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Affidavit

Je soussigné, Roméo Leylekian, déclare par la présente que le travail présenté dans ce manuscrit est mon propre travail, réalisé sous la direction scientifique de François Hamel et Enea Parini, dans le respect des principes d'honnêteté, d'intégrité et de responsabilité inhérents à la mission de recherche. Les travaux de recherche et la rédaction de ce manuscrit ont été réalisés dans le respect à la fois de la charte nationale de déontologie des métiers de la recherche et de la charte d'Aix-Marseille Université relative à la lutte contre le plagiat.

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Liste de publications et participation aux conférences

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Résumé et mots clés

La fréquence principale d'une plaque vibrante serrée aux extrémités peut être modélisée d'un point de vue mathématique par la première valeur propre du bilaplacien avec conditions au bord de Dirichlet. En 1877, Rayleigh conjectura qu'à aire prescrite, la plaque vibrante de plus petite fréquence principale est circulaire. Formellement, cela revient à dire que la boule minimise la première valeur propre du bilaplacien de Dirichlet sous contrainte de volume. En 1995, Nadirashvili démontra l'assertion en dimension 2, puis Ashbaugh et Benguria la démontrèrent en dimensions 2 et 3. Depuis lors, la conjecture de Rayleigh est restée ouverte en dimension $d \geq 4$. Dans cette thèse, nous explorons la question sous plusieurs angles. Entre autres, nous montrons que le problème de minimisation de la première valeur propre du bilaplacien est bien posé jusqu'en dimension 8. Nous formulons également des conditions suffisantes de nature variée permettant de démontrer la conjecture. D'autre part, dans la perspective de nouvelles stratégies de preuves, nous exhibons des formules inédites pour la première valeur propre. Enfin, nous étudions des quantités reliées à la première valeur propre, telles que la rigidité torsionnelle biharmonique ou encore la première valeur propre du p-bilaplacien.

Mots clés Analyse des EDPs · Optimisation de forme · Bilaplacien · Valeurs propres · Conjecture de Rayleigh

Abstract and keywords

A mathematical model for the principal frequency of a vibrating plate with clamped edge is provided by the first eigenvalue of the bilaplacian with Dirichlet boundary conditions. In 1877, Rayleigh conjectured that, upon prescribing the area, the vibrating clamped plate with least principal frequency is circular. Formally, the conjecture claims that the ball minimizes the first eigenvalue of the Dirichlet bilaplacian under volume constraint. In 1995, Nadirashvili proved the Rayleigh Conjecture in dimension 2 and, subsequently, Ashbaugh and Benguria proved it in dimensions 2 and 3. Since then, the conjecture has remained open in dimension $d \geq 4$. In this thesis, we explore the conjecture from various points of view. Among others, we prove that the problem of minimizing the first eigenvalue of the bilaplacian under volume constraint is well-posed up to dimension 8. We also provide several interesting sufficient conditions of diverse nature for the conjecture to hold. Besides, in view of potential new approaches, we investigate different formulations for the first eigenvalue. Lastly, we analyse quantities related to the first eigenvalue such as the biharmonic torsional rigidity and the first eigenvalue of the p-bilaplacian.

Keywords Analysis of PDEs · Shape optimization · Bilaplacian · Eigenvalues · Rayleigh Conjecture

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Synthèse

Quelle plaque d'aire prescrite, maintenue aux extrémités, produit par résonance le son le plus grave ?

En 1877, Rayleigh conjectura que c'est la plaque circulaire qui doit être la plus grave : « When the edges are clamped, the form of gravest pitch is doubtless the circle » [Ray45, Volume I, p. 382]. C'est à l'étude de cette célèbre conjecture que je me suis intéressé au cours de ma thèse.

Modèle de plaque

Avant de s'atteler à la conjecture de Rayleigh à proprement dit, il convient de présenter un modèle permettant de décrire la dynamique des plaques. Pour cela, considérons une plaque (rigide) fixée aux extrémités, et représentons par $\Omega \subseteq \mathbb{R}^2$ l'ensemble des points du plan occupés par la plaque au repos. Le déplacement vertical de la plaque à l'instant $t \geq 0$ et au point $x \in \Omega$, noté $v(t,x)$, satisfait en l'absence de force extérieur (et après normalisation de toutes les constantes physiques) l'équation

$$\partial_t^2 v + \Delta^2 v = 0.$$

L'opérateur $\Delta^2 v = \Delta(\Delta v)$ est appelé bi-Laplacien, ou plus simplement bilaplacien, dans la mesure où $\Delta = \partial_x^2 + \partial_y^2$ est communément appelé Laplacien. Un calcul élémentaire montre alors que $\Delta^2 = \partial_x^4 + 2\partial_x^2\partial_y^2 + \partial_y^4$ (au moins formellement). Le fait que la plaque soit fixée horizontalement se traduit par la condition supplémentaire $v(t,x) = 0$ pour tout $x \in \partial\Omega$, où $\partial\Omega$ représente la frontière de Ω.

Mais cela ne suffit pas à décrire correctement la dynamique. En effet, la rigidité de la plaque entraîne par exemple que toute torsion au niveau du système de fixation aura un impact sur le déplacement à l'intérieur de Ω. Le système de fixation utilisé a donc ici son importance. Dans ce manuscrit, je me suis intéressé au cas de plaques « serrées » aux extrémités. Le terme « serré » suppose des entraves maintenant horizontalement la frontière et empêchant tout type de mouvement au niveau du bord. De cette façon, la contrainte supplémentaire $\nabla v(t,x) = 0$ s'applique en chaque point $x \in \partial\Omega$, forçant la plaque à ployer afin de se maintenir horizontale le long de sa frontière dès lors qu'elle subit une déformation intérieure. Le terme consacré à ce type de configuration en anglais est « clamped », qui se rapporte à une situation où le serrage est réalisé par des pinces. En français, le verbe « clamper » a un sens similaire mais il n'est utilisé que dans le cadre de la médecine chirurgicale,

et il semble que le terme le plus courant soit « encastrer » [HP05, p. 11]. Cependant, cette terminologie insiste plus à mon sens sur la nature que sur la fonction du système de fixation. C'est pourquoi dans la suite j'ai préféré parler de plaque « serrée » ou « maintenue » aux extrémités. Remarquons enfin qu'en vertu de la condition $v = 0$, la dérivée tangentielle de v s'annule au bord. Il s'ensuit que la condition $\nabla v = 0$ est équivalente à $\partial_n v = 0$, où ∂_n est la dérivée dans la direction de la normale à la frontière $\partial\Omega$. Pour résumer, la déformation verticale v satisfait la loi d'évolution

$$\begin{cases} \partial_t^2 v + \Delta^2 v &= 0 \quad \text{dans } \Omega, \\ v &= 0 \quad \text{sur } \partial\Omega, \\ \partial_n v &= 0 \quad \text{sur } \partial\Omega. \end{cases}$$

Ainsi, nous voyons que le bilaplacien Δ^2 et les conditions au bord $v = \partial_n v = 0$ (dite conditions de Dirichlet) occupent une place centrale dans la dynamique des plaques serrées aux extrémités. Avant d'aller plus loin, soulignons un phénomène surprenant typique de la physique de ces systèmes. Pour cela, considérons une plaque serrée horizontalement décrivant au repos la forme Ω. A l'équilibre, lorsque la plaque est soumise à une force verticale f, le déplacement v résultant satisfait l'équation

$$\begin{cases} \Delta^2 v &= f \quad \text{dans } \Omega, \\ v &= 0 \quad \text{sur } \partial\Omega, \\ \partial_n v &= 0 \quad \text{sur } \partial\Omega. \end{cases}$$

Si la force est appliquée en direction du haut en chaque point de la plaque, nous pouvons nous attendre à ce que celle-ci réponde par une déformation en direction du haut. En termes mathématiques, si $f \geq 0$ le modèle devrait garantir $v \geq 0$. Seulement, contrairement au Laplacien, le bilaplacien ne vérifie pas de principe du maximum. Autrement dit $f \geq 0$ n'implique pas nécessairement $v \geq 0$. Physiquement cela signifie que même si la force s'exerce en direction du haut en chaque point de la plaque, il est possible que celle-ci réponde par un déplacement en direction du bas quelque part dans Ω. En fait ce phénomène dépend non seulement de la force appliquée, mais aussi de la géométrie de la plaque. Par exemple, si Ω est circulaire, la réponse de la plaque se fera bien vers le haut comme attendu. Mais si au contraire Ω est carré, il est alors certain de pouvoir trouver un point de la plaque où appliquer une force permettant d'obtenir un déplacement antagoniste en un autre point de la plaque.

Après cette parenthèse (qui a son importance), revenons à Rayleigh et à sa conjecture. Les harmoniques du son produit par une plaque Ω serrée aux extrémités correspondent mathématiquement aux valeurs propres du bilaplacien avec conditions au bord de Dirichlet. Ces valeurs sont les nombres $\Gamma \in \mathbb{R}$ pour lesquels il existe une fonction $u : \Omega \to \mathbb{R}$ non identiquement nulle, appelée fonction propre, tel que

$$\begin{cases} \Delta^2 u &= \Gamma u \quad \text{dans } \Omega, \\ u &= 0 \quad \text{sur } \partial\Omega, \\ \partial_n u &= 0 \quad \text{sur } \partial\Omega. \end{cases} \tag{0.1}$$

La hauteur du son correspond à la plus grave des harmoniques, et donc à la plus petite de ces valeurs propres, aussi appelée valeur propre principale ou fondamentale. Ainsi, la question de Rayleigh revient à déterminer pour quel $\Omega \subseteq \mathbb{R}^2$ d'aire prescrit la valeur propre principale du bilaplacien (avec conditions au bord) de Dirichlet est minimale.

Maintenant que les objets d'importance pour la conjecture de Rayleigh sont identifiés, inscrivons-les dans un contexte mathématique plus formel. Pour commencer, plaçons-nous dans la situation générale d'un ouvert $\Omega \subseteq \mathbb{R}^d$ où $d \in \mathbb{N}^*$. Le bon cadre fonctionnel pour donner un sens à l'équation (0.1) est l'espace de Sobolev $H_0^2(\Omega)$, défini comme l'adhérence de $C_c^\infty(\Omega)$, dans $H^2(\Omega)$. Une fonction u est alors déclarée solution de (0.1) dès que u appartient à $H_0^2(\Omega)$ et vérifie

$$\int_\Omega \Delta u \Delta \varphi = \Gamma \int_\Omega u\varphi, \quad \forall \varphi \in H_0^2(\Omega).$$

Pour prouver que valeurs propres et fonctions propres existent bel et bien étant donné un ouvert Ω, la façon la plus commode de procéder est probablement d'introduire l'opérateur résolvant (ou plus simplement résolvante) $R_{\Delta^2} : L^2(\Omega) \to L^2(\Omega)$ qui, à une donnée $f \in L^2(\Omega)$, associe la solution $v \in H_0^2(\Omega)$ de l'équation

$$\begin{cases} \Delta^2 v &= f \quad \text{dans } \Omega, \\ v &= 0 \quad \text{sur } \partial\Omega, \\ \partial_n v &= 0 \quad \text{sur } \partial\Omega. \end{cases}$$

Lorsque Ω est de mesure finie, il est possible de démontrer que R_{Δ^2} définit un opérateur compact et auto-adjoint sur $L^2(\Omega)$. En lui appliquant le théorème spectral, nous concluons que le bilaplacien de Dirichlet admet une suite de valeurs propres $(\Gamma_n)_{n\in\mathbb{N}^*}$ et que les fonctions propres associées $(u_n)_{n\in\mathbb{N}^*}$ constituent une base hilbertienne de $L^2(\Omega)$. En outre, on obtient sans difficulté que $\Gamma_n > 0$ pour tout $n \in \mathbb{N}^*$. En résumé,

Lemme. *Si Ω est de mesure finie, les valeurs propres du bilaplacien avec conditions de Dirichlet forment une suite*

$$0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \to \infty.$$

En particulier l'existence d'une plus petite valeur propre est garantie. Dans la suite, sauf mention contraire, nous noterons cette valeur propre particulière

$$\Gamma(\Omega) := \Gamma_1$$

afin de mettre en exergue sa dépendance par rapport au domaine Ω sous-jacent. Pour pouvoir manipuler plus simplement $\Gamma(\Omega)$, il est pratique de disposer d'une caractérisation variationnelle adaptée. De façon générale, ceci peut être obtenu grâce au théorème de min-max qui fournit la formule suivante.

Lemme. *Si Ω est de mesure finie, la première valeur propre du bilaplacien de Dirichlet satisfait*

$$\Gamma(\Omega) = \min_{v \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (\Delta v)^2}{\int_\Omega v^2}. \tag{0.2}$$

De plus, toute fonction u réalisant le minimum est fonction propre associée à $\Gamma(\Omega)$. La formule (0.2) est absolument incontournable pour une compréhension correcte des termes de la conjecture de Rayleigh. Elle a été utilisée à de nombreuses reprises tout au long de cette thèse. Elle permet notamment de mettre en évidence plusieurs propriétés élémentaires de $\Gamma(\Omega)$. En premier lieu, un simple changement de variable montre que pour tout $r > 0$,

$$\Gamma(r\Omega) = r^{-4}\Gamma(\Omega). \tag{0.3}$$

Autrement dit, toute dilatation du domaine résulte en un abaissement de la fréquence fondamentale, tandis qu'une contraction rend la fréquence fondamentale plus aigüe. Cette dernière remarque est en fait le cas particulier de la propriété de monotonie suivante.

$$\Omega_1 \subseteq \Omega_2 \quad \Rightarrow \quad \Gamma(\Omega_2) \leq \Gamma(\Omega_1)$$

Toujours par changement de variable, (0.2) montre également que $\Gamma(\Omega)$ est invariant par isométries. Pour conclure, mentionnons que d'autres manipulations plus techniques préservent $\Gamma(\Omega)$. Celles-ci font appel à la notion de capacité, qui est explorée en détail au chapitre 4. Nous y voyons notamment que deux domaines égaux à un ensemble de H^2-capacité nulle près génèrent les mêmes espaces H_0^2, et donc les mêmes valeurs propres.

Conjecture de Rayleigh et état de l'art

Après avoir introduit les concepts nécessaires, nous pouvons désormais énoncer la conjecture de Rayleigh sous sa formulation moderne. Dans ce qui suit, la mesure de Lebesgue d-dimensionnelle d'un ensemble mesurable Ω est notée $|\Omega|$.

Conjecture (Conjecture de Rayleigh sur les plaques vibrantes)**.** *Soit $d \in \mathbb{N}^*$. Alors pour tout ouvert borné $\Omega \subseteq \mathbb{R}^d$,*

$$\Gamma(\Omega) \geq \Gamma(B), \tag{0.4}$$

où B est une boule euclidienne tel que $|B| = |\Omega|$. De plus, en cas d'égalité, $\Omega = B$ à isométries et ensembles de H^2-capacité nulle près.

Dans la mesure où $\Gamma(\Omega)$ constitue un modèle mathématique pour la fréquence fondamentale d'une plaque Ω serrée aux extrémités, (0.4) est bien une reformulation de la conjecture telle qu'initialement énoncée par Rayleigh, mis à part que nous considérons des dimensions $d \in \mathbb{N}^*$ arbitraires là où Rayleigh ne s'intéressait qu'au

cas $d = 2$. Mais il est également possible de présenter la conjecture sous l'angle légèrement différent de l'optimisation de forme. En effet, le problème posé par Rayleigh consiste à minimiser la fonctionnelle $\Gamma : \Omega \mapsto \Gamma(\Omega)$ sous contrainte de volume. Ainsi, en définissant

$$\min\{\Gamma(\Omega) : \Omega \subseteq \mathbb{R}^d \text{ ouvert borné, } |\Omega| = c\}, \tag{0.5}$$

démontrer la conjecture revient à prouver que (modulo des ensembles de H^2-capacité nulle) les boules de volume c, et elles seules, résolvent (0.5).

Il est de notoriété publique que la conjecture de Rayleigh pour le bilaplacien est un problème ardu, en témoigne l'état de l'art que nous présentons plus bas. Mais mettons dès à présent en évidence quelques premières difficultés. Pour traiter des problèmes d'optimisation de forme semblables à la conjecture de Rayleigh, la stratégie classique consiste à utiliser les techniques de symétrisation mises en œuvre par Faber et Krahn dans la première moitié du XXème siècle [Fab23 ; Kra25] et permettant, à partir d'une fonction définie dans un ouvert Ω, de fabriquer une fonction à symétrie radiale définie dans une boule B de même volume que Ω. Cette méthode leur permit de prouver que les boules minimisent la première valeur propre du Laplacien de Dirichlet sous contrainte de volume (un résultat qui avait aussi été conjecturé par Rayleigh dans [Ray45]). Seulement, les techniques de symétrisation sont surtout pertinentes dans le cas où les fonctions mises en jeu sont positives. Malheureusement, comme évoqué plus haut, le bilaplacien ne satisfait en général pas de principe du maximum. Or la perte de principe du maximum s'avère problématique en ce qu'elle écarte la possibilité d'appliquer le théorème de Krein-Rutman [Hen06, Theorem 1.2.6]. Dans ces conditions, non seulement rien n'empêche la première fonction propre de changer de signe, mais rien n'empêche non plus la première valeur propre d'être multiple. Cette situation a effectivement été observée pour la première fois en 1952 dans des anneaux de rayon interne suffisamment petit [DS52 ; CDS79]. Mais outre le changement de signe potentiel d'une fonction propre principale $u \in H^2_0(\Omega)$, nous aimerions soulever un problème supplémentaire. Car quand bien même u serait de signe fixé, sa symétrisation u^* risque de ne pas appartenir à l'espace $H^2_0(B)$. Cela provient du fait que les dérivées d'ordre deux réagissent mal aux symétrisations. Cette difficulté est discutée en détail en section 3.3.

Les explications précédentes ont mis en exergue au moins deux écueils concernant la conjecture de Rayleigh. Pour autant, des résultats majeurs ont bien été obtenus à son sujet au cours du siècle dernier. Tout d'abord, après sa publication en 1877, l'un des premiers résultats concernant la conjecture de Rayleigh est dû à Szegő qui démontra dans [Sze50] l'inégalité (0.4) sous l'hypothèse d'une première fonction propre sur Ω ne changeant pas de signe. Mais cela ne suffit pas à résoudre complètement la conjecture dans la mesure où, comme rappelé plus haut, les fonctions propres principales sont en général de signe variable. Après le premier résultat partiel de Szegő, Mohr établit dans [Moh75] les conditions

d'optimalité de premier et de second ordre satisfaites par une hypothétique forme optimale. En utilisant ces conditions d'optimalité de façon originale, il conclut alors que les seules solutions régulières au problème (0.5) en dimension $d = 2$ sont les disques admissibles. Malheureusement, l'approche de Mohr repose non seulement sur l'hypothèse d'existence d'une forme optimale, mais également sur l'hypothèse bidimensionnelle. Ainsi, son résultat fut finalement éclipsé par une série de papiers initiée par [Tal76] dans lequel Talenti y exposait son désormais fameux principe de comparaison. Le principe de comparaison de Talenti a eu par la suite des répercussions importantes, et est encore de nos jours largement étudié (voir section 2.2 pour plus de détails). Quelques années plus tard, Talenti raffina son principe de comparaison en l'appliquant à la formulation variationnelle (0.2). Cette démarche lui permit de borner inférieurement la première valeur propre du bilaplacien par un problème auxiliaire, communément appelé « problème aux deux boules » de nos jours. Le problème aux deux boules a l'intérêt d'être posé, comme son nom l'indique, dans deux boules couplées par une certaine équation, et de ne dépendre de l'ouvert initial qu'à travers la taille des deux boules. Il s'agit donc d'un problème tangible. Dans cette thèse, nous avons eu l'occasion de l'étudier en détail et de rencontrer à plusieurs reprises des généralisations de ce problème (voir section 3.2, proposition 5.22, section 6.5). Quoiqu'il en soit, c'est ainsi que Talenti obtenait pour la première fois une borne inférieure non triviale et inconditionnelle sur Γ.

A la suite de Talenti, Nadirashvili parvint finalement dans [Nad95] à résoudre le problème aux deux boules et par la même la conjecture de Rayleigh dans $\mathbb{R}^2$. Ainsi, plus d'un siècle après la publication de [Ray45], l'article de Nadirashvili résolvait la conjecture de Rayleigh sous sa formulation initiale. La même année, toujours suivant la stratégie de Talenti, Ashbaugh et Benguria démontrèrent la conjecture de Rayleigh en dimensions 2 et 3. Leur approche diffère singulièrement de celle de Nadirashvili. Elle repose sur des calculs explicites fastidieux mais néanmoins minutieux (analogues à ceux de l'annexe G). Enfin, le problème aux deux boules fut résolu en toute dimension dans [AL96]. Dans ce dernier article, les auteurs mirent en évidence qu'en dimension $d \geq 4$, le problème aux deux boules est minimal lorsque les deux boules sont de même volume, là où en dimension $d = 2, 3$ [AB95] montrait que le problème est minimal lorsqu'une des deux boules disparaît. Ce phénomène surprenant conduit Ashbaugh et Laugesen à conclure que l'approche de Talenti, basée sur le problème aux deux boules, ne permettrait pas de démontrer la conjecture de Rayleigh en dimension $d \geq 4$. Néanmoins, leur résultat n'est pas seulement négatif puisqu'il permit de dégager la meilleure borne inférieure sur Γ connue à ce jour.

Depuis cette époque, peu de progrès ont été faits. Mentionnons cependant les récents articles [Kri20 ; Kri22], dans lesquels Kristály étudie l'analogue de la conjecture de Rayleigh dans des variétés riemanniennes à courbure strictement négative et strictement positive respectivement. Dans le cas d'espaces courbés négativement, la restriction dimensionnelle $d < 4$ est de nouveau nécessaire pour adapter la stratégie

de Talenti. Dans le cas d'espaces courbés positivement, curieusement, la conjecture peut être résolue en toute dimension, avec en contrepartie une restriction sur la contrainte de volume. En effet, seuls les volumes suffisamment grands sont autorisés. Ainsi, toute stratégie consistant à se placer dans des variétés à courbure positive puis à faire tendre la courbure vers 0 en vue d'une preuve de la conjecture de Rayleigh dans le cadre euclidien semble compromise.

Contributions

Au cours de ma thèse, je me suis emparé de la conjecture de Rayleigh en dimension $d \geq 4$. Après plusieurs attaques de front soldées par des résultats intéressants bien que partiels, j'ai exploré la conjecture de façon plus détournée. J'ai ainsi soulevé des questions de nature diverses. Rappelons que, de façon générale, plusieurs questions se posent naturellement dans le cadre d'un problème d'optimisation de forme tel que la conjecture de Rayleigh. La première, bien sûr, consiste à s'interroger sur la géométrie de la forme optimale. Mais il est bien souvent difficile d'identifier cette forme de façon catégorique. Dans ces circonstances, une façon alternative de traiter le problème peut être d'établir des estimations acceptables de la valeur optimale de la fonctionnelle en jeu. Enfin, il apparaît central de se demander si une forme optimale existe bel et bien. Le cas échéant, sa régularité peut alors éventuellement être analysée. Ainsi, les problématiques qui ont guidé ma thèse sont les suivantes :

1. Que peut-on dire de la géométrie d'une forme optimale ?
2. Est-il possible d'estimer la valeur propre optimale ?
3. Existe-t-il une forme optimale ? Si tel est le cas, quelles sont ses propriétés de régularité ?

La plus stimulante de ces trois questions est à mes yeux la première, et c'est celle à laquelle je me suis attelé en premier lieu. Ainsi, j'ai proposé des démonstrations de la conjecture de Rayleigh sous réserve de conditions simples bien qu'implicites portant sur une hypothétique forme optimale. Pour ce faire, j'ai commencé par établir les conditions d'optimalité satisfaites par la forme optimale. En général, ces conditions sont exploitées pour profiler sinon identifier sa géométrie. Mais une première difficulté s'est rapidement posée. Car si les conditions d'optimalité relatives à une valeur propre simple peuvent être obtenues par dérivation de forme, les valeurs propres principales du bilaplacien sont en général multiples. C'est pourquoi il a tout d'abord fallu démontrer la simplicité de la première valeur du bilaplacien sur un ouvert optimal. Cette démarche m'a conduit au résultat suivant.

Théorème (Théorème 5.4)**.** *Soit Ω un ouvert C^4 optimal pour le problème (0.5). Alors $\Gamma(\Omega)$ est simple. De plus, si u est une fonction propre associée normalisée dans L^2, Δu est constante égale à $\pm\alpha$ sur chaque composante connexe de $\partial\Omega$, avec*

$$\alpha := \sqrt{\frac{4\Gamma(\Omega)}{d|\Omega|}}.$$

Le résultat précédent était encourageant dans la mesure où il inscrit l'équation aux valeurs propres de l'ouvert optimal dans la classe des problèmes surdéterminés. Or ces problèmes ont été étudiés en détail depuis l'article fondateur [Ser71]. Pourtant, la théorie des problèmes surdéterminés pour les opérateurs d'ordre quatre est bien moins fournie (voir [Ben86 ; PS89 ; Dal90]) que celle concernant les opérateurs d'ordre deux comme le Laplacien. En outre, les résultats de surdétermination nécessitent le plus souvent des hypothèses fortes sur le signe des solutions. Le changement de signe potentiel des premières fonctions propres du bilaplacien était donc là encore un frein.

Néanmoins, le théorème précédent fournit une information intéressante sur la première fonction propre optimale, et il s'agissait de l'exploiter au mieux. C'est ainsi que j'en suis venu à développer une méthode de réduction d'ordre grâce à laquelle j'ai ramené le problème surdéterminé satisfait par la première fonction propre optimale u à un problème d'ordre inférieur de la forme $\Delta z = \sqrt{\Gamma(\Omega)}z + \alpha$ où $z \in H_0^1(\Omega)$ s'exprime de façon explicite en fonction de u. J'ai ainsi pu mettre en œuvre un certain nombre d'outils disponibles uniquement dans le contexte d'équations elliptiques de second ordre. En particulier, l'utilisation de techniques de symétrisation a permis de démontrer le résultat suivant (voir chapitre 5).

Théorème (Théorème 5.1)**.** *Soit Ω un ouvert C^4 de frontière connexe, optimal pour le problème (0.5). Soit u une première fonction propre sur Ω normalisée dans L^2 et u_B une première fonction propre sur B (une boule de volume $|\Omega|$) normalisée dans L^2 également. Alors,*

$$\left|\int_\Omega u\right| \geq \left|\int_B u_B\right|. \tag{0.6}$$

De plus, en cas d'égalité ci-dessus, Ω est une boule.

Grâce au théorème précédent, nous réalisons qu'il suffirait de démontrer la réciproque de l'inégalité (0.6) pour prouver (une version affaiblie de) la conjecture de Rayleigh. Quelques pistes ont été explorées à ce sujet (voir section 5.4), mais nous ne sommes pas parvenu à démontrer un tel résultat. En revanche, ce théorème nous a par la suite permis d'établir un lien insoupçonné entre la première valeur propre du bilaplacien et la rigidité torsionnelle biharmonique, qui est la quantité définie, pour un ouvert Ω de mesure finie, par

$$T(\Omega) := \min_{\substack{u\in H_0^2(\Omega)\\ u\neq 0}} \frac{\int_\Omega (\Delta u)^2}{\left(\int_\Omega u\right)^2}.$$

C'est alors que je me suis penché sur le problème d'optimisation de cette rigidité torsionnelle biharmonique T sous contrainte de volume. En reprenant la stratégie générale de Talenti, j'ai alors pu démontrer (voir section 5.5) que la boule minimise T en toute dimension.

Théorème (Théorème 5.18)**.** *Soit Ω un ouvert borné de $\mathbb{R}^d$ et B une boule de volume $|\Omega|$. Alors,*

$$T(\Omega) \geq T(B).$$

De plus, en cas d'égalité, Ω est une boule à un ensemble de H^1-capacité nulle près.

Outre ces travaux issus du principe de réduction d'ordre, j'ai tenté de mettre en œuvre une démarche tout à fait différente, basée sur une étude approfondie de la structure des fonctions propres principales sur un ouvert optimal. Cette stratégie est partie de l'observation de ce que la condition $\Delta u = \textit{constante}$ sur chaque composante connexe du bord empêche l'ensemble nodal, s'il existe, de rencontrer la frontière. Il est alors possible de se convaincre que l'un des sous-domaines $\{u > 0\}$ et $\{u < 0\}$ doit nécessairement posséder un trou contenant l'autre. Cette information est extrêmement précieuse et a permis, en améliorant le principe de comparaison de Talenti sur les domaines troués, de dégager un problème aux deux boules *asymétrique*. Ce travail, expliqué en détail au chapitre 6, a finalement donné lieu au théorème suivant.

Théorème (Théorème 6.1)**.** *Soit Ω un ouvert C^4 optimal pour le problème (0.5). Si la première fonction propre sur Ω n'admet aucune valeur critique (dans Ω) hormis ses extrema globaux et si $4 \leq d \leq 9$, alors Ω est une boule.*

Dans ce théorème, la restriction dimensionnelle n'est pas vraiment problématique. Elle provient en effet d'inégalités vérifiées numériquement pour les dimensions appartenant à cette plage particulière, et rien ne laisse penser que ces inégalités pourraient être mises en défaut au delà de la plage en question. En revanche, l'hypothèse d'absence de valeur critique de niveau intermédiaire semble, elle, incontournable. Elle est utilisée de façon capitale dans le cadre du principe de comparaison de Talenti adapté aux domaines troués. La preuve du théorème ci-dessus est à mon avis, bien que technique en certains points, globalement plutôt élégante. Pour autant, le résultat lui-même apparaît finalement moins pertinent que celui du théorème 5.1 dans la mesure où il nécessite une connaissance particulièrement fine de la fonction propre optimale, qui est somme toute inconnue.

Dans les résultats précédents, nous avons exploité les formulations traditionnelles de la valeur propre fondamentale du bilaplacien. Cependant, une certaine prise de recul permet de s'apercevoir que d'autres types de caractérisation existent, notamment concernant la valeur propre fondamentale du Laplacien. L'une d'entre elles permet d'exprimer cette valeur propre sur un domaine arbitraire par des convolutions multiples faisant intervenir des noyaux gaussiens. J'ai tenté de généraliser ce point de vue au cas de la première valeur propre du bilaplacien. Au chapitre 8,

en tirant partie d'une formule de Trotter pour les semi-groupes, je suis finalement parvenu à la formulation suivante.

Théorème (Théorème 8.1)**.** *Soit $\Omega \subseteq \mathbb{R}^d$ un ouvert Lipschitz borné. Soit p_s la solution fondamentale de l'équation de la chaleur biharmonique (voir équation (8.1) pour $m = 2$). Alors la première valeur propre du bilaplacien de Dirichlet sur Ω vérifie*

$$\Gamma(\Omega) = -\lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{t}\log\int_\Omega\dots\int_\Omega p_{\frac{t}{n}}(x_1-x_2)\dots p_{\frac{t}{n}}(x_{n-1}-x_n)dx_1\dots dx_n.$$

L'intérêt de cette formule réside en ce que la dépendance de $\Gamma(\Omega)$ vis-à-vis de Ω ne se retrouve qu'au travers de domaines d'intégration. En particulier, contrairement à (0.2), aucune fonction test n'intervient dans cette caractérisation. De plus, certains outils, comme en premier lieu les inégalités de Brascamp-Lieb-Luttinger (voir section 2.5), fournissent des résultats intéressants concernant de telles produits d'intégrales. Malheureusement, en raison du changement de signe de la solution fondamentale p_s, ces outils s'avèrent le plus souvent inefficients, et je ne suis donc pas parvenu à déduire de résultat significatif de cette nouvelle formulation.

En parallèle des approches précédentes visant à résoudre la conjecture de Rayleigh de manière directe, j'ai tenté d'estimer par diverses méthodes la première valeur propre optimale du bilaplacien. L'objectif que j'espérais atteindre était alors de pouvoir améliorer les bornes inférieures fournies par [AL96]. En réalité, obtenir des bornes inférieures de bonne qualité est presque aussi difficile que de résoudre la conjecture puisque celle-ci consiste justement à déterminer la borne inférieure optimale. Ainsi, les estimations que j'ai pu obtenir (section 5.5, section 7.3) se sont révélées insatisfaisantes. Néanmoins, cette démarche m'a conduit à développer un principe de comparaison de Talenti concernant les fonctions de signe variable. Étonnamment, ce nouvel outil s'est révélé particulièrement approprié dans l'étude d'un problème non linéaire en dimension 2. Le problème en question fait intervenir la notion de rigidité torsionnelle p-biharmonique et de première valeur propre du p-bilaplacien de Dirichlet, des analogues non linéaires à la rigidité torsionnelle biharmonique et à la première valeur propre du bilaplacien, respectivement définis par

$$T(\Omega) := \inf_{\substack{u\in C_c^\infty(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\Delta u|^p}{|\int_\Omega u|^p}, \qquad \Gamma_p(\Omega) := \inf_{\substack{u\in C_c^\infty(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\Delta u|^p}{\int_\Omega |u|^p}.$$

Le principe de comparaison de Talenti pour fonctions signées a ainsi donné lieu à la borne inférieure suivante sur la rigidité torsionnelle p-biharmonique.

Théorème (Théorème 7.5 et Proposition 7.8)**.** *Soit $\Omega \subseteq \mathbb{R}^d$ un ouvert borné de classe C^2 et B une boule de volume $|\Omega|$. Si $d = 2$, alors pour $1 \leq p < \infty$,*

$$T_p(\Omega) \geq \Gamma_p(B)|B|^{1-p}.$$

Pour conclure l'étude de la conjecture de Rayleigh, je me suis intéressé à la question de l'existence d'une forme optimale pour la première valeur propre du bilaplacien. En adaptant le célèbre principe de concentration-compacité de P.-L. Lions, j'ai prouvé l'existence d'une forme optimale dans la classe généralisée des H^1-quasi-ouverts. Puis, en analysant la régularité d'une première fonction propre optimale, je suis parvenu à exhiber jusqu'en dimension 8 un ouvert optimal.

Théorème (Théorème 4.1)**.** *Si* $2 \leq d \leq 8$*, il existe un ouvert minimisant la première valeur propre du bilaplacien de Dirichlet sous contrainte de volume. De plus, toute première fonction propre sur cet ouvert prolongée par zéro sur* $\mathbb{R}^d$ *est* $C^{1,\alpha}$ *pour tout* $0 < \alpha < 1$*.*

Bien évidemment, si les travaux réalisés pendant cette thèse ont apporté quelques réponses à la conjecture de Rayleigh, ils ont également soulevé bien des questions, et ils nécessitent d'être approfondis. Donnons très brièvement quelques exemples de perspectives possibles. Tout d'abord, le principe de réduction d'ordre a été utilisé de façon intéressante, mais il pourrait probablement encore être exploité sous un angle différent. Un autre travail intéressant consisterait bien sûr à tenter de s'affranchir de l'hypothèse sur les valeurs critiques de la fonction propre optimale dans le théorème 6.1. En outre, la formulation intégrale que nous avons démontrée pour la première valeur propre du bilaplacien n'est pas une fin en soit. Elle nécessite maintenant d'être exploitée. Enfin, la restriction dimensionnelle intervenant dans le cadre du théorème d'existence d'une forme optimale n'est probablement que technique, et il conviendrait de chercher à s'en passer.

1. Introduction

Which plate of prescribed area, clamped along its edge, has the gravest pitch?

This question was raised in 1877 by J.W. Strutt, famous 1904 Nobel laureate in Physics, better known by his title Lord Rayleigh. The question appeared in his equally well-known book entitled (perhaps with a certain cheekiness) "The Theory of Sound" [Ray45]. Based on several explicit computations, Rayleigh was led to claim that the circular plate has the deepest principal frequency: "When the edges are clamped, the form of gravest pitch is doubtless the circle" [Ray45, Volume I, p. 382].

It is probably no exaggeration to say that Rayleigh's book paved the way for the modern theory of spectral shape optimization in mathematical Physics. Indeed, Rayleigh's question turns into a surprisingly concise optimization problem. But from a mathematical point of view his answer resembles a conjecture rather than a proof. Making Rayleigh's claim rigorous has disconcerted mathematicians until now. It is the purpose of the present manuscript to explain more in detail the terms and hopefully to give a new light on the Rayleigh Conjecture for the clamped plate.

1.1. Mathematical models of membranes and plates

1.1.1. Physical context

Actually, one of the primary concerns of Rayleigh in [Ray45] is the study of membranes. Membranes enjoy close links with elastic plates. In particular, a claim analogous to the one for clamped plates was raised in [Ray45, Volume I, page xl]: "Of all membranes of equal area, that of circular form has the gravest pitch ". Even if more arguments were given to support this thesis for membranes (see [Ray45, Volume I, paragraph 210]), a complete proof was yet needed.

As a first step, a mathematical model for describing membranes and plates needs to be recalled. The one-dimensional version of a membrane is a string since a string can be thought of as a very narrow membrane. In this one-dimensional setting, Newton's second principle indicates that the transversal deformation $v(t,x)$ at time $t \geq 0$ and position $x \in (0,L)$ of a string of length L fixed at both ends and with no external force applied obeys (up to physical constants)

$$\partial_t^2 v = \partial_x^2 v,$$

with the boundary conditions $v(t,0) = v(t,L) = 0$. Over time, the scientific community then agreed that the correct generalisation of this evolution law to a membrane Ω fixed horizontally at the boundary $\partial\Omega$ was given by the wave equation

$$\partial_t^2 v = \Delta v,$$

with $v = 0$ on $\partial\Omega$. Here again v corresponds to the transversal deformation of the membrane and Ω is a planar domain representing the membrane at rest. As illustrated above, it is common to conflate the physical device and the mathematical domain used to describe its evolution. In a similar way, the one-dimensional version of a clamped plate is a beam, whose transversal deformation is governed through

$$\partial_t^2 v + \partial_x^4 v = 0.$$

For the boundary conditions, clamping means not only $v(t,0) = v(t,L) = 0$ but also $\partial_x v(t,0) = \partial_x v(t,L) = 0$, L being the length of the beam. This last condition on the derivative of v at the ends expresses the bending of the beam resulting from clamping.

Here, the two-dimensional generalisation seems to have been nonconsensual at the beginning. Indeed, a first model was proposed to be $\partial_t^2 v + \partial_x^4 v + \partial_y^4 v = 0$. But the operator $\partial_x^4 + \partial_y^4$ is not rotation invariant. Indeed, if $R : v \mapsto v(\alpha(x+y, x-y))$ is the rotation of angle $\pi/4$ – for that take $\alpha = \sqrt{2}/2$ –, then $\partial_x \circ R = \alpha R \circ (\partial_x + \partial_y)$ while $\partial_y \circ R = \alpha R \circ (\partial_x - \partial_y)$. Consequently,

$$(\partial_x^4 + \partial_y^4) \circ R = \frac{1}{2} R \circ (\partial_x^4 + 12\partial_x^2\partial_y^2 + \partial_y^4) \neq R \circ (\partial_x^4 + \partial_y^4).$$

Moreover, this model was not coherent with the experimental observations of vibration modes made by physicists, sprinkling some sand or powder onto vibrating clamped plates and recording the drawn patterns (see Chladni figures). On the contrary, the Laplace operator Δ is rotation invariant. As a sole argument, compute

$$\Delta \circ R = \alpha^2 R \circ [(\partial_x + \partial_y)^2 + (\partial_x - \partial_y)^2] = R \circ \Delta.$$

Therefore, a model involving the bi-Laplace operator $\Delta^2 = \Delta \circ \Delta = \partial_x^4 + 2\partial_x^2\partial_y^2 + \partial_y^4$, more conveniently referred to as bilaplacian, was finally proposed to describe the deformation of a plate Ω:

$$\partial_t^2 v + \Delta^2 v = 0.$$

Clamping the plate at the edges translates in the boundary conditions $v = 0$ and $\nabla v = 0$ on $\partial\Omega$. Observe that since the tangential derivative of v at the boundary is zero due to $v = 0$, the condition $\nabla v = 0$ is equivalent to $\partial_n v = 0$, where ∂_n is the derivative in the direction of the normal unit vector to the boundary $\partial\Omega$. We refer to [GGS10, section 1.1] for more details and references on these historical debates.

Anyway, we understand that the Laplacian Δ and the bilaplacian Δ^2 play a

central role in the dynamics of membranes and plates respectively. At this point, one may think that those two types of mechanical systems enjoy similar behaviours, being described by operators of similar nature. We would like to disclaim this belief. For that, consider a horizontally fixed membrane Ω at equilibrium subject to a vertical force f. The vertical displacement v of the membrane satisfies

$$\begin{cases} -\Delta v &= f \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

When the force applies in the upward direction, we expect the displacement of the membrane to be also upwards. In other words, if $f \geq 0$, we expect the mathematical model to guarantee $v \geq 0$. It turns out that the Laplacian has a property called the maximum principle which provides this relation. If now Ω is a clamped plate subject to the force f, the vertical displacement solves

$$\begin{cases} \Delta^2 v &= f \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \\ \partial_n v &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

But unlike the Laplacian, the bilaplacian does not enjoy the maximum principle. In particular $f \geq 0$ does not yield $v \geq 0$. The physical interpretation is that even if the force is exerted upwards at every point of Ω, it may happen that the vertical displacement occurs downwards somewhere inside Ω. Actually, this phenomenon strongly depends (on the force and) on the shape of the plate. For instance, for circular plates the vertical displacement will always remain upwards. But for square plates it is doubtless that a force can be applied upwards at a particular point while the displacement be downwards at another point. Surprisingly, this strange behaviour is consistent with Physics. In this respect, Rémy Mougenot, PhD student at the University of Lorraine, reported to me an anecdote that we may call the panel beater's trick: if some bodywork has been dented, instead of pressing on the dent from the inside – which is often impossible since the inner side may be inaccessible – panel beaters tap carefully around the dent to restore the body to its original shape. This illustrates possible applications to Physics of the lack of maximum principle for plates. However, from a mathematical point of view the lack of this property will be the source of many troubles.

After those preliminaries, let us go back to Rayleigh's question. So far we have not explained what is the "pitch" of a membrane or a plate. This notion is related to the frequency of its *intrinsic* vibrations. An intrinsic vibration is a time-periodic deformation of the form $v(t,x) = w(t)u(x)$. Plugging this separated variables solution into the equation satisfied by v, we observe that w must be of the form $w(t) = A\sin(kt) + B\cos(kt)$ for some $A, B \in \mathbb{R}$, where $k^2 = \lambda$ is coupled with u through the equation

$$-\Delta u = \lambda u,$$

for a membrane, and

$$\Delta^2 u = \lambda u,$$

for a plate. The value $k = \sqrt{\lambda}$ corresponds physically to the frequency of the vibration. Consequently, computing the intrinsic vibrations of a fixed membrane is equivalent to finding all the values $\lambda > 0$ for which the equation $-\Delta u = \lambda u$ admits a nonzero solution u vanishing at the boundary. Analogously, the vibrations of a clamped plate are characterised by the values λ for which $\Delta^2 u = \lambda u$ has a nontrivial solution u vanishing at the boundary together with its gradient. The previous are nothing else but eigenvalue problems, except that for the moment it is not very clear in which functional space they should be read. We will see this in a few lines. For now, we conclude by defining the pitch – also called principal frequency – to be the lowest of the frequencies of the intrinsic vibrations. Therefore, Rayleigh's questions shall be reformulated in the following way.

— *For which $\Omega \subseteq \mathbb{R}^2$ of prescribed area does the Laplacian with vanishing boundary conditions have smallest lowest eigenvalue?*

— *For which $\Omega \subseteq \mathbb{R}^2$ of prescribed area does the bilaplacian with clamped boundary conditions have smallest lowest eigenvalue?*

1.1.2. Mathematical setting

Bilaplacian

In this manuscript, as motivated above, we study the eigenvalues of the bilaplacian – also called bilaplace or biharmonic operator – with clamped boundary conditions over an open subset $\Omega \subseteq \mathbb{R}^d$. Here and in all the manuscript $d \in \mathbb{N}$ is the ambient dimension. These eigenvalues are the numbers $\Gamma \in \mathbb{R}$ for which there exists a nonzero function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \Delta^2 u &= \Gamma u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \\ \partial_n u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Equation (1.1) is only formal since some regularity is needed in order to be able to differentiate u. Therefore, we resort to the Sobolev space $H_0^2(\Omega)$, which is the closure of compactly supported smooth functions $C_c^\infty(\Omega)$ with respect to the space $H^2(\Omega)$ of functions which are $L^2(\Omega)$ together with their derivatives up to order 2. Then, u is said to solve (1.1) if and only if u lies in $H_0^2(\Omega)$ and solves the first equation in the sense of distributions, or in other words

$$\int_\Omega \Delta u \Delta \varphi = \Gamma \int_\Omega u\varphi, \quad \forall \varphi \in H_0^2(\Omega).$$

Such a solution, if not identically zero, is called an eigenfunction associated with Γ. Observe that since $H_0^2(\Omega)$ coincides with the set of functions in $H^2(\Omega)$ whose

restriction to the boundary vanishes together with its gradient, the boundary conditions of (1.1) are fulfilled.

To prove that eigenvalues and eigenfunctions do exist for an arbitrary open set Ω, we first need to have a nice enough linear operator in hand. The more convenient way is probably to introduce the resolvent operator $R_{\Delta^2} : L^2(\Omega) \to L^2(\Omega)$ mapping a data $f \in L^2(\Omega)$ to the solution $v \in H_0^2(\Omega)$ of the equation

$$\left\{ \begin{array}{rcll} \Delta^2 v & = & f & \text{in } \Omega, \\ v & = & 0 & \text{on } \partial\Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

Once again the above system means that $v \in H_0^2(\Omega)$ satisfies the first equation in the space of distributions $\mathcal{D}'(\Omega)$, or equivalently

$$\int_\Omega \Delta v \Delta\varphi = \int_\Omega f\varphi, \quad \forall \varphi \in H_0^2(\Omega).$$

Whenever Ω is of finite measure, it is classic to prove by the Lax-Milgram Theorem that R_{Δ^2} defines a bounded operator over $L^2(\Omega)$. This argument uses the Poincaré inequality

$$\|\varphi\|_{L^2(\Omega)} \leq C_\Omega \|\Delta\varphi\|_{L^2(\Omega)}, \tag{1.2}$$

for all $\varphi \in H_0^2(\Omega)$. Actually, applying twice the standard Poincaré inequality (for sets of finite measure, see [Wil95, Theorem 8.14]) yields $\|\varphi\| \leq C\|D^2\varphi\|$, but for $\varphi \in H_0^2(\Omega)$ we will see in a few lines that this turns into (1.2). For the moment, let us go back to the resolvent operator. Rellich's Theorem ([Wil95, Theorem 7.45]) shows that it is a compact operator. Lastly, R_{Δ^2} is symmetric and bounded hence self-adjoint over the Hilbert space $L^2(\Omega)$. As a result, by the Spectral Theorem (see e.g. [Hen06, Theorem 1.2.1]), there exists a sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ and a Hilbert basis $(u_n)_{n\in\mathbb{N}^*}$ such that $R_{\Delta^2} u_n = \lambda_n u_n$. Since $R_{\Delta^2} u_n$ is in $H_0^2(\Omega)$, we actually have $u_n \in H_0^2(\Omega)$. Setting $\Gamma_n := 1/\lambda_n$, these numbers are the eigenvalues of the bilaplacian, and u_n are the corresponding eigenfunctions. The Spectral Theorem also shows that $\Gamma_n \to \infty$ as $n \to \infty$. Finally, by (1.2) we have $\Gamma_n > 0$ for all $n \in \mathbb{N}^*$. To sum up:

Lemma 1.1. *If Ω is of finite measure, the bilaplacian with clamped boundary conditions over Ω admits a sequence of eigenvalues*

$$0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \to \infty.$$

This shows that the lowest – usually called principal – eigenvalue is well-defined. In the rest of the manuscript, unless otherwise mentioned, we will denote it

$$\Gamma(\Omega) := \Gamma_1$$

in order to emphasise its dependence on the underlying domain. From the previous discussion it is now clear that the functional space $H_0^2(\Omega)$ plays an important role in the understanding of the bilaplacian with clamped boundary conditions, so let us explore it a little bit further. Endowed with the scalar product $(v,w) \mapsto \int_\Omega uv + \int_\Omega D^2v : D^2w$, the space $H^2(\Omega)$ is a Hilbert space. Here, D^2v is the Hessian matrix of v and : denotes the Frobenius scalar product

$$D^2v : D^2w = \sum_{i,j\in\{1,\dots,d\}} \partial_i\partial_j v \partial_i\partial_j w.$$

By Poincaré's inequality, when restricted to $H_0^2(\Omega)$, the scalar product of $H^2(\Omega)$ is equivalent to $(v,w) \mapsto \int_\Omega D^2v : D^2w$. Moreover, we have the following relation for all $v, w \in H_0^2(\Omega)$,

$$\int_\Omega D^2w : D^2w = \int_\Omega \Delta v \Delta w, \tag{1.3}$$

which can be obtained after integration by parts since

$$\int_\Omega \partial_i\partial_j v \partial_i\partial_j w = \int_\Omega \partial_j v \partial_i\partial_i\partial_j w = \int_\Omega \partial_j\partial_j v \partial_i\partial_i w.$$

As a result, the initial scalar product shall and will be replaced by the simpler

$$\langle \cdot, \cdot \rangle_{H_0^2(\Omega)} : (v,w) \mapsto \int_\Omega \Delta v \Delta w.$$

Equation (1.3) is a very important relation that will be used extensively in this manuscript. It explains why the Poincaré inequality turned into (1.2). We now come back to considering eigenvalues of the bilaplacian. We would like to provide a tractable variational characterisation for Γ_n. In general, the min-max Theorem for compact operators applied to R_{Δ^2} yields

$$\Gamma_n = \min_{\substack{S_n \subseteq H_0^2(\Omega) \\ dim S_n = n}} \max_{\substack{v \in S_n \\ v \neq 0}} \frac{\int_\Omega (\Delta v)^2}{\int_\Omega v^2}. \tag{1.4}$$

We recall an elementary proof of this relation, at least for Γ_1, which is our main concern. Notice that we have the equivalence

$$\begin{aligned} & u \text{ is an eigenfunction associated with } \Gamma_n \\ \Leftrightarrow \quad & u \text{ is a critical point of } \mathcal{R} : v \in H_0^2(\Omega) \setminus \{0\} \mapsto \frac{\int_\Omega (\Delta v)^2}{\int_\Omega v^2} \text{ and } \mathcal{R}(u) = \Gamma_n. \end{aligned}$$

This is obtained by direct computation of the differential of $\mathcal{R}$. In particular, Γ_1 is the smallest critical value of $\mathcal{R}$, and since $\mathcal{R}$ admits a minimum, we obtain

Lemma 1.2. *If Ω is of finite measure, the first eigenvalue of the clamped bilaplacian is*

$$\Gamma(\Omega) = \min_{v \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (\Delta v)^2}{\int_\Omega v^2}. \tag{1.5}$$

Furthermore, any function u reaching this minimal value is an eigenfunction associated with $\Gamma(\Omega)$. In other words, (1.1) is the Euler-Lagrange equation associated with problem (1.5). Formulation (1.5) is essential for a correct understanding of the Rayleigh Conjecture.

Polylaplacian

In the above discussion, we focused on the bilaplacian, but we mentioned in paragraph 1.1.1 that Rayleigh was also interested in the eigenvalues of the Laplacian. As we will see, analogous results hold for this operator. More generally, we shall introduce the polylaplacian $(-\Delta)^m$ of order $m \in \mathbb{N}^*$ which allows for a uniform treatment. The boundary conditions generalising the fixed boundary conditions of the Laplacian and the clamped boundary conditions of the bilaplacian are called Dirichlet boundary conditions and read $u = \partial_n u = \cdots = \partial_n^{m-1} u$. Therefore, the eigenvalue problem for the Dirichlet m-polylaplacian is

$$\begin{cases} (-\Delta)^m u &= \Gamma u \quad \text{in } \Omega, \\ \partial_n^i u &= 0 \quad \text{on } \partial\Omega, \text{ for all } 0 \le i < m. \end{cases}$$

Here, the canonical functional space is now $H_0^m(\Omega)$, the closure of $C_c^\infty(\Omega)$ with respect to $H^m(\Omega)$. The scalar product over $H^m(\Omega)$ is given by $(v,w) \mapsto \int_\Omega vw + \int_\Omega D^m v : D^m w$. The notation : stands for the Frobenius product for tensors

$$D^m v : D^m w = \sum_{i \in \{1,\dots,d\}^m} \partial_i v \partial_i w$$

where for $i = (i_1, ..., i_m) \in \{1, ..., d\}^m$, we write $\partial_i v = \partial_{i_1} ... \partial_{i_m} v$. It may also be convenient to remark that, in multi-index notation,

$$D^m v : D^m w = \sum_{\substack{\gamma \in \mathbb{N}^d, \\ |\gamma| = m}} \frac{m!}{\gamma!} \partial^\gamma v \partial^\gamma w,$$

where for $\gamma = (\gamma_1, ..., \gamma_d) \in \mathbb{N}^d$, we write $\partial^\gamma v = \partial_1^{\gamma_1} ... \partial_d^{\gamma_d} v$, $\gamma! = \gamma_1! ... \gamma_d!$, and $|\gamma| = \gamma_1 + ... + \gamma_d$. This comes from the fact that for $\gamma \in \mathbb{N}^d$, counting how many $i \in \{1, .., d\}^m$ are such that, for each $1 \le k \le d$ the number of occurrences of k in the list $(i_1, ..., i_d)$ is γ_k, amounts to

$$\binom{m}{\gamma_1} \binom{m-\gamma_1}{\gamma_2} ... \binom{m-\gamma_1-...\gamma_{m-1}}{\gamma_d} = \frac{m!}{\gamma!}.$$

When restricted to $H_0^m(\Omega)$, by repeated applications of Poincaré's inequality, the scalar product of $H^m(\Omega)$ is equivalent to

$$\langle \cdot, \cdot \rangle_{H_0^m(\Omega)} : (v, w) \mapsto \int_\Omega D^m v : D^m w.$$

Moreover, in the same fashion as (1.3), we have for $u \in H_0^m(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$ the relation

$$\langle (-\Delta)^m u, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \int_\Omega D^m u : D^m \varphi. \tag{1.6}$$

This justifies to read the system

$$\begin{cases} (-\Delta)^m u &= f \quad \text{in } \Omega, \\ \partial_n^i u &= 0 \quad \text{on } \partial\Omega, \text{ for all } 0 \leq i < m, \end{cases}$$

with $f \in L^2(\Omega)$ in the way $u \in H_0^m(\Omega)$ and

$$\int_\Omega D^m u : D^m \varphi = \int_\Omega f\varphi, \quad \forall \varphi \in H_0^m(\Omega).$$

As for the bilaplacian, the previous system allows to define a resolvent operator $R_{(-\Delta)^m}$, which is compact over $L^2(\Omega)$ with range $H_0^m(\Omega)$. The Spectral Theorem still applies and yields a positive nondecreasing sequence of eigenvalues growing to infinity. The first one, that we still denote $\Gamma(\Omega)$, admits the following variational characterisation.

Lemma 1.3. *If Ω is of finite measure, the first eigenvalue of the Dirichlet polylaplacian of order $m \in \mathbb{N}^*$ is*

$$\Gamma(\Omega) = \min_{v \in H_0^m(\Omega)\setminus\{0\}} \frac{\int_\Omega |D^m v|^2}{\int_\Omega v^2}. \tag{1.7}$$

Here $|D^m v|^2 = D^m v : D^m v$. Moreover the functions realising the minimum are eigenfunctions associated with $\Gamma(\Omega)$. When $m = 2$, the previous formula coincides as expected with (1.5) by virtue of (1.3), whereas when $m = 1$ we obtain

$$\Gamma(\Omega) = \min_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2}.$$

This is well-known to be the variational characterisation of the first eigenvalue of the Laplacian with Dirichlet boundary conditions.

Terminology

In the previous discussion, we introduced several notions and several names to refer to it. Here is a short glossary listing interchangeable denominations. It will be valid for the whole manuscript.

— *bilaplacian, bi-Laplacian, bilaplace operator, biharmonic operator.*

— *Laplacian, Laplace operator.*

— *polylaplacian, polyharmonic operator.*

— *polylaplacian of order m, m-polylaplacian, polyharmonic operator of order m, m-polyharmonic operator.*

— [...] *with Dirichlet boundary conditions, Dirichlet* [...].

— [...] *with clamped boundary conditions, clamped* [...].

— [...] *with vanishing boundary conditions*, [...] *with fixed boundary conditions.*

— *principal, lowest, first* (eigenvalue, eigenfunction, etc...).

— *principal eigenvalue, pitch, fundamental frequency, fundamental tone.*

1.2. Basic properties

In this section, we introduce basic properties of eigenvalues and eigenfunctions of the polylaplacian of order $m \in \mathbb{N}^*$. First, thanks to (1.7), we remark that $\Gamma(\Omega)$ enjoys some homogeneity with respect to Ω. Indeed, for any $r > 0$ we have

$$\Gamma(r\Omega) = r^{-2m}\Gamma(\Omega). \tag{1.8}$$

In other words, dilating the domain decreases the pitch while contracting it increases the pitch. This is a particular case of the more general monotonicity property according to which

$$\Omega_1 \subseteq \Omega_2 \quad \Rightarrow \quad \Gamma(\Omega_2) \leq \Gamma(\Omega_1).$$

Those properties are crucial to understand the sensitivity of the principal eigenvalue with respect to the geometry of Ω. Actually, they hold for any eigenvalue, not only the first one. In the same fashion, the spectrum is invariant under isometries of Ω. Recall that isometries and dilations commute so, given some domain, the sets that one shall generate with these operations is limited. Lastly, we should mention that more technical manipulations also leave $\Gamma(\Omega)$ invariant. These involve the notion of capacity (we will explore this further in chapter 4). We will see that two domains coinciding up to a set of zero H^m-capacity share the same Sobolev space H_0^m, and hence yield the same eigenvalues.

Now we would like to highlight some trivial yet useful relation between the principal eigenvalue of polylaplacians of different order. Beforehand, we shall disclaim that polylaplacians are powers of each other in the sense of Spectral Theory. This is due to the fact that the boundary conditions are not compatible. As a result, it is not true in general that the eigenvalues of polylaplacians of different order are powers of each other. To illustrate this fact, consider an eigenfunction u associated with an eigenvalue λ of the Laplacian over a domain Ω. Then $(-\Delta)^m u = \lambda^m u$ for all $m \in \mathbb{N}^*$. However the boundary conditions

are $u = -\Delta u = ... = (-\Delta)^{m-1}u = 0$, which are often called Navier boundary conditions and do not coincide with Dirichlet boundary conditions. Indeed, we do not have $u \in H_0^m(\Omega)$ in general. If Ω is not regular enough, it is not even true that $u \in H^m(\Omega)$. Yet, even if no direct relation exists, comparisons may be obtained thanks to the next lemma.

Lemma 1.4. *Let Ω be an open set and $u \in H_0^m(\Omega)$. Then,*

$$\left(\int_\Omega |\nabla u|^2\right)^m \leq \left(\int_\Omega u^2\right)^{m-1} \int_\Omega |D^m u|^2. \tag{1.9}$$

Proof. After an integration by parts and two Cauchy-Schwarz inequalities, we observe that for any $1 \leq k < m$,

$$\int_\Omega |D^k u|^2 = \sum_{i\in\{1,...,d\}^{k-1}} \int_\Omega \Delta\partial_i u \partial_i u \leq \left(\sum_{i\in\{1,...,d\}^{k-1}} \int_\Omega (\Delta\partial_i u)^2 \sum_{i\in\{1,...,d\}^{k-1}} \int_\Omega (\partial_i u)^2\right)^{1/2}.$$

Since, by (1.3), $\int_\Omega (\Delta\partial_i u)^2 = \int_\Omega |D^2\partial_i u|^2$ for all $i \in \{1, ..., d\}^{k-1}$, the previous reads

$$v_k^2 \leq v_{k-1}v_{k+1},$$

where $v_k = \int_\Omega |D^k u|^2$. As a consequence, we prove by induction on k that $(v_k)_k$ obeys the rule

$$v_{m-k}^{p_k} \leq v_{m-k-1}^{q_k} v_m, \qquad \forall 1 \leq k < m,$$

where $(p_k)_k$ and $(q_k)_k$ are defined by $(p_1, q_1) = (2, 1)$, and for all $k \geq 1$,

$$p_{k+1} = 2p_k - q_k, \qquad q_{k+1} = p_k.$$

The couple $(p_k, q_k)_{k\geq 1}$ forms a constant-recursive sequence of order 1, the solution of which is $(p_k, q_k) = (k + 1, k)$. Consequently,

$$v_1^m \leq v_0^{m-1} v_m,$$

which is precisely (1.9). □

Applying inequality (1.9) to a first eigenfunction u of the m-polylaplacian immediately provides the relation

$$\lambda(\Omega)^m \leq \Gamma(\Omega), \tag{1.10}$$

where $\lambda(\Omega)$ is an alternative notation for the first eigenvalue of the Laplacian with Dirichlet boundary conditions. To conclude, we would like to investigate the regularity of eigenfunctions. For the moment, we not only know that they are in $L^2(\Omega)$, but also in $H_0^m(\Omega)$. They even form an orthogonal family in $H_0^m(\Omega)$ since they are orthonormal in $L^2(\Omega)$. By bootstrap arguments and Sobolev injections, we may improve this in the case where Ω has some regularity.

Lemma 1.5. *Let Ω be a C^{2m} regular bounded open set. Then, the eigenfunctions of the Dirichlet m-polylaplacian are $W^{2m,p}(\Omega)$ for all $1 \leq p < \infty$ and $C^{2m-1,\alpha}(\overline{\Omega})$ for all $0 < \alpha < 1$.*

Proof. Let u be an eigenfunction. Since Ω is regular enough, by elliptic regularity ([GGS10, Theorem 2.20]) we know that u is in $H^{2m}(\Omega) = W^{2m,2}(\Omega)$. Let us assume that $u \in W^{2m,p}(\Omega)$ for some $p \geq 2$. By Sobolev embeddings ([GGS10, Theorem 2.4]) $u \in L^q(\Omega)$ for all $1 < q < p^*$, where $p^* = +\infty$ if $d/p \leq 2m$, and $p^* = dp/(d-2mp)$ otherwise. Then, because Ω is C^{2m}, elliptic regularity yields $u \in W^{2m,q}(\Omega)$. This suggests to define the sequence $(p_n)_n$ by $p_0 := 2$, and

$$p_{n+1} := p_n^* = \begin{cases} +\infty & \text{if } 2mp_n \geq d, \\ dp_n/(d-2mp_n) & \text{if } 2mp_n < d. \end{cases}$$

By induction, the previous discussion shows that for all $n \in \mathbb{N}$ and all $1 \leq q < p_n$, $u \in W^{2m,q}(\Omega)$. But the sequence $(p_n)_n$ is nondecreasing, hence converges in $[2, +\infty]$. If the limit p were finite, it would satisfy $p = dp/(d-2mp)$, which admits no solution in $[2, +\infty)$, hence $p = +\infty$.

In particular, $u \in W^{2m,p}(\Omega)$ for all $1 \leq p < \infty$. Since, for all $0 < \alpha < 1$, there exists $1 \leq p < \infty$ such that $2m - \frac{d}{p} \geq 2m - 1 + \alpha$, we obtain that $u \in C^{2m-1,\alpha}(\overline{\Omega})$. □

For the particular case of the bilaplacian, the previous result tells that whenever Ω is C^4 regular, the eigenfunctions are $W^{4,p}(\Omega)$ for all $1 \leq p < \infty$ and in $C^{3,\alpha}(\overline{\Omega})$. This improved regularity will be used extensively in the manuscript. Regularity issues over less smooth domains are delicate questions, and this will be the source of troubles. In particular, it is unclear whether the eigenfunctions of the m-polylaplacian are bounded on irregular domains. We can only answer in the case $m = 1$. Indeed, any eigenfunction of the Dirichlet Laplacian over a bounded open set is bounded by [Dav90, Example 2.1.8].

The previous discussion concerned global regularity. Lastly, let us recall that when it comes to talk about local interior regularity, the eigenfunctions are known to be very smooth (see [DL88, chapter V, section 4.2]).

Lemma 1.6. *Let Ω be an open set of finite measure. Any eigenfunction of the m-polylaplacian over Ω is real analytic in Ω.*

1.3. The Rayleigh Conjecture, pitfalls and state-of-the-art

It is now time to state and discuss the Rayleigh Conjecture to be addressed in this manuscript. We focus on the biharmonic operator, and $\Gamma(\Omega)$ denotes the first eigenvalue of the Dirichlet bilaplacian over a domain Ω. Furthermore, $|\Omega|$ stands for the Lebesgue measure of Ω.

Conjecture (Rayleigh's Conjecture for clamped plates)**.** *Let $d \in \mathbb{N}^*$. Then, for any bounded open set $\Omega \subseteq \mathbb{R}^d$,*

$$\Gamma(\Omega) \geq \Gamma(B), \tag{1.11}$$

where B is a Euclidean ball such that $|B| = |\Omega|$. Moreover, in case of equality, $\Omega = B$ up to isometries and to sets of zero H^2-capacity.

Since $\Gamma(\Omega)$ is a mathematical model for the pitch of a clamped plate Ω, (1.11) is a reformulation of Rayleigh's question, except that we stated it for any dimension $d \in \mathbb{N}^*$ whereas the original concern was for $d = 2$. But Rayleigh also conjectured a similar result for membranes. In this case the conjecture corresponds to the inequality

$$\lambda(\Omega) \geq \lambda(B), \tag{1.12}$$

whenever $|\Omega| = |B|$. We recall that $\lambda(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian. Since the use of the Laplacian is widespread in Mathematics and Physics, this last inequality is probably more celebrated than (1.11). The full proof of (1.12) has been obtained by Faber and Krahn independently a century ago [Fab23; Kra25]. It relies on a now standard technique of symmetrization. Consequently, (1.12) is commonly referred to as the Faber-Krahn inequality. Nowadays, a wide variety of proofs of (1.12) are known as we shall see in section 2.6.

Due to the fact that (1.12) and (1.11) are of same nature – the only change being the order of the operator involved –, we will also refer to (1.11) as a Faber-Krahn inequality. More generally, any sharp lower bound on a shape functional will be called a Faber-Krahn or Faber-Krahn type inequality in this manuscript. Unlike for the Laplacian, the Faber-Krahn inequality for the bilaplacian in the form stated above is still unproved in full generality. We will understand in this manuscript why the case of plates is more challenging.

Before going further, we would like to mention that another way to express Rayleigh's Conjecture for plates is to see it as a shape optimization problem. Indeed, solving the conjecture boils down to minimizing the shape functional $\Gamma : \Omega \mapsto \Gamma(\Omega)$ under volume constraint. In other words, introducing

$$\min\{\Gamma(\Omega) : \Omega \subseteq \mathbb{R}^d \text{ bounded open set}, |\Omega| = c\}, \tag{1.13}$$

Rayleigh's Conjecture claims that Ω solves (1.13) if and only if it is an admissible ball up to sets of null H^2-capacity. At this point, we shall wonder why Rayleigh focused on minimization rather than on maximization issues. Actually, it turns out that the problem of maximizing Γ under volume constraint is ill-posed. This was already the case for $\lambda : \Omega \mapsto \lambda(\Omega)$. Indeed, for a hyperrectangle $R = (0, l_1) \times \cdots \times (0, l_d)$ we have

$$\lambda(R) = \sum_{i=1}^{d} \frac{\pi^2}{l_i^2}.$$

Therefore as long as one of the dimensions of R goes to 0, $\lambda(R)$ goes to infinity. For the bilaplacian however, $\Gamma(R)$ is not known explicitly. Worse, $\Gamma(\Omega)$ is known for no domain Ω except radially symmetric ones (see Proposition 5.15 in this respect). Fortunately, thanks to relation (1.10), it follows easily that $\Gamma(R)$ also diverges when R shrinks. As a last word on (1.13), observe that due to the homogeneity (1.8), this constrained problem is equivalent to the unconstrained one

$$\min\{\Gamma(\Omega)|\Omega|^{\frac{4}{d}} : \Omega \subseteq \mathbb{R}^d \text{ bounded open set}\}, \tag{1.14}$$

in the sense that any solution to (1.13) solves (1.14) and any solution to (1.14) rescaled for fitting the volume constraint solves (1.13). This remark will be used extensively in this manuscript.

All the material being now introduced, let us explain what have been attempted by mathematicians for addressing Rayleigh's Conjecture for clamped plates. First, we mention that the proof of Faber and Krahn for the Laplacian cannot be adapted to the case of the bilaplacian. This is because second-order derivatives are ill-behaved with respect to symmetrization (see section 3.3). Thus, after its publication in 1877, one of the first results regarding the conjecture goes back to Szegő who proved in [Sze50] that, as long as the first eigenfunction is of fixed sign, the Faber-Krahn type inequality (1.11) holds. Unfortunately – and it is one of the main challenges when dealing with higher order elliptic operators – it turns out that the bilaplacian does not enjoy the maximum principle for arbitrary domains. More precisely, in general it is not true to say that if v solves

$$\begin{cases} \Delta^2 v &= f \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \\ \partial_n v &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $f \geq 0$, then $v \geq 0$. In other words, the fundamental solution of the previous equation may change sign. Of course, this depends on the domain Ω. For balls, such a pathological situation does not occur, but even for apparently nice domains like ellipses this may happen. Comprehensive information on this issue may be found in [GGS10]. As a consequence of the lack of positivity preservation, one cannot apply the Krein-Rutman Theorem (see [Hen06, Theorem 1.2.6]). In particular, the simplicity of the first eigenvalue together with the one-sign property of the first eigenfunction is not guaranteed, and does fail for certain domains. This behaviour was observed for the first time in annuli with small inner radius in 1952, see [DS52; CDS79]. On the contrary, when the maximum principle is satisfied on a given domain, the Krein-Rutman Theorem applied to the resolvent operator with the set of nonnegative functions as the convex cone shows that some first eigenfunction must be nonnegative (moreover the first eigenvalue is simple whenever a strong version of the maximum principle is available). A similar argument is usually invoked to prove that, unlike the bilaplacian, the first eigenvalue of the Laplacian is simple and the first eigenfunction is positive on connected domains.

After the first partial result from Szegő, Mohr computed the optimality conditions of first and second order by making slight perturbations on a hypothetical optimal shape [Moh75]. As a result, he showed that the only possible planar regular solution to (1.13) with $d = 2$ is an admissible disk. Unfortunately, besides the assumption of existence of an optimal shape, the approach of Mohr relied on the fact that the ambient space is $\mathbb{R}^2$. Moreover, his result was finally outshined by a series of papers beginning with [Tal76], in which Talenti proved a nowadays famous comparison principle. Talenti's comparison principle has been used extensively in various ways and is still a central tool (see section 2.2 for more details). A refinement of the comparison principle led Talenti to exhibit from the variational formulation (1.5) an auxiliary problem, today called the "two-ball problem". The two-ball problem has the advantage to be posed, as its name suggests, on two coupled balls and to depend on the original shape only through the relative size of the two balls. Hence it is tangible. We will have the opportunity to study the two-ball problem and its generalisations in detail on several occasions (see section 3.2, Proposition 5.22, section 6.5). This finally allowed Talenti to give a lower bound on the optimal eigenvalue depending on the dimension in [Tal81]. It was the first time that a nontrivial and unconditional lower bound was found on Γ.

Following Talenti's work, Nadirashvili solved the two-ball problem and hence the Rayleigh Conjecture in $\mathbb{R}^2$ in [Nad95]. Thus, more than a century after Rayleigh's publication, the paper of Nadirashvili closed the question in its original statement. Subsequently, still in the wake of Talenti's approach, Ashbaugh and Benguria proved the conjecture in $\mathbb{R}^2$ and $\mathbb{R}^3$ in [AB95]. Their proof is radically different than that of Nadirashvili. It is based on very involved explicit computations (in the fashion of Appendix G). The two-ball problem was finally solved in any dimension in [AL96]. Whereas [AB95] showed that the two-ball problem is minimal when one of the two balls vanishes in dimension $d = 2, 3$, [AL96] concluded that, when $d \geq 4$, it reaches its minimum value when the two balls are of equal volume. As a consequence, Ashbaugh and Laugesen provided a very precise lower bound on the optimal eigenvalue. As a payback, however, they showed that the plain approach of Talenti could not answer the Rayleigh Conjecture when $d \geq 4$.

Since then, few progress has been performed. Let us quote the paper [CL16], addressing an analogue of the Rayleigh Conjecture in Gauss space. There, a lower bound on the first eigenvalue of the bi-Hermite operator is obtained among open sets of prescribed Gaussian measure. It is remarkable that this lower bound does not depend on the dimension. In return, the bound does depend on the volume constraint, and appears to be sharp only asymptotically for large volumes. Finally we mention the recent works [Kri20; Kri22], in which Kristály proves an analogue of the Rayleigh Conjecture in negatively and positively curved Riemanian manifolds. In the setting of negative curvature the dimensional restriction $d < 4$ is still needed to run Talenti's strategy. It is however striking that in positively curved spaces the conjecture is solved in any dimension, yet with a restriction on the volume constraint. Indeed, only large enough volumes are allowed. As a result, one may

not expect to prove the Rayleigh Conjecture in the Euclidean space by considering positively curved manifolds and making the curvature go to zero.

1.4. Related problems

In this section we would like to introduce several problems related to the Rayleigh Conjecture. We focus on problems that will take part in this manuscript. For instance, the problem of the buckling bilaplacian, although closely related to the clamped bilaplacian, will not be addressed here (see for instance [GGS10, section 3.2] and [AL96] for further details). First, we recall that for an open set Ω of finite volume, by density of $C_c^\infty(\Omega)$ in $H_0^2(\Omega)$, the principal eigenvalue of the bilaplacian over Ω is

$$\Gamma(\Omega) := \inf_{v \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega (\Delta v)^2}{\int_\Omega v^2}.$$

Then, one may wonder what occurs if we replace L^2 norms by L^p norms in the above quotient, that is if we consider for $1 \leq p < \infty$,

$$\Gamma_p(\Omega) = \inf_{v \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta v|^p}{\int_\Omega |v|^p}.$$

The first question is to know in which space the minimum is reached. For the moment, let us focus on the simpler case $p > 1$. In this situation, it appears that the appropriate functional space is $W_0^{2,p}(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ in $W^{2,p}(\Omega)$, the space of functions which are in $L^p(\Omega)$ together with their derivatives up to order 2. The assumption $1 < p < \infty$ provides the reflexivity of $W_0^{2,p}(\Omega)$. Furthermore, in case of a C^2 regular bounded domain, it indicates that any test function whose Laplacian is in $L^p(\Omega)$ has controlled norm in $W_0^{2,p}(\Omega)$ by elliptic regularity – observe that the regularity assumption on Ω can be removed when $p = 2$ thanks to (1.3). As a consequence, if $(v_n)_n$ is a minimizing sequence, it converges weakly in $W_0^{2,p}(\Omega)$, and by Rellich's Theorem strongly in $L^p(\Omega)$. Then the limit function, which is $W_0^{2,p}(\Omega)$, minimizes the corresponding quotient. In other words, for $1 < p < \infty$ and a C^2 bounded open set Ω,

$$\Gamma_p(\Omega) = \min_{v \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta v|^p}{\int_\Omega |v|^p}. \tag{1.15}$$

Furthermore, any minimizer u of (1.15) solves the equation

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) &= \Gamma_p(\Omega)|u|^{p-2}u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \\ \partial_n u &= 0 & \text{on } \partial\Omega. \end{cases}$$

As usual, this means that $u \in W_0^{2,p}(\Omega)$ verifies the first equation in a weak

sense. The differential operator defined by $\Delta_{2,p} v := \Delta\left(|\Delta v|^{p-2}\Delta v\right)$ is called the p-bilaplacian, and $\Gamma_p(\Omega)$ is usually referred to as the principal eigenvalue of the (clamped) p-bilaplacian. For $p = 1$, it is more involved to identify the correct functional space and the Euler-Lagrange equation associated with the quotient (see section 7.2).

As for the first eigenvalue of the bilaplacian, one may wonder which shape Ω minimizes $\Gamma_p(\Omega)$ under the constraint $|\Omega| = c$ for some fixed $c \in \mathbb{R}_+^*$. We conjecture that the optimal shape is a ball. This is not only supported by the Rayleigh Conjecture but also by the papers [PRT15] and [KP17]. Indeed, in the former the authors prove in dimension 2 that the ball minimizes Γ_1 under volume constraint as expected. In the later, Katzoukaris and Parini show that in the limit $p \to \infty$, the ball minimizes Γ_p under volume constraint, this time in any dimension. Up to our knowledge, nothing is known for intermediate $p \in (1, \infty)$ except $p = 2$.

Another problem related to the Rayleigh Conjecture is the biharmonic torsional rigidity. This generalises the standard torsional rigidity and is defined through

$$T(\Omega) := \min_{\substack{u \in H_0^2(\Omega) \\ u \neq 0}} \frac{\int_\Omega (\Delta u)^2}{\left(\int_\Omega u\right)^2}, \tag{1.16}$$

for an open set Ω of finite measure. As previously, due to Rellich's Theorem the minimum in (1.16) is attained for a function $u \in H_0^2(\Omega)$ solving

$$\begin{cases} \Delta^2 u &= T(\Omega) \int_\Omega u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \\ \partial_n u &= 0 & \text{on } \partial\Omega. \end{cases}$$

The right-hand member being constant in the first equation, we observe that $T(\Omega)$ is nothing else but the inverse of $\int_\Omega w_\Omega$, where w_Ω is the biharmonic torsion function, which is defined by

$$\begin{cases} \Delta^2 w_\Omega &= 1 & \text{in } \Omega, \\ w_\Omega &= 0 & \text{on } \partial\Omega, \\ \partial_n w_\Omega &= 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

The biharmonic torsion function is a generalisation of the classic torsion function, which plays an important role in the theory of second order elliptic equations. As an example, given a sequence of domains, the convergence of the torsion function yields the convergence of the resolvent operator associated with the Dirichlet Laplacian (see [Buc00]). In the case of fourth order equations, the lack of maximum principle prevents an analogous result from holding. It makes the biharmonic torsion function less useful. Yet the biharmonic torsion function will naturally come into play (see section 5.5). Before proceeding further, we would like to recall that the biharmonic torsion function enjoys some regularity. Indeed, the next lemma provides a high degree of interior regularity (see [DL88, chapter V, section 4.2]).

Lemma 1.7. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set and $m \in \mathbb{N}^*$. Then, if $w \in H^m(\Omega)$ satisfies $(-\Delta)^m w = 1$, w is real analytic in Ω.*

A last quantity related to the principal frequency of the bilaplacian is in some sense a combination of the first eigenvalue of the p-bilaplacian and of the biharmonic torsional rigidity. Indeed for $1 < p < \infty$, we call p-biharmonic torsional rigidity the value

$$T_p(\Omega) := \min_{\substack{u \in W_0^{2,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\Delta u|^p}{|\int_\Omega u|^p}. \tag{1.18}$$

As for (1.15), the above minimization problem is well-posed as soon as Ω is a C^2 regular bounded open set. The Euler-Lagrange equation associated with a minimizer is

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) &= T_p(\Omega)|\int_\Omega u|^{p-2}\int_\Omega u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \\ \partial_n u &= 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, as for the biharmonic torsional rigidity, $T_p(\Omega)$ shall be related to some p-biharmonic torsion function. Since this will not be used in the manuscript, we do not detail this point. As previously, the case $p = 1$ is more involved and the minimum in (1.18) may still not be reached.

1.5. Achievements of the thesis

Now I would like to present a selection of main results obtained during the thesis. These results answer issues of diverse natures that I came across in the study of the Rayleigh Conjecture. Indeed, when addressing any shape optimization problem, several questions naturally arise. The first, of course, is to determine what is the optimal shape. But sometimes, even if a conjecture guides the research, it is difficult to exhibit the optimal shape, or even to collect information on its geometry. In this case, one may try to estimate the optimal value of the functional being considered. Lastly, it is of interest to show that at least an optimal shape exists. Possibly, regularity results might then be obtained. To summarise, the first natural questions that I raised regarding the Rayleigh Conjecture for the clamped plate were the following:

1. What can be said about the geometry of an optimal shape?
2. Is it possible to estimate the optimal eigenvalue?
3. Does an optimal shape exist, and what are its regularity properties?

During the thesis, I mainly focused on the most challenging of the previous questions, which is undoubtedly the first one. Indeed, in chapter 5 and chapter 6, I provide simple but unfortunately implicit sufficient conditions for the conjecture to hold. To do so, I needed to compute the optimality conditions satisfied by a hypothetical

optimal shape. From these conditions, it is usually expected that the optimal shape shall be profiled, if not identified. The optimality conditions related to simple eigenvalues are obtained through shape derivation. However, the principal eigenvalue of the bilaplacian is in general multiple. Therefore, I had to prove simplicity for the optimal first eigenvalue, and this finally led me to the next result.

Theorem (Theorem 5.4)**.** *Let Ω be a C^4 optimal shape for (1.13). Then $\Gamma(\Omega)$ is simple. Moreover, if u denotes an associated L^2-normalized eigenfunction, Δu is constant equal to $\pm\alpha$ on each connected component of $\partial\Omega$, where*

$$\alpha := \sqrt{\frac{4\Gamma(\Omega)}{d|\Omega|}}.$$

This result encourages us to invoke the theory of overdetermined elliptic problems. In the case of the Laplacian, the famous result of Serrin [Ser71] combined with the positivity of the first eigenfunction allows one to deduce from the overdetermination that Ω is a ball. For fourth order operators however, the theory of overdetermined problems is much less developed (see nevertheless [Ben86; PS89; Dal90; Bar08]). Furthermore, the lack of positivity of u prevents us from applying Serrin type results.

Despite the previous issues, this theorem gives crucial information on the optimal eigenfunction. Given this information, I worked in chapter 5 on a new method of reduction of order which allowed me to conclude that if an eigenfunction u satisfies the overdetermined condition $\Delta u = \alpha$ on the boundary, then some explicit function $z \in H^1_0(\Omega)$ satisfies the "affine eigenvalue problem" of reduced order $\Delta z = \sqrt{\Gamma(\Omega)}z + \alpha$ in Ω. This reduction of the order allows for utilising tools available only for second order equations. In particular, thanks to symmetrization techniques, I was able to prove the following.

Theorem (Theorem 5.1)**.** *Let Ω be a C^4 optimal shape for (1.13) with connected boundary. Let u be an L^2 normalized eigenfunction associated with $\Gamma(\Omega)$ and u_B an L^2-normalized eigenfunction associated with $\Gamma(B)$, B being a ball of volume $|\Omega|$. Then,*

$$\left|\int_\Omega u\right| \geq \left|\int_B u_B\right|, \tag{1.19}$$

and Ω is a ball if and only if there is equality above.

With the previous result, a big step towards the resolution of the Rayleigh Conjecture would be to prove that the reverse of inequality (1.19) holds for an optimal shape. Therefore, the previous result paves the way for a new direction of research. Moreover, since inequality (1.19) concerns the mean value of an optimal eigenfunction, it provides a direct connection between the principal eigenvalue of the bilaplacian and the biharmonic torsional rigidity. This justified wondering about a Faber-Krahn type inequality for the biharmonic torsional rigidity.

Theorem (Theorem 5.18). *Let Ω be a bounded open set of $\mathbb{R}^d$ and B a ball such that $|\Omega| = |B|$. Then,*

$$T(\Omega) \geq T(B).$$

Moreover, in case of equality Ω must be a ball up to a set of zero H^1-capacity.

In chapter 6, I followed a different approach that consisted in a fine understanding of the structure of an eigenfunction satisfying the overdetermined condition of the first theorem. Indeed, using the condition $\Delta u = constant$ on each connected component of $\partial\Omega$, I was able to prove that one of the nodal domains $\{u > 0\}$ and $\{u < 0\}$ must have a hole containing the other one. With this new information in hand, I adapted the approach of Talenti and Ashbaugh-Benguria. That is, I refined Talenti's comparison principle for holed domains, and then derived an *asymmetric* two-ball problem. I finally obtained the following result.

Theorem (Theorem 6.1). *Let Ω be a C^4 optimal shape for (1.13) with $4 \leq d \leq 9$ and assume that an eigenfunction associated with $\Gamma(\Omega)$ has no critical value (in Ω) except its global minimum and global maximum. Then, Ω is a ball.*

The proof of the previous theorem, although sometimes technical, is quite nice in my opinion. On the contrary, the result in itself appears less useful than Theorem 5.1 since it requires a very fine knowledge of the optimal eigenfunction.

Besides the sufficient conditions obtained above for the Rayleigh Conjecture to hold, I tried to derive alternative characterisations for the principal eigenvalue. Indeed, working with formulas different than classic Rayleigh quotients could possibly lead to new results. In chapter 8, based on operator theory and on Trotter's formula for semigroups, I proved a new integral formulation generalising an already known one for the Laplacian.

Theorem (Theorem 8.1). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz open set. Let $m \in \mathbb{N}^*$ and p_s be the fundamental solution of the m-polyharmonic heat equation for $s > 0$ (see equation (8.1)). Then, the first eigenvalue of the Dirichlet m-polylaplacian satisfies*

$$\Gamma(\Omega) = -\lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{t}\log\int_\Omega\dots\int_\Omega p_{\frac{t}{n}}(x_1 - x_2)\dots p_{\frac{t}{n}}(x_{n-1} - x_n)dx_1\dots dx_n.$$

Unfortunately, I was unable to deduce from this new formula any interesting result regarding $\Gamma(\Omega)$. This is mainly due to the fact that for $m > 1$, the kernel p_s changes sign.

Since a complete proof of the Rayleigh Conjecture seemed at this point almost out of range, I made do with the second main question, which is to estimate the optimal eigenvalue. In practice this consists in obtaining lower bounds on the first eigenvalue and providing what we call approximate Faber-Krahn inequalities. An approximate Faber-Krahn inequality is an inequality of the form $\Gamma(\Omega) \geq c\Gamma(B)$ with a constant $0 < c \leq 1$ for all bounded open sets Ω of volume $|B|$. From

this point of view the question is similar to addressing the Rayleigh Conjecture, for which we just look for an approximate Faber-Krahn inequality with $c = 1$. Therefore it is hard to get good approximate Faber-Krahn inequalities, and the lower bounds that I obtained (see section 5.5, section 7.3) were not very satisfactory, since they did not improve the state-of-the-art contained in [AL96]. Yet, with the purpose of improving the known lower bounds on the first eigenvalue of the bilaplacian, I worked in chapter 7 on a new version of Talenti's comparison principle for signed functions. Surprisingly, the main consequence was a lower bound on the p-biharmonic torsional rigidity in dimension 2.

Theorem (Theorem 7.5 and Proposition 7.8)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set of class C^2 and B a ball of volume $|\Omega|$. Assume that $d = 2$. Then, for $1 \leq p < \infty$,*

$$T_p(\Omega) \geq \Gamma_p(B)|B|^{1-p}.$$

In order to conclude the study of the Rayleigh Conjecture, I investigated the existence of an optimal shape for (1.13) and more generally for the problem of minimizing the first eigenvalue of Dirichlet polyharmonic operators. Thanks to a concentration-compactness principle, I proved the existence of an optimal shape in the generalised class of *quasi-open* sets. Then performing an analysis of regularity I was able to show that a plain open optimal shape exists up to dimension 8 in the case of the bilaplacian.

Theorem (Theorem 4.1)**.** *Let $2 \leq d \leq 8$. There exists an open set minimizing the first eigenvalue of the Dirichlet bilaplacian under volume constraint. Moreover, any principal eigenfunction over this set extended by zero in $\mathbb{R}^d$ is $C^{1,\alpha}$ for all $0 < \alpha < 1$.*

Content of the manuscript

In the remaining of this manuscript, we would like to explore in detail the above questions and to prove the previous selected theorems. We will also investigate related topics and derive ancillary results. More precisely,

— in chapter 2, we study several specific tools that have been used during the thesis. In particular, we present symmetrization techniques (section 2.1), Talenti's comparison principle (section 2.2), shape derivatives (section 2.3), concentration-compactness principle (section 2.4), and Brascamp-Lieb-Luttinger inequalities (section 2.5). Then, as an application, we provide in section 2.6 six different proofs of the classic Faber-Krahn inequality for the Laplacian.

— in chapter 3, we analyse basic facts related to the Rayleigh Conjecture. We first explain in section 3.1 why the proofs of the Faber-Krahn inequality for the Laplacian do not work in the case of the bilaplacian. In section 3.2 we detail the main steps of the proof of the Rayleigh Conjecture in dimensions 2 and 3. Lastly we focus on one-dimensional second-order Pólya-Szegő inequalities

(section 3.3). In particular, even if general inequalities of this type cannot be expected, we show that a nontrivial Pólya-Szegő type inequality should exist for functions with no critical value of intermediate level.

— in chapter 4, we study the existence of an open optimal shape for the first eigenvalue of polyharmonic operators. For that purpose, we explore the notion of capacity and quasi-open sets in section 4.1. Then, we state in section 4.2 a higher-order concentration-compactness principle from which we prove the existence of a quasi-open optimal shape. We finally focus on regularity issues in section 4.3. We briefly introduce the theory of quasi-minimizers and prove that an open optimal shape must exist in low dimension. Further regularity questions are discussed in section 4.4.

— in chapter 5 we provide a first sufficient condition for the Rayleigh Conjecture to hold. Indeed, we show that if the mean value of a hypothetical optimal eigenfunction is in some sense minimal, then the corresponding optimal shape is a ball. We also show that the same result holds if instead the normal derivative of the Laplacian of the optimal eigenfunction is constant. For that purpose, in section 5.1, we develop the order reduction principle on which the strategy relies. To implement the order reduction principle, we prove that the first eigenvalue over an optimal domain is simple and we derive the corresponding optimality conditions in section 5.2. Then, we show that the sufficient conditions mentioned above yield the Rayleigh Conjecture (section 5.3). In section 5.4, we deduce some consequences from the previous main results. First we estimate the size of the nodal domains of an optimal eigenfunction. We also study more in depth the criterion regarding the mean value. Lastly, in section 5.5 we explore relationships between the first eigenvalue of a clamped plate and the biharmonic torsional rigidity. In particular we derive a Faber-Krahn inequality for the biharmonic torsional rigidity.

— in chapter 6 we provide another sufficient condition implying the Rayleigh Conjecture. Instead of the order reduction principle, the central tool of chapter 6 is a refined version of Talenti's comparison principle for holed domains. The refinement yet requires a new assumption, which is that the eigenfunction under consideration has no critical value except its global minimum and global maximum. To be more precise, in section 6.1 we observe that any optimal eigenfunction, if it exists, does not change sign near the boundary. As a result, we obtain in section 6.2 that one of the nodal domains needs to have a hole containing the other nodal domain. Section 6.3 is devoted to stating and proving the refined version of Talenti's comparison principle on domains with holes. Combining the previous results, we derive in section 6.4 an asymmetric two-ball problem that we solve in section 6.5. This allows to conclude in section 6.6 that the Rayleigh Conjecture holds (under our additional assumptions) at least for $4 \leq d \leq 9$. We also take the time to

discuss the assumption of absence of intermediate critical value.

— in chapter 7 we address another generalisation of Talenti's comparison principle. The purpose is to tackle the case of signed functions. This is performed in section 7.1. The new comparison principle is employed with the aim of obtaining spectral inequalities. It turns out that the comparison principle is mainly useful in dimension 2. In this setting, we apply it to the p-biharmonic torsional rigidity and obtain approximate Faber-Krahn inequalities. This is performed in section 7.2. As a conclusion, in section 7.3, we propose a very elementary way to obtain an approximate Faber-Krahn inequality of different nature for the regular bilaplacian in any dimension.

— in chapter 8 we provide a new formula for the first eigenvalue of the polylaplacian. The proof provided in section 8.1 illustrates the underlying links between the first eigenvalue of the polylaplacian and the polyharmonic heat equation, both in the domain under consideration and in the whole space. As a first attempt to address the Rayleigh Conjecture from this new perspective, we tackle in section 8.2 a Hardy-Littlewood type inequality for sign-changing functions.

2. The tools and their applications

This chapter is devoted to a brief presentation of the main tools used in this thesis, with a focus on techniques encountered in the context of shape optimization. In particular, the standard machinery of Partial Differential Equations (such as Sobolev embeddings, maximum principles, Fourier transform, spectral theory, etc...) will not be detailed. Once equipped with those tools, we will present several proofs of the classic Faber-Krahn inequality for the Laplacian with Dirichlet boundary conditions. This inequality is the archetype of a spectral shape optimization problem.

In section 2.1, we will introduce symmetrization, which is perhaps the most standard method for solving shape optimization problems. We will only address Schwarz symmetrization and try to highlight some interesting features in the case of signed functions. Additionally, we will explore an interesting connection between symmetrization and optimal transport. Section 2.2 presents Talenti's comparison principle, which is a central property of symmetrization. We will evidence that the traditional comparison principle is weaker than the one involving signed functions. Next, the notion of shape derivatives for simple and multiple eigenvalues will be briefly explained in section 2.3. The general setting of polyharmonic operators will be under consideration. Section 2.4 is dedicated to the concentration-compactness framework, which shall be utilised for proving the existence of optimal shapes. For the moment, we will only address the H^1 case. Lastly, the Brascamp-Lieb-Luttinger inequality will be recalled in section 2.5. With these tools, we will be able to present six proofs of the classic Faber-Krahn inequality.

2.1. Symmetrization

Symmetrizations – also called more explicitly level set rearrangements – recover several techniques allowing to build from a given (most of the time real-valued) function a symmetrized function whose level sets have same volume but are "more symmetric" than those of the initial function. As we shall see, such a symmetrized function will then be equimeasurable with the former function, and is expected to enjoy better properties. Many different types of level set rearrangements exist (see [Kaw85]), but we will focus on Schwarz symmetrization. At this point, it has to be noted that, in the case of signed functions, there is no consensus on the definition of Schwarz symmetrization. In this thesis, we will follow the framework of [Kes06], which allows to preserve the property of equimeasurability.

Several equivalent definitions of Schwarz symmetrization shall be used. For instance, an intuitive one is given in [Kaw85]. Here, we present a definition which might appear less intuitive but which will be easier to handle in proofs and computations.

Definition 2.1. *Let $\omega \subseteq \mathbb{R}^d$ be a bounded open set. Let $f : \omega \to \mathbb{R}$ be measurable.*

1. *The Schwarz symmetrization A^* of a measurable set $A \subseteq \mathbb{R}^d$ is the open ball of volume $|A|$ and centre 0.*
2. *The distribution function of f is the function $\mu_f : t \in \mathbb{R} \mapsto |\{f > t\}| \in \mathbb{R}_+$.*
3. *The nonincreasing rearrangement of f is the generalised inverse function of μ_f, which is the function $f^\star : [0, |\omega|) \to \mathbb{R} \cup \{\infty\}$ defined by*

$$\begin{cases} f^\star(s) := \mu_f^{[-1]}(s) = \inf\{\mu_f < s\} = \inf\{t : |\{f > t\}| < s\} & \forall s \in (0, |\omega|), \\ f^\star(0) := \operatorname{ess\,sup} f. \end{cases}$$

4. *The Schwarz symmetrization of f is the measurable function $f^* : \omega^* \to \mathbb{R} \cup \{\infty\}$ such that for all $x \in \omega^*$,*

$$f^*(x) := f^\star(|B_{|x|}|),$$

where B_r denotes the ball of radius r and of same centre as 0.

Remark 2.2. *Our definition of Schwarz symmetrization follows the framework of [Kes06]. In many references, the Schwarz symmetrization of some function f actually corresponds to $|f|^*$. The advantage of our choice is that, even when f changes sign, f and f^* remain equimeasurable (see Lemma 2.4).*

The next lemma collects some basic properties of the objects defined above.

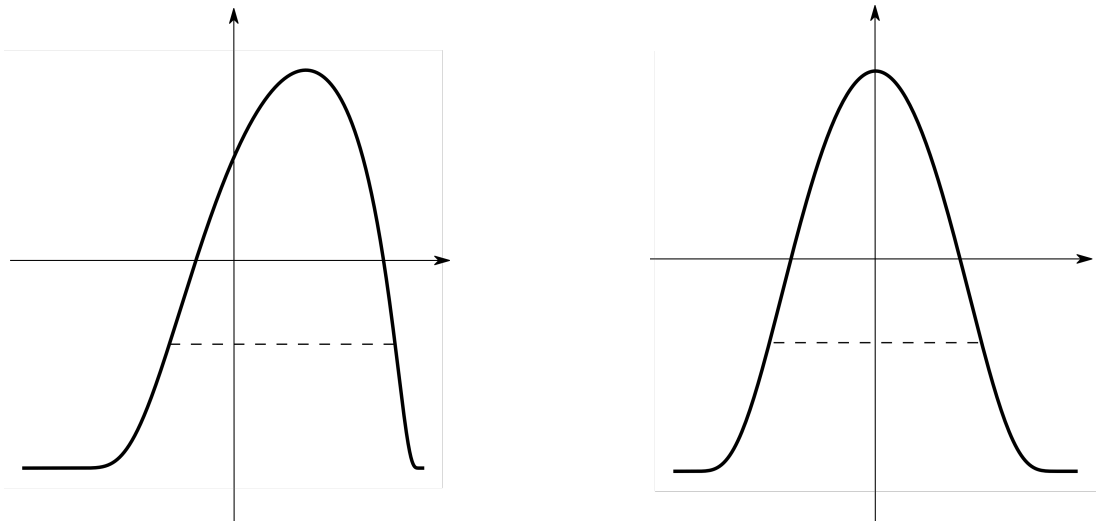
Lemma 2.3. *Let ω be a bounded open set and $f, g : \omega \to \mathbb{R}$ be measurable.*

1. *The nonincreasing rearrangement of f is left-continuous, whereas the distribution function μ_f is right-continuous.*
2. *For all $s \in [0, |\omega|)$ and $t \in \mathbb{R}$, $\mu_f \circ f^\star(s) \leq s$ and $f^\star \circ \mu_f(t) \geq t$.*
3. *If $h, k : [0, |\omega|) \to \mathbb{R}$ are nonincreasing, then $h \leq k$ a.e. if and only if $\mu_h \leq \mu_k$.*
4. *If $f \leq g$ then $f^* \leq g^*$.*

Roughly speaking, the Schwarz symmetrization consists in replacing the level sets of a function f with balls of same volume, as shown in Figure 2.1. As a result, f and f^* have same distribution function, and then are equimeasurable.

Lemma 2.4. *Let ω be a bounded open set and $f : \omega \to \mathbb{R}$ be measurable. Let also $G : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $G \circ f \in L^1(\omega)$. Then,*

$$\int_\omega G \circ f = \int_{\omega^*} G \circ f^* = \int_0^{|\omega|} G \circ f^\star.$$



(a) f (b) f^*

Figure 2.1. – A function f and its Schwarz symmetrization f^*. The two dashed lines represent the upper level sets $\{f > t\}$ and $\{f^* > t\}$ for some $t \in \mathbb{R}$. They have same length.

Remark 2.5. *An immediate consequence of this lemma is that* $\{f^* > t\} = \{f > t\}^*$ *a.e. for any* $t \in \mathbb{R}$*, since those two sets are balls of same volume centred at* 0*.*

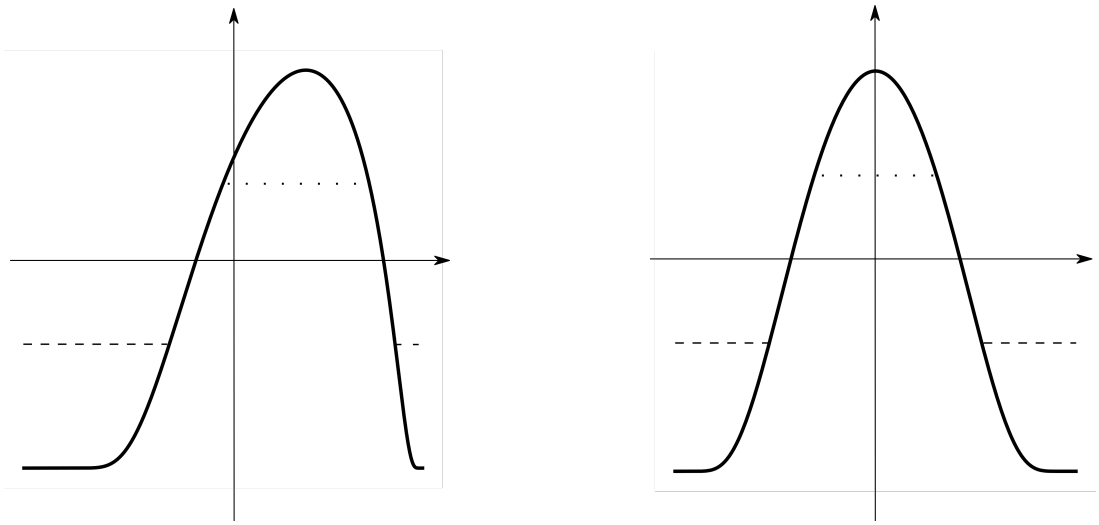
Proof. The maps $A \mapsto \int_\omega \mathbb{1}_A \circ f$ and $A \mapsto \int_\omega \mathbb{1}_A \circ f^*$ define two measures over Borel sets. Since f and f^* have same distribution function (see [Kes06, Proposition 1.1.3]), the two measures coincide on sets of the form $]t, \infty[, t \in \mathbb{R}$ which are stable by intersection and generate all Borel sets. By Sierpiński-Dynkin uniqueness Theorem, the two measures are the same, which means that the identity holds whenever G is an indicator function. If G is nonnegative, it is the limit of an increasing sequence of simple functions, hence the identity also holds. If G is just Borel, consider its positive and negative parts to conclude. The last identity comes after integrating $G \circ f^*$ over spherical sets. □

At this point, we shall mention that in the course of the thesis, we encountered the central paper [Tal81], in which Talenti introduced a special symmetrization for signed functions (see equation (2.1) and Figure 2.2). It is noteworthy that the definition of Talenti coincides with Schwarz symmetrization.

Lemma 2.6. *Let* ω *be a bounded open set and* $f : \omega \to \mathbb{R}$ *be measurable. Then, for almost every* $s \in [0, |\omega|)$*,*

$$f^\star(s) = f^\dagger(s) := f_+^\star(s) - f_-^\star(|\omega| - s). \tag{2.1}$$

Remark 2.7. *By* $f_+^\star$ *and* $f_-^\star$ *we mean* $(f_+)^\star$ *and* $(f_-)^\star$*.*



(a) f (b) $f^\ddagger$

Figure 2.2. – A sign-changing function f and the function $f^\ddagger : x \in \omega \mapsto f^\dagger(|B_{|x|}|)$ with $f^\dagger : [0, |\omega|) \to \mathbb{R}$ defined through equation (2.1). The dashed lines represent the upper level sets $\{f_- > t\}$ and $\{(f^\ddagger)_- > t\}$ for some $t \in \mathbb{R}$. The doted lines represent the upper level sets $\{f_+ > s\}$ and $\{(f^\ddagger)_+ > s\}$ for some $s \in \mathbb{R}$. The dashed lines and the dotted lines have respectively same length on the two figures.

Proof. We are trying to prove that $f^\star = f^\dagger$ almost everywhere over $[0, |\omega|)$. Since $f^\star$ and $f^\dagger$ are nonincreasing, it is enough to prove that their distribution functions $\mu^\star$ and $\mu^\dagger$ coincide (see point 3 of Lemma 2.3). So let's consider a level $t \in \mathbb{R}$. Then, $\mu^\star(t) = |\{f > t\}|$ by Lemma 2.4. For $\mu^\dagger$ it is more delicate. First, observe that for any $s \in (0, |\omega|)$, $f_+^\star(s)$ and $f_-^\star(|\omega| - s)$ cannot be simultaneously nonzero, since by definition and lower-semi continuity of the nonincreasing rearrangement

$$f_+^\star(s) \neq 0 \Rightarrow s < |\{f > 0\}|, \qquad f_-^\star(|\omega| - s) \neq 0 \Rightarrow s > |\omega| - |\{f < 0\}| = |\{f \geq 0\}|.$$

This results in the partition $[0, |\omega|) = \{f_+^\star > 0\} \cup \{f_-^\star(|\omega| - \cdot) > 0\} \cup \{f_+^\star = f_-^\star(|\omega| - \cdot) = 0\}$, which yields

$$|\{f^\dagger > t\}| = |\{f_+^\star > \max(t, 0)\}| + |\{-t > f_-^\star > 0\}| + |\{f_+^\star = 0\} \cap (|\omega| - \{f_-^\star = 0\})| \mathbb{1}_{t<0}.$$

By equimeasurability of Schwarz symmetrization, we immediately have

$$|\{f_+^\star > \max(t, 0)\}| = |\{f_+ > \max(t, 0)\}|, \qquad |\{-t > f_-^\star > 0\}| = |\{-t > f_- > 0\}|.$$

For the remaining term, observe that, by equimeasurability and by the fact that

$f_+^\star$ is nonincreasing, we have the a.e. equality

$$\{f_+^\star = 0\} = \{f_+^\star > 0\}^c = [|\{f_+ > 0\}|, |\omega|],$$

and analogously

$$|\omega| - \{f_-^\star = 0\} = |\omega| - [|\{f_- > 0\}|, |\omega|] = [0, |\{f_- = 0\}|].$$

Their intersection is $[|\{f_+ > 0\}|, |\{f_- = 0\}|] = [|\{f > 0\}|, |\{f \geq 0\}|]$, the measure of which is $|\{f = 0\}|$. Consequently,

$$\mu^\dagger(t) = |\{f > \max(t, 0)\}| + |\{0 > f > t\}| + |\{f = 0\}|\mathbb{1}_{t<0} = |\{f > t\}| = \mu^\star(t),$$

which concludes. □

Note that when $f \in L^\infty(\omega)$, there is a nicer way to prove Lemma 2.6 (see page 184). Besides equimeasurability, one of the main features of Schwarz symmetrization is the fact that it decreases any Lebesgue norm of the gradient. This is the famous Pólya-Szegő inequality.

Theorem 2.8. *Let ω be a bounded open set and $f \in W_0^{1,p}(\omega)$ be nonnegative with $1 \leq p < \infty$. Then,*

$$\int_{\omega^*} |\nabla f^*|^p \leq \int_\omega |\nabla f|^p.$$

Remark 2.9. *The assumption that f is constant on the boundary cannot be removed, as can be seen on the example $f(t) = t$ for $t \in (0, 1)$.*

Proof. Begin with $f \in C_c^\infty(\omega)$. By the coarea formula in appendix (Theorem A.1),

$$\mu_f(t) = \int_\omega \mathbb{1}_{\{u>t\}} = \int_t^\infty ds \int_{\{f=s\}} \frac{1}{|\nabla f|},$$

hence for almost every $t > 0$,

$$\mu_f'(t) = -\int_{\{f=t\}} \frac{1}{|\nabla f|}. \tag{2.2}$$

Then, the result will follow by applying sequentially the coarea formula, Hölder's inequality, and the isoperimetric inequality (see Appendix A). Indeed, for almost every $t > 0$, by Sard's Theorem $\nabla f \neq 0$ on $\{u = t\}$, hence

$$\int_\omega |\nabla f|^p \overset{\text{coarea}}{=} \int_0^\infty dt \int_{\{f=t\}} |\nabla f|^{p-1},$$

$$\int_{\{f=t\}} \frac{|\nabla f|^{1-1/p}}{|\nabla f|^{1-1/p}} \overset{\text{Hölder}}{\leq} \left(\int_{\{f=t\}} |\nabla f|^{p-1}\right)^{1/p} \left(\int_{\{f=t\}} \frac{1}{|\nabla f|}\right)^{1-1/p},$$

$$|\{f^* = t\}| \overset{\text{isop. ineq.}}{\leq} |\{f = t\}|.$$

Let us point out that for the isoperimetric inequality we used the positivity of f and the fact that $\nabla f \neq 0$ on $\{f = t\}$ for a.e. t to assert that $\{f = t\} = \partial\{f > t\}$. Moreover $\{f > t\}$ is regular due to the regularity of f. Lastly, Remark 2.5 was also invoked. Combining those inequalities, we get (recalling that $\mu_f = \mu_{f^*}$)

$$\int_\omega |\nabla f|^p \geq \int_0^\infty |\{f^* = t\}|^p |\mu'_{f^*}(t)|^{1-p} dt.$$

Observe that f^* being radially nonincreasing, it is almost everywhere differentiable, with radially symmetric gradient. Consequently, the previous Hölder and isoperimetric inequalities turn into equalities when applied to f^*, and we get

$$\int_\omega |\nabla f|^p \geq \int_0^\infty dt \int_{\{f^*=t\}} |\nabla f^*|^{p-1}.$$

The last integral shall be identified as the L^p norm of f^* after performing the change of variable $t = f^*(r)$. For a non-smooth function f, the result is obtained by approximation (see [Kes06, Theorem 2.3.1] for details). □

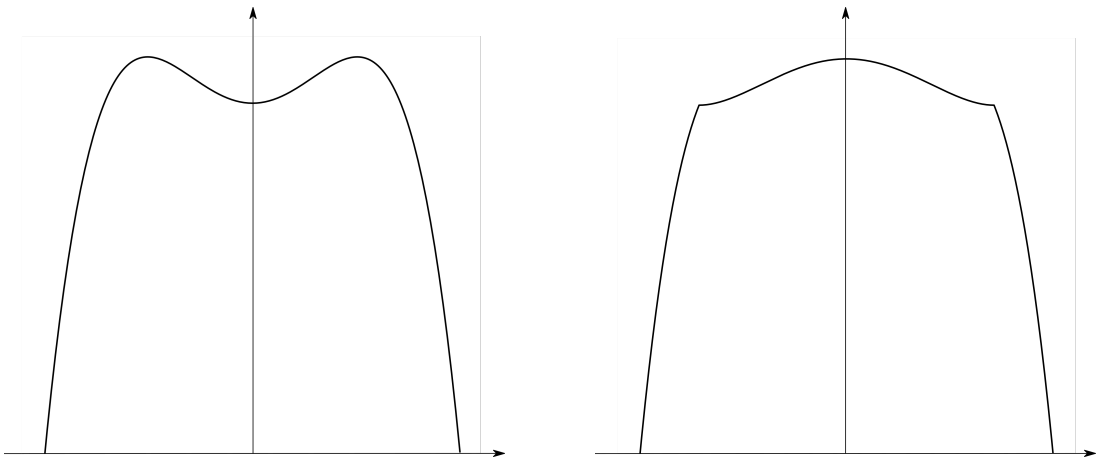
The previous central result shows that Schwarz rearrangement preserves first order regularity. A natural question (in particular when dealing with higher-order operators) is to ask whether, whenever one is given a more regular function f, more regularity can be deduced regarding f^*. In general, the answer is negative. Indeed, even the symmetrization of a C^∞ function might not be very regular. This is the case for instance when f admits a critical value which is not extremal. As an example, consider

$$f : x \in (-1, 1) \mapsto 1 + x^2 - 2x^4, \tag{2.3}$$

for which 0 is a critical point of value $1 < \sup f$ (see Figure 2.3). One can explicitly compute

$$\mu_f(t) = \begin{cases} \sqrt{1 + \sqrt{1 - 8(t-1)}} & \text{if } t < 1, \\ \sqrt{1 + \sqrt{1 - 8(t-1)}} - \sqrt{1 - \sqrt{1 - 8(t-1)}} & \text{if } t > 1, \end{cases}$$

and observe that $\mu'_f(1_-) = -\sqrt{2}$ whereas $\mu'_f(1_+) = -\infty$. From such a jump, one concludes that $f^{*\prime} = 1/[\mu'_f \circ f^*]$ is discontinuous and that there is no chance for $f^{*\prime\prime}$ to define an integrable function. Despite this counterexample, the literature contains some results about Pólya-Szegő type inequalities for second order derivatives. One of them is [Cia00], in which a striking loss of symmetry is evidenced in one dimension. Indeed, it is shown that the total variation of the second derivative of functions vanishing at both ends of a given interval is minimized by a function made of a linear part and a monotonous part, which in general does not coincide with the Schwarz symmetrization (see [Cia00, Theorem 2] for details). From our



(a) f (b) f^*

Figure 2.3. – The function f defined in equation (2.3) together with its Schwarz symmetrization.

counterexample (2.3) and from [Cia00], it seems desperate to wish for higher-order Pólya-Szegő inequalities in full generality. However, it has to be noted that what is responsible for the singularity of $f^{*\prime\prime}$ in (2.3) is the critical value 1. Moreover, Remark 3 and Theorem 3 in [Cia00] seem to indicate that critical values are the only thing that might prevent $f^{*\prime\prime}$ to be more regular. Surprisingly, this condition – the absence of critical values of intermediate level – will also come into play later on (see chapter 6).

Another interesting inequality apparently unrelated to Pólya-Szegő's (the appearance is deceiving as we shall see in paragraph 2.6.5) is the following Hardy-Littlewood inequality (see also [Kes06, Theorem 1.2.2]).

Proposition 2.10. *Let ω be a bounded open set and $f \in L^1(\omega)$. Then, for any measurable set $A \subseteq \omega$,*

$$\int_A f \leq \int_{A^*} f^*,$$

with equality if A is an upper level set of f.

Proof. Thanks to Lemma 2.4, it is enough to prove that

$$(f|_A)^* \leq f^*|_{A^*} \quad a.e., \tag{2.4}$$

with equality when A is an upper level set of f. Since both functions are radially nonincreasing, it is enough to prove a corresponding inequality for their distribution function, that is, for any $t \in \mathbb{R}$,

$$|\{f > t\} \cap A| \leq |\{f > t\}^* \cap A^*|.$$

But this comes from the fact that $(\{f > t\} \cap A)^* \subseteq \{f > t\}^*$ and $(\{f > t\} \cap A)^* \subseteq$

A^*. When A is an upper level set of f, the above inclusion is an equality, which concludes. □

We exposed the main features of Schwarz symmetrization that will be needed in this manuscript. For further properties and a comprehensive study of rearrangements, we refer to [Kaw85; Kes06]. Before closing this section, we would like to briefly discuss an unexpected connection between Schwarz symmetrization and optimal transportation. Even if it is known that optimal transport theory may be used to obtain isoperimetric inequalities (see [Bur09, chapter 5]), the direct link between symmetrization and optimal transport that we intend to highlight here seems to be new although elementary. It is based on the observation that the optimal transport between two one-dimensional probability measures can be computed explicitly, and corresponds a.e. to $\mu_2^{[-1]} \circ \mu_1$, where μ_2 is the distribution function of the target measure and μ_1 the distribution function of the atomless source measure (see Theorem 2.5 and Theorem 2.9 of [San15]). Thanks to this formula, we see that as long as the target measure is the image of the Lebesgue measure by a measurable function f and that the source measure has distribution $s \mapsto 1 - s$, the optimal transport is $s \in (0, 1) \mapsto f^\star(1 - s)$. To be more precise, we have the following result.

Proposition 2.11. *Let ω be a bounded open set such that $|\omega| = 1$ and $f : \omega \to \mathbb{R}$ be measurable. Let m_1 be the Lebesgue measure over $(0, |\omega|)$ and let m_2 be the pushforward of the Lebesgue measure over ω by f, that is*

$$m_2 := f_\# |\cdot|.$$

Then, a measurable function $T : (0, |\omega|) \to \mathbb{R}$ is an admissible transport map between m_1 and m_2 if and only if T and f are equimeasurable. Moreover, $s \in (0, |\omega|) \mapsto f^\star(|\omega| - s)$ coincides a.e. with the unique optimal transport between m_1 and m_2 for any strictly convex cost.

Proof. Observe that m_1 and m_2 define probability measures and that m_1 is atomless. Let $T : (0, |\omega|) \to \mathbb{R}$ be measurable. Since m_1 is defined over measurable sets and m_2 over Borel sets, T is compatible with the source and target measures. The function T is an admissible transport map if and only if $m_2 = T_\# m_1$, or in other words if and only if for any Borel function $G : \mathbb{R} \to \mathbb{R}$,

$$\int_0^{|\omega|} G \circ T = \int_{\mathbb{R}} G \circ T \, dm_1 = \int_{\mathbb{R}} G \, dm_2 = \int_\omega G \circ f,$$

which is exactly the property for T and f to be equimeasurable. For the second point, by [San15], Theorem 2.5 and Theorem 2.9, we know that there exists a unique optimal transport between m_1 and m_2, given by $\mu_2^{[-1]} \circ \mu_1$, where $\mu_2 : t \in \mathbb{R} \mapsto m_2((t, \infty))$ is the distribution function of m_2 and $\mu_1 : t \in (0, |\omega|) \mapsto m_1((t, \infty))$ is that of m_1. Note that there is a mismatch between our definition

of distribution function μ and the traditional definition which is used in [San15] and corresponds to $1-\mu$. However, since $(1-\mu)^{[-1]}(s) = \mu^{[-1]}(1-s)$, we have $(1-\mu_2)^{[-1]} \circ (1-\mu_1) = \mu_2^{[-1]} \circ \mu_1$. There is also a mismatch regarding the definition of the generalised inverse, but this is not an issue since all the generalised inverses agree except at points of discontinuity. Lastly, we compute $\mu_1(t) = |\omega| - t$ and $\mu_2(t) = \mu_f(t)$. Since $f^\star = \mu_f^{[-1]}$, this concludes. □

Such an interpretation of the Schwarz symmetrization as an optimal transport deepens the link between optimal transport theory and geometric inequalities, and might open perspectives. For instance, it would be interesting to see if a proof of the Pólya-Szegő inequality could be obtained through optimal transport. If such a proof exists, we were not able to find it. As a final word on this topic, let us mention that in most cases, geometric inequalities are obtained after applying Brenier's Theorem to measures which are normalized positive test functions (see [Bur09, Theorem 5.4]). The previous proposition suggests that another way to get such inequalities would be to apply Brenier's Theorem to the pushforward of the Lebesgue measure by the test functions involved.

2.2. Talenti's comparison principle

Talenti's comparison principle is a powerful result relating symmetrization and second-order elliptic equations. Although stemming from previous works, it appeared for the first time in [Tal76], and was dealing with Schwarz symmetrization. After this pioneer work, plenty of generalisations burgeoned for other types of equations and other types of rearrangements (as a sole example, see [ALT91]). However, we would like to stress that, once again, the definition of Schwarz symmetrization used in [Tal76] does not match ours. Consequently, slight differences will appear between the original formulation and the one we will state, following [Kes06, chapter 3]. Moreover, for the purpose of clarity, we will not state the result in the full generality of strongly elliptic second-order operators, and will restrict to the archetypal case of the Laplacian. This will be sufficient in the rest of the manuscript.

Theorem 2.12. *Let ω be a bounded open set and $f \in L^2(\omega)$. Consider the solution $u \in H_0^1(\omega)$ of*

$$\begin{cases} -\Delta u &= f & \text{in } \omega, \\ u &= 0 & \text{on } \partial\omega, \end{cases}$$

and the solution $v \in H_0^1(\omega^*)$ of

$$\begin{cases} -\Delta v &= f^* & \text{in } \omega^*, \\ v &= 0 & \text{on } \partial\omega^*. \end{cases}$$

Assume moreover that $u \geq 0$ a.e. in ω. Then, $u^* \leq v$ a.e. in ω^*.

Before proceeding to the proof of Theorem 2.12, let us make a couple of comments. First, we mention that in the original [Tal76], since a different Schwarz symmetrization is used, the function v is replaced with the function $w \in H_0^1(\omega^*)$ solving $-\Delta w = |f|^*$. As a counterpart, [Tal76] does not need to assume that $u \geq 0$. All things considered, it turns out that our statement is a little bit stronger than that given in [Tal76]. Indeed, letting

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \omega, \\ u & = & 0 & \text{on } \partial\omega, \end{array} \right. \qquad \left\{ \begin{array}{rcll} -\Delta w & = & |f|^* & \text{in } \omega^*, \\ w & = & 0 & \text{on } \partial\omega^*, \end{array} \right.$$

we would like to prove that $|u|^* \leq w$ a.e. in ω^* thanks to Theorem 2.12. For that purpose, introduce $z \in H_0^1(\omega)$ such that $-\Delta z = |f|$ in ω. By the maximum principle, we have $|u| \leq z$. As a result, z is nonnegative and thanks to Theorem 2.12 $z^* \leq v$ a.e. in ω^*. Since Schwarz symmetrization is nondecreasing (recall Lemma 2.3), we obtain as desired that

$$|u|^* \leq z^* \leq w. \tag{2.5}$$

As a second comment, let us observe that in [Tal76] a comparison between the L^q norms of the gradients of u and w defined above is claimed for $0 < q \leq 2$. We mention that in the case $q = 2$, a simple proof of this fact follows from (2.5) and from the Hardy-Littlewood inequality:

$$\int_\omega |\nabla u|^2 = \int_\omega uf \leq \int_\omega |u||f| \leq \int_{\omega^*} |u|^*|f|^* \leq \int_{\omega^*} w|f|^* = \int |\nabla w|^2.$$

The same proof does not work when u and v are as in Theorem 2.12, but a corresponding comparison between the gradients does hold in this setting (see [Kes06, Proposition 3.1.1]). It has to be noted that such inequalities are of interest when addressing problems related to the clamped plate as in [ABM16; ABM22]. By the way, let us mention that in those references, it seems that the traditional Schwarz symmetrization is used. Since this rearrangement does not preserve the integral of signed functions, it is not clear in [ABM22] how the compatibility condition between the normal derivatives of v and w below equation (6.13) is achieved. We will come back to this issue in more detail in section 3.2.

As a last remark, we would like to highlight that besides its applications in optimization which will be discussed later, Talenti's comparison principle has also been used (even by Talenti himself, see [Tal76, Theorem 2]) to get elliptic regularity results.

Proof of Theorem 2.12. We reproduce the proof proposed in [Tal76, section 3], which assumes some regularity on u. More precisely, we will need to apply the coarea formula, the Implicit Function Theorem, and Sard's Theorem to u, which requires $u \in C^d(\omega)$. For $0 < s < \sup u$, set $\omega_s := \{u > s\}$, and $\vec{n}_s$ be the outward normal unit vector to ω_s. Recall that ω_s is regular for almost every s by Sard's Theorem and by the Implicit Function Theorem. Therefore $\vec{n}_s$ is well defined and

moreover $\vec{n}_s = -|\nabla u|^{-1}\nabla u$. Then, the divergence formula gives

$$\int_{\omega_s} f = -\int_{\partial\omega_s} \partial_n u = \int_{\partial\omega_s} |\nabla u|.$$

Let also $\mu(s) = |\omega_s|$ be the distribution function of u, the derivative of which exists for almost every s and was computed in (2.2). Recall also that, as in the proof of Theorem 2.8, $\{u = s\} = \partial\omega_s$ for almost every $s > 0$ by positivity of u. Then, Hölder's inequality gives

$$\mathcal{H}^{d-1}(\partial\omega_s)^2 = \left(\int_{\partial\omega_s} \frac{\sqrt{|\nabla u|}}{\sqrt{|\nabla u|}}\right)^2 \leq -\mu'(s)\int_{\omega_s} f.$$

The isoperimetric inequality (Theorem A.2) applied to $A = \omega_s$, and Proposition 2.10 yield

$$1 \leq -\frac{\mu'(s)}{C_d^2 \mu(s)^{\frac{2d-2}{d}}}\int_{\omega_s^*} f^* = -\frac{\mu'(s)}{C_d^2 \mu(s)^{\frac{2d-2}{d}}}\int_0^{\mu(s)} f^\star. \tag{2.6}$$

After integrating between 0 and t, and performing the change of variable $r = \mu(s)$, we obtain

$$t \leq \int_{\mu(t)}^{|\omega|} \frac{1}{C_d^2 r^{\frac{2d-2}{d}}}\int_0^r f^\star.$$

Note that since μ may not be onto $[\mu(t), |\omega|)$, the positivity of the integrand (see (2.6)) is important to justify the above inequality. To conclude, apply the previous relation with $t = u^\star(s)$ and recall that $\mu(u^\star(s)) \leq s$ (see point 2 of Lemma 2.3). Since the integrand is still nonnegative, this leads to

$$u^\star(s) \leq \int_s^{|\omega|} \frac{1}{C_d^2 r^{\frac{2d-2}{d}}}\int_0^r f^\star.$$

Because v is radially symmetric, this is the desired inequality as can be seen from the fact that $\Delta v = r^{1-d}\partial_r[r^{d-1}\partial_r v]$ (see step 4 of the proof of Theorem 3.1.1 in [Kes06] for more details). □

In the above proof, observe that there is another way to conclude whenever the distribution functions of u and v are continuous. This occurs for instance when f does not vanish on a nonnegligible subset of ω (see Lemma 7.3). Indeed, if μ_u and μ_v are both continuous, then for any $r \in [0, |\omega|)$, $\mu_u(u^\star(r)) = r$ and $\mu_v(v^\star(r)) = r$. As a result, from (2.6), setting $s = u^\star(r)$ we get

$$1 \leq -\frac{\mu_u' \circ u^\star(r)}{C_d^2 r^{\frac{2d-2}{d}}}\int_0^r f^\star.$$

But all the inequalities involved until (2.6) turn into equalities as soon as u is

replaced with v and f with f^*. Consequently,

$$-\frac{\mu_v' \circ v^\star(r)}{C_d^2 r^{\frac{2d-2}{d}}} \int_0^r f^\star \leq -\frac{\mu_u' \circ u^\star(r)}{C_d^2 r^{\frac{2d-2}{d}}} \int_0^r f^\star,$$

or in other words

$$-\mu_v' \circ v^\star \leq -\mu_u' \circ u^\star.$$

From the formula of the derivative of the inverse, we obtain

$$-\frac{1}{v^{\star\prime}} \leq -\frac{1}{u^{\star\prime}},$$

that is $|u^\star| \leq |v^\star|$ which immediately gives $u^\star \leq v^\star$ over $[0, |\omega|)$. We did not found any mention of this alternative way to conclude in the literature, although the idea is classic. Indeed, it is standard to resort to differential inequalities for proving comparison principles (see for instance [LL16]).

Lastly, we would like to recall that a proof of Talenti's comparison principle based on the Pólya-Szegő inequality shall be found in [Lio81]. There, the traditional definition of Schwarz symmetrization is used, and we do not know if the proof could be generalised to Theorem 2.12. It has to be said – as mentioned in [Kes06, Remark 3.1.2] and as we shall see in section 2.6 – that the Pólya-Szegő inequality and, due to the previous remark, Talenti's comparison principle, might be obtained without using the isoperimetric inequality and the coarea formula.

2.3. Shape derivatives

In this section, we will present the concept of shape derivatives, with a focus on the case of simple and multiple eigenvalues of biharmonic operators. Most of the time, results will be presented without proof, but references will be given.

A shape derivative is the generalisation of the notion of derivative for functionals of the form $j : \omega \mapsto j(\omega)$, where ω belongs to some class of open sets of $\mathbb{R}^d$ and j takes real values. The main obstacle in the definition of such a derivative is that there is no canonical vectorial structure on shapes (see however [BC14; Bou17]). To overcome this issue, the most standard technique consists in considering instead of j the functional $J^\Omega : \theta \mapsto j((id + \theta)\Omega)$ where θ is a vector field over $\mathbb{R}^d$ with some regularity. Observe that $j(\Omega) = J(0)$, hence we may define the shape derivative of j at Ω to be the differential of J^Ω at 0 when it exists.

In practice, the shape derivability of j depends not only on Ω but also on the class of vector fields used. In the classic reference [HP05, chapter 5], the vector fields are taken in $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. For instance, as a consequence of [HP05, Theorem 5.2.2], we have the following result.

Proposition 2.13. *Let Ω be a bounded open set and*

$$\mathcal{V}^\Omega : \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \mapsto |(\mathrm{id} + \theta)\Omega| \in \mathbb{R} \tag{2.7}$$

be the volume functional. Then, $\mathcal{V}^\Omega$ is Gâteaux-differentiable and its derivative in the direction of $V \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is

$$\partial_V \mathcal{V}^\Omega(0) = \int_\Omega \mathrm{div}\ V. \tag{2.8}$$

Sometimes, as we shall see, more regularity will be demanded on the vector fields. In the following, we would like to address the differentiability of an eigenvalue of the Dirichlet polylaplacian of order $m \in \mathbb{N}^*$. It appears that the theory is less developed for this class of operators than for the Laplacian, for which we may refer to [HP05]. Note however that [HP05] does not deal in detail with the derivation of multiple eigenvalues. For a framework on the derivation of simple and multiple eigenvalues of a general abstract operator see [HR80; LL04]. For the concrete shape derivation of simple and multiple eigenvalues of polyharmonic operators, we came accross the references [BL13; BL15; OZ00; Buo16; ABF19]. Here, we will refer to [BL15], in which the perturbing vector fields are in $C^m(\mathbb{R}^d, \mathbb{R}^d)$. For this, we let $\Gamma_k(\omega)$ be the kth eigenvalue of $(-\Delta)^m$ with Dirichlet boundary conditions over ω, the eigenvalues being counted repeatedly according to their multiplicity. Then, we define, for a bounded open set Ω, the functional

$$\Gamma_k^\Omega : \theta \in C^m(\mathbb{R}^d, \mathbb{R}^d) \mapsto \Gamma_k((id + \theta)\Omega) \in \mathbb{R}. \tag{2.9}$$

Thanks to [BL15], we have the next result regarding the shape derivative of Γ_k.

Proposition 2.14. *Let Ω be a C^{2m} bounded open set and $k, p \in \mathbb{N}^*$. Assume that $\Gamma_k(\Omega) = \cdots = \Gamma_{k+p-1}(\Omega) =: \Gamma$ and that Γ is of multiplicity p. Then the functionals $\Gamma_k^\Omega, \dots, \Gamma_{k+p-1}^\Omega$ defined in (2.9) are Gâteaux-differentiable at 0 both on the left and on the right, and their partial derivatives in the direction of $V \in C^m(\mathbb{R}^d, \mathbb{R}^d)$ (both on the left and on the right) correspond in a bijective way to the eigenvalues (counted with multiplicity) of the matrix*

$$M_V := \left(-\int_{\partial\Omega} \partial_n^m u_{k+i-1} \partial_n^m u_{k+j-1} V \cdot \vec{n} \right)_{1 \le i,j \le p}, \tag{2.10}$$

where $u_k, \dots, u_{k+p-1}$ is any L^2-orthonormal basis of the eigenspace of Γ.

Proof. Apply [BL15, Theorem 3.2] with $k = m$ (see [BL15, equation (3.1)]), $\phi = \mathrm{id}$ and $\phi_\epsilon = \phi + \epsilon V$. Observe that ϕ_ϵ is C^m, bounded over Ω, as well as its derivatives, and satisfies $\det D\phi_\epsilon \neq 0$ for ϵ small enough, hence belongs to $\mathcal{A}_\Omega^k$. □

Let us comment the previous result. First, since M_V is real symmetric, it is diagonalizable, hence has p eigenvalues counted with multiplicity. Moreover, the

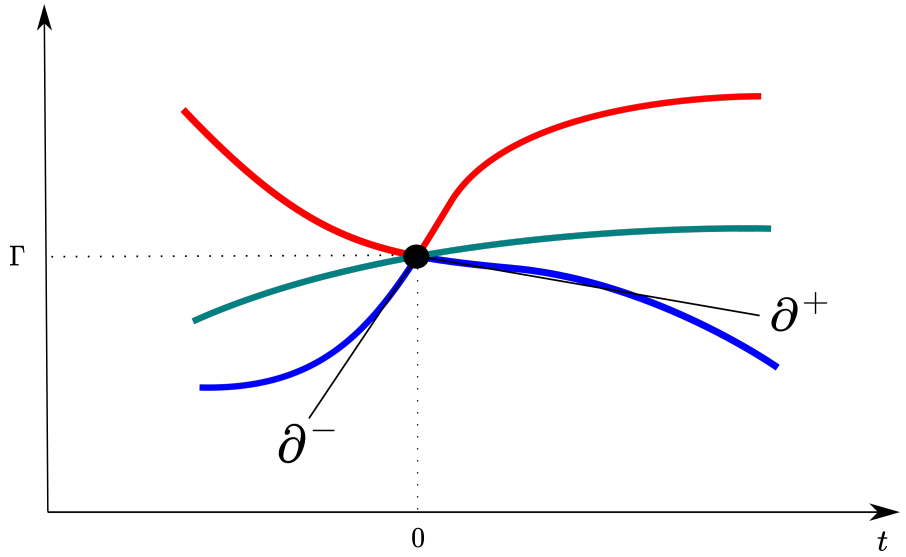


Figure 2.4. – Analytic branches near a multiple eigenvalue Γ on domains of the form $(\mathrm{id} + tV)\Omega$ for a given set Ω and vector field V. The blue, the green, and the red lines are respectively the graphs of $t \mapsto \Gamma_k^\Omega(tV)$, $t \mapsto \Gamma_{k+1}^\Omega(tV)$, and $t \mapsto \Gamma_{k+2}^\Omega(tV)$. The segment ∂^- represents the tangent generated by the left partial derivative of Γ_k^Ω at 0 in the direction of V. The segment ∂^+ represents the tangent generated by the right partial derivative of Γ_k^Ω.

spectrum of M_V does not depend upon the choice of the eigenfunctions $u_k,\dots,u_{k+p-1}$ since changing the basis corresponds to conjugating M_V with an orthogonal matrix.

Second, the crossing of eigenvalue branches (see Figure 2.4) prevents $\Gamma_k^\Omega, \dots, \Gamma_{k+p-1}^\Omega$ from being differentiable in general, even if they are both differentiable on the left and on the right. Indeed, their derivative on the left and on the right might not coincide since the bijection with the spectrum of M_V changes when one derives on the left or on the right. To overcome this lack of differentiability, some authors consider combinations of eigenvalues, called elementary symmetric functions of the eigenvalues (see for instance [BL13, Theorem 3.1]) but this is not relevant if one intends to compute optimality conditions for one precise eigenvalue.

As a result of Proposition 2.14, we shall express the derivative of a simple eigenvalue of the polylaplacian of order m. Indeed, in case of simplicity, M_V is a real number, hence the derivatives of Γ_k^Ω on the left and on the right coincide. This immediately yields the next proposition.

Proposition 2.15. *Let Ω be a C^{2m} bounded open set and $k \in \mathbb{N}^*$. Assume that $\Gamma_k(\Omega)$ is simple and let u_k be an associated L^2-normalized eigenfunction. Then the functional Γ_k^Ω defined in (2.9) is Gâteau-differentiable at 0 and its partial derivative*

in the direction of $V \in C^m(\mathbb{R}^d, \mathbb{R}^d)$ *is*

$$\partial_V \Gamma_k^\Omega(0) := -\int_{\partial\Omega} (\partial_n^m u_k)^2 \, V \cdot \vec{n}. \tag{2.11}$$

Let us briefly discuss how one would prove Proposition 2.14 and Proposition 2.15. The main issue is to show differentiability. This is usually performed thanks to the Implicit Function Theorem (see for instance the proofs of [HP05, Theorem 5.7.1] and [LL04, Theorem 2.11]). In general this gives more than just derivability, and we shall obtain analytic regularity for the eigenvalues and the eigenfunctions (see [LL04, Theorem 2.20] and [OZ00, Theorem 3.5]). Then, as mentioned in [HP05, sections 5.6 and 5.7], it is relatively easy to obtain a formula for the shape derivative, at least in the case of a simple eigenvalue. To see this, consider a simple eigenvalue $\Gamma_k(\Omega)$ and let $u_k[t]$ be an L^2-normalized eigenfunction associated with $\Gamma_k^\Omega(tV)$. Thanks to the previous remark, both depend analytically on t when it is close enough to zero. Set also $u_k = u_k[0]$. Then, for any function $\varphi \in C_c^\infty(\Omega)$, $\varphi \in C_c^\infty((id + tV)\Omega)$ for small t, hence

$$\int_\Omega u_k[t](-\Delta)^m \varphi = \Gamma_k^\Omega(tV) \int_\Omega u_k[t]\varphi.$$

After differentiating with respect to t at zero and setting $u_k' := \frac{d}{dt} u_k[0]$ and $\Gamma_k' := \partial_V \Gamma_k^\Omega(0)$, we obtain

$$\int_\Omega u_k'(-\Delta)^m \varphi = \Gamma_k(\Omega) \int_\Omega u_k'\varphi + \Gamma_k' \int_\Omega u_k\varphi,$$

which means that $(-\Delta)^m u_k' = \Gamma_k(\Omega) u_k' + \Gamma_k' u_k$ in $D'(\Omega)$ and hence in $L^2(\Omega)$. For the boundary conditions, let $0 \leq j < m$, and observe that $D^j u_k[t]((id + tV)x) = 0$ whenever $x \in \partial\Omega$. Therefore, we may differentiate with respect to t and find, by the chain rule

$$D^j u_k' + V \cdot D^{j+1} u_k = 0.$$

In other words $D^j u_k' = 0$ for all $j < m - 1$ and $D^{m-1} u_k' = -V \cdot D^m u_k$ on $\partial\Omega$. In particular, by elliptic regularity (see [GGS10, Theorem 2.20]), u_k' is $H^{2m}(\Omega)$ as long as Ω and V are regular enough. Lastly, observe that the normalization of $u_k[t]$ for all small t yields

$$\int_\Omega u_k' u_k = 0.$$

Combining this with the equation satisfied by u_k', we get

$$\Gamma_k' = \int_\Omega u_k (-\Delta)^m u_k',$$

and after integration by parts,

$$\Gamma_k' = \int_\Omega u_k'(-\Delta)^m u_k + \sum_{j=0}^{m-1} -\int_{\partial\Omega} (-\Delta)^j u_k \partial_n (-\Delta)^{m-1-j} u_k' + \int_{\partial\Omega} \partial_n (-\Delta)^j u_k (-\Delta)^{m-1-j} u_k'.$$

Due to the boundary conditions satisfied by u_k' and u_k, all but one term (corresponding to $j = m/2$ if m is even and $j = (m+1)/2$ if m is odd) vanish in the above sum, and we end up with

$$\Gamma_k' = \int_\Omega u_k'(-\Delta)^m u_k + \int_{\partial\Omega} \partial_n^m u_k \partial_n^{m-1} u_k'.$$

Finally, recall that $\partial_n^{m-1} u_k' = -\partial_n^m u_k V \cdot \vec{n}$, which concludes since

$$\int_\Omega u_k'(-\Delta)^m u_k = \Gamma_k(\Omega) \int_\Omega u_k' u_k = 0.$$

The main purpose of shape derivatives is to deduce necessary conditions for a shape to be optimal. In particular, Proposition 2.15 will allow to obtain such optimality conditions when optimizing some eigenvalue Γ_k. However, the optimization problems that we shall address in this manuscript are considered under volume constraint. In this context, it is not true to say that if Ω minimizes Γ_k, then for all $V \in C^m(\mathbb{R}^d, \mathbb{R}^d)$, $\partial_V \Gamma_k^\Omega(0) = 0$. We rather have the existence of a Lagrange multiplier λ such that for any $V \in C^m \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\partial_V \Gamma_k^\Omega(0) = \lambda \partial_V \mathcal{V}^\Omega(0)$. But if we recall the homogeneity of Γ_k, minimizing Γ_k under volume constraint is equivalent to minimizing $\Gamma_k |\cdot|^{\frac{2m}{d}}$ with no constraint. As a consequence, it seems interesting to introduce, for any bounded open set Ω, the functional

$$G_k^\Omega := \Gamma_k^\Omega \left(\mathcal{V}^\Omega\right)^{\frac{2m}{d}}, \tag{2.12}$$

defined over the intersection $C^m \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, and to compute its derivative. This comes immediately from Proposition 2.13 and Proposition 2.15.

Corollary 2.16. *Let Ω be a C^{2m} bounded open set and $k \in \mathbb{N}^*$. Assume that $\Gamma_k(\Omega)$ is simple and let u_k be an associated L^2-normalized eigenfunction. Then the functional G_k^Ω defined in (2.12) is Gâteau-differentiable at 0 and its partial derivative in the direction of $V \in C^m \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is*

$$\partial_V G_k^\Omega(0) := |\Omega|^{\frac{2m}{d}} \int_{\partial\Omega} \left[\frac{2m\Gamma_k(\Omega)}{d|\Omega|} - \left(\partial_n^m u_k\right)^2 \right] V \cdot \vec{n}. \tag{2.13}$$

2.4. Concentration-compactness

The concentration-compactness principle is a framework used for proving existence results in analysis. Presented for the first time in the seminal paper [Lio84], it characterises the behaviour of a bounded sequence of functions. This tool turns out to be particularly useful when a lack of compactness prevents a priori the sequence to converge. As explained in [Lio84], the concentration-compactness principle does not reduce to a sole theorem. It is rather a method from which a collection of results shall be obtained. In most of those results, the behaviour of the

sequence is categorised into three cases: vanishing, dichotomy, and compactness. The rough idea is that either the sequence disperses more or less uniformly – this is vanishing –, either it essentially splits – this is dichotomy –, or the sequence morally converges locally – and this is compactness. It has to be mentioned that the concentration-compactness principle mainly concerns the "profiles" of the functions involved, that is their behaviour up to translations.

In order to be more precise, we would like to state one of the results that may be obtained through concentration-compactness. Here, we deal with bounded sequences of functions in $H^1(\mathbb{R}^d)$. The statement is as follows.

Proposition 2.17. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $H^1(\mathbb{R}^d)$, and assume that $\|u_n\|_{L^2(\mathbb{R}^d)} \to \lambda \in \mathbb{R}$. Then, up to extracting a subsequence, one of the following occurs:*

1. *(compactness) there exists a sequence of points $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ such that*
$$\forall \epsilon > 0, \exists R > 0, \quad \liminf_{n\to\infty} \|u_n(\cdot - y_n)\|^2_{L^2(B_R)} \geq \lambda - \epsilon.$$
2. *(vanishing)*
$$\forall R > 0, \quad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^d} \|u_n(\cdot - y)\|^2_{L^2(B_R)} = 0.$$
3. *(dichotomy) there exists $\alpha \in (0, \lambda)$ and functions u_n^1 and u_n^2 in $H^1(\mathbb{R}^d)$ such that $|u_n^1|, |u_n^2| \leq |u_n|$ and*
$$\begin{aligned}
&u_n \underset{n\to\infty}{\overset{L^2(\mathbb{R}^d)}{=}} u_n^1 + u_n^2 + o(1),\\
&\|u_n^1\|^2_{L^2(\mathbb{R}^d)} \to \alpha, \quad \|u_n^2\|^2_{L^2(\mathbb{R}^d)} \to \lambda - \alpha,\\
&\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)} \underset{n\to\infty}{\geq} \|\nabla u_n^1\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u_n^2\|^2_{L^2(\mathbb{R}^d)} + o(1),\\
&dist(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \to \infty.
\end{aligned}$$

For a detailed proof of Proposition 2.17 we refer to [Lio84, Lemma III.1] (see also Apendix B), but let us sketch the strategy in few words. The main idea is to study the functions $Q_n : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$Q_n(R) = \sup_{y\in\mathbb{R}^d} \|u_n(\cdot - y)\|^2_{L^2(B_R)}.$$

The maps Q_n form a sequence of nonnegative nondecreasing uniformly bounded functions over $\mathbb{R}_+$. By Helly's selection Theorem, we may extract a subsequence such that Q_n pointwise converges to some Q. Setting

$$\alpha := \lim_{R\to\infty} \uparrow Q(R) \in [0, \lambda],$$

we distinguish the cases $\alpha = 0$, $\alpha = \lambda$ and $\alpha \in (0, \lambda)$, which correspond respectively to the situations of vanishing, compactness, and dichotomy. Indeed, $\alpha = 0$

immediately yields vanishing. If $\alpha = \lambda$, for large enough R and large enough n there exists y_n such that $\|u_n(\cdot - y_n)\|^2_{L^2(B_R)} \geq \lambda - \epsilon$. It remains only to show that the points y_n can be chosen regardless of ϵ and R to get the desired compactness. If $\alpha \in (0, \lambda)$, there is a sequence of radii $(R_n)_{n\in\mathbb{N}}$ and a sequence of points $(y_n)_{n\in\mathbb{N}}$ such that $\|u_n(\cdot - y_n)\|^2_{L^2(B_{R_n})} \to \alpha$. Then, we define

$$u_n^1 := u_n \chi_1 \left(\frac{\cdot}{R_n} + y_n\right), \qquad u_n^2 := u_n \chi_2 \left(\frac{\cdot}{3R_n} + y_n\right),$$

where $0 \leq \chi_1 \leq 1$ is a smooth cutoff function over $B_1(0)$ with support in $B_2(0)$ and $\chi_2 = 1 - \chi_1$. The choice of the dilation factors R_n and $3R_n$ is of importance in the definition of u_n^1 and u_n^2 since it allows to get $dist(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \geq R_n \to \infty$. Thanks to u_n^1 and u_n^2 one shall obtain dichotomy.

In the context of shape optimization, the concentration-compactness principle has been widely used to prove the existence of optimal shapes (see for instance [Buc00; AB03]). In the case where the energy of a shape is given by some state function, a natural strategy would be to apply the concentration-compactness principle to the sequence of state functions associated with an optimizing sequence of shapes. The issue consists in ruling out vanishing and dichotomy, in order to make the sequence converge. Then, it remains to produce a shape of which the state function is the limit function. Such a shape is a candidate for optimality.

The above strategy will be explained in more detail in section 2.6 in the context of the minimization of the first eigenvalue of the Dirichlet Laplacian. Let us mention that, for the Dirichlet Laplacian, comparison principles allow to prove existence for more general spectral functionals than just the first eigenvalue (see [Buc00, Theorem 4.2]). For higher order operators, we will only be able to prove the existence of an optimal shape for the first eigenvalue. For that purpose, a generalised version of Proposition 2.17 will be needed (see section 4).

2.5. Brascamp-Lieb-Luttinger inequalities

Now we would like to briefly introduce a useful tool that we encountered several times, due to its close relation with symmetrization. We already mentioned that, besides preserving the integral, Schwarz symmetrization increases the L^2 scalar product of two nonnegative functions. This is indeed the Hardy-Littlewood inequality (recall Proposition 2.10). Then, one naturally wonders what is the behaviour of the integral of more than two functions under Schwarz symmetrization. When three nonnegative measurable functions f, g, h are involved, the following inequality from [Rie30] answers this question.

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)h(x-y)g(y)\,dx\,dy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^*(x)h^*(x-y)g^*(y)\,dx\,dy \tag{2.14}$$

Brascamp-Lieb-Luttinger inequalities provide a generalisation to the above Riesz inequality, when an arbitrary number of nonnegative measurable functions $f_1, ..., f_k$ are defined on $\mathbb{R}^d$. Indeed, according to [BLL74], we have for any real numbers a_{ij}, with $1 \leq i \leq k$, and $1 \leq j \leq n$,

$$\int_{\mathbb{R}^d} ... \int_{\mathbb{R}^d} \prod_{i=1}^{k} f_k \left(\sum_{j=1}^{n} a_{ij} x_j \right) dx_1 ... dx_n \leq \int_{\mathbb{R}^d} ... \int_{\mathbb{R}^d} \prod_{i=1}^{k} f_k^* \left(\sum_{j=1}^{n} a_{ij} x_j \right) dx_1 ... dx_n \tag{2.15}$$

Let us give some quick hints on how to show (2.15). The proof is based on the case $d = 1$. In this one-dimensional setting, thanks to the layer-cake decomposition and to approximation arguments, one boils down to the situation where the functions $f_1, \ldots, f_k$ are indicators of segments. The ultimate argument is based on an astute application of the Brunn-Minkowski inequality.

2.6. Application: 6 proofs of the Faber-Krahn inequality

In this section, we would like to illustrate how the tools introduced in the previous sections shall be utilised in shape optimization when dealing with spectral quantities. For that purpose, we focus on the simplest and most famous case, which is undoubtedly the problem of minimizing the first eigenvalue of the Dirichlet Laplacian. For a bounded open set Ω, we denote this eigenvalue $\lambda(\Omega)$, and recall that it admits the variational characterisation

$$\lambda(\Omega) = \min_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2}.$$

In the following, we would like to propose six proofs of the celebrated Faber-Krahn inequality, which indicates that for any Ω,

$$\lambda(\Omega) \geq \lambda(\Omega^*). \tag{2.16}$$

The proofs that we will describe are the following:

2.6.1 The classic proof of Faber and Krahn by symmetrization.

2.6.2 A standard proof using Talenti's comparison principle.

2.6.3 A new proof involving the fractional Laplacian.

2.6.4 The incomplete strategy employing shape derivatives.

2.6.5 The proof of Lieb by Fourier transform.

2.6.6 A probabilistic proof.

2.6.1. The proof by direct symmetrization

This is the first and the most popular proof of the Faber-Krahn inequality, as it immediately follows from the Pólya-Szegő inequality (Theorem 2.8). Indeed, consider the first eigenfunction $u \in H^1_0(\Omega)$ of the Dirichlet Laplacian in the bounded open set Ω, which is known to be unique and positive up to renormalization. Then,

$$\lambda(\Omega) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \geq \frac{\int_{\Omega^*} |\nabla u^*|^2}{\int_{\Omega^*} u^{*2}} \geq \min_{\substack{v \in H^1_0(\Omega^*) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2} = \lambda(\Omega^*).$$

In the previous equation, besides the Pólya-Szegő inequality, we used the equimeasurability of u and u^* (Lemma 2.4), and the fact that $u^* \in H^1_0(\Omega^*)$. This last point comes from the positivity of $u \in H^1_0(\Omega)$.

It has to be mentioned here that other similar proofs exist, using different types of symmetrization, for instance Steiner symmetrization, or even polarization (see [Bur09]). In those proofs, the strategy is always to symmetrize the eigenfunction directly. Below we will see other approaches where symmetrization is still playing a crucial role, but is applied to a different function than the eigenfunction itself (except in paragraph 2.6.5).

2.6.2. The proof via Talenti's comparison principle

Now let us turn to another well-known way of proving the Faber-Krahn inequality, which is slightly less popular than the proof by direct symmetrization. Here, the result follows concisely from Talenti's comparison principle. Indeed, take $u \in H^1_0(\Omega)$ to be the first eigenfunction of the Laplacian in Ω, which is still positive. Then, consider $v \in H^1_0(\Omega^*)$ solving

$$\left\{ \begin{array}{rcll} -\Delta v & = & (\lambda(\Omega) u)^* & \text{in } \Omega^*, \\ v & = & 0 & \text{on } \partial\Omega^*. \end{array} \right.$$

By Talenti's comparison principle (Theorem 2.12), we have $0 \leq u^* \leq v$. As a result,

$$\int_{\Omega^*} |\nabla v|^2 = \lambda(\Omega) \int_{\Omega^*} v u^* \leq \lambda(\Omega) \int_{\Omega^*} v^2,$$

and we conclude after dividing by $\int_{\Omega^*} v^2$. In this approach, one may object that, contrary to what was previously announced, the symmetrization has been applied to the eigenfunction u. Although this is almost true in practice, we would like to emphasise that the very spirit was rather to rearrange the Laplacian of u.

2.6.3. A new proof with the fractional Laplacian

The next proof of the Faber-Krahn inequality is similar in many respects to the proof of the Faber-Krahn inequality for the bilaplacian in dimensions 2 and 3.

The strategy originates from a suggestion that Anatole Gaudin, PhD student at Aix-Marseille University, made to me. It appears not to have been published up to my knowledge, although being an easy consequence of existing results. The idea is to resort to a version of Talenti's comparison principle for spectral fractional Laplacians, which shall be found in [DV12].

First, we recall that the spectral fractional Laplacian of order $0 \leq s \leq 2$, noted $(-\Delta_\Omega)^{s/2}$, is defined through

$$(-\Delta_\Omega)^{s/2} u := \sum_{n=1}^{\infty} \lambda_n^{s/2} \langle u, u_n \rangle_{L^2(\Omega)} u_n$$

to be an unbounded self-adjoint operator over $L^2(\Omega)$ with domain

$$\tilde{H}^s := \{u \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^s \langle u, u_n \rangle_{L^2(\Omega)}^2 < \infty\}.$$

Here $(\lambda_n)_{n\in\mathbb{N}^*}$ stand for the eigenvalues of the Dirichlet Laplacian over Ω (counted repeatedly) and $(u_n)_{n\in\mathbb{N}^*}$ for the associated eigenfunctions. We highlight that, in this fashion, $(-\Delta_\Omega)^{s/2}$ is the s power of the Dirichlet Laplacian in the sense of functional calculus. It coincides in no way with the more common restricted fractional Laplacian defined by a singular integral (see [DL22]). In the next lines, we are only interested in the case $s = 1$, for which $(-\Delta_\Omega)^{s/2}$ boils down to the square root of the Dirichlet Laplacian and $\tilde{H}^s = H_0^1(\Omega)$ (see [CT10, equation (1.2)]). The reason comes from the following minmax characterisation

$$\lambda(\Omega) = \min_{\substack{u \in dom(-\Delta_\Omega) \\ u \neq 0}} \frac{\langle -\Delta u, u \rangle_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}^2} = \min_{\substack{u \in dom(-\Delta_\Omega) \\ u \neq 0}} \frac{\|(-\Delta_\Omega)^{1/2} u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2},$$

where $dom(-\Delta_\Omega) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$ is the domain of the Dirichlet Laplacian and the last equality is a consequence of $(-\Delta_\Omega)^{s/2}$ being self-adjoint. Since $dom(-\Delta_\Omega)$ is dense in $H_0^1(\Omega)$, over which the (closed) operator $(-\Delta_\Omega)^{1/2}$ is continuous, we end up with

$$\lambda(\Omega) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\|(-\Delta_\Omega)^{1/2} u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}.$$

Therefore take $u \in H_0^1(\Omega)$ and set $f = (-\Delta_\Omega)^{1/2} u \in L^2(\Omega)$. Then, consider v to be the solution to

$$\begin{cases} (-\Delta_{\Omega^*})^{1/2} v &= |f|^* & \text{in } \Omega^*, \\ v &= 0 & \text{on } \partial\Omega^*. \end{cases}$$

Here v is sought in the functional space $H_{00}^{1/2}(\Omega^*)$ made of functions $w \in H^{1/2}(\Omega^*)$ such that $\int_{\Omega^*} w^2(x)/dist(x, \partial\Omega^*) dx < \infty$. Indeed, it turns out that $H_{00}^{1/2}(\Omega^*)$

is the form domain of $(-\Delta_{\Omega^*})^{1/2}$. Since $|f|^* \in L^2(\Omega^*)$, one can prove by Lax-Milgram Theorem that the solution v exists (see [DV12, p. 2598]). Then, Talenti's comparison principle for the spectral fractional Laplacian [DV12, Theorem 1.1], implies that for all $r > 0$,

$$\int_{B_r} |u|^* \leq \int_{B_r} |v|^*.$$

By [ALT91, Proposition 2], the previous yields

$$\int_{\Omega^*} |u|^2 \leq \int_{\Omega^*} |v|^2.$$

To summarise, using Lemma 2.4,

$$\begin{aligned} &\|(-\Delta_\Omega)^{1/2} u\|_{L^2(\Omega)} = \||f|^*\|_{L^2(\Omega^*)} = \|(-\Delta_{\Omega^*})^{1/2} v\|_{L^2(\Omega^*)}, \\ &\|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega^*)}. \end{aligned}$$

From this one concludes

$$\lambda(\Omega) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\|(-\Delta_\Omega)^{1/2} u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} \geq \min_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\|(-\Delta_{\Omega^*})^{1/2} v\|^2_{L^2(\Omega^*)}}{\|v\|^2_{L^2(\Omega^*)}} = \lambda(\Omega^*).$$

We did not explain above why $v \in H_0^1(\Omega^*)$, but this follows from the fact that $(-\Delta_{\Omega^*})^{1/2} v \in L^2(\Omega^*)$, and from the invertibility of $(-\Delta_{\Omega^*})^{1/2} : H_0^1(\Omega^*) \to L^2(\Omega^*)$ (see [CT10, equation (1.2)]). Let us mention that it is likely for the preceding proof to work also for minimizing the functional

$$\tilde{\lambda}_p(\Omega) := \inf_{\substack{u \in C_c^\infty(\Omega) \\ u \neq 0}} \frac{\|(-\Delta_\Omega)^{1/2} u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}, \tag{2.17}$$

where $1 \leq p < \infty$. However, it is not clear whether (2.17) coincides with a known functional such as the first eigenvalue of the p-Laplacian.

As a final remark, we would like to point out that we recently learned about a very similar proof in [ALT89, Corollary 3.1]. In this paper, the strategy is to symmetrize the norm of the gradient of a first eigenfunction of the Laplacian in order to build a radially symmetric competitor. A comparison principle for Hamilton-Jacobi equations ([ALT89, Theorem 3.1]) is then used to control the norm of the eigenfunction by the norm of the competitor.

2.6.4. The incomplete proof using shape derivatives

In this paragraph we recall the classic but nevertheless incomplete proof of the Faber-Krahn inequality using shape derivatives. The strategy of the proof is perhaps more general than the previous ones in the sense that we do not start

with the assumption that the ball is our optimal shape. Indeed, the approach is as follows. First, we show that there exists an optimal shape. Then, we compute the optimality conditions. The final step is to identify the optimal shape according to these optimality conditions.

To prove that an optimal shape exists, it is convenient to relax the class of admissible shapes, and to allow quasi-open sets to be admissible. Quasi-open sets are related to the notion of capacity (see [HP05, section 3.3] for details). We shall simply keep in mind that the (H^1-)capacity *cap* is an outer measure on $\mathbb{R}^d$ which is finer than the Lebesgue measure, in the sense that $|\cdot| \leq cap$ on measurable sets. From quasi-open sets, one may define the notion of quasi-continuous function, which consists in functions whose preimage of an open set is a quasi-open set. An important property is that any function in $H^1(\mathbb{R}^d)$ admits a q.e. well-defined quasi-continuous representative (q.e. means everywhere but on a set of zero capacity). Moreover, we have the characterisation (see [HP05, Theorem 3.3.42])

$$H_0^1(\Omega) = \{u \in H^1(\mathbb{R}^d) : u = 0 \text{ q.p. in } \Omega^c\} \tag{2.18}$$

Equation (2.18) is a property when Ω is open but it serves as a definition when Ω is a general subset of $\mathbb{R}^d$. It is remarkable that with such a definition, $H_0^1(\Omega)$ coincides with the closure of $C_c^\infty(\Omega)$ with respect to the H^1 norm (see [AH96, Theorem 10.1.1]). Moreover, we shall define the first eigenvalue of the Laplacian in Ω thanks to the variational formula:

$$\lambda(\Omega) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

It has to be noted that the minimum is achieved whenever Ω is a measurable set of finite volume, because of the compact injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ (see Lemma 4.9). More details on quasi-open sets shall be found in chapter 4. As a final word on this topic, we mention that by Lemma 4.12, for any volume constraint $c \in \mathbb{R}_+^*$,

$$\inf_{\substack{\Omega \text{ open} \\ |\Omega| = c}} \lambda(\Omega) = \inf_{\substack{\Omega \text{ quasi-open} \\ |\Omega| = c}} \lambda(\Omega).$$

2.6.4.1. Existence of an optimal shape

After those preliminaries, we shall justify that a quasi-open optimal shape does exist. This will be performed thanks to the concentration-compactness principle. For that purpose, take a minimizing sequence made of open sets Ω_n and consider the first L^2-normalized eigenfunction u_n on Ω_n. We emphasise that Ω_n need not be bounded, we only know that $|\Omega_n| = c$, but this is not an issue. The concentration-compactness principle (Proposition 2.17) applied to the sequence $(u_n)_n$ indicates that either vanishing, dichotomy, or compactness occurs. Vanishing would mean

that for all $R > 0$,

$$\sup_{y\in\mathbb{R}^d} \|u_n(\cdot - y)\|^2_{L^2(B_R)} \underset{n}{\to} 0$$

But thanks to [Lie83, Lemma 6] (see also [BF17, Proposition 3.1]), since $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^d)$ and since $\|u_n\|_{L^2} = 1$ there would exist a sequence of points $y_n \in \mathbb{R}^d$ such that $(u_n(\cdot - y_n))_{n\in\mathbb{N}}$ admit no subsequence converging to 0 in $H^1(\mathbb{R}^d)$. As a result, we would find a small $\epsilon > 0$ and a smooth function φ supported in a ball of radius say $R > 0$ such that for any large n

$$\epsilon < \left(\int \varphi u_n(\cdot - y_n)\right)^2 \leq \sup_{y\in\mathbb{R}^d} \int_{B_R} |u_n(\cdot - y_n)|^2 \int_{B_R} \varphi^2,$$

which would contradict the assumption of vanishing.

If we were in the situation of dichotomy, there would exist $\alpha \in (0, 1)$ and functions $u_n^1, u_n^2 \in H^1(\mathbb{R}^d)$ such that $|u_n^1|, |u_n^2| \leq |u_n|$, and

$$\begin{aligned}
&u_n \overset{L^2(\mathbb{R}^d)}{=} u_n^1 + u_n^2 + o(1),\\
&\|u_n^1\|^2_{L^2(\mathbb{R}^d)} \to \alpha, \quad \|u_n^2\|^2_{L^2(\mathbb{R}^d)} \to \lambda - \alpha,\\
&\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)} \geq \|\nabla u_n^1\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u_n^2\|^2_{L^2(\mathbb{R}^d)} + o(1),\\
&dist(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \to \infty.
\end{aligned}$$

Then thanks to the disjunction of $\operatorname{supp} u_n^1$ and $\operatorname{supp} u_n^2$ for large n, we would get that for any $\epsilon > 0$, and for any large $n \in \mathbb{N}$,

$$\int_{\Omega_n} |u_n|^2 \leq \int_{\Omega_n} |u_n^1|^2 + \int_{\Omega_n} |u_n^2|^2 + 2\epsilon, \qquad \int_{\Omega_n} |\nabla u_n|^2 \geq \int_{\Omega_n} |\nabla u_n^1|^2 + \int_{\Omega_n} |\nabla u_n^2|^2 - 2\epsilon,$$

hence

$$\frac{\int_{\Omega_n} |\nabla u_n|^2}{\int_{\Omega_n} |u_n|^2} \geq \frac{\int_{\Omega_n} |\nabla u_n^1|^2 + \int_{\Omega_n} |\nabla u_n^2|^2 - 2\epsilon}{\int_{\Omega_n} |u_n^1|^2 + \int_{\Omega_n} |u_n^2|^2 + 2\epsilon}.$$

One shall prove (see chapter 4 for details) that $\int_{\Omega_n} |\nabla u_n^1|^2 - \epsilon > 0$ and $\int_{\Omega_n} |\nabla u_n^2|^2 - \epsilon > 0$ for all ϵ sufficiently small and n sufficiently large. Using the relation $\frac{a+b}{c+d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right)$ for $a, b, c, d > 0$, passing to the limit $n \to \infty$ and then $\epsilon \to 0$, we get

$$\liminf_n \lambda(\Omega_n) \geq \min\left(\liminf_n \frac{\int_{\Omega_n} |\nabla u_n^1|^2}{\int_{\Omega_n} |u_n^1|^2}, \liminf_n \frac{\int_{\Omega_n} |\nabla u_n^2|^2}{\int_{\Omega_n} |u_n^2|^2}\right).$$

Assume without loss of generality that the minimum is achieved for the quotient involving u_n^1 and define $\omega_n := \{u_n^1 \neq 0\}$. The set ω_n is quasi-open and $u_n^1 \in H_0^1(\omega_n)$ by (2.18), therefore

$$\liminf_n \lambda(\Omega_n) \geq \liminf_n \lambda(\omega_n).$$

Since $\|u_n^2\|_{L^2} \to 1 - \alpha \neq 0$, one can show (see again chapter 4 for details) that $\limsup |\omega_n| =: c' < c$. Since $\Omega_n' := (c/|\omega_n|)^{\frac{1}{d}}\, \omega_n$ defines a quasi-open set of volume

c, dichotomy gives the contradiction

$$\liminf_n \lambda(\omega_n) \leq \liminf_n \lambda(\Omega_n) \leq \liminf_n \lambda(\Omega'_n) \leq (c'/c)^{\frac{2m}{d}} \liminf_n \lambda(\omega_n).$$

As a consequence, only compactness shall occur. This means that there exists $y_n \in \mathbb{R}^d$ such that for all $k \in \mathbb{N}^*$ for some $R_k > 0$, for all large enough n

$$\|u_n(\cdot - y_n)\|_{L^2(B_{R_k})} \geq 1 - 1/k. \tag{2.19}$$

Up to translation, we shall assume for readability that $y_n = 0$. Now observe that since $(u_n)_n$ is bounded in $L^2(\mathbb{R}^d)$ it converges weakly to some $u \in L^2(\mathbb{R}^d)$. Moreover, since $(u_n)_n$ is bounded in $H^1(\mathbb{R}^d)$, it converges in $L^2(B_{R_k})$ for each $k \in \mathbb{N}^*$ by Rellich-Kondratchov Theorem. As a result, for any $k \in \mathbb{N}^*$

$$\|u\|_{L^2(\mathbb{R}^d)} \geq \|u\|_{L^2(B_{R_k})} = \lim_n \|u_n\|_{L^2(B_{R_k})} \geq 1 - 1/k,$$

hence $\|u\|_{L^2} \geq 1$. This and the weak convergence of $(u_n)_n$ to u implies strong convergence. Finally, $(u_n)_n$ converges to u in $L^2(\mathbb{R}^d)$ strongly and in $H^1(\mathbb{R}^d)$ weakly.

In order to conclude, define $\Omega := \{u \neq 0\}$, and observe that $u \in H^1_0(\Omega)$ by (2.18). Moreover, by the previous two-fold convergence of $(u_n)_n$,

$$\liminf_n \lambda(\Omega_n) \geq \frac{\int_{\mathbb{R}^d} |\nabla u|^2}{\int_{\mathbb{R}^d} u^2} \geq \lambda(\Omega).$$

It remains only to justify that $|\Omega| \leq c$, but this is a consequence of the fact that $u_n \to u$ in $L^2(\mathbb{R}^d)$ and hence almost everywhere. Indeed, from the a.e. convergence we may prove that $|\{u \neq 0\}| \leq \liminf |\{u_n \neq 0\}|$. This property appears to be more complex than it looks at first glance, and since it will serve later on we encapsulate it in Lemma 2.18 below. Anyway, combined with the q.e. inclusion $\{u_n \neq 0\} \subseteq \Omega_n$ provided by (2.18), it allows to conclude that $|\Omega| \leq c$ as expected.

Lemma 2.18. *Let $u_n : \mathbb{R}^d \to \mathbb{R}^N$ where $N \in \mathbb{N}^*$ form a sequence of measurable functions converging a.e. toward $u : \mathbb{R}^d \to \mathbb{R}^N$. Then $|\{u \neq 0\}| \leq \liminf |\{u_n \neq 0\}|$.*

Proof. We shall assume without loss of generality that $c := \liminf |\{u_n \neq 0\}|$ is finite. We proceed by contradiction, assuming that $|\{u \neq 0\}| > c$. Since

$$\{u \neq 0\} = \bigcup_{k \in \mathbb{N}^*} \{|u| \geq 1/k\},$$

there exists $k \in \mathbb{N}^*$ such that $|\{|u| \geq 1/k\}| > c$. In the same fashion, there exists a large enough ball B such that $|\{|u| \geq 1/k\} \cap B| > c$. By Egorov's Theorem, B being of finite volume, there exists $\Omega \subseteq B$ such that $|\{|u| \geq 1/k\} \cap \Omega| > c$ and $u_n \to u$ uniformly over Ω. As a result for all large enough n, $|u_n| > 0$ over $\{|u| \geq 1/k\} \cap \Omega$, which is of volume greater that c. In other words, $|\{u_n \neq 0\}| > c$, contradicting $\liminf |\{u_n \neq 0\}| = c$. □

2.6.4.2. Identification of the optimal shape

Now that an optimal shape Ω is known to exist, we may try to identify it. For that purpose, we shall employ shape derivatives. Since $\lambda(\Omega)$ is simple, we take advantage of Corollary 2.16. Indeed, due to the optimality of Ω, the function G_1^Ω defined in (2.12) reaches its minimum value at 0, hence its derivative vanishes and we have for all $V \in C^1 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$,

$$0 = \int_{\partial\Omega} \left[\frac{2\lambda(\Omega)}{d|\Omega|} - (\partial_n u)^2 \right] V \cdot \vec{n}.$$

As a result, using density arguments, we obtain

$$(\partial_n u)^2 = \frac{2\lambda(\Omega)}{d|\Omega|} \quad \text{on } \partial\Omega. \tag{2.20}$$

Now recall that u may be assumed positive, hence $\partial_n u < 0$ on $\partial\Omega$, and we end up with

$$\partial_n u = -\sqrt{\frac{2\lambda(\Omega)}{d|\Omega|}} \quad \text{on } \partial\Omega.$$

Therefore, we fall into the category of overdetermined problems. The historical and powerful result [Ser71] immediately allows to conclude as desired that Ω is a ball, given the crucial information that $u > 0$ in Ω.

Before closing this section, let us express a comment. The above proof is actually incomplete since we used Corollary 2.16 in an inappropriate way. Indeed, we recall that some regularity is needed in order to apply shape derivation. Here we would need Ω to be open, C^2 and bounded. But our existence result only exhibited a quasi-open optimal shape of bounded volume. We will see in chapter 4 how to build a plain open optimal shape upon the previous quasi-open optimal shape. However, we are unaware of any method to enhance the regularity beyond openness (see chapter 4 for more details on this topic).

Even if the previous strategy did not lead to a complete and thorough proof of the Faber-Krahn inequality, it is worth noting that a similar approach has been achieved in [BF17]. In that work, the concentration-compactness principle was invoked in the same way as above to prove that a quasi-open optimal shape exists. Then, the optimal shape is proved to be a ball thanks to symmetry arguments, avoiding the use of shape derivatives.

2.6.5. Lieb's proof using Fourier transform

In 1977, Lieb proposed an original proof of the Pólya-Szegő inequality (Theorem 2.8) for the case $p = 2$ based on Fourier transform and integral inequalities. As explained in paragraph 2.6.1, this immediately yields a new proof of the Faber-Krahn inequality. According to [Kes06, Remark 2.3.2], Lieb's approach goes as

follows. First, let us agree on the following definition of the unitary Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx.$$

Now take $u \in H_0^1(\Omega)$, and extend it by zero outside Ω. By the properties of the Fourier transform,

$$\int_\Omega |\nabla u|^2 = \int_{\mathbb{R}^d} |\xi|^2 |\hat{u}(\xi)|^2.$$

Use that $|\xi|^2 = \frac{d}{dt} e^{t|\xi|^2}|_{t=0} = \lim_{t\to 0} \frac{1-e^{-t|\xi|^2}}{t}$, and by dominated convergence

$$\int_\Omega |\nabla u|^2 = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^d} \left[|\hat{u}|^2 - |\hat{u}|^2 e^{-t|\xi|^2} \right].$$

Observe that for $t > 0$, $e^{-t|\xi|^2}$ is the Fourier transform of $e^{-\frac{|x|^2}{4t}} =: K_t(x)$. Then, by Plancherel identity and by the properties of the Fourier transform with respect to convolution, we conclude that

$$\int_\Omega |\nabla u|^2 = \lim_{\substack{t\to 0 \\ t>0}} \frac{1}{t} \left[\int_{\mathbb{R}^d} u^2 - \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (u * K_t) u \right]. \tag{2.21}$$

Formula (2.21) is noteworthy since it allows to write the L^2 norm of the gradient in terms of integrals involving only u and the Gaussian function K_t. This is of particular interest in the context of symmetrization. Indeed, by equimeasurability of Schwarz symmetrization (Lemma 2.4) and by the Riesz inequality (equation (2.14)), we get

$$\int_\Omega |\nabla u|^2 \geq \lim_{\substack{t\to 0 \\ t>0}} \frac{1}{t} \left[\int_{\mathbb{R}^d} u^{*2} - \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (u^* * K_t) u^* \right],$$

where we used the fact that $K_t = K_t^*$. After applying again Fourier transform and using Fatou Lemma to put the limit back into the integral, we get as desired

$$\int_\Omega |\nabla u|^2 \geq \int_{\mathbb{R}^d} |\xi|^2 |u^*|^2 = \int_{\Omega^*} |\nabla u^*|^2.$$

This concludes. It is surprising that one has to use Fourier transform to prove equation (2.21), which is the key point in the previous discussion. It would be interesting to see if one could obtain the same formula with other arguments. In this respect, we would like to mention the work [Bro+18], in which an identity with a similar flavour is obtained thanks to finite differences. When only one function is involved, this formula reads (see [Bro+18, equation (3.6)])

$$\int_{\mathbb{R}^d} |\nabla u|^2 = \frac{d}{\mu_2(\phi)} \lim_{\substack{t\to 0 \\ t>0}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{u(x+th) - u(x)}{t} \right]^2 \phi(h)\, dh\, dx,$$

where ϕ is any radial test function and $\mu_2(\phi) = \int_{\mathbb{R}^d} |h|^2 \phi(h) dh$. In the original paper, ϕ is required to be $C_c^\infty(B_1)$ and radially nonincreasing, but we may take more generally ϕ to be any radial Schwarz function. Straightforward manipulations then produce

$$\int_{\mathbb{R}^d} |\nabla u|^2 = \frac{d}{\mu_2(\phi)} \lim_{\substack{t \to 0 \\ t>0}} \frac{2}{t^2} \left[\int_{\mathbb{R}^d} \phi \int_{\mathbb{R}^d} u^2 - \frac{1}{t^d} \int_{\mathbb{R}^d} \left(u * \phi\left(\frac{\cdot}{t}\right)\right) u \right] \tag{2.22}$$

Although diffusely present in [Bro+18], we never encountered formula (2.22) as stated above. If we take $\phi(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$, we observe that (2.22) is a slight variation on (2.21). Despite the discrepancy between the two formulas, (2.22) also leads to the Pólya-Szegő inequality as previously. Moreover, besides allowing to choose the kernel ϕ, (2.22) has the interest to be likely generalisable to the case of the L^p norm of the gradient (at least for even $p \in \mathbb{N}^*$). However, in the case $p \neq 2$, it is expected that the presence of signed coefficients prevents from using the Riesz inequality and hence from providing an alternative proof of the Pólya-Szegő inequality.

2.6.6. A probabilistic proof

The last approach that we would like to mention is related to the previous one. Here, the idea is to give a new formulation of the first eigenvalue in terms of integrals and convolutions. The strategy seems to go back to [Lut73], in which it is explained that it is also possible to address other functionals than the first eigenvalue with the same technique, including the torsional rigidity and the perimeter. To derive our new formula, let us introduce the solution $v \in L_t^\infty(\mathbb{R}_+, L_x^2(\Omega))$ to

$$\begin{cases} \partial_t v &= \Delta v & \text{in } \mathbb{R}_+^* \times \Omega, \\ v &= 0 & \text{on } \mathbb{R}_+^* \times \partial\Omega, \\ v(0) &= v_0 & \text{in } \Omega, \end{cases}$$

with $v_0 \in L^2(\Omega)$. The function v may be expressed thanks to the heat semi-group in the following way

$$v(t) = e^{-t(-\Delta)} v_0 = \sum_{n \in \mathbb{N}^*} e^{-t\lambda_n} \langle v_0, u_n \rangle_{L^2(\Omega)} u_n,$$

where $(u_n)_n$ is an orthonormal Hilbert basis of $L^2(\Omega)$ made of eigenfunctions of the Dirichlet Laplacian in Ω, denoted $-\Delta$, and λ_n are the corresponding eigenvalues, assumed to be ordered nondecreasingly. After integrating over Ω and using that $\lambda(\Omega) = \lambda_1 \leq \lambda_n$ for all $n \in \mathbb{N}^*$, we obtain

$$e^{t\lambda(\Omega)} \int_\Omega v(t) \underset{t \to \infty}{=} O(1),$$

or in other words

$$\lambda(\Omega) = -\lim_{t\to\infty} \frac{1}{t} \log \left| \int_\Omega v(t) \right|. \tag{2.23}$$

The issue in formula (2.23) is that the right-hand side still depends on Ω in an implicit way through v. However, it does not depend on the choice of the function v_0, hence we shall take $v_0 = \mathbb{1}_\Omega$. Then, we invoke a probabilistic interpretation of $v(t)$. Indeed, by the Feynmann-Kac formula (see [BBC02, equation (1.12)]), as long as τ_x represents the first hitting time of $\partial\Omega$ for a Brownian motion $(B_s^x)_{s\in\mathbb{R}_+}$ departed from $x \in \Omega$, we have

$$v(t,x) = P(\tau_x > t). \tag{2.24}$$

Now by (almost sure) continuity of Brownian trajectories, one may discretize the time interval $[0,t]$ to obtain

$$P(\tau_x > t) = P\left(\bigcap_{0\le s\le t} B_s^x \in \Omega \right) = \lim_{n\to\infty} P\left(\bigcap_{0\le k\le n} B_{kt/n}^x \in \Omega \right).$$

The Markov property of the Brownian motion finally provides

$$\begin{aligned} P(\tau_x > t) &= \lim_{n\to\infty} \prod_{0\le k<n} P\left(B_{(k+1)t/n}^x - B_{kt/n}^x \in \Omega - B_{kt/n}^x \right) \\ &= \lim_{n\to\infty} \int_\Omega \dots \int_\Omega p_{t/n}(x_1 - x) \dots p_{t/n}(x_n - x_{n-1}) dx_1 \dots dx_n, \end{aligned} \tag{2.25}$$

where $p_s(x) := (2\pi s)^{-d/2} e^{-\frac{|x|^2}{s}}$ is the probability density of an increment of the Brownian motion corresponding to a time period s. Combining (2.23), (2.24), and (2.25), we finally obtain

$$\lambda(\Omega) = -\lim_{t\to\infty} \lim_{n\to\infty} \frac{1}{t} \int_\Omega \dots \int_\Omega p_{t/n}(x_1 - x_0) \dots p_{t/n}(x_n - x_{n-1}) dx_0 \dots dx_n. \tag{2.26}$$

As in equation (2.21) formula (2.26) has the advantage of involving only convolutions with Gaussian kernels rather than gradient integrals which come into play in the traditional variational formulation of $\lambda(\Omega)$. Furthermore, (2.26) has a purely geometric nature, in the sense that it does not involve any test function. Indeed, Ω only takes part as an integration domain, hence the dependance of $\lambda(\Omega)$ on Ω is rather explicit. As a payback, (2.26) involves two limits and an onerous multiple integral. Consequently, contrary to (2.21), the sole Riesz inequality is not sufficient anymore, and we require the general version of the Brascamp-Lieb-Luttinger inequality to conclude. Indeed, since $p_{t/n} = p_{t/n}^*$ and $\mathbb{1}_\Omega$ are nonnegative,

we may apply inequality (2.15) to get

$$-\lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{t}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}p_{t/n}(x_1-x_0)\dots p_{t/n}(x_n-x_{n-1})\mathbb{1}_\Omega(x_0)\dots\mathbb{1}_\Omega(x_n)dx_0\dots dx_n$$
$$\geq-\lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{t}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}p_{t/n}(x_1-x_0)\dots p_{t/n}(x_n-x_{n-1})\mathbb{1}_{\Omega^*}(x_0)\dots\mathbb{1}_{\Omega^*}(x_n)dx_0\dots dx_n,$$

which immediately concludes by virtue of (2.26). The above proof follows [Car01]. However, Dorin Bucur informed me of the following objection: some argument should be missing in the derivation of (2.26), since this formula cannot hold for a general bounded open set Ω. Indeed, the right-hand side of (2.26) depends only in a measure theoretic sense on Ω. This means that changing Ω a.e. would not vary $\lambda(\Omega)$, which is false in general. For instance, pulling a diameter out of a disk does change its eigenvalue. As a consequence, some regularity must be assumed on Ω for deriving (2.26). In this respect, we mention that the Feynmann-Kac formula used in the proof is unlikely to be true on irregular domains. For instance in [CM15, equation (4.2.15)] a regularity condition is given in terms of Brownian motion for the Feynmann-Kac formula to hold. This condition is satisfied when the boundary is of class C^1.

We would like to point out that there is a completely different way to obtain formula (2.26). This alternative proof involves in no way probabilities but relies on operator theory and Trotter's formula following [ALT91]. We will come back to this issue more extensively in chapter 8. As we shall see, in this approach the lacking regularity assumption will clearly come into play. Let us reveal as of now that a Lipschitz regularity on Ω is enough for (2.26) to hold.

2.6.7. Conclusion

We have presented six substantially different – although related – proofs of the Faber-Krahn inequality. Those proofs may be categorised into three groups according to the tools they utilise. The first group gathers the proof by direct symmetrization, the proof via Talenti's comparison principle, and the proof involving the fractional Laplacian, since they all essentially rely on the isoperimetric inequality (recall the proof of Theorem 2.8 and of Theorem 2.12). Second, the strategy based on shape derivatives is particular and it is no coincidence that it is the only one which is incomplete. Hence it belongs to its own category. Lastly, the proofs by Fourier transform and the probabilistic proof are similar in the sense that they both resort to the Brascamp-Lieb-Luttinger inequality.

However, it is interesting to recall that the Brascamp-Lieb-Luttinger inequality is in turn based on the Brunn-Minkowski inequality (see section 2.5). Moreover, note that the isoperimetric inequality may also be proved thanks to the Brunn-Minkowski inequality. As a result, it appears that the Brunn-Minkowski inequality lies at the root of all the above proofs of the Faber-Krahn inequality except that based on shape derivatives. In other words, the Faber-Krahn inequality is essentially a

convexity inequality.

As a final word, we would like to mention that since a deep link seems to exist between optimal transportation and symmetrization (see section 2.1), and since optimal transport were already successfully applied for proving geometric inequalities ([Bur09, chapter 5]), it is expected that Faber-Krahn inequality might be obtained by direct arguments coming from optimal transport theory.

3. Basic facts

In this chapter, I would like to explain more in detail why the Rayleigh Conjecture for the clamped plate is challenging and how it has been solved in dimensions 2 and 3. Indeed, in the preceding chapter we gave six proofs of the Faber-Krahn inequality for the Laplacian. It is reasonable to think that one of them could be adapted to the bilaplacian. This is not the case, and I will explain why. Then, as a warm up before the more advanced results of the next chapters, I will discuss more in depth second order Pólya-Szegő inequalities and raise some open questions based on partial results.

3.1. Why the proofs of the Faber-Krahn inequality for the Laplacian do not work for the bilaplacian

The ultimate proof of the Faber-Krahn inequality is done by direct symmetrization and is based on the Pólya-Szegő inequality as recalled in section 2.6.1. It uses the positivity of the first eigenfunction, which is crucial to deduce that the symmetrized function vanishes at the boundary. For the bilaplacian the lack of positivity of the first eigenfunction is thus a first issue. Resorting to positive and negative parts does not help for several reasons. But the main problem is the failure of the Pólya-Szegő inequality for higher-order derivatives. We will come back to this question in section 3.3.

The proof 2.6.2 of the Faber-Krahn inequality based on Talenti's comparison principle is interesting, yet positivity of the first eigenvalue is still needed. Unfortunately, I am unaware of any version of Talenti's comparison principle related to the bilaplacian. This prevents from adapting proof 2.6.2 to the setting of the bilaplacian.

Proof 2.6.3 resorts to Talenti's comparison principle for the spectral fractional Laplacian. Since Talenti's comparison principle for the bilaplacian is not known, it is even more uncertain that one may obtain such a result for fractional bilaplacians. But this is not only about ancillary technicalities. Indeed, as illustrated in [DV12], comparison principles for fractional Laplacians are commonly derived from applying comparison principles to their Caffarelli-Silvestre extensions (except in [FV21], but there the *restricted* fractional Laplacian is under consideration). In this context, we understand that a comparison principle for the bilaplacian is needed prior to

getting any comparison principle for the spectral fractional bilaplacian. Despite the previous issues, we mention that proof 2.6.3 was inspired by the traditional proof of the Rayleigh Conjecture for the bilaplacian in dimensions 2 and 3. From this point of view the proof shall certainly and trivially be adapted to the case of the bilaplacian (in low dimension).

The proof of section 2.6.4 is of interest. At least, computing the optimality condition is a first unmissable step toward a better understanding of the Rayleigh Conjecture. The optimality conditions are more or less known throughout the literature, but we will establish them definitively in Theorem 5.4. Proving from these conditions that an optimal shape is a ball is however a difficult task. It was performed laboriously in dimension 2 in [Moh75], but also second order shape derivative crucially came into play in this paper. In particular, it is unknown whether one shall deduce only from first order optimality conditions that an optimal shape is a ball. The reason explaining this issue is again the lack of positivity of the first eigenfunction. Indeed, even with the Laplacian, solving an overdetermined eigenvalue problem is far from trivial without any positivity assumption (this is the so-called Schiffer Conjecture). We imagine that the problem gets worse with a bilaplacian. Note however that as long as the first eigenfunction over a C^2 optimal shape is positive, [Dal90, Theorem 3.1] immediately yields the roundness of the shape. On the side of the existence of an optimal shape, the approach explained in paragraph 2.6.4.1 may be adapted and extended to the case of the bilaplacian with a little bit of extra development, as we shall see in chapter 4. Unfortunately, we were not able to go beyond openness in terms of regularity.

The proof 2.6.5 based on Fourier transform does not help very much with respect to the Rayleigh Conjecture for the clamped plate since the Pólya-Szegő inequality cannot be obtained in full generality for higher-order derivatives. Therefore, there is no hope to adapt Lieb's proof. Nevertheless, it may be enlightening to analyse where the proof fails. It turns out that the breaking point is the symmetrization of the kernel K_t involved in (2.21). In the case of a gradient integral, K_t is radially symmetric and decreasing. However, in the case of the L^2 norm of a Laplacian, K_t would be the inverse Fourier transform of $\xi \mapsto e^{-t|\xi|^4}$, which is not radially decreasing (see [Hoc78, equation (2.5)]). Surprisingly, the issue is quite different for the proof by finite differences also proposed in paragraph 2.6.5 (recall (2.22)). Indeed, in this case, since

$$D^2u(x)\cdot h^2 = \lim_{t\to 0} \frac{u(x+th)-2u(x)+u(x-th)}{t},$$

the problem is that more terms with unwelcome signs would appear in (2.22), preventing from using the Riesz inequality. Let us stress once again that the above issues are not technical since second order Pólya-Szegő inequalities cannot be obtained in general.

The probabilistic proof provided in paragraph 2.6.6 is intriguing. Besides operator theory, it is mainly based on a probabilistic interpretation of the solutions of the

heat equation. As we shall see more in depth in chapter 8, less has been produced concerning a probabilistic interpretation of the biharmonic heat equation compared to the standard heat equation. This is mainly because the fundamental solution of the biharmonic heat equation is not positive by [Hoc78]. As a result, it is not possible to define a process whose transition probabilities would be given by such a fundamental solution. Another way to say this is that the Dirichlet bilaplacian is not the infinitesimal generator of a Feller process due to the failure of the maximum principle. Nevertheless, several authors have defined pseudo-processes whose purpose was to address polyharmonic heat equations. Yet, it seems that the literature is not uniform with respect to defining such pseudo-processes. Furthermore, the Feynmann-Kac type formulas that were derived seem not easy to handle in practice. Anyway, we mentioned at the end of paragraph 2.6.6 that the integral formulation (2.26) of the first eigenvalue of the Laplacian could also be obtained only through operator theory. As we shall see in chapter 8, this works as well in the case of polyharmonic operators. Unfortunately, the kernel involved in the resulting integral formulation is the fundamental solution of the polyharmonic heat equation, which changes sign. This will prevent from applying the Brascamp-Lieb-Luttinger inequality.

3.2. The proof of the Rayleigh Conjecture in dimensions 2 and 3

With the previous explanations we illustrated how challenging is the Rayleigh Conjecture for the clamped plate. Nevertheless, the conjecture has been solved in low dimensions and we would like to detail the corresponding proof. The strategy goes back to [Sze50] and was improved successively in [Tal81; Nad95; AB95]. The main idea is to symmetrize the Laplacian of the first eigenfunction instead of the eigenfunction itself and then to solve a Poisson equation in order to define an appropriate competitor. To compare the Rayleigh quotients for the eigenfunction and for its competitor, Talenti's comparison principle turns out to be a central tool.

Below we give more details on this general strategy. Let u be the first eigenfunction of the bilaplacian with Dirichlet boundary conditions over a bounded domain Ω. In its traditional version, Talenti's comparison principle is not very useful since symmetrizing $|\Delta u|$, loses much information. On the contrary, symmetrizing Δu preserves all the p-moments $\int_\Omega (\Delta u)^p$, $p \in \mathbb{N}$ (and also all the L^p norms). Note that preserving the 1-moment is of particular importance since

$$\int_\Omega \Delta u = \int_{\partial\Omega} \partial_n u.$$

Therefore it allows to keep an information on the normal derivative of u, which plays a role in the boundary conditions. Indeed, a good competitor for u should keep a track on the fact that $\partial_n u = 0$.

For these reasons, we would like to symmetrize Δu, so we need to use Kesavan's version of Talenti's comparison principle (Theorem 2.12). Unfortunately, u needs to be positive in order to apply Theorem 2.12. In this context, why not restricting the study to the nodal domains $\Omega_+ = \{u > 0\}$ and $\Omega_- = \{u < 0\}$ separately? Proceeding in this fashion, we consider the functions v and w solving the Poisson equations

$$\left\{\begin{array}{rcll} -\Delta v & = & (-\Delta u)^* & in \quad \Omega_+^*, \\ v & = & 0 & on \quad \partial\Omega_+^*, \end{array}\right. \qquad \left\{\begin{array}{rcll} -\Delta w & = & (\Delta u)^* & in \quad \Omega_-^*, \\ w & = & 0 & on \quad \partial\Omega_-^*. \end{array}\right. \tag{3.1}$$

In this way, instead of building a single competitor for u we constructed a couple (v, w) competing (u_+, u_-). Indeed, Theorem 2.12 yields the comparisons $v \geq u_+^* \geq 0$ and $w \geq u_-^* \geq 0$. Observe however that, if u were positive, we would get back only one competitor, and this competitor would coincide with that proposed in [Sze50]. To sum up, thanks to the equimeasurability of Schwarz symmetrization, we have

$$\int_\Omega (\Delta u)^2 = \int_{\Omega_+} (-\Delta u)^2 + \int_{\Omega_-} (\Delta u)^2 = \int_{\Omega_+^*} (\Delta v)^2 + \int_{\Omega_-^*} (\Delta w)^2 , \tag{3.2}$$

$$\int_\Omega u^2 = \int_{\Omega_+} u_+^2 + \int_{\Omega_-} u_-^2 \leq \int_{\Omega_+^*} v^2 + \int_{\Omega_-^*} w^2. \tag{3.3}$$

Lastly, we insisted on the necessity of preserving the 1-moment of the Laplacian of u. In the current setting with two competitors this reads

$$\int_\Omega \Delta u = -\int_{\Omega_+} (-\Delta u) + \int_{\Omega_-} \Delta u = -\int_{\Omega_+^*} (-\Delta v) + \int_{\Omega_-^*} (-\Delta w) = \int_{\partial\Omega_+^*} \partial_n v - \int_{\partial\Omega_-^*} \partial_n w. \tag{3.4}$$

Thanks to the assumption $\partial_n u = 0$ and to the radial symmetry of v and w (the data of the Poisson equations being radially symmetric), we conclude that

$$a^{d-1}\partial_n v(a) = b^{d-1}\partial_n w(b),$$

where a is the radius of Ω_+^* and b the radius of Ω_-^*. Putting all together, we have proved that

$$\Gamma(\Omega) \geq \frac{\int_{\Omega_+^*} (\Delta v)^2 + \int_{\Omega_-^*} (\Delta w)^2}{\int_{\Omega_+^*} v^2 + \int_{\Omega_-^*} w^2} \geq \inf_{v,w} \frac{\int_{B_a} (\Delta v)^2 + \int_{B_b} (\Delta w)^2}{\int_{B_a} v^2 + \int_{B_b} w^2}, \tag{3.5}$$

where the infimum is taken among positive radial functions $v \in H_0^1 \cap H^2(B_a)$ and $w \in H_0^1 \cap H^2(B_b)$ such that $a^{d-1}\partial_n v(a) = b^{d-1}\partial_n w(b)$. We stress that the above infimum does not correspond to $\Gamma(B_a \sqcup B_b)$ and cannot be bounded from below by the smallest of $\Gamma(B_a)$ and $\Gamma(B_b)$ (see however [Tal81, Theorem 1]). Indeed, admissible functions v and w defined on the balls B_a and B_b are always coupled by the boundary condition $a^{d-1}\partial_n v(a) = b^{d-1}\partial_n w(b)$. The only trivial case is when

$a = 0$ or $b = 0$. In this situation the problem is somewhat ill-defined, but thanks to [Sze50], we may replace the couple (v, w) by a single $H_0^2(\Omega^*)$ function.

Due to the coupling between the two balls appearing in (3.5) the previous infimum is usually called the *two-ball problem*. Despite this coupling, the two-ball problem provides a very nice lower bound on $\Gamma(\Omega)$ since it only involves positive radial functions and depends upon Ω only through the parameters a and b. The two-ball problem has been extensively studied in [Nad95; AB95; AL96]. First, it turns out that the infimum is attained by admissible functions v and w satisfying for some $\lambda > 0$,

$$\Delta^2 v = \lambda v, \text{ in } B_a, \qquad \Delta^2 w = \lambda w \text{ in } B_b,$$

and the ancillary condition $\Delta v(a) + \Delta w(b) = 0$ (see [AB95, Appendix 2]). Then, through different methods, the value of the two-ball problem with respect to the parameters a and b (subject to the constraint $|B_a| + |B_b| = |\Omega|$) was studied. In dimensions 2 and 3, it was shown to be minimal for either $a = 0$ or $b = 0$, whereas in dimension $d \geq 4$, it is minimal when $a = b$. In other words, the Rayleigh Conjecture is solved in dimensions 2 and 3, but only an approximate Faber-Krahn inequality shall be derived in higher dimension through this strategy.

As a last word on this proof, a few technical details. We were intentionally a little bit vague in the definition of v and w given in (3.1). Indeed, there are actually two possibilities since we could define v and w such that either

$$-\Delta v = f|^*_{\Omega_+}, \qquad -\Delta w = g|^*_{\Omega_-}, \tag{3.6}$$

or

$$-\Delta v = f^*|_{\Omega^*_+}, \qquad -\Delta w = g^*|_{\Omega^*_-}, \tag{3.7}$$

where $f = -\Delta u$ and $g = \Delta u$. By virtue of (2.4) this is not the same, but surprisingly both give rise to the expected two-ball problem. This is because equations (3.2), (3.3) and (3.4) are fulfilled in both cases. Indeed, for definition (3.6) this follows immediately from the equimeasurability of Schwarz symmetrization. For definition (3.7) it is slightly more delicate and relies on Lemma F.1. Moreover, we must point out that with the latter choice, Talenti's comparison principle cannot be used directly since the data are not symmetrizations of Laplacians. Therefore we actually need to go back to definition (3.6) and to combine the maximum principle with (2.4) in order to obtain the desired comparison result. Consequently, (3.7) appears to be more difficult to manipulate. In defence of definition (3.7), we mention that it guarantees by construction the relation $\Delta v(a) + \Delta w(b) = 0$, whereas this is not the case in general with definition (3.6).

3.3. Second order Pólya-Szegő inequality

The purpose of the present section is to discuss more in detail a possible Pólya-Szegő type inequality involving second order derivatives. Typically, it would be

desirable, for a positive function $f \in W_0^{2,p}(\omega)$, where ω is a bounded open set and $1 \leq p < \infty$, to have

$$\int_{\omega^*} |\Delta f^*|^p \leq \int_{\omega} |\Delta f|^p. \tag{3.8}$$

As already explained in section 2.1, relation (3.8) cannot hold in full generality due to eventual critical values (recall the example of equation (2.3)). A natural question is then to ask if (3.8) holds for the restricted class of functions having no critical value of intermediate level – also called intermediate critical value. To explore this question, let us focus in all this section on the one-dimensional setting, which already contains interesting situations.

Since the Schwarz symmetrization of a concave function is concave, Theorem 3 of [Cia00] goes in the direction of such a second-order Pólya-Szegő inequality. On the contrary, the example of Figure 3 in [Cia00] indicates that one might construct a counterexample. Inspired by this example, we build a C^1 and piecewise polynomial function defined by

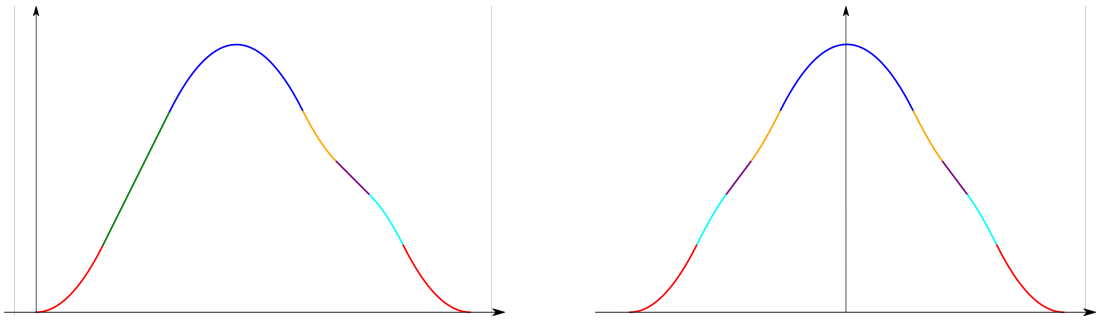
$$f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1, \\ 2x - 1 & \text{if } 1 < x < 2, \\ 4 - (x-3)^2 & \text{if } 2 < x < 4, \\ 2 + (x-5)^2 & \text{if } 4 < x < 4.5, \\ 6.75 - x & \text{if } 4.5 < x < 5, \\ 2 - (x-4.5)^2 & \text{if } 5 < x < 5.5, \\ (x-6.5)^2 & \text{if } 5.5 < x < 6.5, \end{cases} \tag{3.9}$$

over $(0, 6.5)$. Observe that as in [Cia00, Figure 3], the function f is essentially made of affine portions connected by second order polynomials which are intended for replacing the angular junctions of [Cia00, Figure 3]. One then computes its distribution function

$$\mu_f(t) = \begin{cases} 6.5 - 2\sqrt{t} & \text{if } t < 1, \\ 4 + \sqrt{2-t} - t/2 & \text{if } 1 < t < 1.75, \\ 6.25 - 3t/2 & \text{if } 1.75 < t < 2.25, \\ 4.5 - \sqrt{t-2} - t/2 & \text{if } 2.25 < t < 3, \\ 2\sqrt{4-t} & \text{if } 3 < t < 4, \\ 0 & \text{if } 4 < t, \end{cases}$$

and inverts it in order to obtain the Schwarz symmetrization

$$f^*(x) = \begin{cases} 4 - x^2 & \text{if } |x| < 1, \\ (2\sqrt{2 - |x|} - 1)^2 + 2 & \text{if } 1 < |x| < 3/2 - 1/16, \\ 4 + 1/6 - 4|x|/3 & \text{if } 3/2 - 1/16 < |x| < 2 - 3/16, \\ 2 - (1 - \sqrt{4|x| - 5})^2 & \text{if } 2 - 3/16 < |x| < 2 + 1/4, \\ (3 + 1/4 - |x|)^2 & \text{if } 2 + 1/4 < x < 3 + 1/4. \end{cases}$$



(a) f (b) f^*

Figure 3.1. – The function f of equation (3.9) and its Schwarz symmetrization. The different pieces defining f and f^* are represented in different colours.

The graphs of f and of its Schwarz symmetrization are represented in Figure 3.1. Observe that, up to translations, f coincides on $(2, 4)$ with f^* on $\{|x| < 1\}$ whereas f coincides on $(0, 1) \cup (5.5, 6.5)$ with f^* on $\{2 + 1/4 < |x| < 3 + 1/4\}$. As a result,

$$\begin{aligned} & \int_{-3-1/4}^{3+1/4} |f^{*\prime\prime}|^p \leq \int_0^{6.5} |f''|^p \\ \Leftrightarrow \quad & 2\int_1^{3/2-1/16} |f^{*\prime\prime}|^p + 2\int_{2-3/16}^{2+1/4} |f^{*\prime\prime}|^p \leq \int_4^{4.5} |f''|^p + \int_5^{5.5} |f''|^p = 2^p \\ \Leftrightarrow \quad & \frac{3^{2-3p}64^p - 16}{6p - 4} \leq 2^p. \end{aligned}$$

The last inequality is false for $p = 2$ and since the function $p \in (0, \infty) \mapsto 2^p(6p - 4) - 64^p 3^{2-3p} + 16$ is negative between its first two roots which are approximately 0.667 and 2.461, we conclude that f provides a counterexample for (3.8) at least for p in this range.

Although inequality (3.8) does not hold in general even for functions with no intermediate critical value, [Duf67, section 7] shows that it holds in dimension 1 as long as one replaces the Schwarz symmetrization of f with its nonincreasing rearrangement. More precisely [Duf67, Theorem 7.1] tells that for $f \in C^2([a, b])$,

the L^p norm of $f^{\star\prime\prime}$ is smaller than the L^p norm of f'' for any $p \geq 1$. However, as we already discussed, such a statement is false, and it seems that the author forgot to make the ancillary assumption that $f^{\star\prime\prime}$ is already in L^p. Nevertheless, the proof presented is enlightening. It is based on the fact that the $(d-1)$-Hausdorff measure is the counting measure when $d = 1$ and seems not to be generalisable to the case $d > 1$. It turns out that the arguments also work when dealing with the Schwarz symmetrization f^*, but in this case some multiplicative constant pops up. Indeed, we have the following result.

Proposition 3.1. *Let ω be a bounded open subset of $\mathbb{R}^1$ with regular boundary and $f \in W^{2,p} \cap C^1(\omega)$ with $1 \leq p < \infty$ such that $f = 0$ on $\partial\omega$. Assume that f is nonnegative and admits a finite number of critical values. Then, there exists a distribution S over ω^* with finite support such that*

$$\int_{\omega^*} |f^{*\prime\prime} - S|^p \leq 4^p \int_{\omega} |f''|^p.$$

More precisely, if $0 \leq t_1 < ... < t_n < t_{n+1} = \max f$ are the critical values of f, then S is a linear combination of Dirac masses over $\cup_{i=1}^n \partial\{f^* = t_i\}$. In particular, if f has no critical value of level $t \in [0, \max f)$,

$$\int_{\omega^*} |f^{*\prime\prime}|^p \leq 4^p \int_{\omega} |f''|^p. \tag{3.10}$$

Remark 3.2.
1. *Since f shall be extended by continuity up to the boundary by Sobolev embeddings, it makes sense to prescribe its value on the boundary.*
2. *We only consider critical values originating from critical points contained within ω. As a result (3.10) holds even if f' vanishes on $\partial\omega$.*
3. *The assumption $f \in C^1(\omega)$ is necessary since one needs the notion of critical value to be well-defined. It may be removed as long as $p > 1$ thanks to Sobolev embeddings.*

Proof. Set for convenience $t_0 = 0$ and let $0 \leq i \leq n$. The proof uses formula (29) of [LS89], which asserts that as long $t \in (t_i, t_{i+1})$, the second derivative of the distribution function μ of f is

$$\mu''(t) = \int_{\{u=t\}} \frac{\Delta_0 f}{|\nabla f|} d\sigma, \tag{3.11}$$

where $\Delta_0 f = div\left(\frac{\nabla f}{|\nabla f|^2}\right)$ and σ is the $(d-1)$-Hausdorff measure. To apply [LS89], we used that $|\nabla f| > \delta$ over $\{t - \epsilon < f < t + \epsilon\}$, for some $\epsilon, \delta > 0$. Considering the set $\omega_i := \{t_i < f < t_{i+1}\}$, thanks to the coarea formula (Theorem A.1),

$$\int_{\omega_i} |\Delta f|^p = \int_{t_i}^{t_{i+1}} dt \int_{\{f=t\}} \frac{|\Delta f|^p}{|\nabla f|} d\sigma.$$

By the reverse Hölder inequality,

$$\int_{\{f=t\}} \frac{|\Delta f|^p}{|\nabla f|}\,d\sigma = \int_{\{f=t\}} \frac{|\Delta f|^p}{|\nabla f|^p}|\nabla f|^{p-1}\,d\sigma \geq \left(\int_{\{f=t\}} \frac{|\Delta f|}{|\nabla f|}\,d\sigma\right)^p \left(\int_{\{f=t\}} \frac{1}{|\nabla f|}\,d\sigma\right)^{1-p}$$
$$= \left(\int_{\{f=t\}} \frac{|\Delta f|}{|\nabla f|}\,d\sigma\right)^p (-\mu'(t))^{1-p},$$

where we used equation (2.2) – holding as long as the coarea formula is valid. Let us focus on the last integral. We would like to express it in terms of μ''. For that compute

$$\Delta f = \Delta\left(\frac{f}{|\nabla f|^2}|\nabla f|^2\right) = |\nabla f|^2\Delta_0 f + 2\frac{\nabla f D^2 f \nabla f}{|\nabla f|^2}.$$

To carry on, the one-dimensional setting is crucial, since in this case the previous turns into

$$f'' = -|f'|^2\Delta_0 f.$$

In view of (3.11), this leads to the identity

$$\mu''(t) = -\int_{\{f=t\}} \frac{f''}{|f'|^3}\,d\sigma,$$

and to the estimation

$$|\mu''(t)| \leq \left(\max_{\{f=t\}} \frac{1}{|f'|}\right)^2 \int_{\{f=t\}} \frac{|f''|}{|f'|}\,d\sigma.$$

Now, following [Duf67, section 7], the main argument is that when $d = 1$, σ is nothing else but the counting measure, hence

$$\max_{\{f=t\}} \frac{1}{|f'|} \leq \int_{\{f=t\}} \frac{1}{|f'|}\,d\sigma = |\mu'(t)|. \tag{3.12}$$

Putting all together, we finally get

$$\int_{\omega_i} |\Delta f|^p \geq \int_{t_i}^{t_{i+1}} \frac{|\mu''|^p}{|\mu'|^{3p-1}}. \tag{3.13}$$

Now replace f by f^* – which has no critical value of level between t_i and t_{i+1} by (2.2) – and ω_i by the annulus $\overset{*}{\omega}_i := \{t_i < f^* < t_{i+1}\}$. Perform the same discussion as above and observe that all the inequalities turn into equalities except (3.12). Nevertheless, in the case of f^*, since $\sigma(\{f^* = t\}) = 2$ we have

$$|\mu'(t)| = 2\max_{\{f^*=t\}} \frac{1}{|f^{*\prime}|},$$

and consequently

$$\int_{\overset{*}{\omega_i}} |\Delta f^*|^p = 4^p \int_{t_i}^{t_{i+1}} \frac{|\mu''|^p}{|\mu'|^{3p-1}}. \tag{3.14}$$

Combine (3.13) and (3.14) and sum over $i \in \{0, \dots, n\}$ in order to get

$$\int_{\cup_{i=0}^n \overset{*}{\omega_i}} |f^{*\prime\prime}|^p \leq 4^p \int_{\cup_{i=0}^n \omega_i} |f''|^p \leq 4^p \int_\omega |f''|^p.$$

To conclude, set $S := f^{*\prime\prime} - g$, where $g \in L^p(\omega^*)$ is defined as follows. The restriction of the distribution $f^{*\prime\prime}$ to $\cup_{i=0}^n \overset{*}{\omega_i}$ is in L^p by the previous discussion, and we extend it by zero on ω^* to define g.

Observe that S is $W^{-1,p}(\omega^*)$ since g is $L^p(\omega^*)$ and f^* is $W^{1,p}(\omega^*)$ by the Pólya-Szegő inequality – here we used that $f \in W_0^{1,p}(\omega)$. Moreover S is clearly supported in $\cup_{i=1}^{n+1}\{f^* = t_i\}$. Actually, since f^* is constant over $\{f^* = t_i\}$, $f^{*\prime\prime}$ and hence S vanish in its interior. Therefore, S is supported in $\cup_{i=1}^{n+1}\partial\{f^* = t_i\}$, which is a finite set. As a result, S must be a linear combination of Dirac masses and derivatives of Dirac masses. But in one dimension, $\delta^{(k)}$ is in $W^{s,p}$ if and only if $s < -(1-1/p)-k$. Consequently, $S \in W^{-1,p}(\omega^*)$ is only a combination of Dirac masses.

It remains only to show that $\partial\{f^* = t_{n+1}\}$ does not intersect supp S. For this, we use the notation $A_\epsilon := \partial\{f^* = t_{n+1}\} + B_\epsilon(0)$, and show that $\|f^{*\prime}\|_{L^2(A_\epsilon)} \underset{\epsilon\to 0}{=} o(\sqrt{\epsilon})$. Let us explain how this allows to conclude. For some small $\epsilon_0 > 0$, A_ϵ remains disjoint from $\{f^* = t_i\}$ for all $1 \leq i \leq n$ and all $\epsilon < \epsilon_0$. Then, taking an arbitrary test function φ with support in A_{ϵ_0}, we need to prove that $\langle S, \varphi\rangle = 0$. Consider a smooth cutoff function $0 \leq \chi_\epsilon \leq 1$ over a neighbourhood of $\partial\{f^* = t_{n+1}\}$ with support in A_ϵ and such that $\|\nabla\chi_\epsilon\|_{L^\infty} \underset{\epsilon\to 0}{=} O(1/\epsilon)$. Then $\|\nabla\chi_\epsilon\|_{L^2} \underset{\epsilon\to 0}{=} O(1/\sqrt{\epsilon})$ Since the support of $(1-\chi_\epsilon)\varphi$ is disjoint from that of S for all $\epsilon < \epsilon_0$, we get

$$|\langle S, \varphi\rangle| = |\langle S, \varphi\chi_\epsilon\rangle| \leq \int_{A_\epsilon} |g\varphi| + \|f^{*\prime}\|_{L^2(A_\epsilon)}\|\varphi\|_{W^{1,\infty}}\|\chi_\epsilon\|_{H^1}.$$

The first term above is $O(\|g\|_{L^p(A_\epsilon)}) \to 0$. The second term is $o(1) \to 0$. Therefore, we only have to show that $\|f^{*\prime}\|_{L^2(A_\epsilon)} \underset{\epsilon\to 0}{=} o(\epsilon^{1/2})$. This shall be obtained for instance thanks to the Pólya-Szegő inequality. Indeed, if $f^* = t_\epsilon$ on $\partial(\{f^* = t_{n+1}\} + B_\epsilon(0))$, then

$$\|f^{*\prime}\|_{L^2(A_\epsilon)} = \|(f-t_\epsilon)^{*\prime}\|_{L^2(\{f^*>t_\epsilon\})} \leq \|(f-t_\epsilon)'\|_{L^2(\{f>t_\epsilon\})} \leq \|f'\|_{L^\infty(\{t_{n+1}>f>t_\epsilon\})}|A_\epsilon|^{1/2}.$$

Above we used that $|\{t_\epsilon < f < t_{n+1}\}| \leq |A_\epsilon|$ by equimeasurability of Schwarz symmetrization. Recall that $|A_\epsilon| = O(\epsilon)$. Furthermore, since the Hausdorff distance between $\{f > t_\epsilon\}$ and $\{f = t_{n+1}\}$ – where $f' = 0$ – tends to 0 and since f' is continuous, we conclude as desired that $\|f'\|_{L^\infty(\{t_{n+1}>f>t_\epsilon\})} \to 0$. □

Besides possible generalisations to higher dimension, several questions naturally arise from the previous analysis. First, since functions with no intermediate critical

value do not fulfil inequality (3.8) in general, it remains to identify – or at least understand more accurately – the class of functions satisfying (3.8). In this direction, a first step would be to prove that a function with an intermediate critical value never satisfies (3.8). Indeed, even if expected, it has not been proved up to our knowledge that intermediate critical values inescapably lead to singularities in the second order derivative of the Schwarz symmetrization. On the other hand, [Cia00, Theorem 3] indicates that functions having concave Schwarz symmetrization behave properly with regard to (3.8) – at least for $p = 1$. Therefore one could study whether concavity yields (3.8). Of course, in this case it would be needed to replace $W_0^{2,p}$ with $W^{2,p} \cap W_0^{1,p}$ for the question not to be trivial.

A question of different nature arises from the next remark. Observe that, as consequence of inequality (3.10) one has the following relation

$$\sup_{f \in \mathcal{C}_p} \frac{\int_{\omega^*} |f^{*\prime\prime}|^p}{\int_\omega |f''|^p} \leq 4^p,$$

where $\mathcal{C}_p$ is the class of nonzero functions in $W^{2,p} \cap C^1(\omega)$ vanishing on the boundary and having no critical value of level $t \in [0, \max f)$. Moreover, our counterexample of equation (3.9), which does belong to $\mathcal{C}_p$, proves that

$$1 < \sup_{f \in \mathcal{C}_p} \frac{\int_{\omega^*} |f^{*\prime\prime}|^p}{\int_\omega |f''|^p}.$$

It would be of interest to investigate the value in $(1, 4^p]$ of such a supremum and to study if it is attained for some function. In this respect, let us mention that the value 4^p cannot be reached by any function in $\mathcal{C}_p$, since this would lead to an equality in (3.10). But the equality case cannot occur since, in the proof of Proposition 3.1, inequality (3.12) always holds strictly.

4. Existence of an optimal shape for the first eigenvalue of polyharmonic operators

This chapter is an adaptation of the article Leylekian, R. *Existence of an optimal shape for the first eigenvalue of polyharmonic operators.* Preprint at https://arxiv.org/abs/2402.11713. 2024.

The purpose of this section is to prove existence and partial regularity of an optimal shape for the first eigenvalue of the bilaplacian. Actually, since this needs little ancillary effort, we will address the more general case of Dirichlet polylaplacians. We recall that for $m \in \mathbb{N}^*$ (fixed from now on), the eigenvalues of the Dirichlet polylaplacian of order m are the real numbers Γ for which there is a nontrivial function $u \in H_0^m(\Omega)$ solving

$$\left\{ \begin{array}{rcll} (-\Delta)^m u & = & \Gamma u & \text{in } \Omega, \\ \partial_n^i u & = & 0 & \text{on } \partial\Omega, \text{ for all } 0 \leq i < m. \end{array} \right. \tag{4.1}$$

Here $\Omega \subseteq \mathbb{R}^d$ is an open set of bounded volume, which is more general than a bounded open set. The lowest eigenvalue is denoted $\Gamma(\Omega)$, and we are interested in solving

$$\min\{\Gamma(\Omega) : \Omega \subseteq \mathbb{R}^d \text{ open set}, |\Omega| = c\}, \tag{4.2}$$

where c is as usual a given positive real number. For $m = 1$, the Faber-Krahn inequality shows that (4.2) admits the ball as a solution. For $m = 2$ we get back the bilaplacian, hence the ball is the solution in dimension $d = 2, 3$, whereas for $d \geq 4$ the problem is open. For $m \geq 3$, it seems that nothing is known about the solution of (4.2) in any dimension (except dimension 1). In this context, it is a natural question to ask whether there is at least one solution to (4.2). We will give a positive answer to this question for a certain range of dimensions by proving the following.

Theorem 4.1. *Let $2 \leq d \leq 4m$. Problem (4.2) admits a solution Ω. Moreover, any corresponding eigenfunction extended by 0 outside Ω is $C^{m-1,\alpha}(\mathbb{R}^d)$ for all $0 < \alpha < 1$.*

Remark 4.2. *1. We do not claim anything on the boundedness of an optimal shape.*

2. Even if the eigenfunction is somewhat regular, we do not claim any regularity on the optimal shape more than its openness.

3. The regularity of the eigenfunction is not sharp in general (see [Buc+15, Theorem 6.1]).

4. In the case $m = 1$, the restriction $d \leq 4m$ can be removed (see also Remark 4.6).

The proof of Theorem 4.1 will be divided into two steps, following the approach of [Sto23]. First, we prove the existence of an optimal shape in a wider class than the class of open sets. As we shall see, this part of the proof does not require the restriction $2 \leq d \leq 4m$. Then, we show that the generalised optimal shape enjoys some regularity properties allowing to solve (4.2). Let us begin with the existence of a generalised optimal shape.

Existence of an optimal shape

As a first step towards a proof of Theorem 4.1, similar to what was explained in subsection 2.6.4 we would like to provide an optimal shape for (4.2) in the wider class of quasi-open sets. Here a notion of H^m-capacity and its associated toolkit will be needed since we are working with higher-order operators. Surprisingly however, the optimal shape that we intend to exhibit will only belong to the class of H^1-quasi-open sets.

Before plunging into details, let us recall several breakthroughs going in the direction of providing an optimal shape for (4.2). We cite [AB03; Sto23], in which an analogous problem is tackled for the buckling bilaplace eigenvalue problem, using a concentration-compactness technique. Another interesting result is [Buc00, Theorem 3.5], which establishes the existence of a quasi-open optimal shape for a family of spectral optimization problems similar to (4.2). However, the class of admissible shapes is restricted to some design region. Moreover, we emphasise that the definition chosen in [Buc00] for the Sobolev space $H_0^m(\Omega)$ does not match the classic one, unless Ω enjoys some regularity (see [Buc00, p. 109] for details). As a consequence, even if the optimal shape would be open, it would not necessarily solve (4.2). In the same fashion, a similar result but with a different approach shall be found in [Sto21]. There, the existence of an open optimal shape for problem (4.2) in dimension $d \geq 4$ is evidenced, still among sets lying in a design region. However, let us mention that it is unclear in the proof of [Sto21, Theorem 5] whether the function v is well-defined. Therefore, some of the results might need ancillary arguments.

In the present chapter, following the approach of [AB03], the first main result that we would like to prove is stated below.

Theorem 4.3. *Let $d \in \mathbb{N}^*$. The following problem admits a solution.*

$$\min\{\Gamma(\Omega) : \Omega \subseteq \mathbb{R}^d,\ \Omega\ H^1\text{-quasi-open},\ |\Omega| = c\} \tag{4.3}$$

Remark 4.4.
1. *Note that the optimal shape need not be bounded, we only know that it has bounded measure.*
2. *At first glance it should be surprising that for m arbitrary, the appropriate class of quasi-open sets for getting existence is the class of H^1-quasi-open sets instead of H^m-quasi-open sets. This will become clear as soon as we will understand how the optimal shape is constructed.*

The proof of Theorem 4.3 is achieved in paragraph 4.2. Following [AB03], the idea is to apply the concentration-compactness principle (see Lemma 4.13) directly to the first eigenfunctions of a minimizing sequence of open sets. As in paragraph 2.6.4.1, we show that only compactness shall occur, hence we are able to extract a converging subsequence of first eigenfunctions. The optimal shape is identified as the set of points where the limit function or one of its derivatives up to order $m-1$ does not vanish. Such a set is only H^1-quasi-open in general, which explains why we needed to widen the class of admissible sets.

Regularity of the optimal shape

After providing an optimal shape among quasi-open sets, the next step in the proof of Theorem 4.1 is to show that this optimal set is actually open, and to give some hint on its regularity. A wide literature have been produced on this subject, especially in the case $m = 1$ (see for instance [Buc+15; BBV13] and the references given at the end of paragraph 4.4). For the case $m = 2$, the work [Sto21], following [Sto22], gives partial answers, but again with the assumption of a prescribed design region. This assumption was dropped successfully only in the case of the buckling bilaplacian [Sto23]. Therefore, as for the existence, the regularity of an optimal shape seems still open in full generality. Indeed, gaining regularity from scratch is a quite difficult task in general. As a consequence, we will not prove that any optimal shape for (4.3) is a regular open set (since this is false). Alternatively, in the wake of [Buc+15; Sto23], we will obtain regularity for the first eigenfunction, and from this fact build an open set competing with the optimal quasi-open set of Theorem 4.3. More precisely, we will prove the following.

Theorem 4.5. *Assume that $2 \leq d \leq 4m$ and that there exists a measurable set Ω solving*

$$\min\{\Gamma(\omega) : \omega \subset \mathbb{R}^d \text{ measurable},\ |\omega| = c\}. \tag{4.4}$$

Then, any first eigenfunction of Ω extended by 0 outside Ω is $C^{m-1,\alpha}(\mathbb{R}^d)$ for all $0 < \alpha < 1$. As a consequence, there exists an(other) optimal set $\tilde{\Omega}$ for (4.4) which is open.

Remark 4.6. *1. The regularity which is obtained for the eigenfunction is not sharp in general (see [Buc+15, Theorem 6.1]).*

2. The restriction $d \leq 4m$ comes from the fact that Theorem 4.19 is used with $f = \Gamma(\Omega)u$ which needs to belong to $L^{d/m}_{loc}(\mathbb{R}^d)$ (here u is an eigenfunction over Ω). When $d \leq 4m$, this holds by Sobolev embeddings. In dimension greater than $4m$, such an integrability of u is known only when Ω is already very regular (for instance C^{2m}). In the special case $m = 1$, this condition on u is also automatically fulfilled since the eigenfunctions of the Laplacian are bounded [Dav90, Example 2.1.8]. For $m > 1$, it would be of interest to study the boundedness of the eigenfunctions in measurable sets or even in open sets with regularity lower than C^{2m}.

The proof of Theorem 4.5 is given in paragraph 4.3 page 102. The strategy for proving $C^{m-1,\alpha}$ regularity for an eigenfunction u over Ω is to resort to Morrey's Theorem (see [Gia83, Chapter III, Theorem 1.1]) which requires some control on the growth of the L^2 norm of the derivatives of u on small balls. As shown in the proof of Theorem 4.19, such a control is precisely a crucial property enjoyed by quasi-minimizers of Dirichlet energy functionals (see paragraph 4.3 for the definition and properties of quasi-minimizers). As a consequence, one boils down to showing that u is a quasi-minimizer of some Dirichlet energy functional, which will come from the optimality of Ω. After proving the regularity of the eigenfunction, we will define the competitor $\tilde{\Omega}$ as the set of points where u or one of its derivatives up to order $m-1$ does not vanish.

Structure of the chapter

The chapter is structured as follows. Section 4.1 contains preliminaries. In particular the notions of capacities, quasi-open sets, and related concepts are recalled. Then, the existence result is addressed in section 4.2. There, the method of concentration-compactness adapted to our situation (Lemma 4.13) is presented after what the proof of Theorem 4.3 is achieved. Section 4.3 deals with regularity results. Local quasi-minimizers are introduced and briefly studied and the proof of Theorem 4.5 is finally performed. Section 4.4 concludes, providing the proof of Theorem 4.1 and discussing some perspectives.

4.1. Preliminaries

We recall that if Ω is an open set of bounded measure, the eigenvalue $\Gamma(\Omega)$ can be characterised variationally by the formula

$$\Gamma(\Omega) = \min_{v \in H^m_0(\Omega)} \frac{\int_\Omega |D^m v|^2}{\int_\Omega v^2}, \tag{4.5}$$

and that the minimum of (4.5) is achieved thanks to the compact embedding $H_0^m(\Omega) \hookrightarrow L^2(\Omega)$. Moreover, any minimizer is a first eigenfunction over Ω associated with $\Gamma(\Omega)$.

As explained above, we will need to define the first eigenvalue on sets which are not necessarily open. For that, we will use the right-hand side of formula (4.5) as a definition, hence we need to investigate the functional space $H_0^m(\Omega)$ in the case where Ω is a general set. This involves the notions of capacities of orders 1 up to m. For the sake of brevity, we will use the prefix k when talking about capacity of order k and related notions. For instance, the k-capacity, abbreviating H^k-capacity, refers to the capacity associated with the H^k norm.

We already introduced the 1-capacity in subsection 2.6.4. In the same fashion, one may define an m-capacity ($m \in \mathbb{N}$), denoted cap_m and the corresponding notions of m-quasi-open sets and m-quasi-continuous functions (see [AH96] for a very general theory of capacities). Formally, for a set $E \subseteq \mathbb{R}^d$,

$$cap_m(E) := \inf\{\|v\|^2_{H^m(\mathbb{R}^d)} : \ v \in H^m(\mathbb{R}^d),\ v \geq 1 \text{ a.e. on a neighbourhood of } E\}.$$

For our purpose, it is enough to think of the m-capacity as an outer measure over $\mathbb{R}^d$ which is finer than the $(m-1)$-capacity in the sense that $cap_m(E) = 0 \Rightarrow cap_{m-1}(E) = 0$ for any $E \subseteq \mathbb{R}^d$. Then, m-quasi-open sets are defined to be sets which are open up to a residue of arbitrary small m-capacity. Lastly, m-quasi-continuous functions are functions whose preimages of open sets are m-quasi-open sets. In other words,

Definition 4.7. *A set E is m-quasi-open if for any $\epsilon > 0$, there is $\omega \subseteq \mathbb{R}^d$ such that $E \cup \omega$ is open and $cap_m(\omega) < \epsilon$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is m-quasi-continuous if for any open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is open.*

We recall that any $u \in H^m(\mathbb{R}^d)$ has an m-q.e. well-defined m-quasi-continuous representative and we will always identify u with its representative. When Ω is open, the space $H_0^m(\Omega)$ shall be characterised thanks to the capacities up to order m in the following way (see for instance [GLM23, p.2]):

$$H_0^m(\Omega) = \{u \in H^m(\mathbb{R}^d) : D^k u = 0 \ (m-k)\text{-q.e. in } \Omega^c, \text{ for } 0 \leq k < m\}. \tag{4.6}$$

Let us point out that in general $H_0^m(\Omega)$ does not coincide with the set of functions in $H^m(\mathbb{R}^d)$ vanishing a.e. outside Ω. Indeed, such a property requires some regularity on Ω (see [GLM23]). Observe that as the right-hand member of (4.6) makes sense regardless of the regularity of Ω, we shall use it to define $H_0^m(\Omega)$ even when Ω is not open. Moreover, the compact injection

$$H_0^m(\Omega) \hookrightarrow L^2(\Omega) \tag{4.7}$$

holding when Ω was open, remains when Ω is only a measurable set of finite measure, as explained below.

Definition 4.8. *Let Ω be a subset of $\mathbb{R}^d$. Then, $H_0^m(\Omega)$ is defined by (4.6).*

Lemma 4.9. *Let Ω be a measurable set of finite measure. Then, injection (4.7) is compact.*

Proof. Let $(u_n)_n$ be sequence in $H_0^m(\Omega)$. By outer regularity of the Lebesgue measure, there exists an open set $\tilde{\Omega} \supseteq \Omega$ of measure $|\Omega| + 1 < +\infty$. Then,

$$H_0^m(\Omega) \subseteq H_0^m(\tilde{\Omega}) \hookrightarrow L^2(\tilde{\Omega})$$

is compact, so we shall extract from $(u_n)_n$ a subsequence converging in $L^2(\tilde{\Omega})$, hence in $L^2(\Omega)$. □

Besides, we recall that by [AH96, Theorem 10.1.1], for any set Ω, $H_0^m(\Omega)$ coincides with the closure of $C_c^\infty(\Omega)$ with respect to $H^m(\mathbb{R}^d)$. Lastly, as a consequence of Lemma 4.9, formula (4.5) generalises to any measurable set of finite measure, and the minimum in (4.5) is also achieved in this setting. Before going further, let us mention the following useful observation, which follows immediately from (4.6).

Lemma 4.10. *Let $u \in H^m(\mathbb{R}^d)$. Then, $u \in H_0^m(\bigcup_{k=0}^{m-1}\{D^k u \neq 0\})$.*

Remark 4.11. *Although Lemma 4.10 looks obvious in view of the previous definition of H_0^m, it has to be noted that the set $\omega := \bigcup_{k=0}^{m-1}\{D^k u \neq 0\}$ depends upon the choice of a quasi-continuous representative not only for u but also for its derivatives. Consequently, one may find a set $\tilde{\omega}$ coinciding with ω only 1-q.e. and such that $u \in H_0^m(\tilde{\omega})$. This might appear paradoxical at first glance, since the assertion "$u \in H_0^m(\tilde{\omega})$ for all $\tilde{\omega} = \omega$ 1-q.e." is false when $m > 1$. The nuance is that we only claim the existence of **some** $\tilde{\omega} = \omega$ 1-q.e. such that $u \in H_0^m(\tilde{\omega})$. In other words, ω is defined more than just 1-quasi everywhere.*

To conclude those preliminaries, let us explain the relevance of our procedure of relaxing the class of admissible domains for problem (4.2). This is justified by the fact that minimizing Γ among open sets, quasi-open sets, or even measurable sets does not change the value of the infimum, as stated in the next lemma. Its proof relies on the homogeneity of Γ (see equation (1.8)).

Lemma 4.12. *For any $k \in \mathbb{N}^*$,*

$$\inf_{\substack{\Omega \text{ open} \\ |\Omega|=c}} \Gamma(\Omega) = \inf_{\substack{\Omega \text{ k-quasi-open} \\ |\Omega|=c}} \Gamma(\Omega) = \inf_{\substack{\Omega \text{ measurable} \\ |\Omega|=c}} \Gamma(\Omega) > 0.$$

Proof. Clearly,

$$\inf_{\substack{\Omega \text{ open} \\ |\Omega|=c}} \Gamma(\Omega) \geq \inf_{\substack{\Omega \text{ k-quasi-open} \\ |\Omega|=c}} \Gamma(\Omega) \geq \inf_{\substack{\Omega \text{ measurable} \\ |\Omega|=c}} \Gamma(\Omega),$$

since any open set is also quasi-open, and any quasi-open set is measurable (by [HP05, Remark 3.3.39]). For the reverse inequality, observe that if Ω is a measurable

set of volume c, by outer regularity of the Lebesgue measure there exists for every $\epsilon > 0$ an open set Ω_ϵ of volume $c+\epsilon$ such that $\Omega \subseteq \Omega_\epsilon$, and hence $H_0^m(\Omega) \subseteq H_0^m(\Omega_\epsilon)$. Then, by (1.8),

$$\Gamma(\Omega) \geq \Gamma(\Omega_\epsilon) \geq \left(\frac{c}{c+\epsilon}\right)^{\frac{2m}{d}} \inf_{\substack{\Omega \text{ open} \\ |\Omega|=c}} \Gamma(\Omega),$$

and we get the result by taking $\epsilon \to 0$. In order to prove that the infimum does not vanish, one may consider an open set Ω and combine the classic Faber-Krahn inequality with inequality (1.10). □

4.2. Existence of a quasi-open optimal shape

As already mentioned, the concentration-compactness principle is the main tool used for proving Theorem 4.3. Therefore we must adapt to our needs the standard concentration-compactness result (see Lemma 2.17). We use the notation $\tau_y u$ to designate the translation of a given function $u \in H^m(\mathbb{R}^d)$ by a given point $y \in \mathbb{R}^d$, that is the function defined by $\tau_y u(x) = u(x-y)$.

Lemma 4.13. *Let $(u_n)_n$ be a bounded sequence of functions in $H^m(\mathbb{R}^d)$ such that $\|u_n\|_{L^2}^2 \to \lambda$. Then, up to extracting a subsequence, one of the following occurs:*

(i) (compactness) there exists a sequence of points $(y_n)_n \subseteq \mathbb{R}^d$ such that

$$\forall \epsilon > 0, \exists R > 0, \quad \liminf_{n\to\infty} \|\tau_{y_n} u_n\|_{L^2(B_R)}^2 \geq \lambda - \epsilon.$$

(ii) (vanishing)

$$\forall R > 0, \quad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^d} \|\tau_y u_n\|_{L^2(B_R)}^2 = 0.$$

(iii) (dichotomy) there exists $\alpha \in (0,\lambda)$ and functions u_n^1, and u_n^2 in $H^m(\mathbb{R}^d)$ such that

$$\begin{aligned} &\|u_n - (u_n^1 + u_n^2)\|_{L^2} \to 0, \\ &\|u_n^1\|_{L^2}^2 \to \alpha, \qquad \|u_n^2\|_{L^2}^2 \to \lambda - \alpha, \\ &\liminf \left[\|D^m u_n\|_{L^2}^2 - \|D^m u_n^1\|_{L^2}^2 - \|D^m u_n^2\|_{L^2}^2\right] \geq 0. \end{aligned}$$

Moreover, for any $0 \leq k < m$, up to some sets of zero $(m-k)$-capacity,

$$\begin{aligned} &dist\left(\omega_{n,k}^1, \omega_{n,k}^2\right) \to \infty, \\ &\omega_{n,k}^1, \omega_{n,k}^2 \subseteq \omega_{n,k}, \end{aligned}$$

where $\omega_{n,k} := \bigcup_{j=0}^k \{D^j u_n \neq 0\}$ and $\omega_{n,k}^i := \bigcup_{j=0}^k \{D^j u_n^i \neq 0\}$ for $i = 1, 2$.

The delicate proof of Lemma 4.13 is given in Appendix B. Equipped with the concentration-compactness principle, we can now turn to the proof of Theorem 4.3,

which claims the existence of a 1-quasi-open set minimizing Γ under volume constraint. As in [AB03], the strategy is to apply the concentration-compactness procedure to the eigenfunctions of a minimizing sequence of sets, and to show that only compactness occurs. As a consequence, the eigenfunctions are shown to converge in L^2 towards a limit function. The final step is to identify the optimal set from the limit function, which is performed thanks to the next proposition.

Proposition 4.14. *Let Ω_n be a sequence of measurable sets of volume c, and $u_n \in H_0^m(\Omega_n)$ the corresponding eigenfunctions converging to some u in $H^m(\mathbb{R}^d)$ weakly and in $L^2(\mathbb{R}^d)$ strongly. Then there exists a 1-quasi-open set Ω of volume $|\Omega| \leq c$ such that*

$$\liminf_n \Gamma(\Omega_n) \geq \Gamma(\Omega).$$

Proof. Thanks to weak H^m and strong L^2 convergence of $(u_n)_n$, we immediately have

$$\liminf_n \Gamma(\Omega_n) \geq \frac{\int_{\mathbb{R}^d} |D^m u|^2}{\int_{\mathbb{R}^d} u^2}.$$

That's why one would like to define the optimal shape as the domain where u or one of its derivatives does not vanish. Fortunately, the set $\Omega := \cup_{k=0}^{m-1} \{D^k u \neq 0\}$ is 1-quasi-open, and $u \in H_0^m(\Omega)$ by Lemma 4.10. Therefore,

$$\liminf_n \Gamma(\Omega_n) \geq \Gamma(\Omega).$$

It remains only to show that $|\Omega| \leq c$. For that purpose, observe that not only $(u_n)_n$ converges in $L^2(\mathbb{R}^d)$, but also that $(D^k u_n)_n$ converges in $L^2(\mathbb{R}^d)$ by interpolation, for all $0 \leq k < m$ (recall Gagliardo-Nirenberg inequalities). Moreover, up to a subsequence, we shall assume that the convergence holds almost everywhere. Applying Lemma 2.18 to the functions $U_n := (u_n, \nabla u_n, \dots, D^{m-1} u_n)$, we find that $|\Omega| \leq \liminf |\{U_n \neq 0\}|$, and thanks to the 1-q.e. inclusion $\{U_n \neq 0\} \subseteq \Omega_n$, this shows as desired that $|\Omega| \leq c$. □

Let us now apply the concentration-compactness principle and proceed to the proof of Theorem 4.3.

Proof of Theorem 4.3. Let us consider a minimizing sequence Ω_n. Actually, thanks to Lemma 4.12, we can take the same minimizing sequence for problems (4.3) and (4.2), and hence choose Ω_n to be open. Moreover, we have $\inf_n \Gamma(\Omega_n) =: I > 0$. We note $u_n \in H_0^m(\Omega_n)$ the eigenfunction on Ω_n, and assume without loss of generality that u_n is normalized in L^2. Then, by Lemma 4.13, we extract a subsequence from $(u_n)_n$ such that three situations shall occur: compactness, vanishing, or dichotomy.

If we are in the vanishing situation, for any $0 < R < \infty$,

$$\lim_n \sup_{y \in \mathbb{R}^d} \int_{B_R} |\tau_y u_n|^2 = 0.$$

If we assumed that $(u_n)_n$ were bounded in H^1, then thanks to [AB03, Lemma 3.6], there would exist a sequence of points $(y_n)_n \subseteq \mathbb{R}^d$ such that $\tau_{y_n} u_n$ admit no subsequence weakly converging to 0 in H^1. Consequently, there would exist $\epsilon > 0$ and a test function φ with support in a ball of radius say $R > 0$, such that, for n large enough,

$$\epsilon \leq \left(\int_{\mathbb{R}^d} \tau_{y_n} u_n \varphi \right)^2 \leq \sup_{y \in \mathbb{R}^d} \int_{B_R} |\tau_y u_n|^2 \int_{B_R} \varphi^2.$$

This would contradict the fact that $(u_n)_n$ vanishes. Therefore, we can extract a subsequence such that $\int_{\Omega_n} |\nabla u_n|^2 \to \infty$. But thanks to inequality (1.9), we obtain that $\int_{\Omega_n} |D^m u_n|^2 \to \infty$, showing in turn that $\Gamma(\Omega_n) \to \infty$. To conclude, the vanishing situation cannot occur.

If the dichotomy occurs, there exists $\alpha \in (0,1)$ and sequences $(u_n^1)_n$, $(u_n^2)_n$ in $H^m(\mathbb{R}^d)$ with

$$\begin{aligned}
&u_n \overset{L^2}{=} u_n^1 + u_n^2 + o(1),\\
&\|u_n^1\|_{L^2}^2 \to \alpha, \qquad \|u_n^2\|_{L^2}^2 \to \lambda - \alpha,\\
&\|D^m u_n\|_{L^2}^2 \geq \|D^m u_n^1\|_{L^2}^2 + \|D^m u_n^2\|_{L^2}^2 + o(1),\\
&dist\left(\omega_{n,k}^1, \omega_{n,k}^2\right) \to \infty, && \text{up to sets of null } (m-k)\text{-capacity},\\
&\omega_{n,k}^1, \omega_{n,k}^2 \subseteq \omega_{n,k} \subseteq \Omega_n, && \text{up to sets of null } (m-k)\text{-capacity},
\end{aligned}$$

where $\omega_{n,k} := \bigcup_{j=0}^k \{D^j u_n \neq 0\}$ and $\omega_{n,k}^i := \bigcup_{j=0}^k \{D^j u_n^i \neq 0\}$ for $i = 1, 2$. Then, thanks to the disjunction of $\{u_n^1 \neq 0\}$ and $\{u_n^2 \neq 0\}$, we have that for any $\epsilon > 0$, for any n large enough,

$$\int_{\Omega_n} |u_n|^2 \leq \int_{\Omega_n} |u_n^1|^2 + \int_{\Omega_n} |u_n^2|^2 + 2\epsilon, \quad \int_{\Omega_n} |D^m u_n|^2 \geq \int_{\Omega_n} |D^m u_n^1|^2 + \int_{\Omega_n} |D^m u_n^2|^2 - 2\epsilon.$$

Therefore,

$$\frac{\int_{\Omega_n} |D^m u_n|^2}{\int_{\Omega_n} u_n^2} \geq \frac{\int_{\Omega_n} |D^m u_n^1|^2 + \int_{\Omega_n} |D^m u_n^2|^2 - 2\epsilon}{\int_{\Omega_n} |u_n^1|^2 + \int_{\Omega_n} |u_n^2|^2 + 2\epsilon}.$$

Note that $u_n^1, u_n^2 \in H_0^m(\Omega_n)$, hence we have,

$$\int_{\Omega_n} |u_n^i|^2 \leq \frac{1}{\Gamma(\Omega_n)} \int_{\Omega_n} |D^m u_n^i|^2 \leq I^{-1} \int_{\Omega_n} |D^m u_n^i|^2, \qquad i = 1, 2.$$

As a consequence, both $\liminf_n \int_{\Omega_n} |D^m u_n^1|^2 > 0$ and $\liminf_n \int_{\Omega_n} |D^m u_n^2|^2 > 0$, so we may choose ϵ small enough for having $\int_{\Omega_n} |D^m u_n^i|^2 - \epsilon > 0$ for $i = 1, 2$ and any large n. Then, using the algebraic property $\frac{a+b}{c+d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right)$ for $a, b, c, d > 0$, and passing to the limit $n \to \infty$, we obtain that

$$\liminf_n \frac{\int_{\Omega_n} |D^m u_n|^2}{\int_{\Omega_n} u_n^2} \geq \min\left(\liminf_n \frac{\int_{\Omega_n} |D^m u_n^1|^2 - \epsilon}{\int_{\Omega_n} |u_n^1|^2 + \epsilon}, \liminf_n \frac{\int_{\Omega_n} |D^m u_n^2|^2 - \epsilon}{\int_{\Omega_n} |u_n^2|^2 + \epsilon} \right).$$

Shrinking $\epsilon \to 0$ and assuming that the minimum is achieved for the quotient involving u_n^1, we get

$$\liminf_n \Gamma(\Omega_n) \geq \liminf_n \frac{\int_{\Omega_n} |D^m u_n^1|^2}{\int_{\Omega_n} |u_n^1|^2}.$$

Now, for $i = 1, 2$, define $\omega_n^i := \omega_{n,m-1}^i = \bigcup_{0 \leq k < m} \{D^k u_n^i \neq 0\}$. Observe that ω_n^i is 1-quasi-open, and that, by Lemma 4.10, $u_n^i \in H_0^m(\omega_n^i)$ for $i = 1, 2$. In particular,

$$\liminf_n \Gamma(\Omega_n) \geq \liminf_n \Gamma(\omega_n^1).$$

Moreover, since ω_n^1 and ω_n^2 are 1-q.e. disjoint (recall that $dist(\omega_n^1, \omega_n^2) \to \infty$) subsets of Ω_n, we have

$$|\omega_n^1| \leq |\Omega_n| - |\omega_n^2|.$$

But $\liminf_n |\omega_n^2| > 0$. Indeed, since u_n^2 belongs to $H_0^m(\omega_n^2)$, we have thanks to relation (1.8) and to Lemma 4.12,

$$\begin{aligned} |\omega_n^2|^{\frac{2m}{d}} \frac{\int |D^m u_n^2|^2}{\int |u_n^2|^2} \geq |\omega_n^2|^{\frac{2m}{d}} \Gamma(\omega_n^2) &\geq c^{\frac{2m}{d}} \inf\{\Gamma(\Omega) : \Omega \text{ 1-quasi-open}, |\Omega| = c\} \\ &= c^{\frac{2m}{d}} I > 0. \end{aligned}$$

Now because u_n^2 is uniformly bounded in $H^m(\mathbb{R}^d)$, and because $\|u_n^2\|_{L^2}^2 \to 1 - \alpha \neq 0$, we end up with

$$\liminf_n |\omega_n^2| > 0.$$

As a result, we have $\limsup_n |\omega_n^1| =: c' < c$. Consequently, $\Omega_n' := (c/|\omega_n^1|)^{\frac{1}{d}} \omega_n^1$ is an admissible sequence made of 1-quasi-open sets of volume c, hence the following contradiction

$$\liminf_n \Gamma(\omega_n^1) \leq \liminf_n \Gamma(\Omega_n) \leq \liminf_n \Gamma(\Omega_n') = (c'/c)^{\frac{2m}{d}} \liminf_n \Gamma(\omega_n^1).$$

We have shown that neither vanishing nor dichotomy can occur. The only remaining possibility is compactness. This means that there exists a sequence of points $(y_n)_n \subset \mathbb{R}^d$ such that for all $\epsilon > 0$, there exists some radius $R > 0$ such that, for any n large enough,

$$\|\tau_{y_n} u_n\|_{L^2(B_R)} \geq 1 - \epsilon.$$

First, for readability, considering $\Omega_n + y_n$ instead of Ω_n, it is possible to assume that $y_n = 0$. Now, note that $(u_n)_n$ is bounded in $L^2(\mathbb{R}^d)$, hence converges weakly to some $u \in L^2(\mathbb{R}^d)$. We will see that the convergence is also L^2 strong. Indeed, let $(R_k)_k$ be an increasing sequence of radii such that, for each $k \in \mathbb{N}^*$, for each large $n \in \mathbb{N}$, $\|u_n\|_{L^2(B_{R_k})} \geq 1 - 1/k$. Note that for each $k \in \mathbb{N}^*$, the sequence $(u_n)_n$ is bounded in $H^m(B_{R_k})$, hence we may extract diagonally a subsequence

converging strongly in each $L^2(B_{R_k})$ to u. As a consequence, $\|u\|_{L^2} \geq 1$ since for any $k \in \mathbb{N}^*$, $\|u\|_{L^2} \geq \|u\|_{L^2(B_{R_k})} = \lim_n \|u_n\|_{L^2(B_{R_k})} \geq 1 - 1/k$. On the other hand, by weak convergence $\|u\|_{L^2} \leq 1$. Therefore, in $L^2(\mathbb{R}^d)$, the convergence holds simultaneously weakly and in norm, hence strongly.

Weak H^m and strong L^2 convergence of $(u_n)_n$ towards u, together with Proposition 4.14 show that it is possible to build a 1-quasi-open set Ω, of volume lower or equal than c such that $\liminf_n \Gamma(\Omega_n) \geq \Gamma(\Omega)$. Therefore, Ω, or more precisely its dilation fitting the volume constraint, solves (4.3). □

4.3. Regularity of an optimal shape

In optimization, a common strategy for getting regularity is to resort to the notion of local quasi-minimizers * as defined in [Buc+15], based on the ideas of [BHP05; AC81]. Here, we first recall the notion of local quasi-minimizer of a functional defined on $H^m(\mathbb{R}^d)$.

Definition 4.15. *Let $J : H^m(\mathbb{R}^d) \to \mathbb{R}$ be a functional and $u \in H^m(\mathbb{R}^d)$. We say that u is a local quasi-minimizer for J if there exists $C, \alpha, r_0 > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $0 < r < r_0$,*

$$J(u) \leq J(v) + Cr^d, \tag{4.8}$$

for all $v \in \mathcal{A}_{r,\alpha} := \{v \in H^m(\mathbb{R}^d) : u - v \in H_0^m(B_r(x_0)), \int_{B_r(x_0)} |D^m(u-v)|^2 \leq \alpha\}$.

Remark 4.16. *The constants C, α, and r_0 may depend not only on J, but also on u.*

For the archetypal case of Dirichlet energy functionals, a central property of quasi-minimizers is given in the next lemma.

Lemma 4.17. *Let $f \in L^2(\mathbb{R}^d)$ and $J_f : H^m(\mathbb{R}^d) \to \mathbb{R}$ defined by*

$$J_f(v) = \frac{1}{2} \int_{\mathbb{R}^d} |D^m v|^2 - \int_{\mathbb{R}^d} fv.$$

Let $u \in H^m(\mathbb{R}^d)$ be a local quasi-minimizer for J_f. Then, there exists $C, r_0 > 0$ (depending upon d,m,f and u) such that for all $x_0 \in \mathbb{R}^d$ and $0 < r < r_0$,

$$\langle (-\Delta)^m u - f, \varphi \rangle_{H^{-m}, H^m} \leq C r^{d/2} \left(\int_{B_r(x_0)} |D^m \varphi|^2 \right)^{1/2}, \qquad \forall \varphi \in H_0^m(B_r(x_0)). \tag{4.9}$$

Remark 4.18. *Observe that, in the setting of Lemma 4.17, inequality (4.9) is actually equivalent to (4.8) (see Remark 3.3 and Remark 3.4 of [Buc+15]).*

∗ We mention that as far as we know, there is no direct link between capacity and quasi-minimizers. In particular, the prefix "quasi" of quasi-minimizers does not refer to any capacity.

Proof. Let $\psi \in H_0^m(B_r(x_0))$, such that $\int_{B_r(x_0)} |D^m\psi|^2 \leq \alpha$. Because $\psi \in H_0^m(B_r(x_0))$, we may integrate by parts (recall (1.6)) to obtain

$$\langle (-\Delta)^m u - f, \psi \rangle_{H^{-m},H^m} = \int_{\mathbb{R}^d} D^m u : D^m \psi - \int_{\mathbb{R}^d} f\psi.$$

On the other hand, $v := u - \psi \in \mathcal{A}_{r,\alpha}$, hence from the quasi-minimality of u we get

$$J_f(u) \leq J_f(v) + Cr^d = J_f(u) + J_f(-\psi) - \int_{\mathbb{R}^d} D^m u : D^m \psi + Cr^d,$$

from which we deduce

$$\int_{\mathbb{R}^d} D^m u : D^m \psi - \int_{\mathbb{R}^d} f\psi \leq \frac{1}{2} \int_{\mathbb{R}^d} |D^m \psi|^2 + Cr^d.$$

Now if $\varphi \in H_0^m(B_r(x_0))$, set $\psi = \alpha^{1/2} \left(\frac{r}{r_0}\right)^{d/2} \|D^m\varphi\|^{-1}_{L^2(B_r(x_0))} \varphi$ so that $\psi \in H_0^m(B_r(x_0))$ and $\int_{B_r(x_0)} |D^m\psi|^2 \leq \alpha$. The previous discussion shows that

$$\alpha^{1/2} \left(\frac{r}{r_0}\right)^{d/2} \|D^m\varphi\|^{-1}_{L^2(B_r(x_0))} \langle (-\Delta)^m u - f, \varphi \rangle_{H^{-m},H^m} \leq \frac{\alpha}{2} \left(\frac{r}{r_0}\right)^d + Cr^d.$$

In other words, there exists $C' > 0$ such that for all $r < r_0$,

$$\langle (-\Delta)^m u - f, \varphi \rangle_{H^{-m},H^m} \leq C' r^{d/2} \left(\int_{B_r(x_0)} |D^m\varphi|^2 \right)^{1/2},$$

which concludes. □

To prove Theorem 4.5, the first step is to gain regularity on an optimal eigenfunction. For that, the concept of local quasi-minimizer provides an appropriate framework. Indeed, the following result gives the regularity of local quasi-minimizers of Dirichlet energy functionals. The regularity obtained is not sharp in general, as one shall see in [Buc+15, Theorem 3.6].

Theorem 4.19. *Assume $d \geq 2$. Let $f \in L^2 \cap L^p_{loc}(\mathbb{R}^d)$, $p \geq \frac{d}{m}$ and $J_f : H^m(\mathbb{R}^d) \to \mathbb{R}$ defined by*

$$J_f(v) = \frac{1}{2} \int_{\mathbb{R}^d} |D^m v|^2 - \int_{\mathbb{R}^d} fv. \tag{4.10}$$

Let $u \in H^m(\mathbb{R}^d)$ be a local quasi-minimizer for J_f. Then, $u \in C^{m-1,\alpha}(\mathbb{R}^d)$ for all $0 < \alpha < 1$.

Proof. We proceed in a similar way than [Sto22, Theorem 3 to Theorem 4]. More precisely, our goal is to prove that for all $\alpha \in (0,1)$, there exists $R_0, C > 0$ (depending on d,m,p,f,u, and α) such that, for all $x_0 \in \mathbb{R}^d$ and $0 < r < R_0$,

$$\int_{B_r(x_0)} |D^m u|^2 \leq Cr^{d-2+2\alpha}. \tag{4.11}$$

Indeed, due to Morrey's Theorem (see [Gia83, Chapter III, Theorem 1.1]), this yields that $D^{m-1}u \in C^{0,\alpha}$ for all $\alpha \in (0,1)$. In order to show (4.11), we let $x_0 \in \mathbb{R}^d$, $0 < R$ and define $v_{x_0,R}$ such that $u - v_{x_0,R} \in H_0^m(B_R(x_0))$ and $(-\Delta)^m v_{x_0,R} = f$ in $B_R(x_0)$. Such a function may be constructed by minimizing J_f over the set $u - H_0^m(B_R(x_0))$. Now we will estimate the two terms in the right-hand side of

$$\int_{B_r(x_0)} |D^m u|^2 \leq 2 \int_{B_r(x_0)} |D^m(u - v_{x_0,R})|^2 + 2 \int_{B_r(x_0)} |D^m v_{x_0,R}|^2. \tag{4.12}$$

We will control the first term in the larger ball $B_R(x_0)$ proceeding as in the proof of [Buc+15, inequality (A.8)]. Indeed, integrating by parts, and then using the definition of $v_{x_0,R}$, we get

$$\begin{aligned} \int_{B_R(x_0)} |D^m(u - v_{x_0,R})|^2 &= \langle (-\Delta)^m (u - v_{x_0,R}), (u - v_{x_0,R}) \rangle_{H^{-m},H^m} \\ &= \langle (-\Delta)^m u - f, (u - v_{x_0,R}) \rangle_{H^{-m},H^m}. \end{aligned} \tag{4.13}$$

The quasi-minimality of u and Lemma 4.17 show that there exists $C, R_0 > 0$ such that, as long as $R < R_0$,

$$\langle (-\Delta)^m u - f, (u - v_{x_0,R}) \rangle_{H^{-m},H_0^m} \leq C R^{d/2} \left(\int_{B_R(x_0)} |D^m(u - v_{x_0,R})|^2 \right)^{1/2}. \tag{4.14}$$

Finally, combining (4.13) and (4.14), we obtain (still for all $R < R_0$),

$$\int_{B_R(x_0)} |D^m(u - v_{x_0,R})|^2 \leq C R^d. \tag{4.15}$$

For the second term in (4.12), let us decompose $v_{x_0,R}$ into $y_{x_0,R}$ and $z_{x_0,R}$, which are defined in the following way: $y_{x_0,R}$ is the function in $H_0^m(B_R(x_0))$ such that $(-\Delta)^m y_{x_0,R} = f$; and $z_{x_0,R}$ is the m-polyharmonic function such that $v_{x_0,R} - z_{x_0,R} \in H_0^m(B_R(x_0))$. Notice that, using the definition of $y_{x_0,R}$, Cauchy-Schwarz inequality, and relation (1.8), we find

$$\begin{aligned} \int_{B_R(x_0)} |D^m y_{x_0,R}|^2 = \int_{B_R(x_0)} f y_{x_0,R} &\leq \|f\|_{L^2(B_R(x_0))} \|y_{x_0,R}\|_{L^2(B_R(x_0))} \\ &\leq C R^m \|f\|_{L^2(B_R(x_0))} \|D^m y_{x_0,R}\|_{L^2(B_R(x_0))}. \end{aligned}$$

In particular, using Hölder's inequality (with $q = p/2$) and the fact that $f \in L^p(B_{R_0}(x_0))$,

$$\int_{B_R(x_0)} |D^m y_{x_0,R}|^2 \leq C R^{2m} \int_{B_R(x_0)} f^2 \leq C R^{2m+d(1-2/p)} \leq C R^d, \tag{4.16}$$

the last inequality coming from $m - d/p \geq 0$ and $R \leq R_0$. On the other hand, using inequality (3.2) of [GM79] (see also [Sto22, Lemma 2]), we obtain by polyharmonicity

of $z_{x_0,R}$ the existence of $C > 0$ such that, for all $0 < r < R < R_0$,

$$\int_{B_r(x_0)} |D^m z_{x_0,R}|^2 \leq C \left(\frac{r}{R}\right)^d \int_{B_R(x_0)} |D^m z_{x_0,R}|^2.$$

Because $z_{x_0,R}$ minimizes J_0 amongst functions z such that $v_{x_0,R} - z \in H_0^m(B_R(x_0))$, the previous turns into the next

$$\int_{B_r(x_0)} |D^m z_{x_0,R}|^2 \leq C \left(\frac{r}{R}\right)^d \int_{B_R(x_0)} |D^m u|^2. \tag{4.17}$$

Combining (4.16) and (4.17), we find some $C > 0$ such that for all $0 < r < R < R_0$,

$$\int_{B_r(x_0)} |D^m v_{x_0,R}|^2 \leq C \left(R^d + \left(\frac{r}{R}\right)^d \int_{B_R(x_0)} |D^m u|^2 \right). \tag{4.18}$$

Thanks to (4.15) and (4.18), we can now conclude that for some $C > 0$ and for all $0 < r < R < R_0$,

$$\int_{B_r(x_0)} |D^m u|^2 \leq C \left[R^d + \left(\frac{r}{R}\right)^d \int_{B_R(x_0)} |D^m u|^2 \right]. \tag{4.19}$$

In view of [Gia83, Chapter III, Lemma 2.1, p.86] applied to $\phi(r) = \int_{B_r(x_0)} |D^m u|^2$ with $\alpha = d$ and $\beta = d - \gamma$ for some $0 < \gamma < d$, we conclude that, for all $0 < r < R < R_0$,

$$\int_{B_r(x_0)} |D^m u|^2 \leq C \left(\frac{r}{R}\right)^{d-\gamma} \left[\int_{B_R(x_0)} |D^m u|^2 + R^{d-\gamma} \right],$$

where C depends as usual on d,m,p,f,u, but now also on γ. Fixing for instance $R = R_0/2$, we end up with

$$\int_{B_r(x_0)} |D^m u|^2 \leq C r^{d-\gamma},$$

for all $r < R_0/4$, which is exactly (4.11) as long as we set $\gamma = 2 - 2\alpha < 2 \leq d$ where $0 < \alpha < 1$. □

Thanks to Theorem 4.19, it remains only to show that an optimal eigenfunction is a local quasi-minimizer, which is performed below and allows to conclude the proof of Theorem 4.5.

Proof of Theorem 4.5. To begin with, we explain the last part of Theorem 4.5, which asserts that, as long as an eigenfunction u associated with $\Gamma(\Omega)$ is $C^{m-1,\alpha}(\mathbb{R}^d)$, we shall construct an open set $\tilde{\Omega}$ which is optimal. Indeed, we proceed in a similar

way to the proof of [Buc+15, Theorem 6.1], and define

$$\tilde{\Omega} := \bigcup_{0\leq k<m} \{D^k u \neq 0\}.$$

Then, $\tilde{\Omega}$ is open and, by Lemma 4.10, $u \in H_0^m(\Omega)$. In particular, $\Gamma(\tilde{\Omega}) \leq \Gamma(\Omega)$. It remains to check that $\tilde{\Omega}$ satisfies the volume constraint. But since $u \in H_0^m(\Omega)$, we know that, for each $k < m$, $\{D^k u \neq 0\}$ is included in Ω up to a set of null $(m-k)$-capacity, and hence $\tilde{\Omega} \subseteq \Omega$ up to a set of null 1-capacity. As a consequence, $|\tilde{\Omega}| \leq |\Omega|$, and $\tilde{\Omega}$ satisfies the volume constraint up to a dilation, which concludes.

Let us now turn to the proof of the fact that u is $C^{m-1,\alpha}(\mathbb{R}^d)$. In view of Theorem 4.19, it is enough to show that u is a local quasi-minimizer of the functional J_f defined in (4.10) for some $f \in L^{d/m}(\mathbb{R}^d)$. For that purpose, we introduce the concept of quasi-minimizer of a shape functional: we say that Ω is a local quasi-minimizer of the functional Γ if there exists $C, r_0 > 0$ such that, for all $x_0 \in \mathbb{R}^d$ and $0 < r < r_0$, and for any measurable set Ω',

$$\Omega' \Delta \Omega \subseteq B_r(x_0) \quad \Longrightarrow \quad \Gamma(\Omega) \leq \Gamma(\Omega') + Cr^d,$$

where $\Omega' \Delta \Omega$ is the symmetric difference between Ω and Ω'. As shown in [Buc+15, Lemma 4.6], the local quasi-minimality of u with respect to J_f shall be deduced from the local quasi-minimality of Ω with respect to Γ. Indeed, without loss of generality, we assume u to be L^2 normalized. Then, as long as Ω is a local quasi-minimizer, for any $v \in H^m(\mathbb{R}^d)$ with $v - u \in H_0^m(B_r(x_0))$ and $r \in (0, r_0)$, we have that $v \in H_0^m(\Omega \cup B_r(x_0))$, hence, using the variational definition of $\Gamma(\Omega \cup B_r(x_0))$,

$$\Gamma(\Omega) \leq \Gamma(\Omega \cup B_r(x_0)) + Cr^d \leq \frac{\int_{\mathbb{R}^d} |D^m v|^2}{\int_{\mathbb{R}^d} v^2} + Cr^d, \tag{4.20}$$

Now let $\alpha > 0$ and take an arbitrary $v \in \mathcal{A}_{r,\alpha}$ (recall Definition 4.15). Because u is normalized, $r < r_0$ and $\int_{B_r(x_0)} |D^m(u-v)|^2 < \alpha$, we have

$$\int_{\mathbb{R}^d} v^2 \leq 2 \int_{\mathbb{R}^d} u^2 + \frac{2\alpha}{\Gamma(B_{r_0}(x_0))} \leq C,$$

hence after multiplying (4.20) by $\int_{\mathbb{R}^d} v^2$ we obtain

$$\Gamma(\Omega) \int_{\mathbb{R}^d} v^2 \leq \int_{\mathbb{R}^d} |D^m v|^2 + Cr^d.$$

Use that $v^2 \geq 2uv - u^2$ and find

$$-\Gamma(\Omega) \int_{\mathbb{R}^d} u^2 \leq \int_{\mathbb{R}^d} |D^m \varphi|^2 - 2\Gamma(\Omega) \int_{\mathbb{R}^d} uv + Cr^d.$$

Observe that this reads $J_f(u) \leq J_f(v) + Cr^d$ for $f = \Gamma(\Omega)u$. As a result, u is a

local quasi-minimizer of J_f with $f = \Gamma(\Omega)u$. At this point, remark that thanks to Sobolev injections, $u \in L^p(\Omega)$ for $p = 2d/(d-2m)$. Consequently, the restriction $d \leq 4m$ yields $p \geq d/m$. Thanks to Theorem 4.19, u is then $C^{m-1,\alpha}(\mathbb{R}^d)$. Thus, it remains only to show the local quasi-minimality of Ω with respect to Γ. For this, take some $\Omega' \Delta \Omega \subseteq B_r(x_0)$. Since $\Gamma(\Omega' \cup \Omega) \leq \Gamma(\Omega')$ it is actually enough to consider the case $\Omega \subseteq \Omega'$. Then, as in the proof of [Buc+15, Theorem 6.1], setting $t = (|\Omega'|/|\Omega|)^{\frac{1}{d}}$ and recalling (1.8),

$$\begin{aligned}\Gamma(\Omega) \leq \Gamma(\Omega'/t) \leq t^{2m}\Gamma(\Omega') &\leq \Gamma(\Omega') + (t^{2m}-1)\Gamma(\Omega') \\ &\leq \Gamma(\Omega') + C_{r_0}(t^d - 1)\Gamma(\Omega) \leq \Gamma(\Omega') + \Lambda(|\Omega' \setminus \Omega|),\end{aligned}$$

with $\Lambda = C_{r_0}\Gamma(\Omega)/|\Omega|$. Note that, besides L'Hôpital's rule, we used the fact that t is bounded irrespective of $\Omega' \subseteq \Omega \cup B_{r_0}(x_0)$ in order to get the bound $t^{2m} - 1 \leq C_{r_0}(t^d - 1)$. Since $\Omega' \setminus \Omega \subseteq B_r(x_0)$ the previous shows as expected that Ω is a local quasi-minimizer. □

4.4. Proof and perspectives

Fortified by Theorem 4.3 and Theorem 4.5, we shall propose a proof for Theorem 4.1.

Proof of Theorem 4.1. Let $2 \leq d \leq 4m$. Due to Theorem 4.3, there exists a 1-quasi-open set solving (4.3). From Lemma 4.12 we deduce that this set also solves (4.4). Then, as desired, Theorem 4.5 produces an open set Ω solving (4.2). Moreover, Ω also solves (4.4) by Lemma 4.12. Applying again Theorem 4.5, we conclude that any first eigenfunction on Ω is $C^{m-1,\alpha}(\mathbb{R}^d)$. □

Theorem 4.1 is of course an advance in the context of shape optimization. However, as stated in Remark 4.2, it has several limits besides the dimensional restriction. Indeed, as illustrated in paragraph 2.6.4 (see also chapter 5), existence results are used as a preliminary step in the derivation of optimality conditions in view of identifying an optimizer. Nevertheless some regularity is often needed on the optimal shape to apply the theory of shape derivatives in order to derive optimality conditions (see Theorem 2.14). Yet let us recall that shape derivatives were performed in [AB03] without regularity assumptions.

Here, as mentioned in Remark 4.2, the optimal shape of Theorem 4.1 enjoys only little regularity. A first step towards upgrading it would be to prove boundedness. Some results are known in this direction, but only in the case of second order operators, up to our knowledge (see [Buc12; MP13]). After boundedness, the question of finer regularity of the optimal set shall be tackled. See [LP17] for a review on this subject. The main results known are covered by some works of Lamboley et al. ([BL09; De +18; LP23]). However, as far as we know, those results hold only in special situations (for instance $m = 1$ with [BL09]), or when

ancillary constraints allow to gain regularity (see [De +18; LP23]). To conclude let us mention that [Sto21] also contains some hints on this topic in the case $m = 2$ (see for instance [Sto21, Lemma 9]).

5. The Rayleigh Conjecture by order reduction

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In this chapter, we will investigate sufficient conditions yielding the Rayleigh Conjecture for the clamped plate. More precisely, under existence and regularity of an optimal shape, our main result is to show that the conjecture is true in any dimension whenever the principal eigenfunction has minimal mean value. This will be explained in the next lines. First, we need to assume that there exists a solution with C^4 **regular connected boundary** to the problem

$$\min\{\Gamma(\Omega) : \Omega \subseteq \mathbb{R}^d \text{ bounded open set}, |\Omega| = c\}. \tag{5.1}$$

We recall that the existence of an optimal shape is assured up to dimension 8 by chapter 4. However, even for $d \leq 8$, it is not guaranteed for this optimal shape to be bounded, let alone regular. In the rest of the chapter, we will denote Ω a C^4 regular solution to (5.1). The regularity assumption on Ω will be used for invoking shape derivation. On the other hand, the assumption on the geometry of the boundary is technical as we shall see in the proof of our main theorem. We stress the properties of regularity and geometry enjoyed by Ω by stating the assumption

$$\Omega \textit{ is } C^4 \textit{and } \partial\Omega \textit{ is connected.} \tag{RG}$$

Apart from (RG), we will need another special assumption to run our proof. This condition asks for the mean value $|\int_\Omega u|$ of the first L^2-normalized eigenfunction u in Ω to be minimal. Indeed, the main conclusion of the present chapter is the theorem stated below.

Theorem 5.1. *Let Ω be an optimal shape for problem (5.1) satisfying (RG) and B a ball such that $|\Omega| = |B|$. Let u be a first L^2-normalized eigenfunction in Ω and u_B a first L^2-normalized eigenfunction in B. Then,*

$$\left|\int_\Omega u\right| \geq \left|\int_B u_B\right|. \tag{M}$$

Moreover, (M) holds with equality if and only if $\Omega = B$ (up to a translation).

Remark 5.2. *Roughly speaking, Theorem 5.1 tells that an optimal shape of which the mean of the principal eigenfunction is minimal must be a ball. Therefore, one is led to wonder if the minimality of the H_0^2 norm of an eigenfunction implies the minimality of its mean. Among others, this question will be addressed in section 5.4.*

The proof of Theorem 5.1 is based on a procedure that we shall call "order reduction principle". Such a procedure appears to be new, at least in the present context, although ensuing from recurrent ideas (see for instance [GGS10, section 1.1.3] and [ABF19, equation (2.3)]). In essence, the order reduction principle allows to turn the fourth order eigenvalue problem (1.1) into a second order affine problem, for which a more sophisticated machinery is available. In particular, it becomes possible to use symmetrization techniques, which are the other main ingredient for proving Theorem 5.1. However, we would like to emphasise that the order reduction principle paves the way for the utilisation of many other tools coming from the field of second order elliptic operators. In order to illustrate this fact, we derive another main result, which is based on the theory of overdetermined problems stemming from the historical [Ser71]. Before, let us simply recall that very little is known in general on overdetermined problems of fourth order, [Ben86; PS89; Dal90] being the almost exhaustive list of results.

Theorem 5.3. *Let Ω be an optimal shape for problem (5.1) satisfying (RG). Let u be a first eigenfunction on Ω such that $\partial_n \Delta u$ is constant on $\partial\Omega$. Then, Ω is a ball.*

Actually, the proofs of Theorem 5.1 and Theorem 5.3 do not appeal to the order reduction principle as a standalone. Indeed, to reveal its potential, the order reduction principle needs to thrive on the optimality condition satisfied by an optimal shape Ω. Such an optimality condition shall be derived only when the eigenvalue $\Gamma(\Omega)$ is simple. Even if the question of simplicity of the optimal eigenvalue had already been tackled in [Moh75], one of the main results of the present work is to propose a thorough proof of this fact and to derive the subsequent optimality condition, which is precised in the next theorem.

Theorem 5.4. *Let Ω be a C^4 open set solving (5.1). Then, $\Gamma(\Omega)$ is simple. Moreover, if u denotes an L^2-normalized eigenfunction associated with $\Gamma(\Omega)$, Δu is a.e. constant equal to $\pm\alpha$ on any connected component of $\partial\Omega$, where*

$$\alpha := \sqrt{\frac{4\Gamma(\Omega)}{d|\Omega|}}.$$

In the remainder of this chapter we will detail the proofs of Theorem 5.1, Theorem 5.3 and Theorem 5.4. In paragraph 5.1, we present our main tool, which is the order reduction principle, roughly explained in the previous lines. Section 5.2 addresses shape derivatives and contains the proof of Theorem 5.4. Section 5.3 is devoted to the proofs of Theorem 5.1 and Theorem 5.3. Paragraph 5.4 discusses two consequences of Theorem 5.1.

5.1. Order reduction principle

The order reduction principle, from which arise Theorem 5.1 and Theorem 5.3, is an algebraic trick leading to an "eigenvalue problem" involving a differential operator of order lower than the bilaplacian, that is, the Laplacian. The counterpart to the reduction of the order is that the "eigenvalue problem" is not linear anymore. The precise statement is encapsulated in the next proposition.

Proposition 5.5. *Let Ω be a C^4 bounded open set, and $u \in H^2_0(\Omega)$ an eigenfunction of the bilaplacian in Ω associated with an eigenvalue μ, so that Δu has trace in $H^{\frac{3}{2}}(\partial\Omega)$. Finally, let g_u satisfy*

$$\left\{ \begin{array}{rcll} \Delta g_u & = & 0 & in \quad \Omega, \\ g_u & = & \frac{\Delta}{\sqrt{\mu}} u & on \quad \partial\Omega. \end{array} \right.$$

Then, the function $z_u := \frac{\Delta}{\sqrt{\mu}} u + u - g_u$ solves the reduced order equation

$$\left\{ \begin{array}{rcll} \Delta z_u & = & \sqrt{\mu}(z_u + g_u) & in \quad \Omega, \\ z_u & = & 0 & on \quad \partial\Omega. \end{array} \right. \tag{5.2}$$

In particular, z_u solves the following problem, the value of which is $\frac{1}{\sqrt{\mu}}$:

$$\frac{1}{\sqrt{\mu}} = - \min_{\substack{z \in H^1_0(\Omega) \\ z \neq 0}} \frac{\int_\Omega z^2 + \int_\Omega g_u(2z - z_u)}{\int_\Omega |\nabla z|^2} \tag{5.3}$$

Moreover, if $g_u \geq 0$, then $z_u < 0$.

Proof. The eigenfunction u satisfies by definition

$$\left\{ \begin{array}{rcll} (\Delta^2 - \mu)u & = & 0 & in \quad \Omega, \\ u & = & 0 & on \quad \partial\Omega, \\ \partial_n u & = & 0 & on \quad \partial\Omega. \end{array} \right.$$

The idea now relies on observing that $(\Delta^2 - \mu) = (\Delta - \sqrt{\mu})(\Delta + \sqrt{\mu})$. Hence, setting $y = \left(\frac{\Delta}{\sqrt{\mu}} + 1\right) u$, y verifies $\Delta y = \sqrt{\mu} y$ in Ω. Nevertheless, the boundary condition for y is $y = \frac{\Delta}{\sqrt{\mu}} u$ on $\partial\Omega$. Note that $\frac{\Delta}{\sqrt{\mu}} u \in H^{\frac{3}{2}}(\partial\Omega)$ since $\Delta u \in H^2(\Omega)$ thanks to the regularity assumption made on $\partial\Omega$. But if g_u is the solution to the Dirichlet problem $\Delta g_u = 0$ in Ω and $g_u = \frac{\Delta}{\sqrt{\mu}} u$ on the boundary, setting $z_u := y - g_u = \frac{\Delta}{\sqrt{\mu}} u + u - g_u$, one gets that z_u is an $H^1_0(\Omega) \cap H^2(\Omega)$ function satisfying

$$\Delta z_u = \sqrt{\mu}(z_u + g_u).$$

In particular z_u is a critical point of the functional E_μ defined on $H^1_0(\Omega)$ and given

by

$$E_\mu(z) = \int_\Omega |\nabla z|^2 + \sqrt{\mu}\int_\Omega z^2 + 2\sqrt{\mu}\int_\Omega g_u z.$$

Moreover, E_μ being strictly convex, z_u is the unique minimizer. But, from the equation involving z_u, we derive the identity $E_\mu(z_u) = \sqrt{\mu}\int g_u z_u$. In this context, the relation

$$\int_\Omega |\nabla z|^2 + \sqrt{\mu}\int_\Omega z^2 + 2\sqrt{\mu}\int_\Omega g_u z \geq \sqrt{\mu}\int_\Omega g_u z_u,$$

holding for all $z \in H_0^1(\Omega)$, is an equality if and only if $z = z_u$. Moreover, thanks to elementary manipulations, this inequality can be turned into the next one, which, as before, is attained if and only if $z = z_u$.

$$-\frac{\int_\Omega z^2 + \int_\Omega g_u(2z - z_u)}{\int_\Omega |\nabla z|^2} \leq \frac{1}{\sqrt{\mu}}.$$

This completes the proof of (5.3). Finally, if $g_u \geq 0$, the strong maximum principle applied to the operator $\Delta - \sqrt{\mu}$ in (5.2) shows that $z_u < 0$ unless z_u vanishes identically in Ω. But if $z_u = 0$, due to (5.2), $g_u = 0$, and in turn $-\Delta u = \sqrt{\mu}u$ in Ω. As a result, the function $v : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by $v(x,t) := u(x)e^{\sqrt[4]{\mu}t}$ is harmonic in $\Omega \times \mathbb{R}$ and satisfies $v = \partial_n v = 0$ on $\partial(\Omega \times \mathbb{R})$. Thanks to [Tol23], we conclude that v vanishes identically in each connected component of $\Omega \times \mathbb{R}$. Hence u vanishes identically in Ω. □

Remark 5.6. *1. Setting $y = \left(\frac{\Delta}{\sqrt{\mu}} - 1\right)u$ instead of $y = \left(\frac{\Delta}{\sqrt{\mu}} + 1\right)u$, we see that the function $z'_u := \frac{\Delta}{\sqrt{\mu}}u - u - g_u$ is $H_0^1(\Omega) \cap H^2(\Omega)$ and satisfies $-\Delta z'_u = \sqrt{\mu}(z'_u + g_u)$. However, we cannot obtain a variational formulation similar to (5.3) involving z'_u since, unlike E_μ, the energy functional of which z'_u is a critical point is not convex.*

2. Note that the system (5.2) is linear with respect to (z_u, g_u). As a consequence, the variational formula (5.3) remains true when replacing z_u and g_u respectively with γz_u and γg_u for any $\gamma \in \mathbb{R} \setminus \{0\}$.

3. The regularity on Ω can be weakened in some cases. More precisely, one shall run the proof of (5.2) and (5.3) as long as $\Delta u \in H^1(\Omega)$.

Surprisingly, Proposition 5.5 will not only serve proving Theorem 5.1 and Theorem 5.3. Indeed, it has the following consequence which will be very useful to prove the simplicity of the optimal first eigenvalue. First, let us recall that, in the case of fourth order equations, it is not known whether having $u = \partial_n u = \partial_n^2 u = 0$ on some arbitrary portion γ of $\partial\Omega$ yields $u = 0$ in the neighbourhood of γ. The lack of this property (called uniqueness continuation), is due to the fact that neither Hölmgren principle nor Hopf boundary Lemma apply in this framework (see however Theorem 1.1 of [OZ00] and the discussion above and below its statement).

Corollary 5.7. *Let Ω be a C^4 bounded open set, and $u \in H_0^2(\Omega)$ satisfy $\Delta^2 u = \mu u$ for some $\mu > 0$. Assume that $\Delta u = 0$ on $\partial\Omega$. Then, $u = 0$ in Ω.*

Proof. Assume that u does not vanish identically, so that it is an eigenfunction. The assumption $\Delta u = 0$ on $\partial\Omega$ reads $g_u = 0$ on $\partial\Omega$ and then in Ω, where g_u is defined as in Proposition 5.5. Then, the function z_u satisfies $\Delta z_u = \sqrt{\mu} z_u$. This means that either $z_u = 0$, or $-\sqrt{\mu}$ is an eigenvalue of the Dirichlet Laplacian. As the latter cannot hold, $z_u = 0$, and hence $-\Delta u = \sqrt{\mu} u$, so that u is an eigenfunction of the Dirichlet Laplacian. Because $\partial_n u = 0$, we run into a contradiction applying [Tol23] to the harmonic extension of u as in the end of the proof of Proposition 5.5. □

Remark 5.8. *Corollary 5.7 holds under weaker regularity assumptions on Ω. For instance, it is enough that Ω is Lipschitz with small constant (see [Tol23]), and satisfies a uniform outer ball condition. Indeed, under the outer ball condition and the Lipschitz regularity assumption, the system solution of the equation $\Delta^2 v = f$ on Ω, and $v = \Delta v = 0$ on $\partial\Omega$ (see [GGS10, Example 2.33] for the definition of system and energy solutions) is $H_0^1 \cap H^2(\Omega)$ due to [Ado92, Theorem 1.1]. Thus it coincides with the energy solution. In particular, in Corollary 5.7, u being an energy solution, we get that $\Delta u \in H_0^1(\Omega)$. Therefore, Proposition 5.5 still applies (see the remark under its proof) and the proof of Corollary 5.7 works as long as the Lipschitz constant of Ω is small enough for using [Tol23].*

5.2. Simplicity of the eigenvalue and optimality conditions

In order to fully exploit Proposition 5.5, one needs to gain information on the function g_u (defined in the statement of Proposition 5.5) when Ω is an optimal shape. As g_u depends on the value of Δu on $\partial\Omega$, one may use shape derivatives (see section 2.3). Indeed, we recall that the scale-invariant version G_k^Ω – defined in (2.12) – of the the kth eigenvalue $\Gamma_k(\Omega)$ of the Dirichlet bilaplacian has been computed in Corollary 2.16. It was shown that if Ω is a C^4 bounded open set and if $\Gamma_k(\Omega)$ is simple, then for all $V \in C^2 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$

$$\partial_V G_k^\Omega(0) := |\Omega|^{\frac{4}{d}} \int_{\partial\Omega} \left[\frac{4\Gamma_k(\Omega)}{d|\Omega|} - (\Delta u_k)^2 \right] V \cdot \vec{n}, \tag{5.4}$$

where u_k is an L^2-normalized eigenfunction associated with $\Gamma_k(\Omega)$. Indeed, since $u_k = \partial_n u_k = 0$ on $\partial\Omega$, we have by [HP05, Proposition 5.4.12] $\partial_n^2 u_k = \Delta u_k$ as long as u_k is $C^2(\overline{\Omega})$ which holds here since Ω is C^4. Therefore we may use (5.4) to obtain additional information on Δu_k.

The only issue regarding (5.4) is the assumption on the simplicity of $\Gamma_k(\Omega)$. Indeed, as already mentioned, in the context of fourth order elliptic operators, the lack of positivity prevents from using the Krein-Rutman Theorem. As a result, one is unable to prove the simplicity of the first eigenvalue, which actually fails in general (see [GGS10, Theorem 3.9]). Fortunately, as roughly justified in [Moh75],

it can be proved that simplicity holds on a domain with minimal eigenvalue. The proof of this fact is obtained by contradiction, using the derivative of a multiple eigenvalue. It will be a consequence of the next proposition, which describes a phenomenon of generic "eigenvalue splitting", as illustrated in Figure 2.4.

Proposition 5.9. *Let Ω be a C^4 bounded open set and $k, p \in \mathbb{N}^*$, $p > 1$. Assume that $\Gamma_k(\Omega) = ... = \Gamma_{k+p-1}(\Omega) =: \Gamma$ and that Γ is of multiplicity p. Then, there exists $V \in C^2 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that*

$$\partial_V^+ \Gamma_k^\Omega(0) < 0 < \partial_V^+ \Gamma_{k+p-1}^\Omega(0),$$
$$\partial_V \mathcal{V}^\Omega(0) = 0.$$

Here, $\Gamma_k^\Omega, \Gamma_{k+p-1}^\Omega$ and $\mathcal{V}^\Omega$ are the functionals defined in (2.9) – with $m = 2$ – and (2.7) respectively and ∂_V^+ denotes the derivative in the direction V on the right.

Proof. We adapt [Hen06, Lemma 2.5.9]. The idea is to use deformations which are localised around two arbitrary boundary points A_+ and A_-. For that purpose, set $\epsilon > 0$, and take a vector field $V_\epsilon = V_\epsilon^+ + V_\epsilon^-$, where $V_\epsilon^+, V_\epsilon^- \in C^2 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ satisfy:

$$\operatorname{supp} V_\epsilon^\pm \subseteq B(A_\pm, \epsilon), \qquad \int_{\partial\Omega} V_\epsilon^\pm \cdot \vec{n} = \pm 1, \qquad \int_{\partial\Omega} V_\epsilon \cdot \vec{n} = 0.$$

The last condition immediately gives $\partial_{V_\epsilon} \mathcal{V}^\Omega(0) = 0$, in view of (2.8). The other conditions tell that $\pm V_\epsilon^\pm \cdot \vec{n}$ is an approximate identity in $A_\pm$ on $\partial\Omega$. More precisely, if ψ is a continuous function over $\partial\Omega$ with ω as modulus of continuity,

$$\left| \psi(A_\pm) - \left(\pm \int_{\partial\Omega} \psi V_\epsilon^\pm \cdot \vec{n} \right) \right| \leq \int_{\partial\Omega \cap B(A_\pm, \epsilon)} |\psi - \psi(A_\pm)||V_\epsilon^\pm \cdot \vec{n}| \leq C\omega(\epsilon),$$

and hence $V_\epsilon^\pm \cdot \vec{n}$ converges to $\pm\delta_{A_\pm}$ in the dual of $C(\partial\Omega)$. But recall that since Ω is C^4, $u_{k+i-1} \in C^3(\overline{\Omega})$. As a consequence, Δu_{k+i-1} is continuous over $\partial\Omega$. Then, shrinking $\epsilon \to 0$, we observe that the matrix M_{V_ϵ} given in (2.10) – with $m = 2$ – converges to the matrix M with coefficients

$$-\Delta u_{k+i-1}(A_+)\Delta u_{k+j-1}(A_+) + \Delta u_{k+i-1}(A_-)\Delta u_{k+j-1}(A_-),$$

for $1 \leq i, j \leq p$. Note that we used here the property $\Delta u_{k+i-1} = \partial_n^2 u_{k+i-1}$ on $\partial\Omega$ holding thanks to [HP05, Proposition 5.4.12]. Setting X_+ and X_- to be the column vectors with coordinates $\Delta u_{k+i-1}(A_+)$ and $\Delta u_{k+i-1}(A_-)$ respectively, $i = 1, ..., p$, we obtain the relation

$${}^t X M X = -(X \cdot X_+)^2 + (X \cdot X_-)^2,$$

for any column vector $X \in \mathbb{R}^p$. In particular, the matrix M describes a quadratic form of signature $(1, 1)$ as long as X_+ and X_- are not collinear. At this point, recall

that X_+ and X_- depend on A_+ and A_-, which are arbitrary. We will show that we may tweak A_+ and A_- for X_+ and X_- not to be collinear. For that, assume by contradiction that X_+ and X_- are collinear for any choice of A_+ and A_- on the boundary, and let $1 \leq i < j \leq p$. Then, the matrix

$$\begin{pmatrix} \Delta u_{k+i-1}(A_+) & \Delta u_{k+i-1}(A_-) \\ \Delta u_{k+j-1}(A_+) & \Delta u_{k+j-1}(A_-) \end{pmatrix}$$

has determinant 0. Fixing A_-, and setting $c_i := \Delta u_{k+i-1}(A_-)$ and $c_j := \Delta u_{k+j-1}(A_-)$, this means that $c_j \Delta u_{k+i-1}(A_+) - c_i \Delta u_{k+j-1}(A_+) = 0$ for all $A_+ \in \partial\Omega$. In other words, the function $v := c_j u_{k+i-1} - c_i u_{k+j-1}$ is an eigenfunction of the bilaplacian in Ω such that $\Delta v = 0$ on $\partial\Omega$. Thanks to Corollary 5.7, we obtain that u_{k+i-1} and u_{k+j-1} are collinear – since either c_i or c_j may be assumed nonzero by Corollary 5.7 –, a contradiction.

The previous shows that there exists some points $A_+, A_- \in \partial\Omega$ for which the matrix M has signature $(1,1)$. In other words, the spectrum of M admits a positive and a negative eigenvalue. As a result, for small enough ϵ, M_{V_ϵ} also admits both positive and negative eigenvalues. Since, by Proposition 2.14, the lowest eigenvalue of M_{V_ϵ} corresponds to $\partial^+_{V_\epsilon}\Gamma^\Omega_k(0)$ and the greatest eigenvalue to $\partial^+_{V_\epsilon}\Gamma^\Omega_{k+p-1}(0)$, we conclude that $\partial^+_{V_\epsilon}\Gamma^\Omega_k(0) < 0 < \partial^+_{V_\epsilon}\Gamma^\Omega_{k+p-1}(0)$. □

Combining Proposition 5.9 and formula (5.4) provides with little ancillary work a proof of Theorem 5.4. Later on, this will allow to identify the function g_u of Proposition 5.5.

Proof of Theorem 5.4. Let Ω be a C^4 optimal shape for problem (5.1). The simplicity of $\Gamma(\Omega)$ is a direct consequence of Proposition 5.9, as if one had $\Gamma_1(\Omega) = \Gamma_2(\Omega)$, there would exist a vector field V such that $\partial_V \mathcal{V}^\Omega(0) = 0$ and $\partial^+_V \Gamma^\Omega_1(0) < 0$. Then, we would have $\partial^+_V G^\Omega_1(0) < 0$, with G^Ω_1 defined in (2.12). As a result, Ω would not minimize $\omega \mapsto |\omega|^{\frac{4}{d}}\Gamma_1(\omega)$, so it would not solve (5.1).

Now that simplicity has been proved, we shall invoke formula (5.4). Indeed, as Ω is an optimal shape, 0 is a critical point of G^Ω_1, hence we get the optimality condition

$$0 = \int_{\partial\Omega} \left[\frac{4\Gamma(\Omega)}{d|\Omega|} - (\Delta u)^2 \right] V \cdot \vec{n}, \qquad \forall V \in C^2 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d).$$

We conclude that $\Delta u = \pm\alpha$ a.e. on $\partial\Omega$. Moreover, since Δu is continuous over $\overline{\Omega}$ (recall that $u \in C^3(\overline{\Omega})$), it is a.e. constant on each connected component of $\partial\Omega$. □

Note that the optimality condition given in Theorem 5.4 is actually fulfilled by any C^4 regular shape Ω with simple principal eigenvalue and such that 0 is a critical point for G^Ω_1. This motivates the following definition.

Definition 5.10. *An open set Ω is a critical shape (for the principal eigenvalue of the bilaplacian) if any L^2-normalized first eigenfunction u on Ω is such that Δu is a.e. constant equal to $\pm\sqrt{\frac{4\Gamma(\Omega)}{d|\Omega|}}$ on each connected component of $\partial\Omega$.*

Remark 5.11. *Any ball B is a critical shape since, according to Proposition 5.15, the first eigenfunction is radial (derive G_1^B in the direction of a radially symmetric vector field). See also [Buo16] for more general results.*

Considering the order reduction principle proved in paragraph 5.1 and the optimality condition derived in the present section, we are equipped for proving Theorem 5.1 and Theorem 5.3.

5.3. Proofs of Theorem 5.1 and Theorem 5.3

Let us begin with the most straightforward of the two theorems, which is undoubtedly Theorem 5.3. With Theorem 5.4 in mind, we see that it is enough to prove it for critical shapes, which is performed below.

Theorem 5.12. *Let Ω be a critical shape satisfying (RG). Let u be a first eigenfunction on Ω such that $\partial_n\Delta u$ is constant on $\partial\Omega$. Then, Ω is a ball.*

Proof. Without loss of generality, we assume u to be L^2-normalized. Since Ω is a critical shape, we know that Δu is a.e. constant on each connected component of $\partial\Omega$. But $\partial\Omega$ is assumed connected, hence Δu is constant on $\partial\Omega$ equal to $\pm\alpha$, where $\alpha = \sqrt{\frac{4\Gamma(\Omega)}{d|\Omega|}}$. Considering $-u$ if needed, we shall assume that $\Delta u = \alpha$ a.e. on $\partial\Omega$, and, consequently, $g_u = \sqrt{\frac{4}{d|\Omega|}} > 0$ a.e. not only on $\partial\Omega$ but in the whole Ω. Applying the order reduction principle (Proposition 5.5), we obtain that $z_u = \frac{\Delta}{\sqrt{\mu}}u + u - g_u$ is negative and satisfies (5.2). Moreover, the fact that $\partial_n\Delta u$ remains constant on the boundary (combined with the fact that g_u is constant) shows that $\partial_n z_u$ is constant on $\partial\Omega$. Thus $z_u < 0$ satisfies an overdetermined problem of order 2, and we conclude applying [Ser71, Theorem 2]. □

We now turn to the proof of Theorem 5.1. To do so, we use the variational formulation of the first eigenvalue involving z_u given by Proposition 5.5. This new expression is interesting in the sense that it allows using symmetrization techniques available for one-sign $H_0^1(\Omega)$ functions. As previously, the arguments actually hold for any critical shape, and Theorem 5.1 will be a consequence of the following result.

Theorem 5.13. *Let Ω be a critical shape satisfying (RG) and B a ball such that $|\Omega| = |B|$. Let u be a first L^2-normalized eigenfunction on Ω and u_B a first L^2-normalized eigenfunction on B. Assume that*

$$\left|\int_\Omega u\right| \leq \left|\int_B u_B\right|. \tag{5.5}$$

Then,

$$\Gamma(B) \leq \Gamma(\Omega). \tag{5.6}$$

Moreover, if (5.5) is strict, (5.6) is also strict. Finally, if $\Gamma(\Omega) = \Gamma(B)$, Ω *has to be a translation of* B.

Proof. Proceeding as in the beginning of the proof of Theorem 5.12, we may assume that $g_u = \sqrt{\frac{4}{d|\Omega|}} > 0$ a.e. in Ω. Since B is also a critical shape satisfying (RG) (recall the remark below the definition of critical shapes), the same applies to u_B, and we conclude that we shall also take $g_{u_B} = \sqrt{\frac{4}{d|\Omega|}}$ a.e. in B. As a result of $g_{u_B} \geq 0$, observe that u_B (which is one-sign) is positive due to the maximum principle and to Hopf Boundary Lemma.

On the other hand, since $g_u \geq 0$, we get $z_u < 0$. Consequently the function $z_u^\# := -(-z_u)^*$ is a negative function in $H_0^1(B)$. Moreover, the properties of the Schwarz symmetrization ensure that $\int_\Omega |\nabla z_u|^2 \geq \int_B |\nabla z_u^\#|^2$, $\int_\Omega z_u^2 = \int_B (z_u^\#)^2$, and $\int_\Omega z_u = \int_B z_u^\#$. Therefore, thanks to Proposition 5.5,

$$\begin{aligned}\frac{1}{\sqrt{\Gamma(\Omega)}} = -\frac{\int_\Omega |z_u|^2 + g_u \int_\Omega z_u}{\int_\Omega |\nabla z_u|^2} &\leq -\frac{\int_B |z_u^\#|^2 + g_u \int_B z_u^\#}{\int_B |\nabla z_u^\#|^2} \\ &\leq -\min_{z \in H_0^1(B)} \frac{\int_B z^2 + g_{u_B} \int_B (2z - z_u^\#)}{\int_B |\nabla z|^2}.\end{aligned}$$

Note that the numerator in the above quotients is always nonpositive (from the first equality), which justifies the first inequality. Now, we claim that $\int_B z_u^\# \leq \int_B z_{u_B}$. Indeed, if true, this result would lead to

$$\frac{1}{\sqrt{\Gamma(\Omega)}} \leq -\min_{z \in H_0^1(B)} \frac{\int_B z^2 + g_{u_B} \int_B (2z - z_{u_B})}{\int_B |\nabla z|^2} = \frac{1}{\sqrt{\Gamma(B)}},$$

the last equality coming once again from Proposition 5.5 applied to B. This would in turn give the Faber-Krahn inequality (5.6). Note also that if $\int_B z_u^\# \leq \int_B z_{u_B}$ is strict, then (5.6) is also strict.

Hence it remains only to prove that $\int_B z_u^\# \leq \int_B z_{u_B}$. But thanks to equimeasurability, $\int_B z_u^\# = \int_\Omega z_u$. Then, using the expression of z_u combined with the fact that $\int_\Omega \Delta u = 0$, we find $\int_B z_u^\# = \int_\Omega u - |\Omega| g_u$. In the same way, $\int_B z_{u_B} = \int_B u_B - |B| g_{u_B}$. Thus, as $|\Omega| = |B|$ and $g_u = g_{u_B}$, we obtain that $\int_B z_u^\# \leq \int_B z_{u_B}$ if and only if $\int_\Omega u \leq \int_B u_B$, which holds by assumption (recall that $u_B > 0$). Moreover, if one of these inequalities is strict, the other also holds strictly.

Lastly, if $\Gamma(\Omega) = \Gamma(B)$, all our inequalities become equalities. In particular, $\int_\Omega |\nabla z_u|^2 = \int_B |\nabla z_u^\#|^2$, thus we apply [FM86, Theorem 2.2]. This is possible since, on the one hand, as u is real analytic in Ω, z_u is also real analytic, hence $|\{z_u = t\}| = 0$ for all $\inf z_u < t < \sup z_u$ [Mit20]. On the other hand, since Ω is C^4, $u \in C^3(\overline{\Omega})$, hence z_u is Lipschitz in $\mathbb{R}^d$. Then, [FM86, Theorem 2.2] yields that, up

to translation, $z_u = z_u^{\#}$, and in particular that Ω is a ball. □

Proof of Theorem 5.1. If Ω is an optimal shape satisfying (RG), Theorem 5.4 shows that $\Gamma(\Omega)$ is simple and that Ω is a critical shape. Assume that its L^2-normalized principal eigenfunction u does not verify (M). Theorem 5.13 then applies and shows that $\Gamma(\Omega) > \Gamma(B)$, contradicting the optimality of Ω. Therefore, (M) holds. Moreover, in case of equality, Theorem 5.13 still applies and gives $\Gamma(\Omega) \geq \Gamma(B)$. As Ω is optimal, we conclude that $\Gamma(\Omega) = \Gamma(B)$, hence Theorem 5.13 implies that $\Omega = B$ up to a translation. □

Theorem 5.1 relies on the central assumption that equality is attained in (M), or in other words that the converse of (M) holds. Unfortunately, the inequality $|\int_\Omega u| \leq |\int_B u_B|$ seems not easy to check in general. For instance, to estimate the mean value of u on the optimal domain Ω, one could try to use the inequality

$$\int_\Omega u \leq g_u|\Omega| = \sqrt{\frac{4|\Omega|}{d}}, \tag{5.7}$$

coming from the fact that $\int_\Omega z_u \leq 0$ (recall Proposition 5.5). However, as B is a critical shape, u_B satisfies (5.7) as well, hence it is illusory to intend showing the reverse $\sqrt{\frac{4|\Omega|}{d}} \leq \int_B u_B$, since it would mean that $z_{u_B} = 0$, in contradiction with Proposition 5.5. Nevertheless, even if having equality in (M) is a restrictive condition, Theorem 5.1 has two interesting consequences that we shall explain in the next section.

5.4. Consequences of Theorem 5.1

The first immediate corollary of Theorem 5.1 regards the volume of one of the nodal domains of an optimal eigenfunction u. Indeed, (M) implies that the mean of u cannot be zero. This is testament to an asymmetry between its positive and negative parts. The next corollary makes this observation more concrete.

Corollary 5.14. *With the hypotheses of Theorem 5.1, if (without loss of generality)* $\int_\Omega u > 0$, *writing* $\Omega_+ := \{u > 0\}$, *then*

$$|\Omega_+| > \left(\int_B u_B\right)^2. \tag{5.8}$$

Proof. Assume by contradiction that $\sqrt{|\Omega_+|} \leq |\int_B u_B|$. Then, $\int_\Omega u \leq \int_{\Omega_+} u \leq \sqrt{|\Omega_+|}\sqrt{\int_{\Omega_+} u^2} \leq \sqrt{|\Omega_+|} \leq |\int_B u_B| \leq \int_\Omega u$, the last inequality coming from Theorem 5.1. Therefore, all the previous inequalities, in particular Hölder's, are equalities. This means that $u = 1$ in $\Omega_+ = B$, a contradiction. □

This result confirms that it may be interesting to evaluate the mean value of u_B. This is possible since u_B shall be computed explicitly as stated in the next result, the proof of which is detailed in appendix C.

Proposition 5.15. *Let B be the ball $B(0,R)$. The first L^2-normalized eigenfunction u_B is radially symmetric. Moreover, $\Gamma(B)$ and (up to sign) u_B are given by the formulas*

$$\Gamma(B) = \frac{k_\nu^4}{R^4}, \qquad u_B(r) = \frac{1}{\sqrt{d|B|}}\left[\frac{J_\nu(k_\nu r/R)}{J_\nu(k_\nu)} - \frac{I_\nu(k_\nu r/R)}{I_\nu(k_\nu)}\right]\left(\frac{r}{R}\right)^{-\nu}, \tag{5.9}$$

where $\nu := d/2-1$, J_ν and I_ν stand for the Bessel and modified Bessel functions of order ν, and k_ν is the first positive zero of f_ν defined by

$$f_\nu(r) = \left[\frac{J_{\nu+1}}{J_\nu}(r) + \frac{I_{\nu+1}}{I_\nu}(r)\right] r^{d-1}. \tag{5.10}$$

Finally,

$$\int_B u_B = \frac{\sqrt{d|B|}}{k_\nu}\left[\frac{J_{\nu+1}}{J_\nu}(k_\nu) - \frac{I_{\nu+1}}{I_\nu}(k_\nu)\right] = 2\frac{\sqrt{d|B|}}{k_\nu}\frac{J_{\nu+1}}{J_\nu}(k_\nu). \tag{5.11}$$

Note that it is easy to evaluate numerically (5.11). Indeed, in Python 3 for instance, the package `special`, from the module `scipy`, directly provides the functions `jv` and `iv`, corresponding respectively to the Bessel functions J_ν and I_ν. Then it remains to compute k_ν, but this can be done by a bisection method (f_ν being increasing where defined by [AB95, Appendix 1]) thanks to $j_{\nu,1} < k_\nu < j_{\nu,2}$ (see [BPS18, equation (2.2)]) as long as one knows $j_{\nu,1}$ and $j_{\nu,2}$ (where, for $n \in \mathbb{N}^*$, $j_{\nu,n}$ are the positive zeros of J_ν). In this respect observe, as explained in Theorem 2.1 and Theorem 2.2 of [IKF91], that the zeros of J_ν can be approximated by computing the eigenvalues of some matrix. In Table 5.1 is given the value of $|\int_B u_B|$ in the case where B is the ball of volume 1. We also give the minimum volume allowed for Ω_+ to satisfy (5.8), that is $(\int_B u_B)^2$.

d	$\lvert\int_B u_B\rvert$	$(\int_B u_B)^2$
4	0.6056	0.3668
5	0.5643	0.3185
6	0.5308	0.2817
7	0.5028	0.2528
8	0.4790	0.2294
9	0.4583	0.2101

Table 5.1. – Values of $|\int_B u_B|$ and $(\int_B u_B)^2$ for several dimensions. Here, B is chosen to be the ball of unit volume.

Let us now discuss another consequence of Theorem 5.1. In view of proving the Rayleigh Conjecture, Theorem 5.1 tells that the last step would be to show the converse of (M) for an optimal shape Ω. On the other hand, the optimality of Ω means, by definition, that

$$\int_\Omega |\Delta u|^2 \leq \int_B |\Delta u_B|^2,$$

where u (resp. u_B) is an L^2-normalized first eigenfunction on Ω (resp. B). From this inequality, one shall wonder whether it is possible to deduce the converse of (M). This problem is actually not so far from a maximum principle type property, which classically asserts that if $v_1, v_2 \in H_0^1(\omega)$ satisfy $-\Delta v_1 \leq -\Delta v_2$, then $v_1 \leq v_2$ in ω. In our situation, it would be desirable to convert these pointwise inequalities into integral ones. Therefore, even if it does not immediately answer our initial concern, it would be interesting to study to which extent the following L^p norm and mean value formulations of the maximum principle hold: for $v_1 \in W_0^{1,p}(\omega_1) \cap W^{2,p}(\omega_1)$ and $v_2 \in W_0^{1,p}(\omega_2) \cap W^{2,p}(\omega_2)$,

$$\int_{\omega_1} |\Delta v_1|^p \leq \int_{\omega_2} |\Delta v_2|^p \quad \Longrightarrow \quad \int_{\omega_1} |v_1|^p \leq \int_{\omega_2} |v_2|^p, \tag{5.12}$$

$$\int_{\omega_1} (-\Delta v_1)^p \leq \int_{\omega_2} (-\Delta v_2)^p \quad \Longrightarrow \quad \int_{\omega_1} v_1^p \leq \int_{\omega_2} v_2^p. \tag{5.13}$$

At this time, we were not able to answer the above (quite vague) questions, and could only argue that (5.13) cannot hold in full generality for $p = 1$, since it would imply that any H_0^2 function has zero mean value. Anyway, in the remaining, we will state an interesting consequence of Theorem 5.1 using the standard maximum principle combined with Talenti's comparison principle (Theorem 2.12).

Corollary 5.16. *With the hypotheses of Theorem 5.1, assume without loss of generality that $u_B > 0$ and that $\int_\Omega u > 0$. Writing $\Omega_+ := \{u > 0\}$ and Ω_+^* its Schwarz symmetrization, if $(-\Delta u|_{\Omega_+})^* \leq -\Delta u_B$ in Ω_+^*, then, up to a translation, $\Omega = B$.*

Remark 5.17.

1. *The assumption $(-\Delta u|_{\Omega_+})^* \leq -\Delta u_B$ may be replaced by $(-\Delta u)^*|_{\Omega_+^*} \leq -\Delta u_B$ since we have $(-\Delta u|_{\Omega_+})^* \leq (-\Delta u)^*|_{\Omega_+^*}$ by equation (2.4).*
2. *We stress that if Ω is an optimal shape, then*

$$\int_B |(-\Delta u)^*|^2 = \int_\Omega (\Delta u)^2 = \Gamma(\Omega) \leq \Gamma(B) = \int_B (-\Delta u_B)^2.$$

   The assumption of the corollary is then a pointwise version of this inequality.
3. *As we shall see in the proof, the assumption $(-\Delta u|_{\Omega_+})^* \leq -\Delta u_B$ is used for applying the maximum principle, which in turn yields a pointwise inequality between u and u_B, although only an inequality in terms of mean value is actually needed to invoke Theorem 5.1. That's why, if one could prove some*

"mean value maximum principle" as in (5.12) and (5.13), one could hope to drop the assumption.

Proof. We set $f := -\Delta u|_{\Omega_+}$. Let v be the $H^1_0(\Omega^*_+)$ solution of the problem $-\Delta v = f^*$ in Ω^*_+. According to Talenti's comparison principle, since $u \geq 0$ in Ω_+, $v \geq u^*$ in Ω^*_+.

Then, as $-\Delta v = (-\Delta u|_{\Omega_+})^* \leq -\Delta u_B$ in Ω^*_+, we get $-\Delta(u_B - v) \geq 0$ in Ω^*_+. Thus the maximum principle forces $u_B - v$ to reach its minimum value on the boundary of Ω^*_+. Moreover, $u_B \geq 0$ on B and hence on $\partial\Omega^*_+$. Therefore $u_B - v \geq 0$ on $\partial\Omega^*_+$. To conclude, $u_B \geq v$ not only on the boundary, but in the whole Ω^*_+.

We obtained that $u^* \leq u_B$ pointwise in Ω^*_+. In particular, since $u_B \geq 0$,

$$\int_\Omega u \leq \int_{\Omega_+} u = \int_{\Omega^*_+} u^* \leq \int_{\Omega^*_+} u_B \leq \int_B u_B,$$

and we conclude thanks to Theorem 5.1. □

5.5. Link with the biharmonic torsional rigidity

This part is not present in the original paper [Ley24b].

In this paragraph we would like to address a similar although much easier issue than the Rayleigh Conjecture for the clamped plate. From the perspective of the order reduction principle, we will highlight some unexpected links between the conjecture and our new problem. This problem is related to the biharmonic torsional rigidity. We recall (see section 1.4 for details) that the biharmonic torsional rigidity is defined for a bounded open set $\Omega \subseteq \mathbb{R}^d$ according to

$$T(\Omega) := \min_{\substack{u \in H^2_0(\Omega) \\ u \neq 0}} \frac{\int_\Omega (\Delta u)^2}{\left(\int_\Omega u\right)^2}. \tag{5.14}$$

Equivalently, $T(\Omega)$ is the inverse of $\int_\Omega w_\Omega$ where w_Ω is the biharmonic torsion function. See also [GM24] for general considerations with regard to biharmonic torsion functions (with different boundary conditions). Since the definition and properties of $T(\Omega)$ are not so far from those of $\Gamma(\Omega)$, we shall wonder, as in the Rayleigh Conjecture, which shape minimizes $T(\Omega)$. In other words, we would like to solve

$$T_c := \min\{T(\Omega) : \Omega \subseteq \mathbb{R}^d \text{ bounded open set}, |\Omega| = c\} \tag{5.15}$$

Actually, our interest in problem (5.15) is not as naive as it seems at first glance. There is a deeper link between $\Gamma(\Omega)$ and $T(\Omega)$ than a sole analogy. Let us explain a little bit more our motivation. Recall that with the notations of Theorem 5.1, if

Ω is an optimal shape satisfying (RG),

$$\left|\int_\Omega u\right| \geq \left|\int_B u_B\right|.$$

As an immediate consequence, we shall obtain a lower bound on the optimal eigenvalue in the following way.

$$\Gamma(\Omega) = \int_\Omega (\Delta u)^2 = \frac{\int_\Omega (\Delta u)^2}{\left(\int_\Omega u\right)^2} \left(\int_\Omega u\right)^2 \geq T(\Omega) \left(\int_\Omega u\right)^2 \geq T_c \left(\int_B u_B\right)^2 \tag{5.16}$$

In other words, identifying T_c allows to derive an approximate Faber-Krahn inequality for Γ. For the moment, let us focus on T_c and state the main result of this section. This result consists in a Faber-Krahn type inequality for the biharmonic torsional rigidity holding in any dimension.

Theorem 5.18. *Let Ω be a bounded open set of $\mathbb{R}^d$ and B a ball such that $|\Omega| = |B|$. Then,*

$$T(\Omega) \geq T(B).$$

Moreover, in case of equality Ω must be a ball up to a set of zero H^1-capacity.

Remark 5.19. *The equality case is not completely understood since we actually expect to have equality if and only if Ω is a ball up to a set of zero H^2-capacity.*

The proof of Theorem 5.18 is postponed to paragraph 5.5.2. As for the Rayleigh Conjecture in dimensions 2 and 3, it is based on an auxiliary two-ball problem associated with the torsional rigidity. However, unlike Ashbaugh-Benguria's Theorem for the clamped plate, here the two-ball problem provides a sharp lower bound in any dimension. Observe that the proof is less involved than the one in [AB95]. This is due to the fact that for the biharmonic torsion problem on balls, no Bessel function comes into play, and we only deal with polynomials (see (5.17)).

Theorem 5.18 appears not to have been produced in the literature, hence the result is new. The reason might be that, up to our knowledge, there is no direct physical application of the biharmonic torsional rigidity. Nevertheless, in view of equation (5.16), we deduce from Theorem 5.18 the following approximate Faber-Krahn inequality.

Corollary 5.20. *Let Ω be an optimal shape for (5.1) satisfying (RG) and B be a ball such that $|\Omega| = |B|$. Then,*

$$\Gamma(\Omega) \geq c_d \Gamma(B),$$

where

$$c_d = \min\left\{\frac{T(B)\left(\int_B u_B\right)^2}{\Gamma(B)}, 1\right\},$$

only depends on the dimension d. Here u_B is a first L^2-normalized eigenfunction associated with $\Gamma(B)$.

One would hope that Corollary 5.20 could help improve the already known approximate Faber-Krahn inequalities for the clamped plate (see [Tal81; AL96]). However, as shown in paragraph 5.5.1, the constant c_d seems too poor (see Table 5.2), and even $c_d \to 0$. In particular, Corollary 5.20 does not improve the state-of-the-art concerning the Rayleigh Conjecture, but we mention it since the approach is new. Furthermore, it led us to Theorem 5.18, which is of independent interest.

5.5.1. Estimation of the coefficient of Corollary 5.20

To estimate the value of the coefficient c_d in Corollary 5.20, we need to compute the biharmonic torsional rigidity $T(B)$ of the ball B. This is nothing else but the inverse of $\int_B w_B$, where w_B is the biharmonic torsion function as defined in (1.17). If B is of radius R, w_B is radially symmetric and given for $r \in [0, R)$ by (see [Dal90])

$$w_B(r) = \frac{1}{8d(d+2)} \left[R^2 - r^2\right]^2. \tag{5.17}$$

Hence, at the end of the day, we find

$$\int_B w_B = \frac{R^4|B|}{d(d+4)(d+2)^2}. \tag{5.18}$$

With the notations of Proposition 5.15, it follows that the coefficient c_d appearing in Corollary 5.20 is given by

$$c_d = \frac{4d^2(d+2)^2(d+4)}{k_\nu^6} \frac{J_{\nu+1}^2}{J_\nu^2}(k_\nu). \tag{5.19}$$

As expected it does not depend on the size of the ball B. Some values of $T(B)$ and of c_d are reproduced in Table 5.2.

Besides those particular values, one may also compute the asymptotic behaviour of c_d as d goes to infinity. Since it is not straightforward to analyse the behaviour of $J_{\nu+1}(k_\nu)/J_\nu(k_\nu)$ in terms of ν appearing in (5.19), we resort to the estimation given in equation (5.7) to control $|\int_B u_B|$ and obtain

$$c_d \leq \frac{4(d+4)(d+2)^2}{k_\nu^4}.$$

Moreover, due to [Lor94, Corollary 2] and to the known asymptotic behaviour of zeros of Bessel functions with respect to ν, we have $k_\nu \sim \nu$ as $\nu \to \infty$. Consequently, since $\nu = d/2 - 1$, we conclude that

$$c_d = O(d^{-1}) \xrightarrow[d\to\infty]{} 0.$$

d	$T(B)$	c_d
2	$1.895 \cdot 10^3$	0.9770
3	$3.545 \cdot 10^3$	0.9568
4	$5.685 \cdot 10^3$	0.9348
5	$8.326 \cdot 10^3$	0.9121
6	$11.48 \cdot 10^3$	0.8894
7	$15.15 \cdot 10^3$	0.8672
8	$19.34 \cdot 10^3$	0.8457
9	$24.06 \cdot 10^3$	0.8250

Table 5.2. – Values of $T(B)$ and of the coefficient c_d of Corollary 5.20 in terms of dimension d. Here B is the ball of unit volume.

5.5.2. Proof of Theorem 5.18

In order to prove Theorem 5.18, we employ the framework of Talenti, Nadirashvili, Ashbaugh and Benguria in order to derive a "torsional two-ball problem", which is similar to (3.5), but adapted to the Rayleigh quotient (5.14). This is the purpose of the next result.

Proposition 5.21. *Let Ω be a bounded open set of $\mathbb{R}^d$ and let $u \in H^2_0(\Omega)$ minimize (5.14). Set $\Omega_\pm := \{\pm u > 0\}$, and denote by a the radius of Ω^*_+ and by b the radius of Ω^*_-. Then,*

$$T(\Omega) \geq \min_{v,w} \frac{\int_{B_a} (\Delta v)^2 + \int_{B_b} (\Delta w)^2}{\left(\int_{B_a} v\right)^2 + \left(\int_{B_b} w\right)^2} =: T(a,b), \tag{5.20}$$

where the minimum is taken among radial functions $v \in H^2(B_a)$ and $w \in H^2(B_b)$ satisfying for all $x \in \partial B_a$ and $y \in \partial B_b$,

$$v(x) = w(y) = 0, \qquad a^{d-1}\partial_r v(x) = b^{d-1}\partial_r w(y), \qquad \Delta v(x) + \Delta w(y) = 0,$$

when $a, b \neq 0$. If $b = 0$ (resp. $a = 0$) take $v \in H^2_0(B_a)$ (resp. $w \in H^2_0(B_b)$) above. Moreover, in case of equality in (5.20), $\Omega = B_a \sqcup B_b$ up to sets of zero H^1-capacity.

Proof. Consider only the case $a, b \neq 0$ since the other case is similar but easier. Since the function u minimizes (5.14) we have

$$T(\Omega) = \frac{\int_\Omega (\Delta u)^2}{\left(\int_\Omega u\right)^2}.$$

Then, as in [Tal81] and [AB95] (see also paragraph 3.2), consider the sign-changing functions

$$f := -\Delta u, \qquad g := \Delta u.$$

Now denote v in $H^1_0(B_a)$ and $w \in H^1_0(B_b)$, the solutions of the symmetrized equations:

$$\left\{\begin{array}{rcll} -\Delta v &=& f^*|_{B_a} & in \quad B_a, \\ v &=& 0 & on \quad \partial B_a, \end{array}\right. \qquad \left\{\begin{array}{rcll} -\Delta w &=& g^*|_{B_b} & in \quad B_b, \\ w &=& 0 & on \quad \partial B_b. \end{array}\right.$$

We will use v and w to challenge u_+ and u_-. First of all, recall that u is real analytic (by Lemma 1.7) and hence $|\Omega_+| + |\Omega_-| = |\Omega|$ by [Mit20]. Consequently we shall apply Lemma F.1 to the function f with $p = 2$ and $\omega_\pm = \Omega_\pm$. This provides

$$\int_\Omega (\Delta u)^2 = \int_{B_a} (\Delta v)^2 + \int_{B_b} (\Delta w)^2. \tag{5.21}$$

Indeed, with the notations of Appendix F, we have $f^* = f^\ddagger$ and $g^* = (-f)^\ddagger$ (recall Lemma 2.6). We will also need later the version of Lemma F.1 with $p = 1$, that is

$$\int_\Omega \Delta u = \int_{B_a} \Delta v - \int_{B_b} \Delta w. \tag{5.22}$$

On the other hand, we have to bound from above $\left(\int_\Omega u\right)^2$ in terms of v and w. This may be obtained thanks to Talenti's comparison principle (Theorem 2.12), which reads

$$v \geq u_+^* \text{ in } B_a, \qquad w \geq u_-^* \text{ in } B_b.$$

Here u_+^* means $(u_+)^*$ and u_-^* means $(u_-)^*$. More precisely, besides Talenti's comparison principle we used the maximum principle and the relations $f^*|_{B_a} \geq f|^*_{\Omega_+}$ and $g^*|_{B_b} \geq g|^*_{\Omega_-}$ (see equation (2.4)) to obtain the previous relations. As a consequence, thanks to the equimeasurability of Schwarz symmetrization and to the nonnegativity of u_+ and u_-,

$$\left(\int_\Omega u\right)^2 = \left(\int_{B_a} u_+^* - \int_{B_b} u_-^*\right)^2 \leq \left(\int_{B_a} u_+^*\right)^2 + \left(\int_{B_b} u_-^*\right)^2 \leq \left(\int_{B_a} v\right)^2 + \left(\int_{B_b} w\right)^2. \tag{5.23}$$

Combining (5.21) and (5.23), we obtain as desired

$$T(\Omega) \geq \frac{\int_{B_a}(\Delta v)^2 + \int_{B_b}(\Delta w)^2}{\left(\int_{B_a} v\right)^2 + \left(\int_{B_b} w\right)^2}.$$

Moreover, it is clear that for all $x \in \partial B_a$ and $y \in \partial B_b$, $v(x) = w(y) = 0$. The fact that $a^{d-1}\partial_r v(x) = b^{d-1}\partial_r w(y)$ comes from (5.22) and from the fact that v and w are radially symmetric:

$$a^{d-1}\partial_r v(x) - b^{d-1}\partial_r w(y) = \int_{\partial_n B_a} \partial_n v - \int_{\partial B_b} \partial_n w = \int_{B_a} \Delta v - \int_{B_b} \Delta w = \int_\Omega \Delta u = 0.$$

Eventually, by Lemma 2.6 and since $|B_a| + |B_b| = |\Omega|$,

$$-\Delta v(x) - \Delta w(y) = f^*(x) + g^*(y) = f_+^\star(|B_a|) - f_-^\star(|B_b|) + f_-^\star(|B_b|) - f_+^\star(|B_a|) = 0.$$

Therefore, v and w satisfy all the constraints listed in the statement of Proposition 5.21, and hence $T(\Omega) \geq T(a,b)$. We also mention that $T(a,b)$ is well-defined (make obvious changes in [AB95, Appendix 2]). Moreover, we observe that the equality case implies that there are equalities in Talenti's comparison principle $v \geq u_+^*$ and $w \geq u_-^*$, meaning that Ω_+ and Ω_- are equal H^1-q.e. to two balls (see [Kes06, section 3.2]). Since Ω_+ and Ω_- are disjoint, we get $\Omega = B_a \sqcup B_b$. □

The torsional two-ball problem $T(a,b)$ appearing in (5.20) is an adaptation of the classic two-ball problem (3.5) introduced in [Tal81] and studied in detail in [Nad95; AB95; AL96]. We will see in the next result that the unwelcome behaviour observed in [AL96] for dimension $d \geq 4$ does not occur when it comes to talk about $T(a,b)$.

Proposition 5.22. *Let $a, b \in [0,1]$ such that $a^d + b^d = 1$. Then,*

$$T(a,b) \geq T(B_1).$$

The proof is performed following the strategy of [AB95]. It is rather elementary but quite long and tedious, hence we postpone it to Appendix D. Thanks to Proposition 5.22, the proof Theorem 5.18 follows immediately.

Proof of Theorem 5.18. Let Ω be a bounded open set. We may assume that $|\Omega| = |B_1|$. By Proposition 5.21 $T(\Omega) \geq T(a,b)$ where a is the radius of Ω_+^* and b the radius of Ω_-^*. Moreover, since $|\Omega_+| + |\Omega_-| = |\Omega|$ by [Mit20], we have $a^d + b^d = 1$. By Proposition 5.22 we conclude that

$$T(\Omega) \geq T(a,b) \geq T(B_1).$$

Furthermore, in case of equality, we have $T(\Omega) = T(a,b)$ hence $\Omega = B_a \sqcup B_b$ by Proposition 5.22. Since $T(B_a \sqcup B_b) = \min(T(B_a), T(B_b)) > T(B_1)$ if $a, b \neq 0$, it becomes clear that the only way to fulfil $T(\Omega) = T(B_1)$ is that a or b vanishes. In this situation we obtain $\Omega = B_1$ (H^1-q.e.) as expected. □

6. Talenti's comparison principle on domains with holes

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The present chapter aims to prove that the Rayleigh Conjecture holds whenever the first eigenfunction over an hypothetical optimal shape has no intermediate critical value. We shall explain this more in detail. For this, we first need to assume that there exists a C^4 regular shape solving

$$\min\{\Gamma(\Omega) : \Omega \subseteq \mathbb{R}^d \text{ bounded open set}, |\Omega| = c\}. \tag{6.1}$$

Recall that an (a priori unbounded and irregular) optimal shape is guaranteed to exist at least for $d \leq 8$ by Theorem 4.1. We will denote as usual Ω the C^4 regular solution to (6.1). The assumption about the regularity of Ω is made for using shape derivatives (recall section 2.3), but it will also be used at some point to analyse the behaviour of the first eigenfunction near the boundary of Ω (see Proposition 6.5).

Now let us present our main assumption. This condition regards the critical points of a first eigenfunction over Ω (we will denote it u in the rest of the chapter), and is intended to ensure that the topology of the upper level sets of u does not change (see also Remark 6.12). It can be expressed as follows.

$$\forall x \in \Omega, \quad \nabla u(x) = 0 \;\Rightarrow\; u(x) \textit{ is the global minimum or global maximum of } u. \tag{U}$$

Let us emphasise that, although being critical for u, the points of $\partial\Omega$ are not taken into consideration in (U). In other words, (U) is equivalent to ask that u admits only $\min_{\overline{\Omega}} u$ and $\max_{\overline{\Omega}} u$ as possible critical values within Ω. Note however that it allows u to have many critical points. This occurs for instance in annuli with large inner radius, for which the first eigenfunction is radially symmetric and admits only one critical value in Ω, which is its maximum (or minimum) value (see [CDS79]). In this setting, the corresponding critical points form a sphere due to the radial symmetry of u. Anyway, using assumption (U), the main conclusion of the chapter is the theorem stated below.

Theorem 6.1. *Let Ω be a C^4 optimal shape for problem (6.1). Assume that an eigenfunction associated with $\Gamma(\Omega)$ satisfies (U) and that $4 \leq d \leq 9$. Then Ω is a ball.*

Remark 6.2. *The assumption on the dimension d comes from the fact that Theorem 6.1 relies on Theorem 6.14, the assumptions of which are technical inequalities that one may verify numerically. They are not restrictive, and have been checked for dimensions $4 \leq d \leq 9$ (see Appendix H). In other words, in Theorem 6.1, the assumption $4 \leq d \leq 9$ shall be replaced with the assumptions of Theorem 6.14.*

The proof of Theorem 6.1 is based on a refinement of the proof given in [AB95]. Let us briefly explain the structure of our argumentation. The optimality conditions for the optimal shape read in terms of the Laplacian of the eigenfunction on the boundary. In section 6.1, we exploit this particular information to conclude that the eigenfunction does not change sign in a neighbourhood of the boundary. One of our contributions is to translate this observation in terms of the geometry of the nodal domains of the eigenfunction. This is performed in section 6.2, where we show that one of the two nodal domains has a "hole" containing the other one. The purpose is then to inject the special geometry of the nodal domains in Talenti's comparison principle in order to derive a refined version of this principle. This is obtained in section 6.3. Our improved comparison principle is used instead of the standard one in the classic proof and yields an "asymmetric two-ball problem" as shown in section 6.4. The resolution of the new two-ball problem is carried out in section 6.5. The proof of Theorem 6.1 is finally achieved in section 6.6, in which we also discuss our main assumption (U).

Remark 6.3. *Let us point out that the only part of the proof of Theorem 6.1 which crucially relies on assumption (U) is the refined Talenti type comparison principle. Indeed, assumption (U) is made on the eigenfunction u in order to ensure that the special geometry of the nodal domains of u is transmitted to the upper level sets of u (see Remark 6.12), allowing to obtain a quantitative version of Talenti's comparison principle (see Theorem 6.9). However, it is clear that, even without assumption (U), the standard version of Talenti's comparison principle is not optimal when applied to an open set with a hole. Therefore, it would be of particular interest to study to which extent assumption (U) can be removed in Theorem 6.9. This could in turn allow to remove assumption (U) in Theorem 6.1. In this respect, the recent result [Ama+23] is noteworthy.*

6.1. Sign of an eigenfunction near the boundary

Let us recall that in Theorem 5.4 we computed the optimality conditions satisfied by an optimal shape for problem (6.1). This led us to the notion of critical shape (Definition 5.10). In the present paragraph, we will deduce from the optimality conditions an immediate but not less crucial consequence reading in terms of the

sign of the first eigenfunction near any connected component of the boundary. As stated in the next lemma, this property is actually enjoyed by any critical shape. We mention that this result seems quite known in the literature (see [Moh75, equation (20)] and also [ES22, equation (4.26)]).

Lemma 6.4. *Let Ω be a C^1 critical shape and γ be a connected component of $\partial\Omega$. Assume that a first eigenfunction u is $C^2(\overline{\Omega})$. Then there exists a neighbourhood of γ in which the sign of u is the same as the sign of Δu on γ.*

Proof. Considering $-u$ we can assume without loss of generality that $\Delta u = \alpha > 0$ on γ. Thanks to the ancillary information that $u = |\nabla u| = 0$ on the boundary, we get that $u > 0$ in a neighbourhood of γ. This comes from the fact that for each point $p \in \gamma$, the regularity of u allows writing the following Taylor expansion at p: for all x close enough to p such that $(x-p) \parallel \vec{n}(p)$,

$$u(x) \geq u(p) + \nabla u(p)\cdot(x-p) + \frac{1}{2}D^2u(p)\cdot(x-p)^2 - \frac{\alpha}{4}|x-p|^2 = \frac{\alpha}{4}|x-p|^2 > 0. \quad (6.2)$$

For the moment, the closeness of x and p depends on p. But as $\partial\Omega$ is compact, the associated threshold may be chosen uniformly, meaning that there exists $\epsilon > 0$ such that for all $p \in \partial\Omega$ and $x \in \Omega$, if $x \in B_\epsilon(p)$ and $(x-p) \parallel \vec{n}(p)$, then (6.2) holds. As a result, $\{p - t\vec{n}(p) : p \in \gamma, t \in (0,\epsilon)\}$ is a neighbourhood of γ in which u is positive. □

The previous lemma immediately applies to the optimal shape Ω and yields the following result.

Proposition 6.5. *Let Ω be a C^4 optimal shape for problem (6.1). Then any first eigenfunction is of constant sign in a neighbourhood of any connected component of $\partial\Omega$.*

Proof. Theorem 5.4 shows that Ω is a critical shape. Moreover, given that Ω is C^4, any eigenfunction is $C^{3,\alpha}(\overline{\Omega})$ for all $0 < \alpha < 1$. To conclude, apply Lemma 6.4. □

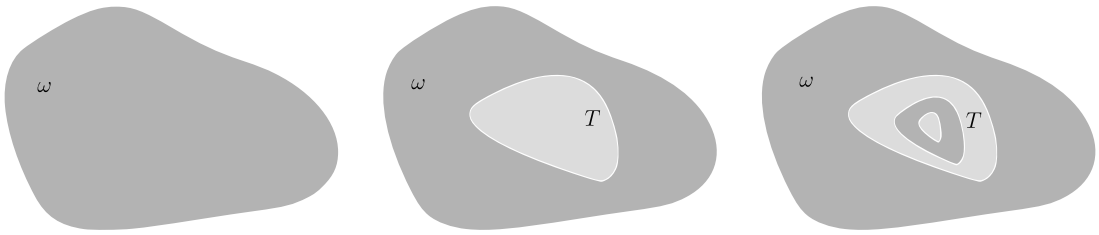
6.2. Geometry of the nodal domains

The proof of Theorem 6.1 is based on a refinement of Talenti's comparison principle which is used in the classic proof given in section 3.2. This refinement needs a crucial new information regarding the geometry of the nodal domains $\Omega_+ := \{u > 0\}$ and $\Omega_- := \{u < 0\}$. For that, one shall use advantageously Proposition 6.5, which precisely tells that Ω_+ and Ω_- cannot meet simultaneously a given connected component of the boundary. Therefore, as we will see in a few moment, if Ω is connected, and if $\Omega_+, \Omega_- \neq \emptyset$, at least one of Ω_+, Ω_- will have a hole. In order to make this observation clearer, we need a precise definition of a hole.

Definition 6.6. *Let ω and T be nonempty bounded open sets. We say that T is a hole for ω if:*

1. *ω and T are disjoint,*
2. *$\partial T \subseteq \partial\omega$.*

Let us remark that a hole need neither be connected nor simply connected (see Figure 6.1c). On the other hand, unlike Figure 6.1b, an open set admitting a hole shall be simply connected. This is the case with annuli in dimension $d > 2$, which admit balls as holes and remain simply connected. Thanks to the next lemma, we see that it is always possible to assume that Ω_+ has such a hole.



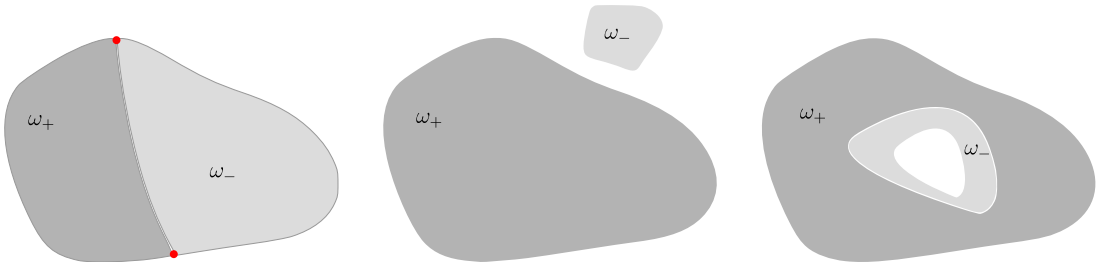
(a) No hole (b) Connected hole (c) Disconnected hole

Figure 6.1. – Holes in different situations. In each case, the open set ω is in dark grey, whereas its hole T, if it exists, is in light grey.

Lemma 6.7. *Let ω_+, ω_- be nonempty disjoint open sets contained in a bounded connected open set ω, such that, $\overline{\omega_+} \cup \overline{\omega_-} = \overline{\omega}$ and $\partial\omega \cap \partial\omega_+ \cap \partial\omega_- = \emptyset$. We also assume that $\omega_\pm$ is C^1 relatively with respect to ω, meaning that $\partial\omega_\pm \cap \omega$ is a C^1 hypersurface of $\mathbb{R}^d$. Then, either ω_+ has a hole containing ω_- or ω_- has a hole containing ω_+.*

The technical proof is given in Appendix E. Let us emphasise that the conclusion of Lemma 6.7 is that the hole of ω_+ (resp. ω_-) contains ω_- (resp. ω_+). Indeed, in general, as shown in Figure 6.2c, the hole is strictly greater than ω_- (resp. ω_+). Let us also briefly discuss the assumption of C^1 regularity of $\partial\omega_\pm \cap \omega$. We believe that this assumption shall be weakened, since it is only needed for applying Jordan-Brouwer Theorem and also for approaching $\partial\omega_\pm \cap \omega$ strictly from one side only (see Lemma E.1 for details). We mention that this last point may be achieved when $\partial\omega_\pm \cap \omega$ is of finite perimeter [Sch15]. However, it is not clear how weak the regularity of $\omega_\pm$ can be assumed to run the whole proof. Anyway, this is not an issue since in the next lines $\partial\omega_\pm \cap \omega$ will turn out to be the nodal set of the first eigenfunction, and hence will enjoy some regularity properties. This is detailed in Corollary 6.8, which explains how Lemma 6.7 is applied to the nodal domains.

Corollary 6.8. *Let Ω be a C^4 optimal shape for (6.1), and u be a first eigenfunction on Ω. Assume that $\Omega_+ := \{u > 0\}$ and $\Omega_- := \{u < 0\}$ are both nonempty and that*



(a) $\partial\omega \cap \partial\omega_+ \cap \partial\omega_- \neq \emptyset$ (b) ω disconnected (c) ω_+ with a hole

Figure 6.2. – Different configurations of the open sets ω_+ and ω_-. In each case, ω_+ is in dark grey, and ω_- is in light grey. In situation 6.2a, the assumption $\partial\omega \cap \partial\omega_+ \cap \partial\omega_- = \emptyset$ of Lemma 6.7 fails since $\partial\omega_+$ and $\partial\omega_-$ meet simultaneously $\partial\omega$ at the red points. In situation 6.2b, the assumption of connectedness of ω fails. In situation 6.2c, all the assumptions of Lemma 6.7 hold, and hence ω_+ admits a hole containing ω_-.

there is no critical point of u of level zero in Ω. Then, either Ω_+ admits a hole containing Ω_- or Ω_- admits a hole containing Ω_+.

Proof. We mention that Ω is connected (otherwise there would exist a connected component ω of Ω such that $\Gamma(\omega) = \Gamma(\Omega)$, and because $|\omega| < |\Omega|$, the dilation of ω would have a lower first eigenvalue than Ω, which would be a contradiction). Moreover, thanks to Proposition 6.5, we have the key property $\partial\Omega \cap \partial\Omega_+ \cap \partial\Omega_- = \emptyset$. The C^1 relative regularity of $\Omega_\pm$ with respect to Ω comes from the assumption regarding the critical points of u and from the Implicit Function Theorem. Therefore, all the hypotheses are fulfilled for applying Lemma 6.7, which gives the result. □

We will see in the next section that this geometric property of Ω_+ can be exploited to show that the symmetrization process used in the classic proof of the Rayleigh Conjecture in dimension $2, 3$ loses too much information if $\Omega_- \neq \emptyset$.

6.3. Improving Talenti's comparison principle

The classic proof of the Rayleigh Conjecture in dimension $d = 2, 3$ relies on a symmetrization technique. The main step is to apply Talenti's comparison principle (Theorem 2.12) on the nodal domains $\Omega_+ := \{u > 0\}$ and $\Omega_- := \{u < 0\}$, giving rise to the well-known two-ball problem. But we have seen in the previous section that for an optimal shape, one of the nodal domains has a hole containing the other one. In this setting, we may try to refine Theorem 2.12. Adapting the arguments proposed in its proof, we obtain the following Talenti type comparison principle on holed domains.

Theorem 6.9. *Let ω be a nonempty bounded open set admitting a (nonempty) hole T. Let $f \in L^2(\omega)$ and let $u \in H_0^1(\omega)$ solve*

$$\begin{cases} -\Delta u &= f \quad in \quad \omega, \\ u &= 0 \quad on \quad \partial\omega. \end{cases}$$

Let $v \in H_0^1(\omega^*)$ be the solution to

$$\begin{cases} -\Delta v &= f^* \quad in \quad \omega^*, \\ v &= 0 \quad on \quad \partial\omega^*. \end{cases}$$

Assume that $u \in C^2(\omega \cup \partial T)$ satisfies (U) in ω, that $\nabla u \neq 0$ on ∂T, and that $u > 0$ in ω. Then,

$$\kappa^2 v \geq u^* \qquad in\ \omega^*,$$

where

$$\kappa := \frac{|\omega|^{\frac{d-1}{d}}}{(|\omega| + |T|)^{\frac{d-1}{d}} + |T|^{\frac{d-1}{d}}} < 1.$$

Moreover, the situation $\kappa^2 v \equiv u^*$ is impossible.

Remark 6.10. *Observing that $\kappa < 1$, we understand that our inequality $\kappa^2 v \geq u^*$ is better than Talenti's one $v \geq u^*$ obtained with Theorem 2.12.*

Before proving Theorem 6.9, we need a technical lemma. Indeed, we have to ensure that the geometric assumption made on $\omega = \{u > 0\}$ in Theorem 6.9 holds more generally on the upper level sets $\{u > t\}$ for all $0 \leq t < \sup_\omega u$. Actually, this fact is not true in general without any assumption on u. However, thanks to our central assumption (U), it will hold as stated below.

Lemma 6.11. *Let ω be a bounded open set admitting a hole T and $u \in C^2(\omega \cup \partial T)$ be positive and satisfying (U) in ω. Assume moreover that $u = 0$ and $\nabla u \neq 0$ on ∂T. Then, for all $0 \leq t < \sup_\omega u$, $\omega_t := \{u > t\}$ has a hole $T_t \supseteq T$.*

Proof. Since $u : \omega \to \mathbb{R}$ verifies (U), it is a Morse function on $\omega \setminus \{u = \sup_\omega u\}$. Then, driven by the gradient near $\partial T \subseteq \partial\omega$, we will build as in Morse theory an open set T_t which satisfies $T \subseteq T_t$, $T_t \cap \omega \subseteq \{u < t\}$, and $\partial T_t \subseteq \{u = t\}$. To do so, set $X = \frac{\nabla u}{|\nabla u|^2}$, well defined on $\omega \cup \partial T \setminus \{u = \sup_\omega u\}$, and let ϕ_t be the flow associated with X. We check that $u \circ \phi_{s_2} - u \circ \phi_{s_1} = s_2 - s_1$, so that, $\forall x \in \partial T$, $u(\phi_s(x)) = s$. Consequently, $T_t := T \bigcup (\cup_{0 \leq s < t} \phi_s(\partial T))$, which is open, satisfies the above requirements.

Indeed, immediately $T \subseteq T_t$ and $T_t \cap \omega_t = \emptyset$. It remains to check that $\partial T_t \subseteq \partial\omega_t$. Let $x \in \partial T_t$ and $\epsilon > 0$. Recall that $u(x) = t$ and that $\nabla u(x) \neq 0$ since (U) is assumed. Then, because u is C^2, we can write a Taylor expansion at x: for all $s > 0$, set $y_s := x + s\frac{\nabla u(x)}{|\nabla u(x)|^2}$, then

$$u(y_s) = u(x) + \nabla u(x) \cdot (y_s - x) + o(|y_s - x|) = t + s + o(s).$$

Consequently, if s is small enough for $y_s \in B_\epsilon(x)$ and $o(s) \geq -s/2$, we get $u(y_s) \geq t + s/2 > t$, or in other words $y_s \in \omega_t$. Therefore, $x \in \partial\omega_t$. □

Remark 6.12. *Roughly speaking, the proof of Lemma 6.11 is based on the fact that the upper level sets of a function do not change topology between critical values. This clarifies the purpose of assumption (U), which precisely rules out the possibility for the function to have critical values (except extremal ones). Thus, we understand that assumption (U) made on the first eigenfunction u of the optimal shape Ω in Theorem 6.1 shall be relaxed as long as it is ensured that the upper level sets of $u|_{\{u>0\}}$ and $-u|_{\{u<0\}}$ do not change topology. For instance, instead of (U), one could ask u to satisfy:*

$$\text{For } v = u \text{ and } v = -u, \overline{\{v > t\}} \text{ is homeomorphic to } \overline{\{v > 0\}} \text{ for each } 0 < t < \max_{\overline{\Omega}} v.$$

Actually, we believe that the points responsible for topology changes in the upper level sets are only ***weak*** *saddle points – a weak saddle point is a critical point which is not a local* ***strict*** *extremum. From this fact, one observes that*

As long as u admits no weak saddle point in Ω, u has empty nodal set, and hence Ω is a ball.

Indeed, if u had a nodal set, without loss of generality Ω_+ would have a hole T by Corollary 6.8. Hence, if moreover u had no weak saddle point, the upper level sets $\{u > t\}$ for $0 < t < \max_{\overline{\Omega}} u$ would also have holes containing T. Eventually, $\{u = \max_{\overline{\Omega}} u\}$ would have a hole containing T (note that the definition of a hole actually makes sense also for sets which are not open). In particular, any point in $\{u = \max_{\overline{\Omega}} u\}$ would not be a strict maximum, hence it would be a weak saddle point, a contradiction with the assumption.

Thanks to Lemma 6.11, it becomes possible to prove Theorem 6.9.

Proof of Theorem 6.9. We follow the proof of Theorem 2.12. Observe that here it is not necessary to assume that $u \in C^d(\omega)$ since assumption (U) replaces Sard's Theorem. Setting for all $0 < t < \sup_\omega u$, $\omega_t := \{u > t\}$, and n_t the outward normal to the boundary of ω_t (note that $\partial\omega_t$ is regular due to assumption (U)), the divergence formula gives

$$\int_{\omega_t} f = -\int_{\omega_t} \Delta u = -\int_{\partial\omega_t} \nabla u \cdot \vec{n_t}.$$

But because of (U), $n_t = -\frac{\nabla u}{|\nabla u|}$, so that

$$\int_{\omega_t} f = \int_{\partial\omega_t} |\nabla u|.$$

Now denote $\mu(t) = |\omega_t|$ the distribution function of u. Then, by (2.2) and by Cauchy-Schwarz inequality,

$$|\partial\omega_t| = \int_{\partial\omega_t} \sqrt{\frac{|\nabla u|}{|\nabla u|}} \leq \left(\int_{\partial\omega_t} \frac{1}{|\nabla u|} \int_{\partial\omega_t} |\nabla u|\right)^{1/2} = \sqrt{-\mu'(t) \int_{\omega_t} f},$$

hence,

$$\mathcal{H}^{d-1}(\partial\omega_t)^2 \leq -\mu'(t) \int_{\omega_t} f. \tag{6.3}$$

At this point the classic proof applies the isoperimetric inequality (Theorem A.2) to ω_t. Here, thanks to the geometric assumption, we are able to improve this inequality. Indeed, because ω has a hole T, Lemma 6.11 implies that ω_t has a hole T_t containing T. In particular, due to the definition of a hole, $\mathcal{H}^{d-1}\left(\partial\left(\omega_t \cup \overline{T_t}\right)\right) = \mathcal{H}^{d-1}(\partial\omega_t) - \mathcal{H}^{d-1}(\partial T_t)$ and $|\omega_t \cup \overline{T_t}| = |\omega_t| + |T_t|$. Consequently, applying Theorem A.2 with $A = \omega_t \cup \overline{T_t}$ one gets

$$\frac{\mathcal{H}^{d-1}(\partial\omega_t) - \mathcal{H}^{d-1}(\partial T_t)}{(|\omega_t| + |T_t|)^{\frac{d-1}{d}}} \geq C_d.$$

Using once again Theorem A.2 with $A = T_t$ this time, and then recalling that $T \subseteq T_t$, we find

$$\begin{aligned}\mathcal{H}^{d-1}(\partial\omega_t) \geq C_d(|\omega_t| + |T_t|)^{\frac{d-1}{d}} + \mathcal{H}^{d-1}(\partial T_t) &\geq C_d\left[(|\omega_t| + |T_t|)^{\frac{d-1}{d}} + |T_t|^{\frac{d-1}{d}}\right] \\ &\geq C_d\left[(\mu(t) + |T|)^{\frac{d-1}{d}} + |T|^{\frac{d-1}{d}}\right].\end{aligned}$$

Plugging this into (6.3),

$$C_d^2 \mu(t)^{\frac{2(d-1)}{d}} \left[\left(1 + \frac{|T|}{\mu(t)}\right)^{\frac{d-1}{d}} + \left(\frac{|T|}{\mu(t)}\right)^{\frac{d-1}{d}}\right]^2 \leq -\mu'(t) \int_{\omega_t} f.$$

Because $\mu(t) \leq |\omega|$, the previous inequality yields

$$1 \leq \frac{1}{\left[\left(1 + \frac{|T|}{|\omega|}\right)^{\frac{d-1}{d}} + \left(\frac{|T|}{|\omega|}\right)^{\frac{d-1}{d}}\right]^2} \frac{-\mu'(t)}{C_d^2 \mu(t)^{\frac{2(d-1)}{d}}} \int_{\omega_t} f = \kappa^2 \frac{-\mu'(t)}{C_d^2 \mu(t)^{\frac{2(d-1)}{d}}} \int_{\omega_t} f.$$

Now use Lemma 2.10:

$$1 \leq \kappa^2 \frac{-\mu'(t)}{C_d^2 \mu(t)^{\frac{2(d-1)}{d}}} \int_0^{\mu(t)} f^\star.$$

Then we integrate both sides between 0 and $u^\star(s)$ (see the proof of Theorem 2.12

for details) to find that

$$u^\star(s) \le \kappa^2 \int_s^{|\omega|} \frac{1}{C_d^2 r^{\frac{2(d-1)}{d}}} \int_0^r f^\star.$$

Finally, since v may be computed explicitly, we conclude as desired

$$u^*(x) \le \kappa^2 \int_{|B_{|x|}|}^{|\omega|} \frac{1}{C_d^2 r^{\frac{2(d-1)}{d}}} \int_0^r f^\star = \kappa^2 v(x).$$

Regarding the equality case, assume by contradiction that $u^* \equiv \kappa^2 v$. Then, all the previous inequalities turn into equalities, and in particular $T = T_t$ for each $0 < t < \sup_\omega u$. Therefore, $\emptyset \neq \partial T_t \subseteq \partial\omega$ for each $0 < t < \sup_\omega u$. Since $u = t$ on ∂T_t and $u = 0$ on $\partial\omega$, we conclude that $t = 0$, which contradicts $0 < t < \sup_\omega u$. Hence the set $(0, \sup_\omega u)$ is empty, or in other words u vanishes identically in ω. Since u were assumed positive in ω, this is a contradiction. □

Thanks to our enhanced comparison principle, it is possible to bound from below the eigenvalue $\Gamma(\Omega)$ by an asymmetric two-ball problem. The next section is devoted to the derivation of this lower bound.

6.4. Derivation of an asymmetric two-ball problem

Proceeding as in section 3.2, but using our modified comparison principle, we get the following.

Proposition 6.13. *Let Ω be a C^4 optimal shape for (6.1). Let a (resp. b) be the radius of Ω_+^* (resp. Ω_-^*). Then, if the first eigenfunction on Ω satisfies (U),*

$$\Gamma(\Omega) \ge \min_{v,w} \frac{\int_{B_a} (\Delta v)^2 + \int_{B_b} (\Delta w)^2}{K(a,b)^4 \int_{B_a} v^2 + \int_{B_b} w^2} =: \mu(a,b), \tag{6.4}$$

where

$$K(a,b) = \frac{a^{d-1}}{(a^d + b^d)^{\frac{d-1}{d}} + b^{d-1}}, \tag{6.5}$$

and the minimum is taken among radial functions $v \in H^2(B_a)$ and $w \in H^2(B_b)$ satisfying, for all $x \in \partial B_a$ and $y \in \partial B_b$,

$$v(x) = w(y) = 0, \quad a^{d-1}\partial_r v(x) = b^{d-1}\partial_r w(y), \quad \Delta v(x) + \Delta w(y) = 0,$$

with the convention that $v \in H_0^2(B_a)$ if $b = 0$ and $w \in H_0^2(B_b)$ if $a = 0$. Moreover, in case of equality in (6.4), Ω is a ball.

Proof. Assume that the first eigenfunction u changes sign (otherwise (6.4) follows directly from Theorem 2.12). As in [Tal81] and [AB95] one symmetrizes the

sign-changing functions

$$f := -\Delta u, \qquad g := \Delta u.$$

Now the solutions of the Laplace equations in B_a with data f^*, denoted v and in B_b with data g^*, say w, are kind of spherical rearrangements of u_+ and u_-:

$$\left\{\begin{array}{rcll} -\Delta v &=& f^*|_{B_a} & in \quad B_a, \\ v &=& 0 & on \quad \partial B_a, \end{array}\right. \qquad \left\{\begin{array}{rcll} -\Delta w &=& g^*|_{B_b} & in \quad B_b, \\ w &=& 0 & on \quad \partial B_b. \end{array}\right.$$

We will compare $\Gamma(\Omega)$ with some quotient involving v and w. For that, recall that with the notations of Appendix F, we have the relations $f^* = f^\ddagger$ and $g^* = g^\ddagger$ a.e. (see Lemma 2.6). Then apply Lemma F.1 with $p = 2$, and $\omega_\pm = \Omega_\pm$. This is possible since u cannot vanish on a set of positive measure due to [Mit20], which justifies that assumption $|\Omega_+| + |\Omega_-| = |\Omega|$ of Lemma F.1 is fulfilled. Then, we find

$$\int_\Omega (\Delta u)^2 = \int_{B_a} (\Delta v)^2 + \int_{B_b} (\Delta w)^2 \,. \tag{6.6}$$

For a later use, we need also to apply Lemma F.1 with $p = 1$, which gives

$$\int_\Omega \Delta u = \int_{B_a} \Delta v - \int_{B_b} \Delta w. \tag{6.7}$$

Thanks to (6.6), the numerator in the Rayleigh quotient $\Gamma(\Omega)$ shall be written in terms of the L^2 norms of Δv and Δw. At this stage, we would like to estimate the denominator in the Rayleigh quotient $\Gamma(\Omega)$ in terms of the L^2 norms of v and w. In the classic proof of the Rayleigh Conjecture, noticing that $\int_\Omega u^2 = \int_{\Omega_+} u_+^2 + \int_{\Omega_-} u_-^2 = \int_{B_a} (u_+^*)^2 + \int_{B_b} (u_-^*)^2$, this is done using Theorem 2.12 (as usual we agree on the convention $u_+^* = (u_+)^*$ and $u_-^* = (u_-)^*$).

More precisely, on the one hand, thanks to equation (2.4), $-\Delta w = g^*|_{B_b} \geq \left(g|_{\Omega_-}\right)^*$ in B_b, and on the other hand, $-\Delta(u_-) = g|_{\Omega_-}$ in Ω_-. Then, Theorem 2.12 combined with the maximum principle yields $w \geq u_-^*$ in B_b. Analogously, we get $v \geq u_+^*$ in B_a. But this last inequality can be improved.

Indeed, with Corollary 6.8, we know that Ω_+ has a hole T containing Ω_-. Moreover, as (U) has been assumed to hold in Ω, clearly it holds in Ω_+ and even $\nabla u \neq 0$ on ∂T. Therefore, we apply Theorem 6.9 with $\omega = \Omega_+$ and combine it as before with the maximum principle to obtain

$$u_+^* \leq \left[\frac{|\Omega_+|^{\frac{d-1}{d}}}{(|\Omega_+| + |T|)^{\frac{d-1}{d}} + |T|^{\frac{d-1}{d}}}\right]^2 v \leq \left[\frac{|\Omega_+|^{\frac{d-1}{d}}}{(|\Omega_+| + |\Omega_-|)^{\frac{d-1}{d}} + |\Omega_-|^{\frac{d-1}{d}}}\right]^2 v.$$

Since $|\Omega_+| = |B_1| a^d$ and $|\Omega_-| = |B_1| b^d$, we get

$$u_+^* \leq \left[\frac{a^{d-1}}{(a^d + b^d)^{\frac{d-1}{d}} + b^{d-1}}\right]^2 v = K(a,b)^2 v.$$

To sum up,

$$\int_\Omega (\Delta u)^2 = \int_{B_a} (\Delta v)^2 + \int_{B_b} (\Delta w)^2,$$
$$\int_\Omega u^2 = \int_{B_a} (u_+^*)^2 + \int_{B_b} (u_-^*)^2 \leq K(a,b)^4 \int_{B_a} v^2 + \int_{B_b} w^2.$$

Therefore,

$$\Gamma(\Omega) \geq \frac{\int_{B_a} (\Delta v)^2 + \int_{B_b} (\Delta w)^2}{K(a,b)^4 \int_{B_a} v^2 + \int_{B_b} w^2}.$$

Before concluding, we emphasise that for every $x \in \partial B_a$ and every $y \in \partial B_b$,

1. $v(x) = 0 = w(y)$;
2. due to (6.7),

$$|\partial B_a| \partial_r v(x) - |\partial B_b| \partial_r w(y) = \int_{\partial B_a} \partial_n v - \int_{\partial B_b} \partial_n w$$
$$= \int_{B_a} \Delta v - \int_{B_b} \Delta w = \int_\Omega \Delta u = 0;$$

3. $\Delta v(x) + \Delta w(y) = f^\dagger(|B_a|) + (-f)^\dagger(|B_b|) = f_+^\star(|B_a|) - f_-^\star(|B_b|) + f_-^\star(|B_b|) - f_+^\star(|B_a|) = 0$, with † defined in (2.1).

Therefore the couple (v, w) is admissible and hence $\Gamma(\Omega) \geq \mu(a,b)$. Eventually, note that $\mu(a,b)$ is well-posed (see [AB95, Appendix 2]). Regarding the rigidity result, we observe that the equality case implies that there are equalities both in Talenti's comparison principle $w \geq u_-^*$ and in our enhanced comparison principle $K(a,b)^2 v \geq u_+^*$. Since equality cannot hold in Theorem 6.9, we obtain a contradiction if both $a \neq 0$ and $b \neq 0$. In other words, we boil down to the setting where u is one-sign. In this situation, it is already known (see [Kes06, section 3.2]) that equality in (6.4) implies that Ω is a ball. □

6.5. Solving the asymmetric two-ball problem

As noted in [AB95], the conjecture will hold if one is able to show that

$$\mu(0,1) = \mu(1,0) = \min_{\substack{a,b \in [0,1] \\ a^d + b^d = 1}} \mu(a,b).$$

Indeed, it is possible to assume without loss of generality that the radius of Ω^* is equal to 1. Then, thanks to Proposition 6.13 (and to [Mit20] yielding $a^d + b^d = 1$),

$$\Gamma(\Omega) \geq \mu(a,b) \geq \min_{\substack{a,b \in [0,1] \\ a^d + b^d = 1}} \mu(a,b) \geq \mu(0,1) = \Gamma(B_1).$$

This is exactly the statement of the next result. Let us mention at this point that it seems difficult to avoid technicalities, since the value of $\mu(a,b)$ has to be clarified, leading to explicit computations. This issue was already present in the resolution of the symmetric two-ball problem (except in [Nad95]), and it will be intensified here, the asymmetry complicating the analysis. In particular, in order to understand the assumptions of the theorem, we need to introduce several notations. As in Proposition 5.15, J_ν and I_ν are the Bessel and modified Bessel functions of order ν, and the function f_ν is defined in accordance with the formula

$$f_\nu(r) = r^{d-1}\left[\frac{J_{\nu+1}(r)}{J_\nu(r)} + \frac{I_{\nu+1}(r)}{I_\nu(r)}\right]. \tag{6.8}$$

The first positive zero of J_ν is denoted j_ν and k_ν is the first positive zero of f_ν. For each $a \in [0,1]$, the number $b \in [0,1]$ such that $a^d + b^d = 1$ is denoted $b(a)$. Then, $K(a)$ is a simplified notation for $K(a,b(a))$ where $K(a,b)$ is defined in (6.5). We define a_I to be the value of a such that $b(a) = j_\nu/k_\nu$. On the other hand, a_S is the value of a when $aK(a) = j_\nu/k_\nu$. Eventually, we set

$$\begin{aligned}
T_1(a) =& (aK(a))^d K'(a)/K(a),\\
g_\nu(r) =& \frac{1}{r^{d-1}}\left[\frac{J_{\nu+1}}{J_\nu}(r) - \frac{I_{\nu+1}}{I_\nu}(r)\right],\\
G_1(a) =& [aK'(a) + K(a)](k_\nu K(a))^{d-1} g_\nu(k_\nu a K(a)),\\
G_2(a) =& k_\nu^{d-1} g_\nu(k_\nu b(a)).
\end{aligned} \tag{6.9}$$

Using the above notations, it becomes possible to state the main result of this paragraph.

Theorem 6.14. *Let $d \geq 4$, set $\nu = d/2 - 1$, and assume that*

$$2\frac{j_\nu^d}{k_\nu^d} + \frac{j_\nu}{k_\nu} > 1. \tag{6.10}$$

Assume moreover that

1. *There exists a sequence $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = a_I$ such that, for each $i \in (0,n]$,*

$$f_\nu(k_\nu K(x_{i+1})x_{i+1}) + K(x_i)^d f_\nu(k_\nu b(x_i)) \leq 0, \tag{6.11}$$

and

$$G_1(x_1) + G_2(x_0) \leq 0. \tag{6.12}$$

2. *There exists a sequence $a_S = y_0 < y_1 < \ldots < y_m < y_{m+1} = 1$ such that for each $i \in [0,m)$,*

$$f_\nu(k_\nu K(y_{i+1})y_{i+1}) + f_\nu(k_\nu b(y_i)) \leq 0, \tag{6.13}$$

and

$$2T_1(y_m) \geq f_\nu(k_v b(y_m))|g_\nu(k_v K(y_m)y_m)| \left[\frac{(d-1)}{y_m^d}\left(1+\frac{1}{b(y_m)}\right)+1\right] + [f_\nu g_\nu](k_v b(y_m)). \tag{6.14}$$

Then,

$$\min_{\substack{a,b\in[0,1]\\ a^d+b^d=1}} \mu(a,b) = \mu(0,1) = \mu(1,0).$$

Remark 6.15. *Let us point out that each of the assumptions of Theorem 6.14 shall be checked numerically. This has been done for $d = 4, 5, 6, 7, 8, 9$ in Appendix H.*

Proof. Fix for the moment an admissible couple (a, b). As explained in [AB95, Appendix 2], the minimum $\mu(a, b)$ is obtained for two H^2 radial functions $v = v(r)$ and $w = w(r)$ satisfying

$$\Delta^2 v = K(a,b)^4\mu(a,b)v \text{ in } B_a, \qquad \Delta^2 w = \mu(a,b)w \text{ in } B_b, \tag{6.15}$$

and the boundary conditions, for all $x \in \partial B_a$ and $y \in \partial B_b$,

$$v(x) = w(y) = 0, \qquad a^{d-1}\partial_r v(x) = b^{d-1}\partial_r w(y), \qquad \Delta v(x) + \Delta w(y) = 0. \tag{6.16}$$

Because v, w are radial, (6.15) is actually an ODE whose solution is of the form

$$v(r) = [AJ_\nu(kKr) + BI_\nu(kKr)]r^{-\nu},$$
$$w(r) = [CJ_\nu(kr) + DI_\nu(kr)]r^{-\nu},$$

where $\nu = d/2 - 1$, $k = k(a) = k(a, b) := \mu(a, b)^{1/4}$ and $K = K(a) = K(a, b) = \frac{a^{d-1}}{1+b^{d-1}}$. Using the previous expressions and the relations $J'_\nu(r) = \frac{\nu J_\nu(r)}{r} - J_{\nu+1}(r)$ and $I'_\nu(r) = \frac{\nu I_\nu(r)}{r} + I_{\nu+1}(r)$, one finds

$$\partial_r v(r) = [-AJ_{\nu+1}(kKr) + BI_{\nu+1}(kKr)]kKr^{-\nu},$$
$$\partial_r w(r) = [-CJ_{\nu+1}(kr) + DI_{\nu+1}(kr)]kr^{-\nu}.$$

Using this time the relation $J'_{\nu+1}(r) = J_\nu(r) - \frac{\nu+1}{r}J_{\nu+1}(r)$ and $I'_{\nu+1}(r) = I_\nu(r) -$

$\frac{\nu+1}{r} I_{\nu+1}(r)$, we obtain

$$\begin{aligned}\partial_r^2 v(r) = &\left(A \left[\frac{(2\nu+1) J_{\nu+1}(kKr)}{kKr} - J_\nu(kKr) \right] \right. \\ &\left. - B \left[\frac{(2\nu+1) I_{\nu+1}(kKr)}{kKr} - I_\nu(kKr) \right] \right) (kK)^2 r^{-\nu}, \\ \partial_r^2 w(r) = &\left(C \left[\frac{(2\nu+1) J_{\nu+1}(kr)}{kr} - J_\nu(kr) \right] \right. \\ &\left. - D \left[\frac{(2\nu+1) I_{\nu+1}(kr)}{kr} - I_\nu(kr) \right] \right) k^2 r^{-\nu}.\end{aligned}$$

Then,

$$\begin{aligned}\Delta v(r) &= \left(-A J_\nu(kKr) + B I_\nu(kKr) \right) (kK)^2 r^{-\nu}, \\ \Delta w(r) &= \left(-C J_\nu(kr) + D I_\nu(kr) \right) k^2 r^{-\nu}.\end{aligned}$$

Eventually, (6.16) yields a system of four linear equations involving (A, B, C, D), that has only the solution $(0, 0, 0, 0)$ unless its determinant is 0. Consequently, for the solution not to be trivial, one gets at the end of the day

$$\begin{aligned}0 =& 2a^{d-1} [I_\nu(kKa) I_\nu(kb) J_{\nu+1}(kKa) J_\nu(kb) + J_\nu(kKa) I_\nu(kb) I_{\nu+1}(kKa) J_\nu(kb)] \\ &+ 2Kb^{d-1} [I_\nu(kKa) I_\nu(kb) J_{\nu+1}(kb) J_\nu(kKa) + J_\nu(kKa) I_\nu(kb) J_{\nu+1}(kb) I_\nu(kKa)],\end{aligned}$$

which is equivalent to

$$0 = f_\nu(kKa) + K^d f_\nu(kb),$$

Therefore $k(a)$ is the first positive zero of $h_\nu : r \mapsto f_\nu(rKa) + K^d f_\nu(rb)$. Hence, to conclude the proof, it remains to show that $k_\nu = k(0) = k(1) \leq k(a)$ for all $a \in [0, 1]$. As shown in [AB95], f_ν has some useful properties: $f_\nu(0) = 0$; f_ν has simple poles at $j_{\nu,i}, i \in \mathbb{N}^*$ (the positive zeros of J_ν); f_ν is continuous and increasing as soon as it is defined. Consequently $f_\nu(j_{\nu,i}^-) = +\infty$ and $f_\nu(j_{\nu,i}^+) = -\infty$. As explained in [ABL97], it is useful to study the function of two variables $F_\nu(k, a) = f_\nu(kK(a)a) + K(a)^d f_\nu(kb)$. The properties of f_ν quoted above justify that F_ν is increasing in its first variable and admits poles on the arcs α_i of equation $kKa = j_{\nu,i}$ and β_i of equation $kb = j_{\nu,i}$. Then, $k(a)$ gives rise to an arc $\kappa = (k(a), a)$ joining the horizontal line $\{a = 0\}$ to the horizontal line $\{a = 1\}$ and lying in a connected component of the complement of the arcs α_i and β_i. More precisely, because k remains between the first and the second pole of h_ν, κ is trapped between α_1 and β_1. Thus one has the qualitative properties summarised in Figure 6.3.

An interesting point in the (k, a)-plane is the point M where α_1 and β_1 intersect, because the previous discussion showed that $M \in \kappa$. At M, one has the relation $Ka = b$, or in other words $2b^d + b - 1 = 0$. Note that the polynomial $P = 2X^d + X - 1$ satisfies $P(0) = -1$, $P(1) = 2$ and is increasing between 0 and 1. If we note b_M

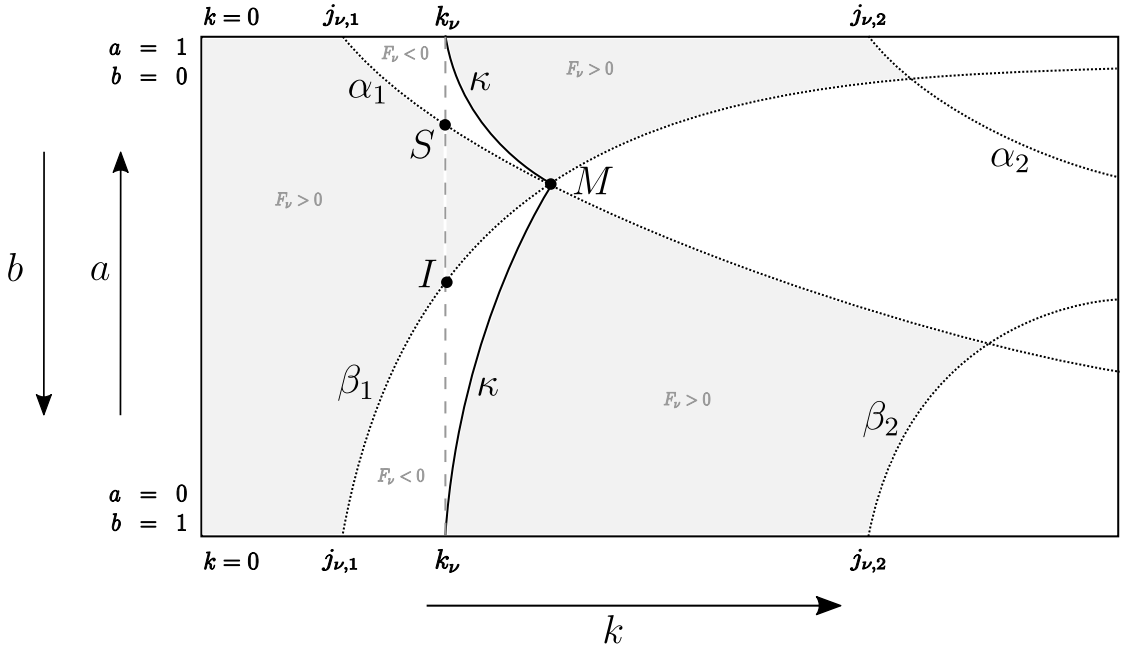


Figure 6.3. – Qualitative properties of the function F_ν in the (k, a)-plane.

the root of P between 0 and 1, and a_M the corresponding value for a, then $k_M := k(a_M) = \frac{j_{\nu,1}}{b_M}$ since $(k_M, a_M) \in \beta_1$. As in [AB95], one wonders whether $k_\nu < k_M$. To see this, one has to check if $b_M < \frac{j_{\nu,1}}{k_\nu}$, that is if $P(\frac{j_{\nu,1}}{k_\nu}) > 0$. Since by definition $j_\nu = j_{\nu,1}$, this condition turns into the next equation.

$$2\frac{j_\nu^d}{k_\nu^d} + \frac{j_\nu}{k_\nu} > 1.$$

This necessary condition for k_ν to be the minimum of k is the counterpart of equation (41) of [AB95], which failed for $d > 3$. In our situation, it is assumed that (6.10) is fulfilled (see also Table H.1). This gives credence to Figure 6.3, where M is at the right of the vertical dashed line $\{k = k_\nu\}$.

Obviously, the fact that $k_\nu < k_M$ holds is not sufficient to claim that k_ν minimizes k. As suggested by Figure 6.3, one checks that each of α_1 and β_1 intersect exactly once the line $\{k = k_\nu\}$. We denote by S and I the intersection points, and by (k_S, a_S, b_I) and (k_I, a_I, b_S) the coordinates of S and I respectively. These coordinates are characterised by the relations

$$b_S = \frac{j_\nu}{k_\nu}, \qquad K(a_S)a_S = \frac{j_\nu}{k_\nu}. \tag{6.17}$$

It is then clear that for $a \in [a_I, a_S]$, $k(a) > k_\nu$ (see Figure 6.3). Hence it remains to show that for $a \in [0, a_I) \cup (a_S, 1] =: A$ we have $k(a) \geq k_\nu$. To do so, thanks to

the fact that F_ν is increasing in the variable k, it is enough to prove that $\forall a \in A$,

$$F_\nu(k_\nu, a) \leq 0. \tag{6.18}$$

The very technical and laboured proof of this inequality is provided by Theorem G.1 in appendix, which relies on equations (6.11), (6.12), (6.13), and (6.14). □

6.6. Proof of Theorem 6.1 and final remarks

Thanks to the analysis performed in sections 6.4 and 6.5, the proof of Theorem 6.1 is rather immediate.

Proof of Theorem 6.1. Up to a dilation, assume that $|\Omega| = |B_1|$. Then, thanks to Proposition 6.13, we know that $\Gamma(\Omega) \geq \mu(a, b)$ (where $|B_a| = |\Omega_+|$ and $|B_b| = |\Omega_-|$). Moreover, since by [Mit20] the first eigenfunction does not vanish on a set of positive measure, $a^d + b^d = 1$. Then, thanks to Theorem 6.14, and to Remark 6.15 we know that $\mu(a, b) \geq \mu(0, 1) = \Gamma(B_1)$ whenever $4 \leq d \leq 9$. Combining the inequalities, we get

$$\Gamma(\Omega) \geq \Gamma(B_1).$$

Since Ω is optimal, all the previous inequalities turn into equalities. As a result, we have $\Gamma(\Omega) = \mu(a, b)$, and we conclude that Ω is a ball thanks to the equality case in Proposition 6.13. □

Before to conclude I should mention that, given the assumptions of Theorem 6.1, Dorin Bucur suggested to me an alternative way to prove the Rayleigh Conjecture. The heuristic argument is similar to that provided in equation (3.12) of [Nad95] for proving that the eigenfunction over a ball is radially decreasing. It goes as follows. Since the nodal domain Ω_+ has a hole (Corollary 6.8), if u has no intermediate critical value, this means that the set $\{u = \max u\}$ is the boundary of a subset ω (see the arguments at the end of Remark 6.12). Replacing u over ω by its inversion $M - (u - M)$, where $M = \max u$ gives rise to a function $v \in H^2(\Omega)$ with same Laplacian as u. Moreover, since $2M - u \geq |u|$, v is nonnegative and the L^2 norm of v is greater or equal than the L^2 norm of u. As a result the eigenfunction over Ω must be positive, which concludes. Unfortunately, this very nice idea suffers from a weakness. Indeed, besides the issue of making rigorous the arguments, it is not true in general that the resulting function v is $H^2_0(\Omega)$ since it may not vanish at the boundary. To see this, consider for instance the radial function over the annulus $A = \{\pi < |x| < 2\pi\}$ defined by $u(x) = \sin|x| \sin 2|x|$, which is in $H^2_0(A)$. Then u reaches its maximum on a circle of radius $3\pi/2 < R < 2\pi$, and it is clear that the function $M - (u - M)$ defined on $\omega = \{\pi < |x| < R\}$ does not vanish on $\{|x| = \pi\}$. Nevertheless, the previous phenomenon does not occur when Ω has connected boundary, hence the argument of Dorin Bucur might work in this setting.

Anyway, Theorem 6.1 mainly relies on the assumption (U). Hence it is essential to understand to which extent (U) has a chance to hold. For instance, one could seek for geometric conditions on Ω yielding (U). The easiest such condition one shall think about is maybe convexity. Indeed, it is known that for second order operators, convexity often implies a unique critical point (hence a unique critical value) for the solutions of certain particular equations (see [Gro21; Gar03] and also the unmissable [Kaw85, chapter III]). However, it is clear that convexity is not enough for showing that the first eigenfunction admits a unique critical value in the case of fourth order equations. To see this, consider classic counterexamples, such as domains with corners, in which it is known that the eigenfunctions of the bilaplacian oscillate ([Cof82]), and consequently do not satisfy (U). Nevertheless, against these examples, one shall put forward that the shapes involved are not critical (see Definition 5.10) and hence not optimal. Therefore, one is led to wonder whether a first eigenfunction of a critical convex shape satisfies (U).

At this time, we were not able to answer this central issue, and could only argue that if u is a first eigenfunction on a C^∞ strictly convex critical shape $\Omega \subseteq \mathbb{R}^2$, then $\Delta u + \sqrt{\Gamma(\Omega)}u$ admits only one critical point. This is a consequence of Proposition 5.5 and of [DGM21, Theorem 1.2]. Indeed, due to the convexity of Ω, $\partial\Omega$ is connected, and due to its criticality, up to a sign change, $\frac{\Delta}{\sqrt{\Gamma(\Omega)}}u = \sqrt{\frac{2}{|\Omega|}}$ on $\partial\Omega$. Then, thanks to Proposition 5.5, the function $z_u := \frac{\Delta}{\sqrt{\Gamma(\Omega)}}u + u - \sqrt{\frac{2}{|\Omega|}}$ satisfies

$$\begin{cases} \Delta z_u &= \sqrt{\Gamma(\Omega)}\left(z_u + \sqrt{\frac{2}{|\Omega|}}\right) & in \quad \Omega, \\ z_u &< 0 & in \quad \Omega, \\ z_u &= 0 & on \quad \partial\Omega, \end{cases}$$

and we conclude that $-z_u$ is a positive semi-stable solution. Hence, it has a unique critical point thanks to [DGM21, Theorem 1.2].

Besides the geometric conditions on Ω that would yield (U), another interesting question is the sensitivity of (U) under perturbations of Ω. Let us consider small C^4 perturbations of the form $(I+\theta)\Omega =: \Omega_\theta$ of a shape Ω such that $\Gamma(\Omega)$ is simple. Then, it is possible to build a first eigenfunction u_θ on Ω_θ varying continuously with θ, in the sense that $\theta \in C^4(\mathbb{R}^d, \mathbb{R}^d) \mapsto u_\theta \circ (I+\theta) \in C^{3,\gamma}(\overline{\Omega})$ is continuous in a neighbourhood of $\theta = 0$. On the other hand, let us define the map $\# : C^2(\overline{\Omega}) \to \mathbb{N} \cup \{\infty\}$, such that $\#v$ is the number of critical points of v in Ω. Then, one can prove that $\#$ is upper-semi continuous at any point v having no degenerate critical point and such that $\Delta v \neq 0$ on $\partial\Omega$. Combining those two results, we obtain that if Ω is a critical shape, if u_0 has only one critical point, and if this critical point is nondegenerate, then also u_θ has only one critical point for all small enough θ. In particular, u_θ satisfies (U) for all small enough θ. This applies to the case $\Omega = B$, and shows, thanks to Theorem 6.1, that small C^4 nontrivial perturbations of the ball cannot be optimal, slightly generalising a known result (see [GGS10, Theorem 6.29]).

As a final remark, we would like to mention the troubling fact that assumption (U) is similar to the condition given in Remark 3 of [Cia00] for the nonincreasing rearrangement to fulfil a second-order Pólya-Szegő inequality. We have seen in section 3.3 that, unlike the nonincreasing rearrangement, the absence of intermediate critical value did not prevent the failure of second-order Pólya-Szegő inequalities for the Schwarz symmetrization. In this chapter, we observed that this condition on critical values is still related to higher-order isoperimetric inequalities. To conclude, the relationship between the Rayleigh Conjecture and the condition of absence of intermediate critical value is not yet well understood.

7. A Talenti-type result for signed functions

In the previous chapter, one of the main achievements was to refine Talenti's comparison principle on holed domains. In this chapter, we will present another generalisation of Talenti's comparison principle developed during the thesis together with some of its applications. The motivation is the following. All the strategies providing a complete proof of the Rayleigh Conjecture for the clamped plate rely on the approach of Talenti, which divides a domain on which the eigenfunction changes sign into two nodal domains. Talenti's comparison principle is then applied separately on each of these two nodal regions, leading to the two-ball problem (see section 3.2 for details). As already mentioned, this approach cannot yield a proof of the Rayleigh Conjecture when $d \geq 4$, the two-ball problem failing to be a sharp lower bound.

The issue in Talenti's approach is really that the comparison principle is applied separately to the positive and negative parts of the eigenfunction. A natural question is the following: what occurs if one applies Talenti's comparison principle to the whole eigenfunction? At this point, we shall recall that the comparison principle stated in Theorem 2.12 requires a positivity assumption, which prevents from applying it to the eigenfunction directly. Of course, one could resort to the traditional version of Talenti's comparison principle, but we already explained in section 3.2 that this was useless in the context of the Rayleigh Conjecture due to the lack of equimeasurability.

7.1. Comparison principle for signed functions

Motivated by the above considerations, we would like to develop a comparison principle for signed functions, still with Definition 2.1 as the underlying choice of Schwarz symmetrization. The next theorem answers this wish.

Theorem 7.1. *Let ω be a bounded open set and let $u \in H_0^2(\omega)$. Assume that the level set $\{u = t\}$ is negligible for all $t \in \mathbb{R}$ and put $f = -\Delta u$. Lastly, let $v \in H_0^1(\omega^*)$ solve*

$$\begin{cases} -\Delta v &= f^* & \text{in } \omega^*, \\ v &= 0 & \text{on } \partial\omega^*. \end{cases}$$

Then, $v \in H_0^2(\omega^*)$ is radially symmetric and nonincreasing, hence nonnegative and

for almost every $s \in [0, |\omega|)$,

$$v^\star(s) \geq \begin{cases} g(s)^2 u^\star(s) + \int_s^{|\omega|} (g^2)' u^\star, & \text{if } s > |\omega_+|, \\ u^\star(s) + \int_{|\omega_+|}^{|\omega|} (g^2)' u^\star, & \text{if } s < |\omega_+|, \end{cases} \tag{7.1}$$

where $\omega_+ = \{u > 0\}$, *and* $g(s) = g_\omega(s) := (|\omega|/s - 1)^{\frac{d-1}{d}}$.

Remark 7.2. *Note that when* $u > 0$, *we get back* $v^\star \geq u^\star$ *as in Theorem 2.12.*

Proof. As usual, we assume for simplicity that u is regular enough, for instance $u \in C^d(\omega)$. Assume also that u changes sign – otherwise the result immediately comes from Talenti's comparison principle – and fix $s \in \mathbb{R} \setminus \{0\}$. Since $u \in H_0^2(\omega)$, $\nabla u = 0$ on $\partial\omega$ and hence

$$\int_{\{u>s\}} f = \int_{\partial\{u>s\}\cap\omega} |\nabla u|.$$

In the previous, we used the fact that $\nabla u \neq 0$ on $\{u = s\}$ by Sard's Theorem (for a.e. $s \in \mathbb{R}$). On the other hand, if μ denotes the distribution function of u, since $\{u = s\} = \partial\{u > s\} \cap \omega$ for almost every s and from (2.2),

$$\mu'(s) = -\int_{\partial\{u>s\}\cap\omega} \frac{1}{|\nabla u|}.$$

As a result, using Cauchy-Schwarz and then Hardy-Littlewood inequality (Proposition 2.10),

$$\mathcal{H}^{d-1}\left(\partial\{u > s\} \cap \omega\right)^2 = \left(\int_{\partial\{u>s\}\cap\omega} \sqrt{\frac{|\nabla u|}{|\nabla u|}}\right)^2 \leq -\mu'(s) \int_0^{\mu(s)} f^\star.$$

But since $u = 0$ on $\partial\omega$, $\partial\{u > s\} \subseteq \omega$ when $s > 0$ whereas $\partial\{u > s\} \supseteq \partial\omega$ when $s < 0$, hence

$$\mathcal{H}^{d-1}\left(\partial\{u > s\} \cap \omega\right) = \begin{cases} \mathcal{H}^{d-1}(\partial\{u > s\}) & \text{if } s > 0, \\ \mathcal{H}^{d-1}\left(\partial(\omega \setminus \{u > s\})\right) & \text{if } s < 0. \end{cases}$$

Consequently, applying the isoperimetric inequality (Theorem A.2) to $A = \{u > s\}$ if $s > 0$ and to $A = \omega \setminus \{u > s\}$ if $s < 0$, we get

$$\mathbb{1}_{s>0} + g(\mu(s))^2 \mathbb{1}_{s<0} \leq -\frac{\mu'(s)}{\left(C_d \mu(s)^{\frac{d-1}{d}}\right)^2} \int_0^{\mu(s)} f^\star. \tag{7.2}$$

Integrate the previous relation between t' and t and perform the change of variable $z = \mu(s)$ (on both sides). Actually, such a change of variable might be problematic since μ is in general not onto $[0, |\omega|)$. However, here we have assumed that $\{u = t\}$

is negligible for all $t \in \mathbb{R}$. Then μ is continuous, hence onto by the intermediate value Theorem, and we conclude as desired

$$\int_{\mu(t)}^{\mu(t')} [\mathbb{1}_{u^\star(z)>0} + g(z)^2 \mathbb{1}_{u^\star(z)<0}] |u^{\star\prime}(z)| dz \leq \int_{\mu(t)}^{\mu(t')} \frac{dz}{\left(C_d z^{\frac{d-1}{d}}\right)^2} \int_0^z f^\star.$$

Now observe that thanks once again to the assumption of negligibility of the level sets of u, $\mu(u^\star(s)) = s$ for all $s \in [0, |\omega|)$. Hence apply the previous inequality with $t' = u^\star(s')$ and $t = u^\star(s)$. Letting $s' \to |\omega|$ gives

$$\int_s^{|\omega|} [\mathbb{1}_{u^\star(z)>0} + g(z)^2 \mathbb{1}_{u^\star(z)<0}] |u^{\star\prime}(z)| dz \leq \int_s^{|\omega|} dz \frac{dz}{\left(C_d z^{\frac{d-1}{d}}\right)^2} \int_0^z f^\star. \tag{7.3}$$

The right-hand side is precisely $v^\star(s)$ (see the proof of Theorem 2.12 for details), and it is nonincreasing by nonnegativity of the integrand (see (7.2)). Let us tackle the left-hand side. Since $|u^{\star\prime}| = -u^\star$, we shall integrate by parts the formula. Beforehand, since $u^\star$ is positive over $[0, |\omega_+|)$ and negative over $(|\omega_+|, |\omega|)$, split the integral in the following way: if $s > |\omega_+|$,

$$\int_s^{|\omega|} [\mathbb{1}_{u^\star(z)>0} + g(z)^2 \mathbb{1}_{u^\star(z)<0}] |u^{\star\prime}(z)| dz = -\int_s^{|\omega|} g(z)^2 u^{\star\prime}(z) dz,$$

and otherwise

$$\int_s^{|\omega|} [\mathbb{1}_{u^\star(z)>0} + g(z)^2 \mathbb{1}_{u^\star(z)<0}] |u^{\star\prime}(z)| dz = -\int_s^{|\omega_+|} u^{\star\prime}(z) dz - \int_{|\omega_+|}^{|\omega|} g(z)^2 u^{\star\prime}(z) dz.$$

This concludes for the comparison result. The fact that $v \in H_0^2(\Omega^*)$ follows from elliptic regularity and from

$$|\partial\Omega^*| \partial_n v = \int_{\partial\Omega^*} v = \int_{\Omega^*} f^* = \int_\Omega f = \int_{\partial\Omega} \partial_n u = 0,$$

since $\partial_n u = 0$. □

Before going further, let us provide a criterion guaranteeing the assumption of negligibility of the level sets appearing in Theorem 7.1.

Lemma 7.3. *Let P be a differential operator of order $m \in \mathbb{N}^*$ with measurable coefficients. Let ω be an open set and $u \in W^{m,1}_{loc}(\omega)$. If $Pu \neq 0$ a.e., then the level sets of u are negligible.*

Proof. The argument is drawn from [HR24, p.10]. Let us prove by induction that for all $i = (i_1, \dots, i_k) \in \{1, \dots, d\}^k$ with $k \in \mathbb{N}^*$, for all $u \in W^{k,1}_{loc}(\omega)$ and all $t \in \mathbb{R}$ we have $\mathbb{1}_{\{u=t\}} \partial_i u = 0$. Here we recall the notation $\partial_i = \partial_{i_1} \dots \partial_{i_k}$. When $k = 1$

and $i \in \{1, \dots, d\}$, this comes from the fact that for $v = u - t$,

$$\mathbb{1}_{\{v=0\}} \partial_i v = \mathbb{1}_{\{v=0\}} (\partial_i v_+ + \partial_i v_-) = \mathbb{1}_{\{v=0\}} (\mathbb{1}_{\{v>0\}} \partial_i v + \mathbb{1}_{\{v<0\}} \partial_i v) = 0.$$

Pretending that the desired identity holds true for $1 \leq \cdots \leq k$, take $u \in W^{k+1,1}_{loc}(\omega)$, $i \in \{1, \dots, d\}^{k+1}$ and write it in the form $i = (i_1, j)$ with $j \in \{1, \dots, d\}^k$. Then

$$\mathbb{1}_{\{u=t\}} \partial_i u = \mathbb{1}_{\{u=t\}} \mathbb{1}_{\{\partial_{i_1} u = 0\}} \partial_j \partial_{i_1} u + \mathbb{1}_{\{u=t\}} \mathbb{1}_{\{\partial_{i_1} u \neq 0\}} \partial_i u.$$

But on the one hand $\mathbb{1}_{\{\partial_{i_1} u = 0\}} \partial_j \partial_{i_1} u = 0$ due to the inductive hypothesis applied to $\partial_{i_1} u$. On the other hand, $\mathbb{1}_{\{u=t\}} \mathbb{1}_{\{\partial_{i_1} u \neq 0\}} = 0$ due to the inductive hypothesis applied to u with $k = 1$. This proves the induction and then the lemma by linearity. □

The main purpose of Theorem 7.1 is to obtain comparison results in terms of L^p norms between the functions u and v defined in Theorem 7.1. Since the infimum of g is zero, we may not obtain such comparison results from rough estimations. Nevertheless, coming back to formulation (7.3) of inequality (7.1), we can prove the following.

Corollary 7.4. *With the assumptions and notations of Theorem 7.1, for any* $1 \leq p < \infty$,

$$\int_{\omega^*} |v|^p \geq \left[\int_0^{|\omega_+|} |u^\star| + \int_{|\omega_+|}^{|\omega|} \left(\frac{(d-2)|\omega|}{zd} + 1 \right) (|\omega|/z - 1)^{\frac{d-2}{d}} |u^\star(z)| \mathrm{d}z \right]^p |\omega|^{1-p}.$$

In particular, if $d = 2$ *we obtain*

$$\int_{\omega^*} |v|^p \geq |\omega|^{1-p} \left(\int_\omega |u| \right)^p.$$

Proof. Let $G(z) := \mathbb{1}_{z<|\omega_+|} + g(z)^2 \mathbb{1}_{z>|\omega_+|}$. By inequality (7.3), we have the pointwise relation

$$\int_s^{|\omega|} G(z) |u^{\star\prime}(z)| dz \leq v^\star(s).$$

Take the L^p norm and use Jensen inequality to find

$$\begin{aligned} \int_0^{|\omega|} |v^\star|^p &\geq |\omega|^{1-p} \left[\int_0^{|\omega|} ds \int_s^{|\omega|} G(z) |u^{\star\prime}(z)| dz \right]^p \\ &= |\omega|^{1-p} \left[\int_0^{|\omega|} \int_0^{|\omega|} \mathbb{1}_{z>s} G(z) |u^{\star\prime}(z)| dz ds \right]^p \\ &= |\omega|^{1-p} \left[\int_0^{|\omega|} z G(z) |u^{\star\prime}(z)| dz \right]^p. \end{aligned}$$

Since $|u^{\star\prime}| = -u^{\star\prime}$, just integrate by parts the last line to get

$$\int_0^{|\omega|} |v^\star|^p \geq |\omega|^{1-p} \left[\int_0^{|\omega_+|} |u^\star(z)|\,dz - \int_{|\omega_+|}^{|\omega|} (g(z)^2 + z(g^2)'(z))|u^\star(z)|\,dz \right]^p.$$

Compute explicitly $(g^2)'(z)$ to obtain the result. Remark that $g(z)^2 + z(g^2)'(z) = -1$ when $d = 2$. Therefore, the last statement follows by equimeasurability of Schwarz symmetrization. □

7.2. Application to spectral inequalities

Thanks to Corollary 7.4, one shall obtain several two-dimensional spectral inequalities. For that purpose, we recall the definitions of the principal eigenvalue of the clamped p-bilaplacian and of the p-biharmonic torsional rigidity. Generalising (1.5) and (5.14), these quantities are respectively obtained through

$$\Gamma_p(\Omega) := \inf_{u \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^p}{\int_\Omega |u|^p}, \qquad T_p(\Omega) := \inf_{u \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^p}{\left|\int_\Omega u\right|^p}. \tag{7.4}$$

Let us postpone the case $p = 1$ and focus on the easier case $1 < p < \infty$ for the moment. In this situation, by reflexivity of $W_0^{2,p}(\Omega)$ and by elliptic regularity, which applies if Ω is for instance C^2, the infimum is reached in $W_0^{2,p}(\Omega)$ (see [PRT14a, section 2] for more details), hence we have

$$\Gamma_p(\Omega) = \min_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^p}{\int_\Omega |u|^p}, \qquad T_p(\Omega) = \min_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^p}{\left|\int_\Omega u\right|^p}. \tag{7.5}$$

A direct consequence of Corollary 7.4 is the following lower bound on the p-biharmonic torsional rigidity in terms of the first eigenvalue of a circular clamped "p-plate".

Theorem 7.5. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set of class C^2 and assume that $d = 2$. Then, for $1 < p < \infty$,*

$$T_p(\Omega)|\Omega|^{p-1} \geq \Gamma_p(\Omega^*). \tag{7.6}$$

Proof. Let $u \in H_0^2(\Omega) \cap W^{2,p}(\Omega)$ and assume that u has negligible level sets (in particular u is nontrivial). Set $f = -\Delta u$ and consider the competitor v defined by

$$\begin{cases} -\Delta v &= f^* & \text{in } \Omega^*, \\ v &= 0 & \text{on } \partial\Omega^*. \end{cases}$$

Thanks to Theorem 7.1, $v \in H_0^2(\Omega)$, and thanks to Corollary 7.4, since $d = 2$ $\left|\int_\Omega u\right|^p \leq |\Omega|^{p-1} \int_{\Omega^*} |v|^p$. Moreover, by equimeasurability of Schwarz symmetrization

$\int_\Omega |\Delta u|^p = \int_{\Omega^*} |\Delta v|^p$. As a result,

$$\frac{\int_\Omega |\Delta u|^p}{|\int_\Omega u|^p} \geq \frac{\int_{\Omega^*} |\Delta v|^p}{\int_{\Omega^*} |v|^p} |\Omega|^{1-p}.$$

Since $v \in H_0^2(\Omega^*)$ has Laplacian in $L^p(\Omega^*)$, actually $v \in W_0^{2,p}(\Omega^*)$ by elliptic regularity. In other words v is admissible for $\Gamma_p(\Omega^*)$ and

$$|\Omega|^{p-1} \frac{\int_\Omega |\Delta u|^p}{|\int_\Omega u|^p} \geq \Gamma_p(\Omega^*).$$

Take the infimum of the above inequality with respect to $u \in H_0^2(\Omega) \cap W_0^{2,p}(\Omega)$ having negligible level sets. By (7.5) it remains to prove that such functions are dense in $W_0^{2,p}(\Omega)$. To get this density result, consider an arbitrary function $u \in H_0^2(\Omega) \cap W_0^{2,p}(\Omega) \cap W^{4,p}(\Omega)$, set $f = \Delta^2 u \in L^p(\Omega)$, and solve

$$\begin{cases} \Delta^2 u_k &= f + \frac{1}{k} \mathbb{1}_{\{f=0\}} & \text{in } \Omega, \\ u_k &= 0 & \text{on } \partial\Omega, \\ \partial_n u_k &= 0 & \text{on } \partial\Omega. \end{cases}$$

Since Ω is bounded, the solution u_k converges to u in $W_0^{2,p}(\Omega)$. Moreover, u_k has negligible level sets by Lemma 7.3 since $f + \frac{1}{k} \mathbb{1}_{\{f=0\}} \neq 0$ almost everywhere. Lastly, $H_0^2(\Omega) \cap W_0^{2,p}(\Omega) \cap W^{4,p}(\Omega)$ is dense in $W_0^{2,p}(\Omega)$, which concludes. □

For $p = 1$, the same idea leads to an analogous result. Furthermore, since $T_1 = \Gamma_1$ over balls, inequality (7.6) provides a Faber-Krahn inequality both for the 1-biharmonic torsional rigidity and for the first eigenvalue of the clamped 1-bilaplacian. Nevertheless, in this case ancillary technicalities come into play. First we briefly recall the state-of-the-art regarding the 1-bilaplacian. Eigenvalue problems and related topics concerning this operator have been considered in [PRT14b; PRT15]. Natural boundary conditions – namely Navier – were studied in the former paper, but it appeared in the latter that clamped boundary conditions are more challenging. Indeed, for Navier boundary conditions a Faber-Krahn inequality was obtained in any dimension in [PRT14b], whereas for clamped boundary conditions, this was proved only in dimension 2 in [PRT15, section 7].

Now some technical elements. According to [PRT14a, section 2], both elliptic regularity and reflexivity fail when $p = 1$, hence the infimums in (7.4) are not reached in $W_0^{2,1}(\Omega)$. Indeed, one has to introduce the wider class $\mathrm{BL}_0(\Omega)$ of functions in $W_0^{1,1}(\Omega)$ whose distributional Laplacian is a Radon measure of finite total variation.

Definition 7.6. *Let Ω be an open set, then*

$$\mathrm{BL}_0(\Omega) := \{v \in W_0^{1,1}(\Omega) : \Delta v \in \mathcal{M}(\Omega), |\Delta v|(\Omega) < \infty\},$$

where $\mathcal{M}(\Omega)$ is the set of (signed) Radon measures over Ω and the total variation of an element $m \in \mathcal{M}(\Omega)$ is given by $|m|(\Omega) = \sup\{\int \phi \mathrm{d}m : \phi \in C_c(\Omega), |\phi| \leq 1\}$. Lastly, $\mathrm{BL}_0(\Omega)$ is endowed with the following norm making it a Banach space.

$$\|\cdot\|_{\mathrm{BL}_0(\Omega)} = \|\cdot\|_{W_0^{1,1}(\Omega)} + |\Delta \cdot|(\Omega)$$

By [PRT15, Proposition 3.1] as long as Ω is $C^{1,1}$ regular, one may define a canonical bounded map $\partial_n : u \in \mathrm{BL}_0(\Omega) \mapsto \partial_n u \in L^1(\partial\Omega)$. Furthermore, equation (8) in [PRT15] reveals that

Lemma 7.7. *If Ω is a $C^{1,1}$ bounded open set,*

$$T_1(\Omega) = \min_{v \in \mathrm{BL}_0(\Omega)\setminus\{0\}} \frac{|\Delta v|(\Omega) + \int_{\partial\Omega} |\partial_n v|}{|\int_\Omega v|}. \tag{7.7}$$

$$\Gamma_1(\Omega) = \min_{v \in \mathrm{BL}_0(\Omega)\setminus\{0\}} \frac{|\Delta v|(\Omega) + \int_{\partial\Omega} |\partial_n v|}{\int_\Omega |v|}. \tag{7.8}$$

Proof. For convenience we recall the main steps of the proof. First, by [PRT15, Proposition 3.2] for any $v \in \mathrm{BL}_0(\Omega)$, $|\Delta v|(\Omega) + \|\partial_n v\|_{L^1(\partial\Omega)}$ corresponds to the total variation $|\Delta \overline{v}|(\mathbb{R}^d)$ of the extension $\overline{v}$ of v by 0 outside Ω. Then by [PRT15, Proposition 4.3], for all $v \in \mathrm{BL}_0(\Omega)$, there exists a sequence of functions $v_n \in C_c^\infty(\Omega)$ such that $\|\Delta v_n\|_{L^1(\Omega)} = |\Delta \overline{v}_n|(\mathbb{R}^d) \to |\Delta v|(\Omega) + \int_{\partial\Omega} |\partial_n v|$, showing that $T_1(\Omega)$ (resp. $\Gamma_1(\Omega)$) coincides with the left-hand side of (7.7) (resp. (7.8)). To prove that the minimums are attained, consider a minimizing sequence $v_n \in \mathrm{BL}_0(\Omega)$ of (7.7) (resp. (7.8)). The sequence $|\Delta \overline{v}_n|(\mathbb{R}^d)$ is bounded, hence $(v_n)_n$ is bounded in $W_0^{1,r}(\Omega)$ for some r slightly greater than 1 by elliptic regularity (see [PRT14b, section 4]). As a result, there exists a subsequence converging weakly in $W_0^{1,r}(\Omega)$ and strongly in $L^1(\Omega)$ to a limit v, hence $\overline{v}_n \to \overline{v}$ in $L^1(\mathbb{R}^d)$. Now take an arbitrary $\phi \in C_c^\infty(\mathbb{R}^d)$ with $|\phi| \leq 1$, and observe that

$$\liminf_n |\Delta \overline{v}_n|(\mathbb{R}^d) \geq \liminf_n \int_{\mathbb{R}^d} \overline{v}_n \Delta\phi = \int_{\mathbb{R}^d} \overline{v} \Delta\phi.$$

Taking the supremum over ϕ, we conclude by density of $C_c^\infty(\Omega)$ in $C_c(\Omega)$ that $\liminf_n |\Delta \overline{v}_n|(\mathbb{R}^d) \geq |\Delta \overline{v}|(\mathbb{R}^d)$. As a result, v belongs to $\mathrm{BL}_0(\Omega)$ and realises the minimum in (7.7) (resp. (7.8)). □

Now that the 1-biharmonic torsional rigidity and the first eigenvalue of the clamped 1-bilaplacian have been correctly defined, we may wonder whether these quantities satisfy Faber-Krahn type inequalities. Corollary 7.3 in [PRT15] indicates that for a $C^{1,1}$ regular bounded open set Ω,

$$\Gamma_1(\Omega) \geq 2^{\frac{2-d}{d}} \Gamma_1(\Omega^*), \tag{7.9}$$

which turns into a Faber-Krahn inequality (only) for $d = 2$. Here, thanks to

Theorem 7.1 and Corollary 7.4, we provide an independent proof of this result. However, unlike (7.9), we are not able to give any nontrivial lower bound for $d > 2$.

Proposition 7.8. *Let Ω be a $C^{1,1}$ bounded open set and assume that $d = 2$. Then,*

$$T_1(\Omega) \geq \Gamma_1(\Omega) \geq \Gamma_1(\Omega^*) = T_1(\Omega^*). \tag{7.10}$$

Proof. The first inequality is immediate. The last one follows from the positivity of the first eigenfunction of the 1-bilaplacian over balls, which is the Green function of the Dirichlet Laplacian (see [PRT15, Proposition 6.2]). For the second inequality, take as before $u \in H^2_0(\Omega)$ with negligible level sets, put $f = -\Delta u$ and consider the competitor v defined by

$$\left\{ \begin{array}{rcll} -\Delta v & = & f^* & \text{in } \Omega^*, \\ v & = & 0 & \text{on } \partial\Omega^*. \end{array} \right.$$

Thanks to Corollary 7.4,

$$\frac{\int_\Omega |\Delta u|}{\int_\Omega |u|} \geq \frac{\int_{\Omega^*} |\Delta v|}{\int_\Omega |v|}.$$

Since $v \in H^2_0(\Omega^*) \subseteq \mathrm{BL}_0(\Omega^*)$, we end up with

$$\frac{\int_\Omega |\Delta u|}{\int_\Omega |u|} \geq \frac{|\Delta v|(\Omega^*) + \int_{\partial\Omega^*} |\partial_n v|}{\int_{\Omega^*} |v|} \geq \Gamma_1(\Omega^*),$$

due to Lemma 7.7. To obtain relation (7.10), thanks to the inclusion $C^\infty_c(\Omega) \subseteq H^2_0(\Omega)$ it remains to prove that $H^2_0(\Omega)$ functions with negligible level sets are dense in $H^2_0(\Omega)$. As previously, consider an arbitrary function $u \in H^2_0(\Omega) \cap H^4(\Omega)$, set $f = \Delta^2 u \in L^2(\Omega)$, and solve

$$\left\{ \begin{array}{rcll} \Delta^2 u_k & = & f + \frac{1}{k}\mathbb{1}_{\{f=0\}} & \text{in } \Omega, \\ u_k & = & 0 & \text{on } \partial\Omega, \\ \partial_n u_k & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

The solution u_k has negligible level sets by Lemma 7.3 and converges to u in $H^2_0(\Omega)$. Lastly, $H^2_0(\Omega) \cap H^4(\Omega)$ is dense in $H^2_0(\Omega)$, which concludes. □

7.3. Approximate Faber-Krahn inequalities

Inequality (7.6) previously obtained through our comparison principle for signed functions provides an approximate Faber-Krahn inequality. Indeed, by homogeneity we have for any bounded open set $\Omega \subseteq \mathbb{R}^2$,

$$T_p(\Omega) \geq \left[\frac{\Gamma_p(B_1)}{|B_1|^{p-1} T_p(B_1)} \right] T_p(\Omega^*),$$

where B_1 is the unit disk. Observe that this is useless for $p = 2$ since in this case a plain Faber-Krahn inequality has been proved in section 5.5. For $p \neq 2$ however, the previous inequality is significant, although it seems quite involved to evaluate the coefficient between brackets. Indeed, $\Gamma_p(B_1)$ and $T_p(B_1)$ cannot be computed explicitly. In this regard, recall that even the first eigenvalue of the p-Laplacian is not known explicitly over balls [BEM09].

In the current section, we would like to take the excuse of the previous inequality to investigate approximate Faber-Krahn inequalities for the first eigenvalue of the (2-)bilaplacian. We recall that inequalities of this taste have been obtained in section 5.5, but unfortunately they did not improve the state-of-the-art contained in [AL96, section 3]. Here, interestingly, the inequalities that we shall derive involve geometric quantities. The approach is based on an elementary dimensional reduction.

More precisely, if Ω is a bounded open set, for some direction $\xi \in \mathbb{S}^{d-1}$ we define the family of moving hyperplanes with parameter $t \in \mathbb{R}$ by $H_t = \{x \in \mathbb{R}^d : \langle x, \xi \rangle = t\}$. We also set $\Omega_t = \Omega \cap H_t$, which we call a section of Ω in the direction ξ. The maximal volume of the sections of Ω, formally defined by

$$\sigma_\xi(\Omega) := \max_{t \in \mathbb{R}} |\Omega_t|,$$

is called the maître-couple of Ω in the direction ξ. This denomination stems from aerodynamic engineering, and appears to have no english translation. The minimal maître-couple of Ω when varying $\xi \in \mathbb{S}^{d-1}$ is denoted $\sigma(\Omega)$. Then, we have the following result.

Proposition 7.9. *Let Ω be a bounded open set and B^{d-1} be a ball of $\mathbb{R}^{d-1}$ of unit volume. Assume that an approximate Faber-Krahn inequality holds in $\mathbb{R}^{d-1}$ for the first eigenvalue of the Dirichlet bilaplacian, meaning that there exists $c_{d-1} > 0$ such that for all bounded open sets $\omega \subseteq \mathbb{R}^{d-1}$,*

$$\Gamma(\omega)|\omega|^{\frac{4}{d-1}} \geq c_{d-1}\Gamma(B^{d-1}).$$

Then,

$$\Gamma(\Omega) \geq \frac{c_{d-1}}{\sigma(\Omega)^{\frac{4}{d-1}}}\Gamma(B^{d-1}).$$

Proof. Take $\xi \in \mathbb{S}^{d-1}$. For an admissible function $\varphi \in C_c^\infty(\Omega)$ consider $\phi(t) = \int_{\Omega_t} \varphi^2$. Since $\varphi|_{\Omega_t} \in C_c^\infty(\Omega_t)$ and $|D^2\varphi|^2 \geq |D^2(\varphi|_{\Omega_t})|^2$, we have

$$\int_{\Omega_t} |D^2\varphi|^2 \geq \int_{\Omega_t} |D^2(\varphi|_{\Omega_t})|^2 \geq \Gamma(\Omega_t) \int_{\Omega_t} \varphi^2.$$

As a result, by Fubini's Theorem,

$$\frac{\int_\Omega |\Delta\varphi|^2}{\int_\Omega \varphi^2} = \frac{\int_{\mathbb{R}^d} |D^2\varphi|^2}{\int_{\mathbb{R}^d} \varphi^2} \geq \frac{\int_{\mathbb{R}} \Gamma(\Omega_t)\phi(t)\,dt}{\int_{\mathbb{R}} \phi(t)\,dt} = \frac{\int_T \Gamma(\Omega_t)\phi(t)\,dt}{\int_T \phi(t)\,dt},$$

where $T = \{t \in \mathbb{R} : |\Omega_t| \neq 0\}$. Using the approximate Faber-Krahn inequality in $\mathbb{R}^{d-1}$ and the definition of maître-couple, we get

$$\frac{\int_\Omega |\Delta\varphi|^2}{\int_\Omega \varphi^2} \geq c_{d-1}\Gamma(B^{d-1})\frac{\int_T \phi(t)/|\Omega_t|^{\frac{4}{d-1}}\,dt}{\int_T \phi(t)\,dt} \geq \frac{c_{d-1}}{\sigma_\xi(\Omega)^{\frac{4}{d-1}}}\Gamma(B^{d-1}).$$

Take the infimum over $\varphi \in C_c^\infty(\Omega)$ and the supremum over $\xi \in \mathbb{S}^{d-1}$ to obtain the desired relation. □

As a corollary of Proposition 7.9 we obtain the next approximate Faber-Krahn inequality in $\mathbb{R}^d$.

Corollary 7.10. *With the assumptions and notations of Proposition 7.9, we have*

$$\Gamma(\Omega) \geq c_{d-1}\left[\left(\frac{|\Omega|}{\sigma(\Omega)^{\frac{d}{d-1}}}\right)^{\frac{4}{d}}\frac{\Gamma(B^{d-1})}{\Gamma(B^d)}\right]\Gamma(\Omega^*). \tag{7.11}$$

In particular, as long as Ω satisfies

$$\frac{\sigma(\Omega)}{|\Omega|^{\frac{d-1}{d}}} \leq \left(c_{d-1}\frac{\Gamma(B^{d-1})}{\Gamma(B^d)}\right)^{\frac{d-1}{4}},$$

we have the Faber-Krahn inequality

$$\Gamma(\Omega) \geq \Gamma(\Omega^*).$$

Lastly, for any Ω such that $\sigma(\Omega) \leq \sigma(\Omega^*)$,

$$\Gamma(\Omega) \geq c_{d-1}\left[\frac{k_{\nu-1/2}}{k_\nu}\right]\Gamma(\Omega^*), \tag{7.12}$$

where $\nu = d/2 - 1$ and k_ν is defined in Proposition 5.15.

Remark 7.11. *1. The quantities $|\Omega|$ and $\sigma(\Omega)^{\frac{d}{d-1}}$ are both homogeneous of degree d with respect to Ω.*

2. We conjecture that the relation $\sigma(\Omega) \leq \sigma(\Omega^*)$ always holds, but it seems not so easy to prove it. For instance, it is not clear whether, given a set Ω, one can find a direction for which σ does not increase after applying Steiner symmetrization to Ω in this direction.

Proof. Apply Proposition 7.9. The only nontrivial inequality is (7.12), but it follows from (7.11), from $\frac{\sigma(\Omega)}{|\Omega|^{\frac{d-1}{d}}} \leq \frac{\sigma(\Omega^*)}{|\Omega^*|^{\frac{d-1}{d}}}$, and from Proposition 5.15. □

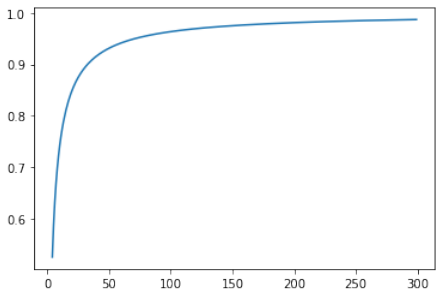
We remark that in dimension $d = 4$ for instance, $\left(\frac{\Gamma(B^{d-1})}{\Gamma(B^d)}\right)^{\frac{d-1}{4}} \geq 0.7813$. But since the Faber-Krahn inequality is known in dimension 3, we have $c_{d-1} = 1$. This means that for all Ω thin enough for its minimal maître-couple to satisfy

$$\frac{\sigma(\Omega)}{|\Omega|^{3/4}} \leq 0.7813,$$

the Faber-Krahn inequality holds for Ω. By way of comparison, we compute $\frac{\sigma(\Omega^*)}{|\Omega^*|^{\frac{d-1}{d}}} \simeq 1.265$ for $d = 4$. On the other hand, the ratio between brackets appearing in (7.12) may be studied both analytically and numerically. Indeed, by [Lor94, Corollary 1], we know that $k_{\nu-1/2}/k_\nu$ ranges in $(0, 1)$ for $\nu \geq 0$. Moreover, as explained in paragraph 5.5.1 we have $k_\nu \sim \nu$ as $\nu \to \infty$. Consequently, $k_{\nu-1/2}/k_\nu$ goes to 1 as $d \to \infty$. On the side of numerics, it seems that the ratio increases with d (see Figure 7.1b), and we present some of its values in Table 7.1a.

d	$k_{\nu-1/2}/k_\nu$
4	0.5259
5	0.5870
6	0.6330
7	0.6690
8	0.6982
9	0.7224

(a)



(b)

Figure 7.1. – Values of the ratio $k_{\nu-1/2}/k_\nu$ where $\nu = d/2 - 1$ with respect to d.

To conclude, even if inequality (7.12) provides a universal nontrivial lower bound on Γ, it may actually not be very useful. Indeed, the best existing lower bound on Γ, given by [AL96, section 3], seems (numerically) to increase with d for $d \geq 8$. Since $k_{\nu-1/2}/k_\nu < 1$, this would mean that (7.12) does not improve the state-of-the-art.

To conclude, the primary interest of the approximate Faber-Krahn inequality derived in this paragraph is not in the universal lower bound that it provides, but in the fact that it involves geometric quantities allowing to assert the Faber-Krahn inequality for asymmetric enough domains. Furthermore the thresholds on the asymmetry is made explicit.

8. An integral formula for the first eigenvalue of polyharmonic operators

The purpose of the present chapter is to provide a new formulation for the first eigenvalue of the bilaplacian and other polyharmonic operators in the fashion of equation (2.26). The interest of equation (2.26) is that it depends on Ω only through domains of integration. An analogous formulation for the first eigenvalue of the bilaplacian would open new perspectives with respect to the Rayleigh Conjecture for the clamped plate. As mentioned in paragraph 3.1 we will not be able to deduce from such a formulation a proof of the conjecture, but we will make some interesting observations going in this direction. Before proceeding further, let us stress that the results obtained in this chapter hold for any Dirichlet polylaplacian of order $m \in \mathbb{N}^*$. In particular, the notation $\Gamma(\Omega)$ refers to the first eigenvalue of the problem

$$\left\{ \begin{array}{rcll} (-\Delta)^m u & = & \Gamma u & \text{in } \Omega, \\ \partial_n^i u & = & 0 & \text{on } \partial\Omega \text{ for all } 0 \leq i < m, \end{array} \right.$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded open set (see paragraph 4.1 for more details concerning $\Gamma(\Omega)$). Without further ado, let us recall that it is problematic to adapt the proof of paragraph 2.6.6 in view of generalising (2.26) to the present setting. This is due to the fact that, unlike the Laplacian, no stochastic process can be easily associated with the Dirichlet m-polylaplacian for $m > 1$. Indeed, to do so one would define the transition probabilities of the process to be the fundamental solution of the polyharmonic heat equation, which is the only function $p : (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d \mapsto p_t(x) \in \mathbb{R}$ in $C^1(\mathbb{R}_+^*, \mathcal{S}(\mathbb{R}^d))$ solving

$$\partial_t p + (-\Delta)^m p = 0, \quad \text{in } \mathbb{R}_+^* \times \mathbb{R}^d,$$

and such that $p_t \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ as $t \to 0$. It turns out that p_t may be expressed explicitly for all $t > 0$ as the following inverse Fourier transform

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^{2m}} e^{i\langle \xi, x \rangle} \, d\xi, \quad \forall x \in \mathbb{R}^d. \tag{8.1}$$

Unfortunately, since (8.1) changes sign in general by [Hoc78], it does not define a probability distribution. In more concise terms, the Dirichlet m-polylaplacian is not

the infinitesimal generator of a Feller process since it does not preserve positivity. Nevertheless, some work have been done in order to define pseudo-processes with the purpose of a probabilistic interpretation of the polyharmonic heat equation. In this respect, let us refer to the following nonexhaustive list of contributions [Kry60; Hoc78; Fun79; GS88; Nis97; Sat02; DeB04; OD11; BM15; BDM19]. Unfortunately, we did not find in the previous references a convenient way to express the solution of the polyharmonic heat equation in a domain in terms of a manageable stochastic process which would have led to a relation similar to (2.26).

Therefore, we resorted to an approach based solely on operator theory and originating from the paper [ALT91]. Besides providing an integral formulation for $\Gamma(\Omega)$ regardless of $m \in \mathbb{N}^*$, this new proof sheds light on the already known case $m = 1$, in particular concerning the regularity required on the domain Ω under consideration. Indeed, we recall that one cannot expect formula (2.26) to hold for a general bounded open set Ω since its right hand side depends on Ω in a measure-theoretic way whereas the left-hand side depends on Ω in a capacitary way.

The alternative strategy for proving (2.26) relies on Trotter's formula. Roughly speaking, this formula allows to interpret the procedure of solving the heat equation over a given domain as iterations of the procedure of solving it in the whole space during a small time interval and then restricting the solution to the domain (see Lemma 8.6). Such an interpretation of the solution to the heat equation is the main step towards (2.26). Indeed, solving the heat equation in the whole space amounts to convolving with p_t. This approach is perfectly generalisable to Dirichlet polylaplacians, and leads to the following result.

Theorem 8.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz open set. Let $m \in \mathbb{N}^*$ and p_s be defined as in (8.1) for $s > 0$. Then, the first eigenvalue of the m-polylaplacian satisfies*

$$\Gamma(\Omega) = -\lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{t}\log\int_\Omega\dots\int_\Omega p_{\frac{t}{n}}(x_1-x_2)\dots p_{\frac{t}{n}}(x_{n-1}-x_n)dx_1\dots dx_n. \tag{8.2}$$

Remark 8.2. *One may refine the regularity assumption required on Ω and only ask it to be stable with respect to the m-capacity (see section 4.1 for the notion of capacity) in the sense that*

$$u \in H_0^m(\Omega) \quad \Leftrightarrow \quad u = 0 \text{ a.e. in } \Omega^c.$$

The proof of Theorem 8.1 is achieved in section 8.1. As a consequence of the new formula (8.2), we see that in purpose of the Faber-Krahn inequality $\Gamma(\Omega) \geq \Gamma(\Omega^*)$ for a bounded open set Ω, it would be enough to prove that for all $s > 0$, and all

$n \geq 2$,

$$\begin{aligned} &\int_\Omega ... \int_\Omega p_s(x_1 - x_2)...p_s(x_{n-1} - x_n)dx_1...dx_n \\ &\leq \int_{\Omega^*} ... \int_{\Omega^*} p_s(x_1 - x_2)...p_s(x_{n-1} - x_n)dx_1...dx_n. \end{aligned} \tag{8.3}$$

In the case where p_s is positive, radially symmetric and decreasing, such a relation comes from the Brascamp-Lieb-Luttinger inequality (2.15), as explained in paragraph 2.6.6. Unfortunately, apart from the case $m = 1$, p_s – although radially symmetric – changes sign and oscillates at infinity for any $s > 0$ by [Hoc78, equation (2.5)']. Moreover, even if we could apply (2.15) in this setting, we would not end up with (8.3) since p_s is not radially decreasing hence $p_s^* \neq p_s$.

Therefore a different strategy needs to be designed. To explore possible proofs of (8.3), we restricted the study to particular situations. First we focused on the case $n = 1$, which is undefined in (8.3), but may be replaced with the Hardy-Littlewood type inequality

$$\int_\Omega p_s \leq \int_{\Omega^*} p_s.$$

As we shall see in section 8.2, it turns out that such an inequality cannot hold in full generality in the context of sign changing kernels. Therefore we considered only convex sets as admissible domains, which led us to the problem

$$\sup \left\{ \int_\Omega q,\ \Omega \subseteq \mathbb{R}^d \text{ bounded open convex set, } |\Omega| = c \right\}, \tag{8.4}$$

where $q \in L^1(\mathbb{R}^d)$ is radially symmetric and $c \in \mathbb{R}_+^*$ is a parameter. We emphasise that we consider here a general situation where the kernel q need not be the fundamental solution of a polylaplacian. Yet we will demand some ancillary assumptions. Indeed, we require that $\int_B q > 0$ for an admissible shape B to guarantee the existence of an optimal shape and that $q \in W^{1,1}(\mathbb{R}^d)$ to perform shape derivatives. Given these assumptions, we were able to partially solve problem (8.4) and to prove the following theorem.

Theorem 8.3. *Let $q \in W^{1,1}(\mathbb{R}^d)$ be such that $\int_B q > 0$ for the ball B of volume c centred at 0. Problem (8.4) admits an optimal shape Ω. Furthermore, if Ω is strictly convex, then $\Omega = B$.*

Remark 8.4.
1. *By strictly convex we mean that all boundary points are extreme points.*
2. *In dimension 2, one may replace the assumption of strict convexity by a C^2 regularity assumption. This is because in this case the curvature is a scalar which is either positive or vanishes at each boundary point.*

We stress that the convexity constraint in problem (8.4) might perhaps be relaxed but not removed in order for Theorem 8.3 to hold. We shall explain this fact in section 8.2, in which the proof of Theorem 8.3 is performed. On the contrary, the

assumption of strict convexity – or regularity in dimension 2 – made on the optimal shape seems technical but delicate to overcome.

In a second stage, we focused on the case $n = 2$ in (8.3), which corresponds to a Riesz-type inequality. In view of the previous discussion, it is reasonable to restrict once again to the class of convex shapes and to consider

$$\sup\left\{\int_\Omega\int_\Omega q(x-y)dxdy,\ \Omega\subseteq\mathbb{R}^d \text{ bounded open convex set, } |\Omega|=c\right\}, \tag{8.5}$$

with $q \in L^1(\mathbb{R}^d)$ and $c \in \mathbb{R}_+^*$ as parameters. In this setting, we were not able anymore to deduce that a regular optimal shape is a ball from the optimality conditions provided by shape derivatives. However, recall that in the context of the Rayleigh Conjecture, (8.3) is only needed for small time intervals (see Theorem 8.1), for which the fundamental solution of the polylaplacian converges to a Dirac delta function. Therefore, one may address (8.5) for a family of kernels $(q_s)_{s>0}$ being an approximate identity. With suitable assumptions on the family $(q_s)_{s>0}$, one may attempt to prove that the ball solves (8.5) *asymptotically*.

To give a precise meaning to this fact, observe that if an optimal shape Ω^s exists for (8.5) with kernel q_s, this shape depends a priori on the parameter s. Then, we tried to show that if Ω^s converges to some regular shape as $s \to 0$, the limit shape must be a ball. For that purpose, we resorted once again to shape derivatives which provided the optimality condition

$$\int_{\Omega^s} q_s(x-y)dx = \textit{constant}, \text{ for all } y \in \partial\Omega^s, \tag{8.6}$$

where the constant depends on s. Then, we tried to combine a moving plane method (see [Ser71]) with a blow-up analysis in order to show that the limit shape is a ball. The work is still in progress. Unfortunately, it is likely that many restrictive assumptions will be needed to achieve this result.

Still with respect to addressing (8.5), I would also like to mention that a simpler strategy could be considered. Indeed, as illustrated in the recent [CMS24], the idea would be to prove that the relaxed functional $\rho \mapsto \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(x)q(x-y)\rho(y)\,dx\,dy$ enjoys some convexity properties for suitable assumptions on the kernel q. This would allow to prove the existence of a unique critical shape for problem (8.5). Then, it would only remain to show that the ball fulfils the associated optimality conditions to conclude.

As a final word let us emphasise that we still believe that the ball solves (8.5) for sufficiently nice kernels q, including p_s for all $s > 0$ and $m \in \mathbb{N}^*$. More generally, we even venture to state the following conjecture.

Conjecture. *For all $m \in \mathbb{N}^*$, $s \in \mathbb{R}_+^*$, $n \geq 2$ and all convex bounded open sets Ω, inequality (8.3) holds.*

8.1. Proof of the integral formula

In order to prove Theorem 8.1, we first need to make the relationship between the first eigenvalue of the polylaplacian and the polyharmonic heat equation evident. As in paragraph 2.6.6, the procedure purely relies on operator theory and leads to the following result.

Proposition 8.5. *Let Ω be a bounded open set and $v_0 \in L^2(\Omega)$ be some initial data. Let $m \in \mathbb{N}^*$ and $v \in C_t(\mathbb{R}_+, L^2_x(\Omega)) \cap C^1_t(\mathbb{R}^*_+, H^m_0(\Omega))$ solve the polyharmonic heat equation*

$$\begin{array}{rcll} \partial_t v + (-\Delta)^m v & = & 0 & \text{in } \mathbb{R}^*_+ \times \Omega, \\ \partial_n^i v & = & 0 & \text{on } \mathbb{R}^*_+ \times \partial\Omega \text{ for all } 0 \leq i < m, \\ v(0) & = & v_0 & \text{in } \Omega. \end{array} \tag{8.7}$$

Then,

$$v(t) \underset{t\to\infty}{=} e^{-t\Gamma(\Omega)} E_\Omega v_0 + o(e^{-t\Gamma(\Omega)}), \tag{8.8}$$

where E_Ω is the orthogonal projection onto the eigenspace associated with $\Gamma(\Omega)$, and where the convergence is meant in $L^2(\Omega)$. As a consequence,

$$\Gamma(\Omega) = -\lim_{t\to\infty} \frac{1}{t} \log \left| \int_\Omega v(t) \right|. \tag{8.9}$$

Proof. We may express the solution v thanks to the semi-group generated by the Dirichlet polylaplacian $(-\Delta)^m_\Omega$ in the following way:

$$v(t) = e^{-t(-\Delta)^m_\Omega} v_0.$$

Projecting over the eigenspaces of $(-\Delta)^m_\Omega$, we get for all $t > 0$,

$$v(t) = \sum_{n\in\mathbb{N}^*} e^{-t\Gamma_n} E_n v_0,$$

where Γ_n is the nth eigenvalue of $(-\Delta)^m_\Omega$ and E_n is the projection on the associated eigenspace (here the eigenvalues are counted **without** repetition, in the sense that $\Gamma_n < \Gamma_{n+1}$ for all $n \in \mathbb{N}^*$). Since $\Gamma(\Omega) = \Gamma_1$ and $E_\Omega = E_1$, we see that it remains to prove that

$$\sum_{n\geq 2} e^{-t\Gamma_n} E_n v_0 \underset{t\to\infty}{=} o(e^{-t\Gamma_1}),$$

which comes from $\Gamma_1 < \Gamma_n$ for all $n \geq 2$, and gives (8.8). Integrating over Ω, taking the log, and passing to the limit, we find (8.9). □

Formula (8.9) looks interesting since it gives a new way to express the eigenvalue $\Gamma(\Omega)$, hence we shall work on this new formulation. However, (8.9) depends on the solution v, which, as the first eigenfunction, depends implicitly on the domain Ω.

Hopefully, it is possible to write v in such a way that the dependence over Ω becomes explicit. Unlike paragraph 2.6.6, this is performed thanks to Trotter's formula (see [KM78]).

Lemma 8.6. *Let Ω be a bounded Lipschitz open set and $t > 0$. Then, for any $u \in L^2(\Omega)$,*

$$\left[\mathbb{1}_\Omega e^{-\frac{t}{n}(-\Delta)^m_{\mathbb{R}^d}}\right]^n \overline{u} \xrightarrow[n\to\infty]{} e^{-t(-\Delta)^m_\Omega} u, \tag{8.10}$$

where $(-\Delta)^m_\omega$ denotes the m-polylaplacian with Dirichlet boundary conditions over the (possibly unbounded) domain ω and $\overline{u}$ is the extension of u by 0 over $\mathbb{R}^d$.

Remark 8.7. *It is possible to relax the regularity assumption demanded on Ω. Indeed, as we shall see in the proof, it is enough for Ω to be stable with regard to the m-capacity (recall paragraph 4.1). In this respect we refer to [GLM23].*

Proof. In the Hilbert space $L^2(\mathbb{R}^d)$, let A be the operator $(-\Delta)^m_{\mathbb{R}^d}$ with associated quadratic form q_A (whose domain is $D_A = H^m(\mathbb{R}^d)$), and B the operator of multiplication by the potential

$$V(x) := \begin{cases} 0 & \text{if } x \in \Omega, \\ +\infty & \text{if } x \in \Omega^c, \end{cases}$$

with associated quadratic form q_B of domain $D_B = \{u \in L^2(\mathbb{R}^d) : u = 0 \text{ a.e. in } \Omega^c\}$. Then, Trotter's formula [KM78, equation (1.1)], gives the strong convergence

$$\left[e^{-\frac{t}{n}B} e^{-\frac{t}{n}(-\Delta)^m_{\mathbb{R}^d}}\right]^n \xrightarrow[n\to\infty]{} e^{-tT} P,$$

where T is the operator associated with the form $q := q_A + q_B$ of domain $D = D_A \cap D_B$, and P is the orthogonal projection onto D. Observe that

$$D = \{u \in H^m(\mathbb{R}^d), u = 0 \text{ a.e. in } \Omega^c\}$$

coincides with $H^m_0(\Omega)$ due to the regularity of Ω (see [GLM23]). Observe also that $q_B = 0$ on D_B so that $q(u) = q_A(u) = \int_\Omega |D^m u|^2$, for all u in D. As a consequence, T is nothing else but $(-\Delta)^m_\Omega$.

On the other hand, it is clear that B coincides with the infinitesimal generator of the constant semi-group $(\mathbb{1}_\Omega)_{t\geq 0}$. In other words $e^{-\frac{t}{n}B} = \mathbb{1}_\Omega$ for all $n \in \mathbb{N}^*$, hence we proved (8.10) for all $u \in D$. But since D is dense in $L^2(\Omega)$, we obtain that (8.10) actually holds for all $u \in L^2(\Omega)$ by boundedness of $e^{-t(-\Delta)^m_\Omega}$. □

The interesting consequence of (8.10) is that the solution of (8.7) may be computed at any time thanks to the semi-group of $(-\Delta)^m_{\mathbb{R}^d}$ on the whole space (which is explicit and does not depend on Ω), and thanks to the projection on $L^2(\Omega)$. More precisely, let us denote $(p_s)_{s>0}$ the fundamental solution of the m-polyharmonic

heat equation given in (8.1). Then, for all $u \in L^2(\mathbb{R}^d)$ and all $s > 0$,

$$e^{-s(-\Delta)^m_{\mathbb{R}^d}} u = p_s * u,$$

hence we get the following result.

Corollary 8.8. *Let Ω be a bounded open set and $v_0 \in L^2(\Omega)$ be some initial data. Let $m \in \mathbb{N}^*$ and $v \in L^\infty_t(\mathbb{R}_+, L^2_x(\Omega))$ solve (8.7). Then, for all $t > 0$ and $x \in \Omega$,*

$$v(t,x) = \lim_{n\to\infty} \int_\Omega ... \int_\Omega p_{\frac{t}{n}}(x-x_1)p_{\frac{t}{n}}(x_1-x_2)...p_{\frac{t}{n}}(x_{n-1}-x_n)v_0(x_n)dx_1...dx_n. \quad (8.11)$$

The previous expression of the solution to the polyharmonic heat equation (8.7) combined with Proposition 8.5 immediately yields a proof of Theorem 8.1.

Proof of Theorem 8.1. Choose $v_0 = \mathbb{1}_\Omega$. Then, combine (8.9) with (8.11), and use the continuity of log. □

8.2. Hardy-Littlewood inequality

According to Theorem 8.1, a major step towards the Rayleigh Conjecture would be to prove inequality (8.3). We already explained why the Brascamp-Lieb-Luttinger inequality (2.15) was useless in this context. Furthermore, we mentioned but did not explain in detail why it was fanciful to expect proving (8.3) in full generality. For that purpose, let us focus on the toy case $n = 1$, which corresponds to

$$\int_\Omega p_s \leq \int_{\Omega^*} p_s. \quad (8.12)$$

Here, we observe that, since p_s changes sign and is positive near the origin by [Hoc78], we shall consider B to be the biggest ball of centre 0 included in $\{p_s > 0\}$, and A some annulus disjoint from B of small volume such that $A \subseteq \{p_s > 0\}$ and $(B \cup A)^* \subseteq B \cup \{p_s < 0\}$. Then, setting $\Omega = B \cup A$, by construction

$$\int_{\Omega^*} p_s - \int_\Omega p_s = \int_{\Omega^* \setminus B} p_s - \int_A p_s < 0.$$

Moreover, by adding a sufficiently thin tube linking the two connected components of Ω, we shall build a connected counterexample to (8.12). We could even ask Ω to have connected boundary by digging a thin tunnel in the annulus A.

The previous discussion shows that strong assumptions on the domain Ω are needed to expect inequality (8.12) to hold. In this respect, the class of convex domains appears to be of interest. On the other hand, the strategy that we plan to borrow allows to treat quite general radially symmetric kernels. This is why we consider problem (8.4). For the purposes of solving (8.4), we will use the framework

of shape derivatives in order to identify the optimal shape, but we first need to prove that an optimal shape does exist.

Lemma 8.9. *If $q \in L^1(\mathbb{R}^d)$ is a radially symmetric kernel such that $\int_B q > 0$ where B is the ball of volume c centred at 0, problem (8.4) admits a solution.*

Proof. Since $q \in L^1(\mathbb{R}^d)$, (8.4) is not infinite. Furthermore, since $\int_B q > 0$ the supremum in problem (8.4) is a strictly positive number M. Let $(\Omega_n)_n$ be a maximizing sequence. We prove that $(\Omega_n)_n$ has to stay confined within some design region. First, let us justify that Ω_n has a uniformly bounded diameter. For that, take a big region D such that

$$\int_{D^c} |q| \leq \epsilon$$

If the diameter of Ω_n were not bounded with respect to n, up to a subsequence we would have $\Omega_n \subseteq R_n$ where R_n is a sequence of hyperrectangles with one side shrinking to zero. As a consequence, we would have $D \cap R_n \to \emptyset$, hence $D \cap \Omega_n \to \emptyset$ a.e., from which we would get

$$\limsup_n \int_{\Omega_n} q \leq \limsup_n \left[\int_{\Omega_n \cap D} |q| + \int_{\Omega_n \cap D^c} |q| \right] \leq \epsilon.$$

Taking $\epsilon < M$, this would contradict the maximality of $(\Omega_n)_n$.

As a result, each Ω_n is contained in a ball $B(x_n, R)$, where R does not depend on n. If the sequence of centres (x_n) were not bounded, once again we could extract a subsequence such that $B(x_n, R) \subseteq D^c$, and hence

$$\int_{\Omega_n} q \leq \int_{D^c} |q| \leq \epsilon,$$

a contradiction as previously. We conclude that $(\Omega_n)_n$ has to remain in some design region D. Then, by convexity the perimeter of Ω_n is uniformly bounded, and thanks to [HP05, Corollary 2.2.24 and Theorem 2.3.10] there exists a domain Ω such that, up to a subsequence $(\Omega_n)_n$ converges toward Ω with respect to Hausdorff distance and characteristic functions. Moreover, Ω is convex due to [HP05, equation (2.19)]. The convergence of characteristic functions shows that not only $|\Omega| = c$, but also

$$\int_\Omega q = \lim_n \int_{\Omega_n} q = M,$$

by dominated convergence, which concludes. □

Contrary to spectral shape functionals where much regularity is needed in general to perform shape derivatives (recall section 2.3), here mild regularity assumptions are enough since the dependence of our functional on the shapes only expresses through the domain of integration of some integral. Moreover, this regularity comes for free from the convexity constraint. Unfortunately, due to this convexity constraint, some more regularity is yet needed to deal with the optimality conditions.

Indeed, we have the next result which, combined with Lemma 8.9, completes the proof of Theorem 8.3.

Theorem 8.10. *Let Ω solve problem (8.4) with $q \in W^{1,1}(\mathbb{R}^d)$. If Ω is strictly convex (or C^2 regular when $d = 2$), then it is the ball of volume c centred at the origin.*

Proof. As announced, we use shape derivatives. For an arbitrary $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ define $\Omega_\theta := (I + \theta)\Omega$ and set

$$I(\theta) := \int_{\Omega_\theta} q, \qquad \mathcal{V}(\theta) := |\Omega_\theta|$$

Then, since $q \in W^{1,1}(\mathbb{R}^d)$, and since Ω is Lipschitz [HP05, Theorem 5.2.2] applies and tells that I and $\mathcal{V}$ are Gâteau-differentiable at 0 and that for any $V \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$,

$$\partial_V I(0) = \int_{\partial\Omega} qV \cdot \vec{n}, \qquad \partial_V \mathcal{V}(0) = \int_{\partial\Omega} V \cdot \vec{n}.$$

Actually, [HP05, Lemma 5.3.3] provides that I and $\mathcal{V}$ are C^1 in a neighbourhood of 0. Then, if there were no convexity constraint, there would exist a Lagrange multiplier λ such that

$$\int_{\partial\Omega} qV \cdot \vec{n} = \lambda \int_{\partial\Omega} V \cdot \vec{n},$$

for all $V \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Here, due to the convexity constraint, this relation only holds for V such that $\Omega_{\epsilon V}$ is convex for small enough ϵ. This is the case if $V|_{\partial\Omega}$ is supported in a strictly convex portion γ of $\partial\Omega$. If Ω is strictly convex, take $\gamma = \partial\Omega$. If $d = 2$ and Ω is C^2, take γ to be the portion where the curvature does not vanish. Then, we have

$$\int_\gamma [q - \lambda] V \cdot \vec{n} = 0,$$

for any V, or in other words

$$q = \lambda \text{ on } \gamma. \tag{8.13}$$

Since q is radially symmetric, the previous means that γ is a union of portions of spheres centred at 0. In the case $\gamma = \partial\Omega$, we immediately conclude that Ω is a ball centred at the origin. In the case of Ω regular in dimension 2, the previous shows that the curvature is piecewise constant. The curvature being continuous and $\partial\Omega$ being connected, the curvature is finally constant, hence $\partial\Omega$ is a sphere – or a hyperplane, but this is impossible. Moreover, this sphere is still centred at 0. □

Conclusion

During the thesis having led to the present manuscript, the Rayleigh Conjecture for clamped plates have been studied under several points of view. In this occasion, spectral optimization problems involving not only the bilaplacian but also more general higher order operators have been addressed. For that purpose, several advanced tools have been introduced. By comparing the behaviour of the bilaplacian to the behaviour of the Laplacian with respect to these tools, we highlighted how challenging is the Rayleigh Conjecture.

Despite the hindrances relative to the conjecture, we implemented a direct method of the Calculus of Variations to prove that (quasi-open) shapes minimizing the first eigenvalue of polylaplacians do exist. An improvement of regularity was performed to show that these shapes are actually open in low dimension.

Then we tackled the conjecture more head-on. We developed an order reduction principle which opened new perspectives. It allowed not only to prove that the first eigenvalue of the bilaplacian over an optimal shape is simple, but also to provide several sufficient conditions for the conjecture to hold. In the course of these developments, we were led to study the biharmonic torsional rigidity, for which we proved that the ball is optimal under volume constraint.

Another avenue of inquiry into the Rayleigh Conjecture emerged from realising that the nodal domains of an optimal eigenfunction of the bilaplacian exhibit a distinctive geometry. A refinement of Talenti's comparison principle for domains with such a geometry led to a proof of Rayleigh's Conjecture, except that an additional assumption has been required. During this work, the thorough analysis of the asymmetric two-ball problem was one of the main ordeals.

As we understood, Talenti's comparison principle is an indispensable tool with respect to Rayleigh's Conjecture, that is why it forms the core of this manuscript. Indeed, after the previous sufficient conditions provided for the conjecture to hold, we devised a Talenti type comparison principle for signed functions. The outcome was of particular significance in two dimensions. In this setting, it led to an interesting lower bound on the p-biharmonic torsional rigidity. Furthermore, the bound appeared to be sharp for $p = 1$.

As a last contribution, we produced a new formula for the principal eigenvalue of any polylaplacian on an arbitrary (yet slightly regular) domain. This integral formula might be the starting point of new directions of research. To illustrate the possible new perspectives, we addressed Hardy-Littlewood type inequalities with signed kernels.

To sum up, the results presented in this manuscript are in favour of the validity of the Rayleigh Conjecture in dimension $d \geq 4$. More precisely, even if no conclusive proof have been provided, several interesting partial results supporting the conjecture were obtained. Furthermore, no rebuttal evidence has been found. For all these reasons, I personally believe that the Rayleigh Conjecture remains true in dimension $d \geq 4$.

To go beyond what have been achieved in this thesis with respect to the conjecture, different strategies must be deployed. Some ideas have already been proposed in the course of the manuscript, such as proving that the mean value of an optimal eigenfunction is minimal in the sense of Theorem 5.1, or working on the formulation given in Theorem 8.1. But new approaches could also be imagined.

For instance, the strategy developed in [Moh75] could be of interest. In that paper, Mohr proved that in dimension 2, if there exists a smooth optimal shape, this shape must be a ball. The work of Mohr was outshined by Talenti's and subsequent publications. However, its ideas could perhaps be generalised in greater dimension. In this respect see [SW15]. Therefore, it would be relevant to study such possible extensions of Mohr's result. More than a generalisation, one of the challenges would be to make rigorous a proof which is performed only formally in the original paper.

A different inquiry would be to study further extensions of Talenti's comparison principle. For instance, the assumption on the critical values in Theorem 6.1 was crucial to prove the Talenti type comparison principle on holed domains (Theorem 6.9). It would be of major interest to see whether one may still obtain a refined version of the comparison principle without this assumption. In this direction the recent paper [Ama+23] could be a nice starting point.

As we understood, Talenti's comparison principle is a fundamental tool. We already studied various versions of this tool, yet it is likely to be generalised in several other directions. One of these could be to develop a fourth order comparison principle. Besides its independent interest, this could be an alternative approach toward the Rayleigh Conjecture. Indeed, recall that Talenti's comparison principle provided a proof of the Faber-Krahn inequality for the Laplacian (see paragraph 2.6.2). Thus an analogous result for the bilaplacian could advance the knowledge concerning the Rayleigh Conjecture. As far as we are aware, no result of this taste has been produced for the moment. Nevertheless, we believe that there are many ways to address this questions. For instance, one could first study the one-dimensional setting. Alternatively one could consider the case of particular data, such as constant functions or Dirac masses. In this respect we should mention that Theorem 5.18 shall be interpreted as a fourth order comparison principle for constant data. Indeed, it tells that for a bounded open set Ω, if

$$\begin{cases} \Delta^2 u &= 1 \quad in \quad \Omega, \\ u &= 0 \quad on \quad \partial\Omega, \\ \partial_n u &= 0 \quad on \quad \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta^2 v &= 1 \quad in \quad \Omega^*, \\ v &= 0 \quad on \quad \partial\Omega^*, \\ \partial_n v &= 0 \quad on \quad \partial\Omega^*, \end{cases}$$

then

$$\int_\Omega u \leq \int_{\Omega^*} v.$$

Still with the Rayleigh Conjecture in mind, one could explore another idea which is natural but delicate to implement. Indeed, as previously illustrated, to obtain isoperimetric inequalities it is a common strategy to symmetrize either a function itself or its Laplacian. In particular, less has been done with respect to symmetrizing the gradient. See however [GN84; ALT89] and recently [Ama+24]. It seems to me that this direction may be investigated further and might possibly yield new results regarding the Rayleigh Conjecture. Our feeling is that there are two issues with respect to symmetrizing the gradient. To illustrate this, consider a ball B. The first problem is that the gradient of a radially symmetric function in $H_0^2(B)$ vanishes not only at the boundary, but also at the centre of B. Therefore, applying for instance Schwarz symmetrization to the norm of the gradient cannot produce good results, since it would not preserve the gradient of the first eigenfunction. Hence one has to think of a different type of symmetrization. The second issue is that since (for radial functions) $\Delta = r^{1-d}\partial_r r^{d-1}\partial_r$, symmetrizing ∂_r and using the Pólya-Szegő inequality might not be the best idea. Indeed, if one were able to build from a radial function $u \in H_0^2(B)$ a radially symmetric competitor v such that $\partial_r v = (\partial_r u)^*$, then $|\nabla \partial_r v| = |\partial_r^2 v| \neq |\Delta v|$. As a result the Pólya-Szegő inequality would not lead to an upper bound on $\int_B (\Delta v)^2$. Instead, it would be better to symmetrize $r^{d-1}\partial_r u$ and to exhibit a radially symmetric function v such that $r^{d-1}\partial_r v = (r^{d-1}\partial_r u)^*$. Then, a Pólya-Szegő type inequality could yield

$$\int_B (\Delta v)^2 = \int_B r^{2-2d} |\nabla (r^{d-1}\partial_r u)^*|^2 \leq \int_B r^{2-2d} |\nabla (r^{d-1}\partial_r u)|^2 = \int_B (\Delta u)^2.$$

In the next years, another perspective could be to study operators which are different although related to the bilaplacian. As an example, a mathematical model for plates under tension or compression is obtained by perturbing the bilaplacian with a Laplacian. In other terms considering the operator $\Delta^2 - \alpha\Delta$ for $\alpha \in \mathbb{R}$ under Dirichlet boundary conditions. Although the Laplacian has good properties, the features of this operator are almost unknown (see [ABF19]). A first undertaking could be to prove that for a domain Ω, the first eigenfunction of the operator $\Delta^2 - \alpha\Delta$ is positive over Ω for all α greater than a certain threshold depending on Ω. Similar issues have been studied in several recent works [CT22; ES22; Eic24]. In dimension 2, it was shown in the first paper that $\Delta^2 - \alpha\Delta$ enjoys the maximum principle for all large enough α. In greater dimension, the question has been addressed in the second paper. There, it was proved the weaker result according to which for each $f \geq 0$, there exists a threshold beyond which for all α, the following relation holds when $u \in H_0^2(\Omega)$.

$$(\Delta^2 - \alpha\Delta)u = f \quad \Rightarrow \quad u \geq 0$$

We highlight that the threshold depends not only on Ω but also on the data f. Consequently, in dimension $d \geq 3$, it is not known whether the first eigenfunction of $\Delta^2 - \alpha\Delta$ is positive for large α, and it would be of interest to explore this question. However, unlike the bilaplacian, such a result would not directly lead to a Faber-Krahn inequality for $\Delta^2 - \alpha\Delta$ since Talenti's strategy does not apply when $\alpha > 0$. See [ABM16; ABM22] for the case $\alpha \leq 0$. Moreover, the other standard approach based on shape derivatives and overdetermined problems seems also delicate in view of [Dal90]. In other words, we shall say that the problem of optimizing the first eigenvalue of the operator $\Delta^2 - \alpha\Delta$ under volume constraint for $\alpha > 0$ is completely open. This could be a direction of research.

Besides those linear operators, nonlinear operators such as the p-bilaplacian have recently been investigated in view of Faber-Krahn type inequalities. Indeed, [PRT14a] addressed the case $p = 1$ and [KP17] the case $p = \infty$, but nothing is known for intermediate p – except $p = 2$. Yet it seems that the proof of [Nad95] for $p = 2$ in dimension 2 might be extended to other values of p. Let us elaborate. Talenti's comparison principle yields that if Ω is a bounded open set

$$\Gamma_p(\Omega) \geq \min_{v,w} \frac{\int_{B_a} |\Delta v|^p + \int_{B_b} |\Delta w|^p}{\int_{B_a} |v|^p + \int_{B_b} |w|^p} =: I_{a,b},$$

where the minimum is taken among positive radial functions $v \in W^{2,p}(B_a)$ and $w \in W^{2,p}(B_b)$ such that for all $x \in \partial B_a$ and $y \in B_b$,

$$v(x) = w(y) = 0, \quad a^{d-1}\partial_n v(x) = b^{d-1}\partial_n w(y).$$

Here a (resp. b) is as usual the radius of Ω_+^* (resp. Ω_-^*) where Ω_+ and Ω_- are the nodal domains of the eigenfunction of the p-bilaplacian over Ω. Assuming without loss of generality that $|\Omega| = |B_1|$, to obtain a Faber-Krahn inequality for Γ_p it is thus enough to study the quantity $I_a := I_{a,b}$ for $0 \leq a \leq b \leq 1$ subject to $a^d + b^d = 1$, and to show that $I_a \geq I_0$. Following [Nad95], we observe that if (v, w) realises the minimum in I_a for some $0 < a < b < 2^{-1/d}$, then $I_a < I_{a+\epsilon}$ for all small $\epsilon > 0$ as soon as

$$\frac{1}{|B_a|}\int_{B_a} v^{p-1} < \frac{1}{|B_b|}\int_{B_b} w^{p-1}. \tag{*}$$

Since v satisfies $\Delta_{2,p} v = I_a |v|^{p-2} v$ in B_a and w satisfies $\Delta_{2,p} w = I_a |w|^{p-2} w$ in B_b, one is led to study the solution u_α to the problem

$$\begin{cases} \Delta_{2,p} u &= \alpha |u|^{p-2} u & in & B_1, \\ u &= 0 & on & \partial B_1, \\ \partial_n u &= -1 & on & \partial B_1, \end{cases}$$

parametrized by $\alpha \in (0, \infty)$. Indeed, condition (*) will hold as long as u_α is unique and increases with respect to α (or at least that its mean increases with α). This last result would be almost enough to obtain a Faber-Krahn inequality for the first

eigenvalue of the p-bilaplacian in dimension 2. Therefore, it would be interesting to work more in depth on this problem, which appears to be more complex than it seems at first glance.

This manuscript has also raised questions which are not directly related to shape optimization and are of independent interest. For instance, we saw (Remark 4.6) that unlike the Laplacian, it is not known whether the first eigenfunction of the bilaplacian is bounded in irregular open sets. In this respect, a possible direction of research could be to study the question in elementary domains such as angular sectors, which usually provide counterexamples. Any result on the boundedness of the eigenfunction in such domains would be of interest in our opinion.

Another example of general issue is the following. We recall that the study of domains over which the bilaplacian enjoys a positivity preserving property has been an active topic of research, but many questions remain open. For instance, negative results are mainly available, and apart from very elementary domains, it is not known in general if an open set has a chance to preserve positivity. This is the case for stadium-like domains which are conjectured to enjoy this property [Swe01]. We believe that a strategy for addressing such questions could be to resort to domain decomposition. It seems that the theory of domain decomposition is not very developed for fourth order operators, and one could contribute in this direction with possible applications to a better understanding of domains which are likely to preserve positivity.

Still concerning this positivity preserving property, a project on which it would be interesting to work regards the following inverse problem. Given an ellipse, can one determine its eccentricity by means of positivity preservation and of mechanical stress? More precisely, the purpose would be to study the operator $\Delta^2 - \alpha\Delta$ over ellipses, and to show that the first value of eccentricity for which positivity preservation fails depends continuously and monotonously on α. Such a study might then be applied in mechanics in the determination of eccentricity for elastic plates.

Lastly, looking further ahead, one could investigate possible connections between isoperimetric inequalities and homogenization. Indeed, in shape optimization open sets are almost always thought as macroscopic domains in which vibrations occur. We think that considering such domains as inclusions in a periodic medium could perhaps be an interesting change of paradigm. As first directions, one could try to prove the Faber-Krahn inequality or Talenti's comparison principle by this means. In case of success, this could be an interesting way to address some of the problems aforementioned.

To conclude this conclusion, this manuscript, and this thesis, I would like to say that I really enjoyed working on this subject. Of course, this was not always simple, and I faced several issues of various nature. In my opinion, one of the greatest hardships was the emotional lift that I have been enduring when thinking to have found a new idea which rarely appeared to be as brilliant as expected. The

emotions curve is a very irregular function that should look like a Dirac comb. But I think that this is not a peculiarity of my experience, and fortunately I gradually learnt to tame these passions. Moreover, this probably illustrates the extent to which I was concerned by the thesis.

Despite (and perhaps thanks to) its challenges, I was very pleased to dedicate myself to this thesis. I sincerely think that the subject perfectly matched my mathematical preferences. Indeed, it is at the crossroads of analysis, optimization, geometry, and even probabilities. In this respect, I would like to thank once again Enea Parini and François Hamel for having introduced me to this domain. In my opinion, mixing analysis and geometry gives rise to a cocktail of refined sophistication which I really savoured. Probabilities added some spice that I was not comfortable with at the beginning, but ultimately created more depth to the mix.

If I had to judge the work carried out during this thesis, I would say that it is made of astute utilisation of existing tools which led to interesting breakthroughs. Unfortunately this was not enough to solve the (notoriously hard) Rayleigh Conjecture. Actually, my main regret is to have been unable to find a new tool tailored to the problem. Yet, the analyses that I proposed were fine enough to make some advances with respect to the conjecture. I presume that this is how Science mostly progresses. Moreover, as explained above I still have a number of ideas, and I hope to continue thinking of clamped plates in the next years.

APPENDICES

A. Coarea and isoperimetry

In this appendix, we recall some well-known properties from Geometric Measure Theory and refer to [Fed69] for further information. First we state the general coarea formula.

Theorem A.1 ([Fed69], Theorem 3.2.11)**.** *Let $\omega \subseteq \mathbb{R}^d$ be an open set and $f : \omega \to \mathbb{R}$ be Lipschitz. Then, for any measurable set $A \subseteq \mathbb{R}^d$,*

$$\int_A |\nabla f| = \int_{\mathbb{R}} \mathrm{d}t \int_{\{f=t\}} \mathbb{1}_A \mathrm{d}\mathcal{H}^{d-1},$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

We will also need to state an isoperimetric inequality. There are various ways to express and to prove it according to the various possible definitions of perimeter. One of the simplest proofs is based on the Brunn-Minkowski inequality and leads to the relation ([Fed69, Theorem 3.2.43])

$$\frac{M^{d-1}(\partial A)}{|\overline{A}|^{\frac{d-1}{d}}} \geq \frac{M^{d-1}(\partial A^*)}{|A^*|^{\frac{d-1}{d}}},$$

for any measurable set A such that $|\overline{A}| < \infty$. Here M^{d-1} is the $(d-1)$-dimensional lower Minkowski content, defined as

$$M^{d-1}(\partial A) = \liminf_{r \to 0} \frac{|\partial A + B_r|}{2r}.$$

But since the Hausdorff measure came into play in Theorem A.1, it will be useful to have an isoperimetric inequality for it. Fortunately, as shown in [Fed69, Theorem 3.2.39], if ∂A is $(d-1)$-rectifiable, then $M^{d-1}(\partial A) = \mathcal{H}^{d-1}(\partial A)$. The rectifiability means that ∂A shall be written as a union of finitely many graphs of Lipschitz functions over bounded sets of $\mathbb{R}^{d-1}$. This is fulfilled when A is a Lipschitz bounded open set. Moreover, when $\mathcal{H}^{d-1}(\partial A) < \infty$ we have $|\partial A| = 0$, hence the following result.

Theorem A.2. *Let A be a Lipschitz bounded open set of $\mathbb{R}^d$. Then,*

$$\frac{\mathcal{H}^{d-1}(\partial A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\mathcal{H}^{d-1}(\partial A^*)}{|A^*|^{\frac{d-1}{d}}} =: C_d,$$

where C_d depends only on the dimension d.

B. Concentration-compactness

Proof of Lemma 4.13. We follow the proof of Lemma III.1 of [Lio84] (see also the explanations in section 2.4) and define

$$Q_n(R) := \sup_{y\in\mathbb{R}^d} \|\tau_y u_n\|^2_{L^2(B_R)}$$

which induces a sequence of nonnegative nondecreasing uniformly bounded functions on $\mathbb{R}_+$ satisfying, for each $n \in \mathbb{N}$, $Q_n(R) \xrightarrow[R\to\infty]{} \lambda_n$, with $\lambda_n \to \lambda$. Then, by Helly's selection Theorem, up to extracting a subsequence, Q_n pointwise converges to some function Q on $\mathbb{R}_+$. Then Q, like Q_n, enjoys the properties of being nonnegative nondecreasing and bounded by λ. Now set

$$\alpha := \lim_{R\to\infty} \uparrow Q(R) \in [0, \lambda].$$

If $\alpha = 0$, we are in the vanishing case, because for any R, $0 \le Q(R) \le \alpha = 0$, hence $Q(R) = 0$ which is the result. If $\alpha = \lambda$, we are in case of compactness. Indeed, let $\epsilon > 0$ and choose r such that $\lambda - \epsilon < Q(r)$. Because $Q_n(r) \to Q(r)$, for all n large enough, $\lambda - \epsilon < Q_n(r)$. Then, there exists $z_n \in \mathbb{R}^d$ such that $\lambda - \epsilon < \|\tau_{z_n} u_n\|^2_{L^2(B_r)}$. Note that the radius $r = r(\epsilon)$ and the points $z_n = z_n(\epsilon)$ depend on ϵ. We need to show that the points z_n actually do not.

But as long as $\epsilon < \lambda/3$, the balls $B(z_n(\epsilon), r(\epsilon))$ and $B(z_n(\lambda/3), r(\lambda/3))$ intersect each other for all n large enough, otherwise we would have, up to a subsequence,

$$\|u_n\|^2_{L^2} \ge \|\tau_{z_n(\epsilon)} u_n\|^2_{L^2(B_{r(\epsilon)})} + \|\tau_{z_n(\lambda/3)} u_n\|^2_{L^2(B_{r(\lambda/3)})} > \lambda - \epsilon + \lambda - \lambda/3 > 4\lambda/3,$$

which is impossible since $\|u_n\|^2_{L^2} \to \lambda$. As a consequence, the distance between $z_n(\epsilon)$ and $z_n(\lambda/3)$ does not exceed $r(\epsilon) + r(\lambda/3)$. Then, setting $R(\epsilon) := r(\lambda/3) + 2r(\epsilon)$, we get that $B(z_n(\epsilon), r(\epsilon))$ is included in $B(z_n(\lambda/3), R(\epsilon))$. Therefore, the sequence $y_n = z_n(\lambda/3)$ satisfies that for any $0 < \epsilon < \lambda/3$, there exists $R := R(\epsilon)$ such that for all n large enough,

$$\|\tau_{y_n} u_n\|^2_{L^2(B_R)} \ge \|\tau_{z_n(\epsilon)} u_n\|^2_{L^2(B_{r(\epsilon)})} > \lambda - \epsilon.$$

If now $\alpha \in (0, \lambda)$, let $\epsilon_n \to 0$ and $R_n \to \infty$ such that $\forall n \in \mathbb{N}$, $\alpha \ge Q(R_n) > \alpha - \epsilon_n$. Because $Q_n \to Q$ once again, we may extract a subsequence $(u_n)_n$ and define a sequence $(y_n)_n$ such that, $\forall n \in \mathbb{N}$, $\alpha - \epsilon_n < \|\tau_{y_n} u_n\|^2_{L^2(B_{R_n})}$ and $Q_n(kR_n) < \alpha + \epsilon_n$ $(k = 2, 6)$.

At this point, we use a smooth cut-off function over B_1 and with support in B_2, say χ^1, and set $\chi^2 := 1 - \chi^1$. Moreover, we define the m-quasi-continuous functions

$$u_n^1 = u_n \tau_{-y_n} \delta_{R_n} \chi^1, \qquad u_n^2 = u_n \tau_{-y_n} \delta_{3R_n} \chi^2,$$

where $\delta_r v$ is the dilation of a function v by r, i.e. $\delta_r v(x) := v(x/r)$. Note that $\delta_r \chi^1$ and $\delta_{3r}\chi^2$, as well as their derivatives, are respectively supported in B_{2r} and in $\mathbb{R}^d \setminus B_{3r}$. As a consequence, we have

$$dist\left(\omega_{n,k}^1, \omega_{n,k}^2\right) \geq R_n \to \infty.$$

Moreover, for $i = 1, 2$, the inclusion $\omega_{n,k}^i \subseteq \omega_{n,k}$ follows from the definition of u_n^1 and u_n^2. Now let us check that

$$\begin{aligned}
\left\|D^m u_n^1\right\|_{L^2}^2 - \left\|(\tau_{-y_n}\delta_{R_n}\chi^1) D^m u_n\right\|_{L^2}^2 &\underset{n\to\infty}{=} O(1/R_n), \\
\left\|D^m u_n^2\right\|_{L^2}^2 - \left\|(\tau_{-y_n}\delta_{3R_n}\chi^2) D^m u_n\right\|_{L^2}^2 &\underset{n\to\infty}{=} O(1/R_n).
\end{aligned}$$

For that purpose, take $\gamma \in \mathbb{N}^d$ a multi-index of length m, and recall that

$$\partial^\gamma (fg) = \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} \partial^\beta f \partial^{\gamma-\beta} g,$$

from which we deduce

$$\begin{aligned}
\left\|\partial^\gamma u_n^1\right\|_{L^2}^2 = \left\|(\tau_{-y_n}\chi_{R_n}^1)\partial^\gamma u_n\right\|_{L^2}^2 &+ \sum_{\substack{\beta \leq \gamma \\ \beta \neq 0}} \binom{\gamma}{\beta}^2 \frac{1}{R_n^{2|\beta|}} \left\|(\partial^{\gamma-\beta} u_n)(\tau_{-y_n}\delta_{R_n}\partial^\beta \chi^1)\right\|_{L^2}^2 \\
&+ \sum_{\substack{\beta,\beta' \leq \gamma \\ \beta \neq \beta'}} \binom{\gamma}{\beta}\binom{\gamma}{\beta'} \frac{1}{R_n^{|\beta|+|\beta'|}} \left\langle (\partial^{\gamma-\beta} u_n)(\tau_{-y_n}\delta_{R_n}\partial^\beta \chi^1) \, , \, (\partial^{\gamma-\beta'} u_n)(\tau_{-y_n}\delta_{R_n}\partial^{\beta'} \chi^1)\right\rangle_{L^2}.
\end{aligned}$$

Since $(u_n)_n$ is bounded in $H^m(\mathbb{R}^d)$, and since $\chi^1 \in W^{m,\infty}(\mathbb{R}^d)$, the two sums above are $O(1/R_n)$, and we obtain as expected

$$\left\|\partial^\gamma u_n^1\right\|_{L^2}^2 \underset{n\to\infty}{=} \left\|(\tau_{-y_n}\chi_{R_n}^1)\partial^\gamma u_n\right\|_{L^2}^2 + O(1/R_n).$$

An analogous discussion leads to an analogous result regarding $\|\partial^\gamma u_n^2\|_{L^2}^2$. Now, using also the inequality $|\delta_r \chi^1|^2 + |\delta_{3r}\chi^2|^2 \leq 1$, the previous justifies that for

all $n \in \mathbb{N}$,

$$\begin{aligned}\|D^m u_n\|_{L^2}^2 - \|D^m u_n^1\|_{L^2}^2 - \|D^m u_n^2\|_{L^2}^2 \\ &\geq \|D^m u_n\|_{L^2}^2 - \|(\tau_{-y_n}\delta_{R_n}\chi^1)D^m u_n\|_{L^2}^2 - \|(\tau_{-y_n}\delta_{3R_n}\chi^2)D^m u_n\|_{L^2}^2 + O(1/R_n) \\ &\geq \|D^m u_n\|_{L^2}^2 - \int_{\mathbb{R}^d} \tau_{-y_n}\left(\left|\delta_{R_n}\chi^1\right|^2 + \left|\delta_{3R_n}\chi^2\right|^2\right)|D^m u_n|^2 + O(1/R_n) \\ &\geq O(1/R_n).\end{aligned}$$

This means that $\liminf_n \left[\|D^m u_n\|_{L^2}^2 - \|D^m u_n^1\|_{L^2}^2 - \|D^m u_n^2\|_{L^2}^2\right] \geq 0$. Moreover,

$$\|u_n^1\|_{L^2}^2 = \|\delta_{R_n}\chi^1\tau_{y_n}u_n\|_{L^2}^2 \geq \|\tau_{y_n}u_n\|_{L^2(B_{R_n})}^2 > \alpha - \epsilon_n \to \alpha.$$

On the other hand, since $\delta_{R_n}\chi^1$ is supported in B_{2R_n},

$$\|u_n^1\|_{L^2}^2 = \|\delta_{R_n}\chi^1\tau_{y_n}u_n\|_{L^2}^2 \leq \|\tau_{y_n}u_n\|_{L^2(B_{2R_n})}^2 \leq Q_n(2R_n) < \alpha + \epsilon_n \to \alpha.$$

Hence $\|u_n^1\|_{L^2}^2 \to \alpha$. In the same spirit, regarding u_n^2,

$$\|u_n^2\|_{L^2}^2 = \|\delta_{3R_n}\chi^2\tau_{y_n}u_n\|_{L^2}^2 \leq \|u_n\|_{L^2}^2 - \|\tau_{y_n}u_n\|_{L^2(B_{3R_n})}^2 \leq \|u_n\|_{L^2}^2 - \alpha + \epsilon_n \to \lambda - \alpha,$$

and

$$\|u_n^2\|_{L^2}^2 \geq \|u_n\|_{L^2}^2 - Q_n(6R_n) > \|u_n\|_{L^2}^2 - \alpha - \epsilon_n \to \lambda - \alpha,$$

showing that $\|u_n^2\|_{L^2}^2 \to \lambda - \alpha$. To conclude,

$$\|u_n - (u_n^1 + u_n^2)\|_{L^2}^2 = \int_{\mathbb{R}^d}\left(\delta_{3R_n}\chi^1 - \delta_{R_n}\chi^1\right)^2|\tau_{y_n}u_n|^2 \leq Q_n(6R_n) - \alpha + \epsilon_n \to 0.$$

□

C. The first eigenfunction of a ball

Proof of Proposition 5.15. For readability, we omit the subscript B in u_B. According to [AB95], the first eigenfunction in B is radially symmetric and of the form $\forall r \in [0, R)$,

$$u(r) = (aJ_\nu(\gamma r) + bI_\nu(\gamma r))\, r^{-\nu},$$

where $\gamma := \Gamma(B)^{\frac{1}{4}}$. Then, using the identities $J'_\nu(x) = \frac{\nu J_\nu(x)}{x} - J_{\nu+1}(x)$ and $I'_\nu(x) = \frac{\nu I_\nu(x)}{x} + I_{\nu+1}(x)$ we find

$$\partial_r u(r) = (-aJ_{\nu+1}(\gamma r) + bI_{\nu+1}(\gamma r))\, \gamma r^{-\nu}.$$

Now, for u to fulfil the condition $u(R) = \partial_r u(R) = 0$ although being non trivial, one observes that the matrix

$$M = \begin{pmatrix} J_\nu(\gamma R) & I_\nu(\gamma R) \\ -J_{\nu+1}(\gamma R) & I_{\nu+1}(\gamma R) \end{pmatrix}$$

needs having a non trivial kernel. In other words, its determinant needs to vanish, hence

$$f_\nu(\gamma R) = 0.$$

Conversely, as soon as γ satisfies this equation, u will be solution of an eigenvalue problem in B with Dirichlet boundary conditions. Consequently, γ is necessarily the lowest positive solution of this equation, meaning that $\gamma = k_\nu/R$. Hence $\Gamma(B) = k_\nu^4/R^4$.

Furthermore, I_ν and $I_{\nu+1}$ being positive, $M \neq 0$ and it has a one-dimensional kernel. By virtue of the identity $u(R) = 0$, the kernel is generated by the vector $(I_\nu(k_\nu), -J_\nu(k_\nu))$ or equivalently $R^\nu(J_\nu(k_\nu)^{-1}, -I_\nu(k_\nu)^{-1})$. In other words, there exists a real number β such that

$$\begin{pmatrix} a \\ b \end{pmatrix} = \beta R^\nu \begin{pmatrix} J_\nu(k_\nu)^{-1} \\ -I_\nu(k_\nu)^{-1} \end{pmatrix}.$$

Finding the values of a and b is thus equivalent to determining β. For that purpose,

we use the normalization of u, i.e.

$$1 = \int_B u^2 = \beta^2 R^{2\nu} |\mathbb{S}^{d-1}| \left[J_\nu(k_\nu)^{-2} \int_0^R J_\nu(\gamma r)^2 r^{d-2\nu-1} \right. \\ + I_\nu(k_\nu)^{-2} \int_0^R I_\nu(\gamma r)^2 r^{d-2\nu-1} \\ \left. - 2 J_\nu(k_\nu)^{-1} I_\nu(k_\nu)^{-1} \int_0^R I_\nu(\gamma r) J_\nu(\gamma r) r^{d-2\nu-1} \right]. \tag{C.1}$$

As $d - 2\nu - 1 = 1$, it turns out that we need to compute the integral of product of Bessel functions against r. That's why we use [GR15, section 6.521, formula 1], that is, for all $\alpha \neq \beta \in \mathbb{C}$ and $\nu > -1$,

$$\int_0^1 x J_\nu(\alpha x) J_\nu(\beta x) = \frac{\beta J_{\nu-1}(\beta) J_\nu(\alpha) - \alpha J_{\nu-1}(\alpha) J_\nu(\beta)}{\alpha^2 - \beta^2} = \frac{\alpha J_{\nu+1}(\alpha) J_\nu(\beta) - \beta J_{\nu+1}(\beta) J_\nu(\alpha)}{\alpha^2 - \beta^2}. \tag{C.2}$$

We apply this formula with $\alpha = ik_\nu$ and $\beta = k_\nu$, and find

$$\int_0^R I_\nu(k_\nu r) J_\nu(k_\nu r) r^{d-2\nu-1} = \frac{R^2}{2k_\nu} [I_{\nu+1}(k_\nu) J_\nu(k_\nu) + J_{\nu+1}(k_\nu) I_\nu(k_\nu)] = 0.$$

For the other integrals, we first remark that when ($\alpha, \beta \in \mathbb{R}$ and) $\alpha \to \beta$ in (C.2), one obtains

$$\int_0^1 x J_\nu(\beta x)^2 = \frac{J_\nu(\beta)^2}{2\beta} \frac{d}{d\beta} \left[\frac{\beta J_{\nu+1}(\beta)}{J_\nu(\beta)} \right] = \frac{1}{2} \left[J_{\nu+1}(\beta)^2 + J_\nu(\beta)^2 - \frac{\nu}{\beta} J_{\nu+1}(\beta) J_\nu(\beta) \right]. \tag{C.3}$$

Hence, with $\beta = k_\nu$, we find

$$\int_0^R J_\nu(k_\nu r)^2 r^{d-2\nu-1} = \frac{R^2}{2} \left[J_{\nu+1}(k_\nu)^2 + J_\nu(k_\nu)^2 - \frac{\nu}{k_\nu} J_{\nu+1}(k_\nu) J_\nu(k_\nu) \right].$$

But because both extremal members in (C.3) depend holomorphicly on β, this formula remains true even when $\beta \in \mathbb{C}$ thanks to the isolation of zeros, hence, we can apply it to $\beta = ik_\nu$:

$$\int_0^R I_\nu(k_\nu r)^2 r^{d-2\nu-1} = \frac{R^2}{2} \left[-I_{\nu+1}(k_\nu)^2 + I_\nu(k_\nu)^2 - \frac{\nu}{k_\nu} I_{\nu+1}(k_\nu) I_\nu(k_\nu) \right].$$

Finally, since $f_\nu(k_\nu) = 0$, the term between the brackets in (C.1) becomes

$$\begin{array}{rll} & \frac{R^2}{2} & \left[J_\nu(k_\nu)^{-2} \left(J_{\nu+1}(k_\nu)^2 + J_\nu(k_\nu)^2 - \frac{\nu}{k_\nu} J_{\nu+1}(k_\nu) J_\nu(k_\nu) \right) \right. \\ & & \left. + I_\nu(k_\nu)^{-2} \left(-I_{\nu+1}(k_\nu)^2 + I_\nu(k_\nu)^2 - \frac{\nu}{k_\nu} I_{\nu+1}(k_\nu) I_\nu(k_\nu) \right) \right] \\ = & \frac{R^2}{2} & \left[2 + \left(\frac{J_{\nu+1}}{J_\nu}(k_\nu) - \frac{I_{\nu+1}}{I_\nu}(k_\nu) - \frac{\nu}{k_\nu} \right) \left(\frac{J_{\nu+1}}{J_\nu}(k_\nu) + \frac{I_{\nu+1}}{I_\nu}(k_\nu) \right) \right] \\ = & R^2 & . \end{array}$$

Using $|\mathbb{S}^{d-1}|R^d = d|B|$, we have that $\beta^{-2} = |\mathbb{S}^{d-1}|R^{2\nu+2} = d|B|$, hence

$$a = \frac{R^\nu}{J_\nu(k_\nu)\sqrt{d|B|}}, \qquad b = -\frac{R^\nu}{I_\nu(k_\nu)\sqrt{d|B|}}. \tag{C.4}$$

In particular,

$$u(r) = \frac{1}{\sqrt{d|B|}}\left(\frac{J_\nu(k_\nu r/R)}{J_\nu(k_\nu)} - \frac{I_\nu(k_\nu r/R)}{I_\nu(k_\nu)}\right)\left(\frac{r}{R}\right)^{-\nu}.$$

which corresponds to (5.9). After having obtained the expression of u, we would like to compute its integral. Observing that

$$\int_B u = \frac{1}{\Gamma(B)}\int_B \Delta^2 u = \frac{1}{\Gamma(B)}\int_{\partial B}\partial_n \Delta u = \frac{R^{d-1}|\mathbb{S}^{d-1}|}{k_\nu^4/R^4}\partial_r \Delta u(R),$$

it remains only to compute

$$\partial_r \Delta u(r) = [aJ_{\nu+1}(\gamma r) + bI_{\nu+1}(\gamma r)]\gamma^3 r^{-\nu},$$

for which we used the identities $J'_{\nu+1}(x) = J_\nu(x) - \frac{\nu+1}{x}J_{\nu+1}(x)$ and $I'_{\nu+1}(x) = I_\nu(x) - \frac{\nu+1}{x}I_{\nu+1}(x)$. As a result, we get

$$\begin{aligned}\int_B u =& \frac{R^{d-1}|\mathbb{S}^{d-1}|}{k_\nu^4/R^4}\frac{k_\nu^3/R^3}{\sqrt{d|B|}}\left[\frac{J_{\nu+1}}{J_\nu}(k_\nu) - \frac{I_{\nu+1}}{I_\nu}(k_\nu)\right] \\ =& \frac{\sqrt{d|B|}}{k_\nu}\left[\frac{J_{\nu+1}}{J_\nu}(k_\nu) - \frac{I_{\nu+1}}{I_\nu}(k_\nu)\right].\end{aligned}$$

Note that the last equality in (5.11) comes from the fact that $f_\nu(k_\nu) = 0$. □

D. The torsional two-ball problem

Proof of Proposition 5.22. For readability, let us write T instead of $T(a,b)$. The first variation of the quotient involved in its definition shows that T is achieved for an admissible couple v, w if and only if $\Delta^2 v = T \fint_{B_a} v$ in B_a and $\Delta^2 w = T \fint_{B_b} w$ in B_b. Since v and w are radially symmetric, these equations turn into ordinary differential equations of the form $r^{1-d}\partial_r r^{d-1}\partial_r r^{1-d}\partial_r r^{d-1}\partial_r u = \textit{constant}$ the solution of which are fourth order polynomials:

$$v(r) = \sum_{i=0}^{4} A_i r^i, \qquad w(r) = \sum_{i=0}^{4} B_i r^i,$$

where $A_i, B_i \in \mathbb{R}$ for $0 \le i \le 4$. Since v and w are radially symmetric, their odd order derivatives must vanish at zero, that is $A_1 = A_3 = B_1 = B_3 = 0$. It remains to identify the coefficients $A_0, A_2, A_4, B_0, B_2, B_4$. For this, we will write six equations involving those coefficients. First, we compute

$$\int_{B_a} v = A_0|B_a| + A_2\frac{d}{d+2}a^2|B_a| + A_4\frac{d}{d+4}a^4|B_a|,$$

and

$$\partial_r v(r) = 2A_2 r + 4A_4 r^3, \quad \Delta v(r) = 2dA_2 + 4(d+2)A_4 r^2, \quad \Delta^2 v(r) = 8d(d+2)A_4. \tag{D.1}$$

Thus, the equation $\Delta^2 v = T(a,b) \fint_{B_a} v$ reads

$$A_0 + \frac{d}{d+2}a^2 A_2 + \left[\frac{d}{d+4}a^4 - \frac{8d(d+2)}{T|B_a|}\right] A_4 = 0 \tag{E1}$$

Analogously, we have

$$B_0 + \frac{d}{d+2}b^2 B_2 + \left[\frac{d}{d+4}b^4 - \frac{8d(d+2)}{T|B_b|}\right] B_4 = 0 \tag{E2}$$

Now the condition $v(a) = w(b) = 0$ leads to two new equations

$$A_0 + a^2 A_2 + a^4 A_4 = 0, \tag{E3}$$

and

$$B_0 + b^2 B_2 + b^4 B_4 = 0. \tag{E4}$$

Using (D.1) and its counterpart for w, we take advantage of the condition $a^{d-1}\partial_r v(a) = b^{d-1}\partial_r w(b)$ to derive the equation

$$a^d A_2 + 2a^{d+2} A_4 = b^d B_2 + 2b^{d+2} B_4 \tag{E5}$$

Eventually, we convert the last condition $\Delta v(a) + \Delta w(b)$ into the equation

$$dA_2 + 2(d+2)a^2 A_4 + dB_2 + 2(d+2)b^2 B_4 = 0 \tag{E6}$$

The linear system built from (E1), (E2), (E3), (E4), (E5), (E6) has only one solution $(A_0, A_2, A_4, B_0, B_2, B_4)$ which is trivial unless the associated matrix

$$\begin{bmatrix} 1 & \frac{d}{d+2}a^2 & \frac{d}{d+4}a^4 - \frac{8d(d+2)}{T|B_a|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{d}{d+2}b^2 & \frac{d}{d+4}b^4 - \frac{8d(d+2)}{T|B_b|} \\ 1 & a^2 & a^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & b^2 & b^4 \\ 0 & a^d & 2a^{d+2} & 0 & -b^d & -2b^{d+2} \\ 0 & d & 2(d+2)a^2 & 0 & d & 2(d+2)b^2 \end{bmatrix}$$

is not invertible. Hence, one is lead to compute the determinant D of this matrix, which is

$$\begin{aligned} D = &- a^2b^2\left(\frac{2}{d+2}\right)^2 D_1 - \left(\frac{4}{d+4}a^4 + \frac{M_a}{T}\right)\left(\frac{4}{d+4}b^4 + \frac{M_b}{T}\right) D_2 \\ &+ \frac{2}{d+2}a^2\left(\frac{4}{d+4}b^4 + \frac{M_b}{T}\right) D_3 + \frac{2}{d+2}b^2\left(\frac{4}{d+4}a^4 + \frac{M_a}{T}\right) D_4, \end{aligned}$$

where $M_r := \frac{8d(d+2)}{|B_r|}$, and

$$D_1 = \begin{vmatrix} 2a^{d+2} & -2b^{d+2} \\ 2(d+2)a^2 & 2(d+2)b^2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a^d & -b^d \\ d & d \end{vmatrix},$$
$$D_3 = \begin{vmatrix} 2a^{d+2} & -b^d \\ 2(d+2)a^2 & d \end{vmatrix}, \qquad D_4 = \begin{vmatrix} a^d & -2b^{d+2} \\ d & 2(d+2)b^2 \end{vmatrix}.$$

Computing D_1, D_2, D_3, D_4 and using that $a^d + b^d = 1$, we get at the end of the day,

$$\begin{aligned} D = &- \frac{2 \cdot 4^3}{(d+2)(d+4)^2}a^4b^4 + \frac{8}{d+2}\left(b^d + \frac{d}{d+4}\right)\frac{a^4 M_b}{T} + \frac{8}{d+2}\left(a^d + \frac{d}{d+4}\right)\frac{b^4 M_a}{T} \\ &- \frac{dM_aM_b}{T^2}. \end{aligned}$$

We conclude that the initial system has a non trivial solution if and only if

$$\begin{aligned} 0 =& P_{a,b}(T) \\ :=& \frac{16T^2}{(d+2)(d+4)^2} - \left[\frac{M_a}{a^4}\left(a^d + \frac{d}{d+4}\right) + \frac{M_b}{b^4}\left(b^d + \frac{d}{d+4}\right)\right]\frac{T}{d+2} + \frac{d}{8}\frac{M_a M_b}{a^4 b^4}. \end{aligned} \tag{D.2}$$

In particular $T(a,b)$ is one of the at most two roots of $P_{a,b}$. Even if this will not serve the rest of the proof, we notice that when $a \to 1$, we have the convergence

$$P_{a,b}(T)\frac{a^4b^4}{M_a M_b} \to -\frac{d}{(d+2)(d+4)M_1}T + d/8,$$

uniformly with respect to T on compact sets. The limit function is linear with a unique root at

$$T = \frac{d(d+2)^2(d+4)}{|B_1|},$$

which corresponds to $T(B_1)$ by (5.18). Consequently if $T(a,b)$ remains bounded it must converge toward $T(B_1)$ as $a \to 1$. Of course the same holds for b by symmetry.

Now in order to prove that $T(B_1) \leq T(a,b)$, for $a, b \in (0,1)$, one has to check that $T(B_1)$ is lower than the least root of $P_{a,b}$. For that, it is enough to check that $P_{a,b}(T(B_1)) \geq 0$ and that $P'_{a,b}(T(B_1)) \leq 0$. Therefore, we compute

$$P'_{a,b}(T(B_1)) = \frac{M_1}{d+4}\left[4 - \frac{1}{d+2}\left(\frac{d}{a^{d+4}} + \frac{d}{b^{d+4}} + \frac{d+4}{a^4} + \frac{d+4}{b^4}\right)\right].$$

Hence, $P'_{a,b}(T(B_1)) \leq 0$ if and only if $f(a,b) \geq 0$, where

$$f(a,b) = \frac{d}{a^{d+4}} + \frac{d}{b^{d+4}} + \frac{d+4}{a^4} + \frac{d+4}{b^4} - 4(d+2).$$

Deriving f with respect to a and taking into account the implicit relation $a^d + b^d = 1$, we see that a couple $(a,b) \in (0,1)^2$ is a critical point for f if and only f

$$da^{-4-2d} + 4a^{-4-d} = db^{-4-2d} + 4b^{-4-d},$$

which does not hold unless $a = b = 2^{-\frac{1}{d}}$ since $x \mapsto dx^{-4-2d} + 4x^{-4-d}$ is decreasing over $(0,1)$. But we check that

$$f(2^{-\frac{1}{d}}, 2^{-\frac{1}{d}}) = (2^{\frac{4}{d}} \cdot 6 - 4)d + 8(2^{\frac{4}{d}} - 1) > 0.$$

Together with the fact that $f(0,1) = f(1,0) = +\infty$, the previous yield $f > 0$ over $(0,1)$ and hence $P'(T(B_1)) \leq 0$. It remains to inspect $P_{a,b}(T(B_1))$. Actually,

$P_{a,b}(T(B_1)) \geq 0$ if and only if $g(a,b) \geq 0$ where

$$g(a,b) = d + 4 + \left(1 - \frac{1}{a^{d+4}}\right)\left(1 - \frac{1}{b^{d+4}}\right) d - (d+4)\left(\frac{1}{a^4} + \frac{1}{b^4}\right).$$

Deriving g with respect to a, we see that g is critical for a couple $(a,b) \in (0,1)^2$ if and only if

$$da^d(a^{d+4} - 1) - db^d(b^{d+4} - 1) + 4a^d b^d(a^{d+4} - b^{d+4}) = 0.$$

As previously, this only holds when $a = b = 2^{-\frac{1}{d}}$. Hence we compute

$$g(2^{-\frac{1}{d}}, 2^{-\frac{1}{d}}) = 2(2^{\frac{d+4}{d}} - 1)\left[(2^{\frac{4}{d}} - 1)d - 2\right].$$

Since the function $h(d) := \left(16^{\frac{1}{d}} - 1\right) d - 2$ is convex on $\mathbb{R}_+^*$, and has first derivative with limit 0 at $+\infty$, h is decreasing on $\mathbb{R}_+^*$ and converges at $+\infty$ toward $\ln(16) - 2 > 0$. Thus, h is positive on $\mathbb{R}_+^*$, and so is $g(2^{-\frac{1}{d}}, 2^{-\frac{1}{d}})$. As a result, $P_{a,b}(T(B_1)) \geq 0$.

To conclude, we have shown that $T(a,b) \geq T(B_1)$ for all $(a,b) \in (0,1)^2$ enjoying the relation $a^d + b^d = 1$. Furthermore, by definition $T(1,0) = T(0,1) = T(B_1)$, hence the result follows. □

E. Holes

This appendix provides a proof for the geometric Lemma 6.7. We recall that Lemma 6.7 claims that if a connected open set ω shall be divided into two open sets of which the boundaries do not meet simultaneously $\partial\omega$, then one of those two sets must have a hole.

Proof of Lemma 6.7. We claim first that if γ_+ and γ_- are connected components respectively of $\partial\omega_+$ and $\partial\omega_-$, such that $\gamma_+ \cap \gamma_- \neq \emptyset$, then $\gamma_+ = \gamma_-$. Indeed, otherwise $\gamma_+ \cap \gamma_-$ would be a proper subset of $\gamma_+ \cup \gamma_-$, hence its (relative) boundary would be nonempty. But any point p of such boundary is not only in $\gamma_+ \cap \gamma_-$, but also in $\partial\omega$ (otherwise, using the regularity of $\omega_\pm$ and looking at a neighbourhood of p, we would obtain a contradiction). To sum up, there would exist a point in $\partial\omega_+ \cap \partial\omega_- \cap \partial\omega$, a contradiction. This means that the connected components of $\partial\omega_+$ and $\partial\omega_-$ either coincide or are disjoint.

But, necessarily, there exists a connected component of $\partial\omega_+$ that intersects a connected component of $\partial\omega_-$, otherwise $\partial\omega_+ \cap \partial\omega_-$ would be empty, and this would in turn mean that $\overline{\omega}$ is not connected (as it would be the disjoint union of the closed sets $\overline{\omega_+}$ and $\overline{\omega_-}$), and hence that ω is not connected (recall that the closure of a connected set is connected), which is impossible. This, and the previous discussion show that one is always able to find a connected component simultaneously of $\partial\omega_+$ and $\partial\omega_-$.

Let us denote Θ the set of unions of connected components shared by $\partial\omega_+$ and $\partial\omega_-$. The arguments above show that Θ is nonempty. In the following, we will construct a maximal element $\tilde{\theta}$ of Θ in the sense that $\tilde{\theta}$ will enclose any other element of Θ. Indeed, as $\omega_\pm$ is regular relatively with respect to ω, any element of Θ (which belongs to ω due to $\partial\omega \cap \partial\omega_+ \cap \partial\omega_- = \emptyset$) form a compact C^1 hypersurface of $\mathbb{R}^d$. Therefore, any connected component of an element of Θ is a C^1 compact connected hypersurface of $\mathbb{R}^d$. Thanks to Jordan-Brouwer Separation Theorem for C^1 hypersurfaces [Lim88], such a connected component is then the boundary of a bounded connected open set. Hence, sending any element $\theta \in \Theta$ to the union of the open sets delimited by the connected components of θ, one builds a well-defined (in general noninjective) map $\phi : \Theta \to \mathcal{T}(\mathbb{R}^d)$ (where $\mathcal{T}(\mathbb{R}^d)$ is the topology of $\mathbb{R}^d$). The map ϕ has the properties that for any $\theta \in \Theta$, $\theta \subseteq \overline{\phi(\theta)}$ and $\partial\phi(\theta) \subseteq \theta$. Moreover, if $\theta \in \Theta$ is connected, $\phi(\theta)$ is also connected. Then, define

$$\tilde{T} := \bigcup_{\theta \in \Theta} \phi(\theta), \quad \text{and} \quad \tilde{\theta} := \partial\tilde{T}.$$

One can show that:

1. $\tilde{\theta} \in \Theta$, in particular $\tilde{\theta}$ is C^1 regular.
2. For any $\theta \in \Theta$, $\theta \subseteq \overline{\tilde{T}}$.
3. $\tilde{T}$ is open and C^1 regular.
4. The connected components of $\tilde{T}$ have connected boundary.

At this point, let us prove that $\tilde{T}^c$ intersects **at most** one of ω_+ and ω_-. Indeed, assume by contradiction that $\tilde{T}^c$ intersects both ω_+ and ω_-. Then, $u_+ := \omega_+ \setminus \overline{\tilde{T}}$ and $u_- := \omega_- \setminus \overline{\tilde{T}}$ are nonempty open sets, and moreover their closures intersect, otherwise the closure of $\omega \setminus \overline{\tilde{T}}$ would be disconnected as the disjoint union of two closed sets. Then, $\omega \setminus \overline{\tilde{T}}$ would in turn be disconnected. But as ω is connected, as $\tilde{T}$ satisfies 3, 4, and as $\partial\tilde{T} \subseteq \omega$ (by 1 and because $\partial\omega \cap \partial\omega_+ \cap \partial\omega_- = \emptyset$), this cannot occur due to Lemma E.1 below. Therefore, $\overline{u_+}$ and $\overline{u_-}$ intersect, meaning that their boundaries intersect (as u_+ and u_- remain disjoint). Let γ be a connected component of $\partial u_+ \cap \partial u_-$. Then, $\gamma \in \Theta$ by 1, hence $\gamma \subseteq \overline{\tilde{T}}$ by 2. We show that this implies $\tilde{T} \cap u_+ \neq \emptyset$ or $\tilde{T} \cap u_- \neq \emptyset$, leading to a contradiction. Indeed, if $p \in \gamma$, by regularity of $\gamma \in \Theta$, there is a small ball B centred at p such that γ divides B into exactly two parts. Since $\gamma \subseteq \partial u_+ \cap \partial u_-$ and since u_+ and u_- are disjoint, those two parts correspond to $B \cap u_+$ and $B \cap u_-$. On the other hand, as $\gamma \subseteq \overline{\tilde{T}}$, B meets $\tilde{T}$, and $\emptyset \neq B \cap \tilde{T} \subseteq (u_+ \cap \tilde{T}) \cup (u_- \cap \tilde{T}) \cup \gamma$. In particular, since $B \cap \tilde{T}$ is open, either $u_+ \cap \tilde{T} \neq \emptyset$ or $u_- \cap \tilde{T} \neq \emptyset$.

We have shown that $\tilde{T}^c$ intersects at most one of ω_+ and ω_-. On the other hand, $\tilde{T}^c$ intersects **at least** one of ω_+ and ω_-, otherwise $\omega \subseteq \tilde{T}$, hence $\tilde{\theta} \subseteq \partial\omega$. Then $\partial\omega \cap \partial\omega_+ \cap \partial\omega_- = \emptyset$ would not hold. Therefore, $\tilde{T}^c$ intersects **exactly** one of ω_+ and ω_-. Changing the notations if needed, one can assume that $\tilde{T}^c$ intersects ω_+, so that $\tilde{T}$ intersects ω_-. Now set $T := \tilde{T} \setminus \overline{\omega_+}$. Because T is disjoint from ω_+ and because $\partial T \subseteq \partial\tilde{T} \cup \partial\omega_+ \subseteq \partial\omega_+$, we see that T is a hole for ω_+.

It remains only to check that T contains ω_-. But this comes from the fact that $\omega_- \subseteq \tilde{T}$ and from the definition of T. □

Lemma E.1. *Let A be a connected open set of $\mathbb{R}^d$, and $B \subseteq \mathbb{R}^d$ a bounded C^1 regular open set such that $\partial B \subseteq A$ and such that the boundary of any connected component of B is connected. Then $A \setminus \overline{B}$ is connected.*

Proof. Since A is open and connected, it is arcwise connected. Fix $x, y \in A \setminus B$ and Σ an arc joining x and y in A. If there exists a connected component b of B such that $\Sigma \cap b \neq \emptyset$, any connected component σ of $\Sigma \cap \overline{b}$ is an arc. Note that $\sigma(0)$ and $\sigma(1)$ belong to ∂b. But, by assumption ∂b is connected, and, as ∂b is a submanifold, it is then arcwise connected. Consequently, $\sigma(0)$ and $\sigma(1)$ might be joined by another arc τ in $\partial b \subseteq A$. Replacing σ by τ in Σ, and doing the same for any connected component of $\Sigma \cap b$ and any connected component b of B intersecting Σ, we build an arc from x to y belonging to $A \setminus B$. Thus, $A \setminus B$ is arc-connected.

But, using the regularity of B, we can do better and show that $A \setminus \overline{B}$ is also arcwise connected. Indeed, if B is C^1, one can define its normal vector $\vec{n}$ (and extend it to the whole $\mathbb{R}^d$). Then, if ϵ is small enough, as A is open, $B_\epsilon := (I + \epsilon\vec{n})B$ satisfies the same hypotheses than B, but also the ancillary condition $A \setminus B_\epsilon \subseteq A \setminus \overline{B}$. Thus, one can find an arc from x to y through $A \setminus B_\epsilon$, hence $A \setminus \overline{B}$ is arcwise connected. □

F. Symmetrization

In this appendix, we prove a property of signed Schwarz symmetrizations which is useful for deriving two-ball problems (see section 3.2, Proposition 5.21, Proposition 6.13). Indeed, Lemma F.1 translates the equimeasurability of f and f^* when the underlying domain is decomposed into two disjoint sets. Moreover, as we shall see it provides an alternative and elementary proof of Lemma 2.6. Before proceeding further, let us introduce for $f : \omega \to \mathbb{R}$ the function $f^\ddagger : \omega^* \to \mathbb{R}$ defined by

$$f^\ddagger(x) = f^\dagger(|B_{|x|}|),$$

where $f^\dagger(s) = f_+^\star(s) - f_-^\star(|\omega| - s)$ was defined in (2.1).

Lemma F.1. *Let $p \in \mathbb{N}$. Let ω be a bounded open set and $f \in L^p(\omega)$. Let $\omega_+, \omega_- \subseteq \omega$ be disjoint sets such that $|\omega| = |\omega_+| + |\omega_-|$. Then*

$$\int_{\omega_+^*} \left(f^\ddagger\right)^p + (-1)^p \int_{\omega_-^*} \left((-f)^\ddagger\right)^p = \int_\omega f^p.$$

Proof. We recall that for almost every $s \in (0, |\omega|)$, it is impossible for $f_+^\star(s)$ and $f_-^\star(|\omega| - s)$ to be simultaneously nonzero (see the proof of Lemma 2.6). Then, using Newton's binomial expansion, we get

$$\int_{\omega_+^*} \left(f^\ddagger\right)^p = \int_0^{|\omega_+|} \left(f^\dagger\right)^p = \int_0^{|\omega_+|} \left(f_+^\star\right)^p (s) + (-1)^p \int_0^{|\omega_+|} \left(f_-^\star\right)^p (|\omega| - s).$$

Due to the assumption $|\omega| = |\omega_+| + |\omega_-|$, a change of variable gives

$$\int_{\omega_+^*} \left(f^\ddagger\right)^p = \int_0^{|\omega_+|} \left(f_+^\star\right)^p (s) + (-1)^p \int_{|\omega_-|}^{|\omega|} \left(f_-^\star\right)^p (s).$$

Replacing f with $-f$ and exchanging ω_+ and ω_- in the previous relation, we find

$$\int_{\omega_-^*} \left((-f)^\ddagger\right)^p = \int_0^{|\omega_-|} \left(f_-^\star\right)^p (s) + (-1)^p \int_{|\omega_+|}^{|\omega|} \left(f_+^\star\right)^p (s).$$

The combination of the last identities turns into the next one

$$\int_{\omega_+^*} \left(f^\ddagger\right)^p + (-1)^p \int_{\omega_-^*} \left((-f)^\ddagger\right)^p = \int_{\omega^*} \left(f_+^\star\right)^p + (-1)^p \int_{\omega^*} \left(f_-^\star\right)^p.$$

Thanks to the equimeasurability of Schwarz symmetrization and using again

Newton's expansion, we eventually get as desired

$$\int_{\omega_+^*} \left(f^\ddagger\right)^p + (-1)^p \int_{\omega_-^*} \left((-f)^\ddagger\right)^p = \int_\omega (f_+)^p + (-1)^p \int_\omega (f_-)^p = \int_\omega f^p.$$

□

Based on Lemma F.1, we propose a simple proof of Lemma 2.6 in the case where f is bounded.

Proof of Lemma 2.6 for $f \in L^\infty(\omega)$. We will prove that f^* and $f^\ddagger$ have same moments. Then, because f^* and $f^\ddagger$ are bounded, this will show that they have same law, and in particular same distribution function. Consequently $f^\star$ and $f^\dagger$ will also enjoy same distribution function. Being nonincreasing they will coincide almost everywhere by Lemma 2.3, point 3. To show that f^* and $f^\ddagger$ share same moments, let us fix some $p \in \mathbb{N}$. First, since f^* and f are equimeasurable,

$$\int_\omega (f^*)^p = \int_\omega f^p.$$

On the other hand, apply Lemma F.1 with $\omega_+ = \omega$ and $\omega_- = \emptyset$ to observe that

$$\int_\omega \left(f^\ddagger\right)^p = \int_\omega f^p,$$

which concludes. □

G. Asymmetric two-ball problem

This appendix is devoted to the technical result needed to complete the proof of Theorem 6.14 (see inequality (6.18)), stating that the asymmetric two-ball problem reaches its minimum when one ball vanishes.

Theorem G.1. *Let $d \geq 4$, set $\nu = d/2 - 1$, and assume that*

1. *There exists a sequence $0 = x_0 < x_1 < ... < x_n < x_{n+1} = a_I$ such that (6.11) holds for each $i \in (0, n]$, and (6.12) holds.*
2. *There exists a sequence $a_S = y_0 < y_1 < ... < y_m < y_{m+1} = 1$ such that (6.13) holds for each $i \in [0, m)$, and (6.14) holds.*

Then, for every $a \in A = [0, a_I) \cup (a_S, 1]$,

$$F_\nu(k_\nu, a) \leq 0. \tag{G.1}$$

Proof. We use the same notations as in the proof of Theorem 6.14. First, let us remark that K is a positive increasing function of a, and that when $a \in A_I := [0, a_I)$, thanks to (6.17), $k_\nu K a \in [0, k_\nu K(a_I) a_I) \subseteq [0, k_\nu K(a_S) a_S) = [0, j_\nu)$. On the other hand, b being positive decreasing with respect to a, then according to (6.17) once again, $k_\nu b \in (k_\nu b_S, k_\nu] = (j_\nu, k_\nu]$. In this context, we see that, on A_I, $F_\nu(k_\nu, a)$ is the sum of two rival terms: the first, $f_\nu(k_\nu K a)$, is positive increasing, and the second, $K^d f_\nu(k_\nu b)$, negative decreasing (as a product of a positive increasing and a negative decreasing function) with respect to the variable a. Then, we can try using a "zigzag argument" (as in Lemma 2, case $n = 3$ of [AB95]) to show that $F_\nu(k_\nu, a) \leq 0$. More precisely, if one is able to exhibit a sequence of points $0 = x_0 < x_1 < ... < x_n < x_{n+1} = a_I$ such that for all $0 \leq i \leq n$,

$$f_\nu(k_\nu K(x_{i+1}) x_{i+1}) + K(x_i)^d f_\nu(k_\nu b(x_i)) \leq 0,$$

then we can conclude that (G.1) holds on A_I. This is because, for all $a \in A_I$, a belongs to some interval $[x_i, x_{i+1}]$, and hence $F_\nu(k_\nu, a) \leq f_\nu(k_\nu K(x_{i+1}) x_{i+1}) + K(x_i)^d f_\nu(k_\nu b(x_i))$ thanks to the monotonic behaviour of each term. Of course, it is impossible to find such sequence since for $i = 0$, $f_\nu(k_\nu K(x_{i+1}) x_{i+1}) > 0$ whereas $K(x_i) = 0$. Nevertheless, apart from the case $i = 0$, one can expect to show this inequality, which explains why equation (6.11) is an assumption of this theorem. This assumption induces that $F_\nu(k_\nu, a) \leq 0$ on $[x_1, a_I)$.

Now if $a \in A_S := (a_S, 1]$, (6.17) implies that $k_\nu K a \in (j_\nu, k_\nu]$ and that $k_\nu b \in [0, k_\nu b_I) \subset [0, j_\nu)$, so that, since $K \leq 1$, $F(k_\nu, a) \leq \tilde{F}_\nu(k_\nu, a) := f_\nu(k_\nu K a) + f_\nu(k_\nu b)$.

Then, $\tilde{F}_\nu(k_\nu, a)$ is the sum of the negative increasing function of a $f_\nu(k_\nu K a)$ and the positive decreasing function of a $f_\nu(k_\nu b)$. Analogously as what have been done for $a \in A_I$, it is clear that (G.1) holds on A_S as soon as one is able to find $a_S = y_0 < y_1 < ... < y_m < y_{m+1} = 1$ such that $\forall 0 \leq i \leq m$,

$$f_\nu(k_\nu K(y_{i+1})y_{i+1}) + f_\nu(k_\nu b(y_i)) \leq 0.$$

Of course, as for (6.11), this inequality always fails for $i = m$. But since (6.13) is assumed to be true for all $i \neq m$, we get that $F_\nu(k_\nu, a) \leq 0$ on $(a_S, y_m]$.

The discussion above shows that it remains to prove $F_\nu(k_\nu, a) \leq 0$ only near $a = 0$ and $a = 1$, that is on $[0, x_1)$ and on $(y_m, 1]$. For this purpose, one can compute the derivative of $a \mapsto F_\nu(k_\nu, a)$. Indeed, since (G.1) is already known for $a = 0$ and $a = 1$, it will hold on the whole $[0, x_1) \cup (y_m, 1]$ if one can show that

$$\partial_a F_\nu(k_\nu, a) < 0, \qquad \forall a \in (0, x_1), \tag{G.2}$$

and that

$$\partial_a F_\nu(k_\nu, a) > 0, \qquad \forall a \in (y_m, 1). \tag{G.3}$$

Actually – as pointed out in [AB95] –, since (G.1) is known asymptotically in a neighbourhood of 0 and 1 (this claim will be proved at the end, but see also Lemma 1 [AB95]), it is enough to show (G.2) and (G.3) only when $F_\nu(k_\nu, a) = 0$. That's why, for the rest of the proof, we make the ancillary assumption that $F_\nu(k_\nu, a) = 0$ unless otherwise mentioned. We have to compute $\partial_a F_\nu(k_\nu, a)$. For readability, we omit from time to time the variable a, and write k instead of k_ν. Moreover, we use, according to Appendix 1 of [AB95] and to the identity $2\nu + 1 = d - 1$, that

$$\begin{aligned} f'_\nu(r) &= \left[\left(\frac{J_{\nu+1}(r)}{J\nu(r)} - \frac{I_{\nu+1}(r)}{I\nu(r)}\right)\left(\frac{J_{\nu+1}(r)}{J\nu(r)} + \frac{I_{\nu+1}(r)}{I\nu(r)}\right) + 2\right] r^{2\nu+1} \\ &= 2r^{d-1} + \left(\frac{J_{\nu+1}(r)}{J_\nu(r)} - \frac{I_{\nu+1}(r)}{I_\nu(r)}\right) f_\nu(r). \end{aligned} \tag{G.4}$$

Then,

$$\begin{aligned} \partial_a F_\nu(k, a) =& 2k(aK' + K)(kKa)^{d-1} \\ &+ k(aK' + K)\left(\frac{J_{\nu+1}}{J_\nu}(kKa) - \frac{I_{\nu+1}}{I_\nu}(kKa)\right) f_\nu(kKa) \\ &+ dK'K^{d-1} f_\nu(kb) + 2kK^d b'(kb)^{d-1} \\ &+ kK^d b'\left(\frac{J_{\nu+1}}{J_\nu}(kb) - \frac{I_{\nu+1}}{I_\nu}(kb)\right) f_\nu(kb). \end{aligned}$$

The hypothesis $F_\nu(k, a) = 0$ made above reads $f_\nu(kKa) = -K^d f_\nu(kb)$. This allows

writing

$$\begin{aligned}\partial_a F_\nu(k,a) =& 2k^d[(aK'+K)(aK)^{d-1} + b'b^{d-1}K^d] \\ &+ dK'K^{d-1}f_\nu(kb) \\ &+ ka^{d-1}G_\nu(a)f_\nu(kKa),\end{aligned} \tag{G.5}$$

where

$$G_\nu(a) = \frac{aK'+K}{a^{2\nu+1}}\left(\frac{J_{\nu+1}}{J_\nu}(kKa) - \frac{I_{\nu+1}}{I_\nu}(kKa)\right) - \frac{b'}{a^{2\nu+1}}\left(\frac{J_{\nu+1}}{J_\nu}(kb) - \frac{I_{\nu+1}}{I_\nu}(kb)\right).$$

We will study each of the three terms involved in (G.5). For that we define

$$\begin{aligned}T_1 &:= (aK'+K)(aK)^{d-1} + b'b^{d-1}K^d, \\ T_2 &:= dK'K^{d-1}f_\nu(kb), \\ T_3 &:= ka^{d-1}G_\nu(a)f_\nu(kKa).\end{aligned}$$

The second term, T_2, is the easiest to analyse. Indeed, if $a \in A_I^* := (0, a_I)$, then $b \in (b_S, 1)$, so that (according to (6.17)) $kb \in (j_\nu, k_\nu)$ and eventually $f_\nu(kb) < 0$. On the other hand, if $a \in A_S^* := (a_S, 1)$, because K is increasing and thanks to (6.17), $kKa \in (j_\nu, k_\nu)$, and as before $f_\nu(kKa) < 0$. The relation $f_\nu(kKa) = -K^d f_\nu(kb)$ shows in turn that $f_\nu(kb) > 0$. In summary, because $K, K' > 0$ on $(0,1)$ (see (G.6)), we have $T_2 < 0$ when $a \in A_I^*$ and $T_2 > 0$ when $a \in A_S^*$.

Let's tackle T_1. With the notation $A = a^{d-1}$ and $B = b^{d-1}$, one computes

$$b' = -\frac{A}{B}, \qquad K = \frac{A}{1+B}, \qquad K' = \frac{(d-1)(b+1)K^2}{abA}, \tag{G.6}$$

and

$$T_1 = (aK)^d \frac{K'}{K}.$$

We see that K' and T_1 are products of positive increasing fonctions on $(0,1)$, so that they are themselves positive increasing. Because the properties of K' and T_1 will be helpful to conclude the proof, we encapsulate it in the following lemma.

Lemma G.2. *K' and T_1 are positive increasing on $(0,1)$, and $K', T_1 \sim \frac{d-1}{b} \to \infty$ when $a \to 1$.*

This result gives that $T_1 > 0$ on A_S which was desirable. The problem is that we do not have $T_1 < 0$ on A_I^*. Hopefully, as we shall see in a few moment, its positivity near $a = 0$ will be compensated by the negativity of T_2. Indeed, recall that $T_2 = dK'K^{d-1}f_\nu(kb) = -d\frac{K'}{K}f_\nu(kKa)$. To analyse the behaviour of $f_\nu(kKa)$ around $a = 0$ and then go further in the analysis of $2k^dT_1 + T_2$, we use the identities

given in [AB95, Lemma 2], which are

$$\frac{J_{\nu+1}}{J_\nu}(r) = 2r \sum_{m\geq 1} \frac{1}{j_{\nu,m}^2 - r^2}, \tag{G.7}$$

$$\frac{I_{\nu+1}}{I_\nu}(r) = 2r \sum_{m\geq 1} \frac{1}{j_{\nu,m}^2 + r^2}. \tag{G.8}$$

Then, for all $r \in (0, j_\nu)$,

$$f_\nu(r) = 2r^d \sum_{m\geq 1} \frac{2j_{\nu,m}^2}{j_{\nu,m}^4 - r^4} > 4r^d \sum_{m\geq 1} \frac{1}{j_{\nu,m}^2}.$$

But thanks to [GM16], $\sum_{m\geq 1} \frac{1}{j_{\nu,m}^2} = \frac{1}{4(\nu+1)} = 1/(2d)$. Therefore, for all $a \in A_I^*$, $kKa \in (0, j_\nu)$ and thus

$$f_\nu(kKa) > \frac{2}{d}(kKa)^d.$$

Hence, for all $a \in A_I^*$,

$$T_2 < -2\frac{K'}{K}(kKa)^d = -2k^d T_1.$$

Then, as desired, $2k^d T_1 + T_2 < 0$ on A_I^*.

We now study $T_3 = ka^{d-1}G_\nu(a)f_\nu(kKa)$, the last term in (G.5). Notice that thanks to the the fact already mentioned that $f_\nu(kKa)$ is positive when $a \in A_I^*$ and negative when $a \in A_S^*$, T_3 will be negative on A_I^* and positive on A_S^* if and only if $G_\nu < 0$ on A_I^* and on A_S^*. Therefore, let us write $G_\nu = G_1 + G_2$ where we recall $G_1 = (aK' + K)(kK)^{d-1}g_\nu(kKa)$ and $G_2 = k^{d-1}g_\nu(kb)$ with

$$g_\nu(r) = \frac{1}{r^{d-1}}\left[\frac{J_{\nu+1}}{J_\nu}(r) - \frac{I_{\nu+1}}{I_\nu}(r)\right].$$

As in [AB95, Lemma 2], one uses, thanks to (G.7) and (G.8), that

$$g_\nu(r) = 4r^{4-d} \sum_{m\geq 1} \frac{1}{j_{\nu,m}^4 - r^4} =: 4r^{4-d}S_\nu(r).$$

Here, we will need a new lemma.

Lemma G.3. *S_ν is positive increasing on $[0, j_\nu)$ and negative increasing on $(j_\nu, k_\nu]$.*

Proof. The fact that S_ν is increasing is obvious as it is the case for each term involved in the sum. The fact that it is positive on $[0, j_\nu)$ is also easy because this property holds for each term. The fact that S_ν is negative on $(j_\nu, k]$ is not as easy. It comes from the identities $4r^3 S_\nu(r) = \frac{J_{\nu+1}}{J_\nu}(r) - \frac{I_{\nu+1}}{I_\nu}(r)$ and $\frac{J_{\nu+1}}{J_\nu}(k) + \frac{I_{\nu+1}}{I_\nu}(k) = f_\nu(k)k^{1-d} = 0$, showing that $S_\nu(k) < 0 \iff \frac{J_{\nu+1}}{J_\nu}(k) < 0$. But

as near $r = 0$, $J_\nu(r) \sim \frac{1}{\Gamma(\nu+1)}(r/2)^\nu$, we have that J_ν is positive near zero. Moreover, it is known that any positive zero of J_ν is simple, so that J_ν is negative between $j_{\nu,1}$ and $j_{\nu,2}$. This allows to conclude that $J_\nu(k) < 0$ (recall $j_{\nu,1} < k < j_{\nu,2}$). On the other hand, thanks to [ABP16, Lemma 3], we have that $k < j_{\nu+1,1}$ and hence that $J_{\nu+1}(k) > 0$ since $J_{\nu+1}$ is positive between 0 and $j_{\nu+1,1}$. Therefore, we conclude that for all $r \in (j_\nu, k)$, $S_\nu(r) < S_\nu(k) < 0$. □

At this point, the proof of (G.1) will differ substantially for $a \in A_I^*$ and for $a \in A_S^*$.

$F_\nu(k,a) \leq 0$ for $a \in (0, x_1)$:

Due to the fact that S_ν is negative increasing on (j_ν, k), g_v is negative increasing on (j_ν, k) as the product of a positive nonincreasing function (recall that here $d \geq 4$) and a negative increasing function. Then, because b is decreasing, G_2 is negative decreasing on A_I^*. On the other hand, K' is positive increasing with respect to a on $(0, 1)$. Moreover,

$$(kK)^{d-1} g_\nu(kKa) = 4K^2 \frac{(ka)^3}{1+B} S_\nu(kKa).$$

The right hand member above is the product of a positive increasing function on A_I^* with $S_\nu(kKa)$, which is positive increasing on A_I. Indeed, we recall that $a \in A_I \Rightarrow kKa \in [0, j_\nu)$. Eventually, G_1 is a product of positive increasing functions on A_I^*, and therefore is itself positive increasing on A_I^*.

To sum up, on A_I^*, G_ν is the sum of the positive increasing function G_1 and the negative decreasing function G_2. Therefore, $G_\nu < 0$ on $(0, x_1)$ if

$$G_1(x_1) + G_2(x_0) \leq 0.$$

As (6.12) is assumed to hold, we get that $\partial_a F_\nu(k,a) < 0$ on $(0, x_1)$ whenever $F_\nu(k,a) = 0$. Now, to conclude that (G.1) holds on $(0, x_1)$, it remains only to show that (G.1) held in some neighbourhood of 0. To do so, we perform an asymptotic analysis. Observe that, thanks to (G.7), (G.8) and [GM16],

$$f_\nu(r) \underset{r\to 0}{\sim} \frac{2}{d} r^d. \tag{G.9}$$

On the other hand, using (G.4), we find that $\frac{d}{d\omega} f_\nu(\omega^{\frac{1}{d}}) \xrightarrow[\omega \to k^d]{} \frac{2}{d}$, from which we get

$$f_\nu(r) \underset{r\to k}{\sim} \frac{2}{d}(r^d - k^d). \tag{G.10}$$

As a consequence, $f_\nu(kKa) \underset{a\to 0}{\sim} -K^d f_\nu(kb)$ and one needs to exhibit the next term in the asymptotic expansions of $f_\nu(kKa)$ and $K^d f_\nu(kb)$ in order to conclude. For

that, plug (G.9) into (G.4), and use the definition of S_ν to find

$$f'_\nu(r) \underset{r\to 0}{=} 2r^{d-1} + \frac{8}{d} S_\nu(0) r^{d+3} + o(r^{d+3}).$$

From this we obtain after integration

$$f_\nu(r) \underset{r\to 0}{=} \frac{2}{d} r^d + \frac{8}{d} S_\nu(0) \frac{r^{d+4}}{d+4} + o(r^{d+4}).$$

Similarly, plug (G.10) into (G.4) and find

$$\frac{d}{d\omega} f_\nu(\omega^{\frac{1}{d}}) \underset{\omega\to k^d}{=} \frac{2}{d} + \frac{2}{d}(\omega - k^d) 4k^3 S_\nu(k) k^{\frac{1-d}{d}} + o(\omega - k^d).$$

After integration, we obtain

$$f_\nu(r) \underset{r\to k}{=} \frac{2}{d}(r^d - k^d) + \frac{4}{d}(r^d - k^d)^2 k^3 S_\nu(k) k^{\frac{1-d}{d}} + o((r^d - k^d)^2).$$

Eventually, since $(Ka)^{d+4} \underset{a\to 0}{=} o(K^d a^{2d})$, we get

$$\begin{aligned} f_\nu(kKa) + K^d f_\nu(kb) &\underset{a\to 0}{=} \frac{8}{d(d+4)} S_\nu(0)(kKa)^{d+4} + \frac{4}{d} S_\nu(k) k^{\frac{2d+1}{d}} (ka)^{2d} K^d \\ &\quad + o((Ka)^{d+4} + K^d a^{2d}) \\ &\underset{a\to 0}{=} \frac{4}{d} S_\nu(k) k^{\frac{2d+1}{d}} (ka)^{2d} K^d + o(K^d a^{2d}). \end{aligned}$$

Since $S_\nu(k) < 0$ due to Lemma G.3, we see that $F_\nu(k, a)$ is negative when a lies in some neighbourhood of 0.

$\underline{F_\nu(k,a) \le 0 \text{ for } a \in (y_m, 1)}$:

In the case $a \in A_S^*$, we are not able to show that $G_\nu < 0$, and indeed at least when $d \ge 6$, $G_\nu(a) \to \infty$ when $a \to 1$. Nevertheless, as we shall see in the next lines, we do have $G_\nu(a) f_\nu(kKa) = -K^d G_\nu(a) f_\nu(kb) \underset{a\to 1}{=} O(b^4 + b^d) \to 0$. But because $T_1 \to \infty$ when $a \to 1$, it will dominate T_3 in (G.5), enforcing $\partial_a F_\nu(k_\nu, a) > 0$ near $a = 1$. However one has to quantify the predominance of T_3 with respect to T_1. For that purpose, one estimates, for $a \in (y_m, 1)$, $|G_1(a)| \le (|K'| + 1) k^{d-1} |g_\nu(kKa)|$. Now g_ν being as S_ν negative increasing on (j_ν, k), we obtain (using also (G.6)) that for $a \in (y_m, 1)$,

$$|G_1(a)| < \left[\frac{d-1}{y_m^d} \left(1 + \frac{1}{b}\right) + 1 \right] k^{d-1} |g_\nu(kK(y_m) y_m)|. \tag{G.11}$$

For G_2, we use again that $g_\nu(r) = 4r^{4-d} S_\nu$, and the fact that S_ν is positive increasing

on $[0, j_\nu)$, yielding, for all $a \in (y_m, 1)$,

$$|G_2(a)| < 4k^{d-1}(kb)^{4-d}S_\nu(kb(y_m)) = k^{d-1}b^{4-d}\frac{g_\nu(kb(y_m))}{b(y_m)^{4-d}}. \tag{G.12}$$

We have also to control $f_\nu(kKa) = -K^d f_\nu(kb)$. To do so, we use again (G.7) and (G.8) to conclude that $r \mapsto r^{-d}f_\nu(r)$ is positive increasing on $[0, j_\nu)$, and hence that, for $a \in (y_m, 1)$,

$$|f_\nu(kb)| < (kb)^d(kb(y_m))^{-d}f_\nu(kb(y_m)) = b^d b(y_m)^{-d}f_\nu(kb(y_m)) \tag{G.13}$$

Combining (G.11), (G.12) and (G.13), we have on $(y_m, 1)$,

$$\begin{aligned}|T_3| \le k|G_\nu||f_\nu(kb)| <& k^d|g_\nu(kK(y_m)y_m)|f_\nu(kb(y_m))b(y_m)^{-d}b^d\left[\frac{d-1}{y_m^d}\left(1+\frac{1}{b}\right)+1\right]\\ &+ k^d(g_\nu f_\nu)(kb(y_m))b(y_m)^{-4}b^4\\ \underset{a\to 1}{=}& O(b^4 + b^{d-1}).\end{aligned}$$

The point is that the right-hand member in the previous inequality is an increasing function of the variable $b < b(y_m)$, so that we get for all $a \in (y_m, 1)$,

$$|T_3| < k^d|g_\nu(kK(y_m)y_m)|f_\nu(kb(y_m))\left[\frac{d-1}{y_m^d}\left(1+\frac{1}{b(y_m)}\right)+1\right] + k^d(g_\nu f_\nu)(kb(y_m)).$$

On the other, because T_1 is positive increasing on A_S, (G.3) will be ensured as long as one is able to tune y_m such that

$$0 \le 2T_1(y_m) - f_\nu(kb(y_m))|g_\nu(kK(y_m)y_m)|\left[\frac{d-1}{y_m^d}\left(1+\frac{1}{b(y_m)}\right)+1\right] - (f_\nu g_\nu)(kb(y_m)).$$

Because (6.14) is an assumption, we conclude that $\partial_a F_\nu(k, a) > 0$ on $(y_m, 1)$ whenever $F_\nu(k, a) = 0$. Hence, to show that (G.1) holds on $(y_m, 1)$, it remains to show that it does asymptotically near $a = 1$. Recalling (G.9), one finds

$$f_\nu(kb) \underset{a\to 1}{\sim} \frac{2}{d}(kb)^d.$$

On the other hand, (G.10) gives

$$f_\nu(kKa) \underset{a\to 1}{\sim} \frac{2}{d}k^d((Ka)^d - 1).$$

Note that since $K \underset{a\to 1}{\longrightarrow} 1$,

$$(Ka)^d - 1 = (Ka-1)\sum_{n=0}^{d-1}(Ka)^n \underset{a\to 1}{\sim} -d\frac{b^d+B}{1+B} \underset{a\to 1}{\sim} -db^{d-1}.$$

From this we deduce that $K^d f_\nu(kb) \underset{a\to 1}{=} o(f_\nu(kKa))$ and that $F_\nu(k,a) \underset{a\to 1}{\sim} -2k^d b^{d-1}$. As a consequence, $F_\nu(k,a) < 0$ in some neighbourhood of $a = 1$, which concludes. □

H. Numerical computations

In this appendix, we would like to check the assumptions (6.10), (6.11), (6.12), (6.13), and (6.14) of Theorem 6.14 for several dimensions. First, it is needed to compute the quantities j_ν, k_ν, a_I and a_S. Hopefully, the m-th first positive zero $j_{\nu,m}$ of the Bessel function J_ν can be approximated by computing the eigenvalues of some matrix as explained in [IKF91] Theorem 2.1 and 2.2. On the other hand, k_ν is the only zero of the increasing function f_ν between $j_{\nu,1}$ and $j_{\nu,2}$, hence it can be obtained by a bisection method. Similarly, a_S being the only zero of the increasing function $a \in [0,1] \mapsto aK(a) - \frac{j_\nu}{k_\nu}$, it shall also be computed by the bisection method. Eventually, a_I is given by the formula $a_I^d = 1 - \frac{j_\nu^d}{k_\nu^d}$.

Then, we need to find, on the one hand, the sequence of points $(x_1, ..., x_n)$ fulfilling hypotheses (6.11) and (6.12); and, on the other hand, the sequence $(y_1, ..., y_m)$ fulfilling hypotheses (6.13) and (6.14). It seems not trivial to give an efficient algorithm exhibiting automatically these points, and we rather find the two sequences "manually". That's why we restricted the study to the dimensions $d = 4, 5, 6, 7, 8, 9$. We point out that the numerical computations have been performed in Python 3, using the package `special` from the module `scipy`, which contains the functions `jv`, and `iv`, corresponding to the Bessel functions J_ν and I_ν.

Let us mention that while checking (6.11), one has to use approximations of a_I and k_ν by the right, namely a_I^+ and k_ν^+, rather than by the left. This is because (see the proof of Theorem G.1 for details) $(k, a) \mapsto f_\nu(kKa)$ is increasing in each variable on $[0, k_\nu^+] \times [0, a_I^+]$ and $(k, a) \mapsto Kf_\nu(kb)$ is increasing in k and decreasing in a on $[0, k_\nu^+] \times [0, a_I^+]$. Therefore, instead of (6.11), one will try to verify

$$F_{\nu,i} := f_\nu(k_\nu^+ K(x_{i+1})x_{i+1}) + K(x_i)^d f_\nu(k_\nu^+ b(x_i)) \leq 0. \tag{6.11'}$$

Analogously, when checking (6.13), one has to use an approximation of a_S by the left, that we denote a_S^-; but still an approximation of k_ν by the right. That's why (6.13) turns into

$$F'_{\nu,i} := f_\nu(k_\nu^+ K(y_{i+1})y_{i+1}) + f_\nu(k_\nu^+ b(y_i)) \leq 0. \tag{6.13'}$$

Similar issues have to be taken into account when validating (6.12) and (6.14). More precisely, in (6.12), G_1 should be replaced by some $\tilde{G}_1$ defined as $\tilde{G}_1(a) = (aK' + K)(k_\nu^+ K)^{d-1} g_\nu(k_\nu^+ Ka)$; G_2 should be replaced by $\tilde{G}_2$ defined as $\tilde{G}_2(a) =$

$(k_\nu^-)^{d-1} g_\nu(k_\nu^+ b)$; and thus (6.12) should be replaced by

$$G_{\nu,0} := \tilde{G}_1(x_1) + \tilde{G}_2(x_0) \leq 0. \tag{6.12'}$$

On the other hand (6.14) should be replaced by the condition

$$G_{\nu,m} \geq 0, \tag{6.14'}$$

where

$$\begin{aligned} G_{\nu,m} :=& 2T_1(y_m) - (f_\nu g_\nu)(k_\nu^+ b(y_m)) \\ & - f_\nu(k_\nu^+ b(y_m))|g_\nu(k_\nu^- K(y_m) y_m)| \left[\frac{d-1}{y_m^d} \left(1 + \frac{1}{b(y_m)} \right) + 1 \right]. \end{aligned}$$

Eventually, since a_S depends on j_ν/k_ν, for computing a_S^- one has to use j_ν^- and k_ν^+. The same conclusion holds for computing a_I^+.

In Table H.1, is given the value $2\frac{j_\nu^d}{k_\nu^d} + \frac{j_\nu}{k_\nu}$ for several dimensions, which shows that (6.10) is fulfilled. Table H.2 contains the values of j_ν^-, k_ν^-, k_ν^+, a_I^+ and a_S^-. In Table H.3 are computed for several dimensions the approximate quantities $G_{\nu,0}$, $F_{\nu,i}$, $F'_{\nu,i}$ and $G'_{\nu,m}$ appearing in (6.11'), (6.12'), (6.13') and (6.14'). We observe that appart from the last column, the value reported in each cell is negative, meaning that inequalities (6.11'), (6.12'), (6.13') are satisfied. Moreover, the fact that the values in the last column are positive show that inequality (6.14') is also true. This justifies that assumptions 1 and 2 of Theorem 6.14 are fulfilled.

d	$2j_\nu^d/k_\nu^d + j_\nu/k_\nu$
4	1.7848
5	1.7563
6	1.7345
7	1.7172
8	1.7029
9	1.6910

Table H.1. – Values of $2\frac{j_\nu^d}{k_\nu^d} + \frac{j_\nu}{k_\nu}$ in terms of dimension d.

d	j_ν^-	k_ν^-	k_ν^+	a_I^+	a_S^-
4	3.831	4.610	4.611	0.8507	0.9836
5	4.493	5.267	5.268	0.8869	0.9879
6	5.135	5.905	5.906	0.9101	0.9907
7	5.763	6.529	6.530	0.9259	0.9927
8	6.380	7.143	7.144	0.9373	0.9940
9	6.987	7.748	7.749	0.9459	0.9951

Table H.2. – Values of j_ν^-, k_ν^-, k_ν^+, a_I^+, and a_S^- in terms of dimension d.

d	$G_{\nu,0}$	$F_{\nu,1}$	$F'_{\nu,0}$	$G'_{\nu,1}$
4	-1.232	-6.234	-2.682	26.92
5	-1.040	-76.82	-52.58	27.50
6	-0.9603	-123.8	-840.3	28.84
7	-0.8131	-1135	$-1336 \cdot 10^1$	30.36
8	-0.7514	-3126	$-2165 \cdot 10^2$	31.83
9	-0.6430	$-2362 \cdot 10^1$	$-3803 \cdot 10^3$	33.12

Table H.3. – Values of $G_{\nu,0}$, $F_{\nu,i}$, $F'_{\nu,i}$ and $G'_{\nu,m}$ in terms of dimension d. For each dimension, it is chosen $n = 1$, $m = 1$, and $y_1 = 0.999$. The point x_1 chosen depends on the dimension: for $d = 4$, $x_1 = 0.83$, for $d = 5$, $x_1 = 0.88$, for $d = 6$, $x_1 = 0.90$, for $d = 7$, $x_1 = 0.92$, for $d = 8$, $x_1 = 0.93$, and for $d = 9$, $x_1 = 0.94$.

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