X^*}} & Q(X^{***})^{\mathrm{op}} \end{array} \qquad \begin{array}{ccc} Q(X) & \xrightarrow{\omega_X} & Q(X^*)^{\mathrm{op}} \\ \uparrow{\scriptstyle Q(\mathtt{j}_X^{-1})} & & \uparrow{\scriptstyle Q(\mathtt{j}_{X^*}^{-1})} \\ Q(X^{**}) & \xrightarrow{\omega_{X^*}} & Q(X^{***})^{\mathrm{op}} \end{array}$$

Using this, we can compute as follows:

$$\begin{aligned} \neg^\omega_{X^*} \circ \omega_X &= Q(\mathtt{j}_{X^*}^{-1}) \circ \omega_{X^{**}} \circ \omega_{X^*} \circ \omega_X \\ &= \omega_X \circ Q(\mathtt{j}_X^{-1}) \circ \omega_{X^*} \circ \omega_X = \omega_X \circ \neg^\omega_X = \omega_X\,, \end{aligned}$$

where the last equality follows from ω_X and $\omega^X := Q(\mathtt{j}_{X^*}^{-1}) \circ \omega_{X^*}$ forming a Galois connection. This proves that every element in the image of ω_X belongs to $\mathsf{Q}^{\neg^\omega}(X^*)$. Consider now the canonical isomorphism $\xi_X : X \longrightarrow [I, X]$. Since ξ_X is the transpose of the canonical map $\lambda_X : I \otimes X \longrightarrow X$, $\int \mathsf{Q}$ is symmetric monoidal closed, and the projection strictly preserves the symmetric monoidal closed structure, ξ_X lifts to the canonical isomorphism $\xi_{(X,\alpha)} : (X, \alpha) \longrightarrow [(I, u), (X, \alpha)]$, for each $\alpha \in Q(X)$ (and where $u \in Q(I)$ is the unit of the monoidal structure of Q). That is, we have the relation

$$Q(\xi_X)(\alpha) = \iota_{I,X}(u, \alpha)\,,$$

for each $\alpha \in Q(X)$. Taking $X = 0$ and $\alpha = \omega \in Q(0)$ above, this yields $Q(\xi_0)(\omega) = \iota_{I,0}(u, \omega) = \omega_X(u)$. Using the first statement, this shows that $Q(\xi_0)(\omega)$ belongs to $\mathsf{Q}^{\neg^\omega}([I, 0])$. Considering now $\xi_0^{-1} : [I, 0] \longrightarrow 0$, we deduce $\omega = Q(\xi_0^{-1})(Q(\xi_0)(\omega)) \in \mathsf{Q}^{\neg^\omega}(0)$, using Lemma 4.8. □

Noticing that $\omega \in \mathsf{Q}^{\neg^\omega}(0)$ and that the nucleus $\neg^\omega$ on $\mathsf{Q}^{\neg^\omega}$ is computed exactly as for Q, the following theorem follows from Proposition 4.3.

THEOREM 4.9. *Let $\mathcal{C}$ be a $*$-autonomous category and $\mathsf{Q} : \mathcal{C} \longrightarrow$ SLATT be a monoidal functor. Let 0 be a dualizing object of $\mathcal{C}$ and pick $\omega \in \mathsf{Q}(0)$. Then $(0, \omega)$ is a dualizing object of $\int \mathsf{Q}^{\neg^\omega}$.*

[1] Notice that since $Q(\mathtt{j}_{X^*})$ is invertible, it preserves all infima, thus it can be considered as a map $Q(X^*)^{\mathrm{op}} \longrightarrow Q(X^{**})^{\mathrm{op}}$ in SLATT.

PROOF. Once more, we add $\neg^{\omega}$ in superscript where needed in order to distinguish the structures of $\int \mathsf{Q}$ and $\int \mathsf{Q}^{\neg^{\omega}}$. Since $\omega \in \mathsf{Q}^{\neg^{\omega}}(0)$, then $\omega_X^{\neg^{\omega}}(\alpha) = \iota_{X,0}^{\neg^{\omega}}(\alpha,\omega) = \iota_{X,0}(\alpha,\omega) = \omega_X(\alpha)$. We also have

$$\begin{aligned}\omega^{\neg^{\omega},X}(\beta) &= \mathsf{Q}^{\neg^{\omega}}(\mathtt{j}_X^{-1})(\omega_{X^*}^{\neg^{\omega}}(\beta)) \\ &= \mathsf{Q}^{\neg^{\omega}}(\mathtt{j}_X^{-1})(\omega_{X^*}(\beta)) = \mathsf{Q}(\mathtt{j}_X^{-1})(\omega_{X^*}(\beta)) = \omega^X(\beta),\end{aligned}$$

using Lemmas 4.14 and 4.8.
Thus $\omega^{\neg^{\omega},X}(\omega_X^{\neg^{\omega}}(\alpha)) = \omega^X(\omega_X(\alpha)) = \neg_X^{\omega}(\alpha) = \alpha$, for every object X of $\mathcal{C}$ and every $\alpha \in \mathsf{Q}^{\neg^{\omega}}(X)$. Therefore, $\int \mathsf{Q}^{\neg^{\omega}}$ is $*$-autonomous, using Proposition 4.3. □

4.5.2. Representation theorem. We end this section by presenting a representation theorem analogous to the well-known representation theorem for Girard quantales that yields the completeness of phase semantics of linear logic—see Section 2.1. The theorem is also meant to illustrate how our construction relates to the focused orthogonality categories introduced in [**HS03**] and further studied in [**FGJ23**]—see Example 4.2.

Consider the functor PUQ defined as the composite

$$\mathcal{C} \xrightarrow{\ \mathsf{Q}\ } \textsc{SLatt} \xrightarrow{\ U\ } \textsc{Set} \xrightarrow{\ P\ } \textsc{SLatt}$$

of Q, of the forgetful functor to SET, and of the covariant powerset functor P (free complete sup-lattice functor). Recall that both U and P are monoidal functors. It follows that PUQ is monoidal.

THEOREM 4.10. *Let $\mathsf{Q} : \mathcal{C} \longrightarrow \textsc{SLatt}$ be a monoidal functor and suppose that $(0,\omega)$ is a dualizing element of $\int \mathsf{Q}$. Define*

$$\Omega := \{\, x \in \mathsf{Q}(0) \mid x \leq \omega \,\} \in \mathsf{PUQ}(0)\,.$$

Then, the maps sending $x \in \mathsf{Q}(X)$ to

$$\downarrow_X(x) := \{\, x' \in \mathsf{Q}(X) \mid x' \leq x \,\} \subseteq \mathsf{Q}(X)$$

form a monoidal natural isomorphism between the monoidal functors $\mathsf{Q} : \mathcal{C} \longrightarrow \textsc{SLatt}$ and $\mathsf{PUQ}^{\neg^{\Omega}} : \mathcal{C} \longrightarrow \textsc{SLatt}$. Consequently, $\int \mathsf{Q}$ and $\int \mathsf{PUQ}^{\neg^{\Omega}}$ are isomorphic.

PROOF. First, note that the multiplication of the functor PUQ is, by composition of natural transformations, given by

$$\begin{aligned}\mu^{\mathsf{PUQ}} : \mathsf{PUQ}(X) \times \mathsf{PUQ}(Y) &\longrightarrow \mathsf{PUQ}(X \otimes Y) \\ (A,B) &\longmapsto \{\, \mu_{X,Y}(x,y) \mid x \in A, y \in B \,\}\,.\end{aligned}$$

Moreover, we have

$$\iota^{\mathsf{PUQ}} : \mathsf{PUQ}(X)^{\mathrm{op}} \times \mathsf{PUQ}(Y) \longrightarrow \mathsf{PUQ}([X,Y])$$
$$(A,B) \longmapsto \{\, \beta \in Q([X,Y]) \mid \langle x, \beta\rangle_{X,Y} \in B \text{ for all } x \in A \,\}.$$

Let us argue that a set $A \subseteq \mathsf{Q}(X)$ belongs to $\mathsf{PUQ}^{\neg^\Omega}(X)$ if and only if it is of the form $\downarrow_X(\alpha)$ for some $\alpha \in \mathsf{Q}(X)$. To this goal, observe that, for $B \in \mathsf{PUQ}(X^*)$, we have

$$\begin{aligned}
\Omega^X(B) &= \mathsf{PUQ}(\mathrm{j}_X^{-1})(\Omega_{X^*}(B)) \\
&= \{\, \mathsf{Q}(\mathrm{j}_X^{-1})(\zeta), \zeta \in \Omega_{X^*}(B) \,\} \\
&= \{\, \alpha \in \mathsf{Q}(X) \mid \mathsf{Q}(\mathrm{j}_X)(\alpha) \in \Omega_{X^*}(B) \,\} \\
&= \{\, \alpha \in \mathsf{Q}(X) \mid \langle \beta, \mathsf{Q}(\mathrm{j}_X)(\alpha)\rangle_{X^*,X^{**}} \in \Omega, \text{ for all } \beta \in B \,\} \\
&= \{\, \alpha \in \mathsf{Q}(X) \mid \langle \alpha, \beta\rangle_{X,X^*} \in \Omega, \text{ for all } \beta \in B \,\} \\
&= \{\, \alpha \in \mathsf{Q}(X) \mid \langle \alpha, \beta\rangle_X \leq \omega, \text{ for all } \beta \in B \,\}.
\end{aligned}$$

Moreover, by general theory of Galois connections, all the elements of $\mathsf{PUQ}^{\neg^\Omega}(X)$ are of this form. Now, it is immediate that if $\alpha \in \Omega^X(B)$ and $\alpha' \leq \alpha$, then $\alpha' \in \Omega^X(B)$. Moreover, since $\langle -, \beta\rangle_X$ is sup-preserving, $\bigvee \Omega^X(B) \in \Omega^X(B)$. Thus, if $A \in \mathsf{PUQ}^{\neg^\Omega}(X)$, then $A = \downarrow \bigvee A$.
Next, for $\alpha \in \mathsf{Q}(X)$, $\downarrow_X(\alpha) \in \mathsf{PUQ}^{\neg^\Omega}(X)$, since

$$\Omega^X(\{\, \omega_X(\alpha) \,\}) = \{\, \alpha' \in \mathsf{Q}(X) \mid \langle \alpha', \omega_X(\alpha)\rangle_X \leq \omega \,\} = \downarrow_X(\neg^\omega_X(\alpha)) = \downarrow_X(\alpha),$$

since we are assuming that $\mathsf{Q}(X) = \mathsf{Q}^{\neg^\omega}(X)$. Thus, $\mathsf{PUQ}^{\neg^\Omega}(X)$ is the set of all the principal downsets of $Q(X)$ and $\downarrow_X(-) : \mathsf{Q}(X) \longrightarrow \mathsf{PUQ}^{\neg^\Omega}(X)$ is an order isomorphism, thus an isomorphism in SLatt.
For any closure operator j on a complete lattice L, $j(x)$ is the greatest element of $\{\, y \in L \mid j(y) = y \,\}$. It follows then that

$$\neg^\Omega_X(A) = \downarrow_X(\bigvee A), \quad \text{for each } A \subseteq \mathsf{Q}(X).$$

For $f : X \longrightarrow Y$ in $\mathcal{C}$, we have

$$\begin{aligned}
\mathsf{PUQ}^{\neg^\Omega}(f)(\downarrow_X(\alpha)) &= \downarrow_Y(\bigvee\{\, \mathsf{Q}(f)(\alpha') \mid \alpha' \leq \alpha \,\}) \\
&= \downarrow_Y(\mathsf{Q}(f)(\bigvee\{\, \alpha' \mid \alpha' \leq \alpha \,\}) = \downarrow_Y(\mathsf{Q}(f)(\alpha)),
\end{aligned}$$

showing that the maps sending $\alpha \in \mathsf{Q}(X)$ to $\downarrow_X(\alpha) \in \mathsf{PUQ}^{\neg^\Omega}(X)$ are natural in X. These maps are also monoidal, as the equality

$$\downarrow_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta)) = \{\, \gamma \in \mathsf{Q}(X \otimes Y) \mid \exists \alpha' \leq \alpha, \beta' \leq \beta \text{ s.t. } \gamma \leq \mu_{X,Y}(\alpha', \beta') \,\}$$

is easily verified. □

4.6. Examples

EXAMPLE 4.1 (Categories of poset-valued sets). A main reason for developing this research was to understand the structure of categories of the form $Q_{\mathtt{id}}$-SET introduced in [**SdP04**] and, possibly, to devise generalization of this kind of constructions.

Recall that for $Q = (Q, 1, *)$ a unital commutative quantale, the category $Q_{\mathtt{id}}$-SET is the total category of $\mathsf{Q} : \text{REL} \to \text{POS}$ defined on sets X by $\mathsf{Q}(X) := Q^X$, and for on relations $R : X \longrightarrow Y$, by

$$\mathsf{Q}(R)(\alpha)(y) := \bigvee_{xRy} \alpha(x)\,, \quad \text{for each } \alpha \in Q^X \text{ and } y \in Y.$$

It is easily verified that $\mathsf{Q}(X)$ is a complete lattice, that $\mathsf{Q}(R)$ is sup-preserving. Moreover, for each set X, $\mathsf{Q}(X)$ is a quantale (with the product structure). For a relation $R : X \longrightarrow Y$, only the inclusions $\mathsf{Q}(R)(1) \leq 1$ and $\mathsf{Q}(R)(\alpha * \beta) \leq \mathsf{Q}(R)(\alpha) * \mathsf{Q}(R)(\beta)$ hold—that is, $\mathsf{Q}(R)$ is comonoidal. Nonetheless, and recalling that $\mathsf{Q}(\{*\})$ can be identified with Q, if we define

$$\eta(\{*\}) := 1 \in \mathsf{Q}(\{*\})\,, \qquad \mu_{X,Y}(\alpha, \beta) := \alpha * \beta\,,$$

then $\mu_{X,Y}$ is natural (and not merely lax-natural), η and μ make Q into a monoidal functor, and $\mu_{X,Y}$ is bilinear. Thus Q *monoidally* factors though SLATT. The monoidal structure of Q-SET exhibited in [**SdP04**] is the one induced by $\mu_{X,Y}$. Recall that REL is, w.r.t. the monoidal structure given by $\times$ and $\{*\}$, compact closed, that is, we have $[X, Y] = X \times Y$ and $0 = I = \{*\}$. It is verified that

$$\iota_{X,Y}(\alpha, \beta)(x, y) = \alpha(x) \backslash \beta(y)\,, \tag{100}$$

thus, for $\omega \in Q \simeq \mathsf{Q}(0)$, $\omega_X(\alpha)(x) = \alpha(x) \backslash \omega$.

In [**SdP04**] this construction was also generalized as follows. Let $F : \text{REL} \longrightarrow \text{REL}$ be a 2-functor that is comonoidal and whose natural arrows $\nu_{X,Y} : F(X \times Y) \longrightarrow F(X) \times F(Y)$ are functional. An object of Q_FSET is a pair (X, α) with $\alpha \in \mathsf{Q}(F(X))(= Q^{F(X)})$. An arrow in Q-SET from (X, α) to (Y, β) is a relation $R : X \longrightarrow Y$ such that, for each $x \in F(X)$ and $y \in F(Y)$, if $xF(R)y$, then $\alpha(x) \leq \beta(y)$. Let $\mathsf{c} : \text{REL} \longrightarrow \text{REL}^{op}$ be the converse functor, identity on objects and picking the converse of a given relation. Given F as above, $\mu^F_{X,Y} := \nu^{\mathsf{c}}_{X,Y} : F(X) \times F(Y) \longrightarrow F(X \times Y)$ makes F into a monoidal functor. Recall next that if $F : \text{REL} \longrightarrow \text{REL}$ and $\mathsf{Q} : \text{REL} \longrightarrow \text{POS}$ are monoidal, then their composition $\mathsf{Q} \circ F$ is monoidal as well , with natural transformation

$$\mu^{\mathsf{Q}\circ F}_{X,Y} = \mathsf{Q}(\mu^F_{X,Y}) \circ \mu^{\mathsf{Q}}_{F(X),F(Y)} : \mathsf{Q}(F(X)) \times \mathsf{Q}(F(Y)) \longrightarrow \mathsf{Q}(F(X \times Y))\,.$$

Clearly $\mathsf{Q} \circ F$ monoidally factors through SLATT and the equality $Q_F\text{SET} = \int \mathsf{Q} \circ F$ holds. The monoidal structure described in [**SdP04**] is the one arising from the monoidal functor $\mathsf{Q} \circ F$.

A substantial generalization of the category Q-SET arises if we replace REL with Q-REL, the category of relations or matrices over Q. An object of the category of Q-REL is a set, while an arrow $X \longrightarrow Y$ is a map $\psi : X \times Y \longrightarrow Q$. Identity is the Kronecker δ, while composition of $\psi : X \longrightarrow Y$ and $\chi : Y \longrightarrow Z$ is defined as follows:

$$(\psi \cdot \chi)(x, z) := \bigvee_{y \in Y} \psi(x, y) * \chi(y, z) \,.$$

The category Q-REL is compact closed— see [**KL80**, **Ros94**]—with $X \otimes Y = [X, Y] = X \times Y$, $I = 0 = \{*\}$, and $X^* = X$ on objects. Q-REL is also a quantaloid, that is, a category enriched over SLATT. From this it follows that Q-REL$(\{*\}, -)$ is a monoidal functor from Q-REL to SLATT. Thus, $\int Q\text{-REL}(\{*\}, (-))$ is monoidal closed. Since formulas as in (100) still hold in this wider setting, then dualizing objects of $\int Q\text{-REL}(\{*\}, -)$ correspond to dualizing elements in Q-REL$(\{*\}, \{*\}) \simeq Q$.

Recall that a fuzzy set is a map $\alpha : X \longrightarrow [0, 1]$. The unit interval $[0, 1]$ has several quantale structures, for example, it is a complete Heyting algebra. A *t-norm* on $[0, 1]$ is a continuous multiplication making $[0, 1]$ into a quantale. Fuzzy sets have been organised in several categories of fuzzy relations—see *e.g.*, [**Win07**, **HWW14**, **Moč15**]. For $Q = [0, 1]$ (with a t-norm), the category $\int Q$-REL$(\{*\}, -)$ accommodates these constructions.

$\Diamond$

EXAMPLE 4.2 (Focused orthogonality categories). The category NUTS is an example of $*$-autonomous category $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{C})$ arising from a focused orthogonality structure [**HS03**]. It was recently recognised [**FGJ23**] that these categories arise as total categories of some functor. Let us reconsider these categories with the help of the theory we have developed.

For a monoidal category $\mathcal{C}$, the hom functor $\mathcal{C}(I, -) : \mathcal{C} \longrightarrow \text{SET}$ is monoidal, since we can associate to $x : I \longrightarrow X$ and $y : I \longrightarrow Y$, the arrow $x \bullet y := I \xrightarrow{\simeq} I \otimes I \xrightarrow{x \otimes y} X \otimes Y$. The powerset functor $\mathcal{P}(-) : \text{SET} \longrightarrow$ SLATT is also monoidal. By composing these functors, we obtain the functor $\mathcal{P}(\mathcal{C}(I, -)) : \mathcal{C} \longrightarrow$ SLATT whose monoidal structure is given by

$$\mu_{X,Y}(\alpha, \beta) = \{\, x \bullet y \mid x \in \alpha,\, y \in \beta \,\} \,.$$

Next, if $\mathcal{C}$ is $*$-autonomous, then there is a natural isomorphism $\sharp : \mathcal{C}(X, 0) \longrightarrow \mathcal{C}(I, X^*)$ with the property that, for $x : I \longrightarrow X$ and $y : X \longrightarrow 0$,

$y \circ x = \mathrm{ev}_{X,0} \circ (x \bullet y^\sharp)$. Let now $\perp\!\!\!\perp \subseteq \mathcal{C}(I,0)$. Using the notation of [**FGJ23**], for $\alpha \subseteq \mathcal{C}(I,X)$, we have

$$\begin{aligned}
\mathcal{P}((\sharp)(\alpha^\perp)) &= \{\, y^\sharp \mid y \in \mathcal{C}(0,X),\ y \circ x \in \perp\!\!\!\perp,\ \text{for all } x \in \alpha \,\} \\
&= \{\, y \in \mathcal{C}(I,X^*) \mid \mathrm{ev}_{X,0} \circ (x \bullet y) \in \perp\!\!\!\perp,\ \text{for all } x \in \alpha \,\} \\
&= \bigcup \{\, \{\, y \,\} \in \mathcal{C}(I,X^*) \mid \mathcal{P}(\mathcal{C}(I,\mathrm{ev}_{X,0}))(\mu_{X,X^*}(\alpha, \{\, y \,\})) \subseteq \perp\!\!\!\perp \,\} \\
&= \bigcup \{\, \beta \subseteq \mathcal{C}(I,X^*) \mid \mathcal{P}(\mathcal{C}(I,\mathrm{ev}_{X,0}))(\mu_{X,X^*}(\alpha,\beta)) \subseteq \perp\!\!\!\perp \,\} = \iota_{X,0}(\alpha, \perp\!\!\!\perp)\,.
\end{aligned}$$

That is, the Galois connection described in [**HS03, FGJ23**] is, up to the isomorphism $P(\sharp) : P(\mathcal{C}(X,0)) \simeq P(\mathcal{C}(I,X^*))$, the one we describe in Section 4.5. The category $\mathcal{O}_\perp(\mathcal{C})$ is, therefore, a special case of the given construction $\int \mathsf{Q}^{\neg^\omega}$, where Q has been instantiated with $P(\mathcal{C}(I,-))$ and $\omega = \perp\!\!\!\perp \in P(\mathcal{C}(I,0))$.

4.7. Frobenius functors

In Section 4.1.2, we have observed that for every monoidal functors $\mathsf{Q} : \mathcal{C} \to \textsc{Pos}$, the total category $\int \mathsf{Q}$ is a monoidal category. Of course, monoidal functors are central in category theory, and they have widely been inspected. Notably, if $\mathcal{C}$ is a small category—meaning the objects and the morphisms of $\mathcal{C}$ are actual sets—and $\mathcal{V}$ has all limits and colimits, monoidal functors $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{V}$ exactly correspond to monoids in the category of functor $[\mathcal{C},\mathcal{V}]$ under the Day's convolution—this has been firstly observed in [**Day70b**, Example 3.2.2]. If, furthermore, $\mathcal{C}$ and $\mathcal{V}$ are $*$-autonomous categories, then $[\mathcal{C},\mathcal{V}]$ has a canonical $*$-autonomous structure. This was shown by Egger in [**Egg08**]. In this section, we present some Frobenius structures in $[\mathcal{C},\mathcal{V}]$—where $\mathcal{C}$ and $\mathcal{V}$ respect the previous conditions. Our objective is to show that, if $\mathcal{V} := \textsc{SLatt}$, the $*$-autonomous structure of $\mathcal{C}$ lifts to the total category of a functor $\mathsf{Q} : \mathcal{C} \to \textsc{SLatt}$ if and only if this functor may be endowed with this specific Frobenius structure.

In the remaining of this section, $\mathcal{C}$ is a small $*$-autonomous category with $0_\mathcal{C}$ its dualizing object and $\mathcal{V}$ is a $*$-autonomous category with all limits and colimits, and $0_\mathcal{V}$ is its dualizing object.

Mainly to settle our notations, we shall quickly introduce the monoidal structure of the functor category $[\mathcal{C},\mathcal{V}]$. We will follow the description given by Egger in [**Egg08**]. We will only provide the structure used in the sequel and we will not enter much into details.

Let $\mathsf{F}, \mathsf{G} : \mathcal{C} \longrightarrow \mathcal{V}$ be two functors. The functor $\mathsf{F} \otimes \mathsf{G} : \mathcal{C} \longrightarrow \mathcal{V}$ is defined on objects X by the colimit of the composite of functors

$$\otimes \downarrow X \xrightarrow{\Pi} \mathcal{C} \times \mathcal{C} \xrightarrow{\mathsf{F} \otimes \mathsf{G}} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}\,,$$

where $\otimes \downarrow X$ is the category whose objects are morphisms of the form $f : P \otimes Q \to X$ in $\mathcal{C}$ and arrows are pairs $(p : P \to P', q : Q \to Q')$, making the obvious diagram commute. The functor Π sends a morphism $f : P \otimes Q \to X$ in $\mathcal{C}$ to the pair (P, Q) in $\mathcal{C} \times \mathcal{C}$ and acts on arrows as expected. For each $f : P \otimes Q \to X$, we write $\iota_f : \mathsf{F}(P) \otimes \mathsf{G}(Q) \to \mathsf{F} \otimes \mathsf{G}(X)$ for the canonical injection. The arrow satisfying the universal property of the colimit is—abusively—written $[-]$.

Notably, if (F, μ, η) is a monoidal functor, then define for each X in $\mathcal{C}$

$$\mathrm{m}_X : \mathsf{F} \otimes \mathsf{F}(X) \to \mathsf{F}(X)$$

as the only arrow making the following diagram commute for each $f : P \otimes Q \to X$

$$\begin{array}{ccc} \mathsf{F} \otimes \mathsf{F}(X) & \overset{\mathrm{m}_X := [\mathsf{F}(f) \circ \mu]}{\dashrightarrow} & \mathsf{F}(X) \\ {\scriptstyle \iota_f}\uparrow & & \uparrow{\scriptstyle \mathsf{F}(f)} \\ \mathsf{F}(P) \otimes \mathsf{F}(Q) & \xrightarrow[\mu]{} & \mathsf{F}(P \otimes Q) \end{array}$$

The family $(\mathrm{m}_X)_{X \in \mathcal{C}}$ is natural and is an associative multiplication

$$\mathrm{m} : \mathsf{F} \otimes \mathsf{F} \to \mathsf{F}\,.$$

Also, for $\xi : \mathsf{F} \to \mathsf{F}'$ and $\zeta : \mathsf{G} \to \mathsf{G}'$, we define the natural transformation $\xi \otimes \zeta : \mathsf{F} \otimes \mathsf{G} \longrightarrow \mathsf{F}' \otimes \mathsf{G}'$ with the following components:

$$\begin{array}{ccc} \mathsf{F} \otimes \mathsf{G}(X) & \overset{\xi \otimes \zeta_X := [\iota_f \circ (\xi_P \otimes \zeta_Q)]}{\dashrightarrow} & \mathsf{F}' \otimes \mathsf{G}(X) \\ {\scriptstyle \iota_f}\uparrow & & \uparrow{\scriptstyle \iota_f} \\ \mathsf{F}(P) \otimes \mathsf{G}(Q) & \xrightarrow[\xi_P \otimes \zeta_Q]{} & \mathsf{F}'(P) \otimes \mathsf{G}'(Q) \end{array}$$

We will not be using the unit of the tensor, however we will need the canonical dualizing object O of $[\mathcal{C}, \mathcal{V}]$. The functor $\mathsf{O} : \mathcal{C} \longrightarrow \mathcal{V}$ is defined on objects X as the limit of

$$\hom(X, 0_{\mathcal{C}}) \xrightarrow{!} \mathbf{1} \xrightarrow{0_{\mathcal{V}}} \mathcal{V}\,,$$

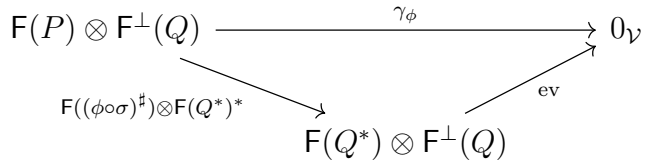
where $\hom(X, 0_{\mathcal{C}})$ is viewed as a discrete category, $\mathbf{1}$ is the terminal category and $0_{\mathcal{V}} : \mathbf{1} \longrightarrow \mathcal{V}$ picks $0_{\mathcal{V}}$ in $\mathcal{V}$. For each $t : X \to 0_{\mathcal{C}}$, we write $\pi_t : \mathsf{O}(X) \to 0_{\mathcal{V}}$ for the canonical projection. The arrow satisfying the universal property of the limit is—abusively—written $\langle - \rangle$.

For each functor $\mathsf{F} : \mathcal{C} \to \mathcal{V}$, the dual of F is written $F^\perp$ and is defined by the composition

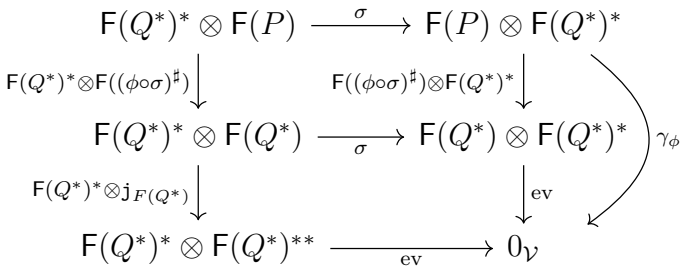
$$\mathcal{C} \xrightarrow{(-)^*} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathsf{F}} \mathcal{V}^{\mathrm{op}} \xrightarrow{(-)^*} \mathcal{V}$$

where we have used the same notation $(-)^*$ for the duality in $\mathcal{C}$ and in $\mathcal{V}$. Egger has shown that under this duality $(-)^\perp$, the functor category $[\mathcal{C}, \mathcal{V}]$ is a $*$-autonomous category.

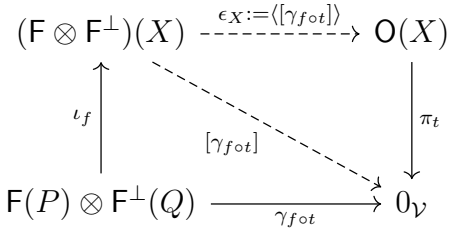
We now fix a functor $\mathsf{F} : \mathcal{C} \to \mathcal{V}$. We define for each $\phi : P \otimes Q \to 0_\mathcal{C}$ in $\mathcal{C}$, an arrow $\gamma_\phi : \mathsf{F}(P) \otimes \mathsf{F}^\perp(Q) \to 0_\mathcal{V}$ in $\mathcal{V}$ by the following composition:



Note that we have the following commutative diagram



We now define a natural transformation $\epsilon : \mathsf{F} \otimes \mathsf{F}^\perp \to \mathsf{O}$ whose components are the only arrows making the following diagram commute for each $f : P \otimes Q \to X$ and each $t : X \to 0_\mathcal{C}$ in $\mathcal{C}$:



Egger in [**Egg08**] constructs dually another natural transformation $\eta : \mathsf{I} \to \mathsf{F}^\perp \oplus \mathsf{F}$, where I is the dual of O, and $\oplus$ is the dual of $\otimes$. In [**Egg08**, Theorem 5.3] he shows that $(\mathsf{F}, \mathsf{F}^\perp, \epsilon, \eta)$ is a linear adjoint. This implies, by Lemma 3.5, that $\epsilon : \mathsf{F} \otimes \mathsf{F}^\perp \to \mathsf{O}$ is a dual pairing in $[\mathcal{C}, \mathcal{V}]$.

After all this preparation, we can finally investigate Frobenius structures in $[\mathcal{C}, \mathcal{V}]$. We now assume that (F, μ, η) is a monoidal functor from $\mathcal{C}$ to $\mathcal{V}$.

LEMMA 4.15. *For each point $\omega : I_{\mathcal{V}} \to \mathsf{F}(0_{\mathcal{C}})^*$, there exist a morphism*

$$l^{\omega} : \mathsf{F} \longrightarrow \mathsf{F}^{\perp},$$

and a morphism

$$\langle -, - \rangle^{\omega} : \mathsf{F} \otimes \mathsf{F} \longrightarrow \mathsf{O}$$

such that the following diagram commutes in $[\mathcal{C}, \mathcal{V}]$

$$\begin{array}{ccc} \mathsf{F} \otimes \mathsf{F} & \xrightarrow{\mathsf{F} \otimes l^{\omega}} & \mathsf{F} \otimes \mathsf{F}^{\perp} \\ & {\scriptstyle \langle -,- \rangle^{\omega}} \searrow \quad \swarrow {\scriptstyle \epsilon} & \\ & \mathsf{O} & \end{array} \tag{101}$$

PROOF. Since $\mathcal{V}$ is a $*$-autonomous category, each $\omega : I_{\mathcal{V}} \to \mathsf{F}(0_{\mathcal{C}})^*$ has an adjoint of type $\omega^{\dagger} : \mathsf{F}(0_{\mathcal{C}}) \to 0_{\mathcal{V}}$—see Definition 3.2. Let us now fix a point $\omega : I_{\mathcal{V}} \to \mathsf{F}(0_{\mathcal{C}})^*$.

The natural transformation $l^{\omega} : \mathsf{F} \to \mathsf{F}^{\perp}$ has components

$$l^{\omega}_X : \mathsf{F}(X) \longrightarrow \mathsf{F}^{\perp}(X) := \mathsf{F}(X^*)^*$$

defined as the only maps making such diagram commutative in $\mathcal{V}$:

$$\begin{array}{ccccc} \mathsf{F}(X^*) \otimes \mathsf{F}(X) & \xrightarrow{\mathsf{F}(X^*) \otimes l^{\omega}_X} & \mathsf{F}(X^*) \otimes \mathsf{F}^{\perp}(X) & \xrightarrow{\quad \mathrm{ev} \quad} & 0_{\mathcal{V}} \\ \downarrow{\scriptstyle \sigma} & & & & \uparrow{\scriptstyle \omega^{\dagger}} \\ \mathsf{F}(X) \otimes \mathsf{F}(X^*) & \xrightarrow[\mu]{\quad} & \mathsf{F}(X \otimes X^*) & \xrightarrow[\mathsf{F}(\mathrm{ev})]{\quad} & \mathsf{F}(0_{\mathcal{C}}) \end{array}$$

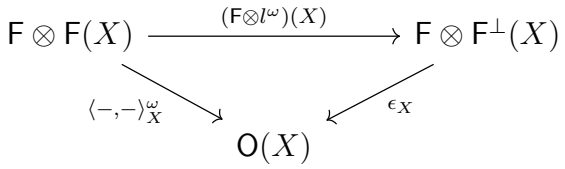
We now define a natural transformation $\mathrm{tr}^{\omega} : \mathsf{F} \longrightarrow \mathsf{O}$ whose components

$$\mathrm{tr}^{\omega}_X : \mathsf{F}(X) \longrightarrow \mathsf{O}(X)$$

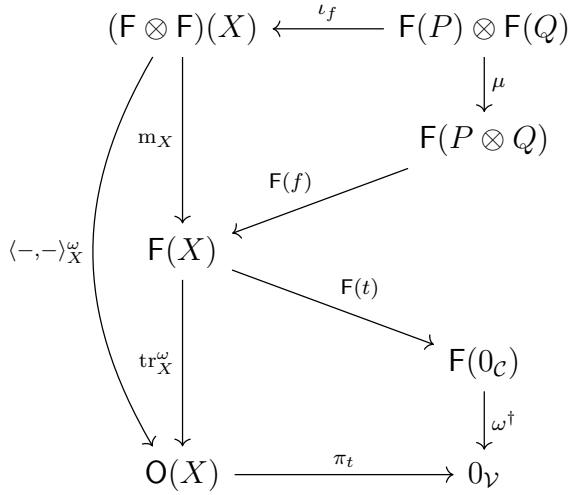
are the universal maps making the following diagram commute for each arrow $t : X \to 0_{\mathcal{C}}$ in $\mathcal{C}$

$$\begin{array}{ccc} \mathsf{F}(X) & \overset{\mathrm{tr}^{\omega}_X := \langle \omega^{\dagger} \circ \mathsf{F}(t) \rangle}{\cdots\cdots\cdots

We want to show that for each object X of $\mathcal{C}$, the following diagram commutes:

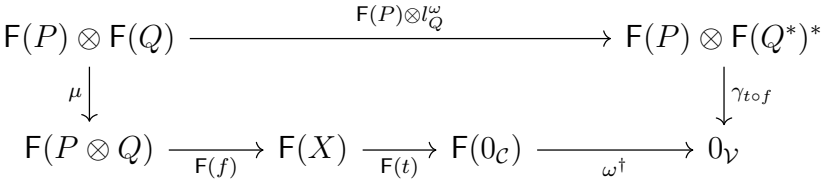


By definition, the bracket $\langle -, - \rangle^{\omega}_{X}$ is the only arrow which makes the following diagram commute for each $f : P \otimes Q \to X$ and each $t : X \to 0_{\mathcal{C}}$ in $\mathcal{C}$:



(102)

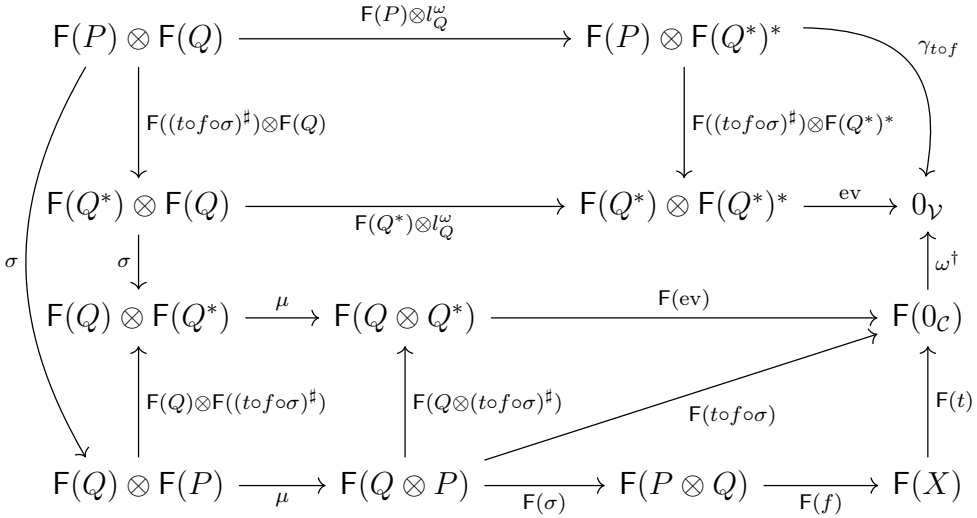
The right path of the latter diagram factorizes as follows:



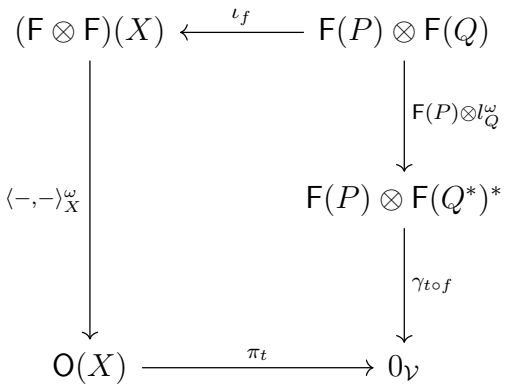
(103)

The commutativity of this diagram is proven by the following chasing diagram—where we have used the naturality of μ, the definition of l^{ω} and of γ, and some

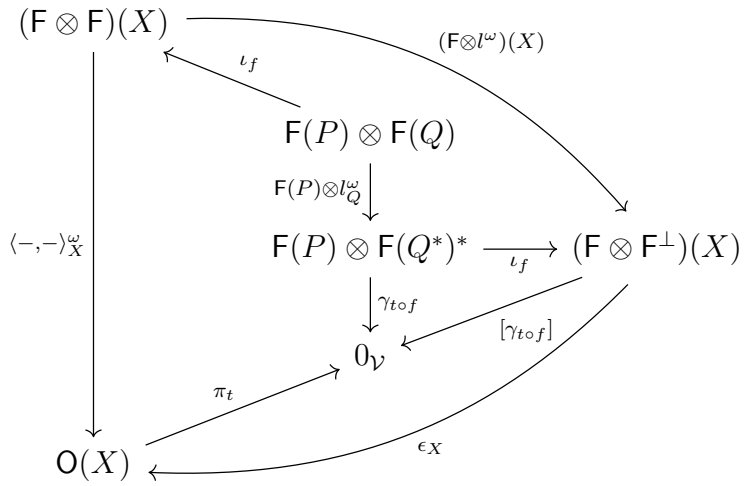
other obvious properties:



We now compose Diagram (102) and Diagram (103), and we obtain the following commutative diagram:



It is now direct to check that for each object X of $\mathcal{C}$, the following diagram commutes for each $f : P \otimes Q \to X$ and $t : X \to 0_{\mathcal{C}}$



This implies that Diagram (101) commutes, and is enough to prove the lemma. □

Diagram (101) exhibits l^ω as the transpose of $\langle -, - \rangle^\omega$ under the dual pairing ϵ. The map l^ω plays the role of an "intuitionistic" left negation, we investigate the right negation in the following lemma.

LEMMA 4.16. *The natural transformation $r^\omega : \mathsf{F} \to \mathsf{F}^\perp$ whose components are defined by the composition*

$$\mathsf{F}(X) \xrightarrow{\mathsf{F}(\mathrm{j}_X)} \mathsf{F}(X^{**}) \xrightarrow{\mathrm{j}_{\mathsf{F}(X^{**})}} \mathsf{F}(X^{**})^{**} \xrightarrow{{l^\omega_{X^*}}^*} \mathsf{F}(X^*)^*$$

makes (l^ω, r^ω) into a Galois connection in $[\mathcal{C}, \mathcal{V}]$.

PROOF. By Lemma 3.4, we know that r^ω is given by the composition

$$\mathsf{F} \xrightarrow{\mathrm{j}_F} \mathsf{F}^{\perp\perp} \xrightarrow{l^{\omega\perp}} \mathsf{F}^\perp .$$

Components of $l^{\omega\perp}$ are easily seen to be ${l^\omega_{X^*}}^*$. We show that the components of j_F are given by the composition

$$\mathsf{F}(X) \xrightarrow{\mathsf{F}(\mathrm{j}_X)} \mathsf{F}(X^{**}) \xrightarrow{\mathrm{j}_{\mathsf{F}(X^{**})}} \mathsf{F}(X^{**})^{**} .$$

Recall that j_F is the only natural transformation making the following diagram commutative:

$$\begin{array}{ccc} \mathsf{F}^\perp \otimes F & \xrightarrow{\mathsf{F}^\perp \otimes \mathrm{j}_F} & \mathsf{F}^\perp \otimes \mathsf{F}^{\perp\perp} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\epsilon_{\mathsf{F}^\perp}} \\ F \otimes \mathsf{F}^\perp & \xrightarrow[\epsilon_F]{} & \mathsf{O} \end{array}$$

On the component X, those maps are the only ones making the following diagram commute for every $f : P \otimes Q \to X$ and $t : X \to 0_{\mathcal{C}}$:

$$\begin{array}{ccccc}
\mathsf{F}^{\perp} \otimes F(X) & \xrightarrow{\sigma_X} & F \otimes \mathsf{F}^{\perp}(X) & \xrightarrow{\epsilon_{FX}} & \mathsf{O}(X) \\
{\scriptstyle \iota_f}\uparrow & & \uparrow{\scriptstyle \iota_{f\circ\sigma}} & & \\
\mathsf{F}^{\perp}(P) \otimes F(Q) & \xrightarrow[\sigma]{} & F(Q) \otimes \mathsf{F}^{\perp}(P) & & \downarrow{\scriptstyle \pi_t} \\
 & & {\scriptstyle F((t\circ f)^{\sharp})\otimes \mathsf{F}^{\perp}(P)}\downarrow & & \\
 & & F(P^*) \otimes \mathsf{F}(P^*)^* & \xrightarrow[\mathrm{ev}]{} & 0_{\mathcal{V}}
\end{array}$$

We want to show that those components are equal to the ones given by the commutativity of the following diagram

$$\begin{array}{ccc}
\mathsf{F}^{\perp} \otimes \mathsf{F}(X) & \xleftarrow{\iota_f} & \mathsf{F}^{\perp}(P) \otimes \mathsf{F}(Q) \\
\vdots & & \downarrow{\scriptstyle \mathsf{F}^{\perp}(P)\otimes \mathsf{F}(\mathsf{j}_Q)} \\
 & & \mathsf{F}^{\perp}(P) \otimes \mathsf{F}(Q^{**}) \\
 & & \downarrow{\scriptstyle \mathsf{F}^{\perp}(P)\otimes \mathsf{j}_{F(Q^{**})}} \\
\mathsf{F}^{\perp} \otimes \mathsf{F}^{\perp\perp}(X) & \xleftarrow{\iota_f} & \mathsf{F}^{\perp}(P) \otimes \mathsf{F}(Q^{**})^{**} \\
 & & \downarrow{\scriptstyle \mathsf{F}^{\perp}((t\circ f\circ\sigma)^{\sharp})\otimes \mathsf{F}(Q^{**})^{**}} \\
{\scriptstyle \epsilon^{\perp}_{\mathsf{F}\,X}}\downarrow & & \mathsf{F}(Q^{**})^* \otimes \mathsf{F}(Q^{**})^{**} \\
 & & \downarrow{\scriptstyle \mathrm{ev}} \\
\mathsf{O}(X) & \xrightarrow[\pi_t]{} & 0_{\mathcal{V}}
\end{array}$$

To do so, we will show the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathsf{F}^{\perp}(P) \otimes \mathsf{F}(Q) & \xrightarrow{\sigma} & F(Q) \otimes \mathsf{F}^{\perp}(P) \\
{\scriptstyle \mathsf{F}^{\perp}(P)\otimes \mathsf{F}(\mathsf{j}_Q)}\downarrow & & \downarrow{\scriptstyle F((t\circ f)^{\sharp})\otimes \mathsf{F}^{\perp}(P)} \\
\mathsf{F}^{\perp}(P) \otimes \mathsf{F}(Q^{**}) & & F(P^*) \otimes \mathsf{F}(P^*)^* \\
{\scriptstyle \mathsf{F}^{\perp}(P)\otimes \mathsf{j}_{F(Q^{**})}}\downarrow & & \\
\mathsf{F}^{\perp}(P) \otimes \mathsf{F}(Q^{**})^{**} & & \downarrow{\scriptstyle \mathrm{ev}} \\
{\scriptstyle \mathsf{F}^{\perp}((t\circ f\circ\sigma)^{\sharp})\otimes \mathsf{F}(Q^{**})^{**}}\downarrow & & \\
\mathsf{F}(Q^{**})^* \otimes \mathsf{F}(Q^{**})^{**} & \xrightarrow[\mathrm{ev}]{} & 0_{\mathcal{V}}
\end{array} \tag{104}$$

By the unicity of the universal arrows, it will be enough to show the lemma. Of course, the following diagram commutes:

$$\begin{array}{ccc}
\mathsf{F}^\perp(P)\otimes\mathsf{F}(Q) & \xrightarrow{\sigma} & \mathsf{F}(Q)\otimes\mathsf{F}^\perp(P) \\
\downarrow{\scriptstyle \mathsf{F}^\perp(P)\otimes\mathsf{F}(\mathsf{j}_Q)} & & \downarrow{\scriptstyle \mathsf{F}(\mathsf{j}_Q)\otimes\mathsf{F}^\perp(P)} \\
\mathsf{F}^\perp(P)\otimes\mathsf{F}(Q^{**}) & & \mathsf{F}(Q^{**})\otimes\mathsf{F}^\perp(P) \\
\downarrow{\scriptstyle \mathsf{F}^\perp(P)\otimes\mathsf{j}_{F(Q^{**})}} & & \downarrow{\scriptstyle \mathsf{j}_{F(Q^{**})}\otimes\mathsf{F}^\perp(P)} \\
\mathsf{F}^\perp(P)\otimes\mathsf{F}(Q^{**})^{**} & & \mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}^\perp(P) \\
\downarrow{\scriptstyle \mathsf{F}^\perp((t\circ f\circ\sigma)^\sharp)\otimes\mathsf{F}(Q^{**})^{**}} & & \downarrow{\scriptstyle \mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}^\perp((t\circ f\circ\sigma)^\sharp)} \\
\mathsf{F}(Q^{**})^*\otimes\mathsf{F}(Q^{**})^{**} & \xleftarrow{\sigma} & \mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}(Q^{**})^* \\
\downarrow{\scriptstyle \mathrm{ev}} & & \\
0_{\mathcal{V}} & &
\end{array}$$

This diagram is also easily seen to commute:

$$\begin{array}{ccc}
\mathsf{F}(Q)\otimes\mathsf{F}^\perp(P) & \xrightarrow{\mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}^\perp((t\circ f\circ\sigma)^\sharp)} & \mathsf{F}(Q)\otimes\mathsf{F}(Q^{**})^* \\
\downarrow{\scriptstyle \mathsf{F}(\mathsf{j}_Q)\otimes\mathsf{F}^\perp(P)} & & \downarrow{\scriptstyle \mathsf{F}(\mathsf{j}_Q)\otimes\mathsf{F}(Q^{**})^*} \\
\mathsf{F}(Q^{**})\otimes\mathsf{F}^\perp(P) & & \mathsf{F}(Q^{**})\otimes\mathsf{F}(Q^{**})^* \\
\downarrow{\scriptstyle \mathsf{j}_{F(Q^{**})}\otimes\mathsf{F}^\perp(P)} & & \\
\mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}^\perp(P) & & \\
\downarrow{\scriptstyle \mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}^\perp((t\circ f\circ\sigma)^\sharp)} & & \\
\mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}(Q^{**})^* & & \\
\downarrow{\scriptstyle \sigma} & & \\
\mathsf{F}(Q^{**})^*\otimes\mathsf{F}(Q^{**})^{**} & \xrightarrow[\mathrm{ev}]{} & 0_{\mathcal{V}}
\end{array}$$

Composing the two previous diagram, we obtain the commutativity of

$$\begin{array}{ccc}
\mathsf{F}^\perp(P)\otimes\mathsf{F}(Q) & \xrightarrow{\sigma} & \mathsf{F}(Q)\otimes\mathsf{F}^\perp(P) \\
\downarrow{\scriptstyle \mathsf{F}^\perp(P)\otimes\mathsf{F}(\mathsf{j}_Q)} & & \downarrow{\scriptstyle \mathsf{F}(Q^{**})^{**}\otimes\mathsf{F}^\perp((t\circ f\circ\sigma)^\sharp)} \\
\mathsf{F}^\perp(P)\otimes\mathsf{F}(Q^{**}) & & \mathsf{F}(Q)\otimes\mathsf{F}(Q^{**})^* \\
\downarrow{\scriptstyle \mathsf{F}^\perp(P)\otimes\mathsf{j}_{F(Q^{**})}} & & \downarrow{\scriptstyle \mathsf{F}(\mathsf{j}_Q)\otimes\mathsf{F}(Q^{**})^*} \\
\mathsf{F}^\perp(P)\otimes\mathsf{F}(Q^{**})^{**} & & \mathsf{F}(Q^{**})\otimes\mathsf{F}(Q^{**})^* \\
\downarrow{\scriptstyle \mathsf{F}^\perp((t\circ f\circ\sigma)^\sharp)\otimes\mathsf{F}(Q^{**})^{**}} & & \downarrow{\scriptstyle \mathrm{ev}} \\
\mathsf{F}(Q^{**})^*\otimes\mathsf{F}(Q^{**})^{**} & \xrightarrow[\mathrm{ev}]{} & 0_{\mathcal{V}}
\end{array}$$

We now again make use of Lemma 3.4. We have

$$\begin{array}{ccc}
\mathsf{F}(Q)\otimes\mathsf{F}^{\perp}(P) & & \xrightarrow{\mathsf{F}((to f^{\sharp})} \\
\downarrow{\scriptstyle \mathsf{F}(\mathrm{j}_Q)\otimes\mathsf{F}(Q^{**})^*} & & \\
\mathsf{F}(Q^{**})\otimes\mathsf{F}^{\perp}(P) & \xrightarrow{\mathsf{F}(((to f\circ\sigma)^{\sharp})^*)} & \mathsf{F}(P^*)\otimes\mathsf{F}^{\perp}(P) \\
\downarrow{\scriptstyle \mathsf{F}(Q^{**})\otimes\mathsf{F}^{\perp}((to f\circ\sigma)^{\sharp})} & & \downarrow{\scriptstyle \mathrm{ev}} \\
\mathsf{F}(Q^{**})\otimes\mathsf{F}(Q^{**})^* & \xrightarrow[\mathrm{ev}]{} & 0_{\mathcal{V}}
\end{array}$$

This is enough to show the commutativity of Diagram (104), and it ends the proof of the lemma. □

PROPOSITION 4.7. *Let* $\mathsf{F}:\mathcal{C}\longrightarrow\mathcal{V}$ *be a monoidal functor and let* $\omega : I_{\mathcal{V}}\to\mathsf{F}(0)^*$.

The tuple $(\mathsf{F},\mathsf{F}^{\perp},\mathrm{m}_F,l^{\omega},r^{\omega})$ *is a Frobenius structure in* $[\mathcal{C},\mathcal{V}]$ *if and only if for each object* X *of* $\mathcal{C}$*, the arrow* l^{ω}_X *of* $\mathcal{V}$ *is invertible.*

PROOF. Since $\langle-,-\rangle^{\omega}$ comes from the trace tr^{ω}, $(F,\mathrm{m}_F,\langle-,-\rangle^{\omega})$ is immediatly an associative bracketed semigroup. Since $(\mathsf{F},\mathsf{F}^{\perp},\epsilon)$ is a dual pairing, if l^{ω} is an isomorphism, then $\langle-,-\rangle^{\omega}$ is a dual pair by Proposition 3.2 and by Proposition 3.9 and Lemma 4.16, $(\mathsf{F},\mathsf{F}^{\perp},\mathrm{m}_F,l^{\omega},r^{\omega})$ is a Frobenius structure. The converse goes by definition of Frobenius structures and the fact that an arrow is an isomorphism if and only if its adjoint an isomorphism. □

We now assume that $\mathcal{V}:=\mathrm{SLATT}$. Recall that the map $\iota_{X,Y}$ was introduced in Section 4.2.2. We notably have shown the importance of $\iota_{X,0}(-,\omega)$ for $(0,\omega)$ to be a dualizing object of $\int\mathsf{Q}$ in Section 4.3. We will show that this map is exactly the negation we have been working on in the beginning of this section.

LEMMA 4.17. *For any monoidal functor* $\mathsf{Q}:\mathcal{C}\longrightarrow\mathrm{SLATT}$*, the natural transformations* $l^{\omega}_X,\iota_{X,0}(-,\omega):\mathsf{Q}(X)\longrightarrow\mathsf{Q}(X^*)^{\mathrm{op}}$ *are equal.*

PROOF. We will prove that the two parallel maps

$$\mathsf{Q}(X^*)\otimes\mathsf{Q}(X)\underset{\mathsf{Q}(X^*)\otimes\iota_{X,0}(-,\omega)}{\overset{\mathsf{Q}(X^*)\otimes l^{\omega}_X}{\rightrightarrows}}\mathsf{Q}(X^*)\otimes\mathsf{Q}(X^*)^{\mathrm{op}}\xrightarrow{\ \epsilon\ }\mathcal{2}^{\mathrm{op}},$$

given by transposition are equal.

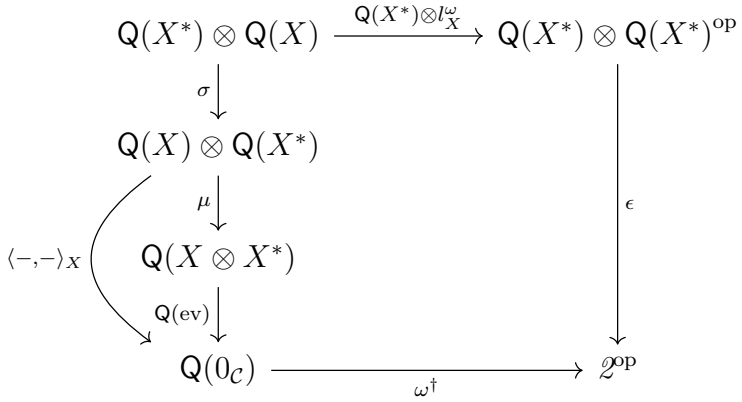
First note that the map $\iota_{X,0}(-,\omega) : \mathsf{Q}(X) \to \mathsf{Q}(X^*)^{\mathrm{op}}$ is actually the composite of

$$\mathsf{Q}(X) \xrightarrow{\rho^{-1}} \mathsf{Q}(X) \otimes \mathcal{Q} \xrightarrow{\mathsf{Q}(X)\otimes\omega} \mathsf{Q}(X) \otimes \mathsf{Q}(0_{\mathcal{C}})^{\mathrm{op}} \xrightarrow{\iota_{X,0_{\mathcal{C}}}(-,-)} \mathsf{Q}(X^*)^{\mathrm{op}}\,.$$

Moreover, in $\mathcal{V} := \textsc{SLatt}$, the natural transformation l^{ω} can be equivalently defined using the dual pairing related to $(-)^{\mathrm{op}}$—see Example 3.1. The components

$$l^{\omega}_X : \mathsf{Q}(X) \longrightarrow \mathsf{Q}(X^*)^{\mathrm{op}}$$

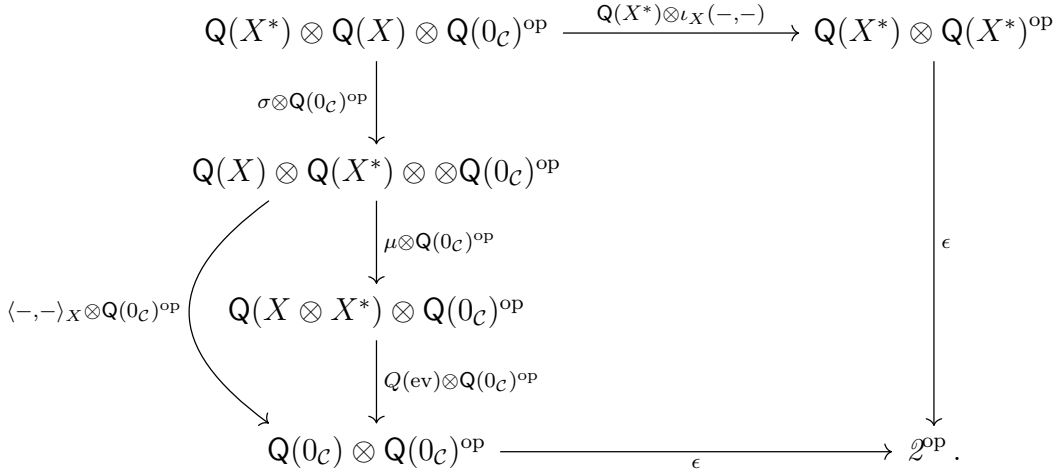
are the only maps making the following diagram commutes.



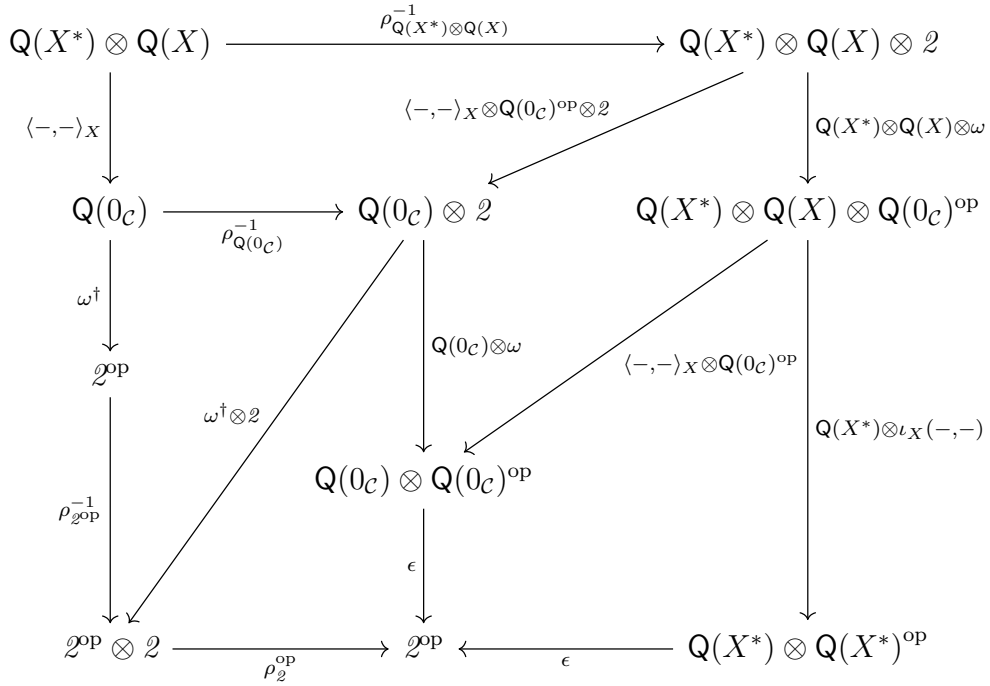
Similarly, we can encode the adjunction

$$\beta \leq \iota_{X,0}(\alpha,\omega) \;\text{ iff }\; \langle \alpha,\beta\rangle_X \leq \omega\,,$$

given in Theorem 4.6, by the following commutative diagram



It is now direct to check the commutativity of the following diagram:



In addition to the diagram we have previously provided, we have used the fact that $\rho_{2^{\mathrm{op}}}$ is a dual pairing in SLATT, the definition of $\omega^\dagger$ and the naturality of ρ. □

We can finally state the main result of this section.

THEOREM 4.11. *Let $\mathsf{Q}:\mathcal{C}\to\mathrm{SLATT}$ be a monoidal functor, and let ω be an element of $\mathsf{Q}(0)$. The object $(0,\omega)$ is a dualizing object of $\int\mathsf{Q}$ if and only if $(\mathsf{Q},\mathsf{Q}^\perp,\mathrm{m}_\mathsf{Q},\epsilon,l^\omega,r^\omega)$ is a Frobenius structure in $[\mathcal{C},\mathrm{SLATT}]$.*

PROOF. By Proposition 4.7, the tuple $(F,F^\perp,\mathrm{m}_F,\epsilon,l^\omega,r^\omega)$ is a Frobenius structure in $[\mathcal{C},\mathrm{SLATT}]$ if and only if l^ω_X is invertible for each object X in $\mathcal{C}$. By the previous lemma, $l^\omega_X=\omega_X$, and by Theorem 4.8, l^ω_X is invertible for each object X in $\mathcal{C}$ if and only if $(0,\omega)$ is a dualizing object of $\int\mathsf{Q}$. □

Conclusion

The first objective, and the starting point of this thesis, was to explore the following theorem from a categorical perspective.

THEOREM. *Let L be a complete lattice. The following statements are equivalent:*

- *L is a completely distributive lattice;*
- *There exists a sup-preserving map $0 : L \to L$ making $([L, L], \circ, 0)$ a Frobenius quantale.*

We wanted to know if this property is proper to the $*$-autonomous category of sup-semilattices SLATT, or if this is an instance of a more general theorem for $*$-autonomous categories. This second possibility was a fruitful axis of research. In order to prove this theorem in a categorical context, we favored a definition of Frobenius quantales that were easily related to its categorical abstraction. For that reason, we developed the concepts of unitless Frobenius quantales and of Frobenius structures.

Unitless Frobenius quantales were also a solution for the conceptual discomfort coming with the use of a dualizing element in the usual definition of Frobenius quantales. The existence of a dualizing element in a quantale necessarily amounts to the existence of a neutral element for the multiplication. We believe that non-unital quantales have an important status in the theory of quantales, and in the theory of Frobenius quantales as we have argued along Chapter 2. In this chapter, we (re)defined Frobenius quantales as quantales equipped with two inverse antitone maps ${}^{\perp}(-)$, $(-)^{\perp}$ working nicely together—see Definition 2.5. We have developed the elementary theory for that structure. Notably, we have shown how to construct a Frobenius quantale from a quantale and two "intuitionistic" negations—formally called a Serre Galois connection. Moreover, we provided a representation theorem for Frobenius quantales via phase quantales, generalizing a classic result due to Rosenthal. We have exhibited several examples of non-unital Frobenius quantales, arising notably from pregroups and C*-algebras. The Frobenius quantale of tight endomaps over a complete lattice L was our main example of Frobenius quantales. It first shows that any complete lattice induces a

(usually non-unital) Frobenius quantale. Moreover, complete distributivity is characterized by means of the existence of a unit in the Frobenius quantale of tight endomaps—see Recapitulation 2.1.

The development of the categorical abstraction of unitless Frobenius quantales, Frobenius structures, was carried out in Chapter 3. To do so, we have developed the notion of dual pairings. They generalize the dual of an object in a $*$-autonomous category and allow working on them "up to iso". They notably permit to encode the order of a complete lattice, and thus to generalize the right adjoint of a sup-preserving map. Dual pairings are a key ingredient in the definition of Frobenius structures: for a dual pair (A, B), if (A, m_A) is a semigroup, then A acts canonically on its dual B on the left and on the right. When A is a quantale—*i.e.*, a semigroup in SLatt—and $B = A^{\mathrm{op}}$, those two actions exactly correspond to the right and left implications of A. With those maps, we defined a Frobenius structure by encoding the Serre equality—or contraposition law—as a commutative diagram involving the two actions and two isomorphisms $l, r : A \to B$ forming a Galois connection—see Definition 3.6. We have provided several properties and equivalent definitions of Frobenius structures, notably in terms of associative bracketed semigroup. We have also compared our definition of Frobenius structures with other notions previously given in the litterature: Frobenius monoids, and H^*-algebras.

More importantly, we have worked out the relation between nuclear objects A and Frobenius structures on the object of endomorphisms $[A, A]$. In any $*$-autonomous category, if A is a nuclear object, then $[A, A]$ carries a canonical Frobenius structure. Under the additional hypothesis that A is pseudo-affine, we proved that if $[A, A]$ comes with a Frobenius structure, then A is nuclear. That is, under the technical hypothesis of pseudoaffine, the theorem which started this thesis holds in any $*$-autonomous category. If the category of semigroup over a $*$-autonomous category has an epi-mono factorization system, then the image of $\mathtt{mix}_A$ always carries a canonical Frobenius structure. In SLatt, this image exactly corresponds to the Frobenius quantale of tight endomaps.

We have further explored the pseudoaffine hypothesis by looking for an object A in a $*$-autonomous category, which is not nuclear but whose object of endomorphisms $[A, A]$ posesses a Frobenius structure—*i.e.*, A is prenuclear. In order to construct such a counterexample, we have presented sufficient conditions for prenuclear objects to be nuclear in the category of poset-valued sets Q-Set. Notably, if the quantale Q has no infinite chain, then prenuclear objects are necessarily nuclear. This led us to construct a quantale Q from the group of integers, and an object in Q-Set which is prenuclear but not nuclear.

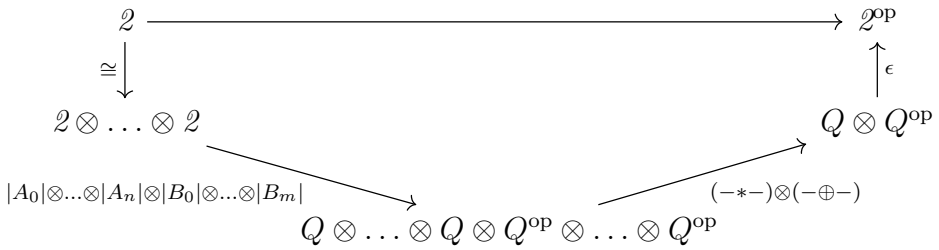
The category Q-SET intrigued us as many of its properties derive from the quantale Q. Notably, if Q is a commutative unital quantale, then Q-SET is a $*$-autonomous category. Since this category arises as a Grothendieck construction of a functor $\mathsf{Q} : \textsc{Rel} \to \textsc{Pos}$, we were interested in total categories over functors of the form $\mathsf{Q} : \mathcal{C} \to \textsc{Pos}$. We wanted to understand better how a categorical structure on $\mathcal{C}$—such as monoidal, monoidal closed, $*$-autonomy—is lifted to the category $\int \mathsf{Q}$ so that the canonical opfibration preserves the structure "on the nose". We worked out this problem in Chapter 4. We have provided a general result relating liftings of functors, liftings of adjunctions with lax natural transformation transformations. Next, we have applied those result to obtain precise conditions to lift the monoidal closed structure of the base category $\mathcal{C}$ to the total category $\int \mathsf{Q}$. We have also characterized the dualizing objects $(0, \omega)$ of $\int \mathsf{Q}$. Moreover, for any element $\omega \in \mathsf{Q}(0)$, we have constructed a functor $\mathsf{Q}^{\neg^\omega} : \mathcal{C} \to \textsc{SLatt}$ such that $(0, \omega)$ is dualizing in $\int \mathsf{Q}^{\neg^\omega}$. To do so, we have developed a theory of nuclei for monoidal functors into SLATT, generalizing the usual theory for quantales. Finally we have shown that, if $\mathcal{C}$ is a small $*$-autonomous category, then monoidal functors $\mathsf{Q} : \mathcal{C} \to \textsc{SLatt}$ such that $(0, \omega)$ is dualizing in $\int \mathsf{Q}$ are exactly the Frobenius structures of the functor category $[\mathcal{C}, \textsc{SLatt}]$ whose negations are induced by the element ω.

After this thesis, several questions remain open. First, from a categorical perspective, one would like that Frobenius structures in a category $\mathcal{V}$ form a category. However, we are still not sure of the definition of morphisms between those structures. In Section 2.6, we have assumed that a morphism between Frobenius quantales $(Q_0, *, {}^\perp(-), (-)^\perp)$ and $(Q_1, *, {}^\perp(-), (-)^\perp)$ should be a quantale homomorphism which preserves the negations, meaning $f(x^\perp) = f(x)^\perp$. However, for typing reasons, this equation does not make sense in the categorical setting. Since Frobenius structures can be equivalently defined with associative bracketed semigroup, it seems reasonable to ask for semigroup morphism which preserves the bracket. This however is still not satisfactory, for instance, in SLATT, such maps do not necessarily preserve the negations. And maps preserving the negations are not necessarily bracketed semigroup homomorphisms...

In the last chapter, we have argued that $*$-autonomous categories arising from a posetal fibration are closely related to Frobenius structures. However, several interesting $*$-autonomous categories do not arise directly from a fibration but are subcategories of a posetal fibration. This is notably the case of the category of coherent spaces, which is quite important in linear logic. We are still searching for ways to deal with such categories.

We have started this thesis with the correspondence between logics and categorical structures. We shall end it with some considerations relating linear

logic and Frobenius structures. Since Frobenius quantales are posetal models of linear logic, we wonder if a Frobenius structure could induce, in some sense, a provability model of a logic. We have been thinking about this problem, and our idea is to interpret a sequent in the non commutative non unital multiplicative fragment of linear logic as a morphism $I \longrightarrow 0$. Indeed, a sequent $A_0, \ldots, A_n \vdash B_0, \ldots, B_m$ is interpreted in a Frobenius quantale by the inequality $|A_0| * \ldots * |A_n| \leq |B_0| \oplus \ldots \oplus |B_m|$. This inequality holds if the sequent is valid in *NCLL*, but we always have an arrow $2 \longrightarrow 2^{\mathrm{op}}$ defined as the composite



where ϵ is the dual pairing induced by the order of the lattice.

For a Frobenius structure $(A, B, \epsilon, \mathrm{m}_A, l, r)$ we could interpret each formula A_i as a morphism $I \xrightarrow{|A_i|} A$ and each the formula B_j as a morphism $I \xrightarrow{|B_j|} B$. That is, as points of A and points of B. Also, we can interpret the list of formulas Γ as a morphism $|\Gamma| : I \to A$ defined by the composite

$$I \cong I \otimes \ldots \otimes I \xrightarrow{|A_0| \otimes \ldots |A_n|} A \otimes \ldots \otimes A \xrightarrow{\mathrm{m}_A^n} A\,.$$

The map $|\Delta| : I \to B$ is similarly defined using m_B.

Moreover, each dual pairing (A, B, ϵ) induces a map

$$\epsilon(-,-) : \hom(I, A) \times \hom(I, B) \to \hom(I, 0)$$

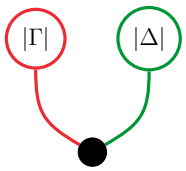
where $\epsilon(x, y)$ is the composite

$$I \cong I \otimes I \xrightarrow{x \otimes y} A \otimes B \xrightarrow{\epsilon} 0\,.$$

Therefore, an interpretation of a sequent $\Gamma \vdash \Delta$ would be a map

$$\epsilon(|\Gamma|, |\Delta|) : I \longrightarrow 0$$

which we represent as a circuit



(105)

This exactly follows the usual interpretation of a sequent in a quantale. However, there are several problems that should be considered.

First, any formula can appear on the left and the right of a sequent. This implies that one should be able to interpret a formula as a point of A or as a point of B. That is, there should be a bijection $\overrightarrow{(-)} : \hom(I, A) \to \hom(I, B)$. This (set-theoretic) bijection occurs in SLatt since elements of L are elements of L^{op}, but this is not the case for every dual pair (A, B).

Moreover, a sequent calculus comes with rules of the form

$$\frac{\Gamma_0 \vdash \Delta_0 \qquad \Gamma_1 \vdash \Delta_1}{\Gamma_2 \vdash \Delta_2 \,.}$$

That is, we should be able to "deduce" a circuit of the form (105) from another and to multiply two such circuits. This indicates that $\hom(I, 0)$ should be equipped with an ordered commutative monoid structure $(\hom(I, 0), \leq, *, 1)$.

Furthermore, the (axiom) rule and the (cut) rule are central, we shall encode them in our "provability model". To do so, the bijection $\overrightarrow{(-)}$ and the ordered monoid $(\hom(I, 0), \leq, *, 1)$ must respect the two equations for every $x, y, z \in \hom(I, A)$:

$$1 \leq \epsilon(x, \overrightarrow{x}) \qquad\qquad \epsilon(x, \overrightarrow{y}) * \epsilon(y, \overrightarrow{z}) \leq \epsilon(x, \overrightarrow{z}) \,.$$

The left inequality encodes the (axiom) rule, and the one on the right one encodes the (cut) rule. Those inequalities actually make $\hom(I, A)$ and $\hom(I, B)$ enriched categories over $\hom(I, 0)$. Lastly, all those data yield two endomaps over $\hom(I, A)$ and two endomaps over $\hom(I, B)$. We ask for them to be equal:

$$\overleftarrow{l \circ x} = r^{-1} \circ \overrightarrow{x} \qquad\qquad \overleftarrow{r \circ x} = l^{-1} \circ \overrightarrow{x} \,.$$

With those assumptions, we were able to recover the more important rules of non commutative non unital multiplicative linear logic. The main example of category in which those hypotheses are satisfied is the category Q-Set with Q a unital commutative Frobenius quantale. A way to tackle this problem could be not to consider the functor $\hom(I, -)$ but a general functor $\mathcal{C} \to \mathcal{V}$, like we did in the last section of the last chapter.

We are aware that this idea uses many *ad hoc* hypotheses, but we believe it could be a fruitful axis of research to understand better provability models from a categorical perspective.

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