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Group actions on irreducible holomorphic symplectic manifolds of dimension 2 and 4

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Résumé

Le but de cette thèse est d'étudier des actions de groupe sur certaines variétés Symplectiques Holomorphes Irréductibles (nous utiliserons l'acronyme anglais IHS). Dans un premier temps, nous nous intéresserons aux surfaces de Kummer qui sont des surfaces K3 spéciales. Une surface de Kummer est obtenue comme désingularisation du quotient d'une surface abélienne par une involution ayant 16 points fixes isolés. Une telle surface admet naturellement une action symplectique de $(\mathbb{Z}/2\mathbb{Z})^4$, signifiant que l'action des éléments de ce groupe sur la 2-forme est triviale. Il est bien connu que les surfaces de Kummer sont des surfaces K3 contenant 16 courbes rationnelles lisses disjointes, appelées *configuration de Nikulin*. Réciproquement, selon Nikulin l'existence d'une telle configuration sur une surface K3 signifie qu'il s'agit d'une surface de Kummer. La question de l'existence de surfaces abéliennes non-isomorphes donnant lieu aux mêmes surfaces de Kummer est une question importante en géométrie algébrique, qui a déjà été étudiée dans la littérature, mais la construction explicite de configurations de 16 courbes rationnelles non-isomorphes reste un problème ouvert. Celui-ci a été traité récemment par X. Roulleau et A. Sarti dans plusieurs travaux, et dans cette thèse nous complétons cette étude. Ensuite nous nous intéresserons à des surfaces K3 admettant une action symplectique du groupe de Mathieu M_{20}. Ce travail a été réalisé en collaboration avec Paola Comparin (Universidad de La Frontera, Chili). Il a été montré par Mukai que l'ordre maximal d'un groupe fini agissant fidèlement et symplectiquement sur une surface K3 est 960 et si un tel groupe est d'ordre 960, alors il est isomorphe à M_{20}. Nous montrons qu'il existe une infinité de surfaces K3 avec cette action. En outre l'un des objectifs de cette thèse est de toutes les décrire et de comprendre quand il est possible de construire explicitement un modèle projectif. Une généralisation naturelle de ces résultats concerne les variétés IHS qui sont équivalentes par déformation au schéma de Hilbert de deux points sur une surface K3, dites de *type K3*$^{[2]}$. Ce dernier point a été réalisé en collaboration avec Paola Comparin et Pablo Quezada Mora (Universidad de La Frontera, Chili). Dans un travail récent G. Höhn et G. Mason ont classifié les groupes qui peuvent agir symplectiquement sur une variété IHS de type K3$^{[2]}$, montrant que $\mathbb{Z}_3^4 : \mathfrak{A}_6$ est celui avec le plus grand ordre. Nous étudions les variétés IHS de type K3$^{[2]}$ pouvant admettre une telle action symplectique ainsi qu'un automorphisme non-symplectique. En particulier, nous prouvons qu'il existe une variété IHS de type K3$^{[2]}$ avec un groupe d'automorphismes fini d'ordre 174960 - qui est le plus grand ordre possible pour le groupe d'automorphismes d'une variété IHS de type K3$^{[2]}$ - et qu'il s'agit de la variété de Fano des droites de la cubique de Fermat.

Mots-clés : Géométrie algébrique, variétés symplectiques holomorphes irréductibles, surfaces K3, surfaces de Kummer, actions de groupe, automorphismes.

Abstract

The aim of this thesis is to study some group actions on some Irreducible Holomorphic Symplectic (IHS) manifolds. First of all we are interested in Kummer surfaces which are special K3 surfaces. A Kummer surface is obtained as the desingularization of the quotient of an abelian surface by an involution with 16 isolated fixed points. Such a surface naturally admits a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$, meaning that the elements of this group act trivially on the 2-form. It is well known that Kummer surfaces are K3 surfaces containing 16 disjoint smooth rational curves, called a *Nikulin configuration*. Conversely, according to Nikulin the existence of such a configuration on a K3 surface means that this is a Kummer surface. The question of the existence of non-isomorphic abelian surfaces giving rise to the same Kummer surface is an important question in algebraic geometry, which has already been studied in the literature, but the explicit construction of configurations of 16 non-isomorphic rational curves remains an open problem. This one was recently treated by X. Roulleau and A. Sarti in several works, and in this thesis we complete this study. In a second part we are interested in K3 surfaces addmitting a symplectic action of the Mathieu group M_{20}. This work is a joint work with Paola Comparin (Universidad de La Frontera, Chile). It was shown by Mukai that the maximum order of a finite group acting faithfully and symplectically on a K3 surface is 960 and if such a group has order 960, then it is isomorphic to M_{20}. We show that there exists infinitely many K3 surfaces with this action and we describe them and their projective models, giving some explicit examples. A natural way to generalize these results is to deal with IHS manifolds which are deformation equivalent to the Hilbert scheme of two points of a K3 surface, called of *K3$^{[2]}$-type*. This last topic is a joint work with Paola Comparin and Pablo Quezada Mora (Universidad de La Frontera, Chile). In a recent work G. Höhn and G. Mason classified the possible symplectic groups acting on an IHS manifold of K3$^{[2]}$-type, finding that $\mathbb{Z}_3^4 : \mathfrak{A}_6$ is the symplectic group with the biggest order. We study the possible IHS manifolds of K3$^{[2]}$-type with such a symplectic action and admitting also a non-symplectic automorphism. In particular we prove that there exists an IHS manifold of K3$^{[2]}$-type with finite automorphism group of order 174960 - which is the biggest possible order for the automorphism group of an IHS manifold of K3$^{[2]}$-type - and that it is the Fano variety of lines of the Fermat cubic fourfold.

Keywords : Algebraic geometry, irreducible holomorphic symplectic manifolds, K3 surfaces, Kummer surfaces, group actions, automorphisms.

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"The pain of your absence is sharp and haunting, and I would give anything not to know it; anything but never knowing you at all (which would be worse)."

From "Outer Wilds"

À Nadine

Contents

Introduction

Compact complex surfaces have been classified in the 20st century in the Enriques-Kodaira classification. In the projective case we can order them in four families thanks to the *Kodaira dimension*. In fact to every projective surfaces X we can associate a rational map $\varphi_n : X \to \mathbb{P}^{p_n-1}$ where $p_n := \dim H^0(X, \left(\Omega_X^2\right)^{\otimes n})$ are the *plurigenera* of X. Then we define the *Kodaira dimension* $\kappa(X)$ to be

$$\kappa(X) := \max_{n\in\mathbb{N}} \{\dim \varphi_n(X)\}.$$

Clearly we have $\kappa(X) \leq 2$ and the four families of surfaces according to their Kodaira dimension are :

- $\kappa(X) = -\infty$: the ruled surfaces and the rational surfaces;
- $\kappa(X) = 0$: the abelian surfaces, the K3 surfaces, the Enriques surfaces and the hyperelliptic surfaces;
- $\kappa(X) = 1$: the elliptic surfaces;
- $\kappa(X) = 2$: the surfaces of general type.

In this thesis we will be interested in Irreducible Holomorphic Symplectic (or just IHS) manifolds. In particular a 2-dimensional IHS manifold is a *K3 surface*. A K3 surface X is a smooth compact complex surface such that the canonical bundle K_X is trivial and its irregularity is zero, i.e. $\dim H^1(X, \mathcal{O}_X) = 0$. Another way to see the first point is the existence of a nowhere vanishing holomorphic 2-form ω_X, unique up to scalar multiplication. In order to understand these surfaces, the group of automorphisms of

X is one of our main interest. We can think about the work of Nikulin in [41] - based on the results of Burns and Rapoport in [11] on the global Torelli theorem for K3 surfaces - where he studied the finite groups of automorphisms of a K3 surface. If G is one of these groups, we have the following exact sequence that we will encounter a lot :

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

where $G_0 := \ker \alpha$ denotes the group of the so-called *symplectic automorphisms*, that is the automorphisms acting as the identity on the 2-form ω_X.

In Chapter 1 we introduce the basic notions we use all along this thesis. The fundamental results listed in this Chapter are stated in the case of K3 surfaces but a lot of them generalized naturally in the case of IHS manifolds of K3$^{[n]}$-type. We also introduce some tools of lattices theory such as the *K3-lattice* and the *invariant lattice*. This last is closely related to the *transcendental lattice* which will be very useful to classify the special K3 surfaces we study in Chapter 3.

In Chapter 2 we are interested in *Kummer surfaces* which are special K3 surfaces obtained as the minimal resolution of the quotient of an abelian surface by an involution. A Kummer surface admits $(\mathbb{Z}/2\mathbb{Z})^4$ as group of symplectic automorphisms and this action was mainly studied by Garbagnati and Sarti in [21]. Another interesting point about these surfaces is that they contain 16 disjoint smooth rational curves. This set of curves is called a *Nikulin configuration* on the Kummer surface and a K3 surface which contains a Nikulin configuration is a Kummer surface. This result strongly reinforces the interest for these surfaces. From now let X be a K3 surface. We call a *Kummer structure* on X an abelian surface $\mathcal{A}$ such that $X \simeq \mathrm{Km}(\mathcal{A})$. Nikulin proved that there is a bijection

$$\{\text{Kummer structures}\} \overset{1:1}{\longleftrightarrow} \{\text{Nikulin configurations}\}/\mathrm{Aut}(X).$$

Moreover we assume that the Picard number of the abelian surface is $\rho(\mathcal{A}) = 1$. Then if M is a generator of $\mathrm{NS}(X)$ such that $M^2 = 2t$, we know by the results of Orlov (see [43, Example 4.16]) that the number of non-isomorphic Kummer structures on $X = \mathrm{Km}(\mathcal{A})$ is 2^ν, where ν is the number of prime divisors of $\frac{1}{2}M^2$ (which is t here). In their works, Roulleau and Sarti constructed explicitly non-isomorphic Nikulin configurations according to some values of t, but it doesn't work for all $t \in \mathbb{N}^*$. For example the first missing case is $t = 4$, and we know that there are two distinct Kummer structures on X. Thanks to the computer algebra system MAGMA [8], we constructed a new Nikulin configuration on X and so we know all the Kummer structures on X in the case $t = 4$.

In Chapter 3 we deal with K3 surfaces admitting a symplectic action of the Mathieu group M_{20}. This topic is motivated by three important results. First, in 1988 Mukai studied the groups G_0 which act symplectically on a K3 surface and he described 11 maximal such groups in [39]. We also notice that $|G_0| \leq 960$ and if $|G_0| = 960$ then it is isomorphic to the Mathieu group M_{20}. Then Xiao in 1996 gave a complete list of 81 such symplectic groups in [54]. Finally, Kondō in 1999 showed that the group $G \subset \mathrm{Aut}(X)$ has order at most 3840. An interesting question is to know the surfaces which admit such an action. All the cases where G strictly contains the group M_{20} have been studied in [9, 32, 39] as follows :

- If $|G| = 3840 = 4\,|M_{20}|$, there exists only one surface which is the Kummer surface $\mathrm{Km}\,(E_i \times E_i)$. This one was studied by Kondō in [32].
- If $|G| = 1920 = 2\,|M_{20}|$, there exists two surfaces admitting such an action of G :
  - (i) $X_{\mathrm{Mu}} \subset \mathbb{P}^3$ defined by a quartic, studied by Mukai in [39];
  - (ii) $X_{\mathrm{BH}} \subset \mathbb{P}^5$ defined by the complete intersection of three quadrics, studied by Brandhorst and Hashimoto in [9].

When $|G| = 960$, that is $G \simeq M_{20}$, we prove the following result.

Theorem 0.1. *There exists infinitely many K3 surfaces admitting a faithful and symplectic action of M_{20}.*

The surfaces we consider have their Picard number equal to 20. This is the biggest possible for a K3 surface and they are called *singular K3 surfaces*. Moreover they are classified in terms of their transcendental lattice. Following the work of Bonnafé and Sarti in [7], we describe them explicitly, studying their projective models and giving some example. The results presented in this Chapter is a joint work with Paola Comparin (see the published article in [13]).

Finally the Chapter 4 can be seen as a natural generalization of the previous Chapter. We are interested in Irreducible Holomorphic Symplectic (IHS) manifolds wihch are deformation equivalent to the Hilbert scheme of 2 points of a K3 surface, called of *K3*$^{[2]}$-*type*. In a recent work in 2019, Höhn and Mason in [25] gave a complete classification of the groups that can act symplectically on an IHS manifold of K3$^{[2]}$-type. Among them, there are 15 maximal such groups and the group with the biggest order is $\mathbb{Z}_3^4 : \mathfrak{A}_6$ - which is a notation for $(\mathbb{Z}/3\mathbb{Z})^4 \rtimes \mathfrak{A}_6$ - that is $\left|\mathbb{Z}_3^4 : \mathfrak{A}_6\right| = 29160$. We will show that if G is a group acting on X whose symplectic part G_0 is $\mathbb{Z}_3^4 : \mathfrak{A}_6$ we get that $|G| \leq 174960$. Moreover in the exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \mu_m \longrightarrow 1$$

we have that $m \in \{2, 3, 6\}$. In other words we study the IHS manifolds of K3$^{[2]}$-type X admitting an action of $G \subset \mathrm{Aut}(X)$, with symplectic part $G_0 = \mathbb{Z}_3^4 : \mathfrak{A}_6$ and a non-symplectic automorphism of order m with $m \in \{2, 3, 6\}$. We show that there are exactly 5 such IHS manifolds in the case $m = 2$. Moreover we prove the Theorem below, which states that there exists an IHS manifold of K3$^{[2]}$-type with the biggest automorphism group.

Theorem 0.2. *Let X be an IHS manifold of K3$^{[2]}$-type admitting an action of G such that*

$$1 \longrightarrow \mathbb{Z}_3^4 : \mathfrak{A}_6 \longrightarrow G \xrightarrow{\alpha} \mu_6 \longrightarrow 1$$

which would mean that G has the biggest possible order (which is 174960) among the IHS manifolds of K3$^{[2]}$-type. Moreover, the Fano variety of lines $F(\mathcal{C})$ of the Fermat cubic fourfold

$$\mathcal{C} := \left\{(x_0 : \cdots : x_5) \in \mathbb{P}^5 \,|\, x_0^3 + \cdots + x_5^3 = 0\right\}$$

is an example of IHS manifold of K3$^{[2]}$-type admitting such action of this G.

Note that we also precisely characterize such an IHS manifold X by giving the transcendental lattice T_X and the polarization L in Theorem 4.9 in Chapter 4. We will end this part by showing that each case we found occurs. This last Chapter is a joint work with Paola Comparin and Pablo Quezada Mora (see the submitted article in [14]).

CHAPTER 1

General facts

1.1 Preliminaries and notations

In this chapter we briefly recall the most important definitions and we present the general objects we use in this thesis. All along the different Chapters, we work over the field of complex numbers. Here an algebraic surface means a compact complex algebraic 2-dimensional variety. In this sense, we can already note that algebraic surfaces means the same as *projective surfaces* (cf. [3, Chapter IV, Corollary 6.6]). The surfaces that will mainly interest us are *K3 surfaces*. We recall here the definition :

Definition 1.1. *A K3 surface X is a smooth compact complex surface such that*

1. *X is simply connected;*
2. *it admits a nowhere vanishing holomorphic 2-form ω_X, unique up to scalar multiplication.*

Note that by definition the first point means that the fundamental group is trivial. Concerning the second point, it is equivalent to say that the canonical divisor K_X is trivial (cf. [3, Chapter VIII, Section 3]).

Remark 1.1. *The name K3 was given by the french mathematician André Weil in 1958 in honor of the three mathematicians Kummer, Kähler and Kodaira and of the the beautiful K2 mountain in Cachemire.*

Example 1.1. *The most common examples of projective K3 surfaces are :*

- *The smooth quartic surfaces in* $\mathbb{P}^3$*;*
- *The complete intersections of a quadric and a cubic in* $\mathbb{P}^4$*;*
- *The complete intersections of three quadrics in* $\mathbb{P}^5$*;*
- *The double cover of* $\mathbb{P}^2$ *ramified on a smooth sextic.*

Another nice example are the Kummer surfaces. We will come back later on it since these are the main topic of Chapter 2.

We call a *polarization* on a K3 surface X a primitive ample class of divisor on it. In general in our cases there will always be a polarization on X, denoted by $L \in \mathrm{Pic}(X)$. We say that X is a *projective K3 surface* if there exists an embedding of X in some projective space $\mathbb{P}^N$. We have the following results which is a direct consequence of [48, Section 4.1] :

Proposition 1.1. *If the linear system* $|L|$ *is base point free, then it defines a map* $\varphi_L : X \to \mathbb{P}^{p_a(L)}$*, where* $p_a(L) = \frac{1}{2}L^2 + 1$ *is the arithmetic genus of* L*.*

Furthermore still in [48, Section 4.1], we know that the degree of $\varphi_L(X)$ is at least $p_a(L) - 1$ and only two cases occur : either L is hyperelliptic or φ_L is birational and its image has degree $2p_a(L) - 2$. By *hyperelliptic*, we mean that the morphism φ_L induced by L is of degree 2 and its image has degree $p_a(L) - 1$. We will see later a condition to check easily if $|L|$ is hyperelliptic or not. It will be very important when we will construct some projective models for K3 surfaces, as in Chapter 3 for example.

1.2 Lattice theory

Lattices are a very important and powerful tool for the study of K3 surfaces due to the *Torelli theorem*. In particular we will define later the *K3 lattice* which is fundamental to understand the *Picard group* of X and its orthogonal complement the *transcendental lattice*, and so to understand the geometry of X.

1.2.1 Definitions and properties

Definition 1.2 ([3, Chapter I, Section 2])**.** *A lattice* $(\mathcal{L}, \langle \cdot, \cdot \rangle)$ *is a finitely generated free* $\mathbb{Z}$*-module* $\mathcal{L}$ *with an integral bilinear form* $\langle \cdot, \cdot \rangle$ *which is symmetric.*

The determinant of the matrix of $\langle \cdot, \cdot \rangle$ written in any basis of $\mathcal{L}$ is called the *discriminant* of $\mathcal{L}$ and denoted by $d(\mathcal{L})$. The lattice is said to be *non-degenerate* if $d(\mathcal{L}) \neq 0$ and *unimodular* if $d(\mathcal{L}) = \pm 1$.

In the case of euclidian lattices (i.e. when the bilinear form is symmetric) we have a quadratic form $q : x \mapsto \langle x, x \rangle$. If q only take even values, the lattice is said to be *even*. Otherwise it is called *odd*. Moreover, if for all $x \in \mathcal{L} \backslash \{0\}$ we have $q(x) > 0$ (resp. $q(x) < 0$), then we said that the lattice is *positive definite* (resp. *negative definite*).

The two following examples come from [3, Part I, Example 2.7] and will be very useful to define the *K3 lattice*.

- The *hyperbolic plane* U. It is the $\mathbb{Z}$-module $\mathbb{Z}^2$ with the bilinear form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is an unimodular and even lattice of rank 2.
- The root lattice E_8. It is the $\mathbb{Z}$-module $\mathbb{Z}^8$ where the matrix $(\langle e_i, e_j \rangle)$ of the canonical basis corresponds to the Cartan matrix of the root system E_8. Explicitly we have

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

  It is an even, unimodular and positive definite lattice. We also denote by $E_8(-1)$ the negative definite lattice obtained by changing all signs in the previous matrix.

Definition 1.3. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two lattices. An isometry between $\mathcal{L}$ and $\mathcal{L}'$ is a morphism $\phi : \mathcal{L} \to \mathcal{L}'$ such that*

$$\forall x, y \in \mathcal{L},\ \langle \phi(x), \phi(y) \rangle = \langle x, y \rangle.$$

We denote by $O(\mathcal{L})$ the group of isometries of $\mathcal{L}$ into itself.

Note that as the image $\phi(\mathcal{L})$ is a sublattice of $\mathcal{L}'$, the isometry ϕ is always injective. Then for isometries of lattices we also talk about *embeddings*.

1.2.2 Discriminant lattice

Let $(\mathcal{L}, \langle\cdot,\cdot\rangle)$ be a lattice. We define the *dual lattice* of $\mathcal{L}$ as

$$\mathcal{L}^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{L}, \mathbb{Z}) = \{v \in \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q} \,|\, \forall x \in \mathcal{L},\, \langle v, x\rangle \in \mathbb{Z}\}\,.$$

The map : $x \mapsto \langle \cdot, x\rangle$ gives a natural embedding of $\mathcal{L}$ in $\mathcal{L}^{\vee}$ and the quotient $\mathcal{L}^{\vee}/\mathcal{L}$ is called the *discriminant lattice* of $\mathcal{L}$. We denote it by $A_{\mathcal{L}}$.

Definition 1.4. *Let A be a finite abelian group. A quadratic form on A is a map*

$$q : A \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

together with a symmetric bilinear form $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ such that :

- $\forall (a,k) \in A \times \mathbb{Z},\, q(ka) = k^2 q(a)$;
- $\forall a, a' \in A,\, q(a+a') - q(a) - q(a') \equiv 2b(a,a') \mod 2\mathbb{Z}$.

If $\mathcal{L}$ is a non-degenerate even lattice, the bilinear form $\langle\cdot,\cdot\rangle$ induces a $\mathbb{Q}$-valued bilinear form b on the dual lattice $\mathcal{L}^{\vee}$ and so a quadratic form

$$q_{\mathcal{L}} : A_{\mathcal{L}} \longrightarrow \mathbb{Q}/2\mathbb{Z}.$$

This quadratic form is called *discriminant form* of $\mathcal{L}$.

Lemma 1.1 ([3, Chapter I, Lemma 2.1]). *Let $(\mathcal{L}, \langle\cdot,\cdot\rangle)$ be a non-degenerate lattice. Then*

1. $[\mathcal{L}^{\vee} : \mathcal{L}] = |d(\mathcal{L})|$.
2. *If $\mathcal{M}$ is a submodule of $\mathcal{L}$ with $\mathrm{rk}(\mathcal{M}) = \mathrm{rk}(\mathcal{L})$ then*

$$[\mathcal{L} : \mathcal{M}]^2 = \left|d(\mathcal{M})d(\mathcal{L})^{-1}\right|.$$

We say that an embedding of lattices $\mathcal{M} \hookrightarrow \mathcal{L}$ is *primitive* if the quotient $\mathcal{L}/\mathcal{M}$ is torsion-free.

Lemma 1.2 ([42, Prop. 1.6.1]). *Let $\mathcal{M} \hookrightarrow \mathcal{L}$ be a primitive embedding of non-degenerate even lattices. We suppose that $\mathcal{L}$ is unimodular. Then there is an isomorphism*

$$\mathcal{M}^{\vee}/\mathcal{M} \simeq (\mathcal{M}^{\perp})^{\vee}/\mathcal{M}^{\perp}$$

where $\mathcal{M}^{\perp}$ denotes the orthogonal complement of $\mathcal{M}$ in $\mathcal{L}$. Moreover $q_{\mathcal{M}^{\perp}} = -q_{\mathcal{M}}$.

We end this subsection with some results on the *isotropic subgroup*, which will be useful in Chapter 4. We say that a subgroup H of $A_{\mathcal{L}}$ is *isotropic* if $q_{\mathcal{L}}|_H = 0$. We also say that an even lattice $\mathcal{N}$ is an *overlattice* of $\mathcal{L}$ if there is an embedding $\mathcal{L} \hookrightarrow \mathcal{N}$ such that $\mathcal{N}/\mathcal{L}$ is a finite abelian group.

Proposition 1.2 ([42, Proposition 1.4.1])**.** *There is a natural one-to-one correspondence between the finite index even overlattices $\mathcal{N} \supset \mathcal{L}$ and the isotropic subgroups $H \subset A_{\mathcal{L}}$.*

Proof. First there is a natural chain of embeddings

$$\mathcal{L} \hookrightarrow \mathcal{N} \hookrightarrow \mathcal{N}^{\vee} \hookrightarrow \mathcal{L}^{\vee}$$

such that we can associate to $\mathcal{N}$ the following isotropic subgroup of $A_{\mathcal{L}}$:

$$H := \mathcal{N}/\mathcal{L} \subset \mathcal{N}^{\vee}/\mathcal{L} \subset \mathcal{L}^{\vee}/\mathcal{L} = A_{\mathcal{L}}.$$

Conversely, to any isotropic subgroup $H \subset A_{\mathcal{L}}$ we can associate the overlattice $\pi^{-1}(H)$ where π is the quotient map from $\mathcal{L}^{\vee}$ to $A_{\mathcal{L}}$. Let us show that $\pi^{-1}(H)$ is integral and even. We denote by $q : \mathcal{L} \to \mathbb{Z}$ the quadratic form of $\mathcal{L}$, $q^{\vee} : \mathcal{L}^{\vee} \to \mathbb{Q}$ the quadratic form which extends q on $\mathcal{L}^{\vee}$ and $q_{\mathcal{L}} : A_{\mathcal{L}} \to \mathbb{Q}/2\mathbb{Z}$ the discriminant form of $\mathcal{L}$. The following diagram is commutative :

$$\begin{tikzcd}
\mathcal{L}^{\vee} \arrow[r, "q^{\vee}"] \arrow[d, "\pi"'] & \mathbb{Q} \arrow[d, "\rho"] \\
A_{\mathcal{L}} \arrow[r, "q_{\mathcal{L}}"'] & \mathbb{Q}/2\mathbb{Z}
\end{tikzcd}$$

and so $\rho \circ q^{\vee} = q_{\mathcal{L}} \circ \pi$. Put $x \in \pi^{-1}(H)$. Since H is an isotropic subgroup of $A_{\mathcal{L}}$ we have $q_{\mathcal{L}} \circ \pi(x) = 0$ and then $\rho \circ q^{\vee}(x) = 0$. This means that $q^{\vee}(x) \in 2\mathbb{Z}$ for any $x \in \pi^{-1}(H)$. Hence $\pi^{-1}(H)$ is an even integral overlattice of $\mathcal{L}$ as expected. $\square$

Remark that this Proposition is a generalization of the point 2 of Lemma 1.1. Finally the following result is an application of [42, Section 4].

Proposition 1.3. *Let $\mathcal{N}$ be an overlattice of $\mathcal{L}$. An element $\phi \in O(\mathcal{L})$ can be extended to $\mathcal{N}$ if an only if $\overline{\phi}(H) = H$, where H is the isotropic subgroup of $A_{\mathcal{L}}$ corresponding to $\mathcal{N}$ and $\overline{\phi} \in O(A_{\mathcal{L}})$ is the image of ϕ under the projection $\pi : \mathcal{L} \to A_{\mathcal{L}}$.*

1.2.3 Néron-Severi group and K3 lattice

Remember that for any complex manifold X of dimension n, we have the *exponential sequence* of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X, $\mathcal{O}_X^*$ is the (multiplicative) sheaf of invertible elements of $\mathcal{O}_X$ and exp represents the function $z \mapsto e^{2i\pi z}$. It induces the long exponential *cohomology sequence*

$$\begin{aligned} \dots \quad & \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \\ & \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow H^2(X,\mathcal{O}_X^*) \longrightarrow \dots \end{aligned}$$

Note that the group $H^1(X,\mathcal{O}_X^*)$ is isomorphic to the *Picard group* $\mathrm{Pic}(X)$. Moreover the map c_1 between $\mathrm{Pic}(X)$ and $H^2(X,\mathbb{Z})$ is called *first Chern class.*

Definition 1.5. *The Néron-Severi group of X, denoted by $\mathrm{NS}(X)$, corresponds to the Picard group of X modulo algebraic equivalence. We also can see it as the image of $\mathrm{Pic}(X)$ by c_1 and so*

$$\mathrm{NS}(X) = H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z}).$$

Remark 1.2. *The equation displayed for $\mathrm{NS}(X)$ in the above definition is known as Lefschetz Theorem on $(1,1)$-classes which states that for any compact surface X, the image of $\mathrm{Pic}(X)$ in $H^2(X,\mathbb{C})$ is $H^{1,1}(X) \cap i^*\left(H^2(X,\mathbb{Z})\right)$, where $i^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ is the natural inclusion (see e.g. [3, Chapter IV, Theorem 2.13]).*

We come back in the case where X is a K3 surface. As we can see in [3, Part VIII, Section 3], we have $H^1(X,\mathcal{O}_X) = 0$. Thus the map c_1 is injective and $\mathrm{Pic}(X)$ is isomorphic to $\mathrm{NS}(X)$. We denote by $\rho(X)$ the rank of $\mathrm{NS}(X)$ which is called the *Picard number of X*. In this thesis, we will be interested in K3 surfaces with a large Picard number. In Chapter 2, the Kummer surfaces have a Picard number at least equal to 17 and in Chapter 3, the surfaces we will study have $\rho(X) = 20$. Since the rank of $H^{1,1}(X,\mathbb{R})$ is 20, this is the maximal possible for a K3 surface.

We also define the *transcendantal lattice* T_X of X as the orthogonal complement of $\mathrm{NS}(X)$ inside $H^2(X,\mathbb{Z})$. Note that $\mathrm{NS}(X)$ and T_X are two primitive sublattices of $H^2(X,\mathbb{Z})$. By Lemma 1.2 we have

$$\mathrm{NS}(X)^\vee/\mathrm{NS}(X) \simeq \mathrm{T}_X^\vee/\mathrm{T}_X.$$

Combining this result with Lemma 1.1 we get

$$|\mathrm{NS}(X)^\vee/\mathrm{NS}(X)| = |\mathrm{T}_X^\vee/\mathrm{T}_X| = |d(\mathrm{NS}(X))|.$$

As K3 surfaces with Picard number 20 are classified in terms of their transcendantal lattice, it will be a powerful tool for the topic of Chapter 3. Keeping the previous notations for the hyperbolic plane U and the root lattice E_8, we define the so-called *K3 lattice*.

Definition 1.6. *We call K3 lattice the following lattice :*

$$\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}\,.$$

For convenience and if there is no ambiguity we will denote it only by Λ.

Theorem 1.1 ([3, Chapter VIII, Prop. 3.3])**.** *The group $H^2(X,\mathbb{Z})$ with the intersection pairing has the structure of a lattice. It is torsion-free of rank 22, with signature $(3,19)$ and unimodular. Moreover it is isometric to the K3 lattice.*

1.2.4 Period of a K3 surface

Definition 1.7. *We call period domain of the K3 surface X the analytic subset of $\mathbb{P}^{21}$*

$$\Omega_\Lambda := \left\{\omega \in \mathbb{P}\left(\Lambda \otimes \mathbb{C}\right) \mid \omega^2 = 0 \text{ and } \langle \omega, \overline{\omega}\rangle > 0\right\}.$$

Remark 1.3. *The period domain Ω_Λ is an analytic open subset of a quadric hypersurface of $\mathbb{P}^{21}$. We will see in Chapter 4 that for IHS manifolds of K3$^{[n]}$-type, the period domain is contained in $\mathbb{P}^{22}$ since the rank of the K3$^{[n]}$-lattice Λ_n is 23.*

Since $H^2(X,\mathbb{Z})$ is isometric to Λ, for each K3 surface X there exists an isomorphism of lattices

$$\eta : H^2(X,\mathbb{Z}) \longrightarrow \Lambda$$

called a *marking* of X. A pair (X,η) is called a *marked K3 surface.* Moreover $\eta\left(\omega_X\right) \in \Omega_\Lambda$ and it is called the *period* of the marked K3 surface (X,η).

Definition 1.8. *Let $\mathfrak{M}$ be the set of isomorphism classes of marked K3 surfaces (X,η), called the moduli space of marked K3 surfaces. The period map of marked K3 surfaces is the map*

$$\mathcal{P} : \mathfrak{M} \longrightarrow \Omega_\Lambda$$

associating (X,η) to $\eta\left(\omega_X\right)$.

Finally, the following Theorem states the surjectivity of the period map.

Theorem 1.2 ([3, Chapter VIII, Section 14])**.** *For any point $\omega \in \Omega_\Lambda$, there exists a marked K3 surface (X,η) such that $\eta\left(\omega_X\right) = \omega$.*

1.3 Fundamental results about algebraic surfaces

In this Section we will recall some classical results about algebraic surfaces (see e.g. [24]). These results will be useful for the whole thesis, in particular for K3 surfaces. Let S be a smooth algebraic surface. Recall that for any coherent sheaf $\mathcal{F}$ its *Euler characteristic* is $\chi(\mathcal{F}) = \sum(-1)^i h^i(\mathcal{F})$, where $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$. In particular

$$\chi\left(S, \mathcal{O}_S\right) = 1 - q(S) + p_g(S)$$

where $q(S) = h^1\left(S, \mathcal{O}_S\right)$ is the *irregularity* of S and $p_g(S) = h^2\left(S, \mathcal{O}_S\right)$ denotes the *geometric genus* of S. With this we can write the famous Riemann-Roch Theorem (see [24, Appendix A]) :

Theorem 1.3 (Riemann-Roch)**.** *For any line bundle L on the surface S we have the formula*

$$\chi(S, L) = \chi\left(S, \mathcal{O}_S\right) + \frac{1}{2} L \cdot \left(L - K_S\right)$$

where K_S denotes the canonical divisor on S.

For X a K3 surface we have $\chi\left(X, \mathcal{O}_X\right) = 2$ and by definition K_X is trivial, so

$$\chi(X, L) = 2 + \frac{1}{2} L^2.$$

Another important famous result is the *Serre duality Theorem* (see [24, Chapter III, Corollary 7.7]) which is used to prove the previous Theorem. In particular, we can state this result in the case of algebraic surfaces.

Theorem 1.4 (Serre duality)**.** *Let X be a smooth variety of dimension n over $\mathbb{C}$. For any locally free sheaf $\mathcal{F}$ on X and any integer i there is a natural isomorphism :*

$$H^i(X, \mathcal{F}) \simeq H^{n-i}\left(X, \mathcal{F}^\vee \otimes \Omega_X^n\right)^*.$$

In particular for surfaces we get the following result where L is a line bundle on S :

$$H^i(S, L) \simeq H^{2-i}\left(S, K_S \otimes L^*\right)^*.$$

From Riemann-Roch theorem and Serre duality applied to the case of K3 surfaces we deduce the following one :

Corollary 1.1. *For any divisor D on a K3 surface X, if $D^2 \geq -2$ then either D or $-D$ is effective.*

Proof. Let D be a divisor on X such that $D^2 \geq -2$. For convenience, we denote by $h^i(D)$ the dimension of $H^i\left(X, \mathcal{O}_X(D)\right)$. By Riemann-Roch Theorem we have

$$h^0(D) - h^1(D) + h^2(D) = 2 + \frac{1}{2}D^2$$

and using Serre duality we get $h^2(D) = h^0(-D)$, so we obtain

$$h^0(D) + h^0(-D) = 2 + \frac{1}{2}D^2 + h^1(D).$$

Since we have $D^2 \geq -2$, then the sum $h^0(D) + h^0(-D)$ is strictly positive. By definition, $h^0(D) > 0$ if and only if D is an effective divisor. Hence either D or $-D$ is effective. $\square$

Now we consider C an irreducible curve on the surface S. The *artihmetic genus* of C is by definition $p_a(C) := 1 - \chi\left(\mathcal{O}_C\right)$.

Theorem 1.5 (Genus formula)**.** *For any irreducible curve C on S we have*

$$p_a(C) = 1 + \frac{1}{2}C \cdot \left(C + K_S\right).$$

In the case of K3 surfaces we get a more simple formula :

$$p_a(C) = 1 + \frac{1}{2}C^2.$$

Moreover, as we mentioned in the previous proof, this formula implies that for every irreducible curve C on a K3 surface we have $C^2 \geq -2$.

The following result is a criterion of ampleness (cf. [24, Chapter V, Theorem 1.10]).

Theorem 1.6 (Nakai-Moishezon criterion)**.** *Let D be a Cartier divisor on a surface S. Then D is ample if and only if it satisfies the following two conditions :*

- $D^2 > 0$*;*
- *for all irreducible curves C we have $D \cdot C > 0$.*

Finally we end this part with the *Hodge decomposition* which gives a direct decomposition of $H^k(X, \mathbb{C})$ into subspaces $H^{p,q}(X)$ with $p, q \geq 0$.

Theorem 1.7 ([3, Chapter I, Corollary 13.4])**.** *For any Kähler compact manifold X there is a direct sum decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

In particular if X is a K3 surface we get the following decomposition of $H^2(X,\mathbb{C})$:

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

1.4 Automorphisms of K3 surfaces

Remember that for any K3 surfaces X, the *group of automorphisms* of X is

$$\mathrm{Aut}(X) := \{f : X \longrightarrow X \text{ s.t. } f \text{ is biholomorphic}\}.$$

1.4.1 Torelli theorem for K3 surfaces

An important point in the study of K3 surfaces is to understand its group of automorphisms. Nikulin produced several results on this subject studying the finite automorphisms groups of K3 surfaces in [41] and Kummer surfaces in [40]. These works are mainly based on the results of Shapiro and Shafarevich about the *Torelli theorem* for algebraic K3 surfaces in [29]. In the particular case of Kummer surfaces, a recent work is the one of Roulleau and Sarti in [46, Section 3.1] where they worked about infinite order automorphisms on Kummer surfaces.

Definition 1.9. *Let X and X' be two surfaces. We say that $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is an Hodge isometry if it is an isomorphism of $\mathbb{Z}$-modules which preserves the cup product (i.e. it is an isometry) and its $\mathbb{C}$-linear extension $H^2(X,\mathbb{C}) \to H^2(X',\mathbb{C})$ preserves the Hodge-decomposition. Moreover we say that ϕ is effective if it sends the cohomology class of any effective divisor on X to the class of an effective divisor on X'.*

Then we can state the *Torelli theorem* for K3 surfaces (cf. [3, Chapter VIII, Theorem 11.1 & Corollary 11.4]).

Theorem 1.8 (Torelli)**.** *Let X and X' be two K3 surfaces and assume that there exists an effective Hodge isometry $\phi : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$. Then $\phi = f^*$ with $f : X \to X'$ a biholomorphic map. Moreover f is the unique isomorphism satisfying $f^* = \phi$.*

Note that we also have the weak version of this theorem given in [3, Chapter VIII, Corollary 11.2]. If we remove the hypothesis "effective" on the Hodge isometry ϕ, then we get that the K3 surfaces X and X' are "only" isomorphic.

Thanks to the definitions given in Section 1.2.4, we can also formulate the global Torelli theorem for K3 surfaces in terms of moduli space as above (cf. [28, Chapter 7]).

Theorem 1.9 (Global Torelli theorem for marked K3 surfaces)**.** *The moduli space $\mathfrak{M}$ of marked K3 surfaces has two connected components interchanged by $(S, \eta) \mapsto (S, -\eta)$. Moreover the period map*

$$\begin{array}{rccc} \mathcal{P}: & \mathfrak{M} & \longrightarrow & \Omega_\Lambda \\ & (S, \eta) & \longmapsto & \eta\left(H^{2,0}(S)\right) \end{array}$$

is generically injective on each of the two components.

Remark 1.4. *Here "generically injective" means injective on the complement of a countable union of proper analytically closed subsets.*

1.4.2 Symplectic automorphisms

We define now some particular automorphisms : the *symplectic automorphisms*. They will be used in Chapter 3 as we will be interested in some symplectic action on K3 surfaces. Let X be a K3 surface and consider G a finite subgroup of order m of $\mathrm{Aut}(X)$. This group induces an action on the vector space $H^{2,0}(X) = \mathbb{C}\omega_X$ through

$$\forall g \in G,\ g^*\omega_X = \alpha(g)\omega_X$$

where $\alpha : G \to \mathbb{C}^\times$ is a morphism sending $g \in G$ to $\alpha(g) \in \mathbb{C}^\times$. Thus we get the following exact sequence :

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mathbb{C}^\times$$

where $G_0 := \ker(\alpha)$ and G/G_0 is isomorphic to $\mathrm{Im}(\alpha) \subset \mathbb{C}^\times$ which is a finite subgroup.

Definition 1.10. *Let $f \in \mathrm{Aut}(X)$ be an automorphism of finte order n. Then f is called*

1. *symplectic if $f^*\omega_X = \omega_X$.*
2. *non symplectic if $f^*\omega_X = \alpha(g)\omega_X$ and $\alpha(g) \neq 1$.*
3. *purely non symplectic if $f^*\omega_X = \zeta_n\omega_X$, with $\zeta_n \in \mu_n$ a primitive n-root of the unity.*

Observe that G_0 corresponds to the group of *symplectic automorphisms*, that is the automorphisms acting as the identity on the 2-form ω_X.

Remark 1.5. *If the group G acts purely non symplectically on X then $G_0 = \{\mathrm{Id}\}$ and $G \simeq \mathrm{Im}(\alpha)$. As $\mathrm{Im}(\alpha)$ is a finite subgroup of $\mathbb{C}^\times$, it is cyclic. Then $G \simeq \mu_m$ where m is the order of G and μ_m the group generated by the primitive m-roots of unity. Moreover the previous exact sequence becomes*

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1 \ .$$

Finally note that all the purely non symplectic automorphisms acting on a K3 surface are classified. Their order is at most 66 *(cf. [41, Corollary 3.2]).*

In regards of the above Remark, we can be more precise on the possible values for the integer m. The following Theorem due to Nikulin gives a bound thanks to the *Euler totient function* φ (cf. [41, Theorem 3.1]).

Theorem 1.10. *Let X be a K3 surface and $G \subset \mathrm{Aut}(X)$. Then :*

- *an element $g \in G$ acts trivially on T_X if and only if $\alpha(g) = 1$ (that is $g \in G_0$);*
- *the representation of μ_m in T_X is isomorphic in $\mathrm{T}_X \otimes \mathbb{Q}$ to the direct sum of irreducible representations over $\mathbb{Q}$ of the cyclic group of order m having maximal rank. Since this maximal rank is $\varphi(m)$, it follows that $\varphi(m) \,|\, \mathrm{rk}\, \mathrm{T}_X$.*

1.4.3 Invariant lattice

As before we consider $G \subset \mathrm{Aut}(X)$ a finite subgroup.

Definition 1.11. *The invariant lattice under the action of G is the following sublattice of $H^2(X, \mathbb{Z})$:*

$$H^2(X, \mathbb{Z})^G := \left\{ x \in H^2(X, \mathbb{Z}) \,|\, \forall g \in G,\ g^* x = x \right\} .$$

Let G_0 be the subgroup of symplectic automorphisms of X. As we will see in Chapter 3, in the case where $G = G_0$ acts symplectically on X, the invariant lattice will be helpful especially to know explicitly the transcendantal lattice T_X. In fact the following result coming from [41, Lemma 4.2] gives a link between these two lattices.

Proposition 1.4. *Assume that $G = G_0$ acts symplectically on X. Then we have*

$$\mathrm{T}_X \subset H^2(X, \mathbb{Z})^{G_0}.$$

Proof. Consider $x \in \mathrm{T}_X$ and $g \in G_0$. Recall that T_X is a sublattice of $H^2(X, \mathbb{Z})$, so it is also equipped with the cup product denoted here by $\langle \cdot, \cdot \rangle$. As G_0 acts symplectically on X we have

$$\langle \omega_X, x \rangle = \langle g^* \omega_X, g^* x \rangle = \langle \omega_X, g^* x \rangle$$

so that $\langle \omega_X, x - g^* x \rangle = 0$. Thus $x - g^* x \in (\mathbb{C}\omega_X)^{\perp} \cap H^2(X, \mathbb{Z})$ which corresponds to $\mathrm{NS}(X)$. In fact, if we take $\delta \in (\mathbb{C}\omega_X)^{\perp} \cap H^2(X, \mathbb{Z})$, then $\delta \cdot \omega_X = 0$. As this equality is invariant by complex conjugation and $\delta \in H^2(X,\mathbb{Z})$, we get $\delta \cdot \overline{\omega_X} = 0$ and so $\delta \in \left(H^{2,0}(X) \oplus H^{0,2}(X)\right)^{\perp}$. Then using Hodge decomposition we have $\delta \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ which is by definition $\mathrm{NS}(X)$. In the same way if $\delta \in \mathrm{NS}(X)$, then $\delta \in (\mathbb{C}\omega_X)^{\perp} \cap H^2(X,\mathbb{Z})$ and so

$$(\mathbb{C}\omega_X)^{\perp} \cap H^2(X,\mathbb{Z}) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}) = \mathrm{NS}(X).$$

Moreover $x \in \mathrm{T}_X$ and $g^*x \in \mathrm{T}_X$ because g^* is a Hodge isometry (so it preserves the lattice T_X). So $x - g^*x \in \mathrm{NS}(X) \cap \mathrm{T}_X$. This last intersection is equal to 0 by [41, Section 3.2] which implies that $g^*x = x$. Hence T_X is pointwise invariant under the action of G_0. $\square$

1.5 Birational transformations

Recall that a *birational map* $X \dashrightarrow Y$ between two (irreducible as always) varieties X and Y is an invertible *rational map*. If such a map exists, X and Y are said to be *birationally equivalent*. The notable birational invariants are the plurigenera, the Kodaira dimension and the fundamental group for smooth complex projective varieties (which is the case for K3 surfaces). In this Section, we present some example of birational transformations which will be useful all along this thesis.

1.5.1 Blowing-up of a surface

In order to introduce easily the process of blowing-up, we will begin with the case of $\mathbb{C}^n$. The *blowing-up* of $\mathbb{C}^n$ at the origin 0 is the following closed subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$:

$$\mathrm{Bl}_0\left(\mathbb{C}^n\right) := \left\{\left(\left(x_1,\ldots,x_n\right),\left(y_1,\ldots,y_n\right)\right) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \,|\, \forall i,j,\, x_i y_j = x_j y_i\right\}.$$

With this subset comes a birational map $\pi_0 : \mathrm{Bl}_0\left(\mathbb{C}^n\right) \to \mathbb{C}^n$. Let us remark that $\pi_0^{-1}(0) = \mathbb{P}^{n-1}$ and this particular divisor is called the *exceptional divisor* of the blowing-up. Following [24, Chapter I, Section 4] we can write the general definition.

Definition 1.12. *Let Y be an affine subvariety of $\mathbb{C}^n$ and consider $p \in \mathbb{C}^n$. We denote by $\mathrm{Bl}_p\left(\mathbb{C}^n\right)$ the blowing-up of $\mathbb{C}^n$ at the point p obtained from $\mathrm{Bl}_0\left(\mathbb{C}^n\right)$ by translating the origin. We consider the associated birational map $\pi_p : \mathrm{Bl}_p\left(\mathbb{C}^n\right) \to \mathbb{C}^n$ so that the blowing-up of Y at the point p is*

$$\mathrm{Bl}_p(Y) := \overline{\pi_p^{-1}\left(Y \backslash \{p\}\right)}.$$

The blowing-up can be naturally extended to algebraic varieties and so to X. This provides us one of the main tool in the resolution of singularities of an algebraic variety.

We will use it in Chapter 2 in order to define *Kummer surfaces* and in Chapter 4 to give an example of construction of the *Hilbert square of a K3 surface.*

1.5.2 Examples of birational embeddings

The first embedding that we define is the *Segre map* (cf. [23, Example 2.11]) :

$$\begin{array}{cccc} \psi: & \mathbb{P}^m \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^{mn+m+n} \\ & (x:y) & \longmapsto & (x_0y_0 : x_0y_1 : \cdots : x_0y_n : x_1y_0 : \cdots : x_my_n) \end{array}.$$

In particular, if $m = n = 1$ the Segre embedding gives a map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ sending $((x:y),(z:t))$ to $(xz : xt : yz : yt)$. The image of this map is the quadric surface $\{XT - YZ = 0\} \subset \mathbb{P}^3$.

The second embedding which will be useful in the next Chapters is the Veronese one. We define the *Veronese embedding* (cf. [23, Example 2.4]) ν_d^n of degree d as

$$\begin{array}{cccc} \nu_d^n: & \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ & (x_0 : \cdots : x_n) & \longmapsto & \left(x_0^d : x_1^d : \cdots : x_n^d : x_0^{d-1}x_1 : x_0^{d-1}x_2 : \ldots\right) \end{array}.$$

Since we use monomials of degree d in n variables, we have $N = \binom{n+d}{d} - 1$. The image of the Veronese map $\nu_d^n(\mathbb{P}^n) \subset \mathbb{P}^N$ is an algebraic variety called *Veronese variety*.

By [23, Example 2.6] we also have a determinantal representation of this variety. Let $(y_{i,j})_{0 \leq i,j \leq n}$ be an element of $\mathbb{P}^N$ and consider $\mathcal{V}$ the $(n+1) \times (n+1)$ symmetric matrix with coefficient $y_{i,j}$ in position (i,j), for $i \leq j$. Then the image of the Veronese embedding ν_2^n of degree 2 corresponds to the zero loci of the 2×2 minors of $\mathcal{V}$. For example, if we take $n = 2$ then $\nu_2^2 : \mathbb{P}^2 \to \mathbb{P}^5$ and the image $\nu_2^2\left(\mathbb{P}^2\right)$ is defined by the intersection of 9 quadrics in $\mathbb{P}^5$. These quadrics corresponds to the zero loci of the 2×2 minors of the matrix

$$\mathcal{V} = \begin{pmatrix} y_0 & y_3 & y_4 \\ y_3 & y_1 & y_5 \\ y_4 & y_5 & y_2 \end{pmatrix}$$

with $(y_0 : \cdots : y_5) \in \mathbb{P}^5$.

CHAPTER 2

Nikulin configurations on Kummer surfaces

2.1 Generalities about Kummer surfaces

In this first part we introduce the main object of the Chapter : the *Kummer surfaces*. This is one well known example of K3 surfaces with big Picard number. We will recall briefly the construction and see some properties of them. In particular we will be interested in some results about the Néron-Severi group of these surfaces.

2.1.1 Construction and Kummer lattice

In order to recall the construction of Kummer surfaces, we follow [12, Exposé 4, Prop 5.3]. We will call *abelian variety* a complex torus together with a polarization. In particular, an abelian surface $\mathcal{A}$ is the quotient of a 2-dimensional $\mathbb{C}$-vector space V (isomorphic to $\mathbb{C}^2$) by a rank 4 lattice Λ. Consider the involution $\iota : \mathcal{A} \to \mathcal{A}$ which sends x to its opposite $-x$. It is called *Kummer involution*. Remark that the fixed locus of ι corresponds to the 2-torsion points in Λ and the number of such fixed points is exactly 16.

Example 2.1. *We will illustrate it with a classical example. Let E_τ be the standard notation for the elliptic curve $\mathbb{C}/\left(\mathbb{Z} \oplus \tau\mathbb{Z}\right)$, where $\tau \in \mathbb{C}$, and assume that $\mathcal{A} = E_\tau \times E_\tau$. There are 4 points of 2-torsion in E_τ which are :*

$$p_1 = 0,\, p_2 = \frac{1}{2},\, p_3 = \frac{\tau}{2},\, p_4 = \frac{1+\tau}{2}.$$

Then for all $1 \leq i, j \leq 4$, the point (p_i, p_j) is a 2-torsion point in $\mathcal{A}$ and there are 16 of them.

Consider the quotient $\mathcal{A}/\langle\iota\rangle$ which is an algebraic surface. This surface contains 16 nodes (i.e. singular points of type A_1) coming from the 2-torsion points in $\mathcal{A}$. Blowing-up $\mathcal{A}$ along these 16 points, we obtain a smooth surface $\widetilde{\mathcal{A}}$ and an involution $\widetilde{\iota}$ (defined locally around 0) which extends ι on $\widetilde{\mathcal{A}}$:

$$\begin{array}{cccc} \widetilde{\iota}: & \widetilde{\mathcal{A}} & \longrightarrow & \widetilde{\mathcal{A}} \\ & (x,u) & \longmapsto & (-x,u) \end{array}$$

where the second coordinate $u \in \mathbb{P}^1$ comes from the blowing-up (see Section 1.5.1). Moreover we get 16 fixed curves on $\widetilde{\mathcal{A}}$ which correspond to the exceptional curves of the blowing-up. Finally the quotient $\widetilde{\mathcal{A}}/\langle\widetilde{\iota}\rangle$ is a smooth algebraic surface isomorphic to $\widetilde{\mathcal{A}/\langle\iota\rangle}$. We are now able to define the *Kummer surface* of $\mathcal{A}$.

Definition 2.1. *Let $\mathcal{A}$ be an abelian surface, ι be the involution defined as above and $\sigma : X \to \mathcal{A}/\langle\iota\rangle$ be the minimal resolution of the 16 singular points of $\mathcal{A}/\langle\iota\rangle$. Then X is called the Kummer surface of $\mathcal{A}$, denoted by $\mathrm{Km}(\mathcal{A})$.*

Remark 2.1. *It is possible to define a Kummer surface only with a 2-dimensional complex torus instead of an abelian surface as it is done in the introduction of [40]. However we will see in the next Sections that we need a polarization L on $\mathrm{Km}(\mathcal{A})$ coming from the one on $\mathcal{A}$.*

By construction the Kummer surface $\mathrm{Km}(\mathcal{A})$ contains 16 disjoint smooth rational curves $C_1, \ldots, C_{16}$ which are the preimages of the singular points of $\mathcal{A}/\langle\iota\rangle$. We summarise this *Kummer process* in the following diagram :

$$\begin{tikzcd} \mathcal{A} \arrow[loop left, "\iota"] \arrow[d] & \widetilde{\mathcal{A}} \arrow[l] \arrow[loop right, "\widetilde{\iota}"] \arrow[d] \\ \mathcal{A}/\langle\iota\rangle & \mathrm{Km}(\mathcal{A}) \arrow[l, "\sigma"'] \end{tikzcd}$$

Finally we define the *Kummer lattice* thanks to the results in [40].

Theorem 2.1. *Let X be a Kummer surface. There exists an even negative definite lattice $\mathcal{K}$ of rank 16 such that :*

1. *There exists a unique (up to automorphism) primitive embedding of $\mathcal{K}$ into the K3 lattice Λ_{K3}. Its orthogonal complement $\mathcal{K}^\perp \cap \Lambda_{K3}$ is isomorphic to $U(2)^{\oplus 3}$.*

2. *The minimal primitive sublattice of $H^2(X, \mathbb{Z})$ containing the classes of the 16 curves $C_1, \ldots, C_{16}$ is isomorphic to $\mathcal{K}$.*

3. *A K3 surface S is Kummer if and only if there is a primitive embedding of $\mathcal{K}$ into the Néron-Severi group* NS(S).

The lattice $\mathcal{K}$ is called Kummer lattice.

Denoting by $\mathcal{A}[2]$ the set of 2-torsion points in $\mathcal{A}$ we have that $\mathcal{A}[2] \simeq \mathbb{Z}_2^4$, where $\mathbb{Z}_2^4$ is a notation for $(\mathbb{Z}/2\mathbb{Z})^4$. Then we can denote the 16 rational curves on $\mathrm{Km}(\mathcal{A})$ by C_a where $a := (a_1, a_2, a_3, a_4) \in \mathbb{Z}_2^4$. The following Proposition gives a set of generators of the Kummer lattice $\mathcal{K}$.

Proposition 2.1 ([21, Remark 2.3.3])**.** *For $i \in \{1, 2, 3, 4\}$ we consider*

$$W_i := \left\{(a_1, a_2, a_3, a_4) \in \mathbb{Z}_2^4 \,|\, a_i = 0\right\}.$$

A set of generators (over $\mathbb{Z}$) of the Kummer lattice is given by the classes : $\frac{1}{2}\sum\limits_{p\in\mathbb{Z}_2^4} C_p$, $\frac{1}{2}\sum\limits_{p\in W_1} C_p$, $\frac{1}{2}\sum\limits_{p\in W_2} C_p$, $\frac{1}{2}\sum\limits_{p\in W_3} C_p$, $\frac{1}{2}\sum\limits_{p\in W_4} C_p$, $C_{0,0,0,0}$, $C_{1,0,0,0}$, $C_{0,1,0,0}$, $C_{0,0,1,0}$, $C_{0,0,0,1}$, $C_{1,1,0,0}$, $C_{1,0,1,0}$, $C_{1,0,0,1}$, $C_{0,1,1,0}$, $C_{0,1,0,1}$ *and* $C_{0,0,1,1}$.

2.1.2 Néron-Severi group of a Kummer surface

From now we denote by X the Kummer surface $\mathrm{Km}(\mathcal{A})$ associated to an abelian surface $\mathcal{A}$. It is well-known that X is a K3 surface and we have just seen that X contains 16 disjoint smooth rational curves $C_1, \ldots, C_{16}$. By the genus formula (cf. Theorem 1.5), as the canonical divisor is trivial, these curves are (-2)-curves. The following result is due to Nikulin.

Theorem 2.2 ([40, Theorem 1])**.** *Let X be a K3 surface containing 16 nonsingular rational curves $C_1, \ldots, C_{16}$ which do not intersect one another. Then there exists a unique (up to isomorphism) complex torus T such that X and $C_1, \ldots, C_{16}$ are obtained from T by the above Kummer process. In particular, X is a Kummer surface.*

Assume that $\rho(\mathcal{A}) = 1$. We denote by M the generator of NS($\mathcal{A}$) (that is the polarization on $\mathcal{A}$) and we suppose that $M^2 = 2t$, with t a positive integer. In addition to the 16 (-2)-curves we obtain a polarization $L \in \mathrm{NS}(X)$ coming from the polarization M (see [38, Proposition 3.2]) and such that $L^2 = 4t$. In order to introduce the following Proposition, recall that for an abelian surface $\mathcal{A}$ we have $H^2(\mathcal{A}, \mathbb{Z}) \simeq U^{\oplus 3}$ (see e.g. [38, Theorem-Definition 1.5]) and the transcendental lattice $\mathrm{T}_{\mathcal{A}}$ of $\mathcal{A}$ is the orthogonal complement of NS($\mathcal{A}$) in $H^2(\mathcal{A}, \mathbb{Z})$.

Proposition 2.2 ([21, Proposition 2.6])**.** *Let* $\mathrm{Km}(\mathcal{A})$ *be the Kummer surface associated to the abelian surface $\mathcal{A}$ and $\mathcal{K}$ the Kummer lattice of* $\mathrm{Km}(\mathcal{A})$*. Then the Picard number* *of* $\mathrm{Km}(\mathcal{A})$ *is* $\rho(\mathrm{Km}(\mathcal{A})) = \rho(\mathcal{A}) + 16$*, in particular* $\rho(\mathrm{Km}(\mathcal{A})) \geq 17$*. Moreover the transcendental lattice of* $\mathrm{Km}(\mathcal{A})$ *is* $\mathrm{T}_{\mathrm{Km}(\mathcal{A})} \simeq \mathrm{T}_{\mathcal{A}}(2)$ *and the Néron-Severi group* $\mathrm{NS}(\mathrm{Km}(\mathcal{A}))$ *is an overlattice of* $\mathrm{NS}(\mathcal{A})(2) \oplus \mathcal{K}$ *such that*

$$[\mathrm{NS}(\mathrm{Km}(\mathcal{A})) : \mathrm{NS}(\mathcal{A})(2) \oplus \mathcal{K}] = 2^{\rho(\mathcal{A})}.$$

This Proposition implies clearly that in our case the Néron-Severi group of $X = \mathrm{Km}(\mathcal{A})$ satisfies

$$\mathbb{Z}L \oplus \mathcal{K} \subset \mathrm{NS}(X).$$

Moreover, since $\rho(\mathcal{A}) = 1$ we have $\rho(X) = 17$ and $\mathrm{NS}(X)$ is an overlattice of index 2 of $\mathbb{Z}L \oplus \mathcal{K}$. The following Theorem allows us to describe $\mathrm{NS}(X)$ precisely.

Theorem 2.3 ([21, Theorem 2.7])**.** *Let* $\mathrm{Km}(\mathcal{A})$ *be a Kummer surface with Picard number 17 and let* L *be a divisor generating* $\mathcal{K}^{\perp} \subset \mathrm{NS}(\mathrm{Km}(\mathcal{A}))$ *such that* $L^2 > 0$*. Let* d *be a positive integer such that* $L^2 = 4d$ *and denote by* $\mathcal{K}_{4d} := \mathbb{Z}L \oplus \mathcal{K}$*. Then* $\mathrm{NS}(\mathrm{Km}(\mathcal{A})) = \mathcal{K}'_{4d}$ *where* $\mathcal{K}'_{4d}$ *is generated by* $\mathcal{K}_{4d}$ *and by a class* $\frac{L+v_{4d}}{2}$ *with :*

- $v_{4d} \in \mathcal{K}$ *and* $\frac{v_{4d}}{2} \in \mathcal{K}^{\vee} \backslash \mathcal{K}$ *(in particular* $v_{4d} \cdot C_i \in 2\mathbb{Z}$*);*
- $L^2 \equiv -v_{4d}^2 \mod 8$ *(in particular* $v_{4d}^2 \in 4\mathbb{Z}$*).*

The lattice $\mathcal{K}'_{4d}$ *is the unique even lattice (up to isometry) such that* $[\mathcal{K}'_{4d} : \mathcal{K}_{4d}] = 2$ *and* $\mathcal{K}$ *is a primitive sublattice of* $\mathcal{K}'_{4d}$*. Hence one can assume the following :*

- *If* $L^2 = 0 \mod 8$ *then*

$$v_{4d} = C_{0,0,0,0} + C_{1,0,0,0} + C_{0,1,0,0} + C_{1,1,0,0}.$$

- *If* $L^2 = 4 \mod 8$ *then*

$$v_{4d} = C_{1,0,0,0} + C_{0,1,0,0} + C_{1,1,0,0} + C_{0,0,0,1} + C_{0,0,1,0} + C_{0,0,1,1}.$$

As a consequence, we have the following Lemma which gives an explicit description of the Néron-Severi group $\mathrm{NS}(X)$ for Kummer surfaces with Picard number 17 (see [21, Remarks 2.3 & 2.10]).

Lemma 2.1. *An element* Γ *in* $\mathrm{NS}(X)$ *can be written as* $\Gamma = \alpha L - \sum_{i=1}^{16} \beta_i C_i$*, with* $\alpha, \beta_1, \ldots, \beta_{16} \in \frac{1}{2}\mathbb{Z}$*. Moreover :*

- *If* α *or one of the* β_i *is in* $\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$*, then at least 4 of the* β_j *are in* $\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$*.*
- *If moreover* $\alpha \in \mathbb{Z}$*, then at least 8 of the* β_j *are in* $\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$*.*

2.2 Nikulin configurations and Kummer structure

In this part we introduce two useful tools for the study of Kummer surfaces : *Nikulin configurations* and *Kummer structure*. We start with the definition of *Nikulin configuration.*

Definition 2.2. *We call Nikulin configuration the data of 16 disjoint smooth rational curves on a K3 surface.*

It is obvious that every Kummer surface contains a Nikulin configuration. Fixing an abelian surface $\mathcal{A}_0$, the one obtained by the Kummer process will be called *trivial Nikulin configuration* of $\mathrm{Km}\left(\mathcal{A}_0\right)$ and denoted usually by $\mathcal{C} := \{C_1, \ldots, C_{16}\}$. The next definition concerns the *Kummer structure.*

Definition 2.3. *Let X be a K3 surface. We call a Kummer structure on X an abelian surface $\mathcal{A}$ (up to isomorphism) such that $X \simeq \mathrm{Km}(\mathcal{A})$.*

The following result is well known to experts (see e.g. [26]).

Proposition 2.3. *Let X be a K3 surface. The Kummer structures on X are in one-to-one correspondence with the orbits of Nikulin configurations under the automorphism group of X :*

$$\{\text{Kummer structures}\} \overset{1:1}{\longleftrightarrow} \{\text{Nikulin configurations}\} / \mathrm{Aut}(X).$$

The proof given below can be found in [46, Proof of Proposition 21].

Proof. Let $\varphi : X \to X$ be an automorphism sending a Nikulin configuration $\mathcal{C}$ to another Nikulin configuration $\mathcal{C}'$. We denote by $\mathcal{A}$ and $\mathcal{A}'$ the abelian surfaces such that $\mathcal{C}$ (resp. $\mathcal{C}'$) is the configuration associated to $X = \mathrm{Km}(\mathcal{A})$ (resp. $X = \mathrm{Km}\left(\mathcal{A}'\right)$). First let us prove that $\mathcal{A}$ is isomorphic to $\mathcal{A}'$. Consider the blowing-up $\widetilde{\mathcal{A}} \to \mathcal{A}$ (resp. $\widetilde{\mathcal{A}'} \to \mathcal{A}'$) at the sixteen 2-torsion points of $\mathcal{A}$ (resp. $\mathcal{A}'$). We get a natural map

$$\widetilde{\mathcal{A}} \longrightarrow \mathrm{Km}(\mathcal{A}) = X \xrightarrow{\varphi} X$$

which is a double cover of X branched over $\mathcal{C}'$ and ramified over the exceptional locus of the blowing-up of $\mathcal{A}$. By the results of Nikulin, there exists a unique (up to isomorphism) double cover $\widetilde{A} \to X$ branched over $\mathcal{C}$ (cf. [40, Theorem 1]). Then $\widetilde{\mathcal{A}}$ is isomorphic to $\widetilde{\mathcal{A}'}$ and so $\mathcal{A} \simeq \mathcal{A}'$. Conversely, suppose that there exists an isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$. It induces an isomorphism $\widetilde{\phi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}'}$ that induces an isomorphism

$$X = \mathrm{Km}(\mathcal{A}) \longrightarrow \mathrm{Km}\left(\mathcal{A}'\right) = X$$

which sends the Nikulin configuration $\mathcal{C}$ associated to $\mathcal{A}$ on the Nikulin configuration $\mathcal{C}'$ associated to $\mathcal{A}'$. $\square$

A first question arising is : "*given a Kummer surface X, is it possible to have more than one Kummer structure on it ?*" If we assume $\rho(X) = 17$ (which is our case here), we can answer positively thanks to the results of Orlov in [43, Example 4.16]. Moreover we know that the number of such structures is exactly 2^{ν} where ν is the number of prime divisor of $\frac{1}{2}M^2$, that is t in our case. Later, Roulleau and Sarti gave a geometrical and explicit construction for some of these surfaces in [46, 47]. The aim of this Chapter is to complete this work, looking for the missing cases.

2.3 New Nikulin configurations

2.3.1 Pell-Fermat equation

Let $X := \mathrm{Km}(\mathcal{A})$ be a Kummer surface and as before we denote by $\mathcal{C}$ the natural Nikulin configuration on X. Following the results of [47], an important preliminary to construct a new Nikulin configuration is the *Pell-Fermat equation.* For $t \in \mathbb{N}^*$ this is the following equation :

$$\alpha^2 - 2t\beta^2 = 1. \tag{2.1}$$

There is a non trivial solution $(\alpha, \beta) \in \mathbb{Z}^2$ if and only if $2t$ is not a square. Then there exists a *fundamental solution* (α_0, β_0) such that for every solution (α, β), there exists $k \in \mathbb{Z}$ such that

$$\alpha + \beta\sqrt{2t} = \pm\left(\alpha_0 + \beta_0\sqrt{2t}\right)^k.$$

Remark that the equation (2.1) implies that α is always an odd integer. The Pell-Fermat equation naturally appears through the construction of another Nikulin configuration (see [47, Section 2.1.3]). In fact let $a, b, \alpha, \beta \in \mathbb{N}^*$ such that

$$\left\{\begin{array}{lcl} C_1' & = & \beta L - \alpha C_1 \\ L' & = & bL - aC_1 \end{array}\right. .$$

Thus C_1' is a (-2)-class and the three conditions $(C_1')^2 = -2$, $L' \cdot C_1' = 0$ and $(L')^2 = L^2 = 4t$ give the following equations :

$$\left\{\begin{array}{ccc} \alpha^2 - 2t\beta^2 & = & 1 \\ 2tb\beta & = & a\alpha \\ 2t\left(b^2-1\right) & = & a^2 \end{array}\right. .$$

Note that the first equation is a Pell-Fermat equation. The following result comes from [47, Proposition 4] and gives a condition to get a solution for these equations.

Proposition 2.4. *There are solutions to the three previous equations if and only if $2t$ is not a square. In this case, if (α, β) is a solution of the first equation, then*

$$(a, b) = (2t\beta, \alpha).$$

Moreover if (α_0, β_0) denotes the fundamental solution of the Pell-Fermat equation (2.1) we have the following result (cf. [47, Lemma 6]).

Lemma 2.2.

- *If $t \neq 0 \mod 4$, then β_0 is even.*
- *There is an infinite number of integers τ such that the fundamental solution (α_0, β_0) of $\alpha^2 - 8\tau^2\beta^2 = 1$ has odd β_0.*
- *There is an infinite number of integers τ such that the fundamental solution (α_0, β_0) of $\alpha^2 - 8\tau^2\beta^2 = 1$ has even β_0.*

The case where β_0 is even has been studied in detail in [47, Section 2.4] thanks to the previous Lemma (we will give more details later) and we know that it covers at least $\frac{3}{4}$ of the possibilities. However the construction of L' and C_1' given at the beggining does not provide a new Nikulin configuration when β_0 is odd. We explain this in the following Section.

2.3.2 Results according to β_0

For all this Section, let (α_0, β_0) be the fundamental solution of the Pell-Fermat equation $\alpha^2 - 2t\beta^2 = 1$ and consider the following two classes :

$$\left\{ \begin{array}{rcl} C_1' & = & \beta_0 L - \alpha_0 C_1 \\ L' & = & \alpha_0 L - 2t\beta_0 C_1 \end{array} \right. .$$

If we suppose at first that β_0 is even, then Roulleau and Sarti proved in [47, Section 2.5] the following result.

Theorem 2.4. *Let C_1' be the class defined above and suppose that $t \geq 2$ and β_0 is even. Then :*

1. *C_1' is the class of a (-2)-curve and $C_1' \cdot C_j = 0$ for all $j \geq 2$.*
2. *There is no automorphism of X sending the configuration $\mathcal{C} = \sum_{i=1}^{16} C_i$ to the configuration $\mathcal{C}' = C_1' + \sum_{i=2}^{16} C_i$.*

Hence we get that these two configurations $\mathcal{C}$ and $\mathcal{C}'$ define two distinct Kummer structures on X by Proposition 2.3. In the case where β_0 is odd, we said before that $\mathcal{C}'$ is not a Nikulin configuration by the following result.

Proposition 2.5. *Suppose that β_0 is odd. Then the (-2)-class $C_1' = \beta_0 L - \alpha_0 C_1$ cannot be the class of an irreducible rational curve.*

Proof. Suppose that C_1' is the class of an irreducible curve. Then

$$\mathcal{C} = \sum_{i=1}^{16} C_i \text{ and } \mathcal{C}' = C_1' + \sum_{i=2}^{16} C_i$$

are two Nikulin configurations on X. Since both configurations are 2-divisible, the class $C_1 + C_1'$ is 2-divisible and so

$$\frac{C_1 + C_1'}{2} = \frac{\beta_0}{2} L - \frac{\alpha_0 - 1}{2} C_1$$

is an integral class. Since α_0 and β_0 are odd integers, we get that $\frac{L}{2} \in \mathrm{NS}(X)$ which is impossible by Lemma 2.1 and since we assume that L is primitive in $\mathrm{NS}(X)$. $\square$

The aim of this Chapter is to study the case β_0 odd and find some examples of new Nikulin configurations. The following Table can be found in [47, Table 1] and contains the first solutions of the Pell-Fermat equation $\alpha^2 - 2t\beta^2 = 1$ for $t \leq 15$. Note that the cases with a box corresponds to the first cases where β_0 is odd.

$2t$	2	4	6	$\boxed{8}$	10	12	14	16	18	20	22	$\boxed{24}$	26	28	30
α_0	3	-	5	3	19	7	15	-	17	9	197	5	51	127	11
β_0	2	-	2	1	6	2	4	-	4	2	42	1	10	24	2

Table 2.1: First solutions of the Pell-Fermat equation for $t \leq 15$

2.4 Case $L^2 = 16$

As we can in see in Table 2.1, the case $L^2 = 16$ is the first one with $\beta_0 = 1$ odd. Remember that the number of Kummer structures on the Kummer surface X is 2^ν where ν is the number of prime divisors of $\frac{1}{2}M^2$, or equivalently the number of prime divisors of $\frac{1}{4}L^2$. Since $t = 4$ admits one prime divisor, we know that X admits two distinct Kummer structures. In [47, Section 3], the authors tried to construct two non-isomorphic Nikulin configurations on $X := \mathrm{Km}(\mathcal{A})$. If $\mathcal{C} := \{C_1, \ldots, C_{16}\}$ denotes the natural Nikulin configuration on X and L a polarization, they considered the following classes :

$$\begin{cases} L_1 & = & 3L - 4\,(C_1 + C_2 + C_3 + C_4) \\ C_1'' & = & \frac{1}{2}\,(L - 3C_1 - C_2 - C_3 - C_4) \\ C_2'' & = & \frac{1}{2}\,(L - C_1 - 3C_2 - C_3 - C_4) \\ C_3'' & = & \frac{1}{2}\,(L - C_1 - C_2 - 3C_3 - C_4) \\ C_4'' & = & \frac{1}{2}\,(L - C_1 - C_2 - C_3 - 3C_4) \end{cases}$$

They proved that L_1 is a big and nef class and that the set $\mathcal{C}' := \{C_1'', C_2'', C_3'', C_4'', C_5, \dots, C_{16}\}$ is a Nikulin configuration on X. However the computations with the MAGMA programs given later show that this configuration defines the same Kummer structure to the one given by $\mathcal{C}$, answering to the question of [47, Remark 16]. In this Section we will try to construct explicitly two non-isomorphic Nikulin configurations on X.

2.4.1 Computations with MAGMA

In order to compute a new Nikulin configuration in the case where $L^2 = 16$, we use the computer algebra system MAGMA [8]. All the programs are given in Appendix A and the steps are the following :

1. First, the program SmoothRationalCurves will look for the smooth rational (-2)-curves of a given degree on X (see Appendix A.1).
2. Then the program FindNikulinConf will find some Nikulin configurations from all the curves computed before (see Appendix A.2).
3. Finally the program IsThereAutoKummer takes two Nikulin configurations and it says if there exists or not an automorphism of X which sends the first Nikulin configuration to the other (see Appendix A.3).

Remark that the first program will always find the natural configuration denoted previously by $\mathcal{C} := \{C_1, \dots, C_{16}\}$. If the last step does not find such an automorphism, then we are done and $\mathcal{C}'$ is a new Nikulin configuration non-isomorphic to $\mathcal{C}$. However, this process can be very long. In fact, let $\mathcal{C}$ and $\mathcal{C}'$ be two Nikulin configurations on X together with the polarizations L and L', where L' is the orthogonal complement of $\mathcal{C}'$. If there exists an automorphism φ sending $\mathcal{C}$ to $\mathcal{C}'$, then it induces an isometry on $\mathrm{NS}(X)$. Using the Torelli theorem for K3 surfaces (cf. Section 1.4.1), one can tests all linear maps

$$\psi : \mathrm{NS}(X) \otimes \mathbb{Q} \longrightarrow \mathrm{NS}(X) \otimes \mathbb{Q}$$

which satisfies the following conditions :

- ψ sends the $\mathbb{Q}$-base $\{L\} \cup \mathcal{C}$ to the $\mathbb{Q}$-base $\{L'\} \cup \mathcal{C}'$;
- ψ preserves the structure of the Néron-Severi lattice $\mathrm{NS}(X)$;
- ψ acts on the discriminant group of $\mathrm{NS}(X)$ by $\pm\mathrm{id}$;
- ψ sends an ample class to an ample class.

Looking at the two first points, we have $16!$ possibilities. If we want to check the computations with MAGMA, it will take too much time. However, using [21, Theorem 2.7] we can be more precise and use the structure of $\mathrm{NS}(X)$ to decrease significantly these computations. We denote by $\gamma_1, \ldots, \gamma_{16}$ the 16 rational curves of $\mathcal{C}'$.

On the one hand, if $L^2 \equiv 0 \mod 8$, we can suppose (up to permutation of the curves C_i) that $\frac{1}{2}\left(L + \sum\limits_{i=1}^{4} C_i\right)$, $\frac{1}{2}\left(L + \sum\limits_{i=5}^{8} C_i\right)$, $\frac{1}{2}\left(L + \sum\limits_{i=9}^{12} C_i\right)$ and $\frac{1}{2}\left(L + \sum\limits_{i=13}^{16} C_i\right)$ are in $\mathrm{NS}(X)$ and the same for the γ_j's. An automorphism has to preserve the relations of divisibility on $\mathrm{NS}(X)$, thus there are $(4!)^5$ possibilities for the automorphism ψ :

- there are $4!$ ways to send 4 blocks of 4 curves A_i's on 4 blocks of 4 curves B_j's;
- given a block, there are $4!$ ways to send the 4 curves A_i's on the 4 curves B_j's and so it adds $(4!)^4$ possibilities.

On the other hand, if $L^2 \equiv 4 \mod 8$ then we can suppose that $\frac{1}{2}\left(L + \sum\limits_{i=1}^{6} C_i\right)$ and $\frac{1}{2}\left(L + \sum\limits_{i=7}^{12} C_i\right)$ are in $\mathrm{NS}(X)$. Similar computations in this case allows us to get $2 \times 4! \times (6!)^2$ possibilities for the linear map ψ :

- there are 2 ways to send 2 blocks of 6 curves A_i's on 2 blocks of 6 curves B_j's;
- given a block, there are $6!$ ways to send the 6 curves A_i's on the 6 curves B_j's and so it adds $(6!)^2$ possibilities;
- and there are $4!$ possibilities for the last 4 curves.

Remark 2.2. *In the two cases, the number of possibilities we have for ψ seems huge. However, is it as huge as $16!$? Actually no and we can quickly see it thanks to the following computations :*

- *For* $L^2 \equiv 0 \mod 8$ *:*
$$\frac{16!}{(4!)^5} = 6 \times 7 \times 10 \times 11 \times 13 \times 14 \sim 10^6\,.$$

- *For* $L^2 \equiv 4 \mod 8$ *:*
$$\frac{16!}{2 \times 4 \times (6!)^2} = 5^3 \times 3 \times 7^2 \times 11 \times 13 \sim 10^6\,.$$

Hence in each case the time of computation is divided by approximately 10^6*. For example if the program took one year to do it, with these considerations it will only take 30 seconds.*

2.4.2 Application to the case $L^2 = 16$

From now we focus on the case $L^2 = 16$ (that is $t = 4$) and we will use the previous scripts to construct a new Nikulin configuration on X. Using Proposition 2.1, the Néron-Severi group $\mathrm{NS}(X)$ is spanned by the 11 (-2)-classes $C_1, \dots, C_{11}$, the 5 following (-2)-classes :

- $\frac{1}{2}\sum\limits_{i=1}^{16} C_i$;
- $\frac{1}{2}(C_1 + C_2 + C_3 + C_4 + C_6 + C_7 + C_9 + C_{12})$;
- $\frac{1}{2}(C_1 + C_2 + C_3 + C_5 + C_6 + C_8 + C_{10} + C_{13})$;
- $\frac{1}{2}(C_1 + C_2 + C_4 + C_5 + C_7 + C_8 + C_{11} + C_{14})$;
- $\frac{1}{2}(C_1 + C_3 + C_4 + C_5 + C_9 + C_{10} + C_{11} + C_{15})$.

and by a class $\frac{L+v}{2}$ where v corresponds to v_{4d} in Theorem 2.3. As $L^2 \equiv 0 \mod 8$ here, then $v = C_1 + C_2 + C_3 + C_6$. Moreover the Gram matrix of $\mathrm{NS}(X)$ is

$$M := \left(\begin{array}{cccccc|c}
2 & -2 & -1 & -1 & -2 & -2 & \\
-2 & -8 & -4 & -4 & -4 & -4 & \\
-1 & -4 & -4 & -2 & -2 & -2 & M_0 \\
-1 & -4 & -2 & -4 & -2 & -2 & \\
-2 & -4 & -2 & -2 & -4 & -2 & \\
-2 & -4 & -2 & -2 & -2 & -4 & \\
\hline
 & & & {}^tM_0 & & & -2\,\mathrm{I}_{12}
\end{array}\right)$$

where M_0 is the following block :

$$M_0 := \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Remark that if we remove the first line and the first column of the matrix M we get the Gram matrix of the Kummer lattice $\mathcal{K}$. Considering [21, Proposition 2.2], we have $\det(\mathcal{K}) = 2^6$ as expected.

As usual we denote by $\mathcal{C} := \{C_1, \ldots, C_{16}\}$ the trivial Nikulin configuration. In order to find a new configuration different from $\mathcal{C}$, the first step is to use the two programs SmoothRationalCurvesKummer (cf. A.1) and FindNikulinConf (cf. A.2). The first program is known as *Vinberg's algorithm* (cf. [52]) and it finds a lot of smooth rational curves on X of degree at most d. For example putting $d = 23$, it finds the 21132 curves of degree at most 23. Among them, the number of curves of degree exactly equal to 23 is 19488. Then we use the second program to find Nikulin configurations on X. Obviously we find a configration with the 16 curves of degree 1 which corresponds to $\mathcal{C}$. But we find other Nikulin configurations which can be good candidates. For example, let us consider the following 10 curves :

- $C'_2 = L - C_4 - C_6 - C_7 - C_8 - C_9 - C_{10} - C_{11} - C_{12} - C_{14}$;
- $C'_4 = L - C_2 - C_6 - C_7 - C_8 - C_9 - C_{10} - C_{11} - C_{12} - C_{14}$;
- $C'_6 = L - C_2 - C_4 - C_7 - C_8 - C_9 - C_{10} - C_{11} - C_{12} - C_{14}$;
- $C'_7 = L - C_2 - C_4 - C_6 - C_8 - C_9 - C_{10} - C_{11} - C_{12} - C_{14}$;
- $C'_8 = L - C_2 - C_4 - C_6 - C_7 - C_9 - C_{10} - C_{11} - C_{12} - C_{14}$;
- $C'_9 = L - C_2 - C_4 - C_6 - C_7 - C_8 - C_{10} - C_{11} - C_{12} - C_{14}$;
- $C'_{10} = L - C_2 - C_4 - C_6 - C_7 - C_8 - C_9 - C_{11} - C_{12} - C_{14}$;
- $C'_{11} = L - C_2 - C_4 - C_6 - C_7 - C_8 - C_9 - C_{10} - C_{12} - C_{14}$;
- $C'_{12} = L - C_2 - C_4 - C_6 - C_7 - C_8 - C_9 - C_{10} - C_{11} - C_{14}$;
- $C'_{14} = L - C_2 - C_4 - C_6 - C_7 - C_8 - C_9 - C_{10} - C_{11} - C_{12}$.

Denoting by $\mathcal{I} := \{1, 3, 5, 13, 15, 16\}$ we can write the previous curves as

$$C'_j := L - \sum_{\substack{k \notin \mathcal{I} \\ k \neq j}} C_k, \text{ for } j \in [\![1, 16]\!] \backslash \mathcal{I}.$$

Then the 6 curves C_i with $i \in \mathcal{I}$ together with the previous 10 curves C'_j give a new Nikulin configuration. We denote it by $\mathcal{C}'$. Thanks to the program IsThereAutoKummer (cf. A.3) we also know that this new configuration is not isomorphic to $\mathcal{C}$.

2.4.3 Explicit new Nikulin configuration

In this last part, we want to be more precise on the configuration we just found. We keep the notations introduced before and we consider

$$L' = 9L - 8 \sum_{i \notin \mathcal{I}} C_i.$$

First we have the following two points :

- $\forall C \in \mathcal{C}',\ L' \cdot C = 0,$
- $(L')^2 = L^2 = 16.$

Hence L' is a generator of the orthogonal complement of $\mathcal{C}'$ in $\mathrm{NS}(X)$.

Proposition 2.6. *The class L' is big and nef, that is for all (-2)-curves Γ, $L' \cdot \Gamma \geq 0$. Moreover $L' \cdot \Gamma = 0$ if and only if Γ is one of the curves in $\mathcal{C}'$.*

Proof. Let Γ be a (-2)-curve. By Lemma 2.1 we can write this curve as follows :

$$\Gamma = \alpha L - \sum_{i=1}^{16} \beta_i C_i, \quad \alpha, \beta_i \in \frac{1}{2}\mathbb{Z}.$$

and suppose that $L' \cdot \Gamma \leq 0$. On one hand, as Γ is an effective (-2)-class we have

$$\Gamma^2 = -2 \Longleftrightarrow \sum_{i=1}^{16} \beta_i^2 = 8\alpha^2 + 1. \tag{$\star$}$$

On the other hand we have

$$L' \cdot \Gamma \leq 0 \Longleftrightarrow \alpha \leq \frac{1}{9} \sum_{i \notin \mathcal{I}} \beta_i. \tag{$\star\star$}$$

If we suppose that $\Gamma \notin \mathcal{C}'$ then we can assume that $\alpha > 0$ and $\beta_i \geq 0$ for all $i = 1, \ldots, 16$. In this case the condition $(\star\star)$ gives $\alpha^2 \leq \frac{1}{81}\left(\sum_{i \notin \mathcal{I}} \beta_i\right)^2$. But

$$\left(\sum_{i \notin \mathcal{I}} \beta_i\right)^2 \leq 10 \sum_{i \notin \mathcal{I}} \beta_i^2 \leq 10 \sum_{i=1}^{16} \beta_i^2$$

and combining with condition $(\star)$ we get $\alpha^2 \leq 10$. Since α is strictly positive and in $\frac{1}{2}\mathbb{Z}$, the only possible values are

$$\alpha \in \left\{1, 2, 3, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right\}.$$

Assume that $\alpha = \frac{1}{2}$, the other cases can be done in the same way. Then $(\star)$ implies that $\sum_{i=1}^{16} \beta_i^2 = 3$ and by Lemma 2.1 we have only 4 possibilities for the β_i's :

(i) 3 β_i's are equal to 1;

(ii) 2 β_i's are equal to 1 and 4 are equal to $\frac{1}{2}$;

(iii) 1 β_i is equal to 1 and 8 are equal to $\frac{1}{2}$;

(iv) 12 β_i's are equal to $\frac{1}{2}$.

The first two cases are impossible due to $(\star\star)$. However we do not get a contradiction for the two last cases. A way to avoid this problem is to use a MAGMA program to check directly that $L' \cdot \Gamma \geq 0$ for all curves Γ in $\mathrm{NS}(X)$, with equality if and only if Γ is a curve in $\mathcal{C}'$. □

Thanks to this Proposition we can prove the following one.

Proposition 2.7. *The linear system $|L'|$ defines a morphism from X to $\mathbb{P}^9$ which is birational onto its image and contracts the 16 divisors in $\mathcal{C}'$.*

Proof. In order to prove this result we want to apply [48, Section 4.1]. To do this, we first need to show that $|L'|$ has no base components. As L' is big and nef, by [45, Section 3.8] either $|L'|$ has no fixed part or $L' = aE + \Gamma$, where E is an elliptic curve and Γ an irreducible (-2)-curve such that $E \cdot \Gamma = 1$. Assume that we are in this second case. Then $(L')^2 = 2a - 2$ and so $a = 9$. It follows that $L' \cdot \Gamma = a - 2 = 7$. By Lemma 2.1 we can write Γ as follows :

$$\Gamma = \alpha L - \sum_{i=1}^{16} C_i.$$

Then

$$L' \cdot \Gamma = 16 \left(9\alpha - \sum_{i \notin \mathcal{I}} \beta_i \right)$$

which can not be equal to 7. So $|L'|$ has no base components. Now we can show that $|L'|$ is not hyperelliptic. By [48, Section 4.1], $|L'|$ is hyperelliptic either if there exists a curve C of genus 2 such that $L' = 2C$ or if there exists an elliptic curve E such that $L' \cdot E = 2$.

In the first case, as $(L')^2 = 16$ we have $C^2 = 4$. Then by the genus formula we get $g(C) = 3$ which is a contradiction. In the second case assume that $E = \alpha L - \sum_{i=1}^{16} C_i$ (cf. Lemma 2.1). Computing $L' \cdot E$ we get

$$8 \left(9\alpha - \sum_{i \notin \mathcal{I}} \beta_i \right) = 1.$$

Since α and all the β_i's are in $\frac{1}{2}\mathbb{Z}$, this last equality is impossible to solve. Hence we get that $|L'|$ defines a birational map from X to $\mathbb{P}^n$ onto its image contracting the (-2)-curves Γ such that $L' \cdot \Gamma = 0$. Moreover $n = h^0(L') - 1 = 9$. $\square$

Hence we know two non-isomorphic Nikulin configurations when $t = 4$. Since this value of t has only one prime divisor, then we know all Nikulin configurations in this case (always up to $\mathrm{Aut}(X)$). In a future work, we can try to use these MAGMA programs to find non-isomorphic Nikulin configurations for the following missing cases. In [47, Table 1] we can see that the next values are 12, 16, 20 and 24. We may hope some similarities between these cases and find a general formula for all the cases when β_0 is odd in the Pell-Fermat equation 2.1. Another work we are able to do is to find more than two non-isomorphic Nikulin configurations in the concerned cases. For example when $t = 6$ (that is $L^2 = 24$) there exists 4 non-isomorphic configurations and we "only" know two of them thanks to [47, Theorem 12]. An idea could be to find the two other configurations thanks to the MAGMA programs.

CHAPTER 3

Symplectic action on K3 surfaces

3.1 Introduction

In this Chapter we will focus on the second big work of my thesis which is about K3 surfaces admitting a symplectic action of the Mathieu group M_{20}. It is the topic of the following article (see the published version in [13]) which is a joint work with Paola Comparin (*Universidad de La Frontera, Chile*) :

K3 surfaces with action of the group M_{20},
https://arxiv.org/abs/2201.02150

The action of finite groups acting on K3 surfaces goes back to the first works by Nikulin and Mukai and a natural first question is : "*Are we able to classify all possible finite groups that can act on K3 surfaces ?*" The answer is yes and depending of the hypothesis put on the group G we have the following results. First, there is the whole work of Nikulin in [41]. He classified finite abelian group acting symplectically on K3 surfaces and he showed that there is only a finite number of them. Removing the assumption of being abelian, Mukai in [39] studied finite groups acting symplectically on K3 surfaces and showed that every such group is a subgroup of the Mathieu group M_{23}. Moreover he gave a complete classification of such groups, showing that they are exactly subgroups of one of the 11 groups in the Table below (cf. [39, Example 0.4]). These groups are called *maximal*. Remark that the order of such a group is at most 960, being equal to 960 if and only if G is the Mathieu group M_{20}.

G	Characterization	Order
T_{48}	$\mathcal{Q}_8 \rtimes \mathfrak{S}_3$	48
N_{72}	$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathcal{D}_8$	72
M_9	$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathcal{Q}_8$	72
$\mathfrak{S}_5$	-	120
$L_2(7)$	$\mathbb{P}\mathrm{SL}_2(\mathbb{F}_7)$	168
N_{72}	$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathcal{D}_8$	72
T_{192}	$(\mathcal{Q}_8 * \mathcal{Q}_8) \rtimes \mathfrak{S}_3$	192
H_{192}	$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{D}_{12}$	192
$\mathfrak{A}_{4,4}$	-	288
$\mathfrak{A}_6$	-	360
F_{384}	$(\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathfrak{S}_4$	384
M_{20}	$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{A}_5$	960

Table 3.1: The 11 maximal groups acting symplectically

Later Xiao in [54] gave an explicit list of all possible finite groups acting symplectically, he found 81 such groups and he classified combinatorial types of the action of them, i.e. computed the number of fixed points in each case. Finally, removing the assumption "symplectically", Kondō showed in [32] that if G contains the Mathieu group M_{20} then $|G| \leq 3840$. Moreover, if G strictly contains M_{20}, then there are three such groups and the possible surfaces admitting such action are explicitly known. So the next question arising is : "*How many K3 surfaces can admit an action by G ?*"

The aim of this Chapter is to prove the following two results.

Theorem 3.1. *There exists infinitely many pairwise non-isomorphic K3 surfaces admitting a faithful and symplectic action of M_{20}.*

All along this Chapter, when we talk about "K3 surfaces" we think of "projective K3 surfaces". Let X be a K3 surface admitting a symplectic action of M_{20}. Since X is smooth and projective, we can always consider a primitive ample class $L \in \mathrm{Pic}(X)$. Moreover by considering the sum $\sum g^*L$ over M_{20} we can assume that L is M_{20}-invariant. We will see that the invariant lattice $H^2(X,\mathbb{Z})^{M_{20}}$ is isometric to a certain lattice denoted by $\mathbb{L}_{20}$ (see Section 3.3.2). Assume for the moment that $L^2 = 4n$ for some $n \in \mathbb{Z}_{\geq 1}$.

Theorem 3.2. *Let X be a K3 surface with a symplectic action of M_{20} and let $L \in \mathrm{Pic}(X)$ be the class defined above. Then there exists an embedding $\langle 4n \rangle \hookrightarrow \mathbb{L}_{20}$ if and only if n is not of the form $4^i(16j+6)$, with i, j some non-negative integers. If it is the case, we can construct a projective model of X in $\mathbb{P}^{2n+1}$ and we have the following results :*

1. *The linear system $|L|$ is base point free and not hyperelliptic.*

2. *For $n > 1$, the projective model is only defined by an intersection of quadrics. Morevover, in this case, the number of quadrics Q_{4n} defining X is*

$$Q_{4n} = 2n^2 - 3n + 1.$$

3.2 Case where G strictly contain M_{20}

As always we consider X a K3 surface and we denote by ω_X the associated holomorphic 2-form. Recall that an automorphism $\sigma \in \mathrm{Aut}(X)$ is said to be *symplectic* if it acts as the identity on ω_X and *non-symplectic* otherwise. In this last case σ^* acts as multiplication by some root of unity $\zeta_m \in \mathbb{C}^\times$. Given a finite group of automorphisms $G \subset \mathrm{Aut}(X)$, we consider $\alpha : G \to \mathbb{C}^\times$ the corresponding action on ω_X. Thus we have the short exact sequence presented in Chapter 1 :

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1$$

where $G_0 = \ker \alpha$ corresponds to the automorphisms acting symplectically on X. As we said previously, when G strictly contains the Mathieu group M_{20} there are three possible cases : one group of order $3840 = 4\,|M_{20}|$ and two other groups of order $1920 = 2\,|M_{20}|$. The K3 surfaces where these groups act first appeared in [7, 9, 32, 39] and we summarise the situation in the following result. Let L be an ample class which is M_{20}-invariant and assume for the moment that $L^2 \in 4\mathbb{Z}$. Recall that E_τ is the standard notation for the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with $\tau \in \mathbb{C}$.

Proposition 3.1 (see [7, 9, 32]). *Let G be a maximal finite group acting faithfully on the K3 surface X which strictly contains the Mathieu group M_{20}. Let $L \in \mathrm{Pic}(X)$ be an ample class which is M_{20}-invariant. There only are three possible cases :*

1. *$|G| = 3840$ and $L^2 = 40$: the K3 surface X_{Ko} is the minimal resolution of the quotient of the Fermat quartic $\left\{x^4 + y^4 + z^4 + t^4 = 0\right\}$ in $\mathbb{P}^3$ by the symplectic involution $(x : y : z : t) \mapsto (x : y : -z : -t)$. So X_{Ko} is the Kummer surface $\mathrm{Km}\,(E_i \times E_i)$ (see [32], [7, Section 3]).*

2. *$|G| = 1920$ and $L^2 = 4$: the K3 surface X_{Mu} is defined as the following quartic in $\mathbb{P}^3$:*

$$\left\{x^4 + y^4 + z^4 + t^4 - 6\left(x^2y^2 + x^2z^2 + x^2t^2 + y^2z^2 + y^2t^2 + z^2t^2\right) = 0\right\}.$$

So X_{Mu} is the Kummer surface $\mathrm{Km}\left(E_{i\sqrt{10}} \times E_{i\sqrt{10}}\right)$ (see [39, p. 190] and [7, Section 4]).

3. $|G| = 1920$ *and* $L^2 = 8$ *: the K3 surface* X_{BH} *is defined as the complete intersection of the three quadrics in* $\mathbb{P}^5$ *:*

$$\begin{cases} x_0^2 + x_3^2 - \phi x_4^2 + \phi x_5^2 = 0 \\ x_1^2 - \phi x_3^2 + x_4^2 - \phi x_5^2 = 0 \\ x_2^2 + \phi x_3^2 - \phi x_4^2 + x_5^2 = 0 \end{cases}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ *(see [7, Section 5]).*

Remark 3.1. *We observe that, according to [16, Corollary 7.3], up to projective transformation there exists a unique octic K3 surface* $X_8 \subset \mathbb{P}^5$ *which admits a faithful symplectic action of the Mukai group* M_{20}*, corresponding to the third case of the previous Proposition.*

3.3 Faithful and symplectic action of M_{20}

We suppose from now and until the end of the Chapter that X admits a faithful and symplectic action by the Mathieu group M_{20}. If we assume that $G = G_0 = M_{20}$, then the previous exact sequence becomes trivial. First we will see some properties of the K3 surfaces we consider in this case, then we will talk about the invariant lattice $H^2(X, \mathbb{Z})^{M_{20}}$.

3.3.1 Singular K3 surfaces

By Xiao's classification [54, Nr. 81, Table 2], the surface X has necessarily Picard number equal to 20. This is the maximal Picard number for K3 surfaces and in this special case we talk about *singular K3 surfaces*. Remember that T_X denotes *transcendental lattice* of X, i.e. the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$. All the results we present here concerning singular K3 surfaces come from [50, Section 4].

Proposition 3.2. *Let* X *be a singular K3 surface. Then the lattice* T_X *has rank 2, it is even and has signature* $(2, 0)$*.*

Denote by $\mathcal{Q}$ the set of 2×2 positive-definite even integral matrices. For $Q \in \mathcal{Q}$ we have

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix},\ a, b, c \in \mathbb{Z}$$

with $a, c > 0$ and $d := 4ac - b^2 > 0$. We define an equivalence relation $\sim$ on $\mathcal{Q}$ by :

$$\forall\, Q_1, Q_2 \in \mathcal{Q},\ Q_1 \sim Q_2 \Longleftrightarrow \exists \gamma \in \mathrm{SL}_2(\mathbb{Z})\ ,\ Q_1 = {}^t\gamma Q_2 \gamma.$$

Let $[Q]$ be the equivalence class of a matrix $Q \in \mathcal{Q}$ and $\mathcal{Q}/\mathrm{SL}_2(\mathbb{Z})$ the set of equivalence classes.

Theorem 3.3 ([50, Theorem 4]). *The map $X \mapsto [\mathrm{T}_X]$ establishes a bijective correspondence from the set of singular K3 surfaces onto $\mathcal{Q}/\mathrm{SL}_2(\mathbb{Z})$.*

In particular, K3 surfaces with Picard number 20 are classified in terms of their transcendental lattice. Moreover, we can choose a representative Q in *reduced form*, that is $-a \leq b \leq a \leq c$ and $b^2 \leq ac \leq \frac{d}{3}$ and we have the following :

Theorem 3.4 ([10, Theorem 2.4]). *No distinct reduced quadratic forms are equivalent, except for the following:*

$$\begin{pmatrix} 2a & a \\ a & 2c \end{pmatrix} \sim \begin{pmatrix} 2a & -a \\ -a & 2c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2a & b \\ b & 2a \end{pmatrix} \sim \begin{pmatrix} 2a & -b \\ -b & 2a \end{pmatrix}.$$

3.3.2 The invariant lattice

Consider $\mathbb{L}_{20}$ to be the lattice defined by the following matrix :

$$\mathbb{L}_{20} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 12 \end{pmatrix}.$$

This lattice has rank 3 and signature $(3,0)$. Denoting by $H^2(X,\mathbb{Z})^{M_{20}}$ the invariant lattice by the action of M_{20} on the K3 lattice, we recall the following result :

Proposition 3.3 ([32, proof of Proposition 2.1]). *Let X be a K3 surface with a faithful symplectic action by M_{20}. Then the invariant lattice $H^2(X,\mathbb{Z})^{M_{20}}$ is isometric to $\mathbb{L}_{20}$.*

As $\mathbb{L}_{20}$ has signature $(3,0)$, its isometry group $O(\mathbb{L}_{20})$ is finite (cf. [32, proof of Prop. 2.1]). We describe it completely thanks to the following result coming from [7, Remark 2.4] :

Proposition 3.4. *Denote by ρ_1 and ρ_2 the isometries of $\mathbb{L}_{20}$ defined by the following matrices :*

$$\rho_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then the isometry group $O(\mathbb{L}_{20})$ is spanned by $-\mathrm{id}_{\mathbb{L}_{20}}$, ρ_1 and ρ_2 and it has order 16.

Since the action is symplectic we get by Proposition 1.4 that $\mathrm{T}_X \subset H^2(X,\mathbb{Z})^{M_{20}} \simeq \mathbb{L}_{20}$. Moreover if (u,v) is a $\mathbb{Z}$-basis of $\mathrm{T}_X \subset \mathbb{L}_{20}$, then by quick computations we get $u^2, v^2 \in 4\mathbb{Z}$ and $u \cdot v \in 2\mathbb{Z}$. Hence

$$\mathrm{T}_X \simeq \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$ satisfy the following reduction conditions :

$$\begin{cases} d := 4ac - b^2 > 0; \\ b^2 \leq ac \leq \frac{d}{3}; \\ -a \leq b \leq a \leq c. \end{cases} \tag{$\star$}$$

Finally observe that $\mathrm{rk}\,(\mathrm{NS}(X) \cap \mathbb{L}_{20}) = 1$. Indeed, as X is smooth and projective it always exists an ample M_{20}-invariant class $L \in \mathrm{NS}(X)$ which is then in $\mathbb{L}_{20}$. Moreover we have $L^2 = 4n$ for some n in $\mathbb{Z}_{\geq 1}$ because $L \in \mathbb{L}_{20}$ as we saw above. One of our main goals is to describe the embedding $\langle 4n \rangle \hookrightarrow \mathbb{L}_{20}$ in order to construct a projective model of X thanks to Proposition 1.1.

As we will see in Section 3.6, the linear system $|L|$ is never hyperelliptic and as $L^2 = 4n$, the projective models of the K3 surfaces that we consider is in $\mathbb{P}^{2n+1}$. By Proposition 3.1 the cases $n \in \{1, 2, 10\}$ were already studied by Bonnafé and Sarti in [7], Brandhorst and Hashimoto in [9] and Kondō in [32], when the surface X admits the symplectic action of a maximal group G containing M_{20} properly.

Remark 3.2. *We assumed at the beginning of this Section that X will always admit a faithful and symplectic action by the Mathieu group M_{20}. However if there exists a polarization L on X with $L^2 \in 4\mathbb{Z}$, $L \in \mathbb{L}_{20}$ and such that $H^2(X,\mathbb{Z}) \supset \mathbb{L}_{20} \supset \mathbb{Z}L \oplus \mathrm{T}_X$, then we are able to show that M_{20} acts (symplectically) on X. This can be done with a Lemma of Kondō (cf. Lemma 3.4) and the arguments are given in the proof of Theorem 3.2 at the end of this Chapter.*

3.4 The case $L^2 = 12$

We now start by studying the first new case, i.e. the polarization with $L^2 = 4n$ with $n = 3$. First we will prove the existence of the projective model computing the transcendental lattice T_X. We will then construct explicitly the K3 surface thanks to the computer algebra system MAGMA [8]. For the following cases ($n > 4$ and $n \neq 10$), the strategy will be to mimic the method used below.

3.4.1 Existence of the projective model

Suppose that $L^2 = 12$. We denote by (e, f, h) the standard basis of $\mathbb{L}_{20}$, that is the vectors in $\mathbb{L}_{20}$ which give the matrix of the bilinear form written before. We search for $\lambda, \mu, \delta \in \mathbb{Z}$ such that

$$L = \lambda e + \mu f + \delta h.$$

Thanks to the Gram matrix of $\mathbb{L}_{20}$ given before we have

$$L^2 = (2\lambda - \delta)^2 + (2\mu - \delta)^2 + 10\delta^2. \tag{3.1}$$

If $L^2 = 12$, then the possible solutions for δ are $\delta \in \{-1, 0, 1\}$ and

- if $\delta = 0$, then $2\lambda^2 + 2\mu^2 = 12$, that is $\lambda^2 + \mu^2 = 6$. This is not possible since 6 is not the sum of two squares;
- if $\delta = 1$, then $(2\lambda - 1)^2 + (2\mu - 1)^2 = 2$, that is $(2\lambda - 1)^2 = (2\mu - 1)^2 = 1$. Hence $\lambda, \mu \in \{0, 1\}$ and then

$$L \in \{h; e + h; f + h; e + f + h\};$$

- if $\delta = -1$, as before one has $\lambda, \mu \in \{0, -1\}$, then

$$L \in \{-h; -e - h; -f - h; -e - f - h\}.$$

Hence there are 8 possibilities for L and they are all in the same orbit, up to isometry (see Prop. 3.4). In fact observe that the possibilities for $\delta = -1$ are the same as in the case $\delta = 1$ up to the isometry $-\mathrm{id}_{\mathbb{L}_{20}}$. Moreover we have :

$$-\mathrm{id} \circ \rho_2(e + f + h) = f + h, \quad \rho_1(f + h) = e + h, \quad -\mathrm{id} \circ \rho_2(e + h) = h.$$

Therefore, up to isometry, there exists a unique embedding of lattices $\langle 12 \rangle \hookrightarrow \mathbb{L}_{20}$ which maps L to h.

We assume now that $L = h$. As $\mathrm{T}_X = (\mathbb{Z}L)^\perp \cap \mathbb{L}_{20}$, the orthogonal complement of the lattice $\mathbb{Z}h$ in $\mathbb{L}_{20}$ is given by vectors $\lambda e + \mu f + \delta h$ such that

$$\langle \lambda e + \mu f + \delta h, h \rangle = 0, \quad \lambda, \mu, \delta \in \mathbb{Z}.$$

One can choose $u_1 = f - e$ and $u_2 = 3e + 3f + h$. Then u_1 and u_2 span the transcendental lattice T_X. Moreover $u_1^2 = 8$, $u_2^2 = 60$ and $\langle u_1, u_2 \rangle = 0$, so that

$$\mathrm{T}_X = \begin{pmatrix} 8 & 0 \\ 0 & 60 \end{pmatrix}.$$

Using the notations introduced before, one has $a = 2$, $b = 0$ and $c = 15$ and these integers satisfy the conditions ($\star$), hence this is the matrix we are looking for.

3.4.2 Construction of the K3 surface

We just proved that there exists a K3 surface polarized with L such that $L^2 = 12$ and admitting a faithful and symplectic action of M_{20}. We now want to construct it explicitly in $\mathbb{P}^7$. To understand the problem, we have the following diagram :

$$\begin{tikzcd} & & & M_{20} \arrow[dl, dashed] \arrow[d, hook] & \\ 1 \arrow[r] & \mathbb{C}^\times \arrow[r] & \mathrm{GL}_8(\mathbb{C}) \arrow[r] & \mathbb{P}\mathrm{GL}_8(\mathbb{C}) \arrow[r] & 1 \end{tikzcd}$$

In general, the arrow $M_{20} \longrightarrow \mathrm{GL}_8(\mathbb{C})$ does not exist. So we must consider a group $\widetilde{M_{20}} \subset \mathrm{GL}_8(\mathbb{C})$ such that :

- $Z(\widetilde{M_{20}}) \simeq \mu_d$;
- $\widetilde{M_{20}}/\mu_d \simeq M_{20}$.

Here μ_d denotes the group of primitive d-th roots of unity and $Z(\widetilde{M_{20}})$ is the center of the group $\widetilde{M_{20}}$ in $\mathrm{GL}_8(\mathbb{C})$. Then we have the following diagram :

$$\begin{tikzcd} 1 \arrow[r] & \mu_d \arrow[r] \arrow[d] & \widetilde{M_{20}} \arrow[r] \arrow[d] & M_{20} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathbb{C}^\times \arrow[r] & \mathrm{GL}_8(\mathbb{C}) \arrow[r] & \mathbb{P}\mathrm{GL}_8(\mathbb{C}) \arrow[r] & 1 \end{tikzcd}$$

We want to understand the non-trivial central extensions of M_{20} by a cyclic group and, thanks to the web version of the Atlas of finite groups [1], we know that there are 6 such extensions :

1. M_{20};
2. H_1, H_2 and H_3, with $Z(H_k) \simeq \mu_2$, $k = 1, 2, 3$;
3. G_1 and G_2, with $Z(G_l) \simeq \mu_4$, $l = 1, 2$.

Denote by G one of these groups. We are looking for the irreducible representations of G in $\mathrm{GL}_8(\mathbb{C})$. In our case, by using the computer algebra system MAGMA [8], we obtain that only the group G_1 gives such a representation. Let $\rho : G \to \mathrm{GL}_8(\mathbb{C})$ be this representation. By abuse of notation, we identify G with the image $\rho(G)$. The following holds :

$$Z(G) \simeq \mu_4 \quad \text{and} \quad G/Z(G) \simeq M_{20}\,.$$

Let X be the K3 surface and assume for the moment that X is defined by quadrics in $\mathbb{P}^7$. Denote $F := \mathbb{C}\left[x_1, \ldots, x_8\right]_2$ the vector space of homogeneous polynomials of degree 2 in 8 variables and let $f_1, \ldots, f_n \in F$ be the equations of the quadrics defining X. Then $\mathbb{C}f_1 + \cdots + \mathbb{C}f_n \subset F$ must be stable by the action of G and so we want to find the stable subspaces of F. By the theory of linear representations (see e.g. [49, Proposition 8]), we have the following decomposition :

$$F = \bigoplus_{S \in \mathrm{Irr}(G)} \mathrm{Im}\left(p_S^F\right)$$

where $p_S^F \in \mathrm{End}_{\mathbb{C}}(F)$ is the projection on the isotypic component associated with S (see below for the explicit definition). Thanks to standard computations with MAGMA, we obtain $F = F_6 \oplus F_{10} \oplus F_{20}$, where the F_i are G-stable, irreducible and of dimension i. Let X_k be the subvariety in $\mathbb{P}^7$ defined by the elements of F_k. By using again standard commands in MAGMA, we compute the dimension of each space : $\dim\left(X_6\right) = 4$, $\dim\left(X_{10}\right) = 2$ and $\dim\left(X_{20}\right) = -1$. This means that $X_{20} = \emptyset$ and X_{10} is a surface, thus it is a good candidate for the surface we are looking for. With standard computations in MAGMA, we check the following conditions :

- the canonical divisor is trivial;
- the surface is smooth.

At this point, it remains two possibilities : either X_{10} is an abelian surface or it is a K3 surface. To conclude, we will compute the Euler characteristic $\chi(X_{10})$ of X_{10} and therefore distinguish if $\chi(X_{10}) = 0$ and X_{10} is abelian or $\chi(X_{10}) = 24$ and X_{10} is a K3 surface. We consider

$$\mathcal{T} : g \in M_{20} \longmapsto \sum_{i \geq 0} (-1)^i \mathrm{Tr}\left(g^*|_{H^i(X_{10})}\right).$$

On the one hand, by the Lefschetz fixed point formula [22, Chapter 3.4], we have $\mathcal{T}(1) = \chi\left(X_{10}\right)$. On the other hand, we can compute $\mathcal{T}(1)$ using the Lemma below.

Lemma 3.1. *Let Γ be a finite group and $f : \Gamma \to \mathbb{C}$ be a function which is the difference of two characters of representations. Then*

$$f(1) \equiv - \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq 1}} f(\gamma) \mod |\Gamma| \,.$$

Before proving this result, we need to recall some definitions and results coming from representation theory (see e.g. [49, Chapters 1 and 2]).

Let (E, ρ_E) be a linear representation of the group Γ. This means that E is a complex vector space and $\rho_E : \Gamma \to \mathrm{GL}(E)$ is a group morphism. We denote by $\rho_{\mathbb{C}}$ the so-called *trivial representation*, that is for all $\gamma \in \Gamma$, $\rho_{\mathbb{C}}(\gamma) = \mathrm{Id}_{\mathbb{C}}$. We call a *character* of the representation ρ_E the morphism $\chi_E : \Gamma \to \mathbb{C}$ defined by

$$\forall \gamma \in \Gamma, \; \chi_E(\gamma) = \mathrm{Tr}\left(\rho_E(\gamma)\right).$$

Finally, we denote by p_S^E the projection on the isotypic component associated with S (cf. [49, Section 2.6]). If S is an irreducible representation of Γ, we have a characterization of this endomorphism which is

$$p_S^E = \frac{\dim S}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_S(\gamma)} \rho_E(\gamma).$$

Proof of Lemma 3.1. Let $S = \mathbb{C}$. For the endomorphism $p_{\mathbb{C}}^E$ we have

$$p_{\mathbb{C}}^E = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_{\mathbb{C}}(\gamma)} \rho_E(\gamma).$$

As $\rho_{\mathbb{C}}$ is the trivial representation of Γ, for all $\gamma \in \Gamma$ we have $\chi_{\mathbb{C}}(\gamma) = \mathrm{Tr}\left(\mathrm{Id}_{\mathbb{C}}\right)$. Then $p_{\mathbb{C}}^E = \frac{1}{|\Gamma|} \sum\limits_{\gamma \in \Gamma} \rho_E(\gamma)$ and taking the trace we get

$$\mathrm{Tr}\left(p_{\mathbb{C}}^E\right) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{Tr}\left(\rho_E(\gamma)\right).$$

Denoting by $f = \chi_E : \Gamma \to \mathbb{C}$ (which is merely $\mathrm{Tr}\left(\rho_E(\gamma)\right)$) and multiplying on each side by $|\Gamma|$ we finally obtain the following :

$$f(1) = |\Gamma| \mathrm{Tr}\left(p_{\mathbb{C}}^E\right) - \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq 1}} f(\gamma).$$

As $\mathrm{Tr}\left(p_{\mathbb{C}}^E\right) = \dim \mathrm{Im}\left(p_{\mathbb{C}}^E\right)$ is a positive integer, we can look at this expression modulo $|\Gamma|$. Similarly, if $f : \Gamma \to \mathbb{C}$ is as in the assumption, we obtain the result. $\square$

Let X_{10}^g be the subvariety of X_{10} where $g \in M_{20}$ acts as the identity and observe that we have only isolated fixed points on it. Then, thanks again to the Lefschetz formula, for $g \neq 1$ we have $\mathcal{T}(g) = \chi(X_{10}) = |X_{10}^g|$. We compute this last term with MAGMA and by Lemma 3.1 we obtain

$$\mathcal{T}(1) \equiv 24 \mod 960.$$

Therefore $\chi(X_{10}) \neq 0$ and thus X must be a K3 surface by the Enriques-Kodaira classification. Finally we put the equations for the quadrics which define the K3 surface in Appendix B.

3.5 Other cases

3.5.1 Cases of non-existence

In order to repeat the construction for other values of L^2, we first observe for which n the equation (3.1) admits solution.

Proposition 3.5. *The equation*

$$4n = (2\lambda - \delta)^2 + (2\mu - \delta)^2 + 10\delta^2. \tag{3.2}$$

admits a solution (λ, μ, δ) *if and only if* n *can not be expressed as* $4^i(16j + 6)$ *for some positive integers* i, j.

Proof. By results of Ramanujan in [44], an even positive integer can be expressed as $x^2 + y^2 + 10z^2$ (called *Ramanujan's ternary quadratic form*) for some $x, y, z \in \mathbb{Z}$ if and only if it is not of the form $4^i(16j + 6)$ for some non-negative integers i, j. Then if $n \neq 4^i(16j + 6)$, there exists $x, y, z \in \mathbb{Z}$ such that

$$4n = x^2 + y^2 + 10z^2. \tag{3.3}$$

Thus we can compute (λ, μ, δ) satisfying (3.2) from the triple (x, y, z):

$$\lambda = \frac{\pm x \pm z}{2},\ \mu = \frac{\pm y \pm z}{2},\ \delta = \pm z.$$

Observe that $\lambda, \mu \in \mathbb{Z}$. In fact, if z is even, then by (3.3) the sum $x^2 + y^2$ has to be congruent to 0 modulo 4. Since 3 is not a square modulo 4, then necessarily

$$x^2 \equiv 0 \mod 4, \quad y^2 \equiv 0 \mod 4.$$

This implies that x^2 and y^2 are even, so x and y are even too. Thus $\lambda, \mu \in \mathbb{Z}$. Similarly if z is odd, then $z^2 \equiv 1 \mod 4$. Thus by (3.3) we get

$$x^2 \equiv 1 \mod 4, \quad y^2 \equiv 1 \mod 4$$

so x and y are odd too and $\lambda, \mu \in \mathbb{Z}$. $\square$

3.5.2 Cases up to $L^2 = 40$

Computations similar to the ones of Section 3.4 allow to study the existence of the K3 surface with $L^2 = 4n$ for any value of n (except for the cases of Proposition 3.5). We show the results for $3 \leq n \leq 10$ in order to give examples in several cases. We will see that some values admit just an embedding, whereas for other values there are more embeddings and we will show why they are not in the same orbit.

In Table 3.2, we compute the values of λ, μ and δ. To get all the possibilities, one can act on the given (λ, μ, δ) with the elements of the isometry group $O(\mathbb{L}_{20})$, i.e. exchange λ and μ and multiply the vector by -1.

L^2	16	20		24	28	32	36		40	
λ	2	1	$-1, 2$	-	2	2	3	$-2, 3$	3	$1, -1$
μ	0	2	$0, 1$	-	-1	2	0	$0, 1$	1	$1, -1$
δ	0	0	1	-	1	0	0	1	0	2

Table 3.2: Cases up to $L^2 = 40$

Remark 3.3. 1. *Two vectors (λ, μ, δ) and $(\lambda', \mu', \delta')$ with $\delta \neq \pm\delta'$ are not in the same orbit by the action of $O(\mathbb{L}_{20})$. This follows from the fact that $\rho_1(\lambda, \mu, \delta) = (\mu, \lambda, \delta)$ and both ρ_2 and $-id_{\mathbb{L}_{20}}$ change the sign of δ.*

2. *Observe that the converse is not true, i.e. there exists vectors $(\lambda, \mu, \delta), (\lambda', \mu', \delta')$ such that $\delta = \delta'$ but not in the same orbit by the action of $O(\mathbb{L}_{20})$. For example for $L^2 = 300$, the two vectors $(\lambda, \mu, \delta) = (3, 6, 5)$ and $(\lambda', \mu', \delta') = (5, 0, 5)$ are not in the same orbit, but $\delta = \delta'$.*

Here we give some details of the computations for the cases listed in Table 3.2.

- $\underline{L^2 = 16}$: There are 4 possibilities for (λ, μ, δ) and they are in the same orbit modulo isometry. Hence there exists a unique embedding $\langle 16 \rangle \hookrightarrow \mathbb{L}_{20}$ which maps L to $2e$. The lattice T_X obtained in this case is the same as the case $L^2 = 4$ and we will see in Section 3.7 the relation between these two examples.

- $\underline{L^2 = 20}$: Up to isometry, we obtained two embedding $\langle 20 \rangle \hookrightarrow \mathbb{L}_{20}$:

$$L \longmapsto e + 2f \quad \text{and} \quad L \longmapsto f - h\,.$$

These embeddings are not in the same orbit, due to Remark 3.3. Let us consider the first embedding $L \mapsto e+2f$; we can choose $u_1 = f-2e$ and $u_2 = e+f+2h$ as generators of the transcendental lattice and we get

$$\mathrm{T}_X = \begin{pmatrix} 20 & 0 \\ 0 & 40 \end{pmatrix}.$$

In the second case with $L \mapsto f - h$, we take $u_1 = 2f + e + h$ and $u_2 = f - 3e$ and we get

$$\mathrm{T}_X = \begin{pmatrix} 20 & 0 \\ 0 & 40 \end{pmatrix}.$$

For both cases the conditions ($\star$) hold.

Remark 3.4. *We observe that for $L^2 = 20$ there are two non isometric vectors (λ, μ, δ) but the matrix for the lattice T_X is the same, therefore the same K3 surface admits two different actions of M_{20}.*

- $\underline{L^2 = 24}$: In this case there is no embedding by Proposition 3.5.
- $\underline{L^2 = 28}$: In this case $\delta = \pm 1$. If $\delta = 1$, then $\lambda, \mu \in \{2, -1\}$ and if $\delta = -1$, then $\lambda, \mu \in \{-2, 1\}$. This gives 8 possibilities for (λ, μ, δ), all in the same orbit. Thus we can choose $L \mapsto e + f - h$. Taking $u_1 = f - e$ and $u_2 = 4e + 4f + 3h$ we compute

$$\mathrm{T}_X = \begin{pmatrix} 8 & 0 \\ 0 & 140 \end{pmatrix}$$

  and the conditions ($\star$) hold for this matrix.
- $\underline{L^2 = 32}$: In this case one obtains the same transcendental lattice T_X as the case $L^2 = 8$. As for $L^2 = 16$, this case will be explained in Section 3.7.
- $\underline{L^2 = 36}$: In this case, there are two possibilities:

$$L \longmapsto 3e \quad \text{and} \quad L \longmapsto 3e + h\,.$$

The first embedding which maps L on $3e$ gives the same transcendental lattice as $L^2 = 4$ because $36 = 3^2 \cdot 4$ (see Section 3.7).

For the second embedding, one can consider $u_1 = 3f + h$ and $u_2 = 2f + e - h$. The matrix then is

$$\mathrm{T}_X = \begin{pmatrix} 36 & 12 \\ 12 & 44 \end{pmatrix}$$

and the conditions $(\star)$ hold for it.

- $\underline{L^2 = 40}$: There are two possibilities, up to isometry. One is $L \mapsto e + f + 2h$ and this one is studied in [32].

  The other one is $L \mapsto 3f + e$. Taking $u_1 = f - e + h$ and $u_2 = 2e + h$ one gets $u_1^2 = u_2^2 = 20, u_1 \cdot u_2 = 0$, i.e.

  $$\mathrm{T}_X = \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}$$

  and the conditions $(\star)$ hold for this matrix.

  Remark 3.5. *Observe that in [32, Lemma 3.1, Prop. 3.3], Kondō proves that the surface $X = \mathrm{Km}(E_i \times E_i)$ is the unique surface admitting the action of a symplectic group G such that $G_0 \simeq M_{20}$ and $G/G_0 \simeq \mathbb{Z}/4\mathbb{Z}$. In this case $\mathrm{T}_X = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. The fact that we find here two different surfaces (with two different transcendental lattices) is not contradicting the unicity proven in [32], since here we are not assuming the action of G such that $G_0 \simeq M_{20}$ and $G/G_0 \simeq \mathbb{Z}/4\mathbb{Z}$.*

  Let X_{Ko} be the surface with $\mathrm{T}_{X_{\mathrm{Ko}}} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ according to the notation of [7, Section 3], and let X' be the K3 surface with transcendental lattice $\mathrm{T}_{X'} = \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}$. We can observe that $\mathrm{T}_{X'} = 5\mathrm{T}_{X_{\mathrm{Ko}}}$ and this relates to the fact that by [51, Section 5], X' is the Kummer surface $\mathrm{Km}\,(E_{5i} \times E_{5i})$.

3.6 About the projective models

In order to describe the projective models of K3 surfaces with an action of M_{20}, we want to understand when the linear system $|L|$ is not hyperelliptic and then when L defines an embedding. Remember that $|L|$ is said to be hyperelliptic if there exists a surface $S \subset \mathbb{P}^n$ such that the map $X \to S$ given by $|L|$ is $2:1$. With the previous notations we prove the first main Theorem given at the beginning of this Chapter :

Theorem 3.5. *Let X be a K3 surface with a symplectic action of M_{20} and let $L \in \mathrm{Pic}(X)$ be a primitive ample class which is M_{20}-invariant and such that $L^2 = 4n$ for some non-zero positive integer n. We have the following results :*

1. *The linear system $|L|$ is base point free and not hyperelliptic.*
2. *If $L^2 \geq 8$, then the projective model is only defined by quadrics. Moreover, in this case, the number of quadrics Q_{4n} defining X is $Q_{4n} = 2n^2 - 3n + 1$.*

3.6.1 Preliminary Lemmas

We introduce the following two Lemmas which will be very useful for the proof of this Theorem.

Lemma 3.2. *Let L be a primitive ample class which is M_{20}-invariant on the K3 surface X. We assume that there exists a curve C such that $C \cdot L = k \neq 0$. Then $4n$ divides kI, where $I := \left[\mathrm{NS}(X) : \mathbb{Z}L \oplus \mathbb{L}_{20}^{\perp}\right]$. Moreover n divides $10k^2$.*

Proof. First we will show that $4n$ divides kI. Note that $(k\mathbb{Z}+4n\mathbb{Z})/4n\mathbb{Z}$ is a subquotient of $\mathrm{NS}(X)/\left(\mathbb{Z}L \oplus \mathbb{L}_{20}^{\perp}\right)$. In fact the map

$$\begin{array}{ccc} \mathrm{NS}(X) & \longrightarrow & \mathrm{NS}(X)\cdot L \quad \subset \mathbb{Z} \\ v & \longmapsto & v \cdot L \end{array}$$

induces a surjection

$$\mathrm{NS}(X)/\mathbb{Z}L \oplus \mathbb{L}_{20}^{\perp} \twoheadrightarrow \mathrm{NS}(X)\cdot L/4n\mathbb{Z}.$$

By assumption there exists a curve C such that $L \cdot C = k$, so $\mathrm{NS}(X)\cdot L \supset k\mathbb{Z} + 4n\mathbb{Z}$. Thus $I' := [k\mathbb{Z}+4n\mathbb{Z} : 4n\mathbb{Z}]$ divides I and, since $I' = \frac{4n}{\gcd\{k,4n\}} \in \frac{4n}{k}\mathbb{Z}$, we obtain $I \in \frac{4n}{k}\mathbb{Z}$. Hence $4n$ divides kI.

Now we can deduce that n divides $10k^2$. By [3, Chapter I, Lemma 2.1] we have

$$I^2 = \frac{\det(\mathbb{Z}L)\cdot \det\left(\mathbb{L}_{20}^{\perp}\right)}{\det \mathrm{NS}(X)} = \frac{640n}{\det \mathrm{T}_X}.$$

Since $\frac{kI}{4n}$ is an integer, so is its square

$$\frac{k^2I^2}{16n^2} = \frac{10k^2}{n\left(\det \mathrm{T}_X/4\right)}.$$

Note that $\det \mathrm{T}_X = 4\left(4ac - b^2\right)$ is always divisible by 4, thus we get the Lemma. $\square$

As a consequence we have the following result.

Lemma 3.3. *Under the assumptions of Lemma 3.2, if we assume* $k \in \{1, 2, 3\}$*, then* $(n, k) \in \{(1, 2), (2, 2), (10, 2)\}$*, and* X *is one of the examples studied in [7] (see Proposition 3.1).*

Proof. Suppose that $k \in \{1, 2, 3\}$ and let us see what happens in each case.

- Case $k = 1$: by Lemma 3.2 we know that n divides 10, i.e. $n \in \{1, 2, 5, 10\}$. Moreover $4n$ should divide I, but if we look at Table C.1 in Appendix C we see that it is impossible for each of the four cases.

- Case $k = 2$: by Lemma 3.2 we know that n divides 40, i.e. $n \in \{1, 2, 4, 5, 8, 10, 20, 40\}$. Moreover $4n$ should divide $2I$, and if we look at Table C.1 in Appendix C we see that it is possible only for $n = 1$, $n = 2$ and $n = 10$ with X_{Ko}.

- Case $k = 3$: by Lemma 3.2 we know that n divides 90, i.e. $n \in \{1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90\}$. Moreover $4n$ should divide $3I$, but if we look at Table C.1 in Appendix C we see that it is impossible.

□

3.6.2 Proof of Theorem 3.2

For the following, let L be a primitive ample M_{20}-invariant class such that $L^2 = 4n$, with $n \in \mathbb{Z}_{\geq 1}$.

Proof of Theorem 3.2. 1. In order to prove that $|L|$ is not hyperelliptic, we first need to prove that $|L|$ has no fixed part. By [45, Section 3.8], either $|L|$ has no fixed part or $L = aE + \Gamma$, where a is a positive integer, E an elliptic curve and Γ an irreducible (-2)-curve such that $E \cdot \Gamma = 1$. If we suppose that we are in the second case, then $E \cdot L = E \cdot \Gamma = 1$ and we are under the assumptions of Lemma 3.2. However by Lemma 3.3 we know that there is no integer n such that $L^2 = 4n$ and $L \cdot C = 1$ for some curve C. So the case $L = aE + \Gamma$ is impossible and we conclude that the linear system $|L|$ has no fixed part.

To show that $|L|$ is not hyperelliptic we use the result of [48, Theorem 5.2] which states that $|L|$ is hyperelliptic only in the following two cases :

- There exists an irreducible curve E of genus 1 such that $E \cdot L = 2$.

- There exists an irreducible curve B of genus 2 such that $L = 2B$.

First, assume that there exists an irreducible curve B of genus 2 such that $L = 2B$. Remark that $B^2 = 2$, so $L^2 = 4B^2 = 8$. However, we already know that in this case the model is not hyperelliptic by [7].

Now suppose that there exists an irreducible curve $E \in \mathrm{NS}(X)$ of genus 1 such that $E \cdot L = 2$. We are another time under the assumptions of Lemma 3.2 and by Lemma 3.3 we know that $n \in \{1, 2, 10\}$. Then X corresponds to one of the three surfaces X_{Mu}, X_{BH} or X_{Ko} already studied (see Proposition 3.1) and so this second point is not possible either. Hence $|L|$ is never hyperelliptic.

2. It remains to prove the second part of Theorem 3.2, i.e. assuming $L^2 \geq 8$ the surface is defined by an intersection of quadrics. By [48, Theorem 7.2] this is true except in the following two cases :

- there exists an irreducible curve E of genus 1 with $E \cdot L = 3$, or
- $L = 2B + F$ where B is an irreducible curve of genus 2 and F is an irreducible rational curve such that $B \cdot F = 1$.

By Lemma 3.3, the first point is impossible. The second point is not possible either, since $(2B + F)^2 = 10 \neq 4n$.

It remains to compute the number of quadrics. Suppose that we are not in the case given by Proposition 3.5 so that $X \subset \mathbb{P}^N$ exists with the embedding given by L such that $L^2 = 4n$. We know that $N = 2n + 1$ by Proposition 1.1. Following [48, Section 6.5.3], one can compute the number of quadrics defining the surface as

$$\dim \mathcal{S}^2 H^0(L) - \dim H^0(2L).$$

On the one hand we have $\dim \mathcal{S}^2 H^0(L)$ which corresponds to the total number of quadrics in the projective space $\mathbb{P}^{2n+1}$, that is $\binom{2n+3}{2}$. On the other hand, by Riemann Roch theorem (cf. Theorem 1.3) we can compute $\dim H^0(2L) = 2 + 8n$. Finally, we conclude that the number of quadrics defining X is

$$Q_{4n} = \binom{2n+3}{2} - (2 + 8n) = 2n^2 - 3n + 1.$$

$\square$

Remark 3.6. *Observe that by [48, §4.1], since $|L|$ is not hyperelliptic and base point free, it is actually very ample.*

3.7 Non primitive polarizations

With the computations of T_X (see Appendix C), one could remark that there are similarities between some cases. For example, we get the same lattice T_X when $L^2 = 4$ and $L^2 = 16$, as well as in cases $L^2 = 8$ and $L^2 = 32$. Also, there is a relation between these embeddings through the following result :

Proposition 3.6. *Given the embedding $\langle 4n \rangle \hookrightarrow \mathbb{L}_{20}$, for all integers $r \geq 1$ there exists an embedding $\langle r^2 4n \rangle \hookrightarrow \mathbb{L}_{20}$ such that the lattice T_X obtained for L is the same as the one for rL.*

Proof. For the existence of the embedding $\left\langle r^2 4n \right\rangle \hookrightarrow \mathbb{L}_{20}$, we can assume that there exists $\lambda_0, \mu_0, \delta_0 \in \mathbb{Z}$ such that $L \mapsto \lambda_0 e + \mu_0 f + \delta_0 h$. Hence

$$rL \longmapsto (r\lambda_0)\, e + (r\mu_0)\, f + (r\delta_0)\, h$$

is the desired embedding. The transcendental lattice T_X obtained is obviously the same. $\square$

The previous Proposition allows us to describe explicitly some *non-primitive* cases, i.e. cases where the polarization is multiple of some other polarization L. Throughout this section, X_{4n} will denote the K3 surface admitting the polarization L with $L^2 = 4n$. Thanks to Theorem 3.2, we are able to compute the number of quadrics defining X_{4n}. In cases which are not primitive, we can be more precise and give explicitly the equations of the quadrics defining the surface. To do this, we first recall the *Veronese embedding* ν_d^n of degree d defined in Section 1.5.2 :

$$\begin{array}{cccc} \nu_d^n : & \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ & (x_0 : \cdots : x_n) & \longmapsto & \left(x_0^d : x_1^d : \cdots : x_n^d : x_0^{d-1} x_1 : x_0^{d-1} x_2 : \ldots \right) \end{array}$$

where $N = \binom{n+d}{d} - 1$.

3.7.1 Cases $L^2 = 4 \cdot (4n)$

We will first show explicitly some examples and then state the general property.

Example 3.1 (Case $L^2 = 16$)**.** *To study the case $L^2 = 16$, we first consider the surface X_{Mu} with the polarisation given by M, with $M^2 = 4$. By Proposition 3.1, X_{Mu} is the zero locus of a quartic in $\mathbb{P}^3$ whose equation is given in Proposition 3.1. Put $L = 2M$ and consider the Veronese embedding of degree 2 :*

$$\nu_2^3 : \mathbb{P}^3 \hookrightarrow \mathbb{P}^9.$$

Let $(y_0 : \ldots : y_9)$ be the coordinates of $\mathbb{P}^9$. The image of the equation defining X_{Mu} in $\mathbb{P}^9$ via ν_2^3 is given by

$$y_0^2 + y_1^2 + y_2^2 + y_3^2 - 6\left(y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2 + y_9^2\right) \tag{3.4}$$

The image $\nu_2^3\left(\mathbb{P}^3\right)$ is defined by the quadrics given by the zero loci of the 2×2 minors of the matrix

$$\mathcal{V} := \begin{pmatrix} y_0 & y_4 & y_5 & y_6 \\ y_4 & y_1 & y_7 & y_8 \\ y_5 & y_7 & y_2 & y_9 \\ y_6 & y_8 & y_9 & y_3 \end{pmatrix}.$$

This gives 20 equations of quadrics. Let us consider the K3 surface X_{16} in $\mathbb{P}^9$ given by the 20 quadrics obtained as minors of the matrix $\mathcal{V}$ plus the quadric defined by the equation (3.4)*. The number of quadrics is $Q_{16} = 21$, as desired. As observed in Proposition 3.6, the surface X_{16} admits a polarization L such that $L^2 = 4M^2 = 16$ and it inherits the action of M_{20}, as well as the action of μ_2 described in [7, Section 4].*

Example 3.2 (Case $L^2 = 64$)**.** *We have previously shown a model of X_{16} in $\mathbb{P}^9$ defined by $Q_{16} = 21$ quadrics. In order to exhibit a projective model of a K3 surface X_{64} with a polarization L with $L^2 = 64$, we consider the Veronese embedding $\nu_2^9 : \mathbb{P}^9 \hookrightarrow \mathbb{P}^{54}$. The 21 quadrics defining $X_{16} \subset \mathbb{P}^9$ give 21 hyperplanes in $\mathbb{P}^{54}$, thus their intersection determines a 33-dimensional space. The desired surface $X_{64} \subset \mathbb{P}^{33}$ is given by the restriction of the quadrics defining $\nu_2^9\left(\mathbb{P}^9\right)$ to this 33-dimensional space.*

Example 3.3 (Case $L^2 = 32$)**.** *By [7, Section 5] the K3 surface X_{BH} admits a polarization M such that $M^2 = 8$. Taking $L = 2M$, we obtain a polarization with $L^2 = 32$ on the same surface. We would like to describe its projective model which is in $\mathbb{P}^{17}$. By Proposition 3.1 (case 3.), X_{BH} is the intersection of 3 quadrics in $\mathbb{P}^5$. In order to exhibit the projective model of X_{BH} in $\mathbb{P}^{17}$ (cf. Proposition 1.1), we consider the Veronese embedding*

$$\nu_2^5 : \mathbb{P}^5 \longhookrightarrow \mathbb{P}^{20}.$$

The image of $\mathbb{P}^5$ via ν_2^5 is the zero locus of quadrics and their number is

$$\frac{1}{2}\left(\binom{6}{2}^2 + \binom{6}{2}\right) - \binom{6}{4} = 105.$$

The last term $\binom{6}{4}$ considers the Plücker relations. The images of q_1, q_2 and q_3 via ν_2^5 give three linear equations, thus q_1, q_2 and q_3 define 3 hyperplanes in $\mathbb{P}^{20}$. Their intersection is a 17-dimensional space and, in this projective space of dimension 17, the equation of X is given by the restrictions of the 105 equations defining $\nu_2^5\left(\mathbb{P}^5\right)$.

In general let $M^2 = 4n$. Recall that the number of quadrics defining X_{4n} by Theorem 3.2 is $Q_{4n} = 2n^2 - 3n + 1$. If we consider $L = 2M$, therefore $L^2 = 4M^2 = 16n$ and we expect to find a projective model of X_{16n} in $\mathbb{P}^{8n+1}$ by Proposition 1.1. We consider the Veronese embedding of degree 2 :

$$\nu_2^{2n+1} : \mathbb{P}^{2n+1} \longhookrightarrow \mathbb{P}^{2n^2+5n+2}.$$

The Q_{4n} quadrics defining X_{4n} give via ν_2^{2n+1} a set of independent hyperplanes in $\mathbb{P}^{2n^2+5n+2}$, whose number is $Q_{4n} = 2n^2 - 3n + 1$. Thus we obtain a subspace of dimension

$$(2n^2 + 5n + 2) - (2n^2 - 3n + 1) = 8n + 1$$

as expected. The surface X_{16n} is given by the restriction of the quadrics defining $\nu_2^{2n+1}\left(\mathbb{P}^{2n+1}\right)$ to this space of dimension $8n+1$. Observe that this argument works for any value of $M^2 = 4n$. We showed explicitly what happens when $M^2 = 4$, $M^2 = 16$ and $M^2 = 8$ in the previous examples.

3.7.2 Cases $L^2 = 4r^2$

Another interesting case is when we take $L = rM$ with $M^2 = 4$. Let X_{Mu} be the K3 surface defined in Proposition 3.1, with $M^2 = 4$ and whose projective model is given by the zero locus of a polynomial f_4 of degree 4 in $\mathbb{P}^3$. We show how to obtain the projective model of the surface with the embedding $L = rM$, that is having $L^2 = 4r^2$.

Example 3.4 (Case $L^2 = 36$, $r = 3$)**.** *We expect a model of X_{36} in $\mathbb{P}^{19}$ by Propositon 1.1. Recall that by Proposition 3.1 we have $X_{\mathrm{Mu}} = \{f_4 = 0\} \subset \mathbb{P}^3$ where f_4 is the following quartic polynomial :*

$$f_4\left(x_0 : \cdots : x_3\right) = \sum_{i=0}^{3} x_i^4 - 6 \sum_{j \neq k} x_j x_k.$$

For $i = 1 \ldots 10$, we denote by q_i one of the 10 elements of the basis of quadric polynomials in $(x_0, \ldots, x_3)$, that is x_0^2, x_1^2, x_2^2, $x_3^2, \ldots, x_2x_3$. According to [23, Example 2.4], one can prove that X_{Mu} can also be defined as the intersection of the zero loci of $q_i \cdot f_4$, that is :

$$X_{\mathrm{Mu}} = V\left(\{q_i\left(x_0, \ldots, x_3\right) f\left(x_0, \ldots, x_3\right) \,;\, i = 1 \ldots 10\}\right).$$

Thus X_{Mu} is defined as the intersection of the zero loci of 10 polynomials $g_1, \ldots, g_{10}$ of degree 6. Let $\nu_3^3 : \mathbb{P}^3 \hookrightarrow \mathbb{P}^{19}$ be the Veronese embedding of degree 3. The image of each g_i is a polynomial of degree 2 in $\mathbb{P}^{19}$. Thus we obtain X_{36} as the intersection of the 10 quadrics in $\mathbb{P}^{19}$ given by the g_i's and the quadrics defining the image of the Veronese embedding.

Example 3.5 (Case $L^2 = 100$, $r = 5$)**.** *If $r = 5$ and we consider $L = 5M$ with $M^2 = 4$, then $L^2 = 100$. As before, let f_4 be the polynomial of degree 4 defining the surface X_{Mu}. The zero locus $\{f_4 = 0\}$ can also be described by the intersection of the zero loci of*

$$x_0 f_4 = x_1 f_4 = x_2 f_4 = x_3 f_4 = 0. \tag{3.5}$$

Via the Veronese embedding $\nu_5^3 : \mathbb{P}^3 \to \mathbb{P}^{\binom{8}{5}-1} = \mathbb{P}^{55}$, the image of the surface X_{Mu} is a K3 surface admitting a polarization $L = 5M$. The equations (3.5) *define 4 hyperplanes in $\mathbb{P}^{55}$, thus the projective model of X_{100} is contained in $\mathbb{P}^{51}$.*

More generally, for any integer $r > 3$, thanks to the Veronese embedding $\nu_r^3 : \mathbb{P}^3 \hookrightarrow \mathbb{P}^{\binom{r+3}{3}-1}$ one obtains a K3 surface admitting a polarization $L = rM$ and so $L^2 = 4r^2$. This surface is obtained as the image of X_{Mu} via ν_r^3. As observed, the surface X_{Mu} can be described by the intersection of the zero loci of

$$q_1 f_4 = \ldots = q_s f_4 = 0 \tag{3.6}$$

with $s = \binom{r-1}{3}$ and the q_i's are monomials of degree $r - 4$. The equations (3.6) define $\binom{r-1}{3}$ hyperplanes in $\mathbb{P}^{\binom{r+3}{3}-1}$. Hence, as expected, we obtain a projective model for the K3 surface X_{4r^2} in a space of dimension

$$\binom{r+3}{3} - 1 - \binom{r-1}{3} = 2r^2 + 1.$$

3.8 Existence of infinitely many K3 surfaces admitting a symplectic action of M_{20}

We end this Chapter by proving the second main Theorem (cf. Theorem 3.1) presented at the beginning.

Theorem 3.6. *There exist infinitely many K3 surfaces admitting a faithful and symplectic action of M_{20}.*

It is already known by [7, Propostion 2.1] that there are at most countably many K3 surfaces with a symplectic and faithful action of M_{20}. According to previous Sections, we can consider the polarization $L^2 = 4p$ with p a prime number. If equation (3.1) admits a solution for $L^2 = 4p$, we can repeat the arguments used before and obtain a K3 surface admitting symplectic action of M_{20}. Moreover, since p is a prime number, this surface will be a new one meaning that it does not admit a polarization $L^2 = 4m$ with $m < p$.

The main tool to prove the Theorem is a Lemma of Kondō given in [33, Lemma 8.24]. In order to enunciate it we need a few definitions.

Definition 3.1. *Consider $d \in \mathrm{NS}(X)$ such that $d^2 = -2$. We define a reflection s_d of $H^{1,1}(X, \mathbb{R})$ by*

$$\forall x \in H^{1,1}(X, \mathbb{R}),\ s_d(x) = x + \langle x, d \rangle d.$$

For what follows we denote by $W(X)$ the subgroup of $O\left(H^{1,1}(X, \mathbb{R})\right)$ generated by all the reflections $\left\{s_d : d \in \mathrm{NS}(X) \text{ and } d^2 = -2\right\}$. Moreover we denote by $\mathcal{C}^+(X)$ the *positive cone* that is the connected component of the cone $\mathcal{C}(X) = \left\{x \in H^{1,1}(X, \mathbb{R}) \mid \langle x, x \rangle > 0\right\}$ containing a Kähler class.

Lemma 3.4 ([33, Lemma 8.24])**.** *Suppose that the K3 surface X is projective and let G be a finite subgroup of $O\left(H^2(X, \mathbb{Z})\right)$. Then there exists $w \in W(X)$ such that the conjugate $w^{-1} \circ G \circ w$ acts on X as automorphisms if and only if*

1. *G preserves the holomorphic 2-form on X;*
2. *$H^2(X, \mathbb{Z})^G \cap \mathcal{C}^+(X) \neq \{0\}$;*
3. *$\left(H^2(X, \mathbb{Z})^G\right)^\perp \cap \mathrm{NS}(X)$ contains no elements of square -2.*

Now we are ready to prove the Theorem and this will be done in two steps. First we will show that there exists infinitely many K3 surfaces with the previous assumptions. Then we will show that these surfaces admit well an action of M_{20} by using the above Lemma.

Proof of Theorem 3.1. In order to prove that there exist infinitely many such K3 surfaces, it suffices to show that there are infinite number of primes p such that the polarization $L^2 = 4p$ defines an embedding and thus a K3 surface. In order to answer this, we need to check that there is an infinite number of primes p such that the equation

$$4p = (2\lambda - \delta)^2 + (2\mu - \delta)^2 + 10\delta^2. \tag{3.7}$$

admits a solution (λ, μ, δ). Observe that if $p \equiv 1\ (4)$, one can take $\delta = 0$. So the equation becomes

$$p = \lambda^2 + \mu^2$$

and Fermat's theorem of two squares [15, Chapter 1.1] ensures the existence of two such integers λ and μ. By the weak form of Dirichlet theorem, there exist infinitely many primes congruent to 1 modulo 4. Then there exists infinitely many primes such that the equation (3.7) admits a solution $(\bar{\lambda}, \bar{\mu}, 0)$. It follows that there is an infinite number of pairs (L, p) such that $L^2 = 4p$ and L gives a singular K3 surface.

It remains to prove that the group M_{20} acts on these surfaces. First we consider an embedding of $\mathbb{L}_{20}$ in Λ_{K3} that is isometric to the embedding of $H^2(X', \mathbb{Z})^{M_{20}}$ in $H^2(X', \mathbb{Z})$ for some K3 surface X' admitting an action of the group M_{20}. We take an element $L \in \mathbb{L}_{20}$ such that $L^2 = 4p$ as above and a line $\mathbb{C}\omega \subset \mathbb{L}_{20} \otimes_Z \mathbb{C}$ that is orthogonal to L and such that $\omega \cdot \omega = 0$ and $\omega \cdot \bar{\omega} > 0$. In this way ω can be viewed as an element of the period domain $\Omega_{\Lambda_{K3}}$. By the surjectivity of the period map of K3 surfaces, there exists a K3 surface X and an ample line bundle L_X on X such that the isometry $H^2(X, \mathbb{Z})$ allows us to identify L_X to L and the 1-dimensional subset $H^0\left(X, \Omega_X^2\right) \subset H^2(X, \mathbb{C})$ to $\mathbb{C}\omega$.

We can now apply Lemma 3.4 checking the three conditions with $G = M_{20}$:

1. The group M_{20} has to preserve $H^0\left(X, \Omega_X^2\right)$ and this is the case by construction.
2. Since the class L_X is in $H^2(X, \mathbb{Z})^G \cap \mathcal{C}^+(X)$, the second condition is clearly satisfied.
3. Finally $\mathbb{L}_{20}^{\perp}$ contains no element of square -2 by [41, Lemma 4.2], so the last condition is also verified.

Hence the Mathieu group M_{20} acts (symplectically) on X with $L_X \in H^2(X, \mathbb{Z})^{M_{20}}$ as desired. $\square$

CHAPTER 4

Irreducible holomorphic symplectic manifolds

4.1 Introduction

This last Chapter can be viewed as a generalization of the previous one. It is the topic of a second article (cf. [14]) which is a joint work with Paola Comparin and Pablo Quezada Mora (*Universidad de La Frontera, Chile*) :

Irreducible Holomorphic Symplectic Manifolds with an action of $\mathbb{Z}_3^4 : \mathcal{A}_6$,
https://arxiv.org/abs/2305.05037

An *Irreducible Holomorphic Symplectic* (or just IHS) *manifold* is a compact Kähler smooth manifold which is simply connected and such that $H^0\left(X, \Omega_X^2\right) = \mathbb{C}\omega_X$ where ω_X is an everywhere non-degenerate holomorphic 2-form (we say that ω_X is a *symplectic 2-form*). This last condition implies that $\dim H^0\left(X, \Omega_X^2\right) = 1$ and that the dimension of such a manifold is even. In particular, the case $\dim X = 2$ corresponds to K3 surfaces. Up to deformation, there are four known examples :

- K3$^{[n]}$-type which are deformation equivalent to *Hilbert schemes of n points of a K3 surface*;
- $\mathrm{Km}(\mathcal{A})^{[n]}$-type which are deformation equivalent to *generalized Kummer manifolds*;
- OG_6 and OG_{10} which are *O'Grady's examples* in dimension 6 and 10 respectively.

As for K3 surfaces, given a finite group G acting on an IHS manifold X, we can study the action of $g \in G$ on the 2-form ω_X, which is the multiplication by a complex scalar $\alpha(g)$. In dimension 2 (i.e. X is a K3 surface) we already know that $H^2(X,\mathbb{Z})$ is unimodular and Nikulin used this fact in [41] to classify finite abelian groups acting faithfully and symplectically on X. In higher dimension, we loose the fact that $H^2(X,\mathbb{Z})$ is unimodular and so we have to find another approach to study finite groups acting symplectically on X.

The first main work we can cite is the one of Beauville in [4] where he generalized several results of Nikulin, and then Mongardi in [36] added new results. In particular these works gave a way to classify finite symplectic groups in the case of K3$^{[n]}$-type. In Chapter 3 we saw that Mukai studied finite groups acting symplectically on K3 surfaces. He proved that every such group is a subgroup of the Mathieu group M_{23} (cf. [39]), giving a complete classification of them and showing that they are exactly subgroups of one of the 11 groups in the Table of [39, Example 0.4]. This work of classification for IHS manifolds of K3$^{[n]}$-type was done by Höhn and Mason in [25] for the case $n = 2$. Thanks to the results of Beauville and Mongardi, they classified all the finite symplectic groups acting on IHS manifold of K3$^{[2]}$-type and they found 15 maximal groups among them.

In the same way of the Mathieu group M_{20} for K3 surfaces (see Section 3.3), we will be interested in the group with biggest order which acts symplectically on IHS manifold of K3$^{[2]}$-type. In this sense, this work can be viewed as a natural generalization of the previous Chapter. Looking at [25, Table 13] the biggest group here is $(\mathbb{Z}/3\mathbb{Z})^4 \rtimes \mathfrak{A}_6$ with order 29160. According to the notations in [25, Table 13] we denote this group by $\mathbb{Z}_3^4 : \mathfrak{A}_6$. The aim of this Chapter is to prove the results below. We start with the main Theorem which summarize the situation.

Theorem 4.1. *Let X be an IHS manifold of K3$^{[2]}$-type admitting a symplectic action of $G_0 = \mathbb{Z}_3^4 : \mathfrak{A}_6$ and a polarization L invariant by G_0. If G_0 can be extended to a group G through a non-symplectic automorphism σ of order m, then $m \in \{2,3,6\}$. Moreover the transcendental lattice T_X and the polarization L are as in the two following Propositions 4.1 and 4.2.*

Note that we can summarize the situation through the following exact sequence :

$$1 \longrightarrow G_0 := \ker\alpha \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1 \ .$$

For what follow we denote by $S^{G_0}(X)$ the invariant lattice $H^2(X,\mathbb{Z})^{G_0}$. The first Proposition is proven in Section 4.4.2.

Proposition 4.1. *We keep the notations and the assumptions of the previous Theorem and we suppose that $m = 2$ so that σ is a purely non-symplectic involution normalizing G_0 and such that $G = \langle\sigma\rangle G_0$. We denote by $\{e, f, h\}$ the standard basis of $S^{G_0}(X)$. Then, up to embedding and isometry of the polarization, we have the following five cases :*

1. $L = h$, $L^2 = 6$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$.
2. $L = e$, $L^2 = 6$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 18 \end{pmatrix}$.
3. $L = e - f$, $L^2 = 6$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 18 \end{pmatrix}$.
4. $L = e + f$, $L^2 = 18$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$.
5. $L = 2e - f$, $L^2 = 18$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$.

Remark 4.1. *According to [25, Table 9], the invariant lattice $S^{G_0}(X)$ is isometric to*

$$S^{G_0}(X) \simeq \begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Thus by "standard basis" we mean the basis obtained from this Gram matrix. We give more details about it in Remark 4.5.

The second Proposition cover the case $m = 3$ and it is proven in Section 4.4.3

Proposition 4.2. *If X admits a non-symplectic automophism of order 3, then the only possibility is $L = h$ and $S^{G_0}(X) = \mathbb{Z}L \oplus \mathrm{T}_X$. Moreover if $X = F(\mathcal{C})$ is the Fano variety of lines of a cubic fourfold, then $\mathcal{C}$ has to be the Fermat cubic fourfold*

$$\mathcal{C} := \left\{(x_0 : \cdots : x_5) \in \mathbb{P}^5 \,|\, x_0^3 + \cdots + x_5^3 = 0\right\},$$

$G_0 = \mathbb{Z}_3^4 : \mathfrak{A}_6$ acts faithfully and symplectically on X and X admits a non-symplectic automorphism of order 3.

In particular, as an analogue of the work of Kondō in [32], we are able to prove that there exists an IHS manifold of K3$^{[2]}$-type with the biggest automorphism group :

Theorem 4.2. *Let X be an IHS manifold of K3$^{[2]}$-type admitting an action of G such that*

$$1 \longrightarrow \mathbb{Z}_3^4 : \mathfrak{A}_6 \longrightarrow G \xrightarrow{\alpha} \mu_6 \longrightarrow 1$$

which would mean that G has the biggest possible order (which is 174960) among the IHS manifolds of K3$^{[2]}$-type. Then the only possibility is $L = h$ and

$$\mathrm{T}_X = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}.$$

Moreover, the Fano variety of lines $F(\mathcal{C})$ of the Fermat cubic fourfold is an example of IHS manifold of K3$^{[2]}$-type admitting such action of this G.

Note that the Theorem does not say anything about the unicity of this manifold. As mentioned in our paper (see [14, Section 3.2]), we suspect that this is the only IHS manifold of K3$^{[2]}$-type with the biggest group of automorphisms.

Remark 4.2. *A part of these results are also proven independently in a paper of Wawak (see [53]) where he computes all the finite groups acting faithfully on IHS manifolds of K3$^{[2]}$-type with a maximal symplectic part.*

4.2 Preliminaries for IHS manifolds

Let X be a manifold. As we mentioned before, X is said to be IHS if it is a smooth compact Kähler manifold which is simply connected and such that $H^0\left(X, \Omega_X^2\right) = \mathbb{C}\omega_X$ where ω_X is a symplectic 2-form on X which vanishes nowhere. The existence of such a 2-form is a strong condition since it implies that X has dimension even and the canonical bundle K_X is trivial. This also implies that the first Chern class $c_1(X)$ is zero. The following theorem gives an important role in the study of manifolds with first Chern class equal to zero (cf. [5, Theorem 2]).

Theorem 4.3 (Beauville-Bogomolov decomposition)**.** *Let X be a compact Kähler manifold with real first Chern class $c_1(X)_{\mathbb{R}}$ equal to zero. Then there exists a finite étale covering $\widetilde{X}$ of X which is :*

$$\widetilde{X} = T \times \prod_i C_i \times \prod_j X_j$$

where T is a complex torus, the C_i's are Calabi-Yau manifolds and the X_j's are IHS manifolds.

4.2.1 Hilbert scheme of points of a K3 surface

As a well-known example of IHS manifold we have the Hilbert scheme of n points of a K3 surface.

Definition 4.1. *The Hilbert scheme of n points of a K3 surface S is the space parametrizing the zero-dimensional subschemes $(Z, \mathcal{O}_Z)$ of length n. It is usually denoted by $S^{[n]}$. In the particular case $n = 2$ we also call $S^{[2]}$ the Hilbert square of S.*

We say that an IHS manifold is of *K3$^{[n]}$-type* if it is deformation equivalent to the Hilbert scheme of n points of a K3 surface. Following the steps of Beauville in [5, Section 6], we can explicitly construct them. Let S be a K3 surface and consider :

- $S^{(n)}$ the *n-th symmetric product* of S, i.e. the quotient of S^n by the symmetric group $\mathfrak{S}_n$;
- $\pi : S^n \longrightarrow S^{(n)}$ the quotient map;
- for $1 \leq i, j \leq n$, the *diagonal* $D_{i,j} \subset S^n$ is the set of $(x_1, \ldots, x_n)$ such that $x_i = x_j$;
- $D = \bigcup\limits_{i,j} D_{i,j}$ and $\Delta = \pi(D)$ that is the zero locus of cycles in the form $p_1 + \cdots + p_n$ such that $p_i = p_j$ for some $i \neq j$;
- $\varepsilon : S^{[n]} \longrightarrow S^{(n)}$ the natural map which associates to every finite scheme the associated 0-cycle, called *Hilbert-Chow morphism*.

The results we are going to give are summarized in [5, Section 6] and come from [17, 18, 30]. The Hilbert scheme $S^{[n]}$ is smooth and the singular locus of $S^{(n)}$ is the diagonal Δ. Moreover the morphism $\varepsilon : S^{[n]} \to S^{(n)}$ is birational, so that it is a desingularization of $S^{(n)}$, and $\varepsilon^{-1}(\Delta)$ is an irreducible divisor on $S^{[n]}$.

Theorem 4.4 ([5, Theorem 3])**.** *Let S be a K3 surface. Then $S^{[n]}$ is an IHS manifold of dimension $2n$.*

Remark 4.3. *This Theorem was initially stated by Beauville in [5, Theorem 3] with the additional assumption that the K3 surface S must be "generic". A complete proof of this Theorem without this assumption can be found for instance in [2, Proof of Proposition 2.25].*

In this Chapter we will be interested in the Hilbert square of some K3 surface S, that is $S^{[2]}$. The geometrical description is well understood in this case and first appear in [20]. We resume it with the following diagram :

$$\begin{tikzcd}
\mathrm{Bl}_D\left(S^2\right) \arrow[r] \arrow[d] & S^{[2]} \arrow[d, "\varepsilon"] \\
S^2 \arrow[r, "\pi"] & S^{(2)}
\end{tikzcd}$$

4.2.2 Marked IHS manifolds

Let X be an IHS manifold. On the lattice $H^2(X,\mathbb{Z})$ we have an integral bilinear form, whose existence is proved in [5, Theorem 5], called *Beauville-Bogomolov-Fujiki form* and denoted $\langle\cdot,\cdot\rangle_{\mathrm{BBF}}$ (or just $\langle\cdot,\cdot\rangle$ when there is no ambiguity). It extends naturally the cup-product that we get in dimension 2. Assume now that X is of K3$^{[n]}$-type ($n \geq 2$) and let S be a K3 surface such that $X = S^{[n]}$. We can fully describe the lattice $H^2(X,\mathbb{Z})$ thanks to the following property coming from [5].

Proposition 4.3. *There exists a primitive embedding of lattices*

$$i : H^2(S,\mathbb{Z}) \longrightarrow H^2(S^{[n]},\mathbb{Z})$$

such that $H^2\left(S^{[n]},\mathbb{Z}\right) = i\left(H^2(S,\mathbb{Z})\right) \oplus \mathbb{Z}\delta$ *for some* $\delta \in H^2\left(S^{[n]},\mathbb{Z}\right)$*. Moreover we have* $\delta^2 = -2(n-1)$*, so that*

$$H^2(S^{[n]},\mathbb{Z}) \simeq \Lambda_n := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1)\rangle.$$

The lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1)\rangle$ is called the *K3$^{[n]}$-lattice*. It is an even, non-degenerate lattice of rank 23 with signature $(3,20)$. In particular for the Hilbert square of S we have

$$H^2(S^{[2]},\mathbb{Z}) \simeq \Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle.$$

In Chapter 1 we defined the notions of marked K3 surface and period of a K3 surface and some results such as the Torelli theorem or the surjectivity of the period map. We naturally find these notions again in the case of IHS manifolds of K3$^{[n]}$-type and we give more details in the following.

Definition 4.2. *Let X be an IHS manifold of K3$^{[n]}$-type. We call a marking of X an isometry $\eta : H^2(X,\mathbb{Z}) \to \Lambda$. The pair (X,η) is called a marked IHS manifold.*

We say that two marked IHS manifolds (X_1,η_1) and (X_2,η_2) are *isomorphic* if there exists an isomorphism $\phi : X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ \phi^*$.

4.2.3 Local Torelli theorem

As we saw in Section 1.4.1, a fundamental result for the study of K3 surfaces is the Torelli theorem. Some generalizations for IHS manifolds of K3$^{[n]}$-type exist, but their formulations are more complicated. In order to enunciate one of them, the *local Torelli Theorem*, we need some preliminary definitions.

Definition 4.3. *The moduli space $\mathfrak{M}$ of marked IHS manifolds of K3$^{[n]}$-type is the set of marked IHS manifolds (Y, η) where Y is deformation equivalent to the Hilbert scheme of n points of a K3 surface, modulo isomorphism of marked IHS manifolds.*

We also introduced in Section 1.2.4 the period domain and the period map of a K3 surface. These definitions naturally extend in the case of IHS manifolds of K3$^{[2]}$-type :

- The *period domain* is the analytic open subset of a quadric of $\mathbb{P}^{22}$ given by

$$\Omega_\Lambda := \left\{\omega \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \,|\, \omega^2 = 0 \text{ and } \langle \omega, \overline{\omega} \rangle > 0\right\}.$$

- The *period* of a marked pair (X, η) is the point $\eta\left(H^{2,0}(X)\right) \in \mathbb{P}(\Lambda \otimes \mathbb{C})$. Note that this point lies in the period domain Ω_Λ.

- The period map $\mathcal{P}$ is defined as

$$\begin{array}{rccc} \mathcal{P}: & \mathfrak{M} & \longrightarrow & \Omega_\Lambda \\ & (X, \eta) & \longmapsto & \eta\left(H^{2,0}(X)\right) \end{array}$$

Recall that in Section 1.4.1 we gave a version of the global Torelli theorem for K3 surfaces in terms of moduli space. In the same way, we state now the local Torelli theorem for marked IHS manifolds (cf. [5, Theorem 5, b)]).

Theorem 4.5 (Local Torelli theorem). *Let (X, η) be a marked IHS manifold. The period map $\mathcal{P} : \mathfrak{M} \to \Omega_\Lambda$ is a local isomorphism.*

And concerning only the surjectivity we have the following Theorem (see [27, Theorem 8.1]).

Theorem 4.6. *The restriction of the period map $\mathcal{P} : \mathfrak{M} \to \Omega_\Lambda$ to a connected component $\mathfrak{M}^0$ of $\mathfrak{M}$ is surjective.*

We are not going to state a global version of the Torelli theorem here, but the reader is invited to check the results of Markman in [35] to have an idea.

4.3 Symplectic automorphisms on IHS manifolds

In this Section we introduce the symplectic group actions on IHS manifolds of K3$^{[n]}$-type. As we may expect, we have a similar vocabulary as for K3 surfaces. Let X be an IHS manifold of K3$^{[n]}$-type. We say that $\sigma \in \mathrm{Aut}(X)$ is *symplectic* if it acts trivially on the 2-form ω_X. Otherwise, we say that it is *non symplectic*. Remember that we have the following exact sequence :

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

where $G_0 := \ker \alpha$ is the symplectic part and $\alpha(G)$ is a finite subgroup of $\mathbb{C}^\times$ so it is cyclic. Hence it is isomorphic to the group of m-roots of the unity $\mu_m \simeq \mathbb{Z}/m\mathbb{Z}$ and it is the purely non symplectic part. Recall that in the case of K3 surfaces, the Theorem 1.10 gives a bound thanks to the *Euler totient function* φ that is $\varphi(m) \mid \mathrm{rk}\, \mathrm{T}_X$ (notice that in this case we have $\mathrm{rk}\, \mathrm{T}_X \leq 2$). This result was subsequently generalized by Beauville in [4, Section 4].

Proposition 4.4. *Let X be an IHS manifold and $G \subset \mathrm{Aut}(X)$ be a finite group. Then :*

- *an element $g \in G$ acts trivially on T_X if and only if $g \in G_0$;*
- *the representation of μ_m in T_X is isomorphic in $\mathrm{T}_X \otimes \mathbb{Q}$ to the direct sum of irreducible representations over $\mathbb{Q}$ of the cyclic group of order m having maximal rank. Since this maximal rank is $\varphi(m)$, it follows that $\varphi(m) \mid \mathrm{rk}\, \mathrm{T}_X$.*

In particular if X is an IHS manifold of K3$^{[n]}$-type, then we have the following condition on m :

$$\varphi(m) \leq 23 - \rho(X) \tag{4.1}$$

where $\rho(X) := \mathrm{rk}\, \mathrm{NS}(X)$ is the Picard number of X.

4.3.1 Invariant and co-invariant lattices

As for K3 surfaces, an important tool in the study of symplectic automorphisms is the *invariant lattice*. In this Chapter the notations will be different in order to be coherent with those used by Höhn and Mason in [25].

Definition 4.4. *Let $\mathcal{L}$ be a lattice (later it will be the K3$^{[2]}$-lattice Λ) and G a subgroup of $O(\mathcal{L})$. The invariant lattice is*

$$S^G(\mathcal{L}) := \{x \in \mathcal{L} \mid \forall g \in G,\ g \cdot x = x\}.$$

We also define the co-invariant lattice $S_G(\mathcal{L})$ as the orthogonal complement of $S^G(\mathcal{L})$ in $\mathcal{L}$.

Remark 4.4. *If (X, η) is a marked IHS manifold of K3$^{[2]}$-type, with a marking $\eta : H^2(X, \mathbb{Z}) \to \Lambda$, and $G \subset \mathrm{Aut}(X)$ a finite group acting on X, then we get an injective map $\nu : \mathrm{Aut}(X) \to O(\Lambda)$. In other words, we can identify G with its image in $O(\Lambda)$. Moreover we denote the invariant lattice (resp. co-invariant lattice) by $S^G(X) := S^G(\Lambda)$ (resp. $S_G(X) := S_G(\Lambda)$).*

Thanks to Mongardi we have the following results on the invariant and co-invariant lattices.

Lemma 4.1 ([36, Lemma 3.5])**.** *Let G be a finite group of symplectic automorphisms of a fourfold X of K3$^{[2]}$-type. Then*

1. *$S_G(X)$ is non degenerate and negative definite;*
2. *$S_G(X)$ contains no element with square -2;*
3. *$S_G(X) \subset \mathrm{NS}(X)$ and $\mathrm{T}_X \subset S^G(X)$;*
4. *G acts trivially on the discriminant group $A_{S_G(X)}$.*

The following Theorem is another result of Mongardi which generalizes the one of Nikulin in [41, Theorem 4.3].

Theorem 4.7 ([36, Theorem 3.6])**.** *Let $G \subset O(\Lambda)$ be a finite subgroup, where Λ is the K3$^{[2]}$-lattice. Suppose that the following holds :*

1. *$S_G(\Lambda)$ is non-degenerate and negative definite;*
2. *$S_G(\Lambda)$ contains no element with square -2.*

Then G is induced by a group of symplectic automorphisms of $\mathrm{Aut}(X)$ *for some marked IHS manifold (X, η) of K3$^{[2]}$-type.*

4.3.2 Maximal group action on K3$^{[2]}$-type manifolds

As we said before, Höhn and Mason classified all symplectic groups that act on $K3^{[2]}$-type IHS manifolds in their recent work [25]. To introduce one of their main result, we need the following notations and definitions :

- The *Leech lattice* Λ_{24} is the unique positive-definite, even, unimodular lattice of rank 24 with no roots (i.e. any element of norm 2).
- A *$\mathcal{S}$-lattice* is a sublattice $\mathcal{M} \subset \Lambda_{24}$ on which all elements are congruent modulo $2\Lambda_{24}$ to an element of $\mathcal{M}$ of norm 0, -4 or -6.

- The *Conway group* Co_0 is the group of automorphisms of the Leech lattice Λ_{24}.

Theorem 4.8 ([25, Theorem A]). *Let X be an IHS manifold of K3$^{[2]}$-type and let $G \subset \mathrm{Aut}(X)$ be a finite symplectic group. Then G is isomorphic to one of the following :*

(a) A subgroup of M_{23} with at least four orbits in its natural action on 24 elements.

(b) A subgroup of one of two subgroups $\left(\mathbb{Z}_3^5 : \mathbb{Z}_2\right) \times \mathbb{Z}_2^2$ and $\mathbb{Z}_3^4 : \mathfrak{A}_6$ of Co_0 associated to $\mathcal{S}$-lattices in the Leech lattice.

For each of these cases, Höhn and Mason computed explicitly the invariant and co-invariant lattices and they summarized all their results in [25, Table 12]. In particular there are 13 isomorphism classes of subgroups of type (a) and 2 subgroups of type (b) that are maximal. Moreover for each maximal case we have $\mathrm{rk}\, S_G(X) = 20$. Let $G \subset \mathrm{Aut}(X)$ be a finite group such that its symplectic part G_0 is maximal and $G/G_0 \simeq \mu_m$ is not trivial (i.e. $m \neq 1$). Then by Lemma 4.1 we have $S_{G_0}(X) \subset \mathrm{NS}(X)$ and it is negative definite. Moreover $\mathrm{NS}(X)$ contains an ample class (and we can assume that this class is G_0-invariant). Hence

$$\mathrm{rk}\, \mathrm{NS}(X) \geq \mathrm{rk}\, S_{G_0}(X) + 1 = 21$$

and with the equation 4.1 we get the following bound for m :

$$\varphi(m) \leq 2 \Longrightarrow m \in \{2, 3, 4, 6\}. \tag{4.2}$$

In [25, Table 12] we check that the group with biggest order is $G_0 := \mathbb{Z}_3^4 : \mathfrak{A}_6$ with $|G_0| = 29160$. Then if $G \subset \mathrm{Aut}(X)$ is a finite group such that G_0 is maximal, we get the following bound :

$$|G| \leq 29160 \times 6 = 174960.$$

In what follows we will be interested only in the action of the group $G_0 := \mathbb{Z}_3^4 : \mathfrak{A}_6$ (which is the group with biggest order) on an IHS manifold of K3$^{[2]}$-type X. As we said before $\rho(X) = 21$, $\mathrm{rk}\, \mathrm{T}_X = 2$ and $S^{G_0}(X)$ contains an ample class L so that $\mathrm{rk}\, S^{G_0}(X) = 3$. According to [25, Table 9], the invariant lattice $S^{G_0}(X)$ is isometric to

$$S^{G_0}(X) \simeq \begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

We will denote by $\{e, f, h\}$ the standard basis where $S^{G_0}(X)$ can be written as above.

Remark 4.5. *If $v := \lambda e + \mu f + \delta h$ is an element of $S^{G_0}(X)$, then*

$$\begin{pmatrix} \lambda & \mu & \delta \end{pmatrix} \begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \delta \end{pmatrix} = 6\left(\lambda^2 + \mu^2 + \delta^2 + \lambda\mu\right).$$

So the square of any element of $S^{G_0}(X)$ is in $6\mathbb{Z}$. A similar computation shows that if $u, v \in S^{G_0}(X)$, then $u \cdot v \in 3\mathbb{Z}$.

Finally we have a characterization of the isometry group of $S^{G_0}(X)$. By [25] we know that it has order 24. We can also compute the generators with MAGMA and we get :

Proposition 4.5. *Denote by ρ_1, ρ_2 and ρ_3 the isometries of $S^{G_0}(X)$ defined by the following matrices :*

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ \rho_2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_3 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The isometry group $O\left(S^{G_0}(X)\right)$ of $S^{G_0}(X)$ is generated by $-\mathrm{id}$, ρ_1, ρ_2 and ρ_3. Moreover, $O(S^{G_0}(X))$ does not contain any element of order 4.

4.3.3 The Fano variety of lines of a cubic fourfold

This last subsection is dedicated to the *Fano variety of lines of a cubic fourfold*. This is the only known example of $\mathrm{K3}^{[2]}$-type manifold admitting an action of the biggest possible finite group. It will be very useful in the following Section.

Definition 4.5. *Let $Y \subset \mathbb{P}^5$ be smooth cubic fourfold. The Fano variety of lines $F(Y)$ of Y is defined as*

$$F(Y) := \{[L] \in \mathbb{G}\mathrm{r}(1,5) \mid L \subset Y\}.$$

The Fano variety of lines of a cubic fourfold was studied by Beauville and Donagi in [6] where they proved that it is an IHS manifold of $\mathrm{K3}^{[2]}$-type (cf. [6, Proposition 2]). Moreover $F(Y)$ comes with a polarization h which is the restriction of the Plücker polarization of $\mathbb{G}\mathrm{r}(1,5)$ to $F(Y)$. Furthermore we have that $h^2 = 6$. Concerning the automorphisms of $F(Y)$, it is clear that an automorphism on Y naturally induces an automorphism on $F(Y)$. As a kind of converse, we have the following result coming from [19, Lemma 1.2 and Corollary 1.3].

Lemma 4.2. *An automorphism ψ of $F(Y)$ is induced by an automorphism of Y if and only if $\psi^*(h) = h$. The natural morphism* $\mathrm{Aut}(Y) \longrightarrow \mathrm{Aut}(F(Y))$ *is injective and its image, denoted by* $\mathrm{Aut}_h(F(Y))$*, consists of the automorphisms of $F(Y)$ which fix the polarization h.*

In particular let $\mathcal{C} := \left\{x_0^3 + \cdots + x_5^3 = 0\right\} \subset \mathbb{P}^5$ be the *Fermat cubic fourfold* in $\mathbb{P}^5$. By [34, Theorem 1.8] we know that $\mathcal{C}$ is the only cubic fourfold with the biggest possible automorphisms group, which is $\mathbb{Z}_3^5 : \mathfrak{S}_6$. Using the previous Lemma , we can consider the Fano variety of lines $F(\mathcal{C})$ and then $F(\mathcal{C})$ admits an action of $(\mathbb{Z}/3\mathbb{Z})^5 \rtimes \mathfrak{S}_6$. Since $\left|\mathbb{Z}_3^5 : \mathfrak{S}_6\right| = 174960$ coincides with the bound we found in (4.2), this group is a good candidate to be G in the exact sequence

$$1 \longrightarrow \mathbb{Z}_3^4 : \mathfrak{A}_6 \longrightarrow G \xrightarrow{\alpha} \mu_6 \longrightarrow 1 \ .$$

Moreover in [37, Example 7.4.2] Mongardi proved that

$$\mathrm{T}_{F(\mathcal{C})} \simeq A_2(3) \text{ and } S^{G_0}(F(\mathcal{C})) \simeq \langle 6 \rangle \oplus A_2(3),$$

where A_2 denotes the lattice associated to the root system of the same name and so

$$A_2(3) \simeq \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}.$$

4.4 Extensions of $\mathbb{Z}_3^4 : \mathfrak{A}_6$ by a non-symplectic automorphism

From now on X will be an IHS manifold of K3$^{[2]}$-type, $G \subset \mathrm{Aut}(X)$ a finite group and $G_0 = \mathbb{Z}_3^4 : \mathfrak{A}_6$ its symplectic part. In this Section, we suppose that there exists a non-symplectic automorphism of order m on X, that is we have the exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \mu_m \longrightarrow 1$$

and we want to describe precisely the manifolds admitting such action of G.

4.4.1 Preliminaries

We denote by σ the non-symplectic automorphism on X. Recall that the order m of σ is such that $m \in \{2, 3, 4, 6\}$ by equation (4.2). We can be more precise showing that the value $m = 4$ is not possible.

Proposition 4.6. *Let X be an IHS manifold of K3$^{[2]}$-type and $G \subset$ Aut(X) with symplectic part $G_0 = \mathbb{Z}_3^4 : \mathfrak{A}_6$. If there exists a non-symplectic automorphism of order m on X, then $m \in \{2, 3, 6\}$.*

Proof. Since G_0 is maximal we have that $m \in \{2, 3, 4, 6\}$ by equation 4.1. To discard the case $m = 4$, we only have to notice that the group $G/G_0 \simeq \mu_m$ acts on $S^{G_0}(X)$ faithfully, but $O\left(S^{G_0}(X)\right)$ does not contain any element of order 4 by Proposition 4.5. $\square$

Remark 4.6. *Observe that we only have to study the cases $m = 2$ and $m = 3$. Since 2 and 3 are coprimes, if we have a non-symplectic automorphism of order 2 and another of order 3 on X, then there will be also a non-symplectic automorphism of order 6 on X.*

For the following we assume $G = \langle\sigma\rangle G_0$ where σ is a non-symplectic automorphism normalizing G_0 and such that $\sigma^m \in G_0$ with $m \in \{2, 3\}$. The non-symplectic part $G/G_0 = \langle\sigma\rangle$ acts on $S^{G_0}(X)$ fixing a polarization L and acting as an isometry of order m on T_X. Note that $L^2 \in 6\mathbb{Z}$ (since L is an element of $S^{G_0}(X)$), it is primitive and $\mathrm{T}_X = (\mathbb{Z}L)^\perp \cap S^{G_0}(X)$.

Proposition 4.7. *Let L be a primitive element of $S^{G_0}(X)$ such that $L^2 = 6n$, with $n \geq 1$ an integer. Then, $\langle 6n\rangle \oplus \mathrm{T}_X$ is a sublattice of $S^{G_0}(X)$ of index m except in the case $n = 1$ where we get the equality*

$$S^{G_0}(X) = \langle 6\rangle \oplus \mathrm{T}_X, \text{ with } \mathrm{T}_X = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}.$$

To prove this Proposition, we will need the following Lemma.

Lemma 4.3. *Let $\mathcal{L}$ be a lattice and $G \subset O(\mathcal{L})$ a finite subgroup. Then the quotient $\mathcal{L}/\left(S^G(\mathcal{L}) \oplus S_G(\mathcal{L})\right)$ is of $|G|$-torsion.*

Proof. Consider $x \in \mathcal{L}$. Then we have the following decomposition :

$$|G|x = \sum_{g\in G} g(x) - \sum_{g\in G}(x - g(x)).$$

To prove the Lemma we first show that $\sum\limits_{g\in G} g(x) \in S^G(\mathcal{L})$ and $\sum\limits_{g\in G}(x - g(x)) \in S_G(\mathcal{L})$. On one hand the term $\sum\limits_{g\in G} g(x)$ is obviously G-invariant, so it lies in $S^G(\mathcal{L})$. On the other hand, consider $y \in S^G(\mathcal{L})$ and $g_0 \in G$. Then

$$\langle y, v\rangle = \langle g_0(y), g_0(v)\rangle = \langle y, g_0(v)\rangle$$

which implies that $\langle y, v - g_0(v)\rangle = 0$ and so $v - g_0(v)$ lies in the orthogonal complement of $S^G(\mathcal{L})$ in $\mathcal{L}$. In other words for all $g \in G$, $v - g(v) \in S_G(\mathcal{L})$. $\square$

Proof of Proposition 4.7. Let L be as in the Proposition. We want to find the values of n for which we have $\mathbb{Z}L \oplus \mathrm{T}_X = S^{G_0}(X)$ and those for which it is a sublattice strictly included in $S^{G_0}(X)$. By Lemma 4.1 we know that $\mathrm{T}_X \subset S^{G_0}(X)$. Moreover recall that the invariant lattice $S^{G_0}(X)$ is isometric to $\begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ in the basis $\{e, f, h\}$. Then T_X can be written as

$$\mathrm{T}_X \simeq \begin{pmatrix} 6a & 3b \\ 3b & 6c \end{pmatrix}$$

with $-a \leq b \leq a \leq c$ and $b \geq 0$ if $a = c$. In particular

$$27b^2 \leq d\left(\mathrm{T}_X\right) = 36ac - 9b^2.$$

It follows that if $S^{G_0}(X) = \mathbb{Z}L \oplus \mathrm{T}_X$, then

$$162 = \det\left(S^{G_0}(X)\right) = \det(L) \cdot \det\left(\mathrm{T}_X\right) = 6n\left(36ac - 9b^2\right)$$

which implies that $3 = n\left(4ac - b^2\right)$. Hence the only possible values for n are 1 or 3. Let's discuss what happens in these two cases.

- If $n = 1$, then $d\left(\mathrm{T}_X\right) = 27$, that is $4ac - b^2 = 3$. The only possibility is $a = b = c = 1$ and so $L = \pm h$.
- If $n = 3$, we get $b^2 + 1 = 4ac$. Since $b^2 + 1$ is not divisible by 4 for all $b \in \mathbb{Z}$, this case is impossible.

We conclude that for all other n, the lattice $\langle 6n \rangle \oplus \mathrm{T}_X$ is a strict sublattice of $S^{G_0}(X)$. In order to compute the index of $\mathbb{Z}L \oplus \mathrm{T}_X$ in $S^{G_0}(X)$ we follow [7, Proof of Lemma 2.9]. Since we are assuming that $\mathbb{Z}L \oplus \mathrm{T}_X$ is different from $S^{G_0}(X)$ and since L is primitive in $S^{G_0}(X)$, by [42, Section 5] the projection

$$S^{G_0}(X) / \left(\mathbb{Z}L \oplus \mathrm{T}_X\right) \longrightarrow A_L$$

is an embedding. This shows in particular that $S^{G_0}(X) / \left(\mathbb{Z}L \oplus \mathrm{T}_X\right)$ is cyclic. So by the previous Lemma it is of m-torsion and m is a prime integer. Hence $\mathbb{Z}L \oplus \mathrm{T}_X$ has index m in $S^{G_0}(X)$. $\square$

Thanks to the following result we can characterize the form of the matrix of the transcendental lattice depending if it admits an order 3 isometry.

Proposition 4.8 ([31, Theorem 51a]). *Let T be a positive definite rank 2 even lattice. Then T admits an automorphism of order 3 if and only if there exists $a \in \mathbb{Z}_{>0}$ such that*

$$T \simeq \begin{pmatrix} 2a & a \\ a & 2a \end{pmatrix}.$$

We are now ready to study the manifolds of K3$^{[2]}$-type admitting an action of G in the cases where σ is a non-symplectic automorphism of order 2 or 3.

4.4.2 Extensions by a non-symplectic involution

We start with the first case, studying the transcendental lattice of an IHS manifold X of K3$^{[2]}$-type such that G_0 acts on X together with a non-symplectic involution σ. In other words, the group G is spanned by G_0 and by σ. Moreover we denote by L a polarization on X fixed by G_0 such that $L^2 = 6n$, with $n \in \mathbb{Z}_{\geq 1}$. The following Proposition gives a condition on the integer n.

Proposition 4.9. *The only possible values of n such that there exists an embedding of $\langle 6n \rangle \oplus \mathrm{T}_X$ in the invariant lattice $S^{G_0}(X)$ are $n \in \{1, 3, 4\}$.*

Proof. To prove the result we use the Proposition 4.7. If $n = 1$ then $S^{G_0}(X) = \langle 6 \rangle \oplus \mathrm{T}_X$ with $\mathrm{T}_X = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$. Then the embedding obviously exists in this case.
Assume now that we are in the other case of Proposition 4.7, that is $L^2 = 6n$ (with $n > 1$) and $\langle 6n \rangle \oplus \mathrm{T}_X$ is a sublattice of index 2 in $S^{G_0}(X)$. Recall that T_X has the form

$$\mathrm{T}_X = \begin{pmatrix} 6a & 3b \\ 3b & 6c \end{pmatrix}$$

where a, b, c are some integers such that $-a \leq b \leq a \leq c$. So $d(\mathrm{T}_X) = 9\left(4ac - b^2\right)$. By [3, Chapter I, Lemma 2.1] we have

$$4 = \left[S^{G_0}(X) : \langle 6n \rangle \oplus \mathrm{T}_X\right]^2 = \frac{\det\left(\langle 6n \rangle \oplus \mathrm{T}_X\right)}{\det\left(S^{G_0}(X)\right)} = \frac{54n\left(4ac - b^2\right)}{162}$$

which implies that $n\left(4ac - b^2\right) = 12$. Then n divides 12, i.e $n \in \{2, 3, 4, 6, 12\}$. However the equation $n\left(4ac - b^2\right) = 12$ does not admit integer solutions if $n = 2, 6$ or 12. In fact :

- If $4ac - b^2 = 2$, then b has to be even that is $b = 2k$. Then we get $2\left(ac - k^2\right) = 1$ which is impossible to solve since $ac - k^2$ is an integer.
- If $4ac - b^2 = 6$, as above we get $2\left(ac - k^2\right) = 3$ for some integer k, and it is also impossible to solve.

- Finally since the integers 2 and 6 do not divide $4ac - b^2$, this is also the case for 12.

For the two remaining cases we have the solutions $a = b = c = 1$ for $n = 3$ and $a = c = 1$ and $b = 0$ for $n = 4$. Hence $n \in \{1, 3, 4\}$. □

Remark 4.7. *With the computations made in the previous proof we get*

$$\det \mathrm{T}_X = 4\left(ac - b^2\right) = \frac{9 \times 12}{n} = \frac{108}{n}.$$

The following result is a part of the main Theorem stated in the introduction (cf. 4.1). It summarizes all possible transcendental lattices (up to isometries of $S^{G_0}(X)$) for each possible value of n.

Proposition 4.10. *Let X be an IHS manifold of $K3^{[2]}$-type and assume that the group $G_0 = \mathbb{Z}_3^5 : \mathfrak{A}_6$ acts symplectically and faithfully on it. Assume also that X admits a purely non-symplectic automorphism σ of order 2 acting on it and normalizing G_0. We denote $G := \langle\sigma\rangle G_0$ and by $\{e, f, h\}$ the standard basis of $S^{G_0}(X)$. Then, up to embedding and isometry of the polarization, we have the following five cases :*

1. $L = h$, $L^2 = 6$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$.

2. $L = e$, $L^2 = 6$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 18 \end{pmatrix}$.

3. $L = e - f$, $L^2 = 6$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 18 \end{pmatrix}$.

4. $L = e + f$, $L^2 = 18$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$.

5. $L = 2e - f$, $L^2 = 18$ *and* $\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$.

Proof. Using Proposition 4.7, we have to study the embeddings $\mathbb{Z}L \hookrightarrow S^{G_0}(X)$ with $L^2 = 6n$, for $n \in \{1, 3, 4\}$. In each case we look for the integers λ, μ, δ such that

$$L = \lambda e + \mu f + \delta h$$

in order to determine the transcendental lattice. According to Remark 4.5, it is equivalent to solve the equation

$$n = \lambda^2 + \mu^2 + \delta^2 + \lambda\mu. \tag{4.3}$$

All the computations are resumed in the Table 4.1 given below. Note that in each case, the embedding $\langle 6n \rangle$ is chosen up to isometry of $S^{G_0}(X)$ (cf. Proposition 4.5).

L^2	Value of δ	Possibles (λ, μ)	Embedding up to isometry
$L^2 = 6$	$\delta = \pm 1$	$\{(0,0)\}$	$L \mapsto h$
	$\delta = 0$	$\{(\pm 1, 0), (0, \pm 1)\}$	$L \mapsto e$
		$\{(1,-1), (-1,1)\}$	$L \mapsto e - f$
$L^2 = 18$	$\delta = 0$	$\{(1,1), (-1,-1)\}$	$L \mapsto e + f$
		$\{(2,-1), (-2,1), (1,-2), (-1,2)\}$	$L \mapsto 2e - f$
$L^2 = 24$	$\delta = \pm 2$	$\{(0,0)\}$	$L \mapsto 2h$
	$\delta = \pm 1$	$\{(1,1), (-1,-1)\}$	$L \mapsto e + f + h$
		$\{(2,-1), (-2,1), (1,-2), (-1,2)\}$	$L \mapsto 2e - f + h$
	$\delta = 0$	$\{\pm 2, \pm 2\}$	$L \mapsto 2e$
		$\{(2,-2), (-2,2)\}$	$L \mapsto 2(e - f)$

Table 4.1: Possible embeddings for $m = 2$

- $\underline{L^2 = 6}$: Solving the equation 4.3 with $n = 1$, the possible values of δ are $\delta \in \{-1, 0, 1\}$. Using the description of $O\left(S^{G_0}(X)\right)$ given in Proposition 4.5 we can check that there are 3 possible orbits, up to the action of the isometry group of $S^{G_0}(X)$ (see Table 4.1).

  In the first case on which $L = h$ (which corresponds to $\delta = \pm 1$) we have the equality $S^{G_0}(X) = \langle 6 \rangle \oplus \mathrm{T}_X$ by Proposition 4.7 and we can write T_X on the basis $\{e, f\}$ as

  $$\mathrm{T}_X = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$$

  while in the other two cases we have that $\mathbb{Z}L \oplus \mathrm{T}_X$ is a sublattice of index 2 (by Proposition 4.7 again) and $\det(\mathrm{T}_X) = 108$ by Remark 4.7.

  Let us assume that $L = e$. It corresponds to one of the two cases with $\delta = 0$. Computing the transcendental lattice as $\mathrm{T}_X = (\mathbb{Z}L)^\perp \cap S^{G_0}(X)$, we obtain that T_X is spanned by h and $e - 2f$ and the intersection matrix is

$$\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 18 \end{pmatrix}.$$

In the other case with $\delta = 0$, we have $L = e - f$. Then the the transcendental lattice is spanned by h and $e + f$ and the intersection matrix is

$$\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 18 \end{pmatrix}.$$

- $\underline{L^2 = 18}$: First let us show that $\lambda^2 + \lambda\mu + \mu^2$ is never equal to 2. If λ and μ are both odd integers, or if one is odd and the other is even, then $\lambda^2 + \lambda\mu + \mu^2$ will be odd and so different from 2. Assume that λ and μ are both even integers (obviously different from 0). If $\lambda\mu > 0$ then $\lambda^2 + \lambda\mu + \mu^2 \geq 3$. So it remains to check what happens when λ and μ have different sign. As the equation is symmetric, we can assume

$$\lambda = 2a \text{ and } \mu = -2b, \text{ with } a, b \in \mathbb{Z}_{>0}.$$

Then $\lambda^2 + \lambda\mu + \mu^2 = 4\left(a^2 + b^2 - ab\right)$ and since $a^2 + b^2 - ab = (a-b)^2 + ab > 0$, it is impossible to have $\lambda^2 + \lambda\mu + \mu^2$ equal to 2.

It follows that the only possible value for δ is $\delta = 0$. There are two orbits of (λ, μ, δ) in $O(S^{G_0}(X))$ (see Table 4.1) and in both cases we have that $\mathbb{Z}L \oplus T_X$ is a sublattice of index 2 on $S^{G_0}(X)$ such that $\det(T_X) = 36$.

On the one hand, assume that $L = e + f$. Then we can compute the transcendental lattice as $\mathrm{T}_X = (\mathbb{Z}L)^{\perp} \cap S^{G_0}(X)$, obtaining the generators $e - f$ and h and the intersection matrix

$$\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

On the other hand, let assume that $L = 2e - f$. Then the transcendental lattice is spanned by f and h and the intersection matrix is also

$$\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

- $\underline{L^2 = 24}$: In this last case, we have $\delta \in \{0, \pm 1, \pm 2\}$ and we compute several possible orbits for the polarization, up to isometry (see Table 4.1). In all cases we obtain that $\mathbb{Z}L \oplus \mathrm{T}_X$ is a sublattice of index 2 in $S^{G_0}(X)$ such that $\det(\mathrm{T}_X) = 27$ thanks to Remark 4.7. The only non primitive embeddings are $L \mapsto e + f + h$ and $L \mapsto 2e - f + h$. In both cases the transcendental lattice is

$$\mathrm{T}_X = \begin{pmatrix} 6 & 0 \\ 0 & 72 \end{pmatrix}$$

whose determinant is different from 27. Hence the case $L^2 = 24$ is impossible.

□

4.4.3 Extensions by a non-symplectic automorphism of order 3

In this second case we assume now that X admits a non-symplectic automorphism of order 3. The following result allows us to describe the invariant and co-invariant lattices of such an IHS manifold of $\mathrm{K3}^{[2]}$-type. Moreover we will see that the Fano variety of lines of the Fermat cubic fourfold is an example of such an IHS manifold.

Proposition 4.11. *If X admits a non-symplectic automorphism of order 3, then $S^{G_0}(X) = \mathbb{Z}L \oplus \mathrm{T}_X$ and $L = h$ in $S^{G_0}(X)$.*

Proof. Recall that for all $u \in S^{G_0}(X)$ we have $u^2 \in 6\mathbb{Z}$. Combining this with Proposition 4.8 we get that there exists $a \in \mathbb{Z}_{>0}$ such that

$$\mathrm{T}_X \simeq \begin{pmatrix} 6a & 3a \\ 3a & 6a \end{pmatrix}.$$

Suppose that $L = \lambda e + \mu f + \delta h$ with $L^2 = 6n$ and where $\{e, f, h\}$ is the standard basis of $S^{G_0}(X)$. Recall that $A_2(3)$ denotes the sublattice of $\mathbb{Z}^2$ defined by the intersection matrix

$$A_2(3) \simeq \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}.$$

If $n = 1$, that is $L^2 = 6$, then the only possibility such that the intersection matrix of T_X get the previous form is $S^{G_0}(X) = \langle 6 \rangle \oplus A_2(3)$ and $L = \pm h$.

Assume now that $n > 1$. Then $\langle 6n \rangle \oplus \mathrm{T}_X$ is a sublattice of index 3 of $S^{G_0}(X)$ by Proposition 4.7. Moreover we have

$$9=\left[S^{G_0}(X):\langle 6n\rangle\oplus \mathrm{T}_X\right]^2=\frac{\det\left(\langle 6n\rangle\oplus \mathrm{T}_X\right)}{\det\left(S^{G_0}(X)\right)}=\frac{6n\left(27a^2\right)}{162}$$

which implies that $n=9$ and so $\mathrm{T}_X=A_2(3)$. In this case $L^2=54$ and according to Remark 4.5 we get the following equation :

$$\lambda^2+\mu^2+\delta^2+\lambda\mu=9. \tag{4.4}$$

The possible values of δ are $\delta\in\{0,\pm1,\pm2,\pm3\}$ and studying each case we get :

- If $\delta=\pm3$ then $\lambda=\mu=0$. So $L=3h$ and this is a non-primitive case.
- If $\delta=\pm2$ then the equation 4.4 has no integers solutions.
- If $\delta=\pm1$ then the equation 4.4 has no integers solutions.
- If $\delta=0$ then $L\in\{\pm3e,\pm3f,\pm3(e-f)\}$. However in each case the polarization is non-primitive.

So the case $n>1$ is impossible. Hence the only possible IHS manifold of K3[2]-type admitting a symplectic action of G_0 and a non-symplectic automorphism of order 3 is for $L=h$ and $S^{G_0}(X)=\mathbb{Z}L\oplus\mathrm{T}_X$ with $\mathrm{T}_X=A_2(3)$. Notice that this case corresponds to the case 1. in Proposition 4.10. □

As a consequence we get the second main Theorem of this Chapter.

Theorem 4.9. *Let X be an IHS manifold of K3[2]-type admitting a symplectic action of G_0. If X admits a non-symplectic automorphism of order 6 - which would means that X admits the action of the biggest possible finite group G with $|G|=174960$ - then the only possibility is $L=h$ and*

$$\mathrm{T}_X=\begin{pmatrix}6&3\\3&6\end{pmatrix}.$$

Moreover the Fano variety of lines $X=F(\mathcal{C})$ of the Fermat cubic fourfold

$$\mathcal{C}:=\left\{(x_0:\dots:x_5)\in\mathbb{P}^5\,|\,x_0^3+\dots+x_5^3=0\right\}$$

is an example of such IHS manifold of K3[2]-type, admitting the action of the biggest possible finite group $G\subset\mathrm{Aut}(X)$.

Proof. We saw in Section 4.3.3 that the Fano variety of lines of the Fermat cubic fourfold $F(\mathcal{C})$ admits a polarization h such that $h^2 = 6$ and has the following properties :

$$\mathrm{T}_{F(\mathcal{C})} \simeq A_2(3) \text{ and } S^{G_0}(F(\mathcal{C})) \simeq \langle 6 \rangle \oplus A_2(3).$$

Then we can take X to be $F(\mathcal{C})$ in Proposition 4.11. Moreover notice that the only possibility we get in Proposition 4.11 corresponds to the case 1. of Proposition 4.10. According to Remark 4.6, since 2 and 3 are coprime integers, the existence of a non-symplectic involution and a non-symplectic automorphism of order 3 ensure the existence of a non-symplectic automorphism of order 6. Since $G_0 = \mathbb{Z}_3^4 : \mathfrak{A}_6$ acts symplectically on X we have the exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \mu_6 \longrightarrow 1 \; .$$

Hence $|G| = 6 \times 29160 = 174960$. □

4.4.4 Existence

In this last subsection we prove the existence of each case we found previously.

Theorem 4.10. *Each case of Proposition 4.10 and Proposition 4.11 occurs as a pair of (X, G) where X is an IHS manifold of $K3^{[2]}$-type and $G \subset \mathrm{Aut}(X)$.*

Before we prove this Theorem, we need to recall the following two results of Nikulin given in Section 1.2.2 (up to some changes of notations to be coherent with the proof below) :

Proposition 4.12 ([42, Proposition 1.4.1])**.** *There is a natural one-to-one correspondence between the finite index even overlattices $M \supset T$ and the isotropic subgroups $H \subset A_T$.*

Proposition 4.13 ([42, Subsection 4])**.** *Let M be an overlattice of T. An isometry $\varphi \in O(T)$ can be extended to M if an only if $\overline{\varphi}(H) = H$, where H is the isotropic subgroup of A_T associated to M and $\overline{\varphi} \in O(A_T)$ is the image of φ under the projection $\pi : T \to A_T$.*

In the case where $T \subset M$ is a primitive sublattice, we can consider its orthogonal lattice $S = T^{\perp} \cap M$. Then M is an overlattice of $S \oplus T$. We denote by

$$H_M := M/(S \oplus T) \subset A_{S \oplus T} \simeq A_S \oplus A_T$$

the isotropic subgroup of $A_S \oplus A_T$ associated to M. We also denote the respective projections by $p_T : A_{T\oplus S} \to A_T$ and $p_S : A_{T\oplus S} \to A_S$ and we define $M_T := p_T(H_M)$ and $M_S := p_S(H_M)$. Since T is primitive in M, the restrictions of the projections to H_M are both isomorphisms. We finally define the map

$$\gamma := p_S \circ \left(p_T^{-1}\right)|_{M_T} : M_T \longrightarrow M_S$$

which is called the *gluing morphism*. The following Lemma is also due to Nikulin.

Lemma 4.4 ([42, Corollary 1.5.2]). *Let M be an even lattice with a primitive sublattice $T \subset M$ and orthogonal lattice S. For every isometry $\varphi \in O(T)$, there exists an isometry $\phi \in O(M)$ such that $\phi|_T = \varphi$ if and only if there exists an isometry $\psi \in O(S)$ such that*

$$\overline{\psi} \circ \gamma = \gamma \circ \overline{\varphi}.$$

Now we are able to prove the Theorem 4.10. It was mainly done by Comparin and Quezada in [14].

Proof of Theorem 4.10. Let Λ denotes the $\mathrm{K3}^{[2]}$-lattice, that is

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle.$$

For what follows we denote by T_X one of the T_X's given in Proposition 4.10. By [25] there is an embedding of $S^{G_0}(\Lambda)$ into Λ such that $S^{G_0}(\Lambda)^{\perp_\Lambda}$ is isometric to $S_{G_0}(\Lambda)$. We can consider the lattices T_Λ and L_Λ in $S^{G_0}(\Lambda)$ which are isometric to T_X and $\mathbb{Z}L$ respectively as in Proposition 4.10 or 4.11. If $\mathrm{T}_\Lambda \oplus L_\Lambda$ is not isometric to $S^{G_0}(\Lambda)$, then it is isometric to a sublattice of index 2, but in both cases $T^{\perp_\Lambda} \simeq S_{G_0}(\Lambda)$, where $T := \mathrm{T}_X \oplus \mathbb{Z}L$. Thus by Theorem 4.7 there exists a marked IHS manifold (X, η) with $G_0 \subset \mathrm{Aut}(X)$ acting symplectically on X via η. We can consider the marking such that $\mathrm{T}_X \simeq \mathrm{T}_\Lambda$ and $S_{G_0}(X) \simeq S_{G_0}(\Lambda)$. Furthermore we have that $S^{G_0}(X) \simeq S^{G_0}(\Lambda)$ and we can consider $L := \eta^{-1}(L_\Lambda) \subset S^{G_0}(X)$.

Now we want to construct the non-symplectic automorphism for each X. The case of Proposition 4.11 is analogous, so we focus only on Proposition 4.10. If $T := \mathbb{Z}L \oplus \mathrm{T}_X$ is not isometric to $S^{G_0}(X)$, there exists an isotropic subgroup $H_{S^{G_0}(X)} \subseteq A_T$ of order 2 associated to $S^{G_0}(X)$. By Lemma 4.4, the action on T_X has to be by roots of unity and preserves ω_X. Therefore we want to consider the following action on T :

$$\varphi := -\mathrm{id}_{2\times 2} \oplus \mathrm{id}_{1\times 1} \curvearrowright \mathbb{Z}L \oplus \mathrm{T}_X$$

that is the identity on L and $-\mathrm{id}$ on T_X. By computing $H_{S^{G_0}(X)}$ in A_T, we have to check that $\overline{\varphi}\left(H_{S^{G_0}(X)}\right) = H_{S^{G_0}(X)}$ and so we can use Proposition 4.13 to extend this automorphism to $S^{G_0}(X)$[1]. We still denote by φ the extension of φ to $S^{G_0}(X)$. We

[1] Notice that if $S^{G_0}(X) = \mathbb{Z}L \oplus \mathrm{T}_X$ then we can define φ directly on $S^{G_0}(X)$. For example this is the case for the point 1. in Proposition 4.10

can then check that L is fixed by the action of $\langle G_0, \varphi\rangle$ on $S^{G_0}(X)$.

Then we want to extend the automorphism φ from $S^{G_0}(X)$ to $H^2(X,\mathbb{Z})$ using [42, Proposition 1.5.1]. In order to do this, we need to consider a gluing morphism

$$\gamma : H_L \longrightarrow A_{S^{G_0}(X)}$$

where H_L is a subgroup of order 81 of $A_{S_{G_0}(X)} = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$. In particular we have that $H_L = A_{S_{G_0}}(X)$. Since the morphism $O\left(S_{G_0}(X)\right) \to O\left(A_{S_{G_0}(X)}\right)$ is surjective (see [25, Section 7 & Table 9]), different choices of γ produce isomorphic embeddings of $S^{G_0}(X)$ into $H^2(X,\mathbb{Z})$ such that the orthogonal lattice is $S_{G_0}(X)$. With the gluing morphism γ we can use Proposition 4.13. For this we need to check if we have

$$\gamma \circ \overline{\psi} = \overline{\varphi} \circ \gamma.$$

If φ is extendable, then we can consider $G = \langle G_0, \varphi\rangle \subset O\left(H^2(X,\mathbb{Z})\right)$. By the Torelli Theorem for IHS manifolds of K3$^{[2]}$-type, we can identify G with a subgroup of $\mathrm{Aut}(X)$ (in this case the Torelli Theorem is similar to the one we saw for K3 surfaces). Moreover the morphism $\nu : \mathrm{Aut}(X) \to O\left(H^2(X,\mathbb{Z})\right)$ is injective. Thus we can consider G acting on X with G_0 the symplectic part and $G/G_0 \simeq \langle\varphi\rangle$ by construction and Proposition 4.4.

We apply this method for each case of Propositions 4.10 and 4.11 checking that the respective automorphism exists and thus proving that each case occurs. The computations for the proof have been done with MAGMA and using part of codes contained in [25, 53]. We give here the details of the computations for the case $L^2 = 6$ with $L = e - f$. The other cases are similar and we give in Table 4.2 at the end of the proof the necessary informations to compute them.

When $L = e - f$, the transcendental lattice T_X is spanned by $e + f$ and h. Let $T = \mathbb{Z}L \oplus \mathrm{T}_X$, then we have

$$S^{G_0}(X)/T = \langle f + T\rangle = \langle e + T\rangle$$

which is a group of order 2 with $e + f \in T$. Moreover

$$A_T = \left\langle \frac{e+f}{18} + T,\ \frac{f}{3} + T,\ \frac{h}{6} + T \right\rangle.$$

Then, by Proposition 4.12, there exists a subgroup $H_{S^{G_0}(X)}$ of A_T which can be easily seen as

$$H_{S^{G_0}(X)} = \left\langle \frac{e+f}{2} + T \right\rangle .$$

Let us consider the following action on T :

$$\varphi := -\mathrm{id}_{2\times 2} \oplus \mathrm{id}_{1\times 1} \curvearrowright \mathbb{Z}L \oplus \mathrm{T}_X$$

and observe that $\overline{\varphi}\left(\frac{e+f}{2}\right) = -\frac{e+f}{2}$. Since $e+f \in T$ we have that $\overline{\varphi}\left(H_{S^{G_0}(X)}\right) = H_{S^{G_0}(X)}$ and thus by Lemma 4.13 we can extend φ to $S^{G_0}(X)$. Then φ is the isometry of $S^{G_0}(X)$ fixing $L = e - f$ and represented by the matrix

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\{\overline{g_1}, \overline{g_2}, \overline{g_3}\}$ be a set of generators of $A_{S_{G_0}(X)}$. For the dual of $S^{G_0}(X)$ we take the basis $S^{G_0}(X)^\vee = \langle f_1, f_2, f_3 \rangle$ where

$$f_1 = \frac{1}{3}(2e - f - h),\ f_2 = \frac{-2e}{3} + f,\ f_3 = \frac{1}{9}(2e - f) - \frac{h}{6}$$

and such that their corresponding images by the quotient projection, denoted by $\overline{f_1}$, $\overline{f_2}$ and $\overline{f_3}$, span $A_{S^{G_0}(X)}$. To extend φ from $S^{G_0}(X)$ to $H^2(X, \mathbb{Z})$ we consider the gluing morphism $\gamma : A_{S_{G_0}(X)} \to A_{S^{G_0}(X)}$ represented as the following matrix thanks to the generators of $A_{S_{G_0}(X)}$:

$$\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Finally we need to find an isometry $\psi \in O\left(S_{G_0}(X)\right)$ such that

$$\gamma \circ \overline{\psi} = \overline{\varphi} \circ \gamma$$

where $\overline{\psi}$ is the corresponding morphism in $O(A_{S_{G_0}(X)})$. In other words, we can compute $\overline{\psi}$ as

$$\overline{\psi} = \gamma^{-1} \circ \overline{\varphi} \circ \gamma.$$

To do this we can compute explicitly the action of φ on the image of the generators by γ using their corresponding pre-image on $S^{G_0}(X)^\vee$. We have

$$\varphi(f_1) = \varphi\left(\frac{1}{3}(2e - f - h)\right) = \frac{1}{3}(e - 2f + h),$$
$$\varphi(f_2) = \varphi\left(\frac{-2}{3}e + f\right) = -e + \frac{2}{3}f,$$
$$\varphi(2f_3) = \varphi\left(\frac{1}{9}(4e - 2f) - \frac{4}{3}h\right) = \frac{2}{9}(e - 2f) + \frac{4}{3}h,$$

which implies that $\overline{\varphi}\left(\overline{f_1}\right) = 2\overline{f_1}$, $\overline{\varphi}\left(\overline{f_2}\right) = \overline{f_2} + 12\overline{f_3}$ and $\overline{\varphi}\left(2\overline{f_3}\right) = \overline{f_2} + 4\overline{f_3}$. Then for example, for the generator $\overline{g_1} \in A_{S_{G_0}(X)}$ we have

$$\overline{\psi}\left(\overline{g_1}\right) = \left(\gamma^{-1} \circ \overline{\varphi} \circ \gamma\right)\left(\overline{g_1}\right) = \left(\gamma^{-1} \circ \overline{\varphi}\right)\left(\overline{f_2}\right) = \gamma^{-1}\left(\overline{f_2} + 12\overline{f_3}\right) = \overline{g_1} + 6\overline{g_3}\,.$$

With similar computations we get for the two other generators of $A_{S_{G_0}(X)}$:

$$\overline{\psi}\left(\overline{g_2}\right) = 2\overline{g_2} \quad \text{and} \quad \overline{\psi}\left(\overline{g_3}\right) = \overline{g_1} + 2\overline{g_3}\,.$$

So $\overline{\psi}$ can be written as the following matrix acting on $A_{S_{G_0}(X)}$:

$$\overline{\psi} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 6 & 0 & 2 \end{pmatrix}.$$

Lastly, since the morphism $O\left(S_{G_0}(X)\right) \to O\left(A_{S_{G_0}(X)}\right)$ is surjective, there exists an isometry $\psi \in O\left(S_{G_0}(X)\right)$. Hence by Proposition 4.13 we can extend φ to $H^2(X,\mathbb{Z})$ such that it acts as ψ on $S_{G_0}(X)$. The existence of the respective IHS manifold has been explained before.

For the other cases, we give the respective isometries of $S^{G_0}(X)$ and gluing morphisms in the Table 4.2 below. $\square$

Polarization L	Generators of T_X	Isometry $\varphi \in O(S^{G_0}(X))$	Order of φ	Gluing morphism γ	Isometry $\overline{\psi} \in O(A_{S_{G_0}(X)})$
$L = h$	$\{e, f\}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	2	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 6 & 2 \end{pmatrix}$
$L = e - f$	$\{h, e + f\}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	2	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 6 & 0 & 2 \end{pmatrix}$
$L = e$	$\{h, e - 2f\}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	2	$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$
$L = e + f$	$\{e - f, h\}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	2	$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 6 & 4 \end{pmatrix}$
$L = 2e - f$	$\{f, h\}$	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	2	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 6 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$L = h$	$\{e, f\}$	$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3	$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 6 & 3 & 7 \end{pmatrix}$
$L = h$	$\{e, f\}$	$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	6	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 6 & 6 & 5 \end{pmatrix}$

Table 4.2: Isometries and gluing morphisms in all cases

APPENDIX A

MAGMA programs for Nikulin configurations

We present here the MAGMA programs used in Chapter 2. They were written mainly by Roulleau - and I really thank him to allow me to put them in my thesis - then modified by me and used in the case that interested us. Recall that $X = \mathrm{Km}(\mathcal{A})$ is a Kummer surface together with a polarization L such that $L^2 = 4t$ with t a positive integer.

A.1 Search of rational curves

This first program is known as *Vinsberg's algorithm* (cf. [52]). It allows us to find all the smooth rational curves on X of degree less than a given integer, called here SEARCHED-DEGREE. Note that the Gram matrix of $\mathrm{NS}(X)$ called INT corresponds to the case $L^2 \equiv 0 \mod 8$ in Theorem 2.3. If we want to use this program in the other case $L^2 \equiv 4 \mod 8$, we just have to change the first line and the first column of INT with the correct v_{4d} given in the Theorem.

```
function SmoothRationalCurves(SearchedDegree,tt)

LSquare:=4*tt;
Int:=Matrix(
[[-2+tt,-2,-1,-1,-2,-2,-1,-1,-1,0,0,-1,0,0,0,0,0],
[-2,-8,-4,-4,-4,-4,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1],
[-1,-4,-4,-2,-2,-2,-1,0,-1,-1,-1,0,0,0,-1,-1,-1],
[-1,-4,-2,-4,-2,-2,-1,-1,0,-1,-1,0,-1,-1,0,0,-1],
[-2,-4,-2,-2,-4,-2,-1,-1,-1,0,-1,-1,0,-1,0,-1,0],
[-2,-4,-2,-2,-2,-4,-1,-1,-1,-1,0,-1,-1,0,-1,0,0],
[-1,-1,-1,-1,-1,-1,-2,0,0,0,0,0,0,0,0,0,0],
[-1,-1,0,-1,-1,-1,0,-2,0,0,0,0,0,0,0,0,0],
```

```
[-1,-1,-1,0,-1,-1,0,0,-2,0,0,0,0,0,0,0,0],
[0,-1,-1,-1,0,-1,0,0,0,-2,0,0,0,0,0,0,0],
[0,-1,-1,-1,-1,0,0,0,0,0,-2,0,0,0,0,0,0],
[-1,-1,0,0,-1,-1,0,0,0,0,0,-2,0,0,0,0,0],
[0,-1,0,-1,0,-1,0,0,0,0,0,0,-2,0,0,0,0],
[0,-1,0,-1,-1,0,0,0,0,0,0,0,0,-2,0,0,0],
[0,-1,-1,0,0,-1,0,0,0,0,0,0,0,0,-2,0,0],
[0,-1,-1,0,-1,0,0,0,0,0,0,0,0,0,0,-2,0],
[0,-1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0,-2]]);

Pol:=[4,-1,0,0,0,0,-2,-2,-2,0,0,-2,0,0,0,0,0];

rank:=#Pol;
Pol:=Matrix(1,rank,Pol);
DegPol:=(Pol*Int*Transpose(Pol))[1][1];
Z:=IntegerRing();
M:=RModule(Z, rank);
h:=ElementToSequence(Pol);
SubModDegh:=sub<M | [DegPol*q : q in Basis(M)] cat [M!h]>;
NU:=NullspaceOfTranspose(Pol*Int);
Lambda:=SubModDegh meet sub< M | [M!q : q in Basis(NU)]>;
LAM:=Matrix([ElementToSequence(M!Basis(Lambda)[k]) : k in [1..rank-1]]);
Q:=-LAM*Int*Transpose(LAM);
Lat:=LatticeWithGram(Q);
ListShort:=ShortVectors(Lat, 2*DegPol^2 + DegPol*SearchedDegree^2);

DegPos:={2*DegPol^2 + DegPol*dd^2 : dd in [0..SearchedDegree]};
ListShort:=[ q : q in ListShort | q[2] in DegPos];
ListDeg:=[IntegerRing()|Sqrt((q[2]-2*DegPol^2 )/DegPol) : q in ListShort];
ListShort:=[<ListShort[jj][1],ListDeg[jj]> : jj in [1..#ListShort]];
ListShort:= &cat [[<q[1],q[2]>,<-q[1],q[2]>] : q in ListShort];
Pol:=ElementToSequence(Pol);
Pol2:=M!Pol;
MBasis:=[M!Basis(Lambda)[k] : k in [1..rank-1]];
ListShort:=[<qq[2]*Pol2 + &+[ElementToSequence(qq[1])[k]*MBasis[k] : k in [1..rank-1]],
qq[2]> :
qq in ListShort];
SubModDegM:=sub<M | [DegPol*q : q in Basis(M)]>;
ListShort:=[q : q in ListShort | q[1] in SubModDegM];
ListShort:=[<ElementToSequence(q[1]),q[2]> : q in ListShort];
ListShort:=[<[IntegerRing()|j/DegPol : j in q[1]],q[2]> : q in ListShort];
if #ListShort eq 0 then
return "No smooth rational curves founded", SearchedDegree;
end if;

FinalList:=[];
while #ListShort gt 0 do
FinalListProv:=[];
deg:=ListShort[1][2];
repeat
Append(~FinalListProv,ListShort[1]);
Remove(~ListShort,1);
until ((#ListShort eq 0) or (not (ListShort[1][2] eq deg)));
FinalList:=FinalList cat FinalListProv;
while (#FinalListProv gt 0) do
ListShortProv:=[];
```

```
q:=FinalListProv[1];
Remove(~FinalListProv,1);
if (#ListShort gt 0) then
qu:=Int*Matrix(rank,1,q[1]);
for aq in ListShort do
if (((Matrix(1,rank,aq[1]))*qu)[1][1] ge 0) then
Append(~ListShortProv,aq);
end if;
end for;
end if;
ListShort:=ListShortProv;
end while;
end while;

NumberC:=&+[T^q[2] : q in FinalList];
return FinalList,NumberC,Int,Pol;
end function;
```

A.2 Find the Nikulin configurations

This second program search for Nikulin configurations from a list (called LL) of smooth rational curves found with the previous program.

```
function FindNikulinConf(Int,LL,Rang,Number,VV1);
PAS2:=Matrix([q[1] : q in LL[Rang]]);
VV:=VV1;
VVens:={{u : u in Q} :Q in VV};
no:=#Rang;
repeat
t1:=#VVens;
Dis:=[];
while #Dis ne 16 do
Dis:=[Random(1,no)];
LIND:={j : j in [1..no] | (Matrix(PAS2[Dis[1]])*Int*Transpose(Matrix(PAS2[j])))[1][1] eq 0};
while #LIND ne 0 do
J:=Random(LIND);
Dis:=Dis cat [J];
LIND:=LIND meet {kk : kk in [1..no] | (Matrix(PAS2[J])*Int*Transpose(Matrix(PAS2[kk])))
[1][1] eq 0};
end while;
end while;
VVens:= VVens join {{q : q in Dis}};
if #VVens gt t1 then t1:=#VVens;
NC:=Sort([q : q in Dis]);
Append(~VV,NC);
end if;
until t1 eq Number;
return Sort(VV);
end function;
```

A.3 Final test

Finally if the previous program find more than one Nikulin configuration, we use this last program. Taking two configurations Seq1 and Seq2 among the list LL found before, it will try to find an automorphism sending Seq1 on Seq2. In order to simplify we use the following three auxiliary functions :

- OrderKummer : take a list of 16 disjoint (-2)-curves and return a new list which is ordered by block of 4 curves such that $(L + \text{block})$ is 2-divisible.
- OrthoComplKummer : return the orthogonal complement of 16 divisors on X.
- IsMatrixIntegers : return True if the given matrix is an integer matrix.

As the number of automorphims can be very big, the program is written by considering Remark 2.2 in order to minimize the time of computations. As we want to find two non-isomorphic Nikulin configurations, we hope that the program say '0' at the end : this will mean that it does not exist such automorphim.

```
function IsThereAutoKummer(Int,LL,Pol,Seq1,Seq2,boul)
bo:=0;
PM1:=[];
Se1:=OrderKummer(Int,LL,Pol,Seq1);
Se2:=OrderKummer(Int,LL,Pol,Seq2);
LL1:=LL[Se1]; LL1:=[q[1] : q in LL1];
LL2:=LL[Se2]; LL2:=[q[1] : q in LL2];
Ort1:=(Matrix([OrthoComplKummer(Int,LL,Pol,Seq1)]));
Ort2:=(Matrix([OrthoComplKummer(Int,LL,Pol,Seq2)]));
tto:=(Ort1*Int*Transpose(Ort1))[1,1];
PAS1:=Matrix(LL1);
PAS1:=ChangeRing(Transpose(VerticalJoin(Ort1,PAS1)),Rationals());
PAS1I:=PAS1^-1;
IntI:=ChangeRing(Int,Rationals())^-1;
qt:=LCM([Denominator(q): q in ElementToSequence(IntI)]);
Discri:=ChangeRing(qt*IntI,Integers(qt));
G4P:=[
    [ 1, 2, 3, 4 ],    [ 1, 4, 2, 3 ],    [ 1, 3, 4, 2 ],    [ 1, 3, 2, 4 ],
    [ 1, 2, 4, 3 ],    [ 1, 4, 3, 2 ],    [ 2, 1, 3, 4 ],    [ 2, 4, 1, 3 ],
    [ 2, 3, 4, 1 ],    [ 2, 3, 1, 4 ],    [ 2, 1, 4, 3 ],    [ 2, 4, 3, 1 ],
    [ 3, 2, 1, 4 ],    [ 3, 4, 2, 1 ],    [ 3, 1, 4, 2 ],    [ 3, 1, 2, 4 ],
    [ 3, 2, 4, 1 ],    [ 3, 4, 1, 2 ],    [ 4, 2, 3, 1 ],    [ 4, 1, 2, 3 ],
    [ 4, 3, 1, 2 ],    [ 4, 3, 2, 1 ],    [ 4, 2, 1, 3 ],    [ 4, 1, 3, 2 ]];
W:=[[1..4],[5..8],[9..12],[13..16]];
W:=[Se2[q]: q in W];
for ord in G4P do
S2:=[W[z] : z in ord];
for g1 in G4P do
for g2 in G4P do
for g3 in G4P do
for g4 in G4P do
G:=[g1,g2,g3,g4];
```

```
Se2a:=S2;
Se2a:=[[Se2a[w][k] : k in G[w]]:w in [1..4]];
Se2a:=&cat Se2a;
LL2:=LL[Se2a];
LL2:=[q[1] : q in LL2];
PAS2:=Matrix(LL2);
PAS2:=ChangeRing(Transpose(VerticalJoin(Ort2,PAS2)),Rationals());
if IsMatrixIntegers(PAS2*PAS1I) then
AU:=PAS2*PAS1I;
if IsMatrixIntegers(AU*IntI - IntI) or IsMatrixIntegers(AU*IntI + IntI) then
Append(~PM1,ChangeRing(AU,Integers()));
if bo eq 0 then
"There is an automorphism"; bo:=1;
if boul eq 1 then return 1,PM1,Se1;
end if;end if; end if; end if;
end for; end for; end for; end for; end for;
if #PM1 eq 0 and boul eq 1 then return 0;
end if;
return PM1,Se1,Se2;
end function;
```

APPENDIX B

Quadrics for $X \subset \mathbb{P}^7$

The surface X of Section 3.4 is described as the intersection of 10 quadrics in $\mathbb{P}^7$. We detail here the equations of the quadrics $F_i(x_1, \dots, x_8)$, with $i \in \{1, \dots, 10\}$, obtained by computations by MAGMA using a program by Bonnafé. In what follows, a is a primitive root of unity of order 20.

$$\begin{aligned}
F_1 :=\ & \tfrac{1}{15}(-736a^7 + 528a^6 - 352a^5 - 528a^4 - 736a^3 - 304)x_1x_7 + \tfrac{1}{15}(736a^7 + 272a^6 + 192a^5 - 272a^4 + 736a^3 - 576)x_1x_8 + \\
& \tfrac{1}{15}(64a^7 + 208a^6 + 368a^5 - 208a^4 + 64a^3 - 64)x_2x_7 + \tfrac{1}{15}(-16a^7 - 192a^6 - 112a^5 + 192a^4 - 16a^3 + 416)x_2x_8 + \\
& \tfrac{1}{15}(-176a^7 - 32a^6 - 112a^5 + 32a^4 - 176a^3 + 176)x_3x_5 + \tfrac{1}{5}(-32a^7 + 16a^6 - 64a^5 - 16a^4 - 32a^3 - 128)x_3x_6 + \\
& \tfrac{1}{5}(48a^7 + 96a^6 + 96a^5 - 96a^4 + 48a^3 - 128)x_4x_5 + \tfrac{1}{5}(48a^7 - 64a^6 + 16a^5 + 64a^4 + 48a^3 + 352)x_4x_6,
\end{aligned}$$

$$\begin{aligned}
F_2 :=\ & (-224a^7 - 144a^6 - 160a^5 + 144a^4 - 224a^3 + 256)x_1x_7 + (32a^7 + 240a^6 + 16a^5 - 240a^4 + 32a^3 - 448)x_1x_8 + \\
& (16a^7 + 96a^6 + 16a^5 - 96a^4 + 16a^3 - 128)x_2x_7 + (64a^7 - 80a^6 + 32a^5 + 80a^4 + 64a^3 + 112)x_2x_8 + \\
& (-16a^7 - 64a^6 - 16a^5 + 64a^4 - 16a^3 + 128)x_3x_5 + (-96a^7 - 16a^6 - 48a^5 + 16a^4 - 96a^3)x_3x_6 + \\
& (-32a^7 + 112a^6 - 112a^4 - 32a^3 - 192)x_4x_5 + (192a^7 - 48a^6 + 96a^5 + 48a^4 + 192a^3 + 144)x_4x_6,
\end{aligned}$$

$$\begin{aligned}
F_3 :=\ & (-96a^7 - 16a^6 - 48a^5 + 16a^4 - 96a^3 + 32)x_1x_5 + \tfrac{1}{3}(-32a^7 + 144a^6 - 80a^5 - 144a^4 - 32a^3 - 176)x_1x_6 + \\
& \tfrac{1}{3}(32a^7 + 112a^6 - 112a^4 + 32a^3 - 240)x_2x_5 + \tfrac{1}{3}(80a^7 + 32a^6 + 64a^5 - 32a^4 + 80a^3 + 16)x_2x_6 + \\
& (-16a^7 - 16a^3)x_3x_7 + \tfrac{1}{3}(32a^7 + 48a^6 + 80a^5 - 48a^4 + 32a^3 - 16)x_3x_8 + (-16a^7 - 16a^5 - 16a^3 + 16)x_4x_7 + \\
& (16a^7 + 32a^6 - 32a^4 + 16a^3 + 16)x_4x_8,
\end{aligned}$$

$$\begin{aligned}
F_4 :=\ & \tfrac{1}{3}(-112a^7 - 256a^6 - 32a^5 + 256a^4 - 112a^3 + 400)x_1x_5 + \tfrac{1}{3}(-144a^7 + 64a^6 - 32a^5 - 64a^4 - 144a^3 - 32)x_1x_6 + \\
& \tfrac{1}{3}(-96a^7 + 112a^6 - 80a^5 - 112a^4 - 96a^3 - 176)x_2x_5 + \tfrac{1}{3}(48a^7 + 32a^6 + 80a^5 - 32a^4 + 48a^3 - 112)x_2x_6 + \\
& \tfrac{1}{3}(-16a^6 - 16a^5 + 16a^4 + 80)x_3x_7 + \tfrac{1}{3}(16a^7 + 32a^6 - 32a^4 + 16a^3 - 96)x_3x_8 + (-16a^7 - 32a^6 + 32a^4 - 16a^3 + 80)x_4x_7 + \\
& (-16a^7 + 32a^6 + 16a^5 - 32a^4 - 16a^3 - 48)x_4x_8,
\end{aligned}$$

$$
\begin{aligned}
F_5 :=&(32a^7 - 160a^6 + 160a^4 + 32a^3 + 192)x_1x_4 + 32x_2x_3 + (-32a^7 + 32a^6 - 64a^5 - 32a^4 - 32a^3 - 32)x_2x_4+\\
&(-64a^6 - 16a^5 + 64a^4 + 128)x_5^2 + (-96a^7 + 32a^6 - 64a^5 - 32a^4 - 96a^3 - 64)x_5x_6 + (64a^6 + 16a^5 - 64a^4 - 32)x_6^2+\\
&(32a^7 - 32a^6 + 16a^5 + 32a^4 + 32a^3 + 32)x_7^2 + (-96a^7 - 32a^6 - 64a^5 + 32a^4 - 96a^3)x_7x_8+\\
&(32a^7 + 32a^6 + 48a^5 - 32a^4 + 32a^3 - 64)x_8^2,
\end{aligned}
$$

$$
\begin{aligned}
F_6 :=&32x_1x_3 + (32a^7 - 32a^6 + 64a^5 + 32a^4 + 32a^3 + 32)x_1x_4 + (-32a^7 - 32a^6 + 32a^4 - 32a^3)x_2x_4+\\
&(16a^7 - 48a^6 + 48a^4 + 16a^3 + 64)x_5^2 + (-64a^7 - 32a^5 - 64a^3)x_5x_6 + (16a^7 + 16a^6 - 32a^5 - 16a^4 + 16a^3 - 32)x_6^2+\\
&(16a^7 + 16a^6 - 16a^4 + 16a^3)x_7^2 + (-32a^5 + 64)x_7x_8 + (16a^7 + 16a^6 - 32a^5 - 16a^4 + 16a^3 - 32)x_8^2,
\end{aligned}
$$

$$
\begin{aligned}
F_7 :=&\tfrac{1}{5}(-304a^7 + 352a^6 - 128a^5 - 352a^4 - 304a^3 - 496)x_1^2 + (64a^7 + 32a^5 + 64a^3 + 32)x_1x_2 + (-16a^7 - 16a^3 + 16)x_2^2+\\
&\tfrac{1}{5}(-48a^7 - 96a^6 - 96a^5 + 96a^4 - 48a^3 + 48)x_3^2 + \tfrac{1}{5}(-656a^7 + 128a^6 + 128a^5 - 128a^4 - 656a^3 + 16)x_4^2+\\
&\tfrac{1}{5}(608a^7 - 64a^6 + 256a^5 + 64a^4 + 608a^3 + 672)x_5x_7 + \tfrac{1}{5}(-384a^7 - 288a^6 - 448a^5 + 288a^4 - 384a^3 + 384)x_5x_8+\\
&\tfrac{1}{5}(-192a^7 - 544a^6 - 64a^5 + 544a^4 - 192a^3 + 512)x_6x_7 + \tfrac{1}{5}(-224a^7 + 192a^6 + 192a^5 - 192a^4 - 224a^3 - 736)x_6x_8,
\end{aligned}
$$

$$
\begin{aligned}
F_8 :=&\tfrac{1}{5}(-256a^7 + 368a^6 - 32a^5 - 368a^4 - 256a^3 - 704)x_1^2 + \tfrac{1}{5}(288a^7 - 224a^6 + 96a^5 + 224a^4 + 288a^3 + 192)x_1x_2+\\
&\tfrac{1}{5}(-32a^7 + 16a^6 - 64a^5 - 16a^4 - 32a^3 + 32)x_2^2 + \tfrac{1}{5}(-128a^7 - 16a^6 + 64a^5 + 16a^4 - 128a^3 - 32)x_3^2+\\
&\tfrac{1}{5}(32a^7 - 96a^6 - 256a^5 + 96a^4 + 32a^3 - 32)x_3x_4 + \tfrac{1}{5}(-224a^7 + 272a^6 - 448a^5 - 272a^4 - 224a^3 - 416)x_4^2+\\
&\tfrac{1}{5}(352a^7 - 576a^6 + 224a^5 + 576a^4 + 352a^3 + 768)x_5x_7 + \tfrac{1}{5}(-608a^7 + 64a^6 - 256a^5 - 64a^4 - 608a^3 + 128)x_5x_8+\\
&\tfrac{1}{5}(-448a^7 - 96a^6 - 96a^5 + 96a^4 - 448a^3 + 448)x_6x_7 + \tfrac{1}{5}(192a^7 + 544a^6 + 64a^5 - 544a^4 + 192a^3 - 512)x_6x_8,
\end{aligned}
$$

$$
\begin{aligned}
F_9 :=&\tfrac{1}{9}(-160a^7 - 352a^5 - 160a^3 - 16)x_1^2 + (32a^7 + 32a^6 - 32a^4 + 32a^3 - 32)x_1x_2 + 16x_2^2 + \tfrac{1}{9}(-32a^6 - 32a^5 + 32a^4 + 16)x_3^2+\\
&80x_4^2 + \tfrac{1}{9}(128a^7 - 64a^6 - 128a^5 + 64a^4 + 128a^3 + 160)x_5x_7 + \tfrac{1}{9}(-32a^7 + 96a^6 - 32a^5 - 96a^4 - 32a^3 - 224)x_5x_8+\\
&\tfrac{1}{9}(-32a^7 + 32a^6 - 96a^5 - 32a^4 - 32a^3 + 96)x_6x_7 + \tfrac{1}{9}(128a^7 - 64a^5 + 128a^3 - 160)x_6x_8,
\end{aligned}
$$

$$
\begin{aligned}
F_{10} :=&(-96a^7 + 96a^6 - 64a^5 - 96a^4 - 96a^3 - 144)x_1^2 + (96a^7 - 32a^6 + 64a^5 + 32a^4 + 96a^3 - 32)x_1x_2 + (-32a^6 - 32a^5 + 32a^4 + 16)x_2^2\\
&+ 80x_3^2 + (-32a^6 + 32a^5 + 32a^4 + 16)x_4^2 - 32x_5x_7 + (32a^7 - 32a^6 - 32a^5 + 32a^4 + 32a^3 + 32)x_5x_8+\\
&(32a^7 + 32a^6 + 32a^5 - 32a^4 + 32a^3 - 32)x_6x_7 + (-64a^6 - 64a^5 + 64a^4 + 32)x_6x_8
\end{aligned}
$$

APPENDIX C

Table of T_X

We recall the notations of Table C.1. Let X be a K3 surface admitting a faithful and symplectic action of the Mathieu group M_{20} and let T_X be its transcendental lattice, with $\mathrm{T}_X = \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$ and a, b, c as in Section 3.3, satisfying the conditions given in ($\star$). We denote by $L \in \mathrm{NS}(X)$ a M_{20}-invariant ample class such that $L^2 = 4n$, with $n \in \mathbb{Z}_{\geq 1}$ and by Q_{4n} the number of quadrics describing the projective model, according to Theorem 3.2. Finally, $I = \sqrt{\frac{160n}{4ac-b^2}}$ is the index of $\mathbb{Z}L \oplus \mathbb{L}_{20}^{\perp}$ in $\mathrm{NS}(X)$. When $n = 1, 2, 10$, the surfaces are described in Proposition 3.1.

n	L^2	Q_{L^2}	T_X	a	b	c	$\langle 4n \rangle \hookrightarrow \mathbb{L}_{20}$	$I = \sqrt{\frac{160n}{4ac-b^2}}$
1	4	-	$\begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$	1	0	10	$L \mapsto e$	2
2	8	3	$\begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$	2	2	3	$L \mapsto e+f$	4
3	12	10	$\begin{pmatrix} 8 & 0 \\ 0 & 60 \end{pmatrix}$	2	0	15	$L \mapsto h$	2
4	16	21	$\begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$	1	0	10	$L \mapsto 2e$	4
5	20	36	$\begin{pmatrix} 20 & 0 \\ 0 & 40 \end{pmatrix}$	5	0	10	$L \mapsto e+2f$ $L \mapsto f-h$	2
6	24	-	-	-	-	-	-	-
7	28	80	$\begin{pmatrix} 8 & 0 \\ 0 & 140 \end{pmatrix}$	2	0	35	$L \mapsto e+f-h$	4

n	L^2	Q_{L^2}	T_X	a	b	c	$\langle 4n\rangle \hookrightarrow \mathbb{L}_{20}$	$I=\sqrt{\frac{160n}{4ac-b^2}}$
8	32	105	$\begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$	2	2	3	$L \mapsto 2e+2f$	8
9	36	136	$\begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$	1	0	10	$L \mapsto 3e$	6
			$\begin{pmatrix} 36 & 12 \\ 12 & 44 \end{pmatrix}$	9	6	11	$L \mapsto 3e+h$	2
10	40	171	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	1	0	1	$L \mapsto e+f+2h$	20
			$\begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}$	5	0	5	$L \mapsto e+3f$	4
15	60	406	$\begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}$	2	0	3	$L \mapsto 2e+2f-h$	5
			$\begin{pmatrix} 20 & 0 \\ 0 & 120 \end{pmatrix}$	5	0	25	$L \mapsto e-2h$	1
18	72	595	$\begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$	2	2	3	$L \mapsto 3e+3f$	12
			$\begin{pmatrix} 8 & 4 \\ 4 & 92 \end{pmatrix}$	2	2	23	$L \mapsto 3e+3f+2h$	4
20	80	741	$\begin{pmatrix} 20 & 0 \\ 0 & 40 \end{pmatrix}$	5	0	10	$L \mapsto 2e+4f$ $L \mapsto 2f-2h$	4
30	120	1711	$\begin{pmatrix} 20 & 10 \\ 10 & 20 \end{pmatrix}$	5	5	5	$L \mapsto 3e+f-2h$	8
40	160	3081	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	1	0	1	$L \mapsto 2e+2f+4h$	40
			$\begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}$	5	0	5	$L \mapsto 2e+6f$	8
45	180	3916	$\begin{pmatrix} 20 & 0 \\ 0 & 40 \end{pmatrix}$	5	0	10	$L \mapsto 3f-3h$ $L \mapsto 3e+6f$	6
			$\begin{pmatrix} 20 & 0 \\ 0 & 360 \end{pmatrix}$	5	0	90	$L \mapsto 4e+6f+h$ $L \mapsto 3e+4f+4h$	2
90	360	15931	$\begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}$	5	0	5	$L \mapsto 3e+9f$	12
			$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	1	0	1	$L \mapsto 3e+3f+6h$	60
			$\begin{pmatrix} 20 & 0 \\ 0 & 180 \end{pmatrix}$	5	0	45	$L \mapsto 3e+7f-2h$	4
			$\begin{pmatrix} 8 & 4 \\ 4 & 20 \end{pmatrix}$	2	2	5	$L \mapsto 3e+3f-4h$	20

Table C.1: Table with all cases

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Résumé

Le but de cette thèse est d'étudier des actions de groupe sur certaines variétés Symplectiques Holomorphes Irréductibles (nous utiliserons l'acronyme anglais IHS). Dans un premier temps, nous nous intéresserons aux surfaces de Kummer qui sont des surfaces K3 spéciales. Une surface de Kummer est obtenue comme désingularisation du quotient d'une surface abélienne par une involution ayant 16 points fixes isolés. Une telle surface admet naturellement une action symplectique de $(\mathbb{Z}/2\mathbb{Z})^4$, signifiant que l'action des éléments de ce groupe sur la 2-forme est triviale. Il est bien connu que les surfaces de Kummer sont des surfaces K3 contenant 16 courbes rationnelles lisses disjointes, appelées *configuration de Nikulin*. Réciproquement, selon Nikulin l'existence d'une telle configuration sur une surface K3 signifie qu'il s'agit d'une surface de Kummer. La question de l'existence de surfaces abéliennes non-isomorphes donnant lieu aux mêmes surfaces de Kummer est une question importante en géométrie algébrique, qui a déjà été étudiée dans la littérature, mais la construction explicite de configurations de 16 courbes rationnelles non-isomorphes reste un problème ouvert. Celui-ci a été traité récemment par X. Roulleau et A. Sarti dans plusieurs travaux, et dans cette thèse nous complétons cette étude. Ensuite nous nous intéresserons à des surfaces K3 admettant une action symplectique du groupe de Mathieu M_{20}. Ce travail a été réalisé en collaboration avec Paola Comparin (Universidad de La Frontera, Chili). Il a été montré par Mukai que l'ordre maximal d'un groupe fini agissant fidèlement et symplectiquement sur une surface K3 est 960 et si un tel groupe est d'ordre 960, alors il est isomorphe à M_{20}. Nous montrons qu'il existe une infinité de surfaces K3 avec cette action. En outre l'un des objectifs de cette thèse est de toutes les décrire et de comprendre quand il est possible de construire explicitement un modèle projectif. Une généralisation naturelle de ces résultats concerne les variétés IHS qui sont équivalentes par déformation au schéma de Hilbert de deux points sur une surface K3, dites de *type K3*$^{[2]}$. Ce dernier point a été réalisé en collaboration avec Paola Comparin et Pablo Quezada Mora (Universidad de La Frontera, Chili). Dans un travail récent G. Höhn et G. Mason ont classifié les groupes qui peuvent agir symplectiquement sur une variété IHS de type K3$^{[2]}$, montrant que $\mathbb{Z}_3^4 : \mathfrak{A}_6$ est celui avec le plus grand ordre. Nous étudions les variétés IHS de type K3$^{[2]}$ pouvant admettre une telle action symplectique ainsi qu'un automorphisme non-symplectique. En particulier, nous prouvons qu'il existe une variété IHS de type K3$^{[2]}$ avec un groupe d'automorphismes fini d'ordre 174960 - qui est le plus grand ordre possible pour le groupe d'automorphismes d'une variété IHS de type K3$^{[2]}$ - et qu'il s'agit de la variété de Fano des droites de la cubique de Fermat.

Mots-clés : Géométrie algébrique, variétés symplectiques holomorphes irréductibles, surfaces K3, surfaces de Kummer, actions de groupes, automorphismes.

Abstract

The aim of this thesis is to study some group actions on some Irreducible Holomorphic Symplectic (IHS) manifolds. First of all we are interested in Kummer surfaces which are special K3 surfaces. A Kummer surface is obtained as the desingularization of the quotient of an abelian surface by an involution with 16 isolated fixed points. Such a surface naturally admits a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$, meaning that the elements of this group act trivially on the 2-form. It is well known that Kummer surfaces are K3 surfaces containing 16 disjoint smooth rational curves, called a *Nikulin configuration*. Conversely, according to Nikulin the existence of such a configuration on a K3 surface means that this is a Kummer surface. The question of the existence of non-isomorphic abelian surfaces giving rise to the same Kummer surface is an important question in algebraic geometry, which has already been studied in the literature, but the explicit construction of configurations of 16 non-isomorphic rational curves remains an open problem. This one was recently treated by X. Roulleau and A. Sarti in several works, and in this thesis we complete this study. In a second part we are interested in K3 surfaces addmitting a symplectic action of the Mathieu group M_{20}. This work is a joint work with Paola Comparin (Universidad de La Frontera, Chile). It was shown by Mukai that the maximum order of a finite group acting faithfully and symplectically on a K3 surface is 960 and if such a group has order 960, then it is isomorphic to M_{20}. We show that there exists infinitely many K3 surfaces with this action and we describe them and their projective models, giving some explicit examples. A natural way to generalize these results is to deal with IHS manifolds which are deformation equivalent to the Hilbert scheme of two points of a K3 surface, called of *K3*$^{[2]}$*-type*. This last topic is a joint work with Paola Comparin and Pablo Quezada Mora (Universidad de La Frontera, Chile). In a recent work G. Höhn and G. Mason classified the possible symplectic groups acting on an Irreducible Holomorphic Symplectic (IHS) manifold of K3$^{[2]}$-type, finding that $\mathbb{Z}_3^4 : \mathfrak{A}_6$ is the symplectic group with the biggest order. We study the possible IHS manifolds of K3$^{[2]}$-type with such a symplectic action and admitting also a non-symplectic automorphism. In particular we prove that there exists an IHS manifold of K3$^{[2]}$-type with finite automorphism group of order 174960 - which is the biggest possible order for the automorphism group of an IHS manifold of K3$^{[2]}$-type - and that it is the Fano variety of lines of the Fermat cubic fourfold.

Keywords : Algebraic geometry, irreducible holomorphic symplectic manifolds, K3 surfaces, Kummer surfaces, group actions, automorphisms.