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Stochastic Epidemic model with varying infectivity and waning immunity : Law of large numbers and Central limit theorem.

Weighted norm inequality in the variable Lebesgue spaces for Bergman Projector on the unit ball of $\mathbb{C}^n$

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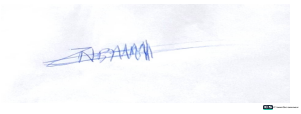

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Liste de publications et participation aux conférences

Liste des publications et/ou brevets réalisées dans le cadre du projet de thèse :

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2. SPA conference 2023, Lisboa, Portugal, speaker : Stochastic epidemic model with varying infectivity and waning immunity.
3. Saint-Flour Summer school, France, July 2023, speaker : Stochastic epidemic model with varying infectivity and warning immunity.

Résumé et mots clés

Cette thèse est organisée en deux parties :

La première partie traite des modèles stochastiques d'épidémies avec perte progressive d'immunité. Plus précisément nous étudions un modèle stochastique individu-centré d'épidémie dans lequel un individu infecté redevient susceptible après guérison et peut à nouveau être infecté avec une probabilité égale à sa susceptibilité courante. Contrairement aux modèles existants qui supposent qu'après rétablissement l'individu redevient immédiatement susceptible, nous supposons que l'individu redevient progressivement susceptible et que l'évolution de sa susceptibilité est décrite par une fonction aléatoire.

Pour ce faire nous avons étudié deux modèles. Pour le premier modèle nous avons établi le comportement limite en grande population, puis nous avons étudié le comportement en temps long. Il s'agit notamment des différents équilibres possibles et enfin nous avons étudié les fluctuations du modèle autour du modèle déterministe obtenu par la loi des grands nombres. La limite en grande population est décrite par un système déterministe d'équations intégrales de type Volterra modélisant la force moyenne d'infection et la susceptibilité moyenne de la population et généralisant le modèle edp de Kermack et McKendrick. De plus, nous avons démontré que le seuil d'endémicité dépend de la susceptibilité des individus et que lorsque le nombre de reproduction de base est au dessous de ce seuil nous avons un équilibre sans maladie asymptotiquement globalement stable et dans le cas contraire il est instable. Nous avons également pu montrer l'existence d'un équilibre endémique lorsque le nombre de reproduction de base est au dessus de ce seuil. Enfin la limite du théorème de la limite centrale est décrit par un système d'équations intégrales de types Volterra avec en entrée un vecteur gaussien.

Pour le second modèle, nous étendons le premier modèle en faisant dépendre à chaque réinfection les valeurs de l'infectivité et de la susceptibilité des individus de celles précédents leur réinfection. Pour cela nous utilisons une approche paramétrique en modélisant par un processus de Markov l'évolution des paramètres et l'âge d'infection des individus au fil du temps. Nous supposons qu'entre deux instants d'infection les traits des individus restent constants et que l'âge d'infection repasse à zéro à chaque nouvelle infection. De plus nous supposons aussi que le nouveau trait est tiré suivant une loi de probabilité décrite par un noyau de transition. Puis nous montrons que la mesure empirique du couple trait et âge converge dans la limite d'une grande population vers une mesure de probabilité solution d'une edp de transport non-linéaire. Par la suite nous démontrons que cette edp admet une solution

stationnaire non nulle, donc l'existence d'un équilibre endémique. Nous obtenons un seuil d'endémicité très général que nous calculons dans certains et enfin nous donnons une conjecture donnant la stabilité locale de notre modèle.

Dans la seconde partie, nous étendons la théorie des poids de Békollè-Bonami B_p en remplaçant la constante p par une fonction mesurable positive $p(\cdot)$ et nous supposons qu'elle est log-Hölder continue et minorée par 1. Puis nous montrons que le projecteur de Bergman dans la boule unité de $\mathbb{C}^n$ est borné sur un espace de Lebesgue à poids à exposant variable $L^{p(\cdot)}(w)$ si et seulement si w appartient à la classe généralisée de poids de Békollè-Bonami $B_{p(\cdot)}$. Pour ce faire nous définissons une fonction maximale et nous montrons qu'elle est bornée sur $L^{p(\cdot)}(w)$ si $w \in B_{p(\cdot)}$. Puis nous prouvons un théorème d'extrapolation à poids qui nous permet de conclure.

Mots clés : Modèle d'épidémie, Infectivité variable, perte progressive d'immunité, Équation ntégral de Volterra stochastique, Mesure aléatoire de Poisson, intégral par rapport à une mesure de Poisson, Modèle en quarentaine, Espaces de Lebesgue à exposant variable, Espaces de Lebesgue à poids à exposant variable, Espaces de Bergman à exposant variable, Inégalités à poids, Projecteur de Bergman.

Abstract and keywords

This thesis is organized in two parts:

The first part deals with stochastic models of epidemics with progressive loss of immunity. More precisely, we study an individual-based stochastic epidemic model in which each infected individual become susceptible again after recovery and can be infected again with a probability equal to his current susceptibility. In contrast to existing models, which assume that after recovery the individuals immediately become susceptible again, we assume that the individuals gradually become susceptible again, and that this is described by a random function.

To do this, we have studied two models. For the first model, we establish the limit behavior in a large population, then we study the long time behavior. Finally, we study the fluctuations of the model around the deterministic model obtained by the law of large numbers. The large-population limit is described by a deterministic system of Volterra-type integral equations modeling the average force of infection and the average susceptibility of the population, and generalizing the edp model of Kermack and Mckendrick. In addition, we have proved that the endemicity threshold depends on the susceptibility of individuals, and that when the basic reproduction number is below this threshold we have an asymptotically globally stable disease-free equilibrium, which is otherwise unstable. We were also able to show the existence of an endemic equilibrium when the basic reproduction number is above this threshold. Finally, the central limit theorem is described by a stochastic system of integral equations of the volterra type with a Gaussian noise vector.

For the second model, we extend the first model by deriving the value of the new individual parameters (infectivity and suceptibility) after each new infection from the previous ones. To do this, we use a parametric approach, using a Markov process to model the evolution of the parameter and the age of infection of individuals over time. We assume that between two times of infection , the traits of the individuals remain constant, and that the age of infection returns to zero with each new infection. We then show that the empirical measure of the pair trait and age converges in the large population limit to a probability measure which solves a nonlinear transport pde. We then show that this pde admits a non-zero stationary solution and hence deduce the existence of an endemic equilibrium. Finally we obtain a very general endemicity threshold that we calculate in certain cases.

In the second part, we extend the theory of Békollè-Bonami B_p weights. Here we

replace the constant p by a non-negative measurable function $p(\cdot)$, which is log-Hölder continuous function with lower bound 1. We show that the Bergman projector on the unit ball of $\mathbb{C}^n$ is continuous on the weighted variable Lebesgue spaces $L^{p(\cdot)}(w)$ if and only if w belongs to the generalised Békollè-Bonami class $B_{p(\cdot)}$. To achieve this, we define a maximal function and show that it is bounded on $L^{p(\cdot)}(w)$ if $w \in B_{p(\cdot)}$. We next state and prove a weighted extrapolation theorem that allows us to conclude.

Keywords: epidemic model, varying infectivity, waning immunity, Gaussian-driven stochastic Volterra integral equations, Poisson random measure, stochastic integral with respect to Poisson random measure, quarantine model,Variable exponent Lebesgue spaces, Variable exponent Bergman spaces, Weighted inequalities, Bergman projector, Muckenhoupt weight.

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«La recherche est une quête perpétuelle de maturité intellectuelle»Zotsa.

DÉDICACE

À mes parents Mr et Mme Ngoufack.

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Première partie

Stochastic epidemic models with varying infectivity and waning immunity

Introduction

Depuis l'antiquité, le monde a fait face à plusieurs crises sanitaires ayant provoqué un nombre important de décès. Nous pouvons citer l'épidémie de paludisme, qui selon le rapport de l'OMS de 2022 aurait déjà atteint 249 millions de décès, de même selon l'OMS le nombre total de décès associé à la pandémie de covid-19 entre le 1er janvier 2020 et le 31 décembre 2021 était d'environ 14.9 millions. Dans le but de contrôler la propagation de ces épidémies, il est apparu urgent de comprendre leurs mécanismes de propagation, et leur comportement sur le long terme. C'est à cette fin qu'est née l'épidémiologie mathématique et l'un des tout premiers modèles mathématique fut proposé par Ross et Hudson en 1915 et 1917, plus précisément ils modélisaient la transmission d'un pathogène du moustique vers l'humain et de l'humain vers le moustique. Plus tard Kermack et McKendrick en 1927 se sont inspirés de ce modèle pour proposer un modèle où le pathogène se transmet d'un individu à un autre. Depuis lors nous avons assisté à la publication de nombreux articles sur le sujet se regroupant principalement en deux grandes écoles : une école axée sur des modèles déterministes et l'autre sur des modèles stochastiques.

Ces deux approches utilisent des modèles compartimentaux. En particulier nous avons le modèle **SEIRS** où **S** représente la classe des individus susceptibles, **E** la classe des individus exposés, **I** celle des infectieux et **R** celle des rétablis. Dans ce modèle on sous-entend qu'un individu initialement susceptible passe de **S** à **E**, puis de **E** à **I**, de **I** à **R** et enfin de **R** à **S**. De même on définit le modèle **SIR, SIS, et SIRS**. Pour que le modèle stochastique soit Markovien il faut que les durées d'exposition, d'infection et d'immunité des individus suivent des lois exponentielles indépendantes, et que les taux de passages d'un compartiment à un autre soit des constantes, voir [15].

Dans l'approche stochastique, pour décrire la dynamique de l'épidémie on part en général d'un nombre fini d'individus, puis on établit un résultat de type loi des grands nombres en faisant tendre la taille de la population vers l'infini. Cette limite est décrite par un système déterministe et est généralement caractérisée soit par un système d'équations différentielles, soit un système d'équations aux dérivées partielles ou un système d'équations intégrales. Ensuite on se pose la question des fluctuations autour du modèle déterministe obtenu par la loi des grands nombres : dans un premier temps on peut regarder le théorème de la limite centrale, puis les grandes déviations de l'épidémie et enfin les déviations modérées afin d'estimer le temps d'extinction.

Dans la suite nous ne considérerons que des modèles homogènes autrement dit nous ne tenons pas compte des inhomogénéités dans la population, par exemple de type spatiale ou différence de niveau d'activité sociale.

1. Modèles déterministes

1.1. Modèle de Kermack-McKendrick

En 1927, Kermack et McKendrick publient un article dans lequel ils proposent l'un des tout premiers modèles d'épidémie [61]. Dans [61] ils décrivent la dynamique de l'épidémie, et la taille finale de l'épidémie. D'autre part ils étudient le comportement en temps long du modèle SIR général sans perte d'immunité, puis déduisent celui du cas particulier où les taux d'infectivité et de rémission sont constants. Plus précisément, ils démontrent l'existence d'un seuil critique au-dessous duquel on a une extinction rapide de l'épidémie et dans le cas contraire l'épidémie touche une large fraction de la population avant de s'éteindre.

1.1.1. Modèle SIR

Dans ce modèle Kermack et McKendrick font les hypothèses suivantes :
- Au début de l'épidémie on a un ou plusieurs individus infectés dans une population d'individus susceptibles au virus ;
- Tous les individus susceptibles ont la même susceptibilité ;
- L'épidémie se propage d'un individu infecté à un non-infecté par un contact infectieux ;
- Chaque individu infecté se remet de la maladie par guérison ou décès ;
- La probabilité d'infecter un susceptible dépend de l'infectivité de l'individu ;
- Après guérison les individus acquièrent une immunité totale ;
- Un individu infecté ne peut être réinfecté pendant sa période infectieuse ;
- La taille de la population reste constante égale à N au cours du temps.

En introduisant les notations suivantes :
- $I(t,\theta)$: Le nombre d'individus infectés à l'instant t depuis θ unités de temps.
- $I(t,0)$ représente le nombre d'individus à l'instant t qui ont été récemment infectés.
- $I(t)$: Le nombre total d'individus infectés à l'instant t.
- y_0 le nombre d'individus qui viennent juste d'être infectés à $t = 0$.
- $v(t)$ le nombre d'individus qui deviennent infectés dans l'intervalle de temps $[t, t+1]$.
- $S(t)$ le nombre d'individus susceptibles à l'instant t.
- $\psi(\theta)$ le taux de rémission (guérisons + morts) après θ unité de temps ;
- $\phi(\theta)$ le taux d'infectivité d'age θ avec $\phi(0) = 0$.
- $R(t)$ le nombre de personnes rémises (guéries et mortes) à l'instant t.

Ils ont donc

$$S(t) + I(t) + R(t) = N. \tag{1}$$

Puis en regardant dans un premier temps chaque intervalle de temps de longueur un comme une unité de temps, ils déduisent que le nombre d'individus rémis dans l'intervalle $[t, t+1]$ après avoir passé θ unités de temps dans l'état infectieux est donné

par $\psi(\theta)I(t,\theta) = I(t,\theta) - I(t+1,\theta+1)$. D'où

$$\begin{aligned} I(t,\theta) &= I(t-1,\theta-1)\big(1-\psi(\theta-1)\big) \\ &= I(t-2,\theta-2)\big(1-\psi(\theta-2)\big)\big(1-\psi(\theta-1)\big) \\ &= B(\theta)I(t-\theta,0) \end{aligned} \tag{2}$$

avec

$$B(\theta) = \prod_{j=1}^{\theta}\big(1-\psi(j-1)\big).$$

De la définition de $I(t)$ ils déduisent que,

$$I(t) = \sum_{\theta=0}^{t} I(t,\theta) = \sum_{\theta=0}^{t} B(\theta)\nu(t-\theta) + B(t)y_0.$$

De même ils en déduisent qu'on a

$$\begin{aligned} \nu(t) &= S(t)\sum_{\theta=1}^{t}\phi(\theta)I(t,\theta) \\ &= S(t)\sum_{\theta=1}^{t}\phi(\theta)B(\theta)I(t-\theta,0) \\ &= S(t)\left(\sum_{\theta=1}^{t}A(\theta)\nu(t-\theta) + A(t)y_0\right), \end{aligned}$$

où $A(\theta) = \phi(\theta)B(\theta)$.

D'autre part, puisqu'on a aussi

$$\nu(t) = S(t) - S(t+1),$$

de (5) ils déduisent que

$$S(t) - S(t+1) = S(t)\left(\sum_{\theta=1}^{t}A(\theta)\nu(t-\theta) + A(t)y_0\right). \tag{3}$$

$$\begin{aligned} R(t+1) - R(t) &= \sum_{\theta=1}^{t}\psi(\theta)I(t,\theta) \\ &= \sum_{\theta=1}^{t}\psi(\theta)B(\theta)\nu(t-\theta) + \psi(t)B(t)y_0 \\ &= \sum_{\theta=1}^{t}C(\theta)\nu(t-\theta) + C(t)y_0. \end{aligned}$$

Ils déduisent de (1) que

$$I(t+1)-I(t)=S(t)\left(\sum_{\theta=1}^{t}A(\theta)\nu(t-\theta)+A(t)y_0\right)-\left(\sum_{\theta=1}^{t}C(\theta)\nu(t-\theta)+C(t)y_0\right).$$

D'où ils déduisent que la dynamique de l'épidémie est décrite par le système discret suivant :

$$\begin{cases} S(t)+I(t)+R(t)=N & (4)\\ \nu(t)=S(t)\left(\sum_{\theta=1}^{t}A(\theta)\nu(t-\theta)+A(t)y_0\right), & (5)\\ S(t)-S(t+1)=\nu(t) & (6)\\ I(t)=\sum_{\theta=0}^{t}B(\theta)\nu(t-\theta)+B(t)y_0 & (7)\\ R(t+1)-R(t)=\sum_{\theta=1}^{t}C(\theta)\nu(t-\theta)+C(t)y_0. & (8) \end{cases}$$

où

$$B(\theta)=\prod_{j=1}^{\theta}\big(1-\psi(j-1)\big),\quad A(\theta)=B(\theta)\phi(\theta),\quad C(\theta)=\psi(\theta)B(\theta).$$

Puis en supposant maintenant que la subdivision de chaque intervalle augmente en nombre de sorte que chaque intervalle devient petit, ils déduisent en passant à la limite dans les équations (1), (6), (3), (7) et (8) le système continu suivant :

$$\begin{cases} S(t)+I(t)+R(t)=N\\ I(t,0)=-S'(t)\\ S'(t)=-S(t)\left(\int_0^t A(\theta)I(t-\theta,0)d\theta+A(t)y_0\right)\\ R'(t)=\int_0^t C(\theta)I(t-\theta,0)d\theta+C(t)y_0\\ I(t)=\int_0^t B(\theta)I(t-\theta,0)d\theta+B(t)y_0, \end{cases} \tag{9}$$

où

$$B(\theta)=\exp\left(-\int_0^{\theta}\psi(a)da\right),\quad A(\theta)=B(\theta)\phi(\theta),\quad C(\theta)=\psi(\theta)B(\theta).$$

Ils déduisent de (9) l'équation suivante :

$$\frac{d\ln S(t)}{dt} = -A(t)N + \int_0^t A'(\theta)S(t-\theta)d\theta, \tag{10}$$

En utilisant le fait que S est une fonction croissante en t et majorée, elle converge vers S_∞ et en passant à la limite dans (10) ils déduisent l'équation suivante

$$\log\left(\frac{S(0)}{S_\infty}\right) = (N - S_\infty)\int_0^\infty A(\theta)d\theta. \tag{11}$$

En notant $\overline{R}_\infty = 1 - \frac{S_\infty}{N}$ et $A = \int_0^\infty A(\theta)d\theta$, ils obtiennent

$$1 - \overline{R}_\infty = \frac{S(0)}{N}e^{-NA\overline{R}_\infty} \tag{12}$$

qui est leur principal résultat et qui permet de décrire la taille finale de l'épidémie. De nos jours cette équation s'écrit autrement. En fait, elle n'a pas de sens lorsque la taille de la population est très grande. Plus précisément, dans ce modèle NA représente le nombre moyen d'infections par un individu infecté dans une population de susceptibles ou tout simplement le nombre de reproduction de base qu'on note aujourd'hui R_0 et qui est supposé indépendant de N. Ainsi (12) se réécrit comme suit :

$$1 - \overline{S}(0)e^{-R_0\overline{R}_\infty} = \overline{R}_\infty \tag{13}$$

où $\overline{S}(0)$ est la proportion d'individus susceptibles à l'instant initial et $\overline{R}_\infty$ la proportion d'individus remis ou ayant été infectés à la fin de l'épidémie.

On en déduit le résultat suivant de Kermack et McKendrick.

Théorème 1. *Si* $\overline{S}(0) = 1$, *on a :*

— *Si* $R_0 \leqslant 1$, $\overline{R}_\infty = 0$.

— *Si* $R_0 > 1$, *il existe* , $\overline{R}_* \in]0,1[$ *solution de* (13).

Corollaire 1.1.

1. *Quand* $R_0 \leqslant 1$, $\overline{R}_\infty$ *peut-être rendue arbitrairement petit en choisissant* $\overline{I}(0)$ *suffisamment petit.*
2. *Quand* $R_0 > 1$, $\overline{R}_\infty > \overline{R}_*$ *pour tout* $\overline{S}(0) \in]0,1[$.

Dans ce même article ils traitent le cas particulier de (9) où ϕ et ψ sont des constantes égales à λ et β respectivement et dans ce cas l'on obtient le célèbre système d'EDO caractérisant le modèle SIR d'épidémie suivant :

$$\begin{cases} S'(t) = -\lambda S(t)I(t) \\ I'(t) = \lambda S(t)I(t) - \beta I(t) \\ R'(t) = \beta I(t). \end{cases} \tag{14}$$

Pour ce cas particulier ils démontrent que lorsque le nombre moyen d'individus infectés dans une population de susceptibles par un individu infecté, noté R_0, qui vaut

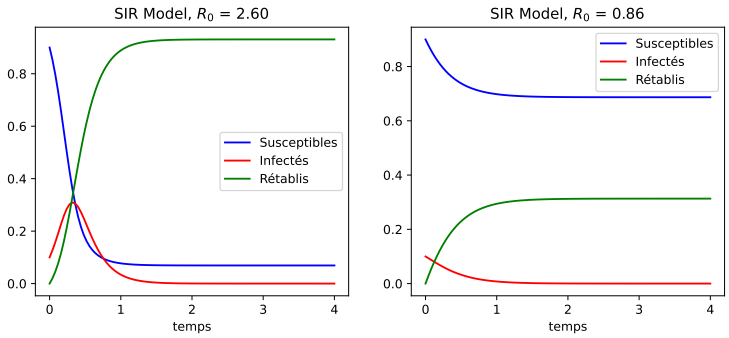

FIGURE 1. – Représentation du modèle SIR déterministe pour différentes valeurs de R_0 avec $\overline{S}(0) = 0.9$, et $\overline{I}(0) = 0.1$.

$\lambda\beta^{-1}$ est inférieur à un, il y a extinction rapide de l'épidémie et dans le cas contraire l'épidémie touche une large fraction de la population avant de s'éteindre (voir la figure 1). C'est le théorème du seuil de Kermack et McKendrick.

1.1.2. Modèle SIRS

En 1932 Kermack et Mckendrick publient un second article [62] dans lequel ils étendent le modèle de [61] en introduisant la perte progressive de l'immunité après rémission des individus et des nouveaux susceptibles dans la population par naissances ou immigration. Ils obtiennent ainsi un modèle avec quatre classes :

— S : la classe des individus qui n'ont jamais été infectés,
— I : la classe des individus qui sont infectieux,
— R : la classe des individus qui ont été infectés au moins une fois,
— D : la classe des individus décédés.

Dans ce modèle un individu est susceptible s'il est dans la classe S ou R.

Dans un premier temps ils étudient un cas particulier où tous les taux sont constants et l'immunité est perdue instantanément. En notant m le taux d'immigration, $\overline{\mu}$ le taux de natalité des individus de la classe S, μ celui des R et ν celui des I. Puis $\overline{k}$ le taux avec lequel un individu de la classe S devient I, k le taux avec lequel un individu de la classe R devient I, d le taux de décès des I et ℓ le taux avec lequel un individu de la classe I devient R, ils obtiennent que la dynamique de l'épidémie est décrite par le système suivant :

$$\begin{cases} S'(t) = m + \overline{\mu}S(t) + \mu R(t) + \nu I(t) - \overline{k}S(t)I(t) \\ R'(t) = \ell I(t) - kR(t)I(t) \\ I'(t) = kR(t)I(t) + \overline{k}S(t)I(t) - \ell I(t) - dI(t). \end{cases}$$

Rappelons qu'une maladie est dit endémique si elle persiste dans la population et lorsqu'elle atteint un équilibre, ce dernier est appelé équilibre endémique. Cela correspond à un équilibre entre le nombre de nouveaux cas infectieux et le nombre de nouvelles guérisons.

Ils en déduisent le théorème d'endémicité suivant :

Théorème 2. *Si $d > \nu$, le triplet (S_*, R_*, I_*) définit par*

$$S_* = \frac{d}{\overline{k}}, \quad R_* = \frac{\ell}{k}, \qquad I_* = \frac{1}{d-\nu}\left(m + \frac{\overline{\mu} d}{\overline{k}} + \mu\frac{\ell}{k}\right).$$

est un équilibre endémique localement stable.

Plus généralement, ils supposent que le taux de guérison, noté ℓ, de décès noté d, et de passage de S vers I, noté k, dépendent de l'âge d'infection, noté θ. De plus, ils supposent que le taux de passage de R vers I, noté k, dépend de l'âge d'infection et de l'âge de guérison, noté τ. Ils supposent aussi que $k(\tau,\theta)$ est de la forme $w(\tau)\phi(\theta)$ et considèrent toujours constant les taux de naissance et d'immigration. En introduisant les notations suivantes :

— $S(t,\tau)$ représente le nombre d'individus susceptibles à l'instant t depuis τ unité de temps,
— $R(t,\tau)$ représente le nombre d'individus remis à l'instant t depuis τ unité de temps,

et en utilisant le fait que

$$\begin{cases} I(t,0) = S(t)\displaystyle\int_0^\infty \overline{k}(\theta) I(t,\theta)\,d\theta + \int_0^\infty\int_0^\infty k(\tau,\theta) I(t,\theta) R(t,\tau)\,d\theta\,d\tau \\ \\ R(t,0) = \displaystyle\int_0^\infty \ell(\theta) I(t,\theta)\,d\theta \\ \\ S(t,0) = m + \overline{\mu} S(t) + \mu R(t) + \nu I(t), \end{cases}$$

où

$$S(t) = \int_0^\infty S(t,\tau)\,d\tau,\ I(t) = \int_0^\infty I(t,\theta)\,d\theta, \text{ et } R(t) = \int_0^\infty R(t,\theta)\,d\theta.$$

Ils obtiennent que la dynamique de l'épidémie est décrite par le système d'équations

suivant :

$$\begin{cases} \dfrac{\partial S(t,\tau)}{\partial t} + \dfrac{\partial S(t,\tau)}{\partial \tau} = -S(t,\tau)\displaystyle\int_0^\infty \overline{k}(\theta) I(t,\theta)\, d\theta \\ \dfrac{\partial I(t,\theta)}{\partial t} + \dfrac{\partial I(t,\theta)}{\partial \theta} = -\left(\ell(\theta) + d(\theta)\right) I(t,\theta) \\ \dfrac{\partial R(t,\tau)}{\partial t} + \dfrac{\partial R(t,\tau)}{\partial \tau} = -R(t,\tau)\displaystyle\int_0^\infty k(\tau,\theta) I(t,\theta)\, d\theta \\ I(t,0) = S(t)\displaystyle\int_0^\infty \overline{k}(\theta) I(t,\theta)\, d\theta + \int_0^\infty \int_0^\infty k(\tau,\theta) I(t,\theta) R(t,\tau)\, d\theta\, d\tau \\ R(t,0) = \displaystyle\int_0^\infty \ell(\theta) I(t,\theta)\, d\theta \\ S(t,0) = m + \overline{\mu} S(t) + \mu R(t) + \nu I(t). \end{cases} \tag{15}$$

Cependant ils laissent pour des travaux futurs l'étude du comportement en temps long.

En 1933 dans [63] Kermack et McKendrick étendent le modèle de [62] en rajoutant les décès qui ne sont pas dus à l'épidémie. À cet effet en notant par $\overline{\pi}$, π et ρ le taux de décès des susceptibles, des rémis et des infectés respectivement, ils obtiennent la dynamique suivante :

$$\begin{cases} \dfrac{\partial S(t,\tau)}{\partial t} + \dfrac{\partial S(t,\tau)}{\partial \tau} = -S(t,\tau)\displaystyle\int_0^\infty \overline{k}(\theta) I(t,\theta)\, d\theta - \overline{\pi} S(t,\tau) \\ \dfrac{\partial I(t,\theta)}{\partial t} + \dfrac{\partial I(t,\theta)}{\partial \theta} = -\left(\ell(\theta) + d(\theta)\right) I(t,\theta) - \rho I(t,\theta) \\ \dfrac{\partial R(t,\tau)}{\partial t} + \dfrac{\partial R(t,\tau)}{\partial \tau} = -R(t,\tau)\displaystyle\int_0^\infty k(\tau,\theta) I(t,\theta)\, d\theta - \pi R(t,\tau) \\ I(t,0) = S(t)\displaystyle\int_0^\infty \overline{k}(\theta) I(t,\theta)\, d\theta + \int_0^\infty \int_0^\infty k(\tau,\theta) I(t,\theta) R(t,\tau)\, d\theta\, d\tau \\ R(t,0) = \displaystyle\int_0^\infty \ell(\theta) I(t,\theta)\, d\theta \\ S(t,0) = m + \overline{\mu} S(t) + \mu R(t) + \nu I(t) \end{cases} \tag{16}$$

Pour ce modèle ils étudient les variations de la solution stationnaire de (16) par rapport aux paramètres de démographies mais laissent pour des travaux futurs l'étude de la stabilité du modèle.

1.2. Comportement en temps long

Les équations (15) et (16) sont appelées généralement équations de Kermack-McKendrick. L'étude du comportement asymptotique de ces équations est à l'origine de l'écriture de plusieurs livres et articles. En effet, dans [50] Hertcote étudie le cas particulier du modèle SIS décrit par le système suivant :

$$\begin{cases} S'(t) = -\beta S(t)I(t) + \gamma I(t) \\ I'(t) = \beta S(t)I(t) - \gamma I(t) \\ S(t) + I(t) = N. \end{cases}$$

Plus précisément il établit le résultat du comportement en temps long en utilisant la technique de Lyapunov voir [99, 12] pour plus de détails sur la technique de Lyapunov. Dans ce cas $R_0 = \frac{\beta}{\gamma}$. En notant $\overline{S}(t)$ et $\overline{I}(t)$ la proportion d'individus susceptibles et infectés respectivement à l'instant t, ce résultat s'énonce ainsi que suit :

Théorème 3. *Si* $R_0 \leqslant 1$,

$$(\overline{S}(t), \overline{I}(t)) \xrightarrow[t\to\infty]{} (1,0),$$

sinon dès que $\overline{I}(0) > 0$, *on a :*

$$(\overline{S}(t), \overline{I}(t)) \xrightarrow[t\to\infty]{} (\frac{1}{R_0}, 1 - \frac{1}{R_0}).$$

Dans [53, 54] Inaba étudie le cas particulier du modèle de Kermack et Mckendrick (16) où $m = 0, \mu = \overline{\mu} = \nu, \rho = \pi = \overline{\pi}, d(\theta) = 0$ et $k(\tau,\theta) = w(\tau)\phi(\theta)$. Dans ce cas le nombre de reproduction de base R_0 est donné par,

$$R_0 = \int_0^\infty \phi(\theta) \exp\left(-\int_0^\theta (\ell(s) + \rho) ds\right) d\theta.$$

Puis il démontre le résultat suivant :

Théorème 4.
1. *Si* $R_0 < \left(\sup_{\theta\geqslant 0} w(\theta)\right)^{-1}$, *l'équilibre sans maladie est asymptotiquement globalement stable.*
2. *Si* $R_0 > \left(\sup_{\theta\geqslant 0} w(\theta)\right)^{-1}$, *l'équilibre sans maladie est instable et il existe un équilibre endémique.*
3. *Si* $R_0 > \left(\sup_{\theta\geqslant 0} w(\theta)\right)^{-1}$, *l'équilibre endémique est unique et est asymptotiquement localement stable si le nombre d'individus infectés à l'équilibre est suffisamment petit.*

Plusieurs autres cas particuliers du modèle de Kermack et McKendrick ont été aussi étudiés. Dans [13] Breda réécrit la force totale d'infection de l'épidémie comme solution d'une équation du renouvellement et en déduit son comportement asymptotique, et dans [68] Magal, McCluskey et Webb étudient le comportement en temps long vu comme modèle de McKendrick-von Foerster. Mieux encore dans [99] Webb présente une théorie assez générale pour l'étude asymptotique des équations de transport non linéaire pour des modèles de populations de naissance et de mort structuré en âge généralement appelé équation de type McKendrick-von Foerster généralisant (16). Il s'agit du système suivant :

$$\begin{cases} \partial_t v(t,a) + \partial_a v(t,a) = -\mu(a,P(t))\, v(t,a) \\ v(t,0) = \displaystyle\int_0^\infty \beta(a,P(t))\, v(a,t)\, da \\ v(0,a) = \phi(a) \\ P(t) = \displaystyle\int_0^\infty v(t,a)\, da \end{cases} \tag{17}$$

Dans ce modèle a représente l'âge de l'individu qui évolue de manière linéaire et qui repasse à 0 à chaque décès. $v(t,0)$ représente le taux de naissance de la population à l'instant t et $\mu(a,P(t))v(t,a)$ le taux de naissance par individu. L'idée principale consiste à transformer notre problème en un problème de Cauchy abstrait de la forme suivante

$$\frac{du(t)}{dt} = \mathscr{A}(u_t) + F(u_t), \tag{18}$$

où $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ est un opérateur différentiel et F un opérateur défini sur un sous-ensemble borné C_0 de X. Puis il définit les quantités suivantes

$$w_0(\mathscr{A}) = \lim_{t\to\infty} \frac{1}{t} \ln \| T_{\mathscr{A}}(t) \|_{\mathscr{L}(X)},$$

et

$$w_{ess}(\mathscr{A}) = \lim_{t\to\infty} \frac{1}{t} \ln \| T_{\mathscr{A}}(t) \|_{ess}. \tag{19}$$

où $\|\cdot\|_{\mathscr{L}(X)}$ désigne la norme d'opérateur continue sur X, $T_{\mathscr{A}}(t)$ le semi-groupe associé à $\mathscr{A}$ et

$$\|L\|_{ess} = \inf_{K\in\mathscr{K}(X)} \|L-K\|_X,$$

avec $\mathscr{K}(X)$ l'espace des opérateurs compacts sur X.

Le premier résultat principal de cette théorie dans [99, Proposition 4.13] est le théorème suivant.

Théorème 5.

$$w_0(\mathscr{A}) = \max\left\{ w_{ess}(\mathscr{A}), \sup\{Re(\alpha),\, \alpha \in \sigma(\mathscr{A})\} \right\},$$

où $\sigma(\mathscr{A})$ désigne le spectre de $\mathscr{A}$.

Il en déduit le résultat suivant, voir [99, Proposition 4.17, page 211] ou [93, Théo-

rème 4.2] qui donne la stabilité locale :

Théorème 6. *Soit F un opérateur continu Fréchet-différentiable sur C_0 et u_* un point d'équilibre de (18). Si $w_0(\mathscr{A}+F'(u_*))<0$, alors pour tout $w>w_0(\mathscr{A}+F'(u_*))$, il existe $\delta>0, c>0$, tel que pour tout $u\in C_0$, $\|u-u_*\|\leqslant\delta$, et $t\geqslant 0$,*

$$\|T_{\mathscr{A}}(t)(u)-u_*\|\leqslant ce^{wt}\|u-u_*\|.$$

Par suite il en déduit le corollaire suivant [99, Théorème 4.12, page 219] ou [93, Corollaire 4.3]

Corollaire 1.2. *Soit F un opérateur continu Fréchet-différentiable sur C_0. Soit u_* un point d'équilibre de (18). Si $w_{ess}(\mathscr{A}+F'(u_*))<0$, alors :*

1. *Si toutes les valeurs propes de $\mathscr{A}+F'(u_*)$, ont leurs parties réelles strictement négatives, il existe $w<0<\delta, c>0$, tel que pour tout $u\in C_0$, $\|u-u_*\|\leqslant\delta$, et $t\geqslant 0$,*

$$\|T_{\mathscr{A}}(t)(u)-u_*\|\leqslant ce^{wt}\|u-u_*\|.$$

2. *Si au moins une des valeurs propes de $\mathscr{A}+F'(u_*)$, a une partie réelle strictement positive, alors u_* est un équilibre instable au sens suivant :*

   Il existe $\epsilon>0$, et une suite $(u_n)_n$ convergeant vers u_* dans C_0 et $t_n\to\infty$, $n\to\infty$ tel que pour tout $n\in\mathbb{N}$,

$$\|T_{\mathscr{A}}(t_n)(u_n)-u_*\|\geqslant\epsilon.$$

Dans [93] Thieme propose une approche un peu différente toujours basée sur les problèmes de Cauchy abstraits afin de traiter des cas où les conditions au bord sont non linéaires. L'idée d'utiliser la méthode des problèmes de Cauchy abstraits pour l'étude du comportement en temps long fut récement reprise par Magal dans [69] où il présente plusieurs exemples. De même dans [21] Chen étudie de la stabilité globale du modèle SIRS avec démographie et âge d'infection.

Dans [44] Gabriel utilise les conditions de minorations de Doeblin pour étudier le comportement asymptotique d'un modèle de population de naissance et mort structuré en âge modélisant la division cellulaire. Il s'agit du système conservatif suivant :

$$\begin{cases}\dfrac{\partial u(t,a)}{\partial t}+\dfrac{\partial u(t,a)}{\partial a}=-\beta(a)u(t,a), & t,a>0,\\ u(t,0)=\displaystyle\int_0^\infty\beta(a)u(t,a)da, & t>0,\\ u(0,a)=u^{int}(a).\end{cases}\tag{20}$$

Il redémontre le résultat suivant (voir aussi [72]),

Théorème 7. *Soit $(T(t))_{t\geq 0}$ un semi-groupe sur $\mathscr{M}(\mathbb{R}_+)$ qui satistifait la condition de Doeblin :*

$$\exists t_0>0,\ c\in]0,1[,\ \nu\in\mathscr{P}(\mathbb{R}_+)\ \text{tels que}\ \forall\mu\in\mathscr{P}(\mathbb{R}_+),\quad \mu T(t_0)\geqslant c\nu.$$

Alors pour tout $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}_+)$, *tels que* $\mu_1(\mathbb{R}_+) = \mu_2(\mathbb{R}_+)$,

$$\|\mu_1 T(t) - \mu_2 T(t)\|_{TV} \leqslant \frac{1}{1-c} \exp(-\alpha(t-t_0)) \|\mu_1 - \mu_2\|_{TV},$$

où

$$\alpha = -\frac{\log(1-c)}{t_0} > 0,$$

et $\|\cdot\|_{TV}$ *désigne la norme en variation totale.*

Il utilise ensuite ce résultat pour démontrer le théorème suivant,

Théorème 8. *S'il existe* $a_* > 0$, $\beta_{min}, \beta_{max} > 0$, $\forall a \geqslant 0$, $\beta_{min} \mathbb{1}_{[a_*,\infty)} \leqslant \beta(a) \leqslant \beta_{max}$, *alors le système d'équation* (20) *admet une unique solution stationnaire à constante près* u_* *définie par :*

$$u_*(a) = \int_0^\infty \beta(s) u_*(s) ds \exp\left(-\int_0^a \beta(s) ds\right).$$

De plus pour tout $t \geqslant 0$, *pour tout* $u_0 \in L^1(\mathbb{R}_+)$,

$$\left\| u_t - \int_{\mathbb{R}_+} u_0(a) \mathrm{d}a u_* \right\|_{TV} \leqslant \frac{1}{1-c} e^{-\alpha(t-a_*)} \|u_0 - u_*\|_{TV},$$

où

$$c = a_* \beta_{min} e^{-a_* \beta_{max}} \text{ et } \alpha = -\frac{\log(1-c)}{a_*} > 0.$$

D'autre part si $a_* = 0$ *pour tout* $t \geqslant 0$,

$$\|u_t - u_*\|_{TV} \leqslant e^{-\beta_{min} t} \|u_0 - u_*\|_{TV}.$$

Dans [73] Michel, Mischler et Perthame présentent la méthode d'entropie généralisée pour étudier le comportement en temps long de différentes EDP. Celle-ci consiste à chercher une fonction de Lyapunov $\mathcal{H}$ et une fonction de dissipation $D_{\mathcal{H}}$ telles que

$$\frac{d}{dt} \mathcal{H}(u_t) = -D_{\mathcal{H}}(u_t) \leqslant 0.$$

Puis d'utiliser une inégalité de type Poincaré et le lemme de Gronwall pour conclure, voir par exemple [83] pour plus d'exemples.

Dans [76] Pakdaman, Perthame et Salort étudient le comportement en temps long du système d'équations suivant modélisant un réseau excitateur composé de neurones

qui se chargent par décharge d'un autre neurone :

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -p(a,X(t))u(t,a), \quad t>0,\, a>0, \\ u(t,0) = N(t) := \displaystyle\int_0^\infty p(a,X(t))u(t,a)da, \quad t>0, \\ u(0,a) = u^{int}(a), \quad a>0 \\ \displaystyle\int_0^\infty u(t,a)da = 1, \quad t\geqslant 0, \end{cases} \tag{21}$$

où $p(a;X)$ représente le taux d'excitation d'un neurone d'âge a situé dans l'environement X. Les auteurs supposent $X(t) = N(t)$ et $p(a,x) = \mathbb{1}_{a>\sigma(x)}$ avec σ une fonction Lipschitzienne strictement positive, minorée et majorée. Puis ils démontrent sous certaines hypothèses que $\frac{d}{dt}N(t)\sigma'(N(t)) < 1$ et $N(t)$ converge lorsque $t\to\infty$ et par conséquent $u(t,\cdot)$ converge lorsque $t\to\infty$ (voir aussi [75]). D'autre part ils construisent une fonction σ et des fonctions périodiques N tels que $\frac{d}{dt}N(t)\sigma'(N(t)) = 1$ et déduisent que le système d'équations aux dérivées partielles (21) admet une solution périodique.

Dans [96] Torres, Perthame et Salort étudient le comportement en temps long du système d'équations suivant modélisant un réseau excitateur composé de neurones qui se chargent par décharge d'un autre neurone :

$$\begin{cases} \partial_t u(t,s,a) + \partial_s u(t,s,a) + \partial_a u(t,s,a) = -p(s,a,X(t))u(t,s,a), \quad t>0,\, a>s>0, \\ u(t,0,a) = N(t,a) := \displaystyle\int_0^\infty p(a,s,X(t))u(t,s,a)ds, \quad t>0,\, a>0 \\ X(t) = \displaystyle\int_0^\infty N(t,a)da, \quad t>0 \\ u(0,s,a) = u^{int}(s,a), \quad a>s>0 \\ \displaystyle\int_0^\infty\int_0^a u(t,s,a)dsda = 1, \quad t\geqslant 0. \end{cases} \tag{22}$$

Ils établissent le résultat suivant :

Théorème 9. *S'il existe σ, p_0, $p_\infty > 0$ tel que $p_0\mathbb{1}_{\{a>s\geqslant\sigma\}} \leqslant p \leqslant p_\infty$, et si p est une fonction Lipshitsienne satisfaisant $\|\partial_X p\|_\infty < 1$ alors le système d'équations (22) admet une unique solution stationnaire (u_*, N_*, X_*). De plus il existe $C,\alpha > 0$ tels que pour tout $t\geqslant 0$,*

$$\|u_t - u_*\|_{TV} \leqslant Ce^{-\alpha t}\|u_0 - u_*\|_{TV},$$

D'autre part $\|N(t) - N_*\|_1$ et $|X(t) - X_*|$ convergent exponentiellement vite vers 0 quand $t\to\infty$.

Pour établir ce théorème ils utilisent les hypothèses de minorations et majorations de p pour vérifier la condition de Doeblin. Ensuite, ils appliquent le Théorème 7 et le fait que $\|\partial_X p\|_\infty < 1$ pour conclure.

Pour en savoir plus sur l'étude des modèles déterministes avec âge d'infection, on pourra consulter [83, 69, 67].

2. Modèles Stochastiques

En 1925 dans [70] McKendrick introduit le modèle stochastique SIR, puis en 1931 et 1946 Greenwood utilise les chaines binomiales pour étudier la distribution des cas multiples de maladie dans les ménages. Nous avons aussi les travaux de Bartlett en 1946 [9] et 1949 [10], et de Bailey en 1950 [2]. Dans ces articles à chaque instant ils décrivent l'évolution de la loi du nombre d'individus infectés et de susceptibles remplaçant ainsi la valeur ponctuelle donnée par le modèle déterministe. Dans [2] Bailey décrit un modèle stochastique d'épidémie dans lequel aucun individu infecté ne se remet de la maladie. D'autre part, dans [3] il traite le cas du modèle stochastique SIR et étudie la loi de la taille finale de l'épidémie. En 1955 dans [17] Bartlett établit la convergence en grande population du modèle stochastique SIR vers le modèle déterminste décrit par le système d'équations différentielles (14). Ce résultat fut étendu à une classe de processus de Markov à temps continu par Kurtz dans [65] en 1970. De même en 1956 dans [60] Kendall étudie le comportement en temps long de l'épidémie pour le modèle stochastique. Les modèles stochastiques d'épidémies furent largement étudiés par la suite et nous avons assisté à de multilples variantes de modèles, voir par exemple [15].

2.1. Modèles Markoviens (SIR et SIRS)

2.1.1. Modèles SIR classique

En 1955 dans [17] et en 1970, dans [65] Bartlett et Kurtz montrent que le système d'équations différentielles (14) décrivant la dynamique déterminste du modèle SIR d'épidémie est la limite en grande population d'un procesus de Markov. Plus précisément, en faisant les hypothèses suivantes :

— On se donne une population de taille N ;
— On suppose qu'il existe un nombre $0 < I^N(0) < N$, d'infectés à l'instant 0 ;
— Chaque paire d'individus a des contacts aux instants de saut d'un processus de poisson d'intensité c/N indépendamment les un des autres.
— Si l'individu contacté est susceptible, alors il devient infecté avec probabilité $p > 0$;
— Chaque individu infecté le reste pendant une durée aléatoire η, de fonction de répartition F, indépendant pour chaque individu. *À* la fin de cette période l'individu est remis.

On note $\lambda = cp$, l'infectivité d'un individu.

Rappellons que $S^N(t)$ représente le nombre de susceptibles à l'instant t, $I^N(t)$ celui des infectés et $R^N(t)$ celui des rémis.

Étant donné qu'à un instant s un individu devient infecté à taux $\lambda \frac{S^N(s)}{N} I^N(s)$, et en supposant que $\eta \sim \mathscr{E}(\gamma)$, le modèle stochastique est décrit par le système fonctionnel

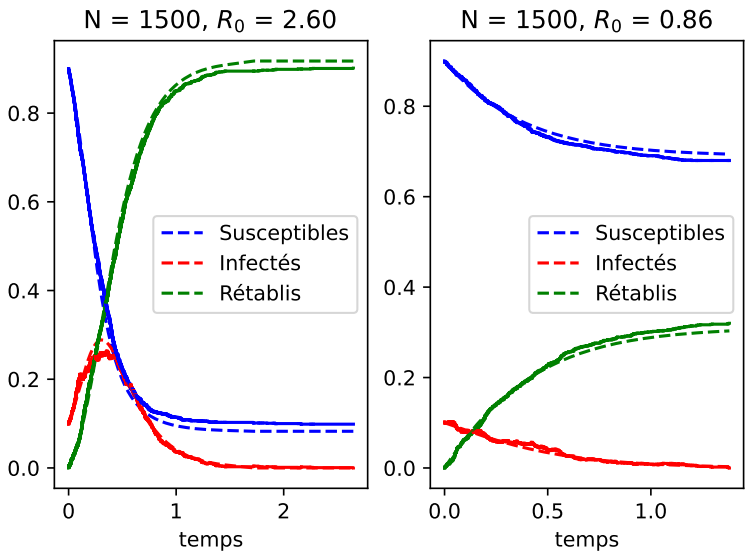

FIGURE 2. – Comparaison du modèle stochastique SIR avec le modèle déterministe pour différentes valeurs de R_0 avec $\overline{S}(0) = 0.9$, et $\overline{I}(0) = 0.1$. La courbe en trait interrompu est celle du modèle déterministe.

d'équations Poissoniennes suivant :

$$\begin{cases} S^N(t) = S^N(0) - P_1\left(\int_0^t \lambda \frac{S^N(s)}{N} I^N(s)ds\right) \\ \\ I^N(t) = I^N(0) + P_1\left(\int_0^t \lambda \frac{S^N(s)}{N} I^N(s)ds\right) - P_2\left(\int_0^t \gamma I^N(s)ds\right) \\ \\ R^N(t) = R^N(0) + P_2\left(\int_0^t \gamma I^N(s)ds\right), \end{cases}$$

où P_1 et P_2 sont deux processus de Poisson indépendants d'intensité 1.

En notant $\overline{S}^N = \frac{S^N}{N}$, $\overline{I}^N = \frac{I^N}{N}$ et $\overline{R}^N = \frac{R^N}{N}$, on déduit de [65, Théorème 3.1] le théorème suivant dit loi des grands nombres (voir aussi [15]).

Théorème 10. *Si* $\left(\overline{S}^N(0), \overline{I}^N(0), \overline{R}^N(0)\right) \to \left(\overline{S}(0), \overline{I}(0), \overline{R}(0)\right)$ *en probabilité lorsque* $N \to \infty$, *alors pour tout* $T \geqslant 0$, *et* $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left\{\sup_{t\leqslant T}\left(|\overline{S}^N(t) - \overline{S}(t)| + |\overline{I}^N(t) - \overline{I}(t)| + |\overline{R}^N(t) - \overline{R}(t)|\right) \geqslant \epsilon\right\} = 0, \tag{23}$$

où $\left(\overline{S}(t), \overline{I}(t), \overline{R}(t)\right)$ *est solution du système d'équations différentielles (ODE) suivant :*

$$\begin{cases} \overline{S}'(t) = -\lambda \overline{S}(t)\overline{I}(t) \\ \\ \overline{I}'(t) = \lambda \overline{S}(t)\overline{I}(t) - \gamma \overline{I}(t) \\ \\ \overline{R}'(t) = \gamma \overline{I}(t). \end{cases}$$

Puis en 1971 dans [66] Kurtz étudie les fluctuactions du modèle stochastique autour de la limite en grande population : c'est le théorème de la limite centrale. Plus précisement, en notant

$$\hat{S}^N = \sqrt{N}\left(\overline{S}^N - \overline{S}\right), \hat{I}^N = \sqrt{N}\left(\overline{I}^N - \overline{I}\right) \text{ et } \hat{R}^N = \sqrt{N}\left(\overline{R}^N - \overline{R}\right),$$

dans [7] Barbour déduit du Théorème 3.1 de [66] le résultat suivant (voir aussi [16]).

Théorème 11. *On suppose que* $(\hat{S}^N(0), \hat{I}^N(0), \hat{R}^N(0)) \to (\hat{S}(0), \hat{I}(0), \hat{R}(0))$*, en loi lorsque* $N \to \infty$*. Alors* $(\hat{S}^N, \hat{I}^N, \hat{R}^N) \to (\hat{S}, \hat{I}, \hat{R})$ *en loi dans* $\mathscr{C}\left([0,T], \mathbb{R}^3\right)$ *quand* $N \to \infty$*, où le triplet* $(\hat{S}, \hat{I}, \hat{R})$ *vérifie le système suivant :*

$$\begin{cases} \hat{S}(t) & = \hat{S}(0) - \lambda \int_0^t \left(\hat{S}(s)\overline{I}(s) + \overline{S}(s)\hat{I}(s)\right) ds - \int_0^t \sqrt{\lambda \overline{S}(s)\overline{I}(s)}\, dB_s^1 \\ \\ \hat{I}(t) & = \hat{I}(0) + \int_0^t \left(\lambda \hat{S}(s)\overline{I}(s) + \lambda \overline{S}(s)\hat{I}(s) - \gamma \hat{I}(s)\right) ds \\ & \qquad + \int_0^t \sqrt{\lambda \overline{S}(s)\overline{I}(s)}\, dB_s^1 - \int_0^t \sqrt{\gamma \overline{I}(s)}\, dB_s^2 \\ \\ \hat{R}(t) & = \hat{R}(0) + \gamma \int_0^t \hat{I}(s) ds + \int_0^t \sqrt{\gamma \overline{I}(s)}\, dB_s^2, \end{cases}$$

et $B = (B^1, B^2)$ *est le mouvement Brownien standard.*

Etant donné que la phase centrale de l'épidémie est décrite par le modèle déterministe obtenu par la loi des grands nombres (Théorème 10), la loi des grands nombres ne permet pas d'étudier les différentes phases d'une épidémie. En fait, le début et la fin de l'épidémie sont des phases intrinsènquement stochastiques, qu'au aucun modèle déterministe ne peut décrire. Ainsi, en 1975 dans [8] Barbour propose une méthode pour étudier les différentes phases d'une épidémie. Plus précisément, il détermine le temps écoulé entre la première infection et la dernière rémission pour le modèle SIR classique. Il établit un théorème limite pour la loi de cette durée lorsque la taille de la population est grande. Plus précisément, de (13) on a

$$\overline{S}_\infty - \frac{1}{R_0} \ln\left(\overline{S}_\infty\right) = 1 - \frac{1}{R_0} \ln\left(\overline{S}(0)\right).$$

Par suite en notant $0 < \theta < 1$ la valeur approchée de la proportion de susceptibles à la

fin de l'épidémie, solution de l'équation suivante :

$$\theta - \frac{1}{R_0}\ln(\theta) = 1 + \overline{I}(0),$$

puis en remarquant que

$$J(a,b) = \frac{1}{\lambda}\int_a^b \frac{du}{u\left[1+\overline{I}(0)+\frac{1}{R_0}\log(u)-u\right]} \tag{24}$$

est la durée approchée pendant laquelle $a \leqslant \overline{S}(t) \leqslant b$ et en posant

$$T^N = \inf\left\{t>0,\, \overline{I}^N(t) = 0\right\},$$

il établit le résultat suivant :

Théorème 12. *1. Si $\overline{I}(0) = h$, $h \in]0,1[$, alors quand $N \to \infty$,*

$$\left(\gamma\left[1-\theta R_0\right]\right)^{-1} T^N - \log(N) - k \tag{25}$$

converge en loi vers la variable aléatoire W de loi définie par :

$$\mathbb{P}(W \leqslant w) = \exp\left(-e^{-w}\right),$$

où

$$k = \lim_{m\to\infty}\left(-\ln(m) + \gamma(1-\theta R_0)\, J\left(\theta\left(1+m^{-1}\left(\gamma\left[1-\theta R_0\right]\right)^{-1}\right),1\right) + \ln(1+\theta R_0)\right).$$

2. Si $R_0 > 1$, et $I^N(0) = 1$. Alors quand $N \to \infty$,

$$\mathbb{P}\left\{T^N \leqslant t\right\} \to \gamma\left(1-e^{-\gamma(R_0-1)t}\right)\left(1-\gamma e^{-\gamma(R_0-1)t}\right)^{-1} \tag{26}$$

$$\mathbb{P}\left\{T^N - \left[\left(\gamma\left[1-\theta R_0\right]\right)^{-1} + \left(\gamma\left[R_0-1\right]\right)^{-1}\right]\log(N) - c \geqslant x\right\} \to \gamma(R_0-1)\,\mathbb{P}\left\{W' \geqslant x\right\}, \tag{27}$$

où

—

$$c = \lim_{m\to\infty}\left(-\left[\left(\gamma\left[1-\theta R_0\right]\right)^{-1} + \left(\gamma\left[R_0-1\right]\right)^{-1}\right]\log(m) + \left(\gamma\left[R_0-1\right]\right)^{-1}\log\left(\gamma\left[R_0-1\right]\right)\right.$$
$$\left. + \left(\gamma\left[1-\theta R_0\right]\right)^{-1}\log(1-\theta R_0) + J\left[\theta\left(1+m^{-1}\left(\gamma\left[1-\theta R_0\right]\right)^{-1}\right), 1-m^{-1}\left(\gamma\left[R_0-1\right]\right)^{-1}\right]\right);$$

— *θ et $J(\cdot,\cdot)$ sont évalués lorsque $\overline{I}(0) = 0$;*
— *$W' = \left(\gamma\left[1-\theta R_0\right]\right)^{-1} W_1 + \left(\gamma\left[R_0-1\right]\right)^{-1} W_2$, où W_1 et W_2 sont deux variables aléatoires indépendantes de même loi que W.*

L'avantage de ce théorème est qu'il permet de donner une bonne approximation de

la durée de l'épidémie à N fixé.

Notons aussi que dans l'énoncé 1 du Théorème 12, Barbour considère que $\overline{S}(t) + \overline{I}(t) + \overline{R}(t) = 1 + h$ et dans le second énoncé 2 il suppose $S^N(t) + I^N(t) + R^N(t) = N + 1$, ce qui diffère légèrement du modèle stochastique décrit ci-haut.

Dans [6] il propose une généralisation de ce résultat pour le modèle SIR général décrit à la section 1.1.1.

2.1.2. Modèles SIRS classique

Si on suppose en plus des hypothèses du modèle SIR qu'après rémission les individus redeviennent susceptibles après une durée aléatoire de loi exponentielle de paramètres β, la dynamique stochastique de l'épidémie est cette fois donnée par

$$\begin{cases} S^N(t) = S^N(0) - P_1\left(\int_0^t \lambda \frac{S^N(s)}{N} I^N(s)ds\right) + P_3\left(\int_0^t \beta R^N(s)ds\right) \\ I^N(t) = I^N(0) + P_1\left(\int_0^t \lambda \frac{S^N(s)}{N} I^N(s)ds\right) - P_2\left(\int_0^t \gamma I^N(s)ds\right) \\ R^N(t) = R^N(0) + P_2\left(\int_0^t \gamma I^N(s)ds\right) - P_3\left(\int_0^t \beta R^N(s)ds\right). \end{cases}$$

où P_1, P_2 et P_3 sont trois processus de Poisson indépendants d'intensité 1.

De [65, 15] on a le théorème limite loi des grands nombres.

Théorème 13. *Si* $\left(\overline{S}^N(0), \overline{I}^N(0), \overline{R}^N(0)\right) \to \left(\overline{S}(0), \overline{I}(0), \overline{R}(0)\right)$ *en probabilité lorsque* $N \to \infty$, *alors pour tout* $T \geqslant 0$, *et* $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left\{\sup_{t\leqslant T}\left(|\overline{S}^N(t) - \overline{S}(t)| + |\overline{I}^N(t) - \overline{I}(t)| + |\overline{R}^N(t) - \overline{R}(t)|\right) \geqslant \epsilon\right\} = 0, \tag{28}$$

où $\left(\overline{S}(t), \overline{I}(t), \overline{R}(t)\right)$ *est solution du système d'équations différentielles ordinaires (ODE) suivant :*

$$\begin{cases} \overline{S}'(t) = -\lambda \overline{S}(t)\overline{I}(t) + \beta \overline{R}(t) \\ \overline{I}'(t) = \lambda \overline{S}(t)\overline{I}(t) - \gamma \overline{I}(t) \\ \overline{R}'(t) = \gamma \overline{I}(t) - \beta \overline{R}(t). \end{cases}$$

Son comportement asymptotique est donnée par le théorème suivant :

Théorème 14.
1. *Le triplet* $(1,0,0)$ *est un équilibre sans maladie. Il est stable si et seulement si* $R_0 \leqslant 1$;
2. *Si* $R_0 > 1$, *le triplet* $(\overline{S}_*, \overline{I}_*, \overline{R}_*)$ *défini par*

$$\overline{S}_* = \frac{1}{R_0}, \quad \overline{I}_* = \frac{\beta}{\gamma+\beta}(1 - \frac{1}{R_0}), \quad \overline{R}_* = \frac{\gamma}{\gamma+\beta}(1 - \frac{1}{R_0}). \tag{29}$$

est l'unique équilibre endémique globalement asymptotiquement stable.

Le théorème de la limite centrale associé est donné par le théorème suivant :

Théorème 15. *On suppose que* $(\hat{S}^N(0), \hat{I}^N(0), \hat{R}^N(0)) \to (\hat{S}(0), \hat{I}(0), \hat{R}(0))$ *en loi lorsque* $N \to \infty$. *Alors* $(\hat{S}^N, \hat{I}^N, \hat{R}^N) \to (\hat{S}, \hat{I}, \hat{R})$ *en loi dans* $\mathscr{C}[0,T]$ *quand* $N \to \infty$, *où le triplet* $(\hat{S}, \hat{I}, \hat{R})$ *vérifie le système suivant :*

$$\begin{cases} \hat{S}(t) &= \hat{S}(0) - \lambda \int_0^t \left(\hat{S}(s)\overline{I}(s) + \overline{S}(s)\hat{I}(s)\right) ds + \beta \int_0^t \hat{R}(s) ds - \int_0^t \sqrt{\lambda \overline{S}(s)\overline{I}(s)}\, dB_s^1 \\ &\qquad + \int_0^t \sqrt{\beta \overline{I}(s)}\, dB_s^3 \\ \hat{I}(t) &= \hat{I}(0) + \lambda \int_0^t \left(\hat{S}(s)\overline{I}(s) + \overline{S}(s)\hat{I}(s)\right) ds - \gamma \int_0^t \hat{I}(s) ds \\ &\qquad + \int_0^t \sqrt{\lambda \overline{S}(s)\overline{I}(s)}\, dB_s^1 - \int_0^t \sqrt{\gamma \overline{I}(s)}\, dB_s^2 \\ \hat{R}(t) &= \hat{R}(0) + \gamma \int_0^t \hat{I}(s) ds - \beta \int_0^t \hat{R}(s) ds + \int_0^t \sqrt{\gamma \overline{I}(s)}\, dB_s^2 - \int_0^t \sqrt{\beta \overline{I}(s)}\, dB_s^3, \end{cases}$$

et (B^1, B^2, B^3) *est le mouvement Brownien standard.*

Du théorème de la limite centrale, dans [81] Pardoux déduit que $\sqrt{N}\left(\overline{I}^N(t) - \overline{I}_*\right)$ converge vers un processus gaussien quand N est grand et il en déduit que pour t suffisament grand, $\overline{I}^N(t) \approx \overline{I}_* + N^{-1/2}\hat{I}(t)$. Puisque pour t grand la loi de $\hat{I}(t)$ est approximativement une gaussienne de moyenne 0 et de variance R_0, il déduit que si $N \sim \frac{1}{\overline{I}_*^2 R_0}$ l'épidémie va s'éteindre en un temps d'ordre 1 avec une probabilité non négligeable.

Dans le but d'étudier le temps d'extinction des épidémies pour le modèle stochastique il apparaît crucial d'étudier les grandes déviations. Les grandes déviations permettent d'étudier la décroissance exponentielle des probabilités d'évènements extrêmes, lorsque le nombre d'observations est arbitrairement grand. Dans [82] Pardoux et Samegni-Kepgnou étudient les grandes déviations pour des modèles stochastiques d'épidémies classiques généralisant [30] et dans [81] Pardoux étudie les déviations modérées. Pour en savoir plus sur les modèles stochastiques d'épidémie markoviens nous renvoyons à [35, 15, 16] et sur les grandes déviations [29].

Bien que ces modèles markoviens nous permettent d'avoir des informations sur l'épidémie, il n'en sont pas pour autant très proches de la réalité. En effet, dans ces modèles, le taux d'infectiosité d'un individu infectieux reste constant durant toute la période de sa maladie. De manière générale les taux avec lesquelles les individus changent de compartiments sont des constantes, ce qui n'est pas réaliste car normalement ces taux devraient évoluer avec le temps. D'autres part les durées de séjour dans chaque compartiment suivent des lois exponentielle, d'où l'absence de mémoire. En effet, dans des situations où il y a changement brusque des paramètres, on observe expérimentalement que le système a une mémoire et ceci est mal reflété par des modèles ODE. De plus lorsque le nombre moyen de personnes qu'un individu peut

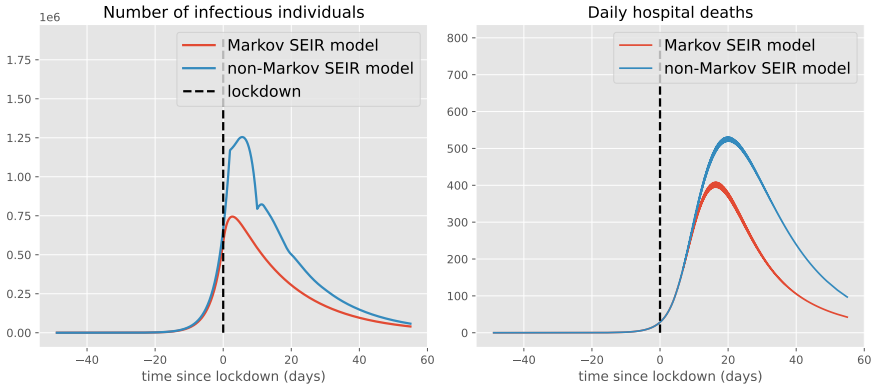

FIGURE 3. – Comparaison du modèle SEIR Markovien et du modèle SEIR non-Markovien (avec durée d'exposition et d'infectiosité déterministe) pour les prédictions de l'effet du confinement sur le nombre de morts dans les hôpitaux. Ces deux modèles supposent que la durée moyenne de la période pré-infectieuse est de 2 jours et celle de l'infection est de 8 jours. Le délai entre l'infection et la mort est en moyenne de 20 jours avec une variance de 7.5 jours.

infecter (noté R_0) devient brutalement inférieur à 1 (par exemple à cause d'un confinement, voir la figure 3) l'épidémie continue de croître pendant un petit moment avant de chuter, cela est dû au changement soudain du taux d'infectiosité et à la mémoire du système. D'autres part l'estimation des paramètres et les prédictions des deux modèles ne sont pas les mêmes. En fait, pour les modèles markoviens en dimension finie, pour estimer les conditions initiales on résout un système d'équations différentielles par contre comme nous le vérons plus tard pour des non-markoviens on résout un système d'équations fonctionnel. Ainsi pour résoudre ces problèmes nous avons trois solutions : soit rester dans le cadre markovien en multipliant le nombre de compartiments, ou en passant en dimension infinie en considérant les modèles structurés en âge (et éventuellement un trait), ou s'affranchir du cadre markovien et passer aux modèles non-Markoviens.

2.2. Modèles structurés en âge

Les processus de branchement classique permettent de mieux décrire le début et la fin de l'épidémie car ils ne supposent pas que les temps de contact sont décrit par un processus de Poisson d'intensité homogène (voir par exemple [4, 16]). Comme nous allons le voir au chapitre 3, lorsqu'on rajoute des interactions aux processus de branchement classique, on est capable de capturer l'évolution entière de l'épidémie. Dans cette section, nous allons introduire les processus de branchement avec interaction et rappeler quelques résultats de limite en grande population.

En 1968 dans [27, 26] Crump, Mode et Jagers introduisent une généralisation des

processus de branchement structurés en âge. Dans [27] ils s'intéressent aux processus décrivant le nombre d'individus vivant à un instant donné qui descendent d'un unique ancêtre à l'instant initial. Leur modèle est le suivant :

— Chaque individu de la population a une durée de vie aléatoire et donne naissance à des instants aléatoires au cours de sa vie ;
— À la mort de l'individu, il se divise en un nombre aléatoire d'individus ;
— La progéniture de l'individu se comporte de la même manière et le comportement de chaque individu est indépendant de tous les autres.

Leur processus de branchement structuré en âge fut appelé plus tard, le processus de Crump-Mode-Jagers (CMJ).

En 1990 dans [74], Oelschlager établit la loi des grands nombres et le théorème de la limite centrale d'une population structurée en âge. Il suppose que le taux de naissance et de décès de chaque individu dépendent de l'âge de l'individu et de la structure d'âge de la population. Les hypothèses de son modèle sont les suivants :

— Il suppose que la taille initiale de la population est N ;
— Il suppose qu'à un instant $t \geqslant 0$ la population est formée d'individus d'âge a_k, $k \in \mathscr{R}_N(t)$, où $\mathscr{R}_N(t)$ est une énumération des individus vivant à l'instant t ;
— Il définit la moyenne empirique du processus décrivant l'évolution de l'âge des individus par
$$A^N(t) = \frac{1}{N} \sum_{k \in \mathscr{R}_N(t)} \delta_{a_k};$$
— Il suppose qu'à un instant $t \geqslant 0$ l'individu j d'âge a_j donne naissance au taux $\beta(a_j, A^N(t))$ et meurt au taux $\gamma(a_j, A^N(t))$. Dans ce cas A^N saute de $A^N(t)$ à $A^N(t) + \frac{1}{N}\delta_0$ pour une naissance et de $A^N(t)$ à $A^N(t) - \frac{1}{N}\delta_{a_j}$ pour un décès. En dehors de ces transitions discontinues, A^N évolue continuellement avec l'âge.

Moyennant des hypothèses techniques, il démontre que la mesure empirique des âges des individus de la population converge en loi vers le processus à valeurs mesure A et $u(t,\cdot)$ est la densité de $A(t)$, décrite par le système d'équation aux dérivées partielles de Von Foerster suivant :

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -\gamma(a, A_t)\, u(t,a) \\ u(t,0) = \displaystyle\int_{\mathbb{R}_+} \beta(\tilde{a}, A(t))\, u(t,\tilde{a}) \mathrm{d}\tilde{a}, \\ A_t = \displaystyle\int_0^\infty u(t,\tilde{a})\, d\tilde{a}. \end{cases} \tag{30}$$

Le même résultat fut obtenu par Hamza, Jagers et Klebaner en 2013 dans [48] via une autre méthode pour le processus de Crump-Mode-Jagers. Pour le modèle CMJ, Hamza, Jagers et Klebaner supposent qu'un individu d'âge a donne naissance au cours de sa vie au taux $b_z(a)$, où z est la taille de la population. De plus ils supposent aussi qu'un individu d'âge a a une durée de vie aléatoire avec un taux de survie $h_z(a)$ (hazard rate). Ils notent $m_z(a) = \mathbb{E}[Y(a)]$ où $Y(a)$ est la variable aléatoire représentant le nombre d'individus produit à la mort d'un individu d'âge a. Ils supposent que les fonctions b, h et m sont uniformément bornées et qu'il existe $C > 0$ tel que pour tout $a \geqslant 0$, A, B

deux mesures quelconque,

$$|b_A(a) - b_B(a)| \leqslant C\rho(A,B), |h_A(a) - h_B(a)| \leqslant C\rho(A,B), |m_A(a) - m_B(a)| \leqslant C\rho(A,B),$$

où ρ est la distance de Levy Prokhorov et ils démontrent que la moyenne empirique de l'âge des individus converge. Dans ce cas on a $\beta(a,A(t)) = b_{A_t}(a) + m_{A_t}(a)h_{A_t}(a)$ et $\gamma(a,A(t)) = h_{A_t}(a)$.

Dans [97], Tran introduit un modèle de reproduction de population structurée en trait et en âge. Il suppose qu'un individu est représenté par une masse de dirac en $(a,\theta) \in \mathbb{R}_+ \times \Theta, \Theta \subset \mathbb{R}^d$ étant l'espace des traits et $\mathbb{R}_+$ l'espace des âges. Il suppose qu'au temps t la population est décrite par

$$Z_t^n = \frac{1}{n}\sum_{k=1}^{N_t^n} \delta_{(a_k^n(t),\theta_k^n(t))},$$

où N_t^n est le nombre d'individus vivant au temps t. Il suppose qu'un individu d'âge a et de traits θ dans une population représentée par Z^n se reproduit de façon asexuée et transmet héréditairement ses traits sauf s'il se produit une mutation, viellit et meurt :

— Il donne naissance à un nouvel individu au taux $b(a,\theta) \in \mathbb{R}_+$. Avec probabilité $p(a,\theta)$, le nouvel individu est un mutant de trait $\widetilde{\theta}$ choisi suivant une distribution de probabilité $K(a,\theta,\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta})$ où ν est une mesure $\sigma-$finie sur Θ,

— Il meurt au taux $d\left(a,\theta,\int_{\mathbb{R}_+\times\Theta} U\left((a,\theta),(\tilde{a},\tilde{\theta})\right) Z^n(\mathrm{d}\tilde{a},\mathrm{d}\widetilde{\theta})\right) \in \mathbb{R}_+$, où $U\left((a,\theta),(\tilde{a},\tilde{\theta})\right)$ est le taux de compétition de l'individu de paramètre $(\widetilde{a},\widetilde{\theta})$ sur l'individu (a,θ),

— Il viellit avec la vitesse $v(a,\theta) \in \mathbb{R}_+$.

Moyennant des hypothèses techniques, il démontre que le processus Z^n converge en probabilité vers le processus à valeurs mesure Z où $u(t,\cdot)$ est la densité de $Z(t)$, décrite par le système d'équations aux dérivées partielles suivant :

$$\begin{cases} \partial_t u_t(a,\theta) + \partial_a u_t(a,\theta) = -d\left(a,\theta,\int_{\mathbb{R}_+\times\Theta} U\left((a,\theta),(\tilde{a},\tilde{\theta})\right) u_t(\tilde{a},\widetilde{\theta})\mathrm{d}\tilde{a}\nu(\mathrm{d}\widetilde{\theta})\right) u_t(a,\theta) \\ v(0,\theta)u(t,0,\theta) = \displaystyle\int_{\mathbb{R}_+} b(a,\theta)(1-p(a,\theta))u_t(a,\theta)\mathrm{d}a \\ \qquad\qquad + \int_{\mathbb{R}_+\times\Theta} b(a,\widetilde{\theta})p(a,\widetilde{\theta})K(a,\widetilde{\theta},\theta)u_t(a,\widetilde{\theta})\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}) \\ u(0,a,\theta) = u_0(a,\theta). \end{cases} \tag{31}$$

2.3. Modèles non-Markoviens

2.3.1. Modèles stochastiques d'épidémies avec infectiosité variable

Dans [78] Guodong Pang et Étienne Pardoux étendent le modèle **SIR, SEIR** et SIRS non-Markovien avec durée d'exposition, d'infection et d'immunité arbitraires et fonction d'infectiosité constante. Puis plus tard, associés à Raphaël Forien, ils définissent dans [39] le modèle SIR et SEIR non-Markovien mais cette fois l'infectiosité de chaque

individu infecté est une fonction aléatoire du temps écoulé depuis l'infection. Ce modèle généralise le modèle de Kermack et McKendrick sans perte d'immunité décrit par (9) et permet de conclure que le modèle de Kermack et McKendrick s'obtient comme limite en grande population d'un modèle stochastique.

Dans ce modèle, le seul aspect de la modélisation qui diffère du modèle Markovien est le fait que la fonction d'infectiosité de chaque individu (noté λ_i pour le i-éme infecté après l'instant 0 et λ_i^0 pour le i-éme infecté à l'instant 0) devient une variable aléatoire dépendant du temps (chacune des familles étant prise i.i.d), puis le fait que les lois des durées d'expositions et d'infectiosités sont quelconques et non plus seulement exponentielles. Ici afin d'établir la loi forte des grands nombres, ils introduisent la notion de fonction force totale d'infection qui n'est autre que la somme des infectiosités des individus infectieux à un instant donné. Ainsi en notant

$$\eta = \sup\{t > 0,\, \lambda(t) > 0\} \text{ et } \eta^0 = \sup\{t > 0,\, \lambda^0(t) > 0\}$$

la variable aléatoire représentant la durée pendant laquelle un individu reste infectieux pour les individus infectés respectivement après l'instant 0, ou déjà infectés à l'instant 0. Posons,

$$F^c(t) = \mathbb{P}\left(\eta > t\right) \text{ et } F_0^c(t) = \mathbb{P}\left(\eta^0 > t\right).$$

Nous faisons l'hypothèse suivante,

Hypothèse 2.1. *Supposons que $F = F_1 + F_2$, où $F_1 = \sum_{i\in\mathscr{I}} a_i \mathbb{1}_{\{t\geqslant t_i\}}$ où $\mathscr{I}$ est un ensemble fini dénombrable ou infini dénombrable, $(a_i)_i$ une suite de réels positifs tel que $\sum_{i\in\mathscr{I}} a_i \leqslant 1$ et $(t_i)_i$ une suite croissante de réels positifs et F_2 une fonction continue de Hölder d'exposant $\theta + \frac{1}{2}$ pour un $\theta > 0$, c'est à dire $|F_2(t+\delta) - F_2(t)| \leqslant c\delta^{\theta+\frac{1}{2}}$.*

De plus on suppose que $\frac{1}{N} I^N(0) \to \overline{I}(0)$ et $\frac{1}{N} R^N(0) \to \overline{R}(0)$ en probabilité lorsque $N \to +\infty$ et on note $\mathfrak{F}^N(t)$ la force totale d'infection à l'instant t et si X^N est un processus on note $\overline{X}^N = N^{-1} X^N$. Les auteurs obtienent que la dynamique de ce modèle est décrite par le système suivant :

$$\begin{aligned}
\overline{\mathfrak{F}}^N(t) &= \frac{1}{N} \sum_{i=1}^{I^N(0)} \lambda_i^0(t) + \frac{1}{N} \sum_{i=1}^{A^N(t)} \lambda_i(t) \\
\overline{I}^N(t) &= \frac{1}{N} \sum_{j=1}^{I^N(0)} \mathbb{1}_{\eta_j^0 > t} + \frac{1}{N} \sum_{i=1}^{A^N(t)} \mathbb{1}_{\tau_i^N + \eta_i > t}, \\
\overline{R}^N(t) &= \overline{R}^N(0) + \frac{1}{N} \sum_{j=1}^{I^N(0)} \mathbb{1}_{\eta_j^0 \leqslant t} + \frac{1}{N} \sum_{i=1}^{A^N(t)} \mathbb{1}_{\tau_i^N + \eta_i \leqslant t}
\end{aligned}$$

où

$$A^N(t) = \int_0^t \int_0^{+\infty} \mathbb{1}_{u \leqslant \Upsilon^N(s)} Q(ds, du)$$

est le nombre d'infectés après l'instant 0,

$$\Upsilon^N(s) = \frac{S^N(s)}{N}\mathfrak{F}^N(s)$$

est le taux avec lequel un nouvel individu est infecté à l'instant s, $(\tau_i^N)_i$ les instants de sauts du processus A^N, c'est-à-dire les instants auxquels les individus sont infectés et Q une mesure de Poisson standard sur $\mathbb{R}_+^2$. Ils établissent alors le théorème suivant représentant la loi forte des grand nombres pour le modèle SIR :

Théorème 16 ([39])**.**

$$(\overline{S}^N, \overline{\mathfrak{F}}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{S}, \overline{\mathfrak{F}}) \text{ dans } \mathbf{D}^2$$

où le couple $(\overline{S}, \overline{\mathfrak{F}})$ *vérifie le système suivant :*

$$\begin{cases} \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}^0(t) + \int_0^t \overline{\lambda}(t-s)\overline{S}(s)\overline{\mathfrak{F}}(s)\,ds \\ \\ \overline{S}(t) = \overline{S}(0) - \int_0^t \overline{S}(s)\overline{\mathfrak{F}}(s)\,ds. \end{cases}$$

De plus étant donné $(\overline{S}, \overline{\mathfrak{F}})$*,*

$$(\overline{I}^N, \overline{R}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{I}, \overline{R}) \text{ dans } \mathbf{D}^2,$$

où $(\overline{I}, \overline{R})$ *est donné par*

$$\begin{cases} \overline{I}(t) = \overline{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\overline{S}(s)\overline{\mathfrak{F}}(s)\,ds \\ \\ \overline{R}(t) = \overline{R}(0) + \overline{I}(0)F_0(t) + \int_0^t F(t-s)\overline{S}(s)\overline{\mathfrak{F}}(s)\,ds. \end{cases}$$

Nous rappelons que **D** *est l'espace des fonctions continues à droite et limitées à gauche et qu'il est muni de la topologie de Skorokhod.*

Plus tard dans [79] Pang et Pardoux proposent une version EDP du Théorème 16

avec infectivité dépendant de l'âge d'infection donnée par le système suivant :

$$\begin{cases} \partial_t i(t,x) + \partial_x i(t,x) = -\frac{i(t,x)}{G^c(x)} \nu(dx) \\ i(t,0) = \overline{S}(t) \int_0^{t+\overline{x}} \frac{\overline{\lambda}(x)}{\frac{G^c(x)}{G^c(x-t)}} i(t,x)dx \\ \overline{S}'(t) = -i(t,0) \\ i(0,x) = \overline{\mathfrak{I}}(0,x) \, \forall x \in [0,\overline{x}] \\ \overline{S}(0) = 1 - \overline{I}(0), \end{cases} \tag{32}$$

où $G(t) = F(t^-)$ et $G^c(t) = 1 - F(t^-)$, ν la loi de η, $i(t,x)$ le nombre moyen d'individus infectés ayant un âge au plus égal à x et où ils supposent qu'il existe $\overline{x}$ tel que $\overline{I}(0) = \overline{\mathfrak{I}}(0,\overline{x})$.

Par la méthode des caractéristiques ils déduisent de (32) que ,

$$i(t,x) = \begin{cases} \dfrac{G^c(x)}{G^c(x-t)} i(0,x-t) \text{ si } x > t \\ \quad G^c(x) i(t-x,0) \text{ si } t \geqslant x. \end{cases}$$

Dans ce cas on a :

$$\overline{I}(t) = \int_0^{t+\overline{x}} i(t,x)dx.$$

Dans [42] Foutel-Rodier et al proposent aussi une version EDP de la loi des grands nombres du modèle de McKendrick-Von Foerster décrit par l'EDP suivante :

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = 0 \\ u(t,0) = \overline{S}(t) \int_0^\infty \lambda(\tilde{a}) u(t,\tilde{a}) d\tilde{a} \\ u(0,a) = x_0 g(a) \\ \overline{S}(t) = 1 - \int_0^\infty u(t,\tilde{a}) d\tilde{a}. \end{cases} \tag{33}$$

où $0 < x_0 < 1$,

$$\int_0^\infty g(a)da = 1, \text{ et } \int_0^\infty \lambda(a)da < \infty.$$

Dans cet équation $u(t,a)$ représente la densité de population infectée à l'instant t ayant un âge a et $\overline{S}(t)$ la proportion de susceptibles à l'instant t.

En effet, en regardant l'évolution de la mesure empirique des âges d'infection des

individus et du processus $(X_x(a))_{a\geqslant 0}$ décrivant l'évolution de l'infection de l'individu x, Ils montrent que la dynamique de l'épidémie lorsque $N\to\infty$ est décrite par l'EDP (34).

Ce modèle suppose qu'une fois rétabli on ne devient plus susceptible, et pour chaque individu x on note $X_x(a)$ le processus à valeurs dans $\{I,R\}$ décrivant l'évolution de l'infection de l'individu x. De plus à chaque individu x on associe un processus ponctuel $\mathscr{P}_x$ dont les atomes représentent les âges auxquels l'individu x infecte d'autres individus. La paire $(\mathscr{P}_x,X_x)$ caractérise l'évolution de l'infection de l'individu x. Ils supposent en plus que la famille $((\mathscr{P}_x,X_x))_x$ est i.i.d et de même loi que $(\mathscr{P},X)$. Afin de simplifier le modèle ils supposent que le processus décrivant les infections est donné par un processus de branchement structuré en âge appelé processus de Crump-Mode-Jagers décrit dans [27] : un individu infecté à l'instant $\tau_x<\infty$ produira N_x infections aux instants $\tau_x+A_1,\tau_x+A_2,\cdots\tau_x+N_x$, D'où

$$\mathscr{P}_x=\sum_{i=1}^{N_x}\delta_{A_i}$$

De plus ils supposent qu'à chaque contact infectieux avec l'individu x à un instant t, l'individu devient infecté avec probabilité $c(t)\in[0,1]$. De plus ils supposent que

$$\lambda(a)da=\mathbb{E}[\mathscr{P}(da)]$$

Ainsi en se donnant une population de taille N et une famille i.i.d de densité g d'âge d'infection des individus initialement infectés, en notant

$$\overline{\mu}_t^N(da\times\{i\})=\frac{1}{N}\sum_{\sigma_x<t}\delta_{t-\sigma_x,X_x(t-\sigma_x)}(da\times\{i\}),$$

la mesure empirique des ages d'infections et des compartiments, et en notant

$$u_i^N(t)=\frac{1}{N}\sum_{\sigma_x<t}\mathbb{1}_{\{X_x(t-\sigma_x)=i\}}=\overline{\mu}_t^N(\mathbb{R}_+\times\{I\}),$$

le nombre d'individus dans le compartiment i à l'instant t, ils établissent le résultat suivant :

Théorème 17. *$\overline{\mu}_t^N(da\times\{i\})\to u(t,a)\mathbb{P}(X(a)=i)da$, quand $N\to\infty$ dans* **D**, *où u est défini par*

$$\begin{cases}\partial_t u(t,a)+\partial_a u(t,a)=0\\ u(t,0)=c(t)\int_0^\infty\lambda(\tilde{a})u(t,\tilde{a})d\tilde{a}\\ u(0,a)=g(a)\end{cases}\tag{34}$$

De plus

$$u_i^N(t) \xrightarrow[N\to\infty]{} \int_0^\infty u(t,a)\mathbb{P}(X(a)=i)\,da$$

Ils démontrent aussi que pour des paramètres bien choisis ce modèle correspond à un cas particulier de celui de Forien et al dans [39] et dans [31] Duchanps, Foutel-Rodier et Schertzer en donnent une généralisation.

Notons que dans [42, 31] ils supposent que le processus décrivant les infections est un processus ponctuel, ce qui est plus général que de supposer que ce sont des processus de Cox comme par exemple dans [39, 79].

Le système d'équations aux dérivées partielles (30) généralise celui de Kermack et McKendrick sans perte d'immunité, de même que le système déterministe (32) de Pang et Pardoux [79] , et (34) de Foutel-Rodier et al [31].

Dans [77] Guodong Pang et Etienne Pardoux établissent le théorème de la limite centrale associé à la loi des grands nombres donnée par le Théorème 16.

3. Résultats de la thèse

Les modèles énumérés dans la sous-section 2.3.1 supposent qu'une fois rétablis, les individus le restent à vie, ce qui n'est pas réaliste. À cet effet, il existe plusieurs modèles stochastiques d'épidémie tenant compte de la perte d'immunité mais supposant qu'une fois rétabli l'individu redevient immédiatement susceptible (voir par exemple [78, 79]), ce qui n'est pas le cas dans la réalité. En fait, on s'attendrait à ce qu'après la première infection, l'immunité de l'individu décroisse progressivement, augmentant ainsi sa probabilité d'être réinfecté. Le but de cette thèse est de définir et d'étudier un modèle d'épidémie tenant compte de la perte progressive d'immunité. Dans le chapitre 1 en collaboration avec Forien, Pang et Pardoux, nous avons décrit un modèle d'épidémie tenant compte de la perte progressive d'immunité. Dans ce chapitre nous faisons les hypothèses suivantes :

- On se donne N individus ;
- On suppose qu'une fraction de la population est infectée à l'instant 0 ;
- On suppose que chaque paire d'individu a des contacts infectieux aux instants de saut d'un processus de Cox d'intensité égal à son infectivité courante sur N et il devient infectieux avec une probabilité qui est égale à sa susceptibilité courante ;
- On suppose de plus qu'à chaque nouvelle infection on tire une paire de fonctions aléatoires càdlàg (λ,γ) indépendante des précédentes, représentant l'infectivité et la susceptibilité de l'individu nouvelllement infecté.

Ainsi en numérotant les individus, on tire deux familles de fonctions aléatoires càdlàg indépendantes, $(\lambda_{k,i},\gamma_{k,i})_{1\leqslant k\leqslant N, i\geqslant 1}$ i.i.d, de même loi que (λ,γ) et $(\lambda_{k,0},\gamma_{k,0})_{1\leqslant k\leqslant N}$ i.i.d de même loi que (λ_0,γ_0). $\lambda_{k,i}$ représente l'infectivité du $k-$ième individu après sa $i-$ième infection, $\lambda_{k,0}$ l'infectivité du $k-$ième individu avant sa première infection, $\gamma_{k,i}$ la susceptibilité du $k-$ième individu après sa $i-$ième infection et $\gamma_{k,0}$ la susceptibilité du $k-$ième individu avant sa première infection. En notant $A_k^N(t)$ le

processus comptant le nombre de réinfection du $k-$ième individu entre l'instant 0 et t, son infectivité à l'instant t est donnée par $\lambda_{k,A_k^N(t)}(\varsigma_k^N(t))$ et sa susceptibilité par $\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))$, où

$$\varsigma_k^N(t) := t - \left(\sup\{s \in [0,t],\, A_k^N(s) = A_k^N(s^-) + 1\} \vee 0\right)$$

est le temps écoulé depuis sa dernière infection.

La force totale d'infection moyenne de la population à l'instant t est alors donnée par

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^{N} \lambda_{k,A_k^N(t)}(\varsigma_k^N(t)),$$

et la susceptibilité moyenne par

$$\overline{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^{N} \gamma_{k,A_k^N(t)}(\varsigma_k^N(t)).$$

De plus nous notons $I^N(t)$ le nombre de personnes infectées à l'instant t et $U^N(t)$ le nombre de personnes non-infectées à l'instant t.

En effet, pour $i \geqslant 0$ et $1 \leqslant k \leqslant N$, en notant,

$$\eta_{k,i} := \sup\{t \geqslant 0 : \lambda_{k,i}(t) > 0\},$$

nous obtenons les expressions suivantes

$$\overline{I}^N(t) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) < \eta_{k,A_k^N(t)}}, \qquad \text{et} \qquad U^N(t) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) \geqslant \eta_{k,A_k^N(t)}}. \tag{35}$$

Nous faisons les hypothèses suivantes :

Hypothèse 3.1. *Il existe une constante déterministe* $\lambda_* > 0$, *telle que* $\sup_{t\geqslant 0}\lambda(t) \leqslant \lambda_*$ *et*

$$\sup\{t \geqslant 0 : \lambda(t) > 0\} \leqslant \inf\{t \geqslant 0 : \gamma(t) > 0\} \text{ presque sûrement.}$$

En s'inspirant des techniques utilisées pour des modèles de propagation du chaos voir par exemple [71, 92, 23] nous établissons le théorème suivant.

Théorème 18. *Sous les hypothèses 3.1,*

$$\left(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}^N\right) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{\mathfrak{S}}, \overline{\mathfrak{F}}) \quad \textit{dans} \quad \mathbf{D}^2 \tag{36}$$

où $(\overline{\mathfrak{S}},\overline{\mathfrak{F}})$ *est solution du système d'équations,*

$$\begin{cases} \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ \qquad\qquad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, & (37) \\ \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. & (38) \end{cases}$$

De plus étant donné $(\overline{\mathfrak{S}},\overline{\mathfrak{F}})$*,*

$$(\overline{U}^N,\overline{I}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{U},\overline{I}) \quad \textit{dans} \quad \mathbf{D}^2$$

où $(\overline{U},\overline{I})$ *est donné par*

$$\begin{aligned} \overline{U}(t) &= \mathbb{E}\left[\mathbb{1}_{t\geqslant\eta_0}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(s)dr\right)\right] \\ &\quad + \int_0^t \mathbb{E}\left[\mathbb{1}_{t-s\geqslant\eta}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, \qquad (39) \\ \overline{I}(t) &= \overline{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \qquad (40) \end{aligned}$$

Notons que $\overline{U}(t)$ peut être vu comme la proportion d'individus non-infectés à l'instant t et $\overline{I}(t)$ celui des infectés à l'instant t.

Le modèle déterministe ainsi obtenu par la loi des grands nombres généralise le modèle de Kermack et McKendrick avec perte d'immunité et sans démographie énoncée ci-dessus (15).

Notons aussi qu'il existe un lien entre ce modèle et les modèles de particules en interactions, en particulier avec les processus de Hawkes multivariés. En effet dans [23] Chevallier établit la loi des grands nombres pour des processus de Hawkes multivariés.

On appelle processus de Hawkes multivarié de paramètres $(n,\mu_F,\nu_F,\Psi,\varsigma_{N_i^-})$ le processus ponctuel $\left(N_1^n,\cdots,N_n^n\right)$ définit sur $\mathbb{R}$ tel que

— $\left(N_i^-\right)_{1\leqslant i\leqslant n}$ est une famille de processus ponctuels i.i.d défini sur $\mathbb{R}_-$, de même loi que $\varsigma_{N_i^-}$, et

— $\left(N_i^{n,+}\right)_i$ est une famille de processus ponctuels défini sur $\mathbb{R}_+$, d'intensité,

$$\lambda_t^i = \Psi_i\left(\varsigma_i^n(t^-), \frac{1}{n}\sum_{j=1}^n\left(\int_0^{t^-} h_{i,j}(t-z)N_j^+(dz) + F_{i,j}(t)\right)\right),$$

avec

$$\varsigma_i^n(t) := t - \left(\sup\{s\in[0,t],\, N_i^n(s) = N_i^n(s^-)+1\}\vee 0\right),$$

le temps écoulé depuis le dernier saut du processus.

Dans cette définition, $\Psi_i:\mathbb{R}\to\mathbb{R}_+$ représente la fonction d'intensité associée à

la particule i, $(h_{i,j})_{ij}$ la matrice de fonctions aléatoires échangeables décrivant les interactions entre les particules après l'instant 0 où $h_{1,1} \sim \nu_h$ et $(F_{i,j})_{ij}$ la matrice de fontions aléatoires i.i.d décrivant l'interaction des particules à l'instant initial et où $F_{1,1} \sim \mu_F$.

Lorsque le nombre de particules n est très grand, Chevallier démontre via des arguments de type propagation du chaos (voir par exemple [71, 92, 45]) que la famille $\left(N_i^n\right)_i$ converge dans **D** vers une famille de variables aléatoires i.i.d. De plus il démontre que la loi du temps écoulé depuis le dernier saut du processus limite est décrite par une équation de type McKean-Vlasov.

En comparaison à notre modèle introduit au chapitre 1 énoncé à la section précédente, le processus ponctuel A_i^N comptant le nombre de réinfections de l'individu i ayant pour intensité

$$\begin{aligned}\Upsilon_i^N(t) &:= \gamma_{i,A_i^N(t)}(\varsigma_i^N(t))\overline{\mathfrak{F}}^N(t)\\ &= \gamma_{i,A_i^N(t)}(\varsigma_i^N(t))\frac{1}{N}\sum_{j=1}^{N}\left(\lambda_{j,0}(t)+\int_0^t \lambda_{j,A_j^N(s^-)}(t-s)A_j^N(ds)\right)\end{aligned}$$

peut être vu comme un cas particulier d'une généralisation du processus de Hawkes avec âge où la fonction d'intensité Ψ_i est aléatoire.

Dans ce même chapitre nous étudions le comportement en temps long du système d'équations (37)-(38) et nous en déduisons celui de (39)-(40). Plus précisément en supposant que la fonction de susceptibilité γ est croissante, en notant γ_* la limite de γ, nous parvenons à montrer que l'équilibre sans maladie est asymptotiquement stable lorsque $R_0 < \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. En supposant de plus que la fonction de susceptibilité est minorée à partir d'un instant aléatoire d'espérence finie, nous parvenons à caractériser l'équilibre endémique lorsque $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$, et au moyen d'hypothèses supplémentaires plus techniques, nous prouvons que l'équilibre sans maladie est instable dans ce cas. Cependant nous n'avons pas pu établir la stabilité globale de l'équilibre endémique. Ces résultats sont résumés dans le théorème suivant :

Théorème 19. 1. *Si* $R_0 < \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ *il existe* $\overline{\mathfrak{S}}_*$ *tel que* $(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t)) \to (0, \overline{\mathfrak{S}}_*)$ *quand* $t \to \infty$.

2. *Si* $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$, *en supposant qu'il existe* $(\overline{\mathfrak{F}}_*, \overline{\mathfrak{S}}_*)$ *tel que* $(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t)) \xrightarrow[t\to+\infty]{} (\overline{\mathfrak{F}}_*, \overline{\mathfrak{S}}_*)$ *alors soit* $\overline{\mathfrak{F}}_* = 0$, *soit*

$$\overline{\mathfrak{S}}_* = \frac{1}{R_0}$$

et $\overline{\mathfrak{F}}_*$ *est l'unique solution positive de l'équation*

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\int_0^s \gamma\left(\frac{r}{\overline{\mathfrak{F}}_*}\right)dr\right)\right] ds = R_0. \tag{41}$$

Dans le second cas, $(\overline{I}(t), \overline{U}(t)) \to (\overline{I}_*, \overline{U}_*)$ *quand* $t \to \infty$*, où* $\overline{U}_* = 1 - \overline{I}_*$ *et*

$$\overline{I}_* = \frac{\mathbb{E}[\eta]\,\overline{\mathfrak{F}}_*}{R_0}. \tag{42}$$

Si $R_0 = \mathbb{E}\left[\frac{1}{\gamma_*}\right]$*, l'énoncé est encore vrai, mais* (41) *n'admet pas de solution positive, donc* $\overline{\mathfrak{F}}_* = 0$ *et ainsi* $\overline{I}_* = 0$.

3. *Si* $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$*, et* $\overline{\mathfrak{F}}(0) > 0$*, sous des hypothèses supplémentaires il existe une constante* $c > 0$*, telle que pour tout* $t \geqslant 0$, $\overline{\mathfrak{F}}(t) > c$.

Ce théorème caractérisant le comportement en temps long nous dit que le seuil d'endémicité de l'épidémie dépend de la loi de l'immunité des individus, ce qui n'était pas le cas dans la plupart des modèles existants.

Dans le chapitre 2 nous étudions les fluctuations

$$\hat{\mathfrak{F}}^N(t) := \sqrt{N}\big(\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\big), \text{ et } \hat{\mathfrak{S}}^N(t) := \sqrt{N}\big(\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}(t)\big).$$

Nous introduisons la définition suivante :

Définition 3.1. *Soit* $(\hat{\mathfrak{J}}, \hat{\mathfrak{M}})$ *un processus gaussien de fonction de covariance : pour* $t, t' \geqslant 0$*, :*

$$\begin{aligned}
Cov\big(\hat{\mathfrak{J}}(t), \hat{\mathfrak{J}}(t')\big) &= Cov\big(\gamma_{1,A_1(t)}(\varsigma_1(t)), \gamma_{1,A_1(t')}(\varsigma_1(t'))\big),\\
Cov\big(\hat{\mathfrak{M}}(t), \hat{\mathfrak{M}}(t')\big) &= Cov\big(\lambda_{1,A_1(t)}(\varsigma_1(t)), \lambda_{1,A_1(t')}(\varsigma_1(t'))\big),\\
Cov\big(\hat{\mathfrak{M}}(t), \hat{\mathfrak{J}}(t')\big) &= Cov\big(\lambda_{1,A_1(t)}(\varsigma_1(t)), \gamma_{1,A_1(t')}(\varsigma_1(t'))\big),
\end{aligned}$$

où

$$A_1(t) = \int_0^t \int_0^{+\infty} \mathbb{1}_{u \leqslant \Upsilon_1(r^-)} Q_1(dr, du),$$

avec

$$\Upsilon_1(t) = \gamma_{1,A_1(t)}(\varsigma_1(t))\overline{\mathfrak{F}}(t),$$

et

$$\varsigma_1^N(t) := t - \big(\sup\{s \in [0,t],\, A_1(s) = A_1(s^-) + 1\} \vee 0\big).$$

Contrairement aux méthodes habituelles, par exemple dans [97, 22] où les auteurs travaillent en dimension infinie dans des espaces de Sobolev, nous abordons le problème directement en dimension finie. Plus précisément, en approximant le modèle décrit au chapitre 1 par un modèle en quarantaine afin de contrôler les interactions entre les individus, en développant un nouveau théorème des moments pour les intégrales par rapport aux mesures de Poisson avec un intégrand non prévisible, en établissant des inégalités des moments, et au moyen d'hypothèses de régularité supplémentaires sur les fonctions d'infectivité et de susceptibilité, nous établissons le théorème de convergence suivant.

Théorème 20.

$$\left(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N\right) \Rightarrow \left(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}\right) \quad \textit{dans} \quad \mathbf{D}^2, \tag{43}$$

où $\left(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}\right)$ *est l'unique solution continue du système d'équations stochastiques intégrales suivant :*

$$\begin{cases} \hat{\mathfrak{S}}(t) = -\displaystyle\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s)ds \\ \qquad -\displaystyle\int_0^t\int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ \qquad +\displaystyle\int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds \\ \qquad +\hat{\mathfrak{J}}(t), \end{cases} \tag{44}$$

$$\hat{\mathfrak{F}}(t) = \int_0^t \overline{\lambda}(t-s)\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds + \hat{\mathfrak{M}}(t), \tag{45}$$

où $\left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right)$ *est un processus gaussien de fonction de covariance donnée par la Définition 3.1.*

Dans le but d'étudier le comportement en temps long, question laissée ouverte dans [41], et de répondre à la question de l'unicité du seuil d'endémicité énoncé dans [43] par Foutel-Rodier, Charpentier et Guérin, dans un travail en cours avec Hélène Guérin, au chapitre 3, nous étendons le modèle décrit dans [41] en faisant dépendre à chaque ré-infection les valeurs de l'infectivité et la susceptibilité des individus de celles précédent leur ré-infection. À cet effet, vu les difficultés rencontrées dans [41] avec une approche complètement non-Markovienne, nous avons opté pour une approche paramétrique. Plus précisément à chaque (ré-)infection nous tirons une variable aléatoire $\theta \in \Theta \subset \mathbb{R}^d$ décrivant les caractéristiques de l'individu et nous suivons aussi le processus décrivant l'évolution de l'âge d'infection des individus. Ainsi étant donné θ, l'infectivité et la susceptibilité d'un individu infecté d'âge a est donnée par la paire $(\lambda(a,\theta), \gamma(a,\theta))$. Ici λ et γ sont des fonctions déterministes et l'aléa du modèle est contenu dans le paramètre θ aussi appelé trait par analogie avec les modèles d'évolution. Par conséquent à un instant t donné, en notant par $\theta_k^N(t)$ le processus décrivant l'évolution des traits de l'individu numéro k et par $a_k^N(t)$ celui de son âge d'infection, nous définissons

$$\mu_t^N = \frac{1}{N}\sum_{k=1}^N \delta_{(a_k^N(t), \theta_k^N(t))}.$$

Puis en supposant $\gamma \leqslant 1$, λ borné, et en supposant que la loi du nouveau trait dépend du précédent par le noyau de transition des traits $K : \Theta \times \Theta \to \mathbb{R}_+$, nous établissons le théorème de convergence suivant :

Théorème 21. $\mu^N \xrightarrow[N\to\infty]{} \mu$, *en probabilité dans* $\mathbb{D}(\mathbb{R}_+; \mathscr{P}(\mathbb{R}_+ \times \Theta))$ *où* μ *est la solution*

faible de

$$\langle \mu_t, f_t \rangle = \langle \mu_0, f_0 \rangle + \int_0^t \langle \mu_s, \partial_a f_s + \partial_s f_s \rangle \mathrm{d}s + \int_0^t \langle \mu_s, \lambda \rangle \langle \mu_s, R f_s \rangle \mathrm{d}s, \tag{46}$$

Pour toute fonction mesurable bornée f sur $\mathbb{R}_+ \times \mathbb{R}_+ \times \Theta$, et de classe $\mathscr{C}^1$ par rapport au deux premières variables, où le terme de saut R est donnée par,

$$Rf(a,\theta) = \int_\Theta \big(f(0,\widetilde{\theta}) - f(a,\theta)\big)\gamma(a,\theta)K(\theta,\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta}),$$

et ν une mesure de probabilité sur Θ.

Notons que si μ_t a pour densité $u(t,\cdot,\cdot)$, par rapport à $\mathrm{d}a\nu(\mathrm{d}\theta)$ alors (46) est équivalent au système d'équations suivant :

$$\begin{cases} \partial_t u_t(a,\theta) + \partial_a u_t(a,\theta) = -\overline{\mathfrak{F}}(t)\gamma(a,\theta)u_t(a,\theta) \\ u(t,0,\theta) = \overline{\mathfrak{F}}(t)\overline{\mathfrak{S}}(t,\theta) \\ u(0,a,\theta) = u_0(a,\theta) \\ \overline{\mathfrak{F}}(t) = \displaystyle\int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)u_t(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta). \\ \overline{\mathfrak{S}}(t,\theta) = \displaystyle\int_{\mathbb{R}_+\times\Theta} \gamma(a,\widetilde{\theta})K(\widetilde{\theta},\theta)u_t(a,\widetilde{\theta})\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}) \end{cases} \tag{47}$$

Notons que le modèle en âge introduit par Tran décrit à la Section 2.2 est similaire au notre. Nous pouvons cependant noter une différence au niveau de la modélisation du taux de naissance. En identifiant avec notre modèle, nous avons $U\big((a,\theta),(\tilde{a},\tilde{\theta})\big) = \gamma(a,\theta)\lambda(\tilde{a},\tilde{\theta})$, et $d(a,\theta,u) = u$. Par conséquent

$$d\left(a,\theta,\int_{\mathbb{R}_+\times\Theta} U\big((a,\theta),(\tilde{a},\tilde{\theta})\big)\,u_t(\tilde{a},\tilde{\theta})\mathrm{d}\tilde{a}\nu(\mathrm{d}\widetilde{\theta})\right) = \overline{\mathfrak{F}}(t)\gamma(a,\theta).$$

Nous pouvons noter que le terme décrivant les nouvelles naissances de l'équation aux dérivées partielles (31) est linéaire en u, par contre celui de (47) est non-linéaire en u.

Nous pouvons aussi noter une légère différence au niveau du modèle stochastique. En effet, dans le modèle stochastique de Tran, le taux de naissance est indépendant de l'état des autres individus, le processus des naissances est indépendant de celui des morts, ce qui n'est pas le cas dans notre modèle car les "naissances" et les "morts" ont lieu simultanément. En fait, implicitement dans notre modèle nous considérons qu'un individu se remet de la maladie par guérison ou décès.

En supposant l'irréductibilité du noyau K, et que pour $\nu-$presque tout $\theta \in \Theta$, la fonction $t \mapsto \gamma(t,\theta)$ est non intégrable, en passant par la théorie des opérateurs compacts et une généralisation du théorème de Perron-Frobenius nous établissons l'existence

et l'unicité d'une fonction $\overline{\mathfrak{S}}_* \in L^1(\Theta)$, telle que

$$\overline{\mathfrak{S}}_*(\theta) = \int_\Theta K(\tilde{\theta},\theta)\overline{\mathfrak{S}}_*(\tilde{\theta})\nu(\mathrm{d}\tilde{\theta}) \text{ et } \int_{\mathbb{R}_+\times\Theta} \lambda(a,\tilde{\theta})\overline{\mathfrak{S}}_*(\tilde{\theta})\mathrm{d}a\nu(\mathrm{d}\tilde{\theta}) = 1.$$

Puis en supposant que pour $\nu-$presque tout $\theta \in \Theta$,

$$\gamma_*(\theta) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \gamma(s,\theta)ds,$$

nous établissons le théorème suivant décrivant l'existence d'une mesure stationnaire non nulle, donc l'existence d'un équilibre endémique.

Théorème 22. *Si*

$$\int_\Theta \frac{1}{\gamma_*(\theta)}\overline{\mathfrak{S}}_*(\theta)\mathbb{P}(d\theta) < 1, \tag{48}$$

il existe $\overline{\mathfrak{F}}_* > 0$, *tel que* u_* *défini par*

$$u_*(a,\theta) = \overline{\mathfrak{F}}_*\overline{\mathfrak{S}}_*(\theta)\exp\left(-\overline{\mathfrak{F}}_*\int_0^a \gamma(s,\theta)\mathrm{d}s\right).$$

est une solution stationnaire de l'équation (47).

Si, on suppose en plus que ν*-p.s,*

$$\forall a \geqslant 0 \quad \gamma(a,.) \geqslant \frac{1}{a}\int_0^a \gamma(s,.)\mathrm{d}s, \tag{49}$$

alors u_* *est l'unique solution stationnaire non nulle de* (47).

Ce théorème généralise le résultat d'endémicité obtenu au chapitre 1 car il ne suppose pas que la fonction de susceptibilité est croissante et par conséquent prend en considération le modèle avec vaccination introduit dans [43]. Dans [43] Foutel-Rodier, Charpentier et Guérin supposent qu'après guérison suite à une infection on tire un temps de vaccination T_V décrivant un processus de renouvellement et une nouvelle fonction de susceptibilité indépendante de la précédente qui vaudra 0 initialement car suite au vaccin l'individu sera totalement immunisé au début. En comparaison au modèle décrit au chapitre 1, dans ce cas la fonction de susceptibilité considérée n'est pas une fonction croissante puisqu'elle repasse à zéro après chaque vaccination.

De plus en supposant qu'il existe un instant déterministe à partir duquel la fonction de susceptibilité γ est minorée par une constante déterministe σ, en réécrivant notre problème sous forme de problème de Cauchy abstrait, nous linéarisons le semigroupe associé au voisinage de u_*, puis nous démontrons que le taux de croissance de ce semigroupe w_{ess} est majoré par $-\sigma\overline{\mathfrak{F}}_*$. Grâce aux travaux de Webb [99] (voir Théorème 5) pour conclure il suffit de démontrer que la partie réelle de toutes les valeurs propres du semigroupe associé est strictement négative. Pour ce faire, nous avons déterminé l'équation caractéristique des valeurs propres de notre semigroupe. Plus

précisément pour une valeur propre $\alpha \in \mathbb{C}, \mathscr{R}e > -\sigma\overline{\mathfrak{F}}_*$ associée à la fonction propre $\phi \in L^1(\mathbb{R}_+ \times \Theta)$ tel que $\int_{\mathbb{R}_+\times\Theta} \phi(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta) = 0$, dans le cas particulier où $K = 1$ nous obtenons l'équation suivante :

$$\begin{aligned}&\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\\ &= \int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\int_0^a \gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a\nu(\theta)\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta). \quad (50)\end{aligned}$$

Rappelons que pour $K = 1, \overline{\mathfrak{S}}_* = \frac{1}{R_0}$. Puis nous introduisons la conjecture suivante.

Conjecture .1. *L'équation (50) n'admet pas de solution avec $\mathscr{R}e(\alpha) \geqslant 0$.*

Si cette conjecture est vraie, nous déduisons du Théorème 6 de Webb énoncé ci-dessus, le résultat suivant.

Théorème 23. *Pour λ et γ pris tels que l 'équation (50) n'admet pas de solution telle que $\mathscr{R}e(\alpha) \geqslant 0$. Si*

$$R_0 > \int_\Theta \frac{1}{\gamma_*(\theta)}\nu(\mathrm{d}\theta),$$

alors il existe $\delta > 0$, et $c > 0$, tels que, pour tout $t \geqslant 0$,

$$\|u_0 - u_*\| \leqslant \delta \implies \|u_t - u_*\| \leqslant ce^{-\sigma\overline{\mathfrak{F}}_* t}\|u_0 - u_*\|.$$

Autrement dit u_* est localement asymptotiquement stable.

Nous avons démontré la conjecture dans le cas particulier

$$\gamma(a,\theta) = \mathbb{1}_{t\geqslant T(\theta)} \text{ et } \lambda(a,\theta) = \lambda_*\mathbb{1}_{t<T(\theta)}$$

où $T = \eta \wedge a_*$ avec a_* un réel positif, et η une variable aléatoire exponentelle de paramètre $\rho > 0$.

La preuve de ce résultat est très technique. Dans la littérature plusieurs auteurs ont étudié le comportement en temps long du modèle de Kermack et McKendrick ou des modèles avec des conditions aux bords à peu près comme la notre. Ils l'ont traité la plupart du temps comme un modèle de reproduction en supposant l'équivalent de notre γ (le taux de naissance ou mortalité) minorée à l'instant 0. Voir par exemple Magal et Ruan [69, Chapitre 8, section 8.2.2], Perthame [83, 68], et Roget [87]. En fait supposer γ minoré à $t = 0$ pour une épidémie revient à dire qu'à chaque contact on a une probabilité non nulle d'être infecté et par conséquent il n'y aura jamais de période où l'on soit complètement immunisé. D'autre part γ minoré à $t = 0$ facilite la majoration du coefficient de linéarité du semi groupe w_{ess} lorsqu'on utilise la théorie des problèmes de Cauchy abstraits. Dans des modèles avec démographie (naissance de nouveaux susceptibles), le coefficient de linéarité du semigroupe w_{ess} est majoré sans minoration de la fonction de susceptibilité et dans certains cas on obtient un résultat de stabilité locale voir par exemple [21, 83, 68].

La méthode utilisant les conditions de minorations de Doeblin comme Gabriel [44], et Torres et al [96] ne fonctionne pas ici. En effet, il n'était tout simplement pas possible d'obtenir une inégalité du type

$$\|u_t - u_*\|_{TV} \leqslant e^{\alpha t}\|u_0 - u_*\|_{TV}$$

avec un α négatif via cette méthode. Les conditions de minorations de Doeblin nous permettent de contrôler l'edp linéarisée, et pour la partie non-linéaire elle dépend essentiellement de l'estimation de $\left|\overline{\mathfrak{F}}(t) - \overline{\mathfrak{F}}_*\right|$. En fait, nous avons $\left|\overline{\mathfrak{F}}(t) - \overline{\mathfrak{F}}_*\right| \leqslant \lambda_*\|u_t - u_*\|_{TV}$ et il est impossible de faire mieux que cette inégalité. Cet inégalité entraine que l'on obtient un α positif. Dans [96] pour résoudre le problème de l'estimation de la partie non-linéaire, les auteurs imposent que leur norme de Lipschitz soit assez petite devant 1 (voir le théorème 9, voir aussi [32]). De même la technique de Doeblin-Lyapunov utilisant le théorème de Harris ne fonctionne pas ici car la condition de Lyapunov sur le semigroupe n'est pas satisfaite [47].

4. Organisation du document

Dans le chapitre 1 nous introduirons les modèles d'épidémies avec perte progressive d'immunité. Dans le chapitre 2 nous étudierons le théorème central limite et dans le chapitre 3 nous généraliserons le modèle du chapitre 1 en introduisant de la dépendence entre les paramètres récents et les précédents.

Le chapitre 1 et le chapitre 2 sont déjà sous formes d'articles soumis.

Le chapitre 1 correspond à [41] et le chapitre 2 à [100].

1. Law of large numbers for stochastic epidemic models with varying infectivity and waning immunity

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1.1. Introduction

Many infectious diseases are such that the immunity acquired after recovery from the illness is eventually lost, after a period whose length varies both with the individual and with the illness. The huge majority of the literature on mathematical epidemiology which considers models with loss of immunity assume that this loss happens instantaneously. However, it is rather clear that, in reality, immunity wanes progressively over some period, which can vary from one individual to another. In their pioneering 1927 paper [61], Kermack and McKendrick introduced the first mathematical model of epidemic propagation in which the infectivity of individuals depends on the time elapsed since their infection (this time is often called the age of infection). This model was deterministic and assumed that recovered individuals can no longer be infected. In two subsequent papers, [62, 63], Kermack and McKendrick introduced another

deterministic model in which recovered individuals eventually loose their immunity. They studied the conditions under which an infectious disease can become endemic, i.e., reach a stable equilibrium where some macroscopic fraction of the population is infected. This kind of equilibrium for the deterministic system is called an endemic equilibrium, as opposed to the disease-free equilibrium, in which there are no infected individuals.

Our aim in the present chapter is to revisit these questions with the use of a general stochastic model, following the recent developments carried out by the first three authors. In [39], the 1927 model of Kermack and McKendrick was obtained as the infinite population limit of an individual-based stochastic model in which the infectivity of individuals is a random function of their age of infection. More precisely, it was shown that the fraction of susceptible individuals and the average infectivity (also called the force of infection) converge to a deterministic limit which solves a system of non-linear integral equations already introduced in [61]. The widespread ordinary differential equations (ODEs) compartmental SIR model is only a very special case of these equations, in which neither the infectivity nor the recovery rate depends upon the age of infection. Such models are less realistic as they do not reproduce the dependence of the dynamics of the epidemic on its past [90, 38], and thus neglect the inertia of the evolution of the epidemic. Note that ODE models are law of large numbers limit of Markov stochastic models, in which individuals move from one compartment to the next after an exponentially distributed time.

In the present chapter, we introduce a very general model in which individuals are characterized by their infectivity and susceptibility, which are assumed to be given as random functions of their age of infection. These random functions are assumed to be i.i.d. among the various individuals in the population, and a new independent pair of random functions is drawn at each new infection. See Figure 2.1 for a realization of the infectivity and susceptibility of an individual after his/her infection.

Note however that, in contrast with the model in [62, 63], we consider a closed population, in which there are no birth and no death (we do not exclude deaths due to infections, represented by individuals whose susceptibility remains equal to zero after the fatal infection).

We also assume (as in [62, 63]) that all pairs of individuals have contacts at the same rate. In other words, we assume a situation of homogeneous mixing. Of course, this is not quite satisfactory. One may wish to take into account the spatial distribution of the population, as well as the variety of social behaviors of the individuals. However, mathematical models involve necessarily a simplification of the complex reality. We believe that the results presented here constitute a significant progress over the classical models where all rates are constant and immunity is lost instantaneously. In future works, we do intend to combine the complexity of the present work with that of inhomogeneous models, such as spatial models or models on graphons.

Besides the fact that we prefer to present our model in the form of a system of integral equations rather than a system of partial differential equations, our deterministic model is more general than the model of [62, 63]. The reason is the following. While the law of large numbers limit of the model with infection age dependent infectivity

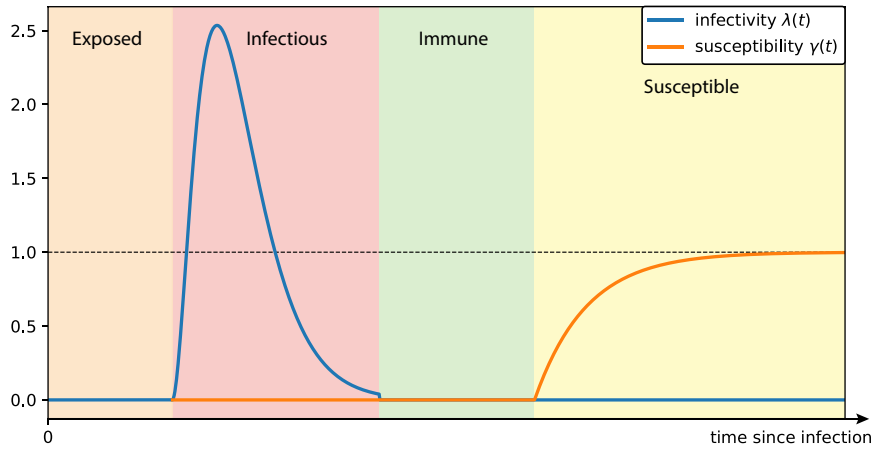

Figure 1.1. – Illustration of a typical realization of the random infectivity and susceptibility functions of an individual from the time of infection to the time of recovery, and then to the time of losing immunity and becoming fully susceptible (or in general, partially susceptible).

depends only on the mean of the random infectivity function [39], the deterministic limit of the model with random susceptibility depends on the distribution of this random susceptibility function in a much more complicated way. To see this, suppose that we wish to compute the average susceptibility of the individuals at some time t. This average can be obtained by summing the contributions of individuals that have not been infected on the interval $[0, t)$, and of individuals that have been infected at some time $s \in [0, t)$ and have not yet been reinfected again on the interval $[s, t)$. But the probability that such an individual has not been reinfected by time t depends on the full trajectory of their susceptibility, as well as on the trajectory of the force of infection, on the interval $[s, t)$ (see (1.15)–(1.16) below). This explains why the deterministic limit is more complex than that in [39], and shows that, in their 1932-33 model, Kermack and McKendrick implicitly assumed that the only random component of the susceptibility function is the time of recovery (see the discussion in Section 1.5.2). As a consequence, we obtain a strict generalization of the model in [62, 63] as the deterministic limit of our stochastic model.

Since in our model there is a flux of new susceptibles, due to waning of immunity, one expects that under certain conditions, there may be a stable endemic equilibrium. We manage to study the existence, uniqueness and some stability properties of an endemic equilibrium (as well as the stability or instability of the disease–free equilibrium) in our deterministic limit. We identify a threshold, which is the harmonic mean of the large time limit of the susceptibility (which is 1 in Figure 1, but may be less than 1 in our general model). If the basic reproduction number R_0 (defined below by (1.22)) is smaller than this threshold, then the process converges to the disease-free equilibrium. We also show that under appropriate assumptions, if R_0 is larger than the threshold, and the model converges as $t \to \infty$ to some limit, then this limit is

the unique endemic equilibrium, which is fully characterized. We conjecture that under appropriate assumptions, when R_0 is larger than the threshold, any solution of the deterministic limit starting with a non zero force of infection at time $t = 0$ does converge to the endemic equilibrium.

Let us comment on the method used to prove our law of large numbers result. One key argument is based upon a coupling of the processes which count the numbers of infections of the various individuals up to time t with i.i.d. counting processes, where the renormalized force of infection is replaced by its deterministic limit. This approach, for which the inspiration came from [23], is an application of ideas from the theory of propagation of chaos, see Sznitman [92]. Note both that we were unable to adapt the methodology of [39] to the setting of the present paper, and that the assumptions made here on the random infectivity function are weaker than those made in [39]. Recently, the methodology of the present paper has been applied to the model in [39], thus resulting in the same result under weaker assumptions, see [40]. We also note that our approach differs significantly from the techniques classically used for age-structured population models, as in [74, 58, 98, 48, 49, 36]. In these papers, the authors describe the model as a branching process (sometimes with interaction), which becomes an infinite dimensional Markov process if one adds all the age structure in the present state of the system, in which the lifespan, birth rate and death rate depend on the ages of all individuals in the population. The state of such a model is then described by the empirical measure of the ages of the individuals. These works combine infinite-dimensional stochastic calculus and measure-valued Markov processes analysis in order to prove the convergence of the model. In contrast, our stochastic models are non Markovian, and their deterministic limit does have a memory.

Using random infectivity and susceptibility functions allows us to build a very general model which is both versatile and tractable. It captures the effect of a progressive loss of immunity, and this loss is allowed to be different from one individual to another. The integral equations that we obtain to describe the large population limit of our model are both compact and extremely general, since most epidemic models with homogeneous mixing and a fixed population size can be written in this form. The effect of the variability of susceptibility on the endemic threshold has received very little attention in the literature, despite some profound implications which we outline in the present work. The fact that the threshold depends on the harmonic mean of the susceptibility reached after an infection shows that the heterogeneity of immune responses in real populations should not be neglected in public health decisions. Similarly, the variability of the immune response after vaccination (both in time and between individuals) should affect the efficacy of vaccination policies in non-trivial ways, although these questions are outside the scope of the present work.

We want to comment on the terminology which can be used for our model as a compartmental model. Recall that the compartments most classically used in epidemic models include S for Susceptible individuals, E for Exposed (those infected individuals who are not yet infectious), I for infectious and R for Recovered. We claim that our model can be classified as a SEIRS, SIRS, or SIS model. Indeed, referring

to Figure 2.1, we can consider that a given individual passes from the S to the E compartment when he/she becomes infected, then into the I compartment when the attached infectivity first becomes positive, into the R compartment when the infectivity reaches 0 and remains null, and into the S compartment when the attached susceptibility becomes positive. However, without modifying the dynamics of the epidemic, we can merge the E and I compartments into a compartment I (for infected), where the infectivity need not be positive all the time. Similarly, we can merge the R and S compartments into the S compartment (or U, for uninfected, as we suggest below), whose members may have a susceptibility equal to zero. As a matter of fact, our model is vey general and includes most of the existing homogeneous epidemic models without demography as particular examples. It is non Markov. All the rates are not only infection age dependent, but also random, i.e., different from one individual to another, which we believe reflects the reality of epidemics.

Models with gradual waning of immunity have been studied since Kermack and McKendrick by only a handful of authors, including Inaba, who in a series of works, see [53, 56, 54, 55], has performed a careful mathematical study of the PDE model from [62, 63], as well as Breda et al. [13] who have considered an integral equation version of the same model. Other authors have pursued the study of the system of ODE/PDEs, see in particular Thieme and Yang [94], Barbarossa and Röst [5] and Carlsson et al. [20]. More recently Khalifi and Britton [64], compare the level of immunity in a model with gradual vs. sudden loss of immunity.

Organization of the paper

The rest of the paper is organized as follows. In Section 1.2, we define the model in detail. In Section 1.3, we state the assumptions and the functional law of large numbers (FLLN) and discuss how the results reduce to already known results when we restrict ourselves to the classical SIS and SIRS models. The results on the endemic equilibrium are presented in Section 1.4. In Section 1.5, we focus on the generalized SIRS model with a particular set of infectivity and susceptibility random functions and initial conditions, and show how the limit relates to the Kermack and McKendrick PDE model with the corresponding infection-age dependent infectivity and recovery-age dependent susceptibility. The proofs for the FLLN are given in Section 1.6 and those for the endemic equilibrium in Section 1.7.

Notation

Throughout the paper, all the random variables and processes are defined on a common complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We use $\xrightarrow[N\to+\infty]{\mathbb{P}}$ to denote convergence in probability as the parameter $N \to \infty$. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}^k(\mathbb{R}_+^k)$ the space of k-dimensional vectors with real (nonnegative) numbers, with $\mathbb{R}(\mathbb{R}_+)$ for $k = 1$. We use $\mathbb{1}_{\{\cdot\}}$ for the indicator function. Let $D = D(\mathbb{R}_+;\mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg functions defined on $\mathbb{R}_+$, with convergence in D meaning

convergence in the Skorohod J_1 topology (see, e.g., [11, Chapter 3]). Also, we use D^k to denote the k-fold product with the product J_1 topology. Let C be the subset of D consisting of continuous functions and D_+ the subset of D of càdlàg functions with values on $\mathbb{R}_+$

1.2. Model description

1.2.1. Definition of the model

Let (λ_0, γ_0) and (λ, γ) be two random elements of D^2. We consider a population of fixed size N. To each individual k, $1 \le k \le N$, is attached a sequence of random elements $(\lambda_{k,i}, \gamma_{k,i})_{i\ge 0}$ of D^2, such that the sequence $(\lambda_{k,i}, \gamma_{k,i})_{1\le k\le N, i\ge 0}$ is independent. The sequence $(\lambda_{k,0}, \gamma_{k,0})_{1\le k\le N}$ is i.i.d., their common law being that of (λ_0, γ_0), while the sequence $(\lambda_{k,i}, \gamma_{k,i})_{1\le k\le N, i\ge 1}$ is i.i.d., their common law being that of (λ, γ).

$\lambda_{k,0}$ (resp. $\gamma_{k,0}$) is the infectivity (resp. susceptibility) of individual k between time 0 and the time of his/her first (re)-infection. We can think of the law of (λ_0, γ_0) as being a mixture of the law for the initially infected individuals, who have been infected before time 0 and for which $\lambda_0(0) \ge 0$ and $\gamma_0(0) = 0$, and the law for the initially susceptible individuals, for which $\lambda_0(0) = 0$ and $\gamma_0(0) > 0$ (possibly $\gamma_0(0) = 1$).

On the other hand, $\lambda_{k,i}(t)$ denotes the infectivity of the k-th individual, at time t after his/her i-th infection (where we count here only the infections after time 0), and $\gamma_{k,i}(t)$ denotes the susceptibility of the k-th individual, at time t after his/her i-th infection, given that this is his/her most recent infection.

We assume that (λ_0, γ_0) and (λ, γ) satisfy the following assumption.

Assumption 1.1. *We assume that:*

1. *There exists a constant $\lambda_* < \infty$ such that for all $t \geqslant 0$, $0 \leqslant \lambda_0(t), \lambda(t) \leqslant \lambda_*$ almost surely, and $0 \leqslant \gamma_0(t), \gamma(t) \leqslant 1$ almost surely.*
2. *Almost surely,*

$$\sup\{t \geqslant 0, \lambda_0(t) > 0\} \leqslant \inf\{t \geqslant 0, \gamma_0(t) > 0\} \text{ and } \sup\{t \geqslant 0, \lambda(t) > 0\} \leqslant \inf\{t \geqslant 0, \gamma(t) > 0\}. \tag{1.1}$$

Condition (1.1) implies that, while an individual is infected, he/she cannot be reinfected. Hence in each infected-immune-susceptible cycle, the infected and susceptible periods do not overlap. Note that (1.1) is not necessary for the process to be well defined, and we discuss the consequences of removing this assumption in Remark 1.4 below. It would be rather natural to assume that $\gamma_0(t)$ and $\gamma(t)$ are a.s. non decreasing. We do not make this restriction at this stage, since we do not need it. We shall make this assumption in section 1.4 below.

For $i \geqslant 0$ and $1 \leqslant k \leqslant N$, we define

$$\eta_{k,0} := \sup\{t \geqslant 0 : \lambda_{k,0}(t) > 0\}, \quad \text{and} \quad \eta_{k,i} := \sup\{t \geqslant 0 : \lambda_{k,i}(t) > 0\}.$$

We will also use the notations $\eta_0 = \sup\{t > 0,\ \lambda_0(t) > 0\}$ and $\eta = \sup\{t > 0,\ \lambda(t) > 0\}$.

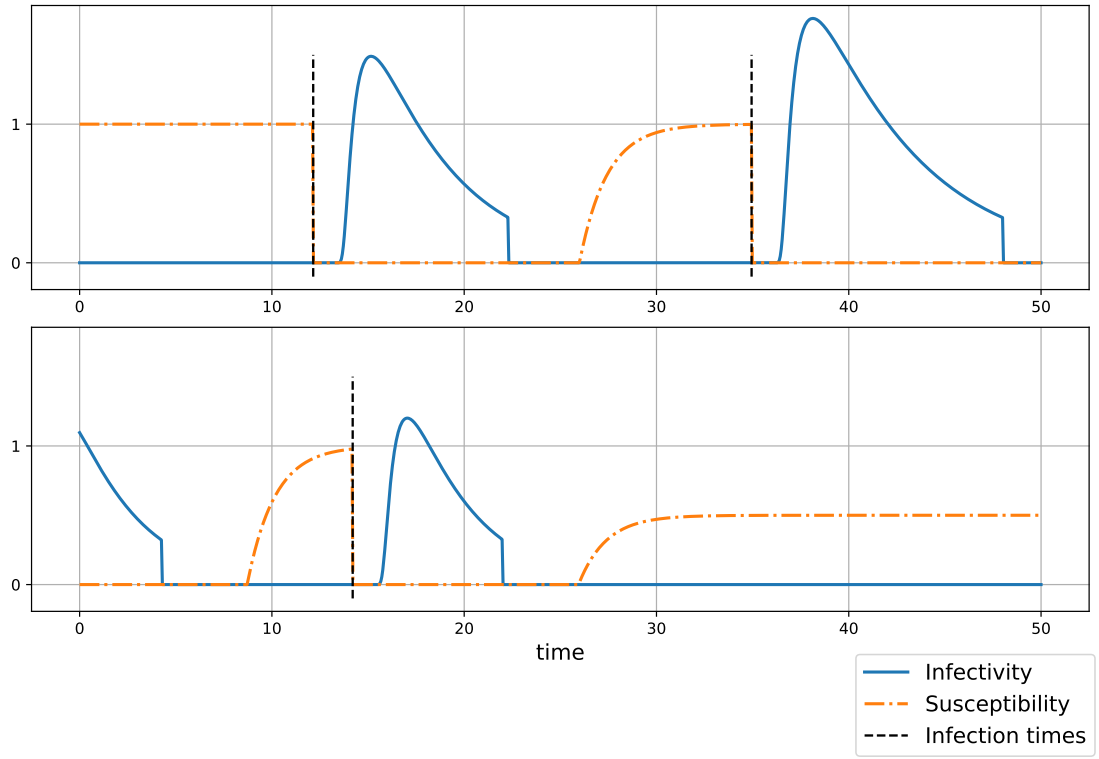

Figure 1.2. – Illustration of the evolution of an individual's infectivity and susceptibility through time. Each graphic shows the dynamics of an individual's infectivity (blue) and susceptibility (orange). The top graphic corresponds to an individual which is initially susceptible, and the bottom one to an initially infectious individual. Note that, after being reinfected, the second individual remains partially immune even a long time after infection.

Let $A_k^N(t)$ be the number of times that the individual k has been (re)–infected on the time interval $(0, t]$. The time elapsed since this individual's last infection (or since time 0 if no such infection has occurred), is given by

$$\varsigma_k^N(t) := t - \big(\sup\{s \in (0, t] : A_k^N(s) = A_k^N(s^-) + 1\} \vee 0\big),$$

where we use the convention $\sup \emptyset = -\infty$. With this notation, the current infectivity and susceptibility of the k-th individual are given by

$$\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)), \qquad \text{and} \qquad \gamma_{k,A_k^N(t)}(\varsigma_k^N(t)).$$

Figure 1.2 shows a realization of these processes for two individuals. Let us now define $\overline{\mathfrak{F}}^N(t)$ and $\overline{\mathfrak{S}}^N(t)$ as the average infectivity and susceptibility in the population, i.e.,

$$\overline{\mathfrak{F}}^N(t) := \frac{1}{N}\sum_{k=1}^{N} \lambda_{k,A_k^N(t)}(\varsigma_k^N(t)), \qquad \overline{\mathfrak{S}}^N(t) := \frac{1}{N}\sum_{k=1}^{N} \gamma_{k,A_k^N(t)}(\varsigma_k^N(t)).$$

The instantaneous rate at which the ℓ-th individual infects the k-th individual is

$$\frac{1}{N}\lambda_{\ell,A_\ell^N(t)}(\varsigma_\ell^N(t))\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))$$

where the $\frac{1}{N}$ factor comes from the probability that the k-th individual is chosen as the target of the infectious contact. Summing over the index ℓ, the instantaneous rate at which the k-th individual is (re)infected is given by

$$\Upsilon_k^N(t) := \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\,\overline{\mathfrak{F}}^N(t). \tag{1.2}$$

Let now $(Q_k, 1 \leqslant k \leqslant N)$ be an i.i.d. family of standard Poisson random measures (PRMs) on $\mathbb{R}_+^2$, independent of the sequence $(\lambda_{k,i}, \gamma_{k,i})_{1\leq k\leq N, i\geq 0}$. The family of counting processes $(A_k^N(t), t \geqslant 0, 1 \leqslant k \leqslant N)$ is then defined as the solution of

$$A_k^N(t) = \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{u\leqslant \Upsilon_k^N(r^-)} Q_k(dr, du)\,. \tag{1.3}$$

Note that we construct A_k^N by induction on the jumps times. Part (i) of Assumption 1.1 implies that the rate $\Upsilon_k^N(t)$ is bounded almost surely by the constant λ_*. Consequently the jump times do not accumulate, and the above induction defines $A_k^N(t)$ for all $t \geqslant 0$.

1.2.2. Numbers of infected and uninfected individuals

For $i \geqslant 1$, $\eta_{k,i}$ is the duration of the infected period of the k-th individual following its i-th infection, while $\eta_{k,0} = 0$ if the k-th individual is initially susceptible and $\eta_{k,0} > 0$ is the remaining infected period of the k-th individual if it is initially infected. We shall say that the k-th individual is currently infected (resp. uninfected) if $\varsigma_k^N(t) < \eta_{k,A_k^N(t)}$ (resp. $\varsigma_k^N(t) \geqslant \eta_{k,A_k^N(t)}$). Note that, with this definition, an individual need not be infectious when he/she is infected. In the same way, an individual is called uninfected if he/she is *no longer infected* or has never been infected, hence this group comprises both recovered and susceptible individuals. This choice of two broadly defined compartments allows us to keep the notations tractable, but the equations obtained below can be generalized in a straightforward way to keep track of the number of individuals at different stages of their infection and susceptibility in more detail.

Let $I^N(t)$ (resp. $U^N(t)$) denote the number of infected (resp. uninfected) individuals

in the population at time $t \geqslant 0$. Then

$$I^N(t) = \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) < \eta_{k,A_k^N(t)}}, \qquad \text{and} \qquad U^N(t) = \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) \geqslant \eta_{k,A_k^N(t)}}. \tag{1.4}$$

Note that, quite obviously, $U^N(t) + I^N(t) = N$ for all $t \geqslant 0$. We then define

$$\overline{I}(0) := \mathbb{P}(\eta_0 > 0), \qquad \overline{U}(0) := \mathbb{P}(\eta_0 = 0) = 1 - \overline{I}(0).$$

Recall that $\{(\lambda_{k,0}, \gamma_{k,0}), 1 \leqslant k \leqslant N\}$, and hence $(\eta_{k,0}, 1 \leqslant k \leqslant N)$, are independent and identically distributed. Thus by the law of large numbers,

$$\left(\frac{1}{N} I^N(0), \frac{1}{N} U^N(0)\right) = \left(\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,0}>0}, \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,0}=0}\right) \to \left(\overline{I}(0), \overline{U}(0)\right)$$

almost surely as $N \to \infty$.

Remark 1.1. ***The classical compartmental models***

Most compartmental epidemic models can be obtained as special cases of our model, as long as no population structure is assumed and birth and death are disregarded. Here are a few examples.

1. *The SIS model considers that infected individuals instantly become infected with some deterministic infectivity β, and remain infected for a random time η, after which they become instantly fully susceptible again. Thus, this model is obtained by assuming that,*

$$\lambda(t) = \beta \mathbb{1}_{0 \leqslant t < \eta}, \qquad \gamma(t) = \mathbb{1}_{t \geqslant \eta}, \tag{1.5}$$

and η is a non-negative random variable, which follows an exponential distribution in the case of the Markov model. Let us also specify the distribution of (λ_0, γ_0). For this, fix $\overline{I}(0) \in [0,1]$, set $\overline{S}(0) = 1 - \overline{I}(0)$ and let χ be a random variable taking values in $\{S, I\}$ such that $\mathbb{P}(\chi = S) = \overline{S}(0)$ and $\mathbb{P}(\chi = I) = \overline{I}(0)$. Then set

$$\lambda_0(t) = \begin{cases} 0 & \text{if } \chi = S, \\ \beta \mathbb{1}_{0 \leqslant t < \eta_0} & \text{if } \chi = I, \end{cases} \qquad \gamma_0(t) = \begin{cases} 1 & \text{if } \chi = S, \\ \mathbb{1}_{t \geqslant \eta_0} & \text{if } \chi = I, \end{cases} \tag{1.6}$$

where η_0 is a positive random variable (which follows an exponential distributions in the case of the Markov model). (Note that, with this definition, the model starts from a random initial condition, which is not always assumed in the literature, but does not greatly affect its behaviour.)

2. *The SIR model instead considers that, at the end of the infectious period, infected individuals recover from the disease, and can no longer be infected again. This model can be obtained from the above by proceeding as for the SIS model, but*

assuming instead that

$$\forall\, t \geqslant 0, \quad \gamma(t) = 0, \qquad \text{and} \qquad \gamma_0(t) = \begin{cases} 1 & \text{if } \chi = S, \\ 0 & \text{if } \chi = I. \end{cases} \tag{1.7}$$

Note that, in this case, if we keep a general distribution for the random functions λ and λ_0, this model reduces to the one studied in [39].

3. *The SIRS model assumes that, at the end of their infectious period, individuals stay immune to the disease for a random time θ, after which they become fully susceptible again. This model is obtained by assuming that,*

$$\lambda(t) = \beta \mathbb{1}_{0 \leqslant t < \eta}, \qquad \gamma(t) = \mathbb{1}_{t \geqslant \eta + \theta}, \tag{1.8}$$

where (η, θ), is a pair of independent random variables taking values in $\mathbb{R}_+^2$ (which are distributed as pairs of independent exponential variables in the Markov models). The generalization of the definition of the distribution of (λ_0, γ_0) to this case is straightforward.

Note that, in the last two cases, the quantity $U^N(t)$ defined in (1.4) counts both susceptible and removed individuals. The actual number of susceptible and removed individuals in the SIRS model is given by

$$S^N(t) := \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,A_k^N(t)} + \theta_{k,A_k^N(t)} \leqslant \varsigma_k^N(t)}, \quad R^N(t) := \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,A_k^N(t)} \leqslant \varsigma_k^N(t) < \eta_{k,A_k^N(t)} + \theta_{k,A_k^N(t)}}, \tag{1.9}$$

where

$$\theta_{k,i} = \inf\{\theta > 0, \gamma_{k,i}(\eta_{k,i} + \theta) > 0\},$$

following (1.8).

In the SIR model, the above expression remains exact provided we set $\theta_{k,i} = +\infty$ for $i \geqslant 1$, $\theta_{k,0} = 0$ if $\chi_k = S$ and $\theta_{k,0} = +\infty$ if $\chi_k = I$.

It is also common to assume that infected individuals do not become infectious right after being infected, but first become exposed (i.e. infected but not yet infectious) before becoming infectious. This results in an additional compartment E, which can also be included in the above examples without difficulty.

Remark 1.2. *In [23], Chevallier studied a model formulated as a system of age-dependent random Hawkes processes. This model considers a system of N neurons which fire at a rate depending both on the times of the previous firings of other neurons and on the time elapsed since their last firing (called the age process). The author proves in [23] a propagation of chaos result for the empirical measure of the point processes corresponding to the firing times of the neurons, and for the empirical measure of the age processes of the neurons, as N tends to infinity. Although neither our model or that of Chevallier can be formulated as a special case of the other, the two are closely related (if firing is understood as an analogous of being infected). The main difference between*

the two frameworks is in the assumptions on the randomness in the interaction between individuals after each firing/infection. For instance, our model would be closer to that of Chevallier if, instead of choosing a different infectivity function after each infection for each individual, we chose a different infectivity function for each directed pair *of individuals at the* beginning*, and kept the same infectivity function for this pair of individual after each infection. Thus, the law of large numbers limit that we prove below can be seen as an extension of Chevallier's result, and indeed some steps of the proof are adapted from [23]. See also Theorem 1.5 in Section 1.6 whose formulation is closer to the propagation of chaos result of [23]. In [22], the same author also proves a central limit theorem for the empirical measures mentioned above, something which we do not do here, but will be the subject of future work.*

1.3. Functional law of large numbers

In this section we present the FLLN for the scaled processes $\left(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}^N, \overline{U}^N, \overline{I}^N\right)$ where $\overline{U}^N := N^{-1}U^N$ and $\overline{I}^N := N^{-1}I^N$.

Let

$$\overline{\lambda}_0(t) := \mathbb{E}\left[\lambda_{1,0}(t)\middle|\eta_{1,0} > 0\right], \quad \overline{\lambda}(t) := \mathbb{E}\left[\lambda_{1,1}(t)\right],$$
$$F_0^c(t) := \mathbb{P}\left(\eta_{1,0} > t\middle|\eta_{1,0} > 0\right), \quad F^c(t) := \mathbb{P}\left(\eta > t\right),$$

and recall that $\overline{I}(0) = \mathbb{P}\left(\eta_{1,0} > 0\right) = \mathbb{P}(\chi_1 = 1)$. We shall denote below by μ the law of γ.

To describe the limits of the FLLN, we introduce the following two-dimensional integral equation. Observe that the solution (x, y) of this system depend on the laws of λ_0 and λ only through expectations but for γ_0 and γ it is much more complex.

We consider the following system of integral equations for which we look for a solution $(x, y) \in D_+^2$:

$$\begin{cases} x(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] \\ \qquad\qquad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right] x(s)y(s)ds, & (1.10) \\ y(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)x(s)y(s)ds. & (1.11) \end{cases}$$

In (1.10) the only random elements are γ_0 and γ.

Our first result establishes existence and uniqueness of the solution (x, y) of (1.10)-(1.11). The proof is given in Section 1.6.

Theorem 1.1. *Under Assumption 1.1, the set of equations* (1.10)–(1.11) *has a unique solution* $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}) \in D^2(\mathbb{R}_+, \mathbb{R}_+)$. *The solution belongs to* $C^2(\mathbb{R}_+)$ *if* $t \mapsto (\gamma_0(t), \overline{\lambda}_0(t))$ *is continuous.*

We have the following convergence result which we prove in Section 1.6.

Theorem 1.2. *Under Assumption 1.1,*

$$(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{\mathfrak{S}}, \overline{\mathfrak{F}}) \quad in \quad D^2 \tag{1.12}$$

where $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ *is the unique solution of the system of equations* (1.10)–(1.11).
Given the solution $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$,

$$(\overline{U}^N, \overline{I}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{U}, \overline{I}) \quad in \quad D^2$$

where $(\overline{U}, \overline{I})$ *is given by*

$$\begin{aligned}\overline{U}(t) &= \mathbb{E}\left[\mathbb{1}_{t\geqslant\eta_0} \exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ &\quad + \int_0^t \mathbb{E}\left[\mathbb{1}_{t-s\geqslant\eta} \exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, \qquad (1.13)\\ \overline{I}(t) &= \overline{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \qquad (1.14)\end{aligned}$$

We rewrite (1.10)–(1.11) as

$$\begin{cases}\overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ \qquad\qquad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, \qquad (1.15)\\ \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \qquad (1.16)\end{cases}$$

We prove the following Lemma in Section 1.6.

Lemma 1.2. *If the pair* (x, y) *is a solution to the set of equations* (1.10)-(1.11)*, then for all* $t \geqslant 0$,

$$\mathbb{E}\left[\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right] x(s)y(s)ds = 1. \tag{1.17}$$

Remark 1.3. *Since* $\overline{U}^N(t) + \overline{I}^N(t) = 1$ *for all* $t \geqslant 0$ *and* $N \geqslant 1$, *it follows from the above convergence that* $\overline{U}(t) + \overline{I}(t) = 1$ *as well. Let us check that this follows also from the set of equations* (1.15)-(1.16) *satisfied by* $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$.

First we note that, by (1.1) *in Assumption 1.1,* $\gamma(t) = 0$ *for all* $t \in [0, \eta)$ *and* $\gamma_0(t) = 0$ *for* $t \in (0, \eta_0)$, *hence*

$$F_0^c(t) = \mathbb{E}\left[\mathbb{1}_{t<\eta_0} \exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right], \quad F^c(t-s) = \mathbb{E}\left[\mathbb{1}_{t-s<\eta} \exp\left(\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right].$$

Hence, summing (1.13) *and* (1.14)*, we obtain*

$$\overline{U}(t)+\overline{I}(t)=\mathbb{E}\left[\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]+\int_0^t\mathbb{E}\left[\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \tag{1.18}$$

Therefore, as $(\overline{\mathfrak{S}},\overline{\mathfrak{F}})$ *is a solution of the set of equations* (1.10)-(1.11)*, by Lemma 1.2 we conclude that* $\overline{U}(t)+\overline{I}(t)=1$ *for all* $t\geqslant 0$*. The identity* (1.17) *can be seen as stating the conservation of the population size.*

Remark 1.4. *Without condition* (1.1) *of Assumption 1.1, the limit obtained in the functional law of large numbers satisfies a different set of equations. More precisely, equation* (1.16) *is replaced by*

$$\begin{aligned}\overline{\mathfrak{F}}(t)=\mathbb{E}\left[\lambda_0(t)\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\\+\int_0^t\mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds,\end{aligned} \tag{1.19}$$

and (1.14) *by*

$$\begin{aligned}\overline{I}(t)=\mathbb{E}\left[\mathbb{1}_{\eta_0>t}\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(s)dr\right)\right]\\+\int_0^t\mathbb{E}\left[\mathbb{1}_{\eta>t-s}\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds.\end{aligned} \tag{1.20}$$

Remark 1.5. *We can check that, in the special cases mentioned in Remark 1.1, the limiting system of equations obtained in Theorem 1.2 coincides with the corresponding models in the literature.*

1. *In the case of the SIS model, we note that, by* (1.5) *and* (1.6)*,*

$$\overline{\lambda}(t)=\beta\mathbb{P}(\eta>t),\qquad\qquad\overline{\lambda}_0(t)=\beta\mathbb{P}(\eta_0>t\,|\,\eta_0>0).$$

It thus follows from (1.16) *and* (1.14) *that* $\overline{\mathfrak{F}}(t)=\beta\overline{I}(t)$ *for all* $t\geqslant 0$*. Moreover, comparing* (1.15) *and* (1.13) *and using* (1.5) *and* (1.6)*, we see that* $\overline{\mathfrak{S}}(t)=\overline{U}(t)=\overline{S}(t)$*. Combining this with the fact that* $\overline{I}(t)+\overline{U}(t)=1$*, we obtain*

$$\overline{I}(t)=\overline{I}(0)F_0^c(t)+\beta\int_0^t F^c(t-s)(1-\overline{I}(s))\overline{I}(s)ds, \tag{1.21}$$

as stated in Theorem 2.3 of [78].

2. *In the case of the SIR model, from the definition of* γ *and* γ_0 *in* (1.7) *and* (1.15)*, combined with the fact that* $\overline{\mathfrak{F}}(t)=\beta\overline{I}(t)$*, we see that*

$$\overline{S}(t)=\overline{\mathfrak{S}}(t)=\overline{S}(0)\exp\left(-\beta\int_0^t\overline{I}(r)dr\right)\quad\text{and}\quad\overline{U}(t)=\overline{S}(t)+\overline{R}(t).$$

This yields the statement of Theorem 2.1 in [78].

3. *In the case of the SIRS model, we note that,* $\overline{\mathfrak{F}}(t) = \beta \overline{I}(t)$, $\overline{\mathfrak{S}}(t) = \overline{S}(t)$, *in view of* (1.13), $\overline{U}(t) = \overline{S}(t) + \overline{R}(t)$, *where*

$$\overline{S}(t) = \mathbb{E}\left[\mathbb{1}_{\eta_0+\theta_0 \leqslant t} \exp\left(-\beta \int_0^t \gamma_0(r)\overline{I}(r)dr\right)\right] + \beta \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta+\theta \leqslant t-s} \exp\left(-\beta \int_s^t \gamma(r-s)\overline{I}(r)dr\right)\right] \overline{S}(s)\overline{I}(s)ds,$$

and

$$\overline{R}(t) = \mathbb{E}\left[\mathbb{1}_{\eta_0 \leqslant t < \eta_0+\theta_0} \exp\left(-\beta \int_0^t \gamma_0(r)\overline{I}(r)dr\right)\right] + \beta \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta \leqslant t-s < \eta+\theta} \exp\left(-\beta \int_s^t \gamma(r-s)\overline{I}(r)dr\right)\right] \overline{S}(s)\overline{I}(s)ds.$$

In fact, $\overline{S}$ *and* $\overline{R}$ *are the limit of* $\frac{1}{N}S^N$ *and* $\frac{1}{N}R^N$, *respectively, where* S^N *and* R^N *are defined in* (1.9) *(recall that, for simplicity, we assumed that no individual is initially in the R compartment). But, from the definition of* γ *in* (1.8), $\gamma(t) = 0$ *for all* $t \in [\eta, \eta+\theta)$ *and* $\gamma_0(t) = 0$ *for all* $t \in [\eta_0, \eta_0+\theta_0)$, *hence*

$$\overline{S}(t) = \overline{S}(0)\exp\left(-\beta \int_0^t \overline{I}(r)dr\right) + \overline{I}(0)\mathbb{E}\left[\mathbb{1}_{\eta_0+\theta_0 \leqslant t} \exp\left(-\beta \int_{\eta_0+\theta_0}^t \overline{I}(r)dr\right)\right] + \beta \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta+\theta \leqslant t-s} \exp\left(-\beta \int_{s+\eta+\theta}^t \overline{I}(r)dr\right)\right] \overline{I}(s)\overline{S}(s)ds,$$

and

$$\overline{R}(t) = \overline{I}(0)\mathbb{P}(\eta_0 \leqslant t < \eta_0+\theta_0 \,|\, \eta_0 > 0) + \beta \int_0^t \mathbb{P}(\eta \leqslant t-s < \eta+\theta)\overline{I}(s)\overline{S}(s)ds.$$

As $\overline{S}(t) + \overline{I}(t) + \overline{R}(t) = \overline{U}(t) + \overline{I}(t) = 1$, *this yields the result stated in Theorem 3.3 in [78].*

1.4. The endemic equilibrium

When the disease persists in the population, we say that the disease becomes endemic, and if it reaches an equilibrium, it is called the endemic equilibrium. This corresponds to a balance between the number of new infections and new recoveries. In this section, we study the conditions that lead to an endemic equilibrium in the FLLN limit. We first recall the endemic equilibrium behavior of the classical SIS model discussed in Remark 1.1. Given the infectivity rate β, the basic reproduction number is given by $R_0 = \beta\mathbb{E}[\eta]$. It is well known (see, e.g., [78, Section 4.3], and also [15] for a

discussion of the Markov model) that if $R_0 \leq 1$, $\overline{I}(t) \to 0$ as $t \to \infty$, and if $R_0 > 1$ and $\overline{I}(0) > 0$, $\overline{I}(t) \to 1 - R_0^{-1}$ as $t \to \infty$. This can be easily obtained from the expression of $\overline{I}(t)$ in (1.21). In words, in the case $R_0 \leq 1$, the disease-free steady state is globally asymptotically stable, and in the case $R_0 > 1$, the disease-free steady state is unstable and there is exactly one endemic steady state, which is globally stable. Recall that in this model, $\overline{\lambda}(t) = \beta F^c(t)$. Thus, since $\int_0^\infty F^c(t)\,dt = \mathbb{E}[\eta]$, we have

$$R_0 = \int_0^\infty \overline{\lambda}(t)\,dt. \tag{1.22}$$

In fact, the expression of R_0 in (1.22) is the definition of the basic reproduction number in the Kermack and McKendrick model with an average infectivity function $\overline{\lambda}(t)$, because it represents the average number of individuals infected by an infectious individual in a fully susceptible population.

Remark 1.6. *Note that to ensure that $\overline{\lambda}(t) \to 0$ as $t \to \infty$ we could assume that $\eta < \infty$ a.s. However, we shall not need that assumption.*

We make the following assumptions on the random susceptibility and infectivity functions in order to study the equilibria of our model.

Assumption 1.3.

1. *The random functions $t \mapsto \gamma(t)$ and $t \mapsto \gamma_0(t)$ are non-decreasing a.s.*
2. *The pair (λ_0, γ_0) is distributed as follows. Let $\xi \geqslant 0$ be a random variable such that $\xi \leqslant \eta$ almost surely. Let χ be a Bernoulli random variable with $\mathbb{P}(\chi = 1) = \overline{I}(0)$, independent of $(\lambda, \gamma, \eta, \xi)$. Then*

$$\lambda_0(t) = \begin{cases} 0 & \text{if } \chi = 0, \\ \lambda(t+\xi) & \text{if } \chi = 1. \end{cases} \quad \text{and} \quad \gamma_0(t) = \begin{cases} 1 & \text{if } \chi = 0, \\ \gamma(t+\xi) & \text{if } \chi = 1. \end{cases}$$

The random variable ξ represents the age of infection at time zero of the initially infectious individuals.

We define

$$\gamma_* := \sup_{t \geqslant 0} \gamma(t) = \lim_{t \to +\infty} \gamma(t), \quad \text{and} \quad \gamma_{0,*} := \lim_{t \to +\infty} \gamma_0(t).$$

In the classical SIS model, the susceptibility functions are given by $\gamma(t) = \mathbb{1}_{t \geqslant \eta}$ and $\gamma_0(t) = \mathbb{1}_{t \geqslant \eta_0}$ so that $\gamma_* = 1$ a.s. However, in general, after being infected and recovered, individuals lose immunity gradually and do not necessarily reach "full" susceptibility (being equal to 1). Thus, γ_* may take any value in $[0,1]$ and is a priori random.

It turns out that the classification of the endemic equilibria depends on the law of γ_*, more specifically on whether R_0 is smaller than or larger than $\mathbb{E}[1/\gamma_*]$. Note that this expectation $\mathbb{E}[1/\gamma_*]$ may be infinite in general (for example if $\mathbb{P}(\gamma_* = 0) > 0$). We first prove the following result under the condition that $R_0 < \mathbb{E}[1/\gamma_*]$.

Theorem 1.3. *Under Assumptions 1.1 and 1.3, if* $R_0 < \mathbb{E}\left[1/\gamma_*\right]$, *there exists* $\overline{\mathfrak{S}}_* \in [0,1]$ *such that*

$$(\overline{\mathfrak{S}}(t), \overline{\mathfrak{F}}(t)) \to (\overline{\mathfrak{S}}_*, 0) \quad as \quad t \to \infty,$$

where

$$\overline{\mathfrak{S}}_* = \lim_{t\to+\infty} \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_{0,*} \exp\left(-\int_0^{+\infty} \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] + \int_0^{+\infty} \mathbb{E}\left[\gamma_* \exp\left(-\int_s^{+\infty} \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \quad (1.23)$$

As a consequence, $(\overline{U}(t), \overline{I}(t)) \to (1,0)$ *as* $t \to \infty$. *In the special case* $\gamma_* = 1$ *a.s., we also have* $\overline{\mathfrak{S}}_* = 1$.

Remark 1.7. *Without the assumption on the monotonicity of the function* γ, *when* $R_0 < \mathbb{E}\left[\left(\sup_t \gamma(t)\right)^{-1}\right]$ *the same proof as that of Theorem 1.3 shows that as* $t \to +\infty$, $\overline{\mathfrak{F}}(t) \to 0$ *and* $\overline{I}(t) \to 0$.

Note that in this theorem, we do not assume $\mathbb{E}\left[1/\gamma_*\right] < +\infty$. However, we do assume that $R_0 < \infty$, that is, $\overline{\lambda}(t)$ is integrable.

Remark 1.8. *In [55, Proposition* 8.9*] one can find a similar result for a model with demography.*

The case $R_0 \geqslant \mathbb{E}[1/\gamma_*]$ is more complex. We make the following additional assumptions.

Assumption 1.4. *There exists a non-negative random variable* t_* *such that* $\mathbb{E}[t_*] < +\infty$ *and for* $t \geqslant t_*$, $\gamma(t) \geqslant \frac{\gamma_*}{2}$ *a.s.*

Assumption 1.5. γ_* *is deterministic, positive and for any* $\delta \in (0,1)$, *there exists a deterministic* $t_\delta > 0$ *such that*

$$\gamma_0(t_\delta) \wedge \gamma(t_\delta) \geqslant (1-\delta)\gamma_* \text{ almost surely.} \quad (1.24)$$

Assumption 1.6. *There exists a positive decreasing function* h *such that* $h(0) = 1$ *and for all* $s, t \in \mathbb{R}_+$, $\overline{\lambda}(s+t) \geqslant h(s)\overline{\lambda}(t)$. *The same holds for* $\overline{\lambda}_0$. *In addition,* $\overline{\lambda}_0$ *is continuous and* $\overline{\lambda}$ *is of bounded total variation.*

Theorem 1.4. *1. Suppose that Assumptions 1.1, 1.3 and 1.4 hold and* $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. *If there exists* $(\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$ *such that* $(\overline{\mathfrak{S}}(t), \overline{\mathfrak{F}}(t)) \xrightarrow[t\to+\infty]{} (\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$, *then either* $\overline{\mathfrak{F}}_* = 0$, *or else*

$$\overline{\mathfrak{S}}_* = \frac{1}{R_0}$$

and $\overline{\mathfrak{F}}_*$ *is the unique positive solution of the equation*

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\int_0^s \gamma\left(\frac{r}{\overline{\mathfrak{F}}_*}\right) dr\right)\right] ds = R_0. \tag{1.25}$$

In the second case, $(\overline{I}(t), \overline{U}(t)) \to (\overline{I}_*, \overline{U}_*)$ *as* $t \to \infty$, *where* $\overline{U}_* = 1 - \overline{I}_*$ *and*

$$\overline{I}_* = \frac{\mathbb{E}[\eta]\overline{\mathfrak{F}}_*}{R_0}. \tag{1.26}$$

If $R_0 = \mathbb{E}\left[\frac{1}{\gamma_*}\right]$, *the same statement holds but* (1.25) *does not admit any positive solution, so, necessarily,* $\overline{\mathfrak{F}}_* = 0$ *and thus* $\overline{I}_* = 0$.

2. *Assume in addition that Assumption 1.5 and Assumption 1.6 hold and that* $\overline{\mathfrak{F}}(0) > 0$. *Then there exists* $c > 0$ *such that for all* $t > 0$, $\overline{\mathfrak{F}}(t) \geqslant c$. *In particular* $\overline{\mathfrak{F}}(t)$ *cannot tend to zero as* $t \to \infty$.

Remark 1.9. *Assumption 1.4 controls the time required for an individual's susceptibility to increase beyond some level, Assumption 1.5 ensures that after some time the average susceptibility returns above* $\frac{1}{R_0}$ *if there are not too many re-infections and Assumption 1.6 ensures that the force of infection does not decrease too rapidly.*

Corollary 1.1. *Suppose that Assumptions 1.1, 1.3 and 1.4 hold and that* $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. *Assume that* $\gamma(t) = \gamma_* \mathbb{1}_{t \geqslant \zeta}$, *where* ζ *is a random variable satisfying* $\zeta \geqslant \eta$ *almost surely and* $\mathbb{E}[\zeta] < +\infty$ *and* γ_* *is a random variable taking values in* $(0, 1]$. *Then,*

$$\overline{\mathfrak{F}}_* = \frac{R_0 - \mathbb{E}\left[\frac{1}{\gamma_*}\right]}{\mathbb{E}[\zeta]}, \tag{1.27}$$

is the solution of equation (1.25) *and*

$$\overline{I}_* = \frac{\mathbb{E}[\eta]\overline{\mathfrak{F}}_*}{R_0} = \frac{\mathbb{E}[\eta]}{\mathbb{E}[\zeta]}\left(1 - R_0^{-1}\mathbb{E}\left[\frac{1}{\gamma_*}\right]\right) \tag{1.28}$$

where R_0 *is given by* (1.22).

Proof. In this case, equation (1.25) becomes

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\gamma_* \int_0^s \mathbb{1}_{\frac{r}{\overline{\mathfrak{F}}_*} \geq \zeta} dr\right)\right] ds = R_0.$$

By a change of variables and Fubini's theorem, the left hand side is equal to

$$\begin{aligned}\overline{\mathfrak{F}}_* \int_0^{+\infty} \mathbb{E}\left[\exp\left(-\overline{\mathfrak{F}}_*\gamma_* \int_0^s \mathbb{1}_{r\geq\zeta} dr\right)\right] ds &= \overline{\mathfrak{F}}_* \int_0^{+\infty} \mathbb{E}\left[\mathbb{1}_{\zeta>s} + \mathbb{1}_{\zeta\leqslant s}\exp\left(-\overline{\mathfrak{F}}_*\gamma_*(s-\zeta)\right)\right] ds \\ &= \overline{\mathfrak{F}}_*\mathbb{E}[\zeta] + \overline{\mathfrak{F}}_*\mathbb{E}\left[\int_\zeta^{+\infty} \exp\left(-\overline{\mathfrak{F}}_*\gamma_*(s-\zeta)\right) ds\right] \\ &= \overline{\mathfrak{F}}_*\mathbb{E}[\zeta] + \mathbb{E}\left[\frac{1}{\gamma_*}\right].\end{aligned}$$

This gives the expression in (1.27). □

Remark 1.10. *For the classical models discussed in Remark 1.1, the above allows us to recover previously known results.*

1. *In the SIS model, we have $\gamma_* = 1$ and $\eta = \zeta$ almost surely, so we obtain that $\overline{I}(t) \to 0$ as $t \to \infty$ if $R_0 \leqslant 1$, and that, if $R_0 > 1$, the only other possible limit for $\overline{I}(t)$ is given by*

$$\overline{I}_* = \frac{\mathbb{E}[\eta]}{\mathbb{E}[\zeta]}\left(1 - R_0^{-1}\mathbb{E}\left[\frac{1}{\gamma_*}\right]\right) = 1 - \frac{1}{R_0}.$$

2. *For the SIRS model, we have $\gamma_* = 1$ and $\zeta = \eta + \theta$. As a result, applying Corollary 1.1, we see that, when $R_0 > 1$, the only possible positive limit for $\overline{I}(t)$ is*

$$\overline{I}_* = \frac{\mathbb{E}[\eta]}{\mathbb{E}[\eta] + \mathbb{E}[\theta]}\left(1 - \frac{1}{R_0}\right).$$

In the same way, we can also deduce that $\overline{R}_* := \lim_{t\to+\infty} \overline{R}(t)$ is given by

$$\overline{R}_* = 1 - \overline{\mathfrak{S}}_* - \overline{I}_* = \frac{\mathbb{E}[\theta]}{\mathbb{E}[\eta] + \mathbb{E}[\theta]}\left(1 - \frac{1}{R_0}\right).$$

We refer to Proposition 4.2 of [78] for a previous derivation of this equilibrium in the particular case where $\lambda(t) = \beta\mathbb{1}_{t<\eta}$ and $\gamma(t) = \mathbb{1}_{t\geqslant\eta+\theta}$ with η and θ independent, the laws of η and θ being arbitrary.

Our results in (1.27)–(1.28) thus extend those for classical compartmental models. Note that γ_* is random and takes values in $[0,1]$, indicating potentially partial susceptibility after recovery.

For the SIS and SIRS models discussed in Remark 1.10, it is known that, if $R_0 > 1$ and $\overline{I}(0) > 0$, then $\overline{I}(t)$ does indeed converge to the endemic equilibrium $\overline{I}_*$ as $t \to \infty$. We are not yet able to prove such a result for our general model.

We make the following conjecture on the convergence to the equilibrium in the case $R_0 > \mathbb{E}\left[1/\gamma_*\right]$.

Conjecture 1.1. *Under Assumptions 1.1, 1.3–1.6, if* $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ *and* $\overline{\mathfrak{F}}(0) > 0$, *then*

$$(\overline{\mathfrak{S}}(t), \overline{\mathfrak{F}}(t)) \to (\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*) \quad as \quad t \to \infty,$$

where $\overline{\mathfrak{S}}_* = 1/R_0$ *and* $\overline{\mathfrak{F}}_*$ *is the unique positive solution of* (1.25).

This conjecture is equivalent to another. Indeed, by applying the Arzelà-Ascoli Theorem to the set of functions $\{(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}}),\ j \in \mathbb{N}\}$ where $\tau_j x(t) := x(t+t_j)$ and $(t_j)_j \subset \mathbb{R}_+$, $t_j \to +\infty$ as $j \to +\infty$, there exists a subsequence of pairs $(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}})$ denoted again $(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}})$ such that $(x_j, y_j) := (\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}}) \to (x, y)$ uniformly on compact sets as $j \to +\infty$. Note that the pair $(x_j(t), y_j(t))$ satisfies the following system of equations: for $t \geqslant -t_j$,

$$\begin{cases} x_j(t) = \mathbb{E}\left[\gamma_0(t+t_j)\exp\left(-\displaystyle\int_0^{t+t_j}\gamma_0(r)y_j(r)dr\right)\right] \\ \qquad\qquad + \displaystyle\int_{-t_j}^{t}\mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)y_j(r)dr\right)\right]x_j(s)y_j(s)ds, & (1.29) \\ y_j(t) = \overline{I}(0)\overline{\lambda}_0(t+t_j) + \displaystyle\int_{-t_j}^{t}\overline{\lambda}(t-s)x_j(s)y_j(s)ds. & (1.30) \end{cases}$$

As a result, as the first terms of the right hand side of (1.29) and (1.30) tend to zero when $j \to +\infty$, and $(x_j(t), y_j(t)) \to (x(t), y(t))$ for all $t \in \mathbb{R}$, we deduce by the dominated convergence theorem that the pair (x, y) satisfies the following set of equations,

$$\begin{cases} y(t) = \displaystyle\int_{-\infty}^{t}\overline{\lambda}(t-s)x(s)y(s)ds, & (1.31) \\ \displaystyle\int_{-\infty}^{t}\mathbb{E}\left[\exp\left(-\int_s^t\gamma(r-s)y(r)dr\right)\right]x(s)y(s)ds = 1. & (1.32) \end{cases}$$

We can remark that the constant pair $(\frac{1}{R_0}, \overline{\mathfrak{F}}_*)$ where $\overline{\mathfrak{F}}_*$ is the unique solution of (1.25) is a solution of (1.31)-(1.32). Hence if this solution is unique, all converging subsequences of $(\tau_j\overline{\mathfrak{S}}, \tau_j\overline{\mathfrak{F}})$ have the same limit, from which we can easily conclude the convergence of $(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t))$ as $t \to \infty$.

Thus, Conjecture 1.1 is equivalent to the following.

Conjecture 1.2. *Under Assumptions 1.3–1.6, if* $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ *and* $\overline{\mathfrak{F}}(0) > 0$, *the set of equations* (1.31)-(1.32) *has a unique positive and bounded solution on* $\mathbb{R}$.

1.5. Relating to the Kermack and McKendrick PDEs for the SIRS model

1.5.1. The FLLN limits with a special set of susceptibility/infectivity functions and initial conditions

We consider the special family of susceptibility and infectivity functions:

$$\lambda(t) = \widetilde{\lambda}(t)\mathbb{1}_{t<\eta} \quad \text{and} \quad \gamma(t) = \widetilde{\gamma}(t-\eta)\mathbb{1}_{t>\eta}$$

where $\widetilde{\lambda}(t)$ and $\widetilde{\gamma}(t)$ are deterministic functions, representing the infectivity and susceptibility functions, and the infected period η is a random variable with cumulative distribution function F. Then we have $\overline{\lambda}(t) = \widetilde{\lambda}(t)F^c(t)$. In addition, let f be the density function of F and $\mu_F(t) := \frac{f(t)}{F^c(t)}$ be the hazard rate function of F. Then one can also write

$$F^c(t) = \exp\left(-\int_0^t \mu_F(s)ds\right), \tag{1.33}$$

We assume that the infectivity function of an initially infected individual is the same as that of a newly infected individual but shifted by the time elapsed since infection before time zero, that is,

$$\lambda_0(t) = \widetilde{\lambda}(t+\xi)\mathbb{1}_{t<\eta_0}$$

where η_0 is the duration of the remaining infectious period after time zero, whose distribution depends on ξ, the age of infection at time 0. To specify the distribution of ξ, we assume that there exists a function $\overline{I}(0,\tau)$ such that $\overline{I}(0) = \int_0^\infty \overline{I}(0,\tau)d\tau$, that is, $\overline{I}(0,\tau)$ is the density of the distribution of $\overline{I}(0)$ over the ages of infection. Then we can specify the distribution of ξ as

$$\mathbb{P}(\xi > x) = \frac{1}{\overline{I}(0)}\int_x^{+\infty} \overline{I}(0,r)dr. \tag{1.34}$$

And the conditional distribution of η_0 given ξ is given by

$$\mathbb{P}(\eta_0 > t\big|\xi) = \frac{F^c(\xi+t)}{F^c(\xi)} = \exp\left(-\int_\xi^{\xi+t} \mu_F(r)dr\right). \tag{1.35}$$

It is then clear that

$$\overline{\lambda}_0(t) = \mathbb{E}\left[\widetilde{\lambda}(t+\xi)\mathbb{1}_{t<\eta_0}\right] = \frac{1}{\overline{I}(0)}\int_0^{+\infty} \widetilde{\lambda}(t+\tau)\overline{I}(0,\tau)\exp\left(-\int_\tau^{t+\tau}\mu_F(s)ds\right)d\tau\,. \tag{1.36}$$

Next, to specify the initial susceptibility $\gamma_0(t)$ associated with each individual, we consider three groups of individuals at time zero, susceptible, infected, and recovered, whose respective proportions are $\overline{S}(0)$, $\overline{I}(0)$ and $\overline{R}(0)$ with $\overline{S}(0)+\overline{I}(0)+\overline{R}(0) = 1$. Note that we have combined the fully susceptible and initially recovered individuals as one group in the model description in Section 1.2. Thus, the model discussed in

this section is in fact a generalized SIRS model. Moreover, we shall in this section consider that individuals in the R compartment have recovered from the disease but are not necessarily immune. In accordance with [53] (see Section 1.5.2 below), once an individual recovers from the infection, after some potential immune period, the immunity is gradually lost, the individual may become infected again, and the process is repeated at each new infection with a new realization of the random infectivity and susceptibility functions. This means that, the susceptible individuals are those who have never been infected. For the initially fully susceptible individuals, their susceptibility γ_0 equals 1. For the initially infected individuals, their susceptibility starts to become positive after time η_0, which is the end of the infected period η_0, i.e. $\gamma_0(t) = \mathbb{1}_{t\geqslant\eta_0}\widetilde{\gamma}(t-\eta_0)$. For the initially recovered individuals, their susceptibility depends on the elapsed time since the last recovery, which we denote by ϑ, so that, for these individuals, $\gamma_0(t) = \widetilde{\gamma}(t+\vartheta)$. Let χ be a random variable indicating the group of an individual at time $t = 0$, with the following law

$$\mathscr{L}(\chi) = \overline{S}(0)\delta_S + \overline{I}(0)\delta_I + \overline{R}(0)\delta_R\,.$$

Then the initial susceptibility $\gamma_0(t)$ of an individual can be written as

$$\gamma_0(t) = \mathbb{1}_{\chi=S} + \widetilde{\gamma}(t-\eta_0)\mathbb{1}_{t\geqslant\eta_0,\chi=I} + \widetilde{\gamma}(t+\vartheta)\mathbb{1}_{\chi=R}\,.$$

To specify the distribution of ϑ, we assume that there exists a function $\overline{R}(0,\theta)$ such that $\overline{R}(0) = \int_0^\infty \overline{R}(0,\theta)d\theta$, that is, $\overline{R}(0,\theta)$ is the density of the distribution of $\overline{R}(0)$ over the age of recovery. Then we can specify the distribution of ϑ by

$$\mathbb{P}(\vartheta > x) = \frac{1}{\overline{R}(0)}\int_x^{+\infty}\overline{R}(0,r)dr. \tag{1.37}$$

Thus, we have

$$\begin{aligned}\mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] &= \overline{S}(0)\exp\left(-\int_0^t\overline{\mathfrak{F}}(r)dr\right)\\ +\int_0^t\int_0^\infty\widetilde{\gamma}(t-\tau)\mu_F(\tau+r)\overline{I}(0,r)\exp\left(-\int_r^{\tau+r}\mu_F(s)ds\right)&\exp\left(-\int_\tau^t\widetilde{\gamma}(s-\tau)\overline{\mathfrak{F}}(s)ds\right)drd\tau\\ &+\int_0^\infty\widetilde{\gamma}(t+\tau)\overline{R}(0,\tau)\exp\left(-\int_0^t\widetilde{\gamma}(s+\tau)\overline{\mathfrak{F}}(s)ds\right)d\tau\,.\end{aligned}$$

Note that, concerning the second term, we have to condition upon ξ in order to compute its expression.

We denote by $\overline{S}(t)$ the proportion of susceptible individuals who have never been infected, $\overline{I}(t)$ the proportion of infected individuals and $\overline{R}(t)$ the proportion of recovered individuals at time t.

We obtain the following expression of the limit $(\overline{\mathfrak{S}},\overline{\mathfrak{F}})$.

Proposition 1.1. *For the generalized SIRS model described above, the limit* $(\overline{\mathfrak{S}},\overline{\mathfrak{F}})$ *from*

Theorem 1.2 solves the following system integral equations:

$$\begin{aligned}\overline{\mathfrak{S}}(t) &= \overline{S}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(r)dr\right)\\ &+\int_0^t\int_0^\infty \widetilde{\gamma}(t-\tau)\frac{f(\tau+r)}{F^c(\tau)}\overline{I}(0,r)\exp\left(-\int_\tau^t \widetilde{\gamma}(s-\tau)\overline{\mathfrak{F}}(s)ds\right)drd\tau\\ &+\int_0^\infty \widetilde{\gamma}(t+\tau)\overline{R}(0,\tau)\exp\left(-\int_0^t \widetilde{\gamma}(s+\tau)\overline{\mathfrak{F}}(s)ds\right)d\tau\\ &+\int_0^t\left(\int_r^t \widetilde{\gamma}(t-\tau)f(\tau-r)\exp\left(-\int_\tau^t \widetilde{\gamma}(s-\tau)\overline{\mathfrak{F}}(s)ds\right)d\tau\right)\overline{\mathfrak{S}}(r)\overline{\mathfrak{F}}(r)dr, \qquad (1.38)\end{aligned}$$

and

$$\overline{\mathfrak{F}}(t) = \int_t^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(0,\tau-t)\frac{F^c(\tau)}{F^c(\tau-t)}d\tau + \int_0^t \widetilde{\lambda}(t-\tau)\overline{\mathfrak{S}}(\tau)\overline{\mathfrak{F}}(\tau)F^c(t-\tau)d\tau. \qquad (1.39)$$

In addition,

$$\begin{aligned}\overline{S}(t) &= \overline{S}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(r)dr\right),\\ \overline{I}(t) &= \int_0^{+\infty}\frac{F^c(t+\tau)}{F^c(\tau)}\overline{I}(0,\tau)d\tau + \int_0^t F^c(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds,\\ \overline{R}(t) &= \int_0^{+\infty}\exp\left(-\int_0^t \widetilde{\gamma}(\tau+r)\overline{\mathfrak{F}}(r)dr\right)\overline{R}(0,\tau)d\tau\\ &+\int_0^{+\infty}\int_0^t \exp\left(-\int_s^t \widetilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu_F(\tau+s)\frac{F^c(\tau+s)}{F^c(\tau)}ds\overline{I}(0,\tau)d\tau\\ &+\int_0^t\int_s^t \exp\left(-\int_u^t \widetilde{\gamma}(r-u)\overline{\mathfrak{F}}(r)dr\right)\mu_F(u-s)\exp\left(-\int_0^{u-s}\mu_F(r)dr\right)du\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds.\end{aligned}$$

1.5.2. Connection with the Kermack and McKendrick PDE model

In [53], the model introduced by Kermack and McKendrick in [62, 63] was reformulated as follows. Let $\overline{S}(t)$ denote the proportion of susceptible individuals who have never been infected, let $\overline{I}(t,\tau)$ denote the density of infectious individuals with infection-age $\tau \geqslant 0$ and $\overline{R}(t,\theta)$ the density of recovered individuals with recovery-age $\theta \geqslant 0$. Then, given an initial condition $(\overline{S}(0), \overline{I}(0,\cdot), \overline{R}(0,\cdot))$ such that

$$\overline{S}(0) + \int_0^\infty \overline{I}(0,\tau)d\tau + \int_0^\infty \overline{R}(0,\theta)d\theta = 1,$$

the system evolves according to the following set of partial differential equations:

$$\begin{cases} \dfrac{d\overline{S}}{dt}(t) = -\overline{S}(t)\displaystyle\int_0^{+\infty} \widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau \\ \dfrac{\partial \overline{I}}{\partial t}(t,\tau) + \dfrac{\partial \overline{I}}{\partial \tau}(t,\tau) = -\mu_F(\tau)\overline{I}(t,\tau) \\ \dfrac{\partial \overline{R}}{\partial t}(t,\tau) + \dfrac{\partial \overline{R}}{\partial \tau}(t,\tau) = -\overline{R}(t,\tau)\widetilde{\gamma}(\tau)\displaystyle\int_0^{+\infty} \widetilde{\lambda}(r)\overline{I}(t,r)dr \\ \overline{I}(t,0) = \left(\overline{S}(t) + \displaystyle\int_0^{+\infty} \widetilde{\gamma}(\theta)\overline{R}(t,\theta)d\theta\right)\int_0^{+\infty} \widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau \\ \overline{R}(t,0) = \displaystyle\int_0^{+\infty} \mu_F(\tau)\overline{I}(t,\tau)d\tau. \end{cases} \tag{1.40}$$

We will use the following notion of a solution of the system of ODE-PDEs (1.40). For us, a solution is an element $(\overline{S}(t), \overline{I}(t,\cdot), \overline{R}(t,\cdot), t \geqslant 0) \in \mathscr{C}\left(\mathbb{R}_+, \mathbb{R}_+ \times L^1(\mathbb{R}_+) \times L^1(\mathbb{R}_+)\right)$ satisfying (1.40), where the derivatives are taken in the distributional sense, and (1.40) is understood as follows. If $(\overline{S}(t), \overline{I}(t,\cdot), \overline{R}(t,\cdot)$ is a solution, then the continuous function $\overline{S}(t)$ is the unique solution of a linear equation which is given explicitly in terms of $\overline{I}$ and $\widetilde{\lambda}$, and the equation is satisfied pointwise. It follows from the second equation that $s \mapsto \overline{I}(t+s,s)$ also solves a linear equation, whose explicit solution is at least continuous, hence we can define $\overline{I}(t,0) = \lim_{s\to 0} \overline{I}(t+s,s)$, and it is this quantity which is assumed to satisfy the boundary condition stated on the fourth line of (1.40). A similar argument can be applied to $\overline{R}$. Note that it is essential that $\widetilde{\lambda}$ and $\widetilde{\gamma}$ be bounded, as well as μ_F. This last point will be an assumption in our statements below The classical reference for systems of PDEs of the type of (1.40) is the book of Webb [99], which establishes existence and uniqueness, but with a notion of solution which is a bit tricky, and we prefer to avoid presenting it. We will show below how to associate to any solution of (1.40) the solution to the system of integral equations (1.38)-(1.39), and vice versa how to construct a solution to (1.40) from the solution of (1.38)-(1.39). This establishes the connection betwen our results, in the particular case of the model described in the present section, and those of Kermack and McKendrick. We shall then deduce existence and uniqueness of a solution of (1.40) as a Corollary.

Proposition 1.2. *We assume that* $\|\mu_F\|_\infty < \infty$. *If* $(\overline{S}(t), \overline{I}(t,\cdot), \overline{R}(t,\cdot), t \geqslant 0)$ *is a solution of the system* (1.40), *and* $(\overline{\mathfrak{S}}(\cdot), \overline{\mathfrak{F}}(\cdot))$ *is given by*

$$\begin{cases} \overline{\mathfrak{S}}(t) = \overline{S}(t) + \displaystyle\int_0^{+\infty} \widetilde{\gamma}(\tau)\overline{R}(t,\tau)d\tau, \\ \overline{\mathfrak{F}}(t) = \displaystyle\int_0^{+\infty} \widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau\,, \end{cases} \tag{1.41}$$

then $(\overline{\mathfrak{S}}(\cdot), \overline{\mathfrak{F}}(\cdot))$ *solves* (1.38)-(1.39). *Conversely, given* $(\overline{\mathfrak{S}}(\cdot), \overline{\mathfrak{F}}(\cdot))$ *the unique solution of*

(1.38)-(1.39), *the following is a solution to* (1.40).

$$\begin{cases} \overline{S}(t) = \overline{S}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(s)ds\right), \\ \overline{I}(t,\tau) = \overline{I}(0,\tau-t)\dfrac{F^c(\tau)}{F^c(\tau-t)}\mathbb{1}_{\tau>t} + \overline{\mathfrak{F}}(t-\tau)\overline{\mathfrak{S}}(t-\tau)F^c(\tau)\mathbb{1}_{t\geqslant\tau}, \\ \overline{R}(t,\tau) = \overline{R}(t-\tau,0)\exp\left(-\int_0^\tau \widetilde{\gamma}(s)\overline{\mathfrak{F}}(t+s-\tau)ds\right)\mathbb{1}_{\tau<t} \\ \qquad\qquad + \overline{R}(0,\tau-t)\exp\left(-\int_{\tau-t}^\tau \widetilde{\gamma}(s)\overline{\mathfrak{F}}(t-\tau+s)ds\right)\mathbb{1}_{\tau\geqslant t}, \\ \overline{R}(t,0) = \int_0^{+\infty}\overline{I}(0,\tau)\dfrac{f(t+\tau)}{F^c(\tau)}d\tau + \int_0^t f(t-\tau)\overline{\mathfrak{S}}(\tau)\overline{\mathfrak{F}}(\tau)d\tau, \end{cases} \tag{1.42}$$

with F given by (1.33) *and f is the density of F.*

Proof. Assuming that $(\overline{S}(t),\overline{I}(t,\cdot),\overline{R}(t,\cdot),t\geqslant 0)$ is a solution of the system (1.40), we now show that $(\overline{\mathfrak{S}}(\cdot),\overline{\mathfrak{F}}(\cdot))$ given by (1.41) satisfies (1.38) and (1.39). Integrating $\overline{I}$ and $\overline{R}$ along the characteristics, which are the parallels of the first diagonal of $\mathbb{R}_+^2$, we obtain:

$$\begin{cases} \overline{S}(t) = \overline{S}(0)\exp\left(-\int_0^t\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(s,\tau)d\tau ds\right), \\ \overline{I}(t,\tau) = \overline{I}(0,\tau-t)\dfrac{F^c(\tau)}{F^c(\tau-t)}\mathbb{1}_{\tau>t} + \overline{I}(t-\tau,0)F^c(\tau)\mathbb{1}_{t\geqslant\tau}, \\ \overline{R}(t,\tau) = \overline{R}(t-\tau,0)\exp\left(-\int_0^\tau\widetilde{\gamma}(s)\int_0^{+\infty}\widetilde{\lambda}(r)\overline{I}(t+s-\tau,r)drds\right)\mathbb{1}_{\tau<t} \\ \qquad\qquad + \overline{R}(0,\tau-t)\exp\left(-\int_{\tau-t}^\tau\widetilde{\gamma}(s)\int_0^{+\infty}\widetilde{\lambda}(r)\overline{I}(t-\tau+s,r)drds\right)\mathbb{1}_{\tau\geqslant t}, \\ \overline{I}(t,0) = \left(\overline{S}(t) + \int_0^{+\infty}\widetilde{\gamma}(\theta)\overline{R}(t,\theta)d\theta\right)\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau, \\ \overline{R}(t,0) = \int_0^{+\infty}\overline{I}(0,\tau)\dfrac{f(t+\tau)}{F^c(\tau)}d\tau + \int_0^t f(t-\tau)\overline{I}(\tau,0)d\tau. \end{cases} \tag{1.43}$$

By (1.41), this can be written

$$\begin{cases} \overline{S}(t) = \overline{S}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(s)ds\right), \\ \overline{I}(t,\tau) = \overline{I}(0,\tau-t)\dfrac{F^c(\tau)}{F^c(\tau-t)}\mathbb{1}_{\tau>t} + \overline{I}(t-\tau,0)F^c(\tau)\mathbb{1}_{t\geqslant\tau}, \\ \overline{R}(t,\tau) = \overline{R}(t-\tau,0)\exp\left(-\int_0^\tau \widetilde{\gamma}(s)\overline{\mathfrak{F}}(t+s-\tau)ds\right)\mathbb{1}_{\tau<t} \\ \qquad\qquad + \overline{R}(0,\tau-t)\exp\left(-\int_{\tau-t}^\tau \widetilde{\gamma}(s)\overline{\mathfrak{F}}(t-\tau+s)ds\right)\mathbb{1}_{\tau\geqslant t}, \\ \overline{I}(t,0) = \overline{\mathfrak{S}}(t)\overline{\mathfrak{F}}(t) \\ \overline{R}(t,0) = \int_0^{+\infty}\overline{I}(0,\tau)\dfrac{f(t+\tau)}{F^c(\tau)}d\tau + \int_0^t f(t-\tau)\overline{I}(\tau,0)d\tau, \end{cases}$$

Substituting those expressions for $\overline{S}(t)$, $\overline{R}(t,\tau)$ and $\overline{I}(t,\tau)$ in (1.41) using the above expressions yields (1.38) and (1.39).

Conversely let $(\overline{\mathfrak{S}}(\cdot),\overline{\mathfrak{F}}(\cdot))$ be a solution of the set of equations (1.38)-(1.39). We now show that $(\overline{S}(t),\overline{I}(t,\cdot),\overline{R}(t,\cdot),\, t\geqslant 0)$ given by (1.42) solves the system (1.40). It suffices in fact to show that $(\overline{S}(t),\overline{I}(t,\cdot),\overline{R}(t,\cdot),\, t\geqslant 0)$ satisfies (1.43) (see Theorem 2.2 in [99]). Comparing (1.42) and (1.43), we see that it is enough to show that, if $\overline{I}(t,\tau)$, $\overline{S}(t)$ and $\overline{R}(t,\tau)$ are given by (1.42), then

$$\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau = \overline{\mathfrak{F}}(t), \tag{1.44}$$

and

$$\overline{S}(t) + \int_0^{+\infty}\widetilde{\gamma}(\tau)\overline{R}(t,\tau)d\tau = \overline{\mathfrak{S}}(t). \tag{1.45}$$

Indeed, replacing $\overline{I}(t,\tau)$, $\overline{S}(t)$ and $\overline{R}(t,\tau)$ by their expressions in (1.42), we obtain

$$\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau = \int_0^{+\infty}\widetilde{\lambda}(\tau+t)\overline{I}(0,\tau)\frac{F^c(\tau+t)}{F^c(\tau)}d\tau + \int_0^t \widetilde{\lambda}(\tau)F^c(\tau)\overline{\mathfrak{F}}(t-\tau)\overline{\mathfrak{S}}(t-\tau)d\tau,$$

and

$$\overline{S}(t)+\int_0^{+\infty}\widetilde{\gamma}(\tau)\overline{R}(t,\tau)d\tau=\overline{S}(0)\exp\left(-\int_0^{+\infty}\overline{\mathfrak{F}}(s)ds\right)$$
$$+\int_0^{+\infty}\widetilde{\gamma}(\tau+t)\overline{R}(0,\tau)\exp\left(-\int_\tau^{\tau+t}\widetilde{\gamma}(s)\overline{\mathfrak{F}}(s-\tau)ds\right)d\tau$$
$$+\int_0^{t}\widetilde{\gamma}(\tau)\exp\left(-\int_0^{\tau}\widetilde{\gamma}(s)\overline{\mathfrak{F}}(t+s-\tau)ds\right)\overline{R}(t-\tau,0)d\tau.$$

Replacing $\overline{R}(t-\tau,0)$ by its expression in (1.42) and using the fact that $(\overline{\mathfrak{S}},\overline{\mathfrak{F}})$ solves (1.38)-(1.39), we obtain (1.44) and (1.45).

This completes the proof of the equivalence. □

Corollary 1.2. *We assume that $\|\mu_F\|_\infty<\infty$. Then the system (1.40) has a unique solution.*

Proof. From the solution of (1.38)-(1.39), we define a solution of (1.40) with the help of (1.42), hence existence. Let now $(\overline{S}^1,\overline{I}^1,\overline{R}^1)$ and $(\overline{S}^2,\overline{I}^2,\overline{R}^2)$ be two solutions of (1.40) with the same initial condition at time $t=0$. Through (1.41), they define the same solution of (1.38)-(1.39). Hence for all $t\geq 0$,

$$\int_0^\infty\widetilde{\lambda}(\tau)\overline{I}^1(t,\tau)d\tau=\int_0^\infty\widetilde{\lambda}(\tau)\overline{I}^2(t,\tau)d\tau.$$

Consequently from the first line of (1.42), $\overline{S}^1(t)=\overline{S}^2(t)$, for all $t\geq 0$. This, combined with the first line of (1.41), implies that for all $t\geq 0$,

$$\int_0^\infty\widetilde{\gamma}(\tau)\overline{R}^1(t,\tau)d\tau=\int_0^\infty\widetilde{\gamma}(\tau)\overline{R}^2(t,\tau)d\tau\,.$$

The above three facts combined with the fourth line of (1.42) imply that $\overline{I}^1(t,0)=\overline{I}^2(t,0)$ for all $t\geq 0$, and hence also $\overline{R}^1(t,0)=\overline{R}^2(t,0)$ for all $t\geq 0$. The second and third lines of (1.42) allow us to conclude that $\overline{I}^1(t,\tau)=\overline{I}^2(t,\tau)$ and $\overline{R}^1(t,\tau)=\overline{R}^2(t,\tau)$ for all $t,\tau\geq 0$. Hence uniqueness. □

1.6. Proofs for the FLLN

In this section, we prove the FLLN. We start with the proof of Lemma 1.2.

Proof of Lemma 1.2. We can note that, if we multiply the equation for $x(t)$ by $y(t)$, we

obtain

$$y(t)x(t) = \mathbb{E}\left[y(t)\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] + \int_0^t \mathbb{E}\left[y(t)\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right]x(s)y(s)ds.$$

As a result,

$$\frac{d}{dt}\left(\mathbb{E}\left[\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right]x(s)y(s)ds\right) = 0.$$

Note that this function may not be differentiable everywhere since there may be an at most countable set of points where its right and left derivatives do not coincide. Nevertheless this function is almost everywhere differentiable and since its derivative is equal to zero almost everywhere, it is constant.

Hence, integrating between 0 and t, we obtain the result. □

We next prove Theorem 1.1.

Proof of Theorem 1.1. Let $(x,y) \in D_+^2$ be a solution to the set of equation (1.10)–(1.11). From Assumption 1.1, $\gamma_0(t) \leqslant 1$ and $\gamma(t-s) \leqslant 1$. Hence the combination of (1.10) and (1.17) implies that $x(t) \leqslant 1$. Moreover, from Assumption 1.1 if $\lambda(t) > 0$ ($\lambda_0(t) > 0$) then $\gamma(s) = 0$ ($\gamma_0(s) = 0$) for $0 \leqslant s \leqslant t$, we have

$$\begin{aligned} y(t) &= \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)x(s)y(s)ds \\ &= \mathbb{E}\left[\lambda_0(t)\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] \\ &\qquad + \int_0^t \mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right]x(s)y(s)ds \\ &\leqslant \lambda^*, \end{aligned}$$

where the last inequality follows from (1.17) and the fact that $\lambda_0(t), \lambda(t-s) \leq \lambda^*$. Consequently, if (x,y) solves (1.10)–(1.11), then for $t \geqslant 0$, $x(t) \leqslant 1$ and $y(t) \leq \lambda^*$.

Suppose now that we have two solutions (x^1, y^1) and (x^2, y^2) of equations (1.10)–

(1.11). Then we have

$$\begin{aligned}
x^1(t)-x^2(t) &= \mathbb{E}\left[\gamma_0(t)\left(\exp\left(-\int_0^t \gamma_0(r)y^1(r)dr\right)-\exp\left(-\int_0^t \gamma_0(r)y^2(r)dr\right)\right)\right] \\
&\quad+\int_0^t \mathbb{E}\left[\gamma(t-s)\left(\exp\left(-\int_s^t \gamma(r-s)y^1(r)dr\right)x^1(s)y^1(s)\right.\right. \\
&\qquad\left.\left.-\exp\left(-\int_s^t \gamma(r-s)y^2(r)dr\right)x^2(s)y^2(s)\right)\right]ds, \\
y^1(t)-y^2(t) &= \int_0^t \overline{\lambda}(t-s)\left[x^1(s)y^1(s)-x^2(s)y^2(s)\right]ds.
\end{aligned}$$

We first deduce from the second relation, taking into account that for $i=1,2$, $x^i(s)\leq 1$ and $y^i(s)\leq \lambda^*$, that for any $t>0$,

$$\begin{aligned}
\left|y^1(t)-y^2(t)\right| &\leq \lambda^*\int_0^t\left[\left|x^1(s)-x^2(s)\right|\left|y^1(s)\right|+\left|y^1(s)-y^2(s)\right|\left|x^2(s)\right|\right]ds \\
&\leqslant \lambda^*\max(\lambda^*,1)\int_0^t\left[\left|x^1(s)-x^2(s)\right|+\left|y^1(s)-y^2(s)\right|\right]ds. \qquad (1.46)
\end{aligned}$$

We now exploit the first relation. Since $\left|\exp(-a)-\exp(-b)\right|\leqslant|a-b|$, $\forall a,b\in\mathbb{R}_+$, $\gamma\leqslant 1$ and $x^1y^1\leqslant\lambda_*$, we have for $0\leqslant t\leqslant T$,

$$\begin{aligned}
\left|x^1(t)-x^2(t)\right| &\leqslant \int_0^t\left|y^1(s)-y^2(s)\right|ds \\
&\quad+\int_0^t\mathbb{E}\left[\left|\exp\left(-\int_s^t\gamma(r-s)y^1(r)dr\right)x^1(s)y^1(s)\right.\right. \\
&\qquad\left.\left.-\exp\left(-\int_s^t\gamma(r-s)y^2(r)dr\right)x^2(s)y^2(s)\right|\right]ds \\
&\leqslant \int_0^t\left|y^1(s)-y^2(s)\right|ds \\
&\quad+\int_0^t\mathbb{E}\left[\left|\exp\left(-\int_s^t\gamma(r-s)y^1(r)dr\right)\right.\right. \\
&\qquad\left.\left.-\exp\left(-\int_s^t\gamma(r-s)y^2(r)dr\right)\right|\right]x^1(s)y^1(s)ds \\
&\quad+\int_0^t\mathbb{E}\left[\exp\left(-\int_s^t\gamma(r-s)y^2(r)dr\right)\right]\left|x^1(s)y^1(s)-x^2(s)y^2(s)\right|ds \\
&\leqslant \int_0^t\left|y^1(s)-y^2(s)\right|ds+T\lambda_*\int_0^t\left|y^1(s)-y^2(s)\right|ds \\
&\quad+\int_0^t\left|x^1(s)y^1(s)-x^2(s)y^2(s)\right|ds.
\end{aligned}$$

Hence, again, given $T > 0$, there exists a constant C such that for any $0 \le t \le T$,

$$\left|x^1(t) - x^2(t)\right| \le C \int_0^t \left[\left|x^1(s) - x^2(s)\right| + \left|y^1(s) - y^2(s)\right|\right] ds. \tag{1.47}$$

Uniqueness follows from (1.46), (1.47) and the Gronwall Lemma.

Now local existence follows by an approximation procedure, which exploits the estimates (1.46) and (1.47). Indeed, Picard iteration procedure works for an existence of integral equation as for the more technical ODEs. For detail, see Brunner [18]. Global existence then follows from the above a priori estimates, which forbids explosion. Theorem 1.1 is established. □

We next prove Theorem 1.2. We first construct a system of stochastic equations driven by Poisson random measures (PRMs), and then use an approach of the type of propagation of chaos as in [92].

Let $m \in D_+$, $(\lambda_0, \gamma_0) \in D_+^2$ and $(\lambda_i, \gamma_i)_{i\ge 1}$ a collection of i.i.d. random elements of D_+ independent of (λ_0, γ_0), Q a standard PRM on $\mathbb{R}_+^2$ independent of the previous collection. We define for $t \geqslant 0$,

$$\begin{cases} A^{(m)}(t) := \displaystyle\int_0^t \int_0^{+\infty} \mathbb{1}_{u \leqslant \Upsilon^{(m)}(r^-)} Q(dr, du) \\ \Upsilon^{(m)}(t) := \gamma_{A^{(m)}(t)}(\varsigma^{(m)}(t)) m(t) \end{cases} \tag{1.48}$$

where $\varsigma^{(m)}$ is defined in the same manner as ς_1^N with $A^{(m)}$ instead of A_1^N.

Let

$$\overline{\mathfrak{F}}^{(m)}(t) = \mathbb{E}\left[\lambda_{A^{(m)}(t)}(\varsigma^{(m)}(t))\right], \quad \text{and} \quad \overline{\mathfrak{S}}^{(m)}(t) = \mathbb{E}\left[\gamma_{A^{(m)}(t)}(\varsigma^{(m)}(t))\right].$$

Lemma 1.7. *There exists a unique function $m^* \in D_+$ such that*

$$\overline{\mathfrak{F}}^{(m^*)} = m^*.$$

Moreover, $(\overline{\mathfrak{S}}^{(m^*)}, \overline{\mathfrak{F}}^{(m^*)})$ solves the set of equations (1.10)–(1.11).

Proof. Let us denote the jump times of the process $A^{(m)}$ by $(\tau_i^{(m)})_{i\geqslant 0}$ with $\tau_0^{(m)} = 0$. From Assumption 1.1, for $t \geqslant \tau_i^{(m)}$, if $\lambda_i(t - \tau_i^{(m)}) \neq 0$, we have

$$\int_{\tau_i^{(m)}}^t \gamma_i(r - \tau_i^{(m)}) \overline{\mathfrak{F}}^{(m)}(r) dr = 0,$$

hence $A^{(m)}(t) = A^{(m)}(\tau_i^{(m)}) = i$. Therefore for $0 \leqslant i \leqslant A^{(m)}(t)$, $\lambda_i(t - \tau_i^{(m)}) = \lambda_i(t - \tau_i^{(m)}) \mathbb{1}_{A^{(m)}(t)=i}$ a.s. We also recall that $\overline{\lambda}_0(t) = \mathbb{E}\left[\lambda_0(t) \middle| \eta_0 > 0\right]$ and $\overline{I}(0) = \mathbb{P}(\eta_0 > 0)$ and

we deduce that

$$\begin{aligned}\overline{\mathfrak{F}}^{(m)}(t) &= \mathbb{E}\left[\lambda_{A^{(m)}(t)}(\varsigma^{(m)}(t))\right] \\ &= \mathbb{E}\left[\sum_{i\geqslant 0}\lambda_i(t-\tau_i^{(m)})\mathbb{1}_{A^{(m)}(t)=i}\right] \\ &= \mathbb{E}\left[\lambda_0(t)+\sum_{i=1}^{A^{(m)}(t)}\lambda_i(t-\tau_i^{(m)})\right] \\ &= \mathbb{E}\left[\lambda_0(t)\,\middle|\,\eta_0>0\right]\mathbb{P}\left(\eta_0>0\right)+\sum_{i\geqslant 1}\mathbb{E}\left[\lambda_i(t-\tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)}\leqslant t}\right] \\ &= \overline{I}(0)\overline{\lambda}_0(t)+\sum_{i\geqslant 1}\mathbb{E}\left[\lambda_i(t-\tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)}\leqslant t}\right],\end{aligned}$$

and as λ_i and $\tau_i^{(m)}$ are independent, we further obtain that

$$\begin{aligned}\overline{\mathfrak{F}}^{(m)}(t) &= \overline{I}(0)\overline{\lambda}_0(t)+\sum_{i\geqslant 1}\mathbb{E}\left[\overline{\lambda}(t-\tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)}\leqslant t}\right] \\ &= \overline{I}(0)\overline{\lambda}_0(t)+\mathbb{E}\left[\int_0^t\overline{\lambda}(t-s)A^{(m)}(ds)\right].\end{aligned}$$

Note that the process $t\mapsto A^{(m)}(t)-\int_0^t\Upsilon^{(m)}(s)ds$ is a martingale. The integral of a bounded deterministic function over a compact interval w.r.t. such a martingale has zero expectation, hence

$$\mathbb{E}\int_0^t\overline{\lambda}(t-s)\left[A^{(m)}(ds)-\Upsilon^{(m)}(s)ds\right]=0.$$

As a result,

$$\overline{\mathfrak{F}}^{(m)}(t)=\overline{I}(0)\overline{\lambda}_0(t)+\int_0^t\overline{\lambda}(t-s)m(s)\overline{\mathfrak{S}}^{(m)}(s)ds. \tag{1.49}$$

In addition,

$$\gamma_{A^{(m)}(t)}(\varsigma^{(m)}(t))=\gamma_0(t)\mathbb{1}_{A^{(m)}(t)=0}+\sum_{i=1}^{+\infty}\gamma_i(t-\tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)}\leqslant t}\mathbb{1}_{A^{(m)}(t)=i}\,.$$

Let

$$\mathscr{F}_t=\sigma\left(\left\{(\lambda_i)_{0\leqslant i\leqslant A^{(m)}(t)},(\gamma_i)_{0\leqslant i\leqslant A^{(m)}(t)},A^{(m)}(t'),0\leqslant t'\leqslant t\right\}\right).$$

Since $Q\big|_{]\tau_i^{(m)},t]}$ is independent of $\mathscr{F}_{\tau_i^{(m)}}$, we have

$$\begin{aligned}
&\mathbb{P}\left(A^{(m)}(t)=i\middle|\mathscr{F}_{\tau_i^{(m)}}\right)\mathbb{1}_{\tau_i^{(m)}\leqslant t}\\
&=\mathbb{P}\left(Q\left(\{(s,u)\in\mathbb{R}_+^2,\,\tau_i^{(m)}<s\leq t,\,\gamma_i((s-\tau_i^{(m)})^-)m(s^-)\geqslant u\}\right)=0\middle|\mathscr{F}_{\tau_i^{(m)}}\right)\mathbb{1}_{\tau_i^{(m)}\leqslant t}\\
&=\exp\left(-\int_{\tau_i^{(m)}}^t\gamma_i(r-\tau_i^{(m)})m(r)dr\right)\mathbb{1}_{\tau_i^{(m)}\leqslant t}.
\end{aligned}$$

Thus, since γ_0 and Q are also independent, we obtain

$$\begin{aligned}
\overline{\mathfrak{S}}^{(m)}(t)&=\mathbb{E}\left[\gamma_0(t)\mathbb{1}_{A^{(m)}(t)=0}\right]+\sum_{i\geqslant 1}\mathbb{E}\left[\gamma_i(t-\tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)}\leqslant t}\mathbb{P}\left(A^{(m)}(t)=i\middle|\mathscr{F}_{\tau_i^{(m)}}\right)\right]\\
&=\mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t\gamma_0(r)m(r)dr\right)\right]\\
&\quad+\sum_{i\geqslant 1}\mathbb{E}\left[\gamma_i(t-\tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)}\leqslant t}\exp\left(-\int_{\tau_i^{(m)}}^t\gamma_i(r-\tau_i^{(m)})m(r)dr\right)\right].
\end{aligned}$$

Moreover, since γ_i and $\tau_i^{(m)}$ are independent, recalling that the law of γ is denoted by μ, we further obtain

$$\begin{aligned}
\overline{\mathfrak{S}}^{(m)}(t)&=\mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t\gamma_0(r)m(r)dr\right)\right]\\
&\quad+\sum_{i\geqslant 1}\mathbb{E}\left[\int_D\gamma(t-\tau_i^{(m)})\exp\left(-\int_{\tau_i^{(m)}}^t\gamma(r-\tau_i^{(m)})m(r)dr\right)\mu(d\gamma)\mathbb{1}_{\tau_i^{(m)}\leqslant t}\right].
\end{aligned}$$

In addition, as $(\tau_i^{(m)})_i$ are the jump time of $A^{(m)}$, by Fubini's theorem, we have

$$\begin{aligned}
\overline{\mathfrak{S}}^{(m)}(t)&=\mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t\gamma_0(r)m(r)dr\right)\right]\\
&\qquad+\int_D\mathbb{E}\left[\sum_{i=1}^{A^{(m)}(t)}\gamma(t-\tau_i^{(m)})\exp\left(-\int_{\tau_i^{(m)}}^t\gamma(r-\tau_i^{(m)})m(r)dr\right)\right]\mu(d\gamma)\\
&=\mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t\gamma_0(r)m(r)dr\right)\right]\\
&\qquad+\int_D\mathbb{E}\left[\int_0^t\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)m(r)dr\right)A^{(m)}(ds)\right]\mu(d\gamma).
\end{aligned}\tag{1.50}$$

For any given $\gamma\in D$, we define

$$g_\gamma(s,t):=\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)m(r)dr\right).$$

Since

$$A^{(m)}(t) = \int_0^t \int_0^{+\infty} \mathbb{1}_{u \le \Upsilon^{(m)}(s^-)} Q(ds, du),$$

by the same argument as the one leading to (1.49) we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^t g_\gamma(s,t) A^{(m)}(ds)\right] &= \mathbb{E}\left[\int_0^t g_\gamma(s,t)\Upsilon^{(m)}(s)ds\right] \\
&= \int_0^t g_\gamma(s,t)\mathbb{E}\left[\Upsilon^{(m)}(s)\right]ds \\
&= \int_0^t g_\gamma(s,t)\overline{\mathfrak{S}}^{(m)}(s)m(s)ds. \qquad (1.51)
\end{aligned}$$

Then, from (1.50) and (1.51), we deduce that

$$\begin{aligned}
\overline{\mathfrak{S}}^{(m)}(t) &= \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right] \\
&\quad + \int_0^t \int_D \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)m(r)dr\right)\mu(d\gamma)m(s)\overline{\mathfrak{S}}^{(m)}(s)ds \\
&= \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right] \\
&\quad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)m(r)dr\right)\right]m(s)\overline{\mathfrak{S}}^{(m)}(s)ds. \qquad (1.52)
\end{aligned}$$

Hence from (1.49) and (1.52), $\overline{\mathfrak{F}}^{(m)} = m$ if and only if $(\overline{\mathfrak{S}}^{(m)}, m)$ solves (1.10)–(1.11). Consequently, by Theorem 1.1, there exists a unique element $m^* \in D_+$ such that $\overline{\mathfrak{F}}^{(m^*)} = m^*$ and that $(\overline{\mathfrak{S}}^{(m^*)}, m^*)$ solves (1.10)–(1.11). This concludes the proof of the lemma. □

We next consider the sequence $(Q_k)_{k\geqslant 1}$ of Poisson random measures introduced in Section 1.2 and for each $k \geqslant 1$, we define the process $\{A_k(t),\, t \geqslant 0\}$:

$$A_k(t) = \int_0^t \int_0^{+\infty} \mathbb{1}_{u \leqslant \Upsilon_k(r^-)} Q_k(dr, du),$$

where

$$\Upsilon_k(t) = \gamma_{k,A_k(t)}(\varsigma_k(t))\overline{\mathfrak{F}}(t),$$

and ς_k is defined in the same manner as ς_1^N with A_k instead of A_1^N. (This definition follows a similar idea to Lemma 1.7 as in [23].) In this definition we use the same $(\lambda_{k,i}, \gamma_{k,i}, Q_k)$ as in the definition of the model in Section 1.2. Moreover, since $\left((\lambda_{k,i})_i, (\gamma_{k,i})_i, Q_k\right)_{k\geqslant 1}$ are i.i.d, $(A_k)_{k\geqslant 1}$ are also i.i.d.

Remark 1.11. *From Lemma 1.7 we have*

$$\overline{\mathfrak{F}}(t) = \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right] \text{ and } \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_{1,A_1(t)}(\varsigma_1(t))\right].$$

Now for each $k \geqslant 1$, we compare the process $A_k^N(t)$, $t \geqslant 0$ with the process $A_k(t)$, $t \geqslant 0$.

Lemma 1.8. *For $k \in \mathbb{N}$ and $T \geqslant 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|A_k^N(t) - A_k(t)\right|\right] \leqslant \int_0^T \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right|\Big] dt =: \delta^N(T) \tag{1.53}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\varsigma_k^N(t) - \varsigma_k(t)\right|\right] \leqslant T\delta^N(T).$$

Moreover,

$$\delta^N(T) \leqslant \frac{\lambda^*}{\sqrt{N}} T \exp\big(2\lambda^* T\big). \tag{1.54}$$

Proof. We adapt here the proof of Theorem IV.1 in [23] to our setting. Since

$$\left|A_k^N(t) - A_k(t)\right| \leqslant \int_0^t \int_0^{+\infty} \mathbb{1}_{\min\left(\Upsilon_k^N(r^-),\Upsilon_k(r^-)\right)<u\leqslant\max\left(\Upsilon_k^N(r^-),\Upsilon_k(r^-)\right)} Q_k(dr,du),$$

we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|A_k^N(t) - A_k(t)\right|\right] \leqslant \int_0^T \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right|\Big] dt = \delta^N(T).$$

We recall that

$$\Upsilon_k^N(t) = \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\overline{\mathfrak{F}}^N(t) \quad \text{and} \quad \Upsilon_k(t) = \gamma_{k,A_k(t)}(\varsigma_k(t))\overline{\mathfrak{F}}(t).$$

However, since $\gamma_{k,i} \leqslant 1$ and $0 \leqslant \overline{\mathfrak{F}}^N(t), \overline{\mathfrak{F}}(t) \leqslant \lambda^*$, we obtain

$$\begin{aligned}\mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right|\Big] \leqslant \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right| \mathbb{1}_{A_k^N(t)=A_k(t),\varsigma_k(t)=\varsigma_k^N(t)}\Big] + \\ \lambda^* \mathbb{P}\left(A_k^N(t) \neq A_k(t) \text{ or } \varsigma_k(t) \neq \varsigma_k^N(t)\right).\end{aligned} \tag{1.55}$$

On the other hand, using $\overline{\mathfrak{F}}(t) = \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right]$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right| \mathbb{1}_{A_k^N(t)=A_k(t),\varsigma_k(t)=\varsigma_k^N(t)}\right] \\
&\leqslant \mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\right|\right] \\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N \left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right]\right)\right|\right] \\
&\leqslant \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N \left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \lambda_{j,A_j(t)}(\varsigma_j(t))\right)\right|\right] \\
&\qquad + \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N \left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right]\right)\right|\right]. \qquad (1.56)
\end{aligned}$$

Since $((\lambda_{k,i})_i, A_k, \varsigma_k)_k$ are i.i.d, $(\lambda_{k,A_k(t)}(\varsigma_k(t))_k$ are i.i.d., hence, by Hölder's inequality we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N \left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right]\right)\right|\right] \\
&\leqslant \frac{1}{N}\left(\mathbb{E}\left[\left(\sum_{j=1}^N \left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{j,A_j(t)}(\varsigma_j(t))\right]\right)\right)^2\right]\right)^{\frac{1}{2}} \\
&= \frac{1}{N}\left(\sum_{j=1}^N \mathbb{E}\left[\left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{j,A_j(t)}(\varsigma_j(t))\right]\right)^2\right]\right)^{\frac{1}{2}} \\
&\leqslant \frac{\lambda^*}{\sqrt{N}}. \qquad (1.57)
\end{aligned}$$

Here the equality holds because $(\lambda_{k,A_k(t)}(\varsigma_k(t)) - \mathbb{E}\left[\lambda_{k,A_k(t)}(\varsigma_k(t))\right])_k$ are i.i.d. and the last inequality holds since $\lambda_{k,i}$ is bounded by λ^*.

In addition, as $(A_j^N)_j$ are exchangeable we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N \left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \lambda_{j,A_j(t)}(\varsigma_j(t))\right)\right|\right] \\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N \left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \lambda_{j,A_j(t)}(\varsigma_j(t))\right) \mathbb{1}_{A_j(t)\neq A_j^N(t) \text{ or } \varsigma_j(t)\neq\varsigma_j^N(t)}\right|\right] \\
&\leqslant \frac{\lambda^*}{N}\sum_{j=1}^N \mathbb{P}\left(A_j(t) \neq A_j^N(t) \text{ or } \varsigma_j(t) \neq \varsigma_j^N(t)\right) \\
&= \lambda^* \mathbb{P}\left(A_k(t) \neq A_k^N(t) \text{ or } \varsigma_k(t) \neq \varsigma_k^N(t)\right). \qquad (1.58)
\end{aligned}$$

Hence from (1.56), (1.57) and (1.58) we have

$$\mathbb{E}\left[\left|\Upsilon_k^N(t)-\Upsilon_k(t)\right|\mathbb{1}_{A_k^N(t)=A_k(t),\varsigma_k(t)=\varsigma_k^N(t)}\right]\leqslant\frac{\lambda^*}{\sqrt{N}}+\lambda^*\mathbb{P}\left(A_k(t)\neq A_k^N(t)\text{ or }\varsigma_k(t)\neq\varsigma_k^N(t)\right).$$

On the other hand, since

$$\left\{A_k^N(t)\neq A_k(t)\text{ or }\varsigma_k(t)\neq\varsigma_k^N(t)\right\}\subset\left\{\sup_{r\in[0,t]}|A_k^N(r)-A_k(r)|\geqslant 1\right\},$$

we have

$$\mathbb{P}\left(A_k(t)\neq A_k^N(t)\text{ or }\varsigma_k(t)\neq\varsigma_k^N(t)\right)\leqslant\mathbb{E}\left[\sup_{r\in[0,t]}|A_k^N(r)-A_k(r)|\right]\leqslant\delta^N(t).$$

Thus, from (1.55), we have

$$\mathbb{E}\Big[\left|\Upsilon_k^N(t)-\Upsilon_k(t)\right|\Big]\leqslant\frac{\lambda^*}{\sqrt{N}}+2\lambda^*\delta^N(t).$$

Combining this with (1.53), we deduce that for any $T\geqslant 0$,

$$\delta^N(T)\leqslant\frac{\lambda^*}{\sqrt{N}}T+2\lambda^*\int_0^T\delta^N(t)dt,$$

hence by Gronwall's lemma, we have

$$\delta^N(T)\leqslant\frac{\lambda^*}{\sqrt{N}}T\exp\left(2\lambda^*T\right).$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\varsigma_k^N(t)-\varsigma_k(t)\right|\right]&=\mathbb{E}\left[\mathbb{1}_{\{\exists t\in[0,T],\varsigma_k^N(t)\neq\varsigma_k(t)\}}\sup_{t\in[0,T]}\left|\varsigma_k^N(t)-\varsigma_k(t)\right|\right]\\&\leqslant T\mathbb{P}\left(\exists t\in[0,T],\varsigma_k^N(t)\neq\varsigma_k(t)\right)\\&=T\mathbb{P}\left(\sup_{t\in[0,T]}\left|A_k^N(t)-A_k(t)\right|\neq 0\right)\\&\leqslant T\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_k^N(t)-A_k(t)\right|\right]\\&\leqslant T\delta^N(T).\end{aligned}$$

This concludes the proof of the lemma. □

From the proof of Lemma 2.15, we deduce the following Remark.

Remark 1.12. *For $k \in \mathbb{N}$ and $t \geqslant 0$ we have*

$$\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\right|\right] \leqslant \frac{\lambda^*}{\sqrt{N}}\left(1 + t\exp\left(2\lambda^* t\right)\right), \qquad \mathbb{E}\left[\left|\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}(t)\right|\right] \leqslant \frac{1}{\sqrt{N}}\left(1 + \lambda^* t\exp\left(2\lambda^* t\right)\right)$$

and $\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right|\right] \leqslant \dfrac{\lambda^*}{\sqrt{N}}\left(1 + 2\lambda^* t\exp\left(2\lambda^* t\right)\right).$

From Lemma 2.15, we deduce the following Lemma.

Lemma 1.9. *For $k \in \mathbb{N}$ and $T \geqslant 0$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leqslant \frac{\lambda^*}{\sqrt{N}} T\exp\left(2\lambda^* T\right), \tag{1.59}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leqslant \frac{\lambda^{*2}}{\sqrt{N}} T\exp\left(2\lambda^* T\right), \tag{1.60}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right|\right] \leqslant \frac{\lambda^*}{\sqrt{N}} T\exp\left(2\lambda^* T\right). \tag{1.61}$$

Proof. From (1.53) and the fact that $\gamma_{k,i} \leq 1$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right|\right] \\
&\qquad = \mathbb{E}\left[\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right| \mathbb{1}_{\sup_{t\in[0,T]}\left|A_k^N(t) - A_k(t)\right| \geqslant 1}\right] \\
&\qquad \leqslant \mathbb{P}\left(\sup_{t\in[0,T]}\left|A_k^N(t) - A_k(t)\right| \geqslant 1\right) \\
&\qquad \leqslant \delta^N(T).
\end{aligned}$$

Similarly we also have

$$\begin{cases}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leqslant \lambda^*\delta^N(T), \\
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right|\right] \leqslant \delta^N(T).
\end{cases}$$

Hence the claims follow from (1.54). □

Completing the proof of Theorem 1.2. For $t \geqslant 0$, we have

$$\begin{aligned}
\overline{\mathfrak{F}}^N(t) &= \frac{1}{N}\sum_{k=1}^N \lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) \\
&= \frac{1}{N}\sum_{k=1}^N \left(\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right) + \frac{1}{N}\sum_{k=1}^N \lambda_{k,A_k(t)}(\varsigma_k(t)),
\end{aligned}$$

and

$$\begin{aligned}\overline{\mathfrak{S}}^N(t) &= \frac{1}{N}\sum_{k=1}^{N}\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\\ &= \frac{1}{N}\sum_{k=1}^{N}\left(\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))-\gamma_{k,A_k(t)}(\varsigma_k(t))\right)+\frac{1}{N}\sum_{k=1}^{N}\gamma_{k,A_k(t)}(\varsigma_k(t)).\end{aligned}$$

From (1.59) and (1.60), we have

$$\begin{cases}\mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N}\sup_{t\in[0,T]}\left|\lambda_{k,A_k^N(t)}(\varsigma_k^N(t))-\lambda_{k,A_k(t)}(\varsigma_k(t))\right|\right]\leqslant\frac{\lambda^{*2}}{\sqrt{N}}T\exp\left(2\lambda^*T\right),\\ \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N}\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))-\gamma_{k,A_k(t)}(\varsigma_k(t))\right|\right]\leqslant\frac{\lambda^{*}}{\sqrt{N}}T\exp\left(2\lambda^*T\right).\end{cases}$$

Hence,

$$\left(\frac{1}{N}\sum_{k=1}^{N}\left(\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))-\gamma_{k,A_k(t)}(\varsigma_k(t))\right),\frac{1}{N}\sum_{k=1}^{N}\left(\lambda_{k,A_k^N(t)}(\varsigma_k^N(t))-\lambda_{k,A_k(t)}(\varsigma_k(t))\right)\right)\xrightarrow[N\to+\infty]{}(0,0)$$

locally uniformly in t.

Moreover, as $\left(\gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)),\lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot))\right)_k$ is a collection of i.i.d. random variables in D^2, by the law of large numbers in D^2 [84, Theorem 1],

$$\left(\frac{1}{N}\sum_{k=1}^{N}\gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)),\frac{1}{N}\sum_{k=1}^{N}\lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot))\right)\xrightarrow[N\to+\infty]{\mathbb{P}}\left(\mathbb{E}\left[\gamma_{1,A_1(\cdot)}(\varsigma_1(\cdot))\right],\mathbb{E}\left[\lambda_{1,A_1(\cdot)}(\varsigma_1(\cdot))\right]\right)\text{ in }D^2.$$

We have shown in the proof on Lemma 1.7 that the pair $(\overline{\mathfrak{S}},\overline{\mathfrak{F}})$ given in Remark 1.11 solves the set of equations (1.10)–(1.11). This proves the convergence (1.12).

For $(\overline{U}^N,\overline{I}^N)$, we have for $t\geqslant 0$,

$$\begin{aligned}\overline{I}^N(t) &= \frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}}\\ &= \frac{1}{N}\sum_{k=1}^{N}\left(\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}}-\mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right)+\frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}},\end{aligned}$$

and

$$\begin{aligned}\overline{U}^N(t) &= \frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\varsigma_k^N(t)\geqslant\eta_{k,A_k^N(t)}}\\ &= \frac{1}{N}\sum_{k=1}^{N}\left(\mathbb{1}_{\varsigma_k^N(t)\geqslant\eta_{k,A_k^N(t)}}-\mathbb{1}_{\varsigma_k(t)\geqslant\eta_{k,A_k(t)}}\right)+\frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\varsigma_k(t)\geqslant\eta_{k,A_k(t)}}.\end{aligned}$$

As above, from (2.46), we deduce that

$$\left(\frac{1}{N}\sum_{k=1}^{N}\left(\mathbb{1}_{\varsigma_k^N(t)\geqslant\eta_{k,A_k^N(t)}}-\mathbb{1}_{\varsigma_k(t)\geqslant\eta_{k,A_k(t)}}\right),\frac{1}{N}\sum_{k=1}^{N}\left(\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}}-\mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right)\right)\xrightarrow[N\to+\infty]{}(0,0)$$

locally uniformly in t.

Moreover, as $\left(\varsigma_k(t),\eta_{k,A_k(t)}\right)_k$ is a collection of i.i.d. random variables in D^2, by the law of large numbers in D^2 [84, Theorem 1],

$$\left(\frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\varsigma_k(t)\geqslant\eta_{k,A_k(t)}},\frac{1}{N}\sum_{k=1}^{N}\mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right)\xrightarrow[N\to+\infty]{\mathbb{P}}\left(\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)\geqslant\eta_{1,A_1(t)}}\right],\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right]\right)\text{ in }D^2.$$

Recall the formula (1.13) and (1.14). In order to complete the proof of Theorem 1.2, it remains to verify that:

$$\overline{I}(t)=\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right]\text{ and }\overline{U}(t)=\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)\geqslant\eta_{1,A_1(t)}}\right].$$

Denote the jump times of the process A_1 by $(\tau_i)_{i\geqslant1}$, if $t-\tau_i<\eta_{1,i}$, $\lambda_{1,i}(t-\tau_i)\neq0$. From Assumption 1.1, we deduce that

$$\int_{\tau_i}^{t}\gamma_{1,i}(r-\tau_i)\overline{\mathfrak{F}}(r)dr=0.$$

So $A_1(t)=A_1(\tau_i)$. Since $\mathbb{1}_{t<\eta_{1,0}}=\mathbb{1}_{t<\eta_{1,0}}\mathbb{1}_{A_1(t)=0}$ a.s., we have

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right]&=\mathbb{E}\left[\mathbb{1}_{t<\eta_{1,0}}\mathbb{1}_{A_1(t)=0}+\sum_{i\geqslant1}\mathbb{1}_{t-\tau_i<\eta_{1,i}}\mathbb{1}_{A_1(t)=i}\right]\\&=\mathbb{P}\left(\eta_{1,0}>t\middle|\eta_{1,0}>0\right)\mathbb{P}\left(\eta_{1,0}>0\right)+\mathbb{E}\left[\sum_{i\geqslant1}\mathbb{1}_{t-\tau_i<\eta_{1,i}}\mathbb{1}_{\tau_i\leqslant t}\right]\\&=\overline{I}(0)F_0^c(t)+\sum_{i\geqslant1}\mathbb{E}\left[\mathbb{P}\left(t-\tau_i<\eta_{1,i}\middle|\tau_i\right)\mathbb{1}_{\tau_i\leqslant t}\right].\end{aligned}$$

Moreover, since $\eta_{1,i}$ and τ_i are independent, we obtain

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right]&=\overline{I}(0)F_0^c(t)+\sum_{i\geqslant1}\mathbb{E}\left[F^c(t-\tau_i)\mathbb{1}_{\tau_i\leqslant t}\right]\\&=\overline{I}(0)F_0^c(t)+\mathbb{E}\left[\int_0^t F^c(t-s)A_1(ds)\right]\\&=\overline{I}(0)F_0^c(t)+\int_0^t F^c(t-s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds.\end{aligned}\tag{1.62}$$

If $\eta_{1,0}>t$, $\lambda_{1,0}(t)>0$ ($\eta_{1,1}>t$, $\lambda_{1,1}(t)>0$), from Assumption 1.1 by using the fact that $\lambda_{1,1}(t)>0$ ($\lambda_{1,0}(t)>0$) implies that $\gamma_{1,1}(s)=0$ ($\gamma_{1,0}(s)=0$) for $0\leqslant s\leqslant t$, from (1.62) we

also have,

$$\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right] = \mathbb{E}\left[\mathbb{1}_{\eta_{1,0}>t}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta_{1,1}>t-s}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds,$$

which combined with (1.17) yields

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)\geqslant\eta_{1,A_1(t)}}\right] &= 1-\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right]\\ &= \mathbb{E}\left[\mathbb{1}_{t\geqslant\eta_{1,0}}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\\ &\quad + \int_0^t \mathbb{E}\left[\mathbb{1}_{t-s\geqslant\eta_{1,1}}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds.\end{aligned}$$

This completes the proof of Theorem 1.2. □

We also have the following convergence result on the empirical measure of the processes $(A_k^N(t),\varsigma_k^N(t))_{t\geqslant 0}$. It is not used in our analysis, but a worth to be established.

Theorem 1.5.

$$\frac{1}{N}\sum_{k=1}^N \delta_{(A_k^N(t),\varsigma_k^N(t))_{t\geqslant 0}} \xrightarrow[N\to+\infty]{\mathbb{P}} \mathscr{L}\left((A_1(t),\varsigma_1(t))_{t\geqslant 0}\right) \text{ in } \mathscr{P}(D^2). \tag{1.63}$$

Proof. By Lemma 2.15, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(\left|A_1^N(t)-A_1(t)\right|+\left|A_2^N(t)-A_2(t)\right|+\left|\varsigma_1^N(t)-\varsigma_1(t)\right|+\left|\varsigma_2^N(t)-\varsigma_2(t)\right|\right)\right] \xrightarrow[N\to+\infty]{} 0.$$

Hence

$$\left(\left\{(A_1^N(t),\varsigma_1^N(t))\right\}_{t\geqslant 0},\left\{(A_2^N(t),\varsigma_2^N(t))\right\}_{t\geqslant 0}\right) \xrightarrow[N\to+\infty]{\mathbb{P}} \left(\{(A_1(t),\varsigma_1(t))\}_{t\geqslant 0},\{(A_2(t),\varsigma_2(t))\}_{t\geqslant 0}\right)$$

in $D^2\times D^2$. Thus since the processes are exchangeable and D is a separable metric space, by [92, Proposition 2.2 i), page 177], the convergence in (1.63) holds. □

1.7. Proofs for the endemic equilibrium

In this section, we prove the results on the endemic equilibrium behaviors. We proceed in two subsections to prove the results in the scenarios $R_0 < \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ and $R_0 \geqslant \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. We have a complete theory in the first scenario as stated in Theorem 1.3, which we prove first. We then establish some of the partial results in the second scenario.

1.7.1. Proof of Theorem 1.3

Proof of Theorem 1.3. This theorem is proved in two cases: $\mathbb{P}\left(\gamma_* = 0\right) > 0$ and $\mathbb{P}\left(\gamma_* = 0\right) = 0$.

Case 1: $\mathbb{P}\left(\gamma_* = 0\right) > 0$. Recall (1.17). Note that, since $\gamma_* = 0$ implies $\gamma(t) = 0$ for all t,

$$\mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\,dr\right)\right] \geqslant \mathbb{P}(\gamma_* = 0).$$

As a result, from (1.17), for all $t \geqslant 0$,

$$\int_0^t \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\,ds \leqslant \frac{1}{\mathbb{P}(\gamma_* = 0)}. \tag{1.64}$$

Consequently,

$$\int_0^{+\infty} \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\,ds < +\infty. \tag{1.65}$$

Since $\lambda(t) \leqslant \lambda_*$ a.s. and $\lambda(t) = 0$ for all $t \geqslant \eta$, from Remark 1.6 and the dominated convergence theorem $\overline{\lambda}_0(t) \to 0$, $\overline{\lambda}(t) \to 0$ as $t \to +\infty$. By the dominated convergence theorem applied to (1.16), using (1.65), we obtain

$$\lim_{t\to+\infty} \overline{\mathfrak{F}}(t) = \lim_{t\to+\infty} \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\,ds = 0.$$

Next by the dominated convergence theorem applied to (1.15) using (1.65) again, $\overline{\mathfrak{S}}_* = \lim_{t\to+\infty} \overline{\mathfrak{S}}(t)$ as given in (1.23). This concludes the proof of the first case of Theorem 1.4.

Case 2: $\mathbb{P}\left(\gamma_* = 0\right) = 0$. We first note that

$$\int_0^{+\infty} \overline{\mathfrak{F}}(u)\,du < +\infty \Leftrightarrow \int_0^{+\infty} \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\,ds < +\infty. \tag{1.66}$$

Indeed, from (1.16), and from Fubuni's theorem

$$\int_0^{+\infty} \overline{\mathfrak{F}}(u)\,du = \overline{I}(0)\int_0^{+\infty} \overline{\lambda}_0(u)\,du + R_0\int_0^{+\infty} \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\,ds. \tag{1.67}$$

Next from Assumption 1.3, we have

$$\int_0^{+\infty} \overline{\lambda}_0(u)\,du \leqslant R_0 < +\infty. \tag{1.68}$$

This combined with (1.67) implies (1.66).

Thus from the proof of Case 1 and (1.66), it suffices to show that $\int_0^{+\infty} \overline{\mathfrak{F}}(s)\,ds < +\infty$. We prove this claim by contradiction. Suppose that

$$\int_0^{+\infty} \overline{\mathfrak{F}}(u)\,du = +\infty. \tag{1.69}$$

By (1.16), using Fubuni's theorem, we obtain

$$\begin{aligned}\int_0^t \overline{\mathfrak{F}}(u)du &= \overline{I}(0)\int_0^t \overline{\lambda}_0(u)du + \int_0^t \left(\int_0^{t-u} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du \\ &= \overline{I}(0)\int_0^t \overline{\lambda}_0(u)du + R_0\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du - \int_0^t \left(\int_{t-u}^{+\infty} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du.\end{aligned}$$

Consequently,

$$\frac{\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} = \frac{1}{R_0} + \frac{\int_0^t \left(\int_{t-u}^{+\infty} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{R_0\int_0^t \overline{\mathfrak{F}}(u)du} - \frac{\overline{I}(0)\int_0^t \overline{\lambda}_0(u)du}{R_0\int_0^t \overline{\mathfrak{F}}(u)du}. \tag{1.70}$$

Thus by (1.68) and (1.69), we have

$$\frac{\int_0^t \overline{\lambda}_0(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \xrightarrow[t\to+\infty]{} 0. \tag{1.71}$$

In addition, since $\int_t^{+\infty} \overline{\lambda}(s)ds \to 0$ as $t \to +\infty$, for $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $\int_{T_\epsilon}^{+\infty} \overline{\lambda}(s)ds < \epsilon$. Hence, for $t \geqslant T_\epsilon$,

$$\begin{aligned}&\int_0^t \left(\int_{t-u}^{+\infty} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du \\ &= \int_0^t \left(\int_{u}^{+\infty} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(t-u)\overline{\mathfrak{F}}(t-u)du \\ &\leqslant \int_0^{T_\epsilon} \left(\int_{u}^{+\infty} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(t-u)\overline{\mathfrak{F}}(t-u)du + \epsilon\int_{T_\epsilon}^t \overline{\mathfrak{S}}(t-u)\overline{\mathfrak{F}}(t-u)du \\ &\leqslant R_0\lambda_* T_\epsilon + \epsilon\int_0^t \overline{\mathfrak{F}}(u)du.\end{aligned}$$

Thus by (1.69), we have

$$\frac{\int_0^t \left(\int_{t-u}^{+\infty} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \xrightarrow[t\to+\infty]{} 0. \tag{1.72}$$

Hence under the assumption (1.69), from (1.70), (1.71) and (1.72), we obtain

$$\frac{\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \xrightarrow[t\to+\infty]{} \frac{1}{R_0}. \tag{1.73}$$

On the other hand, from (1.17) and the fact that $\gamma \leqslant \gamma_*$ we have

$$\int_0^u \mathbb{E}\left[\exp\left(-\gamma_*\int_s^u \overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < 1. \tag{1.74}$$

Next, multiplying by $\overline{\mathfrak{F}}(u)$ and integrating from 0 to t both sides of (1.74), we have

$$\int_0^t \left(\int_0^u \mathbb{E}\left[\overline{\mathfrak{F}}(u) \exp\left(-\gamma_* \int_s^u \overline{\mathfrak{F}}(r) dr \right) \right] \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds \right) du < \int_0^t \overline{\mathfrak{F}}(u) du,$$

and by Fubuni's theorem,

$$\int_0^t \mathbb{E}\left[\int_s^t \overline{\mathfrak{F}}(u) \exp\left(-\gamma_* \int_s^u \overline{\mathfrak{F}}(r) dr \right) du \right] \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds < \int_0^t \overline{\mathfrak{F}}(u) du,$$

from which we obtain

$$\int_0^t \mathbb{E}\left[\frac{1 - \exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r) dr \right)}{\gamma_*} \right] \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds < \int_0^t \overline{\mathfrak{F}}(u) du.$$

Hence for $0 < \epsilon \leqslant 1$, since $\mathbb{1}_{\gamma_* \geqslant \epsilon} \leqslant 1$,

$$\int_0^t \mathbb{E}\left[\frac{1 - \exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r) dr \right)}{\gamma_*} \mathbb{1}_{\gamma_* \geqslant \epsilon} \right] \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds < \int_0^t \overline{\mathfrak{F}}(u) du.$$

Thus,

$$\mathbb{E}\left[\frac{1}{\gamma_*} \mathbb{1}_{\gamma_* \geqslant \epsilon} \right] \int_0^t \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds < \int_0^t \overline{\mathfrak{F}}(u) du + \int_0^t \mathbb{E}\left[\frac{1}{\gamma_*} \mathbb{1}_{\gamma_* \geqslant \epsilon} \exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r) dr \right) \right] \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds. \tag{1.75}$$

Moreover, from (1.74),

$$\int_0^t \mathbb{E}\left[\frac{1}{\gamma_*} \mathbb{1}_{\gamma_* \geqslant \epsilon} \exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r) dr \right) \right] \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds \leqslant \frac{1}{\epsilon}.$$

Consequently, under the assumption (1.69), we have

$$\frac{\int_0^t \mathbb{E}\left[\frac{1}{\gamma_*} \mathbb{1}_{\gamma_* \geqslant \epsilon} \exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r) dr \right) \right] \overline{\mathfrak{S}}(s) \overline{\mathfrak{F}}(s) ds}{\int_0^t \overline{\mathfrak{F}}(u) du} \xrightarrow[t \to +\infty]{} 0.$$

This implies that, by (1.75), for all $0 < \epsilon \leqslant 1$,

$$\limsup_{t \to +\infty} \frac{\int_0^t \overline{\mathfrak{S}}(u) \overline{\mathfrak{F}}(u) du}{\int_0^t \overline{\mathfrak{F}}(u) du} \leqslant \left(\mathbb{E}\left[\frac{1}{\gamma_*} \mathbb{1}_{\gamma_* \geqslant \epsilon} \right] \right)^{-1}.$$

Since $\mathbb{P}\left(\gamma_* = 0\right) = 0$, we deduce by the monotone convergence theorem that

$$\limsup_{t \to +\infty} \frac{\int_0^t \overline{\mathfrak{S}}(u) \overline{\mathfrak{F}}(u) du}{\int_0^t \overline{\mathfrak{F}}(u) du} \leqslant \left(\mathbb{E}\left[\frac{1}{\gamma_*} \right] \right)^{-1},$$

However, this contradicts (1.73) since $R_0 < \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ by the assumption of Theorem 1.3.

This completes the proof of the second case. □

1.7.2. Proofs in the case $R_0 \geqslant \mathbb{E}\left[\frac{1}{\gamma_*}\right]$

In this subsection, we prove Theorem 1.4.

Proof of Theorem 1.4 (i). As $\overline{\mathfrak{S}}(t) \to \overline{\mathfrak{S}}_*$ and $\overline{\mathfrak{F}}(t) \to \overline{\mathfrak{F}}_*$ as $t \to \infty, R_0 < +\infty$ and $\overline{\mathfrak{F}}(t)\overline{\mathfrak{S}}(t) \leqslant \lambda_*$, by the dominated convergence theorem, we have

$$\lim_{t\to+\infty}\int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds = \lim_{t\to+\infty}\int_0^{+\infty} \overline{\lambda}(s)\overline{\mathfrak{S}}(t-s)\overline{\mathfrak{F}}(t-s)\mathbb{1}_{[0,t]}(s)ds = \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_*R_0. \tag{1.76}$$

Thus by (1.76) and (1.16), using the fact that $\overline{\lambda}_0(t) \to 0$ as $t \to \infty$,

$$\overline{\mathfrak{F}}_* = \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_*R_0.$$

As a result, either $\overline{\mathfrak{F}}_* = 0$ or else $\overline{\mathfrak{S}}_* = \frac{1}{R_0}$.

In the following, we assume that $\overline{\mathfrak{F}}_* > 0$, then $\overline{\mathfrak{S}}_* = \frac{1}{R_0}$.

Recall (1.17). Since $\mathbb{E}\left[\frac{1}{\gamma_*}\right] \leqslant R_0 < +\infty$ and $\mathbb{P}(\gamma_* = 0) = 0, \gamma(t+\xi)\overline{\mathfrak{F}}(t) \to \gamma_*\overline{\mathfrak{F}}_* > 0$ with probability one when $t \to +\infty$ and hence

$$\lim_{t\to+\infty}\int_0^t \gamma(r+\xi)\overline{\mathfrak{F}}(r)dr = +\infty \text{ and } \lim_{t\to+\infty}\int_0^t \overline{\mathfrak{F}}(r)dr = +\infty \text{ almost surely.}$$

It follows that

$$\lim_{t\to+\infty}\mathbb{E}\left[\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] = 0.$$

Hence from (1.17) we have

$$\lim_{t\to+\infty}\int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds = 1. \tag{1.77}$$

Since $\overline{\mathfrak{F}}(t) \to \overline{\mathfrak{F}}_*$ when $t \to +\infty$, there exists $t_0 > 0$ such that for all $t \geqslant t_0, \overline{\mathfrak{F}}(t) \geqslant \frac{\overline{\mathfrak{F}}_*}{2}$. Then,

$$\begin{aligned}\int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds \\ = \int_0^{t_0} \mathbb{E}\left[\exp\left(-\int_0^{t-s} \gamma(r)\overline{\mathfrak{F}}(s+r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds \\ + \int_0^{t-t_0} \mathbb{E}\left[\exp\left(-\int_0^{s} \gamma(r)\overline{\mathfrak{F}}(t+r-s)dr\right)\right]\overline{\mathfrak{S}}(t-s)\overline{\mathfrak{F}}(t-s)ds.\end{aligned} \tag{1.78}$$

The first term on the right hand side converges to zero as $t \to \infty$ by the dominated

convergence theorem. On the other hand, for $0 < s < t - t_0$,

$$\exp\left(-\int_0^s \gamma(r)\overline{\mathfrak{F}}(r+t-s)dr\right) \leqslant \exp\left(-\frac{\overline{\mathfrak{F}}_*}{2}\int_0^s \gamma(r)dr\right), \tag{1.79}$$

and by Assumption 1.4, we deduce that

$$\begin{aligned}\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\frac{\overline{\mathfrak{F}}_*}{2}\int_0^s \gamma(r)dr\right)\right] ds &\leqslant \mathbb{E}[t_*] + \mathbb{E}\left[\int_0^{+\infty} \exp\left(-\frac{\overline{\mathfrak{F}}_*\gamma_*}{4}s\right) ds\right] \\ &= \mathbb{E}[t_*] + \frac{4}{\overline{\mathfrak{F}}_*}\mathbb{E}\left[\frac{1}{\gamma_*}\right] < +\infty \end{aligned} \tag{1.80}$$

Thus, applying the dominated convergence theorem to the second term on the right-hand-side of (1.78) and using (1.77), we obtain

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\overline{\mathfrak{F}}_*\int_0^s \gamma(r)dr\right)\right] \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_* ds = 1. \tag{1.81}$$

Next by a change of variables in (5.5) and the fact that $\overline{\mathfrak{S}}_* = \frac{1}{R_0}$, we obtain (1.25).

To conclude, Lemma 1.10 below implies that the equation $H(x) = R_0$ has a unique positive solution if and only if $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$, which yields the result.

On the other hand, as $F_0^c(t) \to 0$ as $t \to +\infty$ and $(\overline{\mathfrak{S}}(t), \overline{\mathfrak{F}}(t)) \to (\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$ as $t \to +\infty$, applying the dominated convergence theorem to (1.14) we obtain

$$\overline{I}_* = \lim_{t\to+\infty} \int_0^t F^c(t-s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds = \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_*\mathbb{E}[\eta] = \frac{\overline{\mathfrak{F}}_*}{R_0}\mathbb{E}[\eta].$$

This proves the claims of Theorem 1.4 (i). □

Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ as follows,

$$\text{for } x > 0,\ H(x) := \int_0^{+\infty} \mathbb{E}\left[\exp\left(-\int_0^s \gamma\left(\frac{r}{x}\right)dr\right)\right] ds; \quad H(0) = \mathbb{E}\left[\frac{1}{\gamma_*}\right].$$

Lemma 1.10. *If γ is nondecreasing and $\mathbb{E}\left[\frac{1}{\gamma_*}\right] < +\infty$, then the function $H : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and strictly increasing. Moreover,*

$$H(0) = \mathbb{E}\left[\frac{1}{\gamma_*}\right], \quad \textit{and} \quad \lim_{x\to+\infty} H(x) = +\infty.$$

Proof. Note that $H(x)$ is also given by the following equivalent formula

$$H(x) = x\int_0^{\infty} \mathbb{E}\left[\exp\left(-x\int_0^s \gamma(r)dr\right)\right] ds.$$

This formula and monotone convergence imply the continuity of H. The fact that H is

nondecreasing follows readily from the first formula and the nondecreasing property of γ. Moreover we have

$$H(0) = \mathbb{E}\left[\frac{1}{\gamma_*}\right], \quad \text{and} \quad \lim_{x\to+\infty} H(x) = +\infty.$$

To prove that H is strictly increasing, assume that $H(x) = H(y)$ for some $y > x$. Then, for a.e. r, $\gamma(r/x) = \gamma(r/y)$ almost surely, which implies that γ is constant.

Moreover, from the second formula for H, it follows readily that $x \mapsto H(x)/x$ is non-increasing. From the continuity at 0, we deduce that there exists $x > 0$ such that $H(x) < \infty$. Now for any $y > x$, $H(y) \le y\frac{H(x)}{x} < \infty$. Hence H takes values in $\mathbb{R}_+$. The rest of the statement is easy to verify. □

Proof of Theorem 1.4 (ii). The goal of this Lemma is to prove that, if γ_* is deterministic, $R_0 > \frac{1}{\gamma_*}$, for all $\delta > 0$ there exists t_δ such that $\gamma(t_\delta) \geqslant (1-\delta)\gamma_*$ and if there exists a positive decreasing function h such that for all $0 \leqslant s, t$, $\overline{\lambda}(t+s) \geqslant h(t)\overline{\lambda}(s)$, and $\overline{\mathfrak{F}}(0) > 0$, then there exists $c > 0$ such that for all $t \geqslant 0$, $\overline{\mathfrak{F}}(t) \geqslant c$.

Let $\delta > 0$ be such that $(1-\delta)\gamma_* > \frac{1}{R_0}$. From Assumption 1.5, there exists $s_1 \geqslant 0$ deterministic such that $\gamma_0(s_1) \wedge \gamma(s_1) \geqslant (1-\delta)\gamma_*$ a.s. Let $\epsilon > 0$ be such that $(1-\delta)\gamma_* > \frac{1+\epsilon}{R_0}$.

Let x be the solution of the following Volterra equation

$$x(t) = h(s_1+t) + (1+\epsilon)\int_0^t p(t-s)x(s)ds, \; t \geqslant 0, \tag{1.82}$$

with

$$p(t) = \frac{\overline{\lambda}(t)}{R_0}.$$

As

$$(1+\epsilon)\int_0^{+\infty} p(t)dt = 1+\epsilon > 1, \qquad \int_0^{+\infty} h(t)dt < +\infty$$

where the integrability of h results from Assumption 1.6 and the integrability of $\overline{\lambda}$. Moreover, since h and p are bounded and non-negative, by [37, Remark following Theorem 4, page 253], $x(t) \to +\infty$ as $t \to +\infty$, hence there exists $s_2 \geqslant 0$ such that $x(s_2) > 2$.

Let $c_2 > 0$ be such that

$$(1-\delta)\gamma_* \exp(-c_2(s_1+s_2)) \geqslant \frac{1+\epsilon}{R_0}. \tag{1.83}$$

Let

$$c_1 = \frac{1}{2}\min(c_2, \overline{\mathfrak{F}}(0)), \quad \text{and} \quad c_0 = \frac{c_1}{2}h(s_1+s_2),$$

$$t_0 = \inf\{t \geqslant 0, \overline{\mathfrak{F}}(t) \leqslant c_0\}, \quad \text{and} \quad t_1 = \sup\{t \leqslant t_0, \overline{\mathfrak{F}}(t) \geqslant c_1\}.$$

Since h is decreasing and $h(0) = 1$ we have $c_0 < c_1$. We want to show that $t_0 = \infty$, and will prove it by contradiction. Let us thus suppose that $t_0 < +\infty$, which implies that $t_1 < +\infty$. From Assumption 1.6, by the continuity of $\overline{\mathfrak{F}}$ and the definition of t_1, for all $t \geqslant t_1$, we obtain

$$\begin{aligned}
\overline{\mathfrak{F}}(t) &= \overline{\lambda}_0(t)\overline{I}(0) + \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds \\
&\geqslant \overline{\lambda}_0(t-t_1+t_1)\overline{I}(0) + \int_0^{t_1} \overline{\lambda}(t-t_1+t_1-s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds \\
&\geqslant h(t-t_1)\left(\overline{\lambda}_0(t_1)\overline{I}(0) + \int_0^{t_1} \overline{\lambda}(t_1-s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds\right) \\
&= h(t-t_1)\overline{\mathfrak{F}}(t_1) \\
&\geqslant c_1 h(t-t_1).
\end{aligned}$$

The definition of t_0 and the continuity of $\overline{\mathfrak{F}}$, implies that $\overline{\mathfrak{F}}(t_0) \leqslant c_0$. Combining with the last inequality evaluated at $t = t_0$, we have $c_0 \geqslant c_1 h(t_0 - t_1)$. Hence, by the definition of c_0 and the fact that h is decreasing, we deduce that, $t_0 - t_1 > s_1 + s_2$. So $t_0 > t_1 + s_1 + s_2$ and for all $t \in [t_1, t_0]$

$$\overline{\mathfrak{F}}(t) \leqslant c_1 < c_2. \tag{1.84}$$

On the other hand, as $\gamma_0(t) \leqslant 1$ and $\gamma(t) \leqslant 1$, we have, for all $t \geqslant t_1$,

$$\begin{aligned}
\overline{\mathfrak{S}}(t) &= \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\
&\qquad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds \\
&\geqslant \exp\left(-\int_{t_1}^t \overline{\mathfrak{F}}(r)dr\right)\left(\mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^{t_1} \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\right. \\
&\qquad \left. + \int_0^{t_1} \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^{t_1} \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds\right).
\end{aligned}$$

But, as for $t \in [t_1+s_1, t_1+s_1+s_2]$ and $s \in [0, t_1]$, $\gamma_0(t) \wedge \gamma(t-s) \geqslant \gamma_0(s_1) \wedge \gamma(s_1) \geqslant (1-\delta)\gamma_*$, and using (1.17) at time t_1 we deduce that, for all $t \in [t_1 + s_1, t_1 + s_1 + s_2]$,

$$\overline{\mathfrak{S}}(t) \geqslant (1-\delta)\gamma_* \exp\left(-\int_{t_1}^t \overline{\mathfrak{F}}(r)dr\right).$$

Moreover, since from (1.84) for $t \in [t_1 + s_1, t_1 + s_1 + s_2]$, $\overline{\mathfrak{F}}(t) \leqslant c_2$,

$$\overline{\mathfrak{S}}(t) \geqslant (1-\delta)\gamma_* \exp(-c_2(s_2 + s_1)).$$

Then from (1.83)

$$\forall t \in [t_1 + s_1, t_1 + s_1 + s_2], \quad \overline{\mathfrak{S}}(t) \geqslant \frac{1+\epsilon}{R_0}. \tag{1.85}$$

Let $y(t) = \overline{\mathfrak{F}}(t + t_1 + s_1)$ and define g as follows:

$$g(t) = \overline{I}(0)\overline{\lambda}_0(t_1 + s_1 + t) + \int_0^{t_1+s_1} \overline{\lambda}(t_1 + s_1 + t - s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds,$$

where we recall that

$$p(t) = \frac{\overline{\lambda}(t)}{R_0}.$$

Then using (1.85) for any $t \geqslant 0$,

$$\begin{aligned} y(t) &\geqslant g(t) + (1+\epsilon)\int_{t_1+s_1}^{t_1+s_1+t} p(t_1 + s_1 + t - s)y(s - t_1 - s_1)ds \\ &= g(t) + (1+\epsilon)\int_0^t p(t-s)y(s)ds. \end{aligned}$$

However, from Assumption 1.6 we deduce that

$$g(t) \geqslant \overline{\mathfrak{F}}(t_1)h(s_1 + t)$$

and as $\overline{\mathfrak{F}}(t_1) = c_1$ by continuity, we deduce that

$$y(t) \geqslant c_1 h(s_1 + t) + (1+\epsilon)\int_0^t p(t-s)y(s)ds. \tag{1.86}$$

Thus by Theorem 1.2.19 in [18] applied to (1.86), we have

$$y(t) \geqslant c_1 x(t)$$

where x is given by (1.82). However $x(s_2) > 2$. Hence $\overline{\mathfrak{F}}(t_1 + s_1 + s_2) > 2c_1 > c_1$ and $t_0 \geqslant t_1 + s_1 + s_2$, this contradicts the definition of t_1. Hence $t_0 = +\infty$. This concludes the proof. □

2. Functional central limit theorems for epidemic models with varying infectivity and waning immunity

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2.1. Introduction

Many infectious diseases become endemic over a long time horizon, for which waning of immunity plays a critical role in addition to the infection process. The classical compartment model, SIRS (susceptible-infectious-recovered-susceptible), assumes that immunity at the individual level is binary, that is, each individual is either fully immune or fully susceptible. However, that is largely unrealistic since it does not allow for partial immunity or gradual waning of immunity. Various models have been developed to study the effects of the waning of immunity and the associated vaccination policies [51, 52, 95, 5, 33, 20, 91, 24, 88, 64, 43, 34]. All the models except [20, 43] start from an ODE model with the additional waning immunity characteristic. In particular, El Khalifi and Britton [64] recently studied an extension of the ODE for the classical SIRS model with a linear or exponential waning function. They started with an approximation using a fixed number of immunity levels and then discussed the corresponding ODE-PDE limiting model (similar to [95]) associated with the age of immunity as the number of immunity level goes to infinity. See also [34] for a perturbation analysis of a model with an arbitrarily large number of discrete compartments with varying levels of disease immunity. Carlsson et al. [20] study an age-structured PDE model that takes into account waning immunity. Despite the interesting findings, there is a lack of individual-based stochastic epidemic models that take into account waning immunity.

In chapter 1 we introduced an individual-based stochastic epidemic model that captures waning immunity as well as varying infectivity [39]. More precisely, we proposed a general stochastic epidemic model which takes into account a random infectivity and a random and gradual loss of immunity (also referred to as waning immunity or varying susceptibility). See Figure 2.1 for a realization of the infectivity and susceptibility of an individual after an infection. Individuals experience the susceptible-infected-immune-susceptible cycle. When an individual becomes infected, the infected period may include a latent exposed period and then an infectious period. Once an individual recovers from the infection, after some potential immune period (whose duration can be zero), the immunity is gradually lost, and the individual progressively becomes susceptible again. Then the individual may be infected again, and repeat the process at each new infection with a different realization of the random infectivity and susceptibility functions. This model can be regarded as a generalized SIS model. When “I" is interpreted as “infected" including exposed and infectious periods, and “S" is interpreted as including immune and susceptible periods. It can of course also be regarded as a generalized SEIRS model. We also mention the recent work [43], where a similar stochastic model of varying infectivity and waning immunity with vaccination is studied, where the focus is on the effect of vaccination policies

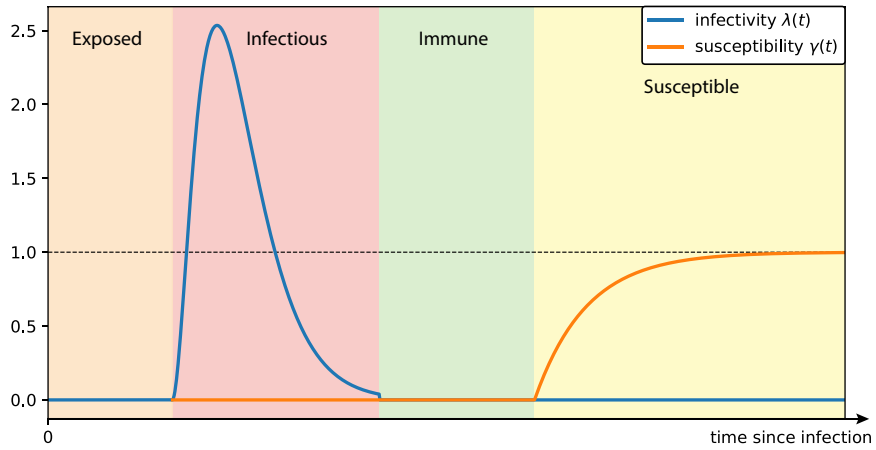

Figure 2.1. – Illustration of a typical realization of the random infectivity and susceptibility functions of an individual from the time of infection to the time of recovery, and then to the time of losing immunity and becoming fully susceptible (or in general, partially susceptible).

to prevent endemicity. We notice the difference from our modeling approach besides the vaccination aspect: the random susceptibility function and varying infectivity function are taken independently in each infection. However, we do not impose the independence between the random infectivity and susceptibility functions in each infection.

In chapter 1, we have proved a functional law of large numbers (FLLN) in which both the average susceptibility and the force of infection converge, when the size of the population goes to infinity, to a deterministic limiting model given by a system of integral equations depending on the law of the susceptibility and on the mean of the infectivity function (see Theorem 1.2 below). Under a particular set of random infectivity and susceptibility functions and initial conditions, we also show that a PDE model with infection-age can be derived from the limiting model, which reduces to the model introduced by Kermack and McKendrick in [62, 63] (see the reformulation in [53]). We also characterize the threshold of endemicity which depends on the law of susceptibility and not only on the mean, and prove the global asymptotical stability of the disease-free steady state when the basic reproduction number is lower than the above-mentioned threshold. When the basic reproduction number is larger than this threshold, we prove existence and uniqueness of the endemic equilibrium and under additional assumptions, we prove that the disease-free-steady state is unstable.

The goal of this work is to study the stochastic fluctuations of the dynamics around the deterministic limits for the stochastic epidemic model with random varying infectivity and a random and gradual loss of immunity, see the result in Theorem 2.1. More precisely we study jointly the fluctuation of the average total force of infection and average susceptibility and then deduce the fluctuations of the proportions of the compartment counting processes. The limiting fluctuations of the average to-

tal force of infection and average susceptibility are given by a system of stochastic non-linear integral equations driven by a two-dimensional Gaussian process. Given these, the limits of the compartment counting processes are expressed in terms of the solutions of the above non-linear stochastic integral equation driven by another two-dimensional Gaussian process. This result extends the functional central limit theorem (FCLT) results of Pang-Pardoux in [77, 78] for the non-Markovian models without gradual loss of immunity and in [15, 66] for the Markovian case.

To prove the FCLT (Theorem 2.1), we first obtain a decomposition for the scaled infectivity and susceptibility processes, each of which has two component processes. We employ the central limit theorems for **D**-valued random variables [46] to prove the convergence of one component since it can be regarded as a sum of i.i.d. **D**-valued random variables. The convergence of the other component is much more challenging, and we must develop novel methods to prove tightness and convergence. We need more assumptions on the pair of random function (λ, γ) than the ones used to establish the FLLN, and these assumptions are crucial to establish tightness. For that purpose, we need to establish moment estimates and maximal inequalities for the increments of the processes. This is extremely difficult because of the complicated interactions among the individuals, as well as the randomness in the infectivity and susceptibility. Some of the expressions involve stochastic integrals with respect to Poisson random measures, with integrands which are not predictable but depend on the future. The classical result for moment calculations of stochastic integrals cannot be used in our setting, for example, [25, Theorem 6.2]. Thus, we establish a new theorem to calculate the moments for such stochastic integrals (see Theorem 2.3).

In addition, we develop an approximation technique by introducing a quarantine model, in which one infected individual is quarantined so that the number of infected descendants of that individual can be bounded conveniently (this scheme can be extended to more than one quarantined individual). Using this approximation, we compare the processes counting the number of infections of each individual for the original process to the number of infections of each individual for the quarantine model, and as a consequence, we obtain the moment estimates and maximal inequalities to prove tightness (see Lemmas 2.30 and 2.31).

Finally it is worth noting that our individual-based stochastic model resembles the recent studies of models with interactions, for instance, interacting age-dependent Hawkes process in [23, 22], age-structured population model in [97] and stochastic excitable membrane models in [85]. In the proof of the FLLN in chapter 1, we adapted the tools of the theory of propagation of chaos (see Sznitman [92]) by constructing a family of i.i.d. processes with a well-chosen coupling. A similar approach was taken in [23, 97]. However, for the FCLT, we depart from the approach of studying fluctuations from the mean limit that were taken in [23, 97], since it is more challenging for our non-Markovian model. In that approach one has to work with processes taking values in a Hilbert space (dual of some Sobolev space of test functions) and the limit is characterized by an SDE in infinite dimension driven by a Gaussian noise. On the contrary, we work directly with the real-valued processes and prove their convergence with the conventional tightness criteria, which leads to a finite-dimensional stochastic

integral equation driven by Gaussian processes.

Organization of the chapter

The rest of the chapter is organized as follows. In Section 2.2, we describe the model and recall the FLLN results from chapter 1. Next, we state the assumptions and the FCLT result. The proof for the FCLT is presented in Section 2.3. In Section 2.4, we present some preliminary results that will be used in the proofs. In Section 2.5 we characterize the limit of the convergent subsequences. In Section 2.6 we approximate the limit, and finally we prove tightness in Section 2.7.

Notation

Throughout the chapter, all the random variables and processes are defined on a common complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We use $\xrightarrow[N\to+\infty]{\mathbb{P}}$ to denote convergence in probability as the parameter $N \to \infty$. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}^k(\mathbb{R}_+^k)$ the space of k-dimensional vectors with real (nonnegative) coordinates, with $\mathbb{R}(\mathbb{R}_+)$ for $k = 1$. We use $\mathbb{1}\{\cdot\}$ for the indicator function. Let $\mathbf{D} = \mathbf{D}(\mathbb{R}_+;\mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg functions defined on $\mathbb{R}_+$, with convergence in $\mathbf{D}$ meaning convergence in the Skorohod J_1 topology (see, e.g., [11, Chapter 3]). Also, we use $\mathbf{D}^k$ to denote the k-fold product with the product J_1 topology. Let $\mathbf{C}$ be the subset of $\mathbf{D}$ consisting of continuous functions and $\mathbf{D}_+$ the subset of $\mathbf{D}$ of càdlàg functions with values in $\mathbb{R}_+$. We use $\Rightarrow$ to denote the weak convergence in $\mathbf{D}$.

2.2. Model and Results

2.2.1. Model description

We start with a population with a fixed finite size N, and enumerate the individuals of the population with the parameter k, $1 \leqslant k \leqslant N$.

Let $(\lambda_{k,i}, \gamma_{k,i})_{k\geqslant 1, i\geqslant 1}$ be a collection of i.i.d. random functions and also, $(\lambda_{k,0}, \gamma_{k,0})_{k\geqslant 1}$ be a collection of i.i.d. random functions taking values in the same space, independent from the previous one. Let $(Q_k)_{k\geqslant 1}$ be a family of independent standard Poisson random measures on $\mathbb{R}_+^2$, independent from the two previously defined families. $\lambda_{k,i}$ represents the infectivity of the k-th individual after its i-th infection and $\gamma_{k,i}$ represents the susceptibility of the k-th individual after its i-th infection. Similarly, $\lambda_{k,0}$ (resp. $\gamma_{k,0}$) represents the infectivity (resp. susceptibility) of the k-th individual in the beginning of the epidemic.

We assume that each infected individual has infectious contacts at a rate equal to its current infectivity. At each infectious contact, an individual is chosen uniformly in the population and this individual becomes infected with probability given by its susceptibility. Thus if we let $A_k^N(t)$ be the number of times that the k-th individual has

been infected between time 0 and t, then the infectivity of the k-th individual at time t is given by $\lambda_{k,A_k^N(t)}(\varsigma_k^N(t))$ and its susceptibility is $\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))$ where

$$\varsigma_k^N(t) := t - \left(\sup\{s \in [0,t] : A_k^N(s) = A_k^N(s^-) + 1\} \vee 0\right) \tag{2.1}$$

is the time elapsed since the last time when it was infected or since the start of the epidemic if it has not been infected yet (we use the convention $\sup \emptyset = -\infty$).

Hence, let $\{A_k^N(t),\, t \geq 0, 1 \leqslant k \leqslant N\}$ be the solution of

$$A_k^N(t) = \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{u \leqslant \Upsilon_k^N(r^-)} Q_k(dr, du)$$

where

$$\Upsilon_k^N(t) = \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\overline{\mathfrak{F}}^N(t)$$

is the instantaneous infectivity rate function at time t with

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^N \lambda_{k,A_k^N(t)}(\varsigma_k^N(t))\,. \tag{2.2}$$

The total force of infection $\mathfrak{F}^N(t)$ at time t is the sum of the infectivities of all the infected individuals at time t.

We also define the average susceptibility of the population by

$$\overline{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^N \gamma_{k,A_k^N(t)}(\varsigma_k^N(t)). \tag{2.3}$$

Let

$$\Upsilon^N(t) = \sum_{k=1}^N \Upsilon_k^N(t) = N\overline{\mathfrak{S}}^N(t)\overline{\mathfrak{F}}^N(t)$$

be the total instantaneous infection rate function in the population at time t.

Define

$$\eta_{k,i} = \sup\{t > 0,\, \lambda_{k,i}(t) > 0\}$$

for each $i \in \mathbb{N}_0$ and $k = 1, \ldots, N$, representing the duration of the i-th infection of the k-th individual. By the i.i.d. assumption on $(\lambda_{k,i})_{k\geq 1, i\geq 1}$, the variables $(\eta_{k,i})_{k\geq 1, i\geq 1}$ are i.i.d., similarly for $(\eta_{k,0})_{k\geq 1}$. Also, the two families of random variables are independent. We denote their cumulative distribution functions by

$$F_0(t) = \mathbb{P}\left(\eta_{1,0} \leq t\right), \qquad F(t) = \mathbb{P}\left(\eta_{1,1} \leq t\right), \quad t \geq 0.$$

Let $F_0^c = 1 - F_0(t)$ and $F^c(t) = 1 - F(t)$ for $t \geq 0$.

We define the number of infectious individuals at time t by

$$I^N(t) = \sum_{k=1}^N \mathbb{1}_{\varsigma_k^N(t) < \eta_{k,A_k^N(t)}}\,, \tag{2.4}$$

and the number of uninfected individuals at time t by

$$U^N(t) = \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) \geqslant \eta_{k,A_k^N(t)}} = N - I^N(t). \tag{2.5}$$

From Lemma 6.1 in chapter 1, we recall that there exists a unique $\overline{\mathfrak{F}} \in \mathbf{D}(\mathbb{R}_+)$ such that

$$\overline{\mathfrak{F}}(t) = \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right] \text{ and } \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_{1,A_1(t)}(\varsigma_1(t))\right],$$

where for $k \geqslant 1$ the process A_k is defined as:

$$A_k(t) = \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{u \leqslant \Upsilon_k(r^-)} Q_k(dr, du),$$

with

$$\Upsilon_k(t) = \gamma_{k,A_k(t)}(\varsigma_k(t))\overline{\mathfrak{F}}(t),$$

and ς_k is defined in the same manner as ς_1^N with A_k instead of A_1^N, see (2.1). In this definition we use the same $(\lambda_{k,i}, \gamma_{k,i}, Q_k)$ as in the definition of the model in subsection 2.2.1. Moreover, note that, as the $\big((\lambda_{k,i})_i, (\gamma_{k,i})_i, Q_k\big)_{k\geqslant 1}$ are i.i.d, the $(A_k)_k$ are also i.i.d.

2.2.2. Main Results

The purpose of this section is to establish a FCLT for the fluctuations of the stochastic sequence around its deterministic limit. More precisely, we define the following fluctuation processes: for all $t \geqslant 0$,

$$\hat{\mathfrak{F}}^N(t) := \sqrt{N}\big(\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\big), \text{ and } \hat{\mathfrak{S}}^N(t) := \sqrt{N}\big(\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}(t)\big), \tag{2.6}$$

and we want to find the limiting law of the pair $(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N)$.

2.2.2.1. Assumptions

We introduce the following Assumptions.

Assumption 2.1. *There exists a deterministic constant $\lambda_* < \infty$ such that $0 \leqslant \lambda_{k,i}(t) \leqslant \lambda_*$ almost surely, and $0 \leqslant \gamma_{k,i}(t) \leqslant 1$ for all $t \geqslant 0$, for all $i \geqslant 0$ and $1 \leqslant k \leqslant N$, almost surely. Moreover,*

$$\sup\{t \geqslant 0, \lambda_{k,i}(t) > 0\} \leqslant \inf\{t \geqslant 0, \gamma_{k,i}(t) > 0\}, \tag{2.7}$$

almost surely for all $1 \leqslant k \leqslant N$ and $i \geqslant 0$.

Let us define

$$\overline{\lambda}_0(t) = \mathbb{E}\big[\lambda_{1,0}(t)\big|\eta_{1,0} > 0\big], \quad \textit{and} \quad \overline{\lambda}(t) = \mathbb{E}\left[\lambda_{1,1}(t)\right]$$

and let μ be the law of $\gamma_{1,1}$, which is in $\mathscr{P}(\mathbf{D})$.

For simplicity, we write γ and γ_0 as random functions with the same law as $\gamma_{1,1}$ and $\gamma_{1,0}$ respectively.

Assumption 2.2. *The random functions (λ,γ), of which $(\lambda_{k,i},\gamma_{k,i})_{k\geq 1,i\geq 1}$ are i.i.d. copies, satisfy the following properties: There exist a number $\ell \in \mathbb{N}^*$, two random sequences $0=\xi^0<\xi^1<\cdots<\xi^\ell=+\infty$ and $0=\zeta^0<\zeta^1<\cdots<\zeta^\ell=+\infty$ and random functions $\lambda^j \in \mathbf{C}$, $\gamma^j \in \mathbf{C}$, $1\leq j\leq \ell$ such that*

$$\lambda(t)=\sum_{j=1}^{\ell}\lambda^j(t)\mathbf{1}_{[\xi^{j-1},\xi^j)}(t) \quad \text{and} \quad \gamma(t)=\sum_{j=1}^{\ell}\gamma^j(t)\mathbf{1}_{[\zeta^{j-1},\zeta^j)}(t). \tag{2.8}$$

In addition, for any $T>0$, there exists deterministic nondecreasing function $\varphi_T \in \mathbf{C}$ with $\varphi_T(0)=0$ such that $|\lambda^j(t)-\lambda^j(s)|\leq \varphi_T(t-s)$ and $|\gamma^j(t)-\gamma^j(s)|\leq \varphi_T(t-s)$ almost surely, for all $0\leq t,s\leq T$, $1\leq j\leq k$.

Assumption 2.3. *There exists $\alpha>1/2$ such that for all $0\leqslant t\leqslant T$, the function φ_T from Assumption 2.2 satisfies*

$$\varphi_T(t)\leq Ct^\alpha, \tag{2.9}$$

for some constant $C>0$. Also, if F_j denotes the c.d.f. of the r.v. ξ^j, and G_j denotes the c.d.f. of the r.v. ζ^j, there exist $C'>0$, and $\rho>1/2$ such that, for any $1\leq j\leq \ell-1$, $0\leq s<t\leqslant T$,

$$F_j(t)-F_j(s)\leq C'(t-s)^\rho \quad \text{and} \quad G_j(t)-G_j(s)\leqslant C'(t-s)^\rho. \tag{2.10}$$

Assumption 2.4. *There exist non-decreasing continuous functions $\phi_1,\phi_2,\psi_1,\psi_2$ and constants $\alpha_1>\frac{1}{2}$, $\alpha_2>\frac{1}{2}$, $\beta_1>1$, $\beta_2>1$ such that for all $0\leqslant s<u<t$,*

i) $\mathbb{E}\left[(\lambda(t)-\lambda(s))^2\right]\leqslant\left(\phi_1(t)-\phi_1(s)\right)^{\alpha_1}$;

ii) $\mathbb{E}\left[\left(\gamma(t)-\gamma(s)\right)^2\right]\leqslant\left(\phi_2(t)-\phi_2(s)\right)^{\alpha_2}$;

iii) $\mathbb{E}\left[(\lambda(t)-\lambda(u))^2(\lambda(u)-\lambda(s))^2\right]\leqslant\left(\psi_1(t)-\psi_1(s)\right)^{\beta_1}$;

iv) $\mathbb{E}\left[\left(\gamma(t)-\gamma(u)\right)^2\left(\gamma(u)-\gamma(s)\right)^2\right]\leqslant\left(\psi_2(t)-\psi_2(s)\right)^{\beta_2}$.

We note that, Assumptions 2.2, 2.3 and 2.4 are not required to establish the FLLN in chapter 1. These additional assumptions are used to establish the tightness of $\hat{\mathfrak{F}}^N$ and $\hat{\mathfrak{S}}^N$, see the proof of Lemma 2.9. There are many examples of (λ,γ) that satisfy them. A typical example of pair (λ,γ) can be given by:

$$\lambda(t)=\lambda\mathbb{1}_{0\leqslant t<\eta}, \text{ and } \gamma(t)=\mathbb{1}_{t\geqslant\eta} \text{ or } \gamma(t)=\left(1-e^{-(t-\eta)}\right)\mathbb{1}_{t\geqslant\eta}, \tag{2.11}$$

where η is a random variable with density.

For more examples and discussions on $\lambda(\cdot)$ we refer to Section 2.3 in [77], and on $\gamma(\cdot)$ in [64].

2.2.2.2. Statement of the main result

Definition 2.1. *Let $(\hat{\mathfrak{J}}, \hat{\mathfrak{M}})$ be a two-dimensional centered continuous Gaussian process, with covariance functions: for $t, t' \geqslant 0$,*

$$\begin{aligned}
Cov\big(\hat{\mathfrak{J}}(t), \hat{\mathfrak{J}}(t')\big) &= Cov\big(\gamma_{1,A_1(t)}(\varsigma_1(t)), \gamma_{1,A_1(t')}(\varsigma_1(t'))\big), \\
Cov\big(\hat{\mathfrak{M}}(t), \hat{\mathfrak{M}}(t')\big) &= Cov\big(\lambda_{1,A_1(t)}(\varsigma_1(t)), \lambda_{1,A_1(t')}(\varsigma_1(t'))\big), \\
Cov\big(\hat{\mathfrak{M}}(t), \hat{\mathfrak{J}}(t')\big) &= Cov\big(\lambda_{1,A_1(t)}(\varsigma_1(t)), \gamma_{1,A_1(t')}(\varsigma_1(t'))\big).
\end{aligned}$$

Note that, thanks to Assumption 2.4, the process $(\hat{\mathfrak{J}}, \hat{\mathfrak{M}})$ is continuous by applying Kolmogorov's continuity theorem for Gaussian processes.

Remark 2.1. *In subsection 2.3.4 Lemma 2.13 we give another expression for $\big(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\big)$.*

We consider the following system of stochastic integral equations for which we have $(x, y) \in \mathbf{C}^2$:

$$\begin{cases}
x(t) = -\displaystyle\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] y(s)ds \\
\qquad -\displaystyle\int_0^t\int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right] y(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\
\qquad +\displaystyle\int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\Big(x(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)y(s)\Big)ds \\
\qquad +\hat{\mathfrak{J}}(t), \qquad (2.12) \\
\\
y(t) = \displaystyle\int_0^t \overline{\lambda}(t-s)\Big(x(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)y(s)\Big)ds + \hat{\mathfrak{M}}(t). \qquad (2.13)
\end{cases}$$

Lemma 2.5. *The set of equations (2.12)-(2.13) has a unique solution $(x, y) \in \mathbf{C}^2$.*
We denote its solution by $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}) \in \mathbf{C}^2$.

Proof. If we denote by (x^1, y^1) and (x^2, y^2) two solutions of (2.12)-(2.13), as $\overline{\mathfrak{S}} \leqslant 1$ and $\overline{\mathfrak{F}} \leqslant \lambda_*$, we obtain by easy computations that there exists C_T such that for all $t \in [0, T]$,

$$|x^1(t) - x^2(t)| + |y^1(t) - y^2(t)| \leqslant C_T \int_0^t \big(|x^1(s) - x^2(s)| + |y^1(s) - y^2(s)|\big)ds. \qquad (2.14)$$

Uniqueness then follows from Gronwall's Lemma.

Now local existence follows by an approximation procedure, which exploits the estimate (2.14). Global existence then follows from the estimates (2.14), which forbid explosion. Lemma 2.5 is established. □

The following is our main result.

Theorem 2.1. *Under Assumptions 2.1–2.4,*

$$\big(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N\big) \Rightarrow \big(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}\big) \quad in \quad \mathbf{D}^2, \qquad (2.15)$$

where $(\hat{\mathfrak{S}},\hat{\mathfrak{F}})$ *is the unique continuous solution of the system of stochastic integral equations* (2.12)-(2.13)*, that is,*

$$\begin{cases} \hat{\mathfrak{S}}(t) = -\displaystyle\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s)ds \\ \qquad -\displaystyle\int_0^t\int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ \qquad +\displaystyle\int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)-\hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds \\ \qquad +\hat{\mathfrak{J}}(t), \end{cases} \tag{2.16}$$

$$\hat{\mathfrak{F}}(t) = \int_0^t \overline{\lambda}(t-s)\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)-\hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds+\hat{\mathfrak{M}}(t), \tag{2.17}$$

where we recall that $(\hat{\mathfrak{J}},\hat{\mathfrak{M}})$ *is specified by Definition 2.1.*

We define

$$\hat{I}^N(t)=\sqrt{N}\big(\overline{I}^N(t)-\overline{I}(t)\big) \quad \text{and} \quad \hat{U}^N(t)=\sqrt{N}\big(\overline{U}^N(t)-\overline{U}(t)\big), \quad t\geq 0.$$

Replacing $\lambda_{k,i}(t)$ by $\mathbb{1}_{t<\eta_{k,i}}$ and using the fact that $\overline{I}^N(t)+\overline{U}^N(t)=1$ and $\overline{I}(t)+\overline{U}(t)=1$, We obtain the following Corollary from Theorem 2.1 and the system of equations (2.16)-(2.17).

Corollary 2.1. *Under Assumptions 2.1–2.4,*

$$\left(\hat{U}^N,\hat{I}^N\right)\Rightarrow\left(\hat{U},\hat{I}\right) \quad \textit{in} \quad \mathbf{D}^2, \tag{2.18}$$

where $\left(\hat{U},\hat{I}\right)$ *is the unique continuous solution of the system of equations:*

$$\begin{cases} \hat{U}(t) = -\displaystyle\int_0^t \mathbb{E}\left[\mathbb{1}_{t\geqslant\eta_0}\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s)ds \\ \qquad -\displaystyle\int_0^t\int_s^t \mathbb{E}\left[\mathbb{1}_{t-s\geqslant\eta}\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ \qquad +\displaystyle\int_0^t \mathbb{E}\left[\mathbb{1}_{t-s\geqslant\eta}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)-\hat{\mathfrak{J}}_1(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds \\ \qquad +\hat{\mathfrak{J}}_1(t), \end{cases} \tag{2.19}$$

$$\hat{I}(t) = \int_0^t F^c(t-s)\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)-\hat{\mathfrak{J}}_1(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds+\hat{\mathfrak{M}}_1(t), \tag{2.20}$$

where $(\hat{\mathfrak{J}}_1,\hat{\mathfrak{M}}_1)$ *is a centered continuous Gaussian process as given in Definition 2.1 where we replace* $\lambda_{1,A_1(t)}$ *and* $\gamma_{1,A_1(t)}$ *by* $\mathbb{1}_{t<\eta_{1,A_1(t)}}$ *and* $\mathbb{1}_{t\geqslant\eta_{1,A_1(t)}}$ *in the expressions of* $\hat{\mathfrak{J}}$ *and* $\hat{\mathfrak{M}}$*, respectively.*

2.2.3. Relaxing Assumption 2.1

From chapter 1 if we remove condition (2.7) from Assumption 2.1, then individuals can be reinfected even when they are still infected. In this case the FLLN limit satisfies a different set of equations. More precisely, equation (1.16) is replaced by (2.21) and (1.14) by (2.22), where (2.21) and (2.22) are given below:

$$\begin{aligned}\overline{\mathfrak{F}}(t) &= \mathbb{E}\left[\lambda_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ &\quad + \int_0^t \mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, \qquad (2.21)\\ \overline{I}(t) &= \mathbb{E}\left[\mathbb{1}_{\eta_0>t}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(s)dr\right)\right] \\ &\quad + \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \qquad (2.22)\end{aligned}$$

In the same way without condition (2.7) of Assumption 2.1, the limit obtained in the FCLT (Theorem 2.1 and Corollary 2.1) satisfies a different set of equations. More precisely, equation (2.17) is replaced by (2.23) and (2.20) by (2.24), where (2.23) and (2.24) are given below. In that case, the convergence of $\hat{\mathfrak{F}}^N(t)$ follows an analogous argument as that used for $\hat{\mathfrak{S}}^N(t)$ in Theorem 2.1. More precisely, without condition (2.7) of Assumption 2.1, equation (2.28) is replaced by a similar equation in (2.29). In fact, for each fixed k, we replace $\gamma_{k,A_k^N(t)}$ and $\gamma_{k,i}$ by $\lambda_{k,A_k^N(t)}$ and $\lambda_{k,i}$ in the expressions in (2.29). Consequently instead of the expression in (2.33), one gets a different expression, which resembles the expression in (2.34), so that the proof follows from a similar argument.

$$\begin{aligned}\hat{\mathfrak{F}}(t) &= -\int_0^t \mathbb{E}\left[\lambda_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s)ds \\ &\quad -\int_0^t\int_s^t \mathbb{E}\left[\lambda(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ &\quad +\int_0^t \mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)-\hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds \\ &\quad +\hat{\mathfrak{M}}(t), \qquad (2.23)\\ \hat{I}(t) &= -\int_0^t \mathbb{E}\left[\mathbb{1}_{\eta_0>t}\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s)ds \\ &\quad -\int_0^t\int_s^t \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ &\quad +\int_0^t \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)-\hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)ds \\ &\quad +\hat{\mathfrak{M}}_1(t). \qquad (2.24)\end{aligned}$$

2.3. Proof of Theorem 2.1

We recall the definitions of $\hat{\mathfrak{F}}^N$ and $\hat{\mathfrak{S}}^N$, from (2.6) below:

$$\hat{\mathfrak{F}}^N(t) := \sqrt{N}\left(\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\right), \text{ and } \hat{\mathfrak{S}}^N(t) := \sqrt{N}\left(\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}(t)\right).$$

Let us write,

$$\begin{aligned}\hat{\mathfrak{F}}^N(t) &= \sqrt{N}\left(\frac{1}{N}\sum_{k=1}^{N}\left(\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right)\right)\\ &\quad + \sqrt{N}\left(\frac{1}{N}\sum_{k=1}^{N}\lambda_{k,A_k(t)}(\varsigma_k(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right]\right)\\ &=: \hat{\mathfrak{F}}_1^N(t) + \hat{\mathfrak{F}}_2^N(t). \end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\hat{\mathfrak{S}}^N(t) &= \sqrt{N}\left(\frac{1}{N}\sum_{k=1}^{N}\left(\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right)\right)\\ &\quad + \sqrt{N}\left(\frac{1}{N}\sum_{k=1}^{N}\gamma_{k,A_k(t)}(\varsigma_k(t)) - \mathbb{E}\left[\gamma_{1,A_1(t)}(\varsigma_1(t))\right]\right)\\ &=: \hat{\mathfrak{S}}_1^N(t) + \hat{\mathfrak{S}}_2^N(t). \end{aligned} \tag{2.26}$$

Since $\left(\gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)), \lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot))\right)_k$ are i.i.d $\mathbf{D}^2$-valued random variables with each component satisfying Assumption 2.4, by applying the central limit theorem in $\mathbf{D}$, (see Theorem 2 in [46]) it follows that $\hat{\mathfrak{S}}_2^N \Rightarrow \hat{\mathfrak{J}}$ and $\hat{\mathfrak{F}}_2^N \Rightarrow \hat{\mathfrak{M}}$ in $\mathbf{D}$ as $N \to \infty$. Then, as $\mathbf{D}$ is separable from [80, Lemma 5.2] the pair $\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right)$ is $\mathbf{C}-$tight in $\mathbf{D}^2$ and using the uniqueness of the limit of $\hat{\mathfrak{S}}_2^N$ and $\hat{\mathfrak{F}}_2^N$, the convergence in $\mathbf{D}^2$ of the pair $\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right)$ follows. Moreover, given the convergence of $\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right)$, by the continuous mapping theorem $\hat{\mathfrak{S}}_2^N\hat{\mathfrak{F}}_2^N$ converges in $\mathbf{D}$. It follows that for each $t, t' \geqslant 0$, the covariance of $\hat{\mathfrak{S}}_2^N(t')$ and $\hat{\mathfrak{F}}_2^N(t)$, which is given by $\mathrm{Cov}\left(\gamma_{1,A_1(t')}(\varsigma_1(t'), \lambda_{1,A_1(t)}(\varsigma_1(t)))\right)$, converges to the covariance of the limit process of $\hat{\mathfrak{S}}_2^N(t')$ and $\hat{\mathfrak{F}}_2^N(t)$. Hence we have the following Lemma:

Lemma 2.6. *Under Assumption 2.4, as $N \to +\infty$,*

$$\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right) \Rightarrow \left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right) \quad \text{in} \quad \mathbf{D}^2.$$

where $\left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right)$ is a centered continuous 2-dimensional Gaussian process given in Definition 2.1.

Proving the convergence of the pair $\left(\hat{\mathfrak{S}}_1^N, \hat{\mathfrak{F}}_1^N\right)$ is highly nontrivial. We start by the following decomposition of the pair in the next subsection.

2.3.1. Decomposition of the fluctuations $(\hat{\mathfrak{S}}_1^N, \hat{\mathfrak{F}}_1^N)$

In chapter 1 we made a coupling between A_k^N and A_k. We now introduce a slightly different coupling, such that the law of the first time for which $A_k^N \neq A_k$ is the same for both couplings.

We then introduce a sequence of Poisson random measures Q_k on $\mathbb{R}_+ \times \mathbf{D}^2 \times \mathbb{R}_+$, so that the mean measure of the PRM is

$$ds \times \mathbb{P}(d\lambda, d\gamma) \times du,$$

where $\mathbb{P}$ is the law of pair (λ, γ).

We denote by $\overline{Q}_k$ its compensated measure.

For $0 \leq s < t$, and $\gamma \in \mathbf{D}$, $\phi \in \mathbf{D}$ we set

$$P_k(s,t,\gamma,\phi) = \int_{[s,t]\times\mathbf{D}^2\times\mathbb{R}_+} \mathbb{1}_{u<\gamma(r^- - s)\phi(r^-)} Q_k(dr, d\lambda', d\gamma', du).$$

So we define A_k^N as follows

$$A_k^N(t) = \int_{[0,t]\times\mathbf{D}^2\times\mathbb{R}_+} \mathbb{1}_{u\leqslant \Upsilon_k^N(r^-)} Q_k(dr, d\lambda, d\gamma, du)$$

where

$$\Upsilon_k^N(t) = \Bigg(\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} + \int_0^t \int_{\mathbf{D}^2} \int_0^{\Upsilon_k^N(s-)} \gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0} Q_k(ds, d\lambda, d\gamma, du)\Bigg)\overline{\mathfrak{F}}^N(t)$$

and

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^N \Bigg\{\lambda_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} + \int_0^t \int_{\mathbf{D}^2} \int_0^{\Upsilon_k^N(s-)} \lambda(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0} Q_k(ds, d\lambda, d\gamma, du)\Bigg\}. \quad (2.27)$$

Under the conditions on $\lambda_{k,i}$ in Assumption 2.1, (2.27) becomes

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^N \Bigg\{\lambda_{k,0}(t) + \int_0^t \int_{\mathbf{D}^2} \int_0^{\Upsilon_k^N(s^-)} \lambda(t-s) Q_k(ds, d\lambda, d\gamma, du)\Bigg\}. \quad (2.28)$$

On the other hand, we have

$$\overline{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^N \Bigg\{\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0}$$

$$+\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s-)}\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0}Q_k(ds,d\lambda,d\gamma,du)\Big\}. \tag{2.29}$$

Similarly, we define

$$A_k(t)=\int_{[0,t]\times\mathbf{D}^2\times\mathbb{R}_+}\mathbb{1}_{u\leqslant\Upsilon_k(r^-)}Q_k(dr,d\lambda,d\gamma,du)$$

with

$$\Upsilon_k(t)=\left(\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0}+\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}Q_k(ds,d\lambda,d\gamma,du)\right)\overline{\mathfrak{F}}(t)$$

where $\overline{\mathfrak{F}}$ is given by (1.16).

Consequently we have

$$\sum_{k=1}^N\lambda_{k,A_k(t)}(\varsigma_k(t)):=\sum_{k=1}^N\left\{\lambda_{k,0}(t)+\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s^-)}\lambda(t-s)Q_k(ds,d\lambda,d\gamma,du)\right\}, \tag{2.30}$$

and

$$\begin{aligned}\sum_{k=1}^N\gamma_{k,A_k(t)}(\varsigma_k(t)):=\sum_{k=1}^N\Big\{&\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0}\\&+\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}Q_k(ds,d\lambda,d\gamma,du)\Big\}.\end{aligned} \tag{2.31}$$

Using the fact that

$$\Upsilon_k^N(s)-\Upsilon_k(s)=\gamma_{k,A_k^N(s)}(\varsigma_k^N(s))\overline{\mathfrak{F}}^N(s)-\gamma_{k,A_k(s)}(\varsigma_k(s))\overline{\mathfrak{F}}(s),$$

from expression (2.26), it follows that,

$$\frac{1}{\sqrt{N}}\sum_{k=1}^N\left(\Upsilon_k^N(s)-\Upsilon_k(s)\right)=\hat{\mathfrak{S}}^N(s)\overline{\mathfrak{F}}^N(s)-\hat{\mathfrak{S}}_2^N(s)\overline{\mathfrak{F}}^N(s)+\tilde{\mathfrak{S}}^N(s)\hat{\mathfrak{F}}^N(s), \tag{2.32}$$

where

$$\tilde{\mathfrak{S}}^N(t)=\frac{1}{N}\sum_{k=1}^N\gamma_{k,A_k(t)}(\varsigma_k(t)),$$

and from expressions (2.25) and (2.26), we obtain the following Proposition.

Proposition 2.1. *Under the condition (2.7) in Assumption 2.1, for every $t\geqslant 0$,*

$$\hat{\mathfrak{F}}_1^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(s-)\wedge\Upsilon_k^N(s-)}^{\Upsilon_k(s-)\vee\Upsilon_k^N(s-)}\lambda(t-s)sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(ds,d\lambda,d\gamma,du)$$

$$+\int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}^N(s)\overline{\mathfrak{F}}^N(s)ds-\int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}_2^N(s)\overline{\mathfrak{F}}^N(s)ds+\int_0^t \overline{\lambda}(t-s)\tilde{\mathfrak{S}}^N(s)\hat{\mathfrak{F}}^N(s)ds, \tag{2.33}$$

and

$$\begin{aligned}\hat{\mathfrak{S}}_1^N(t) &= \frac{1}{\sqrt{N}}\sum_{k=1}^N \gamma_{k,0}(t)\left(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0}-\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0}\right)\\ &+\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s)}\gamma(t-s)\left(\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}^N\right)=0}-\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}\right)Q_k(ds,d\lambda,d\gamma,du)\\ &+\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(s)\wedge\Upsilon_k^N(s)}^{\Upsilon_k(s)\vee\Upsilon_k^N(s)}\gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))Q_k(ds,d\lambda,d\gamma,du)\\ &=:\hat{\mathfrak{S}}_{1,0}^N(t)+\hat{\mathfrak{S}}_{1,1}^N(t)+\hat{\mathfrak{S}}_{1,2}^N(t).\end{aligned} \tag{2.34}$$

Remark 2.2. *Note that, in the following $Q_k(dr,du)$ can be seen as the projection of $Q_k(dr,d\lambda,d\gamma,du)$ on $\mathbb{R}_+\times\mathbb{R}_+$.*

2.3.2. Two continuous integral mappings

Let $\phi_1:\mathbb{R}_+^3\to\mathbb{R}_+$ and $\phi_2:\mathbb{R}_+^2\to\mathbb{R}$ be bounded Borel measurable functions and let $\Psi_1:\mathbf{D}_+^7\to\mathbf{D}$ be given by

$$\begin{aligned}\Psi_1(f)(t) &= f_2(t)-f_3(t)-f_4(t)+\int_0^t\int_s^t\phi_1(t,s,r)f_1(r)f_5(s)f_6(s)drds\\ &-\int_0^t\phi_2(t,s)f_5(s)f_2(s)ds+\int_0^t\phi_2(t,s)f_5(s)f_4(s)ds-\int_0^t\phi_2(t,s)f_1(s)f_7(s)ds,\end{aligned} \tag{2.35}$$

for all t, where $f=(f_1,f_2,f_3,f_4,f_5,f_6,f_7)\in\mathbf{D}_+^7$.

Lemma 2.7. *If $f^n\to f$ in $\mathbf{D}_+^7$, as $n\to\infty$ and f is continuous on $\mathbb{R}_+$, then $\Psi_1(f^n)\to\Psi_1(f)$ in $\mathbf{D}$ as $n\to\infty$.*

Proof. Since $f^n\to f$ in $\mathbf{D}_+^7$, as $n\to\infty$ and f is continuous, $\|f^n-f\|_T\to 0$ as $n\to\infty$. Consequently $(\|f^n\|_T)_n$ is bounded and it follows easily that there exists $C_T>0$, such that

$$\|\Psi_1(f^n)-\Psi_1(f)\|_T\leqslant C_T\|f^n-f\|_T.$$

Where for $f,g\in\mathbf{D}_+^7$, we define,

$$\|f-g\|_T=\sup_{0\leq t\leq T}|f(t)-g(t)|,$$

with $|\cdot|$ the Euclidian norm with respect to the dimension of the space. □

On the other hand, let $\Psi_2 : \mathbf{D}_+^6 \to \mathbf{D}$, be given by

$$\Psi_2(f)(t) = f_1(t) - f_2(t) - \int_0^t \overline{\lambda}(t-s) f_3(s) f_4(s) ds - \int_0^t \overline{\lambda}(t-s) f_5(s) f_4(s) ds + \int_0^t \overline{\lambda}(t-s) f_6(s) f_1(s) ds, \tag{2.36}$$

for any $f = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathbf{D}_+^6$.

Lemma 2.8. *If $f^n \to f$ in $\mathbf{D}_+^6$, as $n \to \infty$ and f is continuous on $\mathbb{R}_+$, then $\Psi_2(f^n) \to \Psi_2(f)$ in $\mathbf{D}$ as $n \to \infty$.*

The proof is similar that of Lemma 2.7.

Remark 2.3. *If $\Psi_1\left(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{J}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\right) = 0$ and $\Psi_2\left(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{J}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\right) = 0$, then taking*

$$\phi_1(t,s,r) = \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right],$$

and

$$\phi_2(t,s) = \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right],$$

in (2.35) then $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}})$ is a solution of (2.16)-(2.17).

2.3.3. Proof of Theorem 2.1

We refer to Section 2.7 for the proof of the following tightness result.

Lemma 2.9. *The sequence $\left(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_{1,0}^N, \hat{\mathfrak{S}}_{1,1}^N\right)_{N\geq 1}$ is tight in $\mathbf{D}^6$.*

We have the following characterisation for the limit of any converging subsequence, of $\hat{\mathfrak{S}}_{1,0}^N$ in $\mathbf{D}$ and we refer to Section 2.5 for the proof.

Lemma 2.10. *Let $\left(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}_{1,0}\right)$ be a limit of a converging subsequence of $\left(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}_{1,0}^N\right)$. Then, almost surely,*

$$\hat{\mathfrak{S}}_{1,0}(t) = -\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s)ds.$$

In the following we consider

$$\phi_1(t,s,r) = \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right],$$

and

$$\phi_2(t,s) = \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right]$$

in (2.35).

We now establish the following Lemma:

Lemma 2.11. *Under Assumptions 2.1–2.4, as* $N \to \infty$,

$$\Psi_1\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{S}}^N_{1,0}, \hat{\mathfrak{S}}^N_2, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big) \Rightarrow 0 \quad \textit{in} \quad \mathbf{D},$$

and

$$\Psi_2\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N_2, \hat{\mathfrak{S}}^N_2, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big) \Rightarrow 0 \quad \textit{in} \quad \mathbf{D}.$$

We refer to Section 2.6 for the proof.
Hence the following characterisation follows:

Lemma 2.12. *Let* $\big(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}, \hat{\mathfrak{S}}_{1,0}\big)$ *be a limit of a converging subsequence of* $\big(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N_{1,0}\big)$ *in* $\mathbf{D}^3$. *Then, almost surely,*

$$\begin{cases} \Psi_1\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{J}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0, \\ \Psi_2\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{J}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0, \end{cases} \tag{2.37}$$

where we recall that the pair $(\hat{\mathfrak{J}}, \hat{\mathfrak{M}})$ *is given by Definition 2.1.*

Proof. As the space $\mathbf{D}$ is separable, and $\overline{\mathfrak{F}}^N$, $\overline{\mathfrak{S}}^N$, and $\tilde{\mathfrak{S}}^N$ are tight in $\mathbf{D}$, from Lemma 2.9.

$$\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{S}}^N_{1,0}, \hat{\mathfrak{S}}^N_2, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big)$$

is tight in $\mathbf{D}^7$. We can extract a subsequence denoted again $\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{S}}^N_{1,0}, \hat{\mathfrak{S}}^N_2, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big)$ that converges to $\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{J}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big)$ in law in $\mathbf{D}^7$. By Lemma 2.11, and the continuous mapping theorem, (2.37) follows. □

Note that

$$\begin{aligned} &\Psi_1(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{J}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \tilde{\mathfrak{S}})(t) = \hat{\mathfrak{S}}(t) - \hat{\mathfrak{S}}_{1,0}(t) - \hat{\mathfrak{J}}(t) \\ &\quad + \int_0^t \int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\right] \hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ &\quad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\left(\overline{\mathfrak{F}}(s)\hat{\mathfrak{J}}(s) - \overline{\mathfrak{F}}(s)\hat{\mathfrak{S}}(s) - \hat{\mathfrak{F}}(s)\tilde{\mathfrak{S}}(s)\right)ds, \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} &\Psi_2(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{J}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \tilde{\mathfrak{S}})(t) \\ &= \hat{\mathfrak{F}}(t) - \hat{\mathfrak{M}}(t) - \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds - \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s)ds + \int_0^t \overline{\lambda}(t-s)\tilde{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)ds. \end{aligned} \tag{2.39}$$

Combining Lemma 2.12 with Lemma 2.10, it follows that the pair $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}})$ satisfies the set of equations (2.12)-(2.13) and by Lemma 2.5 this concludes the proof of Theorem 2.1.

2.3.4. Alternative expressions of $(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N)$ and their limits

Note that the following expression follows from (2.30) and (2.29)

$$\hat{\mathfrak{S}}_2^N(t) = \hat{\mathfrak{S}}_{2,0}^N(t) + \hat{\mathfrak{S}}_{2,1}^N(t) + \hat{\mathfrak{S}}_{2,2}^N(t) + \int_0^t \mathbb{E}\left[\gamma_{1,1}(t-s)\exp\left(-\int_s^t \gamma_{1,1}(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\hat{\mathfrak{S}}_2^N(s)ds,$$

where

$$\hat{\mathfrak{S}}_{2,0}^N(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N\left(\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} - \mathbb{E}\left[\gamma_{1,0}(t)\exp\left(-\int_0^t \gamma_{1,0}(s)\overline{\mathfrak{F}}(s)ds\right)\right]\right),$$

$$\hat{\mathfrak{S}}_{2,1}^N(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\Big[\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0} - \int_{\mathbf{D}}\tilde{\gamma}(t-s)\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\tilde{\gamma})\Big]Q_k(ds,d\lambda,d\gamma,du),$$

$$\hat{\mathfrak{S}}_{2,2}^N(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\int_{\mathbf{D}}\tilde{\gamma}(t-s)\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\tilde{\gamma})\overline{Q}_k(ds,d\lambda,d\gamma,du).$$

Similarly, we have

$$\hat{\mathfrak{F}}_2^N(t) = \hat{\mathfrak{F}}_{2,0}^N(t) + \hat{\mathfrak{F}}_{2,1}^N(t) + \int_0^t\overline{\lambda}(t-s)\overline{\mathfrak{F}}(s)\hat{\mathfrak{S}}_2^N(s)ds,$$

where

$$\hat{\mathfrak{F}}_{2,0}^N(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N\left(\lambda_{k,0}(t) - \mathbb{E}\left[\lambda_{1,0}(t)\right]\right),$$

$$\hat{\mathfrak{F}}_{2,1}^N(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\lambda(t-s)\overline{Q}_k(ds,d\lambda,d\gamma,du).$$

Let

$$\hat{\mathfrak{S}}_{2,1}(t) := \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_1(s-)}\Big[\gamma(t-s)\mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}})=0} - \int_{\mathbf{D}}\tilde{\gamma}(t-s)\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\tilde{\gamma})\Big]Q_1(ds,d\lambda,d\gamma,du),$$

$$\hat{\mathfrak{S}}_{2,2}(t) := \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_1(s-)}\int_{\mathbf{D}}\tilde{\gamma}(t-s)\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\tilde{\gamma})\overline{Q}_1(ds,d\lambda,d\gamma,du),$$

$$\hat{\mathfrak{F}}_{2,1}(t) := \int_0^t\int_D\int_0^{\Upsilon_1(s-)}\lambda(t-s)\overline{Q}_1(ds,d\lambda,du).$$

Recall that $\left(\gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)),\lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot))\right)_k$ are i.i.d. $\mathbf{D}^2$-valued random variables and

that each component satisfies Assumption 2.4 and $(\overline{Q}_k)_k$ are also i.i.d. (Note that the processes P_k depends only on Q_k and Υ_k depends only on γ_{k,A_k}.) As a result, the processes $\hat{\mathfrak{S}}^N_{2,0}(t)$, $\hat{\mathfrak{S}}^N_{2,1}(t)$, $\hat{\mathfrak{S}}^N_{2,2}(t)$, $\hat{\mathfrak{F}}^N_{2,0}(t)$ and $\hat{\mathfrak{F}}^N_{2,1}(t)$ can all be regarded as sums of i.i.d. random processes in $\mathbf{D}$ with mean equals to zero. Thus, applying the central limit theorem in $\mathbf{D}$ to each term (see Theorem 2 in [46]), we obtain the convergence of these processes: $\left(\hat{\mathfrak{S}}^N_{2,0}, \hat{\mathfrak{S}}^N_{2,1}, \hat{\mathfrak{S}}^N_{2,2}, \hat{\mathfrak{F}}^N_{2,0}, \hat{\mathfrak{F}}^N_{2,1}\right) \Rightarrow \left(\hat{\mathfrak{J}}_{0,1}, W^\gamma_1, W^\gamma_2, \hat{\mathfrak{M}}_{0,1}, W^\lambda\right)$ in $\mathbf{D}^5$, where the limits are Gaussian processes as defined below: $\hat{\mathfrak{J}}_{0,1}$ has covariance function, for $t, t' \geq 0$,

$$\mathrm{Cov}(\hat{\mathfrak{J}}_{0,1}(t), \hat{\mathfrak{J}}_{0,1}(t')) = \mathrm{Cov}\left(\gamma_{1,0}(t)\mathbb{1}_{P_1(0,t,\gamma_{1,0},\overline{\mathfrak{F}})=0}, \gamma_{1,0}(t')\mathbb{1}_{P_1(0,t',\gamma_{1,0},\overline{\mathfrak{F}})=0}\right),$$

and $\hat{\mathfrak{M}}_{0,1}$ has covariance function, for $t, t' \geq 0$,

$$\mathrm{Cov}(\hat{\mathfrak{M}}_{0,1}(t), \hat{\mathfrak{M}}_{0,1}(t')) = \mathrm{Cov}\left(\lambda_{1,0}(t), \lambda_{1,0}(t')\right),$$

and the covariances between any two processes can be obtained,

$$\mathrm{Cov}(\hat{\mathfrak{J}}_{0,1}(t), W^\gamma_1(t')) = \mathrm{Cov}\left(\gamma_{1,0}(t)\mathbb{1}_{P_1(0,t,\gamma_{1,0},\overline{\mathfrak{F}})=0}, \hat{\mathfrak{S}}_{2,1}(t')\right)$$

and so on. Then by the continuous mapping theorem, we obtain the convergence of $\hat{\mathfrak{S}}^N_2$, and then given its convergence and the convergence of $\left(\hat{\mathfrak{F}}^N_{2,0}, \hat{\mathfrak{F}}^N_{2,1}\right)$, we obtain the convergence of $\hat{\mathfrak{F}}^N_2$. This leads to the following lemma.

Lemma 2.13. *Under Assumption 2.4, as $N \to +\infty$,*

$$\left(\hat{\mathfrak{S}}^N_2, \hat{\mathfrak{F}}^N_2\right) \Rightarrow \left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right) \quad \text{in} \quad \mathbf{D}^2.$$

where $\left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right)$ is a centered continuous 2-dimensional Gaussian process given in Definition 2.1. More precisely, $\left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right)$ satisfies the following system of equations:

$$\begin{cases} \hat{\mathfrak{J}}(t) = \hat{\mathfrak{J}}_{0,1}(t) + W^\gamma_1(t) + W^\gamma_2(t) + \displaystyle\int_0^t \mathbb{E}\left[\gamma_{1,1}(t-s)\exp\left(-\int_s^t \gamma_{1,1}(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\hat{\mathfrak{J}}(s)ds, \\ \hat{\mathfrak{M}}(t) = \hat{\mathfrak{M}}_{0,1}(t) + W^\lambda(t) + \displaystyle\int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{F}}(s)\hat{\mathfrak{J}}(s)ds. \end{cases}$$

where $\left(\hat{\mathfrak{J}}_{0,1}, W^\gamma_1, W^\gamma_2, \hat{\mathfrak{M}}_{0,1}, W^\lambda\right)$ is a centered continuous 5-dimensional Gaussian process whose covariance functions are described above.

2.4. Some preliminary results

To establish the tightness of $\mathfrak{F}^N$, $\mathfrak{S}^N$ we need a $\mathbf{C}$-tightness criterion, a new result on stochastic integrals with respect to Poisson random measures, the moment estimates,

an approximation of the original model by a quarantine model, and an estimate on the pair (λ, γ), which are given in the next five subsections.

2.4.1. Tightness criterion

We recall the following theorem used to prove of **C**-tightness in **D** (see Theorem 3.21 in [57, page 350]).

Theorem 2.2. *Let $(X^N)_N$ be a sequence of r.v. taking values in* **D**. *It is* **C**-*tight in* **D**, *if*
(i) for any $T > 0, \epsilon > 0$ there exist $N_0 \in \mathbb{N}$ and $C > 0$ such that

$$N \geqslant N_0 \Longrightarrow \mathbb{P}\left(\sup_{0 \leqslant s \leqslant T} |X_s^N| > C\right) \leqslant \epsilon. \tag{2.40}$$

(ii) for any $T > 0, \epsilon > 0, \theta > 0$ there exist $N_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$N \geqslant N_0 \Longrightarrow \mathbb{P}\left(w_T(X^N, \delta) \geqslant \theta\right) \leqslant \epsilon. \tag{2.41}$$

where

$$w_T(\alpha, \delta) = \sup_{0 \leqslant s < t < T, |t-s| \leqslant \delta} |\alpha(t) - \alpha(s)|.$$

Since we work with processes in **D**, we will simply write **C**-tightness below for brevity. In fact, **C**-tightness means that the limits of subsequences are continuous. We also recall the following result from [77, Lemma 3.1].

Lemma 2.14. *Let $\{X^N\}_{N \geq 1}$ be a sequence of random elements in* **D** *such that $X^N(0) = 0$. If for all $T > 0, \epsilon > 0$, as $\delta \to 0$,*

$$\limsup_{N \to \infty} \sup_{0 \leq t \leq T} \frac{1}{\delta} \mathbb{P}\Big(\sup_{0 \leq u \leq \delta} |X^N(t+u) - X^N(t)| > \epsilon\Big) \to 0,$$

then the sequence X^N is **C**-*tight.*

2.4.2. A property of stochastic integrals with respect to Poisson random measures

Theorem 2.3. *Let $(E, \mathscr{B}(E), \nu)$ be a $\sigma-$finite measured space and let Q be a Poisson random measure on $\mathbb{R}_+ \times E$ of intensity $ds\nu(du)$ and $(\mathscr{F}_t)_t$ a filtration such that for all $t \geqslant 0$, $Q|_{[0,t] \times E}$ is $\mathscr{F}_t-$measurable, and for $0 \leqslant s < t$, $\mathscr{F}_s$ and $Q|_{]s,t] \times E}$ are independent. Let $h : \mathbb{R}_+ \times E \to \mathbb{R}$ be a predictable process such that for all $t \in \mathbb{R}_+$,*

$$\mathbb{E}\left[\int_0^t \int_E |h(s,u)| \nu(du) ds\right] < \infty.$$

Let $f : \mathbb{R}_+ \times E \times \mathcal{M}_F(\mathbb{R}_+ \times E) \to \mathbb{R}$, be a bounded and measurable deterministic function. Then

$$\mathbb{E}\left[\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(ds,du)\right] = \mathbb{E}\left[\int_0^t \int_E h(s,u)\overline{f}(s,u,t)\nu(du)ds\right],$$

where

$$\overline{f}(s,u,t) = \mathbb{E}\left[f(s,u,Q|_{]s,t]\times E})\right].$$

Proof. Let $(s_i,u_i)_i$ be an arbitrary ordering of the atoms of the measure Q. We note that

$$\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(ds,du) = \sum_{i\geqslant 1} h(s_i,u_i) f(s_i,u_i,Q|_{]s_i,t]\times E}) \mathbb{1}_{s_i\leqslant t}.$$

As

$$\mathbb{E}\left[\int_0^t \int_E |h(s,u)|\nu(du)ds\right] < \infty,$$

by Fubini's theorem

$$\mathbb{E}\left[\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(ds,du)\right] = \sum_{i\geqslant 1} \mathbb{E}\left[h(s_i,u_i)\mathbb{E}\left[f(s_i,u_i,Q|_{]s_i,t]\times E})\middle|\mathscr{F}_{s_i}\right]\mathbb{1}_{s_i\leqslant t}\right],$$

and as $Q|_{]s_i,t]\times E}$ and $\mathscr{F}_{s_i}$ are independent,

$$\mathbb{E}\left[f(s_i,u_i,Q|_{]s_i,t]\times E})\middle|\mathscr{F}_{s_i}\right] = \overline{f}(s_i,u_i,t),$$

it follows that,

$$\begin{aligned}\mathbb{E}\left[\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(ds,du)\right] &= \sum_{i\geqslant 1} \mathbb{E}\left[h(s_i,u_i)\overline{f}(s_i,u_i,t)\mathbb{1}_{s_i\leqslant t}\right] \\ &= \mathbb{E}\left[\int_{[0,t]\times E} h(s,u)\overline{f}(s,u,t) Q(ds,du)\right].\end{aligned}$$

Consequently, as h is a predictable process, by a classical result on Poisson random measures [25, Theorem 6.2]

$$\mathbb{E}\left[\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(ds,du)\right] = \mathbb{E}\left[\int_0^t \int_E h(s,u)\overline{f}(s,u,t)\nu(du)ds\right].$$

□

2.4.3. Moment Inequalities

We recall the following Lemma from chapter 1.

Lemma 2.15. *For $k \in \mathbb{N}$ and $T \geqslant 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_k^N(t)-A_k(t)\right|\right] \leqslant \int_0^T \mathbb{E}\Big[\left|\Upsilon_k^N(t)-\Upsilon_k(t)\right|\Big]dt =: \delta^N(T) \tag{2.42}$$

and

$$\mathbb{P}\Big((\varsigma_k^N(s))_{t\in[0,T]} \neq (\varsigma_k(t))_{t\in[0,T]}\Big) \leqslant T\delta^N(T).$$

Moreover,

$$\delta^N(T) \leqslant \frac{\lambda_*}{\sqrt{N}} T\exp(2\lambda_* T). \tag{2.43}$$

From Lemma 2.15, we deduce the following Corollary.

Corollary 2.2. *For $k \in \mathbb{N}$ and $T \geqslant 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))-\gamma_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leqslant \frac{\lambda_*}{\sqrt{N}} T\exp(2\lambda_* T), \tag{2.44}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\lambda_{k,A_k^N(t)}(\varsigma_k^N(t))-\lambda_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leqslant \frac{\lambda_*^2}{\sqrt{N}} T\exp(2\lambda_* T), \tag{2.45}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}}-\mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right|\right] \leqslant \frac{\lambda_*}{\sqrt{N}} T\exp\big(2\lambda^* T\big). \tag{2.46}$$

From Remark 1.12 in chapter 1, we deduce the following Corollary.

Corollary 2.3. *For $k \in \mathbb{N}$ and $t \geqslant 0$,*

$$\begin{aligned}
&\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t)-\overline{\mathfrak{F}}(t)\right|\right] \leqslant \frac{\lambda_*}{\sqrt{N}}\big(1+\lambda_* t\exp(2\lambda_* t)\big),\\
&\mathbb{E}\left[\left|\overline{\mathfrak{S}}^N(t)-\overline{\mathfrak{S}}(t)\right|\right] \leqslant \frac{1}{\sqrt{N}}\big(1+\lambda_* t\exp(2\lambda_* t)\big)\\
&\text{and}\quad \mathbb{E}\left[\left|\Upsilon_k^N(t)-\Upsilon_k(t)\right|\right] \leqslant \frac{\lambda_*}{\sqrt{N}}\big(1+2\lambda_* t\exp(2\lambda_* t)\big).
\end{aligned}$$

By exchangeability we have the following Corollary.

Corollary 2.4. *For $t \geq 0$,*

$$\mathbb{E}\left[\left|\overline{\Upsilon}^N(t)-\tilde{\Upsilon}^N(t)\right|\right] \leqslant \frac{\lambda_*}{\sqrt{N}}\big(1+2\lambda_* t\exp(2\lambda_* t)\big), \tag{2.47}$$

where

$$\tilde{\Upsilon}^N(t) = \frac{1}{N}\sum_{k=1}^N \Upsilon_k(t).$$

Now we establish similar inequalities as in Corollary 2.3 for higher moments. Let

$$\chi_N^{(k)}(t) := \mathbb{P}\Big((\varsigma_{k'}^N(s))_{s\in[0,t]} \neq (\varsigma_{k'}(s))_{s\in[0,t]},\ \forall k' = 1,\cdots,k\Big).$$

We establish the following Proposition.

Proposition 2.2. *For all $N \geqslant k$, and $t \in [0,T]$, there are positive constants $C_{k,T}$ and $C'_{k,T}$ depending on k and T, such that*

$$\chi_N^{(k)}(t) \leqslant C_{k,T} N^{-k/2}, \quad \text{and} \quad \xi_N^{(k)}(t) := \mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\right|^k\right] \leqslant C'_{k,T}(t) N^{-k/2}.$$

Proof. We adapt the proof of [22, Proposition 3.1]. Let

$$\Delta_k^N(t) := \int_0^t \int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)} Q_k(ds, du).$$

Observe that for all k', $(\varsigma_{k'}^N(s))_{s\in[0,t]} = (\varsigma_{k'}(s))_{s\in[0,t]}$ if and only if $\Delta_{k'}^N(t) = 0$. As $\Delta_{k'}^N$ takes integer values,

$$\chi_N^{(k)}(t) \leqslant \mathbb{E}\left[\prod_{k'=1}^k \Delta_{k'}^N(t)\right].$$

Let us set, for all $k, p \in \mathbb{N}$ such that $N \geqslant k$,

$$\varepsilon_N^{(k,p)}(t) = \mathbb{E}\left[\prod_{k'=1}^k \left(\Delta_{k'}^N(t)\right)^p\right].$$

We next show by induction on k that

$$\varepsilon_N^{(k,p)}(t) \leqslant C_{T,k,p} N^{-k/2}. \tag{2.48}$$

From Lemma 2.15,

$$\varepsilon_N^{(1,1)}(t) \leqslant C_T N^{-1/2}. \tag{2.49}$$

Note that for all $p \leqslant q$,

$$\varepsilon_N^{(k,p)}(t) \leqslant \varepsilon_N^{(k,q)}(t),$$

because the counting process Δ_k^N takes values in $\mathbb{N}$.

As in [22, Proposition 3.1] we have for all $p \in \mathbb{N}$,

$$\varepsilon_N^{(1,p)}(t) = \mathbb{E}\left[\left(\Delta_1^N(t)\right)^p\right] \leqslant C_{T,p} N^{-1/2}. \tag{2.50}$$

Indeed, noting that, as the process $(\Delta_1^N(t))_t$ jumps at each time from $\Delta_1^N(t^-)$ to $\Delta_1^N(t^-)+1$, the process $((\Delta_1^N(t))^p)_t$ jumps from $(\Delta_1^N(t^-))^p$ to $(\Delta_1^N(t^-)+1)^p$. Consequently, from the fact that

$$(\Delta_1^N(t^-)+1)^p - (\Delta_1^N(t^-))^p = \sum_{p'=0}^{p-1} \binom{p}{p'} \left(\Delta_1^N(t^-)\right)^{p'},$$

it follows that

$$\left(\Delta_1^N(t)\right)^p = \sum_{p'=0}^{p-1} \binom{p}{p'} \int_0^t \left(\Delta_1^N(s^-)\right)^{p'} \Delta_1^N(ds).$$

Moreover, as $\left(\Delta_1^N(s^-)\right)^{p'} \leqslant \left(\Delta_1^N(s^-)\right)^p$ for $p' \leqslant p$, we deduce that

$$\begin{aligned}
\varepsilon_N^{(1,p)}(t) = \mathbb{E}\left[\left(\Delta_1^N(t)\right)^p\right] &\leqslant \mathbb{E}\left[\int_0^t \Delta_1^N(ds)\right] + 2^p \mathbb{E}\left[\int_0^t \left(\Delta_1^N(s^-)\right)^p \Delta_1^N(ds)\right] \\
&= \varepsilon_N^{(1,1)}(t) + 2^p \int_0^t \mathbb{E}\left[\left(\Delta_1^N(s)\right)^p |\Upsilon_1(s) - \Upsilon_1^N(s)|\right] ds \\
&\leqslant C_{T,p} N^{-1/2} + 2^p \lambda_* \int_0^t \varepsilon_N^{(1,p)}(s) ds,
\end{aligned}$$

where the last inequality comes from (2.49) and the fact that $|\Upsilon_1(s) - \Upsilon_1^N(s)| \leqslant \lambda^*$. The conclusion (2.50) follows by Gronwall's Lemma.

When $k \geqslant 2$ and $p \geqslant 1$, as above, noting that

$$\prod_{i=1}^k \left(\Delta_i^N(t^-)\right)^p = \sum_{j=1}^k \sum_{p'=0}^{p-1} \binom{p}{p'} \int_0^t \left(\prod_{i\neq j, i=1}^k \left(\Delta_i^N(s^-)\right)^p \right) \left(\Delta_j^N(s^-)\right)^{p'} \Delta_j^N(ds),$$

almost surely, and using exchangeability of the processes $(\Delta_i^N)_i$ and the fact that the integrand is predictable, it follows that

$$\begin{aligned}
\varepsilon_N^{(k,p)}(t) &= \sum_{j=1}^k \sum_{p'=0}^{p-1} \binom{p}{p'} \mathbb{E}\left[\int_0^t \prod_{i\neq j, i=1}^k \left(\Delta_i^N(s^-)\right)^p \left(\Delta_j^N(s^-)\right)^{p'} \Delta_j^N(ds)\right] \\
&= k \sum_{p'=0}^{p-1} \binom{p}{p'} \int_0^t \mathbb{E}\left[\left(\Delta_1^N(s)\right)^{p'} \prod_{i=2}^k \left(\Delta_i^N(s)\right)^p |\Upsilon_1(s) - \Upsilon_1^N(s)|\right] ds \\
&\leqslant k \int_0^t \left(\mathbb{E}\left[\prod_{i=2}^k \left(\Delta_i^N(s)\right)^p |\Upsilon_1(s) - \Upsilon_1^N(s)|\right] + 2^p \lambda_* \varepsilon_N^{(k,p)}(s) \right) ds.
\end{aligned}$$

However, as

$$\begin{aligned}
|\Upsilon_1(s) - \Upsilon_1^N(s)| &\leqslant |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)| + \lambda_* |\gamma_{1,A_1^N(s)}(\varsigma_1^N(s)) - \gamma_{1,A_1(s)}(\varsigma_1(s))| \\
&\leqslant |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)| + \lambda_* \mathbb{1}_{\varsigma_1^N(s)\neq\varsigma_1(s) \text{ or } A_1^N(s)\neq A_1(s)} \\
&\leqslant |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)| + \lambda_* \Delta_1^N(s),
\end{aligned}$$

it follows that,

$$\varepsilon_N^{(k,p)}(t) \leqslant k \int_0^t \left(M^N(s) + 2^{p+1} \lambda_* \varepsilon_N^{(k,p)}(s) \right) ds,$$

with $M^N(s) = \mathbb{E}\left[\prod_{i=2}^k \left(\Delta_i^N(s)\right)^p |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)|\right]$. Using exchangeability we can replace

each term $\left(\Delta_i^N(s)\right)^p$ in the expression of M^N by the following sum

$$\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=(i-1)\lfloor \frac{N}{k} \rfloor+1}^{i\lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p$$

without changing the value of M^N, since the sums are taken on disjoint indices. Then, from Hölder inequality

$$\begin{aligned} M^N(s) &= \mathbb{E}\left[\prod_{i=2}^{k}\left(\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=(i-1)\lfloor \frac{N}{k} \rfloor+1}^{i\lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p\right)\left|\overline{\mathfrak{F}}^N(s)-\overline{\mathfrak{F}}(s)\right|\right] \\ &\leqslant \left(\prod_{i=2}^{k}\mathbb{E}\left[\left(\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=(i-1)\lfloor \frac{N}{k} \rfloor+1}^{i\lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p\right)^k\right]\right)^{1/k} \left(\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(s)-\overline{\mathfrak{F}}(s)\right|^k\right]\right)^{1/k}. \end{aligned}$$

Consequently, using Young's inequality, it follows that

$$\varepsilon_N^{(k,p)}(t) \lesssim \int_0^t \left(\frac{k-1}{k} E_{N,k,p}(s) + \frac{1}{k}\xi_N^{(k)}(s) + \varepsilon_N^{(k,p)}(s)\right) ds, \tag{2.51}$$

where

$$\begin{aligned} E_{N,k,p}(s) &:= \mathbb{E}\left[\left(\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=1}^{\lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p\right)^k\right] \\ &\leqslant C_k\left(\sum_{k'=1}^{k-1} N^{k'-k}\varepsilon_N^{(k',kp)}(s) + \varepsilon_N^{(k,p)}(s)\right), \end{aligned}$$

and

$$\begin{aligned} \xi_N^{(k)}(t) &:= \mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t)-\overline{\mathfrak{F}}(t)\right|^k\right] \\ &\leqslant 2^{k-1}\left(\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_{i,A_i^N(t)}(\varsigma_i^N(t)) - \lambda_{i,A_i(t)}(\varsigma_i(t))\right)^2\right] + \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_{i,A_i(t)}(\varsigma_i(t)) - \overline{\mathfrak{F}}(t)\right)^2\right]\right) \\ &\leqslant 2^{k-1}\left(\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\Delta_i^N(t)\right)^2\right] + \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_{i,A_i(t)}(\varsigma_i(t)) - \overline{\mathfrak{F}}(t)\right)^2\right]\right) \\ &\leqslant \sum_{k'=1}^{k-1} N^{k'-k}\varepsilon_N^{(k',k)}(t) + \varepsilon_N^{(k,p)}(t) + C_k N^{-k/2}, \end{aligned}$$

We refer to [22, Proposition 3.1] for more details. The conclusion of the proof of this Proposition follows by induction and Gronwall's Lemma. □

The proof of the following Corollary is similar to the proof of Proposition 2.2 replac-

ing $\lambda_{i,j}$ by $\gamma_{i,j}$.

Corollary 2.5. *For all ℓ, k, $T>0$, There exists $C_{T,k}>0$, such that for all $t\in[0,T]$, $\forall N\geqslant k$,*

$$\mathbb{E}\left[\left|\overline{\mathfrak{S}}^N(t)-\overline{\mathfrak{S}}(t)\right|^k\right]\leqslant C_{T,k}N^{-k/2},\ \textit{and}\ \mathbb{E}\left[\left|\Upsilon^N_\ell(t)-\Upsilon_\ell(t)\right|^k\right]\leqslant C_{T,k}N^{-k/2}. \tag{2.52}$$

Let us recall the following result, which follows from Borel-Cantelli Lemma.

Proposition 2.3. *Let $(X^N)_N$ be a family of real-valued random variables and X be a random variable. If for each $\epsilon>0$,*

$$\sum_{N\geqslant 1}\mathbb{P}\left(|X^N-X|\geqslant\epsilon\right)<\infty,$$

then $(X^N)_N$ converges almost surely to X.

The following results is a consequence of Proposition 2.3 and Corollary 2.5.

Corollary 2.6. *For each $t\geqslant 0$, $\overline{\mathfrak{F}}^N(t)$ and $\overline{\mathfrak{S}}^N(t)$ converge almost surely respectively to $\overline{\mathfrak{F}}(t)$ and $\overline{\mathfrak{S}}(t)$.*

2.4.4. Quarantine model

Fix $k\in\mathbb{N}$, let us consider the model above where the k-th individual is quarantined. Then we define

$$\overline{\mathfrak{F}}^N_{(k)}(t)=\frac{1}{N}\sum_{\ell=1,\ell\neq k}^N\lambda_{\ell,A^N_{\ell,(k)}(t)}(\varsigma^N_{\ell,(k)}(t)),\ \text{and}\ \overline{\mathfrak{S}}^N_{(k)}(t)=\frac{1}{N}\sum_{\ell=1,\ell\neq k}^N\gamma_{\ell,(k)}(t)(\varsigma^N_{\ell,(k)}(t)),$$

where

$$A^N_{\ell,(k)}(t)=\int_0^t\int_0^{+\infty}\mathbb{1}_{\gamma_{\ell,A^N_{\ell,(k)}(s^-)}(\varsigma^N_{\ell,(k)}(s^-))\overline{\mathfrak{F}}^N_{(k)}(s^-)>u}Q_\ell(ds,du),\ \ell\neq k,$$

and we recall that Q_k is a standard Poisson random measure on $\mathbb{R}^2_+$. In the original model, the number of individuals infected by the k-th individual can be described by

$$\sum_{\ell=1,\ell\neq k}^N\int_0^t\int_0^{+\infty}\mathbb{1}_{\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\frac{1}{N}\sum_{i=1}^{k-1}\lambda_{i,A^N_i(s^-)}(\varsigma^N_i(s^-))<u\leqslant\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\frac{1}{N}\sum_{i=1}^{k}\lambda_{i,A^N_i(s^-)}(\varsigma^N_i(s^-))}Q_\ell(ds,du).$$

Since $\lambda_{i,A^N_i(s^-)}(\varsigma^N_i(s^-))\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\leqslant\lambda_*$, we have

$$\sum_{\ell=1,\ell\neq k}^N\int_0^t\int_0^{+\infty}\mathbb{1}_{\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\frac{1}{N}\sum_{i=1}^{k-1}\lambda_{i,A^N_i(s^-)}(\varsigma^N_i(s^-))<u\leqslant\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\frac{1}{N}\sum_{i=1}^{k}\lambda_{i,A^N_i(s^-)}(\varsigma^N_i(s^-))}Q_\ell(ds,du)$$

$$\leqslant \int_0^t \int_0^{\lambda_*} Q'_k(ds, du),$$

where Q'_k is a standard Poisson random measure on $\mathbb{R}_+^2$ independent of Q_k.

So we can bound the number of infected descendants of the individual k by a pure birth process with birth rate λ_*, denoted by $Y(t-\tau_{k,1}^N)$ (where $\tau_{k,1}^N$ denotes the first time of infection of the individual k in the full model). Note that $Y(t-\tau_{k,1}^N)) \leq Y(t)$, and that $Y(t)$ follows a geometric distribution with parameter $\exp(-\lambda_* t)$ and is independent of Q_k. As the random variable $Y(t)$ bounds the number of individuals who do not have the same state in the two models, we have

$$\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}^N_{(k)}(t)\right| \leqslant \frac{\lambda_*}{N} Y(t) \quad \text{and} \quad \left|\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}^N_{(k)}(t)\right| \leqslant \frac{Y(t)}{N}.$$

Similarly we can define a quarantine model $(\overline{\mathfrak{F}}^N_{(k,\ell)}, \overline{\mathfrak{S}}^N_{(k,\ell)})$ where the k-th and ℓ-th individuals are quarantined such that

$$\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}^N_{(k,\ell)}(t)\right| \leqslant \frac{\lambda_*}{N} \tilde{Y}(t) \quad \text{and} \quad \left|\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}^N_{(k,\ell)}(t)\right| \leqslant \frac{\tilde{Y}(t)}{N}, \tag{2.53}$$

where $\tilde{Y}(t)$ follows a geometric distribution with parameter $\exp(-2\lambda_* t)$ and is independent of Q_k and Q_ℓ.

2.4.5. Useful inequalities for λ and γ

We establish the following Lemma where λ and γ are given in Assumption 2.2-2.3.

Lemma 2.16. *For $t \geq s \geq 0$, with $\alpha > 1/2$ as in Assumption 2.2-2.3,*

$$\left|\lambda(t) - \lambda(s)\right| \leq C(t-s)^\alpha + \lambda_* \sum_{j=1}^{\ell-1} \mathbf{1}_{s<\xi^j \leq t},$$

$$\left|\gamma(t) - \gamma(s)\right| \leq C(t-s)^\alpha + \sum_{j=1}^{\ell-1} \mathbf{1}_{s<\zeta^j \leq t},$$

and

$$|\overline{\lambda}(t) - \overline{\lambda}(s)| \leq C(t-s)^\alpha + \lambda^* \sum_{j=1}^{\ell-1} (F_j(t) - F_j(s)).$$

Proof. We have

$$\lambda(t) - \lambda(s) = \sum_{j=1}^{\ell} \left(\lambda^j(t) - \lambda^j(s)\right) \mathbf{1}_{\xi^{j-1} \leq s,\, t<\xi^j} + \left(\lambda(t) - \lambda(s)\right) \sum_{j=1}^{\ell-1} \mathbf{1}_{s<\xi^j \leq t}.$$

Thus the statement follows from Assumption 2.3. □

2.5. Characterization of the limit of converging subsequences of $(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N_{1,0})$

The aim of this section is to prove Lemma 2.10. We recall that

$$\hat{\mathfrak{S}}^N_{1,0}(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \gamma_{k,0}(t) \left(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} \right).$$

Let

$$\Xi^N_1(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \chi^N_k(t), \tag{2.54}$$

where

$$\chi^N_k(t) = \gamma_{k,0}(t) \Big(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} - \exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}^N(r)dr \right) + \exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}(r)dr \right) \Big).$$

Lemma 2.17. *Under Assumption 2.4, for all $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\Xi^N_1(t) \right)^2 \right] = 0. \tag{2.55}$$

Proof. By exchangeability, we have

$$\mathbb{E}\left[\left(\Xi^N_1(t) \right)^2 \right] = \mathbb{E}\left[\left(\chi^N_1(t) \right)^2 \right] + (N-1)\mathbb{E}\left[\chi^N_1(t)\chi^N_2(t) \right]. \tag{2.56}$$

From Theorem 1.2, as $N \to \infty$, $\overline{\mathfrak{F}}^N \Rightarrow \overline{\mathfrak{F}}$, in $\mathbf{D}$, consequently as $N \to \infty$, $\chi^N_1(t) \to 0$ in probability and as $|\chi^N_1(t)| \leqslant 2$, it follows that, as $N \to \infty$,

$$\mathbb{E}\left[\left(\chi^N_1(t) \right)^2 \right] \to 0. \tag{2.57}$$

To obtain (2.55) it remains to show that

$$\left| \mathbb{E}\left[\chi^N_1(t)\chi^N_2(t) \right] \right| = o\left(\frac{1}{N} \right). \tag{2.58}$$

We consider a quarantine model where the first and second individuals are in quarantine (see subsection 2.4.4). We denote by $\overline{\mathfrak{F}}^N_{(1,2)}(t)$ the force of infection in the population at time t in this model. As in (2.53), for any $T \geqslant 0$, there exists a geometric random variable Y independent from Q_1 and Q_2 such that almost surely for all

$t \in [0,T]$,

$$\overline{\mathfrak{F}}^N_{(1,2)}(t) - \frac{Y}{N} \leqslant \overline{\mathfrak{F}}^N(t) \leqslant \overline{\mathfrak{F}}^N_{(1,2)}(t) + \frac{Y}{N}.$$

Therefore,

$$P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} - \frac{Y}{N}) \leqslant P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N) \leqslant P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} + \frac{Y}{N}).$$

Let

$$E^N_k = \left\{ P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} - \frac{Y}{N}) = P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} + \frac{Y}{N}) \right\}.$$

Then, for $k \in \{1,2\}$ as Q_k is independent of $(Q_i, \gamma_{k,0}, Y, \overline{\mathfrak{F}}^N_{(1,2)})$ for $k \neq i$, it follows by Markov's inequality that

$$\mathbb{P}\left(\left(E^N_k\right)^c \middle| \gamma_{k,0}, \overline{\mathfrak{F}}^N_{(1,2)}, Y, Q_i \right) \leqslant \frac{2tY}{N}. \tag{2.59}$$

Set

$$\tilde{\chi}^N_k(t) = \gamma_{k,0}(t) \Bigg(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)})=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} \\ - \exp\left(- \int_0^t \gamma_{k,0}(r) \overline{\mathfrak{F}}^N_{(1,2)}(r)\, dr \right) + \exp\left(- \int_0^t \gamma_{k,0}(r) \overline{\mathfrak{F}}(r)\, dr \right) \Bigg).$$

For $k \in \{1,2\}$ as Q_k, $\gamma_{k,0}$, and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, it follows that

$$\mathbb{E}\left[\tilde{\chi}^N_k(t) \middle| \overline{\mathfrak{F}}^N_{(1,2)}, \gamma_{k,0} \right] = 0.$$

Moreover, as Q_1, $\gamma_{1,0}$, Q_2, $\gamma_{2,0}$ and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, it follows that

$$\mathbb{E}\left[\tilde{\chi}^N_1(t) \tilde{\chi}^N_2(t) \right] = \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t) \middle| \overline{\mathfrak{F}}^N_{(1,2)} \right] \mathbb{E}\left[\tilde{\chi}^N_2(t) \middle| \overline{\mathfrak{F}}^N_{(1,2)} \right] \right] = 0. \tag{2.60}$$

We have

$$\mathbb{E}\left[\chi^N_1(t)\chi^N_2(t) - \tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t) \right] = \mathbb{E}\left[\tilde{\chi}^N_1(t) \left(\chi^N_2(t) - \tilde{\chi}^N_2(t) \right) \right] + \mathbb{E}\left[\left(\chi^N_1(t) - \tilde{\chi}^N_1(t) \right) \tilde{\chi}^N_2(t) \right] \\ + \mathbb{E}\left[\left(\chi^N_1(t) - \tilde{\chi}^N_1(t) \right) \left(\chi^N_2(t) - \tilde{\chi}^N_2(t) \right) \right]. \tag{2.61}$$

However,

$$\left| \chi^N_k(t) - \tilde{\chi}^N_k(t) \right| \leqslant \mathbb{1}_{\left(E^N_k\right)^c} + \frac{Yt}{N}. \tag{2.62}$$

Hence from (2.59) and (2.62),

$$\left| \mathbb{E}\left[\tilde{\chi}^N_1(t) \left(\chi^N_2(t) - \tilde{\chi}^N_2(t) \right) \right] \right| \leqslant \mathbb{E}\left[\left| \tilde{\chi}^N_1(t) \right| \mathbb{1}_{(E^N_2)^c} \right] + \frac{t}{N} \mathbb{E}\left[\left| \tilde{\chi}^N_1(t) \right| Y \right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left|\tilde{\chi}_1^N(t)\right| \mathbb{P}\left(\left(E_2^N\right)^c \middle| \gamma_{2,0}, \overline{\mathfrak{F}}^N_{(1,2)}, Y, Q_1\right)\right] + \frac{t}{N}\mathbb{E}\left[\left|\tilde{\chi}_1^N(t)\right| Y\right] \\
&\leqslant \frac{3t}{N}\mathbb{E}\left[\left|\tilde{\chi}_1^N(t)\right| Y\right] \\
&= \frac{3t}{N}\mathbb{E}\left[\left|\tilde{\chi}_1^N(t)\right|\right]\mathbb{E}[Y],
\end{aligned}
$$

where we use the fact that Y and $\tilde{\chi}_1^N$ are independent.

From (2.3) as $\overline{\mathfrak{F}}^N \to \overline{\mathfrak{F}}$, and $|\overline{\mathfrak{F}}^N - \overline{\mathfrak{F}}^N_{(1,2)}| \leqslant \frac{Y}{N}$, it follows that as $N \to \infty$,

$$
\begin{aligned}
\mathbb{E}\left[\left|\tilde{\chi}_1^N(t)\right|\right] &\leqslant \mathbb{E}\left[\int_0^t \int_{\gamma_{1,0}(r^-)\left(\overline{\mathfrak{F}}^N_{(1,2)}(r^-)\wedge\overline{\mathfrak{F}}(r^-)\right)}^{\gamma_{1,0}(r^-)\left(\overline{\mathfrak{F}}^N_{(1,2)}(r^-)\vee\overline{\mathfrak{F}}(r^-)\right)} Q_1(dr,du)\right] + \int_0^t \mathbb{E}\left[\gamma_{1,0}(r)\left|\overline{\mathfrak{F}}^N_{(1,2)}(r) - \overline{\mathfrak{F}}(r)\right|\right] dr \\
&= 2\int_0^t \mathbb{E}\left[\gamma_{1,0}(r)\left|\overline{\mathfrak{F}}^N_{(1,2)}(r) - \overline{\mathfrak{F}}(r)\right|\right] dr \to 0.
\end{aligned}
$$

Thus

$$
\left|\mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]\right| = o\left(\frac{1}{N}\right). \tag{2.63}
$$

Similarly, we show that,

$$
\left|\mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]\right| = o\left(\frac{1}{N}\right). \tag{2.64}
$$

Moreover, from (2.62), by the fact that Q_1 and Q_2 are independent, and exchangeability,

$$
\begin{aligned}
&\left|\mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]\right| \\
&\leqslant \mathbb{E}\left[\left(\mathbb{1}_{(E_1^N)^c} + \frac{Yt}{N}\right)\left(\mathbb{1}_{(E_2^N)^c} + \frac{Yt}{N}\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\left(E_1^N\right)^c \middle| \overline{\mathfrak{F}}^N_{(1,2)}, Y\right)\mathbb{P}\left(\left(E_2^N\right)^c \middle| \overline{\mathfrak{F}}^N_{(1,2)}, Y\right)\right] + \frac{2t}{N}\mathbb{E}\left[Y\mathbb{P}\left(\left(E_1^N\right)^c \middle| \overline{\mathfrak{F}}^N_{(1,2)}, Y\right)\right] + \frac{t^2}{N^2}\mathbb{E}\left[Y^2\right] \\
&\leqslant \frac{10t^2}{N^2}\mathbb{E}\left[Y^2\right],
\end{aligned}
$$

where we use (2.59).

Consequently,

$$
\left|\mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]\right| = o\left(\frac{1}{N}\right). \tag{2.65}
$$

Thus from (2.65), (2.64), (2.63), (2.61) and (2.60), (2.58) holds. □

Corollary 2.7. *Under Assumption 2.4, for any $t \geqslant 0$, as $N \to +\infty$,*

$$
\Xi_1^N(t) \to 0 \quad \textit{in probability.}
$$

We set,

$$\mathbb{V}^N_0(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \gamma_{k,0}(t) \left(\exp\left(-\int_0^t \gamma_{k,0}(r) \overline{\mathfrak{F}}^N(r) dr \right) - \exp\left(-\int_0^t \gamma_{k,0}(r) \overline{\mathfrak{F}}(r) dr \right) \right).$$

Consequently, from (2.54) for any $t \geqslant 0$, $\hat{\mathfrak{S}}^N_{1,0}(t) = \Xi^N_1(t) + \mathbb{V}^N_0(t)$. Therefore from Corollary 2.7 as $\Xi^N_1(t) \to 0$ in probability, to conclude the proof of Lemma 2.10, it suffices to establish the following Lemma.

Lemma 2.18. *Let $\hat{\mathfrak{F}}$ be a limit of a converging subsequence of $\hat{\mathfrak{F}}^N$. Then, for any $t \geqslant 0$, as $N \to \infty$, along this subsequence*

$$\left(\mathbb{V}^N_0(t), \hat{\mathfrak{F}}^N \right) \to \left(Z(t), \hat{\mathfrak{F}} \right)$$

in law, where

$$Z(t) = -\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s) \exp\left(-\int_0^t \gamma_0(r) \overline{\mathfrak{F}}(r) dr \right) \right] \hat{\mathfrak{F}}(s) ds.$$

Proof. By Taylor's formula we have

$$\mathbb{V}^N_0(t) = -Z^N(t) + \frac{1}{N} \sum_{k=1}^{N} r^N_k(t), \tag{2.66}$$

where

$$Z^N(t) = \int_0^t \left(\frac{1}{N} \sum_{k=1}^{N} \gamma_{k,0}(t)\gamma_{k,0}(s) \exp\left(-\int_0^t \gamma_{k,0}(r) \overline{\mathfrak{F}}(r) dr \right) \right) \hat{\mathfrak{F}}^N(s) ds.$$

and

$$|r^N_k(t)| \leqslant \frac{t}{2} \sqrt{N} \int_0^t \left(\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s) \right)^2 ds,$$

From Proposition 2.2, it follows that,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \frac{1}{N} \sum_{k=1}^{N} r^N_k(t) \right| \right] \leqslant T\sqrt{N} \int_0^T \mathbb{E}\left[\left(\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s) \right)^2 \right] ds \leqslant \frac{C_T}{\sqrt{N}}. \tag{2.67}$$

Hence $\frac{1}{N} \sum_{k=1}^{N} r^N_k(t) \to 0$ in probability.

On the other hand, for any fixed t, as $(\gamma_{k,0})_k$ are i.i.d by the classical law of large numbers it follows that, for all $s \in [0, t]$, as $N \to \infty$,

$$\frac{1}{N} \sum_{k=1}^{N} \gamma_{k,0}(t)\gamma_{k,0}(s) \exp\left(-\int_0^t \gamma_{k,0}(r) \overline{\mathfrak{F}}(r) dr \right) \to \mathbb{E}\left[\gamma_0(t)\gamma_0(s) \exp\left(-\int_0^t \gamma_0(r) \overline{\mathfrak{F}}(r) dr \right) \right] \quad a.s.$$

and as $N \to \infty$, $\hat{\mathfrak{F}}^N \Rightarrow \hat{\mathfrak{F}}$ in $\mathbf{D}$. By Slutsky's theorem,

$$\left(\frac{1}{N}\sum_{k=1}^{N}\gamma_{k,0}(t)\gamma_{k,0}(s)\exp\left(-\int_0^t\gamma_{k,0}(r)\overline{\mathfrak{F}}(r)dr\right)\right)\hat{\mathfrak{F}}^N(s)$$
$$\to\mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s),$$

in law. For any fixed t, since the mapping $f\to\int_0^t f(s)ds$ is continuous from $\mathbf{D}$ into $\mathbb{R}$, as $N\to\infty$,

$$Z^N(t)\to\int_0^t\mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\hat{\mathfrak{F}}(s)ds.$$

This concludes the proof.

□

We have thus proved Lemma 2.10.

2.6. Approximation of the limit

In this section we establish the proof of Lemma 2.11. We organise this section as follows: In subsection 2.6.1 we give the proof of Lemma 2.11 with the proofs of the supporting lemmas in subsection 2.6.2.

2.6.1. Proof of Lemma 2.11

We first make the following observation related to the process $\hat{\mathfrak{S}}_0^N$.

Lemma 2.19. *For $t\geqslant 0$,*

$$\sqrt{N}\int_0^t\int_{\mathbf{D}}\gamma(t-s)\left(\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}^N(r)dr\right)-\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right)\overline{\Upsilon}^N(s)\mu(d\gamma)ds$$
$$=-\int_0^t\int_s^t\int_{\mathbf{D}}\gamma(t-s)\gamma(r-s)\hat{\mathfrak{F}}^N(r)\exp\left(-\int_s^t\gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\overline{\Upsilon}^N(s)\mu(d\gamma)drds+r^N(t),$$

where as $N\to\infty$, $r^N\to 0$ in probability in $\mathbf{D}$.

Proof. By Taylor's formula we have

$$\sqrt{N}\int_0^t\int_{\mathbf{D}}\gamma(t-s)\left(\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}^N(r)dr\right)-\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right)\overline{\Upsilon}^N(s)\mu(d\gamma)ds$$
$$=-\int_0^t\int_s^t\int_{\mathbf{D}}\gamma(t-s)\gamma(r-s)\hat{\mathfrak{F}}^N(r)\exp\left(-\int_s^t\gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\overline{\Upsilon}^N(s)\mu(d\gamma)drds+r^N(t),$$

where

$$|r^N(t)|\leqslant\frac{\lambda_*\sqrt{N}}{2}\int_0^t\int_{\mathbf{D}}\left(\int_s^t\gamma(r-s)\left(\overline{\mathfrak{F}}^N(r)-\overline{\mathfrak{F}}(r)\right)dr\right)^2\mu(d\gamma)ds. \tag{2.68}$$

By Hölder's inequality and Proposition 2.2 it follows that,

$$\mathbb{E}\left[\sup_{0\le t\le T}|r^N(t)|\right]\leqslant\frac{T\lambda_*\sqrt{N}}{2}\int_0^T\int_s^T\mathbb{E}\left[\left(\overline{\mathfrak{F}}^N(r)-\overline{\mathfrak{F}}(r)\right)^2\right]drds\leqslant\frac{T^3\lambda_*}{2\sqrt{N}}.$$

□

Hence, from (2.26), (2.34), (2.35) and Lemma 2.19, it follows that

$$\begin{aligned}
&\Psi_1(\hat{\mathfrak{F}}^N,\hat{\mathfrak{S}}^N,\hat{\mathfrak{S}}^N_{1,0},\hat{\mathfrak{S}}^N_2,\overline{\mathfrak{F}}^N,\overline{\mathfrak{S}}^N,\tilde{\mathfrak{S}}^N)(t)\\
&=\hat{\mathfrak{S}}^N(t)-\hat{\mathfrak{S}}^N_{1,0}(t)-\hat{\mathfrak{S}}^N_2(t)\\
&+\int_0^t\int_s^t\int_{\mathbf{D}}\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t\gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\mu(d\gamma)\hat{\mathfrak{F}}^N(r)\overline{\mathfrak{F}}^N(s)\overline{\mathfrak{S}}^N(s)drds\\
&+\int_0^t\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\gamma)\left(\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N_2(s)-\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N(s)-\hat{\mathfrak{F}}^N(s)\tilde{\mathfrak{S}}^N(s)\right)ds\\
&:=\Xi^N_2(t)+r^N(t)+\Xi^N_3(t),
\end{aligned}$$

where

$$\begin{aligned}
\Xi^N_2(t)=\hat{\mathfrak{S}}^N_{1,1}(t)-\sqrt{N}\int_0^t\int_{\mathbf{D}}\gamma(t-s)\Bigg(&\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}^N(r)dr\right)\\
&-\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\Bigg)\overline{\Upsilon}^N(s)\mu(d\gamma)ds,
\end{aligned}\tag{2.69}$$

$$\Xi^N_3(t)=\hat{\mathfrak{S}}^N_{1,2}(t)-\int_0^t\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\gamma)\sqrt{N}\left(\overline{\Upsilon}^N(s)-\tilde{\Upsilon}^N(s)\right)ds,\tag{2.70}$$

and where we use the fact that,

$$\sqrt{N}(\overline{\Upsilon}^N(s)-\tilde{\Upsilon}^N(s))=\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N_2(s)-\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N(s)-\hat{\mathfrak{F}}^N(s)\tilde{\mathfrak{S}}^N(s).$$

On the other hand, from (2.25) and (2.33),

$$\begin{aligned}
&\Psi_2(\hat{\mathfrak{F}}^N,\hat{\mathfrak{S}}^N,\hat{\mathfrak{F}}^N_2,\hat{\mathfrak{S}}^N_2,\overline{\mathfrak{F}}^N,\overline{\mathfrak{S}}^N,\tilde{\mathfrak{S}}^N)(t)\\
&=\hat{\mathfrak{F}}^N(t)-\hat{\mathfrak{F}}^N_2(t)-\int_0^t\overline{\lambda}(t-s)\hat{\mathfrak{S}}^N(s)\overline{\mathfrak{F}}^N(s)ds-\int_0^t\overline{\lambda}(t-s)\hat{\mathfrak{S}}^N_2(s)\overline{\mathfrak{F}}^N(s)ds\\
&\qquad+\int_0^t\overline{\lambda}(t-s)\tilde{\mathfrak{S}}^N(s)\hat{\mathfrak{F}}^N(s)ds\\
&=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon^N_k(r^-)\wedge\Upsilon_k(r^-)}^{\Upsilon^N_k(r^-)\vee\Upsilon_k(r^-)}\lambda(t-s)\mathrm{sign}(\Upsilon^N_k(s^-)-\Upsilon_k(s^-))\overline{Q}_k(ds,d\lambda,d\gamma,du)\\
&=:\Xi^N_4(t).
\end{aligned}\tag{2.71}$$

To conclude the proof of Lemma 2.11 it suffices to establish the following three lemmas,

whose proofs are given in the next subsection.

Lemma 2.20. *Under Assumption 2.4, as $N \to +\infty$,*

$$\Xi_4^N \Rightarrow 0 \quad in \quad \mathbf{D},$$

in probability.

Lemma 2.21. *As $N \to +\infty$,*

$$\Xi_2^N \to 0 \quad in \quad \mathbf{D},$$

in probability.

Lemma 2.22. *As $N \to \infty$,*

$$\Xi_3^N \to 0 \quad in \quad \mathbf{D},$$

in probability.

2.6.2. Proofs of Lemmas 2.20-2.22

2.6.2.1. Proof of Lemma 2.20

We recall that

$$\Xi_4^N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k^N(r^-)\wedge\Upsilon_k(r^-)}^{\Upsilon_k^N(r^-)\vee\Upsilon_k(r^-)} \lambda(t-s)\mathrm{sign}(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(ds,d\lambda,d\gamma,du). \tag{2.72}$$

As $(\overline{Q}_k)_k$ are i.i.d, by exchangeability and from (2.44), it follows that

$$\begin{aligned}\mathbb{E}\left[\left(\Xi_4^N(t)\right)^2\right] &= \mathbb{E}\left[\int_0^t \lambda(t-s)^2 |\Upsilon_1^N(s) - \Upsilon_1(s)| ds\right] \\ &\leqslant (\lambda_*)^2 \int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(s) - \Upsilon_1(s)\right|\right] ds \to 0 \text{ as } N \to +\infty.\end{aligned}$$

To conclude, it suffices to prove tightness of $(\Xi_4^N)_N$. By the expression in (2.72), tightness of the sequence processes $(\Xi_4^N)_N$ can be deduced from the tightness of the following sequence processes since $\mathrm{sign}(\Upsilon_k^N(s^-) - \Upsilon_k(s^-))$ does not modify the setting:

$$\begin{aligned}\mathbb{V}_1^N(t) &= \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k^N(r^-)\wedge\Upsilon_k(r^-)}^{\Upsilon_k^N(r^-)\vee\Upsilon_k(r^-)} \lambda(t-s) Q_k(ds,d\lambda,d\gamma,du), \\ \mathbb{V}_2^N(t) &= \sqrt{N} \int_0^t \overline{\lambda}(t-s) \left|\tilde{\Upsilon}^N(s) - \overline{\Upsilon}^N(s)\right| ds,\end{aligned}$$

where we recall that

$$\overline{\Upsilon}^N(t) = \frac{1}{N} \sum_{k=1}^{N} \Upsilon_k^N(t) \text{ and } \tilde{\Upsilon}^N(t) = \frac{1}{N} \sum_{k=1}^{N} \Upsilon_k(t).$$

The tightness of $(\mathbb{V}_1^N)_N$ will be established in Lemma 2.26 below. Hence it remains to prove the tightness of $(\mathbb{V}_2^N)_N$.

From (2.47) we have

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\mathbb{V}_2^N(t)\right]\leqslant\lambda_*\int_0^T\mathbb{E}\left[\left|\sqrt{N}\left(\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right)\right|\right]dr\leqslant\lambda_* C_T. \tag{2.73}$$

From Lemma 2.16, for any $0\leqslant s\leqslant t\leqslant T$,

$$\begin{aligned}\left|\mathbb{V}_2^N(t)-\mathbb{V}_2^N(s)\right|&\leqslant\lambda_*\sqrt{N}\int_s^t\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr\\&\qquad+\sqrt{N}\int_0^s\left|\overline{\lambda}(t-r)-\overline{\lambda}(s-r)\right|\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr\\&\leqslant\lambda_*\sqrt{N}\int_s^t\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr+\sqrt{N}(t-s)^\alpha\int_0^s\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr\\&\qquad+\lambda_*\sqrt{N}\sum_{j=1}^{\ell-1}\int_0^s\left(F_j(t-r)-F_j(s-r)\right)\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr.\end{aligned} \tag{2.74}$$

By Markov's inequality,

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant s<t<T,|t-s|\leqslant\delta}\sqrt{N}\int_s^t\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr\geqslant\theta\right)\\&\leqslant\frac{N}{\theta^2}\mathbb{E}\left[\sup_{0\leqslant s<t<T,|t-s|\leqslant\delta}(t-s)\int_s^t\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|^2dr\right]\\&\leqslant\frac{\delta}{\theta^2}N\int_0^T\mathbb{E}\left[\left|\overline{\Upsilon}^N(r)-\tilde{\Upsilon}^N(r)\right|^2\right]dr\\&\leqslant\frac{\delta C_T}{\theta^2},\end{aligned} \tag{2.75}$$

where the last line follows from applying (2.52) with $k=2$.

On the other hand,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leqslant s<t<T,|t-s|\leqslant\delta}(t-s)^\alpha\sqrt{N}\int_0^s\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr\geqslant\theta\right)&\leqslant\frac{\delta^\alpha}{\theta}\sqrt{N}\int_0^T\mathbb{E}\left[\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|\right]dr\\&\leqslant\frac{C_T}{\theta}\delta^\alpha,\end{aligned} \tag{2.76}$$

where the last line follows from applying (2.52) with $k=1$.

Applying again (2.52) with $k=1$, by Markov's inequality and Assumption 2.3,

$$\mathbb{P}\left(\sup_{0\leqslant s<t<T,|t-s|\leqslant\delta}\sqrt{N}\sum_{j=1}^{\ell-1}\int_0^s\left(F_j(t-r)-F_j(s-r)\right)\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|dr\geqslant\theta\right)$$

$$\begin{aligned} &\leqslant \frac{\ell\delta^\rho}{\theta}\sqrt{N}\int_0^T \mathbb{E}\left[\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|\right]dr \\ &\leqslant \frac{\ell C_T}{\theta}\delta^\rho. \end{aligned} \tag{2.77}$$

Thus from (2.74), (2.75), (2.76)and (2.77),

$$\lim_{\delta\to 0}\limsup_N \mathbb{P}\left(w_T(\mathbb{V}_2^N,\delta)\geqslant\theta\right)=0,$$

combined with (2.73), it follows from Theorem 2.2 that $(\mathbb{V}_2^N)_N$ is **C**-tight.

2.6.2.2. Proof of Lemma 2.21

We recall that

$$\hat{\mathfrak{S}}_{1,1}^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s^-)}\gamma(t-s)\left(\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0}-\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right)Q_k(ds,d\lambda,d\gamma,du).$$

Define

$$\Xi_5^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\chi_k^N(t).$$

where

$$\begin{aligned}\chi_k^N(t)=&\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s^-)}\left\{\gamma(t-s)\left(\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0}-\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right)\right.\\&\left.-\int_{\mathbf{D}}\tilde{\gamma}(t-s)\left(\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}^N(r)dr\right)-\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\right)\mu(d\tilde{\gamma})\right\}\\&Q_k(ds,d\lambda,d\gamma,du).\end{aligned}$$

Lemma 2.23. *For any $t\geqslant 0$, as $N\to+\infty$,*

$$\mathbb{E}\left[\left(\Xi_5^N(t)\right)^2\right]\to 0.$$

Proof. By exchangeability we have

$$\mathbb{E}\left[\left(\Xi_5^N(t)\right)^2\right]=\mathbb{E}\left[\left(\chi_1^N(t)\right)^2\right]+(N-1)\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)\right]. \tag{2.78}$$

Set

$$\begin{aligned}\tilde{\chi}_k^N(t)=&\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s^-)}\left\{\gamma(t-s)\left(\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N_{(1,2)})=0}-\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right)\right.\\&\left.-\int_{\mathbf{D}}\tilde{\gamma}(t-s)\left(\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}^N_{(1,2)}(r)dr\right)-\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\right)\mu(d\tilde{\gamma})\right\}\\&Q_k(ds,d\lambda,d\gamma,du),\end{aligned}$$

and

$$\hat{\chi}_k^N(t) := \int_0^t \int_{\mathbf{D}^2} \int_0^{\lambda_*} \left\{ \gamma(t-s) \left| \mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N)=0} - \mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0} \right| \right.$$
$$\left. + \int_{\mathbf{D}} \tilde{\gamma}(t-s) \left| \exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}_{(1,2)}^N(r)dr \right) - \exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr \right) \right| \mu(d\tilde{\gamma}) \right\}$$
$$Q_k(ds, d\lambda, d\gamma, du).$$

For $0 \leqslant s \leqslant t$, and $\gamma \in D$, let $E_k^N(s,\gamma)$ denote the event

$$E_k^N(s,\gamma) = \left\{ P_k\left(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N - \frac{Y}{N}\right) = P_k\left(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N + \frac{Y}{N}\right) \right\},$$

where Y is a geometric random variable independent from Q_1 and Q_2 chosen as in (2.53), such that, almost surely, for all $t \in [0,T]$,

$$\overline{\mathfrak{F}}_{(1,2)}^N(t) - \frac{Y}{N} \leqslant \overline{\mathfrak{F}}^N(t) \leqslant \overline{\mathfrak{F}}_{(1,2)}^N(t) + \frac{Y}{N}.$$

We define

$$\mathscr{F}_t^N = \sigma\{(\lambda_{k,i}(\varsigma_k^N(\cdot)))_{1\leqslant k\leqslant N, i\leqslant A_k^N(t)}, (\gamma_{k,i}(\varsigma_k^N(\cdot)))_{1\leqslant k\leqslant N, i\leqslant A_k^N(t)}, (Q_k|_{[0,t]\times E})_{k\leqslant N}\}.$$

As

$$P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)\mathbb{1}_{E_k^N(s,\gamma)} = P_k\left(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N\right)\mathbb{1}_{E_k^N(s,\gamma)},$$

and $\Upsilon_k^N(s^-) \leqslant \lambda_*$, it follows that

$$\left|\chi_k^N(t) - \tilde{\chi}_k^N(t)\right| \leqslant \int_0^t \int_{\mathbf{D}^2} \int_0^{\lambda_*} \mathbb{1}_{(E_k^N(s,\gamma))^c} Q_k(ds, d\lambda, d\gamma, du) + \frac{Yt}{N} B_k(t), \tag{2.79}$$

where

$$B_k(t) = \int_0^t \int_{\mathbf{D}^2} \int_0^{\lambda_*} Q_k(ds, d\lambda, d\gamma, du).$$

Moreover, as $Q_k|_{]s,t]}$ is independent of $(\mathscr{F}_s^N, \overline{\mathfrak{F}}_{(1,2)}^N, Y, Q_\ell)$, for $\ell \neq k$ and $k \in \{1,2\}$,

$$\mathbb{P}\left((E_k^N(s,\gamma))^c \middle| \mathscr{F}_s^N, \overline{\mathfrak{F}}_{(1,2)}^N, Y, Q_\ell\right) \leqslant \frac{2tY}{N}. \tag{2.80}$$

We have

$$\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t) - \tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right] = \mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right] + \mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]$$
$$+ \mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]. \tag{2.81}$$

Note that

$$|\chi_k^N| \leqslant \hat{\chi}_k^N \text{ and } |\tilde{\chi}_k^N| \leqslant \hat{\chi}_k^N.$$

As Q_2 is independent of $(\overline{\mathfrak{F}}^N_{(1,2)}, Y, Q_1)$, from (2.79), (2.80) and Theorem 2.3,

$$\begin{aligned}\left|\mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t)-\tilde{\chi}^N_2(t)\right)\right]\right| &\leqslant \lambda_*\mathbb{E}\left[\hat{\chi}^N_1(t)\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E^N_2(s,\gamma))^c\middle|\mathscr{F}^N_s, \overline{\mathfrak{F}}^N_{(1,2)}, Y, Q_1\right)\mu(d\gamma)ds\right]\\ &\qquad + \frac{t}{N}\mathbb{E}\left[\hat{\chi}^N_1(t)YB_2(t)\right]\\ &\leqslant \frac{3t^2\lambda_*}{N}\mathbb{E}\left[Y\hat{\chi}^N_1(t)\right],\end{aligned}$$

where as $N\to\infty$, $\mathbb{E}\left[Y\hat{\chi}^N_1(t)\right]\to 0$, because $\hat{\chi}^N_1(t)\to 0$ in probability as $N\to\infty$ and $\hat{\chi}^N_1$ is bounded.

Thus it follows that

$$\left|\mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t)-\tilde{\chi}^N_2(t)\right)\right]\right| = o\left(\frac{1}{N}\right).$$

Similarly we show that

$$\left|\mathbb{E}\left[\left(\chi^N_1(t)-\tilde{\chi}^N_1(t)\right)\tilde{\chi}^N_2(t)\right]\right| = o\left(\frac{1}{N}\right).$$

In addition from (2.79),

$$\begin{aligned}&\mathbb{E}\left[\left|\chi^N_1(t)-\tilde{\chi}^N_1(t)\right|\left|\chi^N_2(t)-\tilde{\chi}^N_2(t)\right|\right]\\ &\leqslant \mathbb{E}\left[\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E^N_1(s,\gamma))^c}Q_1(ds,d\lambda,d\gamma,du)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E^N_2(s,\gamma))^c}Q_2(ds,d\lambda,d\gamma,du)\right]\\ &+\frac{t}{N}\mathbb{E}\left[YB_2(t)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E^N_1(s,\gamma))^c}Q_1(ds,d\lambda,d\gamma,du)\right]\\ &+\frac{t}{N}\mathbb{E}\left[YB_1(t)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E^N_2(s,\gamma))^c}Q_2(ds,d\lambda,d\gamma,du)\right]+\frac{t^2}{N^2}\mathbb{E}\left[Y^2B_1(t)B_2(t)\right].\end{aligned}$$

Conditionally on $\overline{\mathfrak{F}}^N_{(1,2)}$ and Y, the terms inside the first expectation above are independent because Q_1 and Q_2 are independent, and from (2.80) and Theorem 2.3, it follows that

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E^N_1(s,\gamma))^c}Q_1(ds,d\lambda,d\gamma,du)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E^N_2(s,\gamma))^c}Q_2(ds,d\lambda,d\gamma,du)\right]\\ &=\lambda_*^2\mathbb{E}\left[\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E^N_1(s,\gamma))^c\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right)\mu(d\gamma)ds\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E^N_2(s,\gamma))^c\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right)\mu(d\gamma)ds\right]\\ &\leqslant\frac{4\lambda_*^2t^4}{N^2}\mathbb{E}\left[Y^2\right].\end{aligned}$$

Hence using again (2.79) and the fact that Q_k is independent of $(\overline{\mathfrak{F}}^N_{(1,2)}, Y, Q_\ell)$, with

$k \neq \ell$ and $k, \ell \in \{1,2\}$, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left|\chi_1^N(t) - \tilde{\chi}_1^N(t)\right|\left|\chi_2^N(t) - \tilde{\chi}_2^N(t)\right|\right] \\
&\leqslant \frac{4\lambda_*^2 t^4}{N^2}\mathbb{E}\left[Y^2\right] + \frac{\lambda_* t}{N}\mathbb{E}\left[YB_2(t)\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E_1^N(s,\gamma))^c\middle|\overline{\mathfrak{F}}_{(1,2)}^N, Y, Q_2\right)\mu(d\gamma)ds\right] \\
&\quad + \frac{\lambda_* t}{N}\mathbb{E}\left[YB_1(t)\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E_2^N(s,\gamma))^c\middle|\overline{\mathfrak{F}}_{(1,2)}^N, Y, Q_1\right)\mu(d\gamma)ds\right] + \frac{t^2}{N^2}\mathbb{E}\left[Y^2B_1(t)B_2(t)\right] \\
&\leqslant \frac{4\lambda_*^2 t^4}{N^2}\mathbb{E}\left[Y^2\right] + \frac{2\lambda_* t^2}{N^2}\mathbb{E}\left[YB_2(t)\right] + \frac{2\lambda_* t^2}{N^2}\mathbb{E}\left[YB_1(t)\right] + \frac{t^2}{N^2}\mathbb{E}\left[Y^2B_1(t)B_2(t)\right].
\end{aligned}$$

Hence,

$$\left|\mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]\right| = o\left(\frac{1}{N}\right).$$

In conclusion, coming back to (2.81)

$$\left|\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t) - \tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right]\right| = o\left(\frac{1}{N}\right). \tag{2.82}$$

On the other hand, since Q_1, Q_2 and $\overline{\mathfrak{F}}_{(1,2)}^N$ are independent,

$$\mathbb{E}\left[\tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right] = 0.$$

Hence from (2.78) and (2.82), it follows that

$$\begin{aligned}
\mathbb{E}\left[\left(\Xi_5^N(t)\right)^2\right] &\leqslant \mathbb{E}\left[\left(\hat{\chi}_1^N(t)\right)^2\right] + (N-1)\left|\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t) - \tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right]\right| \\
&= \mathbb{E}\left[\left(\hat{\chi}_1^N(t)\right)^2\right] + o(1).
\end{aligned}$$

However, as Q_1 and $\overline{\mathfrak{F}}_{(1,2)}^N$ are independent, from Theorem 2.3

$$\begin{aligned}
&\mathbb{E}\left[\hat{\chi}_1^N(t)\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right] = \lambda_*\int_0^t\int_{\mathbf{D}}\gamma(t-s)\mathbb{E}\left[\left|\mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N)=0} - \mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}})=0}\right|\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\mu(d\gamma)ds \\
&+\lambda_*\int_0^t\int_{\mathbf{D}}\tilde{\gamma}(t-s)\mathbb{E}\left[\left|\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}_{(1,2)}^N(r)dr\right) - \exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right)\right|\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\mu(d\tilde{\gamma})ds.
\end{aligned}$$

Hence since $\overline{\mathfrak{F}}_{(1,2)}^N \to \overline{\mathfrak{F}}$, as $N \to \infty$, and $\hat{\chi}_1^N$ is bounded, it follows that as $N \to \infty$, $\mathbb{E}\left[\hat{\chi}_1^N(t)\right] \to 0$ and $\mathbb{E}\left[(\hat{\chi}_1^N(t))^2\right] \to 0$. □

Now, let

$$\Psi_0^N(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \chi_k^N(t),$$

where

$$\chi_k^N(t) = \int_0^t \int_0^{\Upsilon_k^N(s^-)} \int_{\mathbf{D}} \tilde{\gamma}(t-s) \left[\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}^N(r)dr \right) - \exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr \right) \right] \mu(d\tilde{\gamma})\overline{Q}_k(ds, d\lambda, d\gamma, du).$$

Lemma 2.24. *For any $t \geqslant 0$, as $N \to \infty$,*

$$\mathbb{E}\left[|\Psi_0^N(t)| \right] \to 0.$$

Proof. Let

$$\tilde{\chi}_k^N(t) = \int_0^t \int_0^{\Upsilon_k^N(s^-)} \int_{\mathbf{D}} \tilde{\gamma}(t-s) \left[\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}_{(1,2)}^N(r)dr \right) - \exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr \right) \right] \mu(d\tilde{\gamma})\overline{Q}_k(ds, d\lambda, d\gamma, du).$$

As in Lemma 2.23 we have

$$\left| \chi_k^N(t) - \tilde{\chi}_k^N(t) \right| \leqslant \frac{tY}{N} \left(B_k(t) + \lambda_* \right).$$

On the other hand, since Q_1, Q_2 and $\overline{\mathfrak{F}}_{(1,2)}^N$ are independent, it follows that

$$\mathbb{E}\left[\tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t) \middle| \overline{\mathfrak{F}}_{(1,2)}^N \right] = 0.$$

The Lemma follows by exchangeability and the fact that as $N \to \infty$, $\tilde{\chi}_k^N(t) \to 0$, proceeding as in (2.78) and (2.81) □

We recall that

$$\Xi_2^N(t) = \hat{\mathfrak{S}}_{1,1}^N(t) - \sqrt{N} \int_0^t \int_{\mathbf{D}} \gamma(t-s) \left(\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}^N(r)dr \right) - \exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr \right) \right) \overline{\Upsilon}^N(s)\mu(d\gamma)ds.$$

From Lemmas 2.23 and 2.24, for each $t \geqslant 0$, as $N \to \infty$, $\Xi_2^N(t) \to 0$ in probability. As a consequence, for all $n \in \mathbb{N}$, $t_0 < t_1 < \cdots < t_n$,

$$\left(\Xi_2^N(t_0), \Xi_2^N(t_1), \cdots, \Xi_2^N(t_n) \right) \to (0, 0, \cdots, 0).$$

To conclude it remains to show the tightness of $(\Xi_2^N)_N$, but from Lemma 2.9 $(\hat{\mathfrak{S}}_0^N)_N$ is **C**-tight, hence using Lemma 2.19, it remains to show the tightness of the following

sequence of processes:

$$\mathbb{V}^N_{3,10}(t) = \int_0^t \int_s^t \int_{\mathbf{D}} \gamma(t-s)\gamma(r-s)\hat{\overline{\mathfrak{F}}}^N(r) \exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)du\right)\overline{\Upsilon}^N(s)\mu(d\gamma)drds. \tag{2.83}$$

Since the pair $(\hat{\overline{\mathfrak{F}}}^N, \overline{\Upsilon}^N)$ is tight in $\mathbf{D}^2$, the result follows using the continuous mapping theorem.

This concludes the proof of Lemma 2.21.

2.6.2.3. Proof of Lemma 2.22

We recall that

$$\hat{\mathfrak{S}}^N_{1,2}(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(s^-)\wedge\Upsilon^N_k(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon^N_k(s^-)} \gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0} \\ \operatorname{sign}(\Upsilon^N_k(s^-)-\Upsilon_k(s^-))Q_k(ds,d\lambda,d\gamma,du).$$

We define for any $k\in\mathbb{N}$,

$$\Delta^N_k(t) = \int_0^t \int_{\Upsilon_k(s^-)\wedge\Upsilon^N_k(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon^N_k(s^-)} Q_k(ds,du)\,. \tag{2.84}$$

and let $(\vartheta^N_{k,i})_{k,i}$ be such that

$$\Delta^N_k(t) = \sum_{i\geqslant 1} \mathbb{1}_{\vartheta^N_{k,i}\leqslant t}.$$

We note that, for some i.i.d $(\gamma_{k,i},\, i\geqslant 1)$,

$$\hat{\mathfrak{S}}^N_{1,2}(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \sum_{i\geqslant 1} \gamma_{k,i}(t-\vartheta^N_{k,i})\mathbb{1}_{P_k(\vartheta^N_{k,i},t,\gamma_{k,i},\overline{\mathfrak{F}})=0}\operatorname{sign}(\Upsilon^N_k(\vartheta^N_{k,i})-\Upsilon_k(\vartheta^N_{k,i}))\mathbb{1}_{\vartheta^N_{k,i}\leqslant t}\,. \tag{2.85}$$

We recall that

$$\overline{\Upsilon}^N(t) = \overline{\mathfrak{S}}^N(t)\overline{\mathfrak{F}}^N(t) \text{ and } \tilde{\Upsilon}^N(t) = \frac{1}{N}\sum_{k=1}^N \gamma_{k,A_k(t)}(\varsigma_k(t))\overline{\mathfrak{F}}(t).$$

We set

$$\mathscr{G}^N_t = \sigma\{(\lambda_{k,i})_{1\leqslant k\leqslant N, i<A^N_k(t)}, (\gamma_{k,i})_{1\leqslant k\leqslant N, i<A^N_k(t)}, (Q_k|_{[0,t]\times E})_{k\leqslant N}\}, \tag{2.86}$$

and from (2.70) we recall

$$\Xi^N_3(t) = \hat{\mathfrak{S}}^N_{1,2}(t) - \int_0^t \int_{\mathbf{D}} \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\gamma)\sqrt{N}\left(\overline{\Upsilon}^N(s)-\tilde{\Upsilon}^N(s)\right)ds. \tag{2.87}$$

Note that from Lemma 2.9 Ξ^N_3 is tight.

We want to show that, as N tends to ∞, $\Xi_3^N \to 0$ in $\mathbf{D}$ in probability. To do this, we define

$$\Xi_{3,1}^N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sum_{i \geqslant 1} \chi_{k,i}^N(t), \tag{2.88}$$

with

$$\chi_{k,i}^N(t) = \left(\gamma_{k,i}(t - \vartheta_{k,i}^N) \mathbb{1}_{P_k(\vartheta_{k,i}^N, t, \gamma_{k,i}, \overline{\mathfrak{F}})=0} - \int_{\mathbf{D}} \gamma(t - \vartheta_{k,i}^N) \exp\left(- \int_{\vartheta_{k,i}^N}^{t} \gamma(r - \vartheta_{k,i}^N) \overline{\mathfrak{F}}(r) dr \right) \mu(d\gamma) \right) \times \operatorname{sign}(\Upsilon_k^N(\vartheta_{k,i}^N) - \Upsilon_k(\vartheta_{k,i}^N)) \mathbb{1}_{\vartheta_{k,i}^N \leqslant t}. \tag{2.89}$$

We establish the following Lemma.

Lemma 2.25. *For any $t \geqslant 0$, as $N \to \infty$,*

$$\mathbb{E}\left[\Xi_{3,1}^N(t)^2\right] \to 0.$$

Proof. For all $k, i \in \mathbb{N}$, $\mathbb{E}\left[\chi_{k,i}^N(t) \middle| \mathcal{G}_{\vartheta_{k,i}^N}^N\right] = 0$. Moreover, for any $k \neq \ell$,

$$\begin{aligned}
\mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t)\right] &= \mathbb{E}\left[\mathbb{1}_{\vartheta_{k,i}^N < \vartheta_{\ell,j}^N} \mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t) \middle| \mathcal{G}_{\vartheta_{\ell,j}^N}^N, Q_k\right]\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{\vartheta_{\ell,j}^N < \vartheta_{k,i}^N} \mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t) \middle| \mathcal{G}_{\vartheta_{k,i}^N}^N, Q_\ell\right]\right] \\
&= \mathbb{E}\left[\chi_{k,i}^N(t)\mathbb{1}_{\vartheta_{k,i}^N < \vartheta_{\ell,j}^N} \mathbb{E}\left[\chi_{\ell,j}^N(t) \middle| \mathcal{G}_{\vartheta_{\ell,j}^N}^N, Q_k\right]\right] \\
&\quad + \mathbb{E}\left[\chi_{\ell,j}^N(t)\mathbb{1}_{\vartheta_{\ell,j}^N < \vartheta_{k,i}^N} \mathbb{E}\left[\chi_{k,i}^N(t) \middle| \mathcal{G}_{\vartheta_{k,i}^N}^N, Q_\ell\right]\right] \\
&= 0
\end{aligned} \tag{2.90}$$

because $\chi_{\ell,j}^N$ and Q_k are independent and similarly for $\chi_{k,i}^N$ and Q_ℓ. Hence, as $\left|\chi_{k,i}^N(t)\right| \leqslant \mathbb{1}_{\vartheta_{k,i}^N \leqslant t}$, from (2.90) and exchangeability,

$$\begin{aligned}
\mathbb{E}\left[\Xi_{3,1}^N(t)^2\right] &= \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\left[\left(\sum_{i \geqslant 1} \chi_{k,i}^N(t)\right)^2\right] + \frac{N-1}{2} \sum_{1 \leqslant k < \ell \leqslant N} \sum_{i,j \geqslant 1} \mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t)\right] \\
&\leqslant \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\left[\left(\Delta_k^N(t)\right)^2\right] \\
&= \mathbb{E}\left[\left(\Delta_1^N(t)\right)^2\right].
\end{aligned}$$

where Δ_k^N is given by (2.84).

However from Corollary 2.3,

$$\begin{aligned}\mathbb{E}\left[\left(\Delta_1^N(t)\right)^2\right] &\leqslant 2\left(\int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|\right]ds+\mathbb{E}\left[\left(\int_0^t\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|ds\right)^2\right]\right)\\ &\leqslant 2\left(\int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|\right]ds+t\int_0^t\mathbb{E}\left[\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|^2\right]ds\right)\\ &\leqslant \frac{C_t}{\sqrt{N}}.\end{aligned}$$

Consequently

$$\mathbb{E}\left[\Xi_{3,1}^N(t)^2\right]\leqslant\frac{C_t}{\sqrt{N}}.$$

□

Note that from (2.87),

$$\begin{aligned}&\Xi_3^N(t)-\Xi_{3,1}^N(t)\\ &\quad=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)}\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\gamma)\\ &\qquad\qquad\times\operatorname{sign}(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(ds,du).\end{aligned}$$

Hence, as $(Q_k)_k$ are i.i.d.,

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)}\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\gamma)\right.\right.\\ &\qquad\qquad\left.\left.\times sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(ds,du)\right)^2\right]\\ &\quad=\frac{1}{N}\sum_{k=1}^N\mathbb{E}\left[\left(\int_0^t\int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)}\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\gamma)\right.\right.\\ &\qquad\qquad\left.\left.\times sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(ds,du)\right)^2\right]\\ &\quad=\int_0^t\left(\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\mu(d\gamma)\right)^2\mathbb{E}\left[\left|\overline{\Upsilon}^N(s)-\tilde{\Upsilon}^N(s)\right|\right]ds.\end{aligned}$$

From Corollary 2.5 and Lemma 2.25 it follows that, as N tends to ∞, $\Xi_3^N(t)\to 0$ in probability for any $t\geqslant 0$. This implies the finite dimensional convergence and as $(\Xi_3^N)_N$ is tight in $\mathbf{D}$, it follows that as $N\to\infty$, $\Xi_3^N\to 0$ in law in $\mathbf{D}$ and hence in probability.

2.7. Proof of Tightness

In this section we prove Lemma 2.9.

To prove Lemma 2.9, since $\mathbf{D}$ is separable, it suffices to establish that each component of $\left(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_{1,0}^N\right)$ is tight in $\mathbf{D}$. Moreover, from Lemma 2.6 the pair $(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N)$ is C-tight in $\mathbf{D}^2$, since it converges in $\mathbf{D}^2$ to a continuous limit. The tightness of the rest follows from the following Lemmas.

Lemma 2.26. *$\hat{\mathfrak{F}}^N$ is $\mathbf{C}$-tight.*

Lemma 2.27. *$\hat{\mathfrak{S}}_{1,2}^N$ is $\mathbf{C}$-tight.*

Lemma 2.28. *$\hat{\mathfrak{S}}_{1,1}^N$ is $\mathbf{C}$-tight.*

Lemma 2.29. *$\hat{\mathfrak{S}}_{1,0}^N$ is $\mathbf{C}$-tight.*

2.7.1. Proof of Lemma 2.26

Since $(\hat{\mathfrak{F}}_2^N)_N$ is $\mathbf{C}-$tight in $\mathbf{D}$, it suffices to prove that $(\hat{\mathfrak{F}}_1^N)_N$ is $\mathbf{C}-$tight in $\mathbf{D}$. By the expression in (2.33), that claim can be deduced from the tightness of the following sequence of processes

$$\mathbb{V}_3^N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(s-)\wedge\Upsilon_k^N(s-)}^{\Upsilon_k(s-)\vee\Upsilon_k^N(s-)} \lambda(t-s) Q_k(ds, d\lambda, d\gamma, du).$$

From Assumption 2.2 and Lemma 2.16, for any $0 \leqslant s \leqslant t \leqslant T$,

$$\begin{aligned}
|\mathbb{V}_3^N(t) - \mathbb{V}_3^N(s)| \leqslant & \frac{\lambda_*}{\sqrt{N}} \sum_{k=1}^{N} \int_s^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(dr, d\lambda, d\gamma, du) \\
& + \frac{\ell(t-s)^\alpha}{\sqrt{N}} \sum_{k=1}^{N} \int_0^s \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(dr, d\lambda, d\gamma, du) \\
& + \frac{\lambda_*}{\sqrt{N}} \sum_{j=1}^{\ell-1} \sum_{k=1}^{N} \int_0^s \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \mathbb{1}_{s-r<\mathbb{V}^j\leqslant t-r} Q_k(dr, d\lambda, d\gamma, du).
\end{aligned} \tag{2.91}$$

By Markov's inequality,

$$\begin{aligned}
\mathbb{P}\left(\frac{\ell\delta^\alpha}{\sqrt{N}} \sum_{k=1}^{N} \int_0^T \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(dr, d\lambda, d\gamma, du) \geqslant \theta \right) \\
\leqslant \frac{1}{\theta^2} \mathbb{E}\left[\left(\frac{\ell\delta^\alpha}{\sqrt{N}} \sum_{k=1}^{N} \int_0^T \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(dr, d\lambda, d\gamma, du) \right)^2 \right].
\end{aligned}$$

However, by exchangeability and Hölder's inequality, it follows that

$$\mathbb{E}\left[\left(\frac{\ell\delta^\alpha}{\sqrt{N}} \sum_{k=1}^{N} \int_0^T \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(dr, d\lambda, d\gamma, du) \right)^2 \right]$$

$$
\begin{aligned}
&\leqslant 2\mathbb{E}\left[\left(\frac{\ell\delta^{\alpha}}{\sqrt{N}}\sum_{k=1}^{N}\int_0^T\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\overline{Q}_k(dr,d\lambda,d\gamma,du)\right)^2\right]\\
&\quad+2\mathbb{E}\left[\left(\frac{\ell\delta^{\alpha}}{\sqrt{N}}\sum_{k=1}^{N}\int_0^T\left|\Upsilon_k^N(r)-\Upsilon_k(r)\right|dr\right)^2\right]\\
&\leqslant 2\ell^2\delta^{2\alpha}\int_0^T\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]dr+2\ell^2\delta^{2\alpha}N\int_0^T\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|^2\right]dr\\
&\leqslant 2\ell^2\delta^{2\alpha}\int_0^T\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]dr+2C_T\ell^2\delta^{2\alpha},
\end{aligned}
$$

where the second term in the last line follows from applying (2.52) with $k=2$. Note that from (2.44) the first term in the last line tends to 0 as $N\to\infty$, while the second term is independent of N.

Then, thanks to $\alpha>\frac{1}{2}$ (Assumption 2.3),

$$
\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\leqslant v\leqslant\delta}\frac{\ell v^{\alpha}}{\sqrt{N}}\sum_{k=1}^{N}\int_0^T\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(dr,d\lambda,d\gamma,du)\geqslant\theta\right)=0. \tag{2.92}
$$

On the other hand, for all $0<t\leqslant T$, since $(Q_k)_k$ are i.i.d, by Hölder's inequality and exchangeability, we obtain

$$
\begin{aligned}
&\mathbb{P}\left(\frac{\lambda_*}{\sqrt{N}}\sum_{k=1}^{N}\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(dr,d\lambda,d\gamma,du)>\theta\right)\\
&\leqslant\frac{1}{\theta^2}\mathbb{E}\left[\left(\frac{\lambda_*}{\sqrt{N}}\sum_{k=1}^{N}\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(dr,d\lambda,d\gamma,du)\right)^2\right]\\
&\leqslant\frac{2\lambda_*^2}{\theta^2}\left\{\mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N}\left(\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\overline{Q}_k(dr,d\lambda,d\gamma,du)\right)^2\right]\right.\\
&\qquad\qquad\left.+\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_t^{t+\delta}\left|\Upsilon_k^N(r)-\Upsilon_k(r)\right|dr\right)^2\right]\right\}\\
&\leqslant\frac{2\lambda_*^2}{\theta^2}\left\{\mathbb{E}\left[\int_t^{t+\delta}\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|dr\right]+N\mathbb{E}\left[\left(\int_t^{t+\delta}\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|dr\right)^2\right]\right\}\\
&\leqslant\frac{2\lambda_*^2}{\theta^2}\left\{\int_0^{T+\delta}\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]dr+C_T\delta^2\right\},
\end{aligned}
$$

where the last line follows from Hölder's inequality and applying (2.52) with $k=2$. Note that the second term is independent of N and from (2.44) the first term tends to 0 as $N\to\infty$. Consequently,

$$
\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\leqslant v\leqslant\delta}\frac{\lambda_*}{\sqrt{N}}\sum_{k=1}^{N}\int_t^{t+v}\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(dr,d\lambda,d\gamma,du)>\epsilon\right)=0. \tag{2.93}
$$

Moreover, by Markov's inequality,

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{t-r<\mathbb{V}^j\leqslant t+\delta-r}Q_k(dr,d\lambda,d\gamma,du)\geqslant\theta\right)\\
&\quad\leqslant\frac{1}{\theta^2}\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{t-r<\mathbb{V}^j\leqslant t+\delta-r}Q_k(dr,d\lambda,d\gamma,du)\right)^2\right]\\
&\quad\leqslant\frac{2}{\theta^2}\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{t-r<\mathbb{V}^j\leqslant t+\delta-r}\overline{Q}_k(dr,d\lambda,d\gamma,du)\right)^2\right]\\
&\qquad+\frac{2}{\theta^2}\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^t\left(F_j(s+\delta-r)-F_j(t-r)\right)\left|\Upsilon_k^N(r)-\Upsilon_k(r)\right|dr\right)^2\right].
\end{aligned}$$

Consequently, since $(Q_k)_k$ are i.i.d., from (2.10) and Hölder's inequality and from exchangeability, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{t-r<\mathbb{V}^j\leqslant t+\delta-r}Q_k(dr,d\lambda,d\gamma,du)\geqslant\theta\right)\\
&\quad\leqslant\frac{2}{N\theta^2}\sum_{k=1}^{N}\mathbb{E}\left[\left(\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{t-r<\mathbb{V}^j\leqslant t+\delta-r}\overline{Q}_k(dr,d\lambda,d\gamma,du)\right)^2\right]\\
&\qquad+\frac{2\ell^2\delta^{2\rho}}{\theta^2}\sum_{k=1}^{N}\mathbb{E}\left[\left(\int_0^t\left|\Upsilon_k^N(r)-\Upsilon_k(r)\right|dr\right)^2\right]\\
&\quad\leqslant\frac{2}{\theta^2}\sum_{j=1}^{\ell-1}\int_0^t\left(F_j(s+\delta-r)-F_j(t-r)\right)\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]dr\\
&\qquad\qquad+\frac{2\ell^2\delta^{2\rho}}{\theta^2}N\int_0^t\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|^2\right]dr\\
&\quad\leqslant\frac{2\ell\delta^{\rho}}{\theta^2}\int_0^t\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]dr+\frac{2C_T\ell^2\delta^{2\rho}}{\theta^2}.
\end{aligned}$$

where the last line follows from Hölder's inequality and applying (2.52) with $k=2$. Note that the second term is independent of N and from (2.44) the first term tends to 0 as $N\to\infty$.

Therefore, as $\rho>\frac{1}{2}$ (Assumption 2.3),

$$\begin{aligned}
&\lim_{\delta\to0}\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\\
&\mathbb{P}\left(\sup_{0\leqslant\nu\leqslant\delta}\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{t-r<\mathbb{V}^j\leqslant t+\nu-r}Q_k(dr,d\lambda,d\gamma,du)\geqslant\theta\right)=0.
\end{aligned}\tag{2.94}$$

Thus, from (2.91), (2.92), (2.93) and (2.94), we deduce that

$$\lim_{\delta\to 0}\limsup_N \sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\leq v\leq\delta}|\mathbb{V}_3^N(t+v)-\mathbb{V}_3^N(t)|\geqslant\theta\right)=0,$$

and thanks to Lemme 2.14, $(\mathbb{V}_3^N)_N$ is $\mathbf{C}-$tight in $\mathbf{D}$.

2.7.2. Proof of Lemma 2.27

We recall that

$$\hat{\mathfrak{S}}_{1,2}^N(t):=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(s^-)\wedge\Upsilon_k^N(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon_k^N(s^-)}\gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))$$
$$Q_k(ds,d\lambda,d\gamma,du). \quad (2.95)$$

By the expression in (2.95), tightness of the processes $(\hat{\mathfrak{S}}_{1,2}^N)_N$ can be deduced from the tightness of the following processes

$$\mathbb{V}_4^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(s^-)\wedge\Upsilon_k^N(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon_k^N(s^-)}\gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}Q_k(ds,d\lambda,d\gamma,du). \quad (2.96)$$

For all $0\leqslant s\leqslant t\leqslant T$,

$$\begin{aligned}
&\left|\mathbb{V}_4^N(t)-\mathbb{V}_4^N(s)\right|\\
&\leqslant\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_s^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(dr,d\lambda,d\gamma,du)\\
&+\frac{\ell(t-s)^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^s\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(dr,d\lambda,d\gamma,du)\\
&+\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^N\int_0^s\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{s-r<\zeta^j\leqslant t-r}Q_k(dr,d\lambda,d\gamma,du)\\
&+\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^s\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\left(\mathbb{1}_{P_k\left(r,s,\gamma,\overline{\mathfrak{F}}\right)=0}-\mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}\right)=0}\right)Q_k(dr,d\lambda,d\gamma,du),
\end{aligned}$$

where we use Lemme 2.16.

From (2.92), (2.93) and (2.94) it remains to prove that, as $\delta\to 0$,

$$\limsup_N\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}$$
$$\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\left(\mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}\right)=0}-\mathbb{1}_{P_k\left(r,t+\delta,\gamma,\overline{\mathfrak{F}}\right)=0}\right)Q_k(dr,d\lambda,d\gamma,du)\geqslant\epsilon\right)$$
$$\to 0. \quad (2.97)$$

We recall that for $0 \leq t \leq T$,

$$\mathscr{F}_t^N = \sigma\left\{ (\lambda_{k,i}, \gamma_{k,i})_{\substack{1\leqslant k\leqslant N \\ 1\leqslant i\leqslant A_k^N(t)}}, (Q_k|_{[0,t]})_{1\leqslant k\leqslant N} \right\}.$$

We set

$$D_k^N(t) = \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(v^-)\wedge\Upsilon_k^N(v^-)}^{\Upsilon_k(v^-)\vee\Upsilon_k^N(v^-)} Q_k(dv, d\lambda, d\gamma, du),$$

and $\overline{D}_k^N$ is given by the same expression as D_k^N but where we replace Q_k by its compensated measure.

Since

$$0 \leqslant \mathbb{1}_{P_k\left(v,t,\gamma,\overline{\mathfrak{F}}\right)=0} - \mathbb{1}_{P_k\left(v,t+\delta,\gamma,\overline{\mathfrak{F}}\right)=0} \leqslant \int_t^{t+\delta}\int_0^{\lambda_*} Q_k(du_1, du_2) = C_k(t, t+\delta),$$

by exchangeability and the fact that $(Q_k)_k$ are independent and $C_k(t,t+\delta)$ and $Q_k|_{[0,t]}$ are independent, it follows that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(v^-)\wedge\Upsilon_k^N(v^-)}^{\Upsilon_k(v^-)\vee\Upsilon_k^N(v^-)} \left(\mathbb{1}_{P_k\left(v,t,\gamma,\overline{\mathfrak{F}}\right)=0} - \mathbb{1}_{P_k\left(v,t+\delta,\gamma,\overline{\mathfrak{F}}\right)=0}\right) Q_k(dv, d\lambda, d\gamma, du)\right)^2\right] \\
&\leqslant \mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N C_k(t,t+\delta) D_k^N(t)\right)^2\right] \\
&= \mathbb{E}\left[C_1^2(t,t+\delta)\left(D_1^N(t)\right)^2\right] + (N-1)\mathbb{E}\left[C_1(t,t+\delta)C_2(t,t+\delta)D_1^N(t)D_2^N(t)\right] \\
&= \mathbb{E}\left[C_1^2(t,t+\delta)\right]\mathbb{E}\left[\left(D_1^N(t)\right)^2\right] + (N-1)\mathbb{E}\left[C_1(t,t+\delta)\right]\mathbb{E}\left[C_2(t,t+\delta)\right]\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] \\
&\leqslant 2(\lambda_*\delta + \lambda_*^2\delta^2)\left(\mathbb{E}\left[\left(\overline{D}_1^N(t)\right)^2\right] + \mathbb{E}\left[\left(\int_0^t |\Upsilon_1^N(v) - \Upsilon_1(v)|\,dv\right)^2\right]\right) + \lambda_*^2\delta^2 N\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] \\
&\leqslant 2(\lambda_*\delta + \lambda_*^2\delta^2)\left(\int_0^t \mathbb{E}\left[|\Upsilon_1(v) - \Upsilon_1^N(v)|\right]dv + T\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v) - \Upsilon_1(v)|^2\right]dv\right) \\
&\quad + \lambda_*^2\delta^2 N\mathbb{E}\left[D_1^N(t)D_2^N(t)\right],
\end{aligned} \tag{2.98}$$

where we use Hölder's inequality in the last inequality.

At this stage, we admit the following inequality holds (and we will show this immediately below):

$$N\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] \leqslant C, \tag{2.99}$$

for some $C > 0$ independent of N.

Consequently from Corollary 2.5, from (2.99) and (2.98), (2.97) follows.

Now we establish the inequality (2.99). To do this, we define the following process

for each $k \in \mathbb{N}$,

$$B_k^N(t) = \int_0^t \int_0^{\Theta_k^N(r^-)} Q_k(dr, du),$$

where

$$\Theta_k^N(t) = \gamma_{k,B_k^N(t)}(\vartheta_k^N(t))\overline{\mathfrak{F}}_{(1,2)}^N(t),$$

and ϑ_k^N is defined in the same manner as ς_1^N with B_k^N instead of A_1^N in (2.1).

Lemma 2.30. *For $k \in \mathbb{N}$ and $T \geqslant 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|A_k^N(t) - B_k^N(t)\right|\right] \leqslant \int_0^T \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|\Big] dt =: \delta^N(T) \tag{2.100}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\varsigma_k^N(t) - \vartheta_k^N(t)\right|\right] \leqslant T\delta^N(T).$$

Moreover,

$$\delta^N(T) \leqslant \frac{\mathbb{E}[Y]}{N} T \exp\big(2\lambda^* T\big). \tag{2.101}$$

Proof. Since

$$\left|A_k^N(t) - B_k^N(t)\right| \leqslant \int_0^t \int_0^{+\infty} \mathbb{1}_{\min\left(\Upsilon_k^N(r^-),\Theta_k^N(r^-)\right)<u\leqslant\max\left(\Upsilon_k^N(r^-),\Theta_k^N(r^-)\right)} Q_k(du, dr),$$

we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|A_k^N(t) - B_k^N(t)\right|\right] \leqslant \int_0^T \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|\Big] dt = \delta^N(T).$$

We recall that

$$\Upsilon_k^N(t) = \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\overline{\mathfrak{F}}^N(t) \quad \text{and} \quad \Theta_k^N(t) = \gamma_{k,B_k^N(t)}(\vartheta_k^N(t))\overline{\mathfrak{F}}_{(1,2)}^N(t).$$

However, since $\gamma_{k,i} \leqslant 1$ and $0 \leqslant \overline{\mathfrak{F}}^N(t), \overline{\mathfrak{F}}_{(1,2)}^N(t) \leqslant \lambda^*$, we obtain

$$\begin{aligned}\mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|\Big] \leqslant \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right| \mathbb{1}_{A_k^N(t)=B_k^N(t),\varsigma_k(t)=\vartheta_k^N(t)}\Big] + \\ \lambda^* \mathbb{P}\left(A_k^N(t) \neq B_k^N(t) \text{ or } \varsigma_k^N(t) \neq \vartheta_k^N(t)\right).\end{aligned} \tag{2.102}$$

On the other hand, as $\gamma_{k,i} \leqslant 1$, from (2.53), we have

$$\begin{aligned}\mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right| \mathbb{1}_{A_k^N(t)=B_k^N(t),\varsigma_k^N(t)=\vartheta_k^N(t)}\Big] &\leqslant \mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}_{(1,2)}^N(t)\right|\right] \\ &\leqslant \frac{1}{N}\mathbb{E}[Y].\end{aligned} \tag{2.103}$$

Moreover, since

$$\{A_k^N(t) \neq B_k^N(t) \text{ or } \varsigma_k^N(t) \neq \vartheta_k^N(t)\} \subset \left\{ \sup_{r\in[0,t]} |A_k^N(r) - B_k^N(r)| \geqslant 1 \right\},$$

we have

$$\mathbb{P}\left(A_k^N(t) \neq B_k^N(t) \text{ or } \varsigma_k^N(t) \neq \vartheta_k^N(t)\right) \leqslant \mathbb{E}\left[\sup_{r\in[0,t]} |A_k^N(r) - B_k^N(r)| \right] \leqslant \delta^N(t).$$

Thus, from (2.102) and (2.103), we have

$$\mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k(t)\right| \Big] \leqslant \frac{\mathbb{E}[Y]}{N} + \lambda^* \delta^N(t).$$

Hence, from (2.100), we deduce that

$$\delta^N(T) \leqslant \frac{\mathbb{E}[Y]}{N} T + \lambda^* \int_0^T \delta^N(t)\, dt,$$

and by Gronwall's lemma, it follows that

$$\delta^N(T) \leqslant \frac{\mathbb{E}[Y]}{N} T \exp\big(\lambda^* T\big).$$

Moreover,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \left|\varsigma_k^N(t) - \vartheta_k^N(t)\right| \right] &= \mathbb{E}\left[\mathbb{1}_{\{\exists t\in[0,T], \varsigma_k^N(t)\neq\vartheta_k^N(t)\}} \sup_{t\in[0,T]} \left|\varsigma_k^N(t) - \vartheta_k^N(t)\right| \right] \\
&\leqslant T\mathbb{P}\left(\exists t \in [0,T], \varsigma_k^N(t) \neq \vartheta_k^N(t)\right) \\
&= T\mathbb{P}\left(\sup_{t\in[0,T]} \left|A_k^N(t) - B_k^N(t)\right| \neq 0 \right) \\
&\leqslant T\mathbb{E}\left[\sup_{t\in[0,T]} \left|A_k^N(t) - B_k^N(t)\right| \right] \\
&\leqslant T\delta^N(T).
\end{aligned}$$

This concludes the proof of the lemma. □

Adapting the proof of Proposition 2.2 we obtain the following Lemma, where we replace A_k by B_k^N and Υ_k by Θ_k^N and using Lemma 2.101 instead of Lemma 2.15.

Lemma 2.31. *For all $p \in \mathbb{N}$, $t \in [0,T]$,*

$$\mathbb{P}\left((\varsigma_{k'}^N(s))_{s\in[0,t]} \neq (\vartheta_{k'}^N(s))_{s\in[0,t]}, \forall k' = 1, \cdots, p\right) \leqslant \frac{C_T}{N^p},$$

and

$$\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|^p \right] \leqslant \frac{C_T}{N^p}.$$

We can now establish the following lemma for inequality (2.99).

Lemma 2.32. *There exists $C>0$ such that inequality (2.99) holds for all N.*

Proof. By exchangeability and the fact that Q_1 and Q_2 are independent,

$$\begin{aligned}\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] &= \mathbb{E}\left[\overline{D}_1^N(t)\overline{D}_2^N(t)\right] + 2\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)|\overline{D}_2^N(t)\right]dv \\ &\qquad + \int_0^t\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]dvdr \\ &= 2\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)|\overline{D}_2^N(v)\right]dv \\ &\quad + \int_0^t\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]dvdr, \end{aligned} \tag{2.104}$$

where the first term in the last equality follows from the fact that

$$\mathbb{E}\left[\overline{D}_2^N(t)-\overline{D}_2^N(v)\middle|\mathscr{F}_v^N\right] = \mathbb{E}\left[\int_v^t\int_{\mathbf{D}^2}\int_{\Upsilon_2(r^-)\wedge\Upsilon_2^N(r^-)}^{\Upsilon_2(r^-)\vee\Upsilon_2^N(r^-)} \overline{Q}_2(dr,d\lambda,d\gamma,du)\middle|\mathscr{F}_v^N\right] = 0.$$

As

$$|\Upsilon_1^N(v)-\Upsilon_1(v)| \leqslant |\Upsilon_1^N(v)-\Theta_1^N(v)| + |\Theta_1^N(v)-\Upsilon_1(v)|,$$

by Hölder's inequality, it follows that

$$\begin{aligned}\mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] &\leqslant \mathbb{E}\left[|\Upsilon_1^N(v)-\Theta_1^N(v)||\overline{D}_2^N(v)|\right] + \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] \\ &\leqslant \left(\mathbb{E}\left[|\Upsilon_1^N(v)-\Theta_1^N(v)|^2\right]\right)^{1/2}\left(\mathbb{E}\left[(\overline{D}_2^N(v))^2\right]\right)^{1/2} \\ &\qquad + \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right].\end{aligned}$$

Moreover, as

$$\begin{aligned}|\overline{D}_2^N(v)| &\leqslant |D_2^N(v)| + \int_0^v |\Upsilon_2^N(r)-\Upsilon_2(r)|dr \\ &\leqslant |D_2^N(v)-\tilde{D}_2^N(v)| + \tilde{D}_2^N(v) + \int_0^v |\Upsilon_2^N(r)-\Upsilon_2(r)|dr,\end{aligned}$$

where

$$\tilde{D}_k^N(t) = \int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(v^-)\wedge\Theta_k^N(v^-)}^{\Upsilon_k(v^-)\vee\Theta_k^N(v^-)} Q_k(dv,d\lambda,d\gamma,du),$$

it follows that,

$$\begin{aligned}\mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] &\leqslant \left(\mathbb{E}\left[|\Upsilon_1^N(v)-\Theta_1^N(v)|^2\right]\right)^{1/2}\left(\mathbb{E}\left[(\overline{D}_2^N(v))^2\right]\right)^{1/2} \\ &\quad + \lambda_*\mathbb{E}\left[|D_2^N(v)-\tilde{D}_2^N(v)|\right] \\ &\quad + \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)|\mathbb{E}\left[\tilde{D}_2^N(v)\middle|\overline{\mathfrak{F}}_{(1,2)}^N, Q_1\right]\right]\end{aligned}$$

$$+\int_0^v \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]dr,$$

where we use the fact that $|\Theta_1^N(v)-\Upsilon_1(v)|\leqslant\lambda_*$.

Consequently from Lemma 2.31 and the fact that Q_2 and $\left(Q_1,\overline{\mathfrak{F}}^N_{(1,2)}\right)$ are independent, it follows that

$$\begin{aligned}
\mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] &\leqslant \frac{C_T}{N^{5/4}}+\lambda_*\mathbb{E}\left[\int_0^v\int_{\mathbf{D}^2}\int_{\Upsilon_2^N(r^-)\wedge\Theta_2^N(r^-)}^{\Upsilon_2^N(r^-)\vee\Theta_2^N(r^-)} Q_2(dr,d\lambda,d\gamma,du)\right]\\
&\quad+\int_0^v\mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Theta_2^N(r)-\Upsilon_2(r)|\right]dr\\
&\quad+\int_0^v\mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]dr\\
&=\frac{C_T}{N^{5/4}}+\lambda_*\int_0^v\mathbb{E}\left[|\Theta_2^N(r)-\Upsilon_2^N(r)|\right]dr+\int_0^v\mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Theta_2^N(r)-\Upsilon_2(r)|\right]dr\\
&\quad+\int_0^v\mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]dr\\
&\leqslant\frac{2C_T}{N^{5/4}}+\frac{C_T(\lambda_*+1)}{N},
\end{aligned}$$

where the last line follows from Lemma 2.31 and Corollary 2.5.

Hence, from (2.104) and Corollary 2.5 it follows that the inequality (2.99) holds. □

2.7.3. Proof of Lemma 2.28

We recall that

$$\hat{\mathfrak{S}}_{1,1}^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(r^-)}\gamma(t-r)\left(\mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}^N\right)=0}-\mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}\right)=0}\right)Q_k(dr,d\lambda,d\gamma,du). \tag{2.105}$$

We set

$$H_k(s,t,\gamma,\phi_1,\phi_2)=\int_s^t\int_{\gamma(v-s)(\phi_1\wedge\phi_2(v-))}^{\gamma(v-s)(\phi_1\vee\phi_2(v-))}Q_k(dv,dw).$$

Using the fact that for all $A,B\in\mathbb{N}$,

$$|\mathbb{1}_{A=0}-\mathbb{1}_{B=0}|\leqslant|A-B|.$$

From Assumptions 2.2 and 2.3 and Lemma 2.16, it follows that

$$\left|\hat{\mathfrak{S}}_{1,1}^N(t)-\hat{\mathfrak{S}}_{1,1}^N(s)\right|$$

$$
\begin{aligned}
&\leqslant + \frac{2}{\sqrt{N}} \sum_{k=1}^{N} \int_s^t \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du) \\
&\quad + \frac{2(t-s)^\alpha}{\sqrt{N}} \sum_{k=1}^{N} \int_0^s \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du) \\
&\quad + \frac{2}{\sqrt{N}} \sum_{j=1}^{\ell-1} \sum_{k=1}^{N} \int_0^s \int_{\mathbf{D}^2} \int_0^{\lambda_*} \mathbb{1}_{s-r<\zeta^j\leqslant t-r} H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du) \\
&\quad + \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_0^s \int_{\mathbf{D}^2} \int_0^{\lambda_*} \left(H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) + H_k(r,s,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) \right) Q_k(dr,d\lambda,d\gamma,du).
\end{aligned}
$$

Lemma 2.33. *As $\delta \to 0$,*

$$
\limsup_{N\to\infty} \sup_{0\leqslant t\leqslant T} \frac{1}{\delta} \mathbb{P}\left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du) \geqslant \epsilon \right) \to 0. \tag{2.106}
$$

Proof. By exchangeability, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du) \right)^2 \right] \\
&\quad = \mathbb{E}\left[\left(\int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_1(dr,d\lambda,d\gamma,du) \right)^2 \right] \\
&\qquad + (N-1)\mathbb{E}\left[\left(\int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_1(dr,d\lambda,d\gamma,du) \right) \right. \\
&\qquad\qquad \left. \left(\int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_2(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_2(dr,d\lambda,d\gamma,du) \right) \right]
\end{aligned} \tag{2.107}
$$

Let

$$
\chi_k^N(t) = \int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du),
$$

and

$$
\tilde{\chi}_k^N(t) = \int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du).
$$

Note that, as $\overline{\mathfrak{F}}_{(1,2)}^N$ and Q_k for $k \in \{1,2\}$ are independent, from Theorem 2.3 we have

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\chi}_k^N(t) \middle| \overline{\mathfrak{F}}_{(1,2)}^N \right] &= \mathbb{E}\left[\int_t^{t+\delta} \int_{\mathbf{D}^2} \int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N,\overline{\mathfrak{F}}) Q_k(dr,d\lambda,d\gamma,du) \middle| \overline{\mathfrak{F}}_{(1,2)}^N \right] \\
&= \lambda_* \mathbb{E}\left[\int_t^{t+\delta} \int_{\mathbf{D}} \int_r^{t+\delta} \gamma(v-r) \left| \overline{\mathfrak{F}}_{(1,2)}^N(v) - \overline{\mathfrak{F}}(v) \right| dv\mu(d\gamma)dr \middle| \overline{\mathfrak{F}}_{(1,2)}^N \right] \\
&\leqslant \lambda_* \int_t^{t+\delta} \int_t^{t+\delta} \left| \overline{\mathfrak{F}}_{(1,2)}^N(v) - \overline{\mathfrak{F}}(v) \right| dvdr
\end{aligned}
$$

$$= \lambda_* \delta \int_t^{t+\delta} \left| \overline{\mathfrak{F}}^N_{(1,2)}(v) - \overline{\mathfrak{F}}(v) \right| dv. \tag{2.108}$$

Consequently, conditioning on $\overline{\mathfrak{F}}^N_{(1,2)}$, and using the fact that Q_1, Q_2 and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right] &= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\mathbb{E}\left[\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\right] \\
&\leqslant \lambda_*^2\delta^2\mathbb{E}\left[\left(\int_t^{t+\delta}\left|\overline{\mathfrak{F}}^N_{(1,2)}(v) - \overline{\mathfrak{F}}(v)\right| dv\right)^2\right] \\
&\leqslant \lambda_*^2\delta^3\int_t^{t+\delta}\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N_{(1,2)}(v) - \overline{\mathfrak{F}}(v)\right|^2\right] dv \\
&\leqslant 2\lambda_*^2\delta^4\left(\frac{1}{N^2}\mathbb{E}\left[Y^2\right] + \frac{C_T}{N}\right),
\end{aligned} \tag{2.109}$$

where we use Hölder's inequality and the fact that $|\overline{\mathfrak{F}}^N_{(1,2)}(v) - \overline{\mathfrak{F}}(v)| \leqslant \frac{Y}{N} + |\overline{\mathfrak{F}}^N(v) - \overline{\mathfrak{F}}(v)|$, $(a+b)^2 \leqslant 2(a^2+b^2)$ and Proposition 2.2.

In addition for $k \in \{1,2\}$, using subsection 2.4.4

$$\begin{aligned}
\left|\chi^N_k(t) - \tilde{\chi}^N_k(t)\right| &= \int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r, t+\delta, \gamma, \overline{\mathfrak{F}}^N_{(1,2)}, \overline{\mathfrak{F}}^N) Q_k(dr, d\lambda, d\gamma, du) \\
&\leqslant \int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)+Y/N)} Q_k(dv, dw)\right) Q_k(dr, d\lambda, d\gamma, du).
\end{aligned} \tag{2.110}$$

Consequently, using Theorem 2.3

$$\mathbb{E}\left[\left|\chi^N_1(t) - \tilde{\chi}^N_1(t)\right|\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right] \leqslant \frac{2Y\lambda_*\delta}{N}, \tag{2.111}$$

and from (2.110) we deduce that as $N \to \infty$, $\chi^N_1(t) \to 0$ in probability and as

$$\chi^N_1(t) \leqslant \left(\int_t^{t+\delta}\int_0^{\lambda_*} Q_1(dv, dw)\right)^2,$$

it follows that, as $N \to \infty$,

$$\mathbb{E}\left[\left(\chi^N_1(t)\right)^2\right] \to 0. \tag{2.112}$$

We have

$$\mathbb{E}\left[\chi^N_1(t)\chi^N_2(t) - \tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right] = \mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t) - \tilde{\chi}^N_2(t)\right)\right] + \mathbb{E}\left[\left(\chi^N_1(t) - \tilde{\chi}^N_1(t)\right)\tilde{\chi}^N_2(t)\right]$$

$$+\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right]. \tag{2.113}$$

As (Q_1, Q_2) and $\left(\overline{\mathfrak{F}}^N_{(1,2)}, Y\right)$ are independent, from Theorem 2.3, (2.110), (2.111) and (2.108) it follows that

$$\begin{aligned}
&\left|\mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right]\right| \\
&\leqslant \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)+Y/N)} Q_2(dv,dw)\right)Q_2(dr,d\lambda,d\gamma,du)\middle|\overline{\mathfrak{F}}^N_{(1,2)},Y\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)},Y\right]\right. \\
&\quad \left.\times\mathbb{E}\left[\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)+Y/N)} Q_2(dv,dw)\right)Q_2(dr,d\lambda,d\gamma,du)\middle|\overline{\mathfrak{F}}^N_{(1,2)},Y\right]\right] \\
&\leqslant \frac{2\lambda_*^2\delta^2}{N}\mathbb{E}\left[Y\int_t^{t+\delta}\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|dv\right] \\
&\leq \frac{2\lambda_*^3\delta^3}{N}\mathbb{E}[Y],
\end{aligned} \tag{2.114}$$

where the last line follows from the fact that $\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\leqslant\lambda_*$.

Similarly, we show that

$$\left|\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]\right| \leq \frac{2\lambda_*^3\delta^3}{N}\mathbb{E}[Y]. \tag{2.115}$$

From (2.110), and as in the setting of (2.114), it follows that

$$\mathbb{E}\left[\left|\chi_1^N(t)-\tilde{\chi}_1^N(t)\right|\left|\chi_2^N(t)-\tilde{\chi}_2^N(t)\right|\right] \leqslant \frac{4\lambda_*^4\delta^4}{N^2}\mathbb{E}\left[Y^2\right]. \tag{2.116}$$

Consequently from (2.116),(2.115), (2.114), (2.113), and (2.109), it follows that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}N\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)\right]=0.$$

Therefore, from (2.112) and (2.107), it follows that (2.106) holds. □

Applying the same method as in the previous lemma, we establish the following Lemma,

Lemma 2.34. *As $\delta\to 0$,*

$$\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\mathbb{P}\left(\frac{\delta^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du)\geqslant\epsilon\right)\to 0,$$

$$\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du)\geqslant\epsilon\right)\to 0,$$

and

$$\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du)\geqslant\epsilon\right)\to 0,$$

To conclude the tightness of $\hat{\mathfrak{S}}^N_{1,1}$, it remains to establish the following Lemma.

Lemma 2.35. *As* $\delta\to 0$,

$$\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du)\geqslant\epsilon\right)\to 0. \tag{2.117}$$

Proof. By exchangeability it follows that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du)\right)^2\right]\\
&=\mathbb{E}\left[\left(\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_1(dr,d\lambda,d\gamma,du)\right)^2\right]\\
&\quad+(N-1)\mathbb{E}\left[\left(\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_1(dr,d\lambda,d\gamma,du)\right)\right.\\
&\qquad\left.\left(\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_2(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_2(dr,d\lambda,d\gamma,du)\right)\right].
\end{aligned}\tag{2.118}$$

Let

$$\chi^N_k(t)=\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du),$$

and

$$\tilde{\chi}^N_k(t)=\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N_{(1,2)},\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du).$$

Since $\overline{\mathfrak{F}}^N_{(1,2)}$ and Q_k for $k\in\{1,2\}$ are independent, from Theorem 2.3, we have

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{\chi}_k^N(t)\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\\
&=\sum_{j=1}^{\ell-1}\mathbb{E}\left[\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du)\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\\
&=\lambda_*\sum_{j=1}^{\ell-1}\mathbb{E}\left[\int_0^t\int_{\mathbf{D}}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}\int_r^{t+\delta}\gamma(v-r)\left|\overline{\mathfrak{F}}_{(1,2)}^N(v)-\overline{\mathfrak{F}}(v)\right|dv\mu(d\gamma)dr\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\\
&\leqslant\lambda_*\sum_{j=1}^{\ell-1}\mathbb{E}\left[\int_0^t\left(G_j(t+\delta-r)-G_j(t-r)\right)\int_r^{t+\delta}\left|\overline{\mathfrak{F}}_{(1,2)}^N(v)-\overline{\mathfrak{F}}(v)\right|dvdr\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\\
&\leqslant\lambda_*\ell T\delta^\rho\int_0^T\left|\overline{\mathfrak{F}}_{(1,2)}^N(v)-\overline{\mathfrak{F}}(v)\right|dv,
\end{aligned}
\tag{2.119}
$$

where we use $\gamma\leqslant 1$, to get the third line.

Consequently, conditioning by $\overline{\mathfrak{F}}_{(1,2)}^N$, and using the fact that Q_1, Q_2 and $\overline{\mathfrak{F}}_{(1,2)}^N$ are independent, we have

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right]&=\mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\mathbb{E}\left[\tilde{\chi}_2^N(t)\middle|\overline{\mathfrak{F}}_{(1,2)}^N\right]\right]\\
&\leqslant\lambda_*^2\ell^2T^2\delta^{2\rho}\mathbb{E}\left[\left(\int_0^T\left|\overline{\mathfrak{F}}_{(1,2)}^N(v)-\overline{\mathfrak{F}}(v)\right|dv\right)^2\right]\\
&\leqslant\lambda_*^2\ell^2T^3\delta^{2\rho}\int_0^T\mathbb{E}\left[\left|\overline{\mathfrak{F}}_{(1,2)}^N(v)-\overline{\mathfrak{F}}(v)\right|^2\right]dv\\
&\leqslant 2\lambda_*^2\ell^2T^3\delta^{2\rho}\left(\frac{1}{N^2}\mathbb{E}\left[Y^2\right]+\frac{C_T}{N}\right),
\end{aligned}
\tag{2.120}
$$

where we use Hölder's inequality and the fact that $|\overline{\mathfrak{F}}_{(1,2)}^N(v)-\overline{\mathfrak{F}}(v)|\leqslant\frac{Y}{N}+|\overline{\mathfrak{F}}^N(v)-\overline{\mathfrak{F}}(v)|$, $(a+b)^2\leqslant 2(a^2+b^2)$ and Proposition 2.2.

In addition for $k\in\{1,2\}$, using subsection 2.4.4

$$
\begin{aligned}
&\left|\chi_k^N(t)-\tilde{\chi}_k^N(t)\right|\\
&=\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N,\overline{\mathfrak{F}})Q_k(dr,d\lambda,d\gamma,du)\\
&\leqslant\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\leqslant t+\delta-r}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}_{(1,2)}^N(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}_{(1,2)}^N(v^-)+Y/N)}Q_k(dv,dw)\right)Q_k(dr,d\lambda,d\gamma,du)\\
&=:\hat{\chi}_k^N(t).
\end{aligned}
\tag{2.121}
$$

Consequently, using Theorem 2.3, it follows that,

$$\mathbb{E}\left[\left|\chi_1^N(t)-\tilde{\chi}_1^N(t)\right|\right] \leqslant \frac{2T^2\lambda_*\ell\delta^\rho}{N}\mathbb{E}[Y]. \tag{2.122}$$

and from (2.121) we deduce that as $N\to\infty$, $\chi_1^N\to 0$ in probability, and as it is bounded by a square-integrable process, it follows that, as $N\to\infty$,

$$\mathbb{E}\left[\left(\chi_1^N(t)\right)^2\right]\to 0. \tag{2.123}$$

We have

$$\begin{aligned}\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)-\tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right] &= \mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right]+\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]\\ &\quad+\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right],\end{aligned} \tag{2.124}$$

As (Q_1, Q_2) and $(\overline{\mathfrak{F}}^N_{(1,2)}, Y)$ are independent from (2.121) and (2.119),

$$\begin{aligned}\left|\mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right]\right| &\leqslant \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\hat{\chi}_k^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\mathbb{E}\left[\hat{\chi}_1^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\right]\\ &\leqslant \frac{2T^3\lambda_*^2\ell^2\delta^{2\rho}}{N}\mathbb{E}\left[Y\int_0^T\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|dv\right]\\ &\leq \frac{2T^3\lambda_*^2\ell^2\delta^{2\rho}}{N^{3/2}}\mathbb{E}[Y].\end{aligned} \tag{2.125}$$

Similarly we show that

$$\left|\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]\right| \leq \frac{2T^3\lambda_*^2\ell^2\delta^{2\rho}}{N^{3/2}}\mathbb{E}[Y]. \tag{2.126}$$

From (2.121), and as in the setting in (2.125), it follows that

$$\mathbb{E}\left[\left|\chi_1^N(t)-\tilde{\chi}_1^N(t)\right|\left|\chi_2^N(t)-\tilde{\chi}_2^N(t)\right|\right] \leqslant \frac{4T^4\lambda_*^4\ell^4\delta^{4\rho}}{N^2}\mathbb{E}\left[Y^2\right]. \tag{2.127}$$

Consequently from (2.127),(2.126), (2.125), (2.124), and (2.120), it follows that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\leqslant t\leqslant T}\frac{1}{\delta}N\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)\right]=0.$$

Therefore, (2.117) follows from (2.123) and (2.118). □

2.7.4. Proof of Lemma 2.29

We recall that

$$\hat{\mathfrak{S}}^N_{1,0}(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \gamma_{k,0}(t) \left(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} \right).$$

Using the fact that for all $A, B \in \mathbb{N}$,

$$|\mathbb{1}_{A=0} - \mathbb{1}_{B=0}| \leqslant |A - B|,$$

from Assumption 2.2 and 2.3 and Lemma 2.16, it follows that

$$\begin{aligned} \left|\hat{\mathfrak{S}}^N_{1,0}(t) - \hat{\mathfrak{S}}^N_{1,0}(s)\right| \leqslant & \frac{(t-s)^\alpha}{\sqrt{N}} \sum_{k=1}^{N} H_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) \\ & + \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sum_{j=1}^{\ell-1} \mathbb{1}_{s-r<\zeta^j_k \leqslant t-r} H_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) \\ & + \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left(H_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) + H_k(0,s,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) \right). \end{aligned} \tag{2.128}$$

Hence as in the setting of (2.92), (2.93) and (2.94) respectively, we establish the tightness of $\hat{\mathfrak{S}}^N_{1,0}$.

3. Stochastic epidemic model with memory of previous infections and waning immunity

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3.1. Introduction

After an individual is first infected by a pathogen, T lymphocytes store information about the pathogen. Consequently, upon reinfection by the same pathogen, there is an immune response. In the first chapter 1 we introduced a stochastic epidemic model which takes into account this immune response. More precisely we introduced a stochastic epidemic model which takes into account the varying infectivity and waning immunity. However, we assume that at each new infection the new infectivity and susceptibility does not depend of their previous values. This is not very realistic. So in the present chapter we extend the individual-based epidemic model to take into account the fact that at each new infection the new infectivity and susceptiblity depend of their previous values. We study a particular case of this model with vaccination policy to understand its impact on the stochastic epidemic model with waning immunity over the long term as in [43]. In [43], the authors assume that at each event (recovery, infection, or vaccination) new curves of infection and susceptibility are randomly chosen independently of the past. Besides, they also assume that there is

no dependence between the individual's level of infectivity following an infection and the resulting level of immunity. They prove that when the basic reproduction number R_0 is greater than some threshold there exists an endemic equilibrium. This threshold depends of the immunity of the individuals in the population.

To do this, we use a parametric approach. More precisely, for each individual, we define his/her *age* as the duration since last infection. We consider a probability space of parameters (also called *traits*) $(\Theta, \mathscr{A}, \nu)$ with $\Theta \subset \mathbb{R}^d$ $(d \geqslant 1)$ and a family of nonnegative functions $(\lambda(\cdot,\theta), \gamma(\cdot,\theta))_{\theta\in\Theta}$ defined on $\mathbb{R}^+$, where $\lambda(a,\theta)$ and $\gamma(a,\theta)$ respectively model the infectivity and the susceptibility at age a of an individual with parameter θ. Infectivity represents the virulence of an infection and susceptibility represents the probability of being reinfected (the lower the susceptibility, the stronger the individual's immunity). In this work, we therefore consider a parametric family of infectivity and susceptibility, satisfying the following assumptions.

Assumption 3.1. *λ and γ are nonnegative measurable functions satisfying the following conditions:*

1. $\lambda \in L^1(\mathbb{R}_+ \times \Theta, \mathrm{d}a\nu(\mathrm{d}\theta))$*, and there is a constant* $\lambda_* > 0$ *such that*
   — $0 \leqslant \lambda \leqslant \lambda_*$*;*
   — $\forall\theta \in \Theta$, $\lambda(\cdot,\theta)$ *has compact support;*
2. $\gamma \in L^1([0,T] \times \Theta, \mathrm{d}a\nu(\mathrm{d}\theta))$ *for any* $T > 0$*, and such that*
   — $0 \leqslant \gamma \leqslant 1$*;*
   — $\forall\theta \in \Theta$, $\gamma(0,\theta) = 0$*;*

In this chapter, we allow the susceptibility to depend on the infectivity (as in [41]). At each reinfection of an individual, a new value of the parameter θ is randomly chosen for this individual and his age drops to zero. Given this parameter, the infectivity and susceptibility curves following the infection are deterministic. We allow the choice of the value of θ to depend on its previous value to keep the memory of the previous infection. Depending on the epidemiological model, we might, for instance, assume that if an individual had one serious infection, the next might be less severe.

Finally, we suppose that the epidemic has been spreading for a long enough time that all individuals have been infected at least once at $t = 0$.

We consider a population of size N. For $k \in \{1,\dots,N\}$, we denote by $(a_k^N(t))_{t\geqslant 0}$ the age process and by $\theta_k^N(t)$ the parameter of the k-th individual at time t. The individuals are in interaction through the force of infection in the population, denoted by $\overline{\mathfrak{F}}^N$. This system is difficult to study for any finite N. However, for each $t \geqslant 0$ and for i.i.d initial values, when the size of the population goes to infinity, we prove that the empirical distribution of $(a_k^N(t), \theta_k^N(t))_{1\leqslant k\leqslant N}$ converges weakly to the solution of a nonlinear equation. Thus there is propagation of chaos. Then we will focus on the long time behaviour of this (deterministic) nonlinear equation. More precisely, we show that this equation admits a non-zero stationary solution and hence there exists of an endemic equilibrium. We obtain a very general endemicity threshold that we compute in some cases. Finally, we show the local stability of our model in the non-Markovian SIRS and SIS model, where the period of infection and recovery follows an exponential law.

Organization of the chapter

The rest of the chapter is organized as follows. In Section 3.2, we define the model in detail. In Section 3.3, we present the main results (functional law of large numbers (FLLN) and endemic equilibrium) of this work. The proofs for the FLLN are given in Section 3.4 and those for the endemic equilibrium in Section 3.5. In Section 3.6, we study some applications.

3.2. The stochastic parametric model with memory

We consider a population of size N. For $k \in \{1,\dots,N\}$, we denote by $(a_k^N(t))_{t\geqslant 0}$ the age process and by $\theta_k^N(t)$ the trait of the k-th individual at time t. We assume that $(a_0^k,\theta_0^k)_{1\leqslant k\leqslant N}$ are i.i.d random variables with distribution μ_0 on $\mathbb{R}^+\times\Theta$ modeling the initial age and parameter of each individual. The individuals are in interaction through the force of infection of the disease in the population, denoted by $\overline{\mathfrak{F}}^N$ and defined hereafter.

Let us denote by μ_t^N the empirical measure of ages and parameters at time t with

$$\mu_t^N = \frac{1}{N}\sum_{k=1}^{N}\delta_{(a_k^N(t),\theta_k^N(t))}. \tag{3.1}$$

At each infection of an individual its age jumps to 0 and a new parameter is randomly chosen. Between two infections, the age grows linearly and the parameter remains unchanged.

We introduce N independent Poisson random measures $(Q_k)_{1\leqslant k\leqslant N}$ on $\mathbb{R}^+\times\Theta\times\mathbb{R}^+$ with intensity $\mathrm{d}z\nu(\mathrm{d}\theta)\mathrm{d}t$. We also introduce a memory kernel $K:\Theta\times\Theta\to\mathbb{R}_+$, satisfying the following.

Assumption 3.2. *K is a nonnegative measurable function on $\Theta\times\Theta$ such that for any $\theta\in\Theta\subset\mathbb{R}^d$,*

$$\int_\Theta K(\theta,\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta}) = 1. \tag{3.2}$$

Then the behavior of $(a_k^N,\theta_k^N)_{1\leqslant k\leqslant N}$ can be described by the following system of stochastic differential equations,

$$\begin{cases} a_k^N(t) &= a_0^k + t - \displaystyle\int_0^t\int_\Theta\int_0^\infty a_k^N(s^-)\mathbb{1}_{\overline{\mathfrak{F}}^N(s^-)\gamma_k^N(s^-)K(\theta_k^N(s^-),\widetilde{\theta})\geqslant z}Q_k\big(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}s\big)\\ \theta_k^N(t) &= \theta_0^k + \displaystyle\int_0^t\int_\Theta\int_0^\infty\big(\widetilde{\theta}-\theta_k^N(s^-)\big)\mathbb{1}_{\overline{\mathfrak{F}}^N(s^-)\gamma_k^N(s^-)K(\theta_k^N(s^-),\widetilde{\theta})\geqslant z}Q_k\big(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}s\big)\\ \gamma_k^N(t) &= \gamma(a_k^N(t),\theta_k^N(t)), \end{cases} \tag{3.3}$$

where the force of infection in the population is given by

$$\overline{\mathfrak{F}}^N(t) = \langle \mu_t^N, \lambda \rangle = \frac{1}{N} \sum_{k=1}^{N} \lambda\big(a_k^N(t), \theta_k^N(t)\big) \tag{3.4}$$

We also denote the mean susceptibility in the population by

$$\overline{\mathfrak{S}}^N(t) = \langle \mu_t^N, \gamma \rangle = \frac{1}{N} \sum_{k=1}^{N} \gamma_k^N(t). \tag{3.5}$$

Consequently, the individual k gets reinfected at time t at rate $\overline{\mathfrak{F}}^N(t)\gamma_k^N(t)$ and if it occurs, his age jumps to 0 and he is assigned a new parameter according to the distribution $K(\theta_k^N(t^-), \widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta})$.

Let us observe that $(a_k, \theta_k)_{1 \leqslant k \leqslant N}$ is a system of interacting piecewise deterministic Markov processes on the Skorohod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+ \times \Theta)$. This stochastic system is well defined and has a unique solution under Assumptions 3.1 and 3.2 (see [28]).

3.3. Main results

The system (3.3) is difficult to study since the individuals are in interaction. However, for each $t \geqslant 0$ and for i.i.d initial values, when the size of the population goes to infinity, we prove that the empirical distribution of $(a_k^N(t), \theta_k^N(t))_{1 \leqslant k \leqslant N}$ converges weakly to the solution of a nonlinear equation. The system being exchangeable, there is propagation of chaos.

We prove the following theorem in Section 3.4.

Theorem 3.1. *Under Assumption 3.2 and Assumption 3.1, as $N \to \infty$, μ^N converges to μ in distribution in $\mathbb{D}$, where $\mu \in \mathbb{D}(\mathbb{R}_+; \mathscr{P}(\mathbb{R}_+ \times \Theta))$ is the weak solution of*

$$\langle \mu_t, f_t \rangle = \langle \mu_0, f_0 \rangle + \int_0^t \langle \mu_s, \partial_a f_s + \partial_s f_s \rangle \mathrm{d}s + \int_0^t \langle \mu_s, \lambda \rangle \langle \mu_s, R f_s \rangle \mathrm{d}s, \tag{3.6}$$

for any bounded function f on $\mathbb{R}_+ \times \mathbb{R}_+ \times \Theta$, and of class $\mathscr{C}^1$ with respect to the first two variables, where the operator R is given by,

$$Rf(a,\theta) = \int_\Theta \big(f(0,\widetilde{\theta}) - f(a,\theta)\big)\gamma(a,\theta)K(\theta,\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta}). \tag{3.7}$$

When μ_0 has a density u_0 with respect to $da\nu(\mathrm{d}\theta)$, the weak solution μ_t of (3.6)

admits a density u_t which solves the following PDE for any $(a,\theta) \in \mathbb{R}_+ \times \Theta$,

$$\begin{cases} \partial_t u_t(a,\theta) + \partial_a u_t(a,\theta) = -\overline{\mathfrak{F}}(t)\gamma(a,\theta)u_t(a,\theta) \\ u(t,0,\theta) = \overline{\mathfrak{F}}(t)\displaystyle\int_{\mathbb{R}_+\times\Theta} \gamma(a,\widetilde{\theta})K(\widetilde{\theta},\theta)u_t(a,\widetilde{\theta})\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}) \qquad (3.8) \\ u(0,a,\theta) = u_0(a,\theta) \\ \overline{\mathfrak{F}}(t) = \displaystyle\int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)u_t(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta). \qquad (3.9) \end{cases}$$

We introduce the deterministic function

$$\overline{\mathfrak{S}}(t,\theta) := \int_{\mathbb{R}_+\times\Theta} \gamma(a,\widetilde{\theta})K(\widetilde{\theta},\theta)u_t(a,\widetilde{\theta})\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}). \qquad (3.10)$$

Then we will focus on the long time behaviour of this (deterministic) nonlinear equation.

It is easy to check the following theorem.

Theorem 3.2. *Given u solution of the system of equation (3.8)-(3.9), the pair $(\overline{\mathfrak{F}},\overline{\mathfrak{S}})$ given by (3.9) and (3.10) respectively, is the unique solution of the following system of integral equations*

$$\begin{cases} \overline{\mathfrak{S}}(t,\theta) = \int_0^t\int_\Theta \gamma(t-b,\widetilde{\theta})\exp\left(-\int_b^t \overline{\mathfrak{F}}(s)\gamma(s-b,\widetilde{\theta})\mathrm{d}s\right)\overline{\mathfrak{S}}(b,\widetilde{\theta})\overline{\mathfrak{F}}(b)K(\widetilde{\theta},\theta)\mathrm{d}b\nu(\mathrm{d}\widetilde{\theta}) \\ \qquad\qquad + \int_0^\infty\int_\Theta \gamma(b+t,\widetilde{\theta})\exp\left(-\int_0^t \overline{\mathfrak{F}}(s)\gamma(b+s,\widetilde{\theta})\mathrm{d}s\right)u_0(b,\widetilde{\theta})K(\widetilde{\theta},\theta)\mathrm{d}b\mathbb{P}(\mathrm{d}\widetilde{\theta}) \\ \overline{\mathfrak{F}}(t) = \int_0^t\int_\Theta \lambda(t-a,\widetilde{\theta})\exp\left(-\int_a^t \overline{\mathfrak{F}}(s)\gamma(s-a,\widetilde{\theta})\mathrm{d}s\right)\overline{\mathfrak{S}}(a,\widetilde{\theta})\overline{\mathfrak{F}}(a)\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}) \\ \qquad\qquad + \int_0^\infty\int_\Theta \lambda(a+t,\widetilde{\theta})\exp\left(-\int_0^t \overline{\mathfrak{F}}(s)\gamma(a+s,\widetilde{\theta})\mathrm{d}s\right)u_0(a,\widetilde{\theta})\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}). \end{cases} \qquad (3.11)$$

Conversely, given $(\overline{\mathfrak{F}},\overline{\mathfrak{S}})$ solution of the system of equations (3.11),

$$u_t(a,\theta) = \begin{cases} u_0(a-t,\theta)\exp\left(-\int_0^t \overline{\mathfrak{F}}(s)\gamma(a-t+s,\theta)\mathrm{d}s\right) & \text{if } a > t \\ \overline{\mathfrak{F}}(t-a)\overline{\mathfrak{S}}(t-a,\theta)\exp\left(-\int_{t-a}^t \overline{\mathfrak{F}}(s)\gamma(s-t+a,\theta)\mathrm{d}s\right) & \text{if } a \leqslant t. \end{cases} \qquad (3.12)$$

is the unique solution of the system of equations (3.8).

Using the fact that u_t is a probability density with respect to $da\nu(\mathrm{d}\theta)$, we get the following Remark.

Remark 3.1. *For all $t \geqslant 0$, $\overline{\mathfrak{F}}$ and $\overline{\mathfrak{S}}$ are such that,*

$$\int_0^\infty\int_\Theta \exp\left(-\int_0^t \overline{\mathfrak{F}}(s)\gamma(r+s,\theta)ds\right)u_0(r,\theta)dr\nu(\mathrm{d}\theta)$$

$$+\int_0^t\int_\Theta \exp\left(-\int_r^t \overline{\mathfrak{F}}(s)\gamma(s-r,\theta)ds\right)\overline{\mathfrak{F}}(r)\overline{\mathfrak{S}}(r,\theta)\mathrm{d}r\nu(\mathrm{d}\theta)=1. \quad (3.13)$$

Then we note that for all $t\geqslant 0$, $\overline{\mathfrak{F}}(t)\leqslant\lambda_*$ and $t\geqslant 0,\theta\in\Theta$, $\overline{\mathfrak{S}}(t,\theta)\leqslant 1$.

Remark 3.2. *In the case $K(\theta,\widetilde{\theta})=K(\widetilde{\theta})$, the system of equations (3.11) is identical to the system of equations (1.15)-(2.21) obtained in chapter 1.*

We make the following Assumptions.

Assumption 3.3. *There exist $\Theta_0\in\mathscr{A}$ with $\mathbb{P}(\Theta_0)=0$, and a positive function $\gamma_*:\Theta\to[0,1]$, such that for any $\theta\in\Theta\setminus\Theta_0$,*

$$\lim_{a\to+\infty}\frac{1}{a}\int_0^a\gamma(s,\theta)\mathrm{d}s=\gamma_*(\theta),$$

exists.

Assumption 3.4.

1. *The kernel K is positive on Θ^2 and the function $\theta\mapsto\sup_{\widetilde{\theta}\in\Theta}K(\widetilde{\theta},\theta)$ is in $L^1(\Theta,\nu)$.*
2. *There exists $c>0$ such that for each $\theta\in\Theta,\gamma_*(\theta)>c$ a.s.*
3. $\forall\theta\in\Theta$,
$$\sup\{t\geqslant 0,\lambda(t,\theta)>0\}\leqslant\inf\{t\geqslant 0,\gamma(t,\theta)>0\}.$$

Assumption 3.3, will be important to prove the existence of a positive $\overline{\mathfrak{F}}_*$. Assumption 3.4-(3) implies that as long as an individual remains infectious, he or she cannot be infected. Assumption 3.4-(2) will be replaced below by a general framework (see Assumption 2).

Then we prove the following theorem in section 3.5.1

Theorem 3.3. *Under Assumptions 3.1, 3.2, 3.8 and 3.3,*

1. *There is a unique eigenfunction $\overline{\mathfrak{S}}_*$, positive ν-a.s., in $L^1(\Theta,\nu)$, such that*
$$\overline{\mathfrak{S}}_*(\theta)=\int_\Theta K(\widetilde{\theta},\theta)\overline{\mathfrak{S}}_*(\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta})\ \text{and}\ \int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)\overline{\mathfrak{S}}_*(\theta)\mathrm{d}a\nu(\mathrm{d}\theta)=1.$$
2. *When*
$$\int_\Theta\frac{1}{\gamma_*(\theta)}\overline{\mathfrak{S}}_*(\theta)\nu(d\theta)<1, \quad (3.14)$$
there exists an endemic equilbrium.

   Moreover, if we also assume that ν-a.s,
$$\forall a\geqslant 0\quad\gamma(a,.)\geqslant\frac{1}{a}\int_0^a\gamma(s,.)\mathrm{d}s, \quad (3.15)$$
this equilibrium is unique, and when $\int_\Theta\frac{1}{\gamma_*(\theta)}\overline{\mathfrak{S}}_*(\theta)\nu(d\theta)>1$ there is no endemic equilibrium.

3.4. Functional law of Large Numbers

Let $(a_0^k,\theta_0^k)_{1\leqslant k\leqslant N}$ be i.i.d random variables with distribution μ_0 on $\mathbb{R}^+\times\Theta$ modeling the initial age and parameter of each individuals, and $(a_k^N,\theta_k^N)_{1\leqslant k\leqslant N}$ solution of the SDEs (3.3). In this section we study the behaviour of the associated empirical measure μ^N, defined by (3.1), when N goes to infinity.

It is easy to check that for all $f:\mathbb{R}_+\times\mathbb{R}_+\times\Theta\to\mathbb{R}_+$, such that for all $\theta\in\Theta$, $(t,a)\mapsto f(t,a,\theta)=:f_t(a,\theta)$ is a continuously differentiable function on $\mathbb{R}_+\times\mathbb{R}_+$, we have

$$\begin{aligned}\langle\mu_t^N,f_t\rangle &= \langle\mu_0^N,f_0\rangle+\int_0^t\langle\mu_s^N,\partial_a f_s+\partial_s f_s\rangle ds\\ &\quad+\frac{1}{N}\sum_{k=1}^N\int_0^t\int_\Theta\int_0^\infty\left(f_s(0,\widetilde{\theta})-f_s(a_k^N(s^-),\theta_k^N(s^-))\right)\mathbb{1}_{\overline{\mathfrak{F}}^N(s^-)\gamma_k^N(s^-)K(\theta_k^N(s^-),\widetilde{\theta})\geqslant z}Q_k(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}s)\\ &= \langle\mu_0^N,f_0\rangle+\int_0^t\langle\mu_s^N,\partial_a f_s+\partial_s f_s\rangle ds+\int_0^t\langle\mu_s^N,\lambda\rangle\langle\mu_s^N,Rf_s\rangle\mathrm{d}s\\ &\quad+\frac{1}{N}\sum_{k=1}^N\int_0^t\int_\Theta\int_0^\infty\left(f_s(0,\widetilde{\theta})-f_s(a_k^N(s^-),\theta_k^N(s^-))\right)\mathbb{1}_{\overline{\mathfrak{F}}^N(s^-)\gamma_k^N(s^-)K(\theta_k^N(s^-),\widetilde{\theta})\geqslant z}\overline{Q}_k(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}s)\end{aligned}\tag{3.16}$$

where $\overline{Q}_k$ is the compensated Poisson measure, and Rf is defined by (3.7).

3.4.1. The deterministic limit

Recall that μ_0 is the initial distribution of ages and parameters in the population.

Remark 3.3. *Note that when $(a_0^k,\theta_0^k)_{1\leqslant k\leqslant N}$ are i.i.d. random variables with distribution μ_0, by the law of large number, it follows that, as $N\to\infty$ a.s μ_0^N converges to μ_0 in distribution in $\mathbb{D}$.*

If μ^N converges weakly to a probability measure $\mu=(\mu_t)_{t\geqslant 0}\in\mathbb{D}(\mathbb{R}_+,\mathscr{P}(\mathbb{R}_+\times\Theta))$, then, from (3.16), μ should satisfy Equation (3.6) for any bounded function f on $\mathbb{R}_+\times\mathbb{R}_+\times\Theta$, and C^1 with respect to the first two variables.

Note that, if μ_0 has a density $u_0(a,\theta)$ with respect to $\mathrm{d}a\nu(\mathrm{d}\theta)$, we have for any $t\geqslant 0$, $\mu_t\ll\mathrm{d}a\nu$ thanks to the Radon-Nikodym theorem. Thus, μ_t has also a density $u(t,a,x):=u_t(a,\theta)$, with respect to $\mathrm{d}a\nu(\mathrm{d}\theta)$ on $\mathbb{R}_+\times\Theta$, and u is a weak solution of (3.8)

We consider the norm in total variation on the space of probability measures on $(\mathbb{R}_+\times\Theta,\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{A},\mathrm{d}a\otimes\nu)$:

$$\|\mu-\nu\|_{\mathrm{TV}}=\sup_{\varphi\in L^\infty(\mathbb{R}_+\times\Theta),\,\|\varphi\|_\infty\leq 1}\left|\langle\mu-\nu,\varphi\rangle\right|.$$

Proposition 3.1. *The solution $\mu=(\mu_t)_{t\geqslant 0}\in\mathbb{D}(\mathbb{R}_+;\mathscr{P}(\mathbb{R}_+\times\Theta))$ to Equation (3.6) is unique.*

Proof. Assume that there are two solutions μ^1 and μ^2 of (3.6) with the same initial value μ_0. Let φ be a test function on $\mathbb{R}_+ \times \Theta$, of class $\mathscr{C}^1$ with respect to its first variable, with $\|\varphi\|_\infty \leq 1$.

For any fixed $t > 0$ and for all $\theta \in \Theta$ the following parametric transport equation,

$$\begin{cases} \partial_s f_s(a,\theta) + \partial_a f_s(a,\theta) = 0 \quad \forall s \in [0,t] \\ f_t(a,\theta) = \varphi(a,\theta), \end{cases}$$

has a unique solution f defined by:

$$\forall t \in \mathbb{R}_+, \forall s \in [0,t], \forall (a,\theta) \in \mathbb{R}_+ \times \Theta, \quad f(s,a,\theta) = \varphi(a-(s-t),\theta).$$

Consequently, from (3.7) and (3.6), for $i \in \{1,2\}$,

$$\langle \mu_t^i, \varphi \rangle = \langle \mu_0, \varphi_t \rangle + \int_0^t \langle \mu_s^i, \lambda \rangle \langle \mu_s^i, R\varphi_{t-s} \rangle \mathrm{d}s, \tag{3.17}$$

where $\varphi_s(a,\theta) = \varphi(a+s,\theta)$. Let us note that for $\|\varphi\|_\infty \leqslant 1, |R\varphi_{t-s}(a,\theta)| \leqslant 2$. Therefore, using Assumption 3.1 and Assumption 3.2 it follows that, for $t \geqslant 0$

$$\begin{aligned} \left|\langle \mu_t^1 - \mu_t^2, \varphi \rangle\right| &\leqslant \int_0^t \left|\langle \mu_s^1 - \mu_s^2, \lambda \rangle\right| \left|\langle \mu_s^1, R\varphi_{t-s} \rangle\right| \mathrm{d}s + \int_0^t \left|\langle \mu_s^2, \lambda \rangle\right| \left|\langle \mu_s^1 - \mu_s^2, R\varphi_{t-s} \rangle\right| \mathrm{d}s \\ &\leqslant 4\lambda_* \int_0^t \|\mu_s^1 - \mu_s^2\|_{TV} \mathrm{d}s. \end{aligned}$$

Since this class of test functions is dense in $\{\varphi \in L^\infty(\mathbb{R}_+ \times \Theta, \mathrm{d}a \otimes \nu) : \|\varphi\|_\infty \leqslant 1\}$, we deduce that

$$\sup_{0 \leqslant u \leqslant t} \|\mu_u^1 - \mu_u^2\|_{\mathrm{TV}} \leqslant 4\lambda_* \int_0^t \sup_{0 \leqslant u \leqslant s} \|\mu_u^1 - \mu_u^2\|_{TV} \mathrm{d}s,$$

and by Gronwall's lemma we obtain $\sup_{0 \leqslant u \leqslant t} \|\mu_u^1 - \mu_u^2\|_{\mathrm{TV}} = 0 \ \forall t > 0$. Then uniqueness is proved.

□

3.4.2. Propagation of chaos

In this section we prove Theorem 3.1. But first, let us make a few comments.

From Proposition 3.1 and Theorem 3.1, a straightforward consequence is the following result.

Corollary 3.1. *There exists a unique solution to Equation (3.6).*

Remark 3.4. *As we will see in the proof of Theorem 3.1, the solution μ of Equation (3.6) is the distribution of a pair of random processes $(a(t),\theta(t))_{t \geqslant 0}$ solution of the following*

stochastic differential system

$$\begin{cases} a(t) = a_0 + t - \displaystyle\int_0^t \int_\Theta \int_0^\infty a(s^-) \mathbb{1}_{\overline{\mathfrak{F}}(s^-)\gamma(s^-)K(\theta(s^-),\widetilde{\theta})\geqslant z} Q(\mathrm{d}z, \mathrm{d}\widetilde{\theta}, \mathrm{d}s) \\ \theta(t) = \theta_0 + \displaystyle\int_0^t \int_\Theta \int_0^\infty \big(\widetilde{\theta} - \theta(s^-)\big) \mathbb{1}_{\overline{\mathfrak{F}}(s^-)\gamma(s^-)K(\theta(s^-),\widetilde{\theta})\geqslant z} Q(\mathrm{d}z, \mathrm{d}\widetilde{\theta}, \mathrm{d}s) \\ \gamma(t) = \gamma(a(t), \theta(t)), \end{cases}$$

where (a_0, θ_0) *is a random variable with distribution* μ_0, $\overline{\mathfrak{F}}$ *is defined by* (3.9)*, and* Q *is a Poisson measure on* $\mathbb{R}_+ \times \Theta \times \mathbb{R}_+$ *with intensity* $\mathrm{d}z\nu(\mathrm{d}\widetilde{\theta})\mathrm{d}t$*. We also note that* $\overline{\mathfrak{F}}(t) = \mathbb{E}[\lambda(a(t), \theta(t))]$.

To prove Theorem 3.1, we will first prove $\mathscr{C}$-tightness of the sequence $\left(\mu^N\right)$, then we will identify the limits as solution of Equation (3.6). By uniqueness of the solution of Equation (3.6), we will deduce the convergence of $\left(\mu^N\right)$ to μ on any time interval $[0, T]$.

For the $\mathscr{C}$-tightness of $\left(\mu^N\right)$, we will use the following criterion (see [77, Lemma 3.1]).

Lemma 3.5. *Let* $\left(X^N\right)_{N\geq 1}$ *be a sequence of random processes taking values in the Skorohod space* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ *such that* $X^N(0) = 0$*. If for all* $T > 0$, $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \sup_{0\leq t\leq T} \frac{1}{\delta} \mathbb{P}\Big(\sup_{0\leq u\leq \delta} |X^N(t+u) - X^N(t)| > \epsilon \Big) = 0,$$

then the sequence X^N *is* $\mathbf{C}$*-tight in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

Lemma 3.6. *The sequence* $\left(\mu^N\right)_{N\geqslant 1}$ *defined by* (3.1) *is* $\mathbf{C}$*-tight in* $\mathbb{D}\left(\mathbb{R}_+, \mathscr{P}(\mathbb{R}_+ \times \Theta)\right)$.

Proof. Let $\mathscr{C}_0(\mathbb{R}_+ \times \Theta)$ be the space of continuous real-valued function converging to 0 at infinity, with the norm $\|f\|_\infty = \sup_{a\geqslant 0, \theta\in\Theta} |f(a,\theta)|$. Fix $T > 0$. From [86, Theorem 2.1], it suffices to prove that $\langle \mu^N, f\rangle$ is tight in $\mathbb{D}([0,T], \mathbb{R})$ for any $f \in \mathscr{C}_0(\mathbb{R}_+ \times \Theta)$ with a derivative with respect to the first variable and $\|\partial_a f\|_\infty < +\infty$, which is a dense family in $\mathscr{C}_0(\mathbb{R}_+ \times \Theta)$.

We have, for all $0 \leqslant s \leqslant t \leqslant T$, as f and $\partial_a f$ are bounded,

$$\begin{aligned} &\left|\langle \mu_t^N, f\rangle - \langle \mu_s^N, f\rangle\right| \\ &\leqslant \int_s^t \langle \mu_r^N, \partial_a f\rangle \mathrm{d}r \\ &\qquad + \frac{1}{N} \sum_{k=1}^N \int_s^t \int_\Theta \int_0^\infty \left|f(0,\widetilde{\theta}) - f(a_k^N(r^-), \theta_k^N(r^-))\right| \mathbb{1}_{\overline{\mathfrak{F}}^N(r^-)\gamma_k^N(r^-)K(\theta_k^N(r^-),\widetilde{\theta})\geqslant z} Q_k\big(\mathrm{d}z, \mathrm{d}\widetilde{\theta}, \mathrm{d}r\big) \\ &\leqslant \|\partial_a f\|_\infty |t-s| + \frac{2\|f\|_\infty}{N} \sum_{k=1}^N \int_s^t \int_\Theta \int_0^\infty \mathbb{1}_{\overline{\mathfrak{F}}^N(r^-)\gamma_k^N(r^-)K(\theta_k^N(r^-),\widetilde{\theta})\geqslant z} Q_k\big(\mathrm{d}z, \mathrm{d}\widetilde{\theta}, \mathrm{d}r\big) \\ &\leqslant \big(\|\partial_a f\|_\infty + 2\lambda_* \|f\|_\infty\big) |t-s| + \frac{2\|f\|_\infty}{N} \sum_{k=1}^N \int_s^t \int_\Theta \int_0^\infty \mathbb{1}_{\overline{\mathfrak{F}}^N(r^-)\gamma_k^N(r^-)K(\theta_k^N(r^-),\widetilde{\theta})\geqslant z} \overline{Q}_k\big(\mathrm{d}z, \mathrm{d}\widetilde{\theta}, \mathrm{d}r\big), \end{aligned}$$

where the last line follows from Assumption 3.2 and that $\overline{\mathfrak{F}}^N(r^-) \leqslant \lambda_*, \gamma \leqslant 1$ from Assumption 3.1.

Hence for δ small enough that $\left(\|\partial_a f\|_\infty + 2\lambda_* \|f\|_\infty\right)\delta \leqslant \frac{\epsilon}{2}$, by Doob's maximal inequality,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq v\leq\delta} \left|\langle \mu^N_{t+v}, f\rangle - \langle \mu^N_t, f\rangle\right| \geqslant \epsilon\right)\\
&\leqslant \mathbb{P}\left(\sup_{0\leq v\leq\delta} \left|\frac{1}{N}\sum_{k=1}^N \int_t^{t+v}\int_\Theta\int_0^\infty \mathbb{1}_{\overline{\mathfrak{F}}^N(r^-)\gamma_k^N(r^-)K(\theta_k^N(r^-),\widetilde{\theta})\geqslant z}\overline{Q}_k\big(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}r\big)\right| \geqslant \frac{\epsilon}{4\|f\|_\infty}\right)\\
&\leqslant \frac{16\|f\|_\infty^2}{\epsilon^2}\mathbb{E}\left[\left(\frac{1}{N}\sum_{k=1}^N \int_t^{t+\delta}\int_\Theta\int_0^\infty \mathbb{1}_{\overline{\mathfrak{F}}^N(r^-)\gamma_k^N(r^-)K(\theta_k^N(r^-),\widetilde{\theta})\geqslant z}\overline{Q}_k\big(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}r\big)\right)^2\right]\\
&= \frac{16\|f\|_\infty^2}{N\epsilon^2}\mathbb{E}\left[\left(\int_t^{t+\delta}\int_\Theta\int_0^\infty \mathbb{1}_{\overline{\mathfrak{F}}^N(r^-)\gamma_1^N(r^-)K(\theta_1^N(r^-),\widetilde{\theta})\geqslant z}\overline{Q}_1\big(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}r\big)\right)^2\right]\\
&= \frac{16\|f\|_\infty^2}{N\epsilon^2}\int_t^{t+\delta}\int_\Theta \mathbb{E}\left[\overline{\mathfrak{F}}^N(r^-)\gamma_1^N(r^-)K(\theta_1^N(r),\theta)\right]\nu(\mathrm{d}\theta)\mathrm{d}r\\
&\leqslant \frac{16\lambda_*\|f\|_\infty^2}{N\epsilon^2}\delta
\end{aligned}$$

where the third line follows from the orthogonality as $(Q_k)_k$ are independent and the family of processes $(a_k^N, \theta_k^N, Q_k)_k$ is exchangeable and the last line follows from Assumption 3.1 and Assumption 3.2.

Then

$$\lim_{\delta\to 0}\limsup_N \sup_{0\leq t\leq T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\leq v\leq\delta}\left|\langle \mu^N_{t+v}, f\rangle - \langle \mu^N_t, f\rangle\right| \geqslant \epsilon\right) = 0.$$

Thus from Lemma 3.5 $\left(\langle \mu_t^N, f\rangle\right)_{N\geqslant 1}$ is $\mathscr{C}$-tight. □

Let f be a bounded measurable function defined on $\mathbb{R}_+\times\mathbb{R}_+\times\Theta$ to $\mathbb{R}$. We introduce

$$\mathscr{Z}^N f(t) = \frac{1}{N}\sum_{k=1}^N \int_0^t\int_\Theta\int_0^\infty \left(f_s(0,\widetilde{\theta}) - f_s(a_k^N(s^-),\theta_k^N(s^-))\right)\mathbb{1}_{\overline{\mathfrak{F}}^N(s^-)\gamma_k^N(s^-)K(\theta_k^N(s^-),\widetilde{\theta})\geqslant z}\overline{Q}_k\big(\mathrm{d}z,\mathrm{d}\widetilde{\theta},\mathrm{d}s\big), \tag{3.18}$$

where $\left(\overline{Q}_k\right)_{1\leqslant k\leqslant N}$ are the compensated Poisson measures.

Lemma 3.7. *For all $0\leqslant t\leqslant T$, for all bounded function f, as $N\to\infty$,*

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left(\mathscr{Z}^N f(t)\right)^2\right] \to 0. \tag{3.19}$$

Proof. Note that $\left(\mathscr{Z}^N f(t)\right)_{t\geqslant 0}$ is a martingale. First by the Burkholder-Davis-Gundy inequality, and then using the fact that the family $(Q_k)_k$ is independent, the family $(\overline{Q}_k)_k$ is orthogonal, then it follows that,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left(\mathcal{Z}^N f(t)\right)^2\right] &\leqslant 4\mathbb{E}\left[\langle \mathcal{Z}^N f\rangle_T\right] \\
&= \frac{4}{N^2}\sum_{k=1}^{N}\int_0^T\int_\Theta \mathbb{E}\left[\left(f_s(0,\theta)-f_s(a_k^N(s),\theta_k^N(s))\right)^2\overline{\mathfrak{F}}^N(s)\gamma_k^N(s)K(\theta_k^N(s),\theta)\right]\nu(\mathrm{d}\theta)\mathrm{d}s \\
&= \frac{4}{N}\int_0^T\int_\Theta \mathbb{E}\left[\left(f_s(0,\theta)-f_s(a_1^N(s),\theta_1^N(s))\right)^2\overline{\mathfrak{F}}^N(s)\gamma_1^N(s)K(\theta_1^N(s),\theta)\right]\nu(\mathrm{d}\theta)\mathrm{d}s \\
&\leqslant \frac{16\|f\|_\infty^2 T\lambda_*}{N},
\end{aligned}
$$

where the third line follows by exchangeability and the last line by Assumption 3.1 and Assumption 3.2. This conclude the proof. □

We deduce from (3.16) that the limit of any converging subsequence of (μ^N) satisfies Equation (3.6), and by the uniqueness stated in Proposition 3.1, Theorem 3.1 is proved.

3.5. Long time behavior

We do not assume that the susceptibility is increasing, unlike in chapter 1.

3.5.1. Existence of a stationary measure

We adapt the proof of [19] to study the equilibria of (3.6). Assume that μ_0 has a density u_0 with respect to $\mathrm{d}a\nu(\mathrm{d}\theta)$ on $\mathbb{R}_+\times\Theta$. Let us recall that the limit measure μ_t has a density u_t solution of (3.8), and given by

$$
u_t(a,\theta)=\begin{cases} u_0(a-t,\theta)\exp\left(-\int_0^t\overline{\mathfrak{F}}(s)\gamma(a-t+s,\theta)ds\right) & \text{if } a>t \\ \overline{\mathfrak{F}}(t-a)\overline{\mathfrak{S}}(t-a,\theta)\exp\left(-\int_{t-a}^t\overline{\mathfrak{F}}(s)\gamma(s-t+a,\theta)ds\right) & \text{if } a\leqslant t\end{cases}
$$

where $\overline{\mathfrak{F}}$, and $\overline{\mathfrak{S}}$ are defined by (3.9) and (3.10).

Consequently, if it exists, a stationary solution u_* of (3.8) is obviously given by the following system:

$$
\begin{cases} u_*(a,\theta)=\overline{\mathfrak{F}}_*\overline{\mathfrak{S}}_*(\theta)\exp\left(-\overline{\mathfrak{F}}_*\int_0^a\gamma(s,\theta)\mathrm{d}s\right) \\ \overline{\mathfrak{F}}_*=\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta) \\ \overline{\mathfrak{S}}_*(\theta)=\int_{\mathbb{R}_+\times\Theta}\gamma(a,\widetilde{\theta})K(\widetilde{\theta},\theta)u_*(a,\widetilde{\theta})\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}).\end{cases} \tag{3.20}
$$

Obviously $u_*\equiv 0$ is solution of (3.20). We are looking for a criterion for the existence of probability density solutions on $\mathbb{R}_+\times\Theta$ of (3.20), i.e such that

$$
\int_0^\infty\int_\Theta u_*(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta)=1. \tag{3.21}
$$

To achieve this we study the existence of a nonnegative solution $\left(x,\overline{\mathfrak{S}}\right)$, denoted by $\left(\overline{\mathfrak{F}}_*,\overline{\mathfrak{S}}_*\right)$, of the following system:

$$\begin{cases} x\displaystyle\int_{\mathbb{R}_+\times\Theta}\overline{\mathfrak{S}}(\theta)\exp\left(-x\int_0^a\gamma(s,\theta)\mathrm{d}s\right)\mathrm{d}a\nu(\mathrm{d}\theta)=1 & (3.22)\\ \overline{\mathfrak{S}}(\theta)=x\displaystyle\int_{\mathbb{R}_+\times\Theta}\gamma(a,\widetilde{\theta})K(\widetilde{\theta},\theta)\overline{\mathfrak{S}}(\widetilde{\theta})\exp\left(-x\int_0^a\gamma(s,\widetilde{\theta})\mathrm{d}s\right)\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta}), & (3.23)\\ x=x\displaystyle\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)\overline{\mathfrak{S}}(\theta)\exp\left(-x\int_0^a\gamma(s,\theta)\mathrm{d}s\right)\mathrm{d}a\nu(\mathrm{d}\theta) & (3.24)\end{cases}$$

where the first equation is related to (3.21) to ensure that u_* is a probability density function, and the second equation is the last condition in (3.20).

We now make the following assumptions to obtain the existence of a density solution u_* of (3.20). The following first set of hypothesis, Assumption 3.8, will be used to prove the existence of $\overline{\mathfrak{S}}_*$.

Assumption 3.8. *1. The kernel K is positive on Θ^2 and the function $\theta\mapsto\sup_{\widetilde{\theta}\in\Theta}K(\widetilde{\theta},\theta)$ is in $L^1(\Theta,\nu)$.*

2. There exist $\Theta_0\in\mathscr{A}$ with $\mathbb{P}(\Theta_0)=0$, such that

a) $\forall\theta\in\Theta\setminus\Theta_0$

$$\int_0^\infty\gamma(a,\theta)\mathrm{d}a=\infty.$$

b) $\forall x>0$,

$$\sup_{\theta\in\Theta\setminus\Theta_0}\int_{\mathbb{R}_+}\exp\left(-x\int_0^a\gamma(s,\theta)\mathrm{d}s\right)\mathrm{d}a<\infty.$$

3. $\forall\theta\in\Theta$,

$$\sup\{t\geqslant 0,\,\lambda(t,\theta)>0\}\leqslant\inf\{t\geqslant 0,\,\gamma(t,\theta)>0\}.$$

Remark 3.5. — *For the existence of a density solution to (3.20), we do not really need K to be positive, we indeed need that K satisfies Condition 2 of Theorem 3.4.*

— *Assumption 3.3 implies Assumption 3.8 condition 2a.*

— *When $\gamma(a,.)$ has a positive limit when $a\to+\infty$ (as assumed in [41, Assumption 4.1]), i.e. there is a function $\gamma_*(\cdot)$ on Θ such that $\lim_{s\to\infty}\gamma(s,.)=\gamma_*(.)$ ν-a.s., then Assumption 3.3 is satisfied.*

— *If $\gamma(.,\theta)$ is a periodic function, then Assumption (3.3) is satisfied.*

— *Under Assumption 3.3, if γ_* has a positive lower bound on Θ_0, then Assumptions 3.8(ii) are satisfied.*

We first note that by Assumption 3.8-(ii)(a), for any $x>0$ and $\widetilde{\theta}\notin\Theta_0$,

$$x\int_{\mathbb{R}_+}\gamma(a,\widetilde{\theta})\exp\left(-x\int_0^a\gamma(s,\widetilde{\theta})\mathrm{d}s\right)\mathrm{d}a=1.$$

Consequently, Equation (3.23) becomes

$$\overline{\mathfrak{S}}(\theta) = \int_\Theta K(\widetilde{\theta},\theta)\overline{\mathfrak{S}}(\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta}).$$

We introduce the linear operator T from $L^1(\Theta,\nu)$ to $L^1(\Theta,\nu)$ defined by

$$T(B)(\theta) = \int_\Theta K(\widetilde{\theta},\theta)B(\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta}). \tag{3.25}$$

Remark 3.6. *We easily observe, by Assumption 3.2, $\forall B \in L^1(\Theta)$,*

$$\|T(B)\|_{L^1(\Theta,\nu)} \leqslant \|B\|_{L^1(\Theta,\nu)} \quad \textit{and} \quad \int_\Theta T(B)(\theta)\nu(\mathrm{d}\theta) = \int_\Theta B(\theta)\nu(\mathrm{d}\theta).$$

It follows from Remark 3.6 that the spectral radius $\rho(T)$ of T is smaller than 1 and the only possible eigenvalue with an integrable nonnegative eigenfunction is 1. Moreover, for eigenvalues different form 1, the associated eigenfunctions B satisfy $\int_\Theta B(\theta)\nu(\mathrm{d}\theta) = 0$.

From (3.23), we are looking for an integrable nonnegative solution $\overline{\mathfrak{S}}$ of $\overline{\mathfrak{S}}(\theta) = T(\overline{\mathfrak{S}})$. We now recall the following spectral property for some integral operators.

Theorem 3.4. *[89, Theorem 6.6, Chapter 6] Let $E := L^p(\Theta,\mathscr{A},\nu)$, where $1 \leqslant p \leqslant \infty$ and $(\Theta,\mathscr{A},\nu)$ is a σ-finite measure space. Let $T \in \mathscr{L}(E)$ be an integral operator given by a measurable kernel $K \geqslant 0$ fulfilling*

1. *some power of T is compact;*
2. *for any $S \in \mathscr{A}$ such that $\mathbb{P}(S) > 0$ and $\mathbb{P}(\Theta \setminus S) > 0$*

$$\int_{\Theta\setminus S}\int_S K(\widetilde{\theta},\theta)\nu(\mathrm{d}\theta)\nu(\mathrm{d}\widetilde{\theta}) > 0.$$

Then the spectral radius $\rho(T)$ of T is strictly positive, with a unique normalized eigenfunction $\overline{\mathfrak{S}}$ satisfying $\overline{\mathfrak{S}}(\theta) > 0$ ν-a.e. Moreover, if $K > 0$ $\nu\otimes\nu$-a.e., then any eigenvalue κ of T different from $\rho(T)$ has modulus $|\kappa| < \rho(T)$.

From Remark 3.6 and Theorem 3.4, we easily deduce the following proposition.

Proposition 3.2. *Under Assumptions 3.1, 3.2 and 3.8, the spectral radius of T defined by (3.25) is $\rho(T) = 1$. There is a unique eigenfunction $\overline{\mathfrak{S}}_*$, positive ν-a.s., in $L^1(\Theta,\nu)$, associated to the eigenvalue 1, such that*

$$\int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)\overline{\mathfrak{S}}_*(\theta)\mathrm{d}a\nu(\mathrm{d}\theta) = 1. \tag{3.26}$$

Proof. From Assumption 3.8-(i), we have

$$\int_\Theta \sup_{\widetilde{\theta}\in\Theta} K(\widetilde{\theta},\theta)\nu(\mathrm{d}\theta) < \infty,$$

consequently the operator T is of Hille-Tamarkin type (see [59, Section 11.3]) and it follows from [59, Theorem 11.9] that the square of the operator T^2 from $L^1(\Theta,\nu)$ to $L^1(\Theta,\nu)$ is compact. Moreover as the application $\theta \mapsto K(.,\theta)$ is a positive density on Θ, Condition (2) of Theorem 3.4 is also satisfied. Then, by Theorem 3.4, the spectral radius $\rho(T)$ of the operator T is a strictly positive, isolated simple eigenvalue of T, and it is the only eigenvalue with a corresponding normalised positive a.e. eigenfunction $\overline{\mathfrak{S}}_*(\theta)$.

Moreover, by Remark 3.6, we note that $\rho(T) = 1$. We thus define $\overline{\mathfrak{S}}_*$ as a positive eigenfunction of T in $L^1(\Theta,\nu)$ associated to the eigenvalue 1. Since $\overline{\mathfrak{S}}_*$ can be choosen up to a multiplicative constant, we take it such that Condition (3.26) is satisfied. We note that $\overline{\mathfrak{S}}_*$ satisfying (3.26) is well defined since $0 \leqslant \lambda \leqslant \lambda_*$. □

Given $\overline{\mathfrak{S}}_*$ defined in Proposition 3.2, we are now looking for $x > 0$ such that (3.22) holds.

We introduce $H : \mathbb{R}_+ \to \mathbb{R}_+$ the function defined by

$$H(x) = x \int_{\mathbb{R}_+\times\Theta} \exp\left(-x \int_0^a \gamma(s,\theta)\mathrm{d}s\right) \overline{\mathfrak{S}}_*(\theta)\mathrm{d}a\nu(\mathrm{d}\theta). \tag{3.27}$$

Lemma 3.9. *Under Assumption 3.3, the function $H : \mathbb{R}_+ \to \mathbb{R}_+$ is well defined and continuous. Moreover*

$$H(0) = \int_\Theta \frac{\overline{\mathfrak{S}}_*(\theta)}{\gamma_*(\theta)}\nu(d\theta), \text{ and } \lim_{x\to+\infty} H(x) = +\infty.$$

Proof. Since $\overline{\mathfrak{S}}_* \in L^1(\Theta,\nu)$, from Assumption 3.8 $(iii)(b)$, H is well defined and from expression (3.27), H is continuous.

Using the change of variables $b = ax$, and then $u = xs$, we have

$$H(x) = \int_{\mathbb{R}_+\times\Theta} \exp\left(-x \int_0^{b/x} \gamma(s,\theta)\mathrm{d}s\right) \overline{\mathfrak{S}}_*(\theta)\mathrm{d}b\nu(\mathrm{d}\theta) \tag{3.28}$$

$$= \int_{\mathbb{R}_+\times\Theta} \exp\left(-\int_0^b \gamma\left(\frac{u}{x},\theta\right)\mathrm{d}u\right) \overline{\mathfrak{S}}_*(\theta)\mathrm{d}b\nu(\mathrm{d}\theta). \tag{3.29}$$

Hence, from Assumption 3.3 for the limit when x goes to 0, and $\gamma \in [0,1]$ with $\gamma(0,\theta) = 0$ for any $\theta \in \Theta$ and by Fatou's Lemma for the limit when x goes to $+\infty$, it follows that,

$$H(0) = \int_\Theta \frac{1}{\gamma_*(\theta)} \overline{\mathfrak{S}}_*(\theta)\nu(d\theta), \text{ and } \lim_{x\to+\infty} H(x) = +\infty.$$

□

Using Lemma 3.9 and by continuity of H, we deduce the following proposition.

Proposition 3.3. *Let H be the function defined by (3.27). Under Assumptions 3.3, when*

$$\int_\Theta \frac{1}{\gamma_*(\theta)} \overline{\mathfrak{S}}_*(\theta)\nu(d\theta) < 1,$$

there exists $\overline{\mathfrak{F}}_* > 0$ *such that* $H(\overline{\mathfrak{F}}_*) = 1$.

Theorem 3.5. *Under Assumptions 3.1, 3.2, 3.8 and 3.3, when*

$$\int_\Theta \frac{1}{\gamma_*(\theta)} \overline{\mathfrak{S}}_*(\theta) \nu(d\theta) < 1, \tag{3.30}$$

there exists an endemic equilbrium.

Moreover, if we also assume that ν*-a.s,*

$$\forall a \geqslant 0 \quad \gamma(a,.) \geqslant \frac{1}{a} \int_0^a \gamma(s,.) \mathrm{d}s, \tag{3.31}$$

this equilibrium is unique. Otherwise if $\int_\Theta \frac{1}{\gamma_*(\theta)} \overline{\mathfrak{S}}_*(\theta) \nu(d\theta) > 1$*, there is no endemic equilibrium.*

Proof. Let $\overline{\mathfrak{S}}_*$ be given by Proposition 3.2 and $\overline{\mathfrak{F}}_*$ by Proposition 3.3, we consider the function u_* defined on $\mathbb{R}_+ \times \Theta$ by

$$u_*(a,\theta) = \overline{\mathfrak{F}}_* \overline{\mathfrak{S}}_*(\theta) \exp\left(-\overline{\mathfrak{F}}_* \int_0^a \gamma(s,\theta) \mathrm{d}s\right).$$

By Assumption 3.8 condition 3, $supp(\lambda) \cap supp(\gamma) = \emptyset$, we then observe that $\forall x \geqslant 0$,

$$\int_{\mathbb{R}_+ \times \Theta} \lambda(a,\theta) \overline{\mathfrak{S}}_*(\theta) \exp\left(-x \int_0^a \gamma(s,\theta) ds\right) \mathrm{d}a\nu(\mathrm{d}\theta) = \int_{\mathbb{R}_+ \times \Theta} \lambda(a,\theta) \overline{\mathfrak{S}}_*(\theta) \mathrm{d}a\nu(\mathrm{d}\theta) = 1. \tag{3.32}$$

Since the pair $(\overline{\mathfrak{F}}_*, \overline{\mathfrak{S}}_*)$ satisfies the system of equation (3.22)-(3.24), and by (3.32), we deduce that u_* is a solution of the system (3.20).

Moreover, we easily compute from (3.28)

$$\begin{aligned} H'(x) &= \int_{\mathbb{R}_+ \times \Theta} \left(-\int_0^{b/x} \gamma(s,\theta) \mathrm{d}s + \frac{b}{x} \gamma\left(\frac{b}{x},\theta\right)\right) \exp\left(-x \int_0^{b/x} \gamma(s,\theta) \mathrm{d}s\right) \overline{\mathfrak{S}}_*(\theta) \mathrm{d}b\mathbb{P}(\mathrm{d}\theta) \\ &= x \int_{\mathbb{R}_+ \times \Theta} \left(-\int_0^a \gamma(s,\theta) \mathrm{d}s + a\gamma(a,\theta)\right) \exp\left(-x \int_0^a \gamma(s,\theta) \mathrm{d}s\right) \overline{\mathfrak{S}}_*(\theta) \mathrm{d}a\mathbb{P}(\mathrm{d}\theta). \end{aligned}$$

Then when ν-a.s., for any $a \geqslant 0$, $a\gamma(a,.) \geqslant \int_0^a \gamma(s,.) \mathrm{d}s$, H is non-decreasing and the conclusion follows. □

Remark 3.7. *Under Assumptions 3.1, 3.2 and 3.8,* $\overline{\mathfrak{S}}_*$ *is uniquely defined. An extra condition, given in Proposition 3.3, is needed for the existence of a positive* $\overline{\mathfrak{F}}_*$*.*

Since $\int_{\mathbb{R}_+ \times \Theta} \lambda(a,\theta) \overline{\mathfrak{S}}_*(\theta) \mathrm{d}a\mathbb{P}(\mathrm{d}\theta) = 1$*, the condition of existence of an endemic equilibrium in Theorem 3.5, can be written*

$$\mathbb{E}^*\left[\frac{1}{\gamma_*}\right] < R_0^* \quad \textit{with } R_0^* = \mathbb{E}^*\left[\int_0^\infty \lambda(a) \mathrm{d}a\right],$$

where $\mathbb{E}^*$ is the expectation on Θ with respect to the measure $\kappa^{-1}\overline{\mathfrak{S}}_*(\theta)\nu(\mathrm{d}\theta)$, with $\kappa = \int_\Theta \overline{\mathfrak{S}}_*(\theta)\nu(\mathrm{d}\theta)$.
Consequently, up to a change of measure, we have a generalisation of the condition of existence of an endemic equilibrium obtained in chapter 1.

Remark 3.8. *When $a \mapsto \gamma(a,.)$ is ν-a.s. non-decreasing, then Condition (3.31) is satisfied, and H given by (3.29) is obviously non-decreasing. But as we will notice in Section 3.6.1, Condition (3.31) is not a necessary condition for H to be non-decreasing.*
When H is non-decreasing, then Condition (3.30) becomes a sufficient and necessary condition for the existence of an endemic equilibrium.

Even if Condition (3.31) is not a necessary condition for H to be non-decreasing, we remark in the following specific example that it can be an optimal condition for the monotony of H.

Example 3.1. *Fix $\alpha, \beta \in (0,1]$, with $\alpha \geqslant \beta$, and T_V, T_R such that $0 < T_R < T_V$. We consider the non-monotone function $\gamma(a) = \alpha \mathbb{1}_{T_R<a<T_V} + \beta \mathbb{1}_{a\geqslant T_V}$, independent of θ. Assumption 3.3 is satisfied with $\gamma_* = \beta$. We easily compute from (3.27) that*

$$H(x) = \kappa\left(xT_R + \frac{1}{\alpha} - \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\mathrm{e}^{-x(T_V-T_R)\alpha}\right),$$

where $\kappa = \int_\Theta \overline{\mathfrak{S}}_*(\theta)\nu(\mathrm{d}\theta)$. We observe that H is non-decreasing if and only if $\beta T_V \geqslant \alpha(T_V - T_R)$.
On the other hand, we have

$$\int_0^a \gamma(s)\mathrm{d}s = \alpha(a - T_R)\mathbb{1}_{T_R<a<T_V} + \big(\alpha(T_V - T_R) + \beta(a - T_V)\big)\mathbb{1}_{a\geqslant T_V},$$

which satisfies $\gamma(a) \geqslant \frac{1}{a}\int_0^a \gamma(s)\mathrm{d}s$ for any $a \geqslant 0$ if and only if $\beta T_V \geqslant \alpha(T_V - T_R)$.

We now compute the value of $\overline{\mathfrak{S}}_*$ for special cases of $K(\cdot,\cdot)$ in the following example.

Example 3.2. *1. When there is no memory of the previous infections, we have $K(\widetilde{\theta}) := K(\theta,\widetilde{\theta})$ independent of θ with $\int_\Theta K(\widetilde{\theta})\nu(\mathrm{d}\widetilde{\theta}) = 1$. Up to a change of probability measure on $(\Theta, \mathscr{A}, \nu)$, we can always assume $K \equiv 1$. Then, we easily deduce that $\overline{\mathfrak{S}}_*$ is a constant, and using condition (3.26), we have*

$$\overline{\mathfrak{S}}_* = \left(\int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta)\right)^{-1} = \frac{1}{R_0}.$$

Let us recall that under Assumptions 1.1, 1.3 and 1.4 of chapter 1, Assumptions 3.8 and 3.3 are satisfied and $\gamma_*(\theta) = \lim_{a\to\infty}\gamma(a,\theta)$. Then, by Proposition 3.3, there is existence of an endemic equilibrium when

$$R_0 > \int_\Theta \frac{1}{\gamma_*(\theta)}\nu(\mathrm{d}\theta) = \mathbb{E}\left[\frac{1}{\gamma_*}\right],$$

where $\mathbb{E}$ is the expectation on Θ with respect to ν. As noted in Remark 3.7, we recover the condition of chapter 1 in the case of a parametric model.

2. *If $K(\cdot,\cdot)$ is symmetric, i.e for each $\theta,\widetilde{\theta}\in\Theta$, $K(\widetilde{\theta},\theta)=K(\theta,\widetilde{\theta})$, then from Assumption 3.2,*
$$\int_\Theta K(\widetilde{\theta},\theta)\nu(d\widetilde{\theta})=\int_\Theta K(\theta,\widetilde{\theta})\nu(d\widetilde{\theta})=1.$$
Consequently, by Proposition 3.2, $\overline{\mathfrak{S}}_*=\frac{1}{R_0}$ and the condition of existence of an endemic equilibrium is
$$R_0>\int_\Theta\frac{1}{\gamma_*(\theta)}\nu(\mathrm{d}\theta)=\mathbb{E}\left[\frac{1}{\gamma_*}\right].$$

3. *We assume that there exists a density π on Θ such that $\pi(\theta)K(\theta,\widetilde{\theta})=\pi(\widetilde{\theta})K(\widetilde{\theta},\theta)$. From Assumption 3.2*
$$\int_\Theta K(\widetilde{\theta},\theta)\pi(\widetilde{\theta})\mathbb{P}(d\widetilde{\theta})=\pi(\theta)\int_\Theta K(\theta,\widetilde{\theta})\mathbb{P}(d\widetilde{\theta})=\pi(\theta),$$
consequently by Proposition 3.2, there exists $c>0$ such that $\overline{\mathfrak{S}}_*(\theta)=c\pi(\theta)$. Therefore since
$$\int_{\mathbb{R}_+\times\Theta}\lambda(s,\widetilde{\theta})\overline{\mathfrak{S}}_*(\widetilde{\theta})\mathbb{P}(d\widetilde{\theta})\,ds=1,$$
it follows that
$$\overline{\mathfrak{S}}_*(\theta)=\pi(\theta)\left(\int_{\mathbb{R}_+\times\Theta}\lambda(a,\widetilde{\theta})\pi(\widetilde{\theta})\mathrm{d}a\mathbb{P}(\mathrm{d}\widetilde{\theta})\right)^{-1}.$$
There is an endemic equilibrium in this case if and only if
$$\mathbb{E}^*\left[\frac{1}{\gamma_*}\right]<R_0^*,$$
with $R_0^*=\mathbb{E}^*\left[\int_0^\infty\lambda(a)\mathrm{d}a\right]$ and $\mathbb{E}^*$ the expectation on Θ with respect to the measure $\pi(\theta)\nu(\mathrm{d}\theta)$.

4. *We take $\Theta=\{\theta_1,\theta_2\}$, with $\nu=p_1\delta_{\theta_1}+p_2\delta_{\theta_2}$, $p_2=1-p_1\in(0,1)$, and $K=\left(K_{ij}\right)_{1\leqslant i,j\leqslant 2}$, where $K_{ij}=K(\theta_i,\theta_j)$ with $K_{11}p_1+K_{12}p_2=K_{21}p_1+K_{22}p_2=1$ (see Assumption 3.2). From (3.25), we consider the matrix*
$$T=\begin{pmatrix}K_{11}p_1 & K_{21}p_2\\ K_{12}p_1 & K_{22}p_2\end{pmatrix}.$$
We notice that $\mathrm{Span}\begin{pmatrix}K_{21}\\K_{12}\end{pmatrix}$ is the eigenspace associated to the eigenvalue 1 and $\mathrm{Span}\begin{pmatrix}-p_2\\p_1\end{pmatrix}$ is the eigenspace associated to the eigenvalue $p_2(K_{22}-K_{12})$. Since $\overline{\mathfrak{S}}_*$

is the eigenfunction associated to the eigenvalue 1 *with* $\int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)\overline{\mathfrak{S}}_*(\theta)\mathrm{d}a\nu(\mathrm{d}\theta) = 1$, *we deduce*

$$\begin{pmatrix}\overline{\mathfrak{S}}_*(\theta_1)\\ \overline{\mathfrak{S}}_*(\theta_2)\end{pmatrix} = \frac{1}{p_1K_{21}\int_0^\infty \lambda(a,\theta_1)\mathrm{d}a + p_2K_{12}\int_0^\infty \lambda(a,\theta_2)\mathrm{d}a}\begin{pmatrix}K_{21}\\ K_{12}\end{pmatrix}.$$

The condition of existence of an endemic equilibrium is then

$$\mathbb{E}^*\left[\frac{1}{\gamma_*}\right] < R_0^* \quad \textit{with } R_0^* = \mathbb{E}^*\left[\int_0^\infty \lambda(a)\mathrm{d}a\right]$$

where $\mathbb{E}^*$ *is the expectation with respect to the measure* $\left(\frac{p_1K_{21}}{p_1K_{21}+p_2K_{12}}, \frac{p_2K_{12}}{p_1K_{21}+p_2K_{12}}\right)$ *on* $\Theta = \{\theta_1,\theta_2\}$.
As noted in Remark 3.6, we observe that the eigenvectors B *associated to* $p_2(K_{22}-K_{12})$ *satisfy* $\int_\Theta B(\theta)\nu(\mathrm{d}\theta) = B_1p_1 + B_2p_2 = 0$.

3.5.2. Local stability of endemic equilibria

To deal with the asymptotic stability of the equilibrium, we use the tools of abstract semi-linear Cauchy problems. We refer the reader to [69, 93, 99] for more details.

We set $\mathbb{X} = L^1(\Theta)\times L^1(\mathbb{R}_+\times\Theta)$, and endow $\mathbb{X}$ with the product norm

$$\|(\psi,\phi)\|_{\mathbb{X}} = \|\psi\|_{L^1(\Theta)} + \|\phi\|_{L^1(\mathbb{R}_+\times\Theta)}.$$

We set $\mathbb{X}_0 = \{0\}\times L^1(\mathbb{R}_+\times\Theta)$.

Let $AC(\mathbb{R}_+\times\Theta)$ be the space of functions on $\mathbb{R}_+\times\Theta$ which are absolutely continuous with respect to their first variable. We set $D(\mathscr{A}) = \{0\}\times AC(\mathbb{R}_+\times\Theta)$. Note that $\overline{D(\mathscr{A})} = \mathbb{X}_0$.

We define, for $(0,\phi) \in D(\mathscr{A})$

$$\mathscr{A}\begin{pmatrix}0\\ \phi\end{pmatrix}(a,\theta) := \begin{pmatrix}-\phi(0,\theta)\\ -\partial_a\phi(a,\theta)\end{pmatrix}, \tag{3.33}$$

and for $\phi \in L^1(\mathbb{R}_+\times\Theta)$

$$F(\phi) =: F\begin{pmatrix}0\\ \phi\end{pmatrix} := \begin{pmatrix}F_0(\phi)\\ F_1(\phi)\end{pmatrix},$$

with

$$F_0(\phi)(\theta) = \langle\lambda,\phi\rangle\langle K(\cdot,\theta),\gamma\phi\rangle,$$
$$F_1(\phi)(a,\theta) = -\langle\lambda,\phi\rangle\gamma(a,\theta)\phi(a,\theta),$$

where $\langle \lambda, \phi \rangle$ and $\langle K(\cdot,\theta), \gamma\phi \rangle$ denote

$$\langle \lambda, \phi \rangle := \int_{\mathbb{R}_+ \times \Theta} \lambda(a,\theta)\phi(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta)$$
$$\langle K(\cdot,\theta), \gamma\phi \rangle := \int_{\mathbb{R}_+ \times \Theta} \gamma(a,\widetilde{\theta})K(\widetilde{\theta},\theta)\phi(a,\widetilde{\theta})\mathrm{d}a\nu(\mathrm{d}\widetilde{\theta})$$

We remark that

$$\int_\Theta \langle K(\cdot,\theta), \gamma\phi \rangle \nu(\mathrm{d}\theta) = \int_{\mathbb{R}_+ \times \Theta} \gamma(a,\theta)\phi(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta),$$

and for $\phi = u_*$, we have $\langle \lambda, u_* \rangle = \overline{\mathfrak{F}}_*$ and $\langle K(\cdot,\theta), \gamma u_* \rangle = \overline{\mathfrak{S}}_*(\theta)$.

We note that $\mathscr{A}$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{X}_0$ (see e.g. [69, Theorem 1.3.1]):

$$T_{\mathscr{A}}(t)\begin{pmatrix} 0 \\ \phi \end{pmatrix}(a,\theta) = \begin{pmatrix} 0 \\ \phi(a-t,\theta)\mathbb{1}_{a \geqslant t} \end{pmatrix}. \tag{3.34}$$

We thus can rewrite the PDE system (3.8) as folllows:

$$\begin{cases} \partial_t v_t = \mathscr{A}(v_t) + F(v_t) \\ v(0,a,\theta) = v_0(a,\theta) \end{cases} \tag{3.35}$$

where

$$v_t(a,\theta) = \begin{pmatrix} 0 \\ u_t(a,\theta) \end{pmatrix} \text{ and } v_0(a,\theta) = \begin{pmatrix} 0 \\ u_0(a,\theta) \end{pmatrix}.$$

The Frechet-derivatives of F_0 and F_1 are given by:

$$F_0'(h)(\phi)(\theta) = \langle \lambda, h \rangle \langle K(\cdot,\theta), \gamma\phi \rangle + \langle \lambda, \phi \rangle \langle K(\cdot,\theta), \gamma h \rangle$$
$$F_1'(h)(\phi)(a,\theta) = -\langle \lambda, h \rangle \gamma(a,\theta)\phi(a,\theta) - \langle \lambda, \phi \rangle \gamma(a,\theta)h(a,\theta).$$

We assume that the assumptions of Theorem 3.5 are satisfied, and we consider u_* an endemic equilibrium. Thus u_* is a probability density on $\mathbb{R}_+ \times \Theta$ solution of the system (3.20). We prove the local stability for susceptibility curves satisfying the following assumption.

Assumption 3.10. *There is $\sigma \in (0,1]$ such that for each $\theta \in \Theta$ there exists $a_*(\theta) > 0$ and bounded such that $\forall\, a \in \mathbb{R}_+$,*

$$\gamma(a,\theta) \geqslant \sigma \mathbb{1}_{(a_*(\theta),+\infty)}.$$

We remark that under this assumption the function γ_* defined in Assumption 3.3 satisfies $\gamma_*(\theta) \geqslant \sigma$ for any $\theta \in \Theta$.

We note that

$$\begin{aligned}\left(\mathscr{A}+F'(u_*)\right)\begin{pmatrix}0\\ \phi\end{pmatrix}(a,\theta) &= \begin{pmatrix}-\phi(0,\theta)+\overline{\mathfrak{S}}_*(\theta)\langle\phi,\lambda\rangle+\langle K(\cdot,\theta),\gamma\phi\rangle\overline{\mathfrak{F}}_*\\ -\partial_a\phi(a,\theta)-\gamma(a,\theta)\phi(a,\theta)\overline{\mathfrak{F}}_*-\gamma(a,\theta)u_*(a,\theta)\langle\phi,\lambda\rangle\end{pmatrix}\\ &= \begin{pmatrix}-\phi(0,\theta)\\ -\partial_a\phi(a,\theta)-\gamma(a,\theta)\phi(a,\theta)\overline{\mathfrak{F}}_*\end{pmatrix}+\begin{pmatrix}\overline{\mathfrak{S}}_*(\theta)\langle\phi,\lambda\rangle+\langle K(\cdot,\theta),\gamma\phi\rangle\overline{\mathfrak{F}}_*\\ -\gamma(a,\theta)u_*(a,\theta)\langle\phi,\lambda\rangle\end{pmatrix}\\ &:= \mathscr{A}_*\begin{pmatrix}0\\ \phi\end{pmatrix}(a,\theta)+\mathscr{B}_*\begin{pmatrix}0\\ \phi\end{pmatrix}(a,\theta)\end{aligned}$$

Using the method of characteristics, the semigroup generated by the operator $\mathscr{A}_*$ is given by

$$T_{\mathscr{A}_*}(t)\begin{pmatrix}0\\ \phi\end{pmatrix}=\begin{pmatrix}0\\ T_*(t)(\phi)\end{pmatrix}$$

with

$$\begin{aligned}T_*(t)\left(\phi\right)(a,\theta) &= \mathbb{1}_{a\geqslant t}\Big[\phi(a-t,\theta)\exp\left(-\overline{\mathfrak{F}}_*\int_0^t\gamma(a-t+r,\theta)dr\right)\\ &\quad-\int_0^t\gamma(a-t+s,\theta)u_*(a-t+s,\theta)\langle\lambda,T_*(s)(\phi)\rangle\exp\left(-\overline{\mathfrak{F}}_*\int_s^t\gamma(a-t+r,\theta)dr\right)ds\Big]\\ &=: T_*^1(t)\left(\phi\right)(a,\theta)-T_*^2(t)\left(\phi\right)(a,\theta).\end{aligned}\tag{3.36}$$

Lemma 3.11. *For all $t\geqslant 0$, $T_*^2(t)$ is a compact operator on $L^1(\mathbb{R}_+\times\Theta)$.*

Proof. For $\phi\in L^1(\mathbb{R}_+\times\Theta)$, we define

$$\hat{T}_2(t)(\phi)(a,\theta)=\mathbb{1}_{a\geqslant t}\int_0^t u_*(a-t+s,\theta)\left|\langle T^*(s)(\phi),\lambda\rangle\right|ds.$$

We note that,

$$\begin{aligned}\int_{\mathbb{R}_+\times\Theta}\hat{T}_2(t)(\phi)(a,\theta)\mathrm{d}av(\mathrm{d}\theta) &= \int_0^t\left|\langle T^*(s)(\phi),\lambda\rangle\right|\int_\Theta\int_t^\infty u_*(a-t+s,\theta)\mathrm{d}av(\mathrm{d}\theta)\mathrm{d}s\\ &= \int_0^t\left|\langle T^*(s)(\phi),\lambda\rangle\right|\int_\Theta\int_s^\infty u_*(a,\theta)\mathrm{d}av(\mathrm{d}\theta)ds\\ &\leqslant \int_0^t\left|\langle T^*(s)(\phi),\lambda\rangle\right|\mathrm{d}s\\ &<\infty.\end{aligned}$$

Hence from Riez-Fréchet-Kolmogorov criterion in $L^1(\mathbb{R}_+\times\Theta)$ [14, Theorem IV.25], and [14, Lemma IV.4] it follows that $\hat{T}_2(t)$ is a compact operator. As a result, as $T_*^2(t)\left(\phi\right)\leqslant\hat{T}_2(t)(\phi)$, for all $\phi\in L^1(\mathbb{R}_+\times\Theta)$ it follows that $T_*^2(t)$ is also a compact operator. □

As in [99, Proposition 4.12, page 169] we introduce

$$w_0(\mathscr{A}+F'(u_*)):=\lim_{t\to\infty}\frac{1}{t}\log\|T_*(t)\|_{op},\tag{3.37}$$

where $\|.\|_{op}$ is the operator norm defined by $\|T\|_{op} := \sup_{\phi:\|\phi\|_{L^1(\mathbb{R}_+\times\Theta)}=1} \|T(\phi)\|_{L^1(\mathbb{R}_+\times\Theta)}$.

We denote by $\mathscr{K}(\mathbb{R}_+\times\Theta)$ the set of compact operator on $\mathbb{R}_+\times\Theta$ and by $\mathscr{L}(\mathbb{R}_+\times\Theta)$ the set of bounded operators on $\mathbb{R}_+\times\Theta$. We then introduce

$$w_{ess}(\mathscr{A}+F'(u_*)) := \lim_{t\to\infty}\frac{1}{t}\log\|T_*(t)\|_{ess}. \tag{3.38}$$

where $\|.\|_{ess}$ is the operators norm on the quotient space $\mathscr{L}(\mathbb{R}_+\times\Theta)/\mathscr{K}(\mathbb{R}_+\times\Theta)$.

Lemma 3.12. *Under Assumption 3.10,*

$$w_{ess}(\mathscr{A}+F'(u_*)) \leqslant -\overline{\mathfrak{F}}_*\sigma.$$

Proof. Using the Riez-Fréchet-Kolmogorov criterion in $L^1(\mathbb{R}_+\times\Theta)$ [14, Theorem IV.25], from [14, Lemma IV.4] it follows that $\mathscr{B}_*$ is a compact operator.

As a result, from Webb [99, Proposition 4.14, page 179] it follows that,

$$w_{ess}(\mathscr{A}+F'(u_*)) = w_{ess}(\mathscr{A}_*).$$

From Lemma 3.11 and from Webb [99, Proposition 4.9, page 166],

$$\|T_*(t)\|_{ess} = \left\|T_*^1(t)-T_*^2(t)\right\|_{ess} = \left\|T_*^1(t)\right\|_{ess} \leqslant \left\|T_*^1(t)\right\|_{op}. \tag{3.39}$$

Note that for $\phi\in L^1(\mathbb{R}_+\times\Theta)$,

$$\left\|T_*^1(t)(\phi)\right\|_{L^1(\mathbb{R}_+\times\Theta)} = \int_\Theta\int_t^\infty \left|T_*^1(t)(\phi)(a,\theta)\right|\mathrm{d}a\nu(\mathrm{d}\theta)$$

By a simple change of variables in the integrals, and using Assumption 3.10 in the inequalities, we observe that

$$\begin{aligned}
&\left\|T_*^1(t)(\phi)\right\|_{L^1(\mathbb{R}_+\times\Theta)} = \int_\Theta\int_t^\infty \left|\phi(a-t,\theta)\right|\mathrm{e}^{-\overline{\mathfrak{F}}_*\int_0^t\gamma(a-t+r,\theta)\mathrm{d}r}\mathrm{d}a\nu(\mathrm{d}\theta)\\
&= \int_\Theta\int_0^\infty \left|\phi(a,\theta)\right|\mathrm{e}^{-\overline{\mathfrak{F}}_*\int_a^{a+t}\gamma(r,\theta)\mathrm{d}r}\mathrm{d}a\nu(\mathrm{d}\theta)\\
&= \int_\Theta\int_0^{a_*(\theta)} \left|\phi(a,\theta)\right|\mathrm{e}^{-\overline{\mathfrak{F}}_*\int_a^{a+t}\gamma(r,\theta)\mathrm{d}r}\mathrm{d}a\nu(\mathrm{d}\theta) + \int_\Theta\int_{a_*(\theta)}^\infty \left|\phi(a,\theta)\right|\mathrm{e}^{-\overline{\mathfrak{F}}_*\int_a^{a+t}\gamma(r,\theta)\mathrm{d}r}\mathrm{d}a\nu(\mathrm{d}\theta)\\
&\leqslant \int_\Theta\int_0^{a_*(\theta)} \left|\phi(a,\theta)\right|\mathrm{e}^{-\overline{\mathfrak{F}}_*\int_a^{a+t}\gamma(r,\theta)\mathrm{d}r}\mathrm{d}a\nu(\mathrm{d}\theta) + \mathrm{e}^{-\overline{\mathfrak{F}}_*\sigma t}\int_\Theta\int_{a_*(\theta)}^\infty \left|\phi(a,\theta)\right|\mathrm{d}a\nu(\mathrm{d}\theta)\\
&\leqslant \mathrm{e}^{-\overline{\mathfrak{F}}_*\sigma t}\left(\int_\Theta \mathbb{1}_{t\geqslant a_*(\theta)}\int_0^{a_*(\theta)} \left|\phi(a,\theta)\right|\mathrm{d}a\nu(\mathrm{d}\theta) + \mathrm{e}^{\overline{\mathfrak{F}}_*\sigma t}\int_\Theta \mathbb{1}_{t\leqslant a_*(\theta)}\int_0^{a_*(\theta)} \left|\phi(a,\theta)\right|\mathrm{d}a\nu(\mathrm{d}\theta)\right.\\
&\qquad\qquad \left. + \int_\Theta\int_0^\infty \left|\phi(a,\theta)\right|\mathrm{d}a\nu(\mathrm{d}\theta)\right)\\
&\leqslant \mathrm{e}^{-\sigma\overline{\mathfrak{F}}_* t}\int_\Theta\int_0^\infty \left|\phi(a,\theta)\right|\mathrm{e}^{\sigma\overline{\mathfrak{F}}_* a_*(\theta)}\mathrm{d}a\nu(\mathrm{d}\theta).
\end{aligned} \tag{3.40}$$

Note that from Assumption 3.10 as a_* is bounded, it follows that

$$\sup_{\|\phi\|_1=1}\int_\Theta\int_0^\infty \left|\phi(a,\theta)\right|\mathrm{e}^{\sigma\overline{\mathfrak{F}}_* a_*(\theta)}\mathrm{d}a\nu(\mathrm{d}\theta)<\infty.$$

Consequently, using Equations (3.40), we deduce

$$\frac{1}{t}\log\left\|T_*^1(t)\right\|_{op}\leqslant -\sigma\overline{\mathfrak{F}}_*+\frac{1}{t}\log\left(\sup_{\|\phi\|_1=1}\int_\Theta\int_0^\infty \left|\phi(a,\theta)\right|\mathrm{e}^{\sigma\overline{\mathfrak{F}}_* a_*(\theta)}\mathrm{d}a\nu(\mathrm{d}\theta)\right).$$

By definition (3.38) of $w_{ess}(\mathscr{A}_*)$ and equation (3.39), we deduce

$$w_{ess}(\mathscr{A}_*)\leqslant -\sigma\overline{\mathfrak{F}}_*.$$

□

Take $\alpha\in\mathbb{C}$, with $\mathscr{R}e(\alpha)>-\sigma\overline{\mathfrak{F}}_*$. We consider the following eigenvalue problem

$$\left(\mathscr{A}+F'(u_*)\right)\begin{pmatrix}0\\ \phi\end{pmatrix}(a,\theta)=\alpha\begin{pmatrix}0\\ \phi(a,\theta)\end{pmatrix}. \tag{3.41}$$

In other words,

$$\begin{cases}-\partial_a\phi(a,\theta)-\gamma(a,\theta)\phi(a,\theta)\overline{\mathfrak{F}}_*-\gamma(a,\theta)u_*(a,\theta)\langle\phi,\lambda\rangle=\alpha\phi(a,\theta) & (3.42)\\ \phi(0,\theta)=\overline{\mathfrak{S}}_*(\theta)\langle\phi,\lambda\rangle+\langle K(\cdot,\theta),\gamma\phi\rangle\overline{\mathfrak{F}}_*. & (3.43)\end{cases}$$

As a result,

$$\begin{aligned}\phi(a,\theta)&=\phi(0,\theta)\exp\left(-\int_0^a\left(\overline{\mathfrak{F}}_*\gamma(s,\theta)+\alpha\right)ds\right)\\ &\qquad-\langle\phi,\lambda\rangle\int_0^a\gamma(b,\theta)u_*(b,\theta)\exp\left(-\int_b^a\left(\overline{\mathfrak{F}}_*\gamma(s,\theta)+\alpha\right)ds\right)db\\ &=\phi(0,\theta)e^{-\alpha a}\frac{u_*(a,\theta)}{\overline{\mathfrak{F}}_*\overline{\mathfrak{S}}_*(\theta)}-\langle\phi,\lambda\rangle u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}db\\ &=\left(\frac{\langle\phi,\lambda\rangle}{\overline{\mathfrak{F}}_*}+\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}\right)u_*(a,\theta)e^{-\alpha a}-\langle\phi,\lambda\rangle u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}db\end{aligned} \tag{3.44}$$

where we use in the second line the fact that

$$u_*(a,\theta)=\overline{\mathfrak{F}}_*\overline{\mathfrak{S}}_*(\theta)\exp\left(-\overline{\mathfrak{F}}_*\int_0^a\gamma(s,\theta)ds\right),$$

and (3.43) in the last line.

Lemma 3.13. *Under Assumption 3.8, if ϕ is the eigenfunction associated to eigenvalue $\alpha \in \mathbb{C}$, with $\mathcal{R}e(\alpha) > -\sigma\overline{\mathfrak{F}}_*$, such that $\int_{\mathbb{R}_+\times\Theta}\phi(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta) = 0$. Then,*

$$\begin{aligned}\left(1-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\right)\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\\ =\left(\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\right)\\ \times\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta).\end{aligned}\tag{3.45}$$

Proof. From (3.44) we have,

$$\begin{aligned}\langle\phi,\lambda\rangle &= \frac{\langle\phi,\lambda\rangle}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\\ &\quad+\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\\ &\quad-\langle\phi,\lambda\rangle\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a\nu(\mathrm{d}\theta)\\ &= \frac{\langle\phi,\lambda\rangle}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\\ &\quad+\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta),\end{aligned}\tag{3.46}$$

where we use in the last line the fact that λ and γ have disjoint support (see Assumption 3.8-(3)).

As a result, as $\langle\phi,\lambda\rangle \neq 0$ by virtue of (3.44), it follows that,

$$\begin{aligned}1-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\\ =\frac{1}{\langle\phi,\lambda\rangle}\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta).\end{aligned}\tag{3.47}$$

We also have

$$\begin{aligned}\int_{\mathbb{R}_+\times\Theta}\phi(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta) &= \frac{\langle\phi,\lambda\rangle}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\\ &\quad+\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\\ &\quad-\langle\phi,\lambda\rangle\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a\nu(\mathrm{d}\theta).\end{aligned}$$

As we consider the eigenvalues such that,

$$\int_{\mathbb{R}_+\times\Theta}\phi(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta)=0,$$

it follows that

$$\langle\phi,\lambda\rangle\left(\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\right)$$
$$=\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta). \tag{3.48}$$

Combining (3.47) and (3.48) it follows that

$$\left(1-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\right)\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)$$
$$=\left(\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\right)$$
$$\times\int_{\mathbb{R}_+\times\Theta}\frac{\langle K(\cdot,\theta),\gamma\phi\rangle}{\overline{\mathfrak{S}}_*(\theta)}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta).$$

□

In particular when $K=1$, recalling that $\overline{\mathfrak{S}}_*=\frac{1}{R_0}$, consequently (3.45) becomes

$$\left(1-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\right)\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)$$
$$=\left(\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a-\frac{1}{\overline{\mathfrak{F}}_*}\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)\right)$$
$$\times\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta). \tag{3.49}$$

This implies that,

$$\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)e^{-\alpha a}da\nu(\mathrm{d}\theta)$$
$$=\int_{\mathbb{R}_+\times\Theta}u_*(a,\theta)\int_0^a\gamma(b,\theta)e^{-\alpha(a-b)}\mathrm{d}b\mathrm{d}a\nu(\mathrm{d}\theta)\int_{\mathbb{R}_+\times\Theta}\lambda(a,\theta)u_*(a,\theta)e^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta). \tag{3.50}$$

From Webb [99, Proposition 4.15] or [93, Corollary 4.3], we can make the following conjecture on the convergence to the equilibrium and we handle some special cases in subsection 3.5.3.

Conjecture 3.1. *We assume that Assumptions 3.1, 3.2, and 3.10 are satisfied. If λ and γ are taken such that equation (3.50) does not have a solution when $\mathcal{R}e(\alpha) \geqslant 0$, when*

$$R_0 > \int_\Theta \frac{1}{\gamma_*(\theta)} \nu(\mathrm{d}\theta),$$

for any $w > w_0(\mathscr{A} + F'(u_*))$, there exist $\delta > 0$, $c > 0$ such that $\|u_0 - u_*\| \leqslant \delta$ implies that for any $t \geqslant 0$,

$$\|T_{\mathscr{A}}(t)(u_0) - u_*\| \leqslant c\mathrm{e}^{wt}\|u_0 - u_*\|.$$

3.5.3. Special cases of local stability of endemicity

We consider the case $K \equiv 1$, and we assume that for any $\theta \in \Theta$, the functions $a \mapsto \gamma(a,\theta)$ are nondecreasing and $\mathrm{Supp}(\gamma(.,\theta)) \subset [\eta(\theta), +\infty)$ with $\theta \mapsto \eta(\theta)$ a positive integrable function on Θ. $\eta(.)$ represents the duration of the infections period.

In this case we recall that the characteristic equation for the eigenvalue α is given by

$$\begin{aligned}
&\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\mathrm{e}^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\\
&= \int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)u_*(a,\theta)\mathrm{e}^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\left(\int_0^a \gamma(b,\theta)\mathrm{e}^{-\alpha(a-b)}\mathrm{d}b\right)\mathrm{d}a\nu(\mathrm{d}\theta)\\
&= \frac{\overline{\mathfrak{F}}_*}{R_0}\int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)\mathrm{e}^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta)\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\left(\int_0^a \gamma(b,\theta)\mathrm{e}^{-\alpha(a-b)}\mathrm{d}b\right)\mathrm{d}a\nu(\mathrm{d}\theta). \quad (3.51)
\end{aligned}$$

Taking the limit in (3.51) when $\alpha \to 0$ we obtain

$$1 = \overline{\mathfrak{F}}_*\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\left(\int_0^a \gamma(b,\theta)\mathrm{d}b\right)\mathrm{d}a\nu(\mathrm{d}\theta). \quad (3.52)$$

In addition , using (3.51),

$$\begin{aligned}
\overline{\mathfrak{F}}_*\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\left(\int_0^a \gamma(b,\theta)\mathrm{d}b\right)\mathrm{d}a\nu(\mathrm{d}\theta) &= \frac{\overline{\mathfrak{F}}_*^2}{R_0}\int_\Theta\int_{\eta(\theta)}^\infty \mathrm{e}^{-\overline{\mathfrak{F}}_*\int_{\eta(\theta)}^a \gamma(s,\theta)\mathrm{d}s}\left(\int_{\eta(\theta)}^a \gamma(b,\theta)\mathrm{d}b\right)\mathrm{d}a\nu(\mathrm{d}\theta)\\
&\leqslant \frac{\overline{\mathfrak{F}}_*}{R_0}\int_\Theta\int_{\eta(\theta)}^\infty \mathrm{e}^{-\overline{\mathfrak{F}}_*\int_{\eta(\theta)}^a \gamma(s,\theta)\mathrm{d}s}\overline{\mathfrak{F}}_*\gamma(a,\theta)\big(a-\eta(\theta)\big)\mathrm{d}a\nu(\mathrm{d}\theta)\\
&= \frac{\overline{\mathfrak{F}}_*}{R_0}\int_\Theta\int_{\eta(\theta)}^\infty \mathrm{e}^{-\overline{\mathfrak{F}}_*\int_{\eta(\theta)}^a \gamma(s,\theta)\mathrm{d}s}\mathrm{d}a\nu(\mathrm{d}\theta)\\
&= \int_\Theta\int_{\eta(\theta)}^\infty u_*(a,\theta)\mathrm{d}a\nu(\mathrm{d}\theta) = 1 - \frac{\overline{\mathfrak{F}}_*}{R_0}\mathbb{E}[\eta] < 1.
\end{aligned}$$

Thus condition (3.52) cannot be satisfied for α close to 0.

For a step susceptibility We still work under the assumption $K \equiv 1$, and we consider a susceptibility γ given by: for $a \geqslant 0$

$$\gamma(a,.) = \mathbb{1}_{[T(.),+\infty)}(a),$$

with T a positive variable such that $T \geqslant \eta$, and $\mathbb{E}[T] := \int_\Theta T(\theta)\nu(\mathrm{d}\theta) < \infty$. In this specific case, we can compute explicitly every quantity: $\gamma_* \equiv 1$, $\overline{\mathfrak{S}}_* \equiv \frac{1}{R_0}$, $\overline{\mathfrak{F}}_* = \frac{R_0-1}{\mathbb{E}[T]}$, and for $(a,\theta) \in \mathbb{R}_+ \times \Theta$

$$u_*(a,\theta) = \frac{\overline{\mathfrak{F}}_*}{R_0}\mathbb{1}_{a<T(\theta)} + \frac{\overline{\mathfrak{F}}_*}{R_0}\mathrm{e}^{-\overline{\mathfrak{F}}_*(a-T(\theta))}\mathbb{1}_{a\geqslant T(\theta)}.$$

We assume $R_0 > 1$ to ensure the existence of an endemic equilibrium. We easily compute that for $\alpha \neq 0$

$$\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\mathrm{e}^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta) = \frac{\overline{\mathfrak{F}}_*}{\alpha R_0} - \frac{\overline{\mathfrak{F}}_*^2}{\alpha R_0(\overline{\mathfrak{F}}_* + \alpha)}\mathbb{E}\left[\mathrm{e}^{-\alpha T}\right].$$

On the other hand, for $\alpha \neq 0$, we have

$$\int_{\mathbb{R}_+\times\Theta} \lambda(a,\theta)u_*(a,\theta)\mathrm{e}^{-\alpha a}\mathrm{d}a\nu(\mathrm{d}\theta) = \overline{\mathfrak{F}}_*\frac{\int_0^\infty \mathbb{E}[\lambda(a)]\mathrm{e}^{-\alpha a}\mathrm{d}a}{R_0}$$
$$\int_{\mathbb{R}_+\times\Theta} u_*(a,\theta)\left(\int_0^a \gamma(b,\theta)\mathrm{e}^{-\alpha(a-b)}\mathrm{d}b\right)\mathrm{d}a\nu(\mathrm{d}\theta) = \frac{1}{R_0(\overline{\mathfrak{F}}_* + \alpha)}.$$

Then Condition (3.51) is equivalent to

$$1 + \frac{\overline{\mathfrak{F}}_*}{\alpha}\left(1 - \mathbb{E}\left[\mathrm{e}^{-\alpha T}\right]\right) = \frac{\int_0^\infty \mathbb{E}[\lambda(a)]\mathrm{e}^{-\alpha a}\mathrm{d}a}{R_0}. \tag{3.53}$$

Since $\overline{\mathfrak{F}}_* = \frac{R_0-1}{\mathbb{E}[T]}$, we note that

$$\lim_{\alpha\to 0}\frac{\overline{\mathfrak{F}}_*}{\alpha}\left(1 - \mathbb{E}\left[\mathrm{e}^{-\alpha T}\right]\right) = R_0 - 1$$
$$\lim_{\alpha\to 0}\frac{\int_0^\infty \mathbb{E}[\lambda(a)]\mathrm{e}^{-\alpha a}\mathrm{d}a}{R_0} = 1.$$

As by hypothesis $R_0 > 1$, we recover that (3.53) cannot be satisfied for small $|\alpha|$.

Take $\alpha = x + iy$ with $x \geqslant 0$. We compute the real part of (3.53). As $\int_0^T \mathrm{e}^{-\alpha a}\mathrm{d}a = \frac{1}{\alpha}\left(1 - \mathrm{e}^{-\alpha T}\right)$, we obtain

$$\mathscr{R}e\left(1 + \frac{\overline{\mathfrak{F}}_*}{\alpha}\left(1 - \mathbb{E}\left[\mathrm{e}^{-\alpha T}\right]\right)\right) = 1 + \overline{\mathfrak{F}}_*\mathbb{E}\left[\int_0^T \mathrm{e}^{-xa}\cos(ya)\mathrm{d}a\right]$$

$$= 1 + \overline{\mathfrak{F}}_* \int_0^\infty \mathrm{e}^{-xa} \cos(ya) \nu(T > a) \mathrm{d}a,$$

$$\mathscr{R}e\left(\frac{\int_0^\infty \mathbb{E}[\lambda(a)] \mathrm{e}^{-\alpha a} \mathrm{d}a}{R_0} \right) = \frac{\int_0^\infty \mathbb{E}[\lambda(a)] \mathrm{e}^{-xa} \cos(ay) \mathrm{d}a}{R_0}.$$

We note that $\left| \frac{\int_0^\infty \mathbb{E}[\lambda(a)] \mathrm{e}^{-xa} \cos(ay) \mathrm{d}a}{R_0} \right| \leqslant 1$.

We recall that

$$\int_0^a \mathrm{e}^{-xr} \cos(yr) \mathrm{d}r = \frac{\mathrm{e}^{-xa}}{x^2+y^2} (y \sin(ay) - x \cos(ay)),$$
$$\int_0^a \mathrm{e}^{-xr} \sin(yr) \mathrm{d}r = \frac{\mathrm{e}^{-xa}}{x^2+y^2} (-x \sin(ay) - y \cos(ay)).$$

We now focus on examples, that are SIS-type and SIRS-type models, where we have been able to conclude. Since the model need to satisfy Assumption 3.10, the classical SIS and SIRS models cannot be included in our study.

Example 3.3. *We assume that $T = E \wedge a_*$ with a_* a positive real, and E is an exponential variable of parameter $\rho > 0$: $\nu(E > a) = \mathrm{e}^{-\rho a}$. This case coincides with a model where T_I is equal to the minimum of T with another variable (in order to ensure that $T \geqslant T_I$). We observe that when $T_I = T$ and $a_* = +\infty$, it coincides with a generalized SIS model (the infectivity curve is not assumed to be a step function).*
We easily compute in this case that

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-xa} \cos(ya) \nu(T > a) \mathrm{d}a &= \int_0^{a_*} \mathrm{e}^{-xa} \cos(ya) \mathrm{e}^{-\rho a} \mathrm{d}a \\
&= \frac{x+\rho}{(x+\rho)^2+y^2} - \frac{\mathrm{e}^{-(x+\rho)a_*}}{(x+\rho)^2+y^2} (y \sin(a_* y) - (x+\rho) \cos(a_* y)) \\
&= \frac{x+\rho}{(x+\rho)^2+y^2} \left(1 - \cos(a_* y) \mathrm{e}^{-(x+\rho)a_*} + \frac{y \sin(a_* y)}{x+\rho} \mathrm{e}^{-(x+\rho)a_*} \right).
\end{aligned} \tag{3.54}$$

When $\lambda(a,\theta) = \lambda_* \mathbb{1}_{T(\theta) \geqslant a}$, $\overline{\mathfrak{F}}_* = \lambda_* \left(1 - \frac{1}{R_0}\right)$. So it follows that the real part of equation (3.53) satisfies

$$1 + (R_0 - 2) \frac{\lambda_*}{R_0} \int_0^\infty \mathrm{e}^{-xa} \cos(ya) \nu(T > a) \mathrm{d}a = 0. \tag{3.55}$$

Note that if $1 < R_0 < 2$, for $x \geqslant 0$,

$$\left| (R_0 - 2) \frac{\lambda_*}{R_0} \int_0^\infty \mathrm{e}^{-xa} \cos(ya) \nu(T > a) \mathrm{d}a \right| \leqslant |R_0 - 2| < 1.$$

So

$$1 + (R_0 - 2) \frac{\lambda_*}{R_0} \int_0^\infty \mathrm{e}^{-xa} \cos(ya) \nu(T > a) \mathrm{d}a > 0.$$

On the other hand if $R_0 > 2$, *from* (3.54), *for* $x \geqslant 0$,

$$1 + (R_0 - 2)\frac{\lambda_*}{R_0}\int_0^\infty \mathrm{e}^{-xa}\cos(ya)v(T > a)\mathrm{d}a > 0.$$

Consequently Equation (3.55) *has no solution* $\alpha = x + i\,y$ *with* $x \geqslant 0$.

3.6. Application to a model with a vaccination policy

3.6.1. Renewal process of vaccination

We define

$$\gamma(a,\theta)=\sum_{n=0}^{\infty}\sigma_n(a-\tau_n(\theta),\theta)\mathbb{1}_{\tau_n(\theta)\leqslant a<\tau_{n+1}(\theta)}=\sum_{n=0}^{N(a,\theta)}\sigma_n(a-\tau_n(\theta),\theta)\mathbb{1}_{\tau_n(\theta)\leqslant a<\tau_{n+1}(\theta)},$$

where

— $(\sigma_n)_{n\geqslant 0}$ be a non-decrasing deterministics curves with respect to its first variable defined on $\mathbb{R}_+\times\Theta$ with values in $[0,1]$.

— $$\tau_n(\theta)=\sum_{k=0}^{n}T_V^k(\theta),$$
where for $\theta\in\Theta$ we construct two independent sequences $(\theta_{1,n})_n$ and $\theta_{2,n})_n$ taking values in Θ, with respect to distribution ν_1 and ν_2 respectively. Then, we set $T_V^0(\theta)=\eta(\theta_{2,0})$ and for $k\geqslant 1$, $T_V^k(\theta)=T_V(\theta_{2,k})$ and $\sigma_n(a,\theta)=\sigma(a,\theta_{1,n})$.

— N is the counting process associated to the stopping times $(\tau_n)_{n\geqslant 0}$: $N(a,\theta)=\sum_{n\geqslant 1}\mathbb{1}_{\tau_n(\theta)\leqslant a}$.

The processes $(N(a,\theta))_{a\geqslant 0}$ is a renewal process.

Let $b>0$, we have

$$\frac{1}{b}\int_0^b\gamma(a,\theta)\mathrm{d}a=\frac{N(b,\theta)}{b}\frac{1}{N(b,\theta)}\sum_{n=1}^{N(b,\theta)}\int_0^{T_V(\theta_{2,n})}\sigma(a,\theta_{1,n-1})\mathrm{d}a+\frac{1}{b}\int_0^{b-\tau_{N(b,\theta)}(\theta)}\sigma(a,\theta_{1,N(b,\theta)})\mathrm{d}a,$$

with $\frac{1}{b}\int_0^{b-\tau_{N(b,\theta)}(\theta)}\sigma_{N(b,\theta)}(a)\mathrm{d}a\leqslant\frac{b-\tau_{N(b,\theta)}(\theta)}{b}$. Let us note that

$$\frac{b-\tau_{N(b,\theta)}(\theta)}{b}=1-\frac{N(b,\theta)}{b}\frac{1}{N(b,\theta)}\sum_{k=1}^{N(b,\theta)}T_V(\theta_{2,k}).$$

A classical result on renewal processes gives $\frac{N(b,\theta)}{b}$ converges a.s. to $\int_\Theta T_V(\theta_2)\nu_2(\mathrm{d}\theta_2)$ when $b\to+\infty$ (see e.g. [1, page 138]). By assumptions $\left(\int_0^{T_V(\theta_{2,n})}\sigma(a,\theta_{1,n-1})\mathrm{d}a\right)_{n\geqslant 1}$ are integrable i.i.d. random variables, we then deduce

$$\lim_{b\to+\infty}\frac{1}{b}\int_0^b\gamma(a,\theta)\mathrm{d}a=\frac{\int_\Theta\int_\Theta\left(\int_0^{T_V(\theta_2)}\sigma(a,\theta_1))\mathrm{d}a\right)\nu_1(d\theta_1)\nu_2(d\theta_2)}{\int_\Theta T_V(\theta_2)\nu_2(\mathrm{d}\theta_2)}=\gamma_*\quad\text{a.s.}$$

We notice that γ_* is a constant. Using Proposition 3.3, there is existence of an endemic

equilibrium if and only if

$$R_0 > \frac{1}{\gamma_*} = \frac{\int_\Theta T_V(\theta_2)\nu_2(\mathrm{d}\theta_2)}{\int_\Theta\int_\Theta\left(\int_0^{T_V(\theta_2)}\sigma(a,\theta_1))\mathrm{d}a\right)\nu_1(d\theta_1)\nu_2(d\theta_2)}.$$

We then recover the result of [43] in the case of parametric distributions for σ and λ.

3.6.2. One shot of vaccination after an infection

We can imagine the case where access to the vaccine is very restricted, and only people vulnerable to the disease (those who have been infected) have the opportunity to receive a dose of vaccine to prevent re-infection.

We assume that the infectious curve λ has a compact support η. We introduce two integrable positive random times T_R and T_V, with $T_R \geqslant \eta$ and $T_V \geqslant \eta$ a.s.: $T_R - \eta$ is the immunity period after an infection, and T_V is the vaccination time after an infection. Let $\alpha, \beta \in (0,1]$ be random variables independent of T_R and T_V. We consider a susceptibility curve given by

$$\gamma(a) = \alpha \mathbb{1}_{T_R \leqslant a < T_V} + \beta \mathbb{1}_{a \geqslant T_V}$$

if $T_V > T_R$, and $\gamma(a) = \beta \mathbb{1}_{a \geqslant T_V}$ otherwise.

We note that Assumption 3.3 is satisfied with $\gamma_* \equiv \beta$. We introduce the expectation $\mathbb{E}^*$ defined in Remark 3.7 and $\kappa = \mathbb{E}\left[\overline{\mathfrak{S}}_*\right]$. Note that when there is no memory of the previous infection, we have $\overline{\mathfrak{S}} = \frac{1}{R_0}$ and $\mathbb{E}^* = \mathbb{E}$.

When $T_V \leqslant T_R$, we have

$$x\int_0^\infty \exp\left(-x\int_0^a \gamma(s)\mathrm{d}s\right)\mathrm{d}a = xT_V + \frac{1}{\beta}.$$

When $T_V > T_R$, we have

$$x\int_0^\infty \exp\left(-x\int_0^a \gamma(s)\mathrm{d}s\right)\mathrm{d}a) = xT_R + \frac{1}{\alpha} - \mathrm{e}^{-x\alpha(T_V - T_R)}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

We note that H is increasing when $\alpha \leqslant \beta$ a.s., which was indeed obvious since γ is increasing.

Let us consider the case $\alpha \geqslant \beta$ a.s., i.e. γ non-monotone, which is more realistic when vaccination gives better immunity.

We have

$$H(x) = \kappa\left(\mathbb{E}^*\left[\frac{1}{\alpha}\right] + x\mathbb{E}^*\left[T_R \wedge T_V\right] - \mathbb{E}^*\left[\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\mathrm{e}^{-x\alpha(T_V - T_R)_+}\right]\right),$$

with $a \wedge b = \min(a,b)$ and $a_+ = \max(a,0)$. We easily compute

$$H'(x) = \kappa\left(\mathbb{E}^*[T_R \wedge T_V] + \mathbb{E}^*\left[\left(1-\frac{\alpha}{\beta}\right)(T_V - T_R)_+ \mathrm{e}^{-x\alpha(T_V-T_R)_+}\right]\right).$$

If $\alpha \geqslant \beta$ a.s., H' is increasing, with $H'(0) = \frac{1}{R_0}\left(\mathbb{E}^*[T_R \wedge T_V] + \mathbb{E}^*\left[1-\frac{\alpha}{\beta}\right]\mathbb{E}^*[(T_V - T_R)_+]\right)$. Consequently,

— if $\alpha \geqslant \beta$ a.s. and $\mathbb{E}^*[T_R \wedge T_V] \geqslant \mathbb{E}^*\left[\frac{\alpha}{\beta} - 1\right]\mathbb{E}^*[(T_V - T_R)_+]$, then H is increasing and then there is a (unique) endemic equilibrium if and only if $R_0 > \frac{1}{\beta}$.

— if $\alpha \geqslant \beta$ a.s. and $\mathbb{E}^*[T_R \wedge T_V] < \mathbb{E}^*\left[\frac{\alpha}{\beta} - 1\right]\mathbb{E}^*[(T_V - T_R)_+]$, then H is decreasing and then increasing, with $H(0) = \kappa\mathbb{E}^*\left[\frac{1}{\beta}\right]$.

Let $x_{\min}$ such that $H'(x_{\min}) = 0$. Then if $\mathbb{E}^*\left[\frac{1}{\beta}\right] < \frac{1}{\kappa}$, there is a (unique) endemic equilibrium. When $\mathbb{E}^*\left[\frac{1}{\beta}\right] > \frac{1}{\kappa}$ and $H(x_{\min}) > 1$, there is extinction of the disease. But when $\mathbb{E}^*\left[\frac{1}{\beta}\right] > \frac{1}{\kappa}$ and $H(x_{\min}) < 1$, there exist two solutions to $H(x) = 1$, and then two possible endemic equilibria.

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Deuxième partie

Bergman Operator on the unit Ball of $\mathbb{C}^n$

4. Introduction

4.1. Intégrales singulières de Calderón-Zygmund

Le développement de la théorie des intégrales singulières fut motivé avec l'apparition de la transformée de Hilbert

$$\mathscr{H}(f)(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}dy$$

en 1905 voir [34, 11, 22]. Plus précisémement, elle fut formalisée en 1970 dans [34] par Stein et en 1971 dans [35] par Stein et Weiss dans des espaces Euclidiens. Puis en 1971 dans [11] par Coifman et Weiss dans le cas des espaces homogènes. On appelle opérateur intégrale singulier tout opérateur intégral de la forme

$$\begin{aligned} Tf :&\mathbb{R}^n \to \mathbb{R} \\ &x \mapsto Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \end{aligned} \tag{4.1}$$

où $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ est un noyau singulier sur la diagonale $x = y$ et $|x-y|^n|K(x,y)|$ est borné lorsque $|x-y| \to 0$.

Cette théorie fut enrichie par l'introduction des temps d'arrêt par Calderón et Zygmund en 1954 dans [8](voir aussi [22, Theorem 4.3.1],[34]).

On rappelle que pour $p > 0$, et w une fonction mesurable positive,

$$L^p(wdx) = \left\{ f, \int_{\mathbb{R}} |f(x)|^p w(x)dx < \infty \right\},$$

est l'espace de Lebesgue de poids w.

Une question qui se posait, était de déterminer toutes les fonctions mesurables positives w tel que pour tout $1 < p < \infty$, il existe une constante $C > 0$ tel que pour tout $f \in L^p(wdx)$,

$$\int_{\mathbb{R}} |\mathscr{H}(f)(x)|^p w(x)dx \leqslant C\int_{\mathbb{R}} |f(x)|^p w(x)dx. \tag{4.2}$$

Dans la section suivante nous donnerons cette caractérisation.

4.2. Inégalités L^p à poids de Muckenhoupt

On rappelle que pour f locallement intégrable, la fonction maximale de Hardy-Littlewood est donnée par,

$$Mf(x) = \sup_I \frac{\chi_I(x)}{|I|} \int_I |f(y)|dy, \tag{4.3}$$

le sup étant pris sur l'ensemble des intervalles ouverts de $\mathbb{R}$.

En 1972 dans [29] Muckenhoupt étudie l'équivalent de l'estimation (4.2) du problème de la transformée de Hilbert pour la fonction maximale de Hardy-Littlewood.

Plus précisément, il introduit la classe A_p définie ainsi que suit :

— pour $1 < p < \infty$, $w \in A_p$ s'il existe une constante $C := C(p,w) > 0$ tel que pour tout intervalle I,

$$\left(\int_I w(x)dx\right)\left(\int_I w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leqslant C|I|^p, \tag{4.4}$$

où $|I|$ est la mesure de Lebesgue de I.

— Pour $p = 1$, $w \in A_1$ s'il existe $C > 0$ tel que pour tout $x \in \mathbb{R}$, $Mw(x) \leqslant Cw(x)$.

— pour $p = \infty$, $A_\infty = \bigcup_{p\geqslant 1} A_p$.

Puis il démontre que pour $p \in]1,\infty[$, les deux propriétés suivantes sont équivalentes :

1. Pour tout $f \in L^p(wdx)$,

$$\int_{\mathbb{R}} |Mf(x)|^p w(x)dx \leqslant C\int_{\mathbb{R}} |f(x)|^p w(x)dx, \tag{4.5}$$

2. $w \in A_p$.

Le cas $w = 1$ avait été traité par Hardy et Littlewood en 1930 dans [23], celui de $w(x) = |x|^\alpha$ avec $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$ par Stein dans sa thèse de doctorat en 1955 et dans [20] en 1971 Fefferman et Stein ont traité le cas $Mw(x) \leqslant Cw(x)$. En fait, dans [20] Fefferman et Stein montre que les deux propriétés suivantes sont vraies :

1. Si $p > 1$ et $Mw(x) \leqslant Cw(x)$ on a :

$$\int_{\mathbb{R}} |Mf(x)|^p w(x)dx \leqslant C\int_{\mathbb{R}} |f(x)|^p w(x)dx. \tag{4.6}$$

2. Si $p = 1$ les deux assertions suivantes sont équivalentes :

   a) pour $f \in L^1(wdx)$, et $t > 0$,

$$\int_0^\infty \chi_{Mf(x)\geqslant t} w(x)dx \leqslant \frac{C}{t}\int_{\mathbb{R}} |f(x)|^p w(x)dx, \tag{4.7}$$

   b) $Mw(x) \leqslant Cw(x)$.

Plus tard dans [25] en 1973, Hunt, Muckenhoupt et Wheeden montrent que l'estimation (4.2) est équivalente à la condition $w \in A_p$. La preuve de ce résultat fut améliorée

dans [10] par Coifman et Fefferman.

En effet, dans [10] pour $n \geqslant 1$ ils rappellent que la fonction Maximale de Hardy-Littlewood est définie sur $\mathbb{R}^n$ par

$$Mf(x) = \sup_B \frac{\chi_B(x)}{|B|} \int_B |f(y)|dy, \tag{4.8}$$

le sup étant pris sur l'ensemble des boules ouvertes de $\mathbb{R}^n$.

De même pour $n \geqslant 1$, ils définissent la classe A_p de Muckenhoupt sur $\mathbb{R}^n$ comme suit :

— pour $1 < p < \infty$, $w \in A_p$ s'il existe une constante $C := C(p,w) > 0$ tel que pour toute boule ouverte B de $\mathbb{R}^n$,

$$\left(\int_B w(x)dx\right)\left(\int_B w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leqslant C|B|^p, \tag{4.9}$$

où $|B|$ est la mesure de Lebesgue de la boule B.

— Pour $p = 1$, $w \in A_1$ s'il existe $C > 0$ tel que pour tout $x \in \mathbb{R}^n$, $Mw(x) \leqslant Cw(x)$.

— pour $p = \infty$, $A_\infty = \bigcup_{p\geqslant 1} A_p$.

Ils considèrent un opérateur singulier à noyau de convolution K vérifiant les conditions standard suivantes,

$$\|\hat{K}\|_\infty < C, \ (\hat{K} \text{ étant la transformée de Fourier de } K) \tag{4.10}$$

$$|K(x)| \leqslant \frac{C}{|x|^n}, \tag{4.11}$$

$$|K(x) - K(x-y)| \leqslant C\frac{|y|}{|x|^{n+1}} \text{ où } |y| < \frac{|x|}{2}. \tag{4.12}$$

Puis ils démontrent que pour $p > 1$, les deux propriétés suivantes sont équivalentes :

1. pour tout $f \in L^p(wdx)$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)dx \leqslant C\int_{\mathbb{R}^n} |f(x)|^p w(x)dx. \tag{4.13}$$

2. $w \in A_p$.

Ce résultat fut étendu à tous les espaces homogènes, voir par exemple [22]. Un espace homogène est un triplet (X,d,μ), où X est un ensemble, d une pseudo-distance et μ une mesure sur X tel que pour tout $r > 0$, $\mu(B(x,2r)) \leqslant C\mu(B(x,r))$, avec

$$B(x,r) = \{y \in X, \ d(x,y) < r\},$$

boule ouverte de X pour la pseudo-distance d.

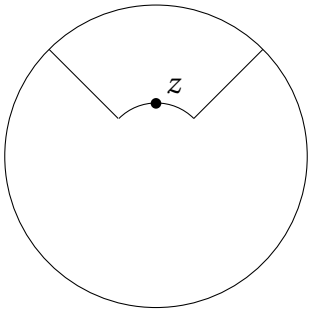

FIGURE 4.1. – Exemple d'ouvert pour cette topologie.

4.3. Intégrales singulières sur les espaces homogènes : le cas de la boule unité de $\mathbb{C}^n$

On note $\mathbb{B} = \{z \in \mathbb{C}^n, |z| < 1\}$ la boule unité du plan complexe $\mathbb{C}^n$.
Pour $z = (z_1, \cdots, z_n)$ et $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{B}$, définissons

$$\langle z, \zeta \rangle = z_1\overline{\zeta_1} + \cdots + z_n\overline{\zeta_n} \qquad \text{et} \qquad |z| = \langle z, z \rangle^{\frac{1}{2}}.$$

Dans la suite nous prendrons

$$d(z,\zeta) = \begin{cases} ||z| - |\zeta|| + \left|1 - \frac{\langle z,\zeta\rangle}{|z||\zeta|}\right| & \text{if } z, \zeta \in \overline{\mathbb{B}} \setminus \{0\} \\ |z| + |\zeta| & \text{if } z = 0 \text{ or } \zeta = 0 \end{cases}.$$

d est une pseudo-distance sur $\mathbb{B}$. Plus précisément, pour tous $z, \zeta, \xi \in \mathbb{B}$, nous avons $d(z,\zeta) \leqslant 2\,(d(z,\xi) + d(\xi,\zeta))$ et $0 \leqslant d(z,\zeta) < 3$.

Un exemple de boule ouverte pour cette topologie sont les fenêtres de Carleson voir la figure 4.1.

Nous noterons μ la mesure de Lebesgue sur $\mathbb{B}$. Pour $\alpha > 0$, $d\mu_\alpha(z) = (1-|\zeta|^2)^{\alpha-1} d\mu(z)$.

Le triplet $\left(\mathbb{B}, d, \mu_\alpha\right)$ est un espace homogène.

Nous noterons $\mathbb{H}(\mathbb{B})$ l'espace des fonctions holomorphes sur $\mathbb{B}$ et on définit $A^p_\alpha(\mathbb{B}) = \mathbb{H}(\mathbb{B}) \cap L^p(\mathbb{B}, d\mu_\alpha)$ l'espace de Bergman. Il fut introduit pour la première fois par S.Bergman en 1895.

Dans le but de caractériser le dual de $A^p_\alpha(\mathbb{B})$ on définit l'opérateur P_α sur $L^2(\mathbb{B}, d\mu_\alpha)$ comme la projection orthogonale de $L^2(\mathbb{B}, d\mu_\alpha)$ sur $A^2_\alpha(\mathbb{B})$. L' opérateur P_α ainsi définit est appelé projecteur de Bergman de $\mathbb{B}$. Il est bien connu que pour $f \in L^2(\mathbb{B}, d\mu_\alpha)$ [34, 24]

$$P_\alpha f(z) = \int_{\mathbb{B}} K(z,\zeta) \frac{f(\zeta)}{(1 - \langle z,\zeta\rangle)^{n+\alpha}} d\mu_\alpha(\zeta),$$

avec

$$K(z,\zeta) = \frac{1}{(1 - \langle z,\zeta\rangle)^{n+\alpha}}$$

appelé noyau de Bergman. Il fut démontré en 1974 par Forelli et Rudin qu'il s'étend en un opérateur continu de $L^p(\mathbb{B}, d\mu_\alpha)$ dans lui même pour $p > 1$ [21].

4.4. Inégalités L^p à poids pour le projecteur de Bergman la boule unité de $\mathbb{C}^n$

La singularité de P_α sur la diagonale n'a lieu qu'à la frontière. Notons aussi que P_α est un opérateur singulier puisque son noyau K vérifie les deux inégalités suivantes :

— Il existe β, $C_1 > 0$ et $C_2 > 0$ tel que

$$|K(z,\zeta) - K(z,\zeta^0)| \leqslant C_1 \frac{d(\zeta,\zeta^0)^\beta}{d(z,\zeta^0)^{n+\alpha+\beta}} \tag{4.14}$$

pour tout $z, \zeta, \zeta^0 \in \mathbb{B}$, tel que $d(z,\zeta^0) > C_2 d(\zeta,\zeta^0)$.

— Pour tout $z, \zeta \in \mathbb{B}$, $|K(z,\zeta)| \leqslant \frac{C_2}{d(z,\zeta)^{n+\alpha}}$.

Des travaux de Coifman et Fefferman [10], pour $p > 1$ on a déduit le résultat suivant :

$$w \in A_p \Longrightarrow \int_{\mathbb{B}} |P_\alpha f(z)|^p w(z) d\mu_\alpha(z) \leqslant C \int_{\mathbb{B}} |f(z)|^p w(z) d\mu_\alpha(z). \tag{4.15}$$

Contrairement au cas de la transformée de Hilbert donnée en (4.13), la réciproque n'est pas vraie. Il se pose donc la question de savoir s'il existe une classe de fonctions mesurables positives sur $\mathbb{B}$ plus grande que A_p rendant l'inégalité (4.15) vraie dans les deux sens. La solution à cette question fut donnée en 1978 par Békollè et Bonami dans le disque unité de $\mathbb{C}$ [3] et en 1982 dans la boule unité de $\mathbb{C}^n$ par Békollè [2]. Plus précisément, ces auteurs introduisent la classe de poids B_p qui porte aujourd'hui leurs noms (classe de Békollè-Bonami). On a l'inclusion $A_p \subset B_p$.

En effet, ils définissent la petite fonction maximale m_α par :

$$m_\alpha f(z) = \sup_{B \in \mathscr{B}} \frac{\chi_B(z)}{\mu_\alpha(B)} \int_B |f(\zeta)| d\mu_\alpha(\zeta), \tag{4.16}$$

où $\mathscr{B}$ est l'ensemble des boules $B \in \mathbb{B}$ telle que $\overline{B} \cap \partial\mathbb{B} \neq \emptyset$.

Puis ils introduisent la définition de la classe B_p suivante :

Définition 4.4.1. *Soit $p > 1$. La classe des poids de Békollè-Bonami B_p est la classe de fonctions positives w sur $\mathbb{B}$ telle que*

$$\sup_{B \in \mathscr{B}} \left(\frac{1}{\mu_\alpha(B)} \int_B w d\mu_\alpha \right) \left(\frac{1}{\mu_\alpha(B)} \int_B w^{-\frac{1}{p-1}} d\mu_\alpha \right)^{p-1} < \infty.$$

D'autre part $w \in B_1$ s'il existe $C > 0$ tel que pour tout $z \in \mathbb{B}$, $m_\alpha w(z) \leqslant C w(z)$.

Leur théorème est le suivant :

Théorème 24. *[2] Soit w une fonction mesurable positive et $1 < p < \infty$ (p une constante réelle). Les deux assertions suivantes sont équivalentes.*

1. *L'opérateur P_α est bien défini et continu sur $L^p(w d\mu_\alpha)$;*
2. *$w \in B_p$.*

Ce résultat fut cité un bon nombre de fois. En particulier dans [32] Pott et Reguera étudient la dépendance du poids dans la norme d'opérateur continue du Projecteur de Bergman dans la boule unité de $\mathbb{C}$ et dans [33] Rahm, Tchoundja et Wick l'étendent à la boule unité de $\mathbb{C}^n$. Dans [31] Peláez et Rättyä étudient les poids de Békollè-Bonami radiaux et récemment dans [4] Békollè, Keumo, Tchoundja et Wick établissent un résultat similaire à celui du Théorème 24 pour une extension du projecteur de Bergman.

4.5. Résultats de la thèse

En 1931 dans [30] Orlicz remplace la constante p des espaces de Lebesgue par une fonction mesurable positive $p(\cdot) : \mathbb{B} \to \mathbb{R}_+$, appellée fonction exposant. C'est la naissance des espaces de Lebesgue à exposant variable. Dans la suite nous noterons $\mathscr{P}(\mathbb{B})$ l'espace des fonctions exposants. Pour tout sous ensemble mesurable E de $\mathbb{B}$, nous introduisons les notations suivantes :

$$p_-(E) = ess\inf_{z\in E} p(z) \quad \text{et} \quad p_+(E) = ess\sup_{z\in E} p(z)$$

et nous utiliserons les notations $p_- = p_-(\mathbb{B})$ and $p_+ = p_+(\mathbb{B})$.

Plus précisément, pour $p(\cdot) \in \mathscr{P}(\mathbb{B})$, on dit que $f \in L^{p(\cdot)}(\nu)$ s'il existe $\lambda > 0$, $\rho_{p(\cdot)}(\frac{f}{\lambda}) < \infty$, où

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{B}} |f(z)|^{p(z)} d\nu(z).$$

Pour $p(\cdot) \in \mathscr{P}(\mathbb{B})$, $p_+ < \infty$ cette définition peut être sinplifiée comme suit : $f \in L^{p(\cdot)}(\nu)$ si $\rho_{p(\cdot)}(f) < \infty$. Quand ν est une mesure σ-finie et $p(\cdot) \in \mathscr{P}(\mathbb{B})$ est tel que $p(\cdot) \geqslant 1$, la fonctionnelle

$$\|f\|_{L^{p(\cdot)}(d\nu)} := \|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \quad \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}$$

est une norme.

Si $d\nu = wd\mu$ pour un poids w, on appelera $L^{p(\cdot)}(\nu)$ l'espace de Lebesgue variable à poids. Dans la suite nous noterons simplement $L^{p(\cdot)}(wd\mu_\alpha)$, par $L^{p(\cdot)}(w)$.

La plupart des propriétés des espaces de Lebesgue à exposant variable furent prouvées par Kovacik et Rakonisk dans [28] en 1991 et récemment par Diening, Harjulehto, Hästö et Ruzicka dans [17] et Cruz-Uribe et Fiorenza dans [15].

En 2014 dans [9] Chacón et Rafeiro introduisent la notion d'espaces de Bergman à exposant variable. Dans ce papier ils démontrent que le projecteur de Bergman est continu de $L^{p(\cdot)}(\mathbb{D})$, dans lui même. Plus précisément, ils établissent le théorème suivant :

Théorème 25. *Soit* $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$, $p(\cdot) \in \mathscr{P}(\mathbb{D})$ *tel que* $1 < p_- \leqslant p_+ < \infty$ *et tel qu'il existe* $c > 0$ *tel que pour tous* $z \neq \zeta \in \mathbb{D}$,

$$|p(z) - p(\zeta)| \leqslant \frac{c}{\log\left(e + \frac{1}{d(z,\zeta)}\right)}.$$

Alors il existe $C>0$ *tel que pour tout* $f\in L^{p(\cdot)}(\mathbb{D})$,

$$\|P_1 f\|_{p(\cdot)}\leqslant C\|f\|_{p(\cdot)}.$$

La preuve de ce résultat uilise le Théorème 24, le fait que la fonction maximale de Hardy-Littlewood donnée par (4.8) est continue sur $L^{p(\cdot)}(\mathbb{D})$ dans lui même et le théorème d'extrapolation de Rubio-de França, voir [15]. En 2017 dans [27] Karapetyants et Sanko proposent une nouvelle preuve du théorème 25 n'utilisant pas l'extrapolation. Plus précisément ils regardent le projecteur de Bergman sur le demi-plan supérieur et l'écrivent comme somme d'opérateur singulière de Calderon-Zygmund.

En effet, dans [27] , en notant $\mathbb{C}_+=\{z=x+iy,\ y\geqslant 0\}$ le demi plan-supérieur, $\mathbb{C}_-$ le demi-plan inférieur et $dA(\zeta)=dxdy$, Karapetyants et Sanko remarquent que :

$$\begin{aligned}P_1 f(z)&=-\frac{1}{\pi}\int_{\mathbb{C}_+}\frac{f(\zeta)}{(z-\overline{\zeta})^2}dA(\zeta)\\&=-\frac{1}{\pi}\int_{\mathbb{C}}\frac{\Omega\left(\frac{z-\zeta}{|z-\zeta|}\right)}{|z-\zeta|^2}g(\zeta)dA(\zeta)\end{aligned}\tag{4.17}$$

où $\Omega(\zeta)=\overline{\zeta}^2$, et $g(\zeta)=E_{\mathbb{C}_-}Qf(\zeta)$ avec $Qf(\zeta)=f(\overline{\zeta})$ l'opérateur de réflexion par rapport à l'axe des abscises et $E_{\mathbb{C}_-}f(\zeta)=\begin{cases}f(\zeta)\text{ si }\zeta\in\mathbb{C}_+\\ 0\text{ sinon .}\end{cases}$

Puis ils déduisent que $P_1 f(z)=\gamma\left(R_1^2 g(z)-R_2^2 g(z)-2R_1R_2 g(z)\right)$ où R_1 et R_2 sont les transformées de Riesz par rapport à $\mathscr{R}e(z)$ et $Im(z)$ respectivement et γ une constante réel.

Étant donné $p(\cdot)\in\mathscr{P}(\mathbb{C}_+)$ tel que $1<p_-\leqslant p_+<+\infty$, et tel qu'il existe $c>0$ et $p_\infty\in\mathbb{R}$ tel que pour tous $z\neq\zeta\in\mathbb{C}_+$,

$$|p(z)-p(\zeta)|\leqslant\frac{c}{\log\left(e+\frac{1}{|z-\zeta|}\right)}\text{ et }|p(z)-p_\infty|\leqslant\frac{c}{\log(e+|z|)}.\tag{4.18}$$

En étendant p sur $\mathbb{C}$ en posant $p(z)=p(\overline{z})$ sur $\mathbb{C}_-$, p vérifie encore les estimations (4.18) et de [19, Corollaire 4.12] qui dit que la transformée de Riesz est bornée sur $L^{p(\cdot)}(\mathbb{R}^2)$ dans ce cas car opérateur singulier, ils déduisent que P_1 est borné sur $L^{p(\cdot)}(\mathbb{C}_+)$.

Dans le même ordre d'idée que Békollè dans [2], dans le chapitre 5 en collaboration avec Békollè et Tchoundja nous nous sommes posés la question de déterminer une classe de fonctions mesurables positives pour laquelle le projecteur de Bergman est continu sur l'espace de Lebesgue variable à poids associé, étendant ainsi le théorème 24 et le théorème 25. Plus précisément, nous avons introduit la définition suivante généralisant la définition 4.4.1 de poids de Békollé-Bonami :

Définition 4.5.1 (Classe de Békollè-Bonami variable)**.** *Soit* $p(\cdot)\in\mathscr{P}(\mathbb{B})$ *tel que* $1<p_-\leqslant p_+<+\infty$ *et* w *une fonction positive. On dit que* w *appartient à la classe de*

Békollè-Bonami variable noté $B_{p(\cdot)}$ si

$$[w]_{B_{p(\cdot)}} := \sup_{B\in\mathscr{B}} \frac{1}{\mu_\alpha(B)} \| w^{\frac{1}{p(\cdot)}} \mathbb{1}_B \|_{p(\cdot)} \| w^{-\frac{1}{p(\cdot)}} \mathbb{1}_B \|_{p'(\cdot)} < \infty. \tag{4.19}$$

où on rappelle que $\mathscr{B}$ est l'ensemble des boules $B \subset \mathbb{B}$, $\overline{B} \cap \partial\mathbb{B} \neq \emptyset$, et $p'(\cdot)$ est la fonction conjugué de $p(\cdot)$, i.e.

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Puis nous avons établi le théorème suivant généralisant le théorème 24 et le théorème 25 et apportant une nouvelle preuve du théorème 25.

Théorème 26. *Soit $p(\cdot) \in \mathscr{P}(\mathbb{B})$ tel que $1 < p_- \leqslant p_+ < \infty$ et tel qu'il existe $c > 0$ tel que pour tout $z \neq \zeta \in \mathbb{B}$,*

$$|p(z) - p(\zeta)| \leqslant \frac{c}{\log\left(e + \frac{1}{d(z,\zeta)}\right)}.$$

Soit w une fonction mesurable positive. Les deux assertions suivantes sont équivalentes.

1. *L'opérateur P_α est bien défini et continu sur $L^{p(\cdot)}(w d\mu_\alpha)$;*
2. *$w \in B_{p(\cdot)}$.*

La preuve de ce résultat est très technique. Pour établir la condition nécessaire nous avons introduit deux nouvelles classes de fonctions mesurables positives équivalentes à la classe $B_{p(\cdot)}$ dont les conditions sont plus faciles à vérifier. Cela se résume dans le théorème suivant :

Théorème 27. *Soit $p(\cdot) \in \mathscr{P}(\mathbb{B})$ tel que $1 < p_- \leqslant p_+ < \infty$ et tel qu'il existe $c > 0$ tel que pour tout $z \neq \zeta \in \mathbb{B}$,*

$$|p(z) - p(\zeta)| \leqslant \frac{c}{\log\left(e + \frac{1}{d(z,\zeta)}\right)}.$$

Soit w une fonction mesurable positive. Les assertions suivantes sont équivalentes.

1. *$w \in B_{p(\cdot)}$;*
2. *$\sup_{B\in\mathscr{B}} \frac{1}{\mu_\alpha(B)^{p_B}} \| w\chi_B \|_1 \| w^{-1}\chi_B \|_{\frac{p'(\cdot)}{p(\cdot)}} < \infty$, C'est la classe $B^+_{p(\cdot)}$;*
3. *$\sup_{B\in\mathscr{B}} \frac{w(B)}{\mu_\alpha(B)} \left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B - 1} < \infty$, C'est la classe $B^{++}_{p(\cdot)}$;*

où

$$p_B = \left(\frac{1}{\mu_\alpha(B)} \int_B \frac{1}{p(x)} d\mu_\alpha(x)\right)^{-1},\ w(B) = \int_B w(z) d\mu_\alpha(z),\ w'(z) = (w(z))^{1-p'(z)}$$

Pour la condition suffisante, nous avons commencé par montrer que la petite fonction maximale m_α donnée par (4.16) est bornée.

Théorème 28. *Soit $p(\cdot) \in \mathscr{P}(\mathbb{B})$ tel que $1 < p_- \leqslant p_+ < \infty$ et tel qu'il existe $c > 0$ tel que pour tous $z \neq \zeta \in \mathbb{B}$,*

$$|p(z) - p(\zeta)| \leqslant \frac{c}{\log\left(e + \frac{1}{d(z,\zeta)}\right)}.$$

. Si $w \in B_{p(\cdot)}$, alors il existe une fonction C définie sur $(0,\infty)$ tel que $f \in L^{p(\cdot)}(w)$ on a

$$\|m_\alpha f\|_{L^{p(\cdot)}(w)} \leqslant C([w]_{B_{p(\cdot)}}) \|f\|_{L^{p(\cdot)}(w)}.$$

La preuve de ce théorème découle de quatre Lemmes fondementaux dont le principal qui est une variante du tour de magie de Diening est le suivant :

Lemme 4.5.1. *Soit $p(\cdot) \in \mathscr{P}(\mathbb{B})$ tel que $1 < p_- \leqslant p_+ < \infty$ et tel qu'il existe $c > 0$ tel que pour tous $z \neq \zeta \in \mathbb{B}$,*

$$|p(z) - p(\zeta)| \leqslant \frac{c}{\log\left(e + \frac{1}{d(z,\zeta)}\right)},$$

et $w \in B_{p(\cdot)}$. Alors pour tout $g \in L^{p(\cdot)}(w)$, $\|g\|_{L^{p(\cdot)}(w)} = 1$, et $z \in B \subset \mathbb{B}$, une pseudo-boule,

$$\left(\frac{1}{\mu_\alpha(B)} \int_B g(\zeta) d\mu_\alpha(\zeta)\right)^{p(z)} \lesssim \frac{1}{\mu_\alpha(B)} \int_B \left(g(\zeta)\right)^{p(\zeta)} d\mu_\alpha(\zeta) + 1.$$

Puis nous avons établi le théorème de factorisation suivant pour la classe B_p,

Théorème 29. *Soit une constante p telle que $1 < p < \infty$, les assertions suivantes sont équivalentes.*

1) *$w \in B_p$;*
2) *il existe $w_1 \in B_1$ et $w_2 \in B_1$ tel que $w = w_1 w_2^{1-p}$.*

Nous utiliserons le Theorème 29 pour prouver le théorème d'extrapolation suivant.

Théorème 30. *Soit $\mathscr{F}$ un sous ensemble de couples de fonctions mesurables positives. Supposons qu'ils existent une constante $p_0 > 1$, et une fonction $C : (0,\infty) \to (0,\infty)$ tel que pour tout $v \in B_{p_0}$ et $(F,G) \in \mathscr{F}$, on a*

$$\int_{\mathbb{B}} F(x)^{p_0} v(x) d\mu_\alpha(x) \leqslant C([v]_{B_{p_0}}) \int_{\mathbb{B}} G(x)^{p_0} v(x) d\mu_\alpha(x). \tag{4.20}$$

Alors pour $p(\cdot) \in \mathscr{P}(\mathbb{B})$ tel que $1 < p_- \leqslant p_+ < \infty$ et tel qu'il existe $c > 0$ tel que pour tous $z \neq \zeta \in \mathbb{B}$,

$$|p(z) - p(\zeta)| \leqslant \frac{c}{\ln\left(e + \frac{1}{d(z,\zeta)}\right)},$$

et $wB_{p(\cdot)}$, nous avons

$$\|F\|_{L^{p(\cdot)}(w)} \leqslant 16 \times 4^{-\frac{1}{p_0}} \left(C([v]_{B_{p_0}})\right)^{\frac{1}{p_0}} \|G\|_{L^{p(\cdot)}(w)} \tag{4.21}$$

pour tout $(F,G) \in \mathscr{F}$ and $F \in L^{p(\cdot)}(w)$

Ceci nous a permis de conclure la preuve du théorème 26 en utilisant le théorème 24.

La même question fut traitée plus tôt dans la littérature pour le cas de l'opérateur singulier T donné par (4.1) qui sont bornés dans un espace de Lebesgue classique [16]. En effet, dans [18], Diening et Hästo introduisent la classe de poids de Muckenhoupt variable sur $\mathbb{R}^n$, et notée $A_{p(\cdot)}$. Ils démontrent que la fonction maximale de Hardy-Littlewood donnée par (4.8) est bornée sur $L^{p(\cdot)}(w)$ si $w \in A_{p(\cdot)}$ pour $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{R}^n)$. Plus tard dans [13] Cruz-Uribe, Diening et Hästo donnent une nouvelle preuve de ce résultat utilisant la décomposition de Calderón-Zygmund et ils prouvent aussi l'implication réciproque. Récemment, dans [12] Cruz-Uribe et Cummings étendent le résultat de [13] aux espaces homogènes. Dans [16] Cruz-Uribe et Wang établissent les théorèmes d'extrapolations pour des espaces de Lebesgue à poids variable, de même dans [14] Cruz-Uribe et Hästö en donnent une version pour des espaces de Orlicz généralisés. Ces deux résultats ont été obtenus pour la topologie euclidienne sur $\mathbb{R}^n$ et non dans le cadre des espaces homogènes quelconques.

4.6. Organisation du document

Cette partie ne contient qu'un seul chapitre. Il a été publié sous forme d'article voir [5].

5. Weighted norm inequalities in the variable Lebesgue spaces for the Bergman projector on the unit ball of $\mathbb{C}^n$

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5.1. Introduction

The purpose of this work is to generalise the Békollè-Bonami theorem [2] for the Bergman projector on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ to the case of weighted variable Lebesgue spaces. The case of the unweighted variable Lebesgue spaces was treated by Chacon and Rafeiro [9, 36]. These authors showed that the Bergman projector is bounded on variable Lebesgue spaces for exponent functions $p(\cdot)$, which are log-Hölder continuous function with lower bound 1 (cf. Definition 5.1 below). The ingredients

of their proof are: the classical Békollè-Bonami theorem, the boundedness of the Hardy-Littlewood maximal function on variable Lebesgue spaces, and an extrapolation theorem. For basic properties of variable Lebesgue spaces, e.g. the boundedness of Hardy-Littlewood maximal function and an extrapolation theorem, we refer to [15, 17].

The σ-algebra on $\mathbb{B}$ is the Borel σ-algebra. Let ν be a positive measure on $\mathbb{B}$. The variable Lebesgue space on $\mathbb{B}$, denoted by $L^{p(\cdot)}(\nu)$, is a generalisation of the classical Lebesgue spaces, obtained by replacing the constant exponent p by a measurable exponent function $p(\cdot):\mathbb{B}\to[0,\infty)$. We shall denote by $\mathscr{P}(\mathbb{B})$ this family of all exponent functions $p(\cdot)$ on $\mathbb{B}$. For a measurable subset E of $\mathbb{B}$, we introduce the following notation:

$$p_-(E)=ess\inf_{z\in E}p(z)\quad\text{and}\quad p_+(E)=ess\sup_{z\in E}p(z)$$

and we will use the notation $p_-=p_-(\mathbb{B})$ and $p_+=p_+(\mathbb{B})$. We shall denote by $\mathscr{P}_+(\mathbb{B})$, the subfamily of $\mathscr{P}(\mathbb{B})$ consisting of those $p(\cdot)$ such that $p_+<\infty$. More precisely, for $p(\cdot)\in\mathscr{P}(\mathbb{B})$, we say that $f\in L^{p(\cdot)}(\nu)$ if for some $\lambda>0$, $\rho_{p(\cdot)}(\frac{f}{\lambda})<\infty$, where

$$\rho_{p(\cdot)}(f):=\int_{\mathbb{B}}|f(z)|^{p(z)}d\nu(z).$$

For $p(\cdot)\in\mathscr{P}_+(\mathbb{B})$, this definition can be simplified as follows: $f\in L^{p(\cdot)}(\nu)$ if $\rho_{p(\cdot)}(f)<\infty$. When ν is a σ-finite measure and $p(\cdot)\in\mathscr{P}(\mathbb{B})$ is such that $p(\cdot)\geqslant 1$, the functional

$$\|f\|_{p(\cdot)}=\inf\left\{\lambda>0:\quad\rho_{p(\cdot)}\left(\frac{f}{\lambda}\right)\leqslant 1\right\}$$

is a norm on the space $L^{p(\cdot)}(\nu)$; equipped with this norm, $L^{p(\cdot)}(\nu)$ is a Banach space.

We shall denote by μ the Lebesgue measure on $\mathbb{B}$. A non-negative locally integrable function on $\mathbb{B}$ is called a weight. If $d\nu=wd\mu$ for a weight w, we call $L^{p(\cdot)}(\nu)$ a weighted variable Lebesgue space. In the sequel, α is a positive number and we set $d\mu_\alpha(z)=(1-|\zeta|^2)^{\alpha-1}d\mu(z)$. We shall focus on the weighted variable Lebesgue space $L^{p(\cdot)}(wd\mu_\alpha)$, which we shall simply denote $L^{p(\cdot)}(w)$.

In this paper, we shall take

$$d(z,\zeta)=\begin{cases}||z|-|\zeta||+\left|1-\frac{\langle z,\zeta\rangle}{|z||\zeta|}\right| & \text{if } z,\zeta\in\overline{\mathbb{B}}\setminus\{0\}\\ |z|+|\zeta| & \text{if } z=0 \text{ or } \zeta=0\end{cases}.$$

Here, for $z=(z_1,\cdots,z_n)$ and $\zeta=(\zeta_1,\cdots,\zeta_n)\in\mathbb{B}$, we have set

$$\langle z,\zeta\rangle=z_1\overline{\zeta_1}+\cdots+z_n\overline{\zeta_n}\qquad\text{and}\qquad|z|=\langle z,z\rangle^{\frac{1}{2}}.$$

This application d is a pseudo-distance on $\mathbb{B}$. Explicitly, for all $z,\zeta,\xi\in\mathbb{B}$, we have $d(z,\zeta)\leqslant 2\,(d(z,\xi)+d(\xi,\zeta))$ and $0\leqslant d(z,\zeta)<3$. In addition, for $z\in\mathbb{B}$ and $r>0$, we denote by

$$B(z,r)=\{\zeta\in\mathbb{B}:\,d(z,\zeta)<r\}$$

the open pseudo-ball centred at z and of radius $r > 0$.

Definition 5.1. *A function $p(\cdot) \in \mathscr{P}(\mathbb{B})$ is log-Hölder continuous on $\mathbb{B}$ if there exists a positive constant $c > 0$ such that for all $z, \zeta \in \mathbb{B}$*

$$|p(z) - p(\zeta)| \leqslant \frac{c}{\log\left(e + \frac{1}{d(z,\zeta)}\right)} \quad if \quad z \neq \zeta. \tag{5.1}$$

We denote by $\mathscr{P}^{\log}(\mathbb{B})$ the space of all log-Hölder continuous functions on $\mathbb{B}$. It is easily checked that $\mathscr{P}^{\log}(\mathbb{B}) \subset \mathscr{P}_+(\mathbb{B})$. As usual, we set $\mathscr{P}^{\log}_{\pm}(\mathbb{B}) = \{p(\cdot) \in \mathscr{P}^{\log}(\mathbb{B}) : \ p_- > 1\}$.

An example of a member of $\mathscr{P}^{\log}_{\pm}(\mathbb{B})$ is $p(z) = 2 + \sin|z|$. We shall denote by $\mathscr{P}_-(\mathbb{B})$ the subfamily of $\mathscr{P}(\mathbb{B})$ consisting of those $p(\cdot)$ such that $p_- > 1$. So $\mathscr{P}^{\log}_{\pm}(\mathbb{B}) = \mathscr{P}^{\log}(\mathbb{B}) \cap \mathscr{P}_-(\mathbb{B})$.

Definition 5.2. *We denote by $\mathscr{B}$ the collection of pseudo-balls B of $\mathbb{B}$ such that $\overline{B} \cap \partial\mathbb{B} \neq \emptyset$. Next we define the maximal function m_α by*

$$m_\alpha f(z) = \sup_{B \in \mathscr{B}} \frac{\chi_B(z)}{\mu_\alpha(B)} \int_B |f(\zeta)| d\mu_\alpha(\zeta).$$

Observe that, from Lemma 5.1, $\mathscr{B}$ is the set of pseudo-balls that touch the boundary of $\mathbb{B}$.

In the classical Lebesgue spaces, we have the following definition.

Definition 5.3. *Let $p > 1$ be a constant exponent. The Békollè-Bonami B_p weight class consists of weights w such that*

$$\sup_{B \in \mathscr{B}} \left(\frac{1}{\mu_\alpha(B)} \int_B w d\mu_\alpha\right) \left(\frac{1}{\mu_\alpha(B)} \int_B w^{-\frac{1}{p-1}} d\mu_\alpha\right)^{p-1} < \infty.$$

This definition of B_p is equivalent to the following definition:

$$\sup_{B \in \mathscr{B}} \frac{1}{\mu_\alpha(B)} \| w^{\frac{1}{p}} \chi_B \|_p \| w^{-\frac{1}{p}} \chi_B \|_{p'} < \infty,$$

where p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

In the same spirit, we introduce a variable generalisation of the B_p weight class. Analogously to the classical case, for $p(\cdot) \in \mathscr{P}_-(\mathbb{B})$, we say that $p'(\cdot)$ is the conjugate exponent function of $p(\cdot)$ if for all $z \in \mathbb{B}$ we have

$$\frac{1}{p(z)} + \frac{1}{p'(z)} = 1.$$

Moreover, we set $\mathscr{P}_\pm(\mathbb{B}) = \mathscr{P}_-(\mathbb{B}) \cap \mathscr{P}_+(\mathbb{B})$, the subfamily of $\mathscr{P}(\mathbb{B})$ consisting of those exponent functions $p(\cdot)$ such that $1 < p_- \leqslant p_+ < \infty$. We now define the variable Békollè-Bonami classes of weights.

Definition 5.4. *Let $p(\cdot) \in \mathscr{P}_{\pm}(\mathbb{B})$. A weight w belongs to the variable Békollè-Bonami class on $\mathbb{B}$, denoted $B_{p(\cdot)}$, if*

$$[w]_{B_{p(\cdot)}} := \sup_{B \in \mathscr{B}} \frac{1}{\mu_\alpha(B)} \| w^{\frac{1}{p(\cdot)}} \chi_B \|_{p(\cdot)} \| w^{-\frac{1}{p(\cdot)}} \chi_B \|_{p'(\cdot)} < \infty. \tag{5.2}$$

We define the operator P_α on $L^1(\mathbb{B}, d\mu_\alpha)$ by

$$P_\alpha f(z) = \int_{\mathbb{B}} \frac{f(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+\alpha}} d\mu_\alpha(\zeta).$$

The restriction to $L^2(\mathbb{B}, d\mu_\alpha)$ of the operator P_α is called the Bergman projector of $\mathbb{B}$. We also define the positive Bergman operator P_α^+ by

$$P_\alpha^+ f(z) = \int_{\mathbb{B}} \frac{f(\zeta)}{|1 - \langle z, \zeta \rangle|^{n+\alpha}} d\mu_\alpha(\zeta).$$

We now recall the classical Békollè-Bonami theorem:

Theorem 5.1. *[2] Let w be a non-negative measurable function and let $1 < p < \infty$ (p is a constant exponent). The following two assertions are equivalent.*

1. *The Bergman operator P_α is well defined and bounded on $L^p(w d\mu_\alpha)$;*
2. *$w \in B_p$.*

Moreover, P_α^+ is well defined and bounded on $L^p(w d\mu_\alpha)$ if $w \in B_p$.

The purpose of this work is to prove the following generalisation of the previous theorem:

Theorem 5.2. *Let w be a non-negative measurable function and $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$. The following two assertions are equivalent.*

1. *The Bergman operator P_α is well defined and bounded on $L^{p(\cdot)}(w d\mu_\alpha)$;*
2. *$w \in B_{p(\cdot)}$.*

Moreover, P_α^+ is well defined and bounded on $L^{p(\cdot)}(w d\mu_\alpha)$ if $w \in B_{p(\cdot)}$.

The problem under study is trivial if $w(\mathbb{B}) = 0$, i.e. $w \equiv 0$ a.e. on $\mathbb{B}$. We shall assume that $w(\mathbb{B}) > 0$.

In [18], Diening and Hästo introduced the variable Muckenhoupt weight class $A_{p(\cdot)}$ on $\mathbb{R}^n$, and showed that for $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function is bounded on $L^{p(\cdot)}(w)$ only if $w \in A_{p(\cdot)}$. In order to manage the necessary condition, they introduced a new class $A_{p(\cdot)}^+$ which coincides with $A_{p(\cdot)}$ when $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{R}^n)$, but whose condition is easier to check. Later, Cruz-Uribe and these two authors [13] gave a new proof of this result using the Calderón-Zygmund decomposition and they also proved the reverse implication. Very recently, Cruz-Uribe and Cummings [12] extended the result of [13] to the spaces of homogeneous type.

In this paper we will use the technique of [18, 13] to manage the proof of the necessary condition in Theorem 5.2. Precisely, we shall introduce a new class denoted $B^+_{p(\cdot)}$ which coincides with $B_{p(\cdot)}$ when $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$. To deal with the sufficient condition, we rely on the result of [12] about the boundedness on $L^{p(\cdot)}(w)$ of the Hardy-Littlewood maximal function on the space of homogeneous type $\mathbb{B}$, for $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$ and w in the corresponding class of weights $A_{p(\cdot)}$. The proof then follows two steps. First, we use this result to show that, for $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$, the maximal function m_α is bounded on $L^{p(\cdot)}(w)$ if $w \in B_{p(\cdot)}$. Secondly, we lean on the first step to define a new extrapolation theorem which allows us to conclude the proof of the sufficient condition in Theorem 5.2.

The rest of our paper is organised as follows. In section 5.2, we shall recall some preliminaries. Next in section 5.3, we review properties of weighted variable Lebesgue spaces, variable Békollè-Bonami and Muckenhoupt classes of weights. In the end of this section, we state the theorem of Cruz-Uribe and Cummings about the boundedness on $L^{p(\cdot)}(w)$ of the Hardy-Littlewood maximal function on $\mathbb{B}$. In section 5.4, we prove the necessity of the conditions $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$ and $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$ in Theorem 5.2. In section 5.5, we define and study the class $B^+_{p(\cdot)}$ and we show the identity $B^+_{p(\cdot)} = B_{p(\cdot)}$. In section 5.6, we prove the necessary condition in Theorem 5.2. In section 5.7, we show that the maximal function m_α is bounded on $L^{p(\cdot)}(w)$ if $w \in B_{p(\cdot)}$. Finally, in section 5.8, we prove a weighted extrapolation theorem from which we deduce a proof of the sufficient condition in Theorem 5.2.

Let a and b be two positive numbers. Throughout the paper, we write $a \lesssim b$ if there exists $C > 0$ independent of a and b such that $a \leqslant Cb$. We write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

5.2. Preliminaries

In this section we present some background material regarding the unit ball of $\mathbb{C}^n$ as a space of homogeneous type and the variable exponent Lebesgue spaces.

5.2.1. The unit ball is a space of homogeneous type

In this subsection we recall some lemmas from [2] where it was first considered $(\mathbb{B}, d, \mu_\alpha)$.

Lemma 5.1. *Let $z \in \mathbb{B}$ and $R > 0$. The pseudo-ball $B(z,r)$ meets the boundary of $\mathbb{B}$ (in other words, $B \in \mathscr{B}$) if and only if $R > 1 - |z|$.*

Lemma 5.2. *Let $z \in \mathbb{B}$ and $0 < R \leqslant 3$, we have*

$$\mu_\alpha(B(z,R)) \simeq R^{n+1} \left(\max(R, 1-|z|)\right)^{\alpha-1}.$$

Remark 5.1. *From Lemma 5.1 and Lemma 5.2 if $B(z,R) \in \mathscr{B}$ we have*

$$\mu_\alpha(B(z,R)) \simeq R^{n+\alpha}.$$

From Lemma 5.2, we deduce that $(\mathbb{B}, d, \mu_\alpha)$ is a space of homogeneous type. Next we have the following lemma.

Lemma 5.3. *[12] There exist two positive constants C and γ such that, for all $\zeta \in B(z,R)$ we have*

$$\mu_\alpha(B(\zeta,r)) \geqslant C\left(\frac{r}{R}\right)^\gamma \mu_\alpha(B(z,R))$$

for every $0 < r \leqslant R < \infty$.

5.2.2. Variable exponent Lebesgue spaces

We denote by $\mathscr{M}$ the space of complex-valued measurable functions defined on $\mathbb{B}$. Let ν be a positive measure on $\mathbb{B}$. The family $\mathscr{P}(\mathbb{B})$ of variable exponents was defined in the introduction. In the rest of the paper, we take $p(\cdot) \in \mathscr{P}(\mathbb{B})$. The next definitions, properties and propositions are stated in [15, 17, 36]. We first recall some properties of the modular functional $\rho_{p(\cdot)} : \mathscr{M} \to [0,\infty]$, defined in the introduction as

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{B}} |f(z)|^{p(z)} d\nu(z).$$

Proposition 5.1. *Let $p(\cdot) \in \mathscr{P}(\mathbb{B})$ be such that $p(\cdot) \geqslant 1$.*

1. *For all $f \in \mathscr{M}$, $\quad \rho_{p(\cdot)}(f) \geqslant 0$, and $\rho_{p(\cdot)}(f) = \rho_{p(\cdot)}(|f|)$.*
2. *For all $f \in \mathscr{M}$ if $\rho_{p(\cdot)}(f) < \infty$ then $|f(z)| < \infty$ a.e. on $\mathbb{B}$.*
3. *$\rho_{p(\cdot)}$ is convex. In particular, for $0 < \alpha \leqslant 1$ and $f \in \mathscr{M}$, $\rho_{p(\cdot)}(\alpha f) \leqslant \alpha \rho_{p(\cdot)}(f)$ and for $\alpha \geqslant 1$, $\quad \alpha \rho_{p(\cdot)}(f) \leqslant \rho_{p(\cdot)}(\alpha f)$.*
4. *$\rho_{p(\cdot)}(f) = 0$ if and only if $f(z) = 0$ a.e. on $\mathbb{B}$.*
5. *If for almost all $z \in \mathbb{B}$, $|f(z)| \leqslant |g(z)|$, then $\rho_{p(\cdot)}(f) \leqslant \rho_{p(\cdot)}(g)$.*
6. *If there exists $\beta > 0$ such that $\rho_{p(\cdot)}(\frac{f}{\beta}) < \infty$, then the function $\lambda \longmapsto \rho_{p(\cdot)}(\frac{f}{\lambda})$ is continuous and non-increasing on $[\beta, \infty[$. In addition*

$$\lim_{\lambda \longrightarrow \infty} \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) = 0$$

.

For $p(\cdot) \in \mathscr{P}(\mathbb{B})$, the variable Lebesgue space $L^{p(\cdot)}(d\nu)$ was defined in the introduction.

Proposition 5.2. *[15, Theorem 2.7.2] Let $p(\cdot) \in \mathscr{P}_+(\mathbb{B})$ be such that $p(\cdot) \geqslant 1$. Then the subspace of continuous functions of compact support in $\mathbb{B}$ is dense in the space $L^{p(\cdot)}(d\nu)$.*

We next recall the Hölder inequality in the variable exponent context.

Proposition 5.3. *[15, Theorem 2.26, Corollary 2.28]*

1. *Let $p(\cdot) \in \mathscr{P}(\mathbb{B})$ be such that $p(\cdot) \geqslant 1$. Then for all $f, g \in \mathscr{M}$, we have*

$$\int_{\mathbb{B}} |fg| d\nu \leqslant 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

2. *Let $r(\cdot),\ q(\cdot) \in \mathscr{P}(\mathbb{B})$ such that $r(\cdot), q(\cdot) \geqslant 1$ and $\frac{1}{q(x)} + \frac{1}{r(x)} \leqslant 1$ for all $x \in \mathbb{B}$. Define $p(\cdot) \in \mathscr{P}(\mathbb{B})$ such that $p(\cdot) \geqslant 1$, by*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a constant K such that for all $f \in L^{q(\cdot)}$ and $g \in L^{r(\cdot)}$, $fg \in L^{p(\cdot)}$ and

$$\|fg\|_{p(\cdot)} \leqslant K\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

We record the following useful remark.

Remark 5.2. *The property $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$ is also true for $p'(\cdot)$.*

The following lemma will be useful.

Lemma 5.4. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{B})$. Let $B = B(x, R)$ be a pseudo-ball of $\mathbb{B}$ such that $R < \frac{1}{4}$. Then*

$$p_+(B) - p_-(B) \leqslant \frac{c}{\log\left(\frac{1}{4R}\right)}.$$

Lemma 5.5. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{B})$. There exist two positive constants $C_1 = C_1(\alpha, n, p(\cdot))$ and $C_2 = C_2(\alpha, n, p(\cdot))$ such that for every pseudo-ball B of $\mathbb{B}$, we have*

$$\mu_\alpha(B)^{p_-(B)-p_+(B)} \leqslant C_1 \text{ and } \mu_\alpha(B)^{p_+(B)-p_-(B)} \leqslant C_2.$$

Proof. Since $p_+(B) - p_-(B) \geqslant 0$, $\mu_\alpha(\mathbb{B}) < \infty$, $B \subset \mathbb{B}$ and $p(\cdot)$ bounded, we have the second inequality. We are going to prove the first inequality. Suppose that $B = B(z, R)$.

1. If $R \geqslant \frac{1}{16}$, from Lemma 5.2, there exists $C > 0$ such that $\mu_\alpha(B) \geqslant CR^{n+\alpha}$ and as $p_-(B) - p_+(B) \leqslant 0$, we obtain:

$$\mu_\alpha(B)^{p_-(B)-p_+(B)} \leqslant (CR^{n+\alpha})^{(p_-(B)-p_+(B))} \leqslant (C(16)^{n+1})^{(p_+(B)-p_-(B))}$$

2. If $R < \frac{1}{16}$, from Lemma 5.3 and Lemma 5.2, there exist positive constants C, C' and γ such that

$$\mu_\alpha(B(z,R)) \geqslant C\,(4R)^\gamma\,\mu_\alpha\left(B\left(z, \frac{1}{16}\right)\right) \geqslant C'R^\gamma. \tag{5.3}$$

Thus, since $p_-(B) - p_+(B) \leqslant 0$, by Lemma 5.4, it follows from (5.3) that

$$\begin{aligned}\mu_\alpha(B)^{p_-(B)-p_+(B)} &\leqslant \left(C'R^\gamma\right)^{p_-(B)-p_+(B)}\\ &\lesssim R^{-\frac{\gamma c}{\log\left(\frac{1}{4R}\right)}}\\ &\leqslant \exp\left(2\gamma c\right).\end{aligned}$$

□

From the previous lemma, we easily deduce the following corollary.

Corollary 5.1. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{B})$. There exists a constant $C = C(\alpha, n, p(\cdot)) > 1$ such that for every pseudo-ball B of $\mathbb{B}$ and every $z \in B$ we have*

$$\frac{1}{C} \leqslant \mu_\alpha(B)^{p_-(B)-p(z)} \leqslant C.$$

5.3. Weighted variable Lebesgue spaces, variable Békollè-Bonami and Muckenhoupt classes of weights

5.3.1. Weighted variable Lebesgue spaces

Let w be a weight and let $p(\cdot) \in \mathscr{P}(\mathbb{B})$ be such that $p(\cdot) \geqslant 1$. The corresponding weighted variable exponent Lebesgue space $L^{p(\cdot)}(w d\mu_\alpha)$ consists of those $f \in \mathscr{M}$ which satisfy the estimate

$$\rho_{p(\cdot),w}(f) = \int_{\mathbb{B}} |f(z)|^{p(z)} w(z) d\mu_\alpha(z) < \infty.$$

We also denote it $L^{p(\cdot)}(\mathbb{B}, w)$ or simply $L^{p(\cdot)}(w)$, and we denote its norm by $\|\cdot\|_{p(\cdot),w}$. It is easy to check that

$$\|f\|_{p(\cdot),w} = \|f w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}.$$

In the sequel, we shall adopt the following notation:

$$w' := w^{1-p'(\cdot)}. \tag{5.4}$$

We recall the notion of subordinate norm on $L^{p(\cdot)}(w)$ defined by

$$\|f\|'_{p(\cdot),w} := \sup_{\|g\|_{p'(\cdot),w'}=1} \left| \int_{\mathbb{B}} f(\zeta)\overline{g(\zeta)} d\mu_\alpha(\zeta) \right|.$$

We next recall the following proposition.

Proposition 5.4. *[15, 17, Corollary 2.7.5] Let $p(\cdot) \in \mathscr{P}(\mathbb{B})$ such that $p(\cdot) \geqslant 1$ and let w be a weight. Then*

$$\|f\|_{p(\cdot),w} \leqslant \|f\|'_{p(\cdot),w} \leqslant 2\|f\|_{p(\cdot),w}.$$

The following lemma will be very useful.

Lemma 5.6. *[15, 17] Let w be a non-negative measurable function and let $p(\cdot) \in \mathscr{P}_+(\mathbb{B})$ be such that $p_- > 0$. Then for every $f \in \mathscr{M}$ whose support is E, the following double inequality holds*

$$\min\left(\rho_{p(\cdot),w}(f)^{\frac{1}{p_-(E)}}, \rho_{p(\cdot),w}(f)^{\frac{1}{p_+(E)}}\right) \leqslant \|f\|_{p(\cdot),w} \leqslant \max\left(\rho_{p(\cdot),w}(f)^{\frac{1}{p_-(E)}}, \rho_{p(\cdot),w}(f)^{\frac{1}{p_+(E)}}\right).$$

It is equivalent to

$$\min\left(\|f\|^{p_-(E)}_{p(\cdot),w}, \|f\|^{p_+(E)}_{p(\cdot),w}\right) \leqslant \rho_{p(\cdot),w}(f) \leqslant \max\left(\|f\|^{p_-(E)}_{p(\cdot),w}, \|f\|^{p_+(E)}_{p(\cdot),w}\right).$$

5.3.2. Variable Békollè-Bonami classes of weights

Concerning the variable Békollè-Bonami weight class $B_{p(\cdot)}$, we record the following elementary remark.

Remark 5.3. *Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$. If $w \in B_{p(\cdot)}$, the following two assertions are valid.*

1) $\|w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} < \infty$ *and* $\|w^{-\frac{1}{p(\cdot)}}\|_{p'(\cdot)} < \infty$.
2) *The functions w and w' are integrable on $\mathbb{B}$.*

Proof of the remark. Otherwise, if $\|w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} = \infty$, then necessarily $\|w^{-\frac{1}{p(\cdot)}}\|_{p'(\cdot)} = 0$ and this would imply that $w \equiv \infty$ a.e. Alternatively, if $\|w^{-\frac{1}{p(\cdot)}}\|_{p(\cdot)} = \infty$, then necessarily $\|w^{\frac{1}{p(\cdot)}}\|_{p'(\cdot)} = 0$ and this would imply that $w \equiv 0$ a.e. Furthermore, by Lemma 5.6, for $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$, the assertions 1) and 2) are equivalent. □

We also have the following proposition.

Proposition 5.5. *Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$. For a weight w, the following two assertions are equivalent.*

1. $w \in B_{p(\cdot)}$;
2. $w' \in B_{p'(\cdot)}$.

Moreover, $[w]_{B_{p(\cdot)}} = [w']_{B_{p'(\cdot)}}$.

The following simple lemma will be useful.

Lemma 5.7. *Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$. For a weight w, the following two assertions are equivalent.*

1. $w \in B_{p(\cdot)}$;
2. $\sup_{B \in \mathscr{B}} \frac{1}{\mu_\alpha(B)} \|\chi_B\|_{p(\cdot),w} \|\chi_B\|_{p'(\cdot),w'} < \infty$.

5.3.3. Variable Muckenhoupt classes of weights

Definition 5.5. *The Hardy-Littlewood maximal function M_α on the space of homogeneous type $(\mathbb{B}, d, \mu_\alpha)$ is defined by*

$$M_\alpha f(z) = \sup_B \frac{\chi_B(z)}{\mu_\alpha(B)} \int_B |f(\zeta)| d\mu_\alpha(\zeta)$$

where the supremum is taken over all pseudo-balls of $\mathbb{B}$.

When p is a constant greater than 1, the Muckenhoupt class A_p consists of weights w which satisfy the estimate

$$\sup_B \left(\frac{1}{\mu_\alpha(B)} \int_B w d\mu_\alpha \right) \left(\frac{1}{\mu_\alpha(B)} \int_B w^{-\frac{1}{p-1}} d\mu_\alpha \right)^{p-1} < \infty,$$

where the sup is taken over all pseudo-balls B of $\mathbb{B}$. This definition is equivalent with the following definition:

$$\sup_B \frac{1}{\mu_\alpha(B)} \| w^{\frac{1}{p}} \chi_B \|_p \| w^{-\frac{1}{p}} \chi_B \|_{p'} < \infty,$$

where the sup is taken again over all pseudo-balls of $\mathbb{B}$.

We next have the following variable generalisation of the variable Muckenhoupt weight classes. This generalisation was given first by Diening and Hästo [18].

Definition 5.6. *Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$. A weight w belongs to the variable Muckenhoupt class $A_{p(\cdot)}$ on $\mathbb{B}$ if*

$$[w]_{A_{p(\cdot)}} := \sup_B \frac{1}{\mu_\alpha(B)} \| w^{\frac{1}{p(\cdot)}} \chi_B \|_{p(\cdot)} \| w^{-\frac{1}{p(\cdot)}} \chi_B \|_{p'(\cdot)} < \infty \tag{5.5}$$

where the sup *is taken over all pseudo-balls of $\mathbb{B}$.*

Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$. The following proposition is similar to Proposition 5.5.

Proposition 5.6. *The following two assertions are equivalent.*

1. $w \in A_{p(\cdot)}$;
2. $w' \in A_{p'(\cdot)}$.

We record the following properties of $A_{p(\cdot)}$ and $B_{p(\cdot)}$.

Proposition 5.7. 1. *The inclusion $A_{p(\cdot)} \subset B_{p(\cdot)}$ holds with $[w]_{B_{p(\cdot)}} \leqslant [w]_{A_{p(\cdot)}}$.*

2. *$[w]_{B_{p(\cdot)}} \geqslant \frac{1}{2}$ and $[w]_{A_{p(\cdot)}} \geqslant \frac{1}{2}$.*

Proof. 1. This follows directly from the definitions of $A_{p(\cdot)}$ and $B_{p(\cdot)}$.

2. We first give the proof for $B_{p(\cdot)}$. Let $B \in \mathscr{B}$. From the Hölder inequality and the definition of $B_{p(\cdot)}$, we have

$$1 = \frac{1}{\mu_\alpha(B)} \int_B w^{\frac{1}{p(\cdot)}} w^{-\frac{1}{p(\cdot)}} d\mu_\alpha \leqslant \frac{2}{\mu_\alpha(B)} \| w^{\frac{1}{p(\cdot)}} \chi_B \|_{p(\cdot)} \| w^{-\frac{1}{p(\cdot)}} \chi_B \|_{p'(\cdot)} \leqslant 2[w]_{B_{p(\cdot)}}. \tag{5.6}$$

The proof for $A_{p(\cdot)}$ then follows from assertion 1.

□

In [12], Cruz-Uribe and Cummings proved the following fundamental result for the maximal Hardy-Littlewood function. This variable theorem generalises a well-known theorem of Muckenhoupt [29] in the Euclidean space $\mathbb{R}^n$. For spaces of homogeneous type, the analogous theorem for constant exponents was proved later by A. P. Calderón [7].

Theorem 5.3. *Let $p(\cdot) \in \mathscr{P}_\pm^{\log}(\mathbb{B})$. The following two assertions are equivalent.*

1. *There exists a positive constant C such that for all $f \in L^{p(\cdot)}(w)$, we have*

$$\| M_\alpha f \|_{p(\cdot),w} \leqslant C \| f \|_{p(\cdot),w}.$$

2. *$w \in A_{p(\cdot)}$.*

In fact, these authors [12] proved their result in the general setting of spaces of homogeneous type. There, in addition to the condition $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{B})$, they have a condition at infinity which has the following expression on the unit ball $\mathbb{B}$: there are two constants c and p_∞ such that

$$|p(z) - p_\infty| \leqslant \frac{c}{\log(e + |z|)}$$

for every $z \in \mathbb{B}$. It is easy to show that this extra condition is satisfied on $\mathbb{B}$.

Since $m_\alpha f \leqslant M_\alpha f$, we deduce the following corollary.

Corollary 5.2. *Let $p(\cdot) \in \mathscr{P}_\pm^{\log}(\mathbb{B})$ and $w \in A_{p(\cdot)}$. For the same constant C as in Theorem 5.3, we have*

$$\| m_\alpha f \|_{p(\cdot),w} \leqslant C \| f \|_{p(\cdot),w}$$

for all $f \in L^{p(\cdot)}(w)$.

5.4. Necessity of the conditions $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$ and $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$ in Theorem 5.2

Proposition 5.8. *Let w be a weight and let $p(\cdot) \in \mathscr{P}_-(\mathbb{B})$. If the operator P_α is well defined on $L^{p(\cdot)}(w)$, then $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$.*

Proof. If the operator P_α is well defined on $L^{p(\cdot)}(w)$, then for every $f \in L^{p(\cdot)}(w)$, we have for all $z \in \mathbb{B}$,

$$|P_\alpha(f)(z)| < \infty.$$

Now, by a contradiction argument, suppose that $w^{-\frac{1}{p(\cdot)}}$ does not belong to $L^{p'(\cdot)}(d\mu_\alpha)$. Then applying the closed graph theorem using Proposition 3.1 above and Proposition 2.67 of [15] [1], there exists a non-negative $g \in L^{p(\cdot)}(d\mu_\alpha)$ such that

$$\int_{\mathbb{B}} g(\zeta) w(\zeta)^{-\frac{1}{p(\zeta)}} d\mu_\alpha(\zeta) = \infty.$$

Let $f = g w^{-\frac{1}{p(\cdot)}}$. We have $f \in L^{p(\cdot)}(w)$ but f does not belong to $L^1(\mathbb{B})$; in other words, $|P_\alpha(f)(0)| = \infty$. This contradicts the fact that $|P_\alpha(f)(0)| < \infty$. □

Proposition 5.9. *Let w be a weight and $p(\cdot) \in \mathscr{P}_-(\mathbb{B})$. If the operator P_α is bounded on $L^{p(\cdot)}(w d\mu)$, then $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$.*

Proof. Let $0 < r < 1$ and define the function $f(z) = (1-|z|^2)^{1-\alpha} \chi_{B(0,r)}(z)$ on $\mathbb{B}$. We have

$$\begin{aligned} P_\alpha f(z) &= \int_{\mathbb{B}} \frac{f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+\alpha}} d\mu_\alpha(\zeta) \\ &= \int_{B(0,r)} \frac{1}{(1-\langle z,\zeta\rangle)^{n+\alpha}} d\mu(\zeta) \\ &= \overline{\int_{B(0,r)} \frac{1}{(1-\langle \zeta,z\rangle)^{n+\alpha}} d\mu(\zeta)}. \end{aligned}$$

Since the function $\zeta \longmapsto \frac{1}{(1-\langle \zeta,z\rangle)^{n+\alpha}}$ is analytic on $\mathbb{B}$ and $B(0,r)$ is the Euclidean ball centred at 0 and of radius r, it follows from the mean value property that $P_\alpha f(z) \equiv C_{r,n}$ and so

$$|C_{r,n}| \| w^{\frac{1}{p(\cdot)}} \|_{p(\cdot)} = \| P_\alpha(f) \|_{p(\cdot),w}. \tag{5.7}$$

In addition,

$$\rho_{p(\cdot),w}(f) = \int_{B(0,r)} w(z)(1-|z|^2)^{(\alpha-1)(1-p(z))} d\mu(z).$$

On the one hand, if $\alpha \leqslant 1$, we have $(1-|z|^2)^{(\alpha-1)(1-p(z))} \leqslant 1$ because $(\alpha-1)(1-p(z)) > 0$ and $1-|z|^2 \leqslant 1$. Consequently

$$\rho_{p(\cdot),w}(f) \leqslant \int_{B(0,r)} w(z) d\mu(z) < \infty$$

because w is locally integrable.

1. Also refer to Exercise 4.7 of [6], for the analogous result for the classical Lebesgue spaces.

On the other hand, if $\alpha > 1$ we have $(1-|z|^2)^{(\alpha-1)(1-p(z))} \leqslant (1-|z|^2)^{(\alpha-1)(1-p_+)}$. So

$$\rho_{p(\cdot),w}(f) \leqslant \sup_{z\in B(0,r)} (1-|z|^2)^{(\alpha-1)(1-p_+)} \int_{B(0,r)} w(\eta)\, d\mu(\eta) < \infty$$

because w is locally integrable and $\sup_{z\in B(0,r)}(1-|z|^2)^{(\alpha-1)(1-p_+)} = (1-r^2)^{(\alpha-1)(1-p_+)}$. Thus, since $\rho_{p(\cdot),w}(f) < \infty$ in both cases, by Lemma 5.6 we obtain $\|f\|_{p(\cdot),w} < \infty$ and as P_α is bounded on $L^{p(\cdot)}(w)$, we deduce from (5.7) that there exists a positive constant $c_{r,\alpha,n}$ such that

$$\|w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \leqslant c_{r,\alpha,n}\|f\|_{p(\cdot),w} < \infty.$$

Hence we have the result. □

In what follows, we will need to calculate the $p(\cdot)-$norm of some functions but is not easy like a constant case. To deal with that, in the following section we introduce the definitions of new spaces whose coincide with the class $B_{p(\cdot)}$.

5.5. The weight classes $B^+_{p(\cdot)}$ and $B^{++}_{p(\cdot)}$

Definition 5.7. *Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$ and let w be a weight. We say that w is in the $B^+_{p(\cdot)}$ class if*

$$[w]_{B^+_{p(\cdot)}} := \sup_{B\in\mathscr{B}} \frac{1}{\mu_\alpha(B)^{p_B}} \|w\chi_B\|_1 \|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} < \infty. \tag{5.8}$$

where

$$p_B = \left(\frac{1}{\mu_\alpha(B)}\int_B \frac{1}{p(x)}\, d\mu_\alpha(x)\right)^{-1}.$$

This class coincides with the B_p class when $p(\cdot) = p$ (p constant). We also adopt the following notation:

$$\langle p\rangle_B = \frac{1}{\mu_\alpha(B)}\int_B p(x)\, d\mu_\alpha(x). \tag{5.9}$$

Remark 5.4. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{B})$ and let B be a pseudo-ball of $\mathbb{B}$. As $p_-(B) \leqslant p_B, \langle p\rangle_B \leqslant p_+(B)$, it follows from Lemma 5.5 and Corollary 5.1 that*

$$\mu_\alpha(B)^{p_-(B)} \simeq \mu_\alpha(B)^{p_B} \simeq \mu_\alpha(B)^{p_B} \simeq \mu_\alpha(B)^{\langle p\rangle_B} \simeq \mu_\alpha(B)^{p_+(B)}.$$

Lemma 5.8. *[17, Theorem 4.5.7]*
Let $p(\cdot) \in \mathscr{P}^{log}(\mathbb{B})$ be such that $p_- > 0$. Let B be a pseudo-ball of $\mathbb{B}$. Then

$$\|\chi_B\|_{p(\cdot)} \simeq \mu_\alpha(B)^{\frac{1}{p_B}}.$$

Proof. From Lemma 5.6 we have

$$\min\left(\mu_\alpha(B)^{\frac{1}{p_-(B)}}, \mu_\alpha(B)^{\frac{1}{p_+(B)}}\right) \leqslant \|\chi_B\|_{p(\cdot)} \leqslant \max\left(\mu_\alpha(B)^{\frac{1}{p_-(B)}}, \mu_\alpha(B)^{\frac{1}{p_+(B)}}\right).$$

Next, from the Remark 5.4, we have $\mu_\alpha(B)^{\frac{1}{p_-(B)}} \simeq \mu_\alpha(B)^{\frac{1}{p_+(B)}} \simeq \mu_\alpha(B)^{\frac{1}{p_B}}$ because $p(\cdot)$ is bounded away from zero. The conclusion follows. □

Lemma 5.9. *Let $p(\cdot) \in \mathscr{P}^{log}_{\pm}(\mathbb{B})$ and let q be a constant exponent greater than $p_+ + 1$. There exists a positive constant C depending only of the log-Hölder constant of $p(\cdot)$ such that*

$$[w]_{B_q} \leqslant C[w]_{B^+_{p(\cdot)}}.$$

Proof. As $p(\cdot) < p_+ + 1 < q$, we have $\frac{q'}{q} < \frac{p'(\cdot)}{p(\cdot)}$. Hence from the Hölder inequality (assertion 2 of Proposition 5.3), we obtain

$$\|w^{-1}\chi_B\|_{\frac{q'}{q}} \leqslant K\|\chi_B\|_{\beta(\cdot)}\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}},$$

where

$$\frac{1}{\beta(\cdot)} = \frac{q}{q'} - \frac{p(\cdot)}{p'(\cdot)} = q - p(\cdot) > 1.$$

It is easy to check that $\beta(\cdot)$ is a member of $\mathscr{P}^{log}(\mathbb{B})$ such that $\beta_- > 0$. Consequently, from Lemma 5.8 and Remark 5.4, we have

$$\|\chi_B\|_{\beta(\cdot)} \simeq \mu_\alpha(B)^{\frac{1}{\beta_B}} \simeq \mu_\alpha(B)^{q-\langle p\rangle_B} \simeq \mu_\alpha(B)^{q-p_B}.$$

Thus there exists a positive constant C such that

$$\begin{aligned}\tfrac{1}{\mu_\alpha(B)^q}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{q'}{q}} &\leqslant C\tfrac{1}{\mu_\alpha(B)^q}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}\mu_\alpha(B)^{q-p_B}\\ &= C\tfrac{1}{\mu_\alpha(B)^{p_B}}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \leqslant C[w]_{B^+_{p(\cdot)}}.\end{aligned}$$

for all pseudo-balls $B \in \mathscr{B}$. The conclusion follows.

□

We recall the following definition.

Definition 5.8. *The weight class B_∞ is defined by $B_\infty = \bigcup_{q\in(1,\infty)} B_q$.*

Remark 5.5. *It follows from Lemma 5.9 that if $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$, we have*

$$B^+_{p(\cdot)} \subset B_\infty.$$

In the rest of this article, to simplify the notation, we denote $w(B) = \|w\chi_B\|_1$.

We next define another class of weights Λ, which contains the class B_∞. For a reference, cf. e.g. [1].

Definition 5.9. *We call Λ the class consisting of those integrable weights w satisfying the following property. There exist two positive constants C and δ such that the following implication holds.*

$$\frac{\mu_\alpha(E)}{\mu_\alpha(B)} \leqslant C\left(\frac{w(E)}{w(B)}\right)^\delta \tag{5.10}$$

whenever $B \in \mathscr{B}$ and E is a measurable subset of B.

Remark 5.6. 1. *For $w \in \Lambda$, the weighted measure $w\,d\mu_\alpha$ is doubling in the following sense. There exists a positive constant C such that for every pseudo-ball B of $\mathbb{B}$ whose pseudo-ball $\widetilde{B}$ of same centre and of double radius is a member of $\mathscr{B}$, we have*

$$w\left(\widetilde{B}\right) \leqslant C\, w(B).$$

This result easily follows from the definition of Λ.

2. *We recall that $w(\mathbb{B}) > 0$. For $w \in \Lambda$, this implies that $w(B) > 0$ for every pseudo-ball B of $\mathbb{B}$. Indeed, take $\mathbb{B}$ for B and B for E in (5.10).*

Lemma 5.10. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{B})$ be such that $p_- > 0$. Let $w \in \Lambda$. Then*

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p(x)}} \simeq w(B)^{\frac{1}{p_B}}$$

for all pseudo-balls B of $\mathbb{B}$ such that $w(B) > 0$ and for all $x \in B$.

Proof. Take $\mathbb{B}$ for B and B for E in Definition 5.9. We have

$$\left(C^{-1}\frac{\mu_\alpha(B)}{\mu_\alpha(\mathbb{B})}\right)^{\frac{1}{\delta}} w(\mathbb{B}) \leqslant w(B) \leqslant w(\mathbb{B}).$$

So

$$w(\mathbb{B})^{p_-(B)-p_+(B)} \leqslant w(B)^{p_-(B)-p_+(B)} \lesssim \mu_\alpha(B)^{\frac{1}{\delta}(p_-(B)-p_+(B))}\, w(\mathbb{B})^{p_-(B)-p_+(B)}.$$

It is easy to check that

$$\min(1, w(\mathbb{B})^{p_- - p_+}) \leqslant w(\mathbb{B})^{p_-(B)-p_+(B)} \leqslant \max(1, w(\mathbb{B})^{p_- - p_+}).$$

Next, combining with Lemma 5.5 gives

$$\min(1, w(\mathbb{B})^{p_- - p_+}) \leqslant w(B)^{p_-(B)-p_+(B)} \lesssim C_\delta \max(1, w(\mathbb{B})^{p_- - p_+}).$$

We have thus proved the estimates $w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p(x)}} \simeq w(B)^{\frac{1}{p_B}}$ for all $x \in B$.

On the other hand, from Lemma 5.6 we have

$$\min\left(w(B)^{\frac{1}{p_-(B)}}, w(B)^{\frac{1}{p_+(B)}}\right) \leqslant \|\chi_B\|_{p(\cdot),w} \leqslant \max\left(w(B)^{\frac{1}{p_-(B)}}, w(B)^{\frac{1}{p_+(B)}}\right).$$

Hence

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p_+(B)}}.$$

□

We recall again the notation $w'(y) = w(y)^{1-p'(y)}$.

Lemma 5.11. *Let $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$ and $w \in B^+_{p(\cdot)}$. Then*

$$\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \left(\rho_{\frac{p'(\cdot)}{p(\cdot)}}(w^{-1}\chi_B)\right)^{p_B-1} = w'(B)^{p_B-1}. \tag{5.11}$$

Proof. Let $w \in B^+_{p(\cdot)}$ and $B \in \mathscr{B}$. By definition we have

$$\frac{1}{\mu_\alpha(B)^{p_B}} w(B)\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \leqslant [w]_{B^+_{p(\cdot)}}. \tag{5.12}$$

On the other hand, by the Hölder inequality (Proposition 5.3, assertion 1) and Lemma 5.10, we have

$$\mu_\alpha(B) = \int_B w(y)^{\frac{1}{p(y)}} w(y)^{-\frac{1}{p(y)}} d\mu_\alpha(y) \leqslant 2\|w^{\frac{1}{p(\cdot)}}\chi_B\|_{p(\cdot)}\|w^{-\frac{1}{p(\cdot)}}\chi_B\|_{p'(\cdot)} \simeq w(B)^{\frac{1}{p_B}}\|w^{-\frac{1}{p(\cdot)}}\chi_B\|_{p'(\cdot)}.$$

Hence

$$\left\| \frac{w(B)^{\frac{1}{p_B}}}{\mu_\alpha(B)} w^{-\frac{1}{p(\cdot)}}\chi_B \right\|_{p'(\cdot)} \gtrsim 1. \tag{5.13}$$

Consequently, from (5.13), Lemma 5.6, Lemma 5.10 and Corollary 5.1, we have:

$$\begin{aligned} 1 \lesssim \rho_{p'(\cdot)}\left(\frac{w(B)^{\frac{1}{p_B}}}{\mu_\alpha(B)} w^{-\frac{1}{p(\cdot)}}\chi_B\right) &= \int_B \left(\frac{w(B)^{\frac{1}{p_B}}}{\mu_\alpha(B)}\right)^{p'(y)} w(y)^{-\frac{p'(y)}{p(y)}} d\mu_\alpha(y) \\ &\simeq \int_B \left(\frac{w(B)}{\mu_\alpha(B)^{p_B}}\right)^{\frac{p'(y)}{p(y)}} w(y)^{-\frac{p'(y)}{p(y)}} d\mu_\alpha(y) \\ &= \rho_{\frac{p'(\cdot)}{p(\cdot)}}\left(\frac{w(B)}{\mu_\alpha(B)^{p_B}} w^{-1}\chi_B\right). \end{aligned}$$

So by Lemma 5.6, we have

$$\left\| \frac{w(B)}{\mu_\alpha(B)^{p_B}} w^{-1}\chi_B \right\|_{\frac{p'(\cdot)}{p(\cdot)}} \gtrsim 1. \tag{5.14}$$

Thus from (5.12) and (5.14), we have

$$\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \frac{\mu_\alpha(B)^{p_B}}{w(B)}. \tag{5.15}$$

Furthermore, from Remark 5.4 and as $p(\cdot) \in \mathscr{P}_{\pm}$, we have the equivalences

$$\begin{aligned} \mu_\alpha(B)^{p_+(B)} \simeq \mu_\alpha(B)^{p_-(B)} &\iff \mu_\alpha(B)^{\frac{1}{p_-(B)-1}} \simeq \mu_\alpha(B)^{\frac{1}{p_+(B)-1}} \\ &\iff \mu_\alpha(B)^{\frac{p_B}{p_-(B)-1}} \simeq \mu_\alpha(B)^{\frac{p_B}{p_+(B)-1}}, \end{aligned}$$

from which we deduce that

$$\mu_\alpha(B)^{\frac{p_B}{p_+(B)-1}} \simeq \mu_\alpha(B)^{\frac{p_B}{p_-(B)-1}} \simeq \mu_\alpha(B)^{\frac{p_B}{p_B-1}}$$

since $p_-(B) \leqslant p_B \leqslant p_+(B)$. Similarly, by Lemma 5.10, we deduce from the estimate

$$w(B)^{p_B} \simeq w(B)^{p_+(B)} \simeq w(B)^{p_-(B)}$$

that

$$w(B)^{\frac{1}{p_-(B)-1}} \simeq w(B)^{\frac{1}{p_+(B)-1}} \simeq w(B)^{\frac{1}{p_B-1}}.$$

So from (5.15) we have

$$\|w^{-1}\chi_B\|^{\frac{1}{p_+(B)-1}}_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \|w^{-1}\chi_B\|^{\frac{1}{p_-(B)-1}}_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \|w^{-1}\chi_B\|^{\frac{1}{p_B-1}}_{\frac{p'(\cdot)}{p(\cdot)}}.$$

Since $\rho_{\frac{p'(\cdot)}{p(\cdot)}}\left(w^{-1}\chi_B\right) = w'(B)$, combining with Lemma 5.6 where $\frac{p'(\cdot)}{p(\cdot)}$ replaces $p(\cdot)$, we obtain the required result. □

Proposition 5.10. *Let $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$ and $w \in B^+_{p(\cdot)}$. Then $w' \in B^+_{p'(\cdot)}$ and*

$$\left(\frac{1}{\mu_\alpha(B)^{p'_B}}\|w'\chi_B\|_1\|w'^{-1}\chi_B\|_{\frac{p(\cdot)}{p'(\cdot)}}\right)^{p_B-1} \simeq \frac{1}{\mu_\alpha(B)^{p_B}}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1}$$

for all pseudo-balls $B \in \mathscr{B}$.

Proof. We recall that $w \in \Lambda$ by Remark 5.5. Hence, from Lemma 5.10 used with $\frac{p(\cdot)}{p'(\cdot)}$ replacing $p(\cdot)$, equation (5.11) and the property $w \in B^+_{p(\cdot)}$, we obtain

$$\begin{aligned}
\frac{1}{\mu_\alpha(B)^{p'_B}}w'(B)\|w'^{-1}\chi_B\|_{\frac{p(\cdot)}{p'(\cdot)}} &= \frac{1}{\mu_\alpha(B)^{p'_B}}w'(B)\|\chi_B\|_{\frac{p(\cdot)}{p'(\cdot)},w} \\
&\simeq \frac{1}{\mu_\alpha(B)^{p'_B}}w'(B)w(B)^{\frac{1}{p_B-1}} \\
&= \left(\frac{w(B)}{\mu_\alpha(B)^{p_B}}w'(B)^{p_B-1}\right)^{\frac{1}{p_B-1}} \qquad (5.16) \\
&\simeq \left(\frac{w(B)}{\mu_\alpha(B)^{p_B}}\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}\right)^{\frac{1}{p_B-1}} \qquad (5.17) \\
&\leqslant [w]^{\frac{1}{p_B-1}}_{B^+_{p(\cdot)}}.
\end{aligned}$$

Hence $w' \in B^+_{p'(\cdot)}$ and from (5.16) and (5.17), we deduce that

$$\frac{w(B)}{\mu_\alpha(B)^{p_B}}\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1}.$$

□

Definition 5.10. *Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$ and let w be a weight. We say that w is in the $B^{++}_{p(\cdot)}$ class if*

$$[w]_{B^{++}_{p(\cdot)}} := \sup_{B\in\mathscr{B}} \frac{w(B)}{\mu_\alpha(B)} \left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1} < \infty. \tag{5.18}$$

It is easy to check the following Proposition.

Proposition 5.11. *Let $p(\cdot) \in \mathscr{P}_\pm(\mathbb{B})$. The following two assertions are equivalent.*

1. $w \in B^{++}_{p(\cdot)}$;
2. $w' \in B^{++}_{p'(\cdot)}$.

Lemma 5.12. *Let $p(\cdot) \in \mathscr{P}^{\log}_\pm(\mathbb{B})$, and $w \in B^{++}_{p(\cdot)}$. Then for all pseudo-balls B of $\mathbb{B}$,*

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p_B}}.$$

Proof. Since $w \in B^{++}_{p(\cdot)}$, we have $w'(\mathbb{B}) < \infty$ and it follows from Lemma 5.6 that $\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'} < \infty$. Hence, by the Hölder inequality, we obtain

$$\begin{aligned}\mu_\alpha(B) &\leqslant 2\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'} \\ &\leqslant 2\|\chi_B\|_{p(\cdot),w}\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}.\end{aligned}$$

Therefore, from Lemma 5.5, we get

$$\begin{aligned}\|\chi_B\|^{p_-(B)-p_+(B)}_{p(\cdot),w} &\lesssim \mu_\alpha(B)^{p_-(B)-p_+(B)}\|\chi_{\mathbb{B}}\|^{p_+(B)-p_-(B)}_{p'(\cdot),w'} \\ &\lesssim \max\left(1, \|\chi_{\mathbb{B}}\|^{p_+-p_-}_{p'(\cdot),w'}\right).\end{aligned} \tag{5.19}$$

On the other hand, using again $w \in B^{++}_{p(\cdot)}$, we have $w(\mathbb{B}) < \infty$ and hence $\|\chi_{\mathbb{B}}\|_{p(\cdot),w} < \infty$. Then

$$\|\chi_B\|^{p_+(B)-p_-(B)}_{p(\cdot),w} \leqslant \left\|\chi_{\mathbb{B}}\right\|^{p_+(B)-p_-(B)}_{p(\cdot),w} \leqslant \max\left(1, \|\chi_{\mathbb{B}}\|^{p_+-p_-}_{p(\cdot),w}\right). \tag{5.20}$$

Thus from (5.19) and (5.20), we have

$$\|\chi_B\|^{p_+(B)}_{p(\cdot),w} \simeq \|\chi_B\|^{p_-(B)}_{p(\cdot),w},$$

and from Lemma 5.6 we conclude the proof. □

Lemma 5.13. *Let $p(\cdot) \in \mathscr{P}^{\log}_\pm(\mathbb{B})$. Then $B^{++}_{p(\cdot)} \subset \Lambda$.*

Proof. Let $w \in B^{++}_{p(\cdot)}$. Let $B \in \mathscr{B}$ and let E be a measurable subset of B. By the Hölder inequality and from Lemma 5.6, we have

$$\begin{aligned}\mu_\alpha(E) &\leqslant 2\|\chi_E\|_{p(\cdot),w}\|\chi_E\|_{p'(\cdot),w'} \\ &\leqslant 2\|\chi_E\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'} \\ &\leqslant 2\max\left(w(E)^{\frac{1}{p_+(B)}}, w(E)^{\frac{1}{p_-(B)}}\right)\|\chi_B\|_{p'(\cdot),w'}.\end{aligned}$$

However, since $w' \in B^{++}_{p'(\cdot)}$ by Proposition 5.11, from Lemma 5.12 we have

$$\|\chi_B\|_{p'(\cdot),w'} \simeq w'(B)^{\frac{1}{p'_+(B)}} \simeq w'(B)^{\frac{1}{p'_-(B)}} \simeq w'(B)^{\frac{1}{p'_B}}.$$

Hence using $w \in B^{++}_{p(\cdot)}$, we deduce that,

$$\begin{aligned}
\mu_\alpha(E) &\lesssim \max\left(w(E)^{\frac{1}{p_+(B)}} w'(B)^{\frac{1}{p'_+(B)}}, w(E)^{\frac{1}{p_-(B)}} w'(B)^{\frac{1}{p'_-(B)}}\right) \\
&\leqslant \max\left([w]_{B^{++}_{p(\cdot)}}^{\frac{1}{p_+(B)}}, [w]_{B^{++}_{p(\cdot)}}^{\frac{1}{p_-(B)}}\right) \max\left(\left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+(B)}}, \left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_-(B)}}\right) \mu_\alpha(B) \\
&\leqslant \max\left([w]_{B^{++}_{p(\cdot)}}^{\frac{1}{p_+}}, [w]_{B^{++}_{p(\cdot)}}^{\frac{1}{p_-}}\right) \left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+(B)}} \mu_\alpha(B) \\
&\leqslant \max\left([w]_{B^{++}_{p(\cdot)}}^{\frac{1}{p_+}}, [w]_{B^{++}_{p(\cdot)}}^{\frac{1}{p_-}}\right) \left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+}} \mu_\alpha(B).
\end{aligned}$$

Therefore,

$$\frac{\mu_\alpha(E)}{\mu_\alpha(B)} \lesssim \left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+}}.$$

□

Proposition 5.12. *Let $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$. Then $B^+_{p(\cdot)} = B^{++}_{p(\cdot)}$.*

Proof. For the inclusion $B^+_{p(\cdot)} \subset B^{++}_{p(\cdot)}$, apply Proposition 5.10. For the reverse inclusion, apply Lemma 5.13 and Lemma 5.10. □

Remark 5.7. *From Proposition 5.10, Remark 5.5 and Lemma 5.10, we have the inclusion $B^+_{p(\cdot)} \subset B_{p(\cdot)}$ for $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$.*

Now we prove the reverse inclusion. In this direction, we first state the following result.

Lemma 5.14. *Let $p(\cdot) \in \mathscr{P}^{\log}_{\pm}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. Then there exists a constant $C > 1$*

$$\frac{1}{C} \leq \|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} \leqslant C$$

for all $B \in \mathscr{B}$.

Proof. By the Hölder inequality, we have

$$\mu_\alpha(B) \leqslant 2\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'}$$

and as $w \in B_{p(\cdot)}$, from Lemma 5.5 and according to the estimate $\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'} < \infty$ given

by Remark 5.3, we have

$$\begin{aligned}\|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} &\lesssim \mu_\alpha(B)^{p_-(B)-p_+(B)}\|\chi_B\|_{p'(\cdot),w'}^{p_+(B)-p_-(B)}\\ &\simeq \|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+(B)-p_-(B)}\\ &\lesssim \max\left(1,\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+-p_-}\right).\end{aligned}$$

On the other hand, according to the estimate $\|\chi_{\mathbb{B}}\|_{p(\cdot),w}<\infty$ given by Remark 5.3, we have

$$\|\chi_B\|_{p(\cdot),w}^{p_+(B)-p_-(B)} \leqslant \max(1,\|\chi_{\mathbb{B}}\|_{p(\cdot),w}^{p_+-p_-}).$$

□

From Lemma 5.6 and Lemma 5.14, we deduce the following corollary.

Corollary 5.3. *Let $p(\cdot)\in\mathscr{P}^{\log}_{\pm}(\mathbb{B})$, and $w\in B_{p(\cdot)}$. Then*

$$\|\chi_B\|_{p(\cdot),w}\simeq w(B)^{\frac{1}{p_+(B)}}\simeq w(B)^{\frac{1}{p_-(B)}}\simeq w(B)^{\frac{1}{p_B}}.$$

for all $B\in\mathscr{B}$,

We next state the following theorem.

Theorem 5.4. *Let $p(\cdot)\in\mathscr{P}^{log}_{\pm}(\mathbb{B})$. Then $B_{p(\cdot)}=B^+_{p(\cdot)}=B^{++}_{p(\cdot)}$.*

Proof. From Remark 5.7, we have $B^+_{p(\cdot)}\subset B_{p(\cdot)}$. Let $w\in B_{p(\cdot)}$. By Proposition 5.5, $w'\in B_{p'(\cdot)}$. It follows from Corollary 5.3 that

$$\frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1}\simeq\left(\frac{1}{\mu_\alpha(B)}\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'}\right)^{p_B}\leqslant\max\left(1,[w]^{p_+}_{B_{p(\cdot)}}\right).$$

□

To end this section, we record with the same proof the following analogous theorem for the variable Muckenhoupt weight classes.

Theorem 5.5. *Let w be a weight and let $p(\cdot)\in\mathscr{P}^{\log}_{\pm}(\mathbb{B})$. The following three assertions are equivalent.*

1. *$w\in A_{p(\cdot)}$;*
2. *$\sup\limits_B \frac{1}{\mu_\alpha(B)^{p_B}}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}<\infty$, where the* sup *is taken over all pseudo-balls of $\mathbb{B}$;*
3. *$\sup\limits_B \frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1}<\infty$, where the* sup *is taken over all pseudo-balls of $\mathbb{B}$.*

5.6. Proof of the necessary condition in Theorem 5.2

The aim of this section is to prove the following result.

Proposition 5.13. *Let w be a weight and let $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$. If the Bergman projector is well defined and bounded on $L^{p(\cdot)}(w d\mu_\alpha)$, then $w \in B_{p(\cdot)}$.*

Proof. According to Theorem 5.4, it suffices to prove that $w \in B_{p(\cdot)}^{++}$, i.e. the following estimate holds

$$\sup_{B \in \mathscr{B}} \frac{w(B)}{\mu_\alpha(B)} \left(\frac{w'(B)}{\mu_\alpha(B)} \right)^{p_B - 1} < \infty. \tag{5.21}$$

From Proposition 5.8, we have that $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$ and from Proposition 5.9 we have $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$. In particular, $w(\mathbb{B}) < \infty$ and $w'(\mathbb{B}) < \infty$. Thus we just have to show the estimate (5.21) for the pseudo-balls of radius smaller than a positive constant R_0, because if the radius of B is larger, then B can be identified with $\mathbb{B}$. We shall use the following lemma.

Lemma 5.15. *[2] There exist three positive numbers R_0, c and C_α such that the following holds. For every pseudo-ball $B^1 \in \mathscr{B}$ of radius $R < R_0$, there exists a pseudo-ball $B^2 \in \mathscr{B}$ of same radius such that $d(B^1, B^2) = cR$, that satisfies the following property: for every non-negative measurable function f supported in B^i and for two distinct superscripts $i, j \in \{1,2\}$, we have*

$$|P_\alpha f| \geqslant C_\alpha \chi_{B^j} \mu_\alpha(B^i)^{-1} \int_{B^i} f \, d\mu_\alpha. \tag{5.22}$$

Thus, by taking $f = \chi_{B^i}$ in (5.22) we obtain :

$$|P_\alpha \chi_{B^i}(z)| \geqslant \chi_{B^j}(z) C_\alpha \mu_\alpha(B^i)^{-1} \int_{B^i} \chi_{B^i} \, d\mu_\alpha \simeq \chi_{B^j}(z).$$

Using the growth of the norm $\|\cdot\|_{p(\cdot),w}$, we obtain

$$\|P_\alpha \chi_{B^i}\|_{p(\cdot),w} = \|w^{\frac{1}{p(\cdot)}} P_\alpha \chi_{B^i}\|_{p(\cdot)} \gtrsim \|\chi_{B^j} w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}$$

So using the fact that P_α is bounded on $L^{p(\cdot)}(w d\mu_\alpha)$, we obtain:

$$\|\chi_{B^j} w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \lesssim \|P_\alpha\| \|\chi_{B^i}\|_{p(\cdot),w}$$

We then deduce that

$$\|\chi_{B^1} w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \simeq \|\chi_{B^2} w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}. \tag{5.23}$$

In the rest of the proof, we shall take $f = w' \chi_{B^1}$. We have $f \in L^{p(\cdot)}(w d\mu_\alpha)$ since

$$\rho_{p(\cdot),w}(f) = \int_{\mathbb{B}} w(z)^{-p'(z)} \chi_{B^1}(z) w(z) d\mu_\alpha(z) = \rho_{p'(\cdot)}(w^{-\frac{1}{p(\cdot)}} \chi_{B^1}) < \infty$$

by Proposition 5.8. However, $\rho_{p(\cdot),w}(f) = \int_{B^1} w' d\mu_\alpha = w'(B^1)$. Also, from (5.22) and the previous equality, we have

$$\chi_{B^2}(z)\, w'(B^1) \leqslant C_\alpha^{-1} \mu_\alpha(B^1) |P_\alpha f(z)|.$$

Moving to the norm $\|\cdot\|_{p(\cdot),w}$, we obtain

$$\| w^{\frac{1}{p(\cdot)}} \chi_{B^2} \|_{p(\cdot)}\, w'(B^1) \leqslant C_\alpha^{-1} \mu_\alpha(B^1) \| P_\alpha f \|_{p(\cdot),w}$$

Then using the boundedness of P_α on $L^{p(\cdot)}(w d\mu_\alpha)$, the previous inequality implies

$$\| w^{\frac{1}{p(\cdot)}} \chi_{B^2} \|_{p(\cdot)}\, w'(B^1) \leqslant C_\alpha^{-1} \mu_\alpha(B^1) \| P_\alpha \| \| f \|_{p(\cdot),w}$$

and combining with (5.23), we obtain the following lemma.

Lemma 5.16 (Main Lemma)**.** *Suppose that P_α is bounded on $L^{p(\cdot)}(w d\mu_\alpha)$. Then*

$$\| w^{\frac{1}{p(\cdot)}} \chi_B \|_{p(\cdot)}\, w'(B) \leqslant C C_\alpha^{-1} \mu_\alpha(B) \| P_\alpha \| \| w' \chi_B \|_{p(\cdot),w} \tag{5.24}$$

for every pseudo-ball $B \in \mathscr{B}$ of radius smaller than R_0. The absolute constants R_0, C and C_α were respectively defined in Lemma 5.15, (5.23) and (5.22).

At this level, we need to calculate $\| w^{\frac{1}{p(\cdot)}} \chi_B \|_{p(\cdot)}$ and $\| w' \chi_B \|_{p(\cdot),w}$. This calculation is not as obvious as in the case where $p(\cdot)$ is constant.

Lemma 5.17. *Let $p(\cdot) \in \mathscr{P}(\mathbb{B})$. If P_α is bounded on $L^{p(\cdot)}(w)$, then P_α is bounded on $L^{p'(\cdot)}(w')$.*

Proof. We first recall that the weighted Bergman projector P_α is the orthogonal projector from the (Hilbert-) Lebesgue space $L^2(d\mu_\alpha)$ to its closed subspace $L^2(d\mu_\alpha) \cap Hol(\mathbb{B})$ (the standard weighted Bergman space). We call $\mathscr{C}_c(\mathbb{B})$ the space of continuous functions with compact support in $\mathbb{B}$. By Proposition 5.2, $\mathscr{C}_c(\mathbb{B})$ is a dense subspace of $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w')$. From Proposition 5.4 and the boundedness of P_α on $L^{p(\cdot)}(w)$, for all $f \in \mathscr{C}_c(\mathbb{B})$, we have

$$\begin{aligned}
\| P_\alpha f \|_{p'(\cdot),w'} &= \sup_{g \in \mathscr{C}_c(\mathbb{B}) : \| g \|_{p(\cdot),w} = 1} \left| \int_{\mathbb{B}} P_\alpha f(\zeta) \overline{g(\zeta)} d\mu_\alpha(\zeta) \right| \\
&= \sup_{g \in \mathscr{C}_c(\mathbb{B}) : \| g \|_{p(\cdot),w} = 1} \left| \int_{\mathbb{B}} f(\zeta) \overline{P_\alpha g(\zeta)} d\mu_\alpha(\zeta) \right| \\
&\leqslant 2 \sup_{g \in \mathscr{C}_c(\mathbb{B}) : \| g \|_{p(\cdot),w} = 1} \| f \|_{p'(\cdot),w'} \| P_\alpha g \|_{p(\cdot),w} \\
&\leqslant 2 \| P_\alpha \| \, \| f \|_{p'(\cdot),w'}.
\end{aligned}$$

We have used the elementary fact that $\mathscr{C}_c(\mathbb{B})$ is contained in $L^2(d\mu_\alpha)$. For the last but one inequality, we used the Hölder inequality. □

Lemma 5.18. *Let $p(\cdot) \in \mathscr{P}(\mathbb{B})$ and let w be a weight. If P_α is bounded on $L^{p(\cdot)}(w)$, then for all $t > 0$,*

$$\|t\chi_{\{|P_\alpha f|>t\}}\|_{p(\cdot),w} \leqslant \|P_\alpha\|\,\|f\|_{p(\cdot),w}.$$

Proof. It suffices to remark that for all $t > 0$, $t\chi_{\{|P_\alpha f|>t\}} \leqslant |P_\alpha f|$. □

Lemma 5.19. *Let $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$ and w be a weight. If P_α is bounded on $L^{p(\cdot)}(w)$, then*

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}}$$

for all pseudo balls B of $\mathbb{B}$.

Proof. If $\|\chi_B\|_{p(\cdot),w} \geqslant 1$, then $\|\chi_{\mathbb{B}}\|_{p(\cdot),w}^{p_- - p_+} \leqslant \|\chi_B\|_{p(\cdot),w}^{p_- - p_+} \leqslant \|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} \leqslant 1$. So

$$\|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} \simeq 1. \tag{5.25}$$

Otherwise, if $\|\chi_B\|_{p(\cdot),w} < 1$, then by the Hölder inequality, we have

$$\begin{aligned}\mu_\alpha(B) &\leqslant 2\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'} \\ &\leqslant 2\|\chi_B\|_{p(\cdot),w}\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}\end{aligned} \tag{5.26}$$

Hence from (5.26) and Lemma 5.5, we have

$$\begin{aligned}\|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} &\leqslant 2^{p_+-p_-}\mu_\alpha(B)^{p_-(B)-p_+(B)}\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+(B)-p_-(B)} \\ &\lesssim \max\left(1, \|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+-p_-}\right).\end{aligned} \tag{5.27}$$

We point out that $\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'} < \infty$ according to Proposition 5.8, since P_α is well defined on $L^{p(\cdot)}(w)$. On the other hand, using the boundedness of P_α on $L^{p(\cdot)}(w)$, we have the estimate $\|\chi_{\mathbb{B}}\|_{p(\cdot),w} < \infty$ according to Proposition 5.9. Then

$$\|\chi_B\|_{p(\cdot),w}^{p_+(B)-p_-(B)} \lesssim \max\left(1, \|\chi_{\mathbb{B}}\|_{p(\cdot),w}^{p_+-p_-}\right) < \infty. \tag{5.28}$$

Thus, from (5.25), (5.27) and (5.28), we deduce that

$$\|\chi_B\|_{p(\cdot),w}^{p_+(B)} \simeq \|\chi_B\|_{p(\cdot),w}^{p_-(B)} \tag{5.29}$$

for all pseudo-balls of $\mathbb{B}$. Applying Lemma 5.6 gives

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}}.$$

□

End of the proof of Proposition 5.13. We go back to the Main Lemma (Lemma 5.16). On the one hand, since P_α is bounded on $L^{p(\cdot)}(w)$, it follows from Lemma 5.17 that P_α

is also bounded on $L^{p'(\cdot)}(w')$. So from Lemma 5.19 with $p'(\cdot)$ in the place of $p(\cdot)$ and w' in the place of w, we have

$$\left\|\chi_B\right\|_{p'(\cdot),w'} \simeq w'(B)^{1-\frac{1}{p_-(B)}} \simeq w'(B)^{1-\frac{1}{p_+(B)}}.$$

This implies the estimate $w'(B)^{\frac{1}{p_-(B)}} \simeq w'(B)^{\frac{1}{p_+(B)}}$. It then follows from Lemma 5.16 that

$$\|w'\chi_B\|_{p(\cdot),w} \simeq w'(B)^{\frac{1}{p_B}}.$$

On the other hand, $\|w^{\frac{1}{p(\cdot)}}\chi_B\|_{p(\cdot)} = \|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_B}}$ by Lemma 5.19. The inequality (5.24) of the Main Lemma takes the following form

$$w(B)^{\frac{1}{p_B}} w'(B) \lesssim \mu_\alpha(B) w'(B)^{\frac{1}{p_B}}.$$

Equivalently,

$$\sup_{B\in\mathscr{B}} \frac{w(B)}{\mu_\alpha(B)} \left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1} < \infty.$$

We have shown the estimate (5.21). This finishes the proof of Proposition 5.13. □

5.7. Boundedness on $L^{p(\cdot)}(w)$ of the maximal function m_α

In this section, we prove the boundedness of the maximal function m_α on $L^{p(\cdot)}(w)$ when $w \in B_{p(\cdot)}$. As in [2], we will use the regularisation operator that we recall here with some of its properties.

Definition 5.11. *For all $k \in (0,1)$, we define the regularisation operator R_k^α of order k by*

$$R_k^\alpha f(z) = \frac{1}{\mu_\alpha(B^k(z))} \int_{B^k(z)} f(w) d\mu_\alpha(w),$$

where $B^k(z) = \{\zeta \in \mathbb{B} : d(z,\zeta) < k(1-|z|)\}$.

Proposition 5.14. *For all $k \in (0,1)$, there exists a constant $C_k > 1$ such that for every non-negative locally integrable function f, the following two estimates hold.*

1) $m_\alpha f \leqslant C_k m_\alpha R_k^\alpha f$;

2) $C_k^{-1} m_\alpha g \leqslant R_k^\alpha m_\alpha g \leqslant C_k m_\alpha g$.

Lemma 5.20. *Let $k \in (0, \frac{1}{2})$. If $z' \in B^k(z)$ then $z \in B^{k'}(z')$ where $k' = \frac{k}{1-k}$, and $\chi_{B^k(z)}(z') \leqslant \chi_{B^{k'}(z')}(z)$. Moreover there exists a constant $C_k > 1$ such that*

$$C_k^{-1} \mu_\alpha(B^k(z)) \leqslant \mu_\alpha(B^{k'}(z')) \leqslant C_k \mu_\alpha(B^k(z)).$$

Lemma 5.21. *Let $k \in \left(0, \frac{1}{5}\right)$ and $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$. For $w \in B_{p(\cdot)}$ there exists a constant $C_k > 1$ such that for all $z, z' \in \mathbb{B}$ such that $z' \in B^k(z)$ we have:*

$$C_k^{-1} w(B^k(z)) \leqslant w(B^{k'}(z')) \leqslant C_k w(B^k(z))$$

Proof. We have $B^k(z) \subset B^{2k'}(z')$ and $B^{k'}(z') \subset B^{6k}(z)$. From Theorem 5.4 and Remark 5.5, we have $w \in \Lambda$. Apply Remark 5.6 to conclude. □

Lemma 5.22. *Let $k \in (0, \frac{1}{2})$. There exists a positive constant C_k such that for all non-negative locally integrable f, g, we have*

$$\int_{\mathbb{B}} f(\zeta) R_k g(\zeta) d\mu_\alpha(\zeta) \leqslant C_k \int_{\mathbb{B}} g(z) R_k f(z) d\mu_\alpha(z).$$

We also recall the following elementary lemma.

Lemma 5.23. *Let $z_0 \in \mathbb{B}$ and $r > 1 - |z_0|$. For $z \in B(z_0, r)$ and $\zeta \in B^k(z)$, we have $\zeta \in B(z_0, ar)$ with $a = 2(2k+1)r$.*

In the rest of this section, to simplify the notation, we write $\sigma = R_k^\alpha w$. The following result is a generalisation to the variable exponent of the analogous result in [2, Lemma 10].

Proposition 5.15. *Let $p(\cdot) \in \mathscr{P}_{\pm}^{log}(\mathbb{B})$, $k \in (0, \frac{1}{2})$ and $w \in B_{p(\cdot)}$. Then $R_k^\alpha w \in A_{p(\cdot)}$ with $[R_k^\alpha w]_{A_{p(\cdot)}} \lesssim [w]_{B_{p(\cdot)}}$.*

Proof. From Theorem 5.5, it suffices to show that

$$\frac{\sigma(B)}{\mu_\alpha(B)} \left(\frac{\sigma'(B)}{\mu_\alpha(B)} \right)^{p(z_0)-1} \lesssim [w]_{B_{p(\cdot)}}$$

for every pseudo-ball B of $\mathbb{B}$.

We write $a = 2k+1$. Let $B = B(z_0, r)$ be a pseudo-ball in $\mathbb{B}$. We set $B' = B(z_0, ar)$. We distinguish two cases: 1. $B \in \mathscr{B}$; 2. B is not a member of $\mathscr{B}$.

1. Suppose first that $B \in \mathscr{B}$. We claim that there exists a positive absolute constant C_k such that
$$\frac{\sigma(B)}{\mu_\alpha(B)} \leqslant C_k \frac{w(B')}{\mu_\alpha(B')}. \tag{5.30}$$

Indeed, from the Fubini-Tonelli theorem and Lemma 5.20 we have

$$\begin{aligned}
\sigma(B) &= \int_B \sigma(z)d\mu_\alpha(z) \\
&= \int_B \left(\frac{1}{\mu_\alpha(B^k(z))} \int_{B^k(z)} w(\zeta)d\mu_\alpha(\zeta) \right) d\mu_\alpha(z) \\
&= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_{B^k(z)}(\zeta)\chi_B(z)}{\mu_\alpha(B^k(z))} d\mu_\alpha(z) \right) w(\zeta)d\mu_\alpha(\zeta) \\
&\lesssim \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_{B^{k'}(\zeta)}(z)\chi_{B'}(\zeta)}{\mu_\alpha(B^{k'}(\zeta))} d\mu_\alpha(z) \right) w(\zeta)d\mu_\alpha(\zeta) \\
&= w(B').
\end{aligned}$$

For the latter inequality, we used Lemma 5.23. Moreover, since $B \subset B'$ and $\mu_\alpha(B) \simeq \mu_\alpha(B')$, we obtain that

$$\frac{\sigma(B)}{\mu_\alpha(B)} \lesssim \frac{\sigma(B')}{\mu_\alpha(B')}.$$

Furthermore, from the Hölder inequality and Lemma 5.10 we have

$$\begin{aligned}
\sigma^{-1}(z) &= \frac{\mu_\alpha(B^k(z))}{w(B^k(z))} \\
&\leqslant \frac{2}{w(B^k(z))} \| w^{\frac{1}{p(\cdot)}} \chi_{B^k(z)} \|_{p(\cdot)} \| w^{-\frac{1}{p(\cdot)}} \chi_{B^k(z)} \|_{p'(\cdot)} \\
&\simeq \frac{1}{w(B^k(z))} w(B^k(z))^{\frac{1}{p(z)}} w'(B^k(z))^{\frac{1}{p'(z)}} \\
&= \left(\frac{w'(B^k(z))}{w(B^k(z))} \right)^{\frac{1}{p'(z)}}.
\end{aligned}$$

Hence

$$\sigma'(z) = \left(\sigma^{-1}\right)^{p'(z)-1}(z) \lesssim \left(\frac{w'(B^k(z))}{w(B^k(z))} \right)^{\frac{1}{p(z)}}.$$

From the Hölder inequality and Lemma 5.10, we have

$$\begin{aligned}
\sigma'(B) &= \int_B \sigma'(z)d\mu_\alpha(z) \\
&\lesssim \int_B \left(\frac{w'(B^k(z))}{w(B^k(z))}w(z)\right)^{\frac{1}{p(z)}} w(z)^{-\frac{1}{p(z)}} d\mu_\alpha(z) \\
&\leqslant 2\| w^{-\frac{1}{p(\cdot)}}\chi_B\|_{p'(\cdot)} \left\| \left(\frac{w'(B^k(.))}{w(B^k(.))}w(.)\right)^{\frac{1}{p(.)}} \chi_B\right\|_{p(\cdot)} \\
&\lesssim w'(B)^{\frac{1}{p'(z_0)}} \left\| \left(\frac{w'(B^k(.))}{w(B^k(.))}w(.)\right)^{\frac{1}{p(.)}} \chi_B\right\|_{p(\cdot)} \qquad (5.31)
\end{aligned}$$

Since $w' \in \Lambda$, from Lemma 5.10 we have $\beta := \| w'^{\frac{1}{p(\cdot)}}\chi_{B'}\|_{p(\cdot)} \simeq w'(B')^{\frac{1}{p(z_0)}}$. Thus as $B \subset B'$, from Lemma 5.20 and Lemma 5.21 we have

$$\begin{aligned}
&\rho_{p(.)}\left(\frac{1}{\beta}\left(\frac{w'(B^k(.))}{w(B^k(.))}w\right)^{\frac{1}{p(.)}}\chi_B\right) \\
&= \int_{\mathbb{B}} \frac{1}{\beta^{p(z)}}\frac{w'(B^k(z))}{w(B^k(z))}w(z)\chi_B(z)d\mu_\alpha(z) \\
&\simeq \int_{\mathbb{B}} \frac{1}{w'(B')}\frac{w'(B^k(z))}{w(B^k(z))}w(z)\chi_B(z)d\mu_\alpha(z) \\
&= w'(B')^{-1} \times \int_{\mathbb{B}} \left(\frac{1}{w(B^k(z))}\int_{\mathbb{B}} w'(\zeta)\chi_{B_k(z)}(\zeta)\chi_B(z)w(z)d\mu_\alpha(\zeta)\right)d\mu_\alpha(z) \\
&\leqslant C_k w'(B')^{-1} \times \int_{\mathbb{B}} \left(\frac{1}{w(B_{k'}(\zeta))}\int_{\mathbb{B}} \chi_{B_{k'}(\zeta)}(z)\chi_{B'}(\zeta)w(z)d\mu_\alpha(z)\right)w'(\zeta)d\mu_\alpha(\zeta) \\
&= C_k.
\end{aligned}$$

For the latter inequality, we used Lemma 5.23. Hence we obtain

$$\left\| \left(\frac{w'(B^k(.))}{w(B^k(.))}w\right)^{\frac{1}{p(.)}}\chi_B\right\|_{p(\cdot)} \lesssim C_k w'(B')^{\frac{1}{p(z_0)}}.$$

Consequently, we deduce from (5.31) that

$$\sigma'(B) \lesssim C_k w'(B)^{\frac{1}{p'(z_0)}} w'(B')^{\frac{1}{p(z_0)}} \leqslant C_k w'(B')$$

because $B \subset B'$. Moreover, as $\mu_\alpha(B) \simeq \mu_\alpha(B')$, we have

$$\frac{\sigma'(B)}{\mu_\alpha(B)} \leqslant C_k \frac{w'(B')}{\mu_\alpha(B')}$$

and hence

$$\left(\frac{\sigma'(B)}{\mu_\alpha(B)}\right)^{p(z_0)-1} \leqslant C'_k \left(\frac{w'(B')}{\mu_\alpha(B')}\right)^{p(z_0)-1}. \tag{5.32}$$

Combining (5.30) and (5.32) gives

$$\frac{\sigma(B)}{\mu_\alpha(B)}\left(\frac{\sigma'(B)}{\mu_\alpha(B)}\right)^{p(z_0)-1} \leqslant \gamma_k \frac{w(B')}{\mu_\alpha(B')}\left(\frac{w'(B')}{\mu_\alpha(B')}\right)^{p(z_0)-1} \leqslant \gamma_k [w]_{B_{p(\cdot)}} \tag{5.33}$$

by Theorem 5.4.

2. Suppose next that the pseudo-ball B is not a member of $\mathscr{B}$, i.e. $r \leqslant 1-|z_0|$. In the case where $k(1-|z_0|) \leqslant r \leqslant 1-|z_0|$, we have $B \subset B(z_0, 1-|z_0|)$ and $\mu_\alpha(B) \simeq (1-|z_0|)^{n+\alpha} \simeq \mu_\alpha(B(z_0,1-|z_0|))$. The pseudo-ball $B(z_0,1-|z_0|)$ is a member of $\mathscr{B}$; so we can apply to it the computations of the first case. We obtain:

$$\begin{aligned}\frac{\sigma(B)}{\mu_\alpha(B)}\left(\frac{\sigma'(B)}{\mu_\alpha(B)}\right)^{p(z_0)-1} &\lesssim \frac{\sigma(B(z_0,1-|z_0|))}{\mu_\alpha(B(z_0,1-|z_0|))}\left(\frac{\sigma'(B(z_0,1-|z_0|))}{\mu_\alpha(B(z_0,1-|z_0|))}\right)^{p(z_0)-1} \\ &\lesssim [w]_{B_{p(\cdot)}}.\end{aligned}$$

Next, if $0 < r < k(1-|z_0|)$, then for $z \in B$ we have $(1-k)(1-|z_0|) \leqslant 1-|z| \leqslant (1+k)(1-|z_0|)$. This shows that $\mu_\alpha(B^k(z_0)) \simeq \mu_\alpha(B^k(z))$.
We also claim that $w(B^k(z_0)) \simeq w(B^k(z))$. Indeed, it is easy to show the inclusions $B^k(z_0) \subset B(z, 4k(1-|z|))$ and $B^k(z) \subset B(z_0, 2k(2+k)(1-|z_0|))$. The claim then follows an application of Remark 5.6. Combining with the estimate $\mu_\alpha(B^k(z_0)) \simeq \mu_\alpha(B^k(z))$ gives

$$\sigma(z) \simeq \sigma(z_0) \tag{5.34}$$

for every $z \in B$. Now, by Remark 5.2, $p'(\cdot)$ is a member of $\mathscr{P}_\pm^{log}(\mathbb{B})$. Then by Lemma 5.10 and Corollary 5.1, we have

$$\sigma(z_0)^{1-p'(z_0)} \simeq \sigma(z)^{1-p'(z_0)} \simeq \sigma(z)^{1-p'(z)}.$$

Combining with (5.34) gives

$$\frac{\sigma(B)}{\mu_\alpha(B)}\left(\frac{\sigma'(B)}{\mu_\alpha(B)}\right)^{p(z_0)-1} \simeq 1. \tag{5.35}$$

The conclusion of the lemma follows a combination of (5.33) and (5.35) with Theorem 5.4.

□

Lemma 5.24. *Let $p(\cdot) \in \mathscr{P}_\pm^{\log}(\mathbb{B})$, $k \in (0, \frac{1}{2})$ and $w \in B_{p(\cdot)}$. Then*

$$(R_k^\alpha g(z))^{p(z)} \lesssim R_k^\alpha(g^{p(\cdot)})(z) + 1$$

for all non-negative functions g such that $\|g\|_{p(\cdot),w} = 1$ and all $z \in \mathbb{B}$.

Proof. As $\|g\|_{p(\cdot),w} = 1$, from the Hölder inequality (assertion 1 of Proposition 5.3), we have

$$\frac{1}{2\|\chi_{B^k(z)}\|_{p'(\cdot),w'}} \int_{B^k(z)} g(\zeta) d\mu_\alpha(\zeta) \leqslant \|g\chi_{B^k(z)}\|_{p(\cdot),w} = 1.$$

Therefore from the usual Hölder inequality and Lemma 5.5, we obtain

$$\begin{aligned}
(R_k^\alpha g(z))^{p(z)} &= \left(\frac{1}{2\|\chi_{B^k(z)}\|_{p'(\cdot),w'}} \int_{B^k(z)} g(\zeta) d\mu_\alpha(\zeta)\right)^{p(z)} \mu_\alpha(B^k(z))^{-p(z)} 2^{p(z)} \|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)} \\
&\leqslant 2^{p(z)} \left(\frac{1}{2\|\chi_{B^k(z)}\|_{p'(\cdot),w'}} \int_{B^k(z)} g(\zeta) d\mu_\alpha(\zeta)\right)^{p_-(B)} \mu_\alpha(B^k(z))^{-p(z)} \|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)} \\
&\lesssim \left(\frac{1}{\mu_\alpha(B^k(z))} \int_{B^k(z)} g(\zeta) d\mu_\alpha(\zeta)\right)^{p_-(B)} \mu_\alpha(B^k(z))^{p_-(B)-p(z)} \|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)-p_-(B)} \\
&\leqslant \mu_\alpha(B^k(z))^{p_-(B)-p(z)} \|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)-p_-(B)} \frac{1}{\mu_\alpha(B^k(z))} \int_{B^k(z)} g(\zeta)^{p_-(B)} d\mu_\alpha(\zeta) \\
&\lesssim \frac{1}{\mu_\alpha(B^k(z))} \int_{B^k(z)} g\chi_{g\geqslant 1}(\zeta)^{p_-(B)} d\mu_\alpha(\zeta) + 1 \\
&\lesssim R_k^\alpha g^{p(\cdot)}(z) + 1.
\end{aligned}$$

For the last but one inequality, we also used the following inequality

$$\|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)-p_-(B)} \leqslant \max(1, \|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+-p_-}).$$

□

Lemma 5.25. *Let $p(\cdot) \in \mathscr{P}_\pm^{\log}(\mathbb{B})$, $k \in (0, \frac{1}{2})$ and $w \in B_{p(\cdot)}$. Then*

$$\|R_k^\alpha g \cdot w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \lesssim \|g \cdot (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}$$

for all non-negative functions g belonging to $L^{p(\cdot)}(R_k^\alpha w d\mu_\alpha)$.

Proof. From Proposition 5.15, we have $\sigma = R_k^\alpha w \in A_{p(\cdot)} \subset B_{p(\cdot)}$ because $w \in B_{p(\cdot)}$. Without loss of generality, we assume that $\|g\|_{p(\cdot),\sigma} = 1$. Thus from Lemma 5.24 and Lemma 5.22 we have

$$\begin{aligned}
\rho_{p(\cdot)}(R_k^\alpha g \cdot w^{\frac{1}{p(\cdot)}}) &= \int_{\mathbb{B}} (R_k^\alpha g(z))^{p(z)} w(z) d\mu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}} R_k^\alpha(g^{p(\cdot)})(z) w(z) d\mu_\alpha(z) + w(\mathbb{B}) \\
&\lesssim \int_{\mathbb{B}} g(z)^{p(z)} \sigma(z) d\mu_\alpha(z) + w(\mathbb{B}) \\
&\lesssim 1 + w(\mathbb{B}).
\end{aligned}$$

For the last inequality, apply Lemma 5.6. □

We still use the notation $\sigma = R_k^\alpha w$, with $k \in (0, \frac{1}{2})$.

Lemma 5.26. *Let $k \in (0, \frac{1}{2})$, $p(\cdot) \in \mathscr{P}_\pm^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. Then there exists a positive constant $C([w']_{B_{p'(\cdot)}})$ depending on $[w']_{B_{p'(\cdot)}}$ such that for every non-negative function $g \in L^{p'(\cdot)}(\sigma')$, we have*

$$\|R_k^\alpha g\|_{p'(\cdot),w'} \leqslant C([w']_{B_{p'(\cdot)}})\|g\|_{p'(\cdot),\sigma'}$$

for all non-negative functions $g \in L^{p'(\cdot)}(\sigma')$.

Proof. Without loss of generality, we assume that $\|g\|_{p'(\cdot),\sigma'} = 1$. From Proposition 5.6 and Proposition 5.15, we have $\sigma' \in A_{p'(\cdot)} \subset B_{p'(\cdot)}$ because $w \in B_{p(\cdot)}$. Thus since $\|g\|_{p'(\cdot),\sigma'} = 1$, using Lemma 5.24 and Lemma 5.22 we have

$$\begin{aligned}
\rho_{p'(\cdot),w'}(R_k^\alpha g) &= \int_{\mathbb{B}} (R_k^\alpha g(z))^{p'(z)} w'(z) d\mu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}} R_k^\alpha (g^{p'(\cdot)})(z) w'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&\lesssim \int_{\mathbb{B}} g(z)^{p'(z)} R_k^\alpha w'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&= \int_{\mathbb{B}} g(z)^{p'(z)} \sigma'(z)\sigma'(z)^{-1} R_k^\alpha w'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&\leqslant [w']_{B_{p'(\cdot)}} \int_{\mathbb{B}} g(z)^{p'(z)} \sigma'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&\lesssim [w']_{B_{p'(\cdot)}} + w'(\mathbb{B}).
\end{aligned}$$

Indeed, the last inequality follows from Lemma 5.6; for the last but one inequality, using Theorem 5.4 and Lemma 5.10 for w' in the place of w, we get

$$\sigma'(z)^{-1} R_k^\alpha w'(z) = (R_k^\alpha w(z))^{\frac{p'(z)}{p(z)}} R_k^\alpha w'(z) \lesssim [w']_{B_{p'(\cdot)}}$$

because $B^k(z)$ is 'almost' a member of $\mathscr{B}$, as it is a subset of the member $B(z, 1-|z|)$ of $\mathscr{B}$, $\mu_\alpha(B^k(z)) \simeq \mu_\alpha(B(z, 1-|z|))$ and $w' \in B_{p'(\cdot)}$. So by Lemma 5.6, we get

$$\|R_k^\alpha g\|_{p'(\cdot),w'} \lesssim C([w']_{B_{p'(\cdot)}}).$$

□

Lemma 5.27. *Let $p(\cdot) \in \mathscr{P}_\pm^{\log}(\mathbb{B})$, f non-negative in $L^{p(\cdot)}(w)$, $k \in \left(0, \frac{1}{2}\right)$ and $w \in B_{p(\cdot)}$. Then*

$$\|R_k^\alpha f (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \leqslant C([w']_{B_{p'(\cdot)}})\|f\|_{p(\cdot),w}.$$

Proof. We still write $\sigma = R_k^\alpha w$. By duality (Proposition 5.4), there exists a non-negative function g satisfying $\|g\|_{p'(\cdot),\sigma'} = 1$ and such that

$$\|R_k^\alpha f\|_{p(\cdot),\sigma} \leqslant 2 \int_{\mathbb{B}} g(z) R_k^\alpha f(z) d\mu_\alpha(z).$$

Next, from Lemma 5.22, the Hölder inequality and Lemma 5.26, we obtain

$$\begin{aligned}\|R_k^\alpha f\|_{p(\cdot),\sigma} &\lesssim \int_{\mathbb{B}} f(z)R_k^\alpha g(z)d\mu_\alpha(z)\\ &\leqslant 2\|f\|_{p(\cdot),w}\|R_k^\alpha g\|_{p'(\cdot),w'}\\ &\leqslant 2C([w']_{B_{p'(\cdot)}})\|f\|_{p(\cdot),w}\|g\|_{p'(\cdot),\sigma'}\\ &= 2C([w']_{B_{p'(\cdot)}})\|f\|_{p(\cdot),w}.\end{aligned}$$

Hence we have the result. □

Theorem 5.6. *Let $p(\cdot)\in\mathscr{P}_{\pm}^{\log}(\mathbb{B})$. If $w\in B_{p(\cdot)}$, there exists a non-negative function C defined on $(0,\infty)$ such that for all $f\in L^{p(\cdot)}(w)$ we have*

$$\|m_\alpha f\|_{p(\cdot),w}\leqslant C([w]_{B_{p(\cdot)}})\|f\|_{p(\cdot),w}.$$

Proof. By Proposition 5.5, we have the equality $[w]_{B_{p(\cdot)}}=[w']_{B_{p'(\cdot)}}$. So from Lemma 5.27, we have

$$\|R_k^\alpha f\cdot(R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}\leqslant C([w]_{B_{p(\cdot)}})\|f\|_{p(\cdot),w}. \tag{5.36}$$

Hence $R_k^\alpha f\in L^{p(\cdot)}(\sigma)$. Next, since $\sigma\in A_{p(\cdot)}$ by Proposition 5.15, Corollary 5.2 gives

$$\|m_\alpha(R_k^\alpha f)\cdot(R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}\lesssim\|(R_k^\alpha f)\cdot(R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \tag{5.37}$$

Hence from (5.36) and (5.37), we have $m_\alpha(R_k^\alpha f)\in L^{p(\cdot)}(R_k^\alpha w d\mu_\alpha)$. Now, Lemma 5.25 gives

$$\|R_k^\alpha(m_\alpha(R_k^\alpha f))w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}\lesssim\|m_\alpha(R_k^\alpha f)\cdot(R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}.$$

Next, by Proposition 5.14, there exists a positive constant C such that

$$m_\alpha f\leqslant CR_k^\alpha(m_\alpha(R_k^\alpha f)).$$

This implies that

$$\|m_\alpha f\|_{p(\cdot),w}\lesssim\|m_\alpha(R_k^\alpha f)\cdot(R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}.$$

Finally, applying (5.37) and (5.36) successively, we have the result. □

5.8. A weighted extrapolation Theorem and the proof of the sufficient condition in Theorem 5.2

We are now ready to prove the sufficient condition in Theorem 5.2, we will adapt the strategy used in [16].

5.8.1. Preliminary results

We recall the B_1 class of weights. A weight w belongs to B_1 if

$$[w]_{B_1} := ess\sup_{z\in\mathbb{B}} \frac{m_\alpha w(z)}{w(z)} < \infty. \tag{5.38}$$

In $\mathbb{R}^n$, the analogous of the following factorisation theorem was proved for the Muckenhoupt classes $A_p, 1 < p < \infty$, by Jones [26].

Theorem 5.7. *For a constant exponent p such that $1 < p < \infty$, the following two assertions are equivalent.*

1) $w \in B_p$;

2) there exist $w_1 \in B_1$ and $w_2 \in B_1$ such that $w = w_1 w_2^{1-p}$.

Proof. We first show the implication 2) $\Rightarrow$ 1). Suppose that $w = w_1 w_2^{1-p}$ with $w_1, w_2 \in B_1$. For all $B \in \mathscr{B}$ and $z \in B$, we have

$$\frac{1}{\mu_\alpha(B)} \int_B w_i d\mu_\alpha \leqslant [w_i]_{B_1} w_i(z), \quad i = 1,2. \tag{5.39}$$

Thus as $(1-p')(1-p) = 1$, we have $w^{1-p'} = \left(w_1 w_2^{1-p}\right)^{1-p'} = w_1^{1-p'} w_2$. So from (5.39) we have

$$\begin{aligned}
&\left(\frac{1}{\mu_\alpha(B)} \int_B w d\mu_\alpha\right)\left(\frac{1}{\mu_\alpha(B)} \int_B w^{1-p'} d\mu_\alpha\right)^{p-1} \\
&= \left(\frac{1}{\mu_\alpha(B)} \int_B w_1 w_2^{1-p} d\mu_\alpha\right)\left(\frac{1}{\mu_\alpha(B)} \int_B w_1^{1-p'} w_2 d\mu_\alpha\right)^{p-1} \\
&\leqslant [w_1]_{B_1} [w_2]_{B_1}^{p-1} \left(\frac{1}{\mu_\alpha(B)} \int_B w_2 d\mu_\alpha\right)^{1-p} \left(\frac{1}{\mu_\alpha(B)} \int_B w_1 d\mu_\alpha\right) \\
&\times \left(\frac{1}{\mu_\alpha(B)} \int_B w_2 d\mu_\alpha\right)^{p-1} \left(\frac{1}{\mu_\alpha(B)} \int_B w_1 d\mu_\alpha\right)^{-1} \\
&= [w_1]_{B_1} [w_2]_{B_1}^{p-1}.
\end{aligned}$$

Hence $w \in B_p$.

We next show the converse implication 2) $\Rightarrow$ 1). Suppose that $w \in B_p$. Set $q = pp'$ and define the operator S_1 on the space $\mathscr{M}$ by

$$S_1 f(z) = w(z)^{\frac{1}{q}} \left(m_\alpha\left(f^{p'} w^{-\frac{1}{p}}\right)(z)\right)^{\frac{1}{p'}}.$$

By the Minkowski inequality, S_1 is sublinear. Moreover, from the constant exponent version of Theorem 5.6 [2, Proposition 3], we have

$$\int_{\mathbb{B}} S_1 f(z)^q d\mu_\alpha(z) = \int_{\mathbb{B}} \left(m_\alpha\left(f^{p'} w^{-\frac{1}{p}}\right)(z)\right)^p w(z) d\mu_\alpha(z) \lesssim C\left([w]_{B_p}\right) \int_{\mathbb{B}} f^q(z) d\mu_\alpha(z).$$

In other words, $\|S_1\|_q \lesssim \left(C\left([w]_{B_p}\right)\right)^{\frac{1}{q}}$.
Similarly, denote again $w' = w^{1-p'} \in B_{p'}$ and define the operator S_2 on the space $\mathscr{M}$ by

$$S_2 f(z) = w'(z)^{\frac{1}{q}} \left(m_\alpha\left(f^p w'^{-\frac{1}{p'}}\right)(z)\right)^{\frac{1}{p}}.$$

By the Minkowski inequality, S_2 is also sublinear. Moreover,

$$\int_{\mathbb{B}} S_2 f(z)^q d\mu_\alpha(z) \lesssim C\left([w']_{B_{p'}}\right) \int_{\mathbb{B}} f^q(z) d\mu_\alpha(z).$$

In other words, $\|S_2\|_q \lesssim C\left(\left([w']_{B_{p'}}\right)\right)^{\frac{1}{q}} = \left(C\left([w]_{B_p}\right)\right)^{\frac{1}{q}}$.

We use the following lemma.

Lemma 5.28. *Set $S = S_1 + S_2$ and define the operator $\mathscr{R}$ on $\mathscr{M}$ by*

$$\mathscr{R}h(z) = \sum_{k=0}^{\infty} \frac{S^k h(z)}{2^k \|S\|_q^k}$$

with $S^0 h = |h|$. Then

a) $|h| \leqslant \mathscr{R}h$;

b) $\|\mathscr{R}h\|_q \leqslant 2\|h\|_q$;

c) $S(\mathscr{R}h) \leqslant 2\|S\|_q \mathscr{R}h$.

Proof of Lemma 5.28. By the definition of $\mathscr{R}h$ we have $h \leqslant \mathscr{R}h$. Moreover

$$\|\mathscr{R}h\|_q \leqslant \sum_{k=0}^{\infty} \frac{\|S^k h\|_q}{2^k \|S\|_q^k} \leqslant \|h\|_q \sum_{k=0}^{\infty} \frac{1}{2^k} = 2\|h\|_q.$$

Next, the sublinearity of S gives $S(\mathscr{R}h) \leqslant 2\|S\|_q \mathscr{R}h$. □

Applying assertion c) of Lemma 5.28, we obtain

$$w(z)^{\frac{1}{q}} \left(m_\alpha\left((\mathscr{R}h)^{p'} w^{-\frac{1}{p}}\right)(z)\right)^{\frac{1}{p'}} = S_1(\mathscr{R}h)(z) \leqslant S(\mathscr{R}h)(z) \leqslant 2\|S\|_q \mathscr{R}h(z). \tag{5.40}$$

Now set $w_2 = (\mathscr{R}h)^{p'} w^{-\frac{1}{p}}$. By (5.40), we have $w_2 \in B_1$.

Similarly, we have

$$w'(z)^{\frac{1}{q}} \left(m_\alpha\,(\mathscr{R}h)^p\, w'^{-\frac{1}{p'}}(z)\right)^{\frac{1}{p}} = S_2(\mathscr{R}h)(z) \leqslant S(\mathscr{R}h) \leqslant 2\|S\|_q \mathscr{R}h(z). \tag{5.41}$$

Now set $w_1 = (\mathscr{R}h)^p w'^{-\frac{1}{p'}}$. By (5.41), we have $w_1 \in B_1$. Moreover $w_1 w_2^{1-p} = w \in B_p$.
This finishes the proof of Theorem 5.7. □

Lemma 5.29. *Let $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$ and let $w \in B_{p(\cdot)}$. We define the operator R on $L^{p(\cdot)}(w)$ by*

$$Rh(x) = \sum_{k=0}^{\infty} \frac{m_\alpha^k h(x)}{2^k \|m_\alpha\|_{L^{p(\cdot)}(w)}^k}$$

where for $k \geqslant 1$, $m_\alpha^k = \underbrace{m_\alpha \circ m_\alpha \circ \cdots \circ m_\alpha}_{k\text{-times}}$ and $m_\alpha^0 h = |h|$. Then R satisfies the following properties:

a) $|h| \leqslant Rh$;

b) R is bounded on $L^{p(\cdot)}(w)$ and $\|Rh\|_{p(\cdot),w} \leqslant 2\|h\|_{p(\cdot),w}$;

c) $Rh \in B_1$ and $[Rh]_{B_1} \leqslant 2\|m_\alpha\|_{L^{p(\cdot)}(w)}$.

Proof. The proof of assertions a) and b) are the same as for assertions a) and b) of Lemma 5.28. Here, we use the sublinearity of m_α.
Finally, by the definition of Rh, we have

$$\begin{aligned} m_\alpha(Rh)(x) &\leqslant \sum_{k=0}^{\infty} \frac{m_\alpha^{k+1} h(x)}{2^k \|m_\alpha\|_{L^{p(\cdot)}(w)}^k} \\ &\leqslant 2\|m_\alpha\|_{L^{p(\cdot)}(w)} \sum_{k=0}^{\infty} \frac{m_\alpha^{k+1} h(x)}{2^{k+1} \|m_\alpha\|_{L^{p(\cdot)}(w)}^{k+1}} \\ &\leqslant 2\|m_\alpha\|_{L^{p(\cdot)}(w)} Rh(x). \end{aligned}$$

Thus $Rh \in B_1$ and $[Rh]_{B_1} \leqslant 2\|m_\alpha\|_{L^{p(\cdot)}(w)}$. □

Lemma 5.30. *Let $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. Define the operator H on $L^{p'(\cdot)}$ by*

$$Hh = \mathscr{R}'\left(h w^{\frac{1}{p(\cdot)}}\right) w^{-\frac{1}{p(\cdot)}}$$

where

$$\mathscr{R}' g(x) = \sum_{k=0}^{\infty} \frac{m_\alpha^k g(x)}{2^k \|m_\alpha\|_{L^{p'(\cdot)}(w')}^k}.$$

Then

a) $|h| \leqslant Hh$;

b) H is bounded on $L^{p'(\cdot)}$ and $\|Hh\|_{p'(\cdot)} \leqslant 2\|h\|_{p'(\cdot)}$;

c) $Hh \cdot w^{\frac{1}{p(\cdot)}} \in B_1$ and $[Hh \cdot w^{\frac{1}{p(\cdot)}}]_{B_1} \leqslant 2\|m_\alpha\|_{L^{p'(\cdot)}(w')}$.

Proof. The proof is the same as for Lemma 5.29. We replace $p(\cdot)$ by $p'(\cdot)$ and $w \in B_{p(\cdot)}$ by $w' \in B_{p'(\cdot)}$. The property $p'(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$ comes from Remark 5.2. □

5.8.2. A weighted extrapolation theorem

We denote by $\mathscr{F}$ a family of couples of non-negative measurable functions. We are now ready to state and prove the following weighted variable extrapolation theorem.

Theorem 5.8. *Suppose that for some constant exponent $p_0 > 1$, there exists a function $C : (0,\infty) \to (0,\infty)$ such that for all $v \in B_{p_0}$ and $(F,G) \in \mathscr{F}$, we have*

$$\int_{\mathbb{B}} F(x)^{p_0} v(x) d\mu_\alpha(x) \leqslant C([v]_{B_{p_0}}) \int_{\mathbb{B}} G(x)^{p_0} v(x) d\mu_\alpha(x). \tag{5.42}$$

Then given $p(\cdot) \in \mathscr{P}_{\pm}^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}$, we have

$$\|F\|_{p(\cdot),w} \leqslant 16 \times 4^{-\frac{1}{p_0}} \left(C([v]_{B_{p_0}})\right)^{\frac{1}{p_0}} \|G\|_{p(\cdot),w} \tag{5.43}$$

for all $(F,G) \in \mathscr{F}$ and $F \in L^{p(\cdot)}(w)$

Proof. We use the technique of Cruz-Uribe and Wang in [16, Theorem 2.6]. Let $(F,G) \in \mathscr{F}$. If $\|F\|_{p(\cdot),w} = 0$ we have the result. Otherwise, $\|F\|_{p(\cdot),w} > 0$ and hence $\|G\|_{p(\cdot),w} > 0$, because if $\|G\|_{p(\cdot),w} = 0$, then $G = 0$ a.e. and by (5.42) we will have $F = 0$ a.e. Henceforth, we assume $0 < \|F\|_{p(\cdot),w} < \infty$ and $0 < \|G\|_{p(\cdot),w} < \infty$. Define

$$h_1 = \frac{F}{\|F\|_{p(\cdot),w}} + \frac{G}{\|G\|_{p(\cdot),w}},$$

then $\|h_1\|_{p(\cdot),w} \leqslant 2$ and so $h_1 \in L^{p(\cdot)}(w)$.

Since $F \in L^{p(\cdot)}(w)$, by duality (Proposition 5.4), there exists $h_2 \in L^{p'(\cdot)}$ such that $\|h_2\|_{p'(\cdot)} = 1$ and

$$\|F\|_{p(\cdot),w} \leqslant 2 \int_{\mathbb{B}} F w^{\frac{1}{p(\cdot)}} h_2 d\mu_\alpha \leqslant 2 \int_{\mathbb{B}} F(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha \tag{5.44}$$

where the latter inequality comes from assertion *a*) of Lemma 5.30.

Set $\gamma = \frac{1}{p_0'}$. By the usual Hölder inequality, we have

$$\begin{aligned} \int_{\mathbb{B}} F(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha &= \int_{\mathbb{B}} F(Rh_1)^{-\gamma} (Rh_1)^{\gamma} (Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha \\ &\leqslant I_1^{\frac{1}{p_0}} I_2^{\frac{1}{p_0'}}, \end{aligned} \tag{5.45}$$

where

$$I_1 := \int_{\mathbb{B}} F^{p_0} (Rh_1)^{1-p_0} (Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha$$

and

$$I_2 := \int_{\mathbb{B}} (Rh_1)(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha.$$

In addition, from Lemma 5.29 and Lemma 5.30 respectively, R is bounded on $L^{p(\cdot)}(w)$ and H is bounded on $L^{p'(\cdot)}$. Thus by the Hölder inequality, assertions *b*) of Lemma 5.29 and Lemma 5.30, we have

$$I_2 \leqslant 2\|Rh_1\|_{p(\cdot),w} \|Hh_2\|_{p'(\cdot)} \leqslant 8\|h_1\|_{p(\cdot),w} \|h_2\|_{p'(\cdot)} \leqslant 16.$$

By the definition of h_1 and assertion a) of Lemma 5.29, we have

$$\frac{\varphi}{\|\varphi\|_{p(\cdot),w}} \leqslant h_1 \leqslant Rh_1$$

for $\varphi \in \{F,\ G\}$. Next, by the Hölder inequality and assertion b) of Lemma 5.30, we have

$$\begin{aligned}
I_1 &\leqslant \int_{\mathbb{B}} F^{p_0}(\zeta)\left(\frac{F(\zeta)}{\|F\|_{p(\cdot),w}}\right)^{1-p_0} H(\zeta)h_2(\zeta)w^{\frac{1}{p(\zeta)}}\,d\mu_\alpha(\zeta)\\
&= \|F\|_{p(\cdot),w}^{p_0-1}\int_{\mathbb{B}} F(\zeta)H(\zeta)h_2(\zeta)w^{\frac{1}{p(\zeta)}}\,d\mu_\alpha(\zeta)\\
&\leqslant 2\|F\|_{p(\cdot),w}^{p_0-1}\|F\|_{p(\cdot),w}\|Hh_2\|_{p'(\cdot)}\\
&\leqslant 4\|F\|_{p(\cdot),w}^{p_0}\\
&< \infty.
\end{aligned}$$

Since $Rh_1 \in B_1$ and $(Hh_2)w^{\frac{1}{p(\cdot)}} \in B_1$ by Lemma 5.29 and Lemma 5.30 respectively, it follows from Theorem 5.7 that $v := (Rh_1)^{1-p_0}\left(Hh_2 w^{\frac{1}{p(\cdot)}}\right) \in B_{p_0}$. Hence by (5.42) and the same argument as above, we have

$$\begin{aligned}
I_1 &= \int_{\mathbb{B}} F^{p_0}(Rh_1)^{1-p_0}(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&\leqslant C([v]_{B_{p_0}})\int_{\mathbb{B}} G^{p_0}(Rh_1)^{1-p_0}(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&\leqslant C([v]_{B_{p_0}})\int_{\mathbb{B}} G^{p_0}\left(\frac{G}{\|G\|_{p(\cdot),w}}\right)^{1-p_0}(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&= C([v]_{B_{p_0}})\|G\|_{p(\cdot),w}^{p_0-1}\int_{\mathbb{B}} G(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&\leqslant 2C([v]_{B_{p_0}})\|G\|_{p(\cdot),w}^{p_0-1}\|G\|_{p(\cdot),w}\|Hh_2\|_{p'(\cdot)}\\
&\leqslant 4C([v]_{B_{p_0}})\|G\|_{p(\cdot),w}^{p_0}.
\end{aligned}$$

Thus from (5.44) and (5.45), we have the result. □

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